Algorithms: Theory, Design and Implementation

ANALYSIS OF ALGORITHMS

- *Introduction*
- *Observations via exercises*
- *Mathematical models and theory (optional)*
- *Order-of-growth classifications*

RECAP OF PREVIOUS WEEK

- We already studied various problems (e.g., connectivity, multiplication of integer numbers).
- We elaborated various approaches towards algorithmic solution of the problem.
- We discussed the Karatsumba algorithm in the lecture.

THIS WEEK (LW2)

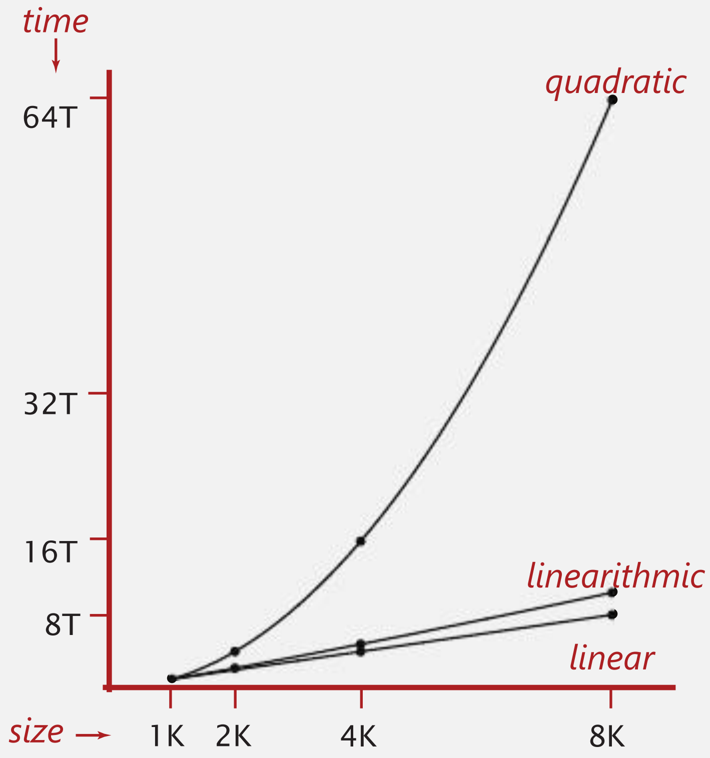
- We will learn what it means to critically compare, evaluate and contrast various algorithmic approaches, designs and implementations, in terms of:
  - Time being spent to computationally resolve the problem
  - Scalability when it comes to operate with different input data sizes
  - Practicality when it comes to keeping things as they are due to their simplicity
- We will start getting an idea of what is considered as **Brute Force strategy** in algorithmic design and implementation

Some more practical algorithmic successes

Discrete Fourier Transformation.

Friedrich Gauss
1805

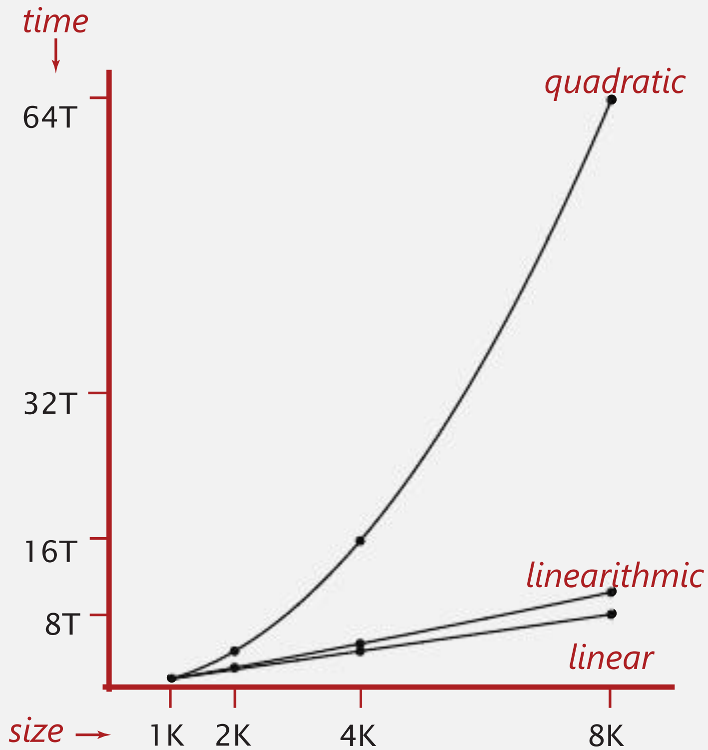
- Break down waveform of N samples into periodic components.
- Applications: DVD, JPEG, MRI, astrophysics, ….
- Brute force: N^2 steps.
- FFT algorithm: $N \log N$ steps, enables new technology.

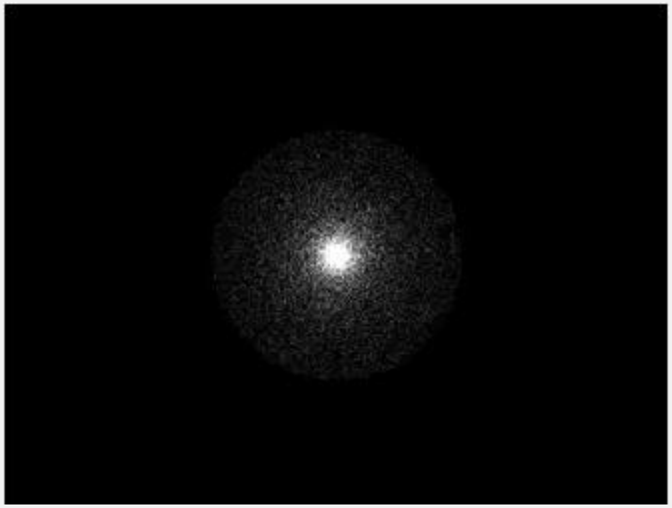


Some algorithmic successes

N-body simulation.

- Simulate gravitational interactions among N bodies.
- Brute force: N^2 steps.
- Barnes-Hut algorithm: $N \log N$ steps, enables new research.





Scientific method applied to analysis of algorithms

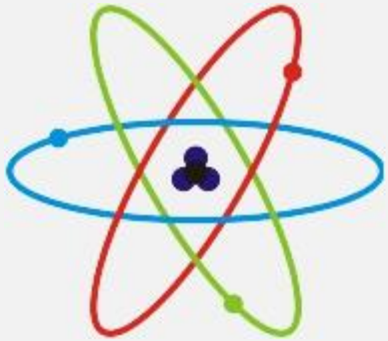
A framework for predicting performance and comparing algorithms.

Scientific method.

- Observe some feature of the natural world.
- Hypothesize a model that is consistent with the observations.
- Predict events using the hypothesis.
- Verify the predictions by making further observations.
- Validate by repeating until the hypothesis and observations agree.

Principles.

- Experiments must be reproducible.
- Hypotheses must be falsifiable.



Feature of the natural world. Computer itself.

Analysis of Algorithms

- *Introduction*
- ***Observations via exercises***
- *Mathematical models*
- *Order-of-growth classifications*

Example: 3-SUM

3-SUM. Given N distinct integers, how many triples sum to exactly zero?

```
% more 8ints.txt
8
30 -40 -20 -10 40 0 10 5

% java ThreeSum 8ints.txt
4
```

	a[i]	a[j]	a[k]	sum
1	30	-40	10	0
2	30	-20	-10	0
3	-40	40	0	0
4	-10	0	10	0

Context. Deeply related to problems in computational geometry.

3-SUM: brute force algorithm

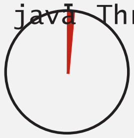
```
public  class  ThreeSum
{
        public  static  int  count(int[]  a)
        {
                int  N  =  a.length;
                int  count  =  0;
                // Check each triple of indices (i, j, k)
                //  We can restrict to i < j < k: re-ordering gives the same sum
                for (int i = 0; i < N; i++)
                        for (int j = i+1; j < N; j++)
                                for (int k = j+1; k < N; k++)
                                        // Ignore possible overflow for simplicity
                                         if (a[i] + a[j] + a[k] == 0)
                                                count++;
        }
        return count;
}
```

Measuring the running time

Q. How to time a program?

A. Manual.

```
% java ThreeSum 1Kints.txt
```



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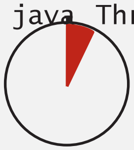
```
70
% java ThreeSum 2Kints.txt
```



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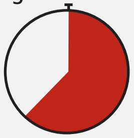
```
528
% java ThreeSum 4Kints.txt
```



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```
4039
```

Observing the running time of a program

Measuring the running time

Q. How to time a program?

A. Automatic.

`public class Stopwatch` (part of `stdlib.jar`)	
`Stopwatch()`	*create a new stopwatch*
`double elapsedTime()`	*time since creation (in seconds)*

```
public static void main(String[] args)
{
   int[] a = In.readInts(args[0]);
   Stopwatch stopwatch = new Stopwatch();
   StdOut.println(ThreeSum.count(a));
   double time = stopwatch.elapsedTime();
}
```

client code

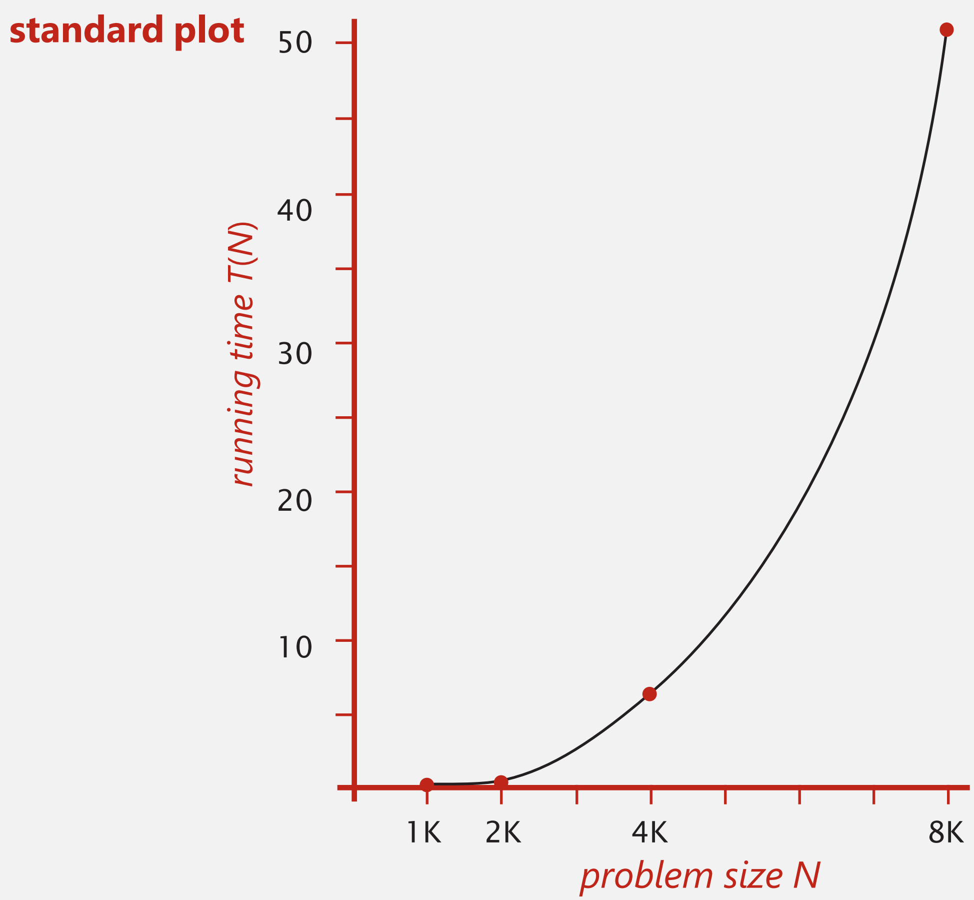
Empirical analysis

Run the program for various input sizes and measure running time.

N	time (seconds) †
250	0.0
500	0.0
1,000	0.1
2,000	0.8
4,000	6.4
8,000	51.1
16,000	?

Data analysis

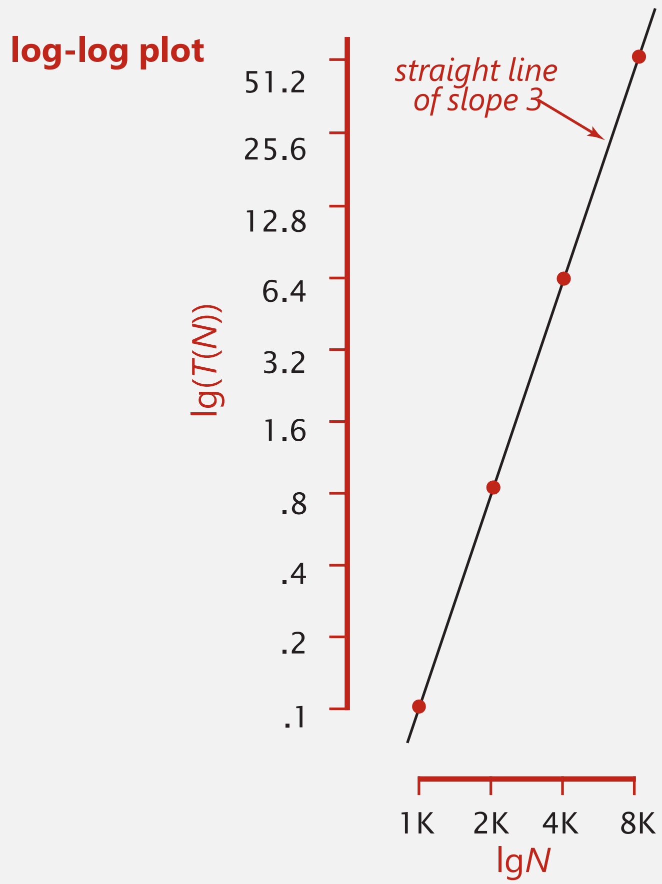
Standard plot. Plot running time $T(N)$ vs. input size N.



Analysis of experimental data (the running time ofThreeSum)

Data analysis

Log-log plot. Plot running time $T(N)$ vs. input size N using log-log scale.



$$\lg(T(N)) = b \lg N + c$$
$$b = 2.999$$
$$c = -33.2103$$

$$T(N) = a\, N^{b}, \textit{ where } a = 2^{c}$$

power law

Regression. Fit straight line through data points: $a\, N^{b}$

Hypothesis. The running time is about $1.006 \times 10^{-10} \times N^{2.999}$ seconds.

slope

Prediction and validation

Hypothesis. The running time is about $1.006 \times 10^{-10} \times N^{2.999}$ seconds.

"order of growth" of running time is about N power 3.... [stay tuned]

Predictions.

- 51.0 seconds for $N = 8{,}000$.
- 408.1 seconds for $N = 16{,}000$.

Observations.

N	time (seconds) †
8,000	51.1
8,000	51.0
8,000	51.1
16,000	410.8

validates hypothesis!

Doubling hypothesis

Doubling hypothesis. Quick way to estimate *b* in a power-law relationship.

Run program, doubling the size of the input.

N	time (seconds) †	ratio	lg ratio
250	0.0		–
500	0.0		
1,000	0.1		
2,000	0.8	7.7	2.9
4,000	6.4	8.0	3.0
8,000	51.1	8.0	3.0

seems to converge to a constant b ≈ 3

3-SUM: sorting-based algorithm

```
public  class  ThreeSum
{
        public  static  int  count(int[]  a)
        {
                // First, sort the array in ascending order
                sort(a);
                int  N  =  a.length;
                int  count  =  0;
                // For each pair of indices (i, j):
                // Check if the array contains -a[i]-a[j]
                // Using binary search, this only takes logarithmic time
                for (int i = 0; i < N; i++)
                        for (int j = i+1; j < N; j++)
                                if (binarySearch (a, -a[i]-a[j]))
                                        count++;
        }
        return count;
}
```

Comparing programs

Hypothesis. The sorting-based $N^2 \log N$ algorithm for 3-SUM is significantly faster in practice than the brute-force N^3 algorithm.

N	time (seconds)
1,000	0.1
2,000	0.8
4,000	6.4
8,000	51.1

ThreeSum.java

N	time (seconds)
1,000	0.14
2,000	0.18
4,000	0.34
8,000	0.96
16,000	3.67
32,000	14.88
64,000	59.16

ThreeSumDeluxe.java

Guiding principle. Typically, better order of growth $\rightarrow$ faster in practice.

How did we improve it?

Hypothesis. The sorting-based $N^2 \log N$ algorithm for 3-SUM is significantly faster in practice than the brute-force N^3 algorithm.

N	time (seconds)
1,000	0.1
2,000	0.8
4,000	6.4
8,000	51.1

ThreeSum.java

N	time (seconds)
1,000	0.14
2,000	0.18
4,000	0.34
8,000	0.96
16,000	3.67
32,000	14.88
64,000	59.16

ThreeSumDeluxe.java

Answer. Sorting + Binary Search [stay tuned….].

Experimental algorithmics

System independent effects.

- Algorithm.
- Input data.

determines exponent b in power law

System dependent effects.

- Hardware: CPU, memory, cache, …
- Software: compiler, interpreter, garbage collector, …
- System: operating system, network, other apps, …

determines constant a in power law

Bad news. Difficult to get precise measurements.

Good news. Much easier and cheaper than other sciences.

e.g., can run huge number of experiments

Analysis of Algorithms

- *introduction*
- *observations*
- *mathematical models (optional)*
- ***order-of-growth classifications***

Common order-of-growth classifications

order of growth	name	typical code framework	description	example	T(2N) / T(N)
1	constant	`a = b + c;`	statement	add two numbers	1
log N	logarithmic	`while (N > 1) { N = N / 2; ... }`	divide in half	binary search	~1
N	linear	`for (int i = 0; i < N; i++) { ... }`	loop	find the maximum	2
N log N	linearithmic	[see mergesort lecture]	divide and conquer	mergesort	~2
N^2	quadratic	`for (int i = 0; i < N; i++) for (int j = 0; j < N; j++) { ... }`	double loop	check all pairs	4
N^3	cubic	`for (int i = 0; i < N; i++) for (int j = 0; j < N; j++) for (int k = 0; k < N; k++) { ... }`	triple loop	check all triples	8
2^N	exponential	[see combinatorial search lecture]	exhaustive search	check all subsets	T(N)

Practical implications of order-of-growth

growth rate	problem size solvable in minutes			
	1970s	1980s	1990s	2000s
1	any	any	any	any
log N	any	any	any	any
N	millions	tens of millions	hundreds of millions	billions
N log N	hundreds of thousands	millions	millions	hundreds of millions
N^2	hundreds	thousand	thousands	tens of thousands
N^3	hundred	hundreds	thousand	thousands
2^N	20	20s	20s	30

Bottom line. Need linear or linearithmic alg to keep pace with Moore's law.

TIME AND SPACE COMPLEXITY

For a deeper understanding of asymptotics, Big-O notation and algorithmic performance analysis, please check the uploaded transcript and videos, as well as recommendation from reading list.