Summary of Results

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1 Equidistribution of the KND and VAL statistics

Let p be a caterpillar parking distribution on $\mathrm{Cat}(m+1, m, \dots)$ ([BGY17]), and

$$\begin{aligned} \mathrm{KND}(p) &:= \text{the number of unique parking preferences of a parking distribution, } p \\ \mathrm{VAL}(d) &:= \text{the number of valleys on Fuss-Dyck path } d. \end{aligned}$$

We use ideas on [Deu99] to study the distribution of the the two statistics. We have the following stronger result.

Theorem 1. *Let m and n be two positive integer. Let $r \in [0, m-1]$ and let $A(r) = \{mt - r + 1 : 0 < t \leq n\}$. Let $\hat{P}(n;r)$ denote the set of elements of $P(mn - r, 1)$ that contains all the numbers in the set $\{1, 2, \dots, mn - r\} - A(r)$ at least once. If $p \in \hat{P}(n;r)$, then there is a bijection θ such that all the valleys of $\theta(p)$ are multiples of m and its last peak has height $m - r$.*

If $r = 0$, we have the relation

$$\mathrm{KND}(p) + \mathrm{VAL}(\theta(p)) = mn - m + 1$$

which proves the two statistic are equi-distributed.

2 Equi-distribution of lucky and ret statistics

Let

$$\begin{aligned} \mathrm{lucky}(p) &:= \text{the number of cars the parked at their prefered spots } p \\ \mathrm{ret}(d) &:= \text{the number of returns (blocks) on Fuss-Dyck path } d. \end{aligned}$$

Here, we devised a bijection σ as follows

1. Let p be parking distribution on $\mathrm{Cat}(m+1, m, \dots)$. Remove the first occurrences of the numbers that are not of the form $mk+1$ for $k = 1, 2, \dots$ Eg. $12334 \to 133$ if $m = 2$.
2. If the new string is $p' = (p_1, \dots, p_n)$ convert it to the string $E, N(p_2 - p_1 \text{ times }), E, N(p_3 - p_2 \text{ times })$... Eg $133 \to ENNEEENNNN$.
3. Reverse the new string $ENNEEENNNN \to NNNNEEENNE$.

The above bijection proves the equi-distribution of the two statistics, as $\mathrm{lucky}(p) - \mathrm{ret}(\sigma(p)) = (m-1)(n-1)$.

$$\mathrm{lucky}(12334) - \mathrm{ret}(NNNNEEENNE) = 2.$$

3 Enumeration of Fuss-Dyck paths with $k+1$ fixed points.

The number of fixed points of a parking distribution on $\mathrm{Cat}(m+1, m, \dots)$ is the number of parking preferences with $p_i = i$. Let

$r_{n,k}^{(m)} :=$ the number of parking distribution on $\mathrm{Cat}(m+1, m, \dots)$ of length n with $k+1$ fixed points.

We prove that

$$\begin{array}{rcl} r_{n,k}^{(m)} & = & \sum_{j \geq 0} r_{n-1,j}^{(m)} \text{ for } 0 \leq k \leq m \\ & = & \\ r_{n,k}^{(m)} & = & \sum_{j \geq k-m} r_{n-1,j}^{(m)} \end{array}$$

We do so by extending the bijection discussed [Deu99] and using the results of [LL16].

4 Symmetric Joint Distribution of lucky and deficit statistics on classical parking distributions

Given the definition of the deficit and ult statistic on classical parking distributions,

$$\mathrm{deficit}(p) = n - \max_{1 \leq i \leq n} p_i$$

$$\mathrm{ult}(p) = \text{the last index } i \text{ such that } p_i = i,$$

We have the following theorem

Theorem 2. *Given the definition*

$$[x^n] I(x; t, s, u) = \sum_p t^{\mathrm{lucky}(p)-1} s^{\mathrm{deficit}(p)} u^{\mathrm{ult}(p)}$$

we have the functional equation

$$I(x; t, s, u, v) = 1 + xtsuC(ux; t)C(x; s),$$

where $C(x; q) = \sum_{n \geq 0} x^n \sum_{p, |p|=n} q^{\mathrm{lucky}(p)}$

As a corollary, we have a q-analog for the Catalan number, $\{q_n\}_{n \geq 0}$ defined by

$$q_0 = 1$$

$$q_n = q \sum_{0 \leq k \leq n-1} q_k C_{n-1-k}$$

.

$$q_n := \sum_{\substack{p \\ |p|=n}} q^{\mathrm{lucky}(p)}$$

where C_n is the classical n-th Catalan number.

5 Symmetric Joint Distribution of lucky and ω_1 statistics on parking distributions on $\mathrm{Cat}(m+1, m, \dots)$

We define the frequency statistic ω_k as

$$\omega_k(p) = |\{i : p_i = k\}|$$

. If

$$\begin{array}{rcl} f_m(p) & := & \text{the first index } i \text{ such that } m(i-2)+2 \leq p_i \leq m(i-1)+1 \\ g_m(p) & := & \text{the first index } i \text{ such that } p_i = m(i-1)+1 \end{array}$$

we have the following theorem

Theorem 3. *Define the polynomial over parking distributions on* Cat$(m+1, m, \dots)$ *with* $mn - m + 1$ *nodes:*

$$\lambda_n^{(m)}(t,s,u,v) = [x^n]\Lambda_m(x;t,s,u,v) := \sum_p t^{\mathrm{lucky}(p)-m} s^{\omega_1(p)-1} u^{f_m(p)} v^{g_m(p)}.$$

Then we have the functional equation

$$\Lambda(x;t,s,u,v) = 1 + xts(uv)^2 \mathcal{B}_m^{m-1}(x)\mathcal{B}_m(uvx;s)\mathcal{B}_m(vx;t)$$

where

$$\mathcal{B}_m(x) = 1 + x\mathcal{B}_m^{m+1}(x)$$

$$[x^n]\mathcal{B}_m(x;q) = \sum_{\substack{p \\ |p| = mn-m+1}} q^{\mathrm{lucky}(p)-m}$$

It follows that when $u = v = 1$, lucky(p) and $\omega_1(p)$ form a symmetric joint distribution, along with other results.

6 A Simplex Symmetry in Parking Distributions

We observe that for parking distributions on Cat$(m+1, m, \dots)$,

$$\omega_i(p) \sim \omega_j(p)$$

for $1 \le i < j \le m$. From this we derive the following theorem:

Theorem 4. *Let*

$$\Gamma_n^{(m)}(t, \zeta_1, \dots, \zeta_m) = \sum_p t^{\mathrm{lucky}(p)-m} \prod_{k=1}^m \zeta_k^{\omega_k(p)-1}$$

Then $\Gamma_n^{(m)}$ *is a symmetric polynomial in* t, ζ_i $1 \le i \le m$.

We call this a simplex symmetry because the coefficients of $t^{i_0}\zeta_1^{i_1}\cdots\zeta_m^{i_m}$ for a $m+1$-dimensional simplex. In other words, the coefficients of $t^{i_0}\zeta_1^{i_1},\cdots,\zeta_m^{i_m}$ is the same for all $m+1$-tuples $i_0, \dots i_m$ if $\sum_{1\le k\le n} i_k = w$ for some non-negative integer w, which resembles the definition of a simplex in $\mathbb{R}^{m+1}$. This remains unexplored direction.

References

[BGY17] Steve Butler, Ron Graham, and Catherine H. Yan. Parking distributions on trees. *European Journal of Combinatorics*, 65:168–185, October 2017.

[Deu99] Emeric Deutsch. An involution on dyck paths and its consequences. *Discrete Mathematics*, 204(1–3):163–166, June 1999.

[LL16] Lily Li Liu and Xiaoli Li. Taylor expansions for the generating function of catalan-like numbers. *Cogent Mathematics*, 3(1):1200305, July 2016.