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Generalized Dyck tilings

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ABSTRACT

Recently, Kenyon and Wilson introduced Dyck tilings, which are certain tilings of the region between two Dyck paths. The enumeration of Dyck tilings is related with hook formulas for forests and the combinatorics of Hermite polynomials. The first goal of this work is to give an alternative point of view on Dyck tilings by making use of the weak order and the Bruhat order on permutations. Then we introduce two natural generalizations: k-Dyck tilings and symmetric Dyck tilings. We are led to consider Stirling permutations, and define an analog of the Bruhat order on them. We show that certain families of k-Dyck tilings are in a bijection with intervals in this order. We also enumerate symmetric Dyck tilings.

1. Introduction

Dyck tilings were recently introduced by Kenyon and Wilson [11] in the study of probabilities of statistical physics model called "the double-dimer model", and independently by Shigechi and Zinn-Justin [16] in the study of Kazhdan–Lusztig polynomials. Dyck tilings also have connection with fully packed loop configurations [7] and representations of the symmetric group [6].

The main purpose of this paper is to give a new point of view on Dyck tilings in terms of the weak order and the Bruhat order on permutations and to consider two natural generalizations of Dyck tilings.

A *Dyck path of length* $2n$ is a lattice path consisting of up steps $(0, 1)$ and right steps $(1, 0)$ from the origin $(0, 0)$ to the point (n, n) which never goes strictly below the line $y = x$. We will also consider

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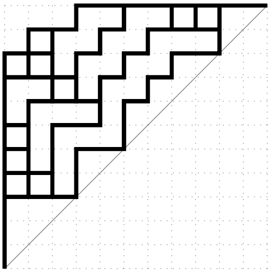


Fig. 1. An example of Dyck tiling.

a Dyck path λ of length $2n$ as the Young diagram whose boundary is determined by λ and the lines $x = 0$ and $y = n$. We note that in the literature a Dyck path, also called a ballot sequence, is often defined as a lattice path from $(0, 0)$ to $(2n, 0)$ consisting of steps $(1, 1)$ and $(1, -1)$ that stays on or above the x-axis. For our purpose it is more convenient to consider Dyck paths from $(0, 0)$ to (n, n).

Suppose that λ and μ are Dyck paths of length $2n$ with μ weakly above λ. A *Dyck tile* is a ribbon such that the centers of the cells form a Dyck path. A *Dyck tiling* of λ/μ is a tiling D of the region between λ and μ with Dyck tiles satisfying the *cover-inclusive property*: if η is a tile of D, then the translation of η by $(1, -1)$ is either completely below λ or contained in another tile of D. See Fig. 1 for an example. We denote by $\mathcal{D}(\lambda/\mu)$ the set of Dyck tilings of λ/μ. For $D \in \mathcal{D}(\lambda/\mu)$ we call λ and μ the *lower path* and the *upper path* of D, respectively. Then the set of Dyck tilings with fixed lower path λ is denoted by $\mathcal{D}(\lambda/*)$ and similarly, the set of Dyck tilings with fixed upper path μ is denoted by $\mathcal{D}(*/\mu)$.

For $D \in \mathcal{D}(\lambda/\mu)$ we have two natural statistics $\mathrm{area}(D)$ and $\mathrm{tiles}(D)$, where $\mathrm{area}(D)$ is the area of the region λ/μ and $\mathrm{tiles}(D)$ is the number of tiles of D. We also consider the statistic $\mathrm{art}(D) = (\mathrm{area}(D) + \mathrm{tiles}(D))/2$.

Kenyon and Wilson [12] conjectured the following two formulas:

$$\sum_{D\in\mathcal{D}(\lambda/*)} q^{\mathrm{art}(D)} = \frac{[n]_q!}{\prod_{x\in F}[h_x]_q}, \tag{1}$$

$$\sum_{D\in\mathcal{D}(*/\mu)} q^{\mathrm{tiles}(D)} = \prod_{u\in \mathrm{UP}(\mu)} [\mathrm{ht}(u)]_q, \tag{2}$$

where F is the plane forest corresponding to λ and, for a vertex $x \in F$, h_x denotes the hook length of x. See Fig. 4 for the correspondence between the plane forests and the Dyck paths. The set of up steps of a Dyck path μ is denoted by $\mathrm{UP}(\mu)$ and for $u \in \mathrm{UP}(\mu)$, $\mathrm{ht}(u)$ is the number of squares between u and the line $y = x$, plus 1. Here we use the standard notation for q-integers and q-factorials: $[n]_q = 1 + q + q^2 + \cdots + q^{n-1}$ and $[n]_q! = [1]_q[2]_q \ldots [n]_q$.

Formula (1) was first proved by Kim [13] non-bijectively and then by Kim, Mészáros, Panova, and Wilson [14] bijectively. In [14], they find a bijection between $\mathcal{D}(\lambda/*)$ and increasing labelings of the plane forest corresponding to λ. Kim [13] and Konvalinka independently proved (2) by finding a bijection between $\mathcal{D}(*/\mu)$ and certain labelings of μ called Hermite histories.

Björner and Wachs showed that the right hand side of (1) is the length generating function for permutations in an interval in the weak order, see [3, Theorem 6.8] and [2, Theorem 6.1].

In this paper we first show that, using the results of Björner and Wachs [2,3], (1) can be interpreted as the length generating function for permutations $\pi \geq_L \sigma$ in the left weak order for a 312-avoiding permutation σ. We also show that (2) is the length generating function for permutations $\pi \geq \sigma$ in the Bruhat order for a 132-avoiding permutation σ. We then consider two natural generalizations of Dyck tilings.

The first generalization are k-Dyck tilings, where we use k-Dyck paths and k-Dyck tiles with the same cover-inclusive property. We generalize (2) by finding a bijection between k-Dyck tilings and k-Hermite histories. We consider k-Stirling permutations introduced by Gessel and Stanley [8]. We define a k-Bruhat order on k-Stirling permutations and show that k-Dyck tilings with fixed upper path are in a bijection with an interval in this order. We also consider a connection with k-regular noncrossing partitions. We generalize (1) to k-Dyck tilings with fixed lower path λ when λ is a zigzag path.

The second generalization are symmetric Dyck tilings, which are invariant under the reflection across a line. We show that symmetric Dyck tilings are in a bijection with symmetric matchings and "marked" increasing labelings of symmetric forests.

This is the full version of an extended abstract published in [10].

2. Dyck tilings as intervals of the Bruhat order and weak order

As we have seen in the introduction, the two natural points of view for enumerating Dyck tilings are when we fix the upper path, and when we fix the lower path. We show in this section that both can be interpreted in terms of permutations, using respectively the Bruhat order and the (left) weak order. Usually, these orders are defined using reflections. For our purpose, we will take the following definitions. See [1] for more details of the Bruhat and weak orders.

We denote by $\mathfrak{S}_n$ the set of permutations of $[n] := \{1, 2, \dots, n\}$. We will consider a permutation $\pi \in \mathfrak{S}_n$ as a word $\pi = \pi_1\pi_2 \dots \pi_n$ and also as a bijection $\pi : [n] \to [n]$ by letting $\pi(i) = \pi_i$.

The *Bruhat order* on $\mathfrak{S}_n$ is given by the cover relation $\sigma \lessdot \pi$ if π is obtained from σ by exchanging the two numbers in positions i and j for some integers $i < j$ such that $\sigma_i < \sigma_j$, and for all $i < \ell < j$, we have either $\sigma_\ell < \sigma_i$ or $\sigma_\ell > \sigma_j$. The *weak order* on $\mathfrak{S}_n$ is given by the cover relation $\sigma \lessdot \pi$ if π is obtained from σ by exchanging the two numbers in positions i and $i + 1$ for some integer i with $\sigma_i < \sigma_{i+1}$.

An *inversion* of $\pi \in \mathfrak{S}_n$ is a pair (i, j) of integers $1 \le i < j \le n$ such that $\pi(i) > \pi(j)$. The number of inversions of π is denoted by $\mathrm{inv}(\pi)$. For permutations $\tau = \tau_1 \dots \tau_k \in \mathfrak{S}_k$ and $\pi = \pi_1 \dots \pi_n \in \mathfrak{S}_n$, we say that π is *τ-avoiding* if there are no integers $1 \le i_1 < i_2 < \cdots < i_k \le n$ such that $\pi_{i_1}, \pi_{i_2}, \dots, \pi_{i_k}$ are order-isomorphic to $\tau_1, \tau_2, \dots, \tau_n$, i.e., $\pi_{i_r} < \pi_{i_s}$ if and only if $\tau_r < \tau_s$ for all $r, s \in [k]$. The set of τ-avoiding permutations in $\mathfrak{S}_n$ is denoted by $\mathfrak{S}_n(\tau)$. For example if $\tau = 132$, then $\sigma \in \mathfrak{S}_n(132)$ if there are no integers $i < j < k$ such that $\sigma_i < \sigma_k < \sigma_j$.

We represent a permutation by a diagram with the "matrix convention", i.e. there is a dot at the intersection of the ith line from the top and the jth column from the left if $\sigma(j) = i$. In these diagrams, we can represent the inversion of a permutation by putting a cross $\times$ in each cell having a dot to its right and a dot below. See the left part of Fig. 2. We need a bijection α between 132-avoiding permutations and Dyck paths. It is easy to see that the inversions of a 132-avoiding permutation are top left justified in its diagram. So we can define a path from the bottom left corner to the top right corner by following the boundary of the region filled with $\times$. This turns out to be a Dyck path and this defines a bijection (this is an easy exercise). The bijection is illustrated in Fig. 2.

Definition 2.1. Let μ be a Dyck path.

- A *Hermite history* of shape μ is a labeling of the up steps of μ with integers such that an up step of height h has a label in $\{0, 1, 2, \dots, h - 1\}$. Here, the *height* of an up step is the number of squares between the up step and the line $y = x$, plus 1.
- A *matching* of shape μ is a partition of $[n]$ in 2-element blocks such that $i \in [n]$ is the minimum of a block if and only if the ith step of μ is an up step. A *crossing* of the matching is a pair of blocks $\{i, j\}$ and $\{k, \ell\}$ such that $i < j < k < \ell$.

The following is well known (see for example [13]).

Proposition 2.2. *There is a bijection between Hermite histories of shape μ and matchings of shape μ. It is such that the sum of weights in the Hermite history is the number of crossings in the matching.*

Theorem 2.3. *Let $\sigma \in \mathfrak{S}_n(132)$ and $\mu = \alpha(\sigma)$, then*

$$\sum_{D \in \mathcal{D}(*/\mu)} q^{\mathrm{tiles}(D)} = \sum_{\pi \ge \sigma} q^{\mathrm{inv}(\pi) - \mathrm{inv}(\sigma)},$$

where $\pi \ge \sigma$ is the Bruhat order on $\mathfrak{S}_n$.

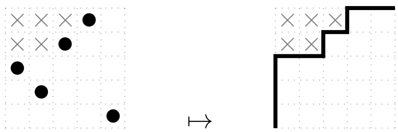


Fig. 2. The bijection from 132-avoiding permutations to Dyck paths. The crosses represent inversions of the permutation 34 215.

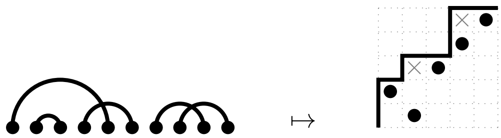


Fig. 3. The bijection proving Theorem 2.3.

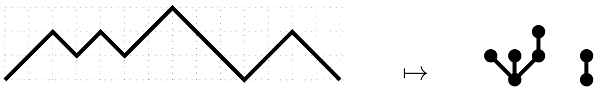


Fig. 4. The bijection from Dyck paths to plane forests. Here the Dyck path is rotated clockwise by an angle of 45°.

Proof. From [14], we know that Dyck tilings with a fixed upper path μ are in a bijection with Hermite histories with shape μ, and the bijection sends the number of tiles to the sum of labels in the Hermite history. Consequently, Dyck tilings with a fixed upper path μ are in a bijection with matchings of shape μ, and the bijection sends the number of tiles to the number of crossings in the matching.

To show the proposition, we give a bijection between matchings with the same shape μ, and permutations above σ in the Bruhat order. It is illustrated in Fig. 3. The idea is to put dots in the grid as follows: if there is a pair (i, j) in the matching (with $i < j$), the ith step in the Dyck path is vertical and the jth step is horizontal, so row to the right of the ith step intersects the column below the jth step in some cell, and we put a dot in this cell. Then we can read these dots as a permutation (with the matrix convention). In the example in Fig. 3 we get 45 321. The crossing in the matchings correspond to inversions of the permutations that lie below the Dyck path.

The next step is the following: we can prove the set of permutations where all dots are below the Dyck path μ is precisely the Bruhat interval $\{\pi : \pi \geq \sigma\}$. First, by construction all the dots of σ are below μ. Suppose all the dots of π are below μ and $\pi' \gtrdot \pi$ in the Bruhat order. It means that π' is obtained from π by transforming a pair of dots arranged as into a pair of dots arranged as , and the new dots cannot be above μ. So the interval $\{\pi : \pi \geq \sigma\}$ is included in the set of permutations where all dots are below the Dyck path μ. Conversely, let π be a permutation where all dots are below the Dyck path μ. If $\pi \neq \sigma$, consider an inversion of π which is as low to the right as possible. This inversion is in a pattern and the cross is below μ. By transforming this pattern into , we obtain π' with $\pi' \lessdot \pi$ and has still the property that all dots are below μ. By repeating this operation we must arrive at a permutation whose inversions are exactly the cells above μ, i.e. σ. So π is in the interval $\{\pi : \pi \geq \sigma\}$. $\square$

Let us turn to the case of a fixed lower path in Dyck tilings. We can identify a Dyck path λ with a plane forest: it is obtained from λ by "squeezing" the path as shown in Fig. 4, a pair of facing up step and right step corresponds to a vertex.

Let F be a plane forest with n vertices. An *increasing labeling* of F is a way of labeling the vertices of F with $1, 2, \ldots, n$ so that the label of a vertex is greater than the label of its parent. Let L be an increasing labeling of F. An *inversion* of L is a pair (j, i) of integers $j > i$ such that j is not a descendant of i and j appears to the left of i. For example, if L is the increasing labeling in Fig. 5, then L has many inversions including $(13, 7)$, $(8, 7)$, $(2, 4)$.

For a plane tree T with labeling L, we denote by $\mathrm{post}(L)$ (respectively, $\mathrm{pre}(L)$) the word obtained by reading L from left to right using post-order (respectively, pre-order). More precisely, if the

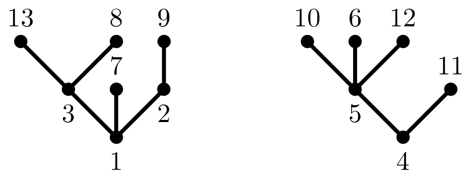


Fig. 5. An increasing labeling L of a forest of size 13. We have $\mathrm{post}(L) = 13, 8, 3, 7, 9, 2, 1, 10, 6, 12, 5, 11, 4$ and $\mathrm{pre}(L) = 1, 3, 13, 8, 7, 2, 9, 4, 5, 10, 6, 12, 11$.

root of T has k children, then $\mathrm{post}(L)$ and $\mathrm{pre}(L)$ are defined recursively as follows: $\mathrm{post}(L) = \mathrm{post}(T_1)\mathrm{post}(T_2)\cdots\mathrm{post}(T_k)r$ and $\mathrm{pre}(L) = r\ \mathrm{pre}(T_1)\mathrm{pre}(T_2)\cdots\mathrm{pre}(T_k)$, where r is the label of the root of T and T_i is the subtree rooted at the ith child of r from the left.

Suppose that a forest F consists of k trees $T_1, T_2, \ldots, T_k$, where T_i is the ith tree from the left. Let L_i be the increasing labeling of T_i obtained by restricting L of F to T_i. Then we define $\mathrm{post}(L) = \mathrm{post}(L_1)\mathrm{post}(L_2)\cdots\mathrm{post}(L_k)$ and $\mathrm{pre}(L) = \mathrm{pre}(L_1)\mathrm{pre}(L_2)\cdots\mathrm{pre}(L_k)$. See Fig. 5.

It is easy to see that there is a unique increasing labeling L_0 of F such that $\mathrm{inv}(\mathrm{post}(L_0))$ is minimal. More specifically, L_0 is the increasing labeling of F with $\mathrm{inv}(L_0) = 0$, or equivalently, L_0 is the increasing labeling of F such that $\mathrm{pre}(L_0) = id_n$, the identity permutation of $[n]$. It is not difficult to see that the permutation $\pi_0 = \mathrm{post}(L_0)$ is 312-avoiding.

Theorem 2.4. *Let λ be a Dyck path with corresponding plane forest F. Let L_0 be the increasing labeling of F such that $\mathrm{pre}(L_0) = id_n$, and $\pi_0 = \mathrm{post}(L_0)$. Then*

$$\sum_{D\in\mathcal{D}(\lambda/*)} q^{\mathrm{art}(D)} = \sum_{\pi \geq_L \pi_0} q^{\mathrm{inv}(\pi)-\mathrm{inv}(\pi_0)},$$

where $\geq_L$ is the left weak order on $\mathfrak{S}_n$.

Proof. It is shown in [14] that

$$\sum_{D\in\mathcal{D}(\lambda/*)} q^{\mathrm{art}(D)} = \sum_{L\in\mathcal{L}(F)} q^{\mathrm{inv}(\pi)},$$

where $\mathcal{L}(F)$ is the set of increasing labelings of F. Thus, it is enough to show that for all $k \geq 0$, there is a bijection between the two sets

$$A_k = \{L \in \mathcal{L}(F) : \mathrm{inv}(L) = k\}, \qquad B_k = \{\pi \in \mathfrak{S}_n : \pi \geq_L \pi_0, \mathrm{inv}(\pi) - \mathrm{inv}(\pi_0) = k\}.$$

We will show that the map $L \mapsto \mathrm{post}(L)$ is a bijection from A_k to B_k for all $k \geq 0$ by induction on k. Since $A_0 = \{L_0\}$ and $B_0 = \{\pi_0\}$, it is true when $k = 0$. Suppose that the claimed statement is true for $k \geq 0$. We need to show that the map $L \mapsto \mathrm{post}(L)$ is a bijection from A_{k+1} to B_{k+1}. Let $L \in A_{k+1}$. Since $\mathrm{inv}(L) = k+1 \geq 1$, we can find an integer i such that $(i+1, i)$ is an inversion of L. Since $i+1$ is not a descendant of i in L, the labeling L' obtained from L by exchanging i and $i+1$ is also an increasing labeling of F. Since $\mathrm{inv}(L') = \mathrm{inv}(L) - 1 = k$ we have $L' \in A_k$. By the induction hypothesis, $\pi' = \mathrm{post}(L') \in B_k$. One can easily see that the permutation $\pi = \mathrm{post}(L)$ is obtained from π' by exchanging i and $i+1$. Since $i+1$ appears to the left of i in π we have $\mathrm{inv}(\pi) = \mathrm{inv}(\pi') + 1$ and $\pi \in B_{k+1}$. Thus $L \mapsto \mathrm{post}(L)$ is a map from A_{k+1} to B_{k+1}. Similarly, we can show that, for given $\pi \in B_{k+1}$, there is $L \in B_{k+1}$ such that $\mathrm{post}(L) = \pi$. Since L is determined by $\mathrm{post}(L)$ for all $L \in \mathcal{L}(F)$, the map $L \mapsto \mathrm{post}(L)$ is a bijection from A_{k+1} to B_{k+1}. $\square$

Note that the inversion generating function of increasing labelings of a plane forest is given by a hook length formula [2]. The fact that some intervals for the weak order have a generating function given by a hook length formula follows from [3].

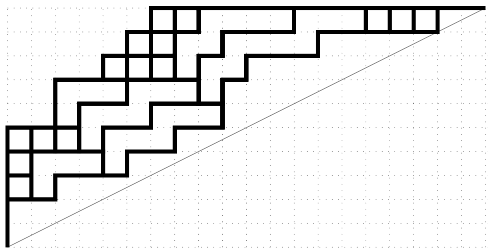


Fig. 6. An example of a 2-Dyck tiling.

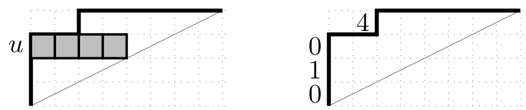


Fig. 7. The up step u on the left has height $\mathrm{ht}(u) = 5$ because there are 4 squares between u and the line $y = x/2$. The diagram on the right is an example of a 2-Hermite history.

3. k-Dyck tilings

For an integer $k \geq 1$, a *k-Dyck path* is a lattice path consisting of *up steps* $(0, 1)$ and *right steps* $(1, 0)$ from the origin $(0, 0)$ to the point (kn, n) which never goes below the line $y = x/k$. Let $\mathrm{Dyck}^{(k)}(n)$ denote the set of k-Dyck paths from $(0, 0)$ to (kn, n). It is well known that the cardinality of $\mathrm{Dyck}^{(k)}(n)$ is the Fuss–Catalan number $\frac{1}{kn+1}\binom{(k+1)n}{n}$ (see for example [5]). As in the case of Dyck path, we denote by $\mathrm{UP}(\mu)$ the set of up steps of a k-Dyck path μ.

A *k-Dyck tile* is a ribbon in which the centers of the cells form a k-Dyck path. Choose $\lambda, \mu \in \mathrm{Dyck}^{(k)}(n)$ such that μ is weakly above λ. A *(cover-inclusive) k-Dyck tiling* is a tiling D of the region λ/μ between $\lambda, \mu \in \mathrm{Dyck}^{(k)}(n)$ with k-Dyck tiles satisfying the cover-inclusive property: if η is a tile in D, then the translation of η by $(1, -1)$ is either completely below λ or contained in another tile of D. See Fig. 6 for an example. We denote by $\mathcal{D}^{(k)}(\lambda/\mu)$ the set of k-Dyck tilings of λ/μ. We also denote by $\mathcal{D}^{(k)}(\lambda/*)$ and $\mathcal{D}^{(k)}(*/\mu)$ the sets of k-Dyck tilings with fixed lower path λ and with fixed upper path μ, respectively.

For $D \in \mathcal{D}^{(k)}(\lambda/\mu)$, there are two natural statistics $\mathrm{tiles}(D)$ and $\mathrm{area}(D)$, where $\mathrm{tiles}(D)$ is the number of tiles in D and $\mathrm{area}(D)$ is the area of the region occupied by D. We also define

$$\mathrm{art}_k(D) = \frac{k \cdot \mathrm{area}(D) + \mathrm{tiles}(D)}{k+1}.$$

Definition 3.1. For an up step u of a k-Dyck path, we define the *height* $\mathrm{ht}(u)$ of u to be the number of squares between u and the line $y = x/k$, plus 1. A *k-Hermite history* is a k-Dyck path in which every up step u is labeled with an integer in $\{0, 1, 2, \ldots, \mathrm{ht}(u) - 1\}$. See for example Fig. 7. Given a k-Dyck path μ, we denote by $\mathcal{H}^{(k)}(\mu)$ the set of k-Hermite histories on μ. The *weight* $\mathrm{wt}(H)$ of a k-Hermite history is the sum of the labels in H.

The following theorem is a generalization of (2). Our proof generalizes the bijective proof in [13].

Theorem 3.2. *For $\mu \in \mathrm{Dyck}^{(k)}(n)$, we have*

$$\sum_{D \in \mathcal{D}^{(k)}(*/\mu)} q^{\mathrm{tiles}(D)} = \prod_{u \in \mathrm{UP}(\mu)} [\mathrm{ht}(u)]_q\,. \tag{3}$$

Proof. It suffices to find a bijection $f : \mathcal{D}^{(k)}(*/\mu) \to \mathcal{H}^{(k)}(\mu)$ such that if $f(D) = H$ then $\mathrm{tiles}(D) = \mathrm{wt}(H)$. We construct such a bijection as follows. Let $D \in \mathcal{D}^{(k)}(*/\mu)$. In order to define the

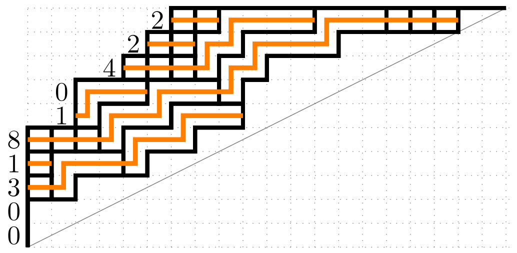


Fig. 8. An illustration of the bijection between k-Dyck tilings and k-Hermite histories.

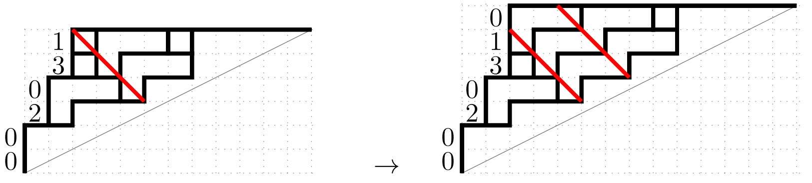


Fig. 9. An example of cutting a k-Dyck tiling and inserting an up step and k right steps.

corresponding $f(D) = H \in \mathcal{H}^{(k)}(\mu)$ we only need to define the labels of each up step in μ. For a k-Dyck tile η, the *entry* of η is the left side of the lowest cell in η and the *exit* of η is the right side of the rightmost cell in η. For each up step u in μ, we travel across the tiles in D in the following way. If u is the entry of a k-Dyck tile η in D, then enter η at the entry and leave η from the exit. If the exit of η is the entry of another k-Dyck tile of D then travel across that tile as well. Continue traveling in this way until we do not reach the entry of any k-Dyck tile in D. Then the label of u is defined to be the number of tiles that we have traveled. See Fig. 8 for an example. It is not difficult to see that $f(D) \in \mathcal{H}^{(k)}(\mu)$. Clearly, we have $\mathrm{tiles}(D) = \mathrm{wt}(H)$.

It remains to show that f is a bijection. Let $H \in \mathcal{H}^{(k)}(\mu)$. We will find the inverse image $D = f^{-1}(H)$ recursively. Suppose that n is the length of μ and m is the number of cells between μ and the line $y = x/k$. If $n = 0$ or $m = 0$, then both $\mathcal{D}^{(k)}(*/\mu)$ and $\mathcal{H}^{(k)}(\mu)$ have a unique element, and $f^{-1}(H)$ is the unique element in $\mathcal{D}^{(k)}(*/\mu)$ without tiles. Now let $n, m \geq 1$ and suppose that we can find $f^{-1}(H')$ for every $H' \in \mathcal{H}^{(k)}(\mu')$ if the length of μ' is smaller than n or the number of cells between μ' and the line $y = x/k$ is smaller than m. There are two cases.

Case 1: H has an up step with label $\ell \geq 1$ followed by a right step. In this case let H' be the k-Hermite history obtained from H by exchanging the up step and the right step following it and decrease the label ℓ by 1. Then the shape μ' of H' has one fewer cells between μ' and the line $y = x/k$. By assumption we can find $f^{-1}(\mu')$. Then $f^{-1}(\mu)$ is the k-Dyck tiling obtained from $f^{-1}(\mu')$ by adding the single square $\mu \setminus \mu'$.

Case 2: H has no up step with label $\ell \geq 1$ followed by a right step. Since μ is a k-Dyck path of positive length, we can find an up step u followed by k right steps. Since u is followed by a right step, its label is 0. Let P be the point where u starts. Let μ' be the k-Dyck path obtained from μ by deleting u and the k right steps following u. By assumption, we can find $f^{-1}(\mu')$. Then $f^{-1}(\mu)$ is the k-Dyck tiling obtained from $f^{-1}(\mu')$ by cutting it with the line of slope -1 passing through P and inserting an up step followed by k right steps. For each k-Dyck tile divided by the line, we attach the two divided pieces by connecting the separated points on the border with an up step and k right steps following it. See Fig. 9.

This gives the inverse map of f. $\square$

It seems unlikely that there is a hook length formula for $\mathcal{D}^{(k)}(\lambda/*)$ when we fix the lower path λ to be arbitrary. If $n = 6,\ k = 2$, and λ is the following path, then we have $|\mathcal{D}_n^{(k)}(\lambda/*)| = 607$, a prime

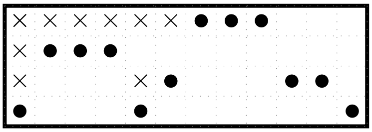
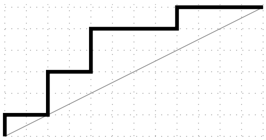
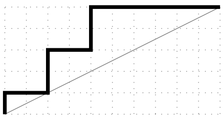


Fig. 10. The matrix corresponding to $422243111334 \in \mathfrak{S}_4^{(3)}$, where ones are represented as dots. The k-inversions are the cells with crosses.

number.



Also $|\mathcal{D}_n^{(k)}(\lambda/*)| = 71$ for the following λ with $n = 5$, $k = 2$.



However, when λ is a zigzag path there is a nice generalization of (1).

Theorem 3.3. *Let λ be the path $u^{n_1}d^{kn_1}u^{n_2}d^{kn_2}\cdots u^{n_\ell}d^{kn_\ell}$, where u means an up step and d means a right step. Then we have*

$$\sum_{D\in\mathcal{D}^{(k)}(\lambda/*)} q^{\mathrm{art}_k(D)} = \begin{bmatrix} kn_1+n_2 \\ n_2 \end{bmatrix}_q \begin{bmatrix} k(n_1+n_2)+n_3 \\ n_3 \end{bmatrix}_q \cdots \begin{bmatrix} k(n_1+\cdots+n_{\ell-1})+n_\ell \\ n_\ell \end{bmatrix}_q,$$

where $\begin{bmatrix} a \\ b \end{bmatrix}_q = \frac{[a]_q!}{[b-a]_q![b]_q!}$.

Proof. This can be proved using the same idea in the inductive proof in [13]. We will omit the details. $\square$

Problem 1. Find a bijective proof of Theorem 3.3.

4. k-Stirling permutations and the k-Bruhat order

In this section we consider k-Stirling permutations which were introduced by Gessel and Stanley [8] for $k = 2$ and studied further for general k by Park [15].

A *k-Stirling permutation of size n* is a permutation of the multiset $\{1^k, 2^k, \ldots, n^k\}$ such that if an integer j appears between two i's then $i > j$. Let $\mathfrak{S}_n^{(k)}$ denote the set of k-Stirling permutations of size n. We can represent a k-Stirling permutation $\pi = \pi_1\pi_2\ldots\pi_{kn}$ as the $n \times kn$ matrix $M = (M_{i,j})$ defined by $M_{i,j} = 1$ if $\pi_j = i$ and $M_{i,j} = 0$ otherwise. Then the entries of M are 0's and 1's such that each column contains exactly one 1, each row contains k 1's, and it does not contain the following submatrix:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

For an example, see Fig. 10.

A *k-inversion* of $\pi \in \mathfrak{S}_n^{(k)}$ is a pair $(i, j) \in [n] \times [kn]$ such that $\pi_j > i$ and the first i appears after π_j. Equivalently, we will think of a k-inversion as an entry (or a cell) in the matrix of π which has k 1's to the right in the same row and one 1 below in the same column, see Fig. 10. We denote the set of k-inversions of π by $\mathrm{INV}_k(\pi)$, and $\mathrm{inv}_k(\pi) = |\mathrm{INV}_k(\pi)|$.

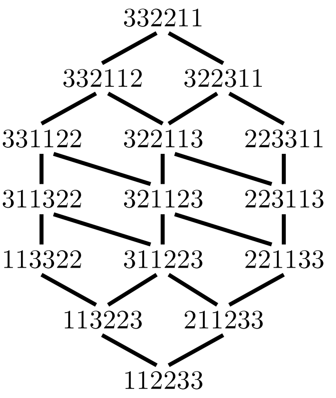


Fig. 11. The Hasse diagram of $\mathfrak{S}_3^{(2)}$.

Proposition 4.1. *We have*

$$\sum_{\pi \in \mathfrak{S}_n^{(k)}} q^{\mathrm{inv}_k(\pi)} = [k+1]_q[2k+1]_q \cdots [(n-1)k+1]_q.$$

Proof. This is an easy induction. Note that all the 1's in a Stirling permutation form a block of consecutive letters. Starting from $\sigma \in \mathfrak{S}_{n-1}^{(k)}$, we build a Stirling permutation in $\mathfrak{S}_n^{(k)}$ by increasing all letters of σ by 1, and inserting a block 1^k at some position. The positions where we can insert this block give the factor $[(n-1)k+1]_q$. $\square$

As we can see in the following lemma, a k-Stirling permutation is determined by its k-inversions.

Lemma 4.2. *For $\sigma, \pi \in \mathfrak{S}_n^{(k)}$, if $\mathrm{INV}_k(\sigma) = \mathrm{INV}_k(\pi)$, we have $\sigma = \pi$.*

Proof. Suppose that $\sigma \neq \pi$. Let r be the smallest index satisfying $\sigma_r \neq \pi_r$. We can assume that $\sigma_r < \pi_r$. Let $m = \sigma_r$. Note that $\pi_j = m$ for some $j > r$. Then π does not have the integer m in the first r positions because otherwise $\pi_i = m$ for $i < r$ and we get $\pi_i\langle\pi_r\rangle\pi_j$ which is forbidden. Then $(m, r) \in \mathrm{INV}_k(\pi)$ but $(m, r) \notin \mathrm{INV}_k(\sigma)$, which is a contradiction. Thus we have $\sigma = \pi$. $\square$

We are now ready to define the k-Bruhat order on k-Stirling permutations.

Definition 4.3. We define the *k-Bruhat order* on $\mathfrak{S}_n^{(k)}$ given by the cover relation $\sigma \lessdot \pi$ if π is obtained from σ by exchanging the two numbers in positions a_1 and a_{k+1} for some integers $a_1 < a_2 < \cdots < a_{k+1}$ satisfying the following conditions:

(1) $\sigma_{a_1} = \sigma_{a_2} = \cdots = \sigma_{a_k} < \sigma_{a_{k+1}}$, and
(2) for all $a_k < i < a_{k+1}$ we have either $\sigma_i < \sigma_{a_1}$ or $\sigma_i > \sigma_{a_{k+1}}$.

In fact, if $k \geq 2$, then for all $a_1 < i < a_{k+1}$ with $i \neq a_j$ we always have $\sigma_i < \sigma_{a_1}$, see Lemma 4.4.

Note that the 1-Bruhat order is the usual Bruhat order. Fig. 11 illustrates the k-Bruhat order.

Remark 1. The elements of $\mathfrak{S}_n^{(k)}$, where all occurrences of i are consecutive for any i, form a subset which is in natural bijection with $\mathfrak{S}_n$. In general ($k > 1$ and $n > 2$), the induced order on this subset is strictly contained in the Bruhat order, and strictly contains the left weak order.

Lemma 4.4. *In this lemma we use the notation in Definition 4.3. If $k \geq 2$, then we have $\sigma_i < \sigma_{a_1}$ for all $a_1 < i < a_{k+1}$ with $i \neq a_j$.*

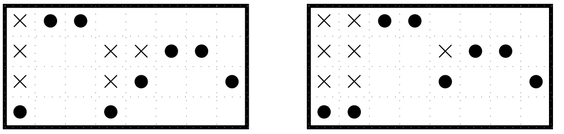


Fig. 12. The matrix on the left represents $\sigma = 41143223 \in \mathfrak{S}_4^{(2)}$ and the matrix on the right represents $\pi = 44113223 \in \mathfrak{S}_4^{(2)}$. We have $\sigma \lessdot \pi$ and $\mathrm{INV}_k(\pi)$ is obtained from $\mathrm{INV}_k(\sigma)$ by adding the cell $(1, 2)$ and moving the cells $(2, 3)$ and $(3, 3)$ to the west.

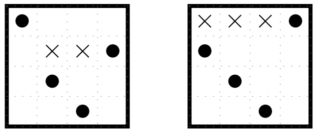


Fig. 13. The matrix on the left represents $\sigma = 1342 \in \mathfrak{S}_4^{(1)}$ and the matrix on the right represents $\pi = 2341 \in \mathfrak{S}_4^{(1)}$. We have $\sigma \lessdot \pi$ and $\mathrm{INV}_k(\pi)$ is obtained from $\mathrm{INV}_k(\sigma)$ by adding the cell $(1, 1)$ and moving the cells $(2, 2)$ and $(2, 3)$ to the north.

Proof. By the definition of a k-Stirling permutation, for all $a_1 < i < a_k$ with $i \neq a_j$ we have $\sigma_i < \sigma_k$, and for all $a_2 < i < a_{k+1}$ with $i \neq a_j$ we have $\pi_i < \pi_k = \sigma_k$. Thus we have $\sigma_i = \pi_i < \sigma_k$ for all $a_1 < i < a_{k+1}$ with $i \neq a_j$. $\square$

Lemma 4.5. *Let $\sigma \lessdot \pi$ in $\mathfrak{S}_n^{(k)}$. Then $\mathrm{INV}_k(\pi)$ is obtained from $\mathrm{INV}_k(\sigma)$ by adding one cell and moving some cells (possibly none) to the west or north. Moreover, if a cell is moved to the west (respectively, north), then the new location of the cell is south (respectively, to east) of the newly added cell in the same column (respectively, row). See Figs. 12 and 13.*

Proof. Suppose that π is obtained from σ as described in Definition 4.3. Then $\mathrm{INV}_k(\pi)$ is obtained from $\mathrm{INV}_k(\sigma)$ as follows. We add the inversion (σ_{a_1}, a_1), and change each k-inversion of the form (r, a_{k+1}) for some $\sigma_{a_1} < r < \sigma_{a_{k+1}}$ to (r, a_1). Furthermore if $k = 1$, we change each k-inversion of the form $(\sigma_{a_{k+1}}, j)$ for some $j < a_{k+1}$ to (σ_{a_1}, j). $\square$

We note that in Lemma 4.5, moving cells to north can happen only when $k = 1$.

By Lemma 4.5 we have $\mathrm{inv}_k(\pi) = \mathrm{inv}_k(\sigma) + 1$ if $\sigma \lessdot \pi$. This proves:

Proposition 4.6. *Endowed with the k-Bruhat order, $\mathfrak{S}_n^{(k)}$ is a graded poset with rank function inv_k.*

The following generalization of 132-avoiding permutations is straightforward and natural.

Definition 4.7. A k-Stirling permutation $\sigma \in \mathfrak{S}_n^{(k)}$ is 132-*avoiding* if there is no $1 \leq i < j < k \leq kn$ such that $\sigma_i < \sigma_k < \sigma_j$. Let $\mathfrak{S}_n^{(k)}(132)$ denote the set of 132-avoiding k-Stirling permutations in $\mathfrak{S}_n^{(k)}$.

In a k-Dyck tilings with 0 tiles, the lower path and upper path are the same. So these tilings are trivially in a bijection with k-Dyck paths. As for the k-Stirling permutations, we have:

Proposition 4.8. *The inversions of a 132-avoiding k-Stirling permutation σ are arranged as the cells of a top-left justified Ferrers diagram. Define a path $\alpha(\sigma)$ from the bottom left to the top right corner, and following the boundary of this Ferrers diagram with up and right steps. Then α is a bijection between 132-avoiding k-Stirling permutations and k-Dyck paths of length n, in particular both are counted by the Fuss–Catalan numbers $\frac{1}{kn+1}\binom{(k+1)n}{n}$.*

The proof is simple and similar to the case $k = 1$. For an example, see Fig. 14.

Proposition 4.9. *Suppose that $\sigma \in \mathfrak{S}_n^{(k)}$ is 132-avoiding. Then, for $\pi \in \mathfrak{S}_n^{(k)}$, we have $\sigma \leq \pi$ if and only if $\mathrm{INV}_k(\sigma) \subseteq \mathrm{INV}_k(\pi)$.*

Proof. Since σ is 132-avoiding, $\mathrm{INV}(\sigma)$ is a Ferrers diagram, say λ. By Lemma 4.5 if $\tau \lessdot \rho$ and $\lambda \subset \mathrm{INV}_k(\tau)$, then we also have $\lambda \subset \mathrm{INV}_k(\rho)$. This implies the "only if" part.

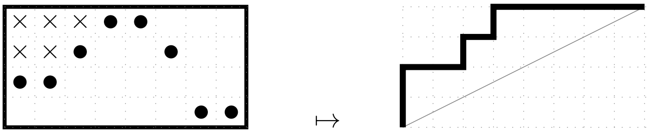


Fig. 14. The bijection α in Proposition 4.8.

We will prove the "if" part by induction on the number $m = \mathrm{inv}_k(\pi) - \mathrm{inv}_k(\sigma)$. If $m = 0$, it is true. Suppose $m > 0$. Let (i, j) be an element in $\mathrm{INV}_k(\pi) \setminus \mathrm{INV}_k(\sigma)$ with j as large as possible. Then there are integers $a_1 < a_2 < \cdots < a_k$ such that $\pi_{a_1} = \pi_{a_2} = \cdots = \pi_{a_k} = i$ and $a_1 > j$. Then we have $\pi_t < i$ for all $j < t < a_1$ by the maximality of j. Let π' be obtained from π by exchanging the two integers in positions j and a_k. Then $\pi' \lessdot \pi$. By Lemma 4.5, $\mathrm{INV}_k(\pi')$ is obtained from $\mathrm{INV}_k(\pi)$ by removing the cell (i, j) and moving some cell located south or east of (i, j). Thus $\mathrm{INV}_k(\pi')$ still contains $\mathrm{INV}_k(\sigma)$, which is a Ferrers diagram. By induction we have $\sigma \leq \pi'$, which completes the proof. $\square$

Proposition 4.10. *Let $\sigma \in \mathfrak{S}_n^{(k)}(132)$, and let $\mu = \alpha(\sigma)$ be the corresponding k-Dyck path. There is a bijection between the interval $\{\pi : \pi \geq \sigma\}$ in $\mathfrak{S}_n^{(k)}$ and the set of k-Hermite histories of shape μ such that $\mathrm{inv}_k(\pi) - \mathrm{inv}_k(\sigma)$ is equal to the sum of weights in the corresponding k-Hermite history.*

Proof. Let $\pi \in \mathfrak{S}_n^{(k)}(132)$ with $\pi \geq \sigma$. Note that by Proposition 4.9, the inversions of σ are inversions of π. Then we define a k-Hermite history as follows. For $i \leq n$, the label of the up step in the ith row from the top is the number of $\times$'s in the matrix of σ that are in the ith row, and below μ. Since $\mathrm{inv}_k(\sigma)$ is the number of $\times$'s above the path μ by definition of the bijection in Fig. 14, the number $\mathrm{inv}_k(\pi) - \mathrm{inv}_k(\sigma)$ is the sum of weights in the k-Hermite history. So it remains only to show that these labels define a k-Hermite history (i.e. they fall in the right range) and that this is a bijection.

Let us start with the first row from the top. If the height of the up step in the first row is h, then there are $h + k$ cells to the right of the up step in the first row. Since there are k consecutive dots in the first row in π, the $\times$'s are located in the cells after the up step and before the k consecutive dots. Thus the number of $\times$'s in the first row is among $0, 1, 2, \ldots, h$. Now consider the second row. If the height of the up step in the first row is h', then there are $h' + 2k$ cells to the right of the up step in the second row. By the condition for k-Stirling permutation, if we delete the columns which have a dot in the first row, then the dots in the second row are consecutive, and the $\times$'s are located in the cells after the up step and before the k consecutive dots. Thus the number of $\times$'s in the second row is among $0, 1, 2, \ldots, h'$. In this manner, we can see that the number of $\times$'s in each row is at most the height of the up step in the same row. This argument also shows that once the number of $\times$'s in each row is determined, we can uniquely construct the corresponding permutation π by putting dots starting from the first row. This implies that the map is a desired bijection. $\square$

We now have a generalization of Theorem 2.3. It is a rewriting of Theorem 3.2 using the bijection from Proposition 4.10.

Theorem 4.11. *If μ is a k-Dyck path corresponding to $\sigma \in \mathfrak{S}_n^{(k)}(132)$, then*

$$\sum_{D \in \mathcal{D}_n^{(k)}(*/\mu)} q^{\mathrm{tiles}(D)} = \sum_{\pi \geq \sigma} q^{\mathrm{inv}_k(\pi) - \mathrm{inv}_k(\sigma)}.$$

5. k-regular noncrossing partitions

In this section, we take another point of view on the k-Stirling permutations studied in the previous section.

Definition 5.1. We denote by $\mathrm{NC}_n^{(k)}$ the set of *k-regular noncrossing partitions* of size n, i.e. set partitions of $[kn]$ such that each block contains k elements (k-regular), and there are no integers $a < b < c < d$

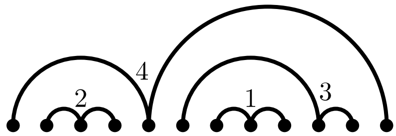


Fig. 15. The bijection proving Proposition 5.2.

such that a, c are in one block, and b, d in another block (noncrossing). To each k-regular noncrossing partition π of $[kn]$, we define its *nesting poset* $\mathrm{Nest}(\pi)$ as follows: the elements of the poset are the blocks of π, and $x \le y$ in the poset when the block x lies between two elements of the block y.

There is a natural way to consider a k-Stirling permutation as a linear extension of the nesting poset of a k-regular noncrossing partition.

Proposition 5.2. *There is a bijection between* $\mathfrak{S}_n^{(k)}$ *and pairs* (π, E) *where* $\pi \in \mathrm{NC}_n^{(k)}$ *and* E *is a linear extension of the poset* $\mathrm{Nest}(\pi)$.

Proof. Let $\sigma \in \mathfrak{S}_n^{(k)}$. We define π by saying that i and j are in the same block if $\sigma_i = \sigma_j$, and E is defined by saying that the label of a block of π is σ_i where i is any element of the block. We can see that π is noncrossing from the definition of Stirling permutations, and E is a linear extension by definition of the nesting poset. In the example $\sigma = 422243111334$, we get the noncrossing partition in Fig. 15 where the labels define the linear extension of the nesting poset. The inverse bijection is simple to describe: σ_i is equal to j if i is in the block with label j. $\square$

The Hasse diagram of the poset $\mathrm{Nest}(\pi)$ is always a forest, so it is possible to consider pairs (π, E) as a decreasing forest. Each block of π is a vertex of the forest, and the forest structure is the order $\mathrm{Nest}(\pi)$. Also, the labeling E naturally gives the decreasing labeling of the forest.

Let b be a block of π, the elements of b being $i_1 < \cdots < i_k$. Then the descendants of b in the forest can be distinguished into $k-1$ categories, depending on the index j such that a descendant of b lies between i_j and i_{j+1}.

So we arrive at the following definition.

Definition 5.3. Let $T_n^{(k-1)}$ denote the set of $(k-1)$-ary plane forests on n vertices defined by the following conditions:

- the descendants of a vertex have a structure of a $(k-1)$-tuple of ordered lists,
- the vertices are labeled with integers from 1 to n and the labels are decreasing from the roots to the leaves.

As a result of the above discussion, we obtain that there is a bijection between $\mathfrak{S}_n^{(k)}$ and $T_n^{(k-1)}$. However we do not insist on this point of view, since the definition of the trees in $T_n^{(k-1)}$ is not particularly natural.

There is a hook length formula for the number of linear extensions of a forest [3]. If $x \in \mathrm{Nest}(\pi)$, let h_x denote the number of elements below x in the nesting poset, then the number of linear extensions of $\pi \in \mathrm{NC}_n^{(k)}$ is

$$\frac{n!}{\prod_{x \in \mathrm{Nest}(\pi)} h_x}.$$

This gives the following formula for the number of k-Stirling permutations.

Proposition 5.4 (*Multifactorial Hook Length Formula*)**.**

$$1(k+1)(2k+1)\dots((n-1)k+1) = \sum_{\pi \in \mathrm{NC}_n^{(k)}} \frac{n!}{\prod_{x \in \mathrm{Nest}(\pi)} h_x}.$$

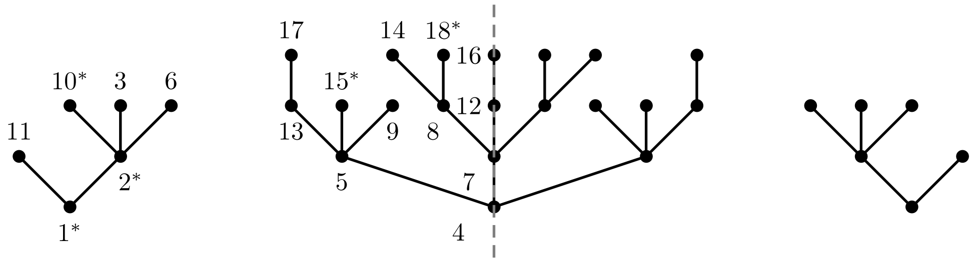


Fig. 16. A symmetric plane tree of size 18 and a marked increasing labeling of it. The center line is the dashed line.

In the case $k > 2$, it is not clear how to choose the q-statistic, but for $k = 2$ we can use the bijection between noncrossing matchings and plane forests (which is just $\pi \mapsto \mathrm{Nest}(\pi)$), and use the q-hook length formula from [2]. We get a hook length formula for $[1]_q[3]_q \dots [2n-1]_q$.

Proposition 5.5 (*q-Double Factorial Hook Length Formula*)**.** *We have*

$$[1]_q[3]_q \dots [2n-1]_q = \sum_F [n]_{q^2}! \prod_{v \in F} \frac{q^{h_v - 1}}{[h_v]_{q^2}},$$

where the sum is over all plane forests F with n vertices.

Proof. We know that the left hand side is the inversion generating function of $\mathfrak{S}_n^{(2)}$. It remains to understand what becomes the number of inversion through the bijection which sends a 2-Stirling permutation to a noncrossing matching with labeled blocks, or equivalently increasing plane forests.

To this end, we distinguish two kinds of inversions. Let $i < j$ and $\sigma \in \mathfrak{S}_n^{(2)}$. If the four letters i, i, j, j appear in σ in this order, there is no inversion. If they appear in the order j, i, i, j, there is one inversion, and this corresponds to the case where the vertex with label i is below the vertex with label j. It means we have to count one inversion for each pair of comparable vertices, and the number of comparable vertices is clearly $\sum_{x \in \pi}(h_x - 1)$. In particular, it does not depend on the labeling. If they appear in the order j, j, i, i, there are two inversions, and this situation corresponds to the case where the vertices with labels i, j form an inversion in the forest.

So for a particular forest F, we get the term

$$[n]_{q^2}! \prod_{v \in F} \frac{q^{h_v - 1}}{[h_v]_{q^2}}.$$

This completes the proof. $\square$

6. Symmetric Dyck tilings and marked increasing forests

A *symmetric plane forest* is a plane forest which is invariant under the reflection across a line, called the *center line*. A *centeral vertex* is a vertex on the center line. The *left part* of a symmetric plane forest is the subforest consisting of vertices on or to the left of the center line. The *size* of a symmetric plane forest is the number of vertices in the left part of it.

Let F be a symmetric plane forest of size n. A *marked increasing labeling* of F is a labeling of the left part of F with $[n]$ such that the labels are increasing from roots to leaves, each integer appears exactly once, and each non-centeral vertex may be marked with $*$. See Fig. 16 for an example of a marked increasing labeling. We denote the set of marked increasing labelings of F by $\mathrm{INC}^*(F)$.

Let $L \in \mathrm{INC}^*(F)$. We denote by $\mathrm{MARK}(L)$ the set of labels of the marked vertices in L. An *inversion* of L is a pair of vertices (u, v) such that $L(u) > L(v)$, u and v are incomparable, and u is to the left of v. Let $\mathrm{INV}(L)$ denote the set of inversions of L.

A Dyck path λ of length $2n$ is called *symmetric* if it is invariant under the reflection across the line $x+y = n$. For two symmetric Dyck paths λ and μ of length $2n$, a Dyck tiling of λ/μ is called *symmetric*

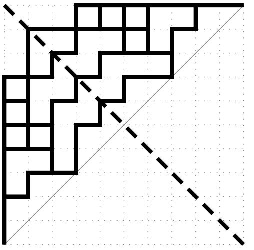


Fig. 17. An example of symmetric Dyck tiling. It is symmetric with respect to the dashed line.

if it is invariant under the reflection across the line $x+y=n$. See Fig. 17 for an example of symmetric Dyck tiling. We denote by $\mathcal{D}_{\mathrm{sym}}(\lambda/\mu)$ the set of symmetric Dyck tilings of shape λ/μ.

For a symmetric Dyck tiling D, a *positive tile* is a tile which lies strictly to the left of the center line, and a *zero tile* is a tile which intersects with the center line. We denote by $\mathrm{tiles}_+(D)$ and $\mathrm{tiles}_0(D)$ the number of positive tiles and zero tiles, respectively. Note that the total number of tiles in D is $\mathrm{tiles}(D) = 2 \cdot \mathrm{tiles}_+(D) + \mathrm{tiles}_0(D)$. We also define $\mathrm{area}_+(D)$ and $\mathrm{area}_0(D)$ to be the total area of positive tiles and zero tiles in D, respectively, and

$$\mathrm{art}_+(D) = \frac{\mathrm{area}_+(D) + \mathrm{tiles}_+(D)}{2}, \qquad \mathrm{art}_0(D) = \frac{\mathrm{area}_0(D) + \mathrm{tiles}_0(D)}{2}.$$

For example, if D is the symmetric Dyck tiling in Fig. 17, we have $\mathrm{tiles}_+(D) = 5$, $\mathrm{tiles}_0(D) = 3$, $\mathrm{area}_+(D) = 7$, $\mathrm{area}_0(D) = 19$, $\mathrm{art}_+(D) = (7+5)/2 = 6$, and $\mathrm{art}_0(D) = (19+3)/2 = 11$.

Theorem 6.1. *Let F be a symmetric plane forest of size n and λ the corresponding Dyck path. Then there is a bijection $\phi : \mathrm{INC}^*(F) \to \mathcal{D}_{\mathrm{sym}}(\lambda/*)$ such that if $\phi(L) = D$, then $\mathrm{tiles}_0(D) = |\mathrm{MARK}(L)|$ and*

$$\mathrm{art}_+(D) + \mathrm{art}_0(D) = |\mathrm{INV}(L)| + \sum_{i \in \mathrm{MARK}(L)} (n+1-i).$$

Proof. This is a generalization of a bijection in [14] constructed recursively. We "spread" and add "broken strips" to all up steps before the center line and to all right steps after the center line. If a vertex is marked, then we also add a square at the center line. See Fig. 18. □

Corollary 6.2. *Let F be a symmetric plane forest of size n with k center vertices and λ the corresponding Dyck path. Then*

$$|\mathcal{D}_{\mathrm{sym}}(\lambda/*)| = 2^{n-k} \frac{n!}{\prod_{x \in F} h_x}.$$

If $k = 0$, we have

$$\sum_{D \in \mathcal{D}_{\mathrm{sym}}(\lambda/*)} q^{\mathrm{art}_+(D) + \mathrm{art}_0(D)} t^{\mathrm{tiles}_0(D)} = (1+tq)(1+tq^2)\cdots(1+tq^n) \frac{[n]_q!}{\prod_{x \in F} [h_x]_q}.$$

7. Symmetric Dyck tilings and symmetric Hermite histories

For a symmetric Dyck path μ of length $2n$, let μ^+ denote the subpath consisting of the first n steps. Note that each up step is matched with a unique right step in a Dyck path. An up step of μ^+ is called *matched* if the corresponding right step in μ lies in μ^+ and *unmatched* otherwise.

A *symmetric Hermite history* is a symmetric Dyck path μ with a labeling of the up steps of μ^+ in such a way that every matched up step of height h has label $i \in \{0, 1, \dots, h-1\}$ and the labels $a_1, a_2, \dots, a_k$ of the unmatched up steps form an involutive sequence. Here, a sequence is called *involutive* if it can be obtained by the following inductive way.

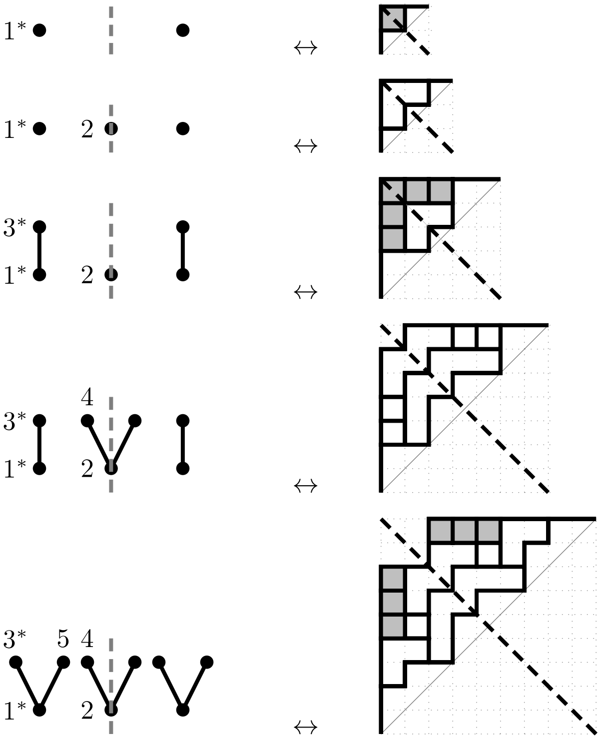


Fig. 18. An illustration of the bijection between marked increasing labeling of a symmetric forest and symmetric Dyck tilings with fixed lower path. The newly added squares in each step are colored gray.

- The empty sequence is defined to be involutive.
- The sequence (0) is the only involutive sequence of length 1.
- For $k \geq 2$, an involutive sequence of length k is either the sequence obtained from an involutive sequence of length $k-1$ by adding 0 at the end or the sequence obtained from an involutive sequence of length $k-2$ by adding an integer r at the end and inserting a 0 before the last r integers, including the newly added integer, for some $1 \leq r \leq k-1$.

See Fig. 19 for an example of a symmetric Hermite history. From the definition it is clear that the number v_k of involutive sequences of length k satisfies the recurrence $v_k = v_{k-1} + (k-1)v_{k-2}$ with initial conditions $v_0 = v_1 = 1$. Thus v_k is equal to the number of involutions in $\mathfrak{S}_n$. We denote by $\mathcal{H}_{\mathrm{sym}}(\mu)$ the set of symmetric Hermite histories on μ. For $H \in \mathcal{H}_{\mathrm{sym}}(\mu)$, let $\|H\|$ be the sum of labels in H and $\mathrm{pos}(H)$ the number of positive labels on unmatched up steps.

Proposition 7.1. *There is a bijection* $\psi : \mathcal{H}_{\mathrm{sym}}(\mu) \to \mathcal{D}_{\mathrm{sym}}(*/\mu)$ *such that if* $\psi(H) = D$*, then* $\|H\| = \mathrm{tiles}_+(D) + \mathrm{tiles}_0(D)$ *and* $\mathrm{pos}(H) = \mathrm{tiles}_0(D)$*. Thus,*

$$\sum_{D \in \mathcal{D}_{\mathrm{sym}}(*/\mu)} q^{\mathrm{tiles}_+(D)+\mathrm{tiles}_0(D)} t^{\mathrm{tiles}_0(D)} = \sum_{H \in \mathcal{H}_{\mathrm{sym}}(\mu)} q^{\|H\|} t^{\mathrm{pos}(H)}.$$

Proof. Given a symmetric Dyck tiling D, considering D as a normal Dyck tiling, we can obtain the Hermite history corresponding to D. By taking only the labels of up steps before the center line, we get a symmetric Hermite history. One can check that this gives a desired bijection. $\square$

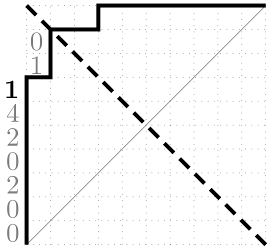


Fig. 19. An example of symmetric Hermite history. The labels of the unmatched up steps are in gray. The sequence (0, 0, 2, 0, 2, 4, 1, 0) of the labels of unmatched up steps is an involutive sequence.

Corollary 7.2. *Let μ be a symmetric Dyck path such that μ^+ has k unmatched up steps. Then*

$$\sum_{D\in\mathcal{D}_{\mathrm{sym}}(*/\mu)} q^{\mathrm{tiles}_+(D)+\mathrm{tiles}_0(D)} t^{\mathrm{tiles}_0(D)} = f_k(q,t) \prod_{u\in\mathrm{UP}(\mu^+)} [\mathrm{ht}(u)]_q,$$

where $\mathrm{UP}(\mu^+)$ *is the set of matched up steps in* μ^+ *and* $f_k(q,t)$ *is defined by* $f_0(q,t) = f_1(q,t) = 1$ *and* $f_k(q,t) = f_{k-1}(q,t) + tq[k-1]_q f_{k-2}(q,t)$ *for* $k \geq 2$.

A *symmetric matching* is a matching on $[\pm n]$ such that if $\{i,j\}$ is an arc, then $\{-i,-j\}$ is also an arc. We denote by $\mathcal{M}_{\mathrm{sym}}(n)$ the set of symmetric matchings on $[\pm n]$. Note that symmetric matchings are in a bijection with fixed-point-free involutions in B_n.

Let $M \in \mathcal{M}_{\mathrm{sym}}(n)$. A *symmetric crossing* of M is a pair of arcs $\{a,b\}$ and $\{c,d\}$ satisfying $a < c < b < d$ and $b, d > 0$. A symmetric crossing $(\{a,b\},\{c,d\})$ is called *self-symmetric* if $\{c,d\} = \{-a,-b\}$. We denote by $\mathrm{cr}(M)$ and $\mathrm{sscr}(M)$ the number of symmetric crossings and self-symmetric crossings of M, respectively.

For a symmetric Dyck path μ of length $2n$, let $\mathcal{M}_{\mathrm{sym}}(\mu)$ denote the set of symmetric matchings M on $[\pm n]$ such that the ith smallest vertex of M is a left vertex of an arc if and only if the ith step of μ is an up step.

Proposition 7.3. *We have*

$$\sum_{H\in\mathcal{H}_{\mathrm{sym}}(\mu)} q^{\|H\|} t^{\mathrm{pos}(H)} = \sum_{M\in\mathcal{M}_{\mathrm{sym}}(\mu)} q^{\mathrm{cr}(M)} t^{\mathrm{sscr}(M)}.$$

For $D \in \mathcal{D}_{\mathrm{sym}}(*/\mu)$, let $\mathrm{ht}(D)$ denote the number of unmatched up steps in μ^+. Using the result in [4,9] on a generating function for partial matchings we obtain the following formula.

Proposition 7.4. *We have*

$$\sum_{D\in\mathcal{D}_{\mathrm{sym}}(n)} q^{\mathrm{tiles}_+(D)+\mathrm{tiles}_0(D)} t^{\mathrm{tiles}_0(D)} s^{\mathrm{ht}(D)}$$
$$= \sum_{m=0}^{n} \frac{s^m f_m(q,t)}{(1-q)^{(n-m)/2}} \sum_{k\geq 0} \left(\binom{n}{\frac{n-k}{2}} - \binom{n}{\frac{n-k}{2}-1} \right) (-1)^{(k-m)/2} q^{\binom{(k-m)/2+1}{2}} \begin{bmatrix} \frac{k+m}{2} \\ \frac{k-m}{2} \end{bmatrix}_q,$$

where $f_m(q,t)$ *is defined in Corollary 7.2.*

If $t = s = 0$ in the above proposition, then we get the generating function for the usual Dyck tilings according to the number of tiles.

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