Properties of Signals

RECALL

→ analog signals typically represented using real-valued functions of a real variable, t, as in

$$x(t),\ u(t),\ y(t)$$

→ we can use the mathematical notation

$$x : \mathbb{R} \to \mathbb{R}$$

as well.

→ discrete-time signals are represented using real-valued functions of an integer variable, n, as in

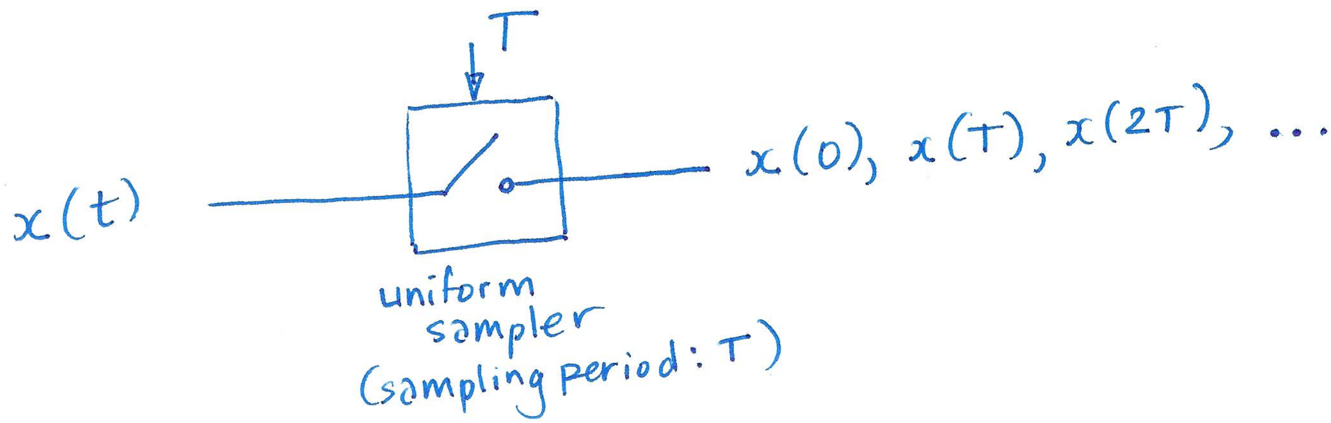
$$x(n),\ u(n),\ y(n)$$

→ mathematically:

$$x : \mathbb{Z} \to \mathbb{R}$$

→ discrete-time signals can also be referred to as sequences.

- while discrete-time signals are things "onto themselves" they are often viewed as being derived from analog signals in a process known as <u>sampling</u>.



and we define

$$x_d(n) = x(nT) \quad \forall\, n$$

"for all"

thus the discrete-time signal, $x_d(n)$, is obtained from the analog signal, $x(t)$.

Recall: Shannon's Sampling Theorem

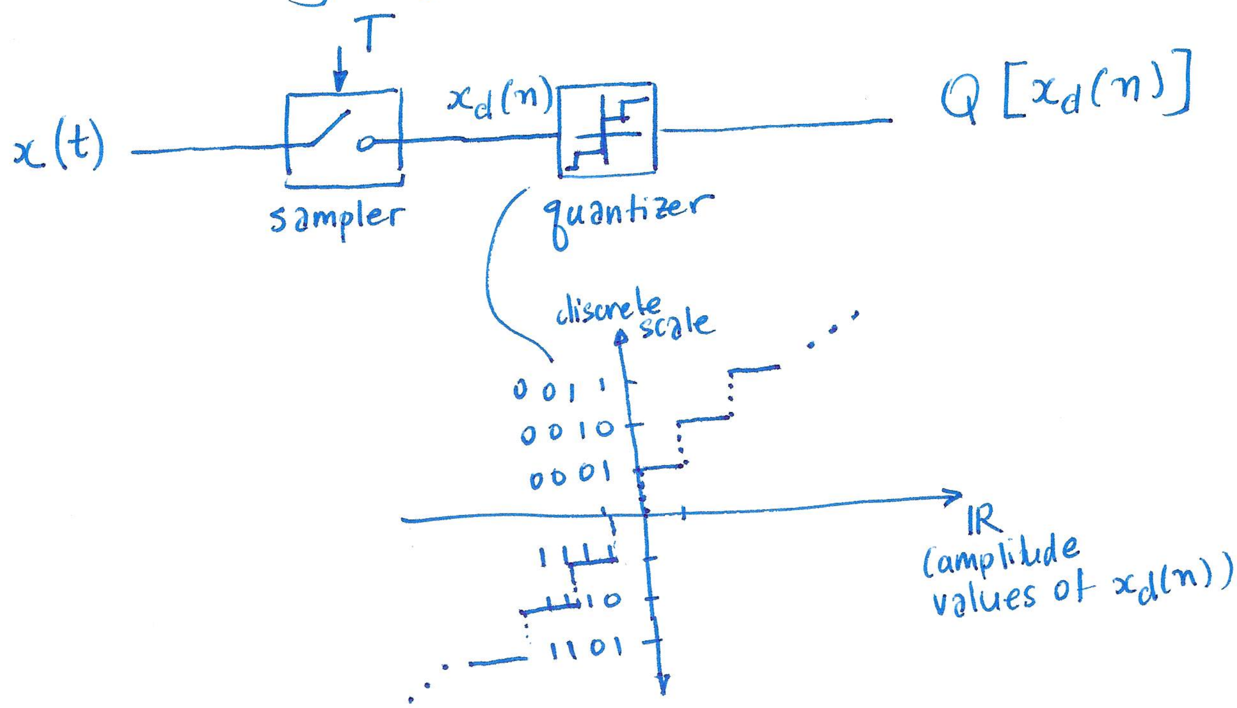
The analog (continuous-time) signal can be recovered ("perfectly reconstructed") from $x_d(n)$ provided that:

i) $x(t)$ is <u>band-limited</u> with band limit, $\Omega_B > 0$
(i.e., if $X(j\Omega) = \mathcal{F}[x(t)]$, then $|X(j\Omega)| = 0 \;\; \forall\, |\Omega| \geq \Omega_B$)

ii) $\Omega_S = \dfrac{2\pi}{T_s} > \Omega_B$, where T_s is the sampling period. ▥

Amplitude Quantization

If, in addition to sampling (or "time quantization" or "temporal quantization") we quantize the amplitude of the analog signal, a digital signal is produced



we are often concerned with the resolution of the discretized amplitude scale, often expressed as the number of bits needed to represent the staircase characteristic.

Elementary Signals

- earlier in the course, we reviewed elementary signals in both continuous- and discrete-time, used for a variety of analyses:

→ continuous-time: delta function, unit step function

$\boxed{\int_{-\varepsilon}^{\varepsilon} \delta(t) = 1 \text{ for all } \varepsilon > 0}$ → $\delta(t)$, $u_s(t)$

→ many ways to define $\delta(t)$!

→ discrete-time: unit sample sequence, unit step sequence

$\boxed{\delta[n] = \begin{cases} 1, & n=0 \\ 0, & n \neq 0 \end{cases} \quad \forall n \in \mathbb{Z}}$ → $\delta(n)$, $u_s(n)$

we can write: $\boxed{u_s(n) = \sum_{k=-\infty}^{n} \delta(n)}$ → homework: prove this

- we also make extensive use of sinusoids, both real- and complex-valued (note: a complex-value can be seen as a pair of real values, the Real and Imaginary components)

→ CT: $\sin(\omega_0 t + \varphi)$, $\cos(\omega_0 t + \varphi)$, $e^{j(\omega_0 t + \varphi)}$

NOTE: Euler's Theorem

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$\therefore \; e^{j(\omega_0 t+\varphi)} = \cos(\omega_0 t+\varphi) + j\sin(\omega_0 t+\varphi)$$

→ DT: $\sin(\omega_0 n + N)$, $\cos(\omega_0 n + N)$, $e^{j(\omega_0 n + N)}$

Consider: $x(n) = Ae^{j(\omega_0 + 2\pi r)n}$, $A \in \mathbb{R}$, $r \in \mathbb{Z}$, for all $n \in \mathbb{Z}$.

So we have:

$$x(n) = A e^{j\omega_0 n} \underbrace{e^{j(2\pi r)n}}$$

$$e^{j(2\pi r)n} = \cos(2\pi r n) + j\sin(2\pi r n)$$

$$= \cos(2\pi L) + j\sin(2\pi L),$$

where $L = rn$, is just an integer

$$\Rightarrow e^{j(2\pi r)n} = 1 + j(0) = 1$$

$$\therefore \; x(n) = A e^{j\omega_0 n}$$

→ so we see that complex exponentials in discrete-time are "periodic in ω", with frequencies differing by integer multiples of 2π, are indistinguishable from each other.

Q: is this true in continuous-time? Why or why not?

Temporal Periodicity

CT: $x(t)$ is periodic with period T if,
for any $t \in \mathbb{R}$,

$$x(t) = x(t+T)$$

→ the period T is also a real number.

DT: $x(n)$ is periodic with period N if,
for any $n \in \mathbb{Z}$,

$$x(n) = x(n+N),$$

where $N \in \mathbb{Z}$.

Ex.// $x(t) = 3\cos(\Omega_0 t + 7)$

- find the period T:

→ if T exists, it satisfies:

$$x(t) = x(t+T) \quad \forall\, t \in \mathbb{R}$$

$$3\cos(\Omega_0 t + 7) = 3\cos(\Omega_0 t + \Omega_0 T + 7)$$

NOTE: $\cos(\theta)$ is periodic with 2π, so that

$$\cos(\theta) = \cos(\theta + 2\pi)$$

setting $\theta = \Omega_0 t + 7$, and

$$2\pi = \Omega_0 T,$$

~~whe~~ we find $\boxed{T = \frac{2\pi}{\Omega_0}}$

setting $T = \frac{2\pi}{\Omega_0}$ yields:

$$x(t) = 3\cos\left[\Omega_0 t + \Omega_0\left(\frac{2\pi}{\Omega_0}\right) + 7\right]$$

$$= 3\cos\left[\Omega_0 t + 7 + 2\pi\right]$$

$$= 3\left[\cos(\Omega_0 t + 7)\underbrace{\cos(2\pi)}_{1} - \sin(\Omega_0 t + 7)\underbrace{\sin(2\pi)}_{0}\right]$$

$$= 3\cos(\Omega_0 t + 7)$$

which proves that the period is $T = \frac{2\pi}{\Omega_0}$ ∎

Ex.//

$$x(n) = 3\cos(\omega_0 n + 7)$$

we want to find N such that

$$x(n) = x(n+N)$$

for all n. Recall that N itself must be an integer.

write: $x(n+N) = 3\cos\left[\omega_0(n+N) + 7\right]$

$$= 3\cos\left[\underbrace{(\omega_0 n + 7)}_{\theta} + \underbrace{\omega_0 N}_{2\pi k,}\right] \quad \text{for some } k \in \mathbb{Z}$$

$\therefore$ we want

$$\omega_0 N = 2\pi k$$

$$\Rightarrow N = \frac{2\pi k}{\omega_0}$$

However, in order for N to be an integer, $\frac{2\pi k}{\omega_0}$ must be an integer;

if we write $\omega_0 = 2\pi f_0$, then

$$\frac{2\pi k}{\omega_0} = \frac{2\pi k}{2\pi f_0} = \frac{k}{f_0}$$

so we need

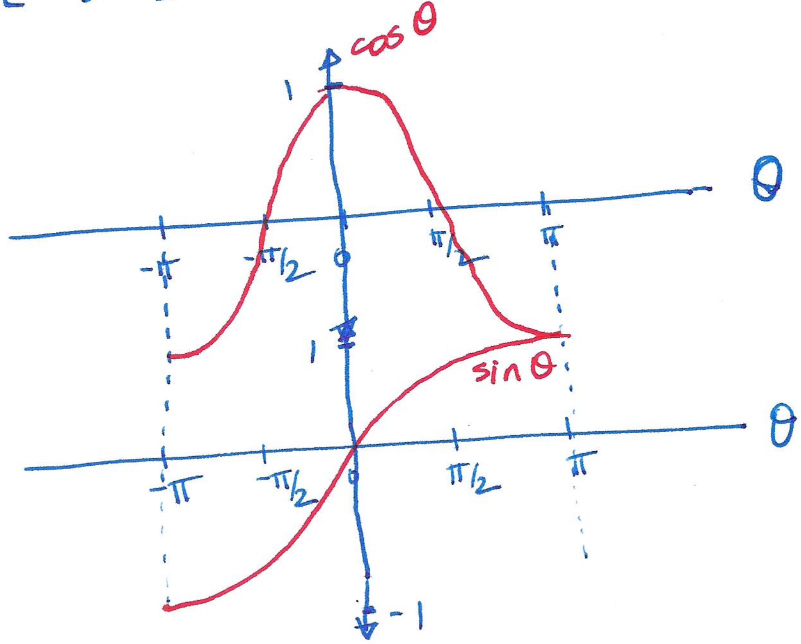
$$\frac{k}{f_0} = m, \quad \text{for some } m \in \mathbb{Z}$$

$$\Rightarrow f_0 = \frac{k}{m}$$

$\therefore f_0$ must be a ratio of two integers,
in otherwords, a rational number. $\blacksquare$

Even & Odd Signals

- a symmetry property:
  - i) $f(x)$ is <u>even</u> if $f(x) = f(-x)$
  - ii) $f(x)$ is <u>odd</u> if $f(x) = -f(-x)$
- consider $\cos(\theta)$ & $\sin(\theta)$ in the domain $\theta \in [-\pi, \pi]$



which is even, which is odd?

Aperiodic Signals

→ signals which are not periodic!

→ in the frequency domain, such signals have continuous (smooth) rather than discrete spectra — recall that only periodic signals can be described exactly by a Fourier Series