6CCS3AIN: Artificial Intelligence Reasoning and Decision Making

Week 9: Consensus Mechanisms Part B: Voter Model

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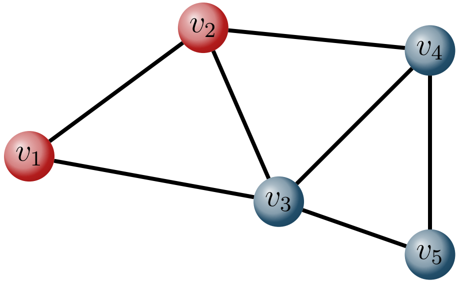
(version 1.1)

Motivation

Motivational Example

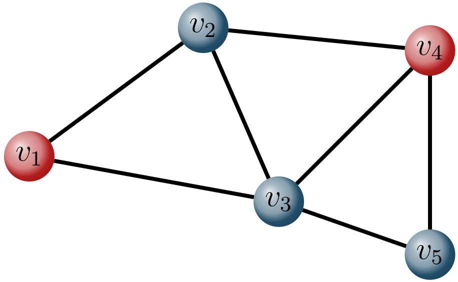


$t = 0$

- Consider a graph such as the one above.
  - Nodes in this graph represent agents.
  - Colour of nodes represent their current opinion. **(it may change over time!)**
  - Edges represent what other agents a node sees.
- Their goal: to reach consensus. **(i.e., all agents with same colour)**

Idea They can learn from their

Motivational Example



$t = 1$

- Consider a graph such as the one above.
    - Nodes in this graph represent agents.
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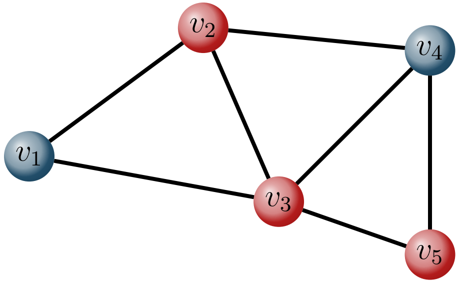
    - Edges represent what other agents a node sees.
- Their goal: to reach consensus. **(i.e., all agents with same colour)**

Idea They can learn from their neighbours

Motivational Example

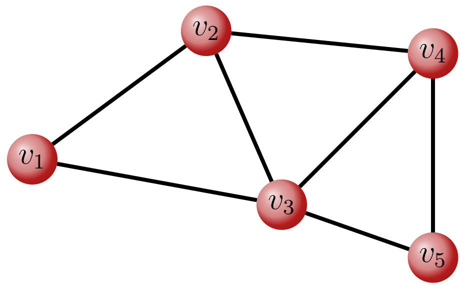


$t = 2$

- Consider a graph such as the one above.
  - Nodes in this graph represent agents.
  - Colour of nodes represent their current opinion. **(it may change over time!)**
  - Edges represent which other agents a node sees.
- Their goal: to reach consensus. **(i.e., all agents with same colour)**

> **Idea** They can learn from their neighbours over

Motivational Example

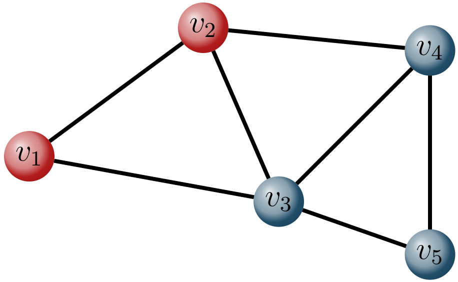


$t = 3$

- Consider a graph such as the one above.
  - Nodes in this graph represent agents.
  - Colour of nodes represent their current opinion. **(it may change over time!)**
  - Edges represent what other agents a node sees.
- Their goal: to reach consensus. **(i.e., all agents with same colour)**

Idea They can learn from their neighbours over time.

Motivational Example



$t = 0$

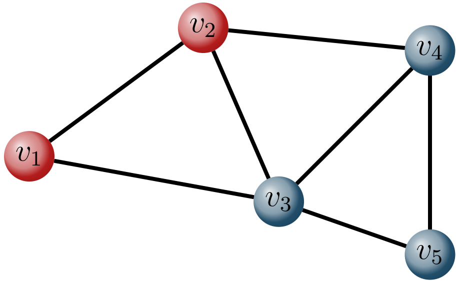
- There are many algorithms that can be considered in this case. Today, we are focusing on the following, usually called **voter model**.
  - The process happens in **rounds**, or time-steps, denoted by t.
  - At each round t, each agent v **selects** one of its neighbours u uniformly at random.
  - Then, v **adopts** u's colours for next round.

> To avoid ambiguity, we consider that they all change at the same time at the end of round t.

Some Notation

- First we assume there is a time variable $t \in T$ that represent rounds. In our case, T is the set of non-negative integers $0, 1, \ldots$.
- We have an undirected connected graph $G = (V, E)$.
  - An edge (v, u) indicates that v sees u and u sees v.
  - The number of neighbours of v is denoted by $\deg(v)$.
- A set of colours or opinions X. **(in our example, $X = \{b, r\}$)**
- For each time $t \in T$, the current state of the process is described by a function $S_t : V \to X$.
- We are interested in the consensus states. When, for a given t, we have $S_t(v) = b$ for all $v \in V$, we say $S_t = \text{blue}$. Analogously for red, we have $S_t = \text{red}$.
  - In these cases, we say the processes **ends**, and we denote this by saying that $S_{end} = \text{blue}$, or $S_{end} = \text{red}$.

Examples of update rules



$t = 0$

1) What is $Pr(S_1(v_1) = r)$?

2) What is $Pr(S_1(v_4) = b)$?

3) What is $Pr(S_1(v_5) = r)$?

4) What is $Pr(S_1 = \text{blue})$?

5) What is $Pr(S_1 = \text{red})$?

Questions

- Is consensus stable? I.e., once consensus is reached, will it be maintained indefinitely? In other words,

$$S_{t_0} = \text{blue} \quad \Rightarrow \quad S_t = \text{blue, for all } t > t_0?$$

  same for red.

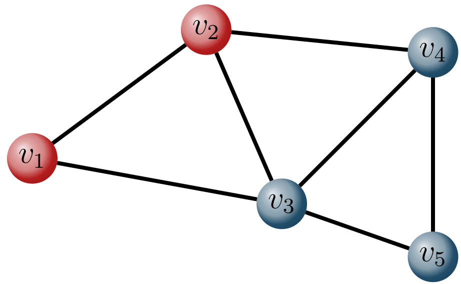
- Can the process ever get stuck and never find a consensus state? **(exercise!)**

- Imagine you are not an agent in this process, but you have the power to flip the colour of an agent before anyone makes a decision in a given round. Which agent is the most influential, i.e., the best choice?

- And our key point:

> **Question** Given an initial configuration S_0, what is the probability that the game ends in blue consensus (same for red)?

Probabilities of Consensus

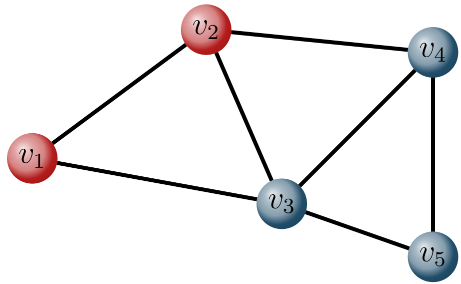


$t = 0$

- What counts for an opinion's advantage?

Option (1) The number of nodes of a given colour.

Probabilities of Consensus



$t = 0$

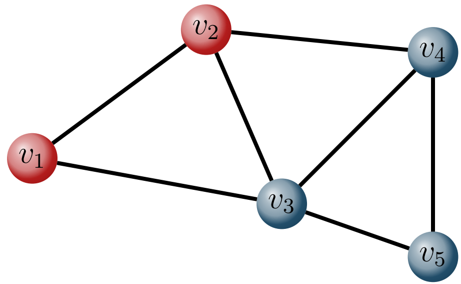
- What counts for an opinion's advantage?

Option (1)	The number of nodes of a given colour.

✗

Option (2)	The sum of degrees of nodes with a given colour.

Probabilities of Consensus

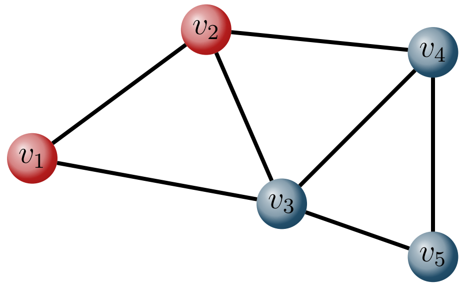


$t = 0$

- What counts for an opinion's advantage?

Option (1)	The number of nodes of a given colour.	✗
Option (2)	The sum of degrees of nodes with a given colour.	✓

Probabilities of Consensus



$t = 0$

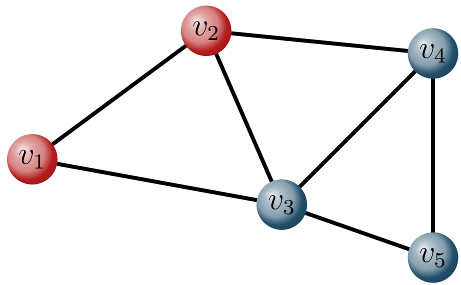
- What counts for an opinion's advantage? **(proof in extra material!)**

Theorem Let $G = (V, E)$ be a graph that do not allow deadlocks in a voter model process, i.e., a consensus is reached with probability 1. Given an initial configuration $S_0 = s$, the probability of blue winning is given by

$$Pr(S_{end} = \text{blue} \mid S_0 = s) = \sum_{v \in V, s(v) = b} \frac{\deg(v)}{2|E|}$$

analogously for red.

Probabilities of Consensus - Example



$t = 0$

- We apply the theorem to the example above.

$$Pr(S_{end} = \text{blue} \mid S_0 = s) = \frac{4+3+2}{14} = \frac{9}{14} \approx 64\%$$

$$Pr(S_{end} = \text{red} \mid S_0 = s) = \frac{2+3}{14} = \frac{5}{14} \approx 36\%$$

Some easy and not so easy follow up questions

- Can we say something about the expected time this will take to converge to consensus? **(see** *Hassin and Peleg, 2001* **and subsequent work)**
- What happens if the graph is not undirected but instead a directed graph with weighted edges? **(see Linear Voting Model** *Cooper and Rivera, 2016***)**
  - We consider that, in this case, the weight of the edge (v, u) represents the probability that v copies colours of u in a given round.
- What is there is some sort of bias toward a colour? For example, if when an agent has a blue and a red neighbour, they may have a higher chance of choosing, say, colour blue. **(see Moran Processes)**

Why not implement this yourself and check if our probabilities check out? Once you are there, make sure to calculate how many rounds it takes on average.

The End

Some references

[Cooper and Rivera, 2016] Cooper, C. and Rivera, N. (2016). The Linear Voting Model: Consensus and Duality.

[Hassin and Peleg, 2001] Hassin, Y. and Peleg, D. (2001). Distributed probabilistic polling and applications to proportionate agreement. *Information and Computation*, 171(2):248–268.