Revision Notes

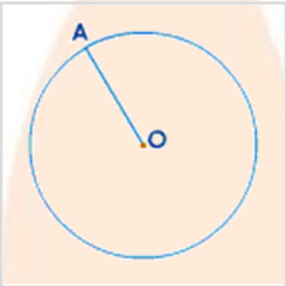
Class - 9 Mathematics

Chapter 9 - Circles

Introduction

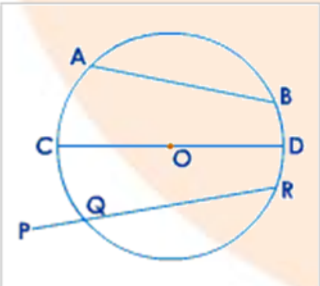
Circle:

- The **locus** of the points at a certain distance from a fixed point is defined as a circle.



Chord:

- A chord is a **straight line that connects any two points on a circle**.

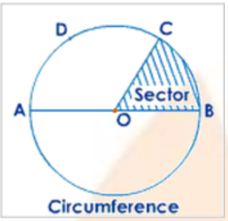


- A chord is represented by the letters AB .
- If the **longest chord** passes through the centre of the circle, it is termed as the **diameter**.
- The **radius** is **twice** as long as **the diameter**.
- A diameter is referred to as a CD.
- A **secant** is a line that **divides a circle in half**.

- PQR is a secant of a circle.

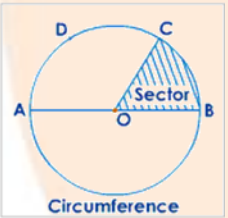
Circumference:

- Circumference refers to the **length of a full circle**.
- The circumference of a circle is defined as the border curve (or perimeter) of the circle.



Arc:

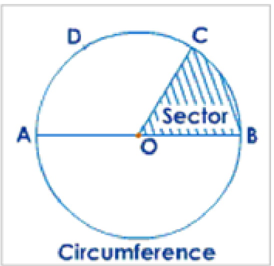
- An **arc** is any **section or a part of the circumference**.
- A diameter divides a circle into two equal pieces.
- A **minor arc** is one that is **smaller than a semicircle**.
- A **major arc** is one that is **larger than a semicircle.**



- ADC is a minor arc, whereas ABC is a major arc.

Sector:

- A **sector** is the area between an arc and the two radii that connects the arc's centre and end points.
- A **segment** is a section of a circle that has been cut off by a chord.

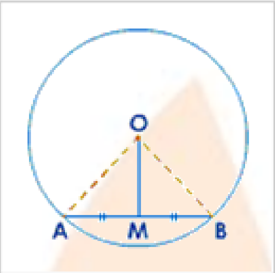


Concentric circles:

Concentric circles are circles with the same centre.

Theorem 1:

A straight line drawn from the centre of a circle to bisect a chord which is not a diameter, is at right angles to the chord.



- **Given Data:**
  - Here, AB is a chord of a circle with the centre O.
  - The midpoint of AB is M.
  - OM is joined.
- **To Prove:**

$\angle AMO = \angle BMO = 90^0$

- **Construction:**

Join AO and BO.

- **Proof:**

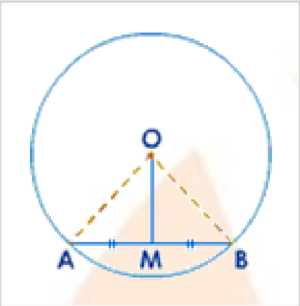
In ΔAOM and ΔBOM

Statement	Reason
$AO = BO$	radii
$AM = BM$	Data
$OM = OM$	Common
$\Delta AOM \cong \Delta BOM$	$(S.S.S)$
$\therefore \angle AMO = \angle BMO$	Statement (4)
But $\angle AMO + \angle BMO = 180^0$	Linear pair

$\therefore \angle AMO = \angle BMO = 90^0$	Statements (5) and (6)

Theorem 2 (Converse of theorem 1):

The perpendicular to a chord from the centre of a circle bisects the chord.



- **Given Data:**
  - Here, AB is a chord of a circle with the centre O.
  - $OM \perp AB$
- **To Prove:**

$AM = BM$

- **Construction:**

Join AO and BO.

- **Proof:**

In ΔAOM and ΔBOM

Statement	Reason
$AMO = \angle BMO$	Each 90^0 (data)
$AO = BO$	Radii
$OM = OM$	Common
$\Delta AOM \cong \Delta BOM$	$(R.H.S)$
$AM = BM$	Statement (4)

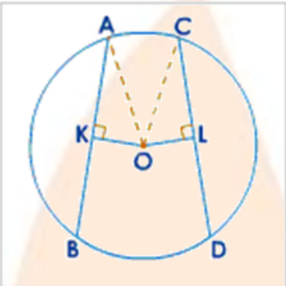
The transposition of a statement consisting of 'data' and 'to prove' is the converse of a theorem.

We can see how it works by looking at the previous two theorems:

Theorem	Converse of theorem
1 Data: M is the mid-point of AB	To prove: M is the mid-point of AB
2 To prove: $OM \perp AB$	Data: $OM \perp AB$

Theorem 3:

Equal chords of a circle are equidistant from the centre.



- **Given Data:**
  - Here, AB and CD are equal chords of a circle with centre O.
  - $OK \perp AB$ and $OL \perp CD$
- **To Prove:**

$OK = OL$

Statement	Reason
$AK = \frac{1}{2}AB$	$\perp$ from the centre bisects the chord.
$CL = \frac{1}{2}CD$	$\perp$ from the centre bisects the chord.
But $AB = CD$	data
$\therefore AK = CL$	Statements $(1),(2)$ and (3)
In ΔAOK and ΔCOL	
$\angle AKO = \angle CLO$	Each 90^0 (data)
$AO = CO$	radii
$AK = CL$	Statements (4)

$\therefore \Delta AOK \cong \Delta COL$	$(R.H.S)$
$\therefore OK = OL$	Statements (8)

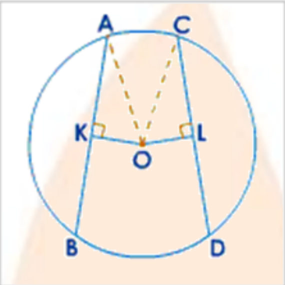
- **Construction:**

Join AO and CO.

- **Proof:**

Theorem 4 (Converse of theorem 3):

Chords which are equidistant from the centre of a circle are equal.



- **Given Data:**
  - Here, AB and CD are equal chords of a circle with centre O.
  - $OK \perp AB$ and $OL \perp CD$
  - $OK = OL$
- **To Prove:**

$AB = CD$

- **Construction:**

Join AO and CO.

Statement	Reason
$\angle AKO = \angle CLO$	Each 90^0 (data)
$AO = CO$	radii
$OK = OL$	data
$\Delta AOK \cong \Delta COL$	$(R.H.S)$
$\therefore AK = CL$	Statements (4)
But $AK = \frac{1}{2} AB$	$\perp$ from the centre bisects the chord.

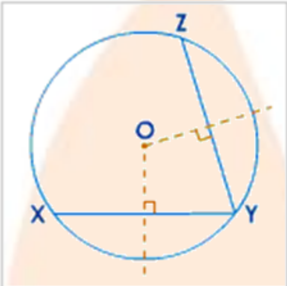
$CL = \frac{1}{2} CD$	$\perp$ from the centre bisects the chord.
$\therefore AB = CD$	Statements $(5),(6)$ and (7)

- **Proof:**

In ΔAOK and ΔCOL

Theorem 5**:**

There is one circle, and only one, which passes through three given points not in a straight line.



- **Given Data:**

Here, X, Y and Z are three points not in a straight line.

- **To Prove:**

A unique circle passes through X, Y and Z.

- **Construction:**

- Join XY and YZ.
- Draw perpendicular bisectors of XY and YZ to meet at O.

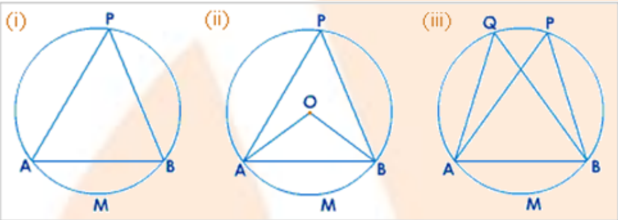
- **Proof:**

Statement	Reason
$OX = OY$	O lies on the $\perp$ bisector of XY
$OY = OZ$	O lies on the $\perp$ bisector of YZ
$OX = OY = OZ$	Statements (1) and (2)
O is the only point equidistant from X, Y and Z.	Statements (3)
With O as centre and radius OX, a circle can be drawn to pass through X, Y and Z.	Statements (4)

Therefore, the circle with centre O is a unique circle passing through X, Y and Z.	Statements (5)

Angle Properties (Angle, Cyclic Quadrilaterals and Arcs):

- In figure (i), the straight line AB students $\angle APB$ on the circumference.

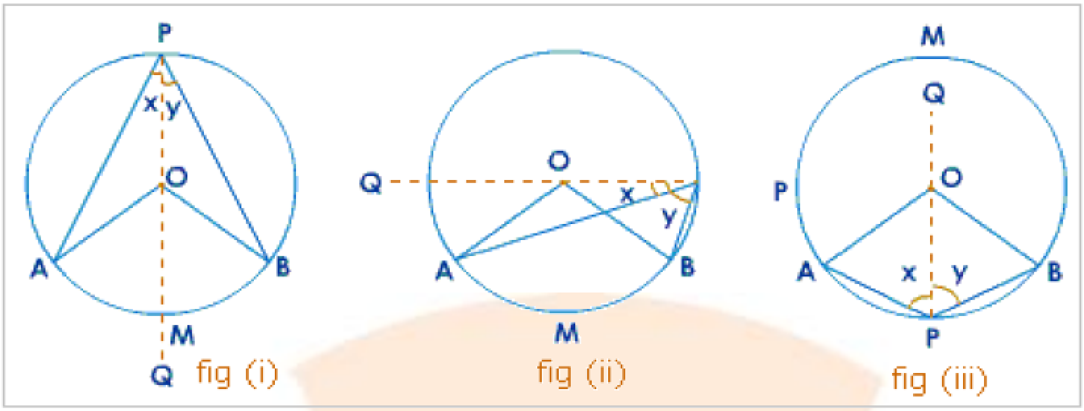


$\angle APB$ can be said to be subtended by arc AMB, on the remaining part of the circumference.

- In fig. (ii), arc AMB subtends $\angle APB$ on the circumference, and it subtends $\angle AOB$ at the centre.
- In fig. (iii), $\angle APB$ and $\angle AQB$ are in the same segment.
- Now we will go through the theorems based on the angle properties of the circles.

Theorem 6:

The angle which an arc of a circle subtends at the centre is double the angle which it subtends at any point on the remaining part of the circumference.



- **Given Data:**

Arc AMB subtends □AOB at the center O of the circle and □APB on the remaining part of circumference.

- **To Prove:**

$\angle AOB = 2\angle APB$

- **Construction:**

Join PO and produce it to Q.

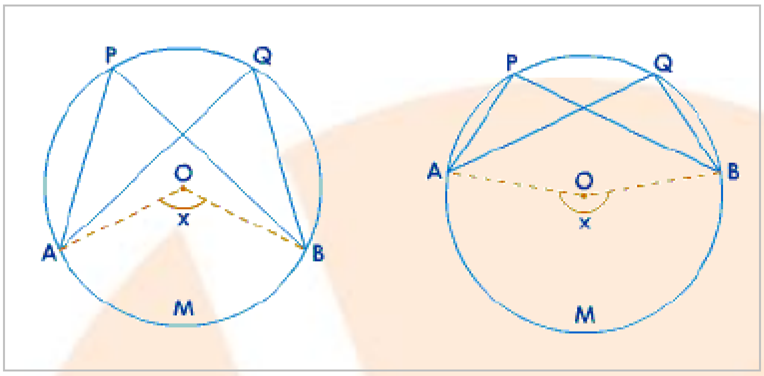
Let, $\angle APQ = x$ and $\angle BPQ = y$

- **Proof:**

Statement	Reason
$\angle AOQ = \angle x + \angle A$	Ext. $\angle =$ sum of the int. opp. $\angle$s
$\angle x = \angle A$	$\because$ $OA = OP$ (Radii)
$\therefore \angle AOQ = 2\angle x$	Statements (1) and (2)
$\therefore \angle BOQ = 2\angle y$	Same way as Statements (3)
From figure (i) and (ii)	
$\angle AOQ + \angle BOQ = 2\angle x + 2\angle y$	Statements (3) and (4)
$\Rightarrow \angle AOB = 2(\angle x + \angle y)$	Statements (5)
From figure (ii)	
$\angle BOQ - \angle AOQ = 2\angle y - 2\angle x$	Statements (3) and (4)
$\angle AOB = 2(\angle y - \angle x)$	Statements (8)
$\because \angle AOB = 2\angle APB$	Statements (9)

Theorem 7 :

Angles in the same segment of a circle are equal.



- **Given Data:**

$\angle APB$ and $\angle AQB$ are in the same segment of a circle with center O.

- **To Prove:**

$\angle APB = \angle AQB$

- **Construction:**

Join AO and BO.

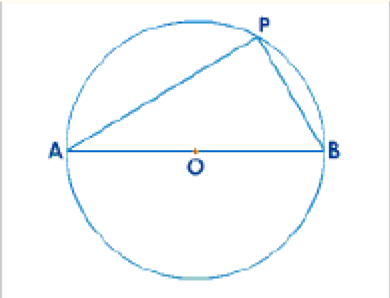
Let, arc AMB subtend angle x at the center O.

- **Proof:**

Statement	Reason
$\angle x = 2\angle APB$	$\angle$at center $= 2 \times \angle$on the circumference
$\angle x = 2\angle AQB$	$\angle$at center $= 2 \times \angle$on the circumference
$\therefore \angle APB = \angle AQB$	Statements (1) and (2)

Theorem 8 :

The angle in a semicircle is a right angle.



- **Given Data:**

AB is a diameter of a circle with center O.
P is any point on the circle.

- **To Prove:**

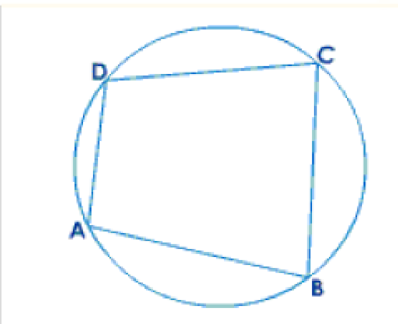
$\angle APB = 90^\circ$

- **Proof:**

Statement	**Reason**
$\angle APB = \dfrac{1}{2}\angle AOB$	$\angle$at center $= 2\times\angle$on the circumference
$\angle AOB = 180^\circ$	AOB is a straight line.
$\therefore \angle APB = \dfrac{1}{2}\times 180^\circ$	Statements (1) and (2)
$\therefore \angle APB = 90^\circ$	Statements (3)

Cyclic Quadrilaterals:

- If the vertices of a quadrilateral lie on a circle, the quadrilateral is called a **cyclic quadrilateral**.
- The vertices are known as **concyclic points**.

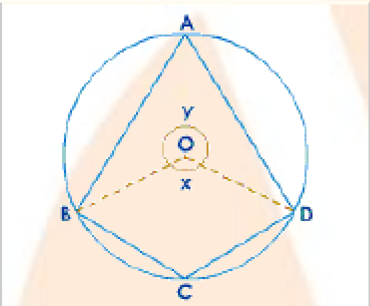


- From the above figure, $ABCD$ is a cyclic quadrilateral.

The vertices A, B, C and D are concyclic points.

Theorem 9**:**

The opposite angles of a quadrilateral inscribed in a circle (cyclic) are supplementary.



- **Given Data:**

$ABCD$ is is a cyclic quadrilateral.

O is a center of a circle.

- **To Prove:**

i. $\angle A+\angle C=180^{\circ}$

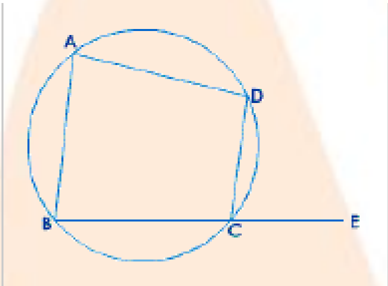
ii. $\angle B+\angle D=180^{\circ}$

- **Proof:**

Statement	Reason
$\angle A = \frac{1}{2}\angle x$	$\angle$at center $=2\times\angle$on the circumference
$\angle C = \frac{1}{2}\angle y$	$\angle$at center $=2\times\angle$on the circumference
$\angle A + \angle C = \frac{1}{2}\angle x + \frac{1}{2}\angle y$	Statements (1) and (2)
$\angle A + \angle C = \frac{1}{2}(\angle x + \angle y)$	Statements (3)

But $\angle x + \angle y = 360^\circ$	$\angle$s at a point
$\therefore \angle A + \angle C = \frac{1}{2} \times 360^\circ$	Statements (4) and (5)
$\therefore \angle A + \angle C = 180^\circ$	Statements (6)
Also, $\angle ABC + \angle ADC = 180^\circ$	Same way as statements (7)

Corollary:
The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.



- **Given Data:**

$ABCD$ is is a cyclic quadrilateral.
BC is produced to E

- **To Prove:**

$\angle DCE = \angle A$

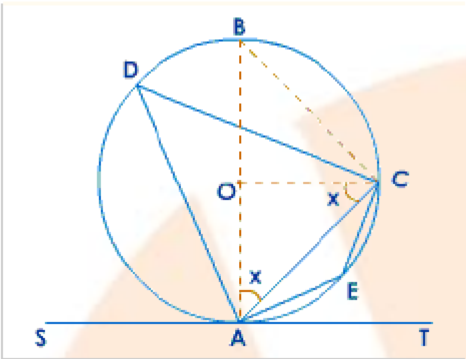
- **Proof:**

Statement	Reason
$\angle A + \angle BCD = 180^\circ$	Opp. $\angle$s of a cyclic quad.
$\angle BCD + \angle DCE = 180^\circ$	Linear pair
$\therefore \angle BCD + \angle DCE = \angle A + \angle BCD$	Statements (1) and (2)
$\therefore \angle DCE = \angle A$	Statements (2)

Alternate Segment Property

Theorem 10**:**

The angle between a tangent and a chord through the point of contact is equal to the angle in the alternate segment.



- **Given Data:**

A straight line SAT touches a given circle with centre O at A. AC is a chord through the point of contact A.

$\angle ADC$ is an angle in the alternate segment to $\angle CAT$ and $\angle AEC$ is an angle in the alternate segment to $\angle CAS$

- **To Prove:**

1. $\angle CAT = \angle ADC$
2. $\angle CAS = \angle AEC$

- **Construction:**

Draw AOB as diameter and join BC and OC.

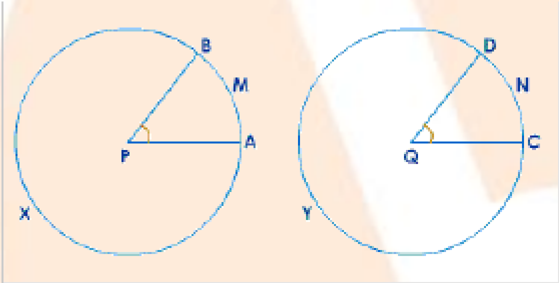
- **Proof:**

Statement	Reason
$\angle OAC = \angle OCA = x$	Since, $OA = OC$ and supposition
$\angle CAT + \angle x = 90^\circ$	Since, tangent-radius property
$\angle AOC + \angle x + \angle y = 180^\circ$	Sum of angles of a triangle
$\angle AOC = 180^\circ - 2\angle x$	Statements (3)
Also, $\angle AOC = 2\angle ADC$	$\angle$at the center $= 2\angle$on the circle
$\angle CAT = 90^\circ - x$	Statements (2)
$2\angle CAT = 180^\circ - 2x$	Statements (6)

$\therefore 2\angle CAT = 2\angle ADC$	Statements (4), (5) and (7)
$\angle CAT = \angle ADC$	Statements (8)
$\angle CAS + \angle CAT = 180^\circ$	Linear pair
$\angle ADC + \angle AEC = 180^\circ$	Opp. Angles of a cyclic quad
$\angle CAS + \angle CAT = \angle ADC + \angle AEC$	Statements (10) and (11)
$\therefore \angle CAS = \angle AEC$	Statements (9) and (12)

Theorem 11:

In equal circles (or in the same circle), if two arcs subtend equal angles at the centres, they are equal.



- **Given Data:**

AXB and CYD are equal circles with centers P and O.
Arcs AMD, CND subtend equal angles APB, CQD.

- **To Prove:**

arc AMD = arc CND

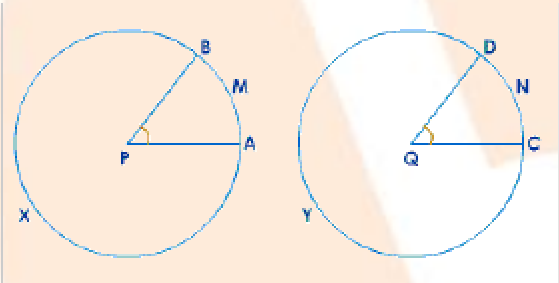
- **Proof:**

Statement	Reason
Apply $\bigcirc$ CYD to $\bigcirc$ AXB so that center Q falls on center P and QC along PA and D on the same side as B. Therefore, $\bigcirc$ CYD overlaps $\bigcirc$ AXB	Since, circles are equal (data)

$\therefore C$ falls on A	Since, $PA = QC$ (data)
$\angle APB = \angle CQD$	data
$\therefore QD$ falls along PB	Statements (1) and (3)
$\therefore D$ falls on B	Since, $QD = PB$ (data)
$\therefore$ arc CND coincides with arc AMD	Statements (2) and (5)
arc AMD = arc CND	Statements (6)

Theorem 12 (Converse of 11):

In equal circles (or in the same circle) if two arcs are equal, they subtend equal angles at the centres.



- **Given Data:**

In equal circles AXB and CYD, equal arcs AMD and CND subtend $\angle APB$ and $\angle CQD$ at the centers P and Q respectively.

- **To Prove:**

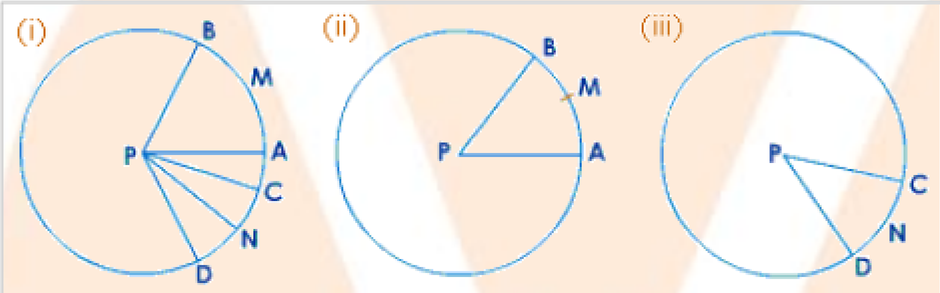
$\angle APB = \angle CQD$

- **Proof:**

Statement	Reason
Apply $\bigcirc$ CYD to $\bigcirc$ AXB so that center Q falls on center P and QC along PA and D on the same side as B. Therefore, $\bigcirc$ CYD overlaps $\bigcirc$ AXB	Since, circles are equal (data)

$\therefore$ C falls on A	Since, $PA = QC$ (data)
arc AMD = arc CND	data
$\therefore$ D falls on B	Statements (1),(2) and (3)
$\therefore$ QD coincides with PB and QC coincides with PA	Statements (1),(2) and (4)
$\angle APB = \angle CQD$	Statements (5)

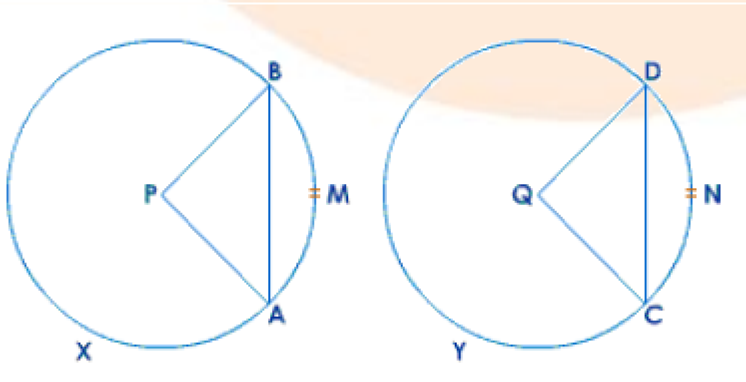
In case of the same circle:



Figures (ii) and (iii) can be considered to be two equal circles which are obtained from figure (i) and then the above proofs may be applied.

Theorem 13:

In equal circles (or in the same circle), if two chords are equal, they cut off equal arcs.



- **Given Data:**

In equal circles AXB, CYD with centers P and Q have

chord AB = chord CD

- **To Prove:**

arc AMB = arc CND

arc AXB = arc CYD

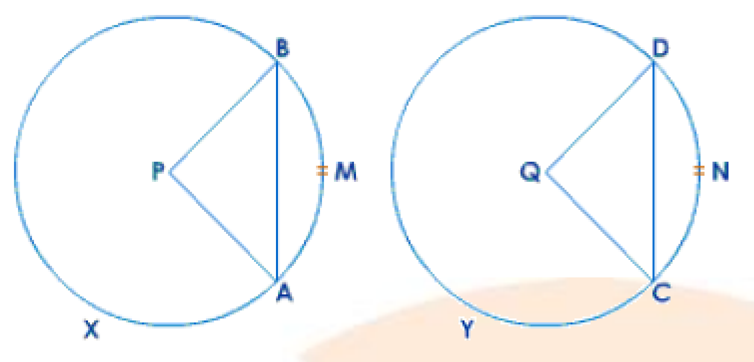
- **Proof:**

In ΔABP and ΔCDQ

Statement	**Reason**
AP = CQ	Radii of equal circles.
BP = DQ	Radii of equal circles.
AB = CD	Radii of equal circles.
$\Delta ABP \cong \Delta CDQ$	(S.S.S)
$\therefore \angle APB = CQD$	Statements (4)
arc AMB = arc CND	Statements (5)
○AXB - arc AMB = ○CYD - arc CND	Equal arcs [Statements (6)]
$\therefore$ arc AXB = arc CYD	Statements (7)

Theorem 14 **(Converse of** 13**):**

In equal circles (or in the same circle) if two arcs are equal, the chords of the arcs are equal.



- **Given Data:**

Equal circles AXB, CYD with centers P and Q have arc AMB = arc CND

- **To Prove:**

chord AB = chord CD

- **Construction:**

Join AP, BP, CQ and DQ.

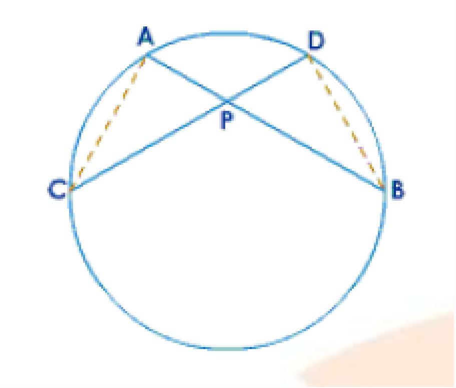
- **Proof:**

In ΔABP and ΔCDQ

Statement	Reason
AP = CQ	Radii of equal circles.
BP = DQ	Radii of equal circles.
$\angle APB = CQD$	$\because$ arc AMB = arc CND
$\therefore \Delta ABP \cong \Delta CDQ$	(S.A.S)
$\therefore AB = CD$	Statements (4)

Theorem 15**:**

If two chords of a circle intersect internally, then the product of the length of the segments are equal.



- **Given Data:**

AB and CD are chords of a circle intersecting externally at P.

- **To Prove:**

$AP \times BP = CP \times DP$

- **Construction:**

Join AC and BD.

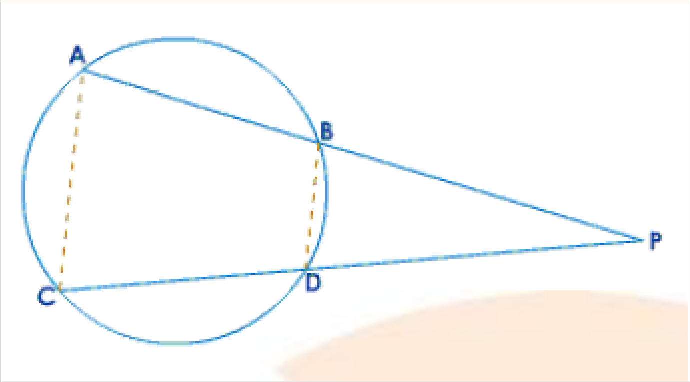
- **Proof:**

In ΔAPC and ΔDPB

Statement	Reason
$\angle A = \angle D$	Angles in same segment.
$\angle C = \angle B$	Angles in same segment.
$\therefore\ \Delta APC \sim \Delta DPB$	AA similarity
$\therefore \dfrac{AP}{DP} = \dfrac{CP}{BP}$	Statements (3)
$\therefore AP \times BP = CP \times DP$	Statements (4)

Theorem 16**:**

If two chords of a circle intersect externally, then the product of the lengths of the segments are equal.



- **Given Data:**

AB and CD are chords of a circle intersecting externally at P.

- **To Prove:**

$AP \times BP = CP \times DP$

- **Construction:**

Join AC and BD.

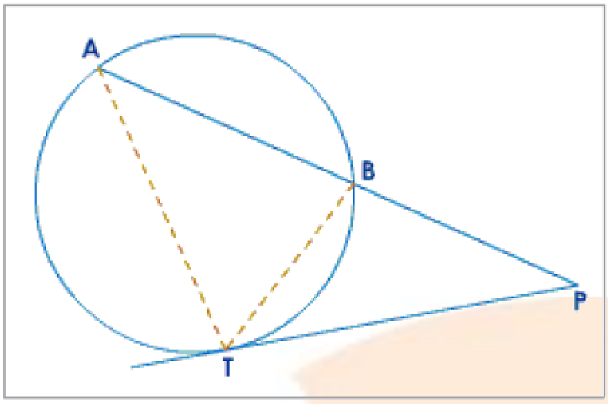
- **Proof:**

In ΔACP and ΔDBP

Statement	Reason
$\angle A = \angle BDP$	Ext. $\angle$of a cyclic quad. = Int. opp. $\angle$
$\angle C = \angle DBP$	Ext. $\angle$of a cyclic quad. = Int. opp. $\angle$
$\therefore\ \Delta ACP \sim \Delta DBP$	AA similarity
$\therefore \dfrac{AP}{DP} = \dfrac{CP}{BP}$	Statements (3)
$AP \times BP = CP \times DP$	Statements (4)

Theorem 17**:**

If a chord and a tangent intersect externally, then the product of the lengths of the segments of the chord is equal to the square on the length of the tangent from the point of contact to the point of intersection.



- **Given Data:**

A chord AB and a tangent TP at a point T on the circle intersect at P.

- **To Prove:**

$AP \times BP = PT^2$

- **Construction:**

Join AT and BT.

- **Proof:**

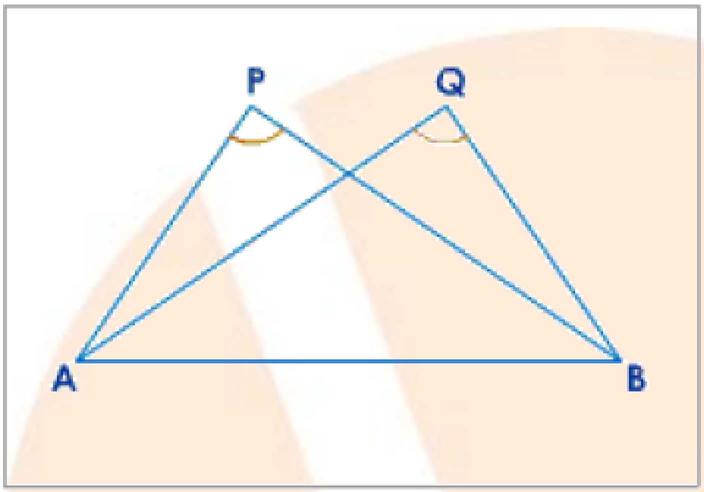
Statement	Reason
In ΔAPT and ΔTPB	Angles in alternate segment
$\angle A = \angle BTP$	
$\angle P = \angle P$	Common
$\therefore\ \Delta APT \sim \Delta TPB$	AA similarity
$\frac{AP}{PT} = \frac{PT}{BP}$	Statements (3)
$AP \times BP = PT^2$	Statements (4)

Test for Concyclic Points:

a) **Converse of the statement**, 'Angles in the same segment of a circle are equal', is one test for **concyclic** points.

We state:

If two equal angles are on the same side of a line and are subtended by it, then the four points are concyclic.
In the figure, if $\angle P = \angle Q$ and the points P, Q are on the same side of AB, then the points A, B, P and Q are **concyclic**.



b) Converse of 'opposite angles of a cyclic quadrilateral are supplementary' is one more test for concyclic points.

We state:

If the opposite angles of a quadrilateral are supplementary, then its vertices are concyclic.
In the figure, if $\angle A + \angle C = 180^\circ$ then A, B, C and D are concyclic points.