Problem: Prove, using mathematical induction, that $x+y$ divides x^n+y^n for all integers x and y is n is odd.

Solution

Induction base: Since any number is divisible by itself the case for $n=1$ is trivial, i.e $x+y$ divides $x+y$.

Induction hypothesis: If x^n+y^n is divisible by $x+y$ for all odd n less than $n+2$ then $x^{n+2}+y^{n+2}$ is divisible by $x+y$. To show this observe that

$$\begin{aligned}(x+y)^{n+2} &= \sum_{k=0}^{n+2}\binom{n+2}{k}x^{(n+2)-k}y^k \\ &= x^{n+2}+y^{n+2}+\sum_{k=1}^{n+1}\binom{n+2}{k}x^{(n+2)-k}y^k \\ &= x^{n+2}+y^{n+2}+\sum_{k=1}^{(n+1)/2}\binom{n+2}{k}x^{(n+2)-k}y^k+\sum_{k=1+(n+1)/2}^{n+1}x^{(n+2)-k}y^k\end{aligned}$$

Let $r=(n+2)-k$. Then the equation becomes

$$(x+y)^{n+2}=x^{n+2}+y^{n+2}+\sum_{k=1}^{(n+1)/2}\binom{n+2}{k}x^{(n+2)-k}y^k+\sum_{r=1}^{(n+1)/2}\binom{n+2}{(n+2)-r}x^ry^{(n+2)-r}$$

Since $\binom{n}{n-k}=\binom{n}{k}$,

$$(x+y)^{n+2}=x^{n+2}+y^{n+2}+\sum_{k=1}^{(n+1)/2}\binom{n+2}{k}x^{(n+2)-k}y^k+\sum_{r=1}^{(n+1)/2}\binom{n+2}{r}x^ry^{(n+2)-r}$$

Since r attains every value k attains, you can substitute r by k. after merging the two sums you get,

$$x^{n+2}+y^{n+2}=(x+y)^{n+2}-\sum_{k=1}^{(n+1)/2}\binom{n+2}{k}(xy)^k(x^{(n+2)-(2k)}+y^{(n+2)-(2k)})$$

. By the induction hypothesis $x^{(n+2)-(2k)}+y^{(n+2)-(2k)}$ is divisible by $x+y$ since $n+2-2k$ is an odd number less than $n+2$ for $1\le k\le(n+1)/2$. Since $(x+y)^{n+2}$ is obviously divisible by $x+y$, $x^{n+2}+y^{n+2}$ is divisble by $x+y$. Mic Drop!!!