Numerical Optimization with Python

Assignment 01 - dry part: Mathematical Review, Optimality Conditions and Convexity

Part 1: Vector differentiation and general preview material

1. Let $a \in \mathbb{R}^n$ be a nonzero constant vector and $f: \mathbb{R}^n \to \mathbb{R}$ defined by: $f(x) = a^T x$. Use explicit scalar differentiation to show the following:
    a. $\nabla f(x) = a$
    b. $\nabla^2 f(x) = 0 \in \mathbb{R}^{n \times n}$
2. Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix and $f: \mathbb{R}^n \to \mathbb{R}$ defined by: $f(x) = \frac{1}{2} x^T A x$. Use explicit scalar differentiation to show the following;
    a. $\nabla f(x) = Ax$
    b. $\nabla^2 f(x) = A$
3. Given a twice differentiable function $f: \mathbb{R}^n \to \mathbb{R}$, use the chain rule to show how ∇f and $\nabla^2 f$ are modified under an affine change of variable $z = Ax + b$, namely: define $g(x) = f\big(z(x)\big)$, and express $\nabla g(x), \nabla^2 g(x)$ using ∇f and $\nabla^2 f$.
4. Let $a \in \mathbb{R}^n$ be a constant, nonzero vector and $b \in \mathbb{R}$. What is the distance of a point $p \in \mathbb{R}^n$ from the hyper-plane $a^T x = b$?

Part 2: Optimality conditions for unconstrained optimization

1. Consider $f(x_1, x_2) = 8x_1 + 12x_2 + x_1^2 - 2x_2^2$.
    a. Write f in vector form, that is: represent $f(x) = x^T Q x + q^T x + c$, for appropriate matrix, vector and scalar Q, q and c, respectively.
    b. Show that f has only one stationary point, and that it is neither a max nor a min, but a saddle.
    c. Provide a rough sketch of the contour lines of f.
2. Define the Rosenbrock function: $f(x) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2$.
    a. Compute $\nabla f(x)$ and $\nabla^2 f(x)$
    b. Show that $x^* = (1,1)^T$ is the only local minimizer of f, and that the Hessian is positive definite at that point.

Part 3: Convex sets and convex functions

1. Prove that the intersection of convex sets is convex.
2. The *sum of two sets* A, B is defined as follows: $A + B := \{a + b : a \in A, b \in B\}$. Prove that the sum of convex sets is a convex set.
3. The set Y is *an affine transformation of a set* X, if it is simply the set of all affine transformations of elements of X, namely: $Y = \{Ax + b : x \in X\}$ where A,b are a constant matrix and vector of the appropriate dimensions. Prove that affine transformations of convex sets are convex sets.
4. If $\|\cdot\|$ is any norm (not necessarily Euclidean) then any ball: $B(x_0,r) = \{x: \|x - x_0\| \leq r\}$ is a convex set (hint: apply the properties of norms. It may also be convenient to prove for the origin and then use the previous question (3)). Does your conclusion change if we were asking about the open ball (strict inequality w.r.t r)?
5. Prove that the solutions of a set of linear equalities and inequalities is a convex set.
6. Let f, g be convex functions. Prove that $\alpha f + \beta g$ is a convex function, non-negative scalars α, β.
7. Pointwise max: Let f, g be convex functions. Prove that $h(x) := \max\{f(x), g(x)\}$ is a convex function.
8. A sub-level set of a function $f: \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}$ is defined as follows: $\{x \in \mathcal{D}: f(x) \leq c\}$ where c is a constant scalar. Prove that sub-level sets of convex functions are convex sets.
9. Composition: let $f: \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}$ be a convex function and let $h: \mathbb{R} \to \mathbb{R}$ be convex and monotone increasing. Prove that $h \circ g: \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}$ is convex. Provide a counter example if we drop only the monotonic requirement on h. Provide another counter example if we drop only the convexity requirement on h.