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COM205T Discrete Structures for Computing-Lecture Notes

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Introduction to Discrete Mathematics and Logic

Objective of this course:

- To have an understanding of the principles of logic.
- To get a flavor of the art of writing proofs.
- To learn to reason and analyze the validity of statements in a given context.
- To get familiar with enumeration and counting problems.
- To have a tour on infinite sets to know about countable and uncountable sets.
- To study the discrete object 'Graphs' and its modeling power in computational problems.

We shall begin this lecture with a set of fundamental questions. And every attempt to answer these questions will take us a long way in understanding this subject better. A deeper understanding of these questions will give a good insight into the relatively young science, namely, computer science.

Some Interesting Questions to Ponder On:

1. How was computer science born?
2. Is there a science in computer science?
3. Like human beings, can a machine think, cook, sing, and drive

Let us first understand, how other fields of science and engineering were discovered over time. A look into this evolution may yield us with some fruitful insights which will help us in answering the above questions. We shall also introduce Logic and its power in expressing facts with no ambiguity.

1 Discoveries (inventions) at a Glance

Zeroth machine age:
This was the time upto early nineteenth century, where there were no noticeable developments in the technology. Preaching and teaching was followed in domestic languages, and knowledge dissemination was not quite high. Some foreign invasions and battles among kingdoms were popular then. Over the years, people became more closer and communications were strengthened. This helped in exchange of knowledge, even though language was a barrier.
Aristotle, a Greek philosopher, who lived in fourth century BC, was one of the pioneers who paved the basics of Logic. He observed general laws of nature which were not immediately perceivable to a common man. Observations from nature helped him to draw probable conclusions, sometimes with undeviating precisions. Perception of physical phenomenon, their causes and their evidence are not derived mathematically. Instead they were outcomes of extensive observations, reasoning and numerous experience of practical verifications. People who do these types

of researches were treated as philosophers in the ancient times. Aristotle was the odd man out, as he not only was interested in giving claims, but also in giving sufficient reasoning. He felt that the observations and conclusions when represented in any natural language will be ambiguous and hence the principles and theorems may not be interpreted in its pure form. Therefore, his investigations were concentrated on the reasoning and representation of principles and theorems (along with the proof), in scientific form. An understanding between the reader and presenter, on the general reasoning and operations presented in the proof, is necessary for its unambiguous reception. Towards this objective, he devised symbols needed for proving claims, mentioned the precise meaning of the symbols and arguments used, and presented the proof. This age is known as zeroth machine age.
In zeroth machine age, knowledge transfer happened among individuals. Precise delivering of knowledge in a human-human interaction required scientific presentation, which was the initial motivation for the study of logic.

First machine age:

Starting from the inventions like nuts, bolts and wheels, this age had witnessed major inventions and discoveries that changed every facet of life. It was an age of globalization. And it was in this age that the major three engineering areas (Electrical, Civil and Mechanical) flourished. Rise of machines necessitated human-machine interaction, which required more precise representation of facts.

After a long gap, in nineteenth century, an English Mathematician and Philosopher George Boole made further research in logic methodologies and took the lead in popularising this area, which was even taught as a subject in Harvard University after his death. He contributed remarkable developments and theories in logical research, that modern logicians are indeed to be inevitably thankful for. His masterpiece includes *Mathematical Analysis of Logic*, *The Laws of Thought*, where he shows that the science of logic is powerful as it is extensively researched upon even today.

Claude Shannon (1916-2001)

"I visualise a time when we will be to robots what dogs are to humans, and I'm rooting for the machines."

Second machine age:

Middle of twentieth century can be considered as the golden time of research, as modern technologies started evolving at this time, where research in atomic particles extended research deep into new specialized areas in technology which included electronics, and computer engineering. Unlike other machines with which humans interacted, computers require extremely precise information. Computers have been used for doing scientific calculations and is considered even now as the greatest technological leap of mankind. Machine-machine interactions observed in this age could not have occurred without the invention of logic gates. As life progresses, automation of each and every processes that we have come across in daily life became the need of the hour. For this purpose, mapping of inputs to logical objects became necessary, so that a practical problem in our daily life is mapped into mathematical domain and is solved using computers. Thus, `Logic` as a language to program computers evolved during this time.

2 Introduction to Logic

Having given motivation for logic in the previous section, we shall now present logic in detail.

Do you know ?

Coloring of regions in a map on a plane such that adjacent regions sharing boundaries (not points) receive different colors can be done with Four colors !!!

Terminologies:
An *Assertion* is a meaningful statement. *Proposition* is an assertion which is either TRUE or FALSE, but not both.
Propositions are denoted using *Proposition variables.*
For example, P : Discrete mathematics is a classical subject in Computer Science
$Q : 2 + 7 = 9.$
If a proposition Q is true, then its *truth value* is TRUE, denoted as truthvalue(Q)=TRUE. Similarly for a FALSE proposition R, truthvalue(R)=FALSE. An *Axiom* is a proposition or statement which is regarded as being established, accepted or self-evidently TRUE.
Theorem is a general proposition which is not self-evident but proved by a chain of reasoning; a truth accepted by means of accepted truths. A logical set of arguments which establishes the theorem to be TRUE is called *Proof. Lemma* is an intermediate theorem in an argument or proof.

Operations on Propositional Variables:
Usually we may handle more than one proposition for a proof which are joined using operators. For ease of representation, propositions are always represented using proposition variables. The commonly used operations on proposition variables are;

- *Negation* of a proposition P negates its truth value, represented as $\neg P$.

- Given two propositions P and Q, *Conjunction* of P and Q, represented as $P \wedge Q$ assumes truth value TRUE when both P and Q are TRUE, otherwise FALSE.

- *Disjunction* of P and Q, denoted as $P \vee Q$ has a truth value TRUE if any one among P or Q is TRUE. Here 'or' is defined in inclusive sense. In the following truth tables, T denotes TRUE and F, FALSE.

- *Implication* or *conditional* of two propositions P, Q is represented as $P \rightarrow Q$, which evaluates to be TRUE only when P is TRUE and Q is FALSE. P is called *premise*, *antecedent* or *hypothesis* of the implication and Q is termed as *conclusion* or *consequent.* Note that the implication assert that *if* the premise is TRUE, *then* the conclusion is TRUE. It also says that if conclusion is FALSE, then premise is also FALSE. Other than the above two assertions, nothing more can be inferred from implication. Observe that $P \rightarrow Q$ is equivalent to the below assertions.

  – if P, then Q
  –Q follows from P
  –Q is necessary for P
  –Q whenever P
  –P only if Q
  –P is sufficient for Q
  –Q if P
  –Q unless $\neg P$

$-Q$ provided P

$-Q$ is a consequence of P

– *Biconditional* of two propositions P and Q, represented as $P \leftrightarrow Q$ is read as "P *if and only if* Q". $P \leftrightarrow Q$ is the conjunction of $P \rightarrow Q$ and $Q \rightarrow P$. Therefore $P \leftrightarrow Q$ has truth value TRUE when both P and Q has same truth values, and FALSE otherwise. The following statements are equivalent.

  – P if and only if Q
  – P is necessary and sufficient for Q
  – if P, then Q and conversely if Q, then P.

Let R denote the proposition $P \rightarrow Q$. *Converse* of R is defined as $Q \rightarrow P$ and *Inverse* of R is defined by the implication $\neg P \rightarrow \neg Q$. *Contrapositive* of a proposition R is defined by the implication $\neg Q \rightarrow \neg P$. Note that the propositions constructed using the above mentioned operations are also propositions. Two propositions are said to be logically equivalent if they have same truth table. Observe that $P{\rightarrow}Q$ is equivalent to $\neg Q{\rightarrow}\neg P$, represented as $P{\rightarrow}Q \equiv \neg Q{\rightarrow}\neg P$. Similarly, $Q{\rightarrow}P \equiv \neg P{\rightarrow}\neg Q$. The above results may be verified using truth tables.

TRUTH TABLE

P	$\neg P$
T	F
F	T

TRUTH TABLE

P	Q	$P \wedge Q$	$P \vee Q$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	F

IMPLICATION

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

BICONDITIONAL

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Compound proposition is a proposition formed from the propositional variables using the above defined operations. A *tautology* is a compound proposition whose truth value is TRUE for all values of its propositional variables contained in it. For example, $P \vee \neg P$ is a tautology. A *contradiction* or *absurdity* is a compound proposition whose truth value is FALSE for all values of its propositional variables contained in it. For example, $P \wedge \neg P$ is a contradiction. A *contingency* is a compound proposition which is neither a tautology nor a contradiction.

Do [1]Exercise 1.5

Reading Assignment 1. Paradox 2. QED
3. Google Aristotle, George Boole, Alan Turing 4. Articles from the text 1^{st}, 2^{nd} machine age, The Laws of Thought.

[1] J.L.Mott, A.Kandel, and T.P.Baker: Discrete Mathematics for Computer Scientists and Mathematicians, PHI.

Logical representation and mechanism of logical reasoning acts as the basics of scientific investigation. It is to be noted that extensive and deep scientific research involves a large number of propositional variables. Compound propositions in these representations can be simplified using logical identities and implications. In this section, we point out some of the well-known logical identities and implications.

Is this right ?

Everyone loves my baby.
My baby only loves me.
$\therefore$ *I am my own baby.*

Logical Identities

Two compound propositions are said to be equivalent or identical if they have same truth values for every possible truth assignment of its propositional variables, i.e., two propositions p and q are identical if $p \leftrightarrow q$ is a tautology ($p \leftrightarrow q$ means p if and only if q).

Equivalence		Name
$p \leftrightarrow (p \vee p)$	$p \leftrightarrow (p \wedge p)$	Idempotence
$p \vee q \leftrightarrow q \vee p$	$p \wedge q \leftrightarrow q \wedge p$	Commutativity
$(p \vee q) \vee r \leftrightarrow p \vee (q \vee r)$	$(p \wedge q) \wedge r \leftrightarrow p \wedge (q \wedge r)$	Associativity
$\neg(p \vee q) \leftrightarrow \neg p \wedge \neg q$	$\neg(p \wedge q) \leftrightarrow \neg p \vee \neg q$	De-morgans Law
$p \wedge (q \vee r) \leftrightarrow (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \leftrightarrow (p \vee q) \wedge (p \vee r)$	Distribution
$p \vee \mathtt{T} \leftrightarrow \mathtt{T}$	$p \wedge \mathtt{F} \leftrightarrow \mathtt{F}$	Domination
$p \vee \neg p \leftrightarrow \mathtt{T}$	$p \wedge \neg p \leftrightarrow \mathtt{F}$	Negation
$p \vee \mathtt{F} \leftrightarrow p$	$p \wedge \mathtt{T} \leftrightarrow p$	Identity
$p \leftrightarrow \neg(\neg p)$		Double negation
$p \vee (p \wedge q) \leftrightarrow p$	$p \wedge (p \vee q) \leftrightarrow p$	Absorption law
$p \rightarrow q \leftrightarrow \neg p \vee q$		Implication
$(p \leftrightarrow q) \leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$		Equivalence
$(p \wedge q) \rightarrow r \leftrightarrow p \rightarrow (q \rightarrow r)$		Exportation
$(p \rightarrow q) \wedge (p \rightarrow \neg q) \leftrightarrow \neg p$		Absurdity
$p \rightarrow q \leftrightarrow \neg q \rightarrow \neg p$		Contrapositive

Implication	Name
$p \rightarrow (p \vee q)$	Addition
$(p \wedge q) \rightarrow p$	Simplification
$[p \wedge (p \rightarrow q)] \rightarrow q$	Modus Ponens
$[(p \rightarrow q) \wedge \neg q] \rightarrow \neg p$	Modus Tollens
$[\neg p \wedge (p \vee q)] \rightarrow q$	Disjunctive Syllogism
$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$	Hypothetical Syllogism
$((p \vee q) \wedge (\neg p \vee r)) \rightarrow (q \vee r)$	Resolution
$(p \rightarrow q) \rightarrow [(q \rightarrow r) \rightarrow (p \rightarrow r)]$	
$[(p \rightarrow q) \wedge (r \rightarrow s)] \rightarrow [(p \wedge r) \rightarrow (q \wedge s)]$	
$[(p \leftrightarrow q) \wedge (q \leftrightarrow r)] \rightarrow p \leftrightarrow r$	

Questions:

– Verify using truth table. $((p \rightarrow q) \rightarrow r) \not\leftrightarrow (p \rightarrow (q \rightarrow r))$

– Prove $\neg(p \leftrightarrow q) \leftrightarrow (p \leftrightarrow \neg q)$

3 Predicate Logic

In the earlier section, we discussed propositional logic (zeroth order logic), its notation, compound expressions involving logical operators and connectives. In this section, we shall discuss predicate logic in detail. Predicate Logic or First order logic are mathematical assertions containing variables which receive values from a specific domain and become proposition once its variables are assigned values from the respective domain. Domain from which predicate variables receive values are termed as *universe of discourse* or *domain of discourse.* Two types of Quantifiers that are of interest are *Universal* and *Existential* quantifiers. *Predicate calculus* is an area in logic that deals with predicates and quantifiers. Note that propositional logic is a special case of predicate logic where there is no quantification done.

Universal quantifiers asserts that for all variables in the universe of discourse a given predicate is to be evaluated. Note that universal quantifier is represented as $\forall\ x$ and is read as $''$ for all x $''$, $''$for each x $''$, $''$ for any x $''$, $''$ for every x $''$. The notation $\forall\ x\ P(x)$ denotes the universal quantification of P(x) and is evaluated to be TRUE if $P(x)$ evaluates to be TRUE for all values of x and FALSE otherwise. As an example,
let $P(x)$: *x is prime*, $x \in \mathbb{N}$
$Q(x)$: *x is non-negative integer*, $x \in \mathbb{N}$
$\forall\ x\ P(x)$ has truthvalue FALSE.
$\forall\ x\ Q(x)$ has truthvalue TRUE.

Existential quantifier ensures that there exists at least one variable x in the universe of discourse such that the predicate can be instantiated on. Note that existential quantifier is represented as $\exists\ x$ and is read as $''$ there exists x $''$. Also note that if the predicate evaluates to be TRUE for at least one value of x, then existential quantification has a truth value TRUE, and FALSE otherwise. For instance, consider the above predicate, $\exists\ x\ P(x)$ has truth value TRUE.

Uniqueness quantifier denoted as $\exists!$ is read as $''$there exist exactly one$''$. A uniqueness quantification $\exists!\ x\ P(x)$ is evaluated to TRUE if there exist a unique x for which $P(x)$ is evaluated to TRUE.
For example, $\exists!\ x\ [4x + 3 = 11]$, where $x \in \mathbb{R}$.

Note 1: Let the elements in the domain be $\{x_1, x_2, x_3 \ldots\}$, then,
$\forall\ x\ P(x) \leftrightarrow P(x_1) \wedge P(x_2) \wedge P(x_3) \wedge \ldots$
$\exists\ x\ P(x) \leftrightarrow P(x_1) \vee P(x_2) \vee P(x_3) \vee \ldots$
$\exists\ !\ x\ P(x) \leftrightarrow [P(x_1) \wedge \neg P(x_2) \wedge \neg P(x_3) \wedge \neg P(x_4) \wedge \ldots] \vee [P(x_2) \wedge \neg P(x_1) \wedge \neg P(x_3) \wedge \neg P(x_4) \wedge \ldots] \vee [P(x_3) \wedge \neg P(x_1) \wedge \neg P(x_2) \wedge \neg P(x_4) \wedge \ldots] \vee \ldots$

Note 2: The scope of a quantifier is that part of an assertion in which variables are bounded by the quantifier.
$\forall\ x\ [P(x) \vee Q(x)] \not\leftrightarrow \forall\ x\ P(x) \vee Q(x) \not\leftrightarrow P(x) \vee \forall\ x\ Q(x)$
$\forall\ x\ [P(x) \vee Q(x)] \not\leftrightarrow \forall\ x\ P(x) \vee \forall\ x\ Q(x)$
$\exists\ x\ [P(x) \wedge Q] \leftrightarrow \exists\ x\ P(x) \wedge Q$

Some examples:
Consider the universe of discourse as integers. We define the following predicates over integers. We present logical statements below, which are in turn represented using quantifiers.

– N(x): x is non-negative integer.
– E(x): x is even
– O(x): x is odd
– P(x): x is prime

1. There exist an even integer $\exists\ x\ E(x)$
2. Every integer is even or odd $\forall\ x\ [E(x) \vee O(x)]$
3. All prime integers are non-negative $\forall\ x\ [P(x) \rightarrow N(x)]$
4. There is one and only one even prime $\exists\ !\ x\ [E(x) \wedge P(x)]$
5. The only even prime is two $\forall\ x\ [[E(x) \wedge P(x)] \rightarrow x = 2]$
6. Not all integers are odd $\exists\ x\ \neg O(x)$ or $\neg\forall\ x\ O(x)$
7. Not all primes are odd $\neg\forall\ x\ [P(x) \rightarrow O(x)]$ or $\exists\ x\ [P(x) \wedge \neg O(x)]$

Note:
$\neg\forall\ x\ P(x) \leftrightarrow \exists\ x\ \neg P(x)$ **and** $\neg\exists\ x\ P(x) \leftrightarrow \forall\ x\ \neg P(x)$

Compound statements involving predicates
For every pair of integers x and y there exists a z such that $x + z = y$
The above predicate is represented as $\forall\ x\ \forall\ y\ \exists\ z\ [x + z = y]$
Note that if universe of discourse is integer, then the predicate's truth value is `TRUE.`
If universe of discourse is $\mathbb{N}$, then the predicate is `FALSE` for some predicate constants.
i.e., $\neg\forall\ x\ \forall\ y\ \exists\ z\ [x + z = y] \implies \exists\ x\ \exists\ y\ \forall\ z\ \neg[x + z = y] \implies \exists\ x\ \exists\ y\ \forall\ z\ [x + z \neq y]$

Logical Identities

1. $\forall\ x\ P(x) \rightarrow P(c)$
2. $P(c) \rightarrow \exists\ x\ P(x)$
3. $\forall\ x\ \neg P(x) \leftrightarrow \neg\exists\ x\ P(x)$
4. $\forall\ x\ P(x) \rightarrow \exists\ x\ P(x)$
5. $\exists\ x\ \neg P(x) \leftrightarrow \neg\forall\ x\ P(x)$
6. $\forall\ x\ P(x) \wedge Q \leftrightarrow \forall\ x\ [P(x) \wedge Q]$
7. $\forall\ x\ P(x) \wedge \forall\ x\ Q(x) \leftrightarrow \forall\ x\ [P(x) \wedge Q(x)]$
8. $\forall\ x\ P(x) \vee \forall\ x\ Q(x) \rightarrow \forall\ x\ [P(x) \vee Q(x)]$
9. $\exists\ x\ [P(x) \wedge Q(x)] \rightarrow \exists\ x\ P(x) \wedge \exists\ x\ Q(x)$
10. $\exists x\ P(x) \vee \exists\ x\ Q(x) \leftrightarrow \exists\ x\ [P(x) \vee Q(x)]$

Rules of Inference for Quantified Statements

RULE OF INFERENCE	NAME
$\forall x\ P(x) \implies P(c)$	Universal Instantiation
$P(c)$ for any arbitrary c $\implies \forall x\ P(x)$	Universal Generalization
$\exists x\ P(x) \implies P(c)$ for some c	Existential Instantiation
$P(c)$ for some c $\implies \exists x\ P(x)$	Existential Generalization

Claim. $\forall\ x\ P(x) \wedge \forall\ x\ Q(x) \leftrightarrow \forall\ x\ [P(x) \wedge Q(x)]$

Proof. Necessity: It follows from definition that $[P(x_0) \wedge P(x_1) \wedge P(x_2) \wedge \ldots] \wedge [Q(x_0) \wedge Q(x_1) \wedge Q(x_2) \wedge \ldots]$ where $\{x_0, x_1, x_2, \ldots\}$ is the universe of discourse.
Apply the below rules inductively. For simplicity we work with the first two terms.
$[P(x_0) \wedge P(x_1)] \wedge [Q(x_0) \wedge Q(x_1)]$

Due to Associativity, $\implies [P(x_0) \wedge P(x_1) \wedge Q(x_0)] \wedge Q(x_1)$
Due to Commutativity, $\implies [P(x_0) \wedge Q(x_0) \wedge P(x_1)] \wedge Q(x_1)$
Due to Associativity, $\implies [P(x_0) \wedge Q(x_0)] \wedge [P(x_1) \wedge Q(x_1)]$
Once inductive application of the above rules is completed in order for all elements in the universe of discourse, we get
$(P(x_0) \wedge Q(x_0)) \wedge (P(x_1) \wedge Q(x_1)) \wedge \ldots$
By definition, $\implies \forall x\ [P(x) \wedge Q(x)]$. Necessity follows.
Sufficiency: $\forall x\ [P(x) \wedge Q(x)]$
By definition, $\implies (P(x_0) \wedge Q(x_0)) \wedge (P(x_1) \wedge Q(x_1)) \wedge \ldots$
Apply the below rules inductively. For simplicity we work with the first two terms.
Consider $[P(x_0) \wedge Q(x_0)] \wedge [P(x_1) \wedge Q(x_1)]$
Due to Associativity, $\implies [P(x_0) \wedge Q(x_0) \wedge P(x_1)] \wedge Q(x_1)$
Due to Commutativity, $\implies [P(x_0) \wedge P(x_1) \wedge Q(x_0)] \wedge Q(x_1)$
Due to Associativity, $\implies [P(x_0) \wedge P(x_1)] \wedge [Q(x_0) \wedge Q(x_1)]$
It follows that $[P(x_0) \wedge P(x_1) \wedge P(x_2) \wedge \ldots] \wedge [Q(x_0) \wedge Q(x_1) \wedge Q(x_2) \wedge \ldots]$
From necessity and sufficiency, the claim follows. $\square$

Claim. $\exists\ x\ [P(x) \vee Q(x)] \leftrightarrow \exists x\ P(x) \vee \exists\ x\ Q(x)$

Proof. From previous claim $\forall\ x\ [P(x) \wedge Q(x)] \leftrightarrow \forall\ x\ P(x) \wedge \forall\ x\ Q(x)$
Inverse, $\implies \neg\forall\ x\ [P(x) \wedge Q(x)] \leftrightarrow \neg[\forall\ x\ P(x) \wedge \forall\ x\ Q(x)]$
De-morgans law, $\implies \exists x\ [\neg[P(x) \wedge Q(x)]] \leftrightarrow \neg\forall\ x\ P(x) \vee \neg\forall\ x\ Q(x)$
De-morgans law, $\implies \exists x\ [\neg P(x) \vee \neg Q(x)] \leftrightarrow \exists\ x\ \neg P(x) \vee \exists\ x\ \neg Q(x)$
$R(x) = \neg P(x)$ and $S(x) = \neg Q(x) \implies \exists x\ [R(x) \vee S(x)] \leftrightarrow \exists\ x\ R(x) \vee \exists\ x\ S(x)$ $\square$

The second proof for Claim 1;

Claim. $\forall\ x\ [P(x) \wedge Q(x)] \leftrightarrow \forall\ x\ P(x) \wedge \forall\ x\ Q(x)$

Proof. Necessity: $\forall x \in UOD^2$, $P(x) \wedge Q(x)$ is TRUE.
$\implies P(x)$ is TRUE and $Q(x)$ is TRUE
$\implies \forall\ x\ P(x) \wedge \forall\ x\ Q(x)$.
Sufficiency: $\forall\ x\ P(x) \wedge \forall\ x\ Q(x)$ is TRUE.
On Simplification, $\implies \forall\ x\ P(x)$ is TRUE.
On Simplification, $\implies \forall\ x\ Q(x)$ is TRUE.
For any $x \in UOD$, $P(x) \wedge Q(x)$ is TRUE.
$\implies \forall x\ [P(x) \wedge Q(x)]$. $\square$

Claim. $\exists\ x\ [R(x) \vee S(x)] \leftrightarrow \exists x\ R(x) \vee \exists\ x\ S(x)$

Proof. $\exists\ x\ [R(x) \vee S(x)]$, let $x = c \in UOD$
Existential instantiation, $\implies R(c) \vee S(c)$ is TRUE.
Existential generalization, $\implies \exists x\ R(x) \vee \exists\ x\ S(x)$
Sufficiency: $\exists x\ R(x) \vee \exists\ x\ S(x)$ let $x = c \in UOD$
Existential instantiation, $\implies R(c) \vee S(c)$ is TRUE. Note that $\exists\ x\ S(x)$ may be true for $x = d$ and may not be true for $x = c$. However, it is certainly true for $\exists\ x\ R(x)$. Because of logical 'or', the truth value is 'True' for the expression even if $S(d)$ is 'false'.
Existential generalization, $\implies \exists x\ [R(x) \vee S(x)]$ $\square$

[2] universe of discourse

Claim. $\forall\ x\ P(x) \vee \forall\ x\ Q(x) \rightarrow \forall\ x\ [P(x) \vee Q(x)]$

Proof. $\forall\ x\ P(x) \vee \forall\ x\ Q(x)$, let $x = c \in UOD$
Universal instantiation, $\implies P(c) \vee Q(c)$
Case 1: $P(c)$ is TRUE and $Q(c)$ is TRUE
$\implies P(c) \vee Q(c)$ is TRUE
Case 2: $P(c)$ is TRUE and $Q(c)$ is FALSE
$\implies P(c) \vee Q(c)$ is TRUE
Case 3: $P(c)$ is FALSE and $Q(c)$ is TRUE
$\implies P(c) \vee Q(c)$ is TRUE
Case 1,2,3, $\implies P(c) \vee Q(c)$ is TRUE
Universal generalization, $\implies \forall\ x\ [P(x) \vee Q(x)]$ □

Converse of the above claim does not hold. i.e., $\forall\ x\ [P(x) \vee Q(x)] \not\rightarrow \forall\ x\ P(x) \vee \forall\ x\ Q(x)$
Counter example:
$P(x) : x$ is an irrational number.
$Q(x) : x$ is a rational number.
UOD: $\mathbb{R}$
Questions:

– Which is correct $\exists y\ \forall x\ [x + y = 0]$ or $\forall x\ \exists y\ [x + y = 0]$ where $x, y \in \mathbb{R}$ **?**

Claim. $\exists x(P(x) \rightarrow Q(x)) \leftrightarrow \exists x\ P(x) \rightarrow \exists x\ Q(x)$. Is the claim TRUE ?

$\exists x(P(x) \rightarrow Q(x)) \leftrightarrow \exists x(\neg P(x) \vee Q(x))$
$\leftrightarrow \exists x(\neg P(x)) \vee \exists x(Q(x))$
$\leftrightarrow \neg \forall x\ P(x) \vee \exists x\ Q(x)$
$\leftrightarrow \forall x\ P(x) \rightarrow \exists x\ Q(x)$

Answering the above question is equivalent to checking the necessary and sufficiency conditions of

$\forall x\ P(x) \rightarrow \exists x\ Q(x) \leftrightarrow \exists x\ P(x) \rightarrow \exists x\ Q(x)$

We shall now verify the validity of the above statement using a truth table. Note that like propositional variables, we treat predicate variables as a variable assuming the value of the variable either true or false.

Truth Table

$\forall x\ P(x)$	$\exists x\ P(x)$	$\exists x\ Q(x)$	$\forall x\ P(x) \rightarrow \exists x\ Q(x)$	$\exists x\ P(x) \rightarrow \exists x\ Q(x)$
0	0	0	1	1
0	0	1	1	1
0	1	0	1	0
0	1	1	1	1
1	0	0	N.A.	N.A.
1	0	1	N.A.	N.A.
1	1	0	0	0
1	1	1	1	1

Also note that in fifth row, it is not possible to have 'true' for $\forall x$ and 'false' for $\exists x$ and this impossibility is written as N.A (not applicable). From the last two columns, it is clear that sufficiency part is TRUE, and necessity is FALSE.

i.e., $[\exists x\ P(x) \rightarrow \exists x\ Q(x)] \rightarrow [\forall x\ P(x) \rightarrow \exists x\ Q(x)]$

$[\forall x\ P(x) \rightarrow \exists x\ Q(x)] \not\rightarrow [\exists x\ P(x) \rightarrow \exists x\ Q(x)]$

For disproving the necessity, consider the following counter example.
P(x): $x = 2$.
Q(x): $x \neq x$.
UOD: integers.
Note that $\forall x\ P(x)$ is FALSE, $\exists x\ P(x)$ is TRUE, and $\exists x\ Q(x)$ is FALSE. It is clear that the premise is true and the conclusion is false, therefore the necessity is false.

Nested Quantifiers:
We shall now discuss the expressions involving multiple quantifiers. In this section, we consider the following case studies and express them using predicate logic.

`Definition of limit:`

$\underset{x\rightarrow c}{Lt}$ f(x)=k $\leftrightarrow \forall\epsilon_{\epsilon>0}\ \exists\delta_{\delta>0}\ \forall x\ [(|x-c| < \delta) \rightarrow (|f(x)-k| < \epsilon)]$.

`Negation of the above limit is defined as ,`

$\underset{x\rightarrow c}{Lt}$ f(x)$\neq$k $\leftrightarrow \exists\epsilon_{\epsilon>0}\ \forall\delta_{\delta>0}\ \exists x\ [(|x-c| < \delta) \wedge (|f(x)-k| \geq \epsilon)]$.

Logical representation of statements using nested quantifiers.

– `The sum of two positive integers is always positive.`

$\forall x\ \forall y\ (x > 0 \wedge y > 0 \rightarrow x + y > 0)$

– `For every real number except 0, there exists a multiplicative inverse.`

$\forall x\ (x \neq 0 \rightarrow \exists y\ (xy = 1))$

– `If a person is female and is a parent, then this person is someone's mother.`

$\forall x\ [female(x) \wedge parent(x) \rightarrow \exists y\ (mother(x, y))]$

– `Every train is faster than some cars.`

$\forall x\ [train(x) \rightarrow \exists y\ (car(y) \wedge faster(x, y))]$

– `Some cars are slower than all trains but at least one train is faster than every car.`

$$\exists x\ [car(x) \wedge \forall y\ (train(y) \rightarrow slower(x,y))] \wedge \exists x\ [train(x) \wedge \forall y\ (car(y) \rightarrow faster(x,y))]$$

– `If it rains tomorrow, then somebody will get wet.`

$$P \rightarrow \exists x\ (person(x) \wedge wet(x))$$

Let A be a 2 dimensional integer array with 20 rows (indexed from 1 to 20) and 30 columns (indexed 1 to 30). Using first order logic make the following assertions.

1. `All entries of` A `are non-negative.`
   $\forall i\ \forall j\ (1 \leq i \leq 20, 1 \leq j \leq 30 \rightarrow A[i][j] \geq 0)$

2. `All entries of` 4^{th} `and` 15^{th} `rows are positive.`
   $\forall j\ (1 \leq j \leq 30 \rightarrow (A[4][j] \geq 1) \wedge (A[15][j] \geq 1))$

3. `Some entries of` A `are zero.`
   $\exists i\ \exists j\ (1 \leq i \leq 20, 1 \leq j \leq 30 \wedge A[i][j] = 0)$

4. `Entries of` A `are sorted in row major order. (i.e., the entries are in order within rows and every entry of the` i^{th} `row is less than or equal to every entry of the` $(i+1)^{th}$ `row).`
   $\{\forall i\ \forall j\ [1 \leq i \leq 20, 1 \leq j \leq 29 \rightarrow A[i][j] \leq A[i][j+1]] \wedge [\forall i\ 1 \leq i \leq 19 \rightarrow A[i][30] \leq A[i+1][1]]\}$

Logical Inference from a given Argument

In this section, we shall determine the truth value of a logical argument using logical identities and logical implications. We first transform the argument into logical notation, followed by validity checking using laws presented in the previous section.

Question 1. If a teacher teaches DM or DSA, then he is considered to be a TCS teacher.
If he is a TCS teacher, then he teaches GT. Therefore, he does not teach DSA.
A: teaches DM
B: teaches DSA
C: TCS teacher
D: teaches GT

$$\begin{array}{ll} (A \vee B) \rightarrow C & \ldots (1) \\ C \rightarrow D & \ldots (2) \\ \neg D & \ldots (3) \\ \therefore \neg B & \end{array}$$

Proof

$$\begin{array}{lll} \text{From} 1,2: & (A \vee B) \rightarrow D & \ldots (4) - \text{Due to Hypothetical Syllogism.} \\ 3,4: & \neg(A \vee B) & \ldots (5) \\ 5: & \neg A \wedge \neg B & \ldots (6) \\ 6: & \neg B & QED \end{array}$$

Therefore, the conclusion, teacher does not teach DSA follows from the given logical argument.

Question 2. Derive a contradiction for the premises 1-5.

$$\begin{array}{ll} A \to B \vee C & \ldots (1) \\ D \to \neg C & \ldots (2) \\ B \to \neg A & \ldots (3) \\ A & \ldots (4) \\ D & \ldots (5) \end{array}$$

$$\begin{array}{lll} 1,4: & B \vee C & \ldots (6) \\ 2,5: & \neg C & \ldots (7) \\ 6: & \neg C \to B & \ldots (8) \\ 7,8: & B & \ldots (9) \\ 3: & A \to \neg B & \ldots (10) \\ 4,10: & \neg B & \ldots (11) \\ 9,11: & B \wedge \neg B & \textit{a contradiction} \end{array}$$

Therefore, the given argument is logically inconsistent.

Question 3. Show that $R \vee S$ follows logically from the premises

$$\begin{array}{ll} C \vee D & \ldots (1) \\ C \vee D \to \neg H & \ldots (2) \\ \neg H \to A \wedge \neg B & \ldots (3) \\ A \wedge \neg B \to R \vee S & \ldots (4) \end{array}$$

$$\begin{array}{lll} 1,2: & \neg H & \ldots (5) \\ 3,5: & A \wedge \neg B & \ldots (6) \\ 4,6: & R \vee S & QED \end{array}$$

Question 4. Show that $S \vee R$ follows logically from the first three premises.

$$\begin{array}{ll} P \vee Q & \ldots (1) \\ P \to R & \ldots (2) \\ Q \to S & \ldots (3) \end{array}$$

$$\begin{array}{lll} 1: & \neg Q \to P & \ldots (4) \\ 2,4: & \neg Q \to R & \ldots (5) \\ 5: & \neg R \to Q & \ldots (6) \\ 3,6: & \neg R \to S & \ldots (7) \\ 7 & S \vee R & QED \end{array}$$

Question 5. If Jack misses many classes through illness, then he fails high school.
If Jack fails high school, then he is uneducated.
If Jack reads a lot of books, then he is not uneducated.
Jack misses many classes through illness and reads a lot of books.

Check whether the argument is consistent.

J: Jack misses many classes through illness.
H: Jack fails high school.

U: Jack is uneducated.
R: Jack reads a lot of books.

$$\begin{array}{lll}
 & J \to H & \ldots (1) \\
 & H \to U & \ldots (2) \\
 & R \to \neg U & \ldots (3) \\
 & J \wedge R & \ldots (4) \\
Proof: & & \\
1,2: & J \to U & \ldots (5) \\
3: & U \to \neg R & \ldots (6) \\
5,6: & J \to \neg R & \ldots (7) \\
7: & R \to \neg J & \ldots (8) \\
4: & R & \ldots (9) \\
8,9: & \neg J & \ldots (10) \\
4: & J & \ldots (11) \\
10,11: & J \wedge \neg J & \textit{a contradiction}
\end{array}$$

Therefore, the given argument is logically inconsistent.

Assertions Involving Quantifiers - Validity Checking

Question 1. *Check validity*

A student in this class has not read the book and everyone in this class passed the first exam. Therefore someone who passed the first exam has not read the book.

Representation using logic variables
C(x): x is in this class.
P(x): x passes the first exam.
B(x): x has read the book.

We want to prove $\exists x\ (P(x) \wedge \neg B(x))$ from 1 and 2.

$$\begin{array}{lll}
 & \exists x\ (C(x) \wedge \neg B(x)) & \ldots (1) \\
 & \forall x\ (C(x) \to P(x)) & \ldots (2) \\
EI\ of\ 1 & C(a) \wedge \neg B(a) & \ldots (3) \\
3 & C(a) & \ldots (4) \\
EI\ of\ 2 & C(a) \to P(a) & \ldots (5) \\
4,5: & P(a) & \ldots (6) \\
3 & \neg B(a) & \ldots (7) \\
6,7: & P(a) \wedge \neg B(a) & \ldots (8) \\
EG\ of\ 8: & \exists x\ (P(x) \wedge \neg B(x)) & QED
\end{array}$$
[3]

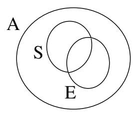
[3] EI: Existential Instantiation
UI: Universal Instantiation
EG: Existential Generalization
UG: Universal Generalization 3

Question 2. *Check validity*

Some scientists are not engineers.
Some astronauts are not engineers.
Hence, some scientists are not astronauts.

Representation using logic variables
$E(x)$: x is an engineer.
$S(x)$: x is a scientist.
$A(x)$: x is an astronaut.
To prove or disprove $\exists x\ (S(x) \wedge \neg A(x))$ from 1 and 2.

$\exists x\ (S(x) \wedge \neg E(x)) \ldots (1)$
$\exists x\ (A(x) \wedge \neg E(x)) \ldots (2)$



A counter example.

Counter example is shown using the Venn diagram. Note that, if we want to prove a claim using Venn diagram then we have to enumerate all possible Venn diagrams in the given context and therefore, such a proof method is not advisable as our listing may not be exhaustive. However, to disprove a claim, the existence of even one Venn diagram suffices. Due to this reasoning, we use Venn diagram to disprove a claim and a proof using logical identities if the claim is correct.

Question 3. *Check validity*

All integers are rational numbers.
Some integers are powers of 2.
Therefore, some rational numbers are powers of 2.
Representation using logic variables
$I(x)$: x is an Integer.
$R(x)$: x is a rational number.
$P(x)$: x is a power of 2.
To prove or disprove $\exists x\ (R(x) \wedge P(x))$ from 1 and 2.

	$\forall x\ (I(x) \rightarrow R(x))$	$\ldots (1)$
	$\exists x\ (I(x) \wedge P(x))$	$\ldots (2)$
$UI\ of\ 1$	$I(a) \rightarrow R(a)$	$\ldots (3)$
$EI\ of\ 2$	$I(a) \wedge P(a)$	$\ldots (4)$
4	$I(a)$	$\ldots (5)$
$3, 5:$	$R(a)$	$\ldots (6)$
4	$P(a)$	$\ldots (7)$
$6, 7:$	$R(a) \wedge P(a)$	$\ldots (8)$
$EG\ of\ 8:$	$\exists x\ (R(x) \wedge P(x))$	QED

Question 4. *Check validity*

Premise: $\exists x\ (F(x) \wedge S(x)) \rightarrow \forall y\ (H(y) \rightarrow W(y))$
$\exists y\ (H(y) \wedge \neg W(y))$
Conclusion: $\forall x\ (F(x) \rightarrow \neg S(x))$

	$\exists x\ (F(x) \wedge S(x)) \rightarrow \forall y\ (H(y) \rightarrow W(y))$	$\ldots(1)$
	$\exists y\ (H(y) \wedge \neg W(y))$	$\ldots(2)$
2	$\neg\neg\exists y\ (H(y) \wedge \neg W(y))$	$\ldots(3)$
3	$\neg\forall y\ \neg(H(y) \wedge \neg W(y))$	$\ldots(4)$
4	$\neg\forall y\ (H(y) \rightarrow W(y))$	$\ldots(5)$
$1,5:$	$\neg\exists x\ (F(x) \wedge S(x))$	$\ldots(6)$
6	$\forall x\ (F(x) \rightarrow \neg S(x))$	QED

Question 5. Show that $\forall x\ (P(x) \vee Q(x)) \rightarrow \forall x\ P(x) \vee \exists x\ Q(x)$

Proof by contradiction: Assume on the contrary that the conclusion is FALSE. i.e., include $\neg$ *Conclusion* as part of premise.

premise	$\forall x\ (P(x) \vee Q(x))$	$\ldots(1)$
premise assumed	$\neg[\forall x\ P(x) \vee \exists x\ Q(x)]$	$\ldots(2)$
2	$\neg\forall x\ P(x) \wedge \neg\exists x\ Q(x)$	$\ldots(3)$
3	$\exists x\ \neg P(x) \wedge \forall x\ \neg Q(x)$	$\ldots(4)$
4	$\exists x\ \neg P(x)$	$\ldots(5)$
EI of 5	$\neg P(a)$	$\ldots(6)$
4	$\forall x\ \neg Q(x)$	$\ldots(7)$
UI of 7	$\neg Q(a)$	$\ldots(8)$
$7,8$	$\neg P(a) \wedge \neg Q(a)$	$\ldots(9)$
9	$\neg[P(a) \vee Q(a)]$	$\ldots(10)$
UI of 1	$P(a) \vee Q(a)$	$\ldots(11)$
$10,11$	$\neg[P(a) \vee Q(a)] \wedge [P(a) \vee Q(a)]$	*a contradiction*

Therefore our assumption is wrong/FALSE and conclusion is TRUE. Therefore $\forall x\ P(x) \vee \exists x\ Q(x)$ follows from $\forall x\ (P(x) \vee Q(x))$. □

Having learnt First Order Logic (FOL), the following Qualities are expected out of the learner by the end of First Order Logic learning.

1. *Be precise and concise.*
2. *Always think before speak.*
3. *Be consistent - always think before you speak.*
4. *Stop believing - start asking/looking for logical reasoning/proof*
5. *Start using necessary, sufficiency, if and only if, during conversations.*

Food for thought
Express the following using First order Logic.

1. *Among the institute faculty, there exist a set of faculty whose expertise is computer science.*
2. *Among the school kids, there exists a set of kids whose IQ level number is five.*

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