Revision Notes

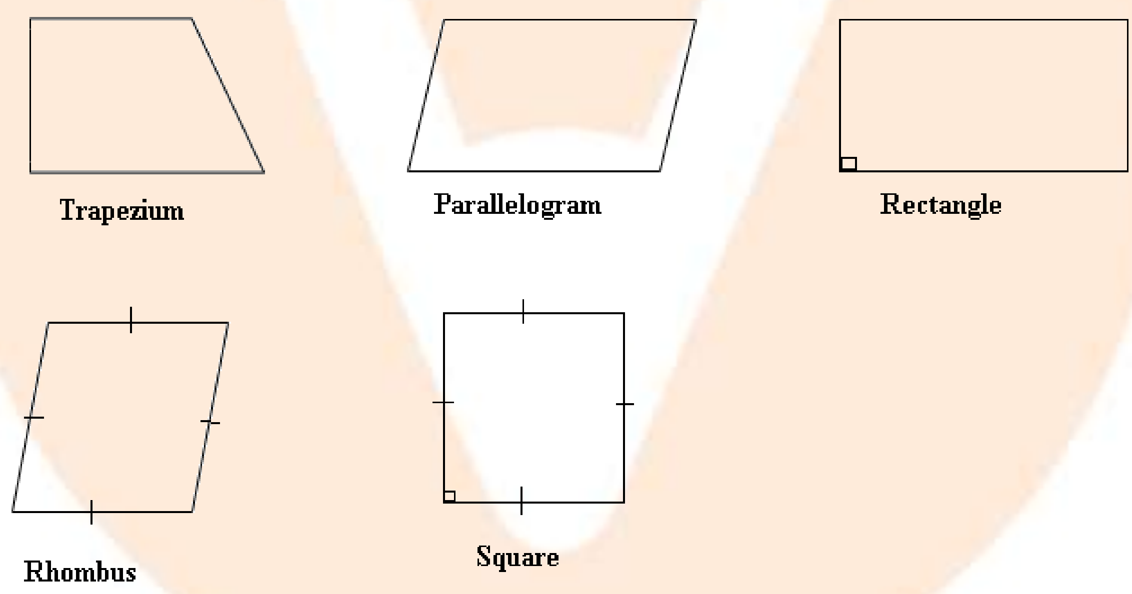
Class 9 Mathematics

Chapter 8 - Quadrilaterals

Introduction:

- We're all familiar with planar figures with sides defined by straight line segments, which are known as **Polygons**.
- The word **polygon** comes from the **Greek language**.
- It refers to a figure with a lot of angles, meaning a lot of sides.
- Quadrilaterals are squares, rectangles, and other four-sided geometric shapes produced by the union of four line segments.
- A **quadrilateral** is a **polygon with four sides**.

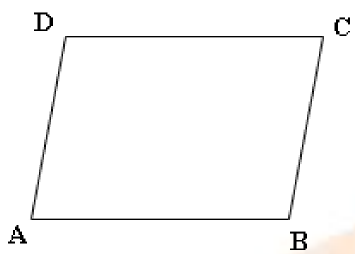
Examples of Quadrilaterals:



(All images are drawn by using the paint)

Parallelograms:

- A **parallelogram** is a quadrilateral with parallel and equal opposite sides.
- Parallelograms include a rectangle, a rhombus, and a square.
- A trapezium is a quadrilateral with one pair of opposite sides that are parallel to one other. As a result, it isn't a parallelogram.
- **The opposite sides of each pair are equal and parallel.**



● In the diagram,

Opposite sides:

$AB \parallel DC$ and $AD \parallel BC$

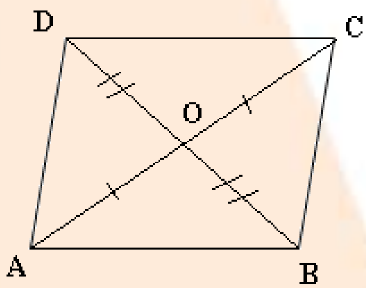
$AB = DC$ and $AD = BC$

● **Opposite angles are equal**.

From figure,

$A = C$ and $B = D$

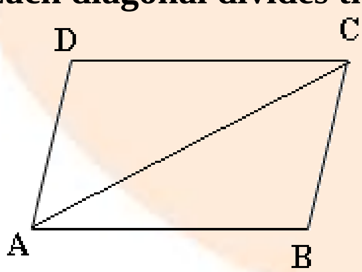
● **Diagonals of a parallelogram bisect each other**.

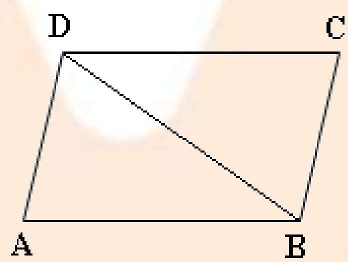


In the diagram,

$OD = OB$ and $OA = OC$

● **Each diagonal divides the parallelogram into two congruent triangles.**



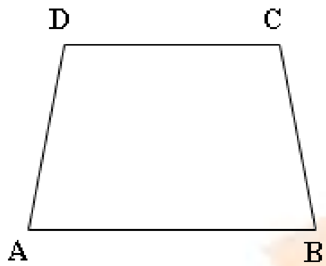


In the diagram,

$\triangle ABC \cong \triangle CDA$

$\triangle ABD \cong \triangle CDB$

Opposite Sides of a quadrilateral:



- Two sides of a quadrilateral, which have **no common point**, are called **opposite sides**.
- In the diagram, AB and DC is one pair of opposite sides.
- DC and BC is the other pair of opposite sides.

Consecutive sides of a quadrilateral:

- Two sides of a quadrilateral, which have a common end point, are called **consecutive sides**.
- In the diagram,

AB and BC is one pair of consecutive sides.

$BC, CD; CD, DA$ and DA, AB are the other three pairs of consecutive sides.

Opposite angles of a quadrilateral:

- Two angles, which do not include a side in their intersection, are called the **opposite angles** of a quadrilateral.
- In the diagram,

A and C is one pair of opposite angles, B and D are another pair of opposite angles.

Consecutive angles of a quadrilateral:

- Two angles of a quadrilateral, which include a side in their intersection, are called **consecutive angles.**
- In the diagram,

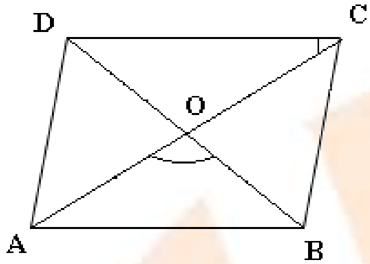
A and B is one pair of consecutive angles, $B, C; CD$ and DA are other three pairs of consecutive angles.

Theorem 1:
Statement:

- The diagonals of a parallelogram bisect each other.
- If two sides of a triangle are unequal, the longer side has the greater angle

opposite to it.

- ABCD is a parallelogram in which diagonals AC and BD intersect each other at O.



To prove:

The diagonals AC and BD bisect each other that is,

$AO = OC$ and

$BO = DO$.

Proof:

$AB \parallel CD$ (By definition of parallelogram)

AC is a transversal.

$\therefore OAB = OCB$ ..... (i) (Alternate angles are equal in a parallelogram)

Also,

$AB = DC$ (Opposite sides are equal in a parallelogram)

Now in ΔAOB and ΔCOD,

$AB = DC$ (Opposite sides of parallelogram are equal)

$OAB = OCD$ (Proved by (i))

$AOB = COD$ (Vertically opposite angles are equal)

Therefore,

$\triangle AOB \cong \triangle COD$ (AAS Congruency condition)

Therefore,

$AO = OC$ and $BO = OD$ (corresponding parts of congruent triangles are congruent) that is the diagonals of a parallelogram bisect each other.

Sufficient Conditions for a Quadrilateral to be a Parallelogram:

We can state the defining property of a parallelogram as follows:

"**If a quadrilateral is a parallelogram, then its opposite sides are equal**".

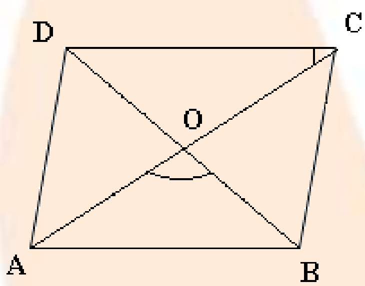
Converse:

"If both pairs of opposite sides of a quadrilateral are equal, then the

quadrilateral is a parallelogram".

- For a quadrilateral to be a parallelogram, the converse assertion given above is a necessary condition.
- Similarly, for a quadrilateral to be a parallelogram, we can establish the following two requirements;
  1. "If the diagonals of a quadrilateral bisect each other, the quadrilateral is a parallelogram".
  2. "If either pair of opposite sides of a quadrilateral are equal and parallel, the quadrilateral is a parallelogram".

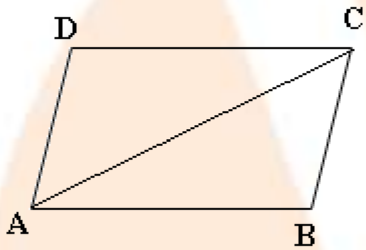
Theorem 2:
Statement:
If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.



Given:
$ABCD$ is a quadrilateral in which diagonals AC and BD intersect at O such that $AO = OC$ and $BO = OD$.
To prove:
$ABCD$ is a parallelogram.
Proof:
In triangles AOB and COD,
$AO = CO$ (Given)
$BO = OD$ (Given)
$AOB = COD$ (Vertically opposite angles are equal)
Therefore,
$\Delta AOB \cong \Delta COD$ (SAS Congruency condition)
Therefore,
$OAB = OCD$ (cpct)
Since these are alternate angles made by the transversal AC intersecting AB and CD

Therefore,
$AB \parallel CD$
Similarly,
$AD \parallel BC$
Hence, $ABCD$ is a parallelogram.

Theorem 3 :
Statement:
A quadrilateral is a parallelogram if one pair of opposite sides are equal and parallel.



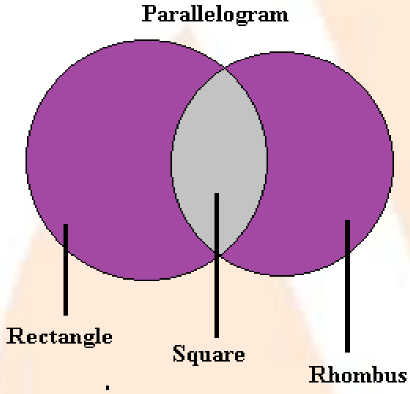
Given:
$ABCD$ is a quadrilateral in which $AB \parallel CD$ and $AB = CD$.
To prove:
$ABCD$ is a parallelogram.
Construction:
Join AC.
Proof:
In triangles ABC and ADC,
$AB = CD$ (Given)
$BAC = ACD$ (Alternate angles are equal)
$AC = AC$ (Common side)
Therefore,
$\Delta ABC \cong \Delta CDA$ (SAS Congruency condition)
$BCA = DAC$ (Corresponding parts of corresponding triangles)
Since these are alternate angles,
$AB \parallel CD$
Thus, in the quadrilateral $ABCD$, $AB \parallel CD$ and $AD \parallel BC$
Therefore, $ABCD$ is a parallelogram.

Special Parallelograms:

- The set of parallelograms includes rectangles, rhombuses, and squares.
- The following is a possible definitions for each of these:

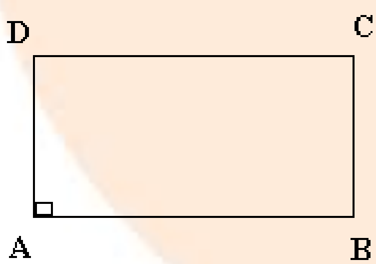
A is a parallelogram that is both equilateral and equiangular. As a result, a square can be both a rectangle and a rhombus.

- The relationships between the special parallelograms can be visualized in the diagram below:



- Because every rectangle and rhombus must be a parallelogram, they're depicted as subsets of a parallelogram, and because a square is both a rectangle and a rhombus, it's represented by the overlapping shaded part.

Rectangle:
A rectangle is a parallelogram with one of its angles as a right angle.



In the above figure,

Let, $A = 90^\circ$

Since,

$AD \parallel BC$,

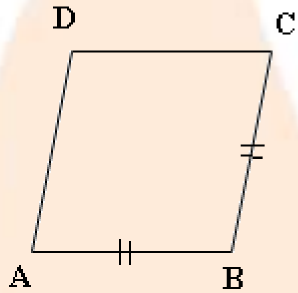
$A + B = 180^\circ$

(Sum of interior angles on the same side of transversal AB)

Therefore,

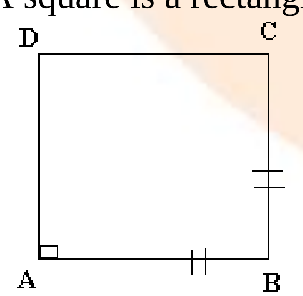
$B = 90^{\circ}$
Here,
$AB \parallel CD$ and $A = 90^{\circ}$ (Given)
Therefore,
$A + D = 90^{\circ}$
$\therefore D = 90^{\circ}$
$\therefore C = 90^{\circ}$
Corollary: Each of the four angles of a rectangle is a right angle.

Rhombus:
A rhombus is a parallelogram with a pair of its consecutive sides equal.



$ABCD$ is a rhombus in which $AB = BC$.
Since a rhombus is a parallelogram,
$AB = DC$ and $BC = AD$
Thus, $AB = BC = CD = AD$
Corollary: All the four sides of a rhombus are equal (congruent).

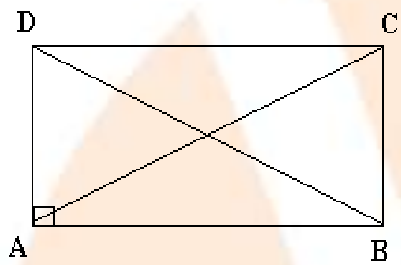
Square:
A square is a rectangle with a pair of its consecutive sides equal.



Since square is a rectangle, each angle of a rectangle is a right angle and $AB = DC$, $BC = CD$.
Thus,

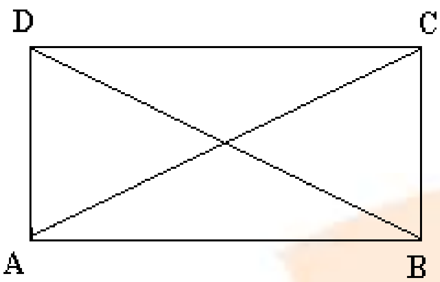
$AB = BC = CD = AD$
Each of the four angles of a square is a right angle and each of the four sides is of the same length.

Theorem 4:
Statement:
The diagonals of a rectangle are equal in length.



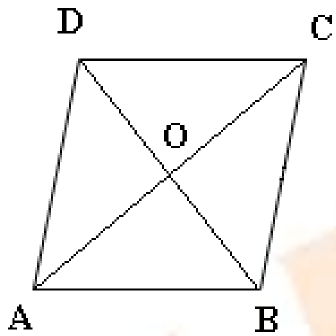
Given:
$ABCD$ is a rectangle.
AC and BD are diagonals.
To prove:
$AC = BD$
Proof:
Let, $A = 90^\circ$ **(**By definition of rectangle**)**
$A + B = 180^\circ$ **(**Consecutive interior angle**)**
$A = B = 90^\circ$
Now in triangles, ABD and ABC,
$AB = AB$ (Common side)
$A = B = 90^\circ$ (Each angle is a right angle)
$AD = BC$ (Opposite sides of parallelogram)
Therefore,
$\Delta ABD \cong \Delta BAC$
Therefore,
$BD = AC$ (Corresponding parts of corresponding triangles)
Hence the theorem is proved.

Converse of Theorem 4:
Statement:
If two diagonals of a parallelogram are equal, it is a rectangle.



Given:
$ABCD$ is a parallelogram in which $AC = BD$.
To prove:
Parallelogram $ABCD$ is a rectangle.
Proof:
In triangles ABC and DBC,
$AB = DC$ (Opposite sides of parallelogram)
$BC = BC$ (Common side)
$AC = BD$ (Given)
Therefore,
$\Delta ABC \cong \Delta DCB$ (SSS congruency condition)
Therefore,
$ABC = DCB$ (Corresponding parts of corresponding triangles)
But these angles are consecutive interior angles on the same side of transversal BC and $AB \parallel DC$.
Therefore,
$ABC + DCB = 180^\circ$
But,
$ABC = DCB$
Therefore,
$ABC = DCB = 90^\circ$
Therefore, by definition of rectangle, parallelogram $ABCD$ is a rectangle.
Hence the theorem is proved.

Theorem 5:
Statement:
The diagonals of a rhombus are perpendicular to each other.



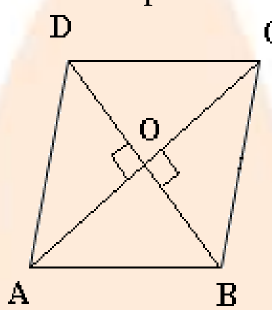
Given:
ABCD is a rhombus.
Diagonal AC and BD intersect at O.
To prove:
AC and BD bisect each other at right angles.
Proof:
A rhombus is a parallelogram such that
$\text{AB} = \text{DC} = \text{AD} = \text{BC} \quad ......(\text{i})$
Also the diagonals of a parallelogram bisect each other.
Hence,
$\text{BO} = \text{DO}$ and $\text{AO} = \text{OC} \quad ......(\text{ii})$
Now, compare triangles AOB and AOD,
$\text{AB} = \text{AD}$ (From (i) above)
$\text{BO} = \text{DO}$ (From (ii) above)
$\text{AO} = \text{AO}$ (Common side)
Therefore,
$\Delta\text{AOB} \cong \Delta\text{AOD}$ (SSS congruency condition)
Therefore,
$\text{AOB} = \text{AOD}$ (Corresponding parts of corresponding parts)
BD is a straight line segment.
Therefore,
$\text{AOB} + \text{AOD} = 180^\circ$
But,
$\text{AOB} = \text{AOD}$ (Proved)
Therefore,

$$AOB = AOD = \frac{180^\circ}{2}$$

$$AOB = AOD = 90^\circ$$

That is, the diagonals bisect at right angles.
Hence the theorem is proved.

Converse of Theorem 5:
Statement:
If the diagonals of a parallelogram are perpendicular then it is a rhombus.
Given:
$ABCD$ is a parallelogram in which AC and BD are perpendicular to each other.



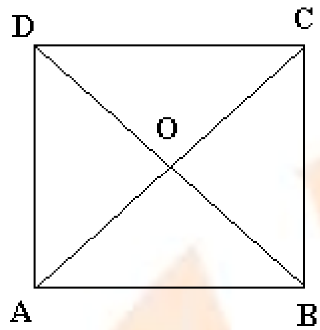
To prove:
$ABCD$ is a rhombus.
Proof:
Let AC and BD intersect at right angles at O.

$AOB = 90^\circ$

In triangles AOD and COD,
$AO = OC$ (Diagonals bisect each other)
$OD = OD$ (Common side)
$AOD = COD = 90^\circ$ (Given)
Therefore,
$\Delta AOD \cong \Delta COD$ (SAS congruency condition)
$AD = DC$
That is, the adjacent sides are equal.
Therefore, by definition, $ABCD$ is a rhombus.
Hence the theorem is proved.

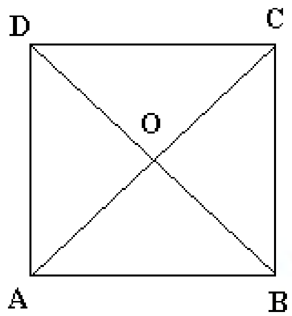
Theorem 6:

Statement:
The diagonals of a square are equal and perpendicular to each other.



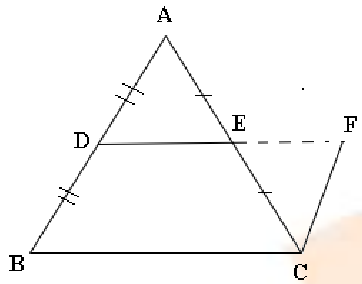
Given:
$ABCD$ is a square.
AC and BD are diagonals intersecting at O.
To prove:
$AC = BD$ and $AC \perp BD$
Proof:
$AB = AD$ (Sides of a square are equal)
$AB \| DC$ (Opposite sides of a square are parallel)
Therefore,
$ABCD$ is parallelogram with consecutive sides equal.
Hence, $ABCD$ is a rhombus. (By definition)
Since, the diagonals of a rhombus are perpendicular to each other,
$AC \perp BD$
Therefore,
$ABCD$ is a parallelogram.
$AB = AD$ and $A = 90^{\circ}$
Therefore, $ABCD$ is a rectangle with a pair of its consecutive sides equal.
Since the diagonals of a rectangle are equal, $AC = BD$.
Therefore,
Diagonal AC = Diagonal BD and $AC \perp BD$
Hence, the theorem is proved.

Converse of Theorem 6:
Statement:
If in a parallelogram, the diagonals are equal and perpendicular, then it is a square.



Given:
$ABCD$ is a parallelogram.
$AC = BD$ and $AC \angle BD$
To prove:
$ABCD$ is a square.
Proof:
Since the diagonals AC and BD are equal,
$ABCD$ is a rectangle $----(\mathrm{i})$
(Diagonal property of rectangle)
Since the diagonals are perpendicular to each other.
$ABCD$ is a rhombus.
Therefore,
$AB = AD ----(\mathrm{ii})$
$ABCD$ is a rectangle. (From (i))
With consecutive sides equal. (From (ii))
Therefore,
$ABCD$ is a square. (By definition of a square)
Hence, the theorem is proved.

The Mid-point Theorem:
Parallel lines and Triangles
So far, we've proved a number of parallelogram theorems. Now let's use these theorems to prove a few interesting and helpful triangle facts.
Statement:
"The line segment joining the mid-points of any two sides of a triangle is parallel to the third side and equal to half of it".



Given:

In ΔABC,

$AB = DB$ and $AE = EC$

To prove:

i. $DE \parallel BC$

ii. $DE = \frac{1}{2}BC$

Construction:

Analysis for construction shows that you need to draw $CF \parallel BD$.

A parallelogram with DB and BC as consecutive sides.

Proof:

In triangles, ADE and CEF,

$AE = EC$ (Given)

$AED = CEF$ (Vertically opposite angles)

$DAE = ECF$ (Alternate angles, $AD \parallel CF$ by construction)

Therefore,

$\Delta ADE \cong \Delta CFE$ (ASA Congruency test)

Therefore,

$AD = CF$ and $DE = EF$ (Corresponding parts of corresponding triangles)

But

$AD = DB$ (Given)

Therefore,

$DB = CF \quad .....(i)$

(AD is equal to both DB and CF)

In quadrilateral $DBCF$,

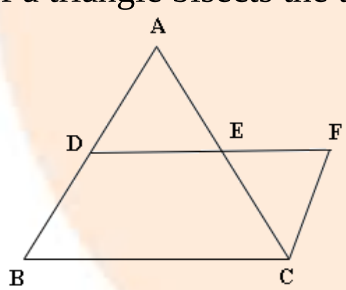
$DB = CF$ and $DB \parallel CF$

Therefore,

$DBCF$ is a parallelogram. (By definition of parallelogram)

Therefore,
$DF = BC$ (Opposite sides of a parallelogram are equal)
and
$DF \parallel BC \quad .....(ii)$
But
$DE = EF$ (Proved above)
and
$DF = DE + EF$
$DF = 2DE$
and $DF = BC$ (From (ii))
Therefore,
$BC = 2DE$ or
$DE = \frac{1}{2}BC$
Hence, the theorem is proved.

Converse of Mid-point Theorem:
Statement:
A straight line drawn through the mid-point of one side and parallel to another side of a triangle bisects the third side.



Given:
ΔABC in which D is the mid-point of AB and $DE \parallel BC$.
To prove:
E is the mid-point of AC. That is, to prove $AE=EC$.
Construction:
Since $DE \parallel BC$, we can complete a parallelogram with DB and BC as consecutive sides.
Hence draw $EF \parallel BD$ to meet DE produced at F.
Proof:

In quadrilateral $DBCF$,

$DB \parallel CF$ (By construction)

$DF \parallel BC$ (Given)

Therefore,

$DBCF$ is a parallelogram.

$DB = CF \quad .....(i)$ (Opposite sides of a parallelogram)

But

$DB = AD \quad .....(ii)$ (Given)

Now, from (i) and (ii),

$AD = CF$

Now compare triangles, AED and CEF,

$AD=CF$

$AED = ECF$ (Vertically opposite angles)

$DAE = ECF$ (Alternate angles, $AD \| CF$ by construction)

Therefore,

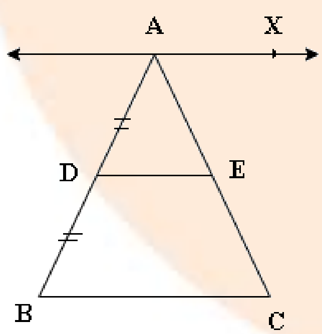
$\Delta AED \cong \Delta CEF$ (ASA Congruency test)

$\angle AE = EC$ CPCT

That is, E is the midpoint of AC.

Hence, the theorem is proved.

Recall the above **theorem** and apply it to the diagram given.

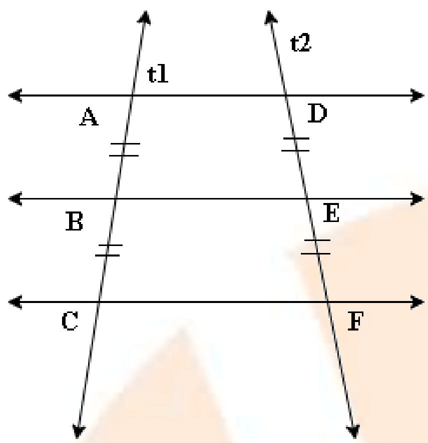


In the diagram if D is the mid-point of AB and DE is drawn parallel to BC, then E will be the midpoint of AC.

That is,

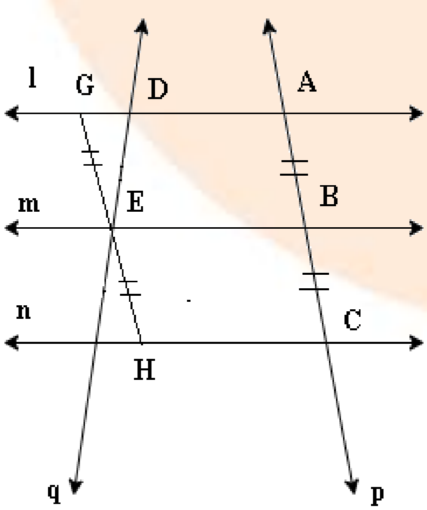
$AE = EC$

Now if AX is drawn parallel to BC, then also $AE = EC$ if $AD = DB$.



Now draw three parallel lines AB, CD, EF as shown in the figure.
Draw a transversal t_1 such that $AB = BC$.
Now draw another transversal t_2.
Measure DE and EF.
We will get find that $DE = EF$ and $AB = BC$.
In the diagram, AD, BE and CF are three parallel lines.
AB and BC are equal intercepts made on t_1.
If any transversal is drawn, the intercepts made on it will also be **equal**.

The Intercept Theorem:
Statement:
If there are three or more parallel lines and the intercepts made by them on one transversal are equal, the corresponding intercepts of any transversal are also equal.



Given:

$l \parallel m \parallel n$

P is a transversal intersecting l, m and n at A, B and C respectively such that $AB = BC$.

Q is another transversal drawn to cut l, m and n at D, E and F respectively.

DE and EF are the intercepts made on q.

To prove:

$DE = EF$

Construction:

Draw a line through E parallel to p intersecting l in G, n in H respectively.

Proof:

$AG \parallel BE$ (Given)

$GE \parallel AB$ (By construction)

Therefore,

$AGEB$ is a parallelogram.

Therefore,

$GE = AB \quad .....(i)$ (Opposite sides of parallelogram)

Similarly, $BEHC$ is a parallelogram.

Therefore,

$EH = BC \quad .....(ii)$ (Opposite sides of parallelogram)

But

$AB = BC$ (Given)

From (i) and (ii),

$GE = EH$

Now, compare triangles GED and EFH,

$GE = EH$ (Proved)

$GED = FEH$ (Vertically opposite angles)

$DGE = FHE$ (Alternate angles, $GD \parallel FH$ by construction)

Therefore,

$\Delta GED \cong \Delta HEF$ (ASA Congruency test)

Therefore,

$DE = EF$ (Corresponding parts of corresponding triangles)

Hence, the theorem is proved.