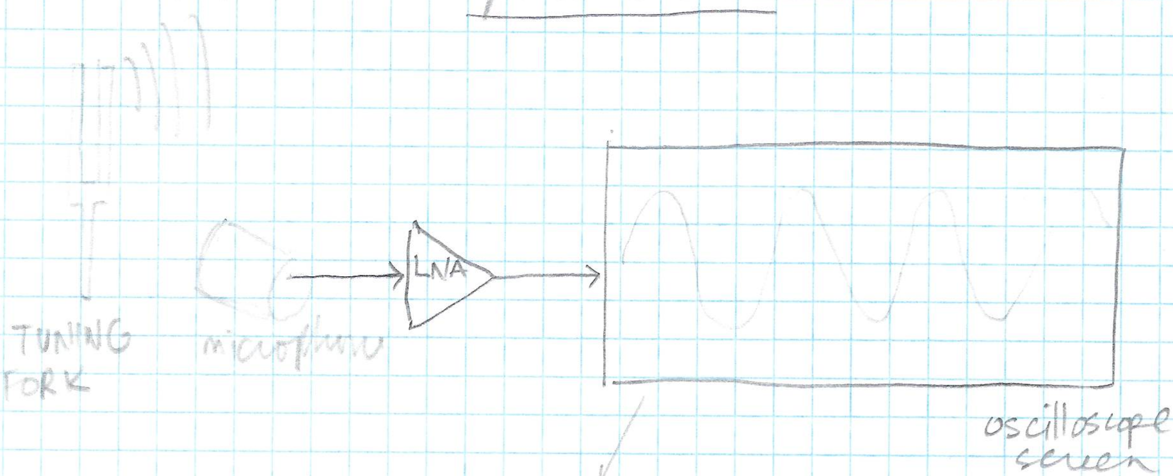
Fourier Transforms

- we distinguish between the time and frequency domain...
- the foundation of all Fourier Transforms is the recognition that many signals can be represented or decomposed into a series of sinusoids or "pure" tones
- What is a pure tone?



we see a sinusoidal variation

- even a tuning fork is not a pure tone, but close!

pure tone at $\Omega = \Omega_0$:

$$\left.\begin{array}{l}\sin(\Omega_0 t + \varphi)\\ \cos(\Omega_0 t + \varphi)\\ e^{j(\Omega_0 t + \varphi)}\end{array}\right\}\text{ all qualify}$$

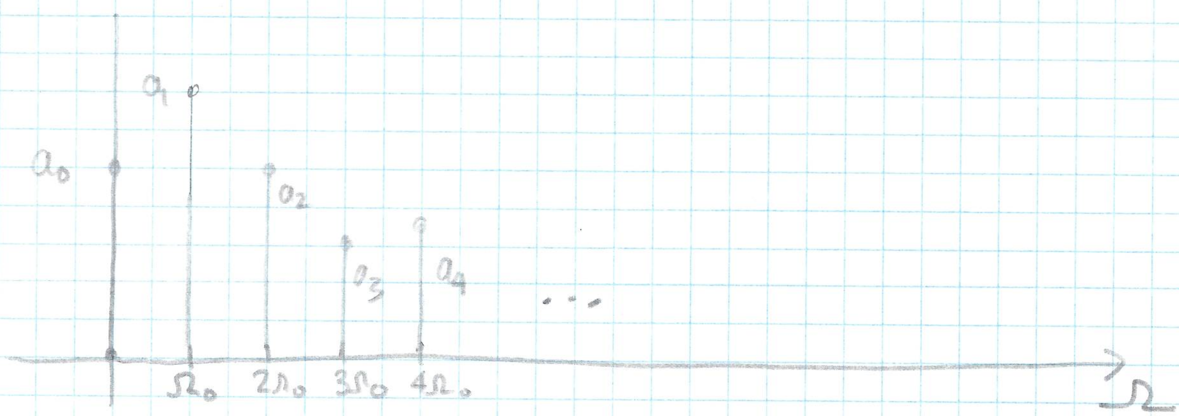
- and, indeed, we can represent signals based on Fourier Series using any of these (one or more)

- in general, given a periodic signal, $r(t)$, we have the expansion

$$r(t) = a_0 + 2\sum_{k=1}^{\infty} b_k \cos(k\Omega_0 t) - 2\sum_{k=1}^{\infty} c_k \sin(k\Omega_0 t)$$

$$= \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t}$$

→ other similar formulas exist



→ note that the spectrum is <u>discrete</u>

→ we associate discrete spectra with <u>periodicity</u> in the time domain

- for aperiodic signals in time we have the Fourier Transform:

$$R(j\Omega) = \int_{-\infty}^{\infty} r(t)\, e^{-j\Omega t}\, dt$$

$$r(t) = \int_{-\infty}^{\infty} R(j\Omega)\, e^{j\Omega t}\, d\Omega$$

The DTFT (the discrete-time Fourier Transform)

- a mathematical tool for investigating the spectral properties of discrete-time signals:

$$(1) \quad X(e^{j\omega}) \triangleq \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

which exists if $x(n)$ is absolutely summable, i.e., if $\sum_{-\infty}^{\infty} |x(n)| < \infty$.

- its inverse, the IDTFT, is given by

$$(2) \quad x(n) \triangleq \mathcal{F}^{-1}\left[X(e^{j\omega})\right] = \left(\frac{1}{2\pi}\right)\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$

- note that (1) is CONTINUOUS in the (digital) frequency variable, ω, even though $x(n)$ is DISCRETE in its independent variable, n.

Some Properties

① PERIODICITY

$X(e^{j\omega})$ is periodic in ω with period 2π, that is,

$$X(e^{j\omega}) = X\left[e^{j(\omega + 2\pi k)}\right], \quad k \in \mathbb{Z}$$

$\Rightarrow$ we only need one period of $X(e^{j\omega})$ for most purposes (since the rest of the function is redundant).

Typically, we use $\omega \in [0, 2\pi]$ or $\omega \in [-\pi, \pi]$

(2) SYMMETRY

for real-valued $x(n)$, $X(e^{j\omega})$ exhibits 'conjugate symmetry', i.e.,

$$X(e^{-j\omega}) = X^*(e^{j\omega})$$

that is,

$$Re[X(e^{-j\omega})] = Re[X(e^{j\omega})] \quad \text{(even symmetry)}$$
$$Im[X(e^{-j\omega})] = -Im[X(e^{j\omega})] \quad \text{(odd symmetry)}$$

useful in plotting
$$\begin{cases} |X(e^{-j\omega})| = |X(e^{j\omega})| & \text{(even symmetry)} \\ \angle X(e^{-j\omega}) = -\angle X(e^{j\omega}) & \text{(odd symmetry)} \end{cases}$$

$\Rightarrow$ when plotting, we only need to view half of the period (for real signals), typically, $\omega \in [0, \pi]$.

other properties: linearity, time/frequency shifting, conjugation, convolution, etc., please see your text.

- If $x(n)$ is of finite duration, then we can compute (1) numerically.

- also, we can compute a <u>sampled-version</u> of $X(e^{j\omega})$, at equi-spaced frequencies

  - recall that if $x(n)$ is periodic, then it is equivalent to a sum of discrete sinusoids — the DTFT becomes discrete, i.e., the DTFT and the <u>Discrete Fourier Series</u> representation of the signal become equivalent.