Rotations and Eigenstuff

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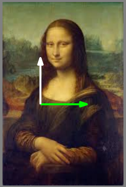
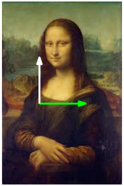
6CCS3AIN

Transformation Matrix



- Let’s say our ’data’ is the Mona Lisa
- Our base vectors are $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (in green) and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Transformation Matrix

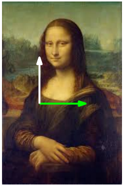


original

transformed

- We can stretch along the y-axis and squish along the x-axis with the matrix
$$\begin{pmatrix} 0.5 & 0 \\ 0 & 2 \end{pmatrix}$$

Transformation Matrix

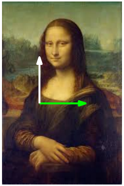


original

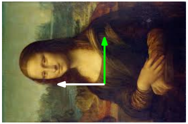
transformed

- We can stretch along the y-axis and squish along the x-axis with the matrix $\begin{pmatrix} 0.5 & 0 \\ 0 & 2 \end{pmatrix}$
- To see this, calculate $\begin{pmatrix} 0.5 & 0 \\ 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0.5x \\ 2y \end{pmatrix}$

Transformation Matrix



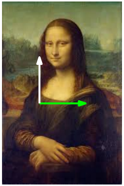
original



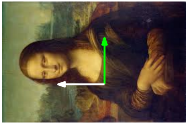
transformed

- We rotate counterclockwise
  $$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Transformation Matrix



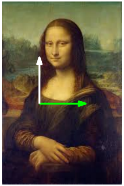
original



transformed

- We rotate counterclockwise
$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$
- To see this, calculate $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$

Transformation Matrix



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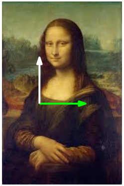
transformed

- We can also perform a so-called shear mapping $\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$. Here $m \approx 1$.

Transformation Matrix



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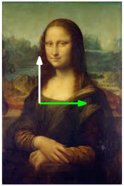
- We can also perform a so-called shear mapping $\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$. Here $m \approx 1$.
- To see this, calculate $\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + my \\ y \end{pmatrix}$

Transformation Matrix

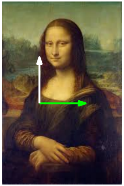


original

transformed

- Notice what happened to the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$? Nothing.

Transformation Matrix

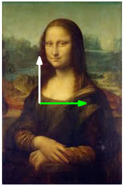


original

transformed

- Notice what happened to the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$? Nothing.
- When a vector doesn't change its direction after multiplying with a matrix, then it's an eigenvector.

Transformation Matrix



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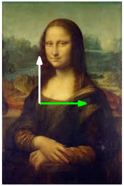
- In our stretching example from earlier, both vectors were actually eigenvectors.
- They didn't change the direction.
- However, they change the length.
- The value by which is changed the length is called the eigenvalue λ.

Transformation Matrix



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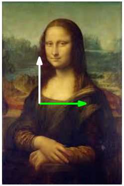
- The largest eigenvalue is $\lambda_1 = 2$ and the second largest here is $\lambda_2 = 0.5$.
- The corresponding eigenvectors are $\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Transformation Matrix



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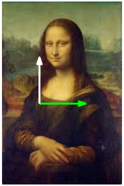
- What is an eigenvector for the matrix $\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$?

Transformation Matrix



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transformed

- What is an eigenvector for the matrix $\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$?
- We can see that $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ must be one. What's the formula?

Formula

- A vector $\mathbf{v}$ is an eigenvector of the matrix $\mathbf{M}$ if

$$\mathbf{M} \cdot \mathbf{v} = \lambda \mathbf{v}.$$

- λ is the corresponding eigenvalue.

Formula

- A vector $\mathbf{v}$ is an eigenvector of the matrix $\mathbf{M}$ if

$$\mathbf{M} \cdot \mathbf{v} = \lambda \mathbf{v}.$$

- λ is the corresponding eigenvalue.
- Consider $\mathbf{M} = \begin{pmatrix} 2 & 2 \\ 5 & -1 \end{pmatrix}$.
- We can verify that $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\lambda_1 = 4$:

$$\mathbf{M} \cdot \mathbf{v}_1 = \begin{pmatrix} 2 & 2 \\ 5 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} = 4 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \lambda_1 \mathbf{v}_1$$

- The second eigenvector is $\mathbf{v}_2 = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$ what's λ_2?