Q) $\sqrt{3}$ is irrational. (Proof by contradiction)

i.e. statement false!

q_1 Assume $\sqrt{3}$ is rational.

q_2 $\sqrt{3} = \frac{a}{b}$

q_3 $\sqrt{3} = \frac{a}{b}$ $GCD(a,b) = 1$

q_4 $3 = \frac{a^2}{b^2}$

q_5 $a^2 = 3b^2$

q_6 a^2 is a multiple of 3.

Claim 1

'a' is a multiple of 3

Proof by contradiction.

suppose not;

'a' is not a multiple of 3.

'a' is either $(3k+1)$ ~~and~~ or $(3k+2)$ for some int k.

$3k+1$ | $3k+2$

$a^2 = 9k^2 + 6k + 1$ | $a^2 = 9k^2 + 12k + 4$

a^2 is not a multiple of 3 | a^2 is not a multiple of 3.

contradicting q_6.

$\Rightarrow$ our assumption that 'a' is not a multiple of 3 is false

$\Rightarrow$ a is multiple of 3.

q_7 $a = 3k'$; $k' \in \mathbb{Z}^+$

$a^2 = 3b^2$

$(3k')^2 = 3b^2$

$3k'^2 = b^2$

q_8 b^2 is a multiple of 3

Claim 2

b^2 is a multiple of 3 $\Rightarrow$ b is a multiple of 3.

q_9 C_1: 'a' is a multiple of 3
C_2: 'b' is a multiple of 3.

q_{10} $GCD(a,b) \geq 3$ [trying to conclude $GCD(a,b) \neq 1$
contradicting q_3

Assumption that $\sqrt{3}$ is rational is false
$\Rightarrow \sqrt{3}$ is irrational.

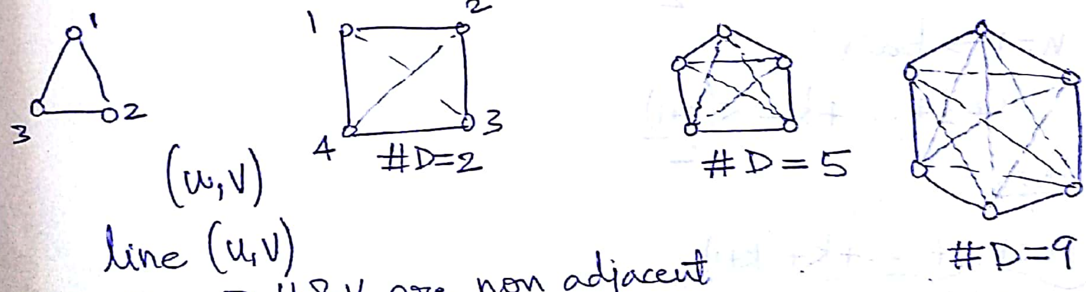
Q) The number of diagonals in an n-sided polygon ___

1) Experimentation
2) Guess # &
   ↳ exact
   ↳ L.B
   ↳ U.B

} observation.

3) Proof of correctness.
   ↳ proof by contradiction.

n=3 n=4 n=5 n=6



(u,v)

line (u,v)

(1,2) adj $\Rightarrow$ u & v are non adjacent
(2,1) adj #diagonals=0
(1,3) adj

0, 2, 5, 9, ?

is false

⟹ Proof by mathematical induction. (o) Induction param...

1) Basis 2) Induction hypothesis 3) Induction step (and...

↳ $n=0, n=1,$ Induction parameter n.

$n=2, n=3,4,5,6$

P_1 $n=0, n=1, n=2, n=5$ $n=0,1,2$ — undef...

P_2 $n=2, n=3$ $n \geq 3$

P_3 $n=17$ [$n<16$ — undefined] $n=3, n=4, n=5$

P_4 $n=108$ [$n<108$ it is ~~undefined~~ random order] $n=3$.

Induction hypothesis.

Assume the claim is true for k.

Induction step.

Prove the claim for $k+1$

Conclusion

The claim is true for all values of $n \geq 3$.

$\frac{1(1+1)}{2} = 1$

$1+2+3+\ldots+n = \frac{n(n+1)}{2}$

$n=1 \rightarrow$ basis.

$1+2+\ldots+k = \frac{k(k+1)}{2}$

Hypo

$\underbrace{1+2+3+\ldots+k}+(k+1)$

$\frac{k(k+1)}{2} + k+1 = \frac{(k+1)(k+2)}{2}$

(*) Write FOL for mathematical induction.

basis,

FOL for the definition of mathematical induction.
To prove claim P.

Basis $\rightarrow$ $P(0) \wedge P(1) \wedge P(2)$
$\wedge$
$\forall k \left(P(k) \rightarrow P(k+1) \right)$

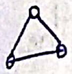
k starts from 2   hypo   induction step.

$\forall n_{\geq 0} P(n)$

Number of diagonals in an n-sized polygon — $\frac{n(n-3)}{2}$

Basis

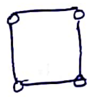
$n = 3$



$D = 0 \quad \frac{n(n-3)}{2} = 0$

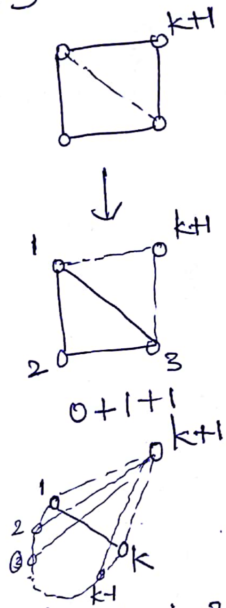
The claim is verified.

Hypo   Assume for a k-sided polygon $(k \geq 3)$



the $\#D = \frac{k(k-3)}{2}$

Step   Consider a polygon of side $k+1$, $k \geq 3$



$(2, 3, \ldots, k-1) = k-1-2+1$

$\frac{k(k-3)}{2} + 1 + k - 2$

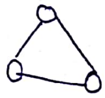
$= \frac{(k+1)(k+1-3)}{2}$

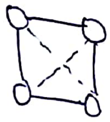
$\ldots + k = \frac{k(k+1)}{2}$

1/Sept/18

The number of diagonals on a polygon of size 'n' is $\frac{n(n-3)}{2}$

Proof by MI on 'the number of nodes'

Base $n=3$  $\#D=0$; $\frac{3(3-3)}{2}=0$

$n=4$  The formula is true for the base case

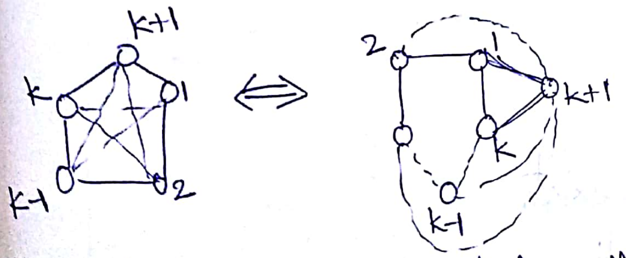
$\#D=2$; $\frac{4(4-3)}{2}=2$

Again, the formula is true.

Hypothesis Assume the formula is true for a k-sided polygon, $k \geq 4$ i.e. $\#D = \frac{k(k-3)}{2}$

Induction step

Consider a polygon of size $k+1$, $k \geq 4$
(Intuitively, to prove the claim for a five-sided polygon, assume the result of a 4-sided polygon (true assumption as it is proved in basis)
A $(k+1)$ sided polygon can be viewed as a k-sided polygon plus the node $(k+1)$ along with edges (links) to k-sided polygon.



$\#D$ in $(k+1) = \#D$ in k-sided + the edge $\{1,k\}$ + edges $\{k+1, 2\}$, $\{k+1, 3\}$, $\{k+1, k-1\}$

$= \frac{k(k-3)}{2} + 1 + k - 2 = \frac{(k+1)(k-2)}{2}$

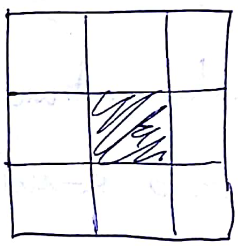
proved for $(k+1)$

(*) Google DeMorgan and paul Erdös

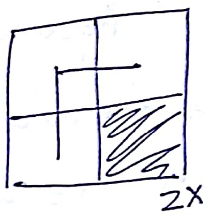
Conclusion

Therefore, the claim is true $\forall n \geq 3$

$$\frac{\begin{array}{l} P(3) \wedge P(4) \quad \text{Basis} \\ \forall k \geq 4 \ (\underbrace{P(k)}_{\text{hypo}} \rightarrow \underbrace{P(k+1)}_{\text{step}}) \end{array}}{\forall n \geq 3 \ P(n) \quad \text{conclusion.}}$$

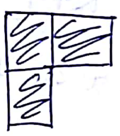


3X3-1



2X2-1

Dominos 

Triominos 

[No overlapping between triominos].

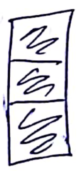
When trying to cover 3X3 with triominos, it doesn't work because it is like covering 8 squares with triominos ie 8/3 not integer.

3 X 3 − 1

4 X 4 − 1

⋮

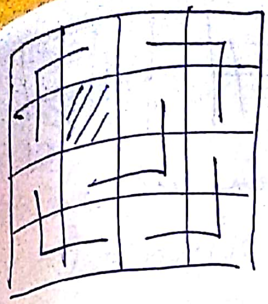
k X k − 1

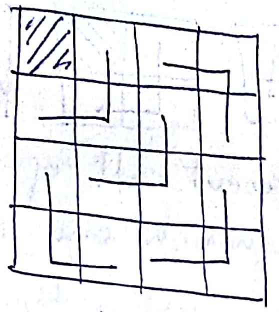
 this is <u>not</u> triominos.

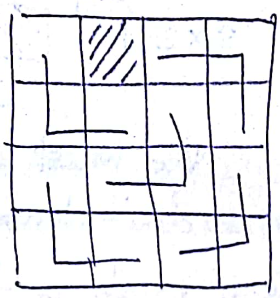
2X2-1 —— One soln exists

3X3-1 —— no solution.

4X4-1

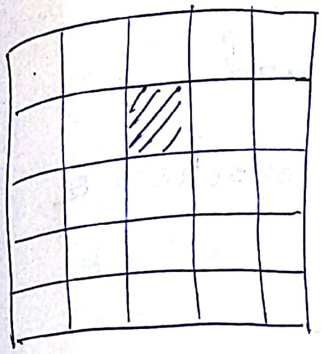






The other cases are already proved by symmetry.

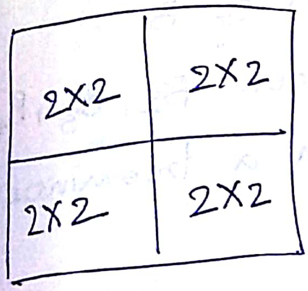
5X5 − 1

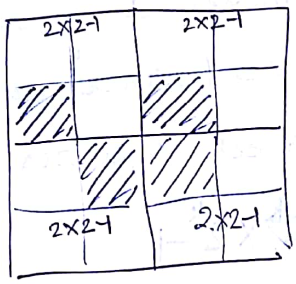


No solution

But you might get a solution for a different configuration.

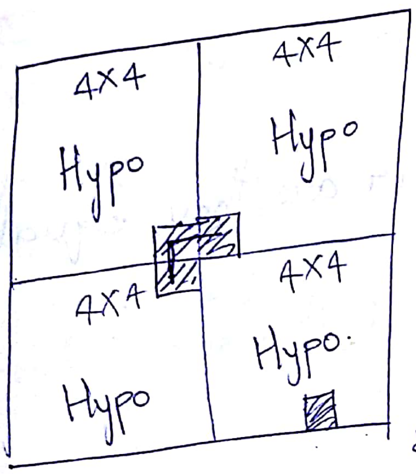
-1





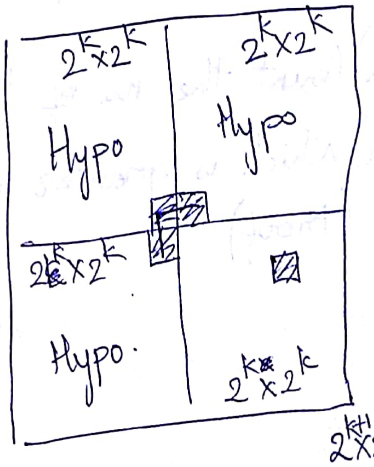
Then the claim is

If given $2^k \times 2^k - 1$ chessboard, then $\exists$ soln.



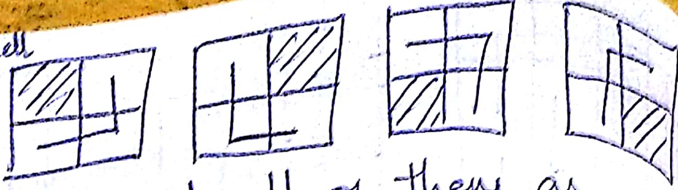
8X8-1

run Induction



$2^{k+1} \times 2^{k+1} - 1$

Base $2 \times 2 - 1$ missing cell



We must present all of them as we don't know which one is removed.

There exists a tiling using triominos.

Hypo Assume for $2^k \times 2^k - 1$, $k \geq 1$

$\exists$ soln.

Indn Consider a $2^{k+1} \times 2^{k+1} - 1$ chessboard

$2^{k+1} \times 2^{k+1}$ can be seen as a collection of four $2^k \times 2^k$ chess boards.

Let B_1, B_2, B_3, B_4 are $2^k \times 2^k$

Assume the tile removed is a part of B_1

Remove one square (corner) each from B_2, B_3, B_4 so that the removed squares form a triominos.

Now each B_i is $2^k \times 2^k - 1$

By hypo, each B_i has a tiling

Hypo + 1 triomino for removed square gives a tiling for $2^{k+1} \times 2^{k+1} - 1$

Q.) Count the no. of triangles.

Q.) Which is greater $n!$ or 2^n. or are they equal. (Proof)

Sept ...

Coin change problem:

Given integer x, denominations Rs5, Rs3.

Q: Can we always give change for any 'x' using the above two denominations.

(or Identify a lower bound for $x \ni$ change (x) using ₹5 or ₹3 from the lower bound).

Claim: for any $x \geq 8$, $\exists$ change.

$x=1$ ×		$x=11$	$5+3+3$
$x=2$ ×		$x=12$	3×4
$x=3$ ✓		$x=13$	$5+5+3+3$
$x=4$ ×		$x=14$	$3\times3+5$
$x=5$ ✓		$x=15$	5×3
$x=6$ ✓		$x=16$	$5\times2+3\times2$
$x=7$ ×		$x=17$	$3\times4+5$
$x=8$ ✓		$x=18$	~~$3\times6=2\times$~~ $8+10$
$x=9$ ✓		$x=19$	$9+10$
$x=10$ ✓		⋮	⋮

Base $x=8$

change 1 ₹5
1 ₹3

Hypothesis

The claim is true for any $x \geq 8$

Induction step

Consider an integer

$x+1$, $x \geq 8$

We need to prove that change $(x+1)$ exists for some $n \in \mathbb{Z}^+$

$\underline{3n}$, $3n+1$, $3n+2$

↳ change using ₹3

↳ $3n+1 = 3(n-3)+10$

$= (3n-9)+10$

$3n+1 = 3(n-3)+10$

eg. $x=9$ $n=3$

$x=10$ $n=3$

$3(3-3)+10=10$

$x=11$ $3(3-1)+5$

$x=12$ $3n$ 3×4

$x=13$ $3n+1$ $3(4-3)+10$

$x=14$ $3n+2$ $3(4-1)+5$

$3n+2 = 3(n-1)+5$ $\forall\ n \geq 8$

$x = n-3$
$y = n-1$ $x,y \geq 0$ $n \geq 3$

<u>Case1</u> (atleast 1 ₹5 exists)

$x = 3a+5b$ (Hypothesis set)

$x+1 = 3a+5b+1$

$= 3a+5b+6-5$

$= 3a+6+5b-5$

$x+1 = 3(\overline{a+2})+5(\overline{b-1})$

from hypothesis set, remove one ₹5 and add two ₹3.

change $(x+1)$ exists using 3 & 5

$\forall x$ chang(x) ✓

but this case would fail when hypothesis is 9, since there are no ₹5 coins to remove.

~~case 2~~ (atleast 3 ₹3 exists)

$x = 3a+5b$

$x+1 = 3a+5b+1$

$= 3a+5b+10-9$

$= 3a-9+5b+10$

$= 3(a-3)+5(b+2)$

Hypo set,

remove ~~₹3~~ 3 ₹3

add 2 ₹5

$x=12$ ~~₹4~~ 4×3

$x=13$ $1\times3+2\times5$

	₹5	₹3
$x=8$	1	1
$x=9$	0	3
$x=10$	2	0
$x=11$	1	2
$x=12$	0	4
$x=13$	2	1

How to prove

no case 3

no case $i \geq 3$

int x, ~~₹3, ₹5~~ ₹5, ₹7

int x, k, l → 2nd denomination

k ↓ 1st denomination

what are the conditions for k & l so that we can give change for any denomination.

5, 7

3n-1, 3n, 3n+1   3n-6+5   3n

7/9/18

Introduction to Relations and Functions

Set: A collection of well-defined objects

Powerset: Set of all subsets of a set

$A = \{1,2,3\}$ $\quad {}^3C_0 + {}^3C_1 + {}^3C_2 + {}^3C_3 = 2^3$

$P(A) = \{\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\},\phi\}$

$|P(A)| = 2^n \; ; \; n = |A|$

$A = \{\underbrace{\{1\},2}_{\text{element}}\}$

$P(A) = \{\{\{1\}\},\{2\},\{\{1\},2\},\phi\}$

$A = \{\phi,\{\phi\}\}$

$P(A) = \{\{\phi\},\{\{\phi\}\},\{\phi,\{\phi\}\},\phi\}$

(a,b) = ordered $\qquad \{a,b\}$ = unordered

$(a,b) \neq (b,a) \qquad \{a,b\} = \{b,a\}$

Given a set A, we define

Cross product = $A \times A$

$A \times A = \{(a,b) \mid a \in A, b \in A\}$

Given A, B

$A \times B = \{(a,b) \mid a \in A, b \in B\}$

Note: Cross product is a set

In particular, a set of ordered pairs.

HW) Prove that empty set $\subset$ set.

~~{0}~~

$\boxed{\{\} = \phi}$

Given a set of size 'n', claim $|P(A)| = 2^n$

Proof by mathematical induction. $|A| = 0$, $A = \{\}$, $P(A) = \{\{\}\}$

Base: $|A| = 1$, $A = \{a\}$, $P(A) = \{\{a\}, \emptyset\}$

$= 2 = 2^1$ Claim is verified.

Hypo: Assume the claim is true for $n \geq 1$

for any set of size n, $P(A) = 2^n$

Step: Consider a set of size $(n+1)$

$A = \{a_1, a_2, a_3, \ldots, a_n, a_{n+1}\}$

eg. $\{1, 2\} \cup \{3\}$

~~$\{1\}$~~
~~$\{1, 2\}$~~
~~$\{2\}$~~
~~$\emptyset$~~

$\{1\}$	$\{1\}$
$\{2\}$	$\{2\}$
$\{3\}$	$\{1, 2\}$
$\{1, 2\}$	$\emptyset$
$\{1, 3\}$	
$\{2, 3\}$	
$\{1, 2, 3\}$	
$\emptyset$	

Direct proof:

$nC_0 + nC_1 + \ldots + nC_n = 2^n$.

$P(A_{a_1 \ldots a_n})$

2^n

Add a_{n+1} to each set in $P()$

Do not add a_{n+1}

$2^n + 2^n = 2^{n+1}$ ~~$P(A_{a_1 a_2}$~~. $P(A_{a_1, a_2, \ldots a_n, a_{n+1}})$

By induction hypothesis $P(A_{a_1, \ldots a_n}) = 2^n$

Not adding a_{n+1} into $P = 2^n$

adding " " " $= 2^n$ $\Big\} 2^{n+1}$

$\forall n \geq P(A) = 2^n$

01, 01, 01, ..., 01

a_1	a_2	a_3	a_4	
0	0	0	0	ϕ
0	0	1	0	$\{a_3\}$
1	1	0	1	$\{a_1, a_2, a_4\}$
1	1	1	1	$\{a_1, a_2, a_3, a_4\}$

$a_1, a_2, ..., a_n$

$2^n = 2^n$

list of all n-bit binary strings
$$\begin{cases} 000...0 \\ 000...1 \\ 00...10 \\ \vdots \\ 11...111 \end{cases}$$

Binary strings $= |P(A)|$

$\underbrace{2 \times 2 \times 2 \times ... 2}_{2^n}$

$0000 \leftrightarrow \phi$

$\Leftrightarrow \quad 1111..1 \leftrightarrow \{a_1, a_2, ... a_n\}$

<u>Relation</u>

Relation $R \subseteq$ cross product

set
$\downarrow$
cross product
$\downarrow$
relation.

CP is $A \times A$ $(A \times B)$

$R \subseteq A \times A$ or $R \subseteq A \times B$

↳ Binary relation.

Given A, B, C

CP: $A \times B \times C$

$R \subseteq A \times B \times C$

↳ Ternary relation.

Given $A_1, A_2, ..., A_n$

CP $A_1 \times A_2 \times ... \times A_n$

$\{(a_1, a_2, ..., a_n) \mid a_i \in A\}$

$R \subseteq A_1 \times A_2 \times \ldots \times A_n$

$\hookrightarrow$ n-ary representation.

$A = \{1, 2, 3\}$

$A \times A = \{(1,1)(2,2)(3,3)(1,2)(1,3)(2,3)(2,1)(3,1)(3,2)\}$

$R \subset A \times A$ $\quad R_1 = \phi$

$R_2 = A \times A$

$R_3 = \{(1,1), (2,2)\}$

$R_4 = \{(1,1), (2,2), (3,3)\}$

$R_5 = \{(1,2), (2,1), (2,2)\}$

$R_6 = \{(1,3)\}$

We shall work with binary relations.

<u>Properties of a relation</u>

<u>Reflexive:</u> R is ref.

$\forall a \in A \quad (a,a) \in R$

<u>Symmetric:</u>

$\forall a, b \in A \left((a,b) \in R \rightarrow (b,a) \in R \right)$

<u>Transitive:</u>

$\forall a, b, c \in A \left(\begin{matrix} (a,b) \in R \\ \wedge \\ (b,c) \in R \end{matrix} \rightarrow (a,c) \in R \right)$

<u>R is antisymmetric</u>

$\forall a, b \in A$

11/Sept/18

⇒ Pick all that are true.

$A = \{1, \{1\}, \emptyset, \{1,2\}, 2, \{\emptyset\}\}$

$1 \in A$ belongs to set (element) $\qquad \emptyset \subset A$

$\{1\} \subset A$ subset $\qquad \emptyset \in A$

$\{1\} \in A \qquad \{\emptyset\} \in A$

$\{1,2\} \in A \qquad \{\emptyset\} \subset A$ X

$\{1,2\} \subset A$

$A = \{1,2,3\}$

$R_1 = \{(1,1), (2,2), (3,3)\}$

$R_2 = \{(1,1)\ (2,2)\}$

$R_3 = \{(1,1)\ (2,1)\ (1,2)\}$

$R_4 = \{(1,2)\}$

$R_5 = \emptyset$

$R_6 = A \times A$

$A \times A = \{(1,1)(2,2)\ (3,3)\ (1,2)(2,1)\ (1,3)\ (3,1)\ (2,3).(3,2)\}$

Reflexive $\quad \forall_{a \in A}\ (a,a) \in R$

Symmetric $\quad \forall_{a,b \in A} ((a,b) \in R \rightarrow (b,a) \in R)$

(or)

$$\forall_{a,b \in A} \left(\begin{matrix} (a,b) \in R \\ \wedge \\ (b,a) \in R \end{matrix} \rightarrow (a,c) \in R \right)$$

Transitive $\quad$ ~~$\forall_{a,b \in A}$~~

$$\forall_{a,b,c \in A} \left(\begin{matrix} (a,b) \in R \\ \wedge \\ (b,c) \in R \end{matrix} \rightarrow (a,c) \in R \right)$$

Antisymmetric

$$\begin{matrix} (a,b) \in R \\ \wedge \\ (b,a) \in R \end{matrix} \longrightarrow a = b$$

Binary relations $\rightarrow \subseteq A \times A$

binary relations = # subsets in $A \times A$

$|A \times A| = n^2$

$\rightarrow 2^{n^2}$

	Reflexive	Symmetric	Transitive	Antisymmetric	Asymm.
R_1	✓	✓	✓	✓	X
R_2	X $(3,3) \notin R$	✓	✓	✓	X
R_3	X	✓	X $(2,2) \notin R$	X $(1,2) \in R, (2,1) \in R$ but $1 \neq 2$	X
R_4	X	X $(2,1) \notin R$	✓	✓	✓
R_5	X	✓	✓	✓	✓
R_6	✓	✓	✓	X	X

$A = \{1, 2, 3\}$

$R = \{(1,3), (2,1) (1,2)\}$

↑

not symmetric

$(3,1) \in R$ but $(1,3) \notin R$

Not symmetric → can we conclude antisymmetric

No $(1,2) \wedge (2,1) \in R$ but $1 \neq 2$

Not antisymmetic.

Asymmetric : $(a,b) \in R \rightarrow (b,a) \notin R$.

not symmetric → antisymmetric

antisymmetric → asymmetric

complete previous class ~~incomplete~~ notes too.

$\Rightarrow$ when $A=\emptyset$

$A \times A = \emptyset$

$R \subseteq A \times A \quad R=\emptyset$

R✓ S✓ T✓ A✓ A✓

$A \neq \emptyset \quad \exists! a \in A, \ (a,a) \in R$

R is reflexive

R is not asymmetric.

$A=\emptyset$

$R=\emptyset$

$A=1 \quad A \times A = \{(1,1)\}$

$\cancel{R \subseteq \{\{(1,1)\}\}} \quad R=\{(1,1)\}$

$A=2 \quad A \times A = \{(1,1)(2,2)(1,2)(2,1)\}$

$R_1 = \{(1,1)(2,2)\}$

$R_2 = \{(1,1)(2,2)(1,2)\}$

$R_3 = \{(1,1)(2,2)(2,1)\}$

$R_4 = \{(1,1)(2,2)(2,1)(1,2)\}$

$A=3 \quad A \times A = \{(1,1)(2,2)(3,3) \dots \}$

$R_1 = \{(1,1)(2,2)(3,3)\}$

$R_2 = \{(1,1)(2,2)(3,3)\} \cup \{(1,2)\}$

$R_3 = R_1 \cup \{(2,1)\}$

$R_4 = R_1 \cup \{(1,3)\}$

$R_5 = R_1 \cup \{(3,1)\}$

$\vdots$

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$$AXA = \{ (1,1)\ (2,2)\ (3,3) \ldots (n,n)$$
$$(1,2)\ (2,1)$$
$$(1,3)\ (3,1) \ldots$$

$$^{n^2-n}C_0 + {}^{n^2-n}C_1 + {}^{n^2-n}C_2 + \ldots + {}^{n^2-n}C_{n^2-n} = 2^{n^2-n}$$

$\cancel{2^{\frac{n^2-n}{2}+n}}$ 2

$\frac{n^2-n+n}{2}$

$\cancel{2^{\frac{n^2+n}{2}}}$

14th Sept 18

Counting

$|A| = n \qquad R \subseteq A \times A$

Count the no. of binary relations ~~those~~ that are reflexive + symmetric.

Recap: # reflexive relations.

(1,1)			
	(2,2)		
		(3,3)	
			(4,4)

In any reflexive relations, all diagonal entries must be there.

$(A \times A) - \text{Diag}$
$= n^2 - n$

$E = \{(x,x) \mid x \in A\}$.

$(R_i \subseteq A\times A - \text{Diag}) \cup \text{Diagonal entries}.$
is a reflexive relation

$\# R_i = \#$ subsets in $A\times A - \text{Diag}$
$= 2^{n^2-n}$

Count no. of symmetric relations.

$(1,2)\ (1,3)\ (2,3)$
$(2,1)\ (3,1)\ (3,2)$

$\cancel{2^{\frac{n^2-n-1+n}{2}}} = \cancel{2^{\frac{n^2+1}{}}} \quad \cancel{2^{\frac{n^2}{2}}}$

Size of lower Δle (upper Δle is forced) $= \frac{n^2-n}{2}$

Any subset of lower Δle is a symmetric relation (upper Δle is forced) $= 2^{\frac{n^2-n}{2}}$

symmetric relat^{n}s $S_1 = \{(1,2),(2,1),(3,2)(2,3)$ $(1,1)$ $(2,2)\}$

$=$ # subsets in Lower Δle $\cup$ Any subset from E (diag pai

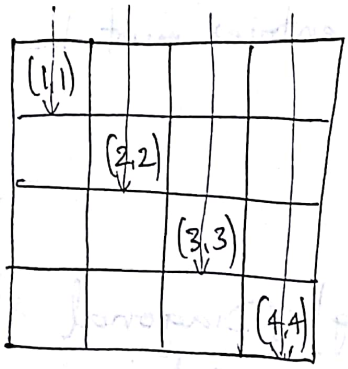
$$2^{\frac{n^2-n}{2}} \times 2^n = 2^{\frac{n^2+n}{2}}$$

(*) # symmetric ~~reat~~ relations = # bin relat^{n}s $-$ # not symme

---

another method for finding the no. of symm. relat^{n}s.

$1+2+3+\dots+n$



entries from col C_i ~~Counting~~ Contributing to 'symm count' is i

$$\# \text{Ent} = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Any subset from 'symm count' is a symm. relations.

$$= 2^{\frac{n^2+n}{2}}$$

# of anti-symmetric relations.

(a,b)

$\wedge$ $\quad a=b$

(b,a)

(a,a)

$\forall a,b \ (a,b)$ $\quad (1,2)$ $(2,3)$

$\neg(b,a)$ $\quad (3,2)$

$\neg(a,b)$ $\quad (b,a)$

$\neg(b,a)$ $\quad \neg(a,b)$

# of antisymmetric relations.

$3C_1$	$3C_1$	$3C_1$		$3C_1$
(1,2)	(1,3)	(1,4)	. . .	$a, \frac{n^2-n}{2}$
(2,1)	(3,1)	(4,1)		$a, \frac{n^2-n}{2}$
none	none	none		none

$R = \{(1,2)(4,1) \dots\}$

~~# Binary~~

$\# \text{Rings} = \frac{n^2-n}{2}$ $\qquad \underbrace{3C_1 \times 3C_1 \times 3C_1 \times \dots \times 3C_1}_{\frac{n^2-n}{2}} = 3^{\frac{n^2-n}{2}} \times 2^n$

Ans. $3^{\frac{n^2-n}{2}} \cdot 2^n \leq 2^{n^2}$

COMBINATORICS — Deals with counting.

18/Sept/18.

Closure of a relation:

reflexive closure — symmetric closure — transitive closure.

Closure : A minimal super relation (super set) which satisfies the property (R, S, T)

Given R is a relation

Reflexive closure of R is a relation R'

1. R' is reflexive
2. $R' \supseteq R$
3. If there is a reflexive relation, $R'' \supseteq R$, then $R'' \supset R'$

Similarly, ~~sth~~ symmetric, closure and transitive closures

* C 1 11

$A = \{1, 2, 3\}$

$R_1 = \{(1,1)\ (2,2)\ (1,3)\ (2,1)\ (1,2)\ (2,3)\}$

$R_2 = \phi$

$R_3 = \{(1,1)(2,2)(3,3)\}$

$R_4 = A \times A.$

eg. R: reflexive.

R: not reflexive.

$R \cup \{(a_1, b_1)\ (a_2, b_2) \dots\}$ is reflexive.

super relation of R.

There are many ~~of~~ super relations that are possible, but we concentrate on the minimal of all the super relations that are possible.

$R = (1,1)$

R_1' (MIN): $(1,1)$, $(2,2)$, $(3,3)$

R_2': $(1,1)$, $(2,2)$, $(3,3)$, $(1,2)$

R_3': $A \times A$

From the above question. (reflexive closure).

(i) $R_1' = R_1 \cup \{(3,3)\}$

(ii) $R_2' = R_2 \cup \{(x,x) \mid x \in A\}$

(iii) $R_3' = R_3$

(iv) $R_4' = R_4$

⇒ Symmetric closure: Symmetric closure of R is a minimal closure ~~good~~ relation R'

(1) R' is symmetric

(2) $R' \supseteq R$

(3) If there is a symmetric relation $R'' \supseteq R$ then $R'' \supset R'$.

⇒ from the given question (symmetric closure).

(i) $R_1' = R_1 \cup \{(3,2)\ (3,1)\}$

(ii) $R_2' = R_2 \Rightarrow S(R_2) = R_2 = R'$

(iii) $S(R_3) = R_3$

(iv) $S(R_4) = R_4$

⇒ Transitive closure:

convert $R_1 \longrightarrow$ transitive relatn.

$T(R_1)$ is a relation R'

$\Rightarrow R' \supseteq R_1$

$T(R_1) = \{(1,1)\ (2,2)\ \cancel{(3,3)}\ (1,3)\ (2,1)\ (1,2)\ (2,3)\}$

Is R_1 transitive ✓

$T(R_2)$ ✓

$T(R_3)$ ✓

$T(R_4)$ ✓

(*) A pure reflexive relation is transitive.

$A = \{1, 2, 3, 4\}$

$R = \{(1,2)(2,3)(3,4)\}$

Is R transitive X

$T(R) = \{(1,2)(2,3)(3,4)(1,3)(2,4)(1,4)\}$

$A = \{1, 2, 3, 4\}$

$R = \{(1,2)(2,3)(3,4)(4,1)\}$.

Is R transitive X

$T(R) = \{(1,2)(2,3)(3,4)(4,1)(1,3)(2,4)(1,4)(4,4)(3,1)(2,1)$
$(1,1)(4,3)(3,3)(3,2)(2,2)(4,2)\}$

Given R.

R is not transitive

$$\left.\begin{array}{l} R \cup \{(a,b)\} \text{ X trans.} \\ R \cup \{(a,b)(b,b)\} \text{ X trans.} \\ R \cup \{(a,b)(b,b)(c,b)\} \text{ X trans.} \\ R \cup \{(a,a)(b,b)(c,b)(c,c)\} \text{ stop.} \end{array}\right\} \begin{array}{l} \text{\# iterations.} \\ = \text{is it finite} \end{array}$$

→ R can always be converted to T(R)?

Can I always find T(R).

Q) How many different graphs are there on 'n' nodes.

Graph models.

Representing a binary relation as a graph.

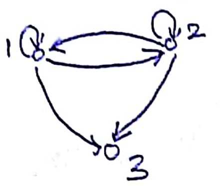
Each element in A gets a vertex/node.

If $(a,b) \in R$, then draw $a \rightarrow b$

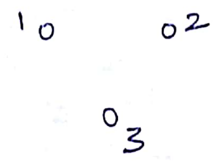
directed edge between 'a' and 'b'..

self arcs.

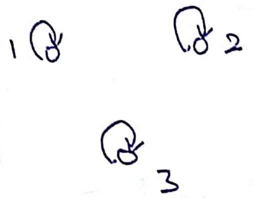
G_1



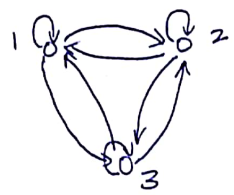
G_2



G_3



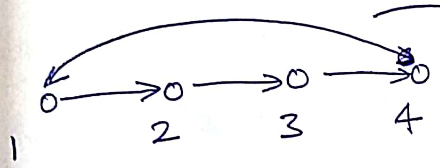
G_4



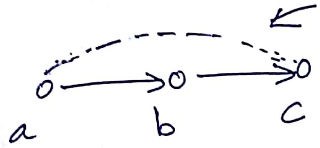
(3,4) (4,1)

R as a graph.

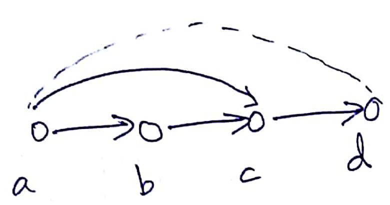
Link/edge



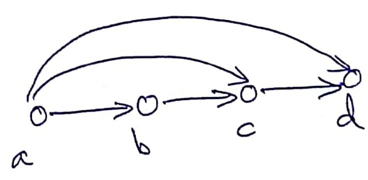
missing if R is not transitive.



$\downarrow T(R)$



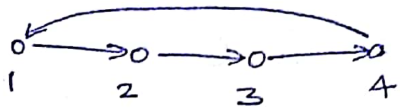
$\xrightarrow{T(R)}$



les.

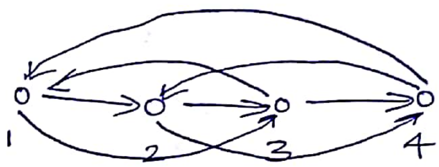
Introduce (a,c) trans arc

Start R



Look for paths of length = 2; Add transitive arcs if missing.

R_1



check if R_1 is trans.

NO

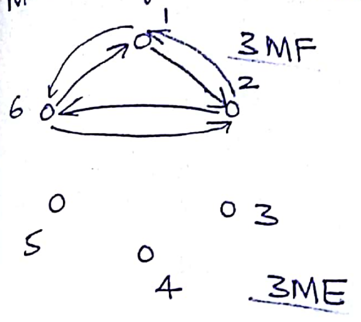
R_2

Work with R_1

Look for P_2 1

ADD trans ABCD

Q) 3 Mutual friends and 3 mutual enemies.



20/9/18

Equivalence relation.

R is an equivalence relation if R is reflexive
symmetric
transitive

Partial order

R is a partial order if R is reflexive
antisymmetric
transitive.

Operations on relations.

R_1 and R_2 are binary relations.

Possible ops. $R_1 - R_2$, $R_1 \cap R_2$, $R_1 \cup R_2$, R_1^c

---

$A = \{1, 2, 3, 4\}$

$R_1 = \{(1,1)(2,2)(3,3)(4,4)\}$

$R_2 = \emptyset$

$R_3 = \{(1,1)(2,2)(3,3)(4,4)(1,2)(2,1)(3,4)(4,3)\}$

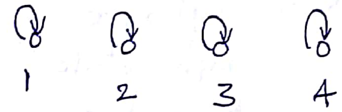
$R_4 = \{(1,1)(2,2)(3,3)(4,4)(1,2)(2,1)(2,3)(3,2)(1,3)(3,1)\}$

$R_5 = \{(1,1)(2,2)(3,3)(4,4)(1,2)(3,4)\}$

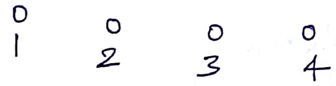
	Equivalence R S T	Partial order R A T
R_1	✓ ✓ ✓	✓ ✓ ✓
R_2	✗	✗
R_3	✓ ✓ ✓	✓ ✗ ✓
R_4	✓ ✓ ✓	✓ ✗ ✓
R_5	✗	✓ ✓ ✓

If R is a partial order, then the underlying set (A) is poset.

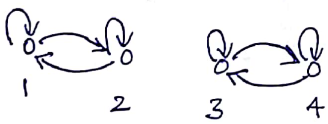
R_1



R_2

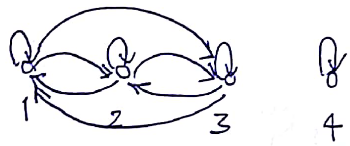


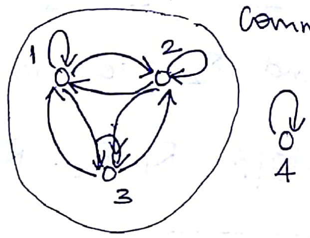
R_3



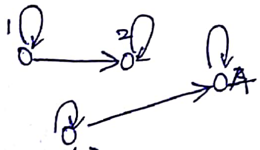
Equivalence class
community

R_4





R_5



q if p $\equiv$ p $\to$ q

eqn rel^n $\to$ RST

$\boxed{\text{eqn rel}^n \leftrightarrow \text{RST}}$

When it comes to definition.

If R is R, S, T $\longrightarrow$ equivalence relation (captures YES cases)

If R $\neg R \vee \neg S \vee \neg T \longrightarrow \neg$equivalence relat^n (captures NO cases)

do not take the contrapositive of this.

Counting equivalence relations

⊛ Do counting for no of partial orders.

$A = \{1\} = \{(1,1)\}$ 1

$A = \{1,2\} = \{(1,1)(2,2)\}$

$\{(1,1)(2,2)(1,2)(2,1)\}$ 2

$A = \{1,2,3\}$ $R_1 = \{(1,1)(2,2)(3,3)\}$

$R_1 \cup \{(1,2)(2,1)\}$

$R_1 \cup \{(2,3)(3,2)\}$

$R_1 \cup \{(3,1)(1,3)\}$ 5

$A \times A$

$A = \{1,2,3,4\}$ $R_1 = \{(1,1)(2,2)(3,3)(4,4)\}$

$R_2 = R_1 \cup \{(1,2)(2,1)\}$

$R_3 = R_1 \cup \{(1,3)(3,1)\}$

$R_4 = R_1 \cup \{(2,3)(3,2)\}$

$R_5 = R_1 \cup \{(1,2)(2,1)(2,3)(3,2)\}$

$R_6 = R_1 \cup \{(1,2)(2,1)\ (1,3)(3,1)\ (1,4)(4,1)(2,4)(4,2)\}$

$R_7 = R_1 \cup \{(1,4)(4,1)\}$

$R_8 = R_1 \cup \{(2,4)(4,2)\}$

$R_9 = R_1 \cup (3,4)(4,3)\}$

$R_{10} = R_1 \cup \{(2,3)(3,2)(3,4)(4,3)(2,4)(4,2)\}$

$R_{11} = A \times A$

$R_{12} = R_1 \cup \{(1,4)(4,1)(3,4)(4,3)(1,3)(3,1)\}$

$\cancel{R_{13} = R_1 \cup \{(1,2)(2,1)(1,3)(3,1)(2,3)(3,2)\}}$

Q) Whose cardinality is greater
(i) integers, natural no.
(ii) integers, rational no.

25th Sept '18

⟹ Counting equivalence relations.

		No.
$\|A\|=1$	$R=\{(1,1)\}$	1
$\|A\|=2$	$R_1=\{(1,1)\ (2,2)\}$ $R_2=\{(1,1)\ (2,2)\ (1,2)\ (2,1)\}$	2
$\|A\|=3$		5
$\|A\|=4$		15
$\|A\|=5$		53

~~$|A|=3$~~

Equivalence class (defined for equivalence relation)

$\forall a \in A$, eqn class $[a]_R$ is defined

$$[a]_R = \{x \mid (a,x) \in R\}$$

$R = \{(1,1)(2,2)(3,3)\}$

$[1]_R = \{1\}$

$[2]_R = \{2\}$ all are distinct.

$[3]_R = \{3\}$

$A = \{1,2,3\}$

$R = \{(1,1)(2,2)(3,3)(1,2)(2,1)\}$

$[1]_R = \{1,2\}$

$[2]_R = \{1,2\}$

$[3]_R = \{$

$R = A \times A$

$[1]_R = \{1, 2, 3, 4\}$

$[2]_R = \{1, 2, 3, 4\}$

$[3]_R = \{1, 2, 3, 4\}$

$[4]_R = \{1, 2, 3, 4\}$

all are same.

Observations

$A = \{1, 2, 3, 4\}$

$R = \{(1,1)(2,2)(3,3)(4,4)(1,2)(2,1)(3,4)(4,3)\}$

$(2,3)\ (3,2)$

$[1]_R = \{1, 2\}$

$[2]_R = \{1, 2\}$

$[3]_R = \{3, 4\}$

$[4]_R = \{3, 4\}$

$1, 2, 3, 4, 5$

$(1,1)(2,2)(3,3)(4,4)(5,5)$

$(1,2)(2,1)(3,4)(4,3)$

$(1,5)(5,1)(2,5)(5,2)$

$(1,4)(4,1)$

$(1,3)(3,1)$

$(5,3)(3,5)$

$(5,4)(4,5)$

$[1]_R = \{1, 2, 5, ^{4,3}\}$

$[2]_R = \{2, 1, 5\}$

$[3]_R = \{3, 4\}$

$[4]_R = \{3, 4\}$

$[5]_R = \{1, 2, 5, ^{3,4}\}$

~~Pf:~~ Ob 1: If $b \in [a]_R$ then $a \in [b]_R$

$b \in [a]_R$

by definition

$(a, b) \in R$

R is an equivalence relation

$\Rightarrow$ R is symmetric

$(a,b) \in R \Rightarrow (b,a) \in R$

By def. $a \in [b]_R$

(*) Prof: Bruce Berndt of UIUC

(*) Ramanujan's 'LOST NOTE BOOK'.

Ob2: $\forall a \in A$ $\quad (a,x)\ (a,y)$

$a \in [a]_R$ $\quad (x,a)$

$\because R$ is reflexive.

Ob3: $b \in [a]_R$

$\wedge$

$c \in [b]_R$

$\Rightarrow c \in [a]_R$

Proof: By def $(a,b) \in R \wedge (b,c) \in R$

$\Rightarrow (a,c) \in R$

$c \in \cancel{\notin} [a]_R$

Ob4: $a_1, a_2, \ldots, a_k \in [a]_R$

$a_i \in [a_j]_R$

$\forall a,b \in A$

Ob5: $[a]_R \cap [b]_R = \emptyset$

or

$[a]_R = [b]_R$

Proof: Given: $[a]_R, [b]_R$

If $[a] \cap [b] = \emptyset$

then DONE

otherwise $[a] \cap [b] \neq \emptyset$

We now claim that

$[a]_R = [b]_R$ $\quad A = B$

$A \subseteq B$

$x \in [a]_R$ $\quad B \subseteq A$

By def: $(a,x) \in R$ $\quad$ x is arbitrary.

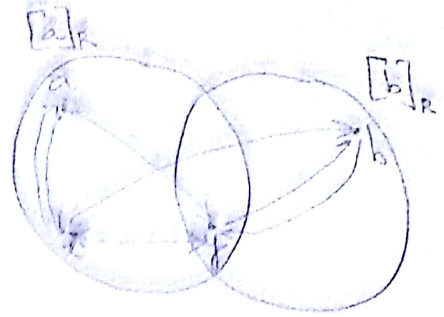
~~since a[a] ∩ b[~~

since $[a] \cap [b] \neq \emptyset$

$\exists y \in [a] \cap [b]$

Clearly, $y \in [a]$
$\wedge$
$y \in [b]$

$(a,y) \in R$



$(a,x) \in R \wedge (a,y) \in R$

$y \in [b]_R$

$(b,y) \in R$

by symm

$(y,b) \in R$

by symmetry

$(x,a) \in R \wedge (a,y) \in R$

by transitivity

$(x,y) \in R \longrightarrow (x,b) \in R \Rightarrow x \in [b]_R$

Universal generalization $[a]_R \subseteq [b]_R$

An argument similar to

$[a] \subseteq [b]$ establishes

$[b] \subseteq [a]$

$\rightarrow [a] = [b]$

$\bigcup [a]_R = A$

→ Think of another observation.

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27th Sept'18

$A = \{1,2,3,4\}$

$R_1 = \{(1,1)(2,2)(3,3)(4,4)\}$

$[1]_{R_1} = \{1\}$

$[2]_{R_1} = \{2\}$

$[3]_{R_1} = \{3\}$

$[4]_{R_1} = \{4\}$

$R_2 = \{(1,1)(2,2)(3,3)(4,4)(1,2)(2,1)\}$

$[1]_{R_2} = \{1,2\}$

$[2]_{R_2} = \{1,2\}$

$[3]_{R_2} = \{3\}$

$[4]_{R_2} = \{4\}$

$R_3 = \{(1,1)(2,2)(3,3)(4,4)$
$(1,2)(2,1)(3,4)(4,3)\}$

$[1]_{R_3} = \{1,2\}$

$[2]_{R_3} = \{1,2\}$

$[3]_{R_3} = \{3,4\}$

$[4]_{R_3} = \{3,4\}$

$R_4 = A \times A$

$[1]_{R_4} = \{1,2,3,4\}$

$[2]_{R_4} =$ "

$[3]_{R_4} =$ "

$[4]_{R_4} =$ "

Discrete Math by Kamala Madan

$\{1,2,3,4\}$

R_1: $\{1\},\{2\},\{3\},\{4\}$

R_2: $\{1,2\},\{3\},\{4\}$

R_3: $\{1,2\},\{3,4\}$

R_4: $\{1,2,3,4\}$

R_5: $\{1,3\}\ \{2,4\}$

R_6: $\{1,4\},\ \{2,3\}$

R_7: $\{1,2,3\},\{4\}$

Model Set 1. All possible integral solutions

$$x_1+x_2+x_3+x_4=4$$

$1+1+1+1$

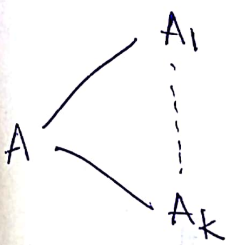
$0+0+2+2$

$3+0+1+0$

$\vdots$

Model 2: Partition.

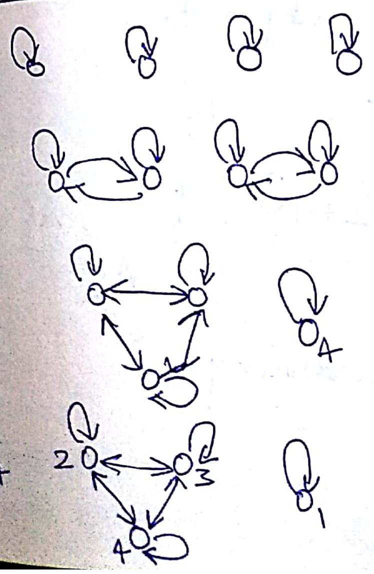
~~$R_4 = A \times A$~~ ~~$[1]_{R_4} = A$~~



(decompose) $\displaystyle\biguplus_{i=1}^{k} A_i = A$

equivalence relation on A $\equiv$ # partitions of A.

Model 3:



R_1 $\{1\}\{2\}\{3\}\{4\}$

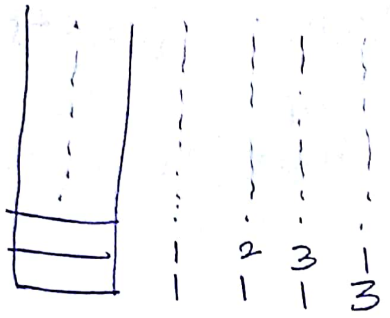
R_2 1,2 3,4

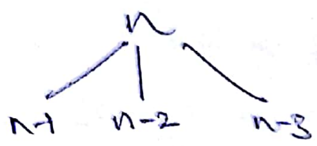
R_3 1 2 3 4

R_4 2 3 4 1

- # equivalence classes = # partitions

Revisit: monkey problem.





$T(n)$: $T(n-1) + T(n-2) + T(n-3)$

Recurrence relation.

slowly the value of n would reduce.

$A = \{1,2,3,4\}$.

$\{1,2\}\ \{3,4\}$ $\quad$ $\{1,2,3\}\ \{4\}$

Take a partition

$\{1,2\}\ \{3,4\}$

FIX 4 $\quad$ Identify the sub partition containing '4'.

$\{1,2\}\ \boxed{\{3,4\}}$

$\{1,2\}\ \{\ ,4\}$

$\{1\}\ \{2\}\ \{\ ,4\}$

FIX 4

Partition: $\{1\}\ \{2,3,4\}$

FIX 4 Subpartition

$\{1\}\ \{\ ,4\}$

Fix a partition: $\underset{A_1}{\{1,2\}}\ \underset{A_2}{\{3,4\}}\ \{5,6,7,n/2,\ n/2+1\} \ldots \underset{A_k}{\{\ ,n\}}$

Fix 'n' and identify the subpart containing 'n'

Let $n \in A_k$ — $A_k = \{n\}$

$A_k = \{ \ldots, n\}$

$|A_k| = l \quad l \geq 1$

$\therefore$ The no. of ways of partitioning 'A' in $\sum_{i=1}^{k-1} A_i = n - l$.

Focus on A_k i $\{ \ldots, n\}$

The number of possibilities of in for $A_k \setminus \{n\} = {}^{(n-1)}C_{(l-1)}$

$\underbrace{A_1 \; A_2 \; \ldots \; A_{k-1}}_{n-l} \quad A_k$

$\{ \underbrace{\quad}, n\}$

${}^{(n-1)}C_{l-1}$

no. of ways of partioning a set of $(n-l)$

$$B_n = \sum_{l=1}^{n} B_{n-l} \times {}^{(n-1)}C_{(l-1)}$$

$B_0 = 0, \; B_1 = 1, \; B_2 = 2, \; B_3 = 4, \; B_4 = 15, \; B_5 = 52.$ (Verify)

28th Sept'18

~~Pro~~ ~~Partial order (poset)~~

Recap

Counting equivalence relations.

$A = \{1, 2, 3, 4, 5\}$

Choose a partition, say

$\{1, 2\} \; \{3\} \quad \{4\cancel{\}}, 5\}$

Fix

$\{\_, 5\}$

count the no of partitions

count the no. of possibilities.

Q) Extra credit

Partial order (poset)

R is a partial order if R is reflexive, anti symmetric, transitive.

$A = \{1, 2, \dots, 10\}$

$R = \{(x,y) \mid x \text{ divides } y\}$

y/x leaves remainder = 0

$R = \{(1,1) \dots (10,10) \quad (2,4)(2,6)(2,8)(2,10)$
$(1,2),(1,3) \dots (1,10) \quad (3,6)(3,9)(4,8)(5,10)\}$

Hasse Diagram (Graphical representation of posets)

- undirected graph obtained from (minimal represe

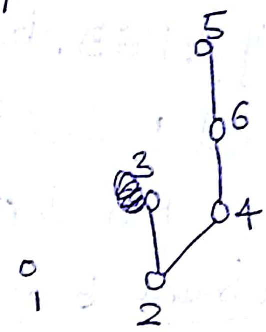
Direct graph rep — reflexive arcs — transitive arcs.

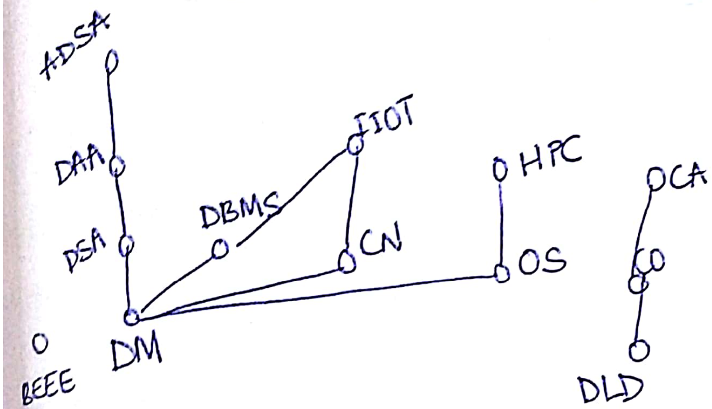
eg. $R = \{(a,b) \mid a \leq b\}$ $a, b \in N$

A = {1 ...

{(1,1) (2,2) ... (6,6)
(2,3) (2,5) (2,4) (6,5) (2,6)
(4,6) (4,5)}

$\cancel{{}^{n-1}C_0 = 1}$

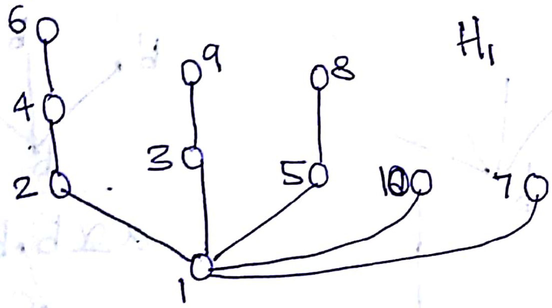


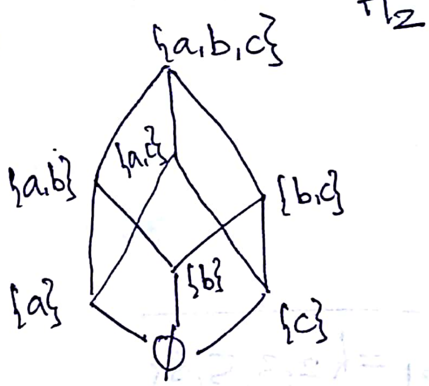


⇒ Posets

Hasse diagram (representation of some relation $\underline{\alpha}$)

refers to some relation.





A poset $B \subseteq A$

<u>Least element</u> $b \in B$ is least if
$\forall b' \in B,\ b \underline{\alpha} b'$

<u>Greatest element</u> $b \in B$ is greatest if
$\forall b' \in B,\ b' \underline{\alpha} b$

<u>Minimum element</u> $b \in B$ is MIN if
$\nexists\ b'_{\in B} \ni b' \underline{\alpha} b$

<u>Maximum element</u> $b \in B$ is MAX if
$\nexists\ b'_{\in B} \ni b \underline{\alpha} b'$

Upper bound $b \in A$ is an upper bound if $\forall b' \in B,\ b' \alpha b$

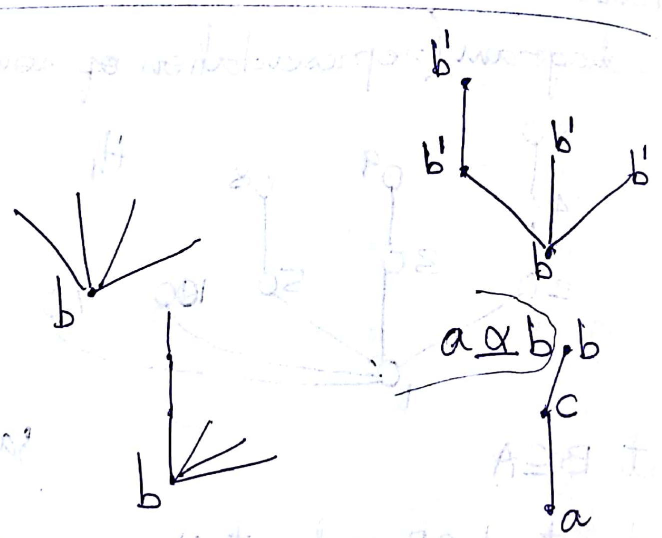
Lower bound $b \in A$ is a lower bound if $\forall b' \in B,\ b \alpha b$

Least upper bound b is least upper bound if b is UB and $\forall$ UB b', $b \alpha b'$

Greatest lower bound b is greatest lower bound if b is LB and $\forall$ LB b', $b' \alpha b$

wrt H_1

$B = \{2, 3, 5, 8\}$



$B_1 = \{2, 3, 5, 8\}$

$2 \not\alpha 3$ no least element

$B_2 = \{1, 3, 8\}$ $1 \alpha 3$

$1 \alpha 8$

least element $= 1$

Greate...

$B_1 = \{2,3,5,8\}$ $3 \not\leq 8$
no greatest element

$B_2 = \{1,3,8\}$ $3 \not\leq 8$
no greatest element

$B_3 = \{1,3,6\}$ least element $= 1$
greatest element $= 6$

---

wrt H_2

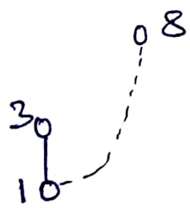
$B_1 = \{\{\phi\}, \{a,b\}, \{b,c\}\}$
least element $= \{\phi\}$ MIN $= \phi$ MAX $= \{\{a,b\},\{b,c\}\}$
greatest element $=$ NIL $\{a,b\} \not\leq \{b,c\}$

$B_2 = P(A)$
least $= \{\phi\}$ MIN $= \phi$ MAX $= \{a,b,c\}$
greatest $= \{a,b,c\}$ ~~UB = NIL LB =~~

$B_1 = \{2,3,5,8\}$ MIN $\{2,3,5\}$ MAX $\{2,3,8\}$

UB = NIL, LB $= \{1\}$

$B_2 = \{1,3,8\}$



MIN $\{1\}$ MAX $\{3,8\}$.
UB = NIL
LB $= \{1\}$.

$B_3 = \{1,2,6\}$
least element $= 1$, greatest element $= 6$, UB $= \{6\}$, LB $= \{1\}$.
$B_4 = \{3,8\}$ UB = NIL, LB $= \{1\}$.
$B_5 = \{4,6\}$ UB $= \{6\}$, LB $= \{1,2,4\}$, GLB $= \{4\}$
$B_6 = \{1,2,4\}$ UB $= \{4,6\}$, LUB $= \{4\}$, LB $= \{1\}$, GLB $= \{1\}$.

Observations.

Claim 1: ~~$A, B \notin$~~

$A, B \subseteq A$

If least exists, then MIN = least

If least exists, then it is unique.

(*) To prove all of this, you will have to focus on anti-symmetric property.