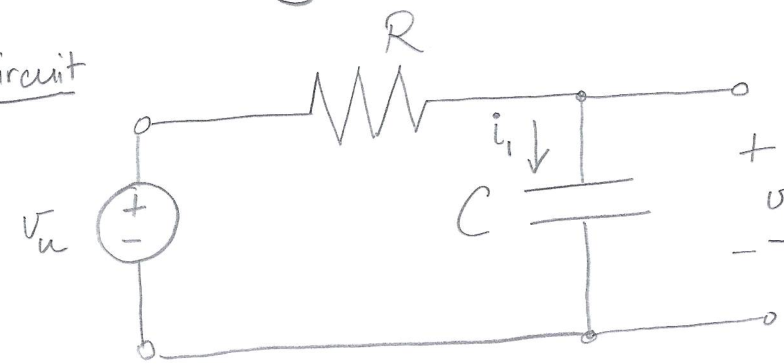
System Examples

Find the ODE (ordinary differential equation) descriptions for the following systems:

① RC circuit



→ "working backwards":

Note: $i_c \downarrow$ C $\overset{+}{\underset{-}{v_c}}$ ⇒ $i_c = C\dot{v}_c$ (capacitor physics)

$$\Rightarrow \quad C\dot{v}_y = i_1 \qquad (1)$$

$$i_1 = \frac{v_u - v_y}{R} \qquad (2)$$

$$\Rightarrow (2) \rightarrow (1): \quad C\dot{v}_y = \frac{v_u - v_y}{R}$$

$$\therefore \quad RC\dot{v}_y + v_y = v_u$$

or $\quad \dot{v}_y + \left(\frac{1}{RC}\right) v_y = \left(\frac{1}{RC}\right) v_u$

$$\sim \boxed{\dot{y} + \alpha_0 y = \beta_0 u}$$

where $\alpha_0 = \beta_0 = \frac{1}{RC}$

linear time-invariant ODE

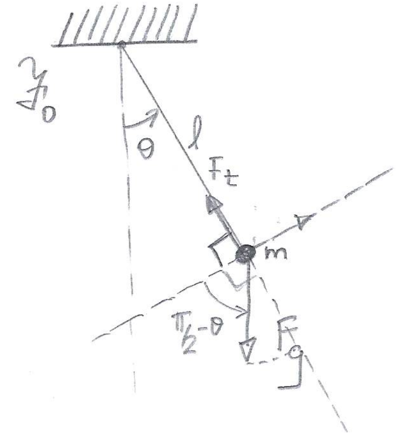
State Model let $x = y$:

$$\dot{x} = -\alpha_0 x + \beta_0 u$$

initial condition → $x(0) = x_0 \in \mathbb{R}$

done!

(2) simple pendulum



rotational version of Netwon's 2nd Law : $ml^2\ddot{\theta} = -\left[|F_g|\cos(\tfrac{\pi}{2}-\theta)\right]l$

$$\Rightarrow ml^2\ddot{\theta} = -mgl\sin\theta$$

$$\Rightarrow l\ddot{\theta} = -g\sin\theta$$

$$\Rightarrow \ddot{\theta} + \left(\frac{g}{l}\right)\sin\theta = 0$$

let $y = \theta$

$\sim \ddot{y} + \left(\frac{g}{l}\right)\sin y = 0$ → nonlinear, time-invariant system w/ no input

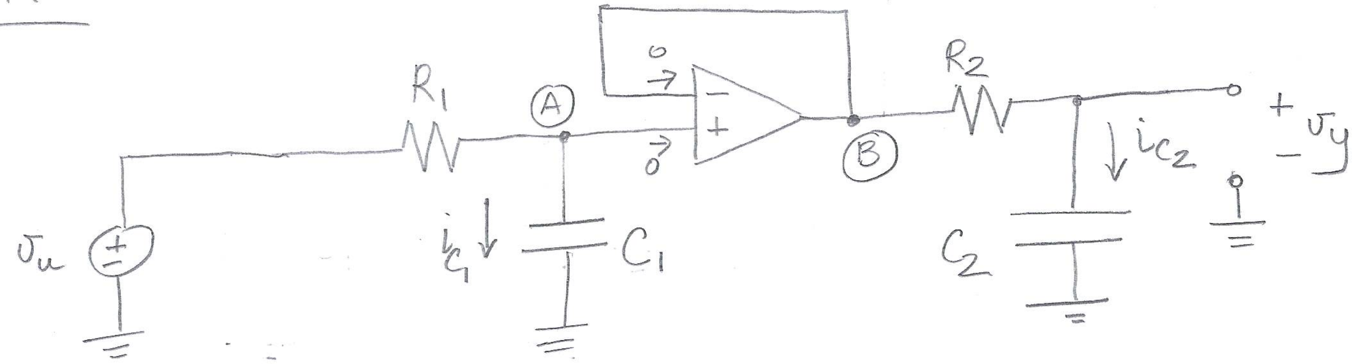
State Model

let $x_1 = y$, $x_2 = \dot{y}$:

$$\Rightarrow \dot{x}_2 = -\left(\frac{g}{l}\right)\sin x_1$$

$$\therefore \boxed{\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\left(\frac{g}{l}\right)\sin x_1 \\ x_1(0) &= x_{1,0} \in \mathbb{R} \\ x_2(0) &= x_{2,0} \in \mathbb{R} \end{aligned}}$$ — state model representation

active-RC



ideal opamp : $v_B = v_A$

$$C_2 \dot{v}_y = i_{C_2} \quad (1)$$

$$i_{C_2} = i_{R_2} = \frac{v_B - v_y}{R_2} = \frac{v_A - v_y}{R_2} \quad (2)$$

(2) → (1) : $C_2 \dot{v}_y = \left(-\frac{1}{R_2}\right) v_y + \left(\frac{1}{R_2}\right) v_A$

$$\Rightarrow \boxed{\dot{v}_y + \left(\frac{1}{R_2 C_2}\right) v_y = \left(\frac{1}{R_2 C_2}\right) v_A \quad (3)}$$

eliminate v_A :

$$\frac{v_u - v_A}{R_1} = C_1 \dot{v}_A$$

$$\Rightarrow C_1 \dot{v}_A = \left(-\frac{1}{R_1}\right) v_A + \left(\frac{1}{R_1}\right) v_u$$

$$\Rightarrow \dot{v}_A + \left(\frac{1}{R_1 C_1}\right) v_A = \left(\frac{1}{R_1 C_1}\right) v_u \quad (4)$$

differentiate (3) : $\ddot{v}_y + \left(\frac{1}{R_2 C_2}\right) \dot{v}_y = \left(\frac{1}{R_2 C_2}\right) \dot{v}_A \quad (5)$

(4) → (5) : $\ddot{v}_y + \left(\frac{1}{R_2 C_2}\right) \dot{v}_y = \left(\frac{1}{R_2 C_2}\right) \left[\left(\frac{1}{R_1 C_1}\right) v_u - \left(\frac{1}{R_1 C_1}\right) v_A\right] \quad (6)$

but, from (3):

$$v_A = (R_2C_2)\dot{v}_y + v_y \quad (7)$$

(7) → (6): $$\ddot{v}_y + \left(\frac{1}{R_2C_2}\right)\dot{v}_y = \left(\frac{1}{R_2C_2}\right)\left\{\left(\frac{1}{R_1C_1}\right)v_u - \left(\frac{1}{R_1C_1}\right)\left[(R_2C_2)\dot{v}_y + v_y\right]\right\}$$

$$\Rightarrow \ddot{v}_y + \left(\frac{1}{R_2C_2}\right)\dot{v}_y = \left(\frac{1}{R_1R_2C_1C_2}\right)v_u - \left(\frac{1}{R_1C_1}\right)\dot{v}_y - \left(\frac{1}{R_2C_2}\right)v_y$$

$$\Rightarrow \ddot{v}_y + \underbrace{\left[\left(\frac{1}{R_2C_2}\right) + \left(\frac{1}{R_1C_1}\right)\right]}_{\alpha_1}\dot{v}_y + \underbrace{\left(\frac{1}{R_2C_2}\right)}_{\alpha_0}v_y = \underbrace{\left(\frac{1}{R_1R_2C_1C_2}\right)}_{\beta_0}v_u$$

letting $y = v_y$, $u = v_u$:

$$\sim \boxed{\ddot{y} + \alpha_1\dot{y} + \alpha_0 y = \beta_0 u}$$

let $x_1 = y$, $x_2 = \dot{y}$

$$\Rightarrow \dot{x}_2 = -\alpha_1 x_2 - \alpha_0 x_1 + \beta_0 u$$

$$\therefore \begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \underbrace{\begin{bmatrix}0 & 1\\ -\alpha_0 & -\alpha_1\end{bmatrix}}_{A}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \underbrace{\begin{bmatrix}0\\ \beta_0\end{bmatrix}}_{b}u$$

$$y = \underbrace{\begin{bmatrix}1 & 0\end{bmatrix}}_{c}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

$$\therefore \left.\begin{aligned}\dot{x} &= Ax + bu\\ y &= cx\end{aligned}\right\} \text{Matrix Form}$$