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Dyck path enumeration

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Dedicated to H.W. Gould, on the occasion of his 70th birthday

Abstract

An elementary technique is used for the enumeration of Dyck paths according to various parameters. For several of the considered parameters the generating function is expressed in terms of the Narayana function. Many enumeration results are obtained, some of which involve the Fine numbers. © 1999 Elsevier Science B.V. All rights reserved

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1. Introduction

The purpose of this paper is three-fold. Firstly, we would like to present an elementary technique, not entirely original, for the algebraic enumeration of certain combinatorial objects. It seems to be a viable alternative to the use of the closely related Schützenberger methodology [43,44]. In this paper this will be done only for the enumeration of Dyck paths according to length and various other parameters but the same systematic approach can be applied to Motzkin paths, Schröder paths, lattice paths in the upper half-plane, various classes of polyominoes, ordered trees, non-crossing partitions, (the last two types of combinatorial objects are in bijection with Dyck paths), etc.

Secondly, we would like to point out the use of the Narayana function (i.e. the generating function of the Narayana numbers) in the enumeration of Dyck paths according to certain parameters.

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Thirdly, in addition to some known results, pointed out by bibliographical references, we shall find several new explicit formulae for the enumeration of Dyck paths according to semilength and various other parameters. Some of the enumeration results are expressed in terms of the Fine sequence and a new manifestation of the Catalan numbers is found.

In the appendices (i) we quote the Lagrange inversion theorem in the form in which it is used throughout the paper; (ii) we define the Catalan function, the Catalan numbers, we give some of its basic properties, and we expand into series various functions that involve the Catalan function; (iii) we define the Fine function and the Fine numbers and we derive some formulas for the computation of the latter (most of them known); (iv) we introduce the Narayana function and the Narayana numbers, we expand into series various functions that involve the Narayana function, and we derive several functional equations satisfied by the Narayana function; and (v) we mention four families of combinatorial objects that are counted by the Catalan numbers and we list some pertinent bijections, most of them known. Results regarding ordered trees, parallelogram polyominoes, similarity relations, and non-crossing partitions follow via these bijections from the results obtained for Dyck paths.

We would like to emphasize that the presented technique cannot be considered original. It has turned out to be close in spirit to Pólya's picture-writing technique [38] as well as to Zeilberger's practice [60] of the Schützenberger methodology.

2. Dyck paths: generalities and terminology

A *Dyck path* is a path in the first quadrant which begins at the origin, ends at $(2n,0)$, and consists of steps $(1,1)$ (North-East), called *rises*, and $(1,-1)$ (South-East), called *falls*. We shall refer to n as the *semilength* of the path. Occasionally, a Dyck path of semilength n will be called a Dyck n-path.

We can encode each rise by the letter u (for up) and each fall by the letter d (for down), obtaining the encoding of Dyck paths by a so-called Dyck word. For example, the Dyck path in Fig. 1 is encoded by the Dyck word $uduuudduuddd$.

A step of a Dyck path having extremities of ordinates $k-1$ and k $(k \geqslant 1)$ is said to be at *level* k. A point of a Dyck path with ordinate k is said to be at *level* k.

In a Dyck path a *peak* is an occurrence of ud, a *valley* is an occurrence of du, and a *doublerise* is an occurrence of uu. By the *level* (or *height*) of a peak or of a valley or of a doublerise we mean the level of the intersection point of its two steps.

By a *return step* we mean a d step at level 1. Dyck paths that have exactly one return step are said to be *primitive*.

An *ascent* (*descent*) of a Dyck path is a maximal string of u's (d's) and its length is the number of steps in it. A *return descent* is a descent that ends with a return step.

If α and β are Dyck paths, then we define

(i) $\alpha\beta$ as the concatenation of α and β;

(ii) $\hat{\alpha} \overset{\text{def}}{=} u\alpha d$ and we call it the *elevation* of α.

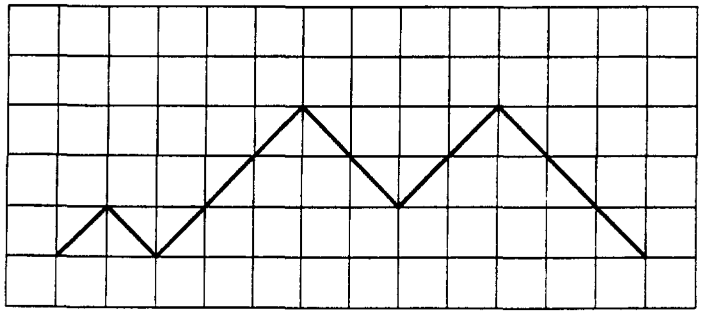

Fig. 1.

By ε we denote the empty path, which, if necessary, may be visualized as a dot.

If $\mathbf{A}$ and $\mathbf{B}$ are finite sets of Dyck paths, then we define the *concatenation* $\mathbf{AB}$ of $\mathbf{A}$ and $\mathbf{B}$ by

$$\mathbf{AB} \stackrel{\text{def}}{=} \{\alpha\beta\colon\ \alpha \in \mathbf{A},\ \beta \in \mathbf{B}\}$$

and the *elevation* $\hat{\mathbf{A}}$ of $\mathbf{A}$ by

$$\hat{\mathbf{A}} \stackrel{\text{def}}{=} \{\hat{\alpha}\colon\ \alpha \in \mathbf{A}\}.$$

Clearly, $\mathbf{A}\{\varepsilon\} = \{\varepsilon\}\mathbf{A} = \mathbf{A}$.

By $\mathbf{D}_n$ we denote the set of all Dyck paths of semilength n. Obviously, $\mathbf{D}_0 = \{\varepsilon\}$.

Every nonempty Dyck path α can be decomposed uniquely in the following manner:

$$\alpha = u\beta_1 d\gamma_1, \quad \text{i.e.} \quad \alpha = \hat{\beta}_1\gamma_1, \tag{2.1}$$

where β_1 and γ_1 are, possibly empty, Dyck paths. This is the so-called *first return decomposition*, since the d step pointed out in the factorization is the first return step of the path. Alternatively, we can write in a unique manner

$$\alpha = \beta_2 u\gamma_2 d, \quad \text{i.e.} \quad \alpha = \beta_2\hat{\gamma}_2, \tag{2.2}$$

where β_2 and γ_2 are, possibly empty, Dyck paths.

Relation (2.1) implies that

$$\mathbf{D}_n = \hat{\mathbf{D}}_0\mathbf{D}_{n-1} \cup \hat{\mathbf{D}}_1\mathbf{D}_{n-2} \cup \cdots \cup \hat{\mathbf{D}}_{n-2}\mathbf{D}_1 \cup \hat{\mathbf{D}}_{n-1}\mathbf{D}_0, \quad n \geqslant 1. \tag{2.3}$$

Similarly, (2.2) implies that

$$\mathbf{D}_n = \mathbf{D}_0\hat{\mathbf{D}}_{n-1} \cup \mathbf{D}_1\hat{\mathbf{D}}_{n-2} \cup \cdots \cup \mathbf{D}_{n-2}\hat{\mathbf{D}}_1 \cup \mathbf{D}_{n-1}\hat{\mathbf{D}}_0, \quad n \geqslant 1. \tag{2.4}$$

In both (2.3) and (2.4) we have disjoint unions. From (2.3) or (2.4) we obtain

$$|\mathbf{D}_n| = |\mathbf{D}_0||\mathbf{D}_{n-1}| + |\mathbf{D}_1||\mathbf{D}_{n-2}| + \cdots + |\mathbf{D}_{n-2}||\mathbf{D}_1| + |\mathbf{D}_{n-1}||\mathbf{D}_0|, \quad n \geqslant 1.$$

Since $|\mathbf{D}_0| = 1$, it follows that the sequence $|\mathbf{D}_n|$ $(n \geqslant 0)$ satisfies the same recurrence relation and the same initial condition as the sequence C_n $(n \geqslant 0)$ of Catalan numbers (see Appendix B). Consequently,

$$|\mathbf{D}_n| = C_n \quad (n \geqslant 0). \tag{2.5}$$

3. Enumeration

Let p be a fixed nonnegative integer-valued parameter of Dyck paths, i.e. a mapping from $\bigcup_{n \geqslant 0} \mathbf{D}_n$ into $\{0, 1, 2, \ldots\}$.

If $\mathbf{A}$ is a finite set of Dyck paths, then by $P_{\mathbf{A}}(t)$ we denote the *enumerating polynomial* of $\mathbf{A}$ relative to the parameter p:

$$P_{\mathbf{A}}(t) \stackrel{\text{def}}{=} \sum_{\alpha \in \mathbf{A}} t^{p(\alpha)}.$$

Obviously, if $\mathbf{A}$ and $\mathbf{B}$ are disjoint finite sets of Dyck paths, then

$$P_{\mathbf{A} \cup \mathbf{B}}(t) = P_{\mathbf{A}}(t) + P_{\mathbf{B}}(t). \tag{3.1}$$

For simplicity, we denote

$$P_n(t) \stackrel{\text{def}}{=} P_{\mathbf{D}_n}(t), \quad \hat{P}_n(t) \stackrel{\text{def}}{=} P_{\hat{\mathbf{D}}_n}(t).$$

Thus, $P_n(t)$ is the enumerating polynomial of the set of all n-Dyck paths, while $\hat{P}_n(t)$ is the enumerating polynomial of the primitive Dyck paths of semilength $n+1$.

We introduce the generating function for the enumeration of Dyck paths according to semilength (coded by z) and the parameter p (coded by t):

$$\Omega(t,z) \stackrel{\text{def}}{=} \sum_{n \geqslant 0} P_n(t) z^n. \tag{3.2}$$

We denote

$$\hat{\Omega}(t,z) \stackrel{\text{def}}{=} \sum_{n \geqslant 0} \hat{P}_n(t) z^n. \tag{3.3}$$

In the considered enumeration problems we shall do the following:

(i) examining the effect of the concatenation of Dyck paths on the parameter and making use of (2.3) (or (2.4)) and (3.1), we express

$$P_n \quad \text{in terms of } P_{n-1}, P_{n-2}, \ldots, P_0, \hat{P}_{n-1}, \hat{P}_{n-2}, \ldots, \hat{P}_0; \tag{3.4}$$

(ii) examining the effect of the elevation of Dyck paths on the parameter, we express

$$\hat{P}_n \quad \text{in terms of } P_n; \tag{3.5}$$

(iii) from (3.2)–(3.5) we obtain two relations connecting Ω and $\hat{\Omega}$ and, eliminating $\hat{\Omega}$ from them, we obtain an equation satisfied by the generating function $\Omega(t,z)$.

In some enumeration problems it is more convenient to use relation (2.3) ((2.4)) than (2.4) ((2.3)).

Remark. Eliminating the $\hat{P}_q$'s from (3.4) and (3.5), we obtain a recurrence relation among the enumerating polynomials P_n.

For a given parameter p, we shall denote by σ_n the sum of the values of the parameter p at all Dyck paths of semilength n. Clearly, if $P_n(t)$ is the enumerating polynomial of $\mathbf{D}_n$ relative to the parameter p, then

$$\sigma_n = P_n'(1)$$

and, consequently, the generating function of the sequence $(\sigma_n)_{n\geqslant 0}$ is $(\partial\Omega/\partial t)_{t=1}$.

If we assume that all n-Dyck paths are equiprobable, then, due to (2.5), the expected value of the parameter p is σ_n/C_n, where the C_n's are the Catalan numbers defined in Appendix B. This equiprobability will be assumed in all subsequent statements about expected values.

Remark. Everything that has been said in this section can be easily extended to the case when instead of one parameter we have several parameters.

Remark. The author is aware that the generating functions to be derived in the sequel can be obtained in a more elegant manner by the so-called symbolic method. It is based, roughly speaking, on a simple correspondence between set-theoretic operations on combinatorial structures and algebraic operations on the corresponding generating functions. For a clear systematic presentation of this method, see [21,45].

4. Some special types of parameters

In order to avoid repetitions at the various parameters to be considered, it is convenient to introduce some classes of parameters on Dyck paths.

A parameter p is said to be *additive* if $p(\alpha\beta) = p(\alpha) + p(\beta)$ for all Dyck paths α, β. A typical example is the parameter 'number of peaks'. Indeed, if α and β are any two Dyck paths, then the peaks of $\alpha\beta$ consist exactly of the peaks of α and those of β. Other examples are: number of doublerises, number of return steps, and number of *udd*'s. Note that the parameter 'number of *duu*'s' is not additive since the concatenation of two Dyck paths may create a new *duu*.

In the case of an additive parameter, for any two finite sets $\mathbf{A}$ and $\mathbf{B}$ of Dyck paths we have $P_{\mathbf{AB}}(t) = P_{\mathbf{A}}(t)P_{\mathbf{B}}(t)$ and then the actual form of (3.4) is

$$P_n(t) = P_{n-1}(t)\hat{P}_0(t) + P_{n-2}(t)\hat{P}_1(t) + \cdots + P_1(t)\hat{P}_{n-2}(t) + P_0(t)\hat{P}_{n-1}(t), \quad n \geqslant 1,$$

which implies

$$\Omega - 1 = z\Omega\hat{\Omega}. \tag{4.1}$$

A parameter p is said to be *quasiadditive* if for any two Dyck paths α and β, α contributes $p(\alpha)$ to $p(\alpha\beta)$, β contributes $p(\beta)$ to $p(\alpha\beta)$, and, in addition, due to the concatenation itself, for some α, β one has $p(\alpha\beta) > p(\alpha) + p(\beta)$. A typical example is the parameter 'number of valleys'. Indeed, the valleys of the paths α and β are valleys also for $\alpha\beta$ but, if both α and β are nonempty, then $\alpha\beta$ has an additional valley created by the concatenation itself. Another example is the parameter 'number of *duu*'s'.

A parameter p is said to be a *left parameter* if for all Dyck paths α, β we have

$$p(\alpha\beta) = \begin{cases} p(\alpha) & \text{if } \alpha \neq \varepsilon, \\ p(\beta) & \text{if } \alpha = \varepsilon. \end{cases}$$

Examples of left parameters are: height of the first peak, height of the first valley, length of the first descent.

In the case of a left parameter, for two finite sets $\mathbf{A}$ and $\mathbf{B}$ of Dyck paths such that $\varepsilon \notin \mathbf{A}$, we have $P_{\mathbf{AB}}(t) = |\mathbf{B}| P_{\mathbf{A}}(t)$. Now, due to $|\mathbf{D}_k| = C_k$, from (2.3) it follows that the actual form of (3.4) is

$$P_n(t) = C_{n-1}\hat{P}_0(t) + C_{n-2}\hat{P}_1(t) + \cdots + C_1\hat{P}_{n-2}(t) + C_0\hat{P}_{n-1}(t), \quad n \geqslant 1,$$

which implies

$$\Omega - 1 = zC(z)\hat{\Omega}. \tag{4.2}$$

Here $C(z)$ is the Catalan function, defined in Appendix B.

5. An illustrative example

In order to highlight the elementary nature of this technique, we intend to find the equation satisfied by the generating function of Dyck paths according to semilength (coded by z), number of peaks (coded by t), sum of peak heights (coded by s), and number of *duu*'s (coded by v). Both parameters 'number of peaks' and 'sum of peak heights' are additive. On the other hand, the parameter 'number of *duu*'s' is quasiadditive since the concatenation of two paths does contain all the *duu*'s of the factors but acquires new *duu*'s for certain paths. More precisely, it acquires an additional *duu* if and only if the first factor ends with a d, which is the case for all members of $\mathbf{D}_k, k \geqslant 1$, and the second factor starts with a *uu*, which is the case for all members of $\hat{\mathbf{D}}_k, k \geqslant 1$. Therefore, (2.4) becomes

$$P_n = P_{n-1}\hat{P}_0 + vP_{n-2}\hat{P}_1 + \cdots + vP_1\hat{P}_{n-2} + P_0\hat{P}_{n-1}, \quad n \geqslant 1.$$

Multiplying this by z^n and summing with respect to n, after elementary manipulations we obtain

$$\Omega - 1 = vz\Omega\hat{\Omega} + ts(1-v)z\Omega + (1-v)z(\hat{\Omega} - ts). \tag{5.1}$$

Examining the effect of elevation on the parameters, we derive

$$\hat{P}_n(t,s,v)=\begin{cases} ts & \text{if } n=0,\\ P_n(ts,s,v) & \text{if } n\geqslant 1.\end{cases} \tag{5.2}$$

Indeed, by elevation, for any path other than the empty one, the number of peaks and the number of *duu*'s remain unchanged, while the sum of the peak heights increases by the number of peaks. From (5.2) we obtain

$$\hat{\Omega}(t,s,v,z)=ts-1+\Omega(ts,s,v,z). \tag{5.3}$$

Eliminating $\hat{\Omega}$ from (5.1) and (5.3), we obtain

$$\begin{aligned}\Omega(t,s,v,z)-1&=vz\Omega(t,s,v,z)\Omega(ts,s,v,z)+(ts-v)z\Omega(t,s,v,z)\\&\quad+(1-v)z(\Omega(ts,s,v,z)-1).\end{aligned}$$

Remark. If we discard some of the parameters by making in (5.4) the corresponding coding variables equal to 1, then, after the substitution $\Omega=1+G$, we obtain, as special cases, Proposition 2 of [8], Proposition 3 of [9], and the first equation of Proposition 4.1 of [3].

6. Enumeration of Dyck paths according to various parameters

In this section we shall derive the generating function for Dyck paths according to semilength and various other parameters and, making use of these generating functions, we shall find several enumeration results.

Within each example, the enumerating polynomials are denoted by P_n, the variable coding the semilength is z, the variable coding the considered parameter is t (or, in the case of two parameters, t and s, respectively). Any auxiliary parameters will be coded by x and/or y.

All the generating functions will refer to the enumeration of Dyck path according to semilength and one or several other parameters. Unless otherwise mentioned, they will be denoted by Ω. Occasionally, when several generating functions are considered within the same subsection, an unambigous subscript will indicate the parameter to which it refers (in addition to semilength).

We use the notation $[z^n]F(z)$ to denote the coefficient of z^n in the series of $F(z)$. Similarly, in the case of several variables, three for example, $[t^is^jz^n]F(t,s,z)$ denotes the coefficient of $t^is^jz^n$ in the series of F. In our situation, this will denote the number of Dyck paths of semilength n for which the parameters take the values i and j, respectively.

The first example is well known.

6.1. Number of peaks

This parameter is additive. Examining the effect of elevation on the number of peaks, we obtain

$$\hat{P}_n = \begin{cases} t & \text{if } n = 0, \\ P_n & \text{if } n \geqslant 1. \end{cases} \tag{6.1}$$

Indeed, by elevation the number of peaks is not changed, with the exception of the empty path which becomes ud. From (6.1) we obtain

$$\hat{\Omega} = \Omega + t - 1. \tag{6.2}$$

Eliminating $\hat{\Omega}$ from (4.1) and (6.2), we derive

$$z\Omega^2 - (1 + z - tz)\Omega + 1 = 0. \tag{6.3}$$

We introduce the Narayana function $\rho(t,z)$ (see Appendix D) by eliminating z from (6.3) and (D.1). After discarding the spurious solution, we find

$$\Omega(t,z) = 1 + t\rho(t,z). \tag{6.4}$$

Now from (D.5) we obtain

$$[t^k z^n]\Omega = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}, \quad n \geqslant 1. \tag{6.5}$$

This is basically Narayana's result [37].

Remark. Formula (6.5) yields the number of ordered trees with n edges and having k leaves (or k internal nodes) [12,13].

Remark. From (6.5) we can obtain easily that the number of parallelogram polyominoes with a columns and b rows is

$$\frac{1}{a+b-1}\binom{a+b-1}{a}\binom{a+b-1}{b}$$

(see [3]). Indeed, a parallelogram polyomino with a columns and b rows has perimeter $2a+2b$. Then the corresponding Dyck path (see Appendix E) has semilength $a+b-1$ and a peaks and the result follows at once.

Remark. Formula (6.5) yields the number of non-crossing partitions of $[n]$ with k blocks [17,28,39].

A simple computation yields

$$\left(\frac{\partial\Omega}{\partial t}\right)_{t=1} = \frac{1-\sqrt{1-4z}}{2\sqrt{1-4z}},$$

which implies (see [59, p. 53, formula (2.5.11)])

$$\sigma_n = \binom{2n-1}{n}, \qquad \frac{\sigma_n}{C_n} = \frac{n+1}{2}.$$

Remark. For trees this result can be found in [7,12,13].

Let $P_n(t)$ $(n=0,1,2,\ldots)$ denote, as usual, the enumerating polynomials in this problem and let $G_k(z)$ $(k=0,1,2,\ldots)$ be the 'column generating functions', i.e.

$$\Omega(t,z) = \sum_{n\geqslant 0} P_n(t)z^n, \qquad \Omega(t,z) = \sum_{k\geqslant 0} G_k(z)t^k. \tag{6.6}$$

We show that the function $G_k(z)$ $(k\geqslant 2)$ can be expressed in terms of the polynomial $P_{k-1}(t)$. Indeed, from (6.4) and (D.7) we obtain

$$\Omega(t,z) = 1 - t + \frac{t}{1-z}\Omega\left(z, \frac{tz}{(1-z)^2}\right)$$

or, making use of (6.6),

$$\sum_{k\geqslant 0} G_k(z)t^k = 1 - t + \sum_{k\geqslant 0} \frac{t^{k+1}z^kP_k(z)}{(1-z)^{2k+1}}$$

Comparing the coefficients of t^k, we obtain

$$G_k(z) = \begin{cases} 1 & \text{if } k=0, \\ \dfrac{z}{1-z} & \text{if } k=1, \\ \dfrac{z^{k-1}P_{k-1}(z)}{(1-z)^{2k-1}} & \text{if } k\geqslant 2. \end{cases}$$

Remark. The parameter 'number of doublerises' can be handled just like the parameter 'number of peaks'. However, since for each n-Dyck path the total number of peaks and doublerises is equal to n (each rise turns either into a peak or else into a doublerise), we have

$$\Omega_{\text{drise}}(t,z) = \Omega_{\text{peaks}}(t^{-1}, tz).$$

Making use of (6.4) and (D.6), we obtain

$$\Omega_{\text{drise}}(t,z) = 1 + \rho(t,z). \tag{6.7}$$

Remark. The parameter 'number of valleys' can be handled just like the parameter 'number of peaks'. However, since for each n-Dyck path the number of valleys is one less than the number of peaks, we can obtain at once from (6.4) that

$$\Omega_{\text{valley}}(t,z) = 1 + \rho(t,z). \tag{6.8}$$

Remark. From (6.7) and (6.8) we can see that the parameters 'number of doublerises' and 'number of valleys' have the same distribution. This is a known result [1,19,27,33,34,55,56,58]. For a recent simple bijective proof, see [15]. It is well known [29–31,60] that this same distribution is shared also by the parameters 'number of rises at even level' and 'number of nonfinal long sequences', i.e. 'number of ascents or nonfinal descents of length greater than 1' (see also [55]). Recently, the author has proved [14] that also the parameter 'number of high peaks' has that same distribution (for another proof see Subsection 6.4).

6.2. *Number of low peaks and number of high peaks*

By definition, a *low peak* is a peak at level 1 and a *high peak* is a peak at a level greater than 1. Both these parameters are additive. Examining the effect of elevation on the parameters, we obtain

$$\hat{P}_n(t,s)=\begin{cases} t & \text{if } n=0,\\ P_n(s,s) & \text{if } n\geqslant 1.\end{cases} \tag{6.9}$$

Indeed, by elevation, the empty path gains a low peak, while in all the other paths each peak turns into a high peak. From (6.9) we derive

$$\hat{\Omega}(t,s,z)=\Omega(s,s,z)+t-1. \tag{6.10}$$

Eliminating $\hat{\Omega}$ from (4.1) and (6.10), we obtain

$$\Omega(t,s,z)-1=z\Omega(t,s,z)(\Omega(s,s,z)+t-1). \tag{6.11}$$

But $\Omega(s,s,z)=1+s\rho(s,z)$ since $\Omega(s,s,z)$ is the generating function of Dyck paths according to semilength and number of peaks. Consequenly,

$$\Omega(t,s,z)=\frac{1}{1-z(t+s\rho(s,z))}. \tag{6.12}$$

After expanding (6.12) into a geometric series, simple manipulations and the use of (D.3) lead to

$$[t^is^jz^n]\Omega=\begin{cases} \displaystyle\sum_{h=i+1}^{\min(i+j,n-j)}\frac{h-i}{n-h}\binom{h}{i}\binom{n-h}{j}\binom{n-h}{i+j-h} & \text{if } i+j<n,\ j>0,\\ 1 & \text{if } i=n,\ j=0,\\ 0 & \text{otherwise.}\end{cases}$$

It is easy to check that

$$[t^is^jz^n]\Omega=[t^is^{n-i-j}z^n]\Omega,$$

i.e. the number of n-Dyck paths with i low peaks and j high peaks is equal to the number of n-Dyck paths with i low peaks and $n-i-j$ high peaks.

6.3. Number of low peaks

Letting $s=1$ in (6.12), we obtain the generating function for Dyck paths according to semilength and number of low peaks:

$$\Omega(t,z)=\frac{1}{1-(t+C-1)z}. \tag{6.13}$$

From here, making use of (B.1), we obtain easily that

$$[t^k]\Omega=\frac{z^k}{(1-z^2C^2)^{k+1}}. \tag{6.14}$$

In particular,

$$[t^0]\Omega=\frac{1}{1-z^2C^2}=F, \tag{6.15}$$

where F is the Fine function (see Appendix C). Consequenly, the number of n-Dyck paths with no low peaks is equal to the Fine number F_n.

Remark. The number of nonsingular similarity relations on $[n]$ is equal to F_n [36,41,46,54].

From (6.13) we obtain

$$[t^kz^n]\Omega=\sum_{h=0}^{\lfloor (n-k)/2\rfloor}\frac{h}{n-k-h}\binom{k+h}{h}\binom{2n-2k-2h}{n-k},\quad k<n \tag{6.16}$$

(see Table 1).

Remark. If in the above equality we let $k=0$ and we take into account (6.15), then we obtain an expression for F_n in terms of binomial coefficients (see (C.6)).

Table 1
Enumeration of Dyck paths according to length (n) and number of low peaks (k)

n/k	0	1	2	3	4	5	6	7	8
0	1								
1	0	1							
2	1	0	1						
3	2	2	0	1					
4	6	4	3	0	1				
5	18	13	6	4	0	1			
6	57	40	21	8	5	0	1		
7	186	130	66	30	10	6	0	1	
8	622	432	220	96	40	12	7	0	1

From (6.13) we obtain

$$\left(\frac{\partial \Omega}{\partial t}\right)_{t=1} = C - 1$$

and so $\sigma_n = C_n$ $(n \geqslant 1)$. Thus, the expected value of the number of low peaks in a random n-Dyck path is equal to 1.

6.4. Number of high peaks

Letting $t=1$ in (6.12) and renaming s by t, we obtain the generating function for Dyck paths according to semilength and number of high peaks:

$$\Omega(t,z) = \frac{1}{1 - z(1 + t\rho(t,z))},$$

or, taking into account (D.1),

$$\Omega(t,z) = 1 + \rho(t,z).$$

Consequently, the parameter 'number of high peaks' has the same distribution as the parameter 'number of doublerises' [14].

6.5. Number of peaks and number of return steps

Both parameters are additive. Since each elevated Dyck path has exactly one return and for $n \geqslant 1$ the number of peaks is preserved by elevation, we have

$$\hat{P}_n(t,s) = \begin{cases} ts & \text{if } n = 0, \\ sP_n(t,1) & \text{if } n \geqslant 1. \end{cases} \tag{6.17}$$

From (6.17) we obtain

$$\hat{\Omega}(t,s,z) = ts - s + s\Omega(t,1,z). \tag{6.18}$$

Eliminating $\hat{\Omega}$ from (4.1) and (6.18), we obtain

$$\Omega(t,s,z) - 1 = sz\Omega(t,s,z)(t - 1 + \Omega(t,1,z)).$$

But $\Omega(t,1,z) = 1 + t\rho(t,z)$, as the generating function of Dyck paths according to semilength and number of peaks. Consequently,

$$\Omega(t,s,z) = \frac{1}{1 - tsz(1 + \rho(t,z))}. \tag{6.19}$$

Expanding (6.19) into a geometric series and making use of (D.4), we obtain

$$[t^i s^j z^n]\Omega = \begin{cases} \dfrac{j}{i}\dbinom{n-1}{i-1}\dbinom{n-1-j}{i-j} & \text{if } i > 0, j < n, \\ 1 & \text{if } i = j = n, \\ 0 & \text{otherwise.} \end{cases} \tag{6.20}$$

Remark. Formula (6.20) gives the number of ordered trees with n edges, i leaves, and root of degree j.

6.6. Number of return steps

If in (6.19) we let $t=1$ and we rename s by t, then we obtain the generating function for the enumeration of Dyck paths according to semilength and number of return steps:

$$\Omega(t,z)=\frac{1}{1-tzC}, \tag{6.21}$$

where $C=C(z)$ is the Catalan function defined in Appendix B. From here, making use of (B.5), we derive

$$[t^k z^n]\Omega=\frac{k}{2n-k}\binom{2n-k}{n} \tag{6.22}$$

From (6.21) we obtain

$$\left(\frac{\partial\Omega}{\partial t}\right)_{t=1}=zC^3$$

and now from (B.5) we derive

$$\sigma_n=\frac{3}{2n+1}\binom{2n+1}{n-1},\quad \frac{\sigma_n}{C_n}=\frac{3n}{n+2}. \tag{6.23}$$

For ordered trees the results (6.22) and (6.23) can be found in [12] (Theorem 4 and Corollary 4.1).

6.7. Number of peaks and height of first peak

The first parameter is additive, while the second one is a left parameter. Then, combining the derivations of (4.1) and (4.2), we obtain

$$\Omega(t,s,z)-1=z\Omega(t,1,z)\hat{\Omega}(t,s,z). \tag{6.24}$$

Since by elevation the height of the first peak is increased by 1 and, for $n\geqslant 1$, the number of peaks is preserved, we have

$$\hat{P}_n(t,s)=\begin{cases} ts & \text{if } n=0,\\ sP_n(t,s) & \text{if } n\geqslant 1,\end{cases}$$

from where

$$\hat{\Omega}(t,s,z)=ts-s+s\Omega(t,s,z). \tag{6.25}$$

Eliminating $\hat{\Omega}$ from (6.24) and (6.25), we obtain

$$\Omega(t,s,z)-1=sz\Omega(t,1,z)[t-1+\Omega(t,s,z)].$$

But $\Omega(t,1,z)=1+t\rho(t,z)$ as the generating function of Dyck paths according to semilength and number of peaks and, consequently,

$$\Omega(t,s,z)=1+\frac{ts\rho(t,z)}{1+(1-s)\rho(t,z)}. \tag{6.26}$$

From here we obtain easily

$$[s^j]\Omega=\frac{t\rho^j}{(1+\rho)^j}=tz^j(1+t\rho)^j$$

and then, making use of (D.5),

$$[t^i s^j z^n]\Omega=\begin{cases}\dfrac{j}{n}\dbinom{n}{i-1}\dbinom{n-1-j}{i-2} & \text{if } i\geqslant n, j\geqslant n,\\ 1 & \text{if } i=1, j=n,\\ 0 & \text{otherwise.}\end{cases} \tag{6.27}$$

Remark. Formula (6.27) gives the number of ordered trees with n edges, i internal nodes, and root of degree j.

6.8. *Height of first peak*

If in (6.26) we let $t=1$ and we rename s by t, then we obtain the generating function of Dyck paths according to semilength and height of the first peak:

$$\Omega(t,z)=\frac{1}{1-tzC}. \tag{6.28}$$

Remark. From (6.21) and (6.28) it follows at once that the parameters 'number of return steps' and 'height of first peak' have the same distribution. This is a known result [1,14,27]. For a recent simple bijective proof, see [15].

6.9. *Number of low valleys and number of high valleys*

By definition, a *low valley* is a valley at level 0 and a *high valley* is a valley at a level greater than 0. The parameter 'number of low valleys' is quasiadditive since the concatenation of two nonempty Dyck paths adds a new low valley to the existing ones. On the other hand, the parameter 'number of high valleys' is additive. Then, (2.3) yields

$$P_n=tP_{n-1}\hat{P}_0+tP_{n-2}\hat{P}_1+\cdots+tP_1\hat{P}_{n-2}+P_0\hat{P}_{n-1}\quad (n\geqslant 1),$$

which in turn implies

$$\Omega-1=tz\Omega\hat{\Omega}+(1-t)z\hat{\Omega}. \tag{6.29}$$

By elevation the low valleys become high valleys and, consequently,

$$\hat{\Omega}(t,s,z)=\Omega(s,s,z). \tag{6.30}$$

A straightforward computation yields

$$\left(\frac{\partial \Omega_1}{\partial t}\right)_{t=1} = \frac{z}{\sqrt{1-4z}}, \qquad \left(\frac{\partial \Omega_2}{\partial t}\right)_{t=1} = \frac{1-2z-\sqrt{1-4z}}{2\sqrt{1-4z}}. \tag{6.43}$$

Denoting by $\sigma_{1,n}(\sigma_{2,n})$ the sum of the values of the parameters 'number of peaks before the first return step' ('number of peaks after the first return step') at all n-Dyck paths, from (6.43) we obtain (see [59, p. 53, formulas (2.5.11) and (2.5.15)])

$$\sigma_{1,n} = \binom{2n-2}{n-1} = nC_{n-1}, \quad \sigma_{2,n} = \binom{2n-2}{n-2} = (n-1)C_{n-1}.$$

6.13. Number of return descents of odd and even length

We introduce two auxiliary parameters on Dyck paths: (i) a parameter, coded by x, that is equal to 1 if last descent is of odd length and to 0 otherwise and (ii) a parameter, coded by y, that is equal to 1 if last descent is of even length and to 0 otherwise. Taking into account that the two parameters of interest are additive, for the concatenations that occur in (2.4) we have

$$P_{\mathbf{D}_q\hat{\mathbf{D}}_{n-1-q}}(t,s,x,y) = P_q(t,s,1,1)\hat{P}_{n-1-q}(t,s,x,y).$$

Indeed, since the second factors in these concatenations are not empty, the last return descents occur in them. Now (2.4) and (3.1)–(3.3) imply

$$\Omega(t,s,x,y,z) - 1 = z\Omega(t,s,1,1,z)\hat{\Omega}(t,s,x,y,z). \tag{6.44}$$

Examining the effect of the elevation on the four parameters, we obtain

$$\hat{P}_n(t,s,x,y) = \begin{cases} tx & \text{if } n=0, \\ P_n(1,1,sy,tx) & \text{if } n \geqslant 1. \end{cases} \tag{6.45}$$

Indeed, an elevated Dyck path has only one return descent and this is of even (odd) length if the nonelevated Dyck path had its last descent of odd (even) length. Relations (6.45) imply

$$\hat{\Omega}(t,s,x,y,z) = tx - 1 + \Omega(1,1,sy,tx,z). \tag{6.46}$$

Eliminating $\hat{\Omega}$ from (6.44) and (6.46), we obtain

$$\Omega(t,s,x,y,z) - 1 = z\Omega(t,s,1,1,z)(tx - 1 + \Omega(1,1,sy,tx,z)). \tag{6.47}$$

However, since x and y code auxiliary parameters, all we need is

$$\Gamma(t,s,z) \stackrel{\text{def}}{=} \Omega(t,s,1,1,z).$$

If in (6.47) we let $t=s=1$, then we obtain

$$\Omega(1,1,x,y,z) - 1 = zC(x - 1 + \Omega(1,1,y,x,z)).$$

Interchanging x and y and then eliminating $\Omega(1,1,y,x,z)$, we derive

$$\Omega(1,1,x,y,z)=\frac{zC(x-1+yzC-zC)}{1-z^2C^2}+\frac{1}{1-zC}. \tag{6.48}$$

From (6.47), letting $x=y=1$, we obtain

$$\Gamma(t,s,z)-1=z\Gamma(t,s,z)(t-1+\Omega(1,1,s,t,z))$$

and now, making use of (6.48), we finally obtain

$$\Gamma(t,s,z)=\frac{1-z^2C^2}{1-tz-sz^2C-z^2C^2}. \tag{6.49}$$

Expanding (6.49) into a series of powers of s, we obtain

$$[s^j]\Gamma=\frac{z^{2j}C^j(1-z^2C^2)}{(1-z^2C^2-tz)^{j+1}},$$

and then

$$[t^is^j]\Gamma=\binom{i+j}{i}\frac{z^{i+2j}C^j}{(1-z^2C^2)^{i+j}},$$

from where

$$[t^is^jz^n]\Gamma=\binom{i+j}{i}[z^{n-i-2j}]\frac{C^j}{(1-z^2C^2)^{i+j}}. \tag{6.50}$$

Before we continue with the evaluation of the right-hand side of (6.50), we obtain from here new manifestations of the Fine numbers F_n (see Appendix C). Indeed, from (6.50) we find that

$$[t^1s^0z^n]\Gamma=[z^{n-1}]\frac{1}{1-z^2C^2}=[z^{n-1}]F=F_{n-1},$$

$$[t^0s^1z^n]\Gamma=[z^{n-2}]\frac{C}{1-z^2C^2}=[z^{n-1}](C-F)=C_{n-1}-F_{n-1}=F_{n-1}+F_{n-2},$$

In other words, the number of primitive n-Dyck paths with a return descent of odd (even) length is equal to the Fine number F_{n-1} ($F_{n-1}+F_{n-2}$). Deleting the first and the last step in these primitive Dyck paths, we conclude (replacing n by $n+1$) that the number of n-Dyck paths having first peak of even (odd) height is equal to $F_n(F_{n-1}+F_n)$.

Remark. The number of ordered trees with n edges and having root of even (odd) degree is equal to F_n ($F_{n-1}+F_n$).

Remark. The number of non-crossing partitions of $[n]$ in which the block containing 1 is of even (odd) size is equal to F_n ($F_{n-1}+F_n$).

Eliminating $\hat{\Omega}$ from (6.29) and (6.30), we obtain

$$\Omega(t,s,z) - 1 = tz\Omega(t,s,z)\Omega(s,s,z) + (1-t)z\Omega(s,s,z). \tag{6.31}$$

But $\Omega(s,s,z) = 1 + \rho(s,z)$, as the generating function of Dyck paths according to semilength and number of valleys (see Subsection 6.1), and so (6.31) yields

$$\Omega(t,s,z) = 1 + \frac{z(1+\rho(s,z))}{1 - tz(1+\rho(s,z))}. \tag{6.32}$$

Expanding (6.32) into a geometric series and then making use of (D.4), we obtain

$$[t^i s^j z^n]\Omega = \frac{i+1}{n}\binom{n}{i+j+1}\binom{n-i-2}{j}, \quad i \neq n-1.$$

6.10. Number of low valleys

Setting $s=1$ into (6.32) and taking into account that $\rho(1,z) = c - 1$, we obtain the generating function for the enumeration of Dyck paths according to semilength and number of low valleys:

$$\Omega(t,z) = 1 + \frac{zC}{1 - tzC}. \tag{6.33}$$

From here, making use of (B.5), we derive

$$[t^k z^n]\Omega = \frac{k+1}{2n-k-1}\binom{2n-k-1}{n}. \tag{6.34}$$

Remark. Since in every nonempty Dyck path the number of low valleys is one less than the number of returns, relations (6.33) and (6.34) can be derived at once from (6.21) and (6.22), respectively.

6.11. Number of high valleys

Setting $t=1$ into (6.32) and renaming s by t, we obtain the generating function for the enumeration of Dyck paths according to semilength and number of high valleys:

$$\Omega(t,z) = \frac{1}{1 - z(1+\rho(t,z))}. \tag{6.35}$$

Expanding this into a geometric series and making use of (D.4), we obtain

$$[t^k z^n]\Omega = \begin{cases} \displaystyle\sum_{h=1}^{n-k-1} \frac{h}{n-h}\binom{n-h}{k}\binom{n-1}{k+h} & \text{if } k \neq 0, \\ 2^{n-1} & \text{if } k = 0 \end{cases}$$

(see Table 2).

Table 2
Enumeration of Dyck paths according to length (n) and number of high valleys (k)

n/k	0	1	2	3	4	5	6
0	1						
1	1						
2	2						
3	4	1					
4	8	5	1				
5	16	17	8	1			
6	32	49	38	12	1		
7	64	129	141	77	17	1	
8	128	321	453	361	143	23	1

A straightforward computation yields

$$\left(\frac{\partial\Omega}{\partial t}\right)_{t=1} = \frac{(1-\sqrt{1-4z})^5}{32z^2\sqrt{1-4z}}$$

from where (see [59, p. 54, formula (2.5.15)]),

$$\sigma_n = \binom{2n-1}{n-3}.$$

Remark. Since in each Dyck path the number of high valleys is the difference between the number of peaks and the number of return steps, we can derive (6.35) by setting $s=1/t$ in (6.19).

6.12. Number of peaks before and after the first return step.

For the concatenations that occur in (2.3) we have

$$P_{\hat{\mathbf{D}}_q\mathbf{D}_{n-1-q}}(t,s) = \hat{P}_q(t,s)P_{n-1-q}(s,s), \quad 0\leqslant q\leqslant n-1$$

since, obviously, the peaks of the second factors in the concatenations occur after the first return. Now (2.3) and (3.1)–(3.3) imply

$$\Omega(t,s,z) - 1 = z\hat{\Omega}(t,s,z)\Omega(s,s,z). \tag{6.36}$$

Since for an elevated Dyck path all the peaks are before the first return, we have

$$\hat{P}_n(t,s) = \begin{cases} t & \text{if } n=0, \\ P_n(t,t) & \text{if } n\geqslant 1, \end{cases}$$

which implies

$$\hat{\Omega}(t,s,z) = t - 1 + \Omega(t,t,z). \tag{6.37}$$

Eliminating $\hat{\Omega}$ between (6.36) and (6.37), we obtain

$$\Omega(t,s,z) = 1 + z\Omega(s,s,z)(t-1+\Omega(t,t,z)). \tag{6.38}$$

Obviously, both $\Omega(t,t,z)$ and $\Omega(s,s,z)$ are the generating functions corresponding to the parameter 'number of peaks', written in the variables t and s, respectively. Therefore,

$$\Omega(t,t,z)=1+t\rho(t,z), \quad \Omega(s,s,z)=1+s\rho(s,z)$$

and (6.38) becomes

$$\Omega(t,s,z)=1+tz(1+\rho(t,z))(1+s\rho(s,z)). \tag{6.39}$$

Let us denote by $\Omega_1(t,z)$ the generating function corresponding to the number of peaks before the first return and by $\Omega_2(t,z)$ the generating function corresponding to the number of peaks after the first return. Then from (6.39) we obtain at once

$$\begin{aligned} \Omega_1(t,z)&=1+tzC(z)(1+\rho(t,z)),\\ \Omega_2(t,z)&=1+zC(z)(1+t\rho(t,z)). \end{aligned} \tag{6.40}$$

From (6.40) we obtain

$$[t^kz^n]\Omega_1=\begin{cases} C_{n-1}+[t^0z^{n-1}](C(z)\rho(t,z)) & \text{if } k=1,\\ [t^{k-1}z^{n-1}](C(z)\rho(t,z)) & \text{if } k\geqslant 2, \end{cases}$$

$$[t^kz^n]\Omega_2=\begin{cases} C_{n-1} & \text{if } k=0,\\ [t^0z^{n-1}](C(z)\rho(t,z)) & \text{if } k=1,\\ [t^{k-1}z^{n-1}](C(z)\rho(t,z)) & \text{if } k\geqslant 2. \end{cases}$$

But

$$\begin{aligned} [t^0z^{n-1}](C(z)\rho(t,z))&=[z^{n-1}](C(z)\rho(0,z))\\ &=[z^{n-1}]\frac{zC(z)}{1-z}=C_0+C_1+\cdots+C_{n-2} \end{aligned}$$

and, making use of (D.3),

$$\begin{aligned} [t^{k-1}z^{n-1}](C(z)\rho(t,z))&=\sum_{i\geqslant 0}C_i[t^{k-1}z^{n-1-i}]\rho(t,z)\\ &=\sum_{i=0}^{n-2}\frac{C_i}{n-1-i}\binom{n-1-i}{k-1}\binom{n-1-i}{k}. \end{aligned}$$

Consequently,

$$[t^kz^n]\Omega_1=\begin{cases} C_0+C_1+\cdots+C_{n-1} & \text{if } k=1,\\ \displaystyle\sum_{i=0}^{n-2}\frac{C_i}{n-1-i}\binom{n-1-i}{k-1}\binom{n-1-i}{k} & \text{if } k\geqslant 2 \end{cases} \tag{6.41}$$

Table 3
Enumeration of Dyck paths according to length (n) and number of peaks before the first return step (k)

n/k	0	1	2	3	4	5	6	7
0	1							
1		1						
2		2						
3		4	1					
4		9	4	1				
5		23	11	7	1			
6		65	27	28	11	1		
7		197	66	87	62	16	1	
8		626	170	239	250	122	22	1

Table 4
Enumeration of Dyck paths according to length (n) and number of peaks after the first return step (k)

n/k	0	1	2	3	4	5	6	7
0	1							
1	1							
2	1	1						
3	2	2	1					
4	5	4	4	1				
5	14	9	11	7	1			
6	42	23	27	28	11	1		
7	132	65	66	87	62	16	1	
8	429	197	170	239	250	122	22	1

(see Table 3),

$$[t^k z^n]\Omega_2 = \begin{cases} C_{n-1} & \text{if } k=0, \\ C_0 + C_1 + \cdots + C_{n-2} & \text{if } k=1, \\ \displaystyle\sum_{i=0}^{n-2} \frac{C_i}{n-1-i} \binom{n-1-i}{k-1} \binom{n-1-i}{k} & \text{if } k \geqslant 2 \end{cases} \tag{6.42}$$

(see Table 4).

In particular, the number of n-Dyck paths having $k \geqslant 2$ peaks before the first return step is equal to the number of n-Dyck paths having k peaks after the first return step.

The expressions in (6.41) and (6.42) can be easily obtained also by using the formula (6.5) for the number of paths with a prescribed semilength and with a prescribed number of peaks.

In order to obtain from (6.50) an explicit expression for $[t^is^jz^n]\Gamma$, we make use of (B.7). After some elementary manipulations for $i \neq n$, $j \neq 0$ we obtain

$$[t^is^jz^n]\Gamma = \binom{i+j}{i} \sum_{h=0}^{\lfloor (n-i-2j)/2 \rfloor} \frac{j+2h}{2n-2i-3j-2h} \binom{i+j+h-1}{h} \binom{2n-2i-3j-2h}{n-i-j}.$$

6.14. Number of return descents of odd length

If in (6.49) we set $s=1$, then, after some elementary manipulations, we obtain the generating function $\Omega(t,z)$ corresponding to the number of return descents of odd length:

$$\Omega(t,z) = \frac{1+zC}{1+z-tzC}. \tag{6.51}$$

Remark. We have

$$[t^0z^n]\Omega = [z^n]\Omega(0,z) = [z^n]\frac{1+zC}{1+z},$$

from where

$$[t^0z^n]\Omega = C_{n-1} - C_{n-2} + C_{n-3} - \cdots + (-1)^nC_1 \quad \text{if } n \geqslant 2. \tag{6.52}$$

This equality gives of course the number of n-Dyck paths having no return descents of odd length.

In order to find $[t^kz^n]\Omega$, first from (6.51) we derive easily

$$[t^k]\Omega = \frac{(1+zC)z^kC^k}{(1+z)^{k+1}}.$$

Now, making use of (B.5), after some computation we find

$$[t^kz^n]\Omega = (-1)^{n-k}\binom{n}{k} + \sum_{h=0}^{n-k-1} (-1)^h \binom{k+h}{k} \binom{2n-k-1-2h}{n-h} \times \left(\frac{k}{n-k-h} + \frac{k+1}{2n-k-1-2h} \right)$$

(see Table 5). Naturally, equality (6.52) can be derived also from here.

From (6.51) a simple computation yields

$$\left(\frac{\partial \Omega}{\partial t}\right)_{t=1} = \frac{zC^3}{1+zC}$$

or, making use of (C.2),

$$\left(\frac{\partial \Omega}{\partial t}\right)_{t=1} = \frac{F-1}{z}.$$

Table 5
Enumeration of Dyck paths according to length (n) and number of return descents of odd length (k)

n/k	0	1	2	3	4	5	6	7	8
0	1								
1	0	1							
2	1	0	1						
3	1	3	0	1					
4	4	4	5	0	1				
5	10	17	7	7	0	1			
6	32	46	34	10	9	0	1		
7	100	155	94	55	13	11	0	1	
8	329	502	335	154	80	16	13	0	1

Consequently, the number σ_n of returns of odd length in all Dyck paths of semilength n is given by

$$\sigma_n = F_{n+1}$$

From here, (B.4) and (C.10) it follows that for the expected value σ_n/C_n of the number of returns of odd length in a random Dyck path of semilength n we have

$$\lim_{n\to\infty} \frac{\sigma_n}{C_n} = \frac{16}{9}.$$

6.15. Number of return descents of even length

If in (6.49) we set $t=1$, then, after some elementary manipulations, and renaming s by t, we obtain the generating function $\Omega(t,z)$ corresponding to the number of return descents of even length:

$$\Omega(t,z) = \frac{1+zC}{1-tz^2C^2}. \tag{6.53}$$

Remark. From (6.53) we can find a manifestation of the Catalan numbers that seems to be new. Indeed, we have

$$[t^0 z^n]\Omega = [z^n]\Omega(0,z) = [z^n](1+zC),$$

from where

$$[t^0 z^n]\Omega = C_{n-1} \quad \text{if } n \geqslant 1.$$

Thus, the number of Dyck paths of semilength n with no returns of even length is equal to the Catalan number C_{n-1}.

In order to find $[t^k z^n]\Omega$, first from (6.53) we derive easily

$$[t^k]\Omega = (1+zC)z^{2k}C^{2k}.$$

Table 6
Enumeration of Dyck paths according to length (n) and number of return descents of even length (k)

n/k	0	1	2	3	4
0	1				
1	1				
2	1	1			
3	2	3			
4	5	8	1		
5	14	23	5		
6	42	70	19	1	
7	132	222	68	7	
8	429	726	240	34	1

Now, making use of (B.5), after some computation we find

$$[t^k z^n]\Omega = \frac{k}{n-k}\binom{2n-2k}{n} + \frac{2k+1}{2n-2k-1}\binom{2n-2k-1}{n} \quad (n \geqslant 1,\ 2k \leqslant n)$$

(see Table 6). In particular, for $k=0$ we obtain again $[t^0 z^n]\Omega = C_{n-1}$ $(n \geqslant 1)$.

From (6.53) a simple computation yields

$$\left(\frac{\partial \Omega}{\partial t}\right)_{t=1} = \frac{z^2 C^4}{1+zC}$$

or, making use of (C.2),

$$\left(\frac{\partial \Omega}{\partial t}\right)_{t=1} = \frac{F-1}{z} - F + 1 - zF.$$

Consequently, the number σ_n of returns of even length in all Dyck paths of semilength n is given by

$$\sigma_n = \begin{cases} 0 & \text{if } n=0, \\ F_{n+1} - F_n - F_{n-1} & \text{if } n \geqslant 1. \end{cases}$$

From here, (B.4) and (C.10) it follows that for the expected value σ_n / C_n of the number of returns of even length in a random Dyck path of semilength n we have

$$\lim_{n\to\infty} \frac{\sigma_n}{C_n} = \frac{11}{9}.$$

6.16. Number of *duu*'s and *ddu*'s

For example, for the Dyck path *uduuudduuddd* (see Fig. 1) the value of this parameter is 3.

Without any additional effort, we can find the generating function corresponding to the parameters 'number of *duu*'s', 'number of *ddu*'s', and 'semilength'. We introduce an auxiliary parameter (coded by x) that is equal to 1 if the path starts with uu and equal to 0 otherwise.

For the concatenations that occur in (2.3) we have

$$P_{\hat{\mathrm{D}}_j \mathrm{D}_{n-1-j}}(t,s,x)=\begin{cases} P_{n-1}(t,s,t) & \text{if } j=0,\\ s\hat{P}_j(t,s,x)P_{n-1-j}(t,s,t) & \text{if } 1\leqslant j\leqslant n-2,\\ \hat{P}_{n-1}(t,s,x) & \text{if } j=n-1. \end{cases}$$

Indeed, the parameters 'number of duu's', 'number of ddu's' are quasiadditive and (i) for $j=0$ a new duu is created only for those paths in D_{n-1} that start with uu, (ii) for $1\leqslant j\leqslant n-2$ a new duu is created only for those paths in D_{n-1-j} that start with uu, (iii) for $j=n-1$ the second factor in each concatenation reduces to the empty path. Now (2.3) and (3.1)–(3.3) imply

$$P_n=\begin{cases} s\hat{P}_0P_0(t,s,t)+(1-s)\hat{P}_0 & \text{if } n=1,\\ (1-s)P_{n-1}(t,s,t)+s\sum_{j=0}^{n-1}\hat{P}_jP_{n-1-j}(t,s,t)+(1-s)\hat{P}_{n-1} & \text{if } n\geqslant 2. \end{cases} \tag{6.54}$$

Multiplying (6.54) by z^n and summing with respect to n, we obtain

$$\Omega-1=(1-s)(\Omega(t,s,t,z)-1)+(1-s)z\hat{\Omega}+sz\hat{\Omega}\Omega(t,s,t,z). \tag{6.55}$$

Examining the effect of elevation on the parameters, we obtain at once

$$\hat{P}_n(t,s,x)=\begin{cases} 1 & \text{if } n=0,\\ xP_n(t,s,1) & \text{if } n\geqslant 1, \end{cases}$$

which implies

$$\hat{\Omega}(t,s,x,z)=1-x+x\Omega(t,s,1,z). \tag{6.56}$$

Eliminating $\hat{\Omega}$ from (6.55) and (6.56), we obtain

$$\begin{aligned}\Omega-1=&(1-s)z(\Omega(t,s,t,z)-1)\\ &+z(1-s+s\Omega(t,s,t,z))(1-x+x\Omega(t,s,1,z)).\end{aligned} \tag{6.57}$$

Since x is an auxiliary parameter, we need only

$$\Gamma(t,s,z)\stackrel{\text{def}}{=}\Omega(t,s,1,z).$$

We also denote $B\stackrel{\text{def}}{=}\Omega(t,s,t,z)$. If in (6.57) we set successively $x=1$ and $x=t$, then we obtain

$$\Gamma-1=(1-s)z(B-1)+z(1-s+sB)\Gamma,$$

$$B-1=(1-s)z(B-1)+z(1-s+sB)(1-t+t\Gamma),$$

Table 7
Enumeration of Dyck paths according to length (n) and number of duu's and ddu's (k)

n/k	0	1	2	3	4	5	6
0	1						
1	1						
2	2						
3	3	2					
4	4	8	2				
5	5	20	15	2			
6	6	40	60	24	2		
7	7	70	175	140	35	2	
8	8	112	420	560	280	48	2

respectively. Eliminating B, we obtain

$$tsz\Gamma^2 - (1 - 2z + z^2 - tz^2 - sz^2 + 2tsz + tsz^2)\Gamma$$
$$+ 1 - z + z^2 + tsz - tz^2 - sz^2 + tsz^2 = 0. \tag{6.58}$$

Note that, as expected, this equation is symmetric in t and s.

Now let us consider the generating function we intended to find, i.e. the one corresponding to the parameters 'number of *duu*'s and *ddu*'s' and 'semilength'. Denoting this generating function by $\Phi(t,z)$, we have

$$\Phi(t,z) = \Gamma(t,t,z)$$

and, consequently, from (6.58) we obtain the equation satisfied by $\Phi(t,z)$:

$$t^2z\Phi^2 - (1 - 2z + z^2 - 2tz^2 + 2t^2z + t^2z^2)\Phi$$
$$+ 1 - z + z^2 + t^2z - 2tz^2 + t^2z^2 = 0. \tag{6.59}$$

If we introduce the Narayana function $\rho(t,z)$ by eliminating t from (6.59) and (D.1), then we obtain a quadratic equation Φ, having the nonspurious solution

$$\Phi(t,z) = 1 + z(1 + \rho(t,z))^2. \tag{6.60}$$

Making use of (D.4), we obtain at once

$$[t^kz^n]\Phi = \frac{2}{n-1}\binom{n-1}{k}\binom{n}{k+2} \quad (n \geqslant 2) \tag{6.61}$$

(see Table 7).

Remark. Taking into account a bijection between Dyck paths and parallelogram polyominoes (see Appendix E), this last result is identical with Theorem 7 of [10], proved in an entirely different manner. Incidentally, in Theorem 7 of [10] there is a misprint: 'perimeter $2n$' should be replaced by 'perimeter $2n + 2$'.

From (6.60), after some manipulation we obtain

$$\left(\frac{\partial \Phi}{\partial t}\right)_{t=1} = \frac{(1-\sqrt{1-4z})^4}{8z\sqrt{1-4z}}$$

from where (see [59, p. 54, formula (2.5.15)])

$$\sigma_n = 2\binom{2n-2}{n-3}. \tag{6.62}$$

Let $P_n(t)$ $(n=0,1,2,\ldots)$ denote, as usual, the enumerating polynomials in this problem, and let $G_k(z)$ $(k=0,1,2,\ldots)$ be the 'column generating functions', i.e.

$$\Phi(t,z) = \sum_{n\geqslant 0} P_n(t)z^n, \qquad \Phi(t,z) = \sum_{k\geqslant 0} G_k(z)t^k. \tag{6.63}$$

We show that the function $G_k(z)$ $(k\geqslant 1)$ can be expressed in terms of the polynomial $P_{k+1}(t)$. Making use of (6.60) and (D.8), we obtain

$$\Phi\left(\frac{1}{z}, \frac{tz^2}{(1-z)^2}\right) - 1 = tz(\Phi(t,z)-1),$$

or, by virtue of (6.63),

$$\sum_{k\geqslant 0} \frac{t^k z^{2k} P_k(\frac{1}{z})}{(1-z)^{2k}} - 1 = tz\left(\sum_{k\geqslant 0} G_k(z)t^k - 1\right).$$

Making equal the coefficients of t^k, we obtain

$$G_k(z) = \begin{cases} \dfrac{1-z+z^2}{(1-z)^2} & \text{if } k=0, \\[2mm] \dfrac{z^{2k+1}P_{k+1}(1/z)}{(1-z)^{2k+2}} & \text{if } k\geqslant 1. \end{cases}$$

6.17. Number of *duu*'s

If in (6.58) we let $s=1$, then we obtain the equation satisfied by the generating function for the enumeration of Dyck paths according to semilength and number of *duu*'s:

$$tz\Omega^2 - (1-2z+2tz)\Omega + 1 - z + tz = 0.$$

Applying the Lagrange inversion theorem to this equation, we obtain

$$[t^k z^n]\Omega = 2^{n-2k-1} C_k \binom{n-1}{2k}, \tag{6.64}$$

giving the number of n-Dyck paths with k noninitial ascents of length at least 2. This yields at once Touchard's identity [57] (see also [47])

$$C_n = \sum_{k\geqslant 0} 2^{n-2k-1} C_k \binom{n-1}{2k}.$$

Due to the symmetry of the configurations *duu* and *ddu*, from (6.62) it follows that in this case

$$\sigma_n = \binom{2n-2}{n-3}.$$

Remark. Formula (6.64) gives the number of ordered trees with n edges having k nonroot nodes of outdegree at least 2 (branch nodes).

Appendix A. The Lagrange inversion theorem

The Lagrange inversion theorem is used in Subsection 6.17, Appendices B and D under the following specialized form.

Assume that a generating function $A(z)$ satisfies the functional equation

$$A(z) = 1 + zH(A(z)), \tag{A.1}$$

where $H(\lambda)$ is a polynomial in λ. Then Eq. (A.1) has a unique solution $A(z)$ and if $G(\lambda)$ is a polynomial in λ, then

$$[z^n]G(A(z)) = \frac{1}{n}[\lambda^{n-1}]G'(1+\lambda)(H(1+\lambda))^n \quad (n \geqslant 1). \tag{A.2}$$

For further information see, for example, [6,22,24,42,45,51,59].

The elementary but sometimes lengthy evaluations of the right-hand side of (A.2) will not be shown explicitly.

Appendix B. The Catalan function and the Catalan numbers

We define the *Catalan function* $C = C(z)$ implicitly by

$$zC^2 - C + 1 = 0, \quad C(0) = 1. \tag{B.1}$$

The coefficient C_n of z^n in the Maclaurin series of $C(z)$ is known as the nth *Catalan number*. From (B.1) we obtain that

$$C_0 = 1,$$

$$C_n = C_0C_{n-1} + C_1C_{n-2} + \cdots + C_{n-2}C_1 + C_{n-1}C_0, \quad n \geqslant 1.$$

This shows, in particular, that the Catalan numbers are positive integers. From (B.1) we obtain the explicit expression of the Catalan function,

$$C(z) = \frac{1-\sqrt{1-4z}}{2z}, \tag{B.2}$$

from where one can derive (see, for example, [5,25,26]) that

$$C_n = \frac{1}{n+1}\binom{2n}{n}, \quad n \geqslant 0 \tag{B.3}$$

(it is sequence M1459 in [50]). The Catalan numbers are probably the most frequently occurring combinatorial numbers after the binomial coefficients (for example, in R. Stanley's forthcoming book [52] there will be given at least 65 combinatorial structures that are counted by the Catalan numbers; see also [32,53,23].

From (B.3) we obtain the recurrence relation

$$C_n = \frac{2(2n+1)}{n+1} C_{n-1}, \quad n \geqslant 1,$$

which implies

$$\lim_{n\to\infty} \frac{C_{n+1}}{C_n} = 4. \tag{B.4}$$

If to (B.1) we apply the Lagrange inversion theorem, then we obtain

$$[z^n]C^s = \frac{s}{2n+s}\binom{2n+s}{n}, \quad (n,s) \neq (0,0) \tag{B.5}$$

(see also [25, p. 203]). In particular, for $s=1$ we reobtain (B.3) and for $s=1$ we obtain $[z^n]C^2 = C_{n+1}$, which, incidentally, follows also directly from (B.1).

Making use of (B.5), one can derive after elementary manipulations

$$[z^n]\frac{C^p}{(1+z)^q} = (-1)^n\binom{q+n-1}{n} + \sum_{i=0}^{n-1}(-1)^i \frac{p}{p+2n-2i}\binom{q+i-1}{i}\binom{p+2n-2i}{n-i}, \tag{B.6}$$

$$[z^n]\frac{C^p}{(1-z^2C^2)^q} = \sum_{i=0}^{\lfloor n/2 \rfloor} \frac{2i+p}{p+2n-2i}\binom{q+i-1}{i}\binom{p+2n-2i}{n-2i} \quad \text{if } (n,p) \neq (0,0), \tag{B.7}$$

Appendix C. The Fine function and the Fine numbers

We define the *Fine function* $F = F(z)$ by

$$F(z) = \frac{1-\sqrt{1-4z}}{z(3-\sqrt{1-4z})}. \tag{C.1}$$

The coefficient F_n of z^n in the Maclaurin series of $F(z)$ is known as the nth *Fine number*. One finds, for example,

$$F_0=1,\ F_1=0,\ F_2=1,\ F_3=2,\ F_4=6,\ F_5=18,$$

$$F_6=57,\ F_7=186,\ F_8=622,\ F_9=2120,\ F_{10}=7338$$

(it is sequence M1624 in [50]).

From (B.2) and (C.1), eliminating the square root, we find

$$F=\frac{C}{1+zC}. \tag{C.2}$$

Multiplying both the numerator and the denominator of the right-hand side of (C.2) by $1-zC$ and making use of (B.1), we obtain

$$F=\frac{1}{1-z^2C^2}. \tag{C.3}$$

Relations (C.2) and (C.3) imply

$$F=\sum_{i\geqslant 0}(-1)^i z^i C^{i+1},\quad F=\sum_{i\geqslant 0} z^{2i}C^{2i}. \tag{C.4}$$

In particular, from the second relation of (C.4) it follows that the Fine numbers F_n are positive integers for $n\geqslant 2$.

Making use of (B.5), from (C.4) we obtain [36]

$$F_n =\frac{1}{n+1}\left[\binom{2n}{n}-2\binom{2n-1}{n}+3\binom{2n-2}{n}-\cdots+(-1)^n(n+1)\binom{n}{n}\right] \tag{C.5}$$

and [36,46,54]

$$F_n=\frac{1}{n-1}\binom{2n-2}{n}+\frac{2}{n-2}\binom{2n-4}{n}+\frac{3}{n-3}\binom{2n-6}{n}+\cdots \quad \text{if} \quad n\geqslant 2. \tag{C.6}$$

From (C.2) one can derive easily

$$2F+zF=1+C. \tag{C.7}$$

This implies the recurrence relation [36,41,46,54]

$$2F_n+F_{n-1}=C_n \quad (n\geqslant 2), \tag{C.8}$$

from where, by telescoping, one derives [36,41,46]

$$F_n=\frac{1}{2}\left[C_n-\frac{1}{2}C_{n-1}+\frac{1}{2^2}C_{n-2}-\cdots+(-1)^{n-2}\frac{1}{2^{n-2}}C_2\right]. \tag{C.9}$$

Incidentally, relation (C.8) shows that F_n and C_{n+1} have the same parity for $n \geqslant 1$. Consequently (see, for example, [18,40,49]), F_n is odd if and only if $n = 2^k - 2$ for some integer $k \geqslant 1$.

From the relation $F(1 + zC) = C$ (see (C.2)), replacing both F and C by their series expansions, multiplying out the two series in the left-hand side, making equal the coefficients of $z, z^2, z^3, \dots, z^n$, and applying Cramer's rule to the obtained system of linear equations, we find

$$F_n = (-1)^n \begin{vmatrix} C_0 & C_1 & C_2 & \cdots & C_{n-1} & C_n \\ 1 & C_0 & C_1 & \cdots & C_{n-2} & C_{n-1} \\ 0 & 1 & C_0 & \cdots & C_{n-3} & C_{n-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & C_0 \end{vmatrix}.$$

If we make use of the expression (C.9) for F_n and to F_n/C_n we apply Tannery's theorem [4] (p. 136), then we can derive

$$\lim_{n\to\infty} \frac{F_n}{C_n} = \frac{4}{9}. \tag{C.10}$$

(see [36, Corollary 5.1.1]). Alternatively, as pointed out by Shapiro (private communication), one can make use of the following theorem found in [2] (p. 496).

Suppose that $A(z) = \Sigma a_n z^n$ and $B(z) = \Sigma b_n z^n$ *are power series with radii of convergence* $\alpha > \beta \geqslant 0$, *respectively. Suppose* b_{n-1}/b_n *approaches a limit* b *as* $n \to \infty$. *If* $A(b) \neq 0$, *then* $c_n \approx A(b) b_n$, *where* $\Sigma c_n z^n = A(z)B(z)$.

By (C.7), we can take $A(z) = 1/(z+2)$, $B(z) = 1 + C$. Then $\alpha = 2$ and, by (B.4), $\beta = b = 1/4$. Now, by the quoted theorem, $F_n/C_n \approx A(1/4) = 4/9$.

The Fine numbers have occurred for the first time in Fine's investigation of similarity relations [20] in connection with his work on extrapolation from restricted data. Additional facts about them have been derived by Shapiro [46], Rogers [41], Strehl [54], and Moon [36] in their investigation of similarity relations. We mention here that the Fine sequence arises in other enumeration problems as well (see, for example, Subsections 6.13–6.15). In addition, one can show, for example by the technique used in this paper, that the number of internal nodes of odd (even) outdegree in all ordered trees with n edges is equal to $\frac{2}{3}\binom{2n-1}{n} + \frac{1}{3}F_{n-1}$ $\left(\frac{1}{3}\binom{2n-1}{n} - \frac{1}{3}F_{n-1}\right)$ (for a closely related result see [35]). Recently, the Fine numbers have come up in the enumeration of the so-called *derangement trees* [16].

Appendix D. The Narayana function and the Narayana numbers

We define the *Narayana function* $\rho(t,z)$ implicitly by

$$(1+\rho)(1+t\rho)z = \rho, \quad \rho(t,0) = 0. \tag{D.1}$$

Its explicit expression is

$$\rho(t,z)=\frac{1-z-tz-\sqrt{1-2z+z^2-2tz-2tz^2+t^2z^2}}{2tz}. \tag{D.2}$$

The *Narayana numbers* $v_{n,k}$ are defined by

$$v_{n,k}=[t^kz^n]\rho.$$

Applying the Lagrange inversion theorem to (D.1), after elementary manipulations we find the following formulas

$$[t^kz^n]\rho^m=\begin{cases}\dfrac{m}{n}\dbinom{n}{k}\dbinom{n}{k+m} & \text{if } n\geqslant 1,\ m\geqslant 1,\\ 1 & \text{if } m=n=k=0,\\ 0 & \text{otherwise.}\end{cases} \tag{D.3}$$

$$[t^kz^n](1+\rho)^m=\begin{cases}\dfrac{m}{n}\dbinom{n}{k}\dbinom{n+m-1}{k+m} & \text{if } n\geqslant 1,\\ 1 & \text{if } n=k=0,\\ 0 & \text{if } n=0,\ k\geqslant 1.\end{cases} \tag{D.4}$$

$$[t^kz^n](1+t\rho)^m=\begin{cases}\dfrac{m}{n}\dbinom{n}{k}\dbinom{n+m-1}{k-1} & \text{if } n\geqslant 1,\\ 1 & \text{if } n=k=0,\\ 0 & \text{if } n=0, k\geqslant 1.\end{cases} \tag{D.5}$$

In particular, the Narayana numbers $v_{n,k}$ are given by

$$v_{n,k}=\frac{1}{n}\binom{n}{k}\binom{n}{k+1},\quad n\geqslant 1.$$

Either from (D.1) or from (D.2) one derives easily

$$\rho(1/t,tz)=t\rho(t,z). \tag{D.6}$$

From (D.6), after a lengthy but straightforward computation, we obtain

$$\rho(t,z)=\frac{z}{1-z}\left(1+\rho\left(z,\frac{tz}{(1-t)^2}\right)\right). \tag{D.7}$$

Finally, if in (D.7) we replace t by z and z by $tz/(1-z)^2$, then, making use of (D.6), we can derive

$$1+\rho\left(\frac{1}{z},\frac{tz^2}{(1-z)^2}\right)=(1-z)(1+\rho(t,z)). \tag{D.8}$$

Appendix E. Other Catalan families and bijections

As mentioned in Appendix B, R. Stanley's forthcoming book will list at least 65 Catalan families. Here we will consider only those that have been mentioned throughout the paper.

E.1. Ordered trees

A *rooted tree* is a finite set of one or more nodes such that

(a) there is one specially designated node called the *root* of the tree and

(b) the remaining nodes are partitioned into a finite number of disjoint sets and each of the sets in turn is a rooted tree, called a *subtree* of the given tree. The number of subtrees of a node is called the *outdegree* of that node.

An *ordered tree* is a rooted tree where the order of the subtrees of a node is significant. A node of outdegree zero is called a *leaf*, while a node of positive outdegree is called an *internal node*. An exception is made in the case of the tree consisting of a single node. This node (the root), although it has outdegree zero, is considered to be an internal node rather than a leaf.

We mention two well-known bijections between the set of ordered trees and the set of Dyck paths.

(i) For a given ordered tree, traverse the tree in preorder (the root is at the top); to each edge passed on the way down there corresponds a NE step and to each edge passed on the way up there corresponds a SE step.

Under this bijection, to the number of edges of the ordered tree there corresponds the semilength of the Dyck path, to the number of leaves there corresponds the number of peaks, and to the outdegree of the root there corresponds the number of return steps.

(ii) For a given ordered tree, traverse the tree in preorder (the root is at the top); to each node of outdegree r there correspond r NE steps followed by 1 SE step; nothing corresponds to the last leaf.

Under this bijection, to the number of edges of the ordered tree there corresponds the semilength of the Dyck path, to the number of internal nodes there corresponds the number of peaks, and to the outdegree of the root there corresponds the height of the first peak.

E.2. Parallelogram polyominoes

A *parallelogram polyomino* is an array of unit squares that is bounded by two lattice paths which use the steps (1,0) and (0,1), and which intersect only initially and finally.

We mention a well-known bijection between parallelogram polyominoes and Dyck paths [11]. Let $a_1, a_2, \ldots, a_p$ denote the number of cells belonging to the ith column and let $b_1, b_2, \ldots, b_{p-1}$ denote the number of edges shared by columns i and $i+1$. Then the Dyck path that corresponds to the given parallelogram polyomino is the Dyck path whose peak heights are $a_1, a_2, \ldots, a_p$ and valley heights are $b_1 - 1$,

$b_2 - 1, \ldots, b_{p-1} - 1$. It is easy to see how one can recapture the parallelogram polyomino from its corresponding Dyck path. It can be also shown that if the perimeter of the parallelogram polyomino is $2n$, then the corresponding Dyck path has semilength $n-1$. From the definition of this bijection it follows at once that the number of columns of the parallelogram polyomino is equal to the number of peaks of the corresponding Dyck path (see also [9,10]).

E.3. Similarity relations

A *similarity relation* π on $[n]$ is a relation that is reflexive, symmetric, and has the property that if $1 \leqslant i < j < k \leqslant n$ and $i\pi k$, then $i\pi j$ and $j\pi k$.

It is known [36,41,46,54] that the number of similarity relations on $[n]$ is c_n. It will follow again from the bijection given below.

Given a similarity relation π on $[n]$, an element $i \in [n]$ is said to be *singular* if $i\pi j$ implies $j = i$. A similarity relation is said to be *nonsingular* if it has no singular elements.

We describe a very simple bijection between Dyck paths and similarity relations (we can almost say that a Dyck path *is* a similarity relation). Consider the subset of the plane enclosed by a Dyck path and its reflection in the horizontal axis. Since this is in fact a subset of $[n] \times [n]$ (the axes are directed NE and SE and the area of the unit square is twice the area of the original unit square), it is a relation on $[n]$. Obviously, it is reflexive and symmetric. Its simple connectedness ensures the last requirement in the definition of a similarity relation. It is easy to see how the Dyck path can be recaptured from the similarity relation.

Under this bijection, the singular elements of a similarity relation correspond to the low peaks of the corresponding Dyck path and, therefore, to nonsingular similarity relations there correspond Dyck paths with no low peaks.

E.4. Non-crossing partitions

A partition of $[n]$ is said to be a *non-crossing partition* if for every four elements $1 \leqslant a < b < c < d \leqslant n$, the following condition is satisfied: if a and c lie in the same block, and b and d lie in the same block, then all four elements lie in the same block (see [17,28,48,49]).

We describe a bijection between non-crossing partitions and Dyck paths. For a given non-crossing partition on $[n]$, traverse $[n]$ from 1 to n; to each i that is not the largest entry in its block there corresponds a u step and to each i that is the largest entry in its block there corresponds a u step followed by q pieces of d steps, q being the size of the block. For the inverse mapping, number the rises of the Dyck path from 1 to n going from left to right. To each fall assign the number of its matching rise (the matching rise of a fall is the first rise to the left situated on the same level). The numbers on the same descent form a block of the non-crossing partition.

Under this bijection, the number of blocks of a non-crossing partition corresponds to the number of peaks of the corresponding Dyck path.

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