On the Sum of Powers of Consecutive Numbers

Amanuel Tewodros Getachew

January 5, 2025

We define the sum of kth powers of the first n positive integers as

$$S_k(n) := \sum_{m=1}^{n} m^k.$$

In this paper we show that $S_k(x)$ is a polynomial in $\mathbb{Q}[x]$ and derive an algebraic expression to calculate the value of $S_k(n)$ for $n \in \mathbb{Z}_+$.

Lemma 1.

$$S_{k+1}(n) = (n+1)S_k(n) - \sum_{j=0}^{n} S_k(j).$$

Proof of Lemma. By definition, we have $S_{k+1}(n) = \sum_{m=1}^{n} m^{k+1} = \sum_{m=1}^{n} mm^k$.

$$\begin{aligned} \sum_{m=1}^{n} mm^k &= \sum_{m=1}^{n}\sum_{p=1}^{m} m^k \\ &= \sum_{m=1}^{n}\sum_{p=1}^{n} f_k(m,p) \end{aligned}$$

where

$$f_k(m,p) = \begin{cases} m^k & \text{if} \quad p \le m \\ 0 & \text{if} \quad p > m \end{cases}$$

Then:

$$\sum_{m=1}^{n}\sum_{p=1}^{n} f_k(m,p) = \sum_{p=1}^{n}\sum_{m=1}^{n} f_k(m,p) = \sum_{p=1}^{n}\sum_{m=p}^{n} f_k(m,p) = \sum_{p=1}^{n}\sum_{m=p}^{n} m^k. \tag{1}$$

But $\sum_{m=p}^{n} m^k = S_k(n) - S_k(p-1)$. Therefore, by definition and (1), we have

$$S_{k+1}(n) = \sum_{p=1}^{n} S_k(n) - S_k(p-1) = \sum_{p=0}^{n} S_k(n) - S_k(p) = (n+1)S_k(n) - \sum_{p=0}^{n} S_k(p). \quad \square$$

It is trivial that $S_0(x) \in \mathbb{Q}[x]$ and has degree 1. Assume $S_k[x] \in \mathbb{Q}(x)$ and $\deg S_k = k+1$ for all $k \in \mathbb{Z}_{\ge 0}$. We can write

$$S_k(x) = \alpha_0(k) + \alpha_1(k)x + \cdots + \alpha_{k+1}(k)x^{k+1}. \qquad k = 1, 2 \ldots \tag{2}$$

Note that $S_k(0) = \alpha_0(k) = 0$ and $\alpha_p(k) = 0$ for $p > k+1$. By assumption and lemma 1, the following theorem follows.

Theorem 1. *Let $S_k(x) = \sum_{0 \le p \le k+1} \alpha_p(k)x^p$ as written in (2). Then*

$$S_{k+1}(n) = \frac{\alpha_{k+1}(k)}{1+\alpha_{k+1}(k)} n^{k+2} + \sum_{1 \le p \le k+1} \frac{\alpha_p(k) + \alpha_{p-1}(k) - \beta_p(k)}{1 + \alpha_{k+1}(k)} n^p,$$

where

$$\beta_p(k) = \sum_{p-1 \le j \le k} \alpha_j(k)\alpha_p(j).$$

Proof. By lemma 1 we know that

$$S_{k+1}(n) = (n+1)S_k(n) - \sum_{j=0}^{n}\sum_{p=0}^{k+1}\alpha_p(k)j^p = (n+1)S_k(n) - \sum_{p=0}^{k+1}\alpha_p(k)\sum_{j=0}^{n}j^p$$

$$\Longrightarrow (1+\alpha_{k+1}(k))S_{k+1}(n) = (n+1)S_k(n) - \sum_{p=0}^{k}\alpha_p(k)S_p(n). \tag{(3)}$$

Now we evaluate the sum on the left:

$$\begin{aligned}
\sum_{j=0}^{k}\alpha_j(k)S_j(n) &= \sum_{j=0}^{k}\alpha_j(k)\sum_{p=0}^{j+1}\alpha_p n^p \\
&= \sum_{j=0}^{k}\sum_{p=1}^{j+1}\alpha_j(k)\alpha_p n^p \\
&= \sum_{j=0}^{k}\sum_{p=0}^{j}\alpha_{p+1}(k)\alpha_p n^{p+1} \\
&= \sum_{p=0}^{k}n^{p+1}\sum_{j=p}^{k}\alpha_{p+1}(k)\alpha_p \\
&= \sum_{p=1}^{k+1}\beta_p(k)n^p
\end{aligned}$$

Therefore equation (3) becomes $(1+\alpha_{k+1}(k))S_{k+1}(n) = (n+1)S_k(n) - \sum_{p=1}^{k+1}\beta_p(k)n^p$. But, $(n+1)S_k(n) = \sum_{p=0}^{k+1}(\alpha_p(k)n^{p+1} + \alpha_p(k)n^p) = \alpha_{k+1}(k)n^{k+2} + \sum_{p=1}^{k+1}(\alpha_p(k)+\alpha_{p-1}(k))n^p$. Thus

$$S_{k+1}(n) = \frac{1}{1+\alpha_{k+1}(k)}\alpha_{k+1}(k)n^{k+2} + \frac{1}{1+\alpha_{k+1}(k)}\left(\sum_{p=1}^{k+1}(\alpha_p(k)+\alpha_{p-1}(k)-\beta_p(k))n^p\right)$$

□

Note that, by induction, this theorem proves that $S_k(x)$ is a degree $k+1$ polynomial in $\mathbb{Q}[x]$ for all $k \in \mathbb{Z}^+$ with coefficients defined as follows;

$$\begin{aligned}
\alpha_{k+1}(k) &= \frac{\alpha_k(k-1)}{1+\alpha_k(k-1)} \\
\alpha_p(k) &= \frac{\alpha_p(k-1)+\alpha_{p-1}(k-1)-\beta_p(k-1)}{1+\alpha_k(k-1)} \qquad \text{for} \quad 1 \le p \le k \\
\alpha_p(k) &= 0 \qquad \text{for } p > k+1 \text{ and } p = 0
\end{aligned}$$