Laws of Boolean Algebra

Boolean Algebra is an algebraic structure defined by set of elements ‘B’ together with two binary operators “+” and “.”

Operator ‘+’	Operator ‘ . ’
1. $x + 0 = x$	$x . 1 = x$
2. $x + x' = 1$	$x . x' = 0$
3. $x + x = x$	$x . x = x$
4. $x + 1 = 1$	$x . 0 = 0$
5. $(x')' = x$	
6. $x + y = y + x$	$x.y = y.x$
7. $x + (y + z) = (x + y) + z$	$x. (y. z) = (x.y) . Z$
8. $x + x.y = x$	$x. (x + y) = x$
9. $(x + y)' = x' . y'$	$(x. y)' = x' + y'$ (DeMorgans Theorem)

The operator precedence for evaluating Boolean expression is (1) parenthesis (2) NOT (3) AND and (4) OR

Boolean Function

- Boolean algebra deals with binary variables and logic operations.
- A Boolean function described by an algebraic expression consist of binary variables and logic operation symbols and constants 0 and 1.
- For a given value of binary variables, the function can be either 0 or 1.
  - ex. $F = x + y'z$

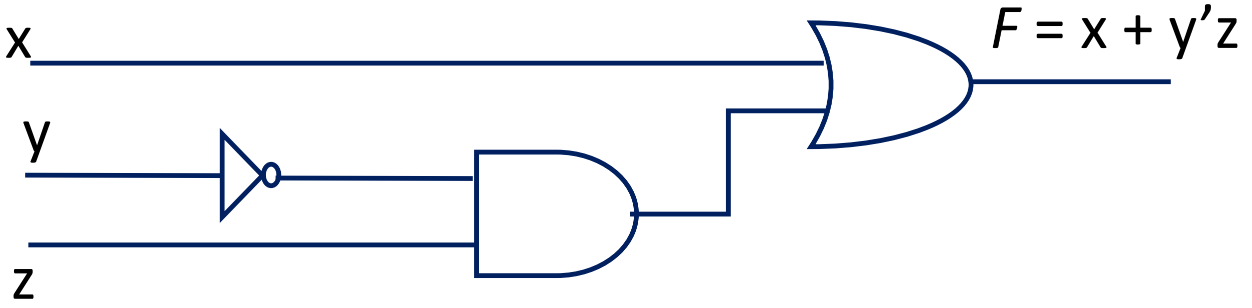
    F is equals to 1 if x =1 or y z = 0 1
- Boolean function expresses logical relationship between binary variables.
- Boolean function can be evaluated by determining the binary value of expression for all possible values of variables
- Boolean function can be transformed from an algebraic expression into circuit diagrams composed of logic gates.

Boolean Function

- There is only one way that a Boolean function can be represented in truth table where as when the function is in algebraic form, it can be expressed in number of ways having equivalent logic.

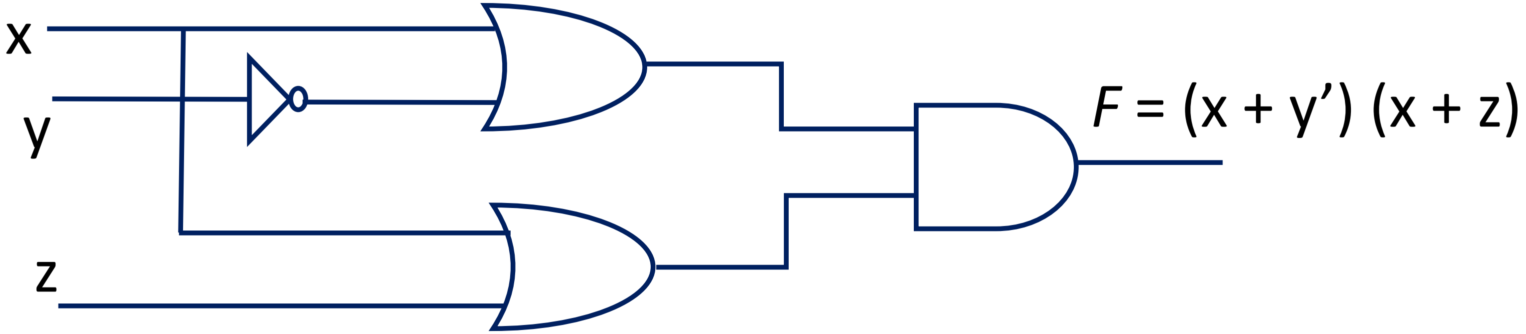
- $F = x + y'z$

x	y	z	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1



- $F = x + y'z$

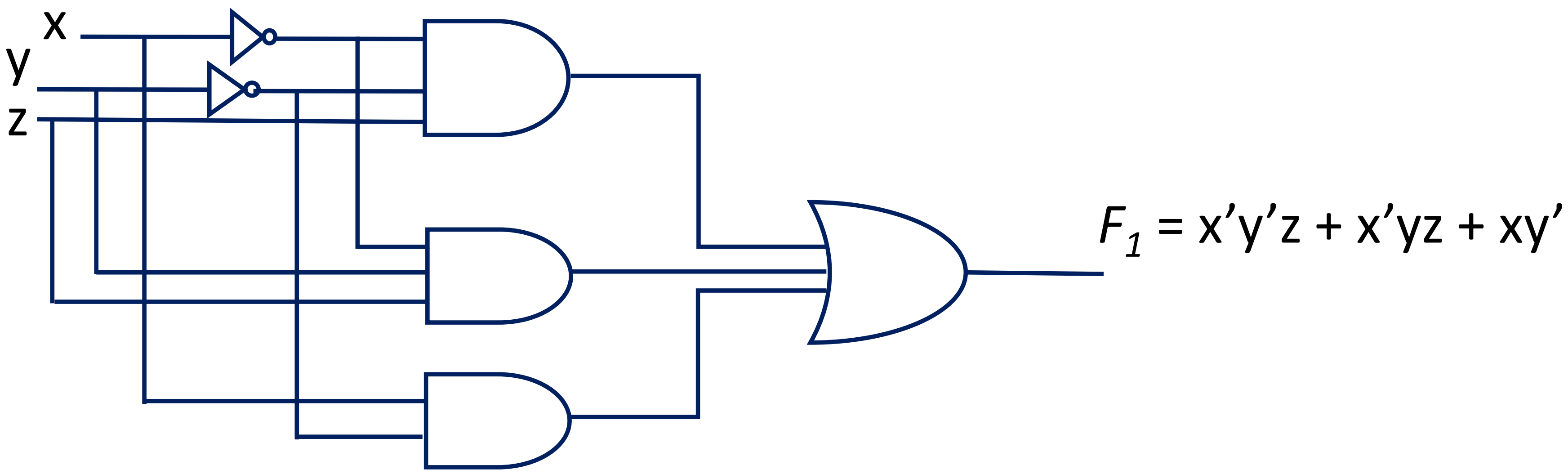
$= (x + y')(x + z)$



Boolean Function

- By manipulating Boolean expression according to rules of Boolean algebra; it is possible to obtained simpler expression for the same function, thus reduce number of gates in circuit and number of inputs to the gate.

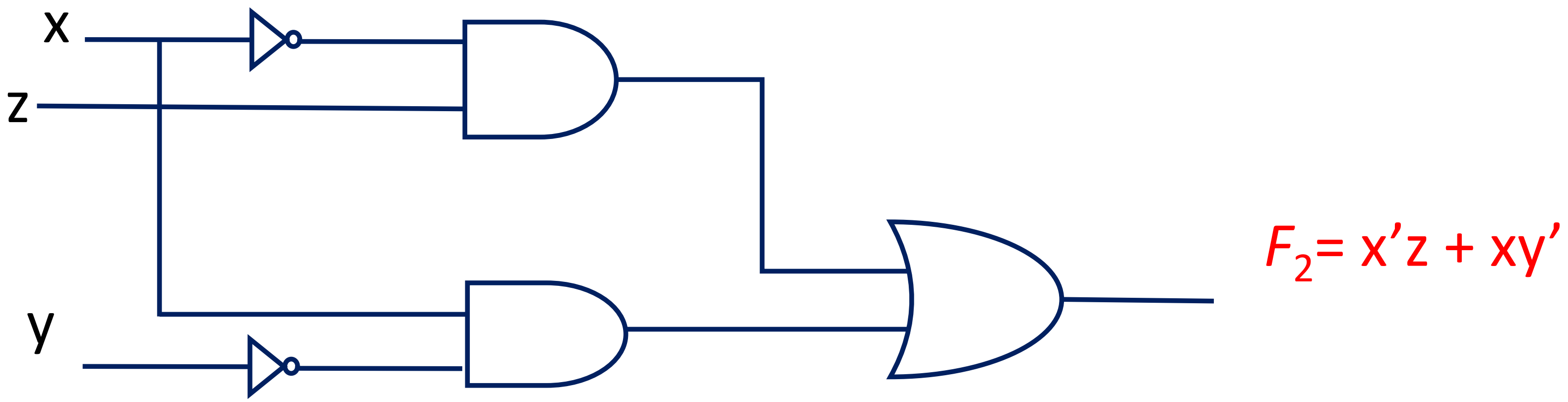
- *Ex.* $F_1 = x'y'z + x'yz + xy'$



- $F_2 = x'y'z + x'yz + xy'$

  $= x'z\,(y' + y) + xy'$

  $= x'z + xy'$



Complement of Function

- Complement of function *F* is *F'* and can be derived algebraically by DeMorgans theorem.
- $(A + B + C)' = A'B'C'$
- The generalized DeMorgans's Theorem: Complement of function is obtained by interchanging AND and OR operator and complementing each literal.

ex. Find the complement of Function $F = x'yz' + x'y'z$

$$F' = (x'yz' + x'y'z)'$$

$$F' = (x'yz')' (x'y'z)'$$

$$F' = (x + y' + z) (x + y + z')$$

Canonical forms

Minterms and Maxterms

- Binary variables may appear in normal (x) or complement forms (x’).
- For two variables x and y combined with AND operator, there are four possible combinations i.e. x’y’, x’y, xy’, xy.
- Each of these four AND-terms is called a *minterm* or standard product.
- *n* variables can be combined to form 2^n minterms.
- Each minterm is obtained from an AND-term of *n* variables, with each variable being primed if the corresponding bit of binary number is 0 and unprimed if it is 1.
- Similarly, *n* variables forming an OR term, with each variable being primed or unprimed, provide 2^n possible combinations called *maxterms* or standard sum.
- Each maxterm is obtained from an OR term of *n* variables, with each variable being primed if the corresponding bit of binary number is 1 and unprimed if it is 0. ex. (x+y, x+y’, x’+y, x’+y’)
- *Each maxterm is complement of its minterm and vice versa.*

Canonical forms

Minterms and Maxterms for three binary variables

			Minterms		Maxterms	
x	y	z	Term	Designation	Term	Designation
0	0	0	x'y'z'	m_0	x+y+z	M_0
0	0	1	x'y'z	m_1	x+y+z'	M_1
0	1	0	x'yz'	m_2	x+y'+z	M_2
0	1	1	x'yz	m_3	x+y'+z'	M_3
1	0	0	xy'z'	m_4	x'+y+z	M_4
1	0	1	xy'z	m_4	x'+y+z'	M_4
1	1	0	xyz'	m_6	x'+y'+z	M_6
1	1	1	xyz	m_7	x'+y'+z'	M_7

Sum of Minterms

- Boolean function can be expressed algebraically, from given truth table by forming minterms of each combination of variable that produces 1 in the function and then taking OR of all the terms (*sum of minterms)*.

x	y	z	F_1	F_2
0	0	0	0	0
0	0	1	1	0
0	1	0	0	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1

- Function F_1 can be determine by expressing combinations 001, 100, and 111 as x'y'z, xy'z' and xyz respectively, since each one of these minterms result in $F_1 = 1$.
- $F_1 = x'y'z + xy'z' + xyz = m_1 + m_4 + m_7$
- Similarly, $F_2 = x'yz + xy'z + xyz' + xyz = m_3 + m_5 + m_6 + m_7$
- *Any Boolean function can be expressed as sum of minterms.*
- Complement of Function can be obtained by forming minterm of each combination that produces 0 in the function and then ORing them.
- Complement of Function F_1 is $F_1' = x'y'z' + x'yz' + x'yz + xy'z + xyz'$

Product of Maxterms

x	y	z	F_1	F_2
0	0	0	0	0
0	0	1	1	0
0	1	0	0	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1

- Complement of Function F_1' is $F_{1.}$
- $F_1 = (x + y + z)(x + y' + z)(x + y' + z')(x' + y + z')(x' + y' + z)$
- $F_1 = M_0 . M_2 . M_3 . M_5 . M_6$
- *Any Boolean function can be expressed as product of maxterms.*
- Boolean function can be expressed algebraically, from given truth table by forming maxterms of each combination of variable that produces 0 in the function and then taking AND of all the terms (*Product of maxterms)*
- *Boolean functions expressed as sum of minterms or product of maxterms are called canonical form.*

Sum of minterms

- Function which is not in sum of minterms form can be made so by

1. first expanding expression into AND-terms.
2. Each term is inspected to see if it contains all the variables.
3. Missing variables are AND ed with an expression (x +x’), where x is missing variable.

Ex. Express function F= A + B’C as sum of minterms.

There are two terms and three variables in the function F.

First term A is missing two variables

A= A (B+B’)

=AB + AB’

=AB(C+C’) + AB’ (C+C’)

=ABC+ABC’+AB’C+AB’C’

Second term B’C is missing one variable

B’C= B’C(A+A’)= AB’C + A’B’C

F = ABC+ABC’+AB’C+AB’C’+ AB’C + A’B’C

F = ABC+ABC’+AB’C+AB’C’ + A’B’C

F = m7 +m6 + m5+m4 + m1

$F(A,B,C) = \Sigma\ (1,4,5,6,7)$

Product of Maxterms

To express Boolean function as a product of maxterm

- Bring the expression in the form of OR terms by distributive law i.e. $x+yz = (x+y)(x+z)$
- Any missing term is then OR ed with $x.x'$, where x is missing variable.

Ex. Express Boolean Function $F = xy + x'z$ as product of maxterms

$$F = xy + x'z$$

$$F = (xy+x')(xy+z)$$

$$F = (x+x')(y+x')(x+z)(y+z)$$

$$F = (y+x')(x+z)(y+z)$$

$(y+x') = (y+x'+zz') = (x'+y+z)(x'+y+z')$

$(x+z) = (x+z+yy') = (x+y+z)(x+y'+z)$

$(y+z) = (y+z+xx') = (x+y+z)(x'+y+z)$

$$F = (x'+y+z)(x'+y+z')(x+y+z)(x+y'+z)(x+y+z)(x'+y+z)$$

$$F = (x'+y+z)(x'+y+z')(x+y+z)(x+y'+z)(x'+y+z)$$

$= M0. M2.M4.M5$

$F(x,y,z) = \Pi(0,2,4,5)$

Standard form

- Canonical forms of Boolean algebra are very seldom with least number of literal because each minterms and maxterms must contain all the variables.
- Standard form is another way to express Boolean function.
- Terms of that forms the function may contain any number of literals.
- Two types of Standard forms: Sum of products and Product of Sum
- Sum of product is Boolean expression containing AND-terms, with one or more literals each.

$F_1 = y' + xy + x'yz'$

- Product of sum : Boolean expression containing OR-terms called sum terms

$F_1 = x\,(y'+z)\,(x'+y+z')$