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COM205T Discrete Structures for Computing-Lecture Notes

Graph Theory

Motivation

1. Given a road network, find a minimum number of policemen so that every road is monitored. (policemen are placed at junctions, and in case of accidents at road r it will be addressed by the policeman standing at junction on any one end of r)
2. Design a router network so that it can handle all 2-node failures. (Fault tolerance level is 2)
3. Consider the interaction between processor and resources. Design an inter-process resource network so that there are no cyclic interactions (deadlock)

Graphs

- An abstract representation of a system under study (system: computer network, road network, router network)
- used as a model to understand the system better
- it is a binary relation
- graphs consist of vertices (nodes) and edges (links/arcs)

Basic Definitions and Simple Counting

$V(G) = \{v_1, v_2, \ldots, v_n\}$, the set of vertices.
$E(G)$, the set of edges
Ex: $V = \{1, 2, 3, 4\}$ $E = \{(1, 1), (1, 3), (3, 4)\}$

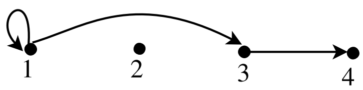


Fig. 1. A Graph

1. How many different graphs on n-vertices are possible? Note that graphs are precisely relations or represents a relation in graphical form.
   Ans. The number of n-vertex graphs = number of binary relations possible on a set of size $n = 2^{n^2}$
   Definition *Simple graphs* are graphs with no self loops and no multiple edges.
   Ex: $V = \{1, 2, 3, 4\}$ $E = \{(1, 3), (3, 4)\}$

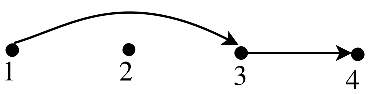


Fig. 2. A Simple Graph

2. How many directed simple graphs are there on n-vertices?
   Ans. The number of such graphs are equivalent to the number of irreflexive relations on a set of size $n = 2^{n^2-n}$
   Largely, we work with simple undirected graphs.
   Ex: $V = \{1,2,3,4\}$ $E = \{\{1,3\},\{2,4\}\}$

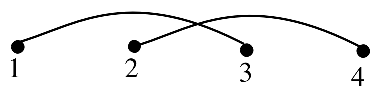


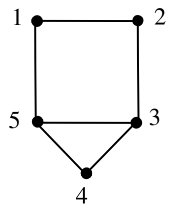
Fig. 3. An Undirected Graph

Note: An undirected graph is a special case of directed graphs

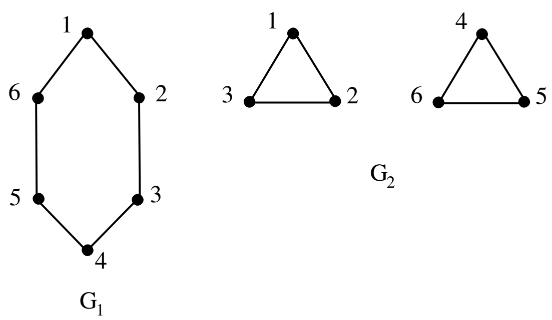
3. How many different undirected simple graphs are there on n-vertices?
   Ans. The number of such graphs are equivalent to the number of irreflexive and symmetric relations on a set of size $n = 2^{\binom{n}{2}}$

Undirected simple graphs and some more definitions

Neighborhood of a vertex v is $N_G(v) = \{u \mid \{u,v\} \in E(G)\}$.
Eg: $N_G(3) = \{2,4,5\}$, $N_G(4) = \{3,5\}$



Degree of a vertex $d_G(v)$ is the number of edges incident on v. $d_G(v) = |N_G(v)|$
The degree sequence of G is the sequence representing the degrees of $V(G)$.
Example: Consider the above graph G, with $d_G(1) = 2, d_G(2) = 2, d_G(3) = 3, d_G(4) = 2, d_G(5) = 3$. The degree sequence is (3 3 2 2 2).
For the degree sequence $(2,2,2,2,2,2)$, the two associated graphs are given below;



Note: Given a degree sequence, one can construct the associated graph in more than one way. The above two graphs G_1 and G_2 are one such example. We shall see the properties of these two graphs in detail. We next introduce *connectedness*.
Definition: *Connectedness* A graph G is connected if for every $u, v \in V(G)$ there exist a path

between u and v
In the above figure, G_1 is connected whereas G_2 is disconnected with two components.
Connected component is a maximal connected subgraph of a graph. Note that maximal is with respect to a property, and here it is connectedness.
We see a natural extension of the previous question as follows. Given a degree sequence $(d_1, d_2, \ldots, d_n)$ can you construct the associated connected graph uniquely?
Ans. NO. Consider the degree sequence $(3, 2, 2, 2, 1)$, there are two associated graphs as shown below;

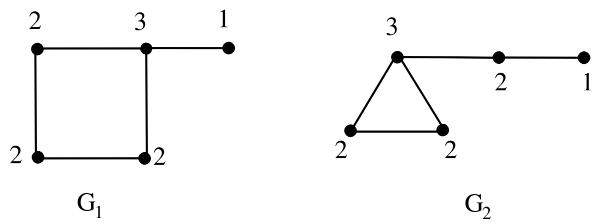


Fig. 4. Two non isomorphic representation of $(3, 2, 2, 2, 1)$

Question: Given two graphs G_1 and G_2, how can you determine that they are different?
Isomorphism
Two graphs G and H are isomorphic if and only if there exist a bijection from $V(G)$ to $V(H)$.
$f : V(G) \rightarrow V(H)$ such that $\{u, v\} \in E(G)$ if and only if $\{f(u), f(v)\} \in E(H)$
In Figure 4, there does not exist such a bijection from $V(G) \rightarrow V(H)$.

Questions

1. Given $(d_1, d_2, \ldots, d_n)$, how will you construct G.
2. Given G and H, how do you check whether they are isomorphic or not? Also, produce the associated bijection, if it exists.
3. In a group of n people, how many handshakes are possible?- Ans: $\binom{n}{2}$
4. Are there graphs with the degree sequence
   (i) (3,3,3,3,3) (ii) (3,3,3,4,4,2) (iii) (1,2,2,2)

5. Does there exist a graphic degree sequence $(5, 4, 3, 2, 1)$- No.

6. Verify: In any group of n-people, there exist at least two with equal number of friends.

Some Structural observations on Graphs

Claim 1: $\sum_{i=1}^{n} d_i = \text{Even}$
Claim 2: $\sum_{i=1}^{n} d_i = 2|E|$
Induction on $m = |E(G)|$

Proof. Base case: $m = 1$, $\sum_{i=1}^{n} d_i = 2$ is even
Induction Hypothesis: Assume that the claim is true for graphs with less than m edges, $m \geq 2$.

Induction Step: Consider the graph with m-edges; $m \geq 2$ and let $\{u,v\} \in E(G)$.
Consider the graph $G - \{u,v\}$. $V(G-\{u,v\}) = V(G)$ and $E(G-\{u,v\}) = E(G)\backslash\{u,v\}$.
Since $|E(G-\{u,v\})| = m-1$, we can bring in the induction hypothesis.
By the Induction hypothesis, in $G-\{u,v\}$, $\sum\limits_{i=1}^{n} d_i = 2m' = 2(m-1)$.
Add $\{u,v\}$ to $G-\{u,v\}$. Consider the degree sequence $d_1 + d_2 + \ldots + d_u + d_v + \ldots + d_n$.
$d_u = d_{u'}+1$, $d_v = d_{v'}+1$, u' and v' are the vertices corresponding to u and v in $G-\{u,v\}$. By introducing the edge $\{u,v\}$, the degree of u' (v') increases by one.
$d_1 + d_2 + \ldots + d_{u'} + 1 + d_{v'} + 1 + \ldots + d_n$.
$= d_1 + d_2 + \ldots + d_{u'} + d_{v'} + \ldots + d_n + 2 = 2(m-1)+2 = 2m.$ □

We shall next present another inductive proof of the above claim; induction on $|V(G)|$

Proof. Base case: $n = 1$, $\sum\limits_{i=1}^{n} d_i = 0$ is even
Induction hypothesis: Assume the claim is true for graphs with $(n-1)$-vertices, $n \geq 2$.
Induction Step: Let G be a graph on n-vertices $n \geq 2$
$V(G) = \{u_1, u_2, u_3, \ldots, u_n\}$
Let u_i be a vertex with minimum degree ($\delta(G)$)
Consider the graph $G - u_i$. $|V(G-\{u_i\})| = n-1$ and $|E(G-\{u_i\})| = |E(G)| - d_{u_i}$
By the induction hypothesis, the claim is true in $G - u_i$
i.e., $d_{u_1} + d_{u_2} + \ldots + d_{u_{i-1}} + d_{u_{i+1}} + \ldots + d_{u_n} = 2m'$
$d_{u_1} + d_{u_2} + \ldots + d_{u_{i-1}} + d_{u_{i+1}} + \ldots + d_{u_n} = 2(m-\delta(G))$, due to the removal $\delta(G)$ edges incident on the minimum degree vertex.
By introducing u_i in $G-u_i$, we can see that every vertex $v \in N_G(u_i)$, $d_G(v)$ is increased by one.
Now, $d_{u_1} + d_{u_2} + \ldots + d_{v_1} + d_{v_2} + \ldots + d_{v_{\delta(G)}} + \ldots + d_{u_n} + d_{u_i}$ where $\{v_1, v_2, \ldots, v_{\delta(G)}\} = N_G(u_i)$.
Also, note that d_{u_i} is added to the sum as u_i is added.
$d_{u_1} + d_{u_2} + \ldots + d_{v'_1} + 1 + d_{v'_2} + 1 + \ldots + d_{v'_{\delta(G)}} + 1 + \ldots + d_{u_n} + d_{u_i}$
$= d_{u_1} + d_{u_2} + \ldots + d_{v'_1} + d_{v'_2} + \ldots + d_{\delta'(G)} + \ldots + d_{u_n} + 1 + 1 \ldots + 1 + d_{u_i}$ [no.of 1's$=\delta(G)$ and $d_{u_i} = \delta(G)$]
By I.H. $\implies 2(m-\delta(G)) + \delta(G) + \delta(G) = 2m$. This completes the proof. □

Based on the above claim, here is an interesting corollary; Let $V_{\text{ODD}} = \{u \mid d_G(u) : 2k+1, k \geq 0\}$ and $V_{\text{EVEN}} = \{u \mid d_G(u) : 2k, k \geq 0\}$
$\sum\limits_{u \in V_{\text{ODD}}} + \sum\limits_{u \in V_{\text{EVEN}}} = 2m$
implies $\sum\limits_{u \in V_{\text{ODD}}}$ is even

Claim 3: The number of odd degree vertices in any graph is always even.
Corollary of claim 2.

Some Special Graphs

Path graph P_n on n vertices.
$|V(P_n)| = n, |E(P_n)| = n-1$
Cycle graph C_n on n vertices
$|V(C_n)| = n, |E(C_n)| = n$
Regular graph: G is k-regular if for every $v \in V(G), d_G(v) = k$
Ex: C_n is 2-regular.

The number of edges in a k regular graph on n vertices $= \frac{nk}{2}$
Are there 3-regular graph on 7 vertices - No
Complete graph K_n
The number of edges in a complete graph on n vertices $= \frac{n(n-1)}{2}$
Tree: is a connected acyclic graph.
Bipartite graph:
G is a bipartite graph if there exist a partition V_1, V_2 of $V(G)$ such that $V(G) = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$. For every edge $e = \{u, v\} \in E(G)$, $u \in V_1$ and $v \in V_2$

Example bipartite graphs include P_n, C_{2n}, and all trees. K_n and C_{2n+1} are not bipartite.

Question: Does there exist a characterization for a graph to be bipartite ?

Claim 4: G is bipartite if and only if G is odd-cycle free.

Proof. Necessity: Since G is bipartite, there exist V_1, V_2 such that $V(G) = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$, and for every edge $e = \{u, v\} \in E(G), u \in V_1$ and $v \in V_2$
Consider $u \in V_1$ and a cycle C starting and ending at u.
Since any cycle C that starts and ends at u visits vertices of V_1 and V_2 alternately, the length of C is clearly even.
Note: for any $\{x, y\} \subseteq V_1$, distance between x and y is $2k+1, k \geq 1$. Therefore, the length of cycle C is $2k+1+1$, which is even.
Sufficiency: G is odd cycle free. To show that G is 2-partite, we need to exhibit a bipartition.

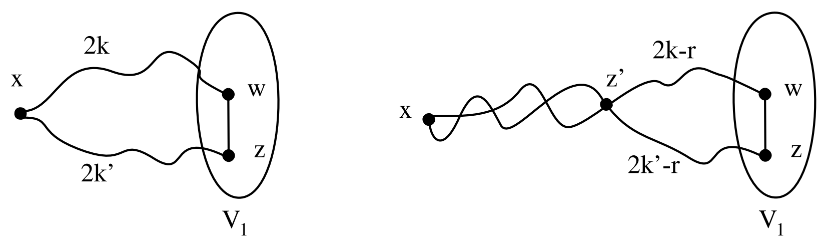


Fig. 5. An illustration for the proof of Claim 3

Let x be a vertex in G.
Consider $V_1 = \{u \mid distance(x, u) = even\}$
$V_2 = \{u \mid distance(x, u) = odd\}$
Claim 1: $V = V_1 \cup V_2$
Claim 2: $V_1 \cap V_2 = \emptyset$
Claim 3: For each $w, z \in V_1, \{w, z\} \notin E(G)$.
Proof: Suppose $\{w, z\} \in E(G)$, then $|P_{xw}| = 2k$ and $|P_{zx}| = 2k'$ for some $k', k \geq 1$ as shown in figure. $(P_{xw}, \{w, z\}, P_{zx})$ is a cycle of length $2k + 2k' - 1 = 2l + 1$ for some $l \geq 1$. Therefore, G contains an odd cycle and this is a contradiction to the premise. Our assumption that there exist $\{w, z\} \in E(G)$ is wrong and $\{w, z\} \notin E(G)$. Therefore, V_1 is an independent set. Similar arguments hold true if V_2 is an independent set.
Suppose, $V(P_{xw}) \cap V(P_{zx}) \neq \emptyset$, then identify the last vertex z' such that $z' \in P_{xw}$ and $z' \in P_{zx}$. Let length of $|P_{xz'}| = r$. Note that $P_{xz'} \subseteq P_{xw}$ and $P_{xz'} \subseteq P_{xz}$ are of length r. Suppose

$|P_{xz'}| < r$, then it contradicts the fact that P_{xz} is a shortest path. It follows that $|P_{zz'}| = 2k' - r$ and $|P_{z'w}| = 2k - r$. Length of cycle $(P_{z'w}, \{w, z\}, P_{zz'})$ is $2k - r + 2k' - r - 1 = 2l + 1$ for some $l \geq 1$. Therefore, there exist an odd cycle which is a contradiction. Therefore, the assumption is wrong, and the claim follows. $\square$

Questions:
A graph is 3-partite if and only if $---$?
What about a necessary and sufficient condition for a graph to be k-partite ?

Graph Coloring

- An assignment of colors to vertices of a graph
- *Proper coloring* - adjacent vertices receive different colors.
- G is k-colorable if and only if there exist $c : V(G) \to \{1, 2, \ldots, k\}$ such that for all $e = \{u, v\}, c(u) \neq c(v)$
- *Chromatic Number* $\chi(G)$ is the minimum number of colors required to properly color a graph.

$\chi(K_n) = n$
$\chi(P_n) = 2$
$\chi(Tree) = 2$
$\chi(C_{2n}) = 2$
$\chi(C_{2n+1}) = 3$
$\chi(\textit{bipartite graph}) = 2$
G is bipartite if and only if G is 2-colorable.
The following statements are equivalent.

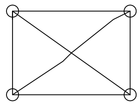
- G is bipartite
- G is 2-colorable
- G is odd-cycle free

Planar Graphs

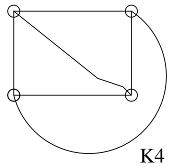
A graph G is a *planar graph* if G has a plane drawing, a drawing with no crossing edges. K_4 is a planar graph as it has a plane drawing which is illustrated below. Interestingly, K_5 is non-planar; there is no plane drawing. It follows that K_n, $n \geq 5$ is non-planar. Also, $K_{3,3}$, the complete bipartite graph with partition size 3 each is non-planar. Further, K_5-e (Remove any one edge from K_5) and $K_{3,3}$-e are planar as there exists a plane drawing.
The following celebrated result is due to Kuratowski.
Theorem: A graph G is planar iff G has no edge subdivisions of K_5 or $K_{3,3}$.
The operation *edge subdivision* of G replaces an edge in G with a path of appropriate length. If a graph G is obtained from K_5 or $K_{3,3}$ through a sequence of edge subdivisions, then G is non-planar. As per the theorem, the converse is also true.

We below mention a few necessary conditions without proofs.
Claim 1: If G is planar, then the number of edges is at most $3n$-6.

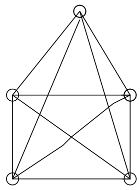
A *face* in a planar graph is a closed region or a cycle. Each planar graph has exactly one exterior face (the plane in which it is drawn) and one or more interior faces. The following formula is due to Euler.



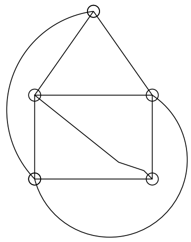
K4
Non−plane drawing



K4
Plane drawing



K5 : Non−planar



K5−e : Planar

Claim 2: If G is planar, then $V - E + F = 2$.

For K_4, $V = 4, E = 6, F = 4$, and for $K_5 - e$, $V = 5, E = 9, F = 6$. For all trees on n vertices, $V = n, E = n - 1, F = 1$. Note that all trees are planar as it is acyclic and has exactly one exterior face and no interior faces.

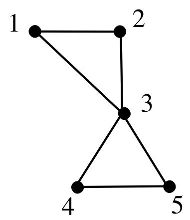
A planar graph with exactly $3n$-6 edges is said to be a *maximal* planar graph. That is, any further edge addition to such a planar graph results in a non-planar graph.

Claim 3: Let G be a planar graph. Then, minimum degree of G is at most 5.
The claim follows from Claim 1; if suppose the degree of all vertices is at least 6, then the number of edges is at most $3n$, contradicting Claim 1.

Claim 4: Any planar graph is 6-colorable.
From Claim 3, we know that there exists a vertex of degree at most 5, say v_1. Color v_1 with the first color, say c_1, its neighbors can be colored with at most 5 colors, $\{c_2, \ldots, c_5\}$. Consider the graph $G - v_1$, the graph obtained from G with v_1 removed. Note that $G - v_1$ is also planar and hence by Claim 3, there exists a vertex degree at most 5, say u. Continue the above process with respect to u by coloring u and its neighbors if it is not colored in previous iterations. This ensures that at each iteration we make use of at most 6 colors and hence any planar graph is 6-colorable.

Problem Session

1. Consider the Adjacency matrix representation of a simple graph. What does entries A^2 denote? What about A^3, A^k?



G

$$A = \begin{bmatrix} 0&1&1&0&0 \\ 1&0&1&0&0 \\ 1&1&0&1&1 \\ 0&0&1&0&1 \\ 0&0&1&1&0 \end{bmatrix} \quad A^2 = \begin{bmatrix} 2&1&1&1&1 \\ 1&2&1&1&1 \\ 1&1&4&1&1 \\ 1&1&1&2&1 \\ 1&1&1&1&2 \end{bmatrix} \quad A^3 = \begin{bmatrix} 2&3&5&2&2 \\ 3&2&5&2&2 \\ 5&5&4&5&5 \\ 2&2&5&2&3 \\ 2&2&5&3&2 \end{bmatrix}$$

Consider the adjacency matrix representation of the above graph.
Note that $A^k[i,j]$ represents the number of paths from i to j of length k.
For example, there exist three paths of length 3 from 1 to 2.
I.e., $A^3[1,2] = 3$ and the paths are $(1-2-1-2), (1-3-1-2), (1-2-3-2)$.

2. If G is a simple graph with 15 edges and $\overline{G}$ has 13 edges, how many vertices does G have?
Total number of edges possible $= \binom{n}{2} = 15 + 13 = 28. \implies n = 8$

3. The maximum number of edges in a simple graph with 10 vertices and 4 components is
Ans: 21. Three components with K_1's and one component with K_7. Number of edges $= \binom{7}{2} = 21$

4. For which value of k an acyclic graph G with 17 vertices, 8 edges and k components exist?
Let $e_i, n_i, 1 \le i \le k$ represents the number of edges, and number of vertices in component i, respectively.
Given $\sum_{i=1}^{k} e_i = 8$
Since the graph is acyclic it follows that $e_i = n_i - 1, 1 \le i \le k$.
Therefore, $\sum_{i=1}^{k}(n_i - 1) = 8 \implies \sum_{i=1}^{k} n_i - k = 8$
Since $\sum_{i=1}^{k} n_i = 17, k = 17 - 8 = 9$

5. Find the minimum number of vertices in a simple graph with 13 edges and having 5 vertices of degree 4 and the rest having degree less than 3.
Ans: 8. $\sum_{i=1}^{n} d_i = 13 \times 2 = 26$. Number of 4 degree vertices $= 5$. This contributes 20 to the degree. For the rest, we can use vertices of degree less than 3. i.e., there should exist three 2-degree vertices. Therefore, the total number of vertices is at least $5 + 3 = 8$.

6. Does there exist a simple graph with degree sequence $(7, 7, 6, 6, 5, 5, 4, 4, 3, 3, 2, 2, 1, 1)$?
Ans: Yes.

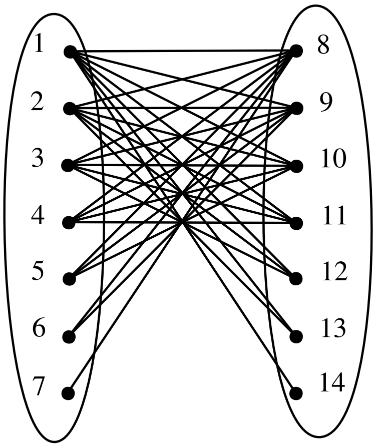


Fig. 6. Bipartite graph with the given degree sequence

7. A simple graph G has degree sequence $(3, 3, 2, 1, 1)$. What is the degree sequence of G^c ?
   Ans: $(1, 1, 2, 3, 3)$

8. If the degree sequence of a simple graph G is $(4, 3, 3, 2, 2)$, what is the degree sequence of G^c ?
   Ans: $(0, 1, 1, 2, 2)$

9. The maximum number of edges in a bipartite graph with n vertices is ...
   Ans: If n is even, then complete bipartite graph $K_{\frac{n}{2},\frac{n}{2}}$ has maximum edges, which equals $\frac{n^2}{4}$. If n is odd, then $K_{\frac{n-1}{2},\frac{n+1}{2}}$ has maximum edges which is $\frac{n^2-1}{4}$.

10. What is the chromatic number of Peterson graph?
    Ans: 3. In the given below graph, $1, 2, 3$ represents three different colors.

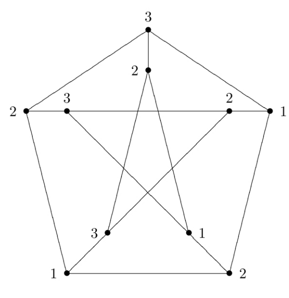


Fig. 7. Coloring of a Peterson graph

11. `Claim:` If G contains K_4 as a subgraph, then G is at least 4-colorable. Is this true? Are there 4-colorable graphs without K_4. Are there triangle free graphs with chromatic number 4? Above claim is true. In the 4-colorable graph shown below, G_1 has no K_3 and G_2 has no K_4 as a subgraph.

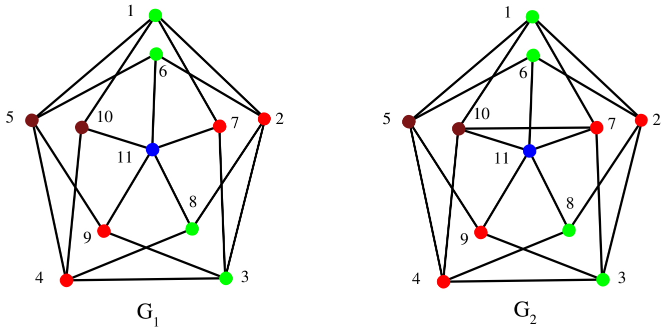


Fig. 8. 4-colorable Graphs

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2. D.F.Stanat and D.F.McAllister, Discrete Mathematics in Computer Science, Prentice Hall, 1977.
3. C.L.Liu, Elements of Discrete Mathematics, Tata McGraw Hill, 1995

Reading assignment

[1]. Shakuntala Devi: "Puzzles to Puzzle you"
"More Puzzles",
"Figuring: The Joy Of Numbers"
[2]. George J. Summers: "The Great Book of Mind Teasers and Mind Puzzlers"

A MATTER OF TIME [1]

Fifty minutes ago if it was four times as many minutes past three o'clock. How many minutes is it to six o'clock