Revision Notes

Class - 9 Maths

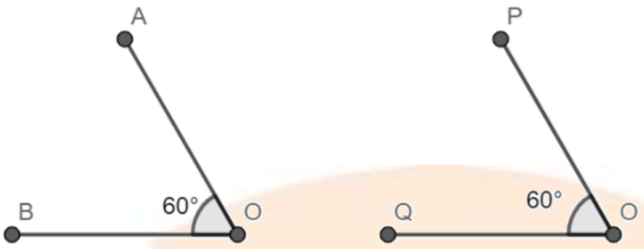
Chapter 7 - Congruent Triangles

- **Introduction:**
  Congruent geometrical figures are those that have the same shape and dimension. To test if two plane figures are congruent, lay one atop the other (superimpose) and compare the results.
  Some examples of congruent figures are:

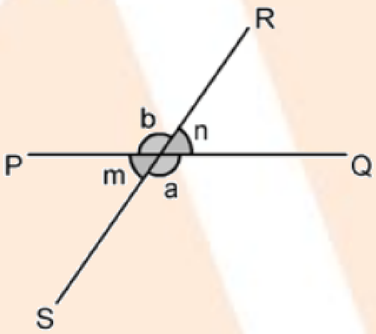
- Drag the shaded portion of each figure over the unshaded portion of the figure. That pair of figures is said to be congruent if the shaded portion matches (coincides) with the corresponding unshaded component.
- Recall the axiom on real numbers "if a and b are two real numbers, then they must follow one of the three relations, $a = b$ or $a > b$ or $a < b$.
  Equality relation: $a = b$
- In the below figure, the two magnitude of the two line segments is the same, hence they are **congruent**.

They are denoted as $AB \cong CD$.

- Angles having equal magnitude are called **congruent angles**.



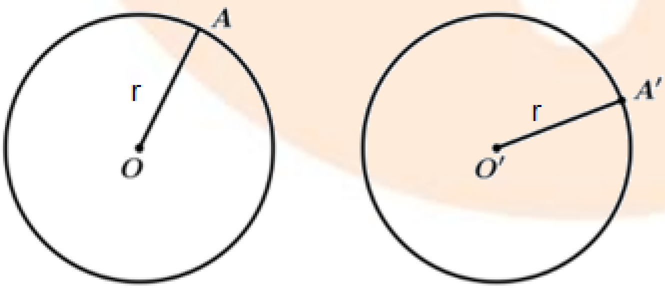
(i)



(ii)

Here, in figure (i), $\angle AOB \cong \angle POQ$, as they are of the same magnitude and for figure (ii), $\angle a = \angle b$ and $\angle m = \angle n$ as they are vertically opposite angles.

- **Congruent Circles:**

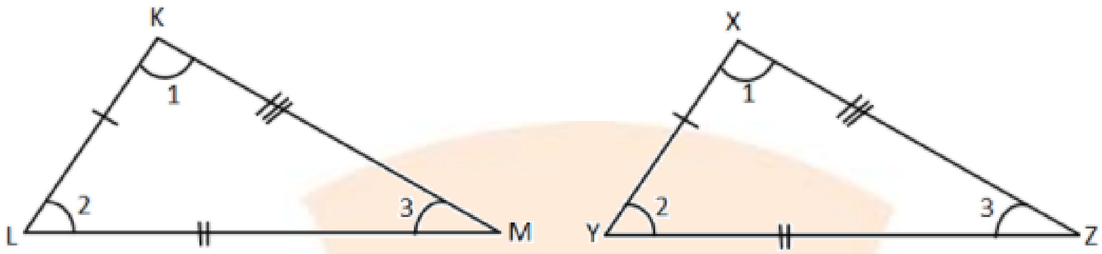


Two circles are said to be congruent if they have the same radius.

In the above figure, both the circles have the same radius $OA = O'A' = r$.

- **Congruent Triangles:**

Triangles with the same shape and size are called congruent triangles. The three angles of a triangle determine the shape of the triangle, while the three sides determine its size.



As a result, the congruency of triangles is determined by angles and sides.

Observe ΔKLM and ΔXYZ. If you superimpose one on top of the other, they will match, hence we can say that they are congruent. You'll notice in both the triangles, the three angles are congruent and the three sides, $KL \cong XY$, $LM \cong YZ$ and $KM \cong XZ$.

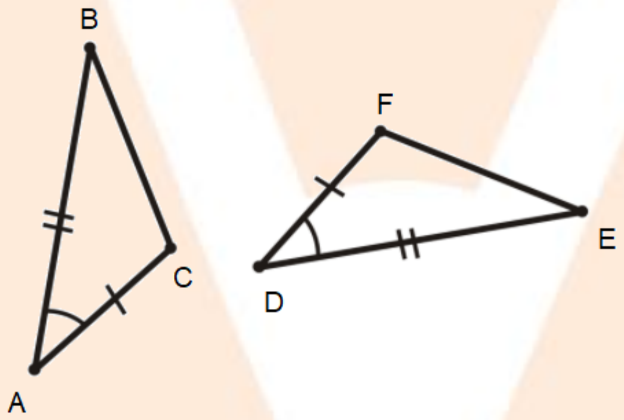
For the two triangles to be congruent, all six elements (3 angles and 3 sides) of one must be congruent with the corresponding six elements of the other.

- **Triangles' Postulates of Congruency:**
  The following definition for congruency of two triangles is based on the preceding observation.
  Definition:
  "Two triangles are congruent if and only if all of one's sides and angles are equal to the corresponding sides and angles of the other," says the definition.
- **Sufficient Condition for Two Triangles to Be Congruent:**
  We can see from the definition of congruence of two triangles that there are six conditions that must be met for two triangles to be congruent (3 sides and 3 angles). However, we shall discover through activities that if three of the six prerequisites are met, the remaining three are met automatically.
- **Activity**
  In the two triangles, ΔABC and ΔDEF, $AB = DE = 4\text{ cm}$
  $AC = DF = 4.5\text{ cm}$ and $\angle BAC = \angle EDF = 95^\circ$
  Here, we can see that the two sides and their included angle of ΔABC is equal in magnitude with the corresponding sides and angle of ΔDEF.

By dragging ΔABC on ΔDEF, we can confirm that they match completely.
Therefore, $\Delta ABC \cong \Delta DEF$.
The activity is known as the SAS congruency postulate, where SAS stands for Side-Angle-Side.

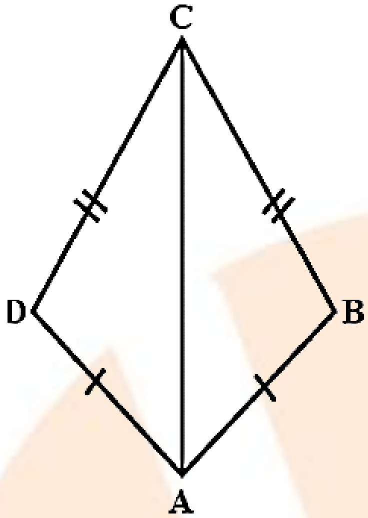
- **SAS Congruency Condition:**
"Two triangles are congruent if two sides and the included angle of one are equal to the corresponding sides and the included angle of the other."

- **Theorem 9:**
"Two triangles are congruent if any two sides and the included angle of one triangle are equal to the corresponding sides and the included angle of the other triangle".



In the two given triangles, ΔABC and ΔDEF, $AB = DE$
$AC = DF$ and
$\angle BAC = \angle EDF$.
To prove: $\Delta ABC \cong \Delta DEF$
Proof:
If we rotate and drag ΔABC on ΔDEF, such the vertices B falls on the vertex of the other triangle E and place BC along EF, we will find that, since $AB = DE$, C falls on F.
Also, $\angle B = \angle E$.
AB falls on DE, A will coincide with the vertex D and C with F.
So, AC coincides with DF.
$\therefore \Delta ABC$ coincides with ΔDEF.
$\therefore \Delta ABC \cong \Delta DEF$

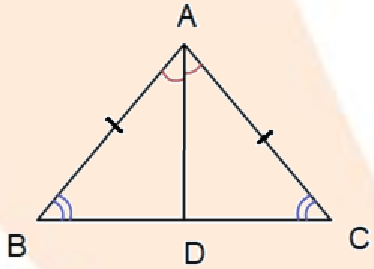
- **Application of SAS Congruency:**



In this figure we cannot apply SAS as the given data is not sufficient. On the basis of angles and sides, the SAS congruency criterion defines the relationship between two triangles. As a result, here we cannot say if AC is the angle bisector of $\angle A$, hence we cannot determine if the two triangles that may be shown are congruent.

- **Theorem 10**

"Angles opposite to equal sides are equal".



In ΔABC, $AB = AC$

To prove:

$\angle ABC = \angle ACB$

Construction: Draw the angle bisector of A, AD.

Proof:

Comparing both the triangles,

Given that $AB = AC$

AD is the common side.

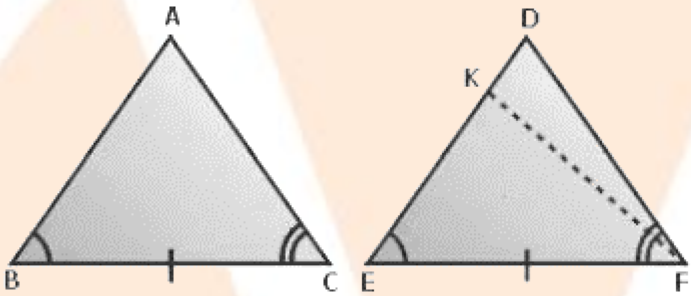
$\angle BAD = \angle DAC$ (as AD is the angle bisector)

$\therefore \Delta BAD = \Delta DAC$ (by SAS congruency rule)

$\therefore \angle ABD = \angle ACD$ (by CPCT)
$\therefore \angle B = \angle C$
Hence, proved.

- **ASA congruence condition:**
“Two triangles are congruent if two angles and the included side of one triangle are equal to the corresponding two angles and the included side of the other triangle”.

- **Theorem 11:**



Given:
In ΔABC and ΔDEF,
$\angle B = \angle E$ and $\angle C = \angle F$
$BC = EF$.
To prove:
$\Delta ABC \cong \Delta DEF$
Proof:
There are three possibilities
Case I: $AB = DE$
Case II: $AB < DE$
Case III: $AB > DE$

Case I: In addition to data, if $AB = DE$ then
$\Delta ABC \cong \Delta DEF$ (by SAS congruence postulate)

Case II: If $AB < DE$ and let K is any point on DE such that $EK = AB$.
Join KF.
Now compare triangles ABC and KCF.
$BC = EF$ (given)
$\angle B = \angle E$ (given)

Let $AB = EK$

$\Delta ABC \cong \Delta KEF$ (SAS criterion)

Hence, $\angle ABC = \angle KEF$

But, $\angle ABC = \angle DEF$ (given)

Hence, K coincides with D.

Therefore, AB must be equal to DE.

Case III: If $AB > DE$, then a similar argument applies.

AB must be equal to DE.

Hence the only possibility is that AB must be equal to DE and from SAS congruence condition

$\Delta ABC \cong \Delta DEF$

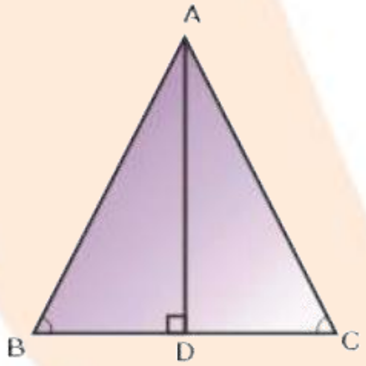
Hence the theorem is proved.

- **Theorem 12:**

"In a triangle the sides opposite to equal angles are equal".

This theorem can also be stated as

"The sides opposite to equal angles of a triangle are equal".



Given:

In ΔABC, $\angle B = \angle C$

To prove:

$\overline{AB} = \overline{AC}$

Construction:

Draw $\overline{AD} \perp \overline{BC}$

Proof:

Construct two right angle triangles, ADB and ADC, right angled at D.

Here, ΔABC, $\angle B = \angle C$

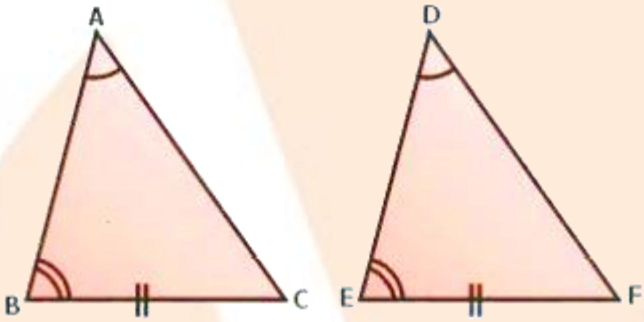
$\angle ADB = \angle ADC = 90^\circ$ (from the construction)

$\overline{AD}$ is common for both the triangles.

$\therefore \Delta ADB \cong \Delta ADC$ (by ASA postulate)

$\overline{AB} = \overline{AC}$ (corresponding sides)

- **AAS Congruence Condition:**
  The two triangles are congruent if instead of two angles and an included side, two angles and a non-included side of one triangle are equal to the corresponding angles and side of another triangle.



Given:

In triangles ABC and DEF,

$BC = EF$ (non-included sides)

$\angle B = \angle E$

$\angle A = \angle D$

To prove:

$\Delta ABC \cong \Delta DEF$

Proof:

$\angle B = \angle E$ (given)

$\angle A = \angle D$ (given)

Now, adding both, $\angle A + \angle B = \angle E + \angle D \ldots(1)$

Since, $\angle A + \angle B + \angle C = \angle E + \angle D + \angle F = 180^\circ$, considering (1) we can say that,

$\angle C = \angle F \ldots(2)$

Now in triangle ABC and DEF,

$\angle B = \angle E$ (given)

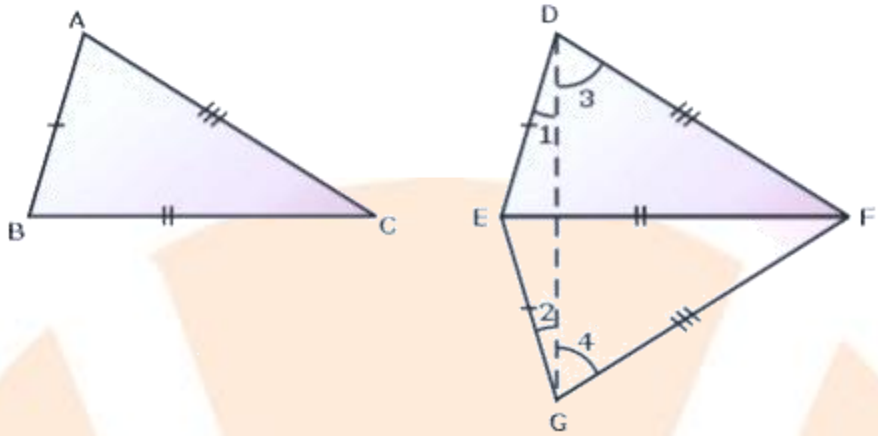
$\angle C = \angle F$ (proved in (2))

$BC = EF$ (given)

$\Delta ABC \cong \Delta DEF$ (by SAS congruency)

- **SSS Congruence Condition:**

"Two triangles are congruent if the three sides of one triangle are equal to the corresponding three sides of the other triangle".



Given:

In triangles ABC and DEF,

$AB = DE$

$BC = EF$

$AC = DF$

Note:

Let BC and EF be the longest sides of triangles ABC and DEF respectively.

To prove:

$\Delta ABC \cong \Delta DEF$

Construction: If BC is the longest side, draw EG such that $EG = AB$ and $\angle GEF = \angle ABC$.

Join GF and DG.

Proof:

In triangles ABC and GEF,

$AB = GE$ (by construction)

$BC = EF$ (given)

$\angle ABC = \angle GEF$ (by construction)

$\therefore \Delta ABC = \Delta GEF$ (SAS congruence condition)

$\angle BAC = \angle EGF$ (by CPCT)

and $AC = GF$ (by construction)

But $AB = DE$ (given)

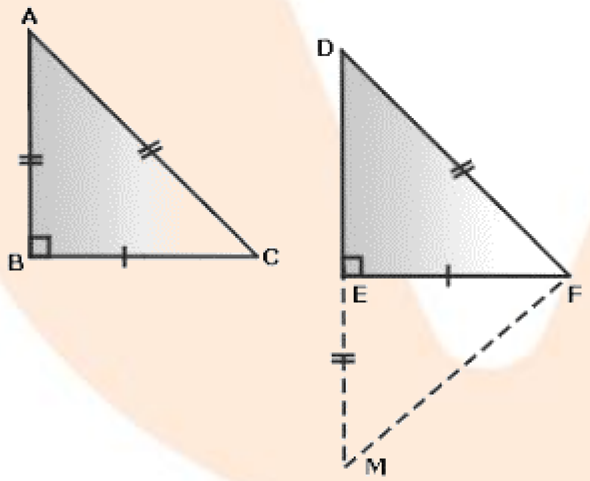
$\therefore DE = GE$

Similarly, $DF = GF$

In ΔEDG,

$DE = GE$ (Proved)
$\therefore \angle 1 = \angle 2 \ldots(1)$ (angles opposite equal sides)
In ΔEGF,
$DF = GF$ (Proved)
$\therefore \angle 3 = \angle 4 \ldots(2)$ (angles opposite equal sides)
By adding (1) and (2), we get,
$\therefore \angle 1 + \angle 3 = \angle 4 + \angle 2$
$\angle EDF = \angle EGF$ but we have proved that $\angle BAC = \angle EGF$
Therefore, $\angle EDF = \angle BAC$
Now in triangles ABC and DEF,
$AB = DE$ (given)
$AC = DF$ (given)
$\angle EDF = \angle BAC$ (proved)
$\Delta ABC \cong \Delta DEF$ (by SAS congruency)

- **Theorem of RHS (Right Angle Hypotenuse Side) Congruence**
  If the hypotenuse and one side of one triangle are equal to the hypotenuse and corresponding side of the other triangle, the two triangles are congruent.



Given:
ABC and DEF are two right-angled triangles such that
I. $\angle B = \angle E = 90^\circ$
II. Hypotenuse $\overline{AC}$ = Hypotenuse $\overline{DF}$ and
III. Side $\overline{BC}$ = Side $\overline{EF}$

To prove:

$\Delta ABC \cong \Delta DEF$

Construction

Produce DE to M so that $EM = AB$. Join MF.

Proof:

In triangles ABC and MEF,

$EM = AB$ (construction)

$BC = EF$ (given)

$\angle ABC = \angle MEF = 90^\circ$

Therefore, $\Delta ABC \cong \Delta MEF$ (SAS congruency)

Hence, $\angle A = \angle M \ldots(1)$ (by CPCT)

$\overline{AC} = \overline{MF} \ldots(2)$ (by CPCT)

Also, $AC = DF \ldots (3)$ (given)

From (1) and (3), $DF = MF$

Therefore, $\angle D = \angle M \ldots(4)$ (Angles opposite to equal sides of ΔDFM)

From (2) and (4),

$\angle A = \angle D \ldots(5)$

Now compare triangles ABC and DEF,

$\angle A = \angle D$ (from 5)

$\angle B = \angle E = 90^\circ$ (given)

$\therefore \angle C = \angle F \ldots(6)$

Compare triangles ABC and DEF,

$\overline{BC} = \overline{EF}$ (given)

$\overline{AC} = \overline{DF}$ (given)

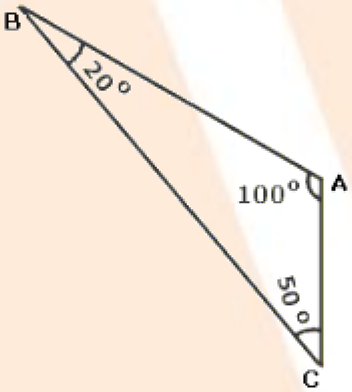
$\angle C = \angle F$ (from 6)

Therefore, $\Delta ABC \cong \Delta DEF$ (by SAS congruency)

- **Inequality of Angles:**

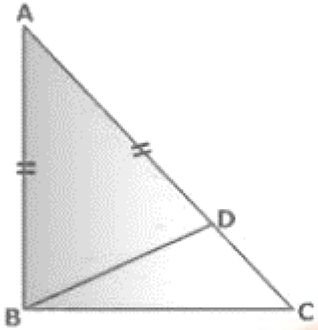
Here we can see that, both these angles are not equal, as $135^\circ < 160^\circ$

- **Inequality in a Triangle:**



Construct a triangle ABC as shown in the figure.
Observe that in triangle ABC,
$\overline{AC}$ is the smallest side (2 cm)
B is the angle opposite to $\overline{AC}$ and $\angle B = 20^\circ$
$\overline{BC}$ is the greatest side (6 cm)
A is the angle opposite to $\overline{BC}$ and $\angle A = 100^\circ$
From the measurements made above of side and angle opposite to it, we can write the relation in the form of a statement.
"If two sides of a triangle are unequal then the longer side has the greater angle opposite to it".

- **Theorem on Inequalities:**
  Theorem 1:
  If two sides of a triangle are unequal, the longer side has the greater angle opposite to it.



Read the statement and draw a triangle as per data.

Draw ΔABC, such that $AC > AB$.

Data:

$AC > AB$

To Prove:

$\angle ABC = \angle ACB$

Construction:

Take a point D on $\overline{AC}$ such that $\overline{AB} = \overline{AD}$. Join B to D.

Proof:

In ΔABC, $\overline{AB} = \overline{AD}$ (by construction)

$\therefore \angle ABD = \angle ADB \ldots(1)$

but $\angle ADB$ is the exterior angle with reference to ΔDBC.

$\angle ADB > \angle DCB \ldots (2)$

From relation (1) and (2) we can write $\angle ABD > \angle DCB$

But $\angle ABD$ is a part of $\angle ACB$.

$\therefore \angle ACB > \angle DCB$ or $\therefore \angle ABC > \angle ACB$

Hence, proved.

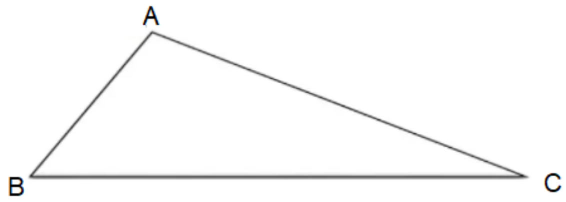
- **Angle Side Relation**

  Theorem 2:

  In a triangle, if two angles are unequal, the side opposite to greater angle is longer than the side opposite to the smaller angle.

- **Theorem 3**

  In a triangle, the greater angle has the longer side opposite to it.



Given:

In $\triangle ABC$, $\angle ABC > \angle ACB$

To prove:

$AC > AB$

Proof:

In $\triangle ABC$, AB and AC are two line segments. So the following are the three possibilities of which
exactly one must be true.

I. either $AB = AC$, then $\angle B = \angle C$ which is contrary to the hypothesis.
$\therefore AB \neq AC$

II. $AB > AC$, then $\angle B < \angle C$ which is contrary to the hypothesis.

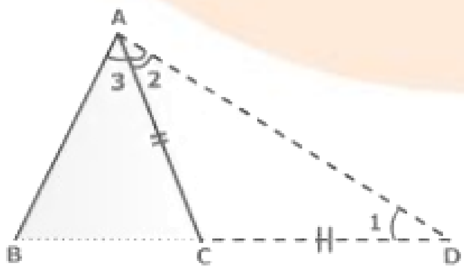
III. $AB < AC$, this is the only condition we are left with, so $AB < AC$ must be true.

Hence, proved.

- **Theorem 4**

Prove that in any triangle the sum of the lengths of any two sides of a triangle is greater than the length of its third side.

Draw $\triangle ABC$.



Data:

ABC is a triangle.

To prove:

$\overline{AB}+\overline{AC}>\overline{BC}$
$\overline{AB}+\overline{BC}>\overline{AC}$
$\overline{AC}+\overline{BC}>\overline{BA}$

Construction:

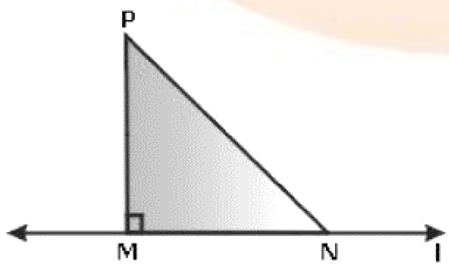
Produce $\overline{BC}$ to D such that $\overline{AC}=\overline{CD}$. Join A to D.

Proof:

$\overline{AC}=\overline{CD}$ (by construction)
$\angle 1=\angle 2 \ldots(1)$
From the figure,
$\angle BAD=\angle 2+\angle 3 \ldots(2)$
$\angle BAD>\angle 2$
$\therefore \angle BAD>\angle 1$ (proved from 1)
$\overline{AB}$ is opposite to $\angle 1$ and $\overline{BD}$ is opposite to $\angle BAD$.
In ΔBAD , $\angle 1<\angle 2+\angle 3$
$BD>AB$ (side opposite to greater angle is greater)
From figure $BD=BC+CD$
$\therefore BC+CD>AB$
$\therefore BC+AC>AB$
Since, $CD=AC$, hence, sum of two sides of a triangle is greater than the third side.
Similarly, $\overline{AB}+\overline{BC}>\overline{AC}$ and $\overline{AC}+\overline{BC}>\overline{BA}$.
Hence, proved.

- **Theorem 5:**

  Of all the line segments that can be drawn to a given line, from a point not lying on it, the perpendicular line segment is the shortest.



Given:

l is a line and P is a point not lying on it. $PM \perp l$. N is any point on l other than M.

To prove:

$PM < PN$

Proof:

In ΔPMN, $\angle M$ is the right angle.

$\therefore \angle N$ is an acute angle, from angle sum property.

$\because \angle M > \angle N$

$PN > PM$ (side opposite to greater angle)

$\therefore PM < PN$.