The Snake Lemma

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Lemma 1. *Let A, B, C, A', B', C' be R-modules that satisfy the following commutative and exact diagram.*

$$\begin{tikzcd}
 & \ker a \arrow[r] \arrow[d] & \ker b \arrow[r] \arrow[d] & \ker c \arrow[d] & \\
 & A \arrow[r] \arrow[d, "a"'] & B \arrow[r] \arrow[d, "b"'] & C \arrow[r] \arrow[d, "c"'] & 0 \\
0 \arrow[r] & A' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & C' \arrow[d] & \\
 & \operatorname{Coker} a \arrow[r] & \operatorname{Coker} b \arrow[r] & \operatorname{Coker} c &
\end{tikzcd}$$

where the the maps associated with the kernels and the cokernels are the natural homomorphisms. Then there is exists a homomorphism $\theta : \ker c \to \operatorname{Coker} a$ and the sequence

$$\ker a \xrightarrow{f^*} \ker b \xrightarrow{g^*} \ker c \xrightarrow{\theta} \operatorname{Coker} a \xrightarrow{\hat{f}} \operatorname{Coker} b \xrightarrow{\hat{g}} \operatorname{Coker} c$$

is exact.

Proof. First, we name the homomorphisms for convenience of the proof. Let f, g, f', g' be the homomorphisms $A \to B$, $B \to C$, $A' \to B'$ and $B' \to C'$ resp.

We next show the existence and well-definition of θ. In fact, θ can be defined as $f'^{-1} \circ d \circ g^{-1}$. To show that θ is well-defined, it suffices to show that for any $z \in \ker c$, we can determine the output of θ, $x \in \operatorname{Coker} a$ regardless of the inverse images. Suppose $z \in \ker c$ and let $y \in B$ such that $g(y) = z$ (by surjectivness of g). Regardless of this choice, we know that $g'(b(y)) = c(g(z)) = 0$ (by communtitivity) and thus $b(y) \in \ker g' = \operatorname{Im} f'$ by exactness. But f' is injective, therefore there is a unique inverse of $b(z)$, $x' \in A'$

For two representatives $y_1, y_2 \in B$ such that $g(y_1), g(y_2) \in \ker c$, we note that $g(y_1 - y_2) = 0$ thus $y_1 - y_2 \in \ker g$. By exactness, $y_1 - y_2 \in \operatorname{Im} f$. Threfore, $b(y_1 - y_2) = b(y_1) - b(y_2) = f'(a(x))$ for some x. If $f'(x'_i) = b(y_i)$, then we have $f'(x'_1) = f'(x'_2 + a(x))$, proving $x'_1 \equiv x'_2 \mod \operatorname{Im} a$ by injectivity of f'. This proves the well-definedness of θ. Clearly, θ is a homomorphisms.

It remains to show the given sequence is exact. We prove it from letf to right.

$\operatorname{Im} f^* \subseteqq \ker g^*$: Let $a(x) = 0$. By exactness of the top sequence $g(f(x)) = 0$, hence $\operatorname{Im} f|_{\ker a} = \operatorname{Im} f^* \subseteqq \ker g^*$.

$\ker g^* \subseteqq \operatorname{Im} f^*$: $g(y) = 0 \implies y = f(x)$ by exactness. By definition of f^*, the x shall be in $\ker a$.

$\operatorname{Im} g^* \subseteqq \ker \theta$: Let $z = g(y)$, for $y \in \ker b$ and let $x' + \operatorname{Im} a = \theta(z)$. By definition, y is a represenative of z and by communtitivity of the diagram and exactness of the top sequence, there exsists $x \in \ker a$ such that $b(y) = f'(a(x))$, this implies $x' = a(x) + \operatorname{Im} a$ and thus $x' = 0 \mod \operatorname{Im} a$.

$\ker \theta \subseteqq \operatorname{Im} g^*$: Let $x' = a(x)$ and let $g(y) = z$ (by surjectivity of g). Then $b(y) = f'(a(x)) = b(f(x)) \implies b(y - f(x)) = 0$. But $g(y - f(x)) = g(y) = z$ by exactness of the top sequence. Hence $y - f(x) \in \ker g^* = \ker g|_{\ker b}$.

Next we consider the induced homomorphisms

$$\hat{f}(x + \operatorname{Im} a) = f'(x) + \operatorname{Im} b$$

$$\hat{g}(y + \operatorname{Im} b) = g'(y) + \operatorname{Im} c$$

$\operatorname{Im} \theta \subseteqq \ker \hat{f}$: $x' + \operatorname{Im} a = \theta(z)$. By exactness, $g(y) = z$ and $f(x) = y$ for some x, y. By commutitivity, $b(y) = b(f(x)) = f'(a(x))$. By exactness, $x' = a(x)$ and thus $z \in \ker \theta$.

$\ker \hat{f} \subseteqq \operatorname{Im} \theta$: Let $\hat{f}(x' + \operatorname{Im} a) = y' + \operatorname{Im} b = 0$ $y' \in \operatorname{Im} b$. Then, $f'(x') = y'$ By definition, $y' = b(y)$. $c(g(y)) = g(b(y)) = g(f(x')) = 0$ and $g(y) \in \ker c$.

$\ker \hat{g} \subseteqq \operatorname{Im} \hat{f}$: Let $\hat{g}(y' + \operatorname{Im} b) = 0 + \operatorname{Im} c$. Then $g'(y') = c(z)$ for some $z \in C$. Since g is on-to, there is a y such that $g(y) = z \implies g'(y') = c(g(y))$. By commutitivity, $g'(y') = g'(b(y))$. But then $y' - b(y) \in \ker g' = \operatorname{Im} f'$ (by exactness). Thus $y' \in \operatorname{Im} \hat{f}$.

$\operatorname{Im} \hat{f} \subseteqq \ker \hat{g}$: Let $y' = f'(x') + b(y)$ for some y. Since $g' \circ f' = 0$ by exactness, $g'(y') = g'(b(y)) = c(g(y))$ (by commutitivity) $\in \operatorname{Im} c$.

$\square$