Projective Transformations

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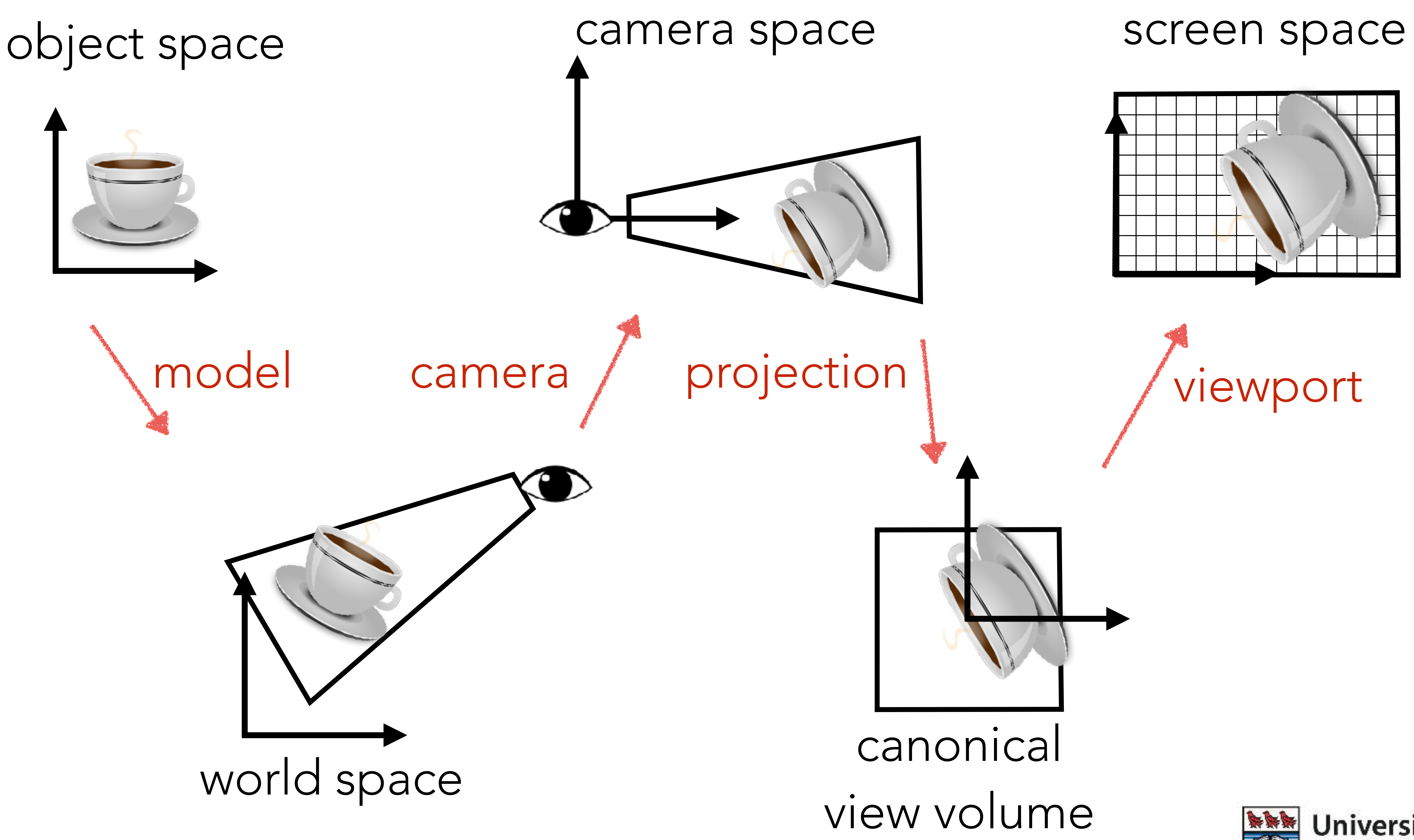
Viewing Transformation



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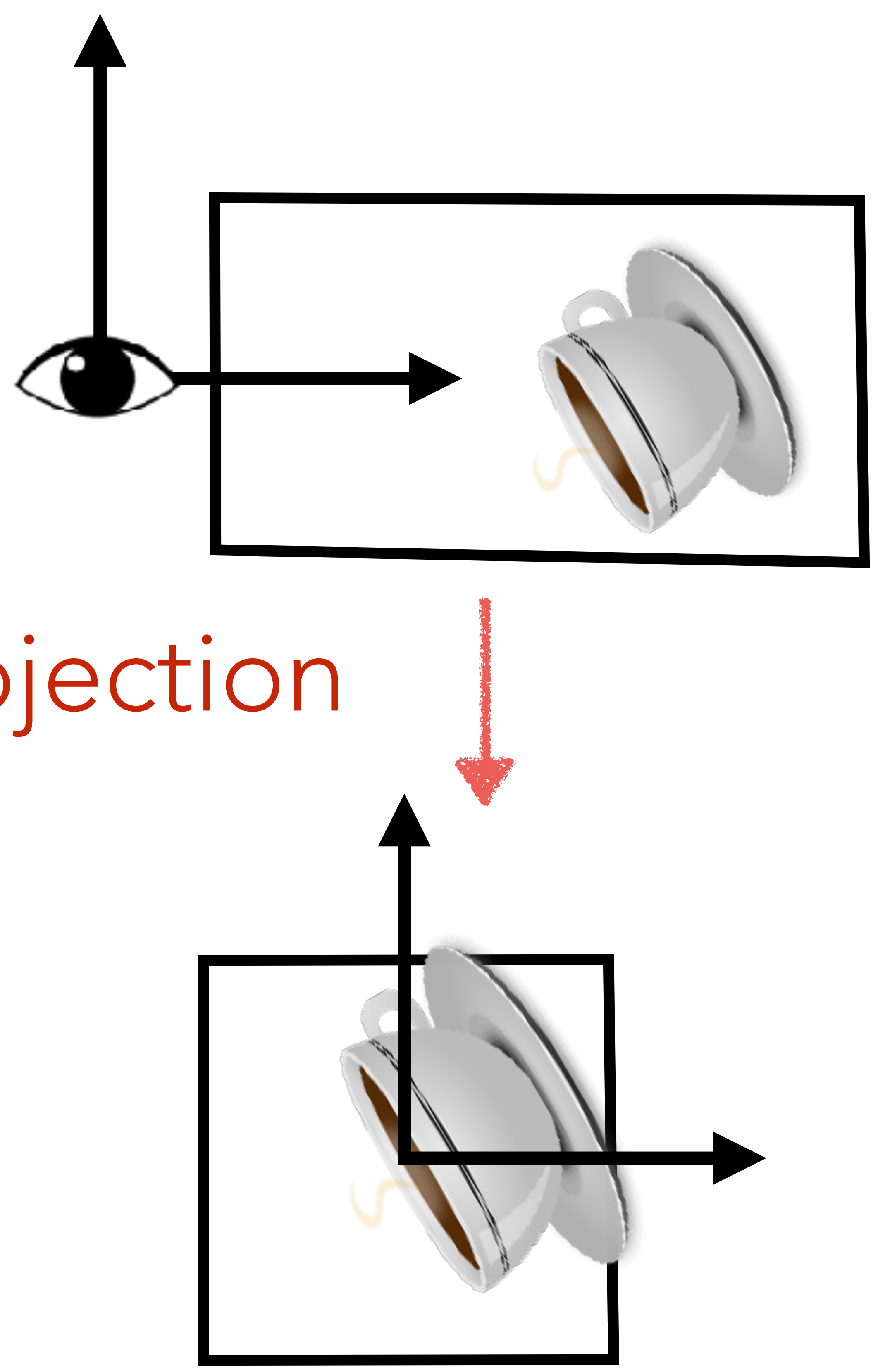
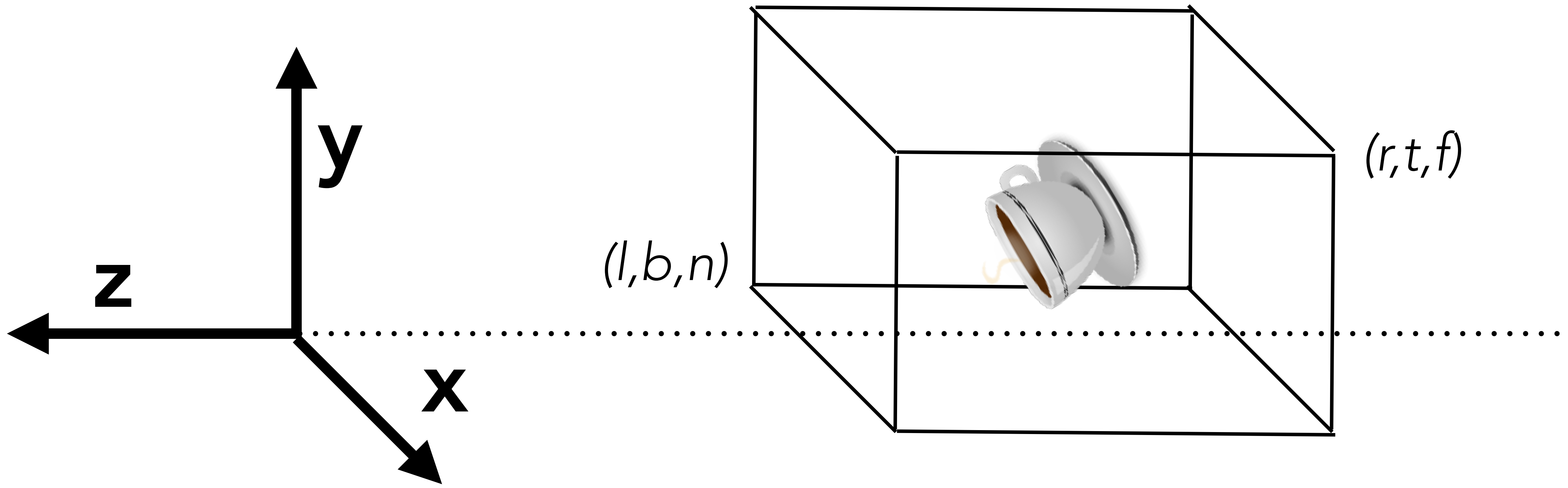
Orthographic Projection

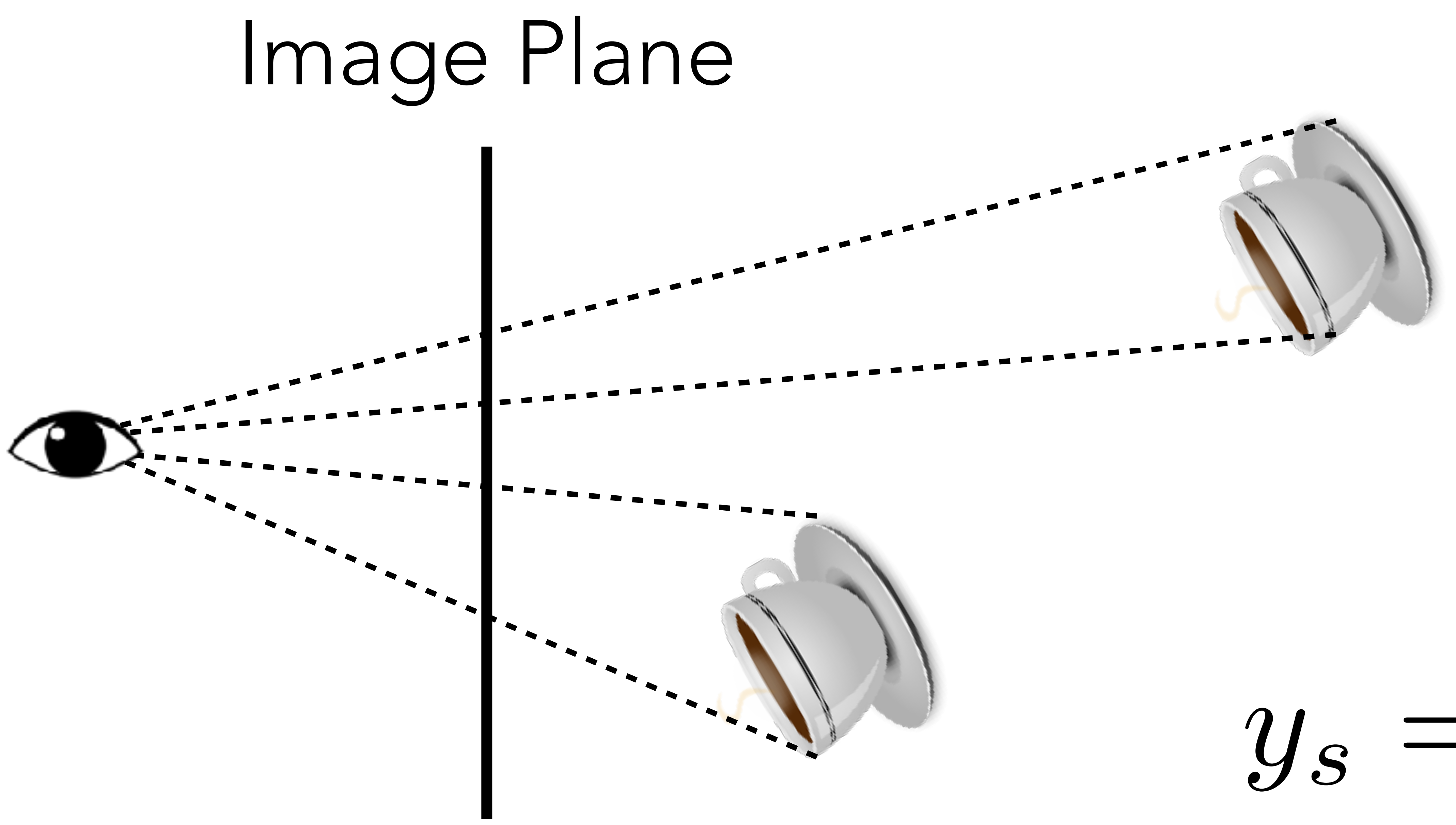




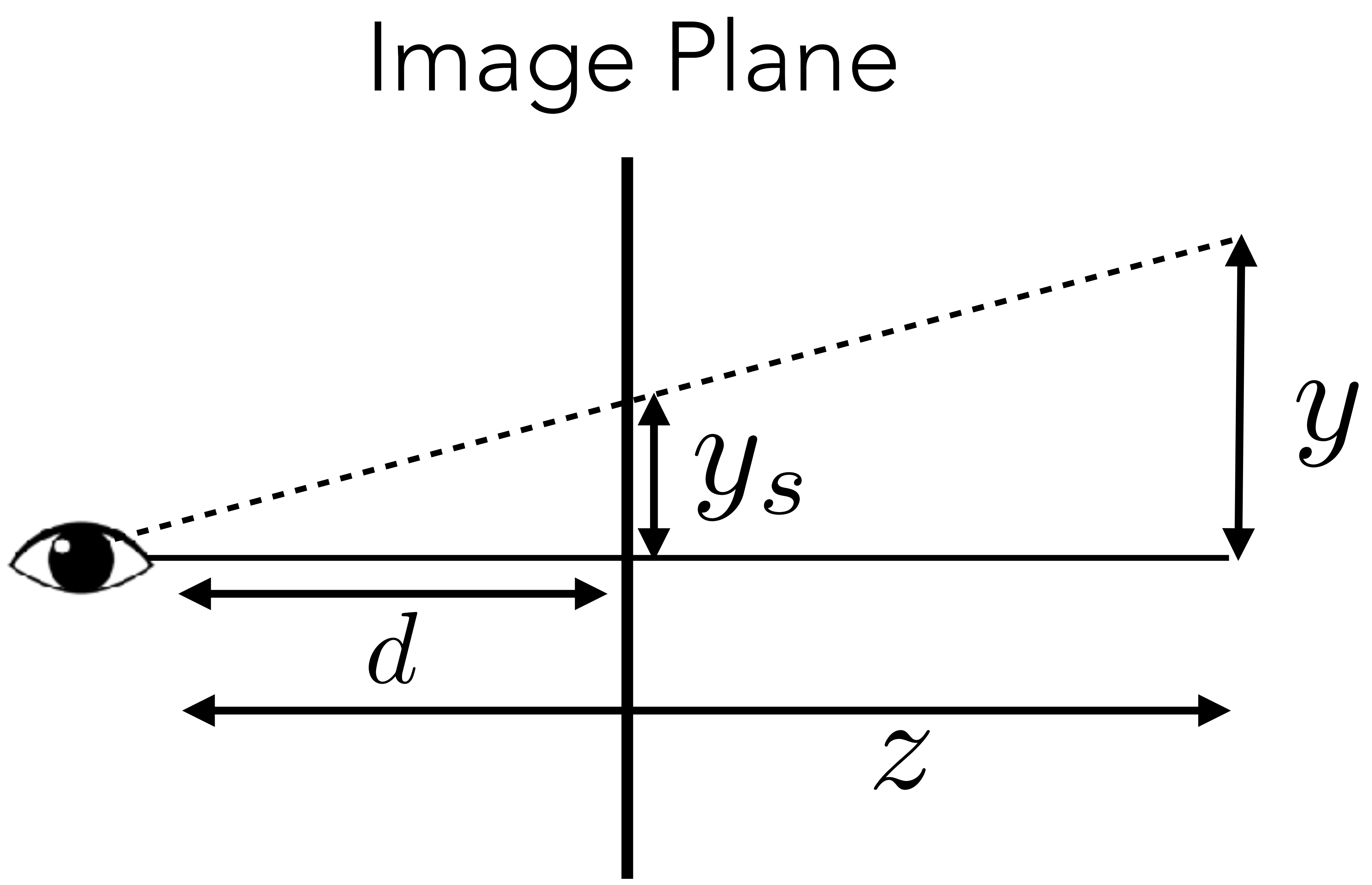
$$
\mathbf{M}_{orth} = \begin{bmatrix} \frac{2}{r-l} & 0 & 0 & -\frac{r+l}{r-l} \\ 0 & \frac{2}{t-b} & 0 & -\frac{t+b}{t-b} \\ 0 & 0 & \frac{2}{n-f} & -\frac{n+f}{n-f} \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Perspective Projection

- In Orthographic projection, the size of the objects does not change with distance
- In Perspective projection, the objects that are far away look smaller

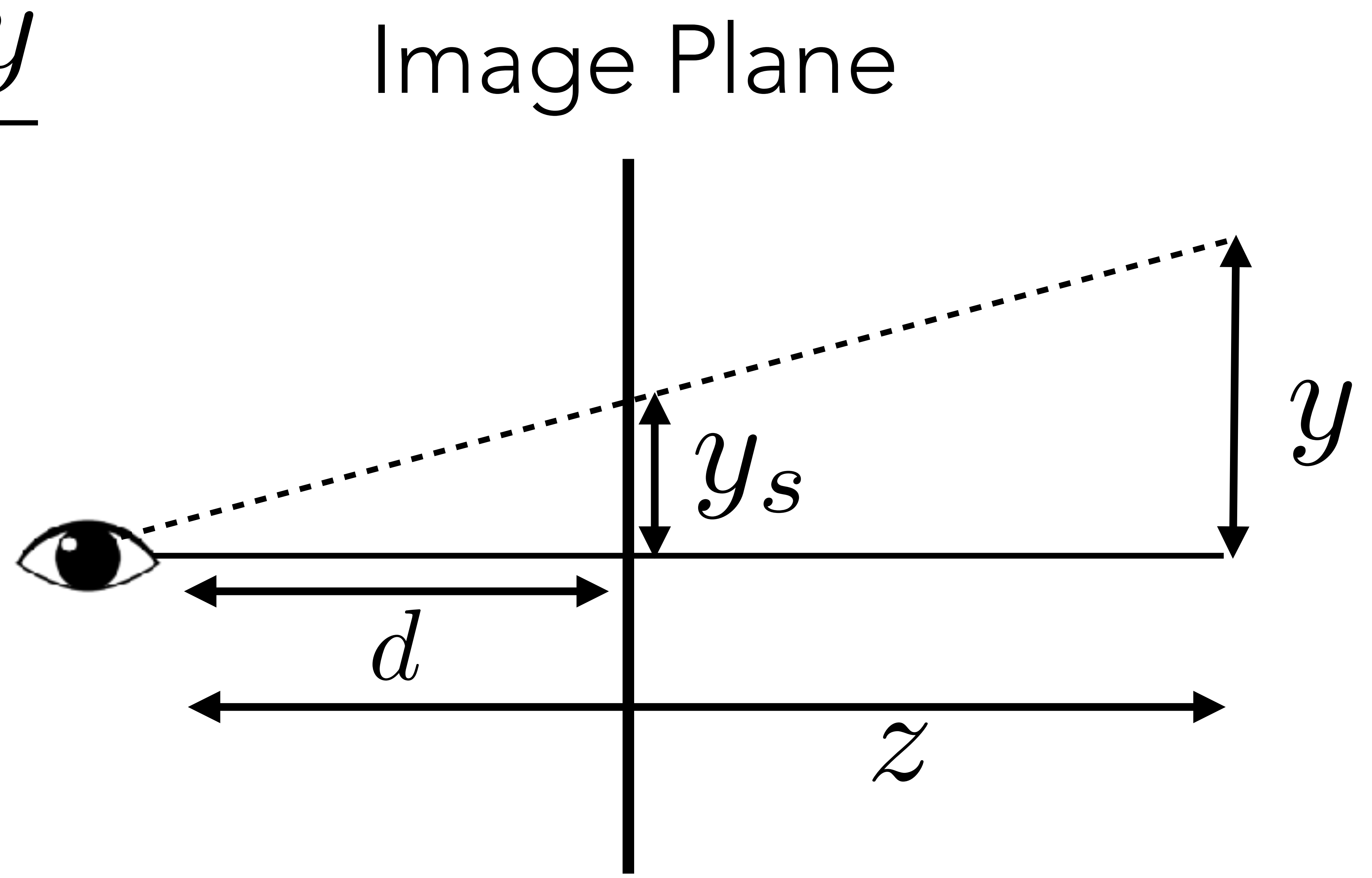


$$y_s = \frac{dy}{z}$$



Divisions in Matrix Form

- We would like to reuse the matrix machinery that we built in the previous lectures
- How do we encode divisions? $y_s = \dfrac{dy}{z}$
- We extend homogeneous coordinates



Until now...

- What do we have left?

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} a_1 x + b_1 y + c_1 \\ a_2 x + b_2 y + c_2 \\ 1 \end{pmatrix}$$

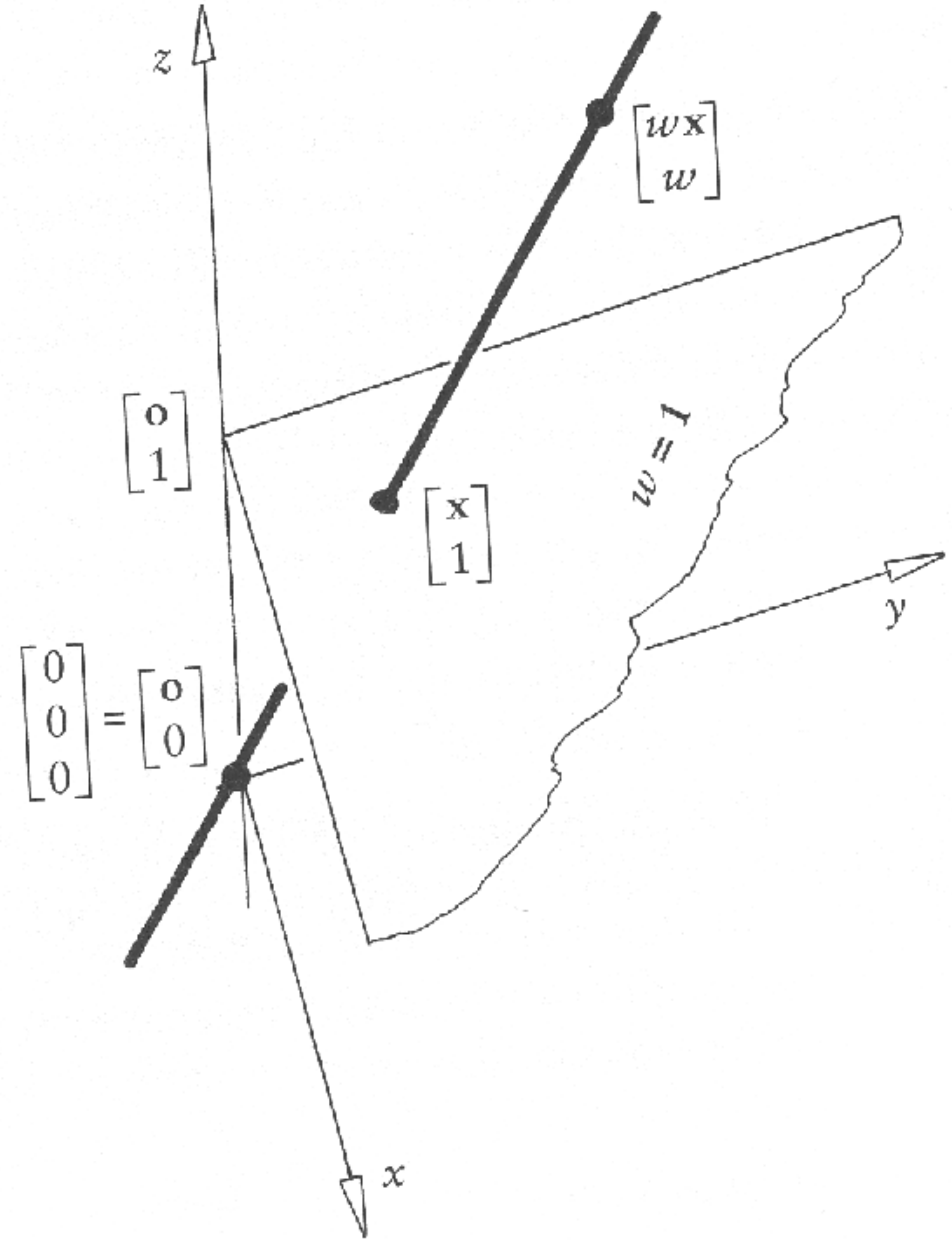
- We can use the last row of the transformation:

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ e & f & g \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} a_1 x + b_1 y + c_1 \\ a_2 x + b_2 y + c_2 \\ ex + fy + g \end{pmatrix} \sim \begin{pmatrix} \frac{a_1 x + b_1 y + c_1}{ex + fy + g} \\ \frac{a_2 x + b_2 y + c_2}{ex + fy + g} \\ 1 \end{pmatrix}$$

Intuition

- Purely algebraic:

$$\begin{pmatrix} x \\ y \\ w \end{pmatrix} \sim \begin{pmatrix} x/w \\ y/w \\ 1 \end{pmatrix}$$



- Or as a projection, where each line is identified by a point on the plane z=1
- Note that in this case, you can think of it as a transformation in a space with one more dimension

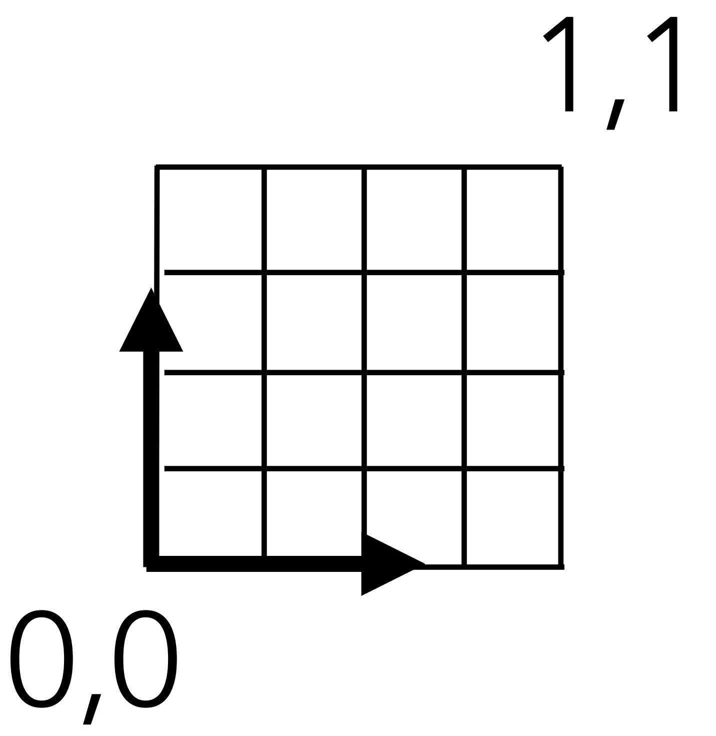
Projective Transformation

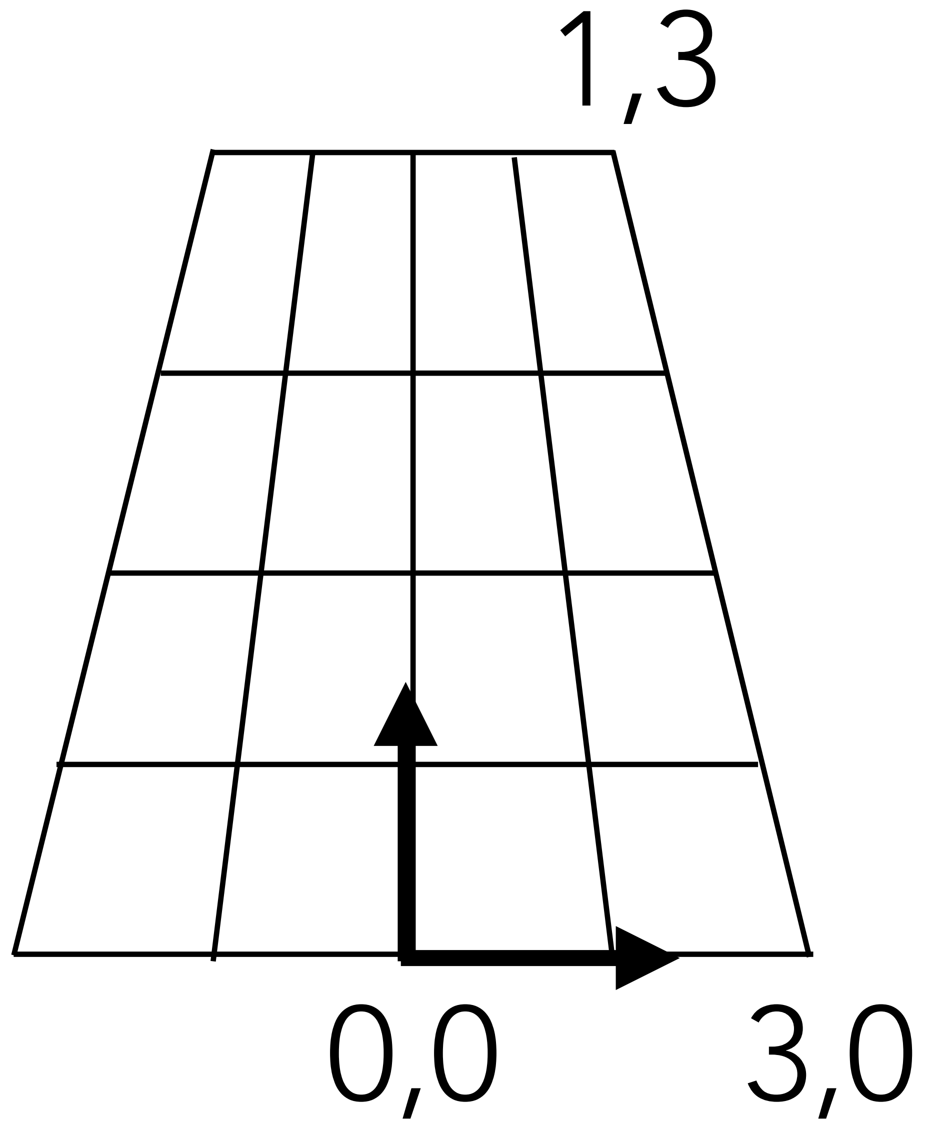
- A transformation of this form is called a *projective transformation* (or a homography)
- The points are represented in *homogeneous coordinates*

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ e & f & g \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} a_1 x + b_1 y + c_1 \\ a_2 x + b_2 y + c_2 \\ ex + fy + g \end{pmatrix} \sim \begin{pmatrix} \frac{a_1 x + b_1 y + c_1}{ex + fy + g} \\ \frac{a_2 x + b_2 y + c_2}{ex + fy + g} \\ 1 \end{pmatrix}$$

Example

$$\mathbf{M} = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 3 & 0 \\ 0 & 2/3 & 1/3 \end{pmatrix}$$





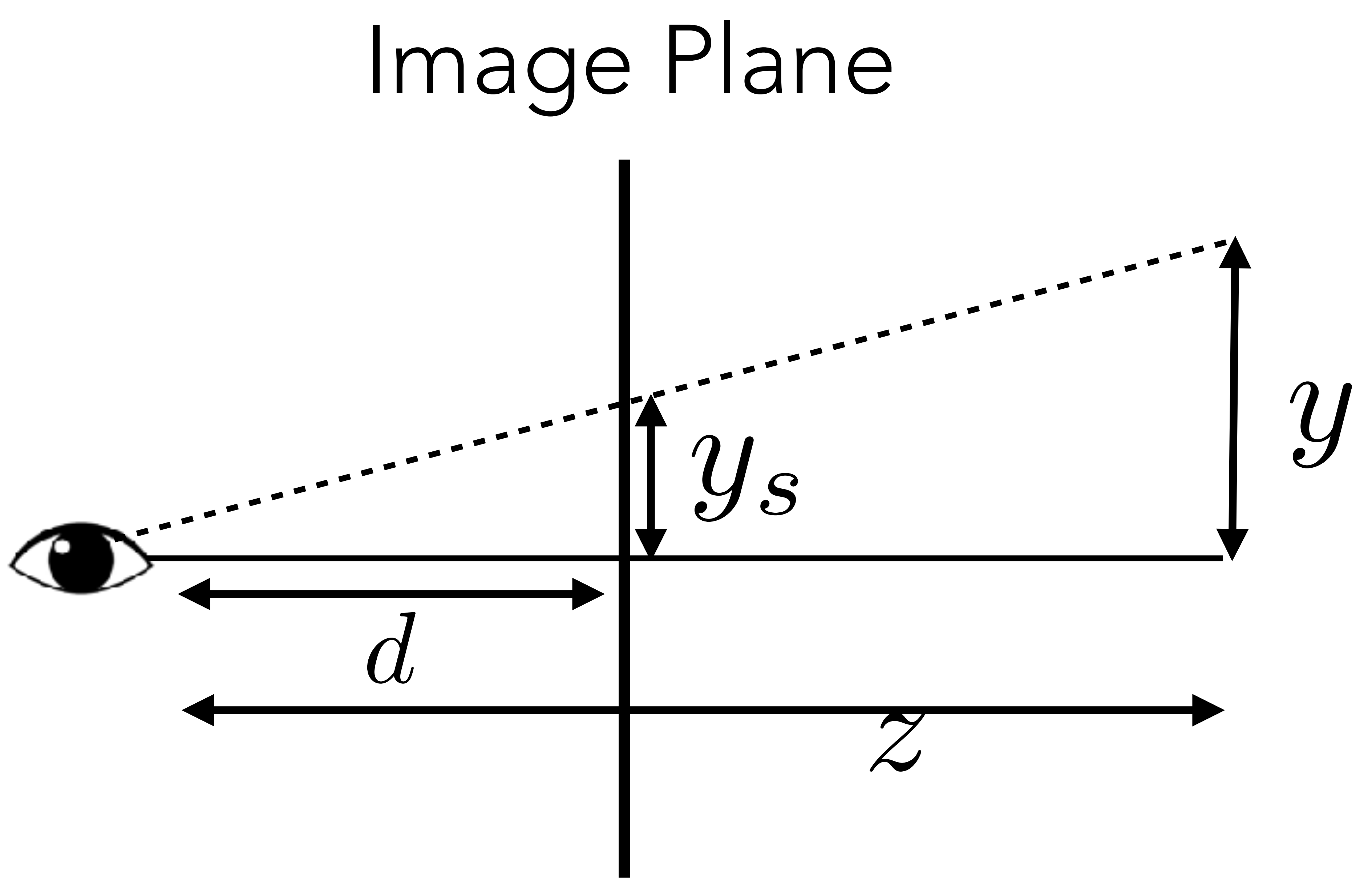
- It transforms a square into a quadrilateral — note that straight lines are preserved, but parallel lines are not!
- Note that you can use homogeneous coordinates for as many transformations as you want, only when you need the cartesian representation you have to normalize

Perspective Projection

- Perspective projection is easily implementable using this machinery

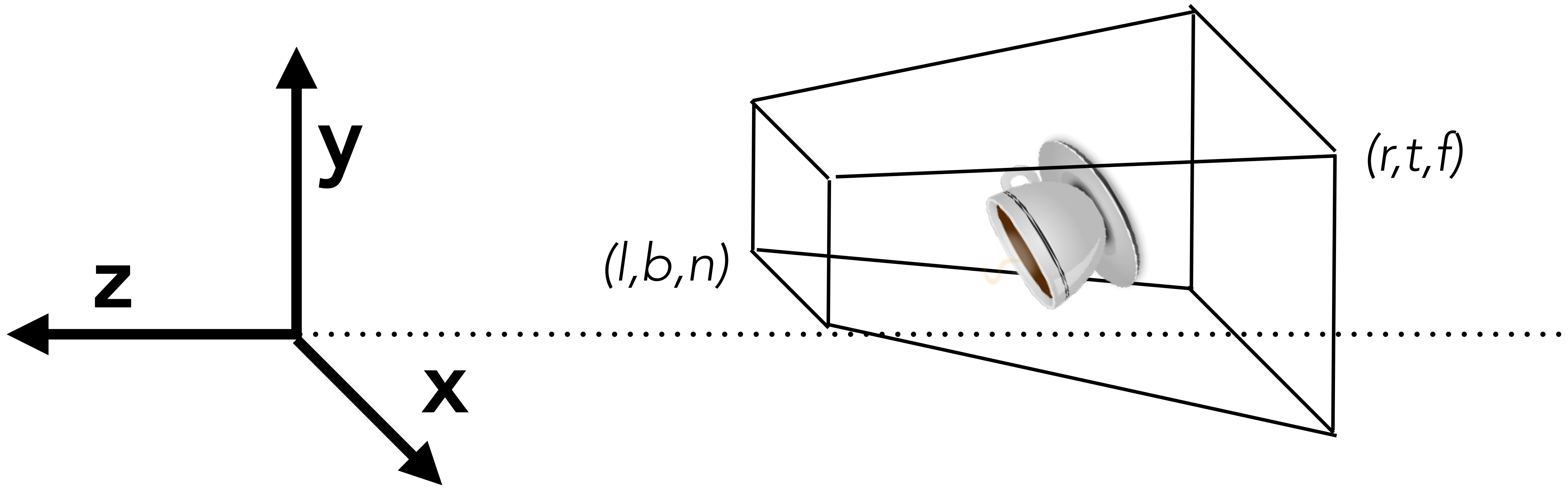
$$y_s = \frac{dy}{z}$$

$$\begin{pmatrix} y_s \\ 1 \end{pmatrix} \sim \begin{pmatrix} d & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} y \\ z \\ 1 \end{pmatrix}$$



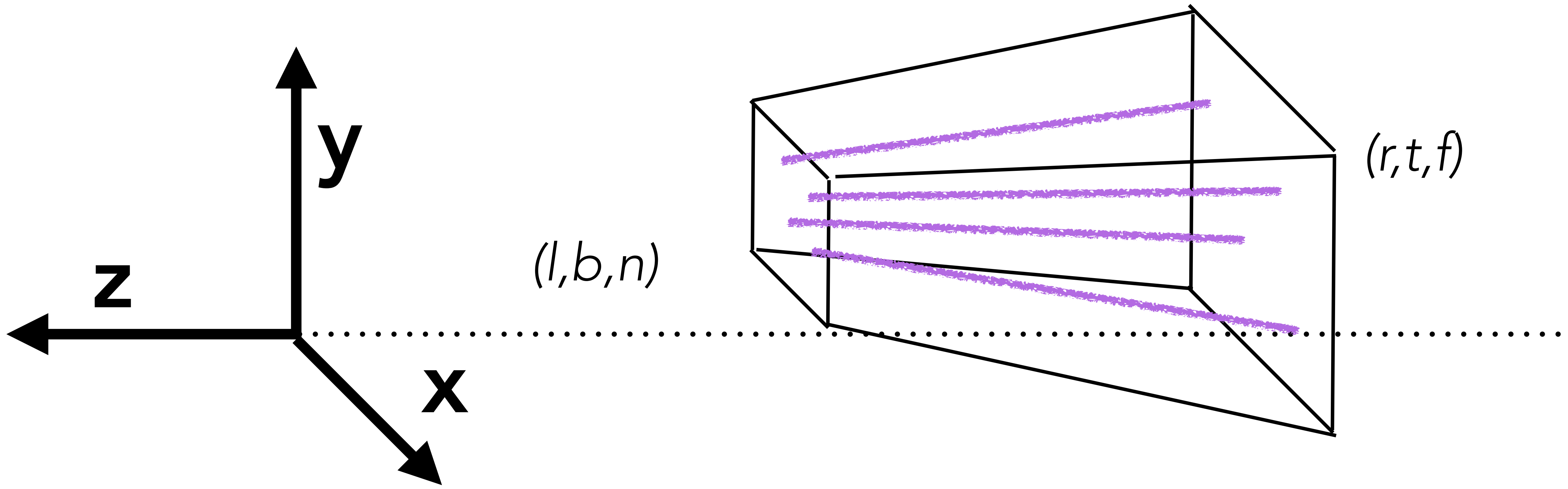
Perspective Projection

- We will use the same conventions that we used for orthographic:
  - Camera at the origin, pointing negative z
  - We scale x, y and "bring along" the z



$$\mathbf{P} = \begin{pmatrix} n & 0 & 0 & 0 \\ 0 & n & 0 & 0 \\ 0 & 0 & n+f & -fn \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

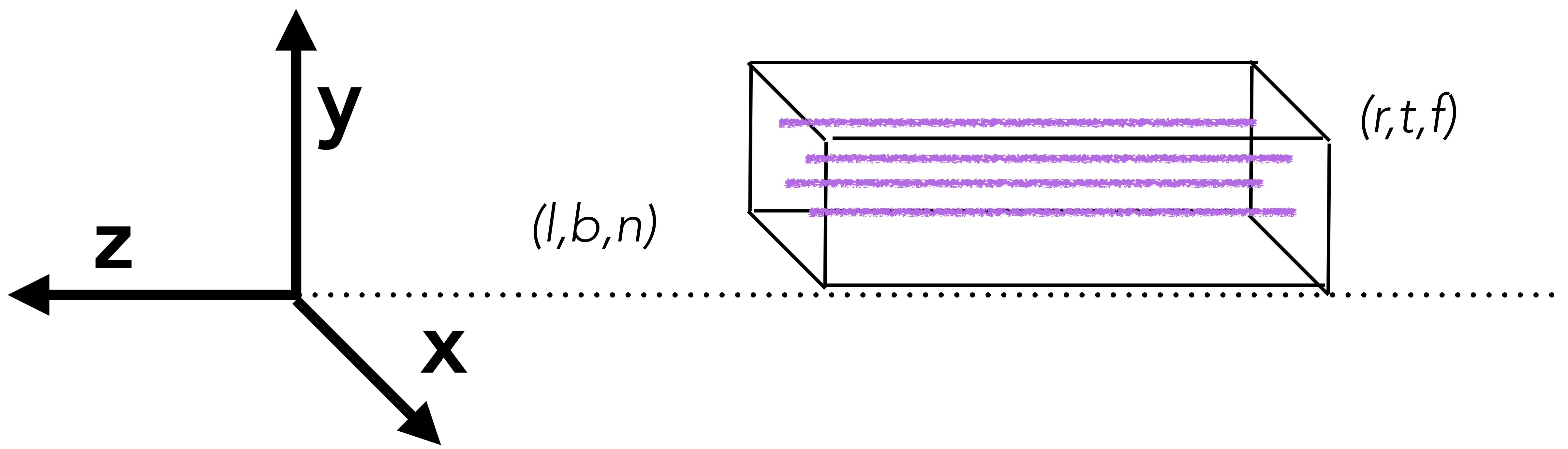
Effect on the points



$$\mathbf{P} = \begin{pmatrix} n & 0 & 0 & 0 \\ 0 & n & 0 & 0 \\ 0 & 0 & n+f & -fn \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\mathbf{P}\begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} nx \\ ny \\ (n+f)z - fn \\ z \end{pmatrix} \sim \begin{pmatrix} \frac{nx}{z} \\ \frac{ny}{z} \\ n+f-\frac{fn}{z} \\ 1 \end{pmatrix}$$

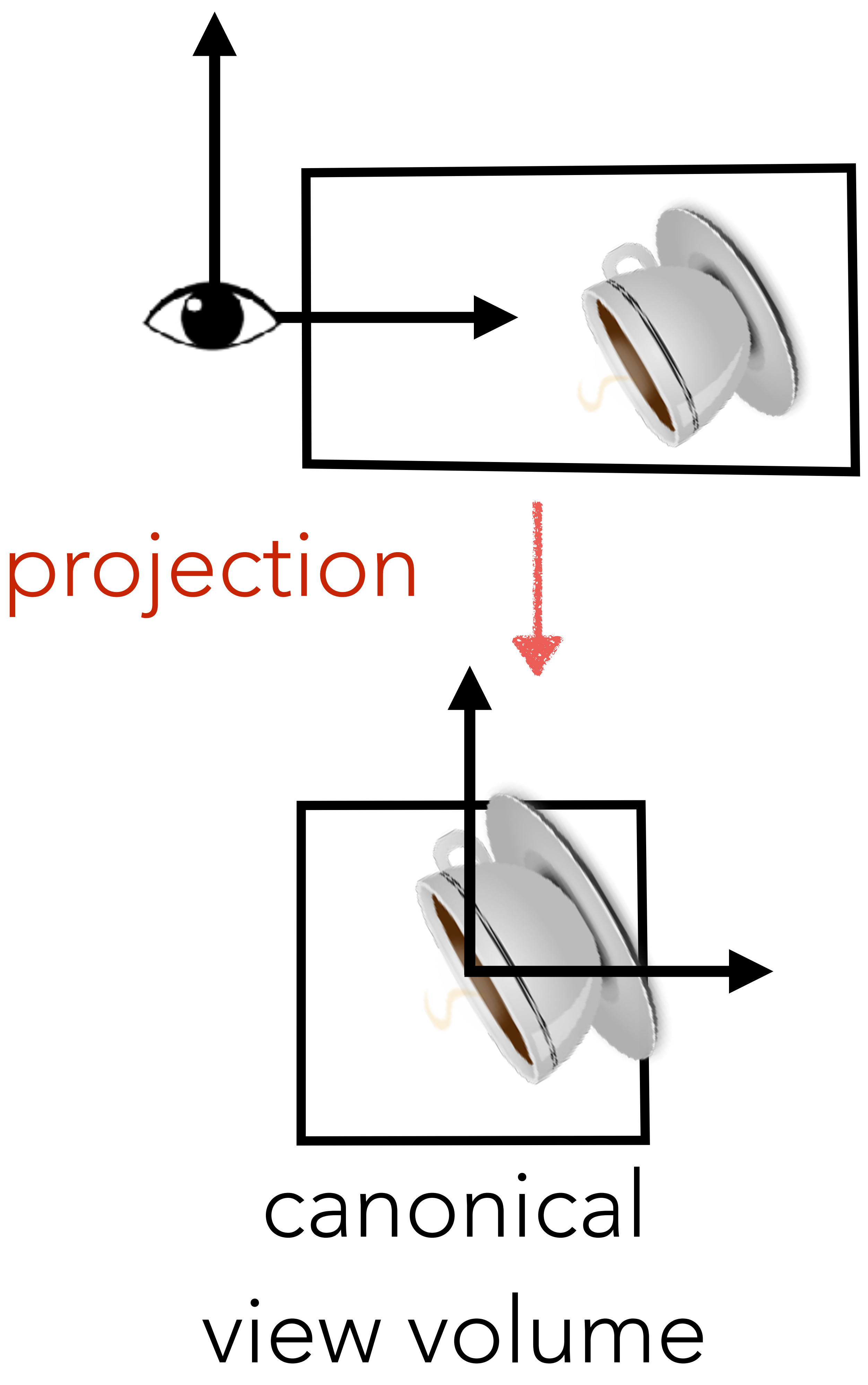
Effect on the points

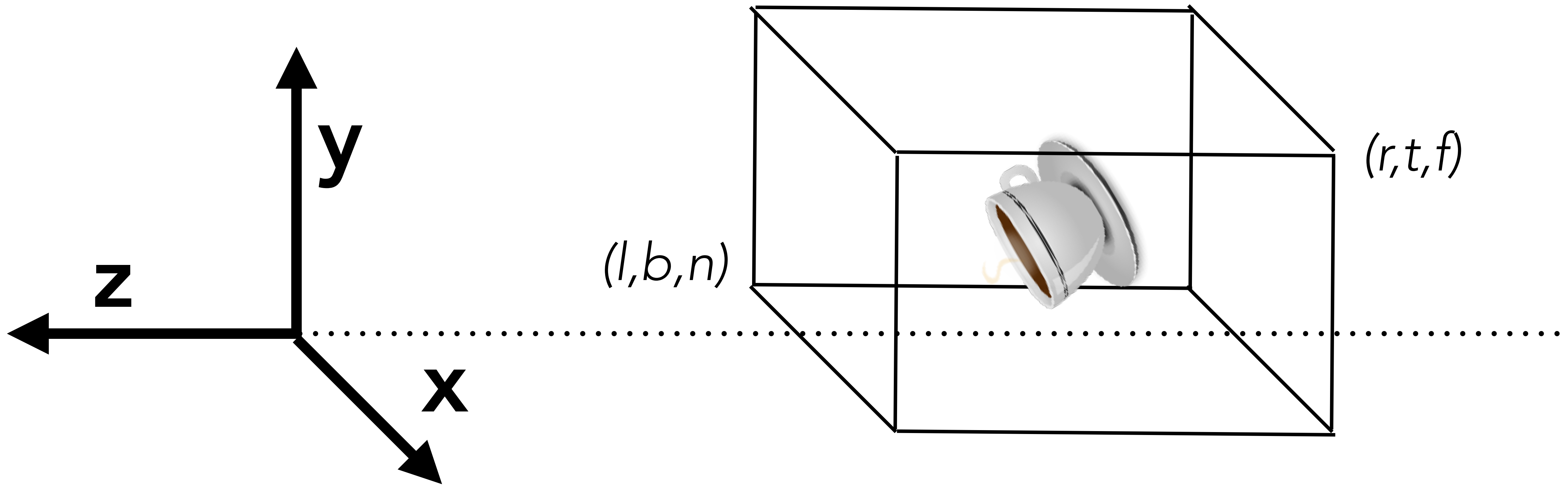


$$\mathbf{P} = \begin{pmatrix} n & 0 & 0 & 0 \\ 0 & n & 0 & 0 \\ 0 & 0 & n+f & -fn \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\mathbf{P}\begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} nx \\ ny \\ (n+f)z - fn \\ z \end{pmatrix} \sim \begin{pmatrix} \frac{nx}{z} \\ \frac{ny}{z} \\ n+f-\frac{fn}{z} \\ 1 \end{pmatrix}$$

Orthographic Projection



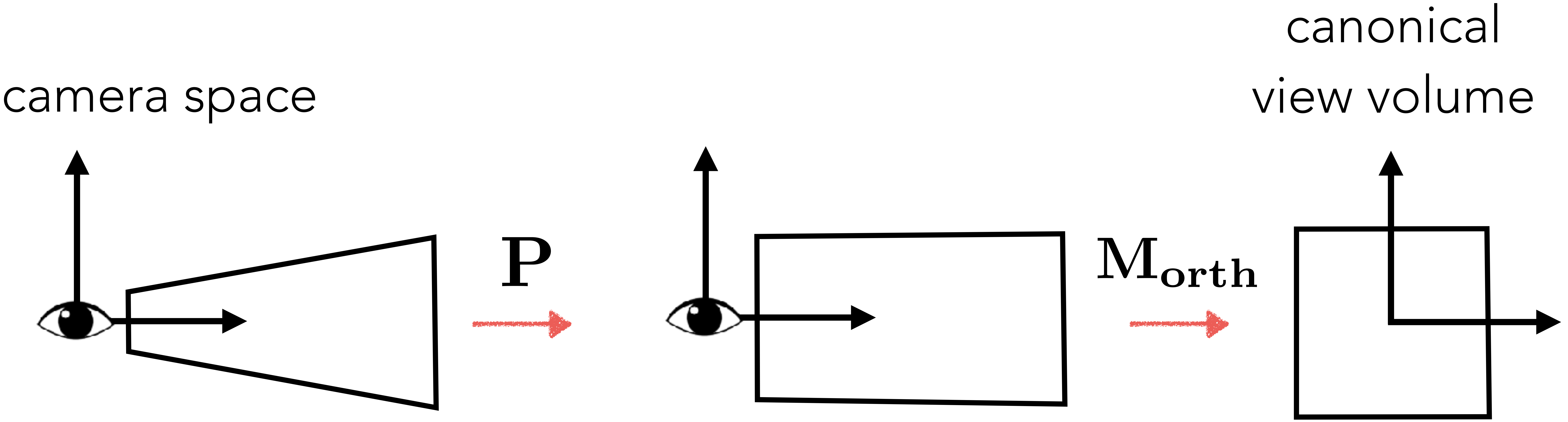


$$\mathbf{M}_{orth} = \begin{bmatrix} \frac{2}{r-l} & 0 & 0 & -\frac{r+l}{r-l} \\ 0 & \frac{2}{t-b} & 0 & -\frac{t+b}{t-b} \\ 0 & 0 & \frac{2}{n-f} & -\frac{n+f}{n-f} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Complete Perspective Transformation

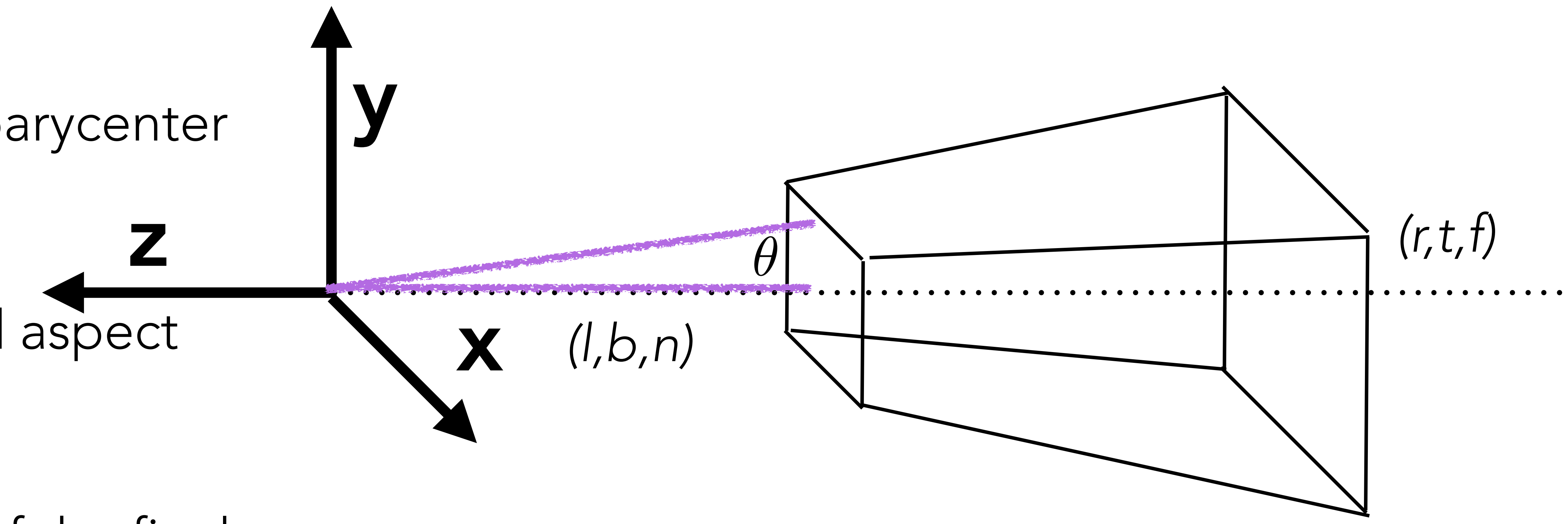
$$\mathbf{P} = \begin{pmatrix} n & 0 & 0 & 0 \\ 0 & n & 0 & 0 \\ 0 & 0 & n+f & -fn \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\mathbf{M}_{orth} = \begin{bmatrix} \frac{2}{r-l} & 0 & 0 & -\frac{r+l}{r-l} \\ 0 & \frac{2}{t-b} & 0 & -\frac{t+b}{t-b} \\ 0 & 0 & \frac{2}{n-f} & -\frac{n+f}{n-f} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Parameters?

- How to set the parameters of the transformation?
- If we look at the center of the center of the window then the barycenter of the front back should be at (0,0,f)
- If we want no distortion on the image we need to keep a fixed aspect ratio:
    - width/height = r/t (width and height are the size in pixels of the final image)
- There is only one degree of freedom left, the field of view angle θ:
    - $tan\frac{\theta}{2} = \frac{t}{|n|}$
- The parameters can thus by found by fixing n and θ. You can then compute t and consequently all the other parameters needed to construct the transformation



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References

Fundamentals of Computer Graphics, Fourth Edition
4th Edition **by Steve Marschner, Peter Shirley**

Chapter 7