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Combinatorics of q, t-parking functions

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Abstract

In [A conjectured combinatorial formula for the Hilbert series for diagonal harmonics, in: Proceedings of FPSAC 2002 Conference, Melbourne, Australia, Discrete Math., in press] Haglund, Haiman, and the present author conjectured a combinatorial formula $CH_n(q,t)$ for the Hilbert series of diagonal harmonics as a weighted sum of parking functions. Another equivalent combinatorial formula was proposed by the present author in [Multivariate analogues of Catalan numbers, parking functions, and their extensions, UCSD doctoral thesis, June 2003]. These formulas involve three statistics on parking functions called *area*, *dinv*, and *pmaj*. In this article, we use the pmaj statistic to solve several combinatorial problems posed in [A conjectured combinatorial formula for the Hilbert series for diagonal harmonics, in: Proceedings of FPSAC 2002 Conference, Melbourne, Australia, Discrete Math., in press]. In particular, we derive a recursion satisfied by the combinatorial Hilbert series and show that $q^{n(n-1)/2}CH_n(1/q,q) = [n+1]_q^{n-1}$.

Keywords: Parking functions; Diagonal harmonics; q-analogue

1. Introduction

Let n be a fixed positive integer. A *parking function* of order n is a function

$$f:\{1,2,\ldots,n\}\to\{1,2,\ldots,n\}$$

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such that

$$\big|\{x\colon\ f(x) \leqslant i\}\big| \geqslant i \quad \text{for } 1 \leqslant i \leqslant n.$$

We let $\mathcal{P}_n$ denote the collection of parking functions of order n. It is well known that $|\mathcal{P}_n| = (n+1)^{n-1}$.

Parking functions were introduced by Konheim and Weiss to study hashing protocols [9]. The name "parking function" arises as follows. There are n cars that wish to park on a one-way street with n spaces. The cars and spaces are each numbered 1 to n. Each car has a particular spot in which it prefers to park. We define $g : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ by setting $g(x) = y$ if and only if car x prefers spot y. In the *original parking policy* of Konheim and Weiss, the cars arrive at the beginning of the street in increasing order. Each car x drives forward to spot $g(x)$ and parks in that spot if it is available. Otherwise, car x continues forward and parks in the next empty spot larger than $g(x)$, if any. It is easy to see that g is a parking function if and only if every car is able to park in this way.

Parking functions have been studied by many authors [1,2,5,10–13]. This article is concerned with the combinatorial properties of a q, t-analogue of parking functions introduced by Haglund, Haiman, and the present author in [5] and studied further in [10]. The rest of this section reviews the relevant definitions and results from those papers. In the remaining sections, we solve two of the open problems posed in [5].

1.1. Diagonal harmonics

We briefly recall the algebraic problem that originally motivated the introduction of q, t-parking functions in [5]. Let $R_n = \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ denote the polynomial ring in $2n$ commuting variables. Define the *diagonal harmonics of order n* by setting

$$DH_n = \left\{ f \in R_n\colon\ \sum_{i=1}^{n} \frac{\partial^h}{\partial x_i^h} \frac{\partial^k}{\partial y_i^k}\ f = 0 \text{ for } 1 \leqslant h + k \leqslant n \right\}.$$

This set is clearly a vector subspace of R_n.

Let $V_{h,k,n}$ denote the set of polynomials $f \in R_n$ such that f is homogeneous of degree h in the x-variables and homogeneous of degree k in the y-variables. By convention, the zero polynomial belongs to every $V_{h,k,n}$. We obtain the decomposition

$$R_n = \bigoplus_{h \geqslant 0} \bigoplus_{k \geqslant 0} V_{h,k,n},$$

which turns R_n into a doubly graded vector space. We can also write

$$DH_n = \bigoplus_{h \geqslant 0} \bigoplus_{k \geqslant 0} (DH_n \cap V_{h,k,n}),$$

since it is easy to see that $f \in DH_n$ if and only if each bihomogeneous component of f is in DH_n. We define the *Hilbert series of diagonal harmonics* by setting

$$H_n(q,t) = \sum_{h \geqslant 0} \sum_{k \geqslant 0} \dim(DH_n \cap V_{h,k,n}) q^h t^k.$$

The Hilbert series $H_n(q,t)$ has the following properties.

(1) $H_n(1,1) = (n+1)^{n-1}$.
(2) $q^{n(n-1)/2} H_n(1/q,q) = [n+1]_q^{n-1}$.
(3) $H_n(q,t) = H_n(t,q)$.

(Here we use the usual notation $[m]_q = \sum_{i=0}^{m-1} q^i$.) The first two properties follow from a hard theorem of Mark Haiman that gives the character formula for DH_n viewed as a doubly graded S_n-module [6–8]. The third property follows from the fact that the x-variables and the y-variables appear symmetrically in the definition of diagonal harmonics. Hence, we have $\dim(DH_n \cap V_{h,k,n}) = \dim(DH_n \cap V_{k,h,n})$ for all h and k. Multiplying by $q^h t^k$ and adding over all h and k gives $H_n(q,t) = H_n(t,q)$. We get a bijection between $DH_n \cap V_{h,k,n}$ and $DH_n \cap V_{k,h,n}$ by simply interchanging x_i and y_i for all i.

Of course, the third property immediately implies that $H_n(q,1) = H_n(1,q)$. We call this last fact *univariate symmetry* of the Hilbert series. In contrast, the stronger statement $H_n(q,t) = H_n(t,q)$ is called *joint symmetry* of the Hilbert series.

1.2. Statistics on parking functions

Since $H_n(1,1) = (n+1)^{n-1}$, the Hilbert series of diagonal harmonics is a sum of $(n+1)^{n-1}$ monomials $q^h t^k$. Recall that there are $(n+1)^{n-1}$ parking functions of order n. This suggests that there may be a combinatorial interpretation of $H_n(q,t)$ as a sum of weighted parking functions. More precisely, we seek two functions $\mathrm{qstat}: \mathcal{P}_n \to \mathbb{N}$ and $\mathrm{tstat}: \mathcal{P}_n \to \mathbb{N}$ such that

$$H_n(q,t) = \sum_{f \in \mathcal{P}_n} q^{\mathrm{qstat}(f)} t^{\mathrm{tstat}(f)}. \tag{1}$$

The weight functions qstat and tstat are often called *statistics* on parking functions. Two such statistics were proposed by Haglund, Haiman, and the present author in [5]. To describe them, we first explain how to regard parking functions as *labelled Dyck paths*.

A *Dyck path of order* n is a sequence of n vertical segments and n horizontal segments going north and east from $(0,0)$ to (n,n) such that no step goes strictly below the line $y = x$. A *labelled Dyck path* is a Dyck path in which the n vertical segments have been labelled 1 to n so that labels of consecutive vertical segments increase from bottom to top. We place each label in the lattice square to the right of the corresponding vertical segment. See Fig. 1 for an example.

There is a simple bijection between labelled Dyck paths and parking functions. Let P be a labelled Dyck path. In the diagram of P, number the columns 1 to n from left to right.

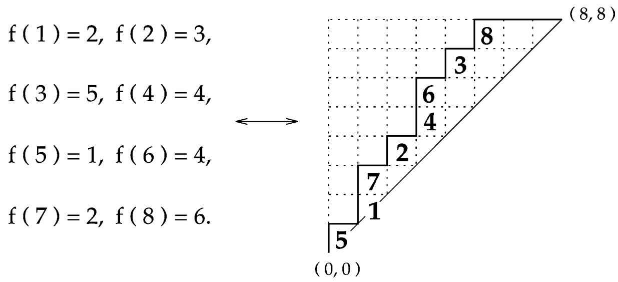

Fig. 1. A parking function and its associated labelled Dyck path.

Define $f:\{1,2,\dots,n\}\to\{1,2,\dots,n\}$ by setting $f(x)=y$ for every label x appearing in column y. In other words, the cars x in column y all prefer to park in spot y. It is easy to see that f is a parking function. Conversely, given a parking function, place all cars that prefer spot y in column y in increasing order, such that cars preferring earlier spots appear in lower rows of the figure. The labels determine a labelled lattice path in the obvious way, which is easily seen to be a Dyck path. Figure 1 gives an example of this bijection.

The *area* of a Dyck path P (labelled or not) is the number of complete lattice squares between the path and the line $y=x$. The *area vector* of a Dyck path is a sequence $g(P)=(g_0,\dots,g_{n-1})$, where g_i is the number of complete lattice squares between the path and the line $y=x$ in row i of the figure. Here we number the rows of the figure 0 to $n-1$ from bottom to top. For example, the labelled path in Fig. 1 has $\operatorname{area}(P)=9$ and

$$g(P)=(0,0,1,1,1,2,2,2).$$

Given a labelled Dyck path P, define a *label vector* $p(P)=(p_0,\dots,p_{n-1})$ by letting p_i be the unique label in row i. It is clear that a labelled Dyck path P is completely determined by the pair of vectors $(g(P),p(P))$. A pair of vectors (g,p) arises from a labelled Dyck path of order n if and only if the following conditions hold.

(1) g and p have length n.
(2) $g_0=0$.
(3) $g_i\geqslant 0$ for $0\leqslant i<n$.
(4) $g_{i+1}\leqslant g_i+1$ for $0\leqslant i<n-1$.
(5) p is a permutation of $\{1,2,\dots,n\}$.
(6) $g_{i+1}=g_i+1$ implies $p_i<p_{i+1}$.

See [10] for a more detailed discussion. If f is a parking function corresponding to the labelled path P, we set $\operatorname{area}(f)=\operatorname{area}(P)$, $g(f)=g(P)$, and so on. With this notation, one can check that

$$\operatorname{area}(P)=\operatorname{area}(f)=n(n+1)/2-\sum_{i=1}^{n}f(i). \tag{2}$$

We can now define $CH_n(q,t)$. For $f\in\mathcal{P}_n$, let

$$\operatorname{dinv}(f) = \sum_{i<j} \big[\chi\big(g_i(f) = g_j(f) \text{ and } p_i(f) < p_j(f)\big) \\ + \chi\big(g_i(f) = g_j(f) + 1 \text{ and } p_i(f) > p_j(f)\big)\big].$$

Here, for any logical statement A, we set $\chi(A) = 1$ if A is true and $\chi(A) = 0$ if A is false. Let qstat = dinv and tstat = area in (1), so that

$$CH_n(q,t) = \sum_{f \in \mathcal{P}_n} q^{\operatorname{dinv}(f)} t^{\operatorname{area}(f)}.$$

It is conjectured in [5] that $CH_n(q,t) = H_n(q,t)$. This conjecture suggests several combinatorial problems, which were posed as open questions in [5].

(1) (*Univariate symmetry*) Prove that $CH_n(q,1) = CH_n(1,q)$.
(2) (*Joint symmetry*) Prove that $CH_n(q,t) = CH_n(t,q)$.
(3) (*Specialization*) Prove that $q^{n(n-1)/2} CH_n(1/q,q) = [n+1]_q^{n-1}$.
(4) (*Recursion*) Find a recursion involving $CH_n(q,t)$.

Of course, the first three statements would all follow from results on diagonal harmonics if we could prove the conjecture that $CH_n(q,t) = H_n(q,t)$. On the other hand, it is clearly desirable to have purely combinatorial proofs of these results.

1.3. *The* pmaj *statistic*

There is an alternate combinatorial definition of $CH_n(q,t)$, first introduced in [10], that involves a third statistic on parking functions called pmaj. In the next few subsections, we summarize the results about pmaj from [10], including the solution of problem 1 (univariate symmetry) using pmaj. In the main body of this paper, we show how to use pmaj to solve problems 3 and 4 above. Problem 2 (joint symmetry) is still open.

First we give the definition of pmaj. Let $f \in \mathcal{P}_n$ be a parking function of order n. Recall that $f(x) = j$ if and only if car x prefers to park in spot j. For $1 \leqslant j \leqslant n$, let $S_j = \{x\colon f(x) = j\}$ be the set of cars that prefer to park in spot j. Let $T_j = \bigcup_{k=1}^{j} S_k$ be the set of cars that want to park at or before spot j. The definition of a parking function states that $|T_j| \geqslant j$ for $1 \leqslant j \leqslant n$.

We introduce the following new *parking policy*. The parking spots $1,\dots,n$ will be filled from left to right with cars $\tau_1,\dots,\tau_n$. The car τ_1 that gets spot 1 is the largest car x in the set $S_1 = T_1$. The car τ_2 that gets spot 2 is the largest car x in $T_2 - \{\tau_1\}$ such that $x < \tau_1$; if there is no such car, then x is the largest car in $T_2 - \{\tau_1\}$. In general, the car τ_i that gets spot i is the largest car x in $T_i - \{\tau_1,\dots,\tau_{i-1}\}$ such that $x < \tau_{i-1}$; if there is no such car, then x is the largest car in $T_i - \{\tau_1,\dots,\tau_{i-1}\}$. Since $|T_i| \geqslant i$, the set $T_i - \{\tau_1,\dots,\tau_{i-1}\}$ is never empty. So this selection process makes sense. At the end of this process, we obtain a *parking order* $\tau = \tau(f) = \tau_1,\dots,\tau_n$, which is a permutation of $1,\dots,n$. We let $\sigma = \sigma(f)$ be the reversal of τ, so that $\sigma_j = \tau_{n+1-j}$ and $\tau_j = \sigma_{n+1-j}$ for $1 \leqslant j \leqslant n$. Finally, we define $\operatorname{pmaj}(f) = \operatorname{maj}(\sigma(f))$, where maj is the ordinary major index statistic on permutations. Recall that $\operatorname{maj}(\sigma_1 \cdots \sigma_n) = \sum_{i=1}^{n-1} i\,\chi(\sigma_i > \sigma_{i+1})$.

For example, if f is the parking function given in Fig. 1, we have $\tau(f) = 5, 1, 7, 6, 4, 3, 2, 8$; $\sigma(f) = 8, 2, 3, 4, 6, 7, 1, 5$; and $\mathrm{pmaj}(f) = \mathrm{maj}(\sigma(f)) = 7$.

1.4. Fermionic formula and univariate symmetry

To relate pmaj to dinv, we first establish a "fermionic formula" for $CH_n(q,t)$. Given $\sigma \in S_n$, we can uniquely factor the word $\sigma_1 \cdots \sigma_n$ into ascending runs R_i separated by descents, say $\sigma_1 \cdots \sigma_n = R_s > \cdots > R_1 > R_0$. Note that the runs are numbered from right to left, with the indexing starting at zero. Let $1 \leqslant k \leqslant n$, and suppose σ_k belongs to run R_i. Define $w_k(\sigma)$ to be the number of symbols greater than σ_k in R_i, plus the number of symbols less than σ_k in R_{i-1}, where we use the convention that R_{-1} consists of a single zero. For example, if $\sigma = 8, 2, 3, 4, 6, 7, 1, 5$, then $R_2 = 8$, $R_1 = 2, 3, 4, 6, 7$, $R_0 = 1, 5$, and

$$\bigl(w_1(\sigma), \ldots, w_n(\sigma)\bigr) = (5, 5, 4, 3, 3, 2, 2, 1).$$

Theorem (Fermionic formula).

$$CH_n(q,t) = \sum_{f \in \mathcal{P}_n} q^{\mathrm{dinv}(f)} t^{\mathrm{area}(f)} = \sum_{\sigma \in S_n} t^{\mathrm{maj}(\sigma)} \prod_{k=1}^{n} \bigl[w_k(\sigma)\bigr]_q = \sum_{f \in \mathcal{P}_n} q^{\mathrm{area}(f)} t^{\mathrm{pmaj}(f)}.$$

Moreover, there is a bijection $\phi : \mathcal{P}_n \to \mathcal{P}_n$ *such that* $\mathrm{pmaj}(\phi(f)) = \mathrm{area}(f)$ *and* $\mathrm{area}(\phi(f)) = \mathrm{dinv}(f)$ *for all* $f \in \mathcal{P}_n$.

Proof. First, we interpret the fermionic formula

$$\sum_{\sigma \in S_n} t^{\mathrm{maj}(\sigma)} \prod_{k=1}^{n} \bigl[w_k(\sigma)\bigr]_q \tag{3}$$

as a sum of weighted objects. Let $\mathcal{I}_n$ be the set of objects $(\sigma; u_1, \ldots, u_n)$, where $\sigma \in S_n$ and $0 \leqslant u_k < w_k(\sigma)$ for all k. Define the t-weight of such an object to be $\mathrm{tstat} = \mathrm{maj}(\sigma)$, and define the q-weight to be $\mathrm{qstat} = \sum_{k=1}^{n} u_k$. Clearly, (3) is the generating function for these weighted objects.

To complete the proof, it suffices to describe bijections $\gamma, \beta : \mathcal{P}_n \to \mathcal{I}_n$ such that β sends $(\mathrm{dinv}, \mathrm{area})$ to $(\mathrm{qstat}, \mathrm{tstat})$, while γ sends $(\mathrm{area}, \mathrm{pmaj})$ to $(\mathrm{qstat}, \mathrm{tstat})$. We then set $\phi = \gamma^{-1} \circ \beta : \mathcal{P}_n \to \mathcal{P}_n$. We merely describe the maps β and γ and their inverses, referring the reader to [10] for the detailed verification that these are well-defined bijections preserving the relevant statistics.

First we define β. Represent $f \in \mathcal{P}_n$ by its area vector and label vector (g, p). Let $D = \max_i g_i$. For each d from D to 0 in this order, list the labels p_i such that $g_i = d$ in increasing order. This list defines a permutation $\sigma_1 \cdots \sigma_n$ such that run R_d consists of the labels p_i with $g_i = d$. Define

$$\begin{aligned} u_k &= \sum_{i<j} \bigl[\chi(g_i = g_j \text{ and } p_i = \sigma_k < p_j) + \chi(g_i = g_j + 1 \text{ and } p_i = \sigma_k > p_j)\bigr] \\ &\in \bigl[0, w_k(\sigma) - 1\bigr]. \end{aligned}$$

To compute β^{-1}, we use $(\sigma; u_1, \dots, u_n)$ to reconstruct the pair of vectors (g, p) that determine f. The run structure of σ determines how to compute g_i for each p_i, via the rule that $g_i = d$ iff p_i is in run R_d. To compute p, scan the symbols $\sigma_1 \cdots \sigma_n$ from right to left, inserting each symbol into an initially empty label vector. For each k, the number u_k uniquely determines the insertion position for σ_k in p that causes the appropriate contribution to dinv.

Now we define γ. Given $f \in \mathcal{P}_n$, let $\sigma = \sigma(f)$ be the reversal of the parking order of f. Set $u_k = n + 1 - k - f(\sigma_k)$. One can show (using the parking rules) that $0 \leqslant u_k < w_k(\sigma)$ for all k. To compute γ^{-1}, map the object $(\sigma; u_1, \dots, u_n) \in \mathcal{I}_n$ to the function f defined by

$$f(\sigma_k) = n + 1 - k - u_k,$$

which is easily seen to be a parking function. $\square$

Corollary (Univariate symmetry). *We have $CH_n(q,1) = CH_n(1,q)$. In fact,*

$$\sum_{f \in \mathcal{P}_n} q^{\operatorname{dinv}(f)} = \sum_{f \in \mathcal{P}_n} q^{\operatorname{area}(f)} = \sum_{f \in \mathcal{P}_n} q^{\operatorname{pmaj}(f)}.$$

These statements all follow from the weight-preserving properties of the bijection ϕ. In particular, we now have a bijective solution of problem 1 (univariate symmetry). However, this result is not strong enough to solve problem 2 (joint symmetry).

In the rest of this paper, we use the pmaj statistic to solve problems 3 and 4 involving specializations and recursions for $CH_n(q,t)$. Section 2 develops a recursion characterizing the polynomials $CH_n(q,t)$. Section 3 uses this recursion to prove the formula for $q^{n(n-1)/2} CH_n(1/q, q)$. Section 4 gives a bijective version of the proof of this formula. In Section 5, we summarize some open problems and possible extensions of the present work.

2. Recursions for parking functions

This section presents a recursion involving the generating function

$$CH_n(q,t) = \sum_{f \in \mathcal{P}_n} q^{\operatorname{area}(f)} t^{\operatorname{pmaj}(f)}.$$

To obtain this recursion, we need to consider some special subcollections of $\mathcal{P}_n$. Throughout this section, we will frequently identify a parking function with its associated labelled Dyck path.

Definition. Let $[n]$ denote the set $\{1, 2, \dots, n\}$. For any set $S \subseteq [n]$, we define

$$\mathcal{P}_{n,S} = \big\{ f \in \mathcal{P}_n\colon \big\{x\colon f(x) = 1\big\} = S\big\}.$$

Also define the generating function

$$P_{n,S}(q,t) = \sum_{f \in \mathcal{P}_{n,S}} q^{\operatorname{area}(f)} t^{\operatorname{pmaj}(f)}.$$

Thus, $\mathcal{P}_{n,S}$ consists of parking functions f such that S is the set of cars preferring spot 1. In terms of labelled paths, $\mathcal{P}_{n,S}$ consists of labelled Dyck paths such that the labels in the first column are exactly the elements of S. For example, the parking function from Fig. 1 is an element of $\mathcal{P}_{8,\{5\}}$. Note that $\mathcal{P}_{n,\emptyset} = \emptyset$ and $P_{n,\emptyset}(q,t) = 0$.

Since $\mathcal{P}_n$ is the disjoint union of all the sets $\mathcal{P}_{n,S}$, we clearly have

$$CH_n(q,t) = \sum_{S \subseteq [n]} P_{n,S}(q,t).$$

We will see shortly that $CH_n(q,t) = P_{n+1,\{n+1\}}(q,t)$, which gives a simpler way to recover $CH_n(q,t)$ from the polynomials $P_{n,S}(q,t)$.

We will present a recursion with initial conditions that characterizes all the polynomials $P_{n,S}(q,t)$. In light of the preceding remark, this solves problem 4 from the Introduction. To obtain our recursion, we need to introduce three basic operations on parking functions.

2.1. *The erasing operation*

First, we define an *erasing operation* $E = E_{n+1} : \mathcal{P}_{n+1,\{n+1\}} \to \mathcal{P}_n$. Given $f \in \mathcal{P}_{n+1,\{n+1\}}$, define $E(f) \in \mathcal{P}_n$ by erasing the leftmost column and lowest row of the labelled path associated to f. Equivalently, $E(f)$ is the function such that $x \mapsto f(x) - 1$ for $x \in [n]$. It is easy to check that the map E is a well-defined, weight-preserving bijection. In particular, the definition of pmaj immediately gives $\tau(f) = n+1, \tau(E(f))$ where the comma denotes concatenation. So $\sigma(f) = \sigma(E(f)), n+1$ and hence

$$\operatorname{pmaj}(f) = \operatorname{maj}\big(\sigma(f)\big) = \operatorname{maj}\big(\sigma\big(E(f)\big), n+1\big) = \operatorname{maj}\big(\sigma\big(E(f)\big)\big) = \operatorname{pmaj}\big(E(f)\big).$$

It is also clear that $\operatorname{area}(f) = \operatorname{area}(E(f))$, since the erased row and column in the diagram of f do not have any area cells.

Since E is a weight-preserving bijection, we have $P_{n+1,\{n+1\}}(q,t) = CH_n(q,t)$ for all n, as claimed earlier.

2.2. *The dragnet operation*

Second, we define the *dragnet operation* $D = D_n : \mathcal{P}_n \to \mathcal{P}_n$. Consider an element $f \in \mathcal{P}_{n,S}$, where $S = \{s_1 < s_2 < \cdots < s_k\}$. Define $D(f)$ to be the function such that $s_k \mapsto 1$, $s_j \mapsto 2$ for $1 \leqslant j < k$, and $x \mapsto f(x)$ for all $x \in [n] - S$. It is easy to check that $D(f)$

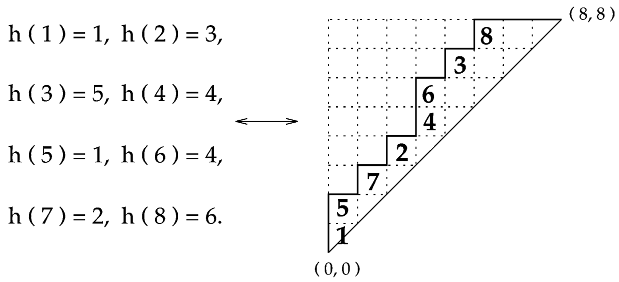

Fig. 2. The parking function h.

is a parking function; indeed, $D(f) \in \mathcal{P}_{n,\{s_k\}}$. If P is the labelled path for f, then we obtain the labelled path for $D(f)$ as follows. Move all the labels in column 1 except s_k into column 2, and then sort all the labels in column 2 into increasing order. The label s_k remains in column 1, falling into the lowest row. We imagine a dragnet dragging across the diagram of f from left to right, letting s_k fall through but pushing the remaining labels in column 1 into column 2.

For example, if h is the parking function shown in Fig. 2, then $D(h)$ is the parking function f shown in Fig. 1. Also, $D(f) = f$. More generally, the fixed points of D are the labelled paths with exactly one label in the first column.

Lemma. *Suppose $f \in \mathcal{P}_{n,S}$ where $|S| = k$. Then*

$$\operatorname{area}(f) = k - 1 + \operatorname{area}\bigl(D(f)\bigr), \qquad \operatorname{pmaj}(f) = \operatorname{pmaj}\bigl(D(f)\bigr).$$

Proof. The labelled path for $D(f)$ is obtained from the labelled path for f by moving $k-1$ labels from the first column into the second column. This causes a loss of $k-1$ area cells in the first column, while the area cells in the other columns are unaffected. Hence, $\operatorname{area}(D(f)) = \operatorname{area}(f) - (k-1)$. This is also clear from (2), since $\sum_{i=1}^{n} D(f)(i) = k - 1 + \sum_{i=1}^{n} f(i)$.

As in Section 1.3, we set $\tau(f) = \tau_1 \cdots \tau_n$ and $T_i = \{x \in [n]\colon\ f(x) \leqslant i\}$. Also define $\tau(D(f)) = \tau'_1 \cdots \tau'_n$ and $T'_i = \{x \in [n]\colon\ D(f)(x) \leqslant i\}$. We claim that $\tau_i = \tau'_i$ for $1 \leqslant i \leqslant n$. Letting $S = \{s_1 < \cdots < s_k\}$, we see that $\tau_1 = s_k = \tau'_1$. Comparing the definitions of $D(f)$ and f, it is clear that $T_i = T'_i$ for $2 \leqslant i \leqslant n$. Assuming that $\tau_j = \tau'_j$ for all $j < i$, it then follows that $\tau_i = \tau'_i$ as well. For, τ_i is the largest element of the set $T_i - \{\tau_1, \ldots, \tau_{i-1}\}$ less than τ_{i-1} (or the largest element of this set if everything in it is greater than τ_{i-1}). Similarly, τ'_i is the largest element of the set $T'_i - \{\tau'_1, \ldots, \tau'_{i-1}\}$ less than τ'_{i-1} (or the largest element of this set if everything in it is greater than τ'_{i-1}). Comparing these two definitions, we see that $\tau_i = \tau'_i$. Since $\operatorname{pmaj}(f)$ only depends on $\tau(f)$, it now follows that $\operatorname{pmaj}(f) = \operatorname{pmaj}(D(f))$. $\square$

2.3. The cyclic-shift operation

Third, we introduce a *cyclic-shift operation* C_n defined on various types of objects. If $x \in [n]$ is an integer, define $C_n(x) = x + 1$ for $x < n$ and $C_n(n) = 1$. If $S \subseteq [n]$ is

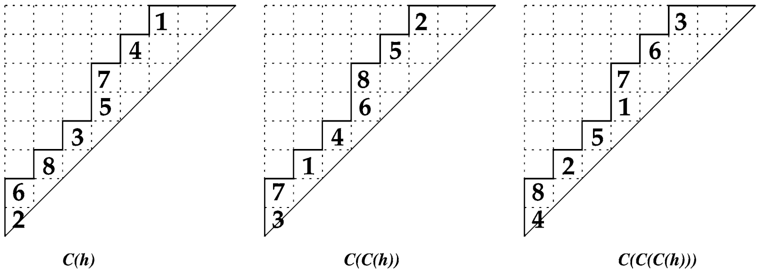

Fig. 3. Applying cyclic shifts to h.

a set, define $C_n(S) = \{C_n(x)\colon\ x \in S\}$. If $\sigma = \sigma_1 \cdots \sigma_n$ is a permutation, define $C_n(\sigma) = C_n(\sigma_1)\cdots C_n(\sigma_n)$. Finally, consider a parking function $f \in \mathcal{P}_{n,S}$ where $n \notin S$. Let $S = \{s_1 < s_2 < \cdots < s_k\}$ with $s_k < n$. We define $C_n(f)$ to be the function such that $1 \mapsto f(n)$ and $x \mapsto f(x-1)$ for $1 < x \leqslant n$. Equivalently, $C_n(f)(C_n(x)) = f(x)$ for all $x \in [n]$. The labelled path for $C_n(f)$ is obtained from the labelled path for f by replacing each label x by $C_n(x)$ and then sorting the labels in the column that now contains 1. It is easy to see that $C_n(f)$ is again a parking function. Indeed, since we have required that $s_k < n$, it follows that $C_n(f) \in \mathcal{P}_{n,S'}$, where $S' = C_n(S) = \{s_1 + 1 < s_2 + 1 < \cdots < s_k + 1\}$. We write $C_n = C$ if n is understood. Note that we choose not to define $C_n(f)$ in the case where $f(n) = 1$.

For example, let h be the parking function from Fig. 2. Figure 3 shows the labelled paths for $C(h)$, $C^2(h) = C(C(h))$, and $C^3(h) = C(C(C(h)))$. Note that $C^4(h)$ is not defined. It is easy to check that

$$
\begin{array}{llll}
\sigma(h) & = 8 > 2\ \ 3\ \ 4\ \ 6\ \ 7 > 1\ \ 5, & \operatorname{pmaj}(h) = 7, \\
\sigma\bigl(C(h)\bigr) & = 1\ \ 3\ \ 4\ \ 5\ \ 7\ \ 8 > 2\ \ 6, & \operatorname{pmaj}\bigl(C(h)\bigr) = 6, \\
\sigma\bigl(C^2(h)\bigr) & = 2\ \ 4\ \ 5\ \ 6\ \ 8 > 1\ \ 3\ \ 7, & \operatorname{pmaj}\bigl(C^2(h)\bigr) = 5, \\
\sigma\bigl(C^3(h)\bigr) & = 3\ \ 5\ \ 6\ \ 7 > 1\ \ 2\ \ 4\ \ 8, & \operatorname{pmaj}\bigl(C^3(h)\bigr) = 4.
\end{array}
$$

Lemma. *Suppose $f \in \mathcal{P}_{n,S}$ where $S = \{s_1 < \cdots < s_k\}$ and $s_k < n$. Then*

$$\operatorname{area}(f) = \operatorname{area}\bigl(C(f)\bigr), \qquad \operatorname{pmaj}(f) = \operatorname{pmaj}\bigl(C(f)\bigr) + 1.$$

Proof. The Dyck path associated to f (disregarding the labels) is the same as the Dyck path associated to $C(f)$. Hence, $\operatorname{area}(f) = \operatorname{area}(C(f))$. This is also clear from (2) and the identity $C(f)(C(x)) = f(x)$.

Now let us compare $\operatorname{pmaj}(f)$ to $\operatorname{pmaj}(C(f))$. As in Section 1.3, we set $\tau(f) = \tau_1 \cdots \tau_n$ and $T_i = \{x \in [n]\colon\ f(x) \leqslant i\}$. Also define $\tau(C(f)) = \tau_1' \cdots \tau_n'$ and $T_i' = \{x \in [n]\colon C(f)(x) \leqslant i\}$.

It is clear from the definitions that $T_i' = C(T_i)$ for $1 \leqslant i \leqslant n$. We further claim that $\tau_i' = C(\tau_i)$ for $1 \leqslant i \leqslant n$. We have $\tau_1 = \max S = s_k$. Since $s_k < n$, we have $\tau_1' = \max C(S) = s_k + 1 = C(s_k) = C(\tau_1)$. Assume that $i > 1$ and that $\tau_j' = C(\tau_j)$ for all $j < i$. Consider once more the definition of τ_i. For $i > 1$, we can describe τ_i as the first element of the set $T_i - \{\tau_1, \ldots, \tau_{i-1}\}$ that is reached by repeatedly applying C^{-1} to τ_{i-1}. If we arrange the integers 1 to n clockwise in a circle, we can find τ_i by going counterclockwise along the circle starting at τ_{i-1}, until an element of $T_i - \{\tau_1, \ldots, \tau_{i-1}\}$ is reached.

Similarly, τ'_i is the first element of $T'_i - \{\tau'_1, \ldots, \tau'_{i-1}\}$ that is reached by repeatedly applying C^{-1} to τ'_{i-1}. Since $T'_i = C(T_i)$ and $\tau'_j = C(\tau_j)$ for $j < i$, it is clear from this description that $\tau'_i = C(\tau_i)$.

So far, we have shown that $\tau(C(f)) = C(\tau(f))$. Reversing these permutations, we have $\sigma(C(f)) = C(\sigma(f))$. Moreover, $\sigma(f)$ does not end in n, since $\tau(f)$ does not begin with n (it begins with $s_k < n$). When we pass from $\sigma(f)$ to $C(\sigma(f)) = \sigma(C(f))$, most of the descents are unchanged. However, consider the descent in $\sigma(f)$ between n and the following symbol, which must exist since n is not the last symbol in $\sigma(f)$. After applying C, this descent disappears, but we now get a descent in $\sigma(C(f))$ between $C(n) = 1$ and the preceding symbol, if any. Thus, the net change in major index when passing from $\sigma(f)$ to $\sigma(C(f))$ is -1. This holds even if n is the first symbol in $\sigma(f)$. A glance at the permutations in the preceding example should make this argument clear. We conclude that $\mathrm{pmaj}(C(f)) = \mathrm{pmaj}(f) - 1$, as desired. $\square$

Corollary. *Suppose $f \in \mathcal{P}_{n,S}$ where $S = \{s_1 < \cdots < s_k\}$. Then*

$$\mathrm{area}(f) = \mathrm{area}\big(C^{n-s_k}(f)\big), \qquad \mathrm{pmaj}(f) = \mathrm{pmaj}\big(C^{n-s_k}(f)\big) + n - s_k.$$

Proof. Just apply the previous lemma $n - s_k \geqslant 0$ times. Note that $n - s_k$ is the maximum number of times we can apply C to f, since n must appear in the first column of the diagram for $C^{n-s_k}(f)$. $\square$

2.4. The recursion

We are now ready to derive the recursion for $P_{n,S}(q,t)$.

Theorem. *Let $n \geqslant 1$ and $S = \{s_1 < \cdots < s_k\} \subseteq [n]$, where $k \geqslant 1$. Then*

$$P_{n,S}(q,t) = q^{k-1} t^{n-s_k} \sum_{T \subseteq [n]-S} P_{n-1, C_n^{n-s_k}(S \cup T - \{s_k\})}(q,t).$$

The initial conditions are $P_{n,\emptyset}(q,t) = 0$ for all n, and $P_{1,\{1\}}(q,t) = 1$.

Proof. For $T \subseteq [n] - S$, define $\mathcal{P}_{n,S,T}$ to be the set $\{f \in \mathcal{P}_n\colon f^{-1}(1) = S, f^{-1}(2) = T\} \subseteq \mathcal{P}_{n,S}$, and define $P_{n,S,T}(q,t) = \sum_{f \in \mathcal{P}_{n,S,T}} q^{\mathrm{area}(f)} t^{\mathrm{pmaj}(f)}$. The elements of $\mathcal{P}_{n,S,T}$ are the parking functions whose diagrams have labels S in column 1 and labels T in column 2. Here S is nonempty, but T may be empty.

Clearly $\mathcal{P}_{n,S}$ is the disjoint union of the sets $\mathcal{P}_{n,S,T}$, so that $P_{n,S}(q,t) = \sum_{T \subseteq [n]-S} P_{n,S,T}(q,t)$. To prove the recursion, it suffices to produce a bijection $\beta : \mathcal{P}_{n,S,T} \to \mathcal{P}_{n-1, C_n^{n-s_k}(S \cup T - \{s_k\})}$ such that $\mathrm{area}(f) = \mathrm{area}(\beta(f)) + k - 1$ and $\mathrm{pmaj}(f) = \mathrm{pmaj}(\beta(f)) + n - s_k$. Note here that $C_n^{n-s_k}(S \cup T - \{s_k\})$ is always a subset of $[n-1]$.

For each T, we can take $\beta = E \circ C^{n-s_k} \circ D$. In words, we obtain $\beta(f)$ from f by applying the dragnet operation, then the cyclic shift operation $n - s_k$ times, and then the

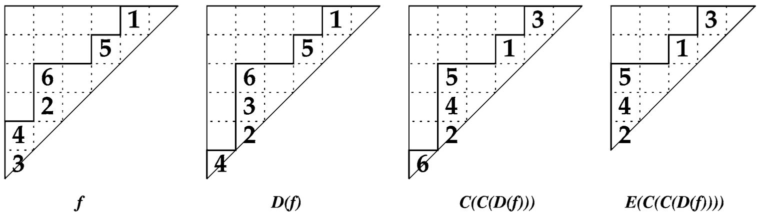

Fig. 4. An example of the map β.

erasing operation. In symbols, $\beta(f) = E(C^{n-s_k}(D(f)))$. Note that $D(f) \in \mathcal{P}_{n,\{s_k\}}$, so that we can apply C to $D(f)$ a total of $n - s_k$ times. Then $C^{n-s_k}(D(f)) \in \mathcal{P}_{n,\{n\}}$, so we can apply E to this object. Thus, $\beta(f)$ is a well-defined parking function. Moreover, the entries in column 2 of $D(f)$ are $S \cup T - \{s_k\}$; the entries in column 2 of $C^{n-s_k}(D(f))$ are $C^{n-s_k}(S \cup T - \{s_k\})$; and so the entries in column 1 of $\beta(f) = E(C^{n-s_k}(D(f)))$ are $C^{n-s_k}(S \cup T - \{s_k\})$. Thus, $\beta(f) \in \mathcal{P}_{n-1,C^{n-s_k}(S\cup T-\{s_k\})}$.

The preceding lemmas show that $\text{area}(\beta(f)) = \text{area}(D(f)) = \text{area}(f) - (k-1)$ and $\text{pmaj}(\beta(f)) = \text{pmaj}(D(f)) - (n - s_k) = \text{pmaj}(f) - (n - s_k)$. Finally, for each T, we show that the map $\beta : \mathcal{P}_{n,S,T} \to \mathcal{P}_{n-1,C_n^{n-s_k}(S\cup T-\{s_k\})}$ is a bijection. Let $f \in \mathcal{P}_{n-1,C_n^{n-s_k}(S\cup T-\{s_k\})}$. First, $E^{-1}(f)$ is well defined. Second, C_n^{-1} can be applied $n - s_k$ times to $E^{-1}(f)$ to produce a labelled path with s_k in column 1 and labels $S \cup T - \{s_k\}$ in column 2. Third, since S is known, we can invert D by moving the labels in $S - \{s_k\}$ from the second column back to the first column and then sorting. The process just described clearly inverts β. This completes the proof of the recursion. The initial conditions are obvious from the definitions. $\square$

Figure 4 gives an example of the map β. The given function f has $n = 6$, $S = \{3, 4\}$, $k = 2$, $s_1 = 3$, $s_2 = 4$, and $T = \{2, 6\}$. We have $n - s_k = 2$, so that $\beta(f) = E(C(C(D(f))))$.

Remark. It is easy to see that $C_n \circ D_n = D_n \circ C_n$ whenever both sides are defined. Thus, we could have also defined $\beta(f) = E(D(C^{n-s_k}(f)))$.

3. The specialized formula

In this section, we prove that $q^{n(n-1)/2} CH_n(1/q, q) = [n+1]_q^{n-1}$. This will follow from a more general formula for $P_{n,S}(1/q, q)$. We recall that $[0]_q = 0$ and $[m]_q = 1 + q + q^2 + \cdots + q^{m-1}$ for integers $m > 0$.

Theorem. *For $n \geqslant 0$ and $S = \{s_1 < s_2 < \cdots < s_k\} \subseteq [n]$, we have*

$$q^{n(n-1)/2} P_{n,S}(1/q, q) = q^{n-k} [n]_q^{n-k-1} \sum_{x \in S} q^{n-x}. \tag{4}$$

Proof. Both sides are zero if $S = \emptyset$ and $n \geqslant 0$. Both sides are 1 if $n = 1$ and $S = \{1\}$. Suppose $S = [n]$, so that $k = n$. There is one object f in $P_{n,[n]}$ given by $f(x) = 1$ for all x.

We have $\mathrm{area}(f) = n(n-1)/2$ and $\mathrm{pmaj}(f) = 0$, so that the left side is 1 in this case. The right side is $[n]_q^{-1}[n]_q = 1$.

We can now assume that $n > 1$, that $0 < k < n$, and that the desired formula holds for all $n' < n$ and all $S' \subseteq [n']$. The recursion for $P_{n,S}(q,t)$ gives

$$\begin{aligned} P_{n,S}(1/q,q) &= q^{n+1-s_k-k} \sum_{T \subseteq [n]-S} P_{n-1, C_n^{n-s_k}(S \cup T - \{s_k\})}(1/q,q) \\ &= q^{n+1-s_k-k} \sum_{u=0}^{n-k} \sum_{\substack{T \subseteq [n]-S \\ |T|=u}} P_{n-1, C_n^{n-s_k}(S \cup T - \{s_k\})}(1/q,q). \end{aligned}$$

The summand where $u = n-k$ involves $P_{n-1,[n-1]}(1/q,q) = q^{-(n-1)(n-2)/2}$. By induction, a typical summand for $0 \leqslant u < n-k$ looks like

$$q^{\mathrm{pow}}[n-1]_q^{n-1-(u+k-1)-1} \sum_{x=1}^{n-1} \chi\bigl(x \in C_n^{n-s_k}\bigl(S \cup T - \{s_k\}\bigr)\bigr) q^{n-1-x},$$

$$\mathrm{pow} = n-1-(u+k-1)-(n-1)(n-2)/2.$$

Here, $u+k-1$ is the size of the set $C_n^{n-s_k}(S \cup T - \{s_k\})$. Using this information in the previous formula, we see that $q^{n(n-1)/2}P_{n,S}(1/q,q)$ is equal to

$$\begin{aligned} &q^{2n-k-s_k} + q^{2n-k-s_k} \sum_{u=0}^{n-k-1} q^{n-k-u}[n-1]_q^{n-k-u-1} \\ &\quad \times \sum_{\substack{T \subseteq [n]-S \\ |T|=u}} \sum_{x=1}^{n-1} \chi\bigl(x \in C_n^{n-s_k}\bigl(S \cup T - \{s_k\}\bigr)\bigr) q^{n-1-x}. \end{aligned} \tag{5}$$

Let us simplify the double sum on the second line of this formula. Interchanging the order of summation gives

$$\begin{aligned} &\sum_{\substack{T \subseteq [n]-S \\ |T|=u}} \sum_{x=1}^{n-1} \chi\bigl(x \in C_n^{n-s_k}\bigl(S \cup T - \{s_k\}\bigr)\bigr) q^{n-1-x} \\ &= \sum_{x=1}^{n-1} q^{n-1-x} \sum_{T} \chi\bigl(T \subseteq [n]-S,\ |T| = u,\ C_n^{s_k-n}(x) \in S \cup T - \{s_k\}\bigr). \end{aligned}$$

Now consider two cases.

Case 1. $x = n - (s_k - s_i)$ for some $i \leqslant k$. Since $1 \leqslant x \leqslant n-1$, we must actually have $i < k$ in this case. Note that $C_n^{s_k - n}(x) = s_i$. The coefficient of q^{n-1-x} is then

$$\left|\left\{T\colon\ T \subseteq [n] - S,\ |T| = u,\ s_i \in S \cup T - \{s_k\}\right\}\right|.$$

The condition $s_i \in S \cup T - \{s_k\}$ automatically holds, so the coefficient is just the number of u-element subsets T of $[n] - S$, which is $\binom{n-k}{u}$. It will be convenient to write this coefficient as $\binom{n-k-1}{u} + \binom{n-k-1}{u-1}$.

Case 2. $x \neq n - (s_k - s_i)$ for all $i \leqslant k$. This condition just says that $x \notin C_n^{n-s_k}(S)$, so that the element $z = C_n^{s_k - n}(x)$ of $[n]$ is not in S. The coefficient of q^{n-1-x} in this case is

$$\left|\left\{T\colon\ T \subseteq [n] - S,\ |T| = u,\ z \in S \cup T - \{s_k\}\right\}\right|.$$

Since $z \notin S$, we must include z in the set T to satisfy the last condition. The remaining $u-1$ elements of T can then be chosen in $\binom{n-k-1}{u-1}$ ways.

We can now evaluate the double sum. The preceding analysis shows that the sum equals

$$\begin{aligned}&\sum_{x \text{ in Case } 1}\left[\binom{n-k-1}{u} + \binom{n-k-1}{u-1}\right] q^{n-1-x}\\&\quad + \sum_{x \text{ in Case } 2}\binom{n-k-1}{u-1} q^{n-1-x}.\end{aligned}\tag{6}$$

Note that x occurs in Case 1 iff $n - 1 - x = s_k - s_i - 1$ for some $i < k$. Using this and combining the terms involving $\binom{n-k-1}{u-1}$ gives

$$\binom{n-k-1}{u-1}[n-1]_q + \binom{n-k-1}{u}\sum_{i<k} q^{s_k - s_i - 1}$$

as the final value for the double sum.

Going back to Eq. (5), we see that $q^{n(n-1)/2} P_{n,S}(1/q, q)$ is the sum of

$$T_1 = q^{2n-k-s_k}\left(1 + \sum_{u=0}^{n-k-1}\binom{n-k-1}{u-1} q^{n-k-u}[n-1]_q^{n-k-u}\right)\tag{7}$$

and

$$T_2 = q^{2n-k-s_k}\left(\sum_{u=0}^{n-k-1}\binom{n-k-1}{u} q^{n-k-u-1}[n-1]_q^{n-k-u-1}\sum_{i=1}^{k-1} q^{s_k - s_i}\right).\tag{8}$$

To evaluate T_1, discard the $u = 0$ summand (which is zero), let $v = u - 1$, and absorb the 1 into the summation as a new summand corresponding to $v = n - k - 1$:

$$T_1 = q^{2n-k-s_k} \sum_{v=0}^{n-k-1} \binom{n-k-1}{v} 1^v \left(q[n-1]_q\right)^{n-k-1-v}.$$

Applying the ordinary binomial theorem (not the q-binomial theorem!), we get

$$T_1 = q^{2n-k-s_k}\left(1 + q[n-1]_q\right)^{n-k-1} = q^{n-k}[n]_q^{n-k-1} q^{n-s_k}.$$

This is the term in (4) corresponding to $x = s_k \in S$.

To evaluate T_2, first use the binomial theorem to see that

$$\sum_{u=0}^{n-k-1} \binom{n-k-1}{u} 1^u \left(q[n-1]_q\right)^{n-k-1-u} = \left(1 + q[n-1]_q\right)^{n-k-1} = [n]_q^{n-k-1}.$$

Now compute

$$T_2 = q^{n-k}[n]_q^{n-k-1} \sum_{i=1}^{k-1} q^{s_k - s_i} q^{n-s_k} = q^{n-k}[n]_q^{n-k-1} \sum_{x \in S - \{s_k\}} q^{n-x}.$$

Thus, T_2 gives us the remaining terms in (4). We conclude that

$$q^{n(n-1)/2} P_{n,S}(1/q, q) = T_1 + T_2 = q^{n-k}[n]_q^{n-k-1} \sum_{x \in S} q^{n-x},$$

as desired. □

Corollary. *For $n \geqslant 1$,*

$$q^{n(n-1)/2} CH_n(1/q, q) = [n+1]_q^{n-1}.$$

Proof. Recall that $CH_n(1/q, q) = P_{n+1,\{n+1\}}(1/q, q)$. We use the formula from the theorem with n replaced by $n + 1$, S replaced by $\{n + 1\}$, and $k = 1$. After simplifying, we obtain the formula in the corollary. □

4. Bijective proof of the specialization

The expression $[n+1]_q^{n-1}$ has a natural combinatorial interpretation. It is the generating function for all sequences $(a_1, \ldots, a_{n-1})$ with $0 \leqslant a_i \leqslant n$ for all i, where the weight of a sequence is $a_1 + \cdots + a_{n-1}$. This suggests the problem of giving a bijection from $\mathcal{P}_n$ to the set of such sequences, so that the specialized weight $n(n-1)/2 - \mathrm{area}(f) + \mathrm{pmaj}(f)$ is mapped to $a_1 + \cdots + a_{n-1}$.

Proceeding more generally, let us interpret each side of (4) as the generating function for a set of weighted objects. For the left side, let the set be $\mathcal{P}_{n,S}$ with weights $wt(f) = n(n-1)/2 - \text{area}(f) + \text{pmaj}(f)$. For the right side, let the set be

$$\mathcal{Q}_{n,S} = \{(a_1, a_2, \ldots, a_{n-|S|-1}; x)\colon 0 \leqslant a_i \leqslant n-1,\ x \in S\}.$$

Define the weight of $(a_1, \ldots, a_{n-|S|-1}; x)$ to be

$$n - |S| + a_1 + a_2 + \cdots + a_{n-|S|-1} + n - x.$$

We want a weight-preserving bijection $\psi_{n,S} : \mathcal{P}_{n,S} \to \mathcal{Q}_{n,S}$ for all n and S.

It is tedious but straightforward to construct such a bijection by following the steps in the proof of (4). Each step in that proof can be done bijectively, and there are no subtractions or divisions to complicate matters. We sketch one definition of $\psi_{n,S}$ in this section. The reader may check that $\psi_{n,S}$ is a well-defined weight-preserving bijection by comparing the construction of $\psi_{n,S}$ to the manipulations in the last section. We also leave the construction of $\psi_{n,S}^{-1}$ to the interested reader.

The construction of $\psi_{n,S}$ is trivial when $S = \emptyset$ and when $S = [n]$, for all $n \geqslant 0$. Now let $n > 1$, $S = \{s_1 < s_2 < \cdots < s_k\}$ with $0 < k < n$, and assume that the bijections $\psi_{n',S'}$ have already been constructed for all $n' < n$ and $S' \subseteq [n']$. Let $f \in \mathcal{P}_{n,S}$, and set $T = \{x\colon f(x) = 2\}$ as usual. Also set $u = |T|$, $S' = C_n^{n-s_k}(S \cup T - \{s_k\}) \subseteq [n-1]$, and $k' = |S'| = u + k - 1$.

The case $u = n - k$ is special and will be handled later. To compute $\psi_{n,S}(f)$ when $0 \leqslant u < n - k$, first calculate $\beta(f) \in \mathcal{P}_{n-1,S'}$, where β is the bijection used to prove the recursion for $P_{n,S}(q,t)$. Then compute

$$\psi_{n-1,S'}(\beta(f)) = (a'_1, a'_2, \ldots, a'_{n-k-u-1}; x')$$

where $0 \leqslant a_i \leqslant n-2$ and $x' \in S'$. (So far, we have reached Eq. (5) in Section 3.) To continue, consider two cases.

Case 1. $x' = n - (s_k - s_i)$ for some $i \leqslant k$. As before, we must have $i < k$. Let the elements of $[n] - S$ be $j_1 < j_2 < \cdots < j_{n-k}$. Define a word w' consisting of u ones and $n - k - u$ zeroes by setting $w'_t = 1$ iff $j_t \in T$ and $w'_t = 0$ iff $j_t \notin T$. Let w be the word obtained by erasing the last symbol of w'.

Case 2. $x' \neq n - (s_k - s_i)$ for all $i \leqslant k$. Compute $z = C_n^{s_k - n}(x') \in [n] - S$. Since $x' \in S'$, we must have $z \in T$. Let the elements of $[n] - S - \{z\}$ be $j_1 < j_2 < \cdots < j_{n-k-1}$. Define a word w consisting of $u - 1$ ones and $n - k - u$ zeroes by setting $w_t = 1$ iff $j_t \in T$ and $w_t = 0$ iff $j_t \notin T$.

We have now reached Eq. (6). To continue the construction, we again must consider cases depending on whether the object being considered is counted by Eq. (7) or Eq. (8).

Case 1. The word w has exactly $u - 1$ ones. We must have $u > 0$ here; setting $v = u - 1$, we have $0 \leqslant v < n - k - 1$. Use the word w to construct a sequence $(a_1, \ldots, a_{n-k-1})$ as follows. Suppose the zeroes in w occur in positions $m_1 < \cdots < m_{n-k-1-v}$. If $w_i = 1$, then define $a_i = 0$. If $w_i = 0$, we must have $i = m_j$ for some j. If $i = m_j$ and $j < n-k-1-v$, then define $a_i = a'_j + 1$ (note that $1 \leqslant j \leqslant n-k-1-u$ here, so that a'_j exists). If $i = m_j$ and $j = n-k-1-v$, then define $a_i = (n-1-x')+1$. (We add one when $w_i = 0$ because the corresponding term in the manipulation of T_1 is $q[n-1]_q$, not $[n-1]_q$.) Finally, define $x = s_k$ and $\psi_{n,S}(f) = (a_1, \dots, a_{n-k-1}; x) \in \mathcal{Q}_{n,S}$.

Before proceeding to Case T2, let us deal with the special case $u = n-k$, which occurs for a unique object $f_0 \in \mathcal{P}_{n,S}$. This object corresponds to the $v = n-k-1$ summand in the manipulation of (7). Accordingly, we must set $\psi_{n,S}(f_0) = (0, 0, \dots, 0; s_k)$.

Case 2. The word w has exactly u ones. Then we must have been in Case 1 earlier in the construction, and so $C_n^{s_k - n}(x') = s_i \in S$ for some $i < k$. Use the word w to construct a sequence $(a_1, \dots, a_{n-k-1})$ as follows. Suppose the zeroes in w occur in positions $m_1 < \cdots < m_{n-k-1-u}$. If $w_i = 1$, then define $a_i = 0$. If $w_i = 0$, we must have $i = m_j$ for some j. If $i = m_j$, then define $a_i = a'_j + 1$ (note that $1 \leqslant j \leqslant n-k-1-u$ here, so that a'_j exists). Finally, define $x = s_i$ and $\psi_{n,S}(f) = (a_1, \dots, a_{n-k-1}; x) \in \mathcal{Q}_{n,S}$. This completes the definition of $\psi_{n,S}$.

The reader, aided by the following remarks, should have no trouble constructing the inverse of $\psi_{n,S}$. When computing $\psi_{n,S}^{-1}((a_1, \dots, a_{n-k-1}; x))$, we must first choose whether to invert Case T1 or Case T2. We choose Case T1 if $x = s_k$ is the maximum element of S; we choose Case T2 if $x = s_i$ is not the maximum element of S.

Continuing the inversion process, we must decide between inverting Case 1 or Case 2. If we chose Case T2 a moment ago, we must proceed by inverting Case 1, with $x' = n - (s_k - s_i)$. If we chose Case T1 instead, we can recover x' by looking at the last nonzero a_i, and it is then possible to decide whether to invert Case 1 or Case 2 next by using the definition of these cases. In Case 1, the erased symbol in w' can be deduced based on whether we inverted Case T1 or Case T2 earlier.

5. Conclusion

The results in this paper may have some bearing on two open problems regarding the combinatorial Hilbert series. First, there is the conjectured joint symmetry $CH_n(q,t) = CH_n(t,q)$. One way to prove joint symmetry would be to give a suitable reinterpretation of the recursion in (4), in which q represented pmaj and t represented area, or in which q represented area and t represented dinv. However, the definition of $P_{n,S}$ would also require modification, since it is false in general that $P_{n,S}(q,t) = P_{n,S}(t,q)$. One way to pursue this problem is to look for set-valued statistics on $\mathcal{P}_n$ that have the same distribution as the statistic $S = f^{-1}(1)$.

The second open problem is the fundamental conjecture that $CH_n(q,t) = H_n(q,t)$. Given the recursion (4), a natural avenue of investigation is to seek a decomposition of DH_n into subspaces $DH_{n,S}$ such that the Hilbert series of $DH_{n,S}$ is $P_{n,S}(q,t)$. One would then try to show that these smaller Hilbert series satisfied the recursion and initial conditions in (4). The operators C_n, D_n, and E_n defined on $\mathcal{P}_n$ should correspond to certain linear operators on DH_n. Although we have no specific operators in mind, it seems that C_n and D_n may be related to the partial differentiation operators relative to the x-variables and y-variables, while E_n may be related to the linear operator sending x_n and y_n to zero.

Another possible approach is to find expressions for $P_{n,S}(q,t)$ in terms of Bergeron and Garsia's nabla operator. If $S=[k]$, computer experiments suggest that

$$P_{n,[k]}(q,t) = q^{k(k-1)/2} t^{n-k} \nabla\big(e_{n-k}\big[X\big(1+q+q^2+\cdots+q^{k-1}\big)\big]\big)\big|_{s_\lambda \to f_\lambda},$$

where we use the notation in [3]. J. Haglund has pointed out that this formula can be deduced from the shuffle conjecture of Haglund, Haiman, Remmel, Ulyanov, and the present author [4]. There are similar formulas for $P_{n,C_n^i([k])}(q,t)$ for $0 \leqslant i \leqslant n-k$. At present, we have no such conjectured formulas for other values of the subset S.

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