IIIT-Bangalore,
Mathematics for ML (GEN 512)
Assignment Set 2

(Orthogonality)

03-07-2018

1. The equation $x - 3y - 4z = 0$ describes a plane P in $\mathbb{R}^3$ (actually a subspace).
   (a) The plane $\mathbf{P}$ is the nullspace of what 1 by 3 matrix A?
   (b) Find a basis s_1, s_2 of special solutions of $x - 3y - 4z = 0$ (these would be the columns of the nullspace matrix N).
   (c) Also find a basis for the line $\mathbf{P}^{\perp}$ that is perpendicular to $\mathbf{P}$.
   (d) Split $v = (6, 4, 5)$ into its nullspace component v_n in $\mathbf{P}$ and its row space component v_r in $\mathbf{P}^{\perp}$.

2. If $\mathbf{P}$ is the plane of vectors in $\mathbb{R}^4$ satisfying $x_1 + x_2 + x_3 + x_4 = 0$, write a basis for $\mathbf{P}^{\perp}$. Construct a matrix that has $\mathbf{P}$ as its nullspace.

3. In $\mathbb{R}^m$, suppose I give you $\mathbf{b}$ and $\mathbf{p}$, and $\mathbf{p}$ is a combination of $\mathbf{a}_1, \ldots, \mathbf{a}_n$. How would you test to see if $\mathbf{p}$ is the projection of $\mathbf{b}$ onto the subspace spanned by the $\mathbf{a}$'s?

4. Suppose A is the 4 by 4 identity matrix with its last column removed. A is 4 by 3. Project $\mathbf{b} = (1, 2, 3, 4)$ onto the column space of A. What shape is the projection matrix P and what is P?

5. Find the parabola $C + Dt + Et^2$ that comes closest (least square error) to the values $\mathrm{b} = (0, 0, 1, 0, 0)$ at the times $t = -2, -1, 0, 1, 2$. First write down the five equations $Ax = \mathbf{b}$ in three unknowns $\mathbf{x} = (C, D, E)$ for a parabola to go through the five points. No solution because no such parabola exists. Solve $A^{\mathsf{T}}A\hat{\mathbf{x}} = A^{\mathsf{T}}\mathbf{b}$.

6. Find the plane that gives the best fit to the four values $\mathbf{b} = (0, 1, 3, 4)$ at the corners $(1, 0)$ and $(0, 1)$ and $(-1, 0)$ and $(0, -1)$ of a square. The equations $C + Dx + Ey = b$ at those four points are $A\mathbf{x} = \mathbf{b}$ with three unknowns $\mathbf{x} = (C, D, E)$. What is A? At the center $(0, 0)$ of the square, show that $C + Dx + Ey =$ average of the b's.

7. Find $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$ (orthonormal) as combinations of $\mathbf{a}, \mathbf{b}, \mathbf{c}$ (independent columns). Then write A as QR:
$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{bmatrix}.$$

8. (a) Find a basis for the subspace S in $\mathbf{R}^4$ spanned by all solutions of $x_1+x_2+x_3-x_4 = 0$.
   (b) Find a basis for the orthogonal complement $S^{\perp}$.
   (c) Find $\mathbf{b}_1$ in S and $\mathbf{b}_2$ in $S^{\perp}$ so that $\mathbf{b}_1 + \mathbf{b}_2 = \mathbf{b} = (1, 1, 1, 1)$.