Mathematical Analysis by Tom M. Apostol(condensed)
part 2

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Chapter 1

The Lebesgue Integral

Definition 1. *Let I denote a general interval (bounded, unbounded, open, closed, or half-open). A function s is called a step function on I if there is a compact subinterval $[a, b]$ of I such that s is a step function on $[a, b]$ and $s(x) = 0$ if $x \in I - [a, b]$. The integral of s over I, denoted by $\int_I s(x)dx$ or by $\int_I s$, is defined to be the integral of s over $[a, b]$, as given by:*

$$\int_a^b s(x)dx = \sum_{k=1}^{n} c_k(x_k - x_{k-1})$$

$\{f_n\}$ is an increasing sequence of functions on S such that $f_n \to f$ almost everywhere on S, we indicate this by writing

$$f_n \nearrow f \text{ a.e on S.}$$

Theorem 1. *Let $\{s_n\}$ be a decreasing sequence of nonnegative step functions such that $s_n \searrow 0$ a.e. on an interval I. Then*

$$\lim_{n\to\infty} \int_I s = 0$$

P.P: Let $I = A \cup B$, where $A := \{x \in I : s_n(x) \to 0\}$ and $B := \{x \in I : s_n(x) \not\to 0 \;\; \& \;\; \text{measure}(B) = 0\}$ Then

$$\int_I s_n = \int_A s_n + \int_B s_n < (b-a)\varepsilon + \max s_n \cdot \varepsilon$$

Theorem 2. *Let $\{t_n\}$ be a sequence of step functions on an interval I such that:*

1. *There is a function f such that $t_n \nearrow f$ a.e. on I.*
2. *$\int_I t_n$ converges.*

Then for any step function t such that $t(x) < f(x)$ a.e. on I, we have

$$\int_I t \leq \lim_{n\to\infty} \int_I t_n.$$

P.P: Define:

$$s_n(x) = \begin{cases} t(x) - t_n(x) \text{ if } t(x) \leq t_n(x) \\ 0 \text{ if otherwise.} \end{cases}$$

Definition 2. *A real-valued function f defined on an interval I is called an upper function on I, and we write $f \in U(I)$, if there exists an increasing sequence of step functions $\{s_n\}$ such that*

1. *$s_n \nearrow f$ a.e on I,*
2. *$\int_I s_n$ is finite.*

The sequence $\{s_n\}$ is said to generate f. The integral of f over I is defined by the

$$\int_I f = \lim_{n\to\infty} \int_I s_n.$$

equation

Theorem 3. *Assume $f, g \in U(I)$, then*

1. *$f + g \in U(I)$ and we have*
$$\int_I f + g = \int_I f + \int_I g$$
2. *If $c \geq 0$, $cf \in U(I)$ and we have*
$$\int_I cf = c \int_I f.$$
3. *$\int_I f \leq \int_I g$ if $f(x) \leq g(x)$ a.e on I.*

Theorem 4. *Let I be an interval which is the union of two subintervals, say $I = I_1 \cup I_2$, where I_1 and I_2 have no interior points in common. Then,*

1. *If $f \in L(I)$, then $f \in L(I_1) and f \in L(I_2)$, and we have,*
$$\int_I f = \int_{I_1} f + \int_{I_2} f$$
2. *If $f_1 \in L(I_1)$, $f_2 \in L(I_2)$ and f is defined as:*
$$f(x) = \begin{cases} f_1(x) \text{ if } x \in I_1 \\ f_2(x) \text{ if } x \in I - I_1 \end{cases}$$
Then
$$\int_I f = \int_{I_1} f_1 + \int_{I_2} f_2$$

Theorem 5. *Let f be defined and bounded on a compact interval $[a, b]$, and assume that f is continuous almost everywhere on $[a, b]$. Then $f \in U([a, b])$ and the integral of f, as a function in $U([a, b])$, is equal to the Riemann integral f a $\int_a^b f(x)dx$.*

P.P: Although the number of discontinuities of s_∞ could uncountable, they have measure 0. In fact $s_n \to f$ at every point of continuity: Let $s_n = m_k(f)$ of a partition P_{2^n}.

Definition 3.

We denote by $L(I)$ the set of all functions f of the form $f = u - v$, where $u, v \in U(I)$. Each function f in $L(I)$ is said to be **Lebesgue integrable on** I, and its integral is defined by the equation

$$\int_I f = \int_I u - \int_I v.$$

Theorem 6. *Assume $f \in L(I)$ and $g \in L(I)$. Then we have:*

1. *$af + bg$ for every real a and b, and we have*
$$\int_I af + bg = a \int_I f + b \int_I g.$$
2. *$\int_I f \leq \int_I g$ if $f(x) \leq g(x)$ a.e on I.*

Definition 4. *If f is a real-valued function, its positive part, denoted by f^+, and its negative part, denoted by f^-, are defined by the equations*

$$f^+ = \max(f, 0) \;\;\&\;\; f^- = \max(-f, 0)$$

Theorem 7. *If f and g are in $L(I)$, then so are the functions $f^+, f^-, |f|, \max(f,g)$ and $\min(f,g)$. Moreover, we have* $\left|\int_I f\right| \leq \int_I |f|$

The next theorem describes the behavior of a Lebesgue integral when the inter. val of integration is translated, expanded or contracted, or reflected through the origin. We use the following notation, where c denotes any real number:

$$I + c = \{x + c | x \in I\} \;\; \& \;\; cI = \{cx | x \in I\}$$

Theorem 8. *Assume $f \in L(I)$, then we have:*

1. ***Invariance under translation**. If $g(x) = f(x - c)$ for x in $I + c$, then $g \in L(I + c)$, and*

$$\int_{I+c} g = \int_I f$$

2. ***Behavior under expansion and contraction**. If $g(x) = f(x/c)$ for $x \in cI$, where $c > 0$, then $g \in L(cI)$ and*

$$\int_{cI} g = c \int_I f$$

3. ***Invariance under reflection**. If $g(x) = f(-x)$ for $x \in -I$. The $g \in L(-I)$ and*

$$\int_{-I} g = \int_I f$$

P.P: (1) and (3) are trivial. (2) $\int g$ expands by c because we are integrating f over c copies of I.

Theorem 9. *Let I be an interval which is the union of two subintervals, say $I = I_1 \cup I_2$, where I_1 and I_2 have no interior points in common. Then,*

1. *If $f \in L(I)$ and if $f \geq 0$, then $f \in U(I_1) and f \in U(I_2)$, and we have,*

$$\int_I f = \int_{I_1} f + \int_{I_2} f$$

2. *If $f_1 \in U(I_1)$, $f_2 \in U(I_2)$ and f is defined as:*

$$f(x) = \begin{cases} f_1(x) \text{ if } x \in I_1 \\ f_2(x) \text{ if } x \in I - I_1 \end{cases}$$

Then

$$\int_I f = \int_{I_1} f_1 + \int_{I_2} f_2$$

Theorem 10. *Assume $f \in L(I)$ and let $\varepsilon > 0$ be given. Then,*

1. *There exist functions u and v in $U(I)$ such that $f = u - v$, where v is non-negative a.e. on I and $\int_I v < \varepsilon$.*
2. *There exists a step function s and a function g in $L(I)$ such that $f = s + g$, where $\int_I |g| < \varepsilon$.*

Levi's monotone convergence theorems

Theorem 11. *Let $\{s_n\}$ be a sequence of step functions such that*

1. *$\{s_n\}$ is increasing everywhere in the interval I. and*
2. *$\lim_{n\to\infty} \int_I s_n$ exists.*

Then s_n converges almost everywhere on I to a limit function f in $U(I)$ and we have

$$\int_I f = \lim_{n\to\infty} \int_I s_n$$

Theorem 12. *Let $\{f_n\}$ be a sequence of upper functions defined on the interval I such that*

1. *$\{f_n\}$ is increasing everywhere in the interval I. and*
2. *$\lim_{n\to\infty} \int_I f_n$ exists.*

Then f_n converges almost everywhere on I to a limit function f in $U(I)$ and we have

$$\int_I f = \lim_{n\to\infty} \int_I f_n$$

Theorem 13. *Let $\{g_n\}$ be a sequence of functions in $L(I)$ such that*

1. *Each function g_n is non-negative a.e on I, and*
2. *$\sum_n^\infty \int_I g_n$ converges on I.*

Then $\sum_n g_n$ converges almost everywhere on I to a sum function g in $L(I)$ and we have

$$\int_I g = \sum_{n=1}^\infty \int_I g_n = \int_I \sum_{n=1}^\infty g_n$$

Theorem 14 (The Lebesgue dominated convergence theorem). *Let $\{f_n\}$ be a sequence of function in $L(I)$ such that*

1. *$\{f_n\}$ converges a.e. to a limit function f and*
2. *there is a non-negative "dominator" function g, such that $|f_n(x)| \leq g(x) \in L(I)$ for all $n > 0$ a.e on I.*

Then the limit function $f \in L(I)$, $\int_I f$ exists and

$$\int_I f = \lim_{n\to\infty} \int_I f_n$$

P.P: The "dominator" function g, provides a bound for $\int_I f_n$.

Theorem 15. *Assume f is Riemann-integrable on $[a,b]$ for every $b \geq a$, and assume there is a positive constant M such that*

$$\int_a^b |f(x)|\, dx \leq M.$$

for every $b \geq a$. Then both f and $|f|$ are Riemann-Integrable on $[a,\infty)$. Also, $f \in L([a,\infty))$

The Cauchy principal value of the improper integral $\int_{-\infty}^{\infty} f(x)\, dx$ is given by

$$\lim_{b\to\infty} \int_{-b}^{b} f(x)\, dx.$$

Definition 5. *A function f defined on I, is called measurable on I,i.e, $f \in M(I)$, if there is a sequence of step functions $\{s_n\}$ on I such that*

$$\lim_{n\to\infty} s_n(x) = f(x)$$

a.e on I.

P.P: The function $f = 1$, is not $L(\mathbb{R})$ as for any $\{s_n\} : s_n \to f$ on $\mathbb{R}$ $\lim_{n\to\infty} \int_{\mathbb{R}} s_n \to \infty$.

Theorem 16. *Let X and Y be two sub-intervals of $\mathbb{R}$, and let f be a function defined on $X \times Y$ satisfying the following conditions,*

1. *For each fixed $y \in Y$, the function f_y defined on X by the equation*

$$f_y(x) = f(x, y)$$

is measureable on X.

2. *There exsists a function g in $L(X)$ such that for each $y \in Y$*

$$|f(x, y)| \leq g(x)$$

a.e on X.

3. *For each fixed $y \in Y$, $\lim_{t \to y} f(x, t) = f(x, y)$ a.e on X.*

Then the Lebesgue integral $\int_X f(x, y)$ exists for each $y \in Y$, and the function defined by $F(y) = \int_X f(x, y)\, dx$ is continuous.

Theorem 17. *Let X and Y be two subintervals of $\mathbb{R}$, and let f be a function defined on $X \times Y$ and satisfying the following conditions:*

1. *For each fixed y in Y, the function f_y defined on X by the equation $f_y(x) = f(x, y)$ is measurable on X, and $f_a \in L(X)$ for some f_a in Y.*
2. *The partial derivative $D_2 f(x, y)$ exists for each interior point (x, y) of $X \times Y$.*
3. *There is a nonnegative function G in $L(X)$ such that*

$$|D_2 f(x, y)| \leq G(x) \text{ for all interior points of } X \times Y.$$

Then the Lebesgue integral $\int_X f(x, y)\, dx$ exists for every y in Y, and the function F defined by

$$F(y) = \int_X f(x, y)\, dx$$

is differentiable at each interior point of Y. Moreover, its derivative is given by the formula

$$F'(y) = \int_X D_2 f(x, y)\, dx$$

Definition 6. *Given any non-empty subset S of $\mathbb{R}$, the function χ_S defined by*

$$\chi_S(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{if } x \notin S \end{cases}$$

is called the characterstic function of S.

Definition 7. *A subset S of $\mathbb{R}$ is called measurable if its characteristic function χ_S is measurable. If, in addition, χ_S is Lebesgue-integrable on $\mathbb{R}$, then the measure $\mu(S)$ of the set S is defined by the equation*

$$\mu(S) = \int_{\mathbb{R}} \chi_S$$

If χ_S is measurable but not Lebesgue-integrable on $\mathbb{R}$, we define $\mu(S) = +\infty$. The function μ so defined is called Lebesgue measure.

Theorem 18. *If $\{A_1, A_2, \dots\}$ is a countable collection of disjoint sets, then*

$$\mu\left(\bigcup_{i=0}^{\infty} A_i\right) = \sum_{i=0}^{\infty} \mu(A_i)$$

Definition 8. *Assume two real-valued functions f and g be Lebesgue integrable on I, whose product $f\dot{g} \in L(I)$. Then the integral*

$$(f, g) = \int_I f(x) g(x)\, dx$$

is called the inner product of f and g. If $f^2 \in L(I)$, then $\|f\| = (f, f)^{1/2}$ is called the L^2-norm of f.

The integral resembles the sum $\sum x_k y_k$ which defines the dot product of two vectors $\mathbf{x} = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ The function values $f(x)$ and $g(x)$ in play the role of the components x_k and y_k, and integration takes the place of summation. The L^2-norm of f is analogous to the length of a vector.

Definition 9. *We denote the set of all measurable real-valued functions f on I such that $f^2 \in L(I)$ by $L^2(I)$. The functions in $L^2(I)$ are called square-integrable.*

The set $L^2(I)$ along with the metric d defined by

$$d(f,g) = \left(\int_I |f-g|^2 \right)^{1/2}$$

is a **semimetric space** as it fails to satisfy $d(x, y \neq x) > 0$

Theorem 19. *Let $\{g_n\}$ be a sequence of square-integrable functions such that*

$$\sum_{n=1}^{\infty} \|g_n\|$$

converges. Then the sum $\sum_{n=1}^{\infty} g_n$ converges a.e on I to a function $g \in L^2(I)$ and we have

$$\|g\| = \lim_{n\to\infty} \left\| \sum_{k=1}^{n} g_k \right\| \leq \sum_{k=1}^{\infty} \|g_k\|.$$

P.P: The convergence of $\sum_n (\int_I g_n^2)^{1/2} \geq \sum_n \int_I |g_n| \geq \int_I \sum_n g_n$ implies the convergence $\sum_n g_n$ to g a.e. By triangle, inequality $\sum \|g_n\| \geq \|\sum_n g_n\|$. Thus $g \in L^2(I)$.

Theorem 20 (Riesz-Fischer). *Let $\{f_n\}$ be a Cauchy sequence of complex-valued function in $L^2(I)$. Then there exists a function f in $L^2(I)$ such that*

$$\lim_{n\to\infty} \|f_n - f\| = 0.$$

Chapter 2

Fourier Series and Fourier Integrals

Definition 10. *Let $S = \{\varphi_0, \varphi_1, \dots\}$ be a collection of functions in $L^2(I)$ for any interval I. If*

$$(\varphi_n, \varphi_m) = 0 \qquad \text{whenever } m \neq m$$

the collection S is called an orthogonal system of functions on I.

Theorem 21. *Let $\{\varphi_0, \varphi_1, \dots\}$ be an* ***orthonormal*** *system on I, and assume that $f \in L^2(I)$. Define two sequences of functions $\{s_n\}$ and $\{t_n\}$ as*

$$s_n(x) = \sum_{k=0}^{n} c_k \varphi_k(x) \;\;\&\;\; t_n(x) = \sum_{k=0}^{n} b_k \varphi_k(x)$$

, where $c_k = (f, \varphi_k)$ and b_k are arbitrary for each $k = 1, 2, \dots, n$. Then we have

$$\|f - s_n\| \leq \|f - t_n\| \quad \text{for each } n.$$

The equality holds iff $c_k = b_k$.

P.P:

$$\|f - t_n\|^2 = \|f\|^2 - \sum_{k=1}^{n} |c_k|^2 + \sum_{k=1}^{n} |b_k - c_k|^2.$$

In other words, the area under $f - t_n$ equals the area under $f-$ $(\sum c_k b_k -$ area under $t_n)$. When $c_k = b_k$ the latter term is minimized.

Definition 11. *Let $S = \{\varphi_0, \varphi_1, \dots\}$ be an orthonormal system on I, and $f \in L^2(I)$. The notation*

$$f(x) \sim \sum_{n=0}^{\infty} c_n \varphi_n(x)$$

means the the coeffecients c_k are given by

$$c_k = \int_I f(x)\overline{\varphi_k(x)}\, dx.$$

Theorem 22. *Let c_k be the given coefficients defined above. Then we have*

$$\sum_{n=0}^{\infty} |c_n|^2 \leq \|f\|^2$$

with equality holding if and only if

$$\lim_{n\to\infty} \left\| f - \sum_{n=0}^{n} c_n \varphi_n \right\| = 0.$$

P.P: ***Parseval's formual***

$$\|f\|^2 = |c_0|^2 + |c_1|^2 + \dots$$

is equivalent to

$$\|\mathbf{x}\|^2 = |x_0|^2 + |x_1|^2 + \dots$$

Theorem 23 (Riesz-Fischer). *Assume $\{\varphi_0, \varphi_1, \dots\}$ be an orthonormal system on I. Let $\{c_k\}$ be any sequence of complex numbers such that $\sum |c_k|^2$ converges. Then there is a function f in $L^2(I)$ such that*

- $c_k = (f, \varphi_k)$.
- $\|f\|^2 = \sum_k |c_k|^2$

Theorem 24 (Riemann-Lebesgue lemma). *Assume $f \in L(I)$. Then for every real β,*

$$\lim_{\alpha\to\infty} \int_I f(t)\sin(\alpha t + \beta)\, dt = 0$$

P.P: As $\alpha \to \infty$, $\sin(\alpha t + \beta)$ attain the small values.

The integrals of the form

$$\int_0^\delta g(t)\frac{\sin \alpha t}{t}\, dt$$

are called Dirichlet Integrals. Consider the following two cases:

1. If g is a constant function, the limit

$$\lim_{\alpha\to\infty} \frac{2}{\pi}\int_0^\delta g(t)\frac{\sin \alpha t}{t}\, dt = g(0)$$

2. For every $0 < \varepsilon < \delta$, by the Riemann-Lebesgue lemma, we have

$$\lim_{\alpha\to\infty} \int_\varepsilon^\delta \frac{g(t)}{t}\sin \alpha t\, dt = 0.$$

Thus, it is fair to have the following theorem:

Theorem 25 (Jordan)**.** *Let g be a function of bounded variation on $[0, \delta]$. Then we have,*

$$\lim_{\alpha\to\infty} \int_0^\delta \frac{g(t)}{t}\sin \alpha t\, dt = g(0^+).$$

Theorem 26 (Dini)**.** *Assume that $g(0+)$ exists and suppose that for some $\delta > 0$ the Lebesgue integral*

$$\int_0^\delta \frac{g(t) - g(0+)}{t} dt$$

exists. Then we have, $\lim_{\alpha\to\infty} \int_0^\delta \dfrac{g(t)}{t}\sin \alpha t\, dt = g(0^+)$.

Dirichlet's Kernel is defined by the formula

$$D_n(x) = \frac{1}{2} + \sum_{k=0}^{n} \cos kt$$

Theorem 27. *Assume a 2π-periodic function $f \in L([0, 2\pi])$. Let $\{s_n\}$ denote the partial sums of the Fourier series generated by f. Then we have the integral representation*

$$s_n(x) = \frac{2}{\pi}\int_0^\pi \frac{f(x-t) + f(x+t)}{2} D_n(t)\, dt$$

Theorem 28 (Riemann's relocalization theorem)**.** *Assume f is 2π-periodic function in $L([0, 2\pi])$. Then the Fourier series generated by f converges at the point x, if and only if, the for some $\delta < \pi$, the limit*

$$\lim_{n\to\infty} \int_0^{\delta} [f(x+t) + f(x-t)] \frac{\sin(n+\frac{1}{2})t}{t}\, dt$$

exists, in which case the value of the limit is the sum of the Fourier series.

P.P: By the Riemann-Lebesgue lemma, we have

$$\lim_{n\to\infty} \int_0^{\delta} \left(\frac{1}{t} - \frac{1}{2\sin(t/2)}\right) \frac{f(x+t)+f(x-t)}{2} \sin(n+\frac{1}{2})t\ dt = 0$$

And we also have

$$D_n(x) = \frac{\sin(n+\frac{1}{2})x}{\sin x/2}$$

Combining these two with the previous theorem gives us the relocaliztion theorem. One can apply Jordan's test and Dini's test to check the existence of the above integral.

Theorem 29 (Fejer). *Assume $f \in L([0, 2\pi])$ and f is 2π-periodic. Define the function s as*

$$s(x) = \lim_{t\to 0^t} \frac{f(x+t) + f(x-t)}{2}$$

wherever it exists. Then for each x where s is defined, the Fourier series generated by f is Cesaro summable and has $(C, 1)$ sum $s(x)$. Furthermore, if f is continuous on $[0, 2\pi]$, $\sigma_n \to f$ uniformly.

P.P: $\sigma_n(x)$ can be calculated as

$$\frac{1}{n\pi} \int_0^{\pi} \frac{f(x+t) + f(x-t)}{2} \frac{\sin^2 \frac{1}{2}nt}{\sin^2 \frac{1}{2}t} dt$$

Consequences of Fejer's theorem include (f continuous):

1. Mean-wise convergence of the Fourier series to f on $[0, 2\pi]$
2. Term by term integration
3. Convergence of the series to the value $f(x)$, not other.
4. Weierstrass Approximation theorem

Now, the Fourier series does not represent a function all over $\mathbb{R}$, the *Fourier integral* does:

Theorem 30 (Fourier Integral). *Assume $f \in L(-\infty, \infty)$, and that there is a real number x and a neighbourhood $I = [x-\delta, x+\delta]$ such that one of the following is true:*

1. *f is of bounded variation on I, or*
2. *both the limits $f(x+)$ and $f(x-)$ exist and both the integrals*

$$\int_0^{\delta} \frac{f(x+t - f(x+)}{t} dt \ \ \& \ \ \int_0^{\delta} \frac{f(x-t) - f(x-)}{t} dt$$

exist

Then we have the formula,

$$\frac{f(x+) + f(x-)}{2} = \frac{1}{\pi} \int_0^{\infty} \int_{-\infty}^{\infty} f(u) \cos v(u-x) du dv$$

For a continuous function we have,

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \int_{-\infty}^{\infty} f(u) \cos v(u-x) du dv$$

Definition 12. *Given two functions f and g, both Lebesgue integrable on $(-\infty, \infty)$, let S denote the set of x for which the Lebesgue integral*

$$h(x) = \int_{\infty}^{\infty} f(t)g(x-t)dt$$

exitsts.This integral defines a function h on S called the convolution of f and g. We also write $h = f * g$ to denote this function.

Theorem 31. *Let f and g be two Lebesgue integrable functions on $\mathbb{R}$. Assume, either f or g is bounded on $\mathbb{R}$. Then the the convolution $h = f * g$ exists at every point $x \in R$. In addition if the bounded function is continuous everywhere, h is also continuous everywhere and $h \in L(\mathbb{R})$.*

Theorem 32. *Let $f, g \in L^2(\mathbb{R})$. Then the convolution $h = f * g$ exists and is bounded on $\mathbb{R}$.*

P.P: If $f, g \in L(\mathbb{R})$, then $f * g$ blows up only if neither f nor g is bounded.

Theorem 33. *Assume $f, g \in L(\mathbb{R})$. If one of these is bounded and continuous on $\mathbb{R}$, then*

$$\mathcal{F}(f * g) = \mathcal{F}(f) \cdot \mathcal{F}(g),$$

where

$$\mathcal{F}(f) = \int_{-\infty}^{\infty} f(t)e^{-ixt}dt$$

Chapter 3

Multivariable Differential Calculus

Definition 13. *The directional derivative of* $\mathbf{f}$ *at* $\mathbf{c}$ *in the direction* $\mathbf{u}$*, denoted by the symbol* $\mathbf{f}$*'*$(\mathbf{c};\mathbf{u})$*, is defined by the equation*

$$\mathbf{f}'(\mathbf{c};\mathbf{u}) = \lim_{h\to 0} \frac{\mathbf{f}(\mathbf{c}+h\mathbf{u}) - \mathbf{f}(\mathbf{c})}{h}.$$

P.P: A Geometric intuition about this would be to think $F(t) = \mathbf{f}(\mathbf{c}+t\mathbf{u})$ as a function from $\mathbb{R}$ to $\mathbb{R}^m$, particularly for $m = 1$, it is the crossection of $\mathbf{f}$ by the hyper-plane parrallel to $\mathbf{u}$ and passes through $\mathbf{c}$. Directional derivative along every direction doesn't guarantee continuity.

Definition 14. *A function* $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ *is said to be differentiable at* $\mathbf{c}$*, if there exists a* ***linear*** *function* $\mathbf{T_c} : \mathbb{R}^n \to \mathbb{R}^m$ *such that*

$$\mathbf{f}(\mathbf{c}+\mathbf{v}) = \mathbf{f}(\mathbf{c}) + \mathbf{T_c}(\mathbf{v}) + \|\mathbf{v}\|\mathbf{E_c}(\mathbf{v})$$

, where $\mathbf{E_c}(\mathbf{v}) \to 0$ *as* $\mathbf{v} \to 0$.

P.P: This is different from the directional derivative because we are considering any point $\mathbf{v}$ in some neighbourhood of $\mathbf{c}$, rather than a point along the direction $\mathbf{u}$. Rearranging gives us

$$\frac{\mathbf{T_c}(\mathbf{v})}{\|\mathbf{v}\|} = \frac{\mathbf{f}(\mathbf{c}+\mathbf{v}) - \mathbf{f}(\mathbf{c})}{\|\mathbf{v}\|} - \mathbf{E_c}(\mathbf{v}).$$

Hence $\mathbf{T_c}$ is called the total **derivative**. It guarantees continuity.

We have $\mathbf{f}'(\mathbf{c};\mathbf{u}) = \mathbf{T_c}(\mathbf{u})$. So the total derivative is in general, the 'better' derivative because the linearity of $\mathbf{T_c}(\mathbf{u}) = \mathbf{f}'(\mathbf{c};\mathbf{u})$ with respect to $\mathbf{u}$ implies the 'smoothness' of $\mathbf{f}$ along any given path in $\mathbb{R}^n$

Theorem 34. *If* $\mathbf{v} = (v_1, \ldots, v_n)$*, then*

$$\mathbf{T_c}(\mathbf{v}) = \sum_{k=1}^{n} v_k \frac{\partial \mathbf{f}(\mathbf{c})}{\partial x_k}.$$

If $m = 1$,

$$\mathbf{f}(\mathbf{c}+\mathbf{v}) = \nabla\mathbf{f}(\mathbf{c}) \cdot \mathbf{v}.$$

P.P: Jacobian Matrix:

$$\mathbf{T_c}(\mathbf{v}) = \begin{bmatrix} D_1 f_1(\mathbf{c}) & D_2 f_1(\mathbf{c}) & \ldots & D_n f_1(\mathbf{c}) \\ D_1 f_2(\mathbf{c}) & D_2 f_2(\mathbf{c}) & \ldots & D_n f_2(\mathbf{c}) \\ \vdots & \vdots & \ddots & \vdots \\ D_1 f_m(\mathbf{c}) & D_2 f_m(\mathbf{c}) & \ldots & D_n f_m(\mathbf{c}) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

Theorem 35. *Assume that* $\mathbf{g}$ *is differentiable at* $\mathbf{a}$*, with total derivative* $\mathbf{g}'(\mathbf{a})$*. Let* $\mathbf{b} = \mathbf{g}(\mathbf{a})$ *and assume that* $\mathbf{f}$ *is differentiable at* $\mathbf{b}$*, with total derivative* $\mathbf{f}'(\mathbf{b})$*. Then the composite function* $\mathbf{h} = \mathbf{f} \circ \mathbf{g}$ *is differentiable at* $\mathbf{a}$*, and the total derivative* $\mathbf{h}'(\mathbf{a})$ *is given by*

$$\mathbf{h}'(\mathbf{a}) = \mathbf{f}'(\mathbf{b}) \circ \mathbf{g}'(\mathbf{a}).$$

Theorem 36. *Let S(open) $\subseteq \mathbb{R}^n$ and $\mathbf{f} : S \to \mathbb{R}^m$ be a function differentiable in S. Let $\mathbf{x}$ and $\mathbf{y}$ be any two points in S, such that $L(\mathbf{x}, \mathbf{y}) = \{t\mathbf{x} + (1-t)\mathbf{y} : 0 \le t \le 1\} \subseteq S$. Then for every vector $\mathbf{a} \in \mathbb{R}^m$, there is a point in $L(\mathbf{x}, \mathbf{y})$ such that*

$$\mathbf{a} \cdot [\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x})] = \mathbf{a} \cdot [\mathbf{f}'(\mathbf{z})(\mathbf{y} - \mathbf{x})]$$

P.P: One dimensional MVT on $F(t) = \mathbf{a} \cdot \mathbf{f}(\mathbf{x} + t\mathbf{y})$.

Theorem 37. *Let one of the n partial derivatives of $\mathbf{f}$ at $\mathbf{c}$, $D_i\mathbf{f}(\mathbf{c})$ exist and let the rest $n-1$ partials exist and be continuous in some n-Ball $B(\mathbf{c})$. Then $\mathbf{f}$ is differentiable at $\mathbf{c}$.*

P.P: $\mathbf{f}(\mathbf{c} + \mathbf{v}) - \mathbf{f}(\mathbf{c})$ can be written as a sum n terms which approximate $\nabla\mathbf{f}(\mathbf{c}) \cdot v$.

Theorem 38. *If both two derivatives of $\mathbf{f}$ D_r and D_k exist in some n-ball, $B(\mathbf{c}; \delta)$, and if both are differentiable at $\mathbf{c}$, then $D_{r,k}\mathbf{f} = D_{k,r}\mathbf{f}$.*

P.P: Consider $f : \mathbb{R}^2 \to \mathbb{R}$ and let $S(h; \mathbf{c}, \mathbf{x})$ be the square $\mathbf{c}$ and $\mathbf{x} = \mathbf{c} + (h, h)$ as its opposite corners. As $\to 0$, $\mathbf{f}(S)/h^2 \to 1$. IOW, f tends to be constant for which the symmetry stated in the theorem is natural and trivial.

Definition 15.

$$f^{(k)}(\mathbf{x}; \mathbf{u}) = \sum_{(i_1, \ldots, i_k) \in [n]^3} D_{i_1, \ldots, i_k} f(\mathbf{x}) \prod_i t_i.$$

Chapter 4

Multiple Integrals

Multiple Riemann Integrals

- The measure of an n-dimensional set $A = A_1 \times \cdots \times A_n$ (or the n-measure A) is given by
$$\mu(A) = \prod_{i=1}^{n} \mu(A_i).$$
- The Riemann integral over an A is given by the limit of
$$S(P, f) = \sum_{k=1}^{m} f(\mathbf{t}_k)\mu(I_k),$$
where $\mathbf{t}_k \in I_k$ (I_k is n dimensional interval) as $P \in \mathbb{R}^n$ gets finer and finer.
- The upper and lower integrals are similarly defined since f is real-valued.
- Let f be defined and bounded on a compact interval I in $\mathbb{R}^n$. Then $f \in \mathbb{R}$ on I if, and only if, the set of discontinuities of f in I has n-measure zero.

Theorem 39 ((Fuibini's Theorem). *Let f be defined and bounded on a rectangular interval $Q = [a, b] \times [c, d]$. The we have:*

1. $\underline{\int}_Q \leq \underline{\int_a^b}\overline{\int_c^d} \leq \overline{\int_a^b \int_c^d} \leq \overline{\int}_Q$.
2. *(1) is true if $\overline{\int_c^d}$ is substituted by $\underline{\int_c^d}$.*
3. $\underline{\int}_Q \leq \underline{\int_c^d}\overline{\int_a^b} \leq \overline{\int_c^d \int_a^b} \leq \overline{\int}_Q$.
4. *(3) is true if $\overline{\int_a^b}$ is substituted by $\underline{\int_a^b}$.*
5. $\int_Q = \int_a^b[\overline{\int_c^d}] = \int_a^b[\underline{\int_c^d}]$.

P.P: It is obvious why the first two inequalities of (1) are true. To see why the third is true, consider a function like
$$f(x) = \begin{cases} 0 \text{ if } (x, y) \neq (0, 0) \\ 1 \text{ if } (x, y) = (0, 0). \end{cases}$$
Now the RHS compute the 'upper' volume under f by taking multiplying areas of small rectangles on the xy-plane with the maximal value of f in the corresponding rectangle. Thus it's value is > 0 for $< \infty$. The LHS works by first fixing x and evaluating the upper integral w.r.t y which is always 0. Thus the LHS = 0.
(5) is true if $f \in R(Q)$ because it then satisfies Riemann's condition. The analogous theorem for n-fold integral is stated below.

Theorem 40. *Let f be defined and bounded on*

$$Q := [a_1, b_1] \times \cdots \times [a_n, b_n].$$

If $f \in R(Q)$, then we have

$$\int_Q f(\mathbf{x}) d\mathbf{x} = \int_{a_i}^{b_i} \left[\overline{\int_{Q_i}} f d(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \right] dx_i.$$

Definition 16. *Let S be a subset of a compact interval $I \in \mathbb{R}^n$. For every partition P of I, define the number $\underline{J}(P, S)$ as the sum of the measures of those sub-intervals of P that cover the interior points of S but not the boundary-points. Similarly, define $\overline{J}(P, S)$ as the sum of the measures of the sub-intervals of P that cover $S \cup \partial S$. The numbers*

$$\underline{c}(S) = \sup\{\underline{J}(P, S) : P \in \mathcal{P}(I)\}$$

$$\overline{c}(S) = \sup\{\overline{J}(P, S) : P \in \mathcal{P}(I)\}$$

are called Jordan's inner content and outer-content respectively. If $\underline{c}(S) = \overline{c}(S)$, we say $c(S) = \underline{c}(S) = \overline{c}(S)$ is the ***(Jordan) content*** *of a Jordan-measurable set S.*

P.P: The content of S, is essentially the 'area' of S.

Theorem 41. *Let S be a bounded set in $\mathbb{R}^n$ and let ∂S denote its boundary. Then we have*

$$\overline{c}(\partial S) = \overline{c}(S) - \underline{c}(S).$$

Definition 17. *Let f be a function defined in a Jordan-measurable set S in $I \in \mathbb{R}^n$. Define g as Then we say f is Riemann-Integrable on S whenever the integral $\int_I g$ exists and we denote it's value as*

$$\int_S f(\mathbf{x})\, d\mathbf{x} = \int_I g(\mathbf{x})\, d\mathbf{x}.$$

P.P: This definition extends the definition of multiple integrals to all Jordan-measurable sets in $\mathbb{R}^n$ by making a connection between $\mu(S)$ and $c(S)$ that provides the most natural extension of the Lebesgue criterion from I to S.

Multiple Lebesgue Integral

Theorem 42. *Every open set $S \in \mathbb{R}^n$ can be expressed as a union of countable collection of disjoint bounded cubes whose closure is contained in S.*

P.P: In the one dimensional case, it is trivial. By extension, this is true when the cubes are rectangular prisms. We can divide any rectangular prism into finite number of cubes. Thus the theorem is true.

Theorem 43. *A set $S \in \mathbb{R}^n$ has an n-measure 0 if and only of there is a countable collection of intervals $J_1, J_2, \ldots,$ whose n-measure is finite and each point in S belong to an infinite J_k's.*

P.P: The n-measure of each interval $< \varepsilon$ as $k \to \infty$, but the collection still covers S.

Theorem 44. *If S is a set in $\mathbb{R}^2$ and has a 2-measure 0, the sets S^y and S_x defined by*

$$S^y := \{x : (x, y) \in S\}$$

$$S_x := \{y : (x, y) \in S\}$$

have 01-measures for ***almost all*** *y and x in $\mathbb{R}^1$ respectively.*

P.P: This can be proved using (43) above.

Theorem 45. *Assume f is lebesgue integrable in $\mathbb{R}$. Then we have:*

1. *There is a set* $T \in \mathbb{R}^1$ *with* 1*-measure 0 such that the intergal* $\int_{\mathbb{R}^1} f(x,y)dx$ *exists for all* x *in* $\mathbb{R}^1 - T$.
2. *The function*
$$G(y) = \begin{cases} \int_{\mathbb{R}^1} f(x,y)dx \text{ if } x \in \mathbb{R}^1 - T0 \text{ if } x \in T, \end{cases}$$
is Lebesgue-integrable in $\mathbb{R}^1$
3. $$\iint_{\mathbb{R}^2} f = \int_{\mathbb{R}^1} G(y)dy.$$

Theorem 46 (Tonelli-Hobson)**.** *Assume* f *is measurable and at least one of the iterated integrals*

$$\int_{\mathbb{R}^1}\int_{\mathbb{R}^1} |f(x,y)|dxdy, \qquad \int_{\mathbb{R}^1}\int_{\mathbb{R}^1} |f(x,y)|dydx$$

exists. Then $f \in L(\mathbb{R}^2)$.

P.P: The existence of the iterated intergals provide the necessary bound for $\int f$ and f is measureable.

$$\int_{\alpha(T)} f(\mathbf{x})d\mathbf{x} = \int_T f(\alpha(t))|J_\alpha(t)|dt$$

Chapter 5

Cauchy's Theorem and the Residue Calculus

Definition 18. *For any $a \in \mathbb{C}$ and $r > 0$, the path γ given by*

$$\gamma(\theta) = a + re^{i\theta}, \quad 0 \le \theta \le 2\pi$$

*is called a **positively oriented** circle.*

Definition 19. *Let γ be a path in the complex plane with domain $[a, b]$, and let f be a complex-valued function defined on the graph of γ. The contour integral of f along γ, denoted by $\int_\gamma f$, is defined by the equation*

$$\int_\gamma f = \int_{\gamma(a)}^{\gamma(b)} f[\gamma(t)] d\gamma(t),$$

whenever the Riemann-Stieltjes integral on the right exists.

Theorem 47. *If γ is rectifiable with path length Λ, and $f \in R(\gamma)$ where $|f| \le M$ then*

$$\left| \int_\gamma f \right| \le M\lambda$$

P.P: $|\int f d\gamma| \le \int |f||\gamma'| \le M \int |\gamma'| = M\Lambda$
A path is called **piece-wise smooth** if it's derivative exists everywhere except for finitely many points. However, the left-hand side and right-hand side derivative need to exist.

Definition 20. *Assume γ_0 and γ_1 are two paths that have a common domain $[a, b]$. Moreover assume that*

1. *$\gamma_0(a) = \gamma_1(a)$ and $\gamma_0(b) = \gamma_1(b)$ **OR***
2. *$\gamma_0(a) = \gamma_0(b)$ and $\gamma_1(a) = \gamma_1(b)$.*

Let D be a subset of $\mathbb{C}$ that contains both paths. Then we say γ_0 and γ_1 are homotopic if there exists a continuous function h defined in $[0, 1] \times [a, b]$ with values in D and $h(s, a) = \gamma_0(a)$ and $h(s, b) = \gamma_0(b)$

P.P: Two paths are homotopic if one can be transformed into the other without getting out of the domain D

Theorem 48 (The Polygonal Interpolation Theorem)**.** *Let γ_0 and γ_1 be two homotopic paths in an open set D. Then there exist a finite number of polygonal paths $\alpha_0, \ldots, \alpha_n$ such that*

1. *$\alpha_0 = \gamma_0$ and $\alpha_n = \gamma_1$*
2. *α_j is polygonal for $j = 1, 2 \ldots, n - 1$.*
3. *α_j is **linearly homotopic** in D to α_{j+1} for all defined j.*

P.P: Two paths are linearly homotopic (in D) if the function

$$h(s,t) = (1-s)\gamma_0(t) + s\gamma_1(t)$$

is defined in D. In other words, the homotopy is linear **for a fixed** t there is a line segment in D joining $\gamma(t)$. If D is convex and **the** line segment joining $\gamma_0(t)$ and $\gamma_1(t)$ does not lie in D then we can always construct *'approximating'* polygonal paths that are linearly homotopic to their neighbouring paths.

Theorem 49 (Cauchy's Theorem). *Assume f is analytic on D except at finitely many points where it is continuous. If $\gamma_{0,1}$ are piece-wise smooth and homotopic in D, then*

$$\int_{\gamma_0} f = \int_{\gamma_1} f$$

Theorem 50. *If f is analytic on an open set $D \supset \gamma$, and z is a point not on γ, then*

$$\int_\gamma \frac{f(w)dw}{w-z} = f(z)\int_\gamma \frac{dw}{w-z}.$$

P.P: Define a function in D as $g(w) = \frac{f(w)-f(z)}{w-z}$ $w \neq z$, $f'(z)$ if $w = z$.

NOTE: Taking $\gamma(t) = z + re^{2\pi i t}$,

$$\frac{1}{2\pi}\int_0^{2\pi} f(z+re^{2\pi i\theta}d\theta = f(z).$$

This can be interpreted as a *Mean- Value Theorem* expressing the value of f at the center of a disk as an average of its values at the boundary of the disk. The function f is assumed to be analytic on the closure of the disk, except possibly for a finite subset on which it is continuous.

Definition 21. *Let γ be a circuit in $\mathbb{C}$ and z be a point not on γ. The integral*

$$n(\gamma, z) = \frac{1}{2\pi i}\int_\gamma \frac{dw}{w-z}$$

is called the winding number of γ around z.

P.P: n is exactly the number of times γ 'winds' around z.

Theorem 51. *Let Γ be a rectifiable path in $\mathbb{C}$ and φ be a continuous function defined on the graph of Γ. Moreover, assume that f is defined on $\mathbb{C} - \Gamma$ by the integral*

$$f(z) = \int_\Gamma \frac{\varphi(w)\, dw}{w-z} \quad \textit{for } \; w \neq z$$

Then

1. *For every $a \in \mathbb{C} - \Gamma$, there is a power series expansion that represents $f(z)$ for z in some neighbourhood of a that is given by*

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n,$$

where

$$c_n = \int_\Gamma \frac{\varphi(w)\, dw}{(w-a)^{n+1}}.$$

2. *The above series converges for a positive radius $\geq R$ where*

$$R = \inf\{|w-a| : w \in \Gamma\}$$

P.P:

$$\frac{w-a}{w-z}=\sum_{n=0}^{k}\left(\frac{z-a}{w-a}\right)^{n}+\left(\frac{z-a}{w-a}\right)^{k+1}\cdot\frac{w-a}{w-z}.$$

Multiplying both sides with $\varphi(w)/(w-a)$ and integrating over Γ,

$$f(z)=\sum_{n=0}^{k}(z-a)^{n}\int_{\Gamma}\frac{\varphi(w)\ dw}{(w-a)^{n+1}}+\int_{\Gamma}\frac{\varphi(w)}{w-z}\left(\frac{z-a}{w-a}\right)^{k+1}dw$$

The rightmost integrand vanishes as $k\to\infty$ because $|z-a|\leq|w-a|$
Equating the above with the Taylor expansion formula,

$$f^{n}(z)=n!\int_{\Gamma}\frac{\varphi(w)\ dw}{w-z}.$$

Theorem 52. ***(Liouville's Theorem)*** *If f is analytic and bounded on $\mathbb{C}$, then f is constant.*

Theorem 53 (Local Maximum modulus principle)**.** *Assume f is analytic and non-constant in an open region S. Then $|f|$ has no local maxima in S.*

P.P: Conside the inequality,

$$|f(a)|\leq\frac{1}{2\pi}\int_{0}^{2\pi}f(r+re^{i\theta})\ d\theta$$

If a is a local maximum, then as $r\to 0$ $\sum|f(a+re^{i\theta})|$ gets closer and closer to $|f(a)$.

Theorem 54 (Open mapping theorem)**.** *If f is analytic and non constant in an open region n, then f maps open sets to open sets.*

P.P: If f is not a constant, then $|df|\neq 0$ and the 'amplitwist' is not the same every where since a non-open function has to amplitwist vectors pointing at the same point.

Theorem 55. *If f is analytic in an annulus centered at a, then f has a Laurent expansion in the annulus in $\{c_n\}$, where*

$$c_n=\frac{1}{2\pi i}\int_{\gamma}\frac{f(w)\ dw}{(w-a)^{n+1}}$$

where γ is any positively oriented circuit with center a and radius r' such that $r<r'<R$.

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Theorem 56. *Assume f is analytic everywhere in an open set S except for a finite number of points, z_k. If γ is a circuit homotopic to a point in S such that $z_k\notin\gamma$, then we have*

$$\int_{\gamma}f(z)\ dz=\sum_{k=1}^{n}n(\gamma,z_k)\operatorname*{Res}_{z=z_k}f(z).$$

P.P: Without the principal parts of f at z_k, f_k, f is analytic everywhere in S, thus $\int f-\sum f_k=0$. Integrating over the laurent expansion of f_k yields the RHS of the theorem. If R is a rational function in two variables, then we have

$$\int_{0}^{2\pi}R(\sin\theta,\cos\theta)\ d\theta=\int_{\gamma}\frac{1}{iz}R\bigg(\frac{z^2-1}{2iz},\frac{z^2+1}{2z}\bigg)\ \ dz$$

where γ is positively oriented unit circle around 0 that does not contain the poles of the integrand in the RHS. This can easily be shown using a substitution.