Numerical Optimization with Python

Assignment 02 - dry part: Constrained Optimization Problems, duality, sensitivity and KKT

1. Consider the following problem:

$$\min |x| + |y|$$

Subject to:

$$(x-1)^2 + (y-1)^2 \leq 1$$

$$y \leq 1$$

   1.1. Draw a sketch of the contour lines of the objective function

   1.2. Add to your sketch the contours of the constraints and show the feasible region

   1.3. Only by inspection of the picture (no calculation or proof needed), find the minimizer.

   1.4. Which constraints are active at the minimizer? Which constraints have slack?

   1.5. At the minimizer, draw arrows that denote the direction of the gradient of the objective function and of each of the constraints.

2. Consider the problem:

$$\min \frac{1}{2}x_1 - x_2$$

Subject to:

$$-x_1 + x_2 - 1 \leq 0$$

$$-\frac{1}{3}x_1 + x_2 - \frac{5}{3} \leq 0$$

$$x_1 - 4 \leq 0$$

$$x_2 - 3 \leq 0$$

$$x_1 \geq 0$$

$$x_2 \geq 0$$

   2.1. Draw the feasible region and contours of the objective function.

   2.2. Only by inspection, no calculation needed, find the minimizer x^* and the optimal value p^*.

   2.3. Which constraints are active at the solution, and which are inactive?

   2.4. Consider the second constraint:

$$-\frac{1}{3}x_1 + x_2 - \frac{5}{3} \leq 0$$

To analyze sensitivity w.r.t this constraint only, we define:

$$-\frac{1}{3}x_1 + x_2 - \frac{5}{3} \leq u$$

Find the minimal and maximal value of u for which the same constraints are active at the solution, as for the unperturbed problem ($u = 0$).

2.5. For the interval $[u_{min}, u_{max}]$ found in the previous part, find a representation of the minimizer $x^*(u)$ as well as a representation of the optimal value function $p^*(u)$.

2.6. Is $p^*(u)$ differentiable at $u = 0$? What is $\frac{\partial p^*}{\partial u}(0)$? From your answer, conclude what will be the value of the λ^*, the optimal dual variable, associated with this constraint.

2.7. Provide two more sketches of the problem, one for the case $u = u_{min}$, and one for the case $u = u_{max}$. In each case, draw the minimizer $x^*(u)$.

2.8. For values $u < u_{min}$: what is the minimizer and what are the active constraints?

2.9. For values $u > u_{max}$: what is the minimizer and what are the active constraints?

2.10. From the above sensitivity analysis, we understand why λ^* expresses the sensitivity only locally. Explain why $p^*(u)$ is not differentiable at u_{min} and at u_{max} (Hint: return to your representation of $x^*(u)$ and $p^*(u)$. Do they apply now? How do they change?)

3. Consider the problem:

$$\min \frac{1}{2}(x_1^2 + x_2^2)$$

$$\text{s.t. } x_1 \geq 1$$

3.1. Write the Lagrangian function for the problem

3.2. Find the dual function $g(\lambda)$

3.3. Formulate the dual problem

3.4. For this problem, does strong duality hold? Justify your answer.

3.5. For the given primal problem, write the KKT optimality conditions

3.6. For this problem, are the KKT conditions necessary? Sufficient? Justify your answer

3.7. Solve the KKT system and conclude the minimizer of the problem.