The next four exercises develop the concept of *direct limits* and the "dual" notion of *inverse limits*. In these exercises I is a nonempty index set with a partial order $\leq$ (cf. Appendix I). For each $i \in I$, let A_i be an additive abelian group. In Exercise 8 assume also that I is a *directed set*: for every $i, j \in I$ there is some $k \in I$ with $i \leq k$ and $j \leq k$.

8. Suppose for every pair of indices i, j with $i \leq j$ there is a map $\rho_{ij} : A_i \to A_j$ such that the following hold:

(i) $\rho_{ik} = \rho_{jk} \circ \rho_{ij}$ whenever $i \leq j \leq k$, and

(ii) $\rho_{ii} = 1$ for all $i \in I$.

Let B be the disjoint union of all the A_i. Define a relation $\sim$ on B by

$$a \sim b \quad \text{if and only if there exists } k \text{ with } i, j \leq k \text{ and } \rho_{ik}(a) = \rho_{jk}(b),$$

for $a \in A_i$ and $b \in A_j$.

(a) Show that $\sim$ is an equivalence relation on B. (The set of equivalence classes is called the *direct* or *inductive limit* of the directed system $\{A_i\}$, and is denoted $\varinjlim A_i$. In the remaining parts of this exercise let $A = \varinjlim A_i$.)

(b) Let $\bar{x}$ denote the class of x in A and define $\rho_i : A_i \to A$ by $\rho_i(a) = \bar{a}$. Show that if each ρ_{ij} is injective, then so is ρ_i for all i (so we may then identify each A_i as a subset of A).

(c) Assume all ρ_{ij} are group homomorphisms. For $a \in A_i$, $b \in A_j$ show that the operation
$$\bar{a} + \bar{b} = \overline{\rho_{ik}(a) + \rho_{jk}(b)}$$
where k is any index with $i, j \leq k$, is well defined and makes A into an abelian group. Deduce that the maps ρ_i in (b) are group homomorphisms from A_i to A.

(d) Show that if all A_i are commutative rings with 1 and all ρ_{ij} are ring homomorphisms that send 1 to 1, then A may likewise be given the structure of a commutative ring with 1 such that all ρ_i are ring homomorphisms.

(e) Under the hypotheses in (c) prove that the direct limit has the following *universal property*: if C is any abelian group such that for each $i \in I$ there is a homomorphism $\varphi_i : A_i \to C$ with $\varphi_i = \varphi_j \circ \rho_{ij}$ whenever $i \leq j$, then there is a unique homomorphism $\varphi : A \to C$ such that $\varphi \circ \rho_i = \varphi_i$ for all i.

Sol.

(a) Let $x \in B$. Then there is s such that $x \in A_s$. Choosing $i = j = k = s$, we see that $\sim$ is *relfexive*. By symmetry of $=$, the symmetry of $\sim$ follows directly. Let $a \sim b$ and $b \sim c$. Let $\rho_{ik}(a) = \rho_{jk}(b)$ and let $\rho_{jt}(b) = \rho_{st}(c)$. WLOG, let $k \leq t$. Then $\rho_{it}(a) = \rho_{kt} \circ \rho_{ik}(a) = \rho_{kt} \circ \rho_{jk}(b) = \rho_{jt}(b) = \rho_{st}(c)$. Thus $\sim$ is transitive.

(b) Let $a, b \in A_i$ with $a \neq b$. By injectivity, $\rho_{ik}(b) \neq \rho_{ik}(b)$ for all $k \geq i, j$. Thus, $a \nsim b$.

(c) For the addition to be well-defined, it should have the same value regardless of the choice of a and b as long as they are picked for their respective equivalence classes. Let $x \sim a$ and $y \sim b$. Let $\rho_{it}(a) = \rho_{st}(x)$ and $\rho_{je}(b) = \rho_{de}(y)$. WLOG, let $t \geq e$. If $k \geq t$, we are done. Otherwise, $\rho_{kt}(\rho_{ik}(a) + \rho_{jk}(b)) = \rho_{it}(a) + \rho_{jt}(b) = \rho_{st}(x) + \rho_{dt}(y) = \rho_{et}(\rho_{se}(x) + \rho_{de}(y))$. Thus $+$ is well-defined.

A is then an abelian group because if $\bar{a}, \bar{b} \in A$, then $\bar{a} - \bar{b} \in A$ and $\bar{0} \supseteq \{0_{A_i}\}_{i \in I} \in A$. It follows that ρ_i are group homomorphisms because $\rho_i(a+b) = \overline{a+b} = \bar{a} + \bar{b}$ (taking $k = i$) $= \rho_i(a) + \rho_i(b)$.

(d) A is still an additive abelian group but now commutitive multiplicative structure is built upon it. The multiplication given by

$$\bar{a} \cdot \bar{b} = \overline{\rho_{ik}(a) \cdot \rho_{jk}(b)}$$

for all $k \geq i, j$ is well defined and the proof is similar to the one given in (c) as ρ_{ij} are ring homomorphisms. Furthermore, $\bar{a} \cdot (\bar{b} + \bar{c}) = \overline{\rho_{ik}(a) \cdot (\rho_{mk}(b) + \rho_{nk}(c))}$ for $k \geq i, m, n$. The distributive property of $(\cdot)$ in A follows from the distributive property $(\cdot)$ in A_i once we note that $a \sim \rho_{ik}(a)$ for all $k \geq i$.

(e) We define $\varphi : A \to C$ as follows,

$$\varphi(\bar{x}) = \varphi_i(x), \quad x \in A_i.$$

We first show that this definition is independent of the choice of the representative x. Let $x \sim y$, i.e., $\rho_{ik}(x) = \rho_{jk}(y)$.

$$\begin{aligned} \varphi(\bar{x}) &= \varphi_i(x) \\ &= \varphi_k(\rho_{ik}(x)) \\ &= \varphi_k(\rho_{jk}(y)) \\ &= \varphi_j(y). \end{aligned}$$

Thus, φ is well defined. Since A is a disjoint union of A_i modulo $\sim$, φ is defined everywhere in A and uniquness follows from definition. $\square$

9. Let I be the collection of open intervals $U = (a, b)$ on the real line containing a fixed real number p. Order these by reverse inclusion: $U \leq V$ if $V \subseteq U$ (note that I is a directed set). For each U let A_U be the ring of continuous real valued functions on U. For $V \subseteq U$ define the *restriction maps*

$$\rho_{UV} : A_U \to A_V \quad \text{by} \quad f \mapsto f|_V,$$

the usual restriction of a function on U to a function on the subset V (which is easily seen to be a ring homomorphism). Let

$$A = \varinjlim A_U$$

be the direct limit. In the notation of the preceding exercise, show that the maps $\rho_U : A_U \to A$ are *not* injective but are all surjective (A is called the ring of *germs of continuous functions at* p).

Sol. First, we need to describe A. A consists of equivalence classes that contain real-valued continuous functions that agree on some open interval containing p. That is, $f \sim g$ iff $f|_X = g|_X$ for some open interval X containg p.

To show that ρ_U is not injective, consider an interval $U = (a,b)$ and let $X = (a, \frac{b+p}{2})$. Define the funtions $f, g \in A_U$ as follows

$$f(x) = \begin{cases} 0 & \text{if } x \in X \\ x - \frac{b+p}{2} & \text{otherwise} \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 0 & \text{if } x \in X \\ 2\left(x - \frac{b+p}{2}\right) & \text{otherwise} \end{cases}$$

Both f and g are continuous and $f \sim g$ because $f|_X = g|_X$. Thus ρ_U is not injective (alternatively, one can show the kernel of the this map contains all elements of A_U that agree with the zero function in some open interval X containing p).

Next, we show surjectivity. Let $F \in A$. Since F is an equivalence class, pick some $f : (a,b) \to \mathbb{R} \in F$. Let $U = (a', b')$. Then define the $f' : U \to \mathbb{R}$ as

$$f'(x) = \begin{cases} f(x) & \text{if } x \in U \cap \left(\frac{a+p}{2}, \frac{b+p}{2}\right) \\ f\left(\frac{a+p}{2}\right) & \text{if } x \in U \cap (-\infty, (a+p)/2] \\ f\left(\frac{b+p}{2}\right) & \text{if } x \in U \cap [(b+p)/2, \infty) \end{cases}$$

It follows that $f' \in A_U$ and $\rho_U(f') = F$. $\square$

We now develop the notion of inverse limits. Continue to assume I is a partially ordered set (but not necessarily directed), and A is a group for all $i \in I$.

10. Suppose for every pair of indices i, j with $i \le j$ there is a map $\mu_{ji} : A_j \to A_i$ such that the following hold:

(i) $\mu_{ji} \circ \mu_{kj} = \mu_{ki}$ whenever $i \le j \le k$, and

(ii) $\mu_{ii} = 1$ for all $i \in I$.

Let P be the subset of elements $(a_i)_{i \in I}$ in the direct product $\prod_{i \in I} A_i$ such that $\mu_{ji}(a_j) = a_i$ whenever $i \le j$ (here a_i and a_j are the i^{th} and j^{th} components respectively of the element in the direct product). The set P is called the *inverse* or *projective limit* of the system $\{A_i\}$, and is denoted $\varprojlim A_i$.

(a) Assume all μ_{ji} are group homomorphisms. Show that P is a subgroup of the direct product group (cf. Exercise 15, Section 5.1).

(b) Assume the hypotheses in (a), and let $I = \mathbb{Z}^+$ (usual ordering). For each $i \in I$ let $\mu_i : P \to A_i$ be the projection of P onto its i^{th} component. Show that if each μ_{ji} is surjective, then so is μ_i for all i (so each A_i is a quotient group of P).

(c) Show that if all A_i are commutative rings with 1 and all μ_{ji} are ring homomorphisms that send 1 to 1, then A may likewise be given the structure of a commutative ring with 1 such that all μ_i are ring homomorphisms.

(d) Under the hypotheses in (a) prove that the inverse limit has the following *universal property*: If D is any group such that for each $i \in I$ there is a homomorphism $\pi_i : D \to A_i$ with $\pi_i = \mu_{ji} \circ \pi_j$ whenever $i \leq j$, then there is a unique homomorphism $\pi : D \to P$ such that $\mu_i \circ \pi = \pi_i$ for all i.

Sol.

(a) Since $P \subseteq \prod_i A_i$, it suffieces to prove that P is a group. Let $(a_i) \in P$. Since $(0, \dots) \in P$ and $(0, \dots) + (a_i)_{i \in I} = (a_i + 0)$, P contains the identity. Let $(a_i), (b_i) \in P$. Then $(a_i) - (b_i) = (a_i - b_i) \in P$ because μ_{ij} are additive.

(b) If μ_{ij} is surjective, for every element $a_i \in A_i$ there is an element $a_{i+1} \in A_{i+1}$ such that $\mu_{i+1,i}(a_{i+1}) = a_i$. Inducting on i, it follows that for every $a_i \in A_i$, there is an element $a \in P$ such that a has a_i at the i-th component. It immediately follows the projection μ_i is onto the A_i.

(c) If $(*)$ is a commutative binary operator such that μ_{ij} is linear in $(*)$, then for two elements $a, b \in P$ $a * b \in P$ and is well defined if $(*)$ is well defined in A_i. To show that consider the expression between $a = (a_i)$ and $b = (b_i)$. If we define $a * b$ as $(a_i * b_i)_{i \in I}$. Since, by assumption, μ_{ij} linear in $(*)$, this product is well-defined. This shows that $(+)$ and $(\cdot)$ are well-defined in A. (For multiplication, the assumption that $\mu_{ij}(1) = 1$ is important to ensure the consisency of the relation $1 + 1 = 2 \cdot 1$ in A_i). Distributive property follows immediately.

(d) We show that $\pi : D \to A$ defined $\pi(d) = (\pi_i(d))_{i \in I}$ satisfies the universal property. The π is clearly homomorphism because each π_i is homomorphism and addtion (and multiplication) are defined component-wise. Let $\pi_i = \mu_i \circ \pi = \mu_i \circ \pi'$. It follows $0 = \mu_i \circ (\pi - \pi')$. This can only happen $\pi = \pi'$, hence uniquness. $\square$

11. Let p be a prime let $I = \mathbb{Z}^+$, let $A_i = \mathbb{Z}/p^i\mathbb{Z}$ and let μ_{ji} be the natural projection maps

$$\mu_{ji} : a \ (\mathrm{mod}\ p^j) \longmapsto a (\mathrm{mod}\ p^i).$$

The inverse limit $\varprojlim \mathbb{Z}/p^i\mathbb{Z}$ is called the ring of *p-adic integers*, and is denoted by $\mathbb{Z}_p$.

(a) Show that every element of $\mathbb{Z}_p$ may be written uniquely as an infinite formal sum $b_0 + b_1 p + b_2 p^2 + b_3 p^3 + \cdots$ with each $b_i \in \{0, 1, \dots, p-1\}$. Describe the rules for adding and multiplying such formal sums corresponding to addition and multiplication in the ring $\mathbb{Z}_p$. [Write a least residue in each $\mathbb{Z}/p^i\mathbb{Z}$ in its base p expansion and then describe the maps μ_{ji}.] (Note in particular that $\mathbb{Z}_p$ is uncountable.)

(b) Prove that $\mathbb{Z}_p$ is an integral domain that contains a copy of the integers.

(c) Prove that $b_0 + b_1 p + b_2 p^2 + b_3 p^3 + \cdots$ as in (a) is a unit in $\mathbb{Z}_p$ if and only if $b_0 \neq 0$.

(d) Prove that $p\mathbb{Z}_p$ is the unique maximal ideal of $\mathbb{Z}_p$ and $\mathbb{Z}_p/p\mathbb{Z}_p \cong \mathbb{Z}/p\mathbb{Z}$ (where $p = 0 + 1p + 0p^2 + 0p^3 + \cdots$). Prove that every ideal of $\mathbb{Z}_p$ is of the form $p^n\mathbb{Z}_p$ for some integer $n \geq 0$.

(e) Show that if $a_1 \not\equiv 0 \pmod{p}$ then there is an element $a = (a_i)$ in the direct limit $\mathbb{Z}_p$ satisfying $a_j^p \equiv 1 \pmod{p^j}$ and $\mu_{j1}(a_j) = a_1$ for all j. Deduce that $\mathbb{Z}_p$ contains $p-1$ distinct $(p-1)^{st}$ roots of 1.