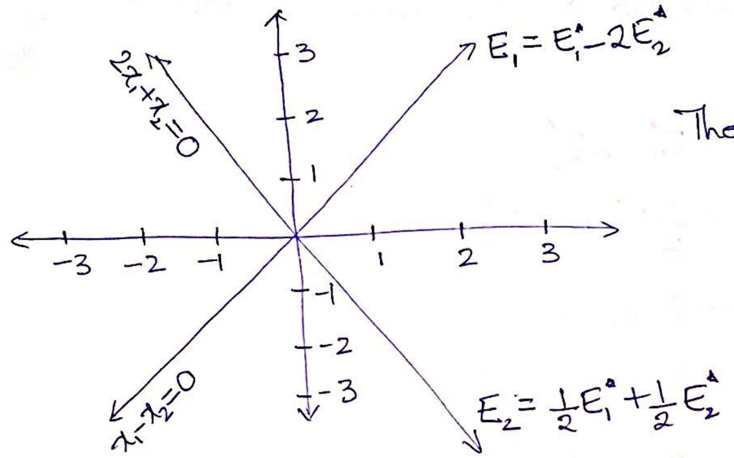
⇒ Systems of linear equations.

Consider two systems of linear equations.

(A)
$$x_1 - x_2 = 0 \text{ --- } E_1$$
$$2x_1 + x_2 = 0 \text{ --- } E_2$$

(B)
$$3x_1 + x_2 = 0 \text{ --- } E_1^*$$
$$x_1 + x_2 = 0 \text{ --- } E_2^*$$

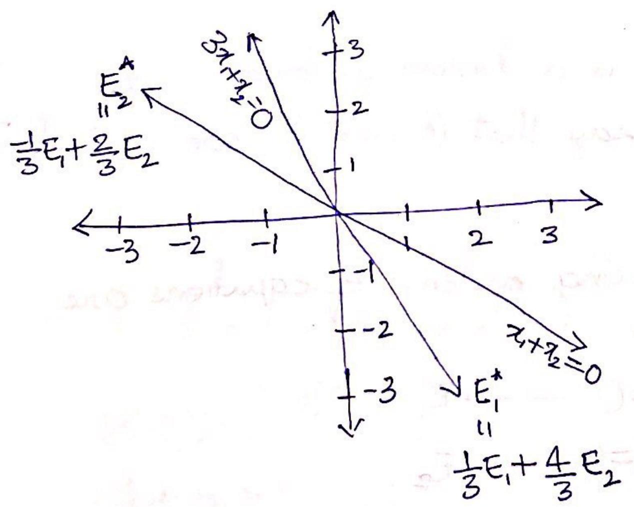
Sketch the system A.



The solution set,

$$S_A = \{(0,0)\}$$

Sketch the system B.



The solution set of B

$$S_B = \{(0,0)\}$$

→ Prove that $E_1^* = \frac{1}{3}E_1 + \frac{4}{3}E_2$

$$E_2^* = \frac{-1}{3}E_1 + \frac{2}{3}E_2$$

$$(3x_1+x_2) = \frac{1}{3}(x_1-x_2) + \frac{4}{3}(2x_1+x_2)$$

$$= \frac{9}{3}x_1 + \frac{3}{3}x_2 = 3x_1+x_2$$

$$(x_1+x_2) = \frac{-1}{3}(x_1-x_2) + \frac{2}{3}(2x_1+x_2)$$

$$= \frac{3}{3}x_1 + \frac{3}{3}x_2 = x_1+x_2$$

<u>Note:</u> Every equation in (B) is a linear combination of equations in (A).

→ Prove that $E_1 = E_1^* - 2E_2^*$

$$E_2 = \frac{1}{2}E_1^* + \frac{1}{2}E_2^*$$

$$(x_1-x_2) = (3x_1+x_2) - 2(x_1+x_2)$$

$$= x_1 - x_2$$

$$(2x_1+x_2) = \frac{1}{2}(3x_1+x_2) + \frac{1}{2}(x_1+x_2)$$

$$= 2x_1+x_2$$

<u>Note:</u> Every ~~solution~~ equation in (A) is a linear combination of equations in (B). So we say that (A) and (B) are equivalent systems.

<u>Question)</u> Test if the following systems of equations are equivalent

1)

(A)
$$-x_1 + x_2 + 4x_3 = 0 \text{ —— } E_1$$
$$x_1 + 3x_2 + 8x_3 = 0 \text{ —— } E_2$$
$$\frac{1}{2}x_1 + x_2 + \frac{5}{2}x_3 = 0 \text{ —— } E_3$$

(B)
$$\left.\begin{array}{l} x_1 - x_3 = 0 \text{ —— } E_1^* \\ x_2 + 3x_3 = 0 \text{ —— } E_2^* \end{array}\right\}$$ Soln space $S_B = \{(a, -3a, a), a \in \mathbb{R}\}$

Soln.

(i) Let $E_1 = aE_1^* + bE_2^*$

$$(-x_1+x_2+4x_3) = a(x_1-x_3)+b(x_2+3x_3)$$
$$= ax_1 + bx_2 + (-a+3b)x_3$$

$\therefore a=-1,\ b=1$

Hence, $E_1 = -E_1^* + E_2^*$

(ii) Let $E_2 = cE_1^* + dE_2^*$

$$(x_1+3x_2+8x_3) = c(x_1-x_3)+d(x_2+3x_3)$$
$$= cx_1+dx_2+(-c+3d)x_3.$$

$\therefore c=1,\ d=3$

Hence, $E_2 = E_1^* + 3E_2^*$

(iii) Let $E_3 = eE_1^* + fE_2^*$

$$\left(\tfrac{1}{2}x_1+x_2+\tfrac{5}{2}x_3\right) = e(x_1-x_3)+f(x_2+3x_3)$$
$$= ex_1+fx_2+(-e+3f)x_3$$

$\therefore e=\frac{1}{2},\ f=1$

Hence, $E_3 = \frac{1}{2}E_1^* + E_2^*$

(iv) Let $E_1^* = aE_1 + bE_2 + cE_3$

$$(x_1-x_3) = a(-x_1+x_2+4x_3)+b(x_1+3x_2+8x_3)+c\left(\tfrac{1}{2}x_1+x_2+\tfrac{5}{2}x_3\right)$$
$$= \left(-a+b+\tfrac{c}{2}\right)x_1+(a+3b+c)x_2+\left(4a+8b+\tfrac{5c}{2}\right)x_3$$

$-a+b+\frac{c}{2}=1$ — ①, $a+3b+c=0$ — ②, $4a+8b+\frac{5c}{2}=-1$ — ③

$\cancel{-2a+2b+c=2}$ $\qquad$ $\cancel{8a+16b+5c=-2}$

$$-a+b+\frac{c}{2}=1$$
$$a+3b+c=0$$
$$\overline{4b+\frac{3c}{2}=1}$$
$$8b+3c=1$$

$$-4a+4b+2c=4$$
$$4a+8b+\frac{5c}{2}=-1$$
$$\overline{12b+\frac{9c}{2}=3}$$

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Notation: A system of 'm' equations in 'n' unknowns is

$$\left.\begin{array}{l} A_{11}x_1 + A_{12}x_2 + A_{13}x_3 + \cdots + A_{1n}x_n = y_1 \\ A_{21}x_1 + A_{22}x_2 + A_{23}x_3 + \cdots + A_{2n}x_n = y_2 \\ \cdots \\ \cdots \\ A_{m1}x_1 + A_{m2}x_2 + A_{m3}x_3 + \cdots + A_{mn}x_n = y_m. \end{array}\right\} \text{I}$$

$A_{ij}, y_i \in \mathbb{R}$ or ϕ $\quad 1 \leq i \leq m$

$1 \leq j \leq n$

$x_j \leftarrow$ unknown $j = 1, 2, \ldots, n$

We can rewrite it as

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} & \cdots & A_{1n} \\ A_{21} & A_{22} & A_{23} & \cdots & A_{2n} \\ \cdots & & & & \\ \cdots & & & & \\ A_{m1} & A_{m2} & A_{m3} & \cdots & A_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}$$

⊛ $$AX = Y$$

HW) Sec 1.2 Q4

Q) Test the following systems as in ex.2.

$$\left.\begin{array}{l} 2x_1 + (-1+i)x_2 + x_4 = 0 \quad \text{—} E_1 \\ 3x_2 - 2ix_3 + 5x_4 = 0 \quad \text{—} E_2 \end{array}\right.$$

$$\left.\begin{array}{l} \left(1+\frac{i}{2}\right)x_1 + 8x_2 - ix_3 - x_4 = 0 \quad \text{—} E_1^* \\ \frac{2}{3}x_1 - \frac{1}{2}x_2 + x_3 + 7x_4 = 0 \quad \text{—} E_2^* \end{array}\right.$$

$E_1 = aE_1^* + bE_2^*$

$$2x_1 + (-1+i)x_2 + x_4 \cancel{=0} = \left[\left(1+\frac{i}{2}\right)x_1 + 8x_2 - ix_3 - x_4\right]a +$$

$$\left[\frac{2}{3}x_1 - \frac{1}{2}x_2 + x_3 + 7x_4\right]b$$

$$= \cancel{\left(\frac{5}{3}+\frac{i}{2}\right)x_1 + \frac{15}{2}x_2 + (1-i)x_3 +}$$

$$= \cancel{\$}\left(a+\frac{a}{2}i+\frac{2b}{3}\right)x_1 + \left(8a-\frac{b}{2}\right)x_2 + (-ai+b)x_3 + (-a+7b)x_4$$

since the coefficient of x_3 on LHS is zero

$-ai+b=0$

$\Rightarrow a=0,\ b=0$

On substituting the values of 'a' and 'b' in the equation, the equality is not maintained.

Therefore, the two systems of equations are not equivalent.

*Soln: These systems are not equivalent. Call the two equations in the first system E_1 and E_2 and the equations in the second system E_1^* and E_2^*. Then if $E_2^* = aE_1 + bE_2$ since E_2 does not have x_1 we must have $a=\frac{1}{3}$. But then to get the coefficient of x_4, we would need $7x_4 = \frac{1}{3}x_4 + 5bx_4$. That forces $b=\frac{4}{3}$. But if $a=\frac{1}{3}$ and $b=\frac{4}{3}$ then the coefficient of x_3 would have to be $-2i\frac{4}{3}$ which does not equal 1. Therefore, the system cannot be equivalent

$$E_2^* = aE_1 + bE_2$$

$$\left(\frac{2}{3}x_1 - \frac{1}{2}x_2 + x_3 + 7x_4\right) = a\left(2x_1 + (-1+i)x_2 + x_4\right) + b\left(3x_2 - 2ix_3 + 5x_4\right)$$

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* $AX = Y$

$A = [A_{ij}]_{m \times n}$

$X = [x_j]_{n \times 1}$

$Y = [y_i]_{m \times 1}$

The solution set,

$S_I = \{X \in R^n : AX = Y\}$

<u>Note:</u> $AX = 0$, i.e. $Y = 0$
is called a homogeneous system.

Let us multiply equations in (I) by $c_1, c_2, c_3, \ldots, c_m$ ~~etc~~ respectively

$$\left.\begin{aligned} c_1A_{11}x_1 + c_1A_{12}x_2 + \cdots + c_1A_{1n}x_n &= c_1y_1 \\ c_2A_{21}x_1 + c_2A_{22}x_2 + \cdots + c_2A_{2n}x_n &= c_2y_2 \\ &\cdots \\ c_mA_{m1}x_1 + c_mA_{m2}x_2 + \cdots + c_mA_{mn}x_n &= c_my_m. \end{aligned}\right\} \text{(II)}$$

<u>Note:</u> Every solution of (I) is a solution of (II)

Let us add all equations in (II)

$$\left.\begin{aligned} &(c_1A_{11} + c_2A_{21} + \cdots + c_mA_{m1})x_1 + \\ &(c_1A_{12} + c_2A_{22} + \cdots + c_mA_{m2})x_2 + \\ &\cdots \\ &(c_1A_{1n} + c_2A_{2n} + \cdots + c_mA_{mn})x_n \\ &= c_1y_1 + c_2y_2 + \cdots + c_my_m \end{aligned}\right\} \text{(III)}$$

(III) is nothing but the linear combination of equations in (I).

Every solution of (I) is a solution of (II), every solution of (II) is a soln. of (III) ~~&~~ by transitivity. But converse may not be true
$\therefore$ every solut^n of (I) is a soln of (III)

<u>Note:</u> Every se
$\Rightarrow$ Every
<u>Note:</u> (III) is cal
(III) $= c_1E_1$
(A) $(E_1, E$
(B) $(E_1^*, E$
<u>Definition:</u> We
<u>equivalent</u> if
combination
<u>Theorem 1:</u> Two
have exactly
<u>Proof:</u> Let (A)
which are eq
Since every
equations in
$\Rightarrow S_A \subseteq$
Since every
equations in
$\Rightarrow S_B \subseteq$
From (1) and

Note: Every solution of (II) is a solution of (III)

$\Rightarrow$ Every solution of (I) is a solution of (III)

Note: (III) is called a linear combination of equations in (I)

$$\text{(III)} = c_1E_1 + c_2E_2 + \dots + c_mE_m$$

(A) $(E_1, E_2, \dots E_m)$ $\qquad S_A \subseteq S_B$

(B) $(E_1^*, E_2^*, \dots E_k^{\wedge})$ $\qquad S_B \subseteq S_A$

Definition: We say two systems (A) and (B) of linear equations are equivalent if every equation in (A) is a linear combination of equations in (B) and vice versa.

Theorem 1: Two equivalent systems of linear equations have exactly same solutions.

Proof: Let (A) and (B) be two systems of linear equations which are equivalent.

Since every equation in (B) is a linear combination of equations in (A),

$\Rightarrow S_A \subseteq S_B$ ——— (1)

Since every equation in (A) is a linear combination of equations in (B),

$\Rightarrow S_B \subseteq S_A$ ——— (2)

From (1) and (2), $S_A = S_B$

Action plan:

To solve a system a linear equations, find a simpler equivalent system

Required mathematical structure ← Field

⇒ Field: A set 'F' with two operations, addition and multiplication, is called a field if it satisfies the following axioms.

Axiom 1) For all $x, y \in F$, $x+y \in F$

2) $x+y = y+x$, $\forall\, x, y \in F$ (commutative)

3) $x+(y+z) = (x+y)+z$, $\forall\, x, y, z \in F$ (associative law)

4) There exists an element 0 (additive identity) in F such that $x+0 = x$, $\forall\, x \in F$.

5) For an element $x \in F$, there exists an element ~~...~~ $-x \in F$ such that $x+(-x) = 0$.

6) For all $x, y \in F$, $xy \in F$

7) $xy = yx$ $\forall\, x, y \in F$

8) $x(yz) = (xy)z$ $\forall\, x, y, z \in F$

9) There is an element 1 (multiplicative identity) in F such that $x.1 = x$ $\forall\, x \in F$

10) For a non-zero element $x \in F$, there exists $\frac{1}{x} \in F$ such that $x\left(\frac{1}{x}\right) = 1$

11) Distributive law

$$x(y+z) = xy + xz, \ \forall x, y, z \in F$$

eg. 1) The set of all rational numbers with the usual addition and multiplication.

2) The set of integers is <u>not a</u> field.

3) Set of real numbers with usual addition and multiplication is a field.

4) Let $F = \{(a,b) : a, b \in R\}$

$(a,b) + (c,d) = (a+c, b+d)$

$(a,b) \cdot (c,d) = (\underset{ac-bd}{\cancel{ad-bc}}, \underset{ad+bc}{\cancel{ac+bd}})$

This field is called complex numbers.

Prove that F is a field.

<u>Note:</u> 1) $(a,b) = a+ib$

2) $i = (0,1)$

There is ordering in the real no. system but not in complex numbers, F. The elements of real no. system can be compared whereas complex numbers cannot be compared.

<u>Note:</u> An element of a field is called a scalar.

$A \times B = \{(a,b) : a \in A, b \in B\}$

relation : $R \subseteq A \times B$.

<u>Definition:</u> An $m \times n$ matrix A over the field F is a function from $\left\{(i,j) : \begin{matrix} 1 \leq i \leq m \\ 1 \leq j \leq n \end{matrix}\right\}$ to F.

i.e $A(i,j) = A_{ij}$

Note 1:
$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= 0 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= 0 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= 0 \end{aligned}\right\} \quad \text{I}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$AX = 0$$

$\boxed{\text{For } c \neq 0}$

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= 0 \\ ca_{21}x_1 + ca_{22}x_2 + ca_{23}x_3 &= 0 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= 0 \end{aligned}\right\} \quad \text{II}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ ca_{21} & ca_{22} & ca_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$e(A)X = 0$$

I and II are equivalent systems.

$$\boxed{\begin{aligned} &A \longrightarrow e(A) \\ &e : R_2 \longrightarrow cR_2, \ c \neq 0 \end{aligned}}$$

'e' is an operation where second equation is multiplied by a constant where $c \neq 0$.

$$\boxed{\begin{aligned} &e(A) \longrightarrow A \\ &e_1 : R_2 \longrightarrow \tfrac{1}{c}R_2 \quad c \neq 0 \end{aligned}}$$

Prove that $e_1(e($

Note 2: $a_{11}x_1 + a$

$a_{21}x_1 + a$

$(a_{31} + ca_{11})$

a_{11} a

a_{21} a

$a_{31} + ca_{11}$ a_{32}

$e(A)X =$

I and III are e

$A \longrightarrow e($

$e : R_3 \longrightarrow R_3$

$e(A) \longrightarrow A$

$e_1 : R_3 \leftarrow$

then $e(e_1(A)) = A$

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Note 3: Interchan

Consider $a_{11}x_1 + a$

$a_{21}x_1 + c$

$a_{31}x_1 + c$

a_{11} a_{12} a

a_{21} a_{22}

a_{31} a_{32}

A

Prove that $e_1(e(A)) = A = e(e_1(A))$

<u>Note 2:</u> $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = 0$ $\quad R_3 \leftarrow R_3 + cR_1$

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = 0$

$(a_{31} + ca_{11})x_1 + (a_{32} + ca_{12})x_2 + (a_{33} + ca_{13})x_3 = 0$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31}+ca_{11} & a_{32}+ca_{12} & a_{33}+ca_{13} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{III}$$

$e(A)X = 0$

I and III are equivalent systems.

$$\boxed{\begin{array}{l} A \longrightarrow e(A) \\ e: R_3 \rightarrow R_3 + cR_1 \end{array}}$$

$$\boxed{\begin{array}{l} e(A) \longrightarrow A \\ e_1: R_3 \leftarrow R_3 - cR_1 \end{array}}$$ then $e(e_1(A)) = A = e_1(e(A))$.

then $e(e_1(A)) = A = e_1(e(A))$

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<u>Note 3:</u> Interchange equations.

Consider
$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = 0 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = 0 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = 0 \end{array}\right\} \text{— I}$$

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad AX = 0$$

ultiplied

Interchange 2^{nd} and 3^{rd} equations.

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= 0 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= 0 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= 0 \end{aligned}\right] \text{— (IV)}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$e(A)$

$$e(A)x = 0$$

Note (I) and (IV) are equivalent systems

We pass from A to $e(A)$ by $R_2 \leftrightarrow R_3$

$A \to e(A)$ $\qquad$ $e(A) \to A$

$e: R_2 \leftrightarrow R_3$ $\qquad$ $e_1: R_2 \leftrightarrow R_3$

$$e_1(e(A)) = A = e(e_1(A))$$

Def: Elementary row operations

Consider an $m \times n$ matrix over the field F. We restrict our attention to the following three operations

Type 1: Multiplying a row by a non-zero scalar in F

i.e. $e: R_i \leftarrow cR_i$, $c \neq 0$

we have an inverse elementary row operations e_1 of Type 1.

i.e. $e_1: R_i \leftarrow \frac{1}{c} R_i$

In addition, $e_1(e(A)) = A = e(e_1(A))$

Type 2: Replace i^{th} ...

j^{th} row

$\Rightarrow e: R_i \leftarrow R_{...}$

~~Type 3~~: Then we h...

operation of type...

$e_1: R_i \leftarrow R...$

$e_1(e(A)) = A = e...$

Type 3: Interchang...

$e: R_i \leftrightarrow R...$

Then we have a...

$e_1: R_i \leftrightarrow R...$

such that $e_1(e...$

Theorem 2: For e...

exists an element...

such that $e_1(e(...$

Proof: (Done ab...

operations)

eg. $A = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}...$

$A_0 = A = ...$

$e_1 = R_2 \leftarrow ...$

$\to \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}...$

Type 2: Replace i^{th} row of A by i^{th} row plus 'c' times of j^{th} row.

$\Rightarrow e: R_i \leftarrow R_i + CR_j$

~~Type 3~~: Then we have an inverse elementary row operation of type 2.

$$e_1 : R_i \leftarrow R_i - CR_j$$

$$e_1(e(A)) = A = e(e_1(A))$$

Type 3: Interchange i^{th} and j^{th} rows of A.

$$e: R_i \leftrightarrow R_j$$

Then we have an inverse elementary row operation

$$e_1 : R_i \leftrightarrow R_j$$

such that $e_1(e(A)) = A = e(e_1(A))$

<u>Theorem 2</u>: For every elementary row operation, there exists an elementary row operation e_1 of same type such that $e_1(e(A)) = A = e(e_1(A))$

<u>Proof:</u> (Done already Type 1, type 2, type 3 has inverse operations).

eg. $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$$A_0 = A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$e_1 = R_2 \leftarrow \frac{1}{2} R_2$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} = A_1 \xrightarrow{e_2 : R_3 \leftarrow \frac{1}{3} R_3} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$= A_2 \xrightarrow{e_3:\ R_1 \leftarrow R_1 - 2R_2}$

$$\overset{e_3}{\sim} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{e_4:\ R_1 \leftarrow R_1 - 3R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A_4$$

Question) Find the solution of $AX=0$, $A_1X=0$, $A_2X=0$, $A_3X=0$ and $A_4X=0$

$$A_4X=0 \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 = 0$, $x_2 = 0$, $x_3 = 0$

$S_4 = \{(0,0,0)\}$

$S_A = S_{A_1} = S_{A_2} = S_{A_3} = S_{A_4} = \{(0,0,0)\}$

Note: $A = A_0 \xrightarrow{e_1} A_1 \xrightarrow{e_2} A_2 \xrightarrow{e_3} A_3 \xrightarrow{e_4} A_4$

Def: Let A and B be two $m \times n$ matrices over the field F. We say B is row-equivalent to A iff B can be obtained from A by a finite sequence of elementary row operations.

Note: $A = A_0 \xrightarrow{e_1} A_1 \xrightarrow{e_2} A_2 \cdots \xrightarrow{e_k} A_k = B$

Q) Prove or disprove that the following pair of matrices are row equivalent

(i)
$$A = \begin{bmatrix} 2 & -1 & 3 & 2 \\ 1 & 4 & 0 & -1 \\ 2 & 6 & -1 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 & -1 & -11/3 \\ 1 & 0 & 0 & 17/3 \\ 0 & 1 & 0 & -5/3 \end{bmatrix}$$

(ii) $A = \begin{bmatrix} -1 & i \\ -i & 3 \\ 1 & 2 \end{bmatrix}$

i) $A = \begin{bmatrix} 2 & -1 & \cdots \\ 1 & 4 & \cdots \\ 2 & 6 & \cdots \end{bmatrix}$

$\xrightarrow{R_1 \to R_1/3} \begin{bmatrix} 0 & -3 & \cdots \\ 1 & 4 & \cdots \\ 2 & 6 & \cdots \end{bmatrix}$

$\xrightarrow{R_3 \to R_3 - 2R_2} \begin{bmatrix} 0 & \cdots \\ 1 & 4 \\ 0 & \cdots \end{bmatrix}$

$\xrightarrow{R_3 \to -R_3/5} \begin{bmatrix} 0 & 2 \\ 1 & \cdots \\ 0 & \cdots \end{bmatrix}$

$\xrightarrow{R_1 \to R_1 - 2R_3} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

(ii) $A = \begin{bmatrix} -1 & i \\ -i & 3 \\ 1 & 2 \end{bmatrix}$

~~Not~~ are row equi

(ii) $A = \begin{bmatrix} -1 & i \\ -i & 3 \\ 1 & 2 \end{bmatrix}$ $\quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$

(i) $$A = \begin{bmatrix} 2 & -1 & 3 & 2 \\ 1 & 4 & 0 & -1 \\ 2 & 6 & -1 & 5 \end{bmatrix} \xrightarrow{R_1 \to R_1 - 2R_2} \begin{bmatrix} 0 & -9 & 3 & 4 \\ 1 & 4 & 0 & -1 \\ 2 & 6 & -1 & 5 \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1/3} \begin{bmatrix} 0 & -3 & 1 & 4/3 \\ 1 & 4 & 0 & -1 \\ 2 & 6 & -1 & 5 \end{bmatrix} \xrightarrow{R_3 \to R_3 + R_1} \begin{bmatrix} 0 & -3 & 1 & 4/3 \\ 1 & 4 & 0 & -1 \\ 2 & 3 & 0 & 19/3 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - 2R_2} \begin{bmatrix} 0 & -3 & 1 & 4/3 \\ 1 & 4 & 0 & -1 \\ 0 & -5 & 0 & 25/3 \end{bmatrix} \xrightarrow{R_1 \to R_1 - R_3} \begin{bmatrix} 0 & 2 & 1 & -7 \\ 1 & 4 & 0 & -1 \\ 0 & -5 & 0 & 25/3 \end{bmatrix}$$

$$\xrightarrow{R_3 \to -R_3/5} \begin{bmatrix} 0 & 2 & 1 & -7 \\ 1 & 4 & 0 & -1 \\ 0 & 1 & 0 & -5/3 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 4R_3} \begin{bmatrix} 0 & 2 & 1 & -7 \\ 1 & 0 & 0 & 17/3 \\ 0 & 1 & 0 & -5/3 \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1 - 2R_3} \begin{bmatrix} 0 & 0 & 1 & -11/3 \\ 1 & 0 & 0 & 17/3 \\ 0 & 1 & 0 & -5/3 \end{bmatrix} = B.$$

(ii) $A = \begin{bmatrix} -1 & i \\ -i & 3 \\ 1 & 2 \end{bmatrix}$ $\quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$

are ~~Not~~ row equivalent

(iii) $A = \begin{bmatrix} -1 & i \\ -i & 3 \\ 1 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$

$$\begin{bmatrix} -1 & i \\ -i & 3 \\ 1 & 2 \end{bmatrix} \xrightarrow{R_1 \to R_1 + iR_2} \begin{bmatrix} 0 & 4i \\ -i & 3 \\ 1 & 2 \end{bmatrix} \xrightarrow{R_2 \to R_2 + iR_3} \begin{bmatrix} 0 & 4i \\ 0 & 3+2i \\ 1 & 2 \end{bmatrix}$$

<u>Theorem 3:</u> If
matrices, then t
equations $AX=$
solutions.

<u>Proof:</u> We pass
of elementary
$A = A_0 \xrightarrow{e_1}$

It suffices to pr
$A_jX=0$ and
for $1 \le j \le k-1$
That is, an elem
disturb the
Let us assume
single element
which type of
every equation
of equations
By theorem 2,
an inverse elem
type.

$AX=0$

Hence, every eq
combination of e
$BX=0$ are equiv
By Theorem 1, A
solutions.

<u>Theorem 3:</u> If A and B are two row equivalent $m \times n$ matrices, then the homogeneous systems of linear equations $AX=0$ and $BX=0$ have exactly the same solutions.

<u>Proof:</u> We pass from A to B by a <u>finite</u> sequence of elementary operations,

$$A = A_0 \xrightarrow{e_1} A_1 \xrightarrow{e_2} A_2 \longrightarrow \cdots \xrightarrow{e_k} A_k = B.$$

It suffices to prove that

$A_j X = 0$ and $A_{j+1} X = 0$ have exactly same solutions for $1 \le j \le k-1$

That is, an elementary row operation does not disturb the ~~solution~~ set of solutions.

Let us assume that B is obtained from A by a single elementary ~~oper~~ row operation. No matter which type of the operation (1), (2) or (3), every equation in $BX=0$ is a linear combination of equations in $AX=0$.

By theorem 2, every elementary row operation has an inverse elementary operation of the ~~can~~ same type.

$$AX=0 \underset{e_1}{\overset{e}{\rightleftarrows}} BX=0 \qquad A \underset{e_1}{\overset{e}{\rightleftarrows}} B$$

Hence, every equation in $AX=0$ is a linear combination of equations in $BX=0$. Then $AX=0$ and $BX=0$ are equivalent systems.

By Theorem 1, $AX=0$ and $BX=0$ have exactly same solutions.

Q) Solve the system $AX=0$ where $A = \begin{pmatrix} -1 & i \\ -i & 3 \\ 1 & 2 \end{pmatrix}$ using elementary row operations.

Soln:)

$$AX=0 \Rightarrow \begin{pmatrix} -1 & i \\ -i & 3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad 3\times 2 \; 2\times 1$$

$$-x_1 + i x_2 = 0$$

$$-i x_1 + 3x_2 = 0$$

$$x_1 + 2x_2 = 0$$

By theorem 3, $AX=$
solutions

$BX=0 \Rightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}$

The solution set S

Q) Solve $AX=0$ w

Soln:)

$$A = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 3 & 0 \\ 2 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

$\begin{pmatrix} -1 & i \\ -i & 3 \\ 1 & 2 \end{pmatrix}$ using

$3\times\boxed{2\ 2}\times 1$

By theorem 3, $AX=0$ and $BX=0$ have exactly same solutions.

$$BX=0 \Rightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The solution set $S_A = S_B = \{(0,0)\}$

Q) Solve $AX=0$ when $A = \begin{pmatrix} 1 & 2 & -1 & 0 \\ -1 & 3 & 0 & 4 \\ 2 & 1 & 3 & 2 \\ 1 & 1 & +1 & -1 \end{pmatrix}$

Soln:)

$$A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ -1 & 3 & 0 & 4 \\ 2 & 1 & 3 & 2 \\ 1 & 1 & 1 & -1 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_1 + R_2} \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 5 & -1 & 4 \\ 2 & 1 & 3 & 2 \\ 1 & 1 & 1 & -1 \end{bmatrix}$$

⟹ Row-reduced matrix.

Defn: An $m \times n$ matrix R is called a row reduced matrix if

(1) the first non-zero entry (leading one) of each non-zero row of R is one

(2) each column of R which contains a leading one has all other enteries '0'.

Examples: (i)

$$R_1 = \begin{pmatrix} \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{pmatrix}$$

leading 1s.

row-reduced matrix.

(ii)

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}$$

row-reduced matrix

(iii)

$$\begin{pmatrix} \boxed{1} & 2 & 3 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

not a row-reduced matrix

(iv)

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

not a row-reduced matrix.

Q) Find an equivalent row-reduced matrix of

$$A = \begin{pmatrix} 3 & 2 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$A = \begin{pmatrix} 3 & 2 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$\begin{pmatrix} \boxed{1} & 2/3 \\ 0 & \boxed{1} \\ 0 & 0 \end{pmatrix}$

The answer m

rows can be in

Theorem: 4 Every

row reduced

⟹ Row reduced

$A = \begin{pmatrix} \boxed{1} & 0 \\ 0 & 0 \\ 0 & \boxed{1} \\ 0 & 0 \end{pmatrix}$

Non-zero row.

k_1 = column co

$k_1 = 1$

k_2 = column c 2nd row.

$k_2 = 3$

k_3 = column co 3rd row

$k_3 = 2$

$$A = \begin{pmatrix} 3 & 2 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 \leftarrow \frac{R_1}{3}} \begin{pmatrix} 1 & 2/3 & 2/3 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_2 \leftarrow \frac{R_2}{2}}$$

$$\begin{pmatrix} \boxed{1} & 2/3 & 2/3 \\ 0 & \boxed{1} & 1/2 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 \leftarrow R_1 - 2/3 R_2} \begin{pmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{pmatrix}$$

The answer may not be unique. The order of the rows can be interchanged.

Theorem: 4 Every $m \times n$ matrix is equivalent to a row reduced ~~matrix~~ $m \times n$ matrix.

$\Longrightarrow$ Row reduced echelon matrix.

$$A = \begin{pmatrix} \boxed{1} & 0 & 0 & 0 & 2 \\ 0 & 0 & \boxed{1} & 0 & 3 \\ 0 & \boxed{1} & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Non-zero rows of A — 1, 2, 3.

k_1 = column corresponding to the leading one in 1st row.

$k_1 = 1$

k_2 = column corresponding to the leading one in 2nd row.

$k_2 = 3$

k_3 = column corresponding to the leading one in 3rd row

$k_3 = 2$

Def: An $m \times n$ matrix R is called a row reduced echlon matrix if

(i) R is row reduced.

(ii) every row R which has all entries '0' occurs below of every row which has all non-zero entry.

(iii) If rows $1, 2, 3, \ldots, r$ be the non-zero rows of R and if leading non-zero entry of row i occurs in column k_i, then $k_1 < k_2 < \ldots < k_r$

Example:)

(i) $A = \begin{pmatrix} 1 & 2 & 0 & 1 & 0 & 2 & 2 \\ 0 & 0 & 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

non-zero rows $\longrightarrow 1, 2, 3.$

$k_1 = 1$

$k_2 = 3 \qquad k_1 < k_2 < k_3$

$k_3 = 5$

Solve $AX = 0$

$$\begin{pmatrix} 1 & 2 & 0 & 1 & 0 & 2 & 2 \\ 0 & 0 & 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix}$$

Consider non-zero rows.

Variables corresponding to leading ones

$$= \{x_1, x_3, x_5\} \rightarrow \text{pivot variables.}$$

rest of the variables.

$$= \{x_2, x_4, x_6, x_7\} \rightarrow \text{free variables.}$$

1^{st} equation.

$$x_1 = -2x_2 - x_4 - 2x_6 - 2x_7$$

$x_1 \leftarrow$ a linear combination of free variables.

$$x_3 = -2x_4 - 3x_7$$

$$x_5 = -3x_6 - 2x_7$$

Every pivot variable can be expressed as a linear combination of free variables.

Give arbitrary values to free variables.

$$x_2 = a,\ x_4 = b,\ x_6 = c,\ x_7 = d.$$

$$x_1 = -2a - b - 2c - 2d$$

$$x_3 = -2b - 3d$$

$$x_5 = -3c - 2d$$

$$S_A = \{(-2a-b-2c-2d,\ a,\ -2b-3d,\ b,\ -3c-2d,\ c,\ d)\ |\ a,b,c,d \in \mathbb{R}.\}$$

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Theorem 5: Every $m \times n$ matrix is row equivalent to row-reduced echelon matrix.

Note:1 Let B be a row reduced echelon matrix. Let $R_1, R_2, R_3, \ldots, R_r$ be the non-zero rows of B. Consider the system of linear equations $BX=0$. Let k_i be the column corresponding to the leading non-zero entry in the i^{th} row of B.

Indeed, $BX=0$ has exactly 'r' non-trivial equations. More over, $x_{k_1}, x_{k_2}, x_{k_3}, \ldots, x_{k_r}$ be the pivot variables and $u_1, u_2, \ldots, u_{n-r}$ are free variables.

The first equation in $BX=0$ is

$$x_{k_1} + \sum_{j=1}^{n-r} C_{1j} u_j = 0$$

2^{nd} equation.

$$x_{k_2} + \sum_{j=1}^{n-r} C_{2j} u_j = 0 \qquad \text{(I)}$$

r^{th} equation.

$$x_{k_r} + \sum_{j=1}^{n-r} C_{rj} u_j = 0$$

To solve (I), we set arbitrary values to free variables $u_1, u_2, \ldots, u_{n-r}$ and compute values of $x_{k_1}, x_{k_2}, \ldots, x_{k_r}$ from equations in (I).

NOTE:2 If $r<n$, the system $BX=0$ has $n-r>0$ free variables and $BX=0$ has a non-trivial solution.

Note:3 If B is an $n \times n$ (square) row reduced echelon matrix. when $r(=n)$ – non-zero rows, then $B=I$

Theorem 6: If A
the homogeneou
has a non-triv

Proof: Let B a
equivalent to A.
have exactly sa
Let 'r' be the nu
Hence, $r \leq m$.
By assumption
So, $BX=0$ ha
a non-trivial
$\Rightarrow AX=0$ has

Theorem 7: Let A
is row-equivale
the ~~sy~~ homogen
$AX=0$ has only

Proof: Case 1: Su
By theorem 3,
solutions.
$IX=0 \Rightarrow X=$
Hence $AX=0$ has

Case 2: Suppose
Let B be the ro
By theorem 3, A
solutions. Hence
Let 'r' be the
an $n \times n$ matrix)

<u>Theorem 6</u>: If A is an $m \times n$ matrix and $m < n$, then the homogeneous system of linear equations $AX = 0$ has a non-trivial solution.

<u>Proof</u>: Let B a row-reduced echelon matrix row-equivalent to A. By theorem 3, $AX=0$ and $BX=0$ have exactly same solutions.
Let 'r' be the number of non-zero rows of B.
Hence, $r \leq m$.
By assumption $m < n$, and hence $r \leq m < n$.
So, $BX=0$ has ~~atleast~~ $(n-r)$ free variables and a non-trivial solution.
$\Rightarrow AX=0$ has a non-trivial solution.

<u>Theorem 7</u>: Let A be an $n \times n$ (square) matrix, then A is row-equivalent to the identity matrix if and only if the ~~sy~~ homogeneous system of linear equations $AX=0$ has only trivial solutions.

<u>Proof</u>: <u>Case 1:</u> Suppose that A is row equivalent to I.
By theorem 3, $AX=0$ and $IX=0$ have exactly same solutions.

$IX = 0 \Rightarrow X = 0$

Hence $AX=0$ has only trivial solution.

<u>Case 2:</u> Suppose that $AX=0$ has only trivial solution.
Let B be the row reduced echelon matrix of A
By theorem 3, $AX=0$ and $BX=0$ have exactly same solutions. Hence, $BX=0$ has only trivial solution.
Let 'r' be the number of non-zero rows of B (B is an $n \times n$ matrix). $\Rightarrow r \leq n$ —— (1)

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Since $BX=0$ has only trivial solutions, then the number of free variables $(n-r) \leq 0$.

$\Rightarrow n \leq r$ ——— (2)

From (1) and (2), $n = r$.

Hence B is an $n \times n$ row reduced echelon matrix with $r = n$ non-zero rows.

So, $B = I$ (By note 3)

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<u>Note</u>: Find the set of all solutions of $AX=Y$ where $Y \neq 0$.

$A, Y \leftarrow$ unknown $\quad X \leftarrow$ known.

Consider the argument matrix $[A/Y]$. We employ a finite sequence of elementary row operations as $[A/Y]$ to get a row equivalent matrix $[B/Z]$.

Then $AX=Y$ and $BX=Z$ have exactly same solutions.

Q) Find all solutions of

$$x_1 - x_2 + 2x_3 = 1$$
$$2x_1 + 2x_3 = 1$$
$$x_1 - 3x_2 + 4x_3 = 2$$

$$\begin{bmatrix} 1 & -1 & 2 \\ 2 & 0 & 2 \\ 1 & -3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$AX = Y$$

$$[A/Y] = \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 2 & 0 & 2 & 1 \\ 1 & -3 & 4 & 2 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 2 & -2 & 1 \\ 0 & -2 & 2 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \leftarrow \frac{1}{2} R_2} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & -1/2 \\ 0 & -2 & 2 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 \leftarrow R_3 + 2R_2]{R_1 \leftarrow R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1/2 \\ 0 & 1 & -1 & -1/2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$B \qquad Z$

$$BX = Z$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1/2 \\ 0 & 1 & -1 & -1/2 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ 0 \end{bmatrix}$$

$$x_1 + x_3 = 1/2$$
$$x_2 - x_3 = -1/2$$

r = no. of non-zeroes = 2

pivot variables = $\{x_1, x_2\}$

free variables = $\{x_3\}$

Let $x_3 = a$ be an arbitrary variable

$$① \Rightarrow \quad x_1 = \frac{1}{2} - a$$
$$x_2 = a - 1/2$$
$$x_3 = a$$

$S = \{(\frac{1}{2} - a,\ a - \frac{1}{2},\ a) : a \in R\}$

2) Find all solutions.

$$x_1 - 2x_2 + x_3 + 2x_4 = 0$$
$$x_1 + x_2 - x_3 + x_4 = 2$$
$$x_1 + 2x_2 - 5x_3 + x_4 = 3$$

$AX = Y$

$[A|Y]$

$$= \left[\begin{array}{ccc|c} 1 & -2 & 1 & 2 \\ 1 & 1 & -1 & 1 \\ 1 & 2 & -5 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 2 \\ 0 & 3 & -2 & -1 \\ 0 & 4 & -6 & -1 \end{array}\right]$$

<u>Elementary oper</u>

let $A = \begin{bmatrix} a_{11} & a \\ a_{21} & \\ a_{31} & \end{bmatrix}$

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

$e_1 : R_2 \leftarrow c$

$e_2 : R_1 \leftarrow R_1$

$e_3 : R_2 \leftrightarrow R$

1) $e_1(A) = \begin{pmatrix} a_{11} \\ ca_{21} \\ a_{31} \end{pmatrix}$

$e_1(I) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$e_1(I).A = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$= \begin{pmatrix} a_{11} \\ ca_{21} \\ a_{31} \end{pmatrix}$

Elementary operations.

let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}_{3\times 4}$

$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}.$

$e_1 : R_2 \leftarrow cR_2$

$e_2 : R_1 \leftarrow R_1 + cR_3$

$e_3 : R_2 \leftrightarrow R_3$

1) $e_1(A) = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ ca_{21} & ca_{22} & ca_{23} & ca_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix}$

$e_1(I) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$e_1(I).A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix}$

$= \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ ca_{21} & ca_{22} & ca_{23} & ca_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix}$

$e_1(I).A = e_1(A)$

2) $e_2(A) = \begin{pmatrix} a_{11}+ca_{31} & a_{12}+ca_{32} & a_{13}+ca_{33} & a_{14}+ca_{34} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix}$

$e_2(I) = \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$e_2(I).A = \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix}$

$= \begin{pmatrix} a_{11}+ca_{31} & a_{12}+ca_{32} & a_{13}+ca_{33} & a_{14}+ca_{34} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix}$

$= e_2(A).I$

3) $\cancel{e_3(A)} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix}$

$e_3(A) = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix}$

$e_3(I) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$e_3(I).A = \begin{bmatrix} 1 & 0 \ldots \\ 0 & \ldots \\ 0 & \ldots \end{bmatrix}$

$= \begin{pmatrix} a_{11} \ldots \\ a_{31} \ldots \\ a_{21} \ldots \end{pmatrix}$

$e_3(A) = e_3(I).A$

4) Suppose A and B ...
So B can be ob...
elementary row ...

$A_0 : A_1 \xrightarrow{e_1} A_1 \ldots$

$A_1 = e_1(A) = \ldots$

$A_2 = e_2(A_1) = \ldots$

$B = A_k = e_k(\ldots$

$B = e_k . e_{k-1} \ldots$

where $E_i = \ldots$

Let $P = E_k E_{k-1} \ldots$

If $B = I$ then ...

Definition: An n...
matrix if it can ...
identity matrix ...

$$e_3(I).A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

$$= \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix}$$

$$e_3(A) = e_3(I).A$$

4) Suppose A and B are row equivalent matrices. So B can be obtained from A by a finite sequence elementary row operations: say $e_1, e_2, \ldots, e_k$

$$A_0 : A_0 \xrightarrow{e_1} A_1 \xrightarrow{e_2} A_2 \longrightarrow \cdots \xrightarrow{e_k} A_k = B$$

$$A_1 = e_1(A) = e_1(I).A$$

$$A_2 = e_2(A_1) = e_2(I).A_1 = e_2(I).e_1(I).A$$

$$B = A_k = e_k(I).e_{k-1}(I) \cdots e_1(I).A$$

$$B = e_k . e_{k-1} \cdots e_1 A$$

where $E_i = e_i(I)$

Let $P = E_k E_{k-1} \cdots E_1$. Then $B = PA$.

If $B = I$ then $P = A^{-1}$

<u>Definition:</u> An $n \times n$ (square) matrix is an elementary matrix if it can be obtained from an $n \times n$ ~~matrix~~ identity matrix by a single elementary row operations.

Q) Find all 2X2 elementary row matrices.

① $R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{e_1(A)} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

② $R_1 \to R_1 + cR_2$

$$\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}$$

③ $R_1 \to cR_1$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}$$

④ $R_2 \to cR_2$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix}$$

⑤ $R_2 \to R_2 + cR_1$

$$\begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$$

HW: Find all 3X3 elementary matrices.

Theorem: Let e be an elementary row operations and E be the $m \times m$ elementary matrix $E = e(I)$. Then for every $m \times n$ matrix A $e(A) = e(I).A = EA$

Proof: Text

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Corollary: Let A and B be $m\times n$ matrices over the field. Then B is row equivalent to A if and only if $B = PA$ where P is a product of $m\times m$ elementary matrices.

Proof: Case 1: Suppose that B is row equivalent to A. We pass from A to B by a ~~set~~ finite sequence of elementary row operations say $e_1, e_2, \dots e_k$.

$$A \xrightarrow{e_1} A_1 \xrightarrow{e_2} A_2 \longrightarrow \dots \xrightarrow{e_k} A_k = B$$

$$A_1 = e_1(A) = e_1(I).A$$

$$A_2 = e_2(A_1) = e_2(I).A_1 = e_2(I).e_1(I).A$$

- - - - - - - - -

$$B = A_k = e_k(I).e_{k-1}(I)\dots e_2(I).e_1(I).A$$

$B = PA$ where $P = e_k(I).e_{k-1}(I).\dots e_1(I)$

product of elementary matrices.

Case 2: Suppose that $B = PA$ where $P = E_k E_{k-1}\dots E_2 E_1$ product of elementary matrices.

We have $E_1 A$ is row equivalent to A.

$E_2(E_1 A)$ is row equivalent to $E_1 A$

$\therefore E_2(E_1 A)$ is row equivalent to A.

|||ly $\overbrace{E_k E_{k-1}\dots E_1}^{P} A$ is row equivalent to A.

$\Rightarrow PA$ is row equivalent to A.

Note 1: Let
$$A = \begin{array}{c} \begin{array}{cc} C_1 & C_2 \end{array} \\ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} \end{array} \qquad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \qquad 3\times\boxed{2\ 2}\times 2$$

$$AB = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \\ a_{31}b_{11} + a_{32}b_{21} & a_{31}b_{12} + a_{32}b_{22} \end{pmatrix}$$

$$= \left(b_{11}\begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} + b_{21}\begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix} \quad b_{12}\begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} + b_{22}\begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix} \right)$$

$$= \left(b_{11}C_1 + b_{21}C_2 \quad b_{12}C_1 + b_{22}C_2 \right)$$

i.e. every column in AB is a linear combination of columns in A.

and every row in AB is a linear combination of rows in B.

<u>Note 2:</u> Let $A \underset{e_1'}{\overset{e_1}{\rightleftarrows}} A_1 \underset{e_2'}{\overset{e_2}{\rightleftarrows}} A_2 \longrightarrow \cdots \underset{e_k'}{\overset{e_k}{\rightleftarrows}} A_k = B$

$$\Rightarrow B = e_k(I)\, e_{k-1}(I) \cdots e_2(I)\, e_1(I) A = PA \quad \text{——} (1)$$

$$A = e_1'(I)\, e_2'(I) \cdots e_{k-1}'(I)\, e_k'(I)\, B = QB. \quad \text{——} (2)$$

Suppose $A = I$

$(1) \Rightarrow B = P$

$(2) \Rightarrow I = QB = QP$

$$\boxed{I = QP}$$

$\Rightarrow$ <u>Inverse of a matrix:</u> Let A and B be two $n \times n$ matrices over the field F.

We say B is the left inverse of A

if $BA = I$

We say B is the right inverse of A

if $AB = I$

We say A is invertible if A has a left inverse and a right inverse.

<u>Note:</u> Let B ...

inverse of A ...

i.e. $BA = I$ ...

$B = BI = B$ ...

$\Rightarrow B = C$

$\therefore$ If left inver...

the same.

<u>Notation:</u> If ...

<u>HW:</u> Q) Let A ...

Show th...

<u>Theorem 11</u>

(i) $A(A^{-1}) = ($...

(ii) $(AB)(B^{-1}A$...

i.e. $(AB$...

$\Rightarrow$ Product of...

<u>Theorem 12:</u> E...

<u>Proof:</u> Let E ...

I by o...

i.e. ...

By theorem ...

operation e...

matrix A.

Note: Let B be a left inverse of A and C be the right inverse of A

i.e. $BA = I$ and $AC = I$

$B = BI = B(AC) = (BA)C = I.C = C$

$\Rightarrow B = C$

$\therefore$ If left inverse and right inverse exist, then they are the same.

Notation: If B is the inverse of A, then

$AB = BA = I$ $\quad$ $A(A^{-1}) = (A^{-1})A = I$

& $B = A^{-1}$

HW: Q) Let A and B be two invertible $n \times n$ matrices.

Show that (i) $(A^{-1})^{-1} = A$ $\longrightarrow$ by symmetry, it is true.

Theorem 11 $\quad$ (ii) $(AB)^{-1} = B^{-1}A^{-1}$

(i) $A(A^{-1}) = (A^{-1})A = I$

(ii) $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$

$= AIA^{-1} = AA^{-1} = I$

i.e. $(AB)^{-1} = B^{-1}A^{-1}$

$\Rightarrow$ Product of invertible matrices are invertible.

Theorem 12: Every elementary matrix is invertible.

Proof: Let E be an elementary matrix obtained from I by a single elementary row operation e.

i.e. $E = e(I)$

By theorem 2, e has an inverse elementary row operation e_1 such that $e(e_1(A)) = e_1(e(A)) = A$ for every matrix A.

Let $E_1 = e_1(I)$

$$EE_1 = e(I)E_1 = e(E_1) = e(e_1(I)) = I$$

$$E_1E = e_1(I)E = e_1(E) = e_1(e(I)) = I$$

Therefore, $EE_1 = I = E_1E$

$\therefore E$ is invertible.

Q) Find inverse of all 2X2 elementary matrices

eg. $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$c \neq 0$ $\begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{c} & 0 \\ 0 & 1 \end{pmatrix}$ (5 of them are there)

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Note: If A is an invertible matrix, then A has non-zero rows

Theorem 12: Let A be an $n \times n$ square matrix, then the following are equivalent.

(i) A is invertible

(ii) A is row equivalent to identity matrix.

(iii) A is a product of elementary matrices.

$$\begin{tikzcd} \text{(i)} \arrow[r, bend left] & \text{(ii)} \arrow[l, bend left] \arrow[dl] \\ \text{(iii)} \arrow[u] & \end{tikzcd}$$

Proof: Let R be a row reduced echelon matrix row-equivalent to A.

By theorem 9, $R = E_kE_{k-1}E_{k-2}\cdots E_2E_1A$ ——— ①

where $E_1, E_2, E_3, \ldots, E_k$ are elementary matrices.

By theorem 11, E_1^{-1},

(1) $\Rightarrow E_k^{-1}R = E_{k-1}$

Similarly

$E_1^{-1}E_2^{-1}\cdots E_k^{-1}R$

(i) $\Rightarrow$ (ii) : Suppo

Since $E_1, E_2,$ -

invertible mat

$R = E_kE_{k-1}\cdots$

$\Rightarrow R$ has no ze

$\Rightarrow R = I$ (R is a

$\Rightarrow A$ is row-equ

(ii) $\Rightarrow$ (i) : Suppose th

② $\Rightarrow E_1^{-1}E_2^{-1}\cdots E$

$A = E_1^{-1}E_2^{-1}\cdots$

~~Since E_1^{-1}, E_2^{-1},~~

Since $E_1, E_2, \ldots E$

$E_1^{-1}, E_2^{-1}, \ldots E$

is invertibl

(ii) $\Rightarrow$ (iii) : Suppose

i.e. $R = I$

② $\Rightarrow A = E_1^{-1}E_2^{-1}$

matrices

(iii) $\Rightarrow$ (i) Every ele

product is

$\Rightarrow A$ is inve

By theorem 11, E_1^{-1}, E_2^{-1}, and E_k^{-1} exist.

$(1) \Rightarrow E_k^{-1} R = E_{k-1} \cdots E_2 E_1 A$

Similarly

$$E_1^{-1} E_2^{-1} \cdots E_k^{-1} R = A \quad \text{—— ②}$$

(i) $\Rightarrow$ (ii) : Suppose that A is invertible.

Since $E_1, E_2, \cdots, E_k$ are invertible and product of invertible matrices is invertible, by ②,

$R = E_k E_{k-1} \cdots E_1 A$ is invertible.

$\Rightarrow$ R has no zero row.

$\Rightarrow R = I$ (R is a row reduced echelon matrix)

$\Rightarrow$ A is row-equivalent to $R = I$.

(ii) $\Rightarrow$ (i) : Suppose that A is row-equivalent to I i.e. $R = I$,

② $\Rightarrow E_1^{-1} E_2^{-1} \cdots E_k^{-1} I = A$

$A = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$

~~Since E_1^{-1}, E_2^{-1},~~

Since $E_1, E_2, \cdots E_k$ are invertible (elementary matrix)

$E_1^{-1}, E_2^{-1}, \cdots E_k^{-1}$ are invertible and its product A is invertible.

(ii) $\Rightarrow$ (iii) : Suppose that A is row equivalent to I, i.e. $R = I$

② $\Rightarrow A = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$, a product of elementary matrices.

(iii) $\Rightarrow$ (i) Every elementary matrix is invertible and its product is also invertible

$\Rightarrow$ A is invertible.

Note: If A is an $n\times n$ invertible matrix and if a sequence of elementary operations reduces A to I, then the same sequence of operations when applied to I, we get A^{-1}.

Q) Find A^{-1} where $A = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$

Soln)
$$(A|I) = \left(\begin{array}{cc|cc} 2 & -1 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cc|cc} 1 & 3 & 0 & 1 \\ 2 & -1 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{R_2 \leftarrow R_2 - 2R_1} \left(\begin{array}{cc|cc} 1 & 3 & 0 & 1 \\ 0 & -7 & 1 & 2 \end{array}\right) \xrightarrow{R_2 \leftarrow \frac{R_2}{-7}} \left(\begin{array}{cc|cc} 1 & 3 & 0 & 1 \\ 0 & 1 & \frac{-1}{7} & \frac{2}{7} \end{array}\right)$$

$$\xrightarrow{R_1 \leftarrow R_1 - 3R_2} \left(\begin{array}{cc|cc} 1 & 0 & 3/7 & 1/7 \\ 0 & 1 & -1/7 & 2/7 \end{array}\right)$$

where the right-hand block is $\underbrace{\begin{array}{cc} 3/7 & 1/7 \\ -1/7 & 2/7 \end{array}}_{A^{-1}}$

$$\therefore A^{-1} = \begin{pmatrix} 3/7 & 1/7 \\ -1/7 & 2/7 \end{pmatrix}$$

Q) Find inverse of $n\times n$

$$A = \begin{pmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{pmatrix}$$

and if a
ces A to I,
when applied

$$(A|I) = \left(\begin{array}{ccc|ccc} 1 & 1/2 & 1/3 & 1 & 0 & 0 \\ 1/2 & 1/3 & 1/4 & 0 & 1 & 0 \\ 1/3 & 1/4 & 1/5 & 0 & 0 & 1 \end{array}\right)$$

$\frac{1}{2} - \frac{2}{3} = \frac{-1}{\cancel{8}}$

$\frac{1}{3} - \frac{1}{2} = \frac{-1}{6}$

$R_2 \leftarrow R_1 - 2R_2$

$\cancel{R_2 \leftarrow \frac{1}{3}R_1 - \frac{1}{2}R_2}$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 1/2 & 1/3 & 1 & \cancel{-2}0 & 0 \\ 0 & -1/6 & -1/6 & 1 & -2 & 0 \\ 1/3 & 1/4 & 1/5 & 0 & 0 & 1 \end{array}\right)$$

$0 \quad 1$

$\frac{2}{7}$

Corollary: Let A and B be two $m\times n$ matrices. Then A and B are row equivalent $\Leftrightarrow B=PA$ where P is an $m\times m$ invertible matrix.

Theorem 13: For an $n\times n$ matrix A, the following are equivalent.

(i) A is invertible

(ii) The system $AX=0$ has only trivial solution.

(iii) The system $AX=Y$ has a solution for any $n\times 1$ matrix Y.

Proof: -

$(i)\Rightarrow(ii)$: Suppose that A is invertible.

By theorem 12, A is row equivalent to I.

By theorem , $AX=0$ and $IX=0$ have exactly same solution. $IX=0 \Rightarrow X=0$

Only solution of $IX=0$ is $X=0$.

Hence, $AX=0$ has only trivial solution.

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$(ii)\Rightarrow(i)$: Suppose that $AX=0$ has only a trivial solution. Let R be a row-reduced echelon matrix row-equivalent to A. By theorem 3, $AX=0$ and $RX=0$ have exactly same solutions. Hence $RX=0$ has a only a trivial solution.

$\Rightarrow RX=0$ has no free variables.

$\Rightarrow R$ has no zero row. Hence $R=I$, since R is a row-reduced echelon matrix

$\Rightarrow A$ is row equivalent to $R=I$

By theorem 12, A is invertible

$$\begin{tikzcd} (i) \arrow[r, shift left] & (ii) \arrow[l, shift left] \\ (iii) \arrow[u] & \end{tikzcd}$$

$(i)\Rightarrow(iii)$: Suppose

$\Rightarrow A^{-1}$ exists.

Consider $AX=Y$

$\Rightarrow X=A^{-1}Y$ is

$(iii)\Rightarrow(i)$: Suppose

any $n\times 1$ mat

Let R be a r

row equivalent

invertible mat

$E=\begin{pmatrix}0\\0\\0\\\vdots\\0\\1\end{pmatrix}$

Consider $RX=E$

By assumption,

$\Rightarrow RX=E$ has

$\Rightarrow RX=\begin{pmatrix}0\\0\\\vdots\\0\\1\end{pmatrix}$ h

$\Rightarrow$ The last ro

$\Rightarrow R$ has no

$\Rightarrow R=I$, since

$\Rightarrow A$ is row e

$\Rightarrow A$ is invertil

$(i)\Leftrightarrow(iii)$

(i) $\Rightarrow$ (iii): Suppose A is invertible.

$\Rightarrow A^{-1}$ exists.

Consider $AX=Y$ where Y is any $n\times 1$ matrix

$\Rightarrow X=A^{-1}Y$ is a solution.

(iii) $\Rightarrow$ (i): Suppose that $AX=Y$ has a solution X for any $n\times 1$ matrix Y.

Let R be a row reduced echelon matrix which is row equivalent to A, then there exists an $n\times n$ invertible matrix P such that $R=PA$

$$E=\begin{pmatrix}0\\0\\0\\\vdots\\0\\1\end{pmatrix}$$

Consider $RX=E \Leftrightarrow (PA)X=E \Leftrightarrow P(AX)=E$

$\Leftrightarrow AX=P^{-1}E$

By assumption, $AX=P^{-1}E$ has a solution.

$\Rightarrow RX=E$ has a solution.

$\Rightarrow RX=\begin{pmatrix}0\\0\\\vdots\\0\\1\end{pmatrix}$ has a solution.

$\Rightarrow$ The last row of R is non-zero

$\Rightarrow R$ has no zero rows

$\Rightarrow R=I$, since R is a row reduced echelon matrix.

$\Rightarrow A$ is row equivalent to RI

$\Rightarrow A$ is invertible.

(i) $\Leftrightarrow$ (iii)

Note: $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$$AX=Y \Rightarrow \begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= \cancel{0}\, y_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= y_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= y_3 \end{aligned}$$

$$x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix} + x_3 \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$x_1C_1 + x_2C_2 + x_3C_3 = Y$$

Any vector in $\mathbb{R}^n$ is a linear combination of columns of A.

Corollary: If a square matrix has either a left inverse or a right inverse, then A is invertible.

Proof: Let A be a square matrix.

Case (i): Let B be a left inverse of A.

i.e. $BA = I$

Consider

$AX = 0 \Rightarrow B(AX) = 0 \Rightarrow (BA)X = 0$

$\Rightarrow IX = 0 \Rightarrow X = 0$

i.e. $AX=0$ has only a trivial solution.

By Theorem B, A is invertible.

Case (ii): Let C be a right inverse of A

$\Rightarrow AC = I$

$\Rightarrow A$ is a left inverse of C.

By case (i), C is invertible and $C^{-1} = A$

$\Rightarrow (C^{-1})^{-1} = A^{-1}$

$\Rightarrow A^{-1} = C$, A is invertible.

Corollary: Let $A = A_1 A_2 \cdots A_k$ where A_i is an $n \times n$ matrix $\forall i$. Then A is invertible if and only if A_i is invertible $\forall i$.

Proof: ($\Leftarrow$): Suppose A_i is invertible $\forall i$.
Since product of invertible matrices is invertible, $A = A_1 A_2 \cdots A_k$ is invertible.

$\Rightarrow$: Suppose that $A = A_1 A_2 \cdots A_k$ is invertible.
Show that $A_1, A_2, A_3, \ldots A_k$ are invertible.
We show that A_k is invertible.
Consider $A_k X = 0 \Rightarrow \cancel{(A_1 A_2 \cdots A_{k-1}(A_k)}$

$$A_1 A_2 \cdots A_{k-1}(A_k X) = 0$$
$$\Rightarrow AX = 0$$
$$\Rightarrow X = 0 \text{ (by theorem 13)}$$

$\Rightarrow A_k$ ~~#~~ is invertible.

We have $A = A_1 A_2 \cdots A_{k-1} A_k$

~~# A_k~~

$A A_k^{-1} \cancel{A_k^{-1} A} = A_1 A_2 \cdots A_{k-1}$ is invertible.

Consider $A_{k-1} X = 0$

$\Rightarrow \underline{A_1 A_2 \cdots A_{k-2}(A_{k-1}} X) = 0$

$(A_k^{-1} A) X = 0 \Rightarrow X = 0$ ($\because A_k^{-1} A$ is invertible)

$\Rightarrow A_{k-1}$ is invertible.

By a similar arguments

$A_1, A_2, A_3, \ldots A_k$ are invertible.

Q) Let A be a $m \times n$ matrix and B be a $n \times m$ matrix and $n < m$, then AB is not invertible.

Soln) $BX = 0$ has a non trivial soln.

$A(BX) = 0$

$(AB)X = 0$ has a non trivial soln

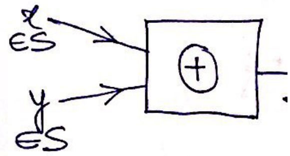
After Quiz 1

31/Aug/18

$\Longrightarrow$ Vector space (

Binary operation

$\oplus : S \times S \rightarrow S$



Examples:

1) $S = \mathbb{R}$

$x \oplus y = x + y$

2) $S = \mathbb{R}$

$x \oplus y = x - y$

3) $S = \mathbb{R}$, $x \oplus y$

4) $S = \mathbb{R}$, $x \oplus y$

5) $S = \mathbb{R}^2$

Let $x = (x_1, x_2)$

$y = (y_1, y_2)$

$x \oplus y = (x_1 + y_1,$

6) $S = \mathbb{R}^n$

$x = (x_1, x_2, \ldots, x$

$y = (y_1, y_2, \ldots, y_n$

$x \oplus y = (x_1 + y_1, x_2 +$

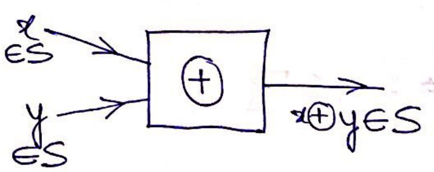
After Quiz 1

31/Aug/18

$\Longrightarrow$ Vector space (Linear space)

Binary operations $\oplus$ on a set S

$\oplus : S \times S \to S$



Examples:

1) $S = \mathbb{R}$

$x \oplus y = x + y \in \mathbb{R} = S$

2) $S = \mathbb{R}$

$x \oplus y = x - y$

3) $S = \mathbb{R}$, $x \oplus y = xy$

4) $S = \mathbb{R}$, $x \oplus y = x^y$

5) $S = \mathbb{R}^2$

Let $x = (x_1, x_2)$

$y = (y_1, y_2)$

$x \oplus y = (x_1 + y_1, x_2 + y_2)$

6) $S = \mathbb{R}^n$

$x = (x_1, x_2, \ldots, x_n)$

$y = (y_1, y_2, \ldots, y_n)$

$x \oplus y = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$

7) $F^{m\times n}$ ← set of all $m\times n$ matrices ~~where e~~ whose elements belong to F.

Let $A=[A_{ij}]_{m\times n}$, $B=[B_{ij}]_{m\times n}$.

$A + B$

$$(A\oplus B)_{ij} = A_{ij}+B_{ij}$$

8) Let $P_n = \{a_0+a_1x+\dots+a_nx^n \cancel{x} : a_i\in F\}$

where F is a field.

Let $p(x) = a_0+a_1x+\dots+a_nx^n$

$q(x) = b_0+b_1x+\dots+b_nx^n$

$$(p\oplus q)(x) = (a_0+b_0)+(a_1+b_1)x+\dots+(a_n+b_n)x^n$$

9) Let $C[a,b]$ be a set of all continuous functions on $[a,b]$

Let $f,g \in C[a,b]$

$$(f\oplus g)(x) = f(x)+g(x)$$

Note: Let F be a field and V be any set.

A function '$\cdot$' : $F\times V\to V$ is called a scalar multiplication.

Example: 1) $V=\mathbb{R}^2$, F

$\cdot : F\times V \to V$ by

$c(x,y)=(cx,cy)$

2) $V=\mathbb{R}^2$, F

$\cdot : F\times V\to V$

by $c(x,y) = (cx,y)$

Definition: (Vector space): A vector space consists of

(i) a field of scalars F.

(ii) a set of vectors, V

(iii) a binary operation, called vector addition, i.e.

$+: V \times V \to V$

which satisfies the ~~foo~~ following.

(A) for $\alpha, \beta \in V$, $\alpha+\beta \in V \longrightarrow$ ~~close~~ closure property.

(a) addition is commutative,

i.e. $\alpha+\beta = \beta+\alpha$, for $\alpha, \beta \in V$

(b) addition is associative

i.e. $\alpha+(\beta+\gamma) = (\alpha+\beta)+\gamma$, $\alpha, \beta, \gamma \in V$

(c) there exists a unique element '0' such that

$\alpha + 0 = \alpha \;\; \forall \alpha \in V$

(d) for every vector $\alpha \in V$,

there exists $-\alpha \in V$ such that

(~~iv) An~~ $\alpha + (-\alpha) = 0$.

(iv) An operation called scalar multiplication,

$\cdot : F \times V \to V$

which satisfies the following.

(B) for $c \in F$, $\alpha \in V$, $c\alpha \in V$

(e) $1.\alpha = \alpha \;\; \forall \alpha \in V$

(f) $(c_1 c_2)\alpha = c_1(c_2\alpha)$ for $c_1, c_2 \in F$, $\alpha \in V$

(g) $c(\alpha+\beta) = c\alpha + c\beta$, $c \in F$, $\alpha, \beta \in V$

(h) $(c_1+c_2)\alpha = c_1\alpha + c_2\alpha$, $c_1, c_2 \in F$, $\alpha \in V$

Note: A vector space $\langle V, F, +, \cdot \rangle$

$V \leftarrow$ vector

$F \leftarrow$ scalar

$+ \leftarrow$ vector addition.

$\cdot \leftarrow$ scalar multiplication.

Example: (~~Then~~ The n-tuple space F^n)

Let F be a field.

Let $V = \{(x_1, x_2, ---, x_n) : x_1, x_2, x_3, --- x_n \in F\}$

Let $x = (x_1, x_2, ..., x_n)$

$y = (y_1, y_2, ..., y_n)$

Vector addition, $+ : V \times V \to V$

$x+y = (x_1+y_1, x_2+y_2, ..., x_n+y_n)$

scalar multiplication $\cdot : F \times V \to V$

$cx = (cx_1, cx_2, ..., cx_n)$

HW Show that $\langle V, F, +, \cdot \rangle$ is a vector space

Example 2: The space of all $m \times n$ matrices over the Field F.

Let $F^{m \times n} = \{[A_{ij}]_{m \times n} : A_{ij} \in F\}$

Let $A = [A_{ij}]_{m \times n}$ $\quad B = [B_{ij}]_{m \times n}$

We define: (i) Vector addition $\quad + : F^{m \times n} \times F^{m \times n} \to F^{m \times n}$

$(A+B)_{ij} = A_{ij} + B_{ij}$

(ii) Scalar multiplication. $\quad \cdot : F \times F^{m \times n} \to F^{m \times n}$

$(cA)_{ij} = c.A_{ij}$, $c, A_{ij} \in F$

Show that $\langle F^{m\times n}, F, +, \cdot \rangle$ is ~~not~~ a vector space.

(A) Let $A, B \in F^{m\times n}$

By definition, $(A+B)_{ij} = A_{ij} + B_{ij} \in F$

$\Rightarrow A+B \in F^{m\times n}$ (since $A_{ij}, B_{ij} \in F$)

(a) $(A+B)_{ij} = A_{ij} + B_{ij}$ (by definition)

$= B_{ij} + A_{ij}$ (since F is a field)

$= (B+A)_{ij}$ (by definition)

$\Rightarrow A+B = B+A$

(b) $(A+B)+C = A+(B+C)$

$[(A+B)+C]_{ij} = (A+B)_{ij} + C_{ij}$ (by definition)

$= (A_{ij}+B_{ij}) + C_{ij}$ (by definition)

$= A_{ij} + (B_{ij} + C_{ij})$ (since F is a field and in a field, vector addition is associative.

$= A_{ij} + (B+C)_{ij}$ (By definition)

$= (A+(B+C))_{ij}$

$\Rightarrow (A+B)+C = A+(B+C)$

(c) Let $O = [0]_{m\times n}$

by definition

$(A+O)_{ij} = A_{ij} + O_{ij} = A_{ij} + 0$

$= A_{ij}$ (since F is a field and 0 is the additive identity in field F

$\Rightarrow A+O = A$

(d) Let $A = [A_{ij}]_{m\times n} \in F^{m\times n}$

since $A_{ij} \in F$, $-A_{ij} \in F \Rightarrow [-A_{ij}]_{m\times n} \in F^{m\times n}$

Let $-A = [-A_{ij}]_{m\times n}$

$(A + -A)_{ij} = A_{ij} + (-A)_{ij} = A_{ij} - A_{ij} = 0_{ij} = 0$ ↘ scalar ↘ matrix

$\Rightarrow A + -A = 0$

(B) Let $A = [A_{ij}]_{m\times n} \in F^{m\times n}$

Since F is a field, $c, A_{ij} \in F \Rightarrow cA_{ij} \in F$

$\Rightarrow [cA_{ij}]_{m\times n} \in F^{m\times n}$

$\Rightarrow cA \in F^{m\times n}$ (by definition)

$\boxed{(cA)_{ij} = cA_{ij}}$

(e) $1.\alpha = \alpha \;\; \forall\, \alpha \in V$

Let $A = [A_{ij}]_{m\times n} \in F^{m\times n}$ $\cancel{\boxed{(cA_{ij}) = cA_{ij}}}$

$(1.A)_{ij} = 1.A_{ij}$ (by definition)

$= A_{ij}$ (since F is a field)

$\Rightarrow 1.A = A$

(f) $(c_1c_2)\alpha = c_1(c_2\alpha)$ $\;c_1, c_2 \in F,\; \alpha \in V$

Let $A = [A_{ij}]_{m\times n} \in F^{m\times n}$

$[(c_1c_2)A]_{ij} = (c_1c_2)A_{ij}$ (by definition of scalar multiplication)

$= c_1(c_2A_{ij})$ (associative property in field F)

(*) If you don't write the reason, you won't get credit for the arguments

$=$

$\Rightarrow (c_1c_2)A$

(g) $c(\alpha+\beta)$

Let $A = [$

$[c(A+B)]_{ij}$

(h) $(c_1+c_2$

Let $A =$

$[(c_1+c_2)$

$(c_1+$

HW: Verify

given in

$$= c_1(c_2A)_{ij} \quad \text{(by definition)}$$

$$= [c_1(c_2A)]_{ij} \quad \text{(by definition)}$$

$$\Rightarrow (c_1c_2)A = c_1(c_2A)$$

(g) $c(\alpha+\beta) = c\alpha + c\beta$, $c \in F$, $\alpha, \beta \in V$

Let $A = [A_{ij}]$, $B = [B_{ij}] \in F^{m\times n}$.

$$[c(A+B)]_{ij} = c(A+B)_{ij} \quad \text{(by definition of scalar)}$$

$$= c(A_{ij}+B_{ij}) \quad \text{(by vector addition)}$$

$$= cA_{ij} + cB_{ij} \quad \text{(distributive law in F)}$$

$$= (cA)_{ij} + (cB)_{ij} \quad \text{(by definition)}$$

$$= (cA + cB)_{ij}$$

(h) $(c_1+c_2)\alpha = c_1\alpha + c_2\alpha$, $c_1, c_2 \in F$, $\alpha \in V$

Let $A = [A_{ij}]_{m\times n}$

$$[(c_1+c_2)A]_{ij} = (c_1+c_2)A_{ij} \quad \text{(by definition of scalar multiplication)}$$

$$= c_1A_{ij} + c_2A_{ij} \quad \text{(F is a field)}$$

$$= (c_1A)_{ij} + (c_2A)_{ij} \quad \text{(by definition of scalar multiplicat}^{n}\text{)}$$

$$= (c_1A + c_2A)_{ij}$$

$$(c_1+c_2)A = c_1A + c_2A.$$

<u>HW</u>: Verify all these 10 axioms for the questions given in the textbook.

Q) Let $V = R^2$ and $F = R$

Define $(x, y) + (x_1, y_1) = (x + x_1, y + y_1)$

$c(x, y) = (cx, y)$

Prove or disprove $\langle V, F, +, \cdot \rangle$ is a vector space

A) $(c_1 + c_2)\alpha = c_1\alpha + c_2\alpha$

$\alpha = (x, y)$

$(c_1 + c_2)(x, y) = ((c_1 + c_2)x, y)$

$c_1\alpha + c_2\alpha = c_1(x, y) + c_2(x, y)$

$= (c_1 x, y) + (c_2 x, y)$

$= ((c_1 + c_2)x, 2y)$

5th Sept 18

Theorem: Let V be a vector space over the field F, then

?

(i) $c0 = 0$, $c \in F$, $0 \in V$

~~(ii) $0\alpha = 0$~~

(ii) $0\alpha = 0$, $0 \in F$, $\alpha, 0 \in V$

(iii) $(-1)\alpha = -\alpha$, $\alpha \in V$

Proof: (i) Clearly $0 \in V$

$0 + 0 = 0$

For $c \in F$, $c(0 + 0) = c0$

$c0 + c0 = c0 \leftarrow$ $\boxed{\begin{array}{c} c(\alpha+\beta) = c\alpha + c\beta \\ c \in F, \alpha, \beta \in V \end{array}}$

since $c0 \in V$, $-(c0) \in V$

$\Rightarrow ((c0 + c0) + (-(c0))) = c0 + (-c0)$

$\Rightarrow c0 + (c0 + (-c0))$

$\Rightarrow c0 + 0 = 0$

$\Rightarrow c0 = 0$

(ii) $0\alpha = 0$, 0

$0 + 0 = 0$

For $\alpha \in V$, 0.

Since $0\alpha \in V$,

$-(0\alpha) + $

0

0

0

(*) Make sure t

step.

(iii) $(-1)\alpha = -\alpha$,

We have, α

0

$1 + (-1) =$

For $\alpha \in V$,

from the

Then

$(1+$

$1.\alpha$

α

$\Rightarrow c0 = 0$

(ii) $0\alpha = 0$, $0 \in F$, $\alpha, 0 \in V$

$0 + 0 = 0$ in F

For $\alpha \in V$, $0.\alpha = (0+0)\alpha$

$= 0\alpha + 0\alpha \leftarrow \boxed{(c_1+c_2)\alpha = c_1\alpha + c_2\alpha}$

Since $0\alpha \in V$, $-(0\alpha) \in V$

$-(0\alpha) + 0\alpha = -(0\alpha) + (0\alpha + 0\alpha)$

$0 = (-(0\alpha) + 0\alpha) + 0\alpha$ $\boxed{\alpha + (\beta + \gamma) = (\alpha+\beta)+\gamma}$

$0 = 0 + 0\alpha$ $(-\alpha + \alpha = 0)$

$0 = 0\alpha$ $(\alpha + 0 = \alpha = 0 + \alpha)$ $\forall \alpha \in F$

(*) Make sure that you write the reason in every step.

(iii) $(-1)\alpha = -\alpha$, $\alpha \in V$

We have, ~~$\alpha+\beta = \beta+\alpha$~~

$\alpha + \beta = 0 \Rightarrow \beta = -\alpha$

$1 + (-1) = 0$ in F

For $\alpha \in V$, $(1+(-1))\alpha = 0\alpha$

from the property obtained from (ii)

$0\alpha = 0$

Then $(1+(-1))\alpha = 0$

$1.\alpha + (-1)\alpha = 0 \leftarrow \boxed{(c_1+c_2)\alpha = c_1\alpha + c_2\alpha}$

$\alpha + (-1)\alpha = 0$ $\boxed{1.\alpha = \alpha}$ *

$\Rightarrow (-1)\alpha = -\alpha$

2023/5/3

⊛ this is a finite set ※

⇒ Linear combination:

Let V be a vector space over the field F.
A vector β is called a linear combination of vectors $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n \overset{⊛}{\in} V$ if there exists scalars $c_1, c_2, \ldots, c_n \in F$ such that

$$\beta = c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n$$
$$= \sum_{i=1}^{n} c_i\alpha_i$$

<u>Note:</u> Let β_1, β_2 be two vectors which are linear combinations of $\alpha_1, \alpha_2, \ldots, \alpha_n \in V$.

$$\Rightarrow \beta_1 = \sum_{i=1}^{n} c_i\alpha_i \;, \quad \beta_2 = \sum_{i=1}^{n} d_i\alpha_i \qquad c_i, d_i \in F.$$

$$\beta_1 + \beta_2 = \sum_{i=1}^{n} (c_i + d_i)\alpha_i$$, a linear combination of $\alpha_1, \alpha_2, \ldots, \alpha_n$.

$$c\beta_1 = c\sum_{i=1}^{n} c_i\alpha_i = \sum_{i=1}^{n} (\underbrace{cc_i}_{\text{scalar}})\alpha_i$$, a linear combination of $\alpha_1, \alpha_2, \ldots, \alpha_n$.

Q1) Find the set of all linear combinations of members of

(i) $S = \{(1,0), (0,1)\}$

(ii) $T = \{(0,1,0), (0,0,1)\}$

over $\mathbb{R}$.

<u>Soln:</u>

(i) $\alpha_1 = (1,0)$, $\alpha_2 = (0,1)$

Consider $c_1\alpha_1 + c_2\alpha_2 = c_1(1,0) + c_2(0,1)$

$= (c_1, c_2)$ where $c_i \in \mathbb{R}$

The set of all linear combinations in

$S = \{(c_1, c_2) : c_1, c_2 \in \mathbb{R}\} = \mathbb{R}^2$

$$S = \{c_1\alpha_1 + c_2\alpha_2 : c_1, c_2 \in R\}$$
$$= \{(c_1, c_2) : c_1, c_2 \in R\}$$
$$= R^2$$

(ii) $T = \{(0,1,0), (0,0,1)\}$ Ans. YZ plane.

Q2) Test whether $(1,2,3)$ is a linear combination of $(0,1,0)$ and $(0,0,1)$

Soln)

$\beta = (1,2,3)$, $\alpha_1 = (0,1,0)$, $\alpha_2 = (0,0,1)$

Does there exist c_1, c_2 such that

$$(1,2,3) = c_1(0,1,0) + c_2(0,0,1) = (0, c_1, c_2)$$

Not possible.

Q) Let $S = \{(1,1,0), (0,1,1)\}$

Prove or disprove

The set of all linear combinations of $S = R^3$

Soln: $\alpha_1 = (1,1,0)$ $\alpha_2 = (0,1,1)$

Consider $c_1\alpha_1 + c_2\alpha_2$

$$= c_1(1,1,0) + c_2(0,1,1)$$
$$= (c_1, c_1 + c_2, c_2)$$

Therefore, the set of all linear combinations of $S = R^3$ cannot be expressed as the second coordinate depends on c_1 and c_2, so only such kind of points can be expressed but not the others.

$$\therefore S \neq R^3$$

Note: Consider $C[a,b]$, the set of all real valued continuous functions defined on $[a,b]$. Let $D[a,b]$ be the set of all real valued differentiable functions defined on ~~$[a,b]$~~ $[a,b]$.

Clearly $D[a,b] \subseteq C[a,b]$

We define two operations.

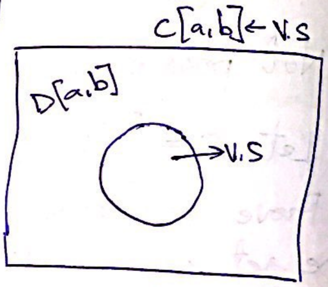
$(f+g)(x) = f(x) + g(x)$

$(cf)(x) = cf(x)$

Show that (i) $\langle C[a,b], \mathbb{R}, +, \cdot \rangle$ and

(ii) $\langle D[a,b], \mathbb{R}, +, \cdot \rangle$ are vector spaces.

$\Rightarrow D[a,b]$ is a subspace of $C[a,b]$ over $\mathbb{R}$.



Subspace: Let V be a vector space over the field F. A subspace of V is a subset W of V is itself a vector space over the field F with operations vector addition and scalar multiplication defined on V.

Theorem 1: Let V be a vector space over the field F. An non-empty ~~set~~ subset W of V is a subspace of V if and only if for each pair of vectors $\alpha, \beta \in W$ and each scalar $c \in F$, the vector $c\alpha + \beta \in W$.

Proof: Case 1: Suppose that W is a sub~~set~~space of V

$\Rightarrow W$ is a vector space over the field F.

Let $\alpha, \beta \in W$, $c \in F$

Since W is a vector space, by axiom (B),

$\alpha \in W, c \in F \Rightarrow c\alpha \in W$

By axiom (A), $c\alpha \in W$, $\beta \in W \Rightarrow c\alpha + \beta \in W$.

Case 2: Suppose that W is a non-empty subset of V such that

$$\forall \alpha, \beta \in W, c \in F \Rightarrow c\alpha + \beta \in W$$

Show that W is a vector space over the field F.

Since $W \neq \emptyset$, there exists $\gamma \in W$.

Then $(-1).\gamma + \gamma \in W$. (choose $c = -1$, $\alpha = \beta = \gamma$)

$\Rightarrow 0 = (-1)\gamma + \gamma \in W$.

We verify all axioms of vector space on W.

Axiom (A): $\forall \alpha, \beta \in W \Rightarrow \alpha + \beta \in W$.

Let $\alpha, \beta \in W$, choose $c = 1 \Rightarrow c\alpha + \beta = 1.\alpha + \beta = \alpha + \beta \in W$

Axiom (B): $\forall \alpha \in W, c \in F \Rightarrow c\alpha \in W$.

Let $\alpha \in W$, $c \in F$

Choose $\beta = 0$, by assumption, $c\alpha + \beta = c\alpha + 0 = c\alpha \in W$.

Axiom (a): $\forall \alpha, \beta \in W$, $\alpha + \beta = \beta + \alpha$

Let $\alpha, \beta \in W \subseteq V$, since V is a vector space,

$\alpha + \beta = \beta + \alpha$.

Axiom (b): $\forall \alpha, \beta, \gamma \in W \subseteq V$. Since V is a vector space

$\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$

Verify all the axioms $\Rightarrow W$ is a vector space.

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Examples: 1) Let V be a vector space over the field F and let $W=\{0\}$.

Show that W is a subspace of V.

(i) $W=\{0\} \neq \emptyset$

(ii) For all $\alpha,\beta \in W$, $c \in F$, $c\alpha+\beta \in W$

$\alpha,\beta \in W=\{0\} \Rightarrow \alpha=\beta=0$, $c\alpha+\beta = c.0+0 = 0+0 = 0 \in W$

2) Let $F^n = \{(x_1, x_2, ..., x_n) : x_i \in F\}$

$(x_1, x_2, ..., x_n)+(y_1, y_2, ..., y_n) = (x_1+y_1, x_2+y_2, ..., x_n+y_n)$

$c(x_1, x_2, ..., x_n) = (cx_1, cx_2, cx_3, ..., cx_n)$

Let $W_1 = \{(0, x_2, x_3, ..., x_n) \in F^n\}$

$W_2 = \{(1+x_2, x_2, x_3, ..., x_n) \in F^n\}$

Prove or disprove that (i) W_1 is a subspace of F^n Ans (yes)

(ii) W_2 is a subspace of F^n (no)

3) Let $F^{n\times n}$ be the set of all $n\times n$ matrices over F

W_1 = the set of all symmetric ~~n×n~~ matrices in $F^{n\times n}$

A matrix $A=[A_{ij}]_{n\times n}$ is hermitian if $A_{ij} = \overline{A_{ji}}$

Prove or disprove

(i) the set of all hermitian matrices over $\mathbb{R}$ is a subspace ~~(Proof)~~ (Prove)

(ii) the set of all hermitian matrices over $\mathbb{C}$ is a subspace. (Disprove)

10th Sept 2018

Q) Show that th

of a system o

a subspace.

Soln) Let A be

The set of all

$S=\{X$

We have, $A0=$

Let $X,Y \in S$, c

$\Rightarrow AX=0=$

Consider $A(cX$

Clearly $S \subseteq F$

Lemma: If A be

$n\times p$ matrices

$A(dB+C$

Proof is avail

Proof: $[A(dB+$

$\therefore [A(dB+C)]$

$= [d(AB)+AC$

10th Sept 2018

Q) Show that the ~~solution space~~ set of all solutions of a system of homogeneous linear equations is a subspace. ~~of~~

Soln) Let A be an $m \times n$ matrix.

The set of all solutions of a system $AX = 0$,

$$S = \{X \in F^{n\times 1} : AX = 0\} \text{ where } F \text{ is a field.}$$

We have, $A0 = 0 \Rightarrow 0 \in S$
$\Rightarrow S \neq \phi$

Let $X, Y \in S$, $c \in F$

$\Rightarrow AX = 0 = AY$

Consider $A(cX+Y) = cAX + AY$

$= c.0 + 0 = 0$

$\Rightarrow cX + Y \in S$

Clearly $S \subseteq F^{n\times 1} \Rightarrow S$ is a subspace of $F^{n\times 1}$

Lemma: If A be an $m\times n$ matrix and B, C are $n \times p$ matrices over the field F, then

$$A(dB+C) = d(AB) + AC \quad \forall d \in F$$

Proof is available in the text book (HW)

Proof: $[A(dB+C)]_{ij} = \sum_k A_{ik}(dB+C)_{kj}$

$= \sum_k (dA_{ik}B_{kj} + A_{ik}C_{kj})$

$= d\sum_k A_{ik}B_{kj} + \sum_k A_{ik}C_{kj}$

$= d(AB)_{ij} + (AC)_{ij}$

$= [d(AB) + (AC)]_{ij}$

$\therefore [A(dB+C)] = [d(AB) + AC]$

Q) Let W_1 and W_2 be two subspaces of a vector space V over the field F.

Prove or disprove

(i) $W_1 \cap W_2$ is a subspace of V.

(ii) $W_1 \cup W_2$ is a subspace of V.

Soln)

(i) Since W_1 and W_2 are subspaces of V

$0 \in W_1$ and $0 \in W_2$

$\Rightarrow 0 \in W_1 \cap W_2 \neq \emptyset$

Let $\alpha, \beta \in W_1 \cap W_2$ and $c \in F$

$\Rightarrow \alpha, \beta \in W_1$, $\alpha, \beta \in W_2$, and $c \in F$

Since W_1 is a subspace of $V \Rightarrow c\alpha + \beta \in W_1$

Since W_2 is a subspace of $V \Rightarrow c\alpha + \beta \in W_2$

$\Rightarrow c\alpha + \beta \in W_1 \cap W_2$

$\Rightarrow W_1 \cap W_2$ is a subspace of V.

(ii) This is not true

So, give a counter example to disprove it.

Let $V = \mathbb{R}^2$,

$W_1 = \{(x, 0) : x \in \mathbb{R}\}$

$W_2 = \{(0, y) : y \in \mathbb{R}\}$

e.g. $\alpha = (1,0)$, $\beta = (0,1) \in W_1 \cup W_2$

$c = 1$, $c\alpha + \beta = (1,0) + (0,1) = (1,1) \notin W_1 \cup W_2$

$W_1 \cup W_2$ is not a subspace.

Theorem 2: Let V be a

Then the intersection

is a subspace of

Proof: (HW). Let $\{W_a\}$

let $W = \bigcap_a W_a$ be their

as the set of all elem

each W_a is a subspace

The zero vector is in the

Let α and β be vector

definition of W, both

because each W_a is a

every W_a. Thus $(c\alpha + \beta)$

is a subspace of V.

From theorem 2, it follo

vectors in V, then there

which contains S, that

S and which is conta

containing S.

Note: i) Let V be a vec

$S \subseteq V$.

· Consider all subsp

Then intersection of

also a subspace (th

Notation: Subspace

$S = \cap \{W : W$ is a

Theorem 2: Let V be a vector space over the field F. Then the intersection of any number of subspaces of V is a subspace of V.

Proof: (HW). Let $\{W_\alpha\}$ be a collection of subspaces of V, & let $W = \bigcap_\alpha W_\alpha$ be their intersection. Recall that W is defined as the set of all elements belonging to every W_α. Since each W_α is a subspace, each contains the zero vector. Thus the zero vector is in the intersection W, and W is non-empty. Let α and β be vectors in W and let c be a scalar. By definition of W, both α and β belong to each W_α, and because each W_α is a subspace, the vector $(c\alpha+\beta)$ is in every W_α. Thus $(c\alpha+\beta)$ is again in W. By theorem 1, W is a subspace of V.

From theorem 2, it follows that if S is any collection of vectors in V, then there is a smallest subspace of V which contains S, that is, a subspace which contains S and which is contained in every other subspace containing S.

Note: 1) Let V be a vector space over the field F. Let $S \subseteq V$.

· Consider all subspaces of V which contain S. Then intersection of all subspaces contains S is also a subspace (theorem 2).

Notation: Subspace spanned by ~~$S = \cap\{$~~

$$S = \cap\{W : W \text{ is a subspace of } V \text{ which contains } S\}$$

2) $L(S)$ = set of all linear combinations of S.

3) If $S = \{\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n\}$, then subspaces spanned by S is called subspace spanned by $\alpha_1, \alpha_2, \ldots, \alpha_n$.

4) $L(S)$ = subspace spanned by S (next theorem).

$$= \cap\{W : W \text{ is a subspace and } S \subseteq W\}$$

~~(1) S~~

Arguments: $\boxed{S \subseteq L(S) \subseteq W}$

(i) $S \subseteq L(S)$

(ii) $L(S)$ is a subspace of V

(iii) For any subspace W which contains S,
$L(S) \subseteq W$

Theorem 3: Let S be a non-empty subset of a vector space V over the field F. Then, the subspace spanned by S is the set of all linear combinations ~~in S~~ of vectors in S.

Proof: It is enough to prove the following

(i) $S \subseteq L(S)$

(ii) $L(S)$ is a subspace of V.

(iii) For a subspace W of V which contains S, $L(S) \subseteq W$.

(i) Let $\alpha \in S$, by definition of $L(S)$,

$1.\alpha \in L(S)$

$\Rightarrow \alpha = 1.\alpha \in L(S)$

$\Rightarrow S \in L(S)$

(ii) Since $S \neq \emptyset$, the

By definition of L

Let $\alpha, \beta \in L(S)$, c

$\Rightarrow \alpha = c_1\alpha_1 + c_2\alpha_2$ -

$\beta = d_1\beta_1 + d_2\beta_2 +$

$c\alpha + \beta = (cc_1)\alpha_1 + ($

$\in L(S)$

$\Rightarrow L(S)$ is a su

(iii) Let W be a

Let $\alpha \in L(S) \Rightarrow$

Since $\alpha_1, \alpha_2, \ldots, \alpha$

$c_1\alpha_1, c_2\alpha_2, \ldots, c_n$

$\alpha = c_1\alpha_1 + c$

$\Rightarrow L(S) \subseteq W$.

(ii) Since $S \neq \phi$, there exists $\alpha \in S$.

By definition of $L(S)$, $0.\alpha \in L(S)$

$$\Rightarrow 0 = 0.\alpha \in L(S) \neq \phi$$

Let $\alpha, \beta \in L(S)$, $c \in F$

$\Rightarrow \alpha = c_1\alpha_1 + c_2\alpha_2 + \dots + c_n\alpha_n$, $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n \in S$, $c_i \in F$

$\beta = d_1\beta_1 + d_2\beta_2 + \dots + d_m\beta_m$, $\beta_1, \beta_2, \dots, \beta_m \in S$, $d_j \in F$

$$c\alpha + \beta = (cc_1)\alpha_1 + (cc_2)\alpha_2 + \dots + (cc_n)\alpha_n + d_1\beta_1 + \dots + d_m\beta_m \in L(S)$$

1 $\Rightarrow L(S)$ is a subspace of V.

(iii) Let W be a subspace of V and $S \subseteq W$.

Let $\alpha \in L(S) \Rightarrow \alpha = c_1\alpha_1 + c_2\alpha_2 + \dots + c_n\alpha_n$, $\alpha_i \in S$, $c_i \in F$

Since $\alpha_1, \alpha_2, \dots, \alpha_n \in S \subseteq W$ and W is a subspace, $c_1\alpha_1, c_2\alpha_2, \dots, c_n\alpha_n \in W$ and hence

$$\alpha = c_1\alpha_1 + c_2\alpha_2 + \dots + c_n\alpha_n \in W$$

$\Rightarrow L(S) \subseteq W$.

12/13 Sept/18 Paper discussion.

Q3) Let $C_1, C_2, \ldots C_n$ be $m \times 1$ column vectors.

(i) $n > m$ Prove or disprove that one of the columns can be a linear combination of other columns.

$$A = \begin{pmatrix} C_1 & C_2 & \cdots & C_n \\ | & | & & | \\ | & | & & | \end{pmatrix}_{m \times n} \qquad AX = 0$$

$n > m$.

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad x_i \neq 0$$

$$x_1 C_1 + x_2 C_2 + \ldots + x_n C_n = 0$$

$x_1 \neq 0$ $\quad C_1 = \frac{-x_2}{x_1} C_2 + \frac{-x_3}{x_1} C_3 + \ldots + \frac{-x_n}{x_1} C_n$

(ii) Prove or disprove that no column can be written as a linear combination of other columns.

$n \leq m$

$$C_1 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad C_2 \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \ldots$$

$\therefore C_1 = C_2 + C_3 + \ldots + C_n$.

14th Sept'18.

Recall: Let V be a v

The subspace spann

Q) Let $V = R^2$ be a v

Let $S = \{(1,0)\}$ a

Find the subspa

Ans) (i) The subspace

$= L(S) = \{c($

$= \{(c,0) : c$

(ii) The subspace

$= L(T) = \{c$

$= \{(c_1, c_2) :$

Q) Let $C_1, C_2, \ldots$

(1) invertible $n \times n$

spanned by ~~A~~

Ans:) Let $S = \{C_1, C_2,$

Subspace spann

Its enough to prov

$R^n \subseteq$ subspac

Let ~~X∈~~ $Y \in R^n$. Si

$AX = Y$ has a se

$\Rightarrow AX = Y$ has a

$x_1 C_1 + x_2 C_2 + \ldots +$

14th Sept'18.

Recall: Let V be a vector space and let $S \subseteq V$.

The subspace spanned by $S = \cap \{W : W$ is a subspace of V and $S \subseteq W\}$.

$$= L(S) = \{c_1\alpha_1 + c_2\alpha_2 + \dots + c_n\alpha_n ; \alpha_i \in S, c_i \in S\}$$

Q) Let $V = R^2$ be a vector space over the field R.
Let $S = \{(1,0)\}$ and $T = \{(1,0), (0,1)\}$
Find the subspaces spanned by (i) S and (ii) T

Ans) (i) The subspace spanned by S

$$= L(S) - \{c(1,0) : c \in R\}$$

$$= \{(c,0) : c \in R\} = \text{The } x\text{-axis in } R^2$$

(ii) The subspace spanned by T

$$= L(T) = \{c_1(1,0), \cancel{d}^{c_2}(0,1) \} : c_1 \in R, c_2 \in R\}$$

$$= \{(c_1, c_2) = c_1, c_2 \in R\} = R^2$$

Q) Let $c_1, c_2, \dots c_n$ be the columns of 'A' an
1 invertible $n \times n$ matrix 'A'. Show that the subspace spanned by ~~A is R^n~~ $\{c_1, c_2, \dots, c_n\}$ is R^n.

Ans:) Let $S = \{c_1, c_2, \dots, c_n\}$

Subspace spanned by $S = L(S)$ (By last theorem)

$$= \{k_1c_1 + k_2c_2 + \dots + k_nc_n : k_i \in R\} \subseteq R^n$$

1

Its enough to prove that

$R^n \subseteq$ subspace spanned by S.

Let ~~$X \in$~~ $Y \in R^n$. Since 'A' is invertible,

$AX = Y$ has a solution for any Y.

$\Rightarrow AX = Y$ has a solution.

$$x_1c_1 + x_2c_2 + \dots + x_nc_n = Y$$

⇒ Y is a linear combination of $c_1, c_2, \ldots, c_n$

⇒ $Y \in L(S)$

⇒ $Y \in$ subspace spanned by S.

$R^n \subseteq$ subspace spanned by S —— (2)

From (1) and (2)

$R^n =$ subspace spanned by $\{c_1, c_2, \ldots, c_n\}$

Def: ~~Proof~~: Let V be a vector space over the field F.

~~i.e.~~ Let S_1, S_2 be two subsets of V.

Then, $S_1 + S_2 = \{x+y : x \in S_1, y \in S_2\}$

Example 1: Let V be the vector space of all 2×2 matrices over the field F.

Let $W_1 = \left\{ \begin{bmatrix} x & y \\ z & 0 \end{bmatrix} : x, y, z \in F \right\}$

$W_2 = \left\{ \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} : x, y \in F \right\}$

Q) Show that (i) W_1 and W_2 are subspaces of $V = F^{2\times 2}$

(ii) $W_1 + W_2 = F^{2\times 2}$.

Soln: (i) Show that W_1 is a subspace of $F^{2\times 2}$.

clearly, $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in W_1 \neq \phi$

Let $\alpha, \beta \in W$, $\alpha = \begin{bmatrix} x_1 & y_1 \\ z_1 & 0 \end{bmatrix}$, $\beta = \begin{bmatrix} x_2 & y_2 \\ z_2 & 0 \end{bmatrix}$

$c \in F$

$$c\alpha + \beta = \begin{bmatrix} cx_1 + x_2 & cy_1 + y_2 \\ cz_1 + z_2 & 0 \end{bmatrix} \in W_1$$

⇒ W_1 is a subspace of $F^{2\times 2}$

Show that W_2

Clearly $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Let $\alpha, \beta \in W$

$c \in F$

$c\alpha + \beta = [c$

⇒ W_2 is a su

(ii) ~~Let~~ We pro

Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

⇒ $\cancel{\begin{pmatrix} a & b \\ c & 0 \end{pmatrix}}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} =$

$\in$

We prove W_1

$\cancel{\begin{pmatrix} a_1 & b_1 \\ c_1 & 0 \end{pmatrix} +}$

Let $\alpha \in W_1 +$

⇒ $\alpha = w$

$\alpha = w_1$

Show that W_2 is a subspace of $F^{2\times2}$

Clearly $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in W_2 \neq \phi$

Let $\alpha, \beta \in W$ $\quad \alpha = \begin{bmatrix} x_1 & 0 \\ 0 & y_1 \end{bmatrix}, \ \beta = \begin{bmatrix} x_2 & 0 \\ 0 & y_2 \end{bmatrix}$

$c \in F$

$$c\alpha + \beta = \begin{bmatrix} cx_1 + x_2 & 0 \\ 0 & cy_1 + y_2 \end{bmatrix} \in W_2$$

$\Rightarrow W_2$ is a subspace of $F^{2\times2}$.

(ii) ~~Let~~ We prove $F^{2\times2} \subseteq W_1 + W_2$

Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in F^{2\times2}$ $\quad a, b, c, d \in F$

$$\cancel{\Rightarrow \begin{pmatrix} a & b \\ c & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}$$

$$\in W_1 + W_2$$

We prove $W_1 + W_2 \subseteq F^{2\times2}$.

$$\cancel{\begin{pmatrix} a_1 & b_1 \\ c_1 & 0 \end{pmatrix} + (\ )}$$

Let $\alpha \in W_1 + W_2 = \{ w_1 + w_2 : w_1 \in W_1, \ w_2 \in W_2 \}$

$\Rightarrow \alpha = w_1 + w_2$, for some $w_1 \in W_1$, $w_2 \in W_2$

$\alpha = w_1 + w_2 \in F^{2\times2}$.

Q) Let $W_1, W_2, \ldots, W_k$ be subspaces of a vector space V over the field F. Then show that

(i) $W = W_1 + W_2 + \ldots + W_k$ is a subspace of V.

(ii) $W =$ subspace spanned by $W_1 \cup W_2 \cup \ldots \cup W_k$

$$= L(W_1 \cup W_2 \cup \ldots \cup W_k)$$

~~Ans) (i)~~

Q) Let V be a vector space of all polynomials over the field F.

Let $f_i(x) = x^i$, $i = 0, 1, 2, \ldots$

and let $S = \{f_0(x), f_1(x), \ldots\}$

Show that subspace spanned by $S = V$

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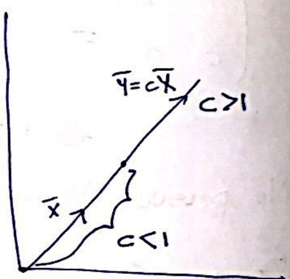
Note: Consider two vectors $\bar{X}$ and $\bar{Y}$ in R such that

$\bar{Y} = c\bar{X}$, $c \neq 0$, $c \in R$.

We say $\bar{X}$ and $\bar{Y}$ are linearly dependent



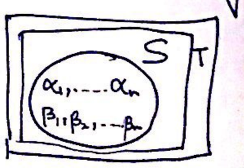
$\Rightarrow c\bar{X} + (-1)\bar{Y} = 0$ i.e. $c_1\bar{X} + c_2\bar{Y} = 0$

(atleast one of $c_i \neq 0$)

Bases and dimension

Definition: Let V be a vector space over the field F. A subset S of V is linearly dependent if there exists distinct vectors $\alpha_1, \alpha_2, \ldots, \alpha_n$ in S and scalars $c_1, c_2, \ldots, c_n$ (atleast one $c_i \neq 0$) in F such that

$$c_1\alpha_1 + c_2\alpha_2 + \ldots + c_n\alpha_n = 0.$$



A set which is n
independent (L.I)

Observations:

1. A super set of
2. A subset of a
3. Since $1.0 = 0$, e
dependent.
4. (A set S is linea
$\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of
for $\alpha_1, \alpha_2, \ldots, \alpha_n$

~~Note: AX=0 ha~~

Note: $AX = 0$ ha

A is invertible $\Leftrightarrow$

$\Leftrightarrow$

$\Leftarrow$

$\Leftarrow$

Q) Show that
is a linearly

Ans) Consider c

$c_1(1,0,0)$

$(c_1, c_2$

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A set which is not linearly dependent is called linearly independent (L.I).

Observations:

1. A super set of a L.D. set is L.D.
2. A subset of a L.I set is L.I.
3. Since $1.0=0$, every set contains "zero" is linearly dependent.
4. A set S is linearly independent $\Leftrightarrow$ every finite subset $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ of S is linearly independent $\Leftrightarrow$ for $\alpha_1, \alpha_2, \dots, \alpha_n \in S$ and $c_1\alpha_1 + c_2\alpha_2 + \dots + c_n\alpha_n = 0 \Rightarrow$

$$c_1 = c_2 = \dots = c_n = 0$$

~~Note: $AX=0$ has only trivial soln solution iff~~

Note: $AX=0$ has only trivial solution $\Leftrightarrow$ A is invertible

A is invertible $\Leftrightarrow$ $AX=c$ has only trivial solution.

$\Leftrightarrow x_1c_1 + x_2c_2 + \dots + x_nc_n = 0$ has ~~and~~ only trivial solution.

$\Leftrightarrow x_1 = x_2 = \dots = x_n = 0$

$\Leftrightarrow$ the columns ~~c_1, c_2, c_n~~ $c_1, c_2, \dots, c_n$ are L.I.

Q) Show that $\{\epsilon_1 = (1,0,0), \epsilon_2 = (0,1,0), \epsilon_3 = (0,0,1)\}$ is a linearly independent set in $\mathbb{R}$.

Ans) Consider $c_1\epsilon_1 + c_2\epsilon_2 + c_3\epsilon_3 = 0$

$$c_1(1,0,0) + c_2(1,0,0) + c_3(0,0,1) = (0,0,0)$$

$$(c_1\ c_2\ c_3) = (0,0,0)$$

$\Rightarrow c_1 = c_2 = c_3 = 0 \Rightarrow \{\epsilon_1, \epsilon_2, \epsilon_3\}$ is linearly independent

Q) Show that $\{\alpha_1 = (1,1,0), \alpha_2 = (1,0,1), \alpha_3 = (0,1,1)\}$ is linearly indendent in R^3?

Consider $c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 = 0$

$$c_1(1,1,0) + c_2(1\ 0\ 1) + c_3(0,1,1) = (0,0,0)$$

$$(c_1 + c_2,\ c_1 + c_3,\ c_2 + c_3) = (0,0,0)$$

$$\left.\begin{array}{l} c_1 + c_2 = 0 \\ c_1 + c_3 = 0 \\ c_2 + c_3 = 0 \end{array}\right\} \begin{array}{l} \rightarrow c_1 + c_2 = 0 \\ \rightarrow c_1 - c_2 = 0 \end{array} \rightarrow 2c_1 = 0 \ \cancel{\rightarrow c_1 = c} \rightarrow c_1 = 0$$

$c_1 = 0 \Rightarrow c_2 = 0 = c_3 \Rightarrow \cancel{c_1 = } \ c_1 = c_2 = c_3 = 0$

Sdn: Prove or disprove that $S = \{\alpha_1 = (3,0,-3), \alpha_2 = (-1,1,2), \alpha_3 = (4,2,-2), \alpha_4 = (2,1,1)\} \in R$ is linearly ~~ine~~ dependent.

Consider $c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 + \dots + c_4\alpha_4 = 0$

Ans: $\boxed{\cancel{2\alpha_1 + 2\alpha_2 - \alpha_3 + 0.\alpha_4 = 0}}$

$\boxed{2\alpha_1 + 2\alpha_2 - \alpha_3 + 0.\alpha_4 = 0}$ ~~is linearly independent~~

Definition (Basis)

A subset S of a vector space V is called a basis for V if

(i) S is ~~lina~~ linearly independent in V. and

(ii) S spans V (i.e., $L(S) = V$).

Q) Show that
a basis for
Ans: Claim (1)
Claim (2): L(
Clearly
Let $a, b, c \in$
$(a,b,c) =$
$=$
$R^3 \subseteq L(\beta)$
From ⓐ and
Q) Show that
for R^3.
Ans: Claim (1):
Claim (2):
Clearly, $L(\beta$

Q) Show that $\beta = \{\epsilon_1 = (1,0,0),\ \epsilon_2 = (0,1,0),\ \epsilon_3 = (0,0,1)\}$ is a basis for R^3. $\longrightarrow$ standard basis for R^3.

<u>Ans:</u> Claim (1): β is linearly independent in R^3 (verified)

Claim (2): $L(\beta) = R^3$

Clearly, $L(\beta) = \{c_1\epsilon_1 + c_2\epsilon_2 + c_3\epsilon_3 : c_i \in R\}$
$\subseteq R^3$ ——— (a)

Let $a,b,c \in R^3$

$$(a,b,c) = a(1,0,0) + b(0,1,0) + c(0,0,1)$$
$$= a\epsilon_1 + b\epsilon_2 + c\epsilon_3 \in L(\beta)$$

$R^3 \subseteq L(\beta)$ ——— (b)

From (a) and (b), $R^3 = L(\beta)$

Q) Show that $\beta' = \{\overset{\alpha_1}{(1,1,0)}, \overset{\alpha_2}{(1,0,1)}, \overset{\alpha_3}{(0,1,1)}\}$ is a basis for R^3.

<u>Ans:</u> Claim (1): β' is linearly independent (verified)

Claim (2): $L(\beta') = R^3$.

Clearly, $L(\beta') = \{c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 : c_i \in R\} \subseteq R^3$.

Q) Let A be an $n\times n$ invertible matrix over the field F and let $c_1, c_2, \ldots, c_n$ be the columns of A. Show that $\beta = \{c_1, c_2, \ldots, c_n\}$ is a basis of F^n.

Ans) <u>Claim (i)</u> $\beta = \{c_1, c_2, \ldots, c_n\}$ is a linearly independent subset of F^n.

For any scalars $x_1, x_2, \ldots, x_n \in F$, consider

$x_1c_1 + x_2c_2 + \cdots + x_nc_n = 0 \Rightarrow AX = 0$

where $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in F^n$

$\Rightarrow X = 0$ since A is invertible.

(A is invertible $\Leftrightarrow AX=0$ has only trivial solution)

$\Rightarrow x_1 = x_2 = \cdots = x_n = 0$

$\Rightarrow \beta$ is a linearly independent subset of F^n.

<u>Claim (ii)</u> $L(\beta) = F^n$

Clearly $L(\beta) \subseteq F^n$ ——— (a)

Let $Y \in F^n$. Then $AX = Y$ has a solution X for any $Y \in F^n$.

$\Rightarrow Y = AX$ (∵ A is invertible).

~~$Y = x_1c_1 + x_2c_2 + \cdots + x_nc_n \quad x_i \in F$~~

$Y = x_1c_1 + x_2c_2 + \cdots + x_nc_n \in L(\beta)$, $x_i \in F$

$\Rightarrow F^n \subseteq L(\beta)$ ——— (b)

From (a) and (b), $F^n = L(\beta)$

By claim (i) and claim (ii), β is a basis for F^n.

Q) Find a basis ...

$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ ...

Ans) Consider $AX =$ ...

$\Rightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ ...

$\Rightarrow x_2 - 3x_3 + \frac{1}{2}$ ...

$x_4 + 2x_5$ ...

Pivot variables ...

Free variables ...

Set arbitrary ...

Let $x_1 = a$, x ...

$\Rightarrow x_2 = 3b - \frac{1}{2}$ ...

The solution spa...

$S = \{X = (x_1, x_2$ ...

$S = \{(a, 3b - \frac{1}{2}$ ...

Note that S i...

Consider the f...

set $x_1 = 1$, x_3 ...

(I) $\Rightarrow x_2 = 0$, ...

$E_1 = (1, 0,$ ...

Q) Find a basis of the solution space of $AX=0$ where

$$A = \begin{pmatrix} 0 & 1 & -3 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Ans) Consider $AX=0$

$$\Rightarrow \begin{pmatrix} 0 & 1 & -3 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \left.\begin{aligned} x_2 - 3x_3 + \tfrac{1}{2}x_5 &= 0 \\ x_4 + 2x_5 &= 0 \end{aligned}\right\} \text{(I)}$$

Pivot variables $= \{x_2, x_4\}$

Free variables $= \{x_1, x_3, x_5\}$

Set arbitrary values to free variables.

Let $x_1 = a, \; x_3 = b, \; x_5 = c$

$\Rightarrow x_2 = 3b - \frac{1}{2}c \;, \; x_4 = -2c$

The solution space S of $AX=0$

$S = \{X = (x_1, x_2, x_3, x_4, x_5) : AX = 0\}$

$S = \{(a, 3b - \frac{1}{2}c, b, -2c, c) : a, b, c \in F\}$

Note that S is a subspace of F^5.

Consider the free variables x_1, x_3, x_5

set $x_1 = 1, \; x_3 = x_5 = 0$ in (I)

(I) $\Rightarrow x_2 = 0, \; x_4 = 0$

$E_1 = (1, 0, 0, 0, 0)$

set $x_3=1$, $x_1=x_5=0$

~~(I) ⇒~~ (I) ⇒ $x_2=3$, $x_4=0$

~~$E_3=(0,\frac{1}{2},0,-2,1)$~~

$E_3=(0,3,1,0,0)$

set $x_5=1$, $x_1=x_3=0$

(I) ⇒ $x_2=-\frac{1}{2}$, $x_4=-2$

$E_5=(0,-\frac{1}{2},0,-2,1)$

We prove that $\beta=\{E_1,E_3,E_5\}$ is a basis of S.

$L(\beta)=\{aE_1+bE_3+cE_5 : a,b,c\in F\}$

$=\{a(1,0,0,0,0)+b(0,3,1,0,0)+c(0,-\frac{1}{2},0,-2,1) : a,b,c\in F\}$

$=\{(a,3b-\frac{1}{2}c,b,-2c,c) : a,b,c\in F\}$

$=S \Rightarrow \beta$ spans S.

Show that β is linearly independent

Consider $aE_1+bE_3+cE_5=0$

$(a,3b-\frac{1}{2}c,b,-2c,c)=(0,0,0,0,0)$

$\Rightarrow a=b=c=0$

$\Rightarrow \beta$ is linearly independent

$\beta=\{E_1,E_3,E_5\}$ is a basis of S.

HW

Q) Let P be the vector space of all polynomials. Show that $\beta=\{f_n(x): n=0,1,2,\dots\}$ where $f_n(x)=x^n$ is a basis of P.

Def: A vector spa

finite dimension

Examples: 1) F^n

~~2) $F^{m\times n}$~~ 2)

HW: Show that P,

a finite dimen

24th Sept'18

Recall: Let A = [

field F.

Consider $AX=0$

$\Rightarrow A_{11}x_1+A_{12}x_2$

$A_{21}x_1+A_{22}x_2$

$A_{m1}x_1+A_{m2}x$

$AX=0$ is same

when $m<n$,

has a non-

such that

Def: A vector space with a finite basis is called a finite dimensional vector space.

Examples: 1) $F^n \leftarrow \beta = \{ \epsilon_1 = (1,0,0\ldots,0)$
$\epsilon_2 = (0,1,0,\ldots 0)$
$\epsilon_3 = (0,0,1,\ldots 0)$
$\epsilon_n = (0,0,0\ldots 1) \}$

~~2) $F^{m\times n}$~~ 2) $F^{2\times 3} \leftarrow \beta = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$

HW: Show that P, vector space of all polynomials is not a finite dimensional vector space.

24th Sept'18

Recall: Let $A = [A_{ij}]_{m\times n}$ be an $m\times n$ matrix over the field F.

Consider $AX=0$

$$\Rightarrow \left.\begin{array}{l} A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n = 0 \\ A_{21}x_1 + A_{22}x_2 + \cdots + A_{2n}x_n = 0 \\ A_{m1}x_1 + A_{m2}x_2 + \cdots + A_{mn}x_n = 0 \end{array}\right\} \Rightarrow \sum_{j=1}^{n} A_{ij}x_j = 0, \quad 1 \le i \le m.$$

$AX=0$ is same as $\sum_{j=1}^{n} A_{ij}x_j = 0$, $1 \le i \le m$.

When $m<n$, $\sum_{j=1}^{n} A_{ij}x_j = 0$, $1 \le i \le m$
has a non-trivial solution $X = (x_1, x_2, \ldots, x_n)$
such that atleast one $x_j \neq 0$.

~~%~~ Very imp.

Theorem 4: Let V be a vector space spanned by the ~~vector~~ a set of finite vectors $\beta_1, \beta_2, \ldots, \beta_m$ in V. Then, any ~~independent~~ linearly independent set in V contains ~~finite number of elements~~ atmost 'n' elements.

Proof: It is enough to prove that every subset S of V contains more than 'n' elements is linearly dependent.

Let $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subseteq V$, where $n > m$.

since $\beta_1, \beta_2, \ldots, \beta_m$ spans V,

$$V = \text{span}\{\beta_1, \beta_2, \ldots, \beta_m\}$$
$$= L(\{\beta_1, \beta_2, \ldots, \beta_m\})$$

Since $\alpha_1 \in V$, $\alpha_1 = A_{11}\beta_1 + A_{21}\beta_2 + \cdots + A_{m1}\beta_m$

$|||^{ly}$ ~~$\alpha_j = A_{1j}B_j$~~

$$\alpha_j = A_{1j}B_1 + A_{2j}B_2 + \cdots + A_{mj}B_m$$
$$= \sum_{i=1}^{m} A_{ij}\beta_i \quad 1 \le j \le n \quad \text{where } A_{ij} \in F$$

Show that S is linearly dependent

$$\boxed{\text{It is enough to prove that } x_1\alpha_1 + x_2\alpha_2 + \cdots + x_n\alpha_n = 0 \text{ has a non-trivial solution.}}$$

Consider $x_1\alpha_1 + x_2\alpha_2 + \cdots + x_n\alpha_n = \sum_{j=1}^{n} x_j\alpha_j$

$$= \sum_{j=1}^{n} x_j\left(\sum_{i=1}^{m} A_{ij}\beta_i\right)$$

$$x_1\alpha_1 + x_2\alpha_2 + \dots + x_n\alpha_n = \sum_{i=1}^{m}\left(\sum_{j=1}^{n} A_{ij}x_j\right)\beta_i \quad \text{——— (1)}$$

Since $m<n$, $\sum_{j=1}^{n} A_{ij}x_j = 0$, $1 \le i \le m$ has a non-trivial solution $X = (x_1, x_2, \dots, x_n)$ such that atleast one $x_j \neq 0$

(1) $\Rightarrow x_1\alpha_1 + x_2\alpha_2 + \dots + x_n\alpha_n = \sum_{i=1}^{m}(0)\beta_i = 0$ has a non-trivial solution.

$\Rightarrow$ S is linearly dependent.

(*) This is an important theorem and many questions can be asked using this.

Corollary: Let V be a finite dimensional vector space. Then any two bases of V have same (finite) number of elements.

Proof: Let $\beta = \{\beta_1, \beta_2, \dots, \beta_m\}$ and

$\beta' = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be two bases of V.

By definition of bases, (i) span β = span $\beta' = V$ and

(ii) β and β' are linearly independent.

Consider span $\beta = V$ and β' is linearly independent.

By theorem 4, $|\beta'| \le |\beta|$ ——— (1)

Consider span $\beta' = V$ and β is linearly independent.

By theorem 4, $|\beta| \le |\beta'|$ ——— (2)

From (1) and (2), $|\beta| = |\beta'|$

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Definition: The dimension of a finite dimensional vector space is number of elements in a basis.

Notation: $\dim V$

Q) 1) Show that

(i) $\dim F^n = n$ → standard basis $\epsilon_1 = (1,0,0,\dots 0)$, $\epsilon_2 = (0,1,0 \dots 0)$, ------, $\epsilon_n = (0,0,\dots,1)$

(ii) $\dim F^{m\times n} = mn$.

*(Answer in the text)

Q) Let A be an $m\times n$ matrix and let 'r' be the number of non-zero rows of a row-reduced echelon matrix equivalent to A.

Let $S = \{X \in F^n : AX = 0\}$

Show that $\dim S = n - r$ (no. of free variables)

Problem: Let V be a finite dimensional vector space and dimension $\dim V = n$.

Show that

(i) any subset of V contains more than 'n' elements is linearly dependent (proof is theorem 4)

(ii) no subset of V which contains less than 'n' vectors spans V (proof in Theorem 4)

Corollary: If W is a proper subspace of a finite dimensional vector space V ($V \setminus W \neq \emptyset$), then W is a finite dimensional vector space and $\dim W < \dim V$.

Proof: If

26th Sept '18

Lemma: Let S be a l

finite dimensional v

$\beta \in V \setminus L(S)$ (i.e.

Then $S \cup \{\beta\}$ is a li

Proof: We have, S is

a ~~finitely~~ finite dim

$\beta \in V \setminus L(S)$

Let $S = S \cup \{\beta\}$. Sho

set in V.

Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be

Consider $c_1\alpha_1 + c_2\alpha_2$

Case 1: $\beta \neq \alpha_i$, $1 \leq i$

$\Rightarrow \alpha_1, \alpha_2, \dots, \alpha_n$

Since S is linear

① $\Rightarrow c_1 = c_2 = \dots =$

Case 2: $\beta = \alpha_i$ for

Without loss of g

~~① $\Rightarrow c_1\alpha_1$~~

~~① $\Rightarrow c_1\beta_1$~~ ① $\Rightarrow c_1$

We claim that

② $\Rightarrow \beta = \left(\frac{-c_2}{c_1}\right)\alpha_2$

a contradi

26th Sept '18

Lemma: Let S be a linearly independent subset of a finite dimensional vector space and

$\beta \in V \setminus L(S)$ (i.e. $\beta \in V$, $\beta \notin L(S) = \text{span } S$)

Then $S \cup \{\beta\}$ is a linearly independent subset of V.

Proof: We have, S is a linearly independent subset of a ~~finitely~~ finite dimensional vector space V and $\beta \in V \setminus L(S)$

Let $S_1 = S \cup \{\beta\}$. Show that S_1 is a linearly independent set in V.

Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be distinct vectors in S_1.

Consider $c_1\alpha_1 + c_2\alpha_2 + \dots + c_n\alpha_n = 0$ ——— (1)

Case 1: $\beta \neq \alpha_i$, $1 \leq i \leq n$

$\Rightarrow \alpha_1, \alpha_2, \dots, \alpha_n \in S$

Since S is linearly independent.

(1) $\Rightarrow c_1 = c_2 = \dots = c_n = 0$

Case 2: $\beta = \alpha_i$ for an $i = 1, 2, \dots, n$

Without loss of generality, $\beta = \alpha_1$

~~(1) $\Rightarrow c_1\alpha_1$~~

~~(1) $\Rightarrow c_1\beta_1$~~ (1) $\Rightarrow c_1\beta + c_2\alpha_2 + \dots + c_n\alpha_n = 0$ ——— (2)

$\beta \in V \setminus L(S)$
$\beta \in V$
$\beta \notin L(S)$

We claim that $c_1 = 0$, if not $c_1 \neq 0$

(2) $\Rightarrow \beta = \left(\frac{-c_2}{c_1}\right)\alpha_2 + \left(\frac{-c_3}{c_1}\right)\alpha_3 + \dots + \left(\frac{-c_n}{c_1}\right)\alpha_n \in L(S)$.

since $\alpha_2, \alpha_3, \dots, \alpha_n \in S$,

a contradiction.

$② \Rightarrow c_2\alpha_2 + c_3\alpha_3 + \dots + c_n\alpha_n = 0$

since $\alpha_2, \alpha_3, \dots, \alpha_n \in S$ and S is LI

$③ \Rightarrow c_2 = c_3 = \dots = c_n = 0$

Hence in both cases, $(1) \Rightarrow c_1 = c_2 = \dots = c_n = 0$

$=0$

28th Sept 16

Corollary: If W is a proper subspace of a finite dimensional vector space V, $(V\setminus W \neq \emptyset)$, then W is a finite dimensional vector space and $\dim W < \dim V$

Proof: If $W=\{0\}$, then $\dim W = 0$

Since $V\setminus W = V\setminus\{0\} \neq \emptyset$, then $\dim V > 0 = \dim W$

So $W \neq \{0\}$. Then there is a non-zero vector $\alpha \in W$.

By thm 5, W has a basis S which contains α. Since S is a L.I set in V, $|S| \leq \dim V$

$\Rightarrow$ W has a finite basis S and W is a finite dimensional vector space

Since S is a basis for W, $W = L(S)$

Since $V\setminus L(S) = V\setminus W \neq \emptyset$, there exists $\beta \in V\setminus W$ such that $S \cup \{\beta\}$ is a L.I set of V.

By theorem 4, $|S\cup\{\beta\}| \leq \dim V$

$\Rightarrow |S|+1 \leq \dim V$

$\Rightarrow \cancel{\dim W+1}$ $\dim W + 1 \leq \dim V$, (since S is a basis for W)

$\rightarrow \dim W < \dim V$.

Note: Let W be a subspace of a finite dimensional vector space V and let $\dim W = \dim V$, then $W = V$.

Corollary 3: Let A be an $n\times n$ matrix over the field F, and suppose that row vectors of A form a linearly independent set of vectors in F^n. Then A is invertible

Proof: Let $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be the rows of A

Given that S is a linearly independent set in F^n.

Let $W = L(S) = \text{span}\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subseteq F^n$.

$\Rightarrow \dim W \leq \dim$

Since S is a L

From ① and ②

$\Rightarrow W = F^n$

i.e $F^n = W = sp$

Consider the st

By ③, $\epsilon_1 = \sum$

i.e $\epsilon_1 = \sum_{j=1}^{n} B$

$\epsilon_n = \sum_{j=1}^{n} B$

$\Rightarrow \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{pmatrix} = \Big(B$

$\Rightarrow I = BA$ wh

$\Rightarrow A$ has a lef

Q) Find three v dependent such independent.

(Find $\alpha_1, \alpha_2, \alpha_3$ -

Ans: there can be

One answer: $\alpha_1 =$

$\Rightarrow \{$

$c_1\alpha_1 + c_2\alpha_2 =$

$\Rightarrow \dim W \leq \dim F^n = n$ ——— ①

Since S is a L.I set in W, $n = |S| \leq \dim W$ ——— ②

From ① and ② $n = \dim W = \dim F^n$.

$\Rightarrow W = F^n$

i.e. $F^n = W = \text{span}\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ ——— ③

Consider the standard basis $\{\epsilon_1, \epsilon_2, \dots, \epsilon_n\}$ of F^n.

By ③, $\epsilon_i = \sum_{j=1}^{n} B_{ij}\alpha_j$, $1 \leq i \leq n$
$B_{ij} \in F$

i.e $\epsilon_1 = \sum_{j=1}^{n} B_{1j}\alpha_j$

$\epsilon_n = \sum_{j=1}^{n} B_{nj}\alpha_j$

$$\Rightarrow \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{pmatrix} = \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1n} \\ \vdots & \vdots & & \vdots \\ B_{n1} & B_{n2} & \cdots & B_{nn} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$$

$\Rightarrow I = BA$ where $B = [B_{ij}]_{n \times n}$

$\Rightarrow A$ has a left inverse $\Rightarrow A$ is invertible.

Q) Find three vectors in $\mathbb{R}^3$ which is linearly dependent such that any of them is linearly independent.

(Find $\alpha_1, \alpha_2, \alpha_3 \longrightarrow$ (i) $\{\alpha_1, \alpha_2, \alpha_3\}$ is linearly dependent
(ii) $\{\alpha_1, \alpha_2\}, \{\alpha_2, \alpha_3\}, \{\alpha_1, \alpha_3\}$ are L.I)

Ans: There can be multiple answers

One answer: $\alpha_1 = (1,0,0)$, $\alpha_2 = (0,1,0)$, $\alpha_3 = \alpha_1 + \alpha_2 = (1,1,0)$

$\Rightarrow \{\alpha_1, \alpha_2, \alpha_3\}$ is L.D.

$c_1\alpha_1 + c_2\alpha_2 = 0 \Rightarrow (c_1, c_2, 0) = (0,0,0) \Rightarrow c_1 = c_2 = 0$

$\Rightarrow \{\alpha_1, \alpha_2\}$ is L.I

$\therefore c_1\alpha_1 + c_2\alpha_3 = 0 \Rightarrow (c_1+c_2, c_2, 0) = (0,0,0) \Rightarrow c_1 = c_2 = 0$

$\Rightarrow \{\alpha_1, \alpha_3\}$ is linearly independent.

HW) Q6, Q2, Q7

4 Oct '18 we did Q5.

Note: 1) Let $A = [A_{ij}]_{m\times n}$ be an $m\times n$ matrix over the field F.

Let $\alpha_i = (A_{i1}\ A_{i2} \dots A_{in})$, $1 \le i \le m$ be the i^{th} row of A.

Let $W = \text{span}\{\alpha_1, \alpha_2, \dots, \alpha_m\} \subseteq F^n$

$\Rightarrow W$ is a subspace of F^n.

$\rightarrow$ Row space of $A = W = \text{span}\{\alpha_1, \alpha_2, \dots, \alpha_m\}$

Row rank of A = dimension of row-space of A.

2) Column space of A = span of columns of A

column rank of A = dimension of column space of A.

3) Let A and B be two row equivalent $m\times n$ matrices, then there exists an invertible $m\times n$ matrix P such that $B = PA$ where $P = [P_{ij}]_{m\times n}$

Let β_i be the i^{th} row of B.

$$\beta_i = P\begin{pmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_m\end{pmatrix}$$

$\beta_i = P_{i1}\alpha_1 + P_{i2}\alpha_2 + \dots + P_{im}\alpha_m \in \text{span}\{\alpha_1, \alpha_2, \dots, \alpha_m\}$

$=$ row space of A.

$\Rightarrow$ every row vector in B is a row space of A.

$\Rightarrow$ linear combination of rows of B is also in row-

space of A.

$\Rightarrow$ row space

Since $A = P^{-1}E$

row space

From ① and (

row space

$\Rightarrow$ Two row e

space.

Theorem 10: I

echelon matrix

row-space e

Note 3) Each m

and only one row

Q) Let $A =$

(a) Find an inv

PA is a

(b) Find a basi

(d) Let $V = \{X \in$

(i) Find a b

(ii) Find all

c) Find dim

space of A.

$\Rightarrow$ row space of $B \subseteq$ row space of A. ——— ①

Since $A = P^{-1}B$

row space of $A \subseteq$ row space of B ——— ②

From ① and ②

row space of A = row space of B.

$\Rightarrow$ Two row equivalent matrices have same row space.

<u>Theorem 10</u>: Let R be an $m \times n$ row reduced echelon matrix. Then its non-zero rows forms a basis for row-space of R.

<u>Note 3</u>) Each $m \times n$ matrix is row equivalent to one and only one row reduced echelon matrix.

Q) Let
$$A = \begin{pmatrix} 1 & 2 & 0 & 3 & 0 \\ 1 & 2 & -1 & -1 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 2 & 4 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(a) Find an invertible matrix P such that PA is a row reduced echelon matrix of A.

(b) Find a basis for row space of A.

(d) Let $V = \{X \in F^5 : AX = 0\}$

(i) Find a basis for V

(ii) Find all vectors Y such that $AX = Y$

c) Find dim of row space of A.

- 5d)

$$A = \begin{pmatrix} 1 & 2 & 0 & 3 & 0 \\ 1 & 2 & -1 & -1 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 2 & 4 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{\neq} \begin{array}{c} \begin{array}{ccccc} & & & A & \end{array} \\ \begin{pmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 2 & 4 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$\rightarrow \begin{pmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & 4 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 2 & 4 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow$$

$$\begin{array}{ccccc} 2 & 4 & 0 & 6 & 0 \\ 2 & 4 & 1 & 4 & 0 \\ \hline \end{array}$$

$$\begin{pmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Art of programming

8th Oct 2018, Paper discuss

$$P_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}$$

$$P_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}$$

$P_2(x) =$

$P_3(x) =$

$C_0P_0(x) + C_1P_1(x)$

8th Oct 2018 Paper discussion Quiz-2

$$P_0(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}$$

$$P_1(x) = \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}$$

$P_2(x) =$

$P_3(x) =$

$P_0(x) = 1$

$P_1(x) = 1$

$P_2(x) = 1$

$P_3(x) = 1$

$c_0 P_0(x) + c_1 P_1(x)$

⇒ <u>Ordered basis</u>: Let V be a finite dimensional vector space over the field F.

A <u>finite sequence</u> of vectors β is an ordered basis for V if β is linearly independent and span $\beta = V$.

<u>Example</u>: 1) $\epsilon_1 = (1,0,0)$, $\epsilon_2 = (0,1,0)$, $\epsilon_3 = (0,0,1)$ is an ordered basis for $\mathbb{R}^3$.

Note that $\epsilon_2, \epsilon_2, \epsilon_3$ is an ordered basis for $\mathbb{R}^3$.

* Unique representation of a vector in V using an ordered basis $\beta = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$

Let $\alpha \in V \neq \text{span } \beta$

⇒ there exists an 'n' vector $(x_1, x_2, \ldots, x_n) \in F^n$ such that

$$\alpha = x_1\alpha_1 + x_2\alpha_2 + \ldots + x_n\alpha_n$$

If there exists $(y_1, y_2, \ldots, y_n) \in F^n$ such that

$$\alpha = y_1\alpha_1 + y_2\alpha_2 + \ldots + y_n\alpha_n \quad \text{———} (2)$$

$(1) - (2) \Rightarrow 0 = (x_1 - y_1)\alpha_1 + (x_2 - y_2)\alpha_2 + \ldots + (x_n - y_n)\alpha_n$

Since 'β' is linearly independent,

$x_1 - y_1 = 0$, $x_2 - y_2 = 0$, $x_3 - y_3 = 0$, $\ldots$, $x_n - y_n = 0$

$\Rightarrow x_1 = y_1$, $x_2 = y_2$, $x_3 = y_3$, $\ldots$, $x_n = y_n$

$\Rightarrow x = (x_1, x_2, \ldots, x_n)$

$= (y_1, y_2, \ldots, y_n) = y.$

For every vector, V, there is a unique representation.

Q) Find the representation of $(1,2,3)$ in $\mathbb{R}^3$ where $\beta = \{(1,1,0), (1,0,1), (0,1,1)\}$ is an ordered basis.

<u>Ans</u>) $\alpha_1 = (1,1,0)$, $\alpha_2 = (1,0,1)$, $\alpha_3 = (0,1,1)$

mensional vector

ordered basis
d span $\beta = V$.
1) is an ordered

for R^3.
V using an

$\in F^n$ such that

$y_n)\alpha_n$

$y_n = 0$
$= y_n$

resentation.
where
basis.

$$(1,2,3) = x_1\alpha_1 + x_2\alpha_2 + x_3\alpha_3$$
$$= x_1(1,1,0) + x_2(1,0,1) + x_3(0,1,1)$$
$$(1,2,3) = (x_1+x_2, x_1+x_3, x_2+x_3)$$

$x_1 + x_2 = 1$

$x_1 + x_3 = 2$

$x_2 + x_3 = 3$

$x_1 = 0$, $x_2 = 1$, $x_3 = \cancel{0}\, 2$

$$(1,2,3) = 0.\alpha_1 + 1.\alpha_2 + \cancel{0}\, 2.\alpha_3$$

$$[\alpha]_\beta = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}_\beta$$

Q) Express $\epsilon_1 = (1,0,0)$, $\epsilon_2 = (0,1,0)$, $\epsilon_3 = (0,0,1)$ in terms of β.

Ans. $\epsilon_1 = x_1\alpha_1 + x_2\alpha_2 + x_3\alpha_3$

$$(1,0,0) = x_1(1,1,0) + x_2(1,0,1) + x_3(0,1,1)$$
$$= (x_1+x_2, x_1+x_3, x_2+x_3)$$

$x_1 + x_2 = 1$

$x_1 + x_3 = 0$ $\qquad x_1 = \frac{1}{2}$, $x_2 = \frac{1}{2}$, $x_3 = -\frac{1}{2}$

$x_2 + x_3 = 0$

$$\epsilon_1 = \frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 - \frac{1}{2}\alpha_3$$

$$\epsilon_2 = \frac{1}{2}\alpha_1 - \frac{1}{2}\alpha_2 + \frac{1}{2}\alpha_3$$

$$\epsilon_3 = -\frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 + \frac{1}{2}\alpha_3$$

$$\begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$$

Note: Let V be a finite dimensional vector space over a field F with ordered basis $\beta = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ and $\beta' = \{\alpha_1', \alpha_2', \dots, \alpha_n'\}$

Since $V = \text{span}\, \beta$, for α_j', there exists scalars $P_{1j}, P_{2j}, \dots, P_{nj}$ such that

$$\alpha_j' = P_{1j}\alpha_1 + P_{2j}\alpha_2 + \dots + P_{nj}\alpha_j$$

i.e. $$\boxed{\alpha_j' = \sum_{i=1}^{n} P_{ij}\alpha_i \quad j = 1, 2, \dots, n}$$

Let $\alpha \in V$. Since β is an ordered basis, there exists a unique 'n' vector $(x_1, x_2, \dots, x_n) \in F^n$ such that $\alpha = x_1\alpha_1 + x_2\alpha_2 + \dots + x_n\alpha_n$

i.e. $$[\alpha]_\beta = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

since β' is an ordered basis for V, there exists a unique vector $(x_1', \dots, n_1') \in F^n$ such that $\alpha = x_1'\alpha_1' + x_2'\alpha_2' + \dots + x_n'\alpha_n' = \sum_{i=1}^{n} x_i\alpha_i$ ——— (a)

$$[\alpha]_{\beta'} = \begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix}$$

$$\alpha = x_1'\alpha_1' + x_2'\alpha_2' + \dots + x_n'\alpha_n'$$

$$= \sum_{j=1}^{n} x_j'\alpha_j'$$

$$= \sum_{j=1}^{n} x_j'\left(\sum_{i=1}^{n} P_{ij}\alpha_i\right)$$

$$= \sum_{i=1}^{n}\left(\sum_{j=1}^{n} P_{ij}x_j'\right)\alpha_i \text{ ——— (b)}$$

From (a) and (b), ...

$$x_i = \sum_{j=1}^{n} P_{ij} \dots$$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} P_{11} & P_{12} & \dots \\ P_{21} & P_{22} & \dots \\ \vdots & & \\ P_{n1} & P_{n2} & \dots \end{pmatrix}$$

$$\boxed{[\alpha]_\beta = P[\alpha]_{\beta'}}$$

1) Show that P is ...

(c) $\Rightarrow X = PX'$ ...

Claim:

$X = 0 \Leftrightarrow X' = 0$

$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ $\quad \alpha = x \dots$

$X = 0 \Rightarrow \alpha = 0$

$\Rightarrow \alpha = x_1'\alpha_1' \dots$

$\Rightarrow x_1 = x_2 = \dots$

$\Rightarrow X' = 0$

Similarly $X' = 0$ ...

2) Show that P is ...

$\cancel{X=0} \Rightarrow X' \neq 0$

$\Rightarrow \cancel{P}X' = 0$

Consider $PX' = 0$

$\Rightarrow$ ...

$\therefore$ P has only tri...

$\Rightarrow$ P is inverti...

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From (a) and (b), by uniqueness,

$$x_i = \sum_{j=1}^{n} P_{ij} x'_j \quad , i = 1, 2, \ldots, n$$

$$\underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} P_{11} & P_{12} & \cdots & P_{1n} \\ P_{21} & P_{22} & \cdots & P_{2n} \\ \vdots & & & \\ P_{n1} & P_{n2} & \cdots & P_{nn} \end{pmatrix}}_{P = (P_{ij})_{n \times n}} \underbrace{\begin{pmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_n \end{pmatrix}}_{X'} \quad \text{——— (c)}$$

$$\boxed{[\alpha]_\beta = P[\alpha]_{\beta'}}$$

1) Show that P is invertible.

(c) $\Rightarrow X = PX'$

Claim:

$X = 0 \Leftrightarrow X' = 0$

$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad \alpha = x_1\alpha_1 + x_2\alpha_2 + \ldots + x_n\alpha_n$

$X = 0 \Rightarrow \alpha = 0$

$\Rightarrow \alpha = x'_1\alpha'_1 + \cdots + x'_n\alpha'_n = 0$

$\Rightarrow x_1 = x_2 = \cdots = x_n = 0$, β' is linearly independent

$\Rightarrow X' = 0$

Similarly $X' = 0 \Rightarrow X = 0$.

2) Show that P is invertible.

$\cancel{X = 0 \Rightarrow X' \neq 0}$

$\cancel{\Rightarrow PX' = 0}$ ~~has only trivial solution.~~

Consider $PX' = 0 \Rightarrow X = 0$

$\Rightarrow X' = 0$

$\therefore$ P has only trivial solution.

$\Rightarrow$ P is invertible.

$$\Rightarrow \boxed{[\alpha]_{\beta'} = P^{-1}[\alpha]_\beta}$$

10th Oct 2018

$\Rightarrow$ Linear transformation/map.

Defn: Let V and W be two vector spaces over a field F. A function $T: V \to W$ is called a linear transformation if

$$T(c\alpha+\beta) = cT\alpha + T\beta \quad \forall\, \alpha,\beta \in V,\ c \in F$$

Eg.) (i) Let V be a vector space.

Let $I: V \to V$ where

$I\alpha = \alpha \quad \forall \alpha \in V.$

Let $\alpha, \beta \in V$, $c \in F \Rightarrow c\alpha+\beta \in V$

$I(c\alpha+\beta) = cI\alpha + I\beta = c\alpha+\beta$

$\Rightarrow I$ is a linear transformation.

Let $O: V \to V$ where

$O\alpha = 0 \quad \forall \alpha \in V$ P.T O is an LT.

(ii) Let V be a set of all polynomials over a field F.

Let $f(x) = c_0 + c_1 x + c_2 x^2 + \dots + c_n x^n \in V$

We define $D: V \to V$

$$(Df)(x) = c_1 + 2c_2 x + \dots + nc_n x^{n-1}$$

Show that D is a linear transformation.

(iii) Let A be an $m\times n$ fixed matrix over a field F.

We define $T: F^{n\times 1} \to F^{m\times 1}$ as

$$TX = AX$$

Show that T is a linear transformation.

Let $X, Y \in F^{n\times 1}$ and $c \in F$

$\Rightarrow cX+Y \in F^{n\times 1}$

$T(cX+Y) = A(cX+Y) = cAX + AY = cTX + TY$

Let $U: F^m \to F^n$

$U\alpha = \alpha A$

Show that U

(iv) Let P be an m

over a field F.

We define $T: F$

by $TA = \underset{m\times m}{P}\underset{m\times n}{A}$

Show that T is

Let $A, B \in F^{m\times n}$, c

$cA+B \in F^{m\times}$

$T(cA+B) = P(cA$

$= cPA$

$= cT$

$\Rightarrow T$ is a linea

Note: Let $T: V \to W$

Then, (i) $T(0) = 0$

(ii) $T(c_1\alpha_1 +$

$T\left(\sum_{i=1}^{n} c_i\right.$

Proof: We have

$T0 = T($

$= T0$

$\Rightarrow T0 = 0$

(ii) $T\left(\sum_{i=1}^{n} c_i\alpha_i\right)$

$=$

$=$

Let $U : F^m \to F^n$ by

$U\alpha = \alpha A$ $\quad$ (α is a row)

Show that U is a linear transformation.

(iv) Let P be an $m \times n$ matrix and Q be an $n \times n$ matrix over a field F.

We define $T : F^{m\times n} \to F^{m\times n}$

by $TA = \underset{m\times m}{P}\,\underset{m\times n}{A}\,\underset{n\times n}{Q}$

Show that T is a linear transformation.

Let $A, B \in F^{m\times n}$, $c \in F$

$cA + B \in F^{m\times n}$

$$T(cA+B) = P(cA+B)Q$$
$$= cPAQ + PBQ$$
$$= cTA + TB$$

$\Rightarrow T$ is a linear transformation.

<u>Note:</u> Let $T : V \to W$ be a linear map.

Then, (i) $T(0) = 0$

(ii) $T(c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n) = c_1T\alpha_1 + c_2T\alpha_2 + \cdots + c_nT\alpha_n$

$$T\left(\sum_{i=1}^{n} c_i\alpha_i\right) = \sum_{i=1}^{n} c_i(T\alpha_i)$$

<u>Proof:</u> We have, $0 = 0 + 0$

$$T0 = T(0+0)$$
$$= T0 + T0$$

(since T is linear
$T(0+0) = T(1.0+0)$
$= 1.T(0) + T(0)$)

$\Rightarrow T0 = 0$ (We add $\cancel{+}$ $-(T_0)$ on both sides)

(ii) $$T\left(\sum_{i=1}^{n} c_i\alpha_i\right) = T\left(c_1\alpha_1 + \sum_{i=2}^{n} c_i\alpha_i\right)$$

$$= c_1T\alpha_1 + T\left(\sum_{i=2}^{n} c_i\alpha_i\right)$$

(since T is linear
$T(c\alpha+\beta) = cT\alpha + T\beta$)

don't sleep

$$= c_1T\alpha_1 + T\left(c_2\alpha_2 + \sum_{i=3}^{n} c_i\alpha_i\right)$$

$$= \sum_{i=1}^{n-1} c_i T\alpha_i + T(c_n\alpha_n) \qquad\qquad T(c_n\alpha_n) = T(c_n\alpha_n + 0)$$

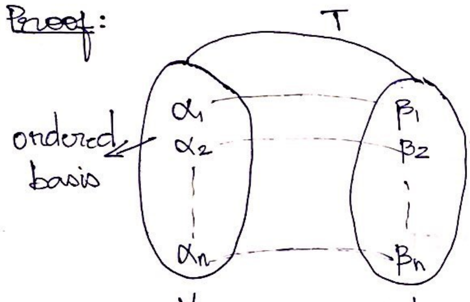
$$= \sum_{i=1}^{n-1} c_i T\alpha_i + c_n T\alpha_n \qquad\qquad = c_n T\alpha_n + T(0)$$

$$= c_n T\alpha_n + 0$$

$$= \sum_{i=1}^{n} c_i T\alpha_i \qquad\qquad = c_n T\alpha_n$$

<u>Theorem</u>: Let V be a finite dimensional vector space over the field F and $\beta = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be an ordered basis for V. Let W be a vector space over the same field and $\beta_1, \beta_2, \dots, \beta_n$ be any 'n' vectors in W. Then, there is precisely one linear transformation $T: V \to W$ such that $T\alpha_j = \beta_j$, $1 \le j \le n$

<u>Proof</u>:



Since 'β' is an ordered basis for V, for $\alpha \in V$, there exists a unique n-vector $(x_1, x_2, \dots, x_n) \in F^n$ such that

$$\alpha = x_1\alpha_1 + x_2\alpha_2 + \dots + x_n\alpha_n.$$

We define $T: V \to W$ as

$$T\alpha = T(x_1\alpha_1 + x_2\alpha_2 + \dots + x_n\alpha_n) = x_1\beta_1 + x_2\beta_2 + \dots + x_n\beta_n$$

Clearly, $T\alpha \in W$. (Since W is a vector space).

Also T is well defined.

We have, $\alpha_1 = 1.\alpha_1 + 0.\alpha_2 + \dots + 0.\alpha_n$

$$T\alpha_1 = T(1.\alpha_1 + 0.\alpha_2 + \dots + 0.\alpha_n) = 1.\beta_1 + 0.\beta_2 + \dots + 0.\beta_n = \beta_1$$

|||^{ly}. $T\alpha_2 = \beta_2, \dots, T\alpha_n = \beta_n$

$= T(c_n\alpha_n+0)$

$= c_nT\alpha_n+T(0)$

$= c_nT\alpha_n+0$

$= c_nT\alpha_n$

space over a

rdered basis

field and

there is

such that

$x_1, x_2, \ldots, x_n) \in F^n$

$+\cdots+x_n\beta_n$

$0.\beta_2+\cdots+0.\beta_n=\beta_1$

Show that T is a linear transformation.

Let $\alpha, \beta \in V$, since β is an ordered basis,

$\alpha = x_1\alpha_1+\cdots+x_n\alpha_n$, $\beta = y_1\alpha_1+\cdots+y_n\alpha_n$

$x_i, y_i \in F$

Let $c \in F$,

$c\alpha+\beta = (cx_1+y_1)\alpha_1+(cx_2+y_2)\alpha_2+\cdots+(cx_n+y_n)\alpha_n$

By defn,

$T(c\alpha+\beta) = (cx_1+y_1)\beta_1+(cx_2+y_2)\beta_2+\cdots+(cx_n+y_n)\beta_n$

$= c(x_1\beta_1+\cdots+x_n\beta_n)+(y_1\beta_1+y_2\beta_2+\cdots+y_n\beta_n)$

$= cT\alpha+T\beta.$

Show that T is unique.

Suppose there exists a linear transformation

$U: V \to W$ such that

$U\alpha_j = \beta_j \quad , j=1,2,\ldots,n$

Let $\alpha \in V$, $\alpha = x_1\alpha_1+x_2\alpha_2+\cdots+x_n\alpha_n$

$U\alpha = U\left(\sum_{i=1}^{n} x_i\alpha_i\right)$

$= \sum_{i=1}^{n} x_i(U\alpha_i)$ (since U is LT)

$= \sum_{i=1}^{n} x_i\beta_i = T\alpha$

$\Rightarrow U\alpha = T\alpha \quad \forall \alpha$

$\therefore U = T$

Therefore, T is unique.

hungry! sleepy!

12th Oct 2018

Q) Let $\beta = \{\alpha_1 = (1,2), \alpha_2 = (3,4)\}$ be an ordered basis for $\mathbb{R}^2$. Let $\beta_1 = (3,2,1)$ and $\beta_2 = (6,5,4)$ be two vectors in $\mathbb{R}^3$. Find a linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^3$ such that $T\alpha_1 = \beta_1$ and $T\alpha_2 = \beta_2$.

Soln) By last theorem, there is a unique L.T. T such that $T\alpha_1 = \beta_1$ and $T\alpha_2 = \beta_2$

$$\boxed{T(x,y) = ?}$$

Let $\alpha = (x,y) \in V = \mathbb{R}^2 = \text{span}\{\alpha_1 = (1,2), \alpha_2 = (3,4)\}$

$$\alpha = c_1\alpha_1 + c_2\alpha_2$$
$$(x,y) = c_1(1,2) + c_2(3,4)$$
$$(x,y) = (c_1 + 3c_2, 2c_1 + 4c_2)$$
$$x = c_1 + 3c_2$$
$$y = 2c_1 + 4c_2$$
$$2x = 2c_1 + 6c_2$$

$$2x - y = 2c_2$$
$$c_2 = x - \frac{y}{2}$$
$$c_1 = x - 3c_2 = x - 3x + \frac{3y}{2} = -2x + \frac{3y}{2}$$

$$\alpha = (x,y) = \left(-2x + \frac{3y}{2}\right)\alpha_1 + \left(x - \frac{y}{2}\right)\alpha_2$$

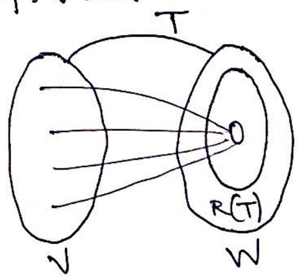
By theorem,

$$T(x,y) = \left(-2x + \frac{3y}{2}\right)\beta_1 + \left(x - \frac{y}{2}\right)\beta_2$$
$$= \left(-2x + \frac{3y}{2}\right)(3,2,1) + \left(x - \frac{y}{2}\right)(6,5,4)$$
$$= \left(\cancel{-6x} + \frac{9y}{2} + \cancel{6x} - 3y, -4x + 3y + 5x - \frac{5y}{2}, \frac{3y}{2} - 2x + 4x - 2y\right)$$
$$= \left(\frac{3y}{2}, x + \frac{y}{2}, 2x - \frac{y}{2}\right)$$

$$T(x,y) = \begin{pmatrix} 0 & 3/2 \\ 1 & 1/2 \\ 2 & -1/2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Note: We consider a linear transformation.

$T: V \to W$



Range of $T = \{w \in W : T\alpha = w \text{ for some } \alpha \in V\} = R(T)$

Null space of $T = \{\alpha \in V : T\alpha = 0\}$ ← set of all vectors mapped to zero.
$= N(T)$

Q) Show that (i) $R(T)$ is a subspace of W.
and (ii) $N(T)$ is a subspace of V.

(i) $R(T) = \{w \in W : T\alpha = w \text{ for some } \alpha \in V\}$

We have $T(0) = 0 \Rightarrow 0 \in R(T) \neq \phi$

Let $x, y \in R(T)$, $c \in F$.

$\Rightarrow$ there exist $\alpha, \beta \in V$ such that $T\alpha = x$, $T\beta = y$

$\Rightarrow c\alpha + \beta \in V$ (V is a vector space).

$\Rightarrow T(c\alpha+\beta) \in R(T) \Rightarrow cT\alpha + T\beta \in R(T)$, ($T$ is a L.T.)

$\Rightarrow cx + y \in R(T) \rightarrow R(T)$ is a subspace of W.

(ii) $N(T) = \{\alpha \in V : T\alpha = 0\}$

We have $T(0) = 0 \Rightarrow 0 \in N(T) \neq \phi$

Let $\alpha, \beta \in N(T)$, $c \in F$

$\Rightarrow T\alpha = 0$, $T\beta = 0$

$T(c\alpha + \beta) = cT\alpha + T\beta$ (T is linear transformation)

$= c \times 0 + 0 = 0$

$\Rightarrow c\alpha + \beta \in N(T)$

Therefore, $N(T)$ is a subspace of V.

⟹ Notation:

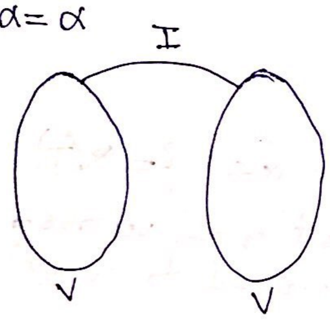
Rank of T $=\dim R(T) = \text{rank}(T)$

nullity of T $= \dim N(T) = \text{nullity}(T)$

Q) Find the null space, range, nullity and rank of (i) identity L.T. and (ii) zero L.T.

(i) $I: V \to V$

$I\alpha = \alpha$



$R(I) = V$, $N(I) = \{0\}$

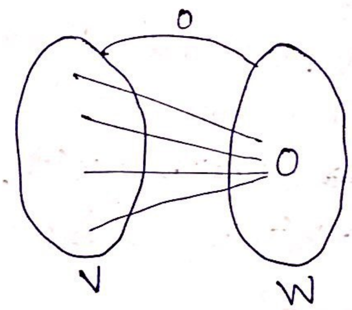
$\text{rank}(I) = \dim V$

$\text{nullity}(I) = 0$

(ii) zero L.T ($T: V \to W$)

$\mathbf{0}: V \to W$

$\mathbf{0}\alpha = 0$



$R(0) = \{0\}$

$N(\mathbf{0}) = V$

$\text{rank}(0) = 0$

$\text{nullity}(0) = 0$

Q1) Which of the following functions $T: R^2 \to R^2$ are L.Ts.

(i) $T(x,y) = (1+x, y)$

(ii) $T(x,y) = (y, x)$

(iii) $T(x,y) = (x^2, y)$

(iv) $T(x,y) = (\sin x, y)$ not a L.T.

Soln)

(i) $T(x,y)$ …

~~T(cx+p~~ …

$T(0,0)$ …

$\therefore$ T is …

(ii) $T(x,y)$ …

$T(0,0)$ …

$T(x,y) =$ …

$\therefore$ T is …

(iii) $T(x,y) =$ …

$T(cx+p$ …

5th Oct 2018.

⟹ Rank- n…

Let V and …

$T: V \to W$ …

vector spa…

~~rank (T)~~ …

Proof: Let …

Let $R(T) =$ …

and let $N(T) =$ …

We have p…

Since $N(T)$ …

$N(T)$ has …

i.e $\dim N($ …

Soln)

(i) $T(x,y) = (1+x, y)$

$\cancel{T(c\alpha+\beta) = T(}$

$T(0,0) = (1,0) \neq (0,0)$

$\therefore$ T is not a linear transformation.

(ii) $T(x,y) = (y,x)$

$T(0,0) = (0,0)$

$T(x,y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow TX = AX \Rightarrow T$ is a L.T.

$\therefore$ T is a linear transformation.

(iii) $T(x,y) = (x^2, y)$ (not a linear transformation)

$\cancel{T(c\alpha+\beta) = cT(\alpha) + T\beta}$

$\cancel{= cT\alpha^2 + T}$

15th Oct 2018.

⇒ Rank-nullity theorem:

Let V and W be two vector spaces over a field F, and let $T: V \to W$ be a linear transformation. If V is a finite dimensional vector space, then

~~rank(T) =~~ $\text{rank}(T) + \text{nullity}(T) = \dim V$.

Proof: Let $\dim V = n$.

Let $R(T) = \{\beta \in W : T\alpha = \beta \text{ for some } \alpha \in V\}$

and let $N(T) = \{\alpha \in V : T\alpha = 0\}$

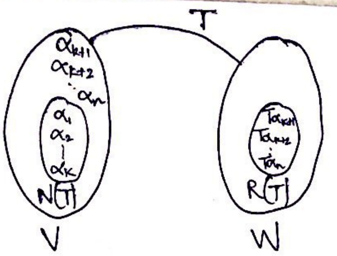
We have proved that (i) $N(T)$ is a subspace of V and

(ii) $R(T)$ is a subspace of W.

Since $N(T) \subseteq V$ and V is a finite dimensional vector space, $N(T)$ has a basis say $\{\alpha_1, \alpha_2, \dots, \alpha_k\}$

i.e $\dim N(T) = \text{nullity}(T) = k$.

Note:



rank(T) = dim R(T)

nullity (T) = dim N(T)

Since $\{\alpha_1, \alpha_2, \dots, \alpha_k\}$ is a linearly independent subset of V, it can be extended to a basis for V say $\{\alpha_1, \alpha_2, \dots \alpha_k, \alpha_{k+1}, \dots \alpha_n\}$

Show that $S = \{T\alpha_{k+1}, T\alpha_{k+2}, \dots, T\alpha_n\}$ is a basis for R(T).

Claim 1: S is a linearly independent subset of R(T).

Consider $C_{k+1}T\alpha_{k+1} + C_{k+2}T\alpha_{k+2} + \dots + C_nT\alpha_n = 0$

$\boxed{\text{Show that } C_{k+1} = C_{k+2} = \dots = C_n = C}$

Since T is linear, $T(C_{k+1}\alpha_{k+1} + C_{k+2}\alpha_{k+2} + \dots + C_n\alpha_n) = 0$

$\Rightarrow C_{k+1}\alpha_{k+1} + \dots + C_n\alpha_n \in N(T) = \text{span}\{\alpha_1, \alpha_2, \dots, \alpha_k\}$

$\Rightarrow C_{k+1}\alpha_{k+1} + \dots + C_n\alpha_n = d_1\alpha_1 + d_2\alpha_2 + \dots + d_k\alpha_k$, $d_i \in F$.

$\Rightarrow d_1\alpha_1 + d_2\alpha_2 + \dots + d_k\alpha_2 + (-C_{k+1})\alpha_{k+1} + (C_{k+2})\alpha_{k+2} + \dots + (C_n)\alpha_n = 0$

Since $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ is a basis for V, $d_1 = d_2 = \dots = d_k = C_{k+1} = \dots C_n = 0$

$\therefore$ S is a linearly independent set of R(T).

Claim 2: R(T) = span S.

We have, $S = \{T\alpha_{k+1}, \dots, T\alpha_n\} \subseteq R(T)$.

and R(T) is a subspace, span $S \subseteq R(T)$ ——— (a)

Let $\underline{\beta \in R(T)}$, by definition, there exists $\alpha \in V$ such that $T\alpha = \beta$.

since $\alpha \in V = \text{span}\{\alpha_1, \alpha_2, \dots, \alpha_k, \alpha_{k+1}, \dots, \alpha_n\}$

$\alpha = x_1\alpha_1 + x_2\alpha_2 + \dots + x_k\alpha_k + x_{k+1}\alpha_{k+1} + \dots + x_n\alpha_n$

$x_i \in F$.

$$\beta = T\alpha = T(x_1\alpha_1 + \dots + x_k\alpha_k + x_{k+1}\alpha_{k+1} + \dots + x_n\alpha_n)$$
$$= x_1 T\alpha_1 + x_2 T\alpha_2 + \dots + x_k T\alpha_k + x_{k+1}T\alpha_{k+1} + \dots + x_n T\alpha_n$$
(T is linear).

$$\beta = x_1 \times 0 + \dots + x_k \times 0 + x_{k+1} T\alpha_{k+1} + \dots + x_n T\alpha_n$$

$$\beta = x_{k+1} T\alpha_{k+1} + \dots + x_n T\alpha_n \in \text{span}\{T\alpha_{k+1}, \dots, T\alpha_n\} = \text{span } S.$$

ie. $\beta = R(T) \Rightarrow \beta \in \text{span } S$

$\Rightarrow R(T) \subseteq \text{span } S$ —— (b)

From (a) and (b), $R(T) = \text{span } S$.

By Claim 1 and Claim 2, S is a basis for $R(T)$.

$S = \{T\alpha_{k+1}, \dots T\alpha_n\}$ is a basis for $R(T)$.

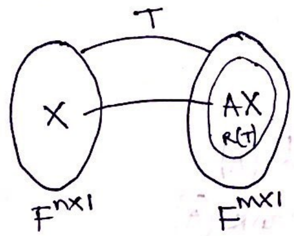
$$\text{rank}(T) = \dim R(T) = n - k$$
$$= \dim V - \text{nullity}(T)$$

$$\Rightarrow \boxed{\text{rank}(T) + \text{nullity}(T) = \dim V}$$

The domain is finite.

<u>Theorem</u>: If A is an $m \times n$ matrix over a field F, then row rank (A) = column rank (A).

<u>Proof</u>: Let $T: F^{n\times 1} \to F^{m\times 1}$ be a linear transformation defined as $TX = AX$.



$$N(T) = \{X \in F^{n\times 1} : TX = 0\}$$
$$= \{X \in F^{n\times 1} : AX = 0\}$$
$= S$, the solution space of $AX = 0$

$$\dim S = \dim N(T) = \text{nullity}(T).$$

$R(T) = \{ Y \in F^{m\times 1} : Y = AX \text{ for some } X \in F^{n\times 1} \}$

$= \{ Y \in F^{m\times 1} : Y = x_1C_1 + x_2C_2 + \dots + x_nC_n$, where $C_1, C_2, \dots C_n$ are columns of $A \}$

$= \{ Y = x_1C_1 + \dots + x_nC_n : x_i \in F \}$

$= \text{span}\{C_1, C_2, \dots, C_n\} =$ column space of A.

$\text{Rank}(T) = \dim R(T) = \dim$ column space of A

$=$ column rank(A). ——— (a)

Since T is a linear transformation and $V = F^{n\times 1}$ is a finite dimensional vector space, by rank nullity theorem,

$\text{rank}(T) + \text{nullity}(T) = \dim V$

$= \dim F^{n\times 1} = n.$

(a) $\Rightarrow$ column rank(A) + nullity$(T) = n$

column rank$(A) = n -$ nullity(T). ——— ①

Let 'r' be the row rank of A.

Then $\dim S = n - r$.

$\Rightarrow r = n - \dim S$.

~~= n~~

row rank $(A) = n -$ nullity(T) ——— ②

From ①, ②,

column rank $(A) =$ row rank (A)

Def: Rank of a matrix $(A) =$ row rank(A)

$=$ column rank (A).

17th Oct'18

Q7) Let $T: F^3 \to F^3$ ...

in text $T(x_1, x_2$ ...

(i) Show that ...

(ii) Find the ra...

$R(T) = \{(a,$ ...

(iii) Find $N(T)$ an...

Solu: i) $T(x_1, x_2,$ ...

T...

This is a lin...

(ii) $R(T) = \{ Y :$ ...

$= \{(a, b$ ...

$= \{ Y = ($ ...

$= \{(a, b$ ...

Consider 1 ... 2 ...

$\begin{bmatrix} 1 & -1 & \dots \\ 2 & 1 & \dots \\ -1 & -2 & \dots \end{bmatrix}$

$R_2 \leftarrow \frac{R_2}{3}, R_3 \leftarrow \frac{R_3}{3}$ ...

The system ha...

$b + c - a = 0$

17th Oct'18

Q1) Let $T: F^3 \to F^3$ be a function defined as

in text $T(x_1,x_2,x_3) = (x_1 - x_2 + 2x_3,\ 2x_1 + x_2,\ -x_1 - 2x_2 + 2x_3)$

(i) Show that T is a linear transformation.

(ii) Find the range of $(a,b,c) \in F^3$ such that $R(T) = \{(a,b,c) : a,b,c \in F\}$

(iii) Find $N(T)$ and nullity (T).

Soln: i) $T(x_1,x_2,x_3) = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ -1 & -2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$TX = AX$. (prove)

This is a linear transformation.

(ii) $R(T) = \{ Y : TX = Y \}$

$\cancel{= \{(a,b,c) :}$

$= \{ Y = (a,b,c) : TX = Y \}$

$= \{ (a,b,c) \in F^3 : AX = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \text{ for some } X \in F^3 \}$

Consider
$$\begin{matrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ -1 & -2 & 2 \end{matrix}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & a \\ 2 & 1 & 0 & b \\ -1 & -2 & 2 & c \end{array}\right] \xrightarrow[R_3 \leftarrow R_3 + R_1]{R_2 \leftarrow R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & -1 & 2 & a \\ 0 & 3 & -4 & b-2a \\ 0 & -3 & 4 & a+c \end{array}\right)$$

$$\xrightarrow{R_2 \leftarrow \frac{R_2}{3},\ R_3 \leftarrow \frac{R_3}{3}} \left(\begin{array}{ccc|c} 1 & -1 & 2 & a \\ 0 & 1 & -4/3 & \frac{b-2a}{3} \\ 0 & -1 & 4/3 & \frac{a+c}{3} \end{array}\right) \xrightarrow[R_3 \leftarrow R_3 + R_2]{R_1 \leftarrow R_1 + R_2} \left(\begin{array}{ccc|c} 1 & 0 & 2/3 & \frac{a+b}{3} \\ 0 & 1 & -4/3 & \frac{b-2a}{3} \\ 0 & 0 & 0 & \frac{b+c-a}{3} \end{array}\right)$$

The system has solution for all (a,b,c) such that $b+c-a=0 \quad a=b+c$.

Scanned with CamScanner

$R(T) = \{(a,b,c) : a = b+c,\ a,b,c \in F\}$

$\text{Rank}(T) = \cancel{\text{Rank}(A)}$

$= \dim R(T) = \text{column rank}(A)$

$= \text{rank}(A).$

Basis for row space of A

$= \{(1,0,2/3), (0,1,-4/3)\}$

$\Rightarrow \text{rank}(A) = 2$

$N(T) = \{X \in F^3 : AX = 0\}$

$$A \sim \begin{pmatrix} 1 & 0 & 2/3 \\ 0 & 1 & -4/3 \\ 0 & 0 & 0 \end{pmatrix}$$

$AX = 0 \rightarrow \quad x_1 + \frac{2}{3}x_3 = 0$

$\qquad\qquad\quad x_2 - \frac{4}{3}x_3 = 0$

The free variable set $= \{x_3\}$

put $x_3 = 1$, $x_1 = \frac{-2}{3}$, $x_2 = \frac{4}{3}$

A basis for $S(=N(T)) = \{(\frac{-2}{3}, \frac{4}{3}, 1)\}$

$N(T) = \{c(\frac{-2}{3}, \frac{4}{3}, 1) : c \in F\}$

nullity $(T) = \dim N(T) = 1$

Q8) Describe explicitly a linear transformation for $R^3 \to R^3$ which has its range as the space spanned by $\{(1,0,-1), (1,2,2)\}$.

Sol) Find $T : R^3 \to R^3$ such that $R(T) = \text{span}\{(1,0,-1)(1,2,2)\}$

Let $\{\epsilon_1 = (1,0,0), \epsilon_2 = (0,1,0), \epsilon_3 = (0,0,1)\}$ be an ordered basis for R^3.

Let $\beta_1 = (1,0,-1)$, $\beta_2 = (1,2,2)$, $\beta_3 = (1,2,2)$

By theorem 1, there is a unique linear transformation such that $T\epsilon_1 = \beta_1$

$$T\epsilon_2 = \beta_2$$

$$T\epsilon_3 = \beta_3.$$

$$(x,y,z) = x\epsilon_1 + y\epsilon_2 + z\epsilon_3.$$

$$T(x,y,z) = \cancel{\$}\,xT\epsilon_1 + yT\epsilon_2 + zT\epsilon_3 \quad (\because T \text{ is linear})$$

$$= x\beta_1 + y\beta_2 + z\beta_3$$

$$= x(1,0,-1) + y(1,2,2) + z(1,2,2)$$

$$T(x,y,z) = (x+y+z,\ 2y+2z,\ -x+2y+2z)$$

$$= \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ -1 & 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$\Rightarrow T$ is a linear transformation.

$R(T)$ = column space of A

$$= \text{span}\{(1,0,-1),(1,2,2),(1,2,2)\}$$

$$= \text{span}\{(1,0,-1),(1,2,2)\}$$

Q11) Let V be the space of all $n \times 1$ matrices over F and let W be the subspace of $m \times 1$ matrices over F. Let A be a fixed $m \times n$ matrix over F. Let $T: V \to W$ defined as $TX = AX$. Prove that T is a zero transformation. $\Leftrightarrow$ A is the zero matrix.

Soln)

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Suppose T is a zero transformation.

i.e. $TX = 0 \quad \forall\ X \in F^{n \times 1}$

i.e. $AX = 0 \quad \forall\ X \in F^{n \times 1}$

$\Rightarrow A\epsilon_1 = 0,\ A\epsilon_2 = 0, \ldots, A\epsilon_n = 0$

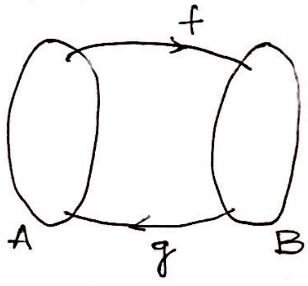
$$A\epsilon_1 = 0 \Rightarrow \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \Rightarrow a_{11}=0,\ a_{21}=0, \ldots, a_{m1}=0$$

$$A\epsilon_2 = 0 \Rightarrow \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \\ \vdots \\ a_{m2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Therefore all the elements of matrix A are equal to zero
Hence, A is a zero matrix.

24th Oct'18

Def: A function $f: A \to B$ is invertible if there exists a function $g: B \to A$ such that $g\circ f: A \to A$ and $f\circ g: B \to B$ are identity functions.



Def: A linear transformation $T: V \to W$ is invertible if and only if
(i) T is one-one. That is $T\alpha = T\beta \Rightarrow \alpha = \beta$
(ii) T is onto. That is $R(T) = W$.

Note: Let $T: V \to W$ be a linear map (transformation). Suppose that T is one-one. We have $T(0) = 0$

Let $\alpha \in N(T) \Rightarrow T\alpha = 0 \Rightarrow T\alpha = T0 \Rightarrow \alpha = 0$ (T is one-one)
$\Rightarrow \alpha \in \{0\}$

That is $\alpha \in N(T) \Rightarrow \alpha \in \{0\}$
$\Rightarrow$ ~~$N(T) \in \{0\}$~~ $N(T) \subseteq \{0\}$
since $\{0\} \subseteq N(T)$, $N(T) = \{0\}$

So, T is one-one $\Rightarrow N(T)=\{0\}$ ——— (i)

Suppose $N(T)=\{0\}$.

Consider $T\alpha=T\beta$ $\Rightarrow T(\alpha-\beta)=0$ (T is linear)

$$\Rightarrow \alpha-\beta \in N(T)=\{0\}$$
$$\Rightarrow \alpha-\beta=0$$
$$\Rightarrow \alpha=\beta \Rightarrow T \text{ is one-one.}$$

Hence, $N(T)=0 \Rightarrow T$ is one-one ——— (ii)

From (i) and (ii),

T is one-one iff $N(T)=\{0\}$

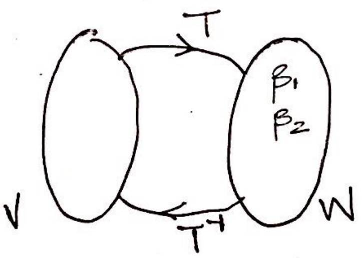
T is one-one $\Leftrightarrow N(T)=\{0\}$.

<u>Def:</u> A linear map $T: V \to W$ is non-singular if $T\alpha=0 \Rightarrow \alpha=0$

i.e. T is non-singular $\Leftrightarrow N(T)=\{0\} \Leftrightarrow T$ is one-one.

<u>Theorem 7:</u> Let V and W be two vector spaces over the field F and $T: V\to W$ is a linear transformation. If T is invertible, then T^{-1} is a linear transformation.

<u>Proof:</u>



Prove that for all $\beta_1, \beta_2 \in W$, and $c\in F$

$$T^{-1}(c\beta_1+\beta_2) = cT^{-1}\beta_1 + T^{-1}\beta_2$$

Since T is invertible, for β_1, β_2, there exists unique vectors $\alpha_1, \alpha_2 \in V$ such that

$T\alpha_1=\beta_1$, $T\alpha_2=\beta_2$

Since V is a vector space over F, $c\alpha_1+\alpha_2 \in V$

Consider $T(c\alpha_1+\alpha_2) = cT\alpha_1 + T\alpha_2$ ($\because T$ is a linear map)

$$T(c\alpha_1+\alpha_2) = c\beta_1+\beta_2$$
$$\Rightarrow c\alpha_1+\alpha_2 = T^{-1}(c\beta_1+\beta_2)$$

$\Rightarrow cT^{-1}\beta_1 + T^{-1}\beta_2 = T^{-1}(c\beta_1+\beta_2)$

Q) Test the non-singularity of the following linear maps.

(i) $T: R^2 \to R^2$ by

$T(x_1, x_2) = (x_2, x_1)$

(ii) $D: P_3 \to P_3$ by ($P_3 \leftarrow$ space of all polynomials of degree atmost 3)

$D(c_0 + c_1x + c_2x^2 + c_3x^3)$

$= c_1 + 2c_2x + 3c_3x^2$

Soln:

(i) $T: R^2 \to R^2$

$T(x_1, x_2) = (x_2, x_1)$

$N(T) = \{(x_1, x_2) : T(x_1, x_2) = (0,0)\}$

$= \{(x_1, x_2) : (x_2, x_1) = (0,0)\}$

$= \{(x_1, x_2) : x_1 = x_2 = 0\}$

$= \{(0,0)\}$

$N(T) = \{0\} \Leftrightarrow T$ is non-singular.

$\therefore T$ is one-one, non-singular.

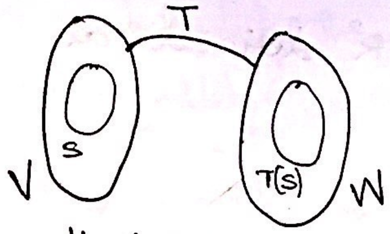
(ii) $N(D) = \{g(x) = c_0 + c_1x + c_2x^2 + c_3x^3 : D(g(x)) = 0\}$

$= \{\text{all constants}\} \neq \{0\}$

$\Rightarrow D$ is singular.

Theorem 8: Let $T: V \to W$ be a linear map, then T is non-singular $\Leftrightarrow T$ carries every independent subset of V onto ~~ax~~ ~~independent~~ linearly independent subset of W.

Proof:



Suppose that T is non-singular.

Hence $N(T) = \{0\}$

Let S be a linearly independent subset of V.
Show that $T(S)$ is a linearly independent subset of W. It is enough to prove that ~~distinct vectors~~ for distinct vectors $\alpha_1, \alpha_2, \dots, \alpha_n \in S$, $\{T\alpha_1, T\alpha_2, \dots, T\alpha_n\}$ is a linearly independent subset of W.

Consider $c_1 T\alpha_1 + c_2 T\alpha_2 + \dots + c_n T\alpha_n = 0$

(Show that $c_1, c_2, \dots, c_n = 0$)

Since T is linear, $T(c_1\alpha_1 + c_2\alpha_2 + \dots + c_n\alpha_n) = 0$

$\Rightarrow c_1\alpha_1 + c_2\alpha_2 + \dots + c_n\alpha_n \in N(T) = \{0\}$.

$\Rightarrow c_1\alpha_1 + c_2\alpha_2 + \dots + c_n\alpha_n = 0$

Since $\alpha_1, \alpha_2, \dots, \alpha_n \in S$ and S is linearly independent,

$c_1 = c_2 = \dots = c_n = 0$.

Suppose that T carries every linearly independent subset of V onto a linearly independent subset of W.

We assume that $N(T) \neq 0$. Let $\alpha \neq 0$ in $N(T)$

$\Rightarrow T\alpha = 0$ and $\alpha \neq 0$.

$\Rightarrow \{\alpha\}$ is linearly independent and $\{T\alpha\} = \{0\}$ is linearly dependent,

a contradiction $\Rightarrow N(T) = \{0\} \Rightarrow T$ is non-singular.

Q1) ~~Let T:~~ A linear map $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined as

$$T(x_1, x_2) = (x_1 + x_2, x_1)$$

Test whether T is invertible.

Find T^{-1} if exists

Q2) Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map defined as

$$T(x_1, x_2, x_3) = (3x_1, x_1 - x_2, 2x_1 + x_2 + x_3)$$

Find T^{-1} if exists.

* We will have to verify if it is one-one and onto. and check N(T).

(i) $N(T) = \{(x_1, x_2) : T(x_1, x_2) = (0,0)\}$

$= \{(x_1, x_2) : (x_1 + x_2, x_1) = (0,0)\}$

$= \{(0,0)\}$

$\therefore$ T is one-one.

Let $(z_1, z_2) \in \mathbb{R}^2$, $T(x_1, x_2) = (z_1, z_2)$

$\Rightarrow (x_1 + x_2, x_1) = (z_1, z_2)$

$x_1 = z_2$

$x_2 = z_1 - z_2$

i.e. $T(z_2, z_1 - z_2) = (z_1, z_2)$

$$\boxed{T^{-1}(z_1, z_2) = (z_2, z_1 - z_2)}$$

26th Oct 2018 * $\boxed{\text{Prove the st.: T is non-singular} \Leftrightarrow N(T) = \{0\} \Leftrightarrow \text{T is one-one}}$

<u>Theorem 9</u>: Let V and W be two finite dimensional vector spaces over the field F such that dim V = dim W. If $T: V \to W$ is a linear transformation, then the following are equivalent.

(i) T is invertible

(ii) T is non-singular

(iii) T is onto.

Proof: By rank-nullity theorem,

$$\text{rank}(T) + \text{nullity}(T) = \dim V \boxed{= \dim W} \quad (A)$$

(i) $\Rightarrow$ (ii)

Suppose T is invertible $\Rightarrow T$ is one-one $\Rightarrow T$ is non-singular

(ii) $\Rightarrow$ (iii)

Suppose T is non-singular $\Rightarrow N(T) = \{0\} \Rightarrow \text{nullity}(T) = 0$

By (A), $\text{rank}(T) = \dim W$

$\Rightarrow \dim R(T) = \dim W$ (we have $R(T) \subseteq W$)

$\Rightarrow R(T) = W \Rightarrow T$ is onto.

(iii) $\Rightarrow$ (i)

Suppose that T is onto. $\Rightarrow R(T) = W$

$\Rightarrow \text{rank}(T) = \dim R(T) = \dim W = \dim V.$

By (A). $\text{nullity}(T) = 0$

$\Rightarrow N(T) = \{0\} \Rightarrow T$ is one-one.

T is one-one and onto $\Rightarrow T$ is invertible.

Q) Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be a function defined as **5M**

$$T(x_1, x_2, x_3) = (3x_1, x_1 - x_2, 2x_1 + x_2 + x_3)$$

~~$\Rightarrow T$ is invertible~~ Is T invertible? If exists, find T^{-1}

Sol.) $V = W = \mathbb{R}^3$ $\dim V = \dim W = 3.$

$$N(T) = \{\underbrace{(x_1, x_2, x_3)}_{\alpha} : \underbrace{T(x_1, x_2, x_3)}_{T\alpha} = (0,0,0)\}$$

$$= \{(x_1, x_2, x_3) : (3x_1, x_1 - x_2, 2x_1 + x_2 + x_3) = (0,0,0)\}$$

$$= \{(x_1, x_2, x_3) : x_1 = x_2 = x_3 = 0\} = \{(0,0,0)\}$$

$\Rightarrow T$ is non-singular.

~~By theorem 9~~ So, $\dim V = \dim W = 3$ and T is non-singular

By thm 9, T is invertible.

Suppose $T(x_1, x_2, x_3) = (z_1, z_2, z_3)$

$$\Rightarrow (3x_1, x_1 - x_2, 2x_1 + x_2 + x_3) = (z_1, z_2, z_3).$$

$x_1 = \frac{z_1}{3}$, $x_2 = \frac{z_1}{3} - z_2$, $x_3 = \frac{-2z_1}{3} - \frac{z_1}{3} + z_2 + z_3$

$= -z_1 + z_2 + z_3$

Since $T(x_1, x_2, x_3) = (z_1, z_2, z_3)$

$\Rightarrow T\left(\frac{z_1}{3}, \frac{z_1}{3} - z_2, -z_1 + z_2 + z_3\right) = (z_1, z_2, z_3)$

$T^{-1}(z_1, z_2, z_3) = \left(\frac{z_1}{3}, \frac{z_1}{3} - z_2, -z_1 + z_2 + z_3\right)$

Q) Let T and U be linear operators on R^2 defined as $T(x_1, x_2) = (x_2, x_1)$ and

$U(x_1, x_2) = (x_1, 0)$

Find (i) $T+U$ (iv) T^2

(ii) TU (v) U^2

(iii) UT

Sol) (i) $T+U$

$(T+U)(x_1, x_2) = T(x_1, x_2) + U(x_1, x_2)$

$= (x_2, x_1) + (x_1, 0) = (x_1 + x_2, x_1)$

(ii) TU

$(TU)(x_1, x_2) = T(U(x_1, x_2)) = T(x_1, 0)$

$= (0, x_1)$

(iii) UT

$(UT)(x_1, x_2) = U(T(x_1, x_2)) = U(x_2, x_1) = (x_2, 0)$

(iv) T^2

$T^2(x_1, x_2) = T(T(x_1, x_2)) = T(x_2, x_1) = (x_1, x_2)$

$T^2 = I$

(v) U^2

$U^2(x_1, x_2) = U(U(x_1, x_2)) = U(x_1, 0) = \cancel{(x_1 + x} \; (x_1, 0)$

$U^2 = U.$

z_3

Q1) Find two linear operators T and U such that $TU = I$ and $UT \neq I$

Q2) Find two linear operators T and U ~~such that~~ on R^2 such that $TU = 0$ and $UT \neq 0$

Sol1) We define $E: P_n \to P_n$, and $D: P_n \to P_n$

$$E(c_0 + c_1x + c_2x^2 + \dots + c_nx^n) = c_0x + c_1\frac{x^2}{2} + \dots + \frac{c_n}{n+1}x^{n+1}$$

$$D(c_0 + c_1x + \dots + c_nx^n) = c_1 + 2c_2x + \dots + nc_nx^{n-1}$$

$$(DE)(c_0 + c_1x + \dots + c_nx^n) = D(E(c_0 + c_1x + \dots + c_nx^n))$$

$$= D\left(c_0x + c_1\frac{x^2}{2} + \dots + \frac{c_n}{n+1}x^{n+1}\right)$$

$$= c_0 + c_1x + c_2x^2 + \dots + c_nx^n$$

$\Rightarrow DE = I$

$$(ED)(c_0 + c_1x + \dots + c_nx^n) = E(D(c_0 + c_1x + \dots + c_nx^n))$$

$$= E(c_1 + 2c_2x + \dots + nc_nx^{n-1})$$

$$= c_1x + 2c_2\frac{x^2}{2} + \dots + \frac{nc_nx^n}{n}$$

$$= c_1 + c_2x^2 + \dots + c_nx^n$$

$\Rightarrow ED \neq I$

defined

Integration and differentiation are linear operators but they ~~Sol2)~~ are not invertible, they are only one way invertible.

Sol2)

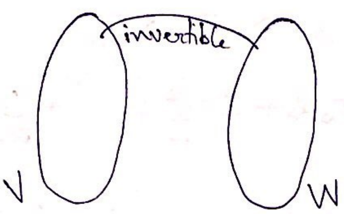
$$\left.\begin{aligned} T(x_1, x_2) &= (0,0) \\ U(x_1, x_2) &= (x_1 + 1, x_2) \end{aligned}\right\} \text{not the soln as} \to \text{it is not a linear operator}$$

X

29th Oct'18

Isomorphism:

Let V and W be two vector spaces over a field F. We say V and W are isomorphic if there exists an invertible linear transformation $T: V \to W$



Theorem: Every n-dimensional vector space over a field F is isomorphic to F^n.

Proof: Let V be a n-dimensional vector space over the field.

Let $\beta = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be an ordered basis for V. By theorem, for every $\alpha \in V$, there exists unique scalars $x_1, x_2, \dots, x_n \in F$ such that

$$\alpha = x_1\alpha_1 + x_2\alpha_2 + \dots + x_n\alpha_n$$

We define $T: V \to F^n$ as

$$T(x_1\alpha_1, x_2\alpha_2, \dots, x_n\alpha_n) = (x_1, x_2, \dots, x_n)$$

Claim 1: T is a linear map.

Let $\alpha = x_1\alpha_1 + x_2\alpha_2 + \dots + x_n\alpha_n$

$\beta = y_1\alpha_1 + y_2\alpha_2 + \dots + y_n\alpha_n \in V$

$T\alpha = (x_1, x_2, \dots, x_n)$, $T\beta = (y_1, y_2, \dots, y_n)$

$\cancel{c\alpha + \beta = (cx_1 + y_1, cx_2 + y_2, \dots, cx_n + y_n)}$

$c\alpha + \beta = (cx_1 + y_1)\alpha_1 + (cx_2 + y_2)\alpha_2 + \dots + (cx_n + y_n)\alpha_n$

By definition,

$$T(c\alpha + \beta) = (cx_1 + y_1, cx_2 + y_2, \dots, cx_n + y_n)$$

$$= c(x_1, x_2, \dots, x_n) + (y_1, y_2, \dots, y_n) = cT\alpha + \beta.$$

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Claim 2: T is one-one.

$$N(T) = \{\alpha = x_1\alpha_1 + x_2\alpha_2 + \cdots + x_n\alpha_n : T\alpha = 0\}$$
$$= \{\alpha = x_1\alpha_1 + \cdots + x_n\alpha_n : (x_1, x_2, \ldots, x_n) = (0, 0, \ldots, 0)\}$$
$$= \{\alpha = x_1\alpha_1 + \cdots + x_n\alpha_n : x_1 = x_2 = \cdots = x_n = 0\}$$
$$= \{0\}$$

$\Rightarrow$ T is one-one.

$\dim V = n = \dim F^n$

By theorem 9, T is an invertible linear transformation

$\Rightarrow$ V is isomorphic to F^n.

Q) Let V and W be two finite dimensional vector spaces over F, Then V is isomorphic to W if and only if $\dim V = \dim W$.

Proof: $\Rightarrow$ suppose that V is isomorphic to W. There exists a linear map $T: V \to W$ which is invertible.

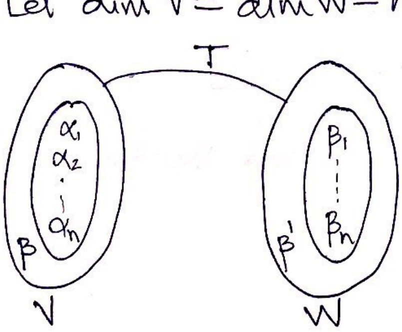
$\Rightarrow$ T is one-one and onto.

$\Rightarrow N(T) = \{0\}$ and $R(T) = W$

By rank-nullity theorem,

$$\text{rank}(T) + \text{nullity}(T) = \dim V.$$
$$\dim R(T) + \dim N(T) = \dim V.$$
$$\dim W + 0 = \dim V.$$

Proof: $\Leftarrow$: Let $\dim V = \dim W = n$. Show that V is isomorphic to W.



Let $\beta = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be an ordered basis for V and $\beta' = \{\beta_1, \beta_2, \ldots, \beta_n\}$ be an ordered basis for W.

Then there exists a unique linear transformation such that

$T\alpha_j = \beta_j \quad j = 1, 2, \ldots, n.$

Also $\alpha = x_1\alpha_1 + \ldots + x_n\alpha_n \in V$

then, $T\alpha = x_1\beta_1 + x_2\beta_2 + \ldots + x_n\beta_n$

Show that T is invertible.

Claim: T is one-one.

$$N(T) = \{\alpha = x_1\alpha_1 + \ldots + x_n\alpha_n : T\alpha = 0\}$$

$$= \{\alpha = x_1\alpha_1 + \ldots + x_n\alpha_n : x_1\beta_1 + \ldots + x_n\beta_n = 0\}$$

$$= \{\alpha = x_1\alpha_1 + \ldots + x_n\alpha_n : x_1 = x_2 = \ldots = x_n = 0\}$$

(β's linearly independent)

$= \{0\} \Rightarrow T$ is one-one.

Given $\dim V = \dim W$.

$\Rightarrow T$ is invertible (theorem 9)

$\Rightarrow V$ is isomorphic to W.

Note: Let V and W be two vector spaces over a field F.

We define,

$$L(V, W) = \{T : T : V \to W \text{ is a linear map}\}$$

Also, $(T_1 + T_2)\alpha = T_1\alpha + T_2\alpha$

$(cT)\alpha = c(T\alpha) \quad \forall c \in F, \ \alpha \in V$

~~every member is a linear tr~~

HW Q) (i) Show that $L(V, W)$ is a vector space

(ii) If V and W are finite dimensional vector spaces, then $\dim L(V, W) = \dim V \cdot \dim W$.

$n \qquad m \qquad T = A_{m \times n}$

Q) Let $T: F^2 \to F^2$ ...

$\beta = \{\epsilon_1 = (1,0), \epsilon_2$ ...

Sol) $T\epsilon_1 = T(1,0) =$ ...

$T(\epsilon_1) = 1.\epsilon_1 + 0\epsilon$ ...

$T(\epsilon_2) = T(0,1) = ($ ...

Q) Let $D: P_3 \to P_3$

$D(c_0 + c_1x + c_2x^2$ ...

Let $\beta = \{1, x, x$ ...

Find $[D]_\beta$

$[T\epsilon_1]_\beta = $ ...

$[T\epsilon_2]_\beta = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ ...

$[T]_\beta = ([T\epsilon_1]_\beta,$ ...

$D(1) = 0 = \cancel{0\epsilon_1}$ ...

$D(x) = 1 = 1 \times 1 +$ ...

$D(x^2) = 2x = 0 \times 1$ ...

$D(x^3) = 3x^2 = 0 \times$ ...

$[D]_\beta = \begin{pmatrix} 0 & 1 & \ldots \\ 0 & 0 & \ldots \\ 0 & 0 & \ldots \\ 0 & 0 & \ldots \end{pmatrix}$

Q) Let $T: F^2 \to F^2$ by $T(x_1, x_2) = (x_1, 0)$ and $\beta = \{\epsilon_1 = (1,0), \epsilon_2 = (0,1)\}$. Find $[T]_\beta$

Sol) $T\epsilon_1 = T(1,0) = \cancel{(1,0)}\ (1,0) = 1(1,0) + 0(0,1)$

$T(\epsilon_1) = 1.\epsilon_1 + 0\epsilon_2$

$T(\epsilon_2) = T(0,1) = (0,0) = 0.\epsilon_1 + 0.\epsilon_2$

$[T\epsilon_1]_\beta = \cancel{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$[T\epsilon_2]_\beta = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$[T]_\beta = \left([T\epsilon_1]_\beta, [T\epsilon_2]_\beta\right) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

Q) Let $D: P_3 \to P_3$ be a linear transform defined on $D(c_0 + c_1 x + c_2 x^2 + c_3 x^3) = c_1 + 2c_2 x + 3c_3 x^2$

Let $\beta = \{1, x, x^2, x^3\}$ be an ordered basis for P_3. ~~Find~~ Find $[D]_\beta$

$D(1) = 0 = \cancel{0x}\ 0\times 1 + 0\times x + 0\times x^2 + 0\times x^3$

$D(x) = 1 = 1\times 1 + 0\times x + 0\times x^2 + 0\times x^3$

$D(x^2) = 2x = 0\times 1 + 2\times x + 0\times x^2 + 0\times x^3$

$D(x^3) = 3x^2 = 0\times 1 + 0\times x + 3\times x^2 + 0\times x^3$

$$[D]_\beta = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \left([D(1)]_\beta, [D(x)]_\beta, [D(x^2)]_\beta, [D(x^3)]_\beta\right)$$

1st Nov'18

Q) Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map defined as ~~$T(x_1, x_2) =$~~ $T(x_1, x_2) = (x_1, 0)$. Let $\beta = \{\epsilon_1 = (1,0), \epsilon_2 = (0,1)\}$ and $\beta' = \{\epsilon_1' = (1,1), \epsilon_2' = (2,1)\}$ be two ordered basis for $\mathbb{R}^2$.

Find (i) $[T]_\beta$, (ii) $[T]_{\beta'}$,

(iii) Establish a connection between $[T]_\beta$ and $[T]_{\beta'}$

Ans.) $T\epsilon_1 = T(1,0) = (1,0) = 1.\epsilon_1 + 0.\epsilon_2 \Rightarrow [T\epsilon_1]_\beta = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$T\epsilon_2 = T(0,1) = (0,0) = 0\epsilon_1 + 0\epsilon_2 \Rightarrow [T\epsilon_2]_\beta = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$[T]_\beta = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$T\epsilon_1' = T(1,1) = (1,0) = c_1\epsilon_1' + c_2\epsilon_2' = c_1(1,1) + c_2(2,1)$

$T\epsilon_1' = (-1)\epsilon_1' + 1.\epsilon_2' \Rightarrow [T\epsilon_1']_{\beta'} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$T\epsilon_2' = T(2,1) = (2,0) = -2\epsilon_1' + 2\epsilon_2'$

$$[T\epsilon_2]_{\beta'} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$$

$c_1 + 2c_2 = 2$

$c_1 + c_2 = 0$

$$[T]_{\beta'} = \begin{pmatrix} -1 & -2 \\ 1 & 2 \end{pmatrix}$$

$c_2 = 2$

$c_1 = -2$

~~$\epsilon_1' = c_1\epsilon_1 + c_2\epsilon_2$~~

$\epsilon_1' = (1,1) = 1(1,0) + 1(0,1)$

$\epsilon_1' = \epsilon_1 + \epsilon_2$

$$[\epsilon_1']_\beta = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

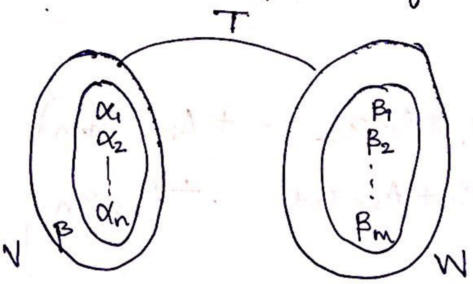
$\epsilon_2' = (2,1) = 2\epsilon_1 + \epsilon_2$

$$[\epsilon_2']_\beta = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$P = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$ ~~$P^{-1} = \begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix}$~~ $P^{-1} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$

Find $P^{-1}[T]_{\beta}P = [T]_{\beta'}$

$\Rightarrow$ Representation of transformations by a matrix.



Let $\dim V = n$ and $\dim W = m$.

Let β and β' are the ordered basis for V and W respectively.

Since T is a linear map.

$$T\alpha_j \in W = \text{span}\{\beta_1, \beta_2, \ldots, \beta_m\} \quad 1 \leq j \leq n.$$

$$\therefore T\alpha_j = A_{1j}\beta_1 + A_{2j}\beta_2 + \cdots + A_{mj}\beta_m \;, \; A_{ij} \in F.$$

$$\cancel{= \sum_{i=1}^{m} A_{ij}\beta_j} = \sum_{i=1}^{m} A_{ij}\beta_i$$

$$[T\alpha_j]_{\beta'} = \begin{pmatrix} A_{1j} \\ A_{2j} \\ \vdots \\ A_{mj} \end{pmatrix}$$

Notation: $[T]_{\beta}^{\beta'} = \left([T\alpha_1]_{\beta'}, [T\alpha_2]_{\beta'}, \ldots, [T\alpha_n]_{\beta'} \right)$

$$= \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & & A_{2n} \\ \vdots & \vdots & & \vdots \\ A_{m1} & A_{m2} & & A_{mn} \end{pmatrix} = A.$$

matrix of T relative to β and β'.

Let $\alpha \in V = \text{span}\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$.

$$\Rightarrow \alpha = x_1\alpha_1 + x_2\alpha_2 + \cdots + x_n\alpha_n \qquad x_i \in F$$

$$[\alpha]_{\beta} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$T\alpha = T\left(\sum_{j=1}^{n} x_j\alpha_j\right) = \sum_{j=1}^{n} x_j T\alpha_j \quad (T \text{ is linear})$$

$$T\alpha = \sum_{j=1}^{n} x_j \left(\sum_{i=1}^{m} A_{ij}\beta_i \right)$$

$$T\alpha = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} A_{ij}x_j \right) \beta_i$$

$$[T\alpha]_{\beta'} = \begin{pmatrix} \sum_{j=1}^{n} A_{1j}x_j \\ \sum_{j=1}^{n} A_{2j}x_j \\ \vdots \\ \sum_{j=1}^{n} A_{mj}x_j \end{pmatrix} = \begin{pmatrix} A_{11}x_1 + A_{12}x_2 + \cdots + A_{1m}x_n \\ A_{21}x_1 + A_{22}x_2 + \cdots + A_{2n}x_n \\ \vdots \\ A_{m1}x_1 + A_{m2}x_2 + \cdots + A_{mn}x_n \end{pmatrix}$$

$$= \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & & & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = A[\alpha]_\beta$$

Theorem 11: Let V be an n-~~directional~~ dimensional vector space over the field F and let W be a m-dimensional vector space over F. Let β and β' be an ordered basis for V and W respectively. For each linear map, there is an m×n matrix A with entries in F such that

$$[T\alpha]_{\beta'} = A[\alpha]_\beta$$

Furthermore, T→A is a one-one corresspondance between the set of all linear transformation from V into W and all m×n matrices with entries from F.

$$\boxed{L(V,W) \cong F^{m\times n}}$$

W.N.W18
Q) Let T be a lin
Let $\beta = \{\epsilon_1 = (1,0)$, ,
ordered basis fo
(i) Find $[T]_\beta$, $[T]$
(ii) Show that T

Ans.) i) $T: R^2 \to R^2$

$T(x_1, x_2) = (-x_2, x$

$T\epsilon_1 = T(1,0) = (0$

$T\epsilon_2 = T(0,1) = (-1$

$[T\epsilon_1]_\beta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$[T]_\beta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

~~$T\alpha_1 = T(1,2) = ($~~

~~$T\alpha_2 = T(1,-1) = ($~~

~~$[T\alpha_1]_{\beta'} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$~~

~~$[T]_\beta = \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix}$~~

$T\alpha_1 = T(1,2) = ($

$T\alpha_1 =$

$[T\alpha_1]_{\beta'} = \begin{pmatrix} -1/3 \\ -5/3 \end{pmatrix}$

$T\alpha_2 = T(1,-1) =$

$=$

$[T\alpha_2]_{\beta'} = \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}$

$[T]_{\beta'} = \begin{pmatrix} -1/3 & 2 \\ -5/3 & 1 \end{pmatrix}$

2WNov18

Q) Let T be a linear operator on $\mathbb{R}^2$ defined by $T(x_1,x_2)=(-x_2,x_1)$. Let $\beta=\{\epsilon_1=(1,0),\ \epsilon_2=(0,1)\}$ and $\beta'=\{\alpha_1=(1,2),\ \alpha_2=(1,-1)\}$ be two ordered basis for $\mathbb{R}^2$.

(i) Find $[T]_\beta$, $[T]_{\beta'}$

(ii) Show that $T-cI$ is invertible for any $c\in\mathbb{R}$.

Ans.) i) $T:\mathbb{R}^2\to\mathbb{R}^2$

$T(x_1,x_2)=(-x_2,x_1)$

$T\epsilon_1=T(1,0)=(0,1)=0\epsilon_1+1\epsilon_2$

$T\epsilon_2=T(0,1)=(-1,0)=-1\epsilon_1+0\epsilon_2$

$[T\epsilon_1]_\beta=\begin{pmatrix}0\\1\end{pmatrix}\qquad [T\epsilon_2]_\beta=\begin{pmatrix}-1\\0\end{pmatrix}$

$[T]_\beta=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$

$T\alpha_1=T(1,2)=(-2,1)=-2\alpha_1+1.\alpha_2$

$T\alpha_2=T(1,-1)=(1,1)=1\alpha_1+1.\alpha_2$

$[T\alpha_1]_{\beta'}=\begin{pmatrix}-2\\1\end{pmatrix}\qquad [T\alpha_2]_{\beta'}=\begin{pmatrix}1\\1\end{pmatrix}$

$[T]_\beta=\begin{pmatrix}-2&1\\1&1\end{pmatrix}$

$T\alpha_1=T(1,2)=(-2,1)=c_1\alpha_1+c_2\alpha_2=c_1(1,2)+c_2(1,-1)$

$T\alpha_1=\frac{-1}{3}\alpha_1-\frac{5}{3}\alpha_2$

$[T\alpha_1]_{\beta'}=\begin{pmatrix}-1/3\\-5/3\end{pmatrix}$

$T\alpha_2=T(1,-1)=(1,1)=c_1\alpha_1+c_2\alpha_2$

$=\frac{2}{3}\alpha_1+\frac{1}{3}\alpha_2$

$[T\alpha_2]_{\beta'}=\begin{pmatrix}2/3\\1/3\end{pmatrix}$

$[T]_{\beta'}=\begin{pmatrix}-1/3&2/3\\-5/3&1/3\end{pmatrix}$

(ii) Show that $T-cI$ is invertible $\forall\, c\in\mathbb{R}$

A matrix corresponds to $T-cI$ is

$$\begin{pmatrix}0&-1\\1&0\end{pmatrix}-c\begin{pmatrix}1&0\\0&1\end{pmatrix}=\begin{pmatrix}-c&-1\\1&-c\end{pmatrix}\xrightarrow{R_1\leftrightarrow R_2}\begin{pmatrix}1&-c\\-c&-1\end{pmatrix}\xrightarrow{R_2\leftarrow R_2+cR_1}$$

$$\cancel{\begin{pmatrix}1&0\end{pmatrix}}\quad\begin{pmatrix}1&-c\\0&-1-c^2\end{pmatrix}$$

$-1-c^2\neq 0\quad\forall c\in\mathbb{R}$

Hence, Rank of the matrix corresponding to $T-cI$ is 2.

$T-cI:\mathbb{R}^2\to\mathbb{R}^2$ is a L.T

$\text{rank}(T-cI)=2$

By rank-nullity theorem,

$$\text{rank}(T-cI)+\text{nullity}(T-cI)=\dim\mathbb{R}^2=2$$

$\Rightarrow \text{nullity}(T-cI)=0$

$\Rightarrow T-cI$ is one-one and

$\dim V=\dim W=\dim\mathbb{R}^2=2$

By theorem 9, $T-cI$ is invertible $\forall c\in\mathbb{R}$.

Note: Let $T:V\to V$ be a linear operator. ~~with~~ Let $\beta=\{\alpha_1,\alpha_2,\dots,\alpha_n\}$ and $\beta'=\{\alpha_1',\alpha_2',\dots,\alpha_n'\}$ be two ordered basis for V. —— (1)

In chapter 2, $[\alpha]_\beta=P[\alpha]_{\beta'}\quad\forall\alpha\in V$

where $P=[P_1,P_2,\dots,P_n]_{n\times n}$ (an invertible matrix).

and $P_j=[\alpha_j']_\beta$

By defn, $[T\alpha]_\beta=[T]_\beta[\alpha]_\beta$ —— (2)

|||ly $[T\alpha]_{\beta'}=[T]_{\beta'}[\alpha]_{\beta'}$ —— (I)

Replace 'α' by $T\alpha$ in (1)

$[T\alpha]_\beta=P[T\alpha]_{\beta'}$ —— (3)

(2) ⇒ $P[T\alpha]_{\beta'}=[T$

$=[T$

$P[T\alpha]_{\beta'}=[T$

Multiply by P^{-1}

$[T$

From (I) and (II),

$[T]_{\beta'}=P^{-1}[$

$B=P^{-1}AP$

Defn: Let A and

over a field F.

an $(n\times n)$ invertible

$B=P^{-1}A$

Q) Show that

on $F^{n\times n}$

(i) A is similar

(ii) If B is simil

and

(iii) $B=P^{-1}AP$, C

Q) Let A and B

Show that AB

Consider,

$(AB)X=0$

$\Rightarrow A(BX)=0$

$\Rightarrow BX=0$ (

$\Rightarrow X=0$ (B

③ $\Rightarrow P[T\alpha]_{\beta'} = [T\alpha]_{\beta}$

$$= [T]_{\beta}[\alpha]_{\beta} \quad \text{by } ②$$

$$P[T\alpha]_{\beta'} = [T]_{\beta} P [\alpha]_{\beta'} \quad \text{by } ①$$

Multiply by P^{-1}

$$[T\alpha]_{\beta'} = \left(P^{-1}[T]_{\beta}P\right)[\alpha]_{\beta'} \text{ ——— } (II)$$

From (I) and (II),

$$[T]_{\beta'} = P^{-1}[T]_{\beta}P \quad \text{(Proof of theorem 14)}$$

$$B = P^{-1}AP$$

<u>Defn</u>: Let A and B be two ~~mxn~~ $n\times n$ (square) matrix over a field F. We say B is similar to A if there is an $(n\times n)$ invertible matrix ~~$(n\times n)$~~. P over the field F such that

$$B = P^{-1}AP$$

Q) Show that ~~the~~ similarity is an equivalence relation on $F^{n\times n}$

(i) A is similar to A, $A = I^{-1}AI$

(ii) If B is similar to A, then $B = P^{-1}AP$,

and $A = (P^{-1})^{-1}BP^{-1} \Rightarrow$ A is similar to B.

(iii) $B = P^{-1}AP$, $C = Q^{-1}BQ \Rightarrow$ ~~Q^{-1}~~ $C = Q^{-1}(P^{-1}AP)Q$

$$C = (PQ)^{-1}A(PQ)$$

Q) Let A and B two ~~inver~~ $n\times n$ invertible matrices. Show that AB is invertible.

<u>Sol</u>

Consider,

$(AB)X = 0$

$\Rightarrow A(BX) = 0$

$\Rightarrow BX = 0$ (A is invertible)

$\Rightarrow X = 0$ (B is invertible)

$\because$ A is invertible

$\Leftrightarrow AX = 0$ has only trivial solution

$\Leftrightarrow AX = 0 \Rightarrow X = 0$

$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $\quad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$\det A = 0$ $\quad \det B \neq 0$

$$AB = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$BA = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

7th Nov'18

Determinant

Inversion: A pair (p, q) of distinct positive integers p & q is called an inversion if $p > q$.

Example: (i) $(5,3)$ is an inversion, $5 > 3$

(ii) $(3,5)$ is not an inversion.

Permutation: A permutation is a bijection from $\{1, 2, \ldots, n\}$ to $\{1, 2, \ldots, n\}$.

Example: $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = (3, 2, 1)$

Notation: S_n = set of all permutations defined on $\{1, 2, \ldots, n\}$

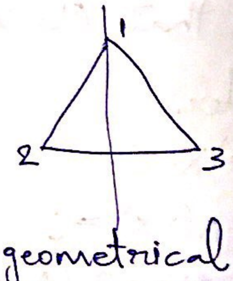
$|S_n| = n!$

Q) Find all elements in S_3

$$S_3 = \Big\{ \overset{\sigma_1\ +1}{(1,2,3)}, \overset{\sigma_2\ -1}{(1,3,2)}, \overset{\sigma_4\ -1}{(2,1,3)}, \overset{\sigma_5\ +1}{(2,3,1)}, \overset{\sigma_3\ +1}{(3,1,2)}, \overset{\sigma_6\ -1}{(3,2,1)} \Big\}$$

Notation: $\sigma = (j_1, j_2, j_3, \ldots, j_n) \in S_n$

$$\phi_\sigma = \left\{ \begin{matrix} (j_1, j_2), (j_1, j_3), (j_1, j_4), \ldots, (j_1, j_n) \\ (j_2, j_3), (j_2, j_4), \ldots (j_2, j_n) \\ \vdots \\ (j_{n-1}, j_n) \end{matrix} \right\}$$



geometrical interpretation.

We say $\sigma \in S_n$ is odd if number of inversions in ϕ_σ is odd.
Similar definition for even permutation.

Signature of a permutation σ,

$$\epsilon(\sigma) = \begin{cases} +1 & \text{if } \sigma \text{ is an even permutation.} \\ -1 & \text{if } \sigma \text{ is an odd permutation.} \end{cases}$$

<u>Example:</u> 1) $\sigma_1 = (1,2,3)$

$$\phi_{\sigma_1} = \{(1,2)(1,3)(2,3)\} \longrightarrow \text{zero inversions.}$$

σ_1 is an even permutation.

$$\epsilon(\sigma_1) = +1$$

2) $\sigma_2 = (1,3,2)$

$$\phi_{\sigma_2} = \{(1,3),(1,2),(3,2)\}$$

no. of permutations in $\phi_{\sigma_2} = 1$

$$\epsilon(\sigma_2) = \cancel{0} -1$$

<u>Note:</u> Consider an $n \times n$ matrix A

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}_{n \times n}$$

Let $\sigma = (j_1, j_2, \ldots, j_n) \in S_n$

We consider a product

$$\epsilon(\sigma_1) . a_{1j_1} . a_{2j_2} . a_{3j_3} \cdots a_{nj_n}$$

~~Let A =~~

$$\operatorname{Det}(A) = \sum \epsilon(\sigma) . a_{1j_1} . a_{2j_2} . a_{3j_3} \cdots a_{nj_n}$$

$$\sigma \in (j_1, j_2, \ldots, j_n) \in S_n$$

<u>Note:</u> ~~Corresponding to~~

Note: corresponding to

$\sigma_1 = (1,2,3)$ $\quad (+1)\, a_{11} a_{22} a_{33}$
$\sigma_2 = (1,3,2)$ $\quad (-1)\, a_{11} a_{23} a_{32}$ I

$\sigma_3 = (3,2,1)$ $\quad (-1)\, a_{13} a_{22} a_{31}$
$\sigma_4 = (2,1,3)$ $\quad (-1)\, a_{12} a_{21} a_{33}$
$\sigma_5 = (2,3,1)$ $\quad (+1)\, a_{12} a_{23} a_{31}$ III — II
$\sigma_6 = (3,1,2)$ $\quad (+1)\, a_{13} a_{21} a_{32}$

$$a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$= \det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Note (i) For two $n \times n$ matrices A & B

$\det(AB) = \det(A)\det(B)$

(ii) For an $n \times n$ matrix A,

A is invertible $\Leftrightarrow \det(A) \neq 0$.

$T: V \to V$

$\beta, \ \beta'$

9th Nov's

Quiz-2 paper discussion

1 $A = \begin{pmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ $\boxed{2M}$

$\boxed{2M}$

Row space of A = span$\{(1,2,0,3,0), (0,0,1,4,0), (0,0,0,0,1)\}$

$AX=0 \Rightarrow \quad x_1+2x_2+3x_4=0$

$$x_3+4x_4=0$$

$$x_5=0$$

Free variables x_2, x_4

$x_2=1,\ x_4=0,\ x_1=-2,\ x_3=0,\ x_5=0$

$$\alpha_1=(-2,1,0,0,0)$$

$x_4=1,\ x_2=0,\ x_1=-3,\ x_3=-4,\ x_5=0$

$$\alpha_2=(-3,0,-4,1,0)$$

Solution space $=$ span $\{\alpha_1, \alpha_2\}$ $\boxed{2M}$

$\boxed{2}$ $\dim V = n$

(a) Any linearly independent set of n vectors in V is a basis for V.

Show that span $S=V$, if not, $\alpha \neq 0 \in V \setminus \text{span} S$.

$\Rightarrow S \cup \{\alpha\}$ is linearly independent

$$\Rightarrow \dim V \geq |S \cup \{\alpha\}| = n+1 \ \times$$

(b) Any spanning set of 'n' vectors is a basis for V

- Let $S=\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ & span $S=V$

Show that S is linearly independent.

If not, there exists scalars $c_1, c_2, \dots, c_n$ (atleast one non-zero) such that $c_1\alpha_1+c_2\alpha_2+\dots+c_n\alpha_n=0$

without loss of generality $c_n \neq 0$

$$\alpha_n = \frac{-c_1}{c_n}\alpha_1 + \frac{-c_2}{c_n}\alpha_2 + \dots + \frac{-c_{n-1}}{c_n}\alpha_{n-1}$$

$$\Rightarrow \text{span}(S \setminus \{\alpha_n\}) = \text{span}\, S = V$$

$$\Rightarrow \dim V \leq |S \setminus \{\alpha_n\}| = n-1$$

$\boxed{3.}$ $\{\beta_1, \beta_2, \ldots, \beta_n\} \leftarrow$ basis for V

$\dim V = n$

$$\alpha_j = \sum_{i=1}^{n} A_{ij}\beta_i$$

$\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is a basis for V

$\Leftrightarrow$ A is invertible.

$\Rightarrow$ Suppose that $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is a basis for V.

$x_1\alpha_1 + x_2\alpha_2 + \ldots + x_n\alpha_n = 0$ has only trivial solution

$$\sum_{j=1}^{n} x_j\alpha_j = 0 \qquad x_1 = x_2 = \ldots = x_n = 0$$

$$\Rightarrow \sum_{j=1}^{n} x_j\left(\sum_{i=1}^{n} a_{ij}\beta_i\right) = 0$$

$$\Rightarrow \sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij}\cancel{\beta} x_j\right)\beta_i = 0$$

Since β is a basis for V,

$\sum_{j=1}^{n} a_{ij}x_j = 0 \quad 1 \le i \le n$, has only trivial solution.

$AX = 0$, has only trivial solution.

$\Rightarrow$ A is invertible. $\boxed{2M}$

$\Leftarrow$ Suppose that A is invertible.

show that $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is linearly independent

$$\Rightarrow \sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij}x_j\right)\beta_i = 0$$

Consider $x_1\alpha_1 + x_2\alpha_2 + \ldots + x_n\alpha_n = 0$

$\boxed{5.}$ $P_i(x_j) = \begin{cases} 1, & i=j \\ 0, & i \ne j \end{cases}$

$c_0P_0(x) + c_1P_1(x) + \ldots + c_3P_3(x) = 0$

12th Nov 2018

Idea: Given a

a field F, fi

$[T]_\beta$

~~Note: If such~~

Note: If such

$T\alpha_1 = c$

$T\alpha_2 = c$

$\vdots$

$T\alpha_n =$

Characteris

Let V be a

a linear o

scalar, $c \in$

$\alpha \in V$, $T\alpha =$

Note: Let c b

T on V. Then

(a) every vec

characteris

(b) the chara

with c is

eigen spa

Q) show that

value of c

Ans) Show th

$T(0) = 0 =$

$\Rightarrow c \in T$

12th Nov 2018

Idea: Given a linear operator T in a vector space V over a field F, find a basis β of V (if exists) such that

$$[T]_\beta = \begin{pmatrix} c_1 & 0 & 0 & \cdots & 0 \\ 0 & c_2 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & c_n \end{pmatrix} \leftarrow \text{diagonal matrix.}$$

~~Note: If such a β exists.~~

Note: If such a $\beta = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ exists, then

$$T\alpha_1 = c_1\alpha_1$$
$$T\alpha_2 = c_2\alpha_2$$
$$\vdots$$
$$T\alpha_n = c_n\alpha_n$$

Characteristic (eigen) value.

Let V be a vector space over the field F. Let $T: V \to V$ be a linear operator. A characteristic (eigen) value of T is a scalar, $c \in F$ such that there exists a non-zero vector $\alpha \in V$, $T\alpha = c\alpha$.

Note: Let 'c' be a characteristic value of a linear operator T on V. Then,

(a) every vector $\alpha \in V$ such that $T\alpha = c\alpha$ is called an characteristic (eigen) vector of T associated with c

(b) the characteristic (eigen) space ~~associated~~ of T associated with 'c' is given by

eigen space $= \{\alpha \in V : T\alpha = c\alpha\}$

Q) Show that eigen space of T corresponding to an eigen value of 'c' is a subspace of V.

Ans.) Show that $T_c = \{\alpha \in V : T\alpha = c\alpha\}$

$T(0) = 0 \Rightarrow T0 = c \cdot 0 = 0$

$\Rightarrow c \in T_c$

Let $\alpha, \beta \in T_c$, $k \in F$

$T\alpha = c\alpha$, $T\beta = c\beta$

$T(k\alpha+\beta) = kT\alpha + T\beta = kc\alpha + c\beta$

$T(k\alpha+\beta) = c(k\alpha+\beta)$

$\Rightarrow k\alpha+\beta \in T_c$

$\Rightarrow T_c$ is a subspace of V.

Note: Let V be a finite dimensional vector space over the field F. Let $T: V \to V$ be a linear operator.

A scalar 'c' is a characteristic value of T

$\Leftrightarrow$ There exists $\alpha \neq 0$ in V such that $T\alpha = c\alpha$

$\Leftrightarrow$ there exists $\alpha \neq 0$ in V such that $T\alpha = (cI)\alpha$

$\Leftrightarrow$ there exists $\alpha \neq 0$ in V such that $(T-cI)\alpha = 0$

$\Leftrightarrow$ $\alpha \neq 0$ in V such that $\alpha \in N(T-cI)$

$\Leftrightarrow$ $N(T-cI) \neq \{0\}$

$\Leftrightarrow$ $T-cI$ is ~~non~~ not one-one.

$\Leftrightarrow$ $T-cI$ is not invertible. ($\dim V < \infty$)

$\Leftrightarrow$ $\det(T-cI) = 0$

Theorem: Let V be a finite dimensional vector space over a field F. Let $T: V \to V$ be a linear operator. The following are equivalent.

(i) A scalar 'c' is a characteristic value of T

(ii) $T-cI$ is not invertible.

(iii) $\det(T-cI) = 0$.

Defn: Let A be an $n \times n$ matrix over the field F. A scalar $c \in F$ is an eigen value of A if $A-cI$ is singular (not invertible).

Note: 1) A scalar 'c' is a[n]

$\Leftrightarrow$ $A-cI$ is singula[r]

$\Leftrightarrow$ $\det(A-cI) = 0$

2) Let $f(x) = \det(xI-A)$

A scalar 'c' is an eigen

$\Leftrightarrow$ $f(c) = \det(cI-A$

$\Leftrightarrow$ 'c' is a root of $f($

3) $f(x) = \det(xI-A)$ is c

of A.

Q) Let $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

(i) Find the characteri

(ii) Find characteristic

(iii) find characteristic

~~$xI=$~~

$xI - A = x\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - ($

$= \begin{pmatrix} x & 1 \\ -1 & x \end{pmatrix}$

(i) $f(x) = \det(xI-A) = x$

(ii) No characteristic v

(iii) $f(x) = 0 \Rightarrow x = \pm i$

Theorem: Similar mat

polynomial.

Proof: Let $B = P^{-1}AP$

$f_B(x) = \det(xI-B)$

$= \det(xP^{-1}P - P^{-1}$

$= \det(P^{-1}(xI-A)$

$= \det(xI-A) =$

Note: 1) A scalar 'c' is an eigen value of A

$\Leftrightarrow$ $A - cI$ is singular (not invertible)

$\Leftrightarrow$ $\det(A - cI) = 0$

2) Let $f(x) = \det(xI - A)$

A scalar 'c' is an eigen value of A

$\Leftrightarrow$ $f(c) = \det(cI - A) = 0$

$\Leftrightarrow$ 'c' is a root of $f(x) = 0$.

3) $f(x) = \det(xI - A)$ is called the characteristic polynomial of A.

Q) Let $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

(i) Find the characteristic polynomial of A.

(ii) Find characteristic values of A in $\mathbb{R}$.

(iii) Find characteristic values of A in $\mathbb{C}$

$$xI - A = x\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} x & 1 \\ -1 & x \end{pmatrix}$$

(i) $f(x) = \det(xI - A) = x^2 + 1$

(ii) No characteristic value for A over $\mathbb{R}$ (roots are $+i, -i$)

(iii) $f(x) = 0 \Rightarrow x = \pm i$, eigen values are $\{i, -i\}$.

Theorem: Similar matrices have same characteristic polynomial.

Proof: Let $B = P^{-1}AP$

$$f_B(x) = \det(xI - B)$$

$$= \det(xP^{-1}P - P^{-1}AP)$$

$$= \det\left(P^{-1}(xI - A)P\right) = \det(P^{-1})\det(xI - A)\det(P)$$

$$= \det(xI - A) = f_A(x)$$

Q) Let $A = \begin{pmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{pmatrix}$

Find (i) characteristic polynomial of A.

(ii) characteristic values of A.

(iii) characteristic space of eigen values of A.

$f(x) = \det(xI - A)$

$$xI - A = \begin{pmatrix} x-3 & -1 & 1 \\ -2 & x-2 & 1 \\ -2 & -2 & x \end{pmatrix}$$

$f(x) = \det(xI - A) = (x-1)(x-2)^2$

(ii) Characteristic values of A

$f(x) = 0 \Rightarrow x = 1, 2, 2$

(iii) Characteristic space of T associated with $c = 1$

$= \{\alpha : A\alpha = c\alpha = \alpha\}$

$= \{\alpha : (A-I)\alpha = 0\}$

$=$ solution space of $(A-I)X = 0$

$$A - I = \begin{pmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & -1 \\ 2 & 1 & -1 \\ 2 & 2 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1/2 & -1/2 \\ 2 & 1 & -1 \\ 2 & 2 & -1 \end{pmatrix}$$

$$\xrightarrow[R_3 \leftarrow R_3 - 2R_1]{R_2 \leftarrow R_2 - 2R_1} \begin{pmatrix} 1 & 1/2 & -1/2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1/2 & -1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

corresponding system, $x_1 - \frac{1}{2}x_3 = 0$

$x_2 = 0$

$x_3 = a$

$\Rightarrow x_1 = \frac{1}{2}a, \; x_2 = 0$

Eigen space $= \{(\frac{1}{2}a, \ldots$

$= \{a(\frac{1}{2} \ldots$

Characteristic spa...

$= \{\alpha : A \ldots$

$= \{\alpha : (A \ldots$

$=$ soluti...

$$A - 2I = \begin{pmatrix} 3 & 1 & - \ldots \\ 2 & 2 & - \ldots \\ 2 & 2 & \ldots \end{pmatrix}$$

13th Nov'18

Q) Let $V = F^n$ be an ...

Find the characteri...

and characteristic...

(i) $I : V \to V$ define...

$I\alpha = \alpha \;\; \forall \alpha \ldots$

(ii) $O : V \to V$ defined...

$O\alpha = 0 \;\; \forall \alpha \in \ldots$

Sol) Let $\beta = \{\alpha_1, \alpha_2 \ldots$

(i) $I\alpha_1 = \alpha_1 = 1.\alpha_1 \ldots$

$I\alpha_2 = \alpha_2 = 0.\alpha_1 \ldots$

$\vdots$

$I\alpha_n = \alpha_n = 0.\alpha \ldots$

$$[I]_\beta = \begin{pmatrix} 1 & 0 & \ldots \\ 0 & 1 & \ldots \\ 0 & 0 & 1 \ldots \\ \vdots & & \\ 0 & 0 & \ldots \end{pmatrix}$$

$$\text{Eigen space} = \{(\tfrac{1}{2}a, 0, a) : a \in R\}$$
$$= \{a(\tfrac{1}{2}, 0, 1) : a \in R\}.$$

Characteristic space of T associated with $c=2$

$$= \{\alpha : A\alpha = c\alpha = 2\alpha\}$$
$$= \{\alpha : (A-2I)\alpha = 0\}$$
$$= \text{solution space of } (A-2I)X = 0$$

$$A-2I = \begin{pmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{pmatrix} - \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

13th Nov'18

Q) Let $V = F^n$ be an n-dimensional vector space over a field F. Find the characteristic polynomial, characteristic values and characteristic space of

(i) $I : V \to V$ defined as
$$I\alpha = \alpha \quad \forall \alpha \in V$$

(ii) $0 : V \to V$ defined as
$$0\alpha = 0 \quad \forall \alpha \in V$$

Sol) Let $\beta = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be an ordered basis

(i) $I\alpha_1 = \alpha_1 = 1.\alpha_1 + 0.\alpha_2 + \dots + 0.\alpha_n$

$I\alpha_2 = \alpha_2 = 0.\alpha_1 + 1.\alpha_2 + \dots + 0.\alpha_n$

$\vdots$

$I\alpha_n = \alpha_n = 0.\alpha_1 + 0.\alpha_2 + \dots + 1.\alpha_n$

$$[I]_\beta = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

Characteristic polynomial of I:

$$f(x) = \det(xI - [I]_\beta) = \det\begin{pmatrix} x-1 & 0 & 0 & \cdots & 0 \\ 0 & x-1 & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & \cdots & \cdots & x-1 \end{pmatrix}$$

$$f(x) = (x-1)^n$$

characteristic values of I = roots of $f(x)=0$

$= 1, 1, \ldots, 1$

characteristic space (corresponding to 1) $= \{\alpha \in V : I\alpha = \alpha\}$

$= V$

(ii) Let β be an ordered basis,

$$\beta = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$$

$$0\alpha_1 = 0 = 0.\alpha_1 + 0.\alpha_2 + \cdots + 0.\alpha_n$$

$$0\alpha_2 = 0 = 0.\alpha_1 + 0.\alpha_2 + \cdots + 0.\alpha_n$$

$$\vdots$$

$$0\alpha_n = 0 = 0.\alpha_1 + 0.\alpha_2 + \cdots + 0.\alpha_n$$

$$[0]_\beta = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

Characteristic polynomial of 0

$$f(x) = \det(xI - [0]_\beta) = \det\begin{pmatrix} x & 0 & 0 & \cdots & 0 \\ 0 & x & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & x \end{pmatrix}$$

$$f(x) = x^n$$

characteristic values of 0 = roots of $f(x)=0$

$= 0, 0, \ldots, 0$

characteristic space (corresponding to 0) $= \{\alpha \in V : 0\alpha = 0\}$

$= V$

Q) Find all characteristic values of

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix}$$

Sol) Characteristic polynomial of I

$$f(x) = \det(xI - A) = \det \begin{pmatrix} x-a_{11} & a_{12} & a_{13} & \cdots & -a_{1n} \\ 0 & x-a_{22} & a_{23} & \cdots & -a_{2n} \\ 0 & 0 & x-a_{33} & \cdots & -a_{3n} \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & x-a_{nn} \end{pmatrix}$$

$$f(x) = (x-a_{11})(x-a_{22}) \cdots (x-a_{nn})$$

characteristic values of A = roots of $f(x) = 0$

$$= a_{11}, a_{22}, a_{33}, \cdots a_{nn}$$

Def: Let V be a finite dimensional vector space over a field F. A linear operator $T: V \to V$ is diagonalizable if there exists a basis for V, each of its vectors are characteristic vectors of T.

Note: 1) Let T be a diagonalizable linear operator. Then there exists a basis $\beta = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ such that

$$T\alpha_i = c_i \alpha_i \;,\; c_i \in F$$

$$\Rightarrow [T]_\beta = \begin{pmatrix} c_1 & 0 & \cdots & 0 \\ 0 & c_2 & & 0 \\ 0 & 0 & & \vdots \\ \vdots & \vdots & & \\ 0 & 0 & & c_n \end{pmatrix}$$ diagonal matrix.

2) $[I]_\beta = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \Rightarrow I$ is diagonalizable.

$[0]_\beta = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \rightarrow 0$ is diagonalizable

Def: Let A be an $n\times n$ matrix,
we say A is diagonalizable if A is similar to diagonal matrix D.
i.e. $D = P^{-1}AP$

Q) Show that $A = \begin{pmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{pmatrix}$ is a diagonalizable matrix.

Sol) Step 1: Find charactoristic polynomial of A.

$$f(x) = \det(xI - A) = \det\begin{pmatrix} x-5 & +6 & 6 \\ 1 & x-4 & -2 \\ -3 & 6 & x+4 \end{pmatrix}$$

$$f(x) = (x-5)\left[x^2-16+12\right] - 6\left[x+4-6\right] + 6\left[6+3(x-4)\right]$$
$$= (x-5)(x^2-4) - 6(x-2) + 18(2+x-4)$$
$$= (x-2)\left[(x-5)(x+2) - 6 + 18\right]$$
$$= (x-2)\left[x^2-3x-10+12\right] = (x-2)(x^2-3x+2)$$
$$= (x-1)(x-2)^2$$

$$f(x) = (x-1)(x-2)^2$$

Step 2: characteristic values of A

$f(x)=0 \Rightarrow 1,2,2 \longrightarrow D=\begin{pmatrix}1&0&0\\0&2&0\\0&0&2\end{pmatrix}$

Step 3: Find character space of A. associated with $c=1$

$$A-cI = A-I = \begin{pmatrix}4&-6&-6\\-1&3&2\\3&-6&-5\end{pmatrix}$$ find row reduced echelon form.

$$\begin{array}{ccc} 12 & -18 & -18 \\ -12 & +24 & +20 \\ \hline 0 & 6 & 2 \end{array}$$

$$\begin{pmatrix}4&-6&-6\\-1&3&2\\3&-6&-5\end{pmatrix} \sim \begin{pmatrix}4&-6&-6\\0&3&1\\3&-6&-5\end{pmatrix} \sim \begin{pmatrix}4&-6&-6\\0&3&1\\0&6&2\end{pmatrix}$$

$$\sim \begin{pmatrix}4&-6&-6\\0&3&1\\0&0&0\end{pmatrix} \sim \begin{pmatrix}4&0&-4\\0&3&1\\0&0&0\end{pmatrix} \sim \begin{pmatrix}1&0&-1\\0&1&1/3\\0&0&0\end{pmatrix}$$

$$\begin{pmatrix}1&0&-1\\0&1&1/3\\0&0&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$x_1 - x_3 = 0$ $\quad \cancel{x_3}$

$x_2 + \frac{1}{3}x_3 = 0$

$x_3 = a$

$x_1 = a$

$x_2 = \frac{-1}{3}a$

$(A-I)X = 0 \Rightarrow$

solution space $= \{a(3,-1,3) : a \in F\}$

$\alpha_1 = (3,-1,3)$

Step 4: Find characteristic space of A.

associated with $c=2$ = solution space of $(A-2I)X=0$

$$(A-2I)X=0 \Rightarrow x_1-2x_2-2x_3=0$$

Free variables x_2, x_3

(a) $x_2=1, x_3=0 \Rightarrow x_1=2, \quad \alpha_2=(2,1,0)$

(b) $x_2=0, x_3=1 \Rightarrow x_1=2, \quad \alpha_3=(2,0,1)$

characteristic space $=\{c_1\alpha_2+c_2\alpha_3 : c_1, c_2 \in F\}$

$$P=\begin{pmatrix} 3 & 2 & 2 \\ -1 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} = (\alpha_1, \alpha_2, \alpha_3)$$

Step 5: Show that $D=P^{-1}AP$

show that $\Rightarrow PD=AP$ Find PD, AP.

$$PD=\begin{pmatrix} 3 & 2 & 2 \\ -1 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 4 & 4 \\ -1 & 2 & 0 \\ 3 & 0 & 2 \end{pmatrix}$$

$$AP=\begin{pmatrix} 4 & -6 & -6 \\ -1 & 3 & 2 \\ 3 & -6 & -5 \end{pmatrix}\begin{pmatrix} 3 & 2 & 2 \\ -1 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} = ($$