The next four exercises develop the concept of *direct limits* and the "dual" notion of *inverse limits*. In these exercises I is a nonempty index set with a partial order $\leq$ (cf. Appendix I). For each $i \in I$, let A_i be an additive abelian group. In Exercise 8 assume also that I is a *directed set*: for every $i, j \in I$ there is some $k \in I$ with $i \leq k$ and $j \leq k$.

8. Suppose for every pair of indices i, j with $i \leq j$ there is a map $\rho_{ij} : A_i \to A_j$ such that the following hold:

(i) $\rho_{ik} = \rho_{jk} \circ \rho_{ij}$ whenever $i \leq j \leq k$, and

(ii) $\rho_{ii} = 1$ for all $i \in I$.

Let B be the disjoint union of all the A_i. Define a relation $\sim$ on B by

$$a \sim b \quad \text{if and only if there exists } k \text{ with } i, j \leq k \text{ and } \rho_{ik}(a) = \rho_{jk}(b),$$

for $a \in A_i$ and $b \in A_j$.

(a) Show that $\sim$ is an equivalence relation on B. (The set of equivalence classes is called the *direct* or *inductive limit* of the directed system $\{A_i\}$, and is denoted $\varinjlim A_i$. In the remaining parts of this exercise let $A = \varinjlim A_i$.)

(b) Let $\bar{x}$ denote the class of x in A and define $\rho_i : A_i \to A$ by $\rho_i(a) = \bar{a}$. Show that if each ρ_{ij} is injective, then so is ρ_i for all i (so we may then identify each A_i as a subset of A).

(c) Assume all ρ_{ij} are group homomorphisms. For $a \in A_i$, $b \in A_j$ show that the operation

$$\bar{a} + \bar{b} = \overline{\rho_{ik}(a) + \rho_{jk}(b)}$$

where k is any index with $i, j \leq k$, is well defined and makes A into an abelian group. Deduce that the maps ρ_i in (b) are group homomorphisms from A_i to A.

(d) Show that if all A_i are commutative rings with 1 and all ρ_{ij} are ring homomorphisms that send 1 to 1, then A may likewise be given the structure of a commutative ring with 1 such that all ρ_i are ring homomorphisms.

(e) Under the hypotheses in (c) prove that the direct limit has the following *universal property*: if C is any abelian group such that for each $i \in I$ there is a homomorphism $\varphi_i : A_i \to C$ with $\varphi_i = \varphi_j \circ \rho_{ij}$ whenever $i \leq j$, then there is a unique homomorphism $\varphi : A \to C$ such that $\varphi \circ \rho_i = \varphi_i$ for all i.

Sol.

(a) Let $x \in B$. Then there is s such that $x \in A_s$. Choosing $i = j = k = s$, we see that $\sim$ is *relfexive*. By symmetry of $=$, the symmetry of $\sim$ follows directly. Let $a \sim b$ and $b \sim c$. Let $\rho_{ik}(a) = \rho_{jk}(b)$ and let $\rho_{jt}(b) = \rho_{st}(c)$. WLOG, let $k \leq t$. Then $\rho_{it}(a) = \rho_{kt} \circ \rho_{ik}(a) = \rho_{kt} \circ \rho_{jk}(b) = \rho_{jt}(b) = \rho_{st}(c)$. Thus $\sim$ is transitive.

(b) Let $a, b \in A_i$ with $a \neq b$. By injectivity, $\rho_{ik}(b) \neq \rho_{ik}(b)$ for all $k \geq i, j$. Thus, $a \nsim b$.

(c) For the addition to be well-defined, it should have the same value regardless of the choice of a and b as long as they are picked for their respective equivalence classes. Let $x \sim a$ and $y \sim b$. Let $\rho_{it}(a) = \rho_{st}(x)$ and $\rho_{je}(b) = \rho_{de}(y)$. WLOG, let $t \geq e$. If $k \geq t$, we are done. Otherwise, $\rho_{kt}(\rho_{ik}(a) + \rho_{jk}(b)) = \rho_{it}(a) + \rho_{jt}(b) = \rho_{st}(x) + \rho_{dt}(y) = \rho_{et}(\rho_{se}(x) + \rho_{de}(y))$. Thus $+$ is well-defined.

A is then an abelian group because if $\bar{a}, \bar{b} \in A$, then $\bar{a} - \bar{b} \in A$ and $\bar{0} \supseteq \{0_{A_i}\}_{i \in I} \in A$. It follows that ρ_i are group homomorphisms because $\rho_i(a+b) = \overline{a+b} = \bar{a} + \bar{b}$ (taking $k = i$) $= \rho_i(a) + \rho_i(b)$.

(d) A is still an additive abelian group but now commutitive multiplicative structure is built upon it. The multiplication given by

$$\bar{a} \cdot \bar{b} = \overline{\rho_{ik}(a) \cdot \rho_{jk}(b)}$$

for all $k \geq i, j$ is well defined and the proof is similar to the one given in (c) as ρ_{ij} are ring homomorphisms. Furthermore, $\bar{a} \cdot (\bar{b} + \bar{c}) = \overline{\rho_{ik}(a) \cdot (\rho_{mk}(b) + \rho_{nk}(c))}$ for $k \geq i, m, n$. The distributive property of $(\cdot)$ in A follows from the distributive property $(\cdot)$ in A_i once we note that $a \sim \rho_{ik}(a)$ for all $k \geq i$.

(e) We define $\varphi : A \to C$ as follows,

$$\varphi(\bar{x}) = \varphi_i(x), \quad x \in A_i.$$

We first show that this definition is independent of the choice of the representative x. Let $x \sim y$, i.e., $\rho_{ik}(x) = \rho_{jk}(y)$.

$$\begin{aligned} \varphi(\bar{x}) &= \varphi_i(x) \\ &= \varphi_k(\rho_{ik}(x)) \\ &= \varphi_k(\rho_{jk}(y)) \\ &= \varphi_j(y). \end{aligned}$$

Thus, φ is well defined. Since A is a disjoint union of A_i modulo $\sim$, φ is defined everywhere in A and uniquness follows from definition.