

Lang's Algebra Chapter 3 Solutions

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- (1) By the second isomorphism theorem, we have

$$\frac{U}{U \cap W} \cong \frac{U + W}{W}.$$

For two vector spaces, $X \supseteq Y$ over a field K , we have $\dim X/Y = \dim X - \dim Y$. Thus $\dim U - \dim U \cap W = \dim U + W - \dim W$.

- (2) Let M be a module over a commutative ring R . Let I be a maximal ideal of R . We first show that for any proper ideal \mathfrak{a} of R and basis set $\{x_1, x_2, \dots\}$, of M ,

Lemma 1.

$$\frac{M}{\mathfrak{a}M} \cong \bigoplus_i \frac{A}{\mathfrak{a}}(x_i + \mathfrak{a}x_i).$$

Proof. $\mathfrak{a}M$ is submodule of M because $\mathfrak{a}M \subseteq M$ by R -closure property of \mathfrak{a} . It immediately follows that $\mathfrak{a}M = \bigoplus_i \mathfrak{a}x_i$. By linear independence of x_i , $(\sum_i r_i x_i) \bmod \mathfrak{a}x_j = (r_j \bmod \mathfrak{a})x_j + \sum_{i \neq j} r_i x_i$. Therefore, $M/\mathfrak{a}M = \bigoplus_i Ax_i/\mathfrak{a}x_i$. By the isomorphism $x_i \mapsto 1_A \mapsto (x_i + \mathfrak{a}x_i)$, $Ax_i/\mathfrak{a}x_i \cong A/\mathfrak{a} \cong A/\mathfrak{a}(x_i + \mathfrak{a}x_i)$. \square

Taking \mathfrak{a} as a maximal ideal of R in the above lemma, we see that $M/\mathfrak{a}M$ is a direct product of vector spaces over the field A/\mathfrak{a} and thus admit a basis of the same cardinality as that of M . Because the dimension of a vector space is independent of the basis choice, M also has a fixed dimension.

- (3) Let $\{x_1, \dots, x_m\}$ form the basis set of R over k and let $1_R = k_1 x_1 + \dots + k_m x_m$ for $k_i \in k$. For any element $a \in R$, define the sequences $\{y_1, \dots, y_m\} \subseteq k$, $\{f_1, f_2, \dots, f_m\} \subseteq R$ as:

$$\begin{aligned} f_1 &= a, & y_1 &= w_{1,1}^{-1} k_1 \\ f_{i+1} &= f_i y_i - k_i x_i, & y_i &= k_i w_{i,i}^{-1}, \end{aligned}$$

,where $f_i = \sum_j w_{i,j} x_j$. By construction, $a^{-1} = \sum_i y_i x_i$. Thus R is a field.

- (4) **Direct Sums**

- (a) First, we show the equivalence of the two statements of the theorem. Suppose there is φ such that $g \circ \varphi = \text{id}$. By the injectivness of the composition, $\text{Im } \varphi \cap \ker g = \{0\}$. But by exactness, $\ker g = \text{Im } f$. We can unambiguously define $\psi(u)$ to be the inverse image of $f^{-1}(u')$ where $u' \equiv u \pmod{\text{Im } \varphi}$ and $u' = f(x)$ for some $x \in M'$ because if $f(x) = f(y) \pmod{\text{Im } \varphi}$, $f(x - y) \in \text{Im } \varphi$ and by injectivity of f , $x = y$. Since $M/\text{Im } f \cong M'' = \text{Im } \varphi$, ψ is defined in all of M . Similarly, if the second statement is true, $\ker \psi \cap \text{Im } f = \{0\}$ because $\psi \circ f$ is injective. By exactness, $\text{Im } f = \ker g$. We can then define $\varphi(u) = u'$ where $u' = y \pmod{\ker \psi}$ and $g(y) = u$ for some y . φ is well-defined because if $g(y_1) = g(y_2)$ for $y_1 \neq y_2$, then $y_1 \neq y_2 \pmod{\ker \varphi}$.

Now suppose $x \in M$. $x - \varphi(u) \in \text{Im } f$ for exactly one u by the argument mentioned previously. Thus we can express $x = r + s$ where $r = \varphi(u) \in \text{Im } \varphi$ and $s = x - \varphi(u) \in \text{Im } f$. This implies $M = \text{Im } f \oplus \text{Im } \varphi$. By bijectivness of $g \circ \varphi$, $\text{Im } \varphi \cong M''$. By contrast, if $M = \text{Im } f \oplus N$ for some N , with isomorphism $t : N \rightarrow M''$. We can define $g : M \rightarrow M''$ as $g(u) = u'$ such that there is $u = y \pmod{N}$ and $t^{-1}(u') = y$. This definition is unambiguous because $N \cap \text{Im } f = \{0\}$. Since $g \circ t^{-1} = \text{id}$, the sequence splits.

Finally, we complete the details of proposition 3.2. We have just shown $M = \text{Im } f \oplus \text{Im } \varphi$. By exactness, $\text{Im } f = \ker g$. Also, $\text{Im } f \cong M'$ and $\text{Im } \varphi \cong M''$ by injectivness of f and φ resp. This proves $M \cong M' \oplus M''$. We can write $x \in M$ as $f(u) + x - f(u)$ where $x - f(u) \in \ker \psi$. u is then uniquely determined by x as $\ker \psi \cap \text{Im } f = \{0\}$ by bijectivness of $\psi \circ f$. This shows $M = \text{Im } f \oplus \ker \psi$.

- (b) First, we note that φ_i is injective because otherwise the composition $\psi_i \circ \varphi_i$ wouldn't be injective, a contradiction. This implies, for every valid i , there is a submodule $E'_i = \text{Im } \varphi_i$ of E that is isomorphic to E_i . Moreover, if $c \in \text{Im } \varphi_i \cap \text{Im } \varphi_j$ for $i \neq j$, then $\psi_i(c) = \psi_j(c) = 0$, forcing c to be 0. These statements prove

$$\bigoplus_{i=1}^n E'_i \subseteq E.$$

The inverse inclusion follows as follows. Let $x \in E$, then $x = \sum_{i=1}^n \varphi_i(\psi_i(x))$, but $\varphi_i(\psi_i(x)) \in E'_i$. Therefore $x \in \bigoplus_i E'_i$.

Let $x = x_1 + \dots + x_m$ where $x_i \in E'_i$. The map defined by $x \mapsto (\psi x_i)_{1 \leq i \leq m}$ is therefore an isomorphism and the inverse map is given by $(\psi x_i)_i \mapsto \sum_i x_i$.

- (5) Let $v'_m = a_1 v_1 + \dots + a_m v_m$. Since $a_m \neq 0$, v'_m , and by the assumption that $\{v_i\}$ is linearly independent over \mathbb{R} , the set $\{v_1, \dots, v_{m-1}, v'_m\}$ is linearly independent over \mathbb{Z} . We also note that, $v'_m - \sum_{i=1}^{m-1} a_i v_i \in A$, thus we can safely assume $a_1 = \dots = a_{m-1} = 0$.

To show, the set spans A , we consider A/A_0 . Suppose, there is $av_m \in A/A_0$ such that $av_m \neq nv'_m$ for all $n \in \mathbb{Z}$. Let r, s be two integers such that $|ra_m + sa| < a_m$. Since contradicts minimality of a_m , it must be the case that $a_m \mid a$. Therefore $A/A_0 = \mathbb{Z}v'_m$.

(6) We induct on the size of S .

First assume that $S = \{w\}$. Then $\mathbb{Z}\langle S \rangle = \{n[w] : n \in \mathbb{Z}\}$. If M is a subgroup of $\mathbb{Z}\langle S \rangle$, then $M = \mathbb{Z}\langle a[w] \rangle$ for some $a \in \mathbb{Z}$. Here we pick $y_w = a[w]$ which is G -linear.

For the induction step, suppose the statement is true for S , $0 \leq |S| \leq m-1$. We shall prove the statement is true for S with m elements. Fix on element $w \in S$, and consider projection map $\pi : \mathbb{Z}\langle S \rangle \rightarrow \mathbb{Z}\langle G \cdot w \rangle$. By correspondence, $\pi(M)$ is a subgroup of $\mathbb{Z}\langle G \cdot w \rangle$ with basis $\{\bar{y}_{gw}\}_{w \in G}$ which satisfy the property for $\sigma \in G$, $\sigma \bar{y}_{gw} = \bar{y}_{\sigma gw}$. We then lift the basis of $\mathbb{Z}\langle \pi(M) \rangle$ to $\mathbb{Z}\langle S \rangle$ by picking a representatives $\mathfrak{R} = \{y_w\}$ in M for \bar{y}_w . The y_w are linearly independent thus form part of the basis for M . Again by hypothesis, $M \cap \mathbb{Z}\langle S - G \cdot w \rangle$ has basis $\mathfrak{B} = \{y_w\}_{w \in S-G \cdot w}$ that satisfy the given property. We finally combine \mathfrak{R} and \mathfrak{B} to get the basis of rank m for M .

(7) For convenience, we identify the properties of a semi-norm as follows

$$\text{SN-1 } |v| \geq 0$$

$$\text{SN-2 } |nv| = |n||v|$$

$$\text{SN-3 } |u+v| \leq |u| + |v|$$

(a) Let $a, b \in M_0$. Then by SN-2 and SN-3, $|u-b| \leq |a| + |b| = 0$. By SN-1, we have $|a-b| \geq 0$, this $a-b \in M_0$. By SN-2, $|0| = |2 \cdot 0| = 2|0|$. This implies $0 \in M_0$. Hence M_0 is a subgroup of M .

(b) If $M_0 \neq \{0\}$, we can make the transformation $x \mapsto x + M_0$ without loss of generality as such map preserves the linear independence of $\{v_i\}$. Thus, we can assume $M_0 = \{0\}$.

Let $N = \langle v_1, \dots, v_r \rangle$. Since M has rank r , the exponent e of M/N is finite and thus eM is a subgroup of N . Moreover, N/eM is torsion group with finite number of elements. Therefore, we can pick the smallest positive integers $n_{i,j}$ such that

$$\sum_{j=1}^i n_{i,j} v_j = dw_i \quad \text{for some } w_i \in M$$

The linear independence follows immediately. Picking $n_{j,k}$ in the range $[0, d-1]$,

$$d|w_i| = |dw_i| \leq \sum_{j=1}^i n_{i,j} |v_j| \leq d \sum_{j=1}^i |v_j|.$$

(8) (a) SN-1 follows immediately because $\log \geq 0$ for all \mathbb{Z}^+ . Since, $h(x^{-1}) = h(x)$, it suffices to prove SN-2 for $n \geq 0$ in which case $h(x^n) = \log \max(|a^n|, |b^n|) = \log \max(|a|, |b|)^n = n \log \max(|a|, |b|) = nh(x)$. Finally, if $y = c/d$, $h(xy) = h(ac/bd)$. Let $e = \gcd(a, d)$ and $f = \gcd(c, b)$. Then

$$\begin{aligned}
h(xy) &= \log \max(|\frac{ac}{ef}|, |\frac{bd}{ef}|) \\
&= \log \left(\frac{1}{|ef|} (\max(|ac|, |bd|)) \right) \\
&= \log \max(|ac|, |bd|) - \log |ef| \\
&\leq \log \max(|ac|, |bd|) \\
&\leq \log \max(|a|, |b|) + \log \max(|c|, |d|)
\end{aligned}$$

Hence SN-3 is satisfied. $\log \max(|a|, |b|) = 0$ if and only if $|a| = |b| = 1$, which makes the kernel of $\ker h = \{\pm 1\}$.

- (b) For a given rational number $x = a/b$, since there are finitely many prime divisors of p, q such that $p|a$ and $q|b$, M can be generated by the set $\{-1, 1\} \cup \{p, 1/q \in \mathbb{Q}^* : p|\text{the numerator of } x_1 \cdots x_m, q|\text{the denominator of } x_1 \cdots x_m\}$. From this we can set upper bound on the norm as

$$h(y) \leq \sum_p \log p$$

where the sum is over all primes p (not necessarily distinct) that divides the numerator or denominator of x_i for some i .

- (9) (a) $S^{-1}M$ can be defined as a subset of $M \times S$ for a commutative ring A , a multiplicative subset S and A -module M such that

$$(m_1, s_1) \sim (m_2, s_2)$$

, if there is an element $s \in S$ that satisfy the equation $s(s_2 m_1 - s_1 m_2) = 0$. As with $S^{-1}A$, we can denote (m, s) with m/s . Since $S^{-1}A$ is a commutative ring, we can define the action of $S^{-1}A$ on $S^{-1}M$ as

$$\frac{a}{s'} \cdot \frac{m}{s} = \frac{a \cdot m}{s' s}.$$

With this definition of the action of $S^{-1}A$ on $S^{-1}M$, we can show that $S^{-1}M$ is an $S^{-1}A$ -module. Let $a_1/b_1, a_2/b_2 \in S^{-1}A$ and let $m_1/s_1, m_2/s_2 \in S^{-1}M$. Then we have

$$\begin{aligned}
\frac{a_1}{b_1} \cdot \left(\frac{m_1}{s_1} + \frac{m_2}{s_2} \right) &= \frac{a_1}{b_1} \cdot \left(\frac{m_1 s_2 + m_2 s_1}{s_1 s_2} \right) \\
&= \frac{a_1 b_1}{b_1 b_1} \cdot \left(\frac{m_1 s_2 + m_2 s_1}{s_1 s_2} \right) \\
&= \frac{a_1 b_1 s_2 m_1 + a_1 b_1 s_1 m_2}{b_1 s_1 b_1 s_2} \\
&= \frac{a_1 m_1}{b_1 s_1} + \frac{a_1 m_2}{b_1 s_2} \\
&= \frac{a_1}{b_1} \cdot \frac{m_1}{s_1} + \frac{a_1}{b_1} \cdot \frac{m_2}{s_2}.
\end{aligned}$$

and

$$\begin{aligned}
\left(\frac{a_1}{b_1} + \frac{a_2}{b_2}\right) \cdot \frac{m_1}{s_1} &= \left(\frac{a_1 b_2 + a_2 b_1}{b_1 b_2}\right) \cdot \frac{m_1}{s_1} \\
&= \left(\frac{a_1 b_2 + a_2 b_1}{a_1 a_2}\right) \cdot \frac{m_1 s_1}{s_1 s_1} \\
&= \frac{a_1 b_2 m_1 s_1 + a_2 b_1 m_1 s_1}{s_1 b_1 s_2 b_2} \\
&= \frac{a_1 m_1}{b_1 s_1} + \frac{a_2 m_1}{b_2 s_1} \\
&= \frac{a_1}{b_1} \cdot \frac{m_1}{s_1} + \frac{a_2}{b_2} \cdot \frac{m_1}{s_1}.
\end{aligned}$$

(b) Let

$$0 \rightarrow M' \xrightarrow{f} M \xrightarrow{f''} M'' \rightarrow 0$$

be exact. Then we have the induced sequence,

$$0 \rightarrow S^{-1}M' \xrightarrow{g} S^{-1}M \xrightarrow{g''} S^{-1}M'' \rightarrow 0,$$

where g is defined as $g(m/s) = f(m)/s$ and g'' is defined as $g''(m/s) = f''(m)/s$. $\ker g = \{m/s : f(m)/s = 0\}$. Since f is injective, $f(m) = 0$ iff $m = 0$, i.e., $\ker g = \{0\}$.

By exactness $\operatorname{Im} f = \ker f''$. Evaluating g'' on $\operatorname{Im} g$, $g''(g(m/s)) = g''(f(m)/s) = f''(f(m))/s = 0/s = 0$. This shows $\operatorname{Im} g \subseteq \ker g''$. Let $g''(x/s) = f''(x)/s = 0$. This implies $f''(x) = 0$ for some x . By exactness, $\ker f'' \subseteq \operatorname{Im} f$, implying $x = f(y)$ for some $y \in M'$. This proves $\operatorname{Im} g \supseteq \ker g''$.

Finally, let $x/s \in S^{-1}M''$. Since $x \in M''$, $x = f''(y)$ for some $y \in M$ by exactness of the first sequence. But then $x/s = f''(y)/s = g''(y/s)$ making g'' surjective.

(10) (a) The natural map under consideration is the map

$$f = x \mapsto (x/1, \dots).$$

If $x/s' \sim 0/1$, for some $s' \in A - \mathfrak{p}$ and $x \in M$, then it means $s'x = 0$ for some $s' \in A - \mathfrak{p}$. Therefore, the kernel of f is the set $\{x : sx = 0, \text{ for some } s \in A - \mathfrak{p} \text{ for all maximal ideals } \mathfrak{p}\}$. If $x \in \ker f$, then $\operatorname{Ann}(x)$ is not contained in any maximal ideal \mathfrak{p} , implying $\operatorname{Ann}(x) = A \implies x = 0$.

(b) Let $f : M'' \rightarrow M$ and $\hat{f} : M''_{\mathfrak{p}} \rightarrow M_{\mathfrak{p}}$. Define g and \hat{g} similarly for the second halves of the sequences.

(\implies) This directly follows from part (b) of exercise 9.

(\Leftarrow) Suppose $0 \rightarrow M'_p \rightarrow M_p \rightarrow M''_p$ is exact sequence for all primes \mathfrak{p} .

Let $f(x) = 0$, then $\hat{f}(x/s) = f(x)/s = 0/1$ for all $s \in \mathfrak{p}$. By exactness, \hat{f} is injective. thus $x/s = 0$. By similar reasoning as part (a) of this problem $x = 0$. Hence f is injective.

Now let $gf(x) = n$. By definition, $\hat{g}\hat{f}(x/s) = n/s$. By exactness, the left-hand side is 0. Thus $s'n = 0$ for $s' \in \mathfrak{p}$ for all prime \mathfrak{p} . Again, by similar reasoning as part (a), n has to be 0 and $\text{Im } f \subseteq \ker g$. To see the converse, suppose $g(y) = 0$. Consequently, $\hat{g}(y/s) = g(y)/s = 0$ for all $s \in \mathfrak{p}$ and by exactness, $y/1 = \hat{f}(x/t_p) = f(x)/t_p$ for some t_p depending on \mathfrak{p} . This implies $s_p(f(x) - t_p y) = 0$ or equivalently $f(s_p x) = r_p y$ for some $x \in M'_p$ and $r_p = s_p t_p$ implying $r_p y \in \text{Im } f$ for all prime \mathfrak{p} . Since $M/\text{Im } f$ is also an A -module, it implies $r_p(x + \text{Im } f) = 0$ for all \mathfrak{p} implying $x + \text{Im } f = 0 + \text{Im } f$ or in other words, $x \in \text{Im } f$. This proves $\text{Im } f = \ker g$.

Finally, suppose $y \in M''$. By surjectivity of \hat{g} , $y/1 = \hat{g}(x/s) = g(x)/s$ for some $x \in M$. By definition, $s_p(g(x) - t_p y) = 0$. By similar argument as above, $y \in \text{Im } g$, proving the exactness of the first sequence.

(c) Let $\phi : M \rightarrow M_p$ be the natural map in question. Then $\phi(x) = x/1$. If $\phi(x) = 0$, then $sx = 0$ for some $s \in A - \mathfrak{p}$. This contradicts the assumption M is torsion-free and since $0 \notin A - \mathfrak{p}$, $x = 0$.