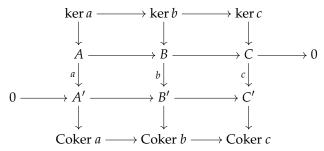
The Snake Lemma

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Lemma 1. Let A, B, C, A', B', C' be R-modules that satisfy the following commutative and exact diagram.



where the the maps associated with the kernels and the cokernels are the natural homomorphisms. Then there is exists a homomorphism θ : ker $c \to \text{Coker } a$ and the sequence

$$\ker a \xrightarrow{f^*} \ker b \xrightarrow{g^*} \ker c \xrightarrow{\theta} \operatorname{Coker} a \xrightarrow{\hat{f}} \operatorname{Coker} b \xrightarrow{\hat{g}} \operatorname{Coker} c$$

is exact.

Proof. First, we name the homomorphisms for convenience of the proof. Let f, g, f', g' be the homomorphisms $A \to B$, $B \to C$, $A' \to B'$ and $B' \to C'$ resp.

We next show the existence and well-definition of θ . In fact, θ can be defined as $f'^{-1} \circ d \circ g^{-1}$. To show that θ is well-defined, it suffices to show that for any $z \in \ker c$, we can determine the output of θ , $x \in \operatorname{Coker} a$ regardless of the inverse images. Suppose $z \in \ker c$ and let $y \in B$ such that g(y) = z (by surjectivness of g). Regardless of this choice, we know that g'(b(y)) = c(g(z)) = 0 (by communitativity) and thus $b(y) \in \ker g' = \operatorname{Im} f'$ by exactness. But f' is injective, therefore there is a unique inverse of b(z), $x' \in A'$

For two representatives $y_1, y_2 \in B$ such that $g(y_1), g(y_2) \in \ker c$, we note that $g(y_1 - y_2) = 0$ thus $y_1 - y_2 \in \ker g$. By exactness, $y_1 - y_2 \in \operatorname{Im} f$. Threfore, $b(y_1 - y_2) = b(y_1) - b(y_2) = f'(a(x))$ for some x. If $f'(x_i') = b(y_i)$, then we have $f'(x_1') = f'(x_2' + a(x))$, proving $x_1' \equiv x_2' \mod \operatorname{Im} a$ by injectivity of f'. This proves the well-definedness of θ . Clearly, θ is a homomorphisms.

It remains to show the given sequence is exact. We prove it from let to right. Im $f^* \subseteq \ker g^*$: Let a(x) = 0. By exactness of the top sequence g(f(x)) = 0, hence Im $f|_{\ker a} = \operatorname{Im} f^* \subseteq \ker g^*$.

 $\ker g^* \subseteq \operatorname{Im} f^*: g(y) = 0 \implies y = f(x)$ by exactness. By definition of f^* , the x shall be in $\ker a$.

Im $g^* \subseteq \ker \theta$: Let z = g(y), for $y \in \ker b$ and let $x' + \operatorname{Im} a = \theta(z)$. By definition, y is a representative of z and by communitarity of the diagram and exactness of the top sequence, there exsists $x \in \ker a$ such that b(y) = f'(a(x)), this implies $x' = a(x) + \operatorname{Im} a$ and thus $x' = 0 \mod \operatorname{Im} a$.

 $\ker \theta \subseteq \operatorname{Im} g^*$: Let x' = a(x) and let g(y) = z (by surjectivity of g). Then $b(y) = f'(a(x)) = b(f(x)) \implies b(y - f(x)) = 0$. But g(y - f(x)) = g(y) = z by exactness of the top sequence. Hence $y - f(x) \in \ker g^* = \ker g|_{\ker b}$.

Next we consider the induced homomorphisms

$$\hat{f}(x + \operatorname{Im} a) = f'(x) + \operatorname{Im} b$$

$$\hat{g}(y + \operatorname{Im} b) = g'(y) + \operatorname{Im} c$$

Im $\theta \subseteq \ker \hat{f}$: $x' + \operatorname{Im} a = \theta(z)$. By exactness, g(y) = z and f(x) = y for some x, y. By commutativity, b(y) = b(f(x)) = f'(a(x)). By exactness, x' = a(x) and thus $z \in \ker \theta$.

 $\ker \hat{f} \subseteq \operatorname{Im} \theta$: Let $\hat{f}(x' + \operatorname{Im} a) = y' + \operatorname{Im} b = 0$ $y' \in \operatorname{Im} b$. Then, f'(x') = y' By definition, y' = b(y). c(g(y)) = g(b(y)) = g(f(x')) = 0 and $g(y) \in \ker c$.

 $\ker \hat{g} \subseteq \operatorname{Im} \hat{f}$: Let $\hat{g}(y' + \operatorname{Im} b) = 0 + \operatorname{Im} c$. Then g'(y') = c(z) for some $z \in C$. Since g is on-to, there is a y such that $g(y) = z \Longrightarrow g'(y') = c(g(y))$. By commutativity, g'(y') = g'(b(y)). But then $y' - b(y) \in \ker g' = \operatorname{Im} f'$ (by exactness). Thus $y' \in \operatorname{Im} \hat{f}$.

Im $\hat{f} \subseteq \ker \hat{g}$: Let y' = f'(x') + b(y) for some y. Since $g' \circ f' = 0$ by exactness, g'(y') = g'(b(y)) = c(g(y)) (by commutativity) $\in \operatorname{Im} c$.