ELSEVIER

Discrete Mathematics 204 (1999) 163–166

DISCRETE MATHEMATICS

www.elsevier.com/locate/disc

An involution on Dyck paths and its consequences

Emeric Deutsch*

Department of Mathematics, Polytechnic University, 333 Jay Street, Brooklyn, NY 11201, USA

Received 31 March 1998; revised 10 April 1998; accepted 3 August 1998

Dedicated to H.W. Gould, on the occasion of his 70th birthday

Abstract

An involution is introduced in the set of all Dyck paths of semilength n from which one re-obtains easily the equidistribution of the parameters 'number of valleys' and 'number of doublerises' and also the equidistribution of the parameters 'height of the first peak' and 'number of returns'.

AMS classification: 05A15

Keywords: Involutions on Dyck paths; Returns of Dyck paths; Height of the first peak of Dyck paths; Valleys of Dyck paths; Doublerises of Dyck paths

1. A *Dyck path* is a path in the first quadrant, which begins at the origin, ends at $(2n,0)$ and consists of steps $(1,1)$ (called rises) and $(1,-1)$ (called falls). We will refer to n as the *semilength* of the path.

We denote by $\boldsymbol{D}_n$ the set of all Dyck paths of semilength n. We denote by $\boldsymbol{D}_0$ the set consisting only of the empty path, denoted by ε. It is well-known that the number of all Dyck paths of semilength n is the nth Catalan number

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

We can encode each rise by a letter u (for up) and each fall by d (for down), obtaining the encoding of a Dyck path by a so-called *Dyck word.*

Numerous nonnegative integer-valued parameters have been defined for a Dyck path. For example, some parameters are: the number of peaks (i.e. ud's), the number of valleys (i.e. du's), the number of doublerises (i.e. uu's; note that the sequence $uuuu$, for example, has three doublerises), the height of the first peak (i.e. the length of the initial maximal sequence of u's), the number of returns (i.e. d's landing on the

* E-mail: deutsch@magnus.poly.edu.

0012-365X/99/$ – see front matter
PII: S0012-365X(98)00370-7

horizontal axis). We will use the symbols DBR, VAL, HFP, and RET for the number of doublerises, the number of valleys, the height of the first peak, and the number of returns, respectively.

We say that two parameters p_1 and p_2 are equidistributed if for each nonnegative integers n and k the sets

$$\{\pi \in \boldsymbol{D}_n\colon p_1(\pi) = k\} \quad \text{and} \quad \{\pi \in \boldsymbol{D}_n\colon p_2(\pi) = k\}$$

are equinumerous.

The equidistribution of the parameters VAL and DBR and that of HFP and RET has been proved in several ways ([1,3,5–8,11–13] and [1–3,6,13], respectively). The actual distributions of these parameters are given by the following statements:

(i) The number of Dyck paths of semilength n and having k returns (or having height of the first peak equal to k) is equal to $k\binom{2n-k}{n}/(2n-k)$ (see, for example, [3]).

(ii) The number of Dyck paths of semilength n and having k valleys (or k doublerises) is equal to the Narayana number $(1/n)\binom{n}{k}\binom{n}{k+1}$ (see, for example, [12]).

In this note we introduce a relatively simple involution $(\ )'\colon \boldsymbol{D}_n \to \boldsymbol{D}_n$ with the aid of which the above equidistributions will follow at once. The fixed points of the involution can be easily described. As an immediate consequence we will obtain a known statement regarding the parity of the Catalan numbers.

2. We define the involution $(\ )'$ inductively. First, we set $\varepsilon' = \varepsilon$. For a nonempty Dyck path δ we write

$$\delta = u\alpha d\beta,$$

where α and β are, possibly empty, Dyck paths. Clearly, this can be done uniquely since the d step pointed out in the above factorization, called *first return decomposition*, is the first return step of the path. We define (see Fig. 1)

$$\delta' = u\beta' d\alpha'.$$

Now, it follows that

$$(ud)' = (u\varepsilon d\varepsilon)' = u\varepsilon' d\varepsilon' = u\varepsilon d\varepsilon = ud,$$

$$(uudd)' = (u(ud)d\varepsilon)' = u\varepsilon' d(ud)' = udud,$$

$$(udud)' = (u\varepsilon d(ud))' = u(ud)' d\varepsilon' = uudd,$$

and so on.

The proof that the mapping $(\ )'$ is an involution uses the following induction: $\varepsilon'' = \varepsilon$ and for an arbitrary nonempty Dyck path $u\alpha d\beta$ we have

$$(u\alpha d\beta)'' = (u\beta' d\alpha')' = u\alpha'' d\beta'' = u\alpha d\beta$$

since both α and β are definitely shorter than $u\alpha d\beta$.

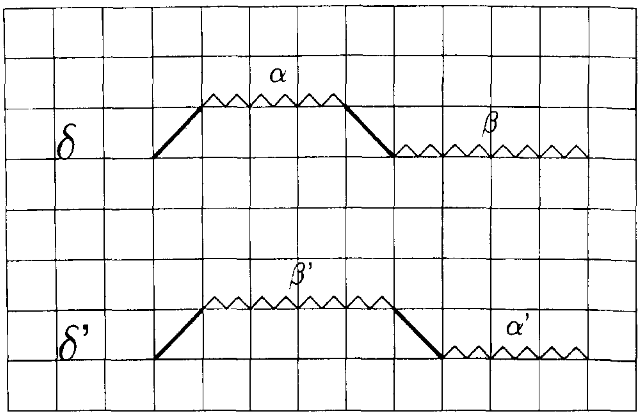

Fig. 1.

3. Now, we intend to show that for each $\delta \in \boldsymbol{D}_n$ we have

$$\mathrm{DBR}(\delta) = \mathrm{VAL}(\delta') \quad \text{and} \quad \mathrm{HFP}(\delta) = \mathrm{RET}(\delta'). \tag{1}$$

Obviously, these equalities hold for $\delta = \varepsilon$. Assume that they hold for all Dyck paths of semilength less than n. Let $\delta \in \boldsymbol{D}_n$, $n \geqslant 1$, and let $\delta = u\alpha d\beta$ be the first return decomposition of δ. From Fig. 1 we see that

$$\mathrm{DBR}(\delta) = \begin{cases} \mathrm{DBR}(\beta) & \text{if } \alpha = \varepsilon, \\ 1 + \mathrm{DBR}(\alpha) + \mathrm{DBR}(\beta) & \text{if } \alpha \neq \varepsilon, \end{cases}$$

$$\mathrm{VAL}(\delta') = \begin{cases} \mathrm{VAL}(\beta') & \text{if} \alpha = \varepsilon, \\ \mathrm{VAL}(\beta') + 1 + \mathrm{VAL}(\alpha') & \text{if} \alpha \neq \varepsilon, \end{cases}$$

$$\mathrm{HFP}(\delta) \ = 1 + \mathrm{HFP}(\alpha),$$

$$\mathrm{RET}(\delta') \ = 1 + \mathrm{RET}(\alpha').$$

Now, making use of the induction hypothesis, equalities (1) follow at once.

4. From the definition of the involution $(\ )'$ it follows that $u\alpha d\beta$ is a fixed point of the involution if and only if $\beta = \alpha'$. Consequently, for n even $(n > 0)$ the involution $(\ )'$ has no fixed points, while for n odd it has $C_{(n-1)/2}$ fixed points, namely $u\alpha d\alpha'$ $(\alpha \in \boldsymbol{D}_{(n-1)/2})$. From here in turn we obtain the well-known result [4,9,10] that c_n is odd if and only if $n = 2^k - 1$, $k = 1, 2, 3, \ldots$.

5. An even simpler involution can be defined by $\varepsilon \to \varepsilon$, $u\alpha d\beta \to u\beta d\alpha$. Making use of this involution, one can derive a connection between the following two parameters defined on Dyck paths: number of peaks before the first return and number of peaks after the first return (for a different approach to this see [3]).

Acknowledgements

The author expresses his appreciation to R.A. Sulanke for his helpful comments. Also, many thanks to an anonymous referee for improving the presentation by suggesting several changes.

References

[1] S. Benchekroun, P. Moszkowski, A new bijection between ordered trees and legal bracketings, European J. Combin. 17 (1996) 605–611.
[2] E. Deutsch, A bijection on Dyck paths and its consequences, Discrete Math. 179 (1998) 253–256.
[3] E. Deutsch, Dyck path enumeration, Discrete Math. 204 (this Vol.) (1999) 167–202.
[4] Ö. Eğecioğlu, The parity of the Catalan numbers via lattice paths, Fibonacci Quart. 21 (1983) 65–66.
[5] J.-M. Fedou, Grammaire et q-énumérations de polyominos, Thèse, Univ. Bordeaux I, 1989.
[6] G. Kreweras, Sur les éventails de segments, Cahiers du B.U.R.O., No. 15 (1970) 3–41.
[7] J.C. Lalanne, Une involution sur les chemins de Dyck, European J. Combin. 13 (1992) 477–487.
[8] J.C. Lalanne, Sur une involution sur les chemins de Dyck, Theoret. Comput. Sci. 117 (1993) 203–215.
[9] J.G. Rey, Problem 2070, Crux Math. 21 (1995) 236; 22 (1996) 279–281.
[10] R. Simion, D. Ullman, On the structure of the lattice of noncrossing partitions, Discrete Math. 98 (1991) 193–206.
[11] R.A. Sulanke, Catalan path statistics having the Narayana distribution, Discrete Math. 180 (1998) 369–389.
[12] R.A. Sulanke, Constraint-sensitive Catalan path statistics having the Narayana distribution, Discrete Math. 204 (this Vol.) (1999) 397–414.
[13] J. Vaillé, Une bijection explicative de plusieurs propriétés remarquables des ponts, European J. Combin. 18 (1997) 117–124.