Mathematical Analysis by Tom M. Apostol(condensed)

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Chapter 1

Real and Complex Number Systems

Definition 1. *A set of real numbers is called inductive if it has the following two properties:*

1. *The number* 1 *is in the set.*
2. *For every* x *in the set, the number* $x+1$ *is also in the set.*

Theorem 1. *Assume* $x \geq 0$*. Then for every integer* $n \geq 1$ *there is a finite decimal* $r_n = a_0.a_1a_2\cdots a_n$ *such that*

$$r_n \leq x < r_n + \frac{1}{10^n}.$$

P.P: for x's decimal representation (finite or infinite) $s_0.s_1s_2\ldots$, we can have $a_k \leq s_k < a_k + 1$ for every k.

Theorem 2. *(Cauchy-Schwarz Inequality): If* $a_1 \ldots a_n$ *and* $b_1 \ldots b_n$ *are arbitrary real numbers, we have*

$$\left(\sum_k a_k b_k\right)^2 \leq \left(\sum_k a_k^2\right)\left(\sum_k b_k^2\right)$$

P.P: Lagrange's identity $(\sum_k a_k b_k)^2 = (\sum_k a_k^2)(\sum_k b_k^2) - \sum_{1\leq j<k\leq n}(a_k b_j - a_j b_k)^2$

Theorem 3. *(Minkowski's inequality)*

$$\left(\sum_k (a_k + b_k)^2\right)^{1/2} \leq \left(\sum_k a_k^2\right)^{1/2} + \left(\sum_k b_k^2\right)^{1/2}$$

Chapter 2

Some Basic Notations of Set Theory

Theorem 4. *Let F be a collection of sets. Then for any set B, we have*

$$B - \bigcup_{A \in F} A = \bigcap_{A \in F} (B - A),$$

and

$$B - \bigcap_{A \in F} A = \bigcup_{A \in F} (B - A),$$

P.P: First statement: B without the whole equals common of $[B$ without the individuals$]$. Second Statement: B equals the sum of $B - A$ and the intersections of the A(since $B - A$ does not contain this intersection).

Theorem 5. *If $F = \{A_1, A_2, \dots\}$ is a countable collection of sets, let $G = \{B_1, B_2, \dots\}$, where $B_1 = A_1$ and for $n > 1$,*

$$B_n = A_n - \bigcup_{k=1}^{n-1} A_k.$$

Then G is a collection of disjoint sets, and we have

$$\bigcup_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty} B_k$$

P.P: B_n is composed of the new members of A_n.

Chapter 3

Elements of Point Set Topology

Definition 2. *The set of all $n-$dimensional point is called $n-$dimensional Euclidean Space or simply $n-$space, and is denoted by $\mathbb{R}^n$.*

Definition 3. *The inner product of two $n-$dimensional points $\mathbf{x}$ and $\mathbf{y}$ is defined as*

$$\mathbf{x} \cdot \mathbf{y} = \sum_{k=1}^{n} x_k y_k$$

Definition 4. *Let $\mathbf{a}$ be a given point in $\mathbb{R}^n$ and let r be a given positive number. The set of all points $\mathbf{x}$ in $\mathbb{R}^n$ such that*

$$\|\mathbf{x} - \mathbf{a}\| < r$$

is called an open $n-$ball of radius r and center $\mathbf{a}$. We denote this set by $B(\mathbf{a})$ or by $B(\mathbf{a}; r)$.

Definition 5. *Let S be a subset of $\mathbb{R}^n$, and assume that $\mathbf{a} \in S$. Then $\mathbf{a}$ is called an interior point of S if there is an open $n-$ball with center $\mathbf{a}$, all of whose point belong to S. The set of all interior points of S is denoted by int S.*

Definition 6. *Let S be an open subset of $\mathbb{R}^1$. An open interval I is called a component of S if $I \subseteq S$and if there is no open interval $J \neq I$ such that $I \subseteq J \subseteq S$*

Theorem 6. *Every point of a nonempty open set S belongs to one and only one component interval of S.*

P.P: A point can not exist in two different non-overlapping intervals.

Theorem 7. ***(Representation theorem for open sets on the real line)**: Every nonempty open set S in $\mathbb{R}^1$ is a union of a countable collection of disjoint component intervals of S.*

Definition 7. *Let S be a subset of $\mathbb{R}^n$, and $\mathbf{x}$ a point in $\mathbb{R}^n$, $\mathbf{x}$ not necessarily in S. Then $\mathbf{x}$ is said to be adherent to S if every $n-$ball $B(\mathbf{x})$ contains at least one point S.*

Definition 8. *If $S \subseteq \mathbb{R}^n$, and $\mathbf{x} \in \mathbb{R}^n$, then $\mathbf{x}$ is called an accumulation point of S if every $n-$ball $B(\mathbf{x})$ contains at least one point S distinct from $\mathbf{x}$. If $\mathbf{x} \in S$ but $\mathbf{x}$ is not an accumulation point of S, then $\mathbf{x}$ is called an isolated point.*

Theorem 8. *If $\mathbf{x}$ is an accumulation point, then every $n-$ball $B(\mathbf{x})$ contains infinitely many points of S.*

P.P: The infinitude of real values of r.

Theorem 9. *A set S in $\mathbb{R}^n$ is closed if, and only if, it contains all its adherent points.*

P.P: a closed set contain the points on its ”boundaries”.

Definition 9. *The set of all adherent points of a set S is called the closure of S and is denoted by $\overline{S}$. The set of all accumulation points of S is called the derived set of S and is denoted by S'*

Theorem 10. ***(Bolzano-Weierstrass Theorem)** If a bounded set S in $\mathbb{R}^n$ contains infinitely many points, then there is at least one point in $\mathbb{R}^n$ which is an accumulation point of S.*

P.P: There should be a place that has infinite density of points in a bounded set, if the number of points in the set is infinite.

Theorem 11. ***(The Cantor Intersection Theorem)****: Let $\{Q_1, Q_2, \dots\}$ be a countable collection of nonempty sets in $\mathbb{R}^n$ such that*

1. $Q_{k+1} \subseteq Q_k$
2. *Each set Q_k is closed and Q_1 is bounded.*

Then the intersection $S = \bigcap_{k=1}^{\infty} Q_k$ is closed and nonempty.

P.P: This happens because all closed infinite sets contain their accumulation points and S should have an accumulation point.

Definition 10. *A collection F of sets is said to be a covering of a given set S if $S \subseteq \bigcup_{A \in F}$. If F is a collection of open sets, then F is called an open covering of S.*

Theorem 12. *Let $G = \{A_1, A_2, \dots\}$ denote a countable collection of all $n-$balls having rational radii and centers at points with rational coordinates. Assume $\mathbf{x} \in \mathbb{R}^n$ and let S be an open set in $\mathbb{R}^n$ which contains $\mathbf{x}$. Then at least one of the $n-$balls of G contains $\mathbf{x}$ and is contained in S. That is, we have*

$$\mathbf{x} \in A_k \subseteq S \quad \text{for some } A_k \text{ in } G$$

P.P: The rationals are so dense that there is at least one arbitrarily close to any other real number such that you can have this trade-off of how small should the radius of A_k should be to be in S and how big it should be to contain $\mathbf{x}$.

Theorem 13. ***(Lindelof covering theoerm)****: Assume $A \subseteq \mathbb{R}^n$ and let F be an open covering of A. Then there is a countable subcollection of F which also covers A.*

P.P: Since G of theorem 12 is countable and as A could not be bounded.

Theorem 14. ***(The Heine-Borel Covering Theorem)*** *Let F be an open covering of a closed and bounded set A in $\mathbb{R}^n$. Then a finite subcollection of F also covers A.*

P.P: Duh! because it is bounded and contains its accumulation point(s).

Definition 11. *A set S in $\mathbb{R}^n$ is said to be compact if, and only if, every open covering of S contains a finite sub-cover, that is, a finite subcollection which also covers S.*

Theorem 15. *Let S be a subset of $\mathbb{R}^n$. Then the following three statments are equivalent:*

1. *S is compact.*
2. *S is closed and bounded.*
3. *Every infinite subset of S has an accumulation point in S.*

P.P: There exists an open finite covering F of an closed bounded set such that the collection does not cover the corresponding open set ($1 \implies 2$).

Definition 12. *A metric space is a nonempty set M of objects(called points) together with a function d from $M \times M$ to $\mathbb{R}$(called the metric of the space) satisfying the following properties for all points x, y, z in M:*

1. $d(x, x) = 0$.
2. $d(x, y) > 0$ *if* $x \neq y$.
3. $d(x, y) = d(y, x)$.
4. $d(x, y) \leq d(x, z) + d(z, y)$

Theorem 16. *Let (S, d) be a metric subspace of (M, d), and let X be a subspace of S. Then X is open in S if, and only if,*

$$X = A \cap S$$

for some set A which is open in M.

P.P: $X \subseteq S$ and X is open in S.

Theorem 17. *Let (S, d) be a metric subspace of (M, d), and let Y be a subspace of S. Then Y is closed in S if, and only if,*

$$Y = B \cap S$$

for some set B which is open in M.

P.P: $Y \subseteq S$ and X is closed in S.

Definition 13. *Let S be a subset of a metric space M. A point x in M is called a boundary point of S if every point of S if every ball $B_M(x; r)$ contains at least one point of S and at least one point of $M - S$. The set of all boundary points of S is called the boundary of S and is denoted by ∂S*

Chapter 4

Limits and Continuity

Theorem 18. *A sequence $\{x_n\}$ in a metric space (S, d) can converge to at most one point in S.*

P.P: The triangle inequality forces the distinct points of converge come closer infinitesimally to each other.

Theorem 19. *In a metric space (S, d), assume $x_n \to p$ and let $T = \{x_1, x_2, \dots\}$ be the range of $\{x_n\}$. Then:*

- *T is bounded.*
- *p is an adherent point of T*

P.P: If not bounded, then infinite x's in infinite space fail to satisfy the definition of convergence. For infinite x's in a limited(finite) space, there has to be a point whose balls contain the x's, i.e the adherent point.

Theorem 20. *Given a metric space (S, d) and a subset $T \subseteq S$. If a point p in S is an accumulation point of T, then there is a sequence of points in T which converges to p.*

P.P: Every ball $B_S(p; \varepsilon)$ contains elements of T

Theorem 21. *In a metric space (S, d) a sequence converges to p if, and only if, every infinite subsequence converges to p.*

P.P: Picking out certain elements does not change the density distribution of the numbers.

Theorem 22. *Assume that converges in a metric space (S, d). Then for every $\varepsilon > 0$ there is an integer N such that*

$$d(x_n, x_m) < \varepsilon \text{ whenever } n \geq N \text{ and } m \geq N$$

P.P: As x_n and x_m get closer to a single point they get closer to each other.

Definition 14. *A sequence $\{x_n\}$ in a metric space (S, d) is called a **Cauchy sequence** if it satisfies the following condition (called the Cauchy condition) :*
For every $\varepsilon > 0$ there is an integer N such that

$$d(x_n, x_m) < \varepsilon \text{ whenever } n \geq N \text{ and } m \geq N$$

Theorem 23. *In Euclidean space $\mathbb{R}^k$ every Cauchy sequence is convergent.*

P.P: In $U = \mathbb{R}^k$, as two points get closer to each other, they get closer to a single point.

Definition 15. *A sequence $\{x_n\}$ in a metric space (S, d) is called a **Cauchy sequence** if it satisfies the following condition (called the Cauchy condition) :*
For every $\varepsilon > 0$ there is an integer N such that

$$d(x_n, x_m) < \varepsilon \text{ whenever } n \geq N \text{ and } m \geq N$$

A metric space (S, d) is called **complete** if every Cauchy sequence in S converges in S

Theorem 24. *In any metric space (S, d) every compact subset T is complete.*

P.P: A compact set is closed and hence contains all of its adherent points.

Definition 16. *If p is an accumulation point of A and if $b \in T$, the notation*

$$\lim_{x \to p} f(x) = b$$

is defined as the following:
For every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$d_T(f(x), b) < \varepsilon \text{ whenever } x \in A, x \neq p, \text{ and } d_S(x, p) < \delta$$

Theorem 25. *Assume p is an accumulation point of A and assume $binT$. Then*

$$\lim_{x \to p} f(x) = b$$

if, and only if,

$$\lim_{n \to \infty} f(x_n) = b$$

for every sequence $\{x_n\}$ of points in $A - \{p\}$ which converges to p.

P.P: The discreetness of the the inputs does not change the value of the function.

Theorem 26. *Let f and g be complex-valued functions defined on a subset A of a metric space (S, d). Let p be an accumulation point of A, and assume that*

$$\lim_{x \to p} f(x) = a \text{ and } \lim_{x \to p} g(x) = b$$

Then we also have:

- $\lim_{x \to p} f(x) \pm g(x) = a \pm b$
- $\lim_{x \to p} f(x) \cdot g(x) = ab$

Theorem 27. *Let p be an accumulation point of A and assume that*

$$\lim_{x \to p} \mathbf{f}(x) = \mathbf{a} \text{ and } \lim_{x \to p} \mathbf{g}(x) = \mathbf{b}$$

Then we also have:

- $\lim_{x \to p} [\mathbf{f}(x) \pm \mathbf{g}(x)] = \mathbf{a} \pm \mathbf{b}$
- $\lim_{x \to p} \lambda \mathbf{f}(x) = \lambda \mathbf{a}$
- $\lim_{x \to p} \mathbf{f}(x) \cdot \mathbf{g}(x) = \mathbf{a} \cdot \mathbf{b}$
- $\lim_{x \to p} \|\mathbf{f}(x)\| = \|\mathbf{a}\|$

P.P: A vector function is a collection of real-valued functions.

Definition 17. *Let (S, d_S) and (T, d_T) be metric spaces and let $f : S \to T$ be a function from S to T. The function f is said to be continuous at a point p in S if for every $\varepsilon > 0$ there is a $\delta > 0$ such that*

$$d_T(f(x), f(p)) < \varepsilon \text{ whenever } d_T(x, p) < \delta$$

P.P: This definition reflects the intuitive idea that points close to p are mapped by f into points close to $f(p)$.

Definition 18. *Let $f : S \to T$ be a function from a set S to a set T. If Y is a subset of T, the inverse image of Y under f, denoted by $f^{-1}(Y)$, is defined to be the largest subset of S which f maps into Y; that is,*

$$f^{-1}(Y) = \{x : x \in S \text{ and } f(x) \in Y\}$$

Theorem 28. . *Let $f : S \to T$ be a function from S to T. If $X \subseteq S$ and $Y \subseteq T$, then we have:*

1. $X = f^{-1}(Y) \implies f(X) \subseteq Y$
2. $Y = f(X) \implies X \subseteq f^{-1}(Y)$

P.P: (1): There might be a y that is not in the map f. (2): There might be $z \notin X$ such that $f(z) \in f(Y)$, i.e, the function f is not one-to-one.

Theorem 29. *Let $f : S \to T$ be a function from one metric space (S, d_S) to another (T, d_T). Then f is continuous on S if, and only if, for every open/closed set Y in T, the inverse image $f^{-1}(Y)$ is open/closed in S.*

P.P: If a function is continuous, all the points near $y = f(x)$ are the image of points near x. If every point in Y is in $intY$, then $f^{-1}(Y) = intf^{-1}(Y)$.

Theorem 30. *Let $f : S \to T$ be a function from one metric space (S, d_S) to another (T, d_T). If f is continuous on a compact subset X of S, then the image $f(X)$ is a compact subset of T; in particular, $f(X)$ is closed and bounded in T.*

P.P: $\#X > \#f(X)$ and if points in vicinity remain close under the mapping, finite covering of X will guarantee the finite covering of $f(X)$.

Theorem 31. *Let $f : S \to T$ be a function from one metric space (S, d_S) to another (T, d_T). Assume that f is one-to-one on S, so that the inverse function f^{-1} exists. If S is compact and if f is continuous on S; then f^{-1} is continuous on $f(S)$.*

P.P: $f^{-1} : f(S) \to S$ is from a compact set to a compact set. If f is one-to-one, so is f^{-1}.

Definition 19. *Let $f : S \to T$ be a function from one metric space (S, d_S) to another (T, d_T). Assume also that f is one-to-one on S, so that the inverse function f^{-1} exists. If f is continuous on S and if f^{-1} is continuous on $f(S)$, then f is called a* ***topological mapping*** *or a* ***homeomorphism****, and the metric spaces (S, d_S) and $(f(S), d_T)$ are said to be* ***homeomorphic****.*

Theorem 32. *Let f be defined on an interval S in $\mathbb{R}$. Assume that f is continuous at a point c in S and that $f(c) \neq 0$. Then there is a 1-ball $B(c; \delta)$ such that $f(x)$ has the same sign as $f(c)$ in $B(c; \delta) \cap S$.*

Theorem 33. ***(Bolzano)****Let f be real-valued and continuous on a compact interval $[a, b]$ in $\mathbb{R}$, and suppose that $f(a)$ and $f(b)$ have opposite signs; that is, assume $f(a)f(b) < 0$. Then there is at least one point c in the open interval (a, b) such that $f(c) = 0$.*

Theorem 34. *A metric space S is called* ***disconnected*** *if $S = A \cup B$, where A and B are disjoint nonempty open sets in S. We call S connected if it is not disconnected.*

Definition 20. *A real-valued function f which is continuous on a metric space S is said to be two-valued on S if $f(S) \subseteq \{0, 1\}$, i.e , maps to the discrete metric space.*

Theorem 35. *A metric space S is connected if, and only if, every two-valued function on S is constant.*

P.P: Points in vicinity should be mapped in vicinity, otherwise they should be disconnected?

Theorem 36. *Let $f : S \to M$ be a function from a metric space S to another metric space M. Let X be a connected subset of S. If f is continuous on X, then $f(X)$ is a connected subset of M.*

P.P: If f is continuous, then points near each other should be mapped near each other, and not separated by gaps like discontinuouities.

Theorem 37. *Let F be a collection of connected subsets of a metric space S such that the intersection $T = \bigcap_{A \in F} A$ is not empty. Then the union $U = \bigcup_{A \in F} A$ is connected.*

P.P: The points in the intersection of the A's connect them.
Components of a set are disjoint connected sets.

Definition 21. *S in $\mathbb{R}^n$ is called* ***arcwise connected*** *if for any two points* $\mathbf{a}$ *and* $\mathbf{b}$ *in S there is a continuous function* $\mathbf{f} : [0, 1] \to S$ *such that*

$$\mathbf{f}(0) = \mathbf{a} \textit{ and } \mathbf{f}(1) = \mathbf{b}.$$

P.P: Arc-wise connected means any two points, $\mathbf{a}$ and $\mathbf{b}$ can be connected by a path.

Theorem 38. *Every arcwise connected set S in $\mathbb{R}^n$ is connected.*

P.P: If $S = A \cup B$ and A and B were disjoint the points $\mathbf{a} \in A$ and $\mathbf{b} \in B$ could not be connected by an arc/path.

Theorem 39. *Every open connected set S in $\mathbb{R}^n$ is arcwise connected.*

P.P: For every x and y in S, since $B(x; r_x)$ and $B(y; r_y)$ are subsets of S, a path of at least length $r_x + r_y$ connect the center of the balls, i.e, x and y.

Theorem 40. *Let $f : S \to T$ be a function from one metric space (S, d_S) to another (T, d_T). Then f is said to be* ***uniformly continuous*** *on a subset A of S if the following condition holds:*
For every $\varepsilon > 0$ there exists a $\delta > 0$ (depending only on ε) such that if $x \in A$ and $p \in A$ then:

$$d_T(f(x), f(p)) < \varepsilon \text{ whenever } d_S(x, p) < \delta_\varepsilon$$

P.P: A function is uniformly continuous if its graph can be "covered" by identical non-overlapping rectangles.

Theorem 41. *Let $f : S \to T$ be a function from one metric space (S, d_S) to another (T, d_T). Let A be a compact subset of S and assume that f is continuous on A. Then f is uniformly continuous on A.*

P.P: If f is continuous on a closed and bounded set S, then it does not undergo a huge amount of increase/decrease.

Let $f : S \to S$ be a function from a metric space (S, d) into itself. A point p in S is called **a fixed point** of f if $f(p) = p$. The function f is called a **contraction** of S if there is a positive number $\alpha < 1$ (called a **contraction constant**), such that

$$d(f(x), f(y)) \leq \alpha d(x, y) \text{ for all } x, y \text{ in } S$$

Theorem 42. ***Fixed-point Theorem****: A contraction f of a complete metric space S has a unique fixed point p.*

P.P: If f is a contraction then it is a concave down function and it crosses $y = x$ only once.

Chapter 5

Derivatives

Theorem 43. *If f is defined on (a, b) and differentiable at a point c in (a, b), then there is a function f^* (depending on f and on c) which is continuous at c and which satisfies the equation*

$$f(x) - f(c) = (x - c)f^*(x)$$

for all x in (a, b), with $f^*(c) = f'(c)$. Conversely, if there is a function f^*, continuous at c, which satisfies the above equation, then f is differentiable at c and $f'(c) = f^*(c)$.

P.P: Continuity forces $f^* = f'$ at $x = c$.

Definition 22. *Let f be defined on a closed interval S and assume that f is continuous at the point c in S. Then f is said to* ***have a righthand derivative at*** c *if the righthand limit*

$$\lim_{x \to ^+} \frac{f(x) - f(c)}{x - c}$$

exists as a finite value, or if the limit is $+\infty$ or $-\infty$. This limit will be denoted by $f'_+(c)$. Lefthand derivatives, denoted by $f'_-(c)$, are similarly defined. In addition if c is an interior point of S, then we say that f has the derivative $f'(c) = +\infty$ both the right- and lefthand derivatives at c are $+\infty$ the derivative $f'(c) = \infty$ id similarly defined.)

Theorem 44. *Let f be defined on an open interval (a, b) and assume that for some c in (a, b) we have $f\prime(c) > 0$ or $f'(c) = +\infty$. Then there is a 1-ball $B(c) \in (a, b)$ in which*

$$f(x) > f(c) \text{ if } x > c \qquad \text{and} \qquad f(x) < f(c) \text{ if } x < c$$

P.P:f is continuous at c and strictly increasing on $B(c)$.

Definition 23. *Let f be a real-valued function defined on a subset S of a metric space M, and assume $a \in S$. Then f is said to have a local maximum at a if there is a ball $B(a)$ such that*

$$f(x) \leq f(a) \qquad \text{for all } x \in B(a) \cap S$$

If $f(x) \geq f(a)$ for all $xinB(a) \cap S$, then f is said to have a local minimum at a.

Theorem 45. *Let f be defined on an open interval (a, b) and assume that f has a local maximum or a local minimum at an interior point c of (a, b). If f has a derivative (finite or infinite) at c, then $f'(c)$ must be 0.*

P.P: The tangent is horizontal at the local extrema.

Theorem 46. ***Rolle*** *Assume f has a derivative (finite or infinite) at each point of an open interval (a, b), and assume that f is continuous at both endpoints a and b. If $f(a) = f(b)$ there is at least one interior point c at which $f'(c) = 0$.*

Theorem 47. ***(Generalized Mean-Value Theorem)****. Let f and g be two functions, each having a derivative (finite or infinite) at each point of an open interval (a, b) and each continuous at the endpoints a and b. Assume also that there is no interior point x at which both $f'(x)$ and $g'(x)$ are infinite. Then for some interior point c we have*

$$f'(c)(g(b) - g(a)) = g'(c)(f(b) - f(a))$$

If $g(x) = x$, we have the mean-value theorem.

P.P Consider the case when $g(x) = x$. The equation

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

is equivalent to saying there is a point $c \in (a, b)$ such that the tangent of f at c is parallel to the line, ℓ, passing through $(a, f(a))$ and $(b, f(b))$. This true since one can apply Rolle's theorem after tilting the whole thing in such a way that ℓ is parallel to the x-axis.

The generalized mean-value theorem can be thought of as x changes as a function of a parameter t, i,e $x = g(t)$ and $y = f(t)$. The $f(t)$ against $g(t)$ graph is continuous.

Theorem 48. ***(Intermediate-value theorem for derivatives)****: Assume that f is defined on a compact interval $[a, b]$ and that f has a derivative (finite or infinite) at each interior point. Assume also that f has finite one-sided derivatives $f'_+(a)$ and $f'_-(b)$ at the endpoints, with $f'_+(a) \neq f'_-(b)$. Then, if c is a real number between $f'_+(a)$ and $f'_-(b)$, there exists at least one interior point x such that $f'(x) = c$.*

P.P: If $f'(x) \neq c$ for all x in (a, b) then there should be a cusp(i.e a sudden change of a tangent) at x which contradicts the fact that f has a derivative in (a, b).s

Theorem 49. ***(Taylor's Formula with Remainder)****:Let f and g be two functions having finite nth derivatives $f^{(n)}$ and $g^{(n)}$ in an open interval (a, b) ,and continuous $(n-1)$st derivatives in the closed interval $[a, b]$. Assume that $c \in [a, b]$. Then, for every x in $[a, b]$, $x \neq c$, there exists a point x_1 interior to the interval joining x and c such that*

$$\left[f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!}(x-c)^k\right] g^{(n)}(x_1) = f^{(n)}(x_1)\left[g(x) - \sum_{k=0}^{n-1} \frac{g^{(k)}(c)}{k!}(x-c)^k\right]$$

P.P: "Young man, in mathematics, you don't understand things, you just get used to them."

Definition 24. *Let $\mathbf{f}(t) = (f_1(t), \ldots, f_n(t))$. Then*

$$\mathbf{f}'(t) = (f'_1(t), \ldots, f'_n(t))$$

Let S be an open set in Euclidean space $\mathbb{R}^n$, and let $f : S \to R$ be a real-valued function defined on S. If $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{c} = (c_1, \ldots, c_n)$ are two points of S having corresponding coordinates equal except for the kth, that is, if $x_i = c_i$; for $i \neq k$ and if $x_k \neq c_k$, then we can consider the limit

$$D_k f(\mathbf{c}) = \lim_{x_k \to c_k} \frac{f(\mathbf{x}) - f(\mathbf{c})}{x_k - c_k}$$

NOTE: Since partial derivative only sees the derivatives along a finite set of co-ordinates, it is possible that there could exist a function not continuous but has all partial derivatives.

Definition 25. *Let f be a complex-valued function defined on an open set S in $\mathbb{C}$, and assume $c \in S$. Then f is said to be differentiable at c if the limit*

$$\lim_{z \to c} \frac{f(z) - f(c)}{z - c}$$

Theorem 1 (The Cauchy-Riemann Equations). *Let $f = u + iv$ be defined on an open set S in $\mathbb{C}$. If $f'(c)$ exists for some c in S, then the partial derivatives $D_1u(c), D_2u(c), D_1v(c)$ and $D_2v(c)$ also exist and we have*

$$f'(c) = D_1u(c) + iD_1v(c) \tag{5.1}$$
$$f'(c) = D_2v(c) - iD_2u(c) \tag{5.2}$$

P.P: In order for $f'(a+bi) = \lim_{(x,y)\to(a,b)} \Delta f/\Delta z$ to exist $\lim_{(a,y)\to(a,b)} \Delta f/\Delta z = \lim_{(x,b)\to(a,b)} \Delta f/\Delta z$.

Chapter 6

Bounded Variations and Rectifiable Curves

Theorem 50. *Let f be an increasing function defined on $[a, b]$ and let x_0, x_1, ... , x_n be $n + 1$ points such that*

$$a = x_0 < x_1 < \cdots < x_n = b$$

Then we have the inequality

$$\sum_{k=1}^{n-1}[f(x_k+) - f(x_k-)] \leq f(b) - f(a)$$

P.P:The sum at the jump of $f(x)$ at $x = x_i$ is within $(f(a), f(b))$ if f is monotonic.

Theorem 51. *If f is monotonic on $[a, b]$, then the set of discontinuities of f is countable.*

P.P: The jumps of f are within an interval that has a finite width. A finite-width interval can only be a union of a countable set of disjoint non-zero width intervals

Definition 26. *If $[a, b]$ is a compact interval, a set of point*

$$P = \{x_0, x_1, \ldots, x_n\}$$

satisfying the inequalities

$$a = x_0 < x_1 < \cdots < x_n = b$$

is called a partition of $[a, b]$. The interval $[x_{k-1}, x_k]$ is called the k-th subinterval of P and we write $\Delta x_k = x_k - x_{k-1}$ so that $\sum_{k=1}^{n} \Delta x_k = b - a$. The collection of all possible partitions of $[a, b]$ will be denoted by $\mathcal{P}([a, b])$.

Definition 27. *Let f be defined on $[a, b]$. If $P = \{x_0, x_1, \ldots, x_n\}$ is a partition of $[a, b]$, write $\Delta f_k = f(x_k) - f(x_{k-1})$, for $k = 1, 2, \ldots, n$. If there exists a positive nuber M such that*

$$\sum_{k=1}^{n} |\Delta f_k| \leq M$$

or all partitions of $[a, b]$, then f is said to be of bounded variation on $[a, b]$.

Counter example:

$$f(x) = \begin{cases} 1/x \text{ if } x \neq 0 \\ 0 \text{ if otherwise} \end{cases}$$

Theorem 52. *If f is monotonic on $[a, b]$, then f is of bounded variation on $[a, b]$*

P.P: The sum telescopes for a monotonic function.

Theorem 53. *If f is continuous on $[a, b]$ and if f' exists and is bounded in the interior, say $|f'(x)| \leq A$ for all x in (a, b), then f is of bounded variation on $[a, b]$.*

P.P : We know $mx+b$ is of bounded variation. Pick $m = A$ and $b = \min(f(x))$. Then $mx+b \geq f(x)$ hence f is of bounded variation.

Theorem 54. *If f is of bounded variation on $[a, b]$, say $\sum |\Delta f_k| \leq M$, for all partitions of $[a, b]$, then f is bounded on $[a, b]$. In fact*

$$|f(x)| \leq |f(a)| + M \text{ for all } x \text{ in } [a, b]$$

P.P: The value of f at x is less than its total jump in $[a, b]$ more than the smallest value of x in $[a, b]$, i.e $f(a)$

Definition 28. *Let f be of bounded variation on $[a, b]$, and let $\sum(P)$ denote the sum $\sum_{k=1}^{n} |\Delta f_k|$ corresponding to the partition $P = \{x_0, x_1, \ldots, x_n\}$, of $[a, b]$. The number*

$$V_f(a, b) = \sup\left\{\sum(P) : P \in \mathcal{P}[a, b]\right\}$$

is called the total variation of f on the interval $[a, b]$.

This can be thought as the maximum jump f can have in $[a, b]$, i.e, for a continuous function, when $P = \{x : x \text{ is a turning point of } f \text{ in } [a, b]\}$

Theorem 55. *Assume that f and g are each of bounded variation on $[a, b]$. Then so are their sum, difference, and product. Also, we have*

$$V_{f \pm g} \leq V_f + V_g$$

and

$$V_{fg} \leq AV_f + BV_g$$

where $A = \sup\{|g(x)| : x \in [a, b]\}$ and $B = \sup\{|f(x)| : x \in [a, b]\}$

P.P: This is in a way the consequence of the fact that $f \pm g$ tends to have lower number of turning points in a fixed interval than the total number of turning points of f and g. And $fg \leq Af$ and $fg \leq Bf$; the equality rarely shows up.

Theorem 56. *Let f be of bounded variation on $[a, b]$. and assume that f is bounded away from zero; that is, suppose that there exists a positive number m such that $0 < m \leq |f(x)|$ or all x in $[a, b]$. Then $g = 1/f$ is also of a bounded variation on $[a, b]$, and $V_g \leq V_f/m^2$.*

Theorem 57. *Let f be of bounded variation on $[a, b]$, and assume that $c \in (a, b)$. Then f is of bounded variation on $[a, c]$ and on $[c, b]$ and we have*

$$V_f(a, b) = V_f(a, c) + V_f(c, d)$$

P.P: If c is a turning point of f, then it is trivial. If otherwise then $|f(c)-f(x_k)|+|f(x_{k+1}-f(c))| = |f(x_{k+1} - f(x_k))|$, where x_k and x_{k+1} are the immediate turning points of f found to the left and to the right of c respectively.

Theorem 58. *Let f be of bounded variation on $[a, b]$. Let V be defined on $[a, b]$ as follows: $V(x) = V_f(a, x)$ if $a < x \leq b$, $V(a) = 0$. Then*

1. *V is an increasing function on $[a, b]$.*
2. *$V - f$ is an an increasing function on $[a, b]$.*

P.P: (1) V_f is additive. (2) $V(x) - V(y) = V(y, x) \geq f(y) - f(x)$.

Theorem 59. *Let f be defined on $[a, b]$. Then f is of bounded variation on $[a, b]$ if, and only if, f can be expressed as the difference of two increasing functions.*

P.P: Theorem 55 and 58.

Theorem 60. *Let f be of bounded variation on $[a, b]$. If $x \in (a, b]$, let $V(x) = V_f(a, x)$ and put $V(a) = 0$. Then every point of continuity of f is also a point of continuity of V. The converse is also true.*

P.P: $\omega_f(x) = 0 \Longleftrightarrow \omega_V(x) = 0$.

Let the set of all polygons that can be inscribed in a curve f be I_f. The curve f is **rectifiable** if the length of the curve is $\sup\{p : p \text{ is the length of a polygon in } I_f\}$. More formally...

Let $\mathbf{f} : [a, b] \to \mathbb{R}^n$ be a path in $\mathbb{R}^n$. For any partition of $[a, b]$ given by $P = \{t_0, t_1, \ldots, t_m\}$, the points $\mathbf{f}(t_0), \mathbf{f}(t_1), \ldots$ are the vertices of an inscribed polygon. The length of this polygon is denoted by $\Lambda_{\mathbf{f}}(P)$ and is defined to be the sum

$$\Lambda_{\mathbf{f}}(P) = \sum_{k=1}^{n} \|\mathbf{f}(t_k) - \mathbf{f}(t_{k-1})\|$$

.

Definition 29. *If the set of numbers $\Lambda_{\mathbf{f}}(P)$ is bounded for all partitions P of $[a, b]$, then the path $\mathbf{f}$ is said to be rectifiable and its arc length, denoted by $\Lambda_{\mathbf{f}}(a, b)$, is defined by the equation*

$$\Lambda_{\mathbf{f}}(a, b) = \sup\{\Lambda_{\mathbf{f}}(P) : P \in \mathcal{P}[a, b]\}$$

If the set of numbers $\Lambda_{\mathbf{f}}(P)$ is unbounded, $\mathbf{f}$ is called nonrectifiable.

Theorem 61. *7. Consider a path $\mathbf{f} : [a, b] \to \mathbb{R}^n$ with components $\mathbf{f} = (f_1, \ldots, f_n)$ is rectifiable if, and only if, each component f_k is of bounded variation on $[a, b]$. If $\mathbf{f}$ is rectifiable, we have the inequalities*

$$V_k(a, b) \leq \Lambda_{\mathbf{f}}(a, b) \leq V_1(a, b) + \cdots + V_n(a, b)$$

where $V_k(a, b)$ is the total variation of f_k in $[a, b]$

P.P: For the $V_k(a, b) \leq \Lambda_{\mathbf{f}}(a, b)$, one can observe that the total variation of f_k is less than the k-th component (in $\mathbb{R}^n$) of a certain increasing function with the same arc-length as $\mathbf{f}$.
For the $\Lambda_{\mathbf{f}}(a, b) \leq V_1(a, b) + \cdots + V_n(a, b)$ observe that any increasing function originating from a "Rook domain"[1] approaches to look like the side of the rook domain which is at most the sum of the total variations $V_1, V_2, \ldots, V_k$, hence less than than this sum. It is easier to visualize on $\mathbb{R}^2$

Theorem 62. *If $c \in (a, b)$ we have*

$$\Lambda(a, b) = \Lambda(a, c) + \Lambda(c, b)$$

Theorem 63. *Consider a rectifiable path $\mathbf{f}$ defined on $[a, b]$. If $x \in (a, b]$, let $s(x) = \Lambda_{\mathbf{f}}(a, x)$ and let $s(a) = 0$. Then we have:*

1. *The function s so defined is increasing and continuous on $[a, b]$*
2. *If there is no subinterval of $[a, b]$ on which $\mathbf{f}$ is constant, then s is strictly increasing on $[a, b]$.*

Two curves $\mathbf{f}$ and $\mathbf{g}$ are called *equivalent* if $\mathbf{g}(t) = \mathbf{f}(u(t))$ for a real-valued monotonic function $u : [c, d] \to [a, b]$ and a vector function $\mathbf{f} : [a, b] \to \mathbb{R}^n$.

[1] a rook domain is a corner of a square, cube or a hypercube along with the sides forming it.

Chapter 7

The Riemann-Stiletjes Integral

Notations:

1. $f, g, \alpha, \beta, \ldots$ are all bounded in the compact interval $[a, b]$.
2. $P = \{x_0, x_1, \ldots, x_n\}$
3. A partition P' of $[a, b]$ is said to be *finer* than P (or *a refinement* of P) if $P \subseteq P'$.
4. $\Delta\alpha_k = \alpha(x_k) - \alpha(x_{k-1})$
5. The norm of a partition P is the length of the largest subinterval of P and is denoted by $\|P\|$.
6. $\alpha \nearrow$ on $[a, b] = \alpha$ is increasing on $[a, b]$.

Definition 30. *Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$ and let t_k be a point in the subinterval $[x_{k-1}, x_k]$. A sum of the form*

$$S(P, f, \alpha) = \sum_{k=1}^{n} f(t_k)\Delta\alpha_k$$

is called a Riemann-Stieltjes sum off with respect to α. We say f is Riemann-Stieltjes integrable with respect to α in $[a, b]$, and we write "$f \in R(\alpha)$ in $[a, b]$" if there exists a number A having the following property. For every $\varepsilon > 0$ there exsists a partiton P_ε of $[a, b]$ such that for every partion P that is finer than P_ε and for every choice of points $t_k \in [x_{k-1}, x_k]$, we have $|S(P, f, \alpha) - A| < \varepsilon$.

For $\alpha(x) = x$, we usually omit α from notations.

Theorem 64. *If $f \in R(\alpha)$ and if $g \in R(\alpha)$ on $[a, b]$, then $c_1 f + c_2 g \in R(\alpha)$ on $[a, b]$ (for any two constants c_l and c_2) and we have*

$$\int_a^b (c_1 f + c_2 g)\, d\alpha = c_1 \int_a^b f\, d\alpha + c_2 \int_a^b g\, d\alpha$$

Theorem 65. *If $f \in R(\alpha)$ and $f \in R(\beta)$ on $[a, b]$, then $f \in R(c_1\alpha + c_2\beta)$ on $[a, b]$ (for any two constants c_l and c_2) and we have*

$$\int_a^b f\, d(c_1\alpha + c_2\beta) = c_1 \int_a^b f\, d\alpha + c_2 \int_a^b f\, d\beta$$

Theorem 66. *Assume that $c \in (a, b)$. If two of the three integrals in (1) exist, then the third also exists and we have*

$$\int_a^c f\, d\alpha + \int_c^b f\, d\alpha = \int_a^b f\, d\alpha.$$

P.P: The sum of the areas under f from a to c and c to b equals that from a to b.

Definition 31. *If $a < b$, we define $\int_b^a f\, d\alpha = -\int_a^b f\, d\alpha$ whenever $\int_a^b f\, d\alpha$ exists. We also define $\int_a^a f\, d\alpha = 0$*

Theorem 67. *If $f \in R(\alpha)$ on $[a, b]$, then $a \in R(f)$ on $[a, b]$ and we have*

$$\int_a^b f(x)\, d\alpha(x) + \int_a^b \alpha(x)\, df(x) = f(b)\alpha(b) - f(a)\alpha(a)$$

Theorem 68. *Let $f \in R(\alpha)$ on $[a, b]$ and let g be a strictly monotonic continuous function defined on an interval S having endpoints c and d. Assume that $a = g(c)$ and $b = g(d)$. Let h and β be the composite functions defined as follows:*

$$h(x) = f(g(x)), \qquad \beta(x) = \alpha(g(x))$$

if $x \in S$. Then

$$\int_a^b f\, d\alpha = \int_c^d h\, d\beta.$$

P.P: The summands $f\Delta\alpha_k$ (in $[a, b]$) and $h\Delta\beta_k$ (in $[c, d]$) are identical.

Theorem 69. *Assume $f \in R(\alpha)$ on $[a, b]$ and assume that α has a continuous derivative α' on $[a, b]$. Then the Riemann integral $\int_a^b f(x)\alpha'(x)\, dx$ exists, and we have:*

$$\int_a^b f(x)\, d\alpha(x) = \int_a^b f(x)\alpha'(x)\, dx$$

P.P: $d\alpha(x) = \alpha'(x)dx$

Theorem 70. *Given $a < c < b$. Define a on $[a, b]$ as follows: The values $\alpha(a), \alpha(c), \alpha(b)$ are arbitrary;*

$$\alpha(x) = \alpha(a) \quad \text{if } a \le x < c,$$

and

$$\alpha(x) = \alpha(b) \quad \text{if } c < x \le c.$$

Let f be defined on $[a, b]$ in such a way that at least one of the functions f or α is continuous from the left at c and at least one is continuous from the right at c. Then $f \in R(\alpha)$ on $[a, b]$ and we have

$$\int_a^b f d\alpha = f(c)(\alpha(c+) - \alpha(c-))$$

P.P: Total area under the graph of f is $w\ell = [\alpha(c+) - \alpha(c-)]f(c)$. If both f is discontinuous at c, there are two cases:

- If α is continuous everywhere in $[a, b]$ then $\Delta\alpha_k = 0$ for all $x_k, x_{k+1} \in P$, making $S(P, f, \alpha) = 0$.
- if α is not continuous at c then $\alpha(c+) - \alpha(c-) \neq 0$. Hence the length of the rectangle does not exist.

Definition 32. *A function, f, a defined on $[a, b]$ is called a step function if there is a partition*

$$a = x_1 < \cdots < x_n = b$$

such that α is constant on each open subinterval (x_{k-1}, x_k). The number $\alpha(x_k+) - \alpha(x_k-)$ is called the jump at x_k if $1 < k < n$. The jump at x_1 is $\alpha(x_1+) - \alpha(x_1)$,and the jump at x_n is $\alpha(x_n) - \alpha(x_n-)$.

Theorem 71. *Let α be a step function defined on $[a, b]$ with jump α_k at x_k, where $x_1, \dots, x_n$ are as described in Definition 32. Let f be defined on $[a, b]$ in such a way that not both f and α are discontinuous from the right or from the left at each x_k. Then $\int_a^b f d\alpha$ exists and we have*

$$\int_a^b f d\alpha = \sum_{k=1}^n f(x_k)\alpha_k$$

P.P: follows from theorem 66 and 70.

Theorem 72. *Every finite sum can be written as a Riemann-Stieltjes integral. In fact, given a sum $\sum_{k=1}^{n} a_k$, define f on $[0,n]$ as follows:*

$$f(x) = a_k \ \text{ if } \ k-1 < x \le k \ \ (k = 1, 2\dots, n), \ \ f(0) = 0.$$

Then

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} f(k) = \int_0^n f(x)\, d[x].$$

where $[x]$ is the greatest integer $\le x$

Theorem 73. *If f has a continuous derivative f' on $[a,b]$, then we have*

$$\sum_{a<n\le b} f(n) = \int_a^b f(x)\, dx + \int_a^b (x - [x]) f'(x)\, dx + f(a)((a-[a])) - f(b)((b-[b])).$$

Definition 33. *Let P be a partition of $[a,b]$ and let*

$$M_k(f) = \sup\{f(x) : x \in [x_{k-1}, x_k]\}$$

$$m_k(f) = \inf\{f(x) : x \in [x_{k-1}, x_k]\}$$

The numbers

$$U(P,f,\alpha) = \sum_{k=1}^{n} M_k(f)\Delta\alpha_k \qquad \text{and} \qquad L(P,f,\alpha) = \sum_{k=1}^{n} m_k(f)\Delta\alpha_k$$

are called, respectively, the upper and lower Stieltjes sums of f with respect to α for the partition P.

Theorem 74. *Assume $\alpha \nearrow$ on $[a,b]$. Then:*

1. *If P' is finer than P, we have*

$$U(P',f,\alpha) \le U(P,f,\alpha) \qquad \text{and} \qquad L(P',f,\alpha) \ge L(P,f,\alpha)$$

2. *For any two partitions P_1 and P_2, we have*

$$L(P_1,f,\alpha) \le U(P_2,f,\alpha)$$

P.P: (1) $2\sup A \cup B \le \sup A + \sup B$. (2)$\sum m_k(f)\Delta\alpha_k \le M_k(f)\sum \Delta\alpha_k$

Definition 34. *Assume that $\alpha \nearrow$ on $[a,b]$. The upper Stieltjes integral of f with respect to α is defined as follows:*

$$\overline{I}(f,\alpha) = \overline{\int_a^b} f d\alpha = \inf\{U(P,f,\alpha) : P \in (P)[a,b]$$

The lower Stieltjes integral is similarly defined:

$$\underline{I}(f,\alpha) = \underline{\int_a^b} f d\alpha = \sup\{L(P,f,\alpha) : P \in (P)[a,b]$$

Theorem 75. *Assume that $\alpha \nearrow$ on $[a,b]$. Then $\underline{I}(f,\alpha) \le \overline{I}(f,\alpha)$.*

P.P: $m_k \le f(t_k) \le M_k$.

Theorem 76. . *Assume that $\alpha \nearrow$ on $[a,b]$ and $c \in (a,b)$. Then:*

$$\overline{\int_a^b} (f+g)\, d\alpha \le \overline{\int_a^b} f\, d\alpha + \overline{\int_a^b} g\, d\alpha$$

and

$$\underline{\int_a^b} (f+g)\, d\alpha \ge \underline{\int_a^b} f\, d\alpha + \underline{\int_a^b} g\, d\alpha$$

P.P: This happens because $\sup\{s+t\} \le \sup\{s\} + \sup\{t\}$

Definition 35. *We say that f satisfies **Riemann's condition** with respect to α on $[a,b]$ if, for every $\varepsilon > 0$, there exists a partition P_ε, such that P finer than P_ε implies*

$$0 \le U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$$

Theorem 77. *lim Assume that $\alpha \nearrow$ on $[a,b]$. Then the following three statements are equivalent:*

1. *$f \in R(\alpha)$ on $[a,b]$*
2. *f satisfies Riemann's condition with respect to α on $[a,b]$.*
3. *$\overline{I}(f,\alpha) = \underline{I}(f,\alpha)$.*

Theorem 78. *Assume that $\alpha \nearrow$ on $[a,b]$. If $f \in R(\alpha)$ and $g \in R(\alpha)$ on $[a,b]$ and if $f(x) \le g(x)$ for all x in $[a,b]$, then we have*

$$\int_a^b f(x)\,d\alpha(x) \le \int_a^b g(x)\,d\alpha(x)$$

Theorem 79. *Assume that $\alpha \nearrow$ on $[a,b]$. If $f \in R(\alpha)$ on $[a,b]$, then $|f| \in R(\alpha)$ on $[a,b]$ and we have the inequality*

$$\left|\int_a^b f(x)\,d\alpha(x)\right| \le \int_a^b |f(x)|\,d\alpha(x)$$

P.P: The area of $|f|$ is The area of f above the $x-$axis + |the area of f under the $x-$axis|. Thus $|f| \in R(\alpha)$. $\int |f|$ is greater because no terms of $|f|$ are bound to cancel out each other as opposed to the terms of f

Theorem 80. *Assume that $\alpha \nearrow$ on $[a,b]$. If $f \in R(\alpha)$ on $[a,b]$, then $f^2 \in R(\alpha)$ on $[a,b]$.*

Theorem 81. *Assume that $\alpha \nearrow$ on $[a,b]$. If $f,g \in R(\alpha)$ on $[a,b]$, then $fg \in R(\alpha)$ on $[a,b]$.*

P.P: The limit of $\sum fg \cdot \Delta\alpha_k$ should exist as the side fg is finite and approaches a certain value as both $f,g \in R(\alpha)$, meaning each approaches a certain value.

Theorem 82. *Assume that α is of bounded variation on $[a,b]$. Let $V(x)$ denote the total variation of α on $[a,x]$ if $a < x \le b$, and let $V(a) = 0$. Let f be defined and bounded on $[a,b]$. If $f \in R(\alpha)$ on $[a,b]$, then $f \in R(V)$ on $[a,b]$.*

P.P: Since α is of bounded variation the sum $\sum f|\alpha_k|$ does not blow up.

Theorem 83. *Let α be of bounded variation on $[a,b]$ and assume that $f \in R(\alpha)$ on $[a,b]$. Then $f \in R(\alpha)$ on every subinterval $[c,d]$ of $[a,b]$.*

P.P: Because the area under f in $[a,b]$ is the sum of some constant + area under f in $[c,d]$.

Theorem 84. *Assume $f \in R(\alpha)$ and $g \in R(\alpha)$ on $[a,b]$, where $\alpha \nearrow$ on $[a,b]$.*
Define

$$F(x) = \int_a^x f(t)\,d\alpha(t)$$

and

$$G(x) = \int_a^x g(t)\,d\alpha(t).$$

Then $f \in R(G)$, $g \in R(F)$, $f \cdot g \in R(\alpha)$ on $[a,b]$ and we have,

$$\int_a^b f(x)g(x)\,d\alpha(x) = \int_a^b f(x)\,dG(x) = \int_a^b g(x)\,dF(x).$$

P.P: $d\int_a^x g\,d\alpha$ means $\int_a^x g\,d\alpha - \int_a^{x-\varepsilon} g\,d\alpha$, where ε is a very small number. This value approaches the area of the rectangle $g(x)d\alpha(x)$.

Theorem 85. *If f is continuous on $[a,b]$ and if α is of bounded variation on $[a,b]$, then $f \in R(\alpha)$ on $[a,b]$.*

P.P: If α is of bounded variation, then $\sum f\Delta\alpha_k$ does not blow up since f is bounded and $f(t_k) \underset{t_k\in[x_{k-1},x_k]}{\longrightarrow} f(x_k) = f(x_{k-1})$ if f is continuous.

Theorem 86. *Assume that $\alpha \nearrow$ on $[a,b]$ and let $a < c < b$. Assume further that both α and f are discontinuous from the right at $x = c$; that is, assume that there exists an $\varepsilon > 0$ such that for every $\delta > 0$ there are values of x and y in the interval $(c, c+\delta)$ for which*

$$|f(x) - f(c)| \geq \varepsilon \;\;\&\;\; |\alpha(y) - \alpha(c)| \geq \varepsilon$$

Then the integral $\int_a^b f\, d\alpha$ cannot exist. The integral also fails to exist if α and f are discontinuous from the left at c.

P.P: $f(t_k)\Delta\alpha_k \to f(c-)(\alpha(c) - \alpha(c-)) \neq 0$ and $f(t_{k+1})\Delta\alpha_{k+1} \not\to f(c-)(\alpha(c) - \alpha(c-))$. In other words,the limit fails to exist because there would be two rectangles with varing area in the neighbourhood of $\alpha(c)$, $f(c-)(\alpha(c) - \alpha(c-))$ and $f(c)(\alpha(c) - \alpha(c-))$.

Theorem 87. *Assume that $\alpha \nearrow$ and let $f \in R(\alpha)$ on $[a,b]$. Let M and m denote, respectively, the* sup *and* inf *of the set $\{f(x) : x \in [a,b]\}$. Then there exists a real number c satisfying $m \leq c \leq M$ such that*

$$\int_a^b f(x)\, d\alpha(x) = c\int_a^b d\alpha = c[\alpha(b) - \alpha(a)].$$

In particular, if f is continuous on $[a,b]$, then $c = f(x_0)$ for some x_0 in $[a,b]$.

P.P: The area of rectangle $M[\alpha(b)-\alpha(a)] >$ Area under f and the area of rectangle $m[\alpha(b)-\alpha(a)] <$ Area under f. So there should be a c that satisfies the equation in the theorem.

Theorem 88. *Assume that a is continuous and that $f \nearrow$ on $[a,b]$. Then there exists a point x_0 in $[a,b]$ such that*

$$\int_a^b f(x)\, d\alpha(x) = f(a)\int_a^{x_0} f(x)d\alpha(x) + f(b)\int_{x_0}^b f(x)\, d\alpha(x)$$

Theorem 89. *Let α be of bounded variation on $[a,b]$ and assume that $f \in R(\alpha)$ on $[a,b]$. Define F by the equation*

$$F(x) = \int_a^x f\, d\alpha \;\; \text{if } x \in [a,b].$$

Then we have:

1. *F is of bounded variation on $[a,b]$,*
2. *Every point of continuity of α is also a point of continuity of F.*
3. *If $\alpha \nearrow$ on $[a,b]$, the derivative $F'(x)$ exists, at each point x in (a,b) where $a'(x)$ exists and f is continuous. For such x, we have*

$$F'(x) = f(x)\alpha'(x)$$

.

P.P: (1) follows from $\sum_{(P)} |\int_{x_{k-1}}^{x_k} f\, d\alpha| \leq \int_{x_{k-1}}^{x_k} |f|\, d\alpha$. which is bounded (Note α is bounded). (2) follows because if $\Delta\alpha_k$ gets infinitesimally smaller and smaller, $F(y) - F(x)$ approaches the area of a line which is 0. (3) follows from the reduction of $\int f\, d\alpha$ to the Riemann integral.

Theorem 90. *Assume $f \in \mathbb{R}$ on $[a,b]$. Let α be a function which is continuous on $[a,b]$ and whose derivative α' is Riemann integrable on $[a,b]$. Then the following integrals exist and are equal:*

$$\int_a^b f(x)\, d\alpha(x) = \int_a^b f(x)\alpha'(x)\, dx.$$

Theorem 91. ***(Change of variable in a Riemann integral).*** *Assume that g has a continuous derivative g' on an interval $[c,d]$. Let f be continuous on $g([c,d])$ and define F by the equation*

$$F(x) = \int_{g(c)}^x f(t)\, dt$$

Then, for each x in $[c,d]$ the integral $\int_c^x f[g(t)]g'(t)dt$ exists and has the value $F[g(x)]$.

P.P: when g is decreasing, $g' < 0$.

Theorem 92. ***(Second MVT)*** *Let g be continuous and assume that $f \nearrow$ on $[a,b]$. Let A and B be two real numbers satisfying the inequalities*

$$A \leq f(a+) \qquad \& \qquad B \geq f(b-)$$

Then there exists a point x_0 in $[a,b]$ such that

1. $\int_a^b f(x)g(x)\,dx = A\int_a^{x_0} g(x)dx + B\int_{x_0}^b g(x)\,dx$
2. *In particular if $f(x) \geq 0$, we have $\int_a^b f(x)g(x)\,dx = B\int_{x_0}^b g(x)\,dx$.*

P.P: $\int B > \int f$ whatsoever. Hence there should be an optimal x_0.

Theorem 93. *Let f be continuous at each point (x,y) of a rectangle*

$$Q = \{(x,y) : a \leq x \leq b,\ c \leq y \leq d\}.$$

Assume that α is of bounded variation on $[a,b]$ and let F be the function defined on $[c,d]$ by the equation

$$F(y) = \int_a^b f(x,y)\,d\alpha(x).$$

Then F is continuous on $[c,d]$. In other words, if $y_0 \in [c,d]$, we have

$$\begin{aligned}\lim_{y\to y_0}\int_a^b f(x,y)\,d\alpha &= \int_a^b \lim_{y\to y_0} f(x,y)\,d\alpha \\ &= \int_a^b f(x,y_0)\,d\alpha\end{aligned}$$

P.P: $f(x,y) = g_x(y)$ and limit is distributive over addition.

Theorem 94. *Let $Q = \{(x,y) : a \leq x \leq b,\ c \leq y \leq d\}$. Assume that α is of bounded variation on $[a,b]$ and, for each fixed y in $[c,d]$, assume that the integral*

$$F(y) = \int_a^b f(x,y)\,d\alpha$$

exists. If the partial derivative $D_2 f$ is continuous on Q, the derivative $F(y)$ exists for each y in (c,d) and is given by

$$F'(y) = \int_a^b D_2 f(x,y)\,d\alpha(x)$$

Theorem 95. *Let $Q = \{(x,y) : a \leq x \leq b, c \leq y \leq d\}$. Assume that α is of bounded variation on $[a,b]$, β is of bounded variation on $[c,d]$, and f is continuous on Q. If (x,y) a Q, define*

$$F(y) = \int_a^b f(x,y)\,d\alpha(x), \qquad G(x) = \int_c^d f(x,y)\,d\beta(y).$$

Then $F \in R(\beta)$ on $[c,d]$, $G \in R(\alpha)$ on $[a,b]$, and we have

$$\int_c^d F(y)\,d\beta(y) = \int_a^b G(x)\,d\alpha(x).$$

P.P: The order of integration can be reversed.

Definition 36. *A set S of real numbers is said to have* ***measure zero*** *if, for every $\varepsilon > 0$, there is a countable covering of S by open intervals, the sum of whose lengths is less than ε.*

This means a set S has measure zero if $S = \bigcup_k (a_k, b_k)$ and $\sum_k b_k - a_k < \varepsilon$

Theorem 96. *Let F be a countable collection of sets in $\mathbb{R}$, say $F = \{F_1, F_2, \dots\}$, each of which has measure zero. Then their union*

$$S = \bigcup_{k=1}^{\infty} F_k$$

also has a measure zero.

P.P:

$$\sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon$$

Definition 37. *Let f be defined and bounded on an interval S. If $T \subseteq S$, the number*

$$\Omega_f(T) = \sup\{f(x) - f(y) : x, y \in T\}$$

is called the oscillation of f on T. The oscillation of f at x is defined to be the number

$$\omega_f(x) = \lim_{h \to 0^+} \Omega_f(B(x; h) \cap S).$$

;

Theorem 97. *Let f be defined and bounded on $[a, b]$, and let $\varepsilon > 0$ be given. Assume that $\omega_f(x) < \varepsilon$ for every x in $[a, b]$. Then there exists a $\delta > 0$ (depending only on ε) such that for every closed subinterval $T \subseteq [a, b]$, we have $\omega(T) < \varepsilon$ whenever the length of T is less than δ.*

P.P: If the maximum jump of f in $[a, b] < \varepsilon$, then the span of $f < \varepsilon$ for every sufficiently small subinterval of $[a, b]$.

Theorem 98. *Let f be defined and bounded on $[a, b]$. For each $\varepsilon > 0$ define the set J_ε as follows:*

$$J_\varepsilon := \{x : x \in [a, b],\ \omega_f(x) \geq \varepsilon\}.$$

Then J_ε is a closed set.

The 'endpoints' of the interval of discontinuity are in J_ε.

Theorem 99. ***(Lebesgue's criterion for Riemann-integrability)**. Let f be defined and bounded on $[a, b]$ and let D denote the set of discontinuities of f in $[a, b]$. Then $f \in R$ on $[a, b]$ if, and only if, D has measure zero.*

P.P: A bounded function f on a compact interval $[a, b]$ is Riemann-integrable on $[a, b]$ if, and only if, f is continuous *almost everywhere* on $[a, b]$. This is the case because if there are enough discontinuities, they can prevent Riemann's condition on integrability from holding: the sum $\sum(M_k - m_k)\Delta x_k = S_1 + S_2$, where S_1 contains points of discontinuities, and $S_1 \geq \sum$ jump of $f \times$ measure of D.

More from the exercise

$$\int_a^b f(x)g(x)\ d\alpha(x) = f(a) \int_a^{x_0} g(x)\ d\alpha(x) + f(b) \int_{x_0}^b g(x)\ d\alpha(x).$$

Cauchy-Schwartz: For $\alpha \nearrow$ we have

$$\left(\int_a^b f(x)g(x)\ dx \right)^2 \leq \int_a^b [f(x)]^2\ dx \int_a^b [g(x)]^2\ dx$$

$$\Lambda_{\mathbf{f}}(a, b) = \int_a^b \|\mathbf{f}'(t)\|\ dt$$

If $f \in R$ and $g \in R$ then it doesn't necessarily follow that $f \circ g \in R$. Example

$$g(x) = \begin{cases} 0 \text{ if } x \text{ is irrational} \\ 1/n \text{ if } x = m/n \text{ is rational} \end{cases} \qquad f(x) = \begin{cases} 0 \text{ if } x = 0 \\ 1 \text{ if otherwise} \end{cases}$$

Chapter 8

Infinite Series and Infinite products

Definition 38. *Let $\{a_n\}$ be a sequence of real numbers. Suppose there is a real number U satisfying the following two conditions:*

1. *For every $\varepsilon > 0$ there exists an integer N such that $n > N$ implies*

$$a_n < U + \varepsilon.$$

2. *Given $\varepsilon > 0$ and given $m > 0$, there exists an integer $n > m$ such that*

$$a_n > U - \varepsilon.$$

Then U is called the limit superior (or upper limit) of $\{a_n\}$ and we write

$$U = \limsup_{n\to\infty} a_n$$

Statement (1) implies that the set $\{a_l, a_2, \dots\}$ is bounded above. If this set is not bounded above, we define

$$\limsup_{n\to\infty} a_n = +\infty.$$

If the set is bounded above but not bounded below and if $\{a_n\}$ has no finite limit superior, then we say $\limsup_{n\to\infty} a_n = -\infty$. The limit inferior (or lower limit) of $\{a_n\}$ is defined as follows:

$$\liminf_{n\to\infty} a_n = -\limsup_{n\to\infty} -a_n$$

P.P: Statement (1) means that ultimately *all* terms of the sequence lie to the left, of $U+\varepsilon$. Statement (2) means that *infinitely many* terms lie to the right of $U - \varepsilon$. The given definition of $\liminf$ follows from the fact that the the maximum(sup) of a_n is the minimum of $-a_n$. In fact let $\limsup -a_n = -L$. Then from the definition $-L-\varepsilon < -a_n < -L+\varepsilon \implies L-\varepsilon < a_n$ for ultimately all terms and $a_n < L+\varepsilon$ for infinitely many terms.

Theorem 100.

1. $\liminf a_n \leq \limsup a_n$.
2. *The sequence $\{a_n\}$ converges if, and only if, $\limsup a_n$ and $\liminf a_n$ are both finite and equal, in which case $\liminf a_n = \limsup a_n = \lim a_n$.*
3. *The sequence diverges to ∞ if, and only if, $\liminf a_n = \limsup a_n = \infty$.*
4. *The sequence diverges to $-\infty$ if, and only if, $\liminf a_n = \limsup a_n = -\infty$.*
5. *Assume that $a_n \leq b_n$ for each $n = 1, 2, \dots$. Then we have*

$$\limsup a_n \leq \limsup b_n \ \ \& \ \ \liminf a_n \leq \liminf b_n$$

Definition 39. *Let $\{a_n\}$ be a sequence of real numbers. We say the sequence is increasing and we write $a_n \neq$ if $a_n \leq a_{n+1}$ for $n = 1, 2, \dots$. If $a_n \geq a_{n+1}$ for all n, we say the sequence is decreasing and we write $a_n \searrow$ A sequence is called monotonic if it is increasing or if it is decreasing.*

Theorem 101. *A monotonic sequence converges if, and only if, it is bounded.*

Definition 40. *The ordered pair of sequences* $(\{a_n\}), \{s_n\})$ *is called an infinite series. The number* s_n *is called the* n*th partial sum of the series. The series is said to converge or to diverge according as* $\{s_n\}$ *is convergent or divergent.*

Theorem 102. *Let* $a = \sum a_n$ *and* $b = \sum b_n$ *be convergent series. Then, for every pair of constants* α *and* β*, the series* $\sum \alpha a_n + \beta b_n$ *converges to the sum* $\alpha a + \beta b$*.*

P.P: $\sum_{k=1}^{n} \alpha a_k + \beta b_k = \alpha \sum_{k=1}^{n} a_k + \beta \sum_{k=1}^{n} b_k$.

Definition 41. *Let* p *be a function whose domain is the set of positive integers and whose range is a subset of the positive integers such that*

$$p(n) < p(m) \quad \textit{if} \quad n < m.$$

Let $\sum a_n$ *and* $\sum b_n$ *be related as follows:*

$$b_1 = a_1 + \cdots + a_{p(1)},$$

$$b_{n+1} = a_{p(n)+1} + \cdots + a_{p(n+1)}.$$

Then we say we obtain $\sum b_n$ *from* $\sum a_n$ ***by inserting parentheses*** *and* $\sum a_n$ *is obtained from* $\sum b_n$ ***by removing parentheses****.*

P.P: Literally! $\sum a_n = (a_1 + \cdots + a_{p(1)}) + (a_{p(1)+1} + \cdots + a_{p(2)}) + \ldots$

Theorem 103. $\sum a_n$ *converges to* s*, every series* $\sum b_n$ *obtained from* $\sum a_n$ *by inserting parentheses also converges to* s*.*

P.P: Inserting parentheses is slightly different from the rearrangement of terms.

Theorem 104. *Let* $\sum a_n, \sum b_n$ *be related as in Definition 41. Assume that there exists a constant* $M > 0$ *such that* $p(n+1) - p(n) < M$ *for all* n*, and assume that* $\lim a_n = 0$*. Then* $\sum a_n$ *converges if, and only if,* $\sum b_n$ *converges, in which case they have the same sum.*

P.P: Let $\sum b_n = t_n \to t$. Then $\sum a_n - s \le t_m - t + |a_j + \cdots + a_k|$

Theorem 105. *If* $\{a_n\}$ *is a decreasing sequence converging to* 0*, the alternating series* $\sum(-1)^{n+1} a_n$ *converges. If* s *denotes its sum and* s_n *its* n*th partial sum, we have the inequality*

$$0 < (-1)^n (s - s_n) < a_{n+1}.$$

P.P: $(-1)^n(s - s_n) = a_{n+1} - (a_{n+2} - a_{n+3}) - \ldots$.

Definition 42. *A series* $\sum a_n$ *is called absolutely convergent if* $\sum |a_n|$ *converges. It is called conditionally convergent if* $\sum a_n$ *converges but* $\sum |a_n|$ *diverges.*

Theorem 106. *Absolute convergence of* $\sum a_n$ *implies convergence.*

Theorem 107. *Let* $\sum a_n$ *be a given series with real terms, and define*

$$p_n = \frac{|a_n| + a_n}{2}, \qquad p_n = \frac{|a_n| - a_n}{2}$$

Then:

1. *If* $\sum a_n$ *is conditionally convergent, both* $\sum p_n$ *and* $\sum q_n$ *diverge.*
2. *If* $\sum |a_n|$ *converge, both both* $\sum p_n$ *and* $\sum q_n$ *converge, and we have*

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} p_n - \sum_{n=1}^{\infty} q_n$$

P.P: p_n are the positive terms of a_n while q_n are the negative terms. Also $a_n = p_n - q_n$ and $|a_n| = p_n + q_n$.

Theorem 108. ***(Comparison Test).*** *If $a_n > 0$ and $b_n > 0$ for $n = 1, 2, \ldots$, and if there exist positive constants c and N such that*

$$a_n < cb_n \qquad \text{for } n > N$$

Then the convergence of $\sum b_n$ implies that of $\sum a_n$.

P.P: $\sum a_n <$ finite sum $+ c(\sum b_n -$ finite sum$)$.

Theorem 109. ***(Limit comparison test)*** *Assume that $a_n > 0$ and $b_n > 0$ for $n = 1, 2, \ldots$, and suppose that*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1.$$

Then $\sum a_n$ converges iff $\sum b_n$ converges.

P.P: If a_n and b_n tend to be equal over the long run, the two sums both tend to converge or both to diverge.

Theorem 110. ***(Integral test).*** *Let f be a positive decreasing function defined on $[1, +\infty)$ such that $\lim\ f(x) = 0$. For $n = 1, 2, \ldots$ define*

$$s_n = \sum_{k=1}^{n} f(k) \qquad t_n = \int_1^n f(t)\, dt \qquad d_n = s_n - t_n.$$

Then we have:

1. $0 < f(n+1) \le d_{n+1} \le d(n) \le f(1)$
2. $\lim d_n$ *exsists*
3. $\{s_n\}$ *converges iff* $\{t_n\}$ *converges.*
4. $0 \le d_k - d_\infty \le f(k)$

P.P: Draw the graph, then observe that d_n are the upper part of the rectangles.

Definition 43. *Given two sequences $\{a_n\}$ and $\{b_n\}$, $b_n \ge 0$ for all $n = 1, 2, \ldots$ we write*

$$a_n = O(b_n)$$

if there exists a constant $M > 0$ such that $|a_n| \le Mb_n$ for all n. We write

$$a_n = o(b_n) \qquad \text{as } \ n \to \infty$$

if $\lim_{n\to\infty} a_n/b_n = 0$

Theorem 111. ***(Ratio test)*** *Given a series $\sum a_n$ non-zero complex terms, let*

$$r = \liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|, \qquad R = \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|.$$

Then:

1. $\sum a_n$ *converges absolutely if* $R < 1$.
2. $\sum a_n$ *diverges if* $r > 1$.
3. *the test is inconclusive if* $r \le 1 \le R$.

P.P: $r, R \ne 1$ means the terms are sufficiently not close to each other.

Theorem 112. *Given a series $\sum a_n$ of complex terms, let*

$$\rho = \limsup_{n\infty} \sqrt[n]{|a_n|}.$$

Then:

1. *$\sum a_n$ absolutely converges if $\rho < 1$.*
2. *$\sum a_n$ diverges if $\rho > 1$.*
3. *the test is inconclusive if $\rho = 1$.*

P.P: If the terms are big enough to diverge the series the nth root tends to be 1 or greater. This relationship between convergence and nth root is due to the fact that series with terms $a_n \to \infty$ that diverge, tend to have greater nth root than the terms themselves.

Theorem 113. *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of complex numbers, define*

$$A_n = a_1 + \cdots + a_n.$$

Then we have the identity

$$\sum_{k=‘1}^{n} a_k b_k = A_{n+1}b_{n+1} - \sum_{k=1}^{n} A_k(b_{k+1} - b_k).$$

P.P: Let $c_k = b_{k+1} - b_k$. Then plot a_k vs c_k graph. The following two test follow from this identity.

Theorem 114. ***(Dirichlet's test)*** *Let $\sum_n a_n$ be a series with partial sums forming bounded sequence and $\{b_n\}$ be a decreasing sequence which converges to 0. Then $\sum a_n b_n$ converges.*

Theorem 115. ***(Abel's test)*** *$\sum a_n b_n$ converges if $\sum a_n$ converges and if $\{b_n\}$ is monotonic convergent.*

Theorem 116. *For every real $x \neq 2\pi m$, we have*

$$\sum_{k=1}^{n} e^{ikx} = \frac{\sin(nx/2)}{\sin(x/2)} e^{i(n+1)} x$$

From this theorem the following relations can be proved

$$\begin{aligned}
\left|\sum_{k=1}^{n} e^{ikx}\right| &\leq \frac{1}{|\sin(x/2)|} \\
.\sum_{k=1}^{n} \cos kx &= -\frac{1}{2} + \frac{1}{2}\sin(2n+1)\frac{x}{2} \Big/ \sin\frac{x}{2} \\
\sum_{k=1}^{n} \sin(2k-1)x &= \frac{\sin^2 nx}{\sin x}
\end{aligned}$$

Definition 44. *Let $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ be a one-to-one function. Let $\sum a_n$ and $\sum b_n$ be two series such that*

$$b_n = a_{f(n)} \qquad \text{for } n = 1, 2, \ldots.$$

Then $\sum_n b_n$ is called the rearrangement of $\sum_n a_n$

Theorem 117. *Let $\sum_n a_n$ be an absolutely convergent series having sum s. Then every rearrangement of $\sum b_n$ also converges absolutely and has sum s.*

P.P: If a_n is absolutely convergent, rearrangement of the terms doesn't change the sum as addition is commutative.

$\longrightarrow\longrightarrow\longleftarrow\equiv\longrightarrow\longleftarrow\longrightarrow\equiv\longrightarrow$

Theorem 118. *Let $\sum a_n$ be a conditionally convergent series with real-valued terms. Let x and y be given numbers in the closed interval $[-\infty, \infty]$, with $x \leq y$. Then there exists a rearrangement $\sum b_n$ of $\sum a_n$ such that*

$$\liminf_{n\to\infty} t_n = x \qquad \& \qquad \limsup_{n\to\infty} t_n = y,$$

where $t_n = b_1 + \ldots b_n$.

P.P: https://demonstrations.wolfram.com/RiemannsTheoremOnRearrangingConditionallyConvergentSeries/

Definition 45. *Let f be a function $\mathbb{Z}^+$ and whose range is an infinite subset of $\mathbb{Z}^+$, and assume that f is one-to-one on $\mathbb{Z}^+$. Let $\sum a_n$ and $\sum b_n$ be two series such that*
$b_n = a_{f(n)}, \qquad if\ \ n \in \mathbb{Z}^+$. *Then $\sum b_n$ is said to be a subseries of $\sum a_n$.*

Theorem 119. *If $\sum a_n$ is absolutely convergent then every subseries $\sum b_n$ of $\sum a_n$ is also absolutely convergent. Moreover we have*

$$\left|\sum_{n=1}^{\infty} b_n\right| \leq \sum_{n=1}^{\infty} |b_n| \leq \sum_{n=1}^{\infty} |a_n|.$$

P.P: The fact that $\sum |b_n|$ is bounded implies absolute convergence and $\sum |b_n| \leq \sum^{\max f} |a_n|$.

Theorem 120. *Let $\{f_1, f_2, \dots$ be a countable collection of functions, each defined on $\mathbb{Z}^+$, having the following properties*

- *Each f_n is one-to-one on $\mathbb{Z}^+$.*
- *The range $f_n(\mathbb{Z}^+)$ is a subset Q_n of $\mathbb{Z}^+$.*
- *$\{Q_1, Q_2, \dots\}$ is a collection of disjoint sets whose union is $\mathbb{Z}^+$.*

Let $\sum a_n$ be absolutely convergent series and define

$$b_k(n) = a_{f_k(n)}, \qquad \text{if } n, k \in \mathbb{Z}^+.$$

Then

1. *For each k, $\sum_n b_k(n)$ is absolutely convergent.*
2. *If $s_k = \sum_n b_k(n)$, then the series $\sum s_n$ converges absolutely and has the same sum as $\sum a_n$.*

P.P: $\sum s_n$ is infinity-sized rearrangement of $\sum a_n$. Treat each s_n as a single terms of $\sum a_n$. (1) says every subseries of $\sum a_n$ converge absolutely. (2) says the sum of "disjoint" subseries of $\sum a_n$ is $\sum a_n$.

Definition 46. *A function f whose domain is $\mathbb{Z}^+ \times \mathbb{Z}^+$ is called a double sequence.*

Definition 47. *If $a \in \mathbb{C}$, we write $\lim_{p,q\to\infty} f(p,q) = a$ and we say that the double sequence f converges to a, provieded that the following condition is satisfied: For every $\varepsilon > 0$, there exists an N such that $|f(p,q) - a| < \varepsilon$ whenever both $p > N$ and $q > N$.*

Theorem 121. *Assume that $\lim_{p,q\to\infty} f(p,q) = a$. For each fixed p. assume that $\lim_{q\to\infty} f(p,q)$ exists. Then the $\lim_{p\to\infty}(\lim_{q\to\infty} f(p,q))$ also exists and has value a. $F(p) := f(p,\infty)$. Then $F(p)$ and a get closer and closer infinitesimally as $p \to \infty$*

Definition 48. *Let f be a double sequence and let s be the double sequence defined by the equation*

$$s(p,q) = \sum_{m=1}^{p} \sum_{n=1}^{q} f(m,n).$$

The pair (f,s) is called a double series and is denoted by the symbol $\sum_{m,n} f(m,n)$ or, more briefly, by $\sum f(m,n)$. The double series is said to converge to the sum a if

$$\lim_{p,q\to\infty} s(p,q) = a.$$

Definition 49. *Let f be a double sequence and let g be a one-to-one function defined on $\mathbb{Z}^+$ with range $\mathbb{Z}^+ \times \mathbb{Z}^+$. Let G be the sequence defined by*

$$G(n) = f[g(n)].$$

Then g is said to be an arrangement of the double sequence f into G.

Theorem 122. *Let $\sum f(m,n)$ be given double series and let g be an arrangment of the double sequence f into a sequence G. Then*

- $\sum G(n)$ *converges absolutely iff* $\sum f(m,n) convergesabsolutely.$
  $noindentAssuming \sum f(m,n) doesconvergeabsolutely, withsumS, wehavefurther:$
  $\sum G(n) = S.$
- $\sum_{n=1}^{\infty} f(m,n)$ *and* $\sum_{m=1}^{\infty} f(m,n)$ *both converge absolutely.*
- *If* $A_m = \sum_{n=1}^{\infty} f(m,n)$ *and* $B_n = \sum_{m=1}^{\infty} f(m,n)$, *both series* $\sum A_m$ *and* $\sum B_n$ *converge absolutely and both have sum* S.

P.P: These are special cases of the theorems under the rearrangement of a "normal" series.

The arrangement/rearrangement of a series does not change the convergence point when no infinite sums are involve since addition is commutative.

Theorem 123. *Let f be a complex-valued double sequence. Assume that $\sum_{n=1}^{\infty}$ converges absolutely for each fixed m and that*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |f(m,n)|,$$

converges. Then:

1. *The double series $\sum_{m,n} f(m,n)$ converges absolutely.*
2. *The series $\sum_{m=1}^{\infty} f(m,n)$ converges absolutely for each n.*
3. $$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} f(m,n) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f(m,n) = \sum_{m,n} f(m,n)$$

Theorem 124. *Let $\sum a_m$ and $\sum b_n$ be two absolutely convergent series with sums A and B. respectively. Let f be the double sequence defined by*

$$f(m,n) = a_m b_n,$$

Then $\sum_{m,n} f(m,n)$ converges and has sum AB.

Definition 50. *Given two series $\sum a_n$ and $\sum b_n$, define $c_n = \sum_{k=0}^{n} a_k b_{n-k}$ for $n \geq 0$. The series $\sum c_n$ is called the Cauchy product of $\sum a_n$ and $\sum b_n$.*

Theorem 125. *Let $\sum a_n$ and $\sum b_n$ be two absolutely convergent series with sums A and B respectively. Then their Cauchy product $\sum c_n$ converge and has the sum AB.*

P.P: $\sum^N c_n = B\sum^N a_n -$ error$_B \sum^N a_n$. $N \to \infty \implies$ error $\to 0$.

Definition 51. *Let s_n denote the nth partial sum of the series $\sum a_n$, and let $\{\sigma_n\}$ be the sequence of the arithmetic means defined by*

$$\sigma_n = \frac{s_1 + \cdots + s_n}{n}, \qquad \text{if } n = 1, 2 \ldots$$

The series $\sum a_n$ is said to be Cesaro summable(or $(C,1)$ summable) if $\{\sigma_n\}$ converges. If $\lim_{n\to\infty} \sigma_n = S$, then S is called the Cesaro sum (or $(C,1)$ sum) of $\sum a_n$, and we write

$$\sum a_n = S \qquad (C,1).$$

Cesaro sum is the "average" of all partial sums of $\sum a_n$.

Theorem 126. *If a series is convergent with sum S, then it is also $(C,1)$ summable with Cesaro sum S.*

P.P: As the number of terms increase the number of s_n close to S increase $\implies$ average $(\sigma_n) \to S$.

Definition 52. *Given a sequence $\{u_n\}$ of real or complex numbers, let*

$$p_1 = u_1, \qquad p_n = u_1 \cdots u_n = \prod_{k=1}^{n} u_k.$$

The ordered pair $(\{u_n\}, \{p_n\})$ is called an infinite product (or simply a product). The number p_n is called the nth partial product and u_n is called the nth factor of the product. The following symbols are used to denote the product defined by the above equalities:

$$u_1 u_2 \cdots u_n \cdots, \qquad \prod_{n=1}^{\infty} u_n$$

.

Definition 53. *Given an infinite product $\prod u_n$, let $p_n = \prod_{k=1}^{n} u_k$.*

1. *If infinitely many factors u_n are zero, we say the product diverges to 0.*
2. *If no factor u_n is zero, we say the product converges if there exists $p \neq 0$ such that $\{p_n\}$ converges to p. In this case, p is called the value of the product and we write $p = \prod_{n=1}^{\infty} u_n$. If $\{p_n\}$ converges to zero, we say the product diverges to zero.*
3. *If there exists an N such that $n > N$ implies $u_n \neq 0$, we say $\prod u_n$ converges provided that $\prod_{n=N+1}^{\infty}$ converges as described in (2). In this case the value of the product $\prod u_n$ is*

$$u_1 u_2 \ldots u_N \prod_{n=N+1}^{\infty} u_n.$$

4. *$\prod u_n$ is called divergent if it does not converge as described in (2) or (3).*

Theorem 127. *The infinite product $\prod u_n$ converges iff for every $\varepsilon > 0$, there is a constant N such that $n > N$ implies:*

$$|u_{n+1} \cdots u_{n+k} - 1| < \varepsilon \qquad \text{for } k = 1, 2 \ldots.$$

P.P: This follows from Cauchy's condition for convergence of a sequence. This theorem says the factors don't get small enough to make the product converge to 0, nor do they get large enough to make the product blow up.

Theorem 128. *Assume $a_n > 0$ The product $\prod(1 + a_n)$ converges iff the series $\sum a_n$ converges.*

P.P: The partial product stays in the threshold of theorem 127 if $a_n \to 0$ since $\prod_n (1 + a_n) = 1 + \sigma_1(a_i) + \sigma_2(a_i) + \ldots$. The convergence of $\sum a_n$ makes the symmetric polynomials $\to 0$.

Definition 54. *The product $\prod(1 + a_n)$ is said to be absolutlely convergent if $\prod(1 + |a_n|)$ converges.*

Theorem 129. *Absolute convergence of $\prod(1 + a_n)$ implies its convergence.*

P.P: If $|a_n|$ is not large enough to diverge the product to ∞, then a_n can not be small enough to diverge the product to 0.

Theorem 130. *Assume $a_n \geq 0$. Then the product $\prod(1 - a_n)$ converges iff $\sum a_n$ converges.*

Chapter 9

Sequences of Functions

Definition 55. *A sequence of functions $\{f_n\}$ is said to converge uniformly to f on a set S, if, for every $\varepsilon > 0$, there is an integer N such that $n > N$ implies*

$$|f_n(x) - f(x)| < \varepsilon \qquad \text{for every} x \in S.$$

We denote this by $f_n \to f$ uniformly on the set S.

P.P: $|f_n(x) - f(x)| < \varepsilon \implies f(x) - \varepsilon \le f_n(x) \le f(x) + \varepsilon$ for all $n > N$: a $f_{n>N}$ lie with in a band of length 2ε.

Theorem 131. *Assume $f_n \to f$ uniformly on S. If each f_n is continuous at a point $c \in S$, then the limit function f is also continuous at point c.*

P.P: $|f(x) - f(c)| \le |f(x) - f_n(x)| + |f_n(x) - f_n(c)| + |f_n(c) - f(c)|$. In other words the **band** can get as small as needed and f_n can get as close to $f_n(c)$.

Theorem 132. *Let $\{f_n\}$ be a sequence of functions defined on S. Then there is a function f such that $f_n \to f$ uniformly on S iff the following condition is satisfied: for every $\varepsilon > 0$, there is an integer N such that $n, m > N$ implies*

$$|f_m(x) - f_n(x)| < \varepsilon \qquad \text{for every } x \in S.$$

P.P: Extension of Cauchy's condition.

Definition 56. *Given a sequence of functions $\{f_n\}$ defined S, let*

$$s_n := \sum_{k=1}^{n} f_k(x).$$

If there is a function f such that $s_n \to f$ uniformly on S, we say the series $\sum f_n(x)$ converges uniformly on S and we write

$$\sum_{n=1}^{\infty} f_n(x) = f(x).$$

Theorem 133. *The infinite series $\sum f_n(x)$ converges uniformly on S, iff, for $\varepsilon > 0$ there is N such that*

$$\left| \sum_{k=n+1}^{n+p} f_k(x) \right| < \varepsilon \qquad \text{for every} \quad p = 1, 2 \ldots \quad \text{and} \quad x \in S.$$

Theorem 134. ***(Weierstrass M-test).*** *Let $\{M_n\}$ be a sequence of non-negative number such that*

$$0 \le |f_n(x)| \le M_n \qquad n = 1, 2 \ldots \quad \text{and} \quad x \in S.$$

Then $\sum f_n$ converges uniformly if $\sum M_n$ converges uniformly.

Theorem 135. *Assume $\sum f_n(x) = f(x)$. If each f_n is continuous at a point $x_0 \in S$, then f is continuous at x_0.*

P.P: Continuity of $f_n \implies$ Continuity of s_n.

Theorem 136. *Let α be of bounded variation on $[a,b]$. Assume each term of the sequence $\{f_n\}$ be a real-valued function such that $f_n \in R(\alpha)$ on $[a,b]$ for each n. Assume $f_n \to f$ uniformly on $[a,b]$ and define $g_n(x) = \int_a^x f_n(t)\, d\alpha(t)$ for $x \in [a,b]$. Then:*

1. *$f \in R(\alpha)$ on $[a,b]$.*
2. *$g_n \to g$ uniformly on $[a,b]$ where $g(x) = \int_a^x f(t)\, d\alpha(t)$.*

P.P: From $|f_n - f| < \varepsilon$, we have $f_n - \varepsilon < f(x) < f_m + \varepsilon \implies -\varepsilon + \int f_n \le \int f \le \varepsilon + \int f_n$. However, uniform convergence is not necessary for Riemann-Integrablity; if f_n is boundedly convergent on $[a,b]$ (uniformly bounded and converges to f), then $\int f$ exists.

Theorem 137. *Let $\{f_n\}$ be a sequence of real-valued function, with each term having a finite derivative on each point $c \in (a,b)$. Assume there is at least one point $x_0 \in (a,b)$ such that the sequence $\{f_n(x_0)\}$ converges and that there is a function g such that $f_n' \to g$ uniformly on $[a,b]$. Then:*

1. *There exists a function f such that $f_n \to f$ uniformly on (a,b).*
2. *For each $x \in (a,b)$, $f'(x)$ exists and equals $g(x)$.*

P.P: $f_n \to g \implies \int f_n \to \int g = f \implies g = f'$.

Theorem 138. *Let $F_n(x)$ be the nth partial sum of the series $\sum f_n(x)$, where f_n is complex valued defined on the set S. Assume the sequence $\{F_n\}$ is bounded on S. Let $\{g_n\}$ be a sequence of functions that satisfy $g_{m+1}(x) \le g_m(x)$ for all $x \in S$ and for all m. Assume furthur that $g_n \to 0$ uniformly on S. Then the $\sum f_n g_n$ converges uniformly on S.*

P.P: Dirichlet's test for sequence of functions.

Definition 57. *Let $\{f_n\}$ be a sequence of Riemann-Intergrable functions on $[a,b]$. Assume $f \in R$ on $[a,b]$. Then we say f_n converges in the mean to f, and we write*

$$\underset{n\to\infty}{\text{l.i.m.}} f_n = f \qquad \textit{on } [a,b]$$

if

$$\lim_{n\to\infty} \int_a^b |f_n(t) - f(t)|^2 dt = 0.$$

P.P: Converges in the mean means the average of all f_n in $[a,b] \to$ the average of f in $[a,b]$.

Theorem 139. *Assume* $\text{l.i.m.}_{n\to\infty}$ *on $[a,b]$. If $g \in R$, define*

$$h(x) = \int_a^x f(t)g(t)\ dt \qquad h_n(x) = \int_a^x f_n(t)g(t)\ dt.$$

if $x \in [a,b]$. Then $h_n \to h$ uniformly on $[a,b]$.

P.P: f_n converges to f, $\int f_n$ converges almost uniformly $\int f$.

Theorem 140. *Assume* $\text{l.i.m.}_{n\to\infty} f_n = f$ *and* $\text{l.i.m.}_{n\to\infty} g_n = g$ *on $[a,b]$. Define*

$$h(x) = \int_a^x f(t)g(t)\ dt \qquad h_n(x) = \int_a^x f_n(t)g_n(t)\ dt.$$

if $x \in [a,b]$. Then $h_n \to h$ uniformly on $[a,b]$.

Theorem 141. *Given a power series $\sum_{n=1}^{\infty} a_n(z-z_0)^n$, let $\lambda = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$, $\quad r = \dfrac{1}{\lambda}$. Then the series $\sum_{n=1}^{\infty} a_n(z-z_0)^n$ converges absolutely if $|z-z_0| < r$, and diverges for $|z-z_0| > r$. r is called the radius of convergence of $\sum a_n(z-z_0)^n$. Furthermore, $\sum a_n(z-z_0)^n$ converges uniformly in every compact subset of the disk of converges.*

Theorem 142. *Assume the power series* $\sum a_n(z-z_0)^n$ *converges for all* $z \in B(z_0;r)$. *Suppose the equation* $f(z) = \sum_{n=1}^{\infty} a_n(z-z_0)^n$ *is known to be valid for every open subset* S *of* $B(z_0;r)$. *Then for each* $z_1 \in S$, *there exists a neighbourhood* $B(z_1;R)$ *in* S *such that* f *has a power series expansion of the form*

$$f(z) = \sum_{n=0}^{\infty} b_n(z-z_1)^n$$

for all $z \in B(z_1;R)$, *where*

$$b_k = \sum_{n=k}^{\infty} \binom{n}{k} a_n(z_1-z_0)^{n-k}.$$

Theorem 143. *Assume the power series* $\sum a_n(z-z_0)^n$ *converges for every* z *in* $B(z_0;r)$. *Then the function defined by* $f(z) = \sum_{n=1}^{\infty} a_n(z-z_0)^n$ *in* $B(z_0;r)$ *has a derivative at each point in* $B(z_0;r)$ *represented by* $f'(z) = \sum_{n=1}^{\infty} na_n(z-z_0)^n - 1$.

For a power series two power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad g(z) = \sum_{n=0}^{\infty} b_n z^n$$

that converge in $B(0;r_f)$ and $B(0;r_g)$ resp. we have the following properties

1. The product $fg(z) = \sum c_n z^n$ for $z \in B(0;r_f) \cap B(0;r_g)$ where $\sum c_n$ is the Cauchy product of $\sum a_n$ and $\sum b_n$.
2. The substitution $f(g(z)) = \sum c_n z^n$ where $c_n = \sum a_n b_n(k)$ and $g^k(z) = \sum b_n(k) z^n$ for $|g(z)| < r_f$ and $|z| < r_g$.
3. The reciprocal of $f(z)$, $q(z)$ has its own power series expansion with $q(0) = 1/f(0)$.

Definition 58. *Let* f *be a real-valued function defined on an interval* $I \subset \mathbb{R}$. *If* f *has derivatives of every order at each point of* I, *we write* $f \in C^{\infty}$ *on* I.

If $f \in C^{\infty}$, the series

$$\sum_{n=0}^{\infty} \frac{f^{(k)}(c)(x-c^n)}{n!}$$

is called **the Taylor series generated about** c **by** f and we can write

$$f(x) \backsim \sum_{n=0}^{\infty} \frac{f^{(k)}(c)(x-c)^n}{n!}$$

However, for an interval containing a real number c on whose neighbourhood f is defined on, it can be proved that

$$\sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k + E_n(x).$$

Where E_n is an error function given by the integral

$$E_n(x) = \frac{1}{n!}\int_c^x (x-t)^n f^{(n+1)}(t) \ dt = \frac{(x-c)^{n+1}}{n!}\int_0^1 u^n f^{(n+1)}[x+(c-x)u] \ du$$

This error representation can be used to prove the following result:

Theorem 144. ***(Bernstein)*** *Assume all* $n+1$ *derivatives of* f *are non-negatives for* $b \le x < r$. *Then for these values of* x:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(b)}(c)(x-b)^n}{n!}$$

Theorem 145. *Assume that we have*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \qquad\qquad -r < x < r$$

If the series also converges at $x = r$, then the limit $\lim_{x \to r^-} f(x)$ exists and we have

$$\lim_{x \to r^-} f(x) = \sum_{n=0}^{\infty} a_n r^n.$$

Theorem 146. *Let $f(x) = \sum_{n=0}^{\infty}$ for $-1 < x < 1$, and assume that $\lim_{n\to\infty} na_n = 0$. If $f(x) \to S$ as $x \to 1^-$, then $\sum_{n=0}^{\infty} a_n$ converges and has sum S.*