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Cagan Model with Memory Effects

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Abstract

This paper considers a generalization of the model that is proposed by Phillip D. Cagan to describe dynamics of the actual inflation. In this generalization the memory effects and memory fading are taken into account. In the standard Cagan model, the indicator of nervousness of economic agents, which characterizes the speed of revising the expectations, is represented as a constant parameter. In general, the speed of revising the expectations of inflation can depend on the history of changes in the difference between the real inflation rate and the rate expected by economic agents. We assume that the nervousness of economic agents can be caused not only by the current state of the process, but also by the history of its changes. The use of the memory function instead of the indicator of nervousness, allows us to take into account the memory effects in the Cagan model. We consider the fractional dynamics of the actual inflation that takes into account memory with power-law fading. The fractional differential equation, which describes the proposed economic model with memory, and the expression of its exact solution are suggested.

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1 Introduction

Inflation and seigniorage are important effects that should be described in economics [1, 2, 3]. Sometimes inflation reaches high levels, and goes into hyperinflations, which are defined as inflation that exceeds 50 percent per month. The basic cause of most cases of high inflation and hyperinflation is government's need to obtain seigniorage, i.e., revenue from printing money [4]. However even governments' need for seigniorage cannot account for hyperinflations (see [1], p.543).

If the public does not immediately adjust its money holdings or its expectations of inflation to changes in the economic environment, then in the short run seigniorage is always increasing in money growth, and the government can obtain more seigniorage than the maximum sustainable amount, S^*. Thus hyperinflations arise when the government's seigniorage needs exceed S^* [4, 1]. Gradual adjustment of money holdings and gradual adjustment of expected inflation have similar implications for the dynamics of inflation.

The Cagan model focuses on the case of gradual adjustment of money holdings. In this model it is assumed that individuals desired money holdings are given by the Cagan money-demand function. The Cagan model was created to describe the processes of hyperinflation (see Section 10.8 in [1], Section 11.9 in [2]). As the only factor in the demand for money, the Cagan model considers inflationary expectations. This assumption corresponds to a situation of lack of economic growth, i.e., the Cagan model describes an economy with constant output.

In the standard economic model, the memory effects and memory fading [5] are neglected. At the same time, it is obvious that to describe expected rate of inflation, it is necessary to take into account memory and memory fading, since amnesia of economic agents is a strong restriction on the standard Cagan model.

In the standard Cagan model, the indicator of nervousness of economic agents, which characterizes the speed of revising the expectations, is represented as a constant parameter [3]. In general case, the nervousness of economic agents can be caused not only by the current state of the process, but also by the history of its changes. The speed of revising the expectations of inflation can depend on the history of changes in the difference between the real inflation rate and the rate expected by economic agents. The use of the memory function instead of the indicator of nervousness, allows us to take into account the memory effects in the Cagan model.

To describe economic processes with memory by using the discrete time approach, C.W.J Granger., R. Joyeux [6, 7, 9], and J.R.M. Hosking [8] proposed the fractional differencing and integrating. Granger, Joyeux, and Hosking independently proposed the so-called autoregressive fractional integrated moving average models. In these models, it is used the difference operator of order, which need not be an integer [6, 8, 10]. In fact, these operators were already well known in mathematics. This operator coincides with the well-known Grunwald-Letnikov fractional difference, which were proposed over a hundred and fifty years ago [11, 12]. In mathematics these fractional differences are actively used. Unfortunately, in economics the application of the well-known Grunwald-Letnikov fractional difference is realized without any direct connection with fractional calculus [13, 14, 15, 16, 17] that exists more than three hundred years [18, 19, 20, 21, 22, 23] and has various application in different sciences (for example, see [24, 25]). At the moment, fractional calculus methods have begun to be used to describe economic phenomena and processes [26].

The fractional calculus of derivatives and integrals of non-integer orders [13, 14, 15, 16, 17] is an effective and powerful tool for describing memory effects. The characteristic properties of fractional differential and integral operators of non-integer order are the violation of standard rules and properties that are fulfilled for operators of integer order [27, 28, 29, 30, 31]. For example, the standard product (Leibniz) rule and the standard chain rule are violated for derivatives of non-integer orders. If the non-standard properties of derivatives of non-integral order [27, 28, 29, 30, 31] are not taken into account in construction of fractional generalizations of classical emodels, then we can get non-self-consistent and incorrect mathematical models in the economics [32]. The non-standard properties lead to difficulties [32] in sequential constructing a fractional generalizations of standard models. At the same time, these non-standard properties of operators allow us to describe non-standard processes and phenomena associated with memory and non-locality.

In this paper we derive a generalization of the economic model that is proposed by Phillip D. Cagan in the work6 [4] (see also [3], p.157-159, [1], p.543-547). In this generalization the memory effects and memory fading are taken into account. Fractional dynamics of the actual inflation, where we take into account memory with power-law fading, is considered. The fractional nonlinear differential equation, which describes the proposed model, and the expression of its exact solution are suggested.

2 Standard Cagan model without memory

In the standard Cagan model was proposed by Phillip D. Cagan in the works [4]. In this model, which does not take into account memory effects [5], the following variable are used (for example, see pages 147-150 and 157-158 in [3], and [1, 2]):

$M(t)$ is the nominal money supply (the money stock);
$P(t)$ is the general price level;
$m = M^{(1)}(t)/M(t)$ is the money supply growth rate;
$z(t) = M(t)/P(t)$ is the real cash reserves (the stocks of money);
$z^D(t)$ is the demand for real cash reserves;
$z^S(t)$ is the supply of real cash reserves;
$\pi(t)$ is the real rate of inflation (the actual inflation);
$\pi^e(t)$ is the expected rate of inflation.

Let us consider the assumptions of the standard Cagan model.

1. As an example of the relation between inflation ans steady-state seigniorage, we will consider the money-demand function proposed by Cagan in 1956 [4] (see also [1], p.540-541, and [3], p.157). The demand function for money has the form

$$\left(\frac{M}{P}\right)^D = f(\pi^e) = \exp(-a\pi^e), \tag{1}$$

where $(M/P)^D$ is the demand for real cash reserves; π^e is the expected inflation rate; a is a parameter characterizing the elasticity of demand for money by inflation, $a > 0$. Note that the elasticity of money demand by inflation rate is $a\pi^e$.

2. The growth rate of the money supply is constant

$$\frac{M^{(1)}(t)}{M(t)} = m = \theta = const. \tag{2}$$

3. The key assumption of the model is that actual money holdings adjust gradually forward desired holdings [1], p.543. The rule for revising expectations in the standard Cagan model is

given by the equation

$$\frac{d\pi^e(t)}{dt} = \beta\left(\pi(t) - \pi^e(t)\right), \tag{3}$$

where $\beta > 0$ is the constant. The idea behind this assumption of gradual adjustment is that it is difficult for individuals to adjust their money holdings; for example, they may have made arrangements to make certain types of purchases using money. As a result, they adjust their money holdings toward the desired level only gradually ([1],p.543). Equation (3) assumes that expectations are adaptive. If the real inflation rate (π) is higher than the rate expected by economic agents (π^e), then they will adjust their expectations for the future towards increasing inflation $\pi^{e(1)} > 0$, and vice versa, if $\pi - \pi^e < 0$, that is, $\pi < \pi^e$, then $\pi^{e(1)} < 0$.

The parameter β is an indicator of nervousness of economic agents that characterizes the speed of revising the expectations [3].

4. Equilibrium condition in the money market:

$$\left(\frac{M}{P}\right)^D = \exp(-a\pi^e) = \left(\frac{M}{P}\right)^S = \frac{M}{P}. \tag{4}$$

Let us logarithm identity (4):

$$-a\pi^e = \ln M - \ln P. \tag{5}$$

Taking the first derivative with respect to t, we obtain the equation for the growth rate

$$-a\pi^{e(1)}(t) = \theta - \pi(t), \tag{6}$$

where we use the definition of θ by equation (2) and that $\pi(t) = P^{(1)}(t)/P(t)$. Multiplying equation (3) by the parameter $-a$, we get

$$-a\pi^{e(1)}(t) = -a\beta\left(\pi(t) - \pi^e(t)\right). \tag{7}$$

Substitution of expression into equation, we have

$$\theta - \pi(t) = -a\beta\left(\pi(t) - \pi^e(t)\right). \tag{8}$$

Solving this algebraic equation we get the expression

$$\pi(t) = \frac{\theta - a\beta\pi^e(t)}{1 - a\beta}. \tag{9}$$

Taking the first derivative of the equation (9), we have

$$\pi^{(1)}(t) = \frac{-a\beta}{1 - a\beta}\pi^{e(1)}(t). \tag{10}$$

Then using (6), we obtain

$$\pi^{(1)}(t) = \frac{\beta(\theta - \pi(t))}{1 - a\beta}. \tag{11}$$

This equation can be writted in the form

$$\pi^{(1)}(t) + \frac{\beta}{1 - a\beta}\pi(t) = \frac{\beta\theta}{1 - a\beta}. \tag{12}$$

The solution to this linear differential equation has the form:

$$\pi(t) = \theta + (\pi(0) - \theta) \exp\left(\frac{-\beta t}{1 - a\beta}\right). \tag{13}$$

Since the high inflationary economy is analyzed, it can be assumed that $\pi(0) > \theta$. It can be seen from equation (9) that if $a\beta < 1$, then $\pi(t) \to \theta$ for $t \to \infty$. This means that in a situation where the coefficients characterizing the elasticity of money demand by inflation (a) and the rate of revision of inflation expectations (β) are not too high, the result of the standard Cagan model is consistent with the conclusion of the quantitative theory of money, t i.e., in equilibrium $\pi = m = \theta$ [3].

If $a\beta > 1$, then $\pi(t) \to \infty$ for $t \to \infty$. In other words, if a or β is large, that is, agents greatly change the demand for money when they revise their expectations or sharply change their expectations, then the economy may not come to an equilibrium state. In the first case, with rising inflation expectations, they sharply reduce the demand for money, which leads to a further increase in inflation. Secondly, with inflation, they sharply increase inflation expectations, which (for the same reasons) strengthens inflation processes. Inflation continues despite the stabilization of the money supply growth rate. To restore equilibrium in such an economy, it is necessary to carry out measures aimed at reducing the nervousness of economic agents. The Cagan model does not take into account the effect on the equilibrium of the dynamics of GDP. This drawback is overcome in the Bruno-Fisher model, which allows deepen the analysis of the equilibrium of the money market and the consequences of monetary policy. This model helps to assess the impact on inflationary equilibrium rate of inflation control measures such as reducing the budget deficit.

3 Generalization: Cagan model with memory

The formal generalization of equation (12) can be suggested on the form

$$\left(D^{\alpha}_{C;0+}\pi\right)(t) + \frac{\beta}{1 - a\beta}\pi(t) = \frac{\beta\theta}{1 - a\beta}, \tag{14}$$

where $D^{\alpha}_{C;0+}$ is the Caputo fractional derivative. For $\alpha = 1$, equation (14) give equation (12) of the standard Cagan model.

However, if we carefully and consistently generalize the process of deriving the equation of the standard model from the basic assumption, then we will not get a fractional differential equation in the form (14).

Let us consider a generalization of the standard equation of revising expectations (3) for the processes with memory.

At the same time, it is obvious that to describe expected rate of inflation, it is necessary to take into account memory and memory fading, since amnesia of economic agents is a strong restriction on the economic models. In the standard Cagan model, the indicator of nervousness of economic agents, which characterizes the speed of revising the expectations, is represented as a constant parameter [3]. In the general case, the nervousness of economic agents can be caused not only by the current state of the process, but also by the history of its changes. We assume thatthe speed of revising the expectations of inflation can depend on the history of changes in the difference between the real inflation rate and the rate expected by economic agents. Therefore we assume that the revising expectations is described by the equation

$$\frac{d\pi^e}{dt} = \int_0^t B(t-\tau)\left(\pi(\tau) - \pi^e(\tau)\right) d\tau. \tag{15}$$

If the function $B(t-\tau)$ is expressed through the Dirac delta-fucntion $B(t-\tau) = \beta\delta(t-\tau)$, then equation (15) gives the standard equation (3) of Cagan model without memory.

The nervousness of economic agents is caused not only by the current state of the process, but also by the history of its changes. The speed $d\pi^e(t)/dt$ of revising the expectations of inflation can depend on the history of changes in the difference between the real inflation rate (π) and the rate expected by economic agents (π^e). The use of the function $B(t-\tau)$ instead of the parameter β, allows us to take into account the memory effects and memory fading in the suggested generalization of the standard Cagan model.

If we assume the power-law form of the function

$$B(t-\tau) = \frac{\beta(\alpha)}{\Gamma(\alpha)}(t-\tau)^{\alpha-1}, \tag{16}$$

where α is the parameter of memory fading [5], then equation (15) can be written in the form

$$\frac{d\pi^e(t)}{dt} = \beta(\alpha)\left(I^{\alpha}_{RL;0+}\left(\pi - \pi^e\right)\right)(t), \tag{17}$$

where $I^{\alpha}_{RL;0+}$ is the Riemann-Liouville fractional integral of the order $\alpha > 0$.

The Riemann-Liouville fractional integral is defined [16], p.69-70, by the equation

$$\left(I^{\alpha}_{RL,0+}f\right)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \tag{18}$$

where $\Gamma(\alpha)$ is the Gamma function. In equation (18) the function $f(t)$ is assumed to be measurable on the interval $(0,t)$ and it must satisfy the condition

$$\int_0^t |f(\tau)|\, d\tau < \infty. \tag{19}$$

We should note that equation (17) with Riemann-Liouville fractional integral can be considered as an approximation of the equations with generalized memory functions (15). In paper [33], we use the generalized Taylor series in the Trujillo-Rivero-Bonilla form for wide class of the memory functions. We proved that the equations with memory functions can be represented through the Riemann-Liouville fractional integrals (and the Caputo fractional derivatives) of non-integer orders.

The action of the Caputo derivative on equation (17) gives

$$\left(D^{\alpha}_{C;0+}\pi^{e(1)}\right)(t) = \beta(\alpha)\left(D^{\alpha}_{C;0+} I^{\alpha}_{RL;0+}(\pi - \pi^{e})\right)(t), \tag{20}$$

where $D^{\alpha}_{C;t_0+}$ is the Caputo fractional derivative of the order $\alpha > 0$. The Caputo fractional derivative is defined by the equation

$$\left(D^{\alpha}_{C;0+} f\right)(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, \tag{21}$$

where $\Gamma(\alpha)$ is the Gamma function, $t > 0$, and $f^{(n)}(\tau)$ is the derivative of the integer order $n = [\alpha] + 1$ with respect to τ. It is assumed that the function $f(\tau)$ has derivatives up to the $(n-1)$th order, which are absolutely continuous functions on the interval $[0, t]$.

Let us use the fact that the Caputo fractional derivative is the left inverse operator for the Riemann-Liouville fractional integral (see equation 2.4.32 of the Lemma 2.21 of [16], p.95), the has the form

$$\left(D^{\alpha}_{C;0+}\left(I^{\alpha}_{RL,0+} f\right)\right)(t) = f(t), \tag{22}$$

if $\alpha > 0$, and $f(t) \in L_{\infty}(t_0, t)$ or $f(t) \in C[t_0, t]$.

As a result, equation (20) is represented in the form

$$\left(D^{\alpha+1}_{C;0+}\pi^{e}\right)(t) = \beta(\alpha)\left(\pi(t) - \pi^{e}(t)\right), \tag{23}$$

where we also use that

$$\left(D^{\alpha}_{C;0+} f^{(1)}\right)(r) = \left(D^{\alpha+1}_{C;0+} f\right)(t), \tag{24}$$

where $\alpha > 0$.

In the model with memory, we will assume the same equilibrium condition (4) in the money market as in the standard model, i.e., we use the expression

$$\exp(-a\pi^{e}) = \frac{M}{P}. \tag{25}$$

First taking the logarithm and then differentiating, we obtain equation (6) that can be written in the form

$$\pi(t) = \theta + a\pi^{e(1)}(t). \tag{26}$$

Substitution of expression (26) into equation (23) gives the equation

$$\left(D^{\alpha+1}_{C;0+}\pi^{e}\right)(t) = \beta(\alpha)\left(\theta + a\pi^{e(1)}(t) - \pi^{e}(t)\right). \tag{27}$$

As a result, we have the fractional differential equation for $\pi^{e}(t)$ in the form

$$\left(D^{\alpha+1}_{C;0+}\pi^{e}\right)(t) - a\beta(\alpha)\pi^{e(1)}(t) + \beta(\alpha)\pi^{e}(t) = \beta(\alpha)\theta. \tag{28}$$

Let us solve fractional differential equation (28). Using the fact that the Caputo derivative of a constant is equal to zero [16], and introducing new variable

$$y(t) = \pi^e(t) - \theta, \tag{29}$$

we can write equation (28) as a homogeneous equation in the form

$$\left(D^{\alpha+1}_{C;0+} y\right)(t) - a\beta(\alpha)y^{(1)}(t) + \beta(\alpha)y(t) = 0. \tag{30}$$

The solution of equation (30) is given by Theorem 5.14, 314, and Corollary 5.9, p.317, of the book [16], where $\alpha \to \alpha + 1$, $\beta \to 1$, $\lambda \to a\beta(\alpha)$, and $\mu \to -\beta(\alpha)$.

Let us consider the case

$$1 < n - 1 < \alpha + 1 \leq n,$$

where $\alpha > 0$.

The solution of equation (30) can be represented in terms of the generalized Wright function (the Fox-Wright function), $\Psi_{1,1}\left[{}^{(a,\alpha)}_{(b,\beta)}|z\right]$, which is defined [16], p. 56, by the equation

$$\Psi_{1,1}\left[{}^{(a,\alpha)}_{(b,\beta)}|z\right] = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha k + a)}{\Gamma(\beta k + b)} \frac{z^k}{k!}. \tag{31}$$

Using Theorem 5.13 of [16], p.314, the solution of equation (30) has the form

$$y(t) = \sum_{j=0}^{n-1} a_j y_j(t), \tag{32}$$

where $y_j(t)$, $j = 0, \ldots, n-1$ are defined by the following equations

$$\begin{aligned} y_0(t) = & \sum_{k=0}^{\infty} \frac{(-1)^k \beta^k(\alpha) t^{k(\alpha+1)}}{\Gamma(k+1)} \Psi_{1,1}\left[{}^{(n+1,1)}_{((\alpha+1)k+1,\alpha)}|a\beta(\alpha)t^{\alpha}\right] - \\ & - a\beta(\alpha) \sum_{k=0}^{\infty} \frac{(-1)^k \beta^k(\alpha) t^{k(\alpha+1)+\alpha}}{\Gamma(k+1)} \Psi_{1,1}\left[{}^{(n+1,1)}_{((\alpha+1)k+1+\alpha,\alpha)}|a\beta(\alpha)t^{\alpha}\right] \end{aligned} \tag{33}$$

and

$$y_j(t) = \sum_{k=0}^{\infty} \frac{(-1)^k \beta^k(\alpha) t^{k(\alpha+1)+j}}{\Gamma(k+1)} \Psi_{1,1}\left[{}^{(n+1,1)}_{((\alpha+1)k+j+1,\alpha)}|a\beta(\alpha)t^{\alpha}\right] \tag{34}$$

for $j = 1, \ldots, n-1$, where $n - 1 = [\alpha + 1]$ for non-integer values of α.

For $1 < \alpha + 1 \leq 2$, $\alpha \in (0, 1)$ the solution is represented by equation

$$y(t) = a_0 y_0(t) + a_1 y_1(t), \tag{35}$$

with $y_0(t)$ in the form

$$y_0(t) = \sum_{k=0}^{\infty} \frac{(-1)^k \beta^k(\alpha) t^{k(\alpha+1)}}{\Gamma(k+1)} \Psi_{1,1}\left[{}^{(n+1,1)}_{((\alpha+1)k+1,\alpha)}|a\beta(\alpha)t^{\alpha}\right] -$$

$$- a\beta(\alpha) \sum_{k=0}^{\infty} \frac{(-1)^k \beta^k(\alpha) t^{k(\alpha+1)+\alpha}}{\Gamma(k+1)} \Psi_{1,1} \left[{}^{(n+1,1)}_{((\alpha+1)k+1+\alpha,\alpha)} | a\beta(\alpha) t^{\alpha} \right]. \tag{36}$$

and $y_1(t)$ is defined by the equation

$$y_1(t) = \sum_{k=0}^{\infty} \frac{(-1)^k \beta^k(\alpha) t^{k(\alpha+1)+1}}{\Gamma(k+1)} \Psi_{1,1} \left[{}^{(n+1,1)}_{((\alpha+1)k+2,\alpha)} | a\beta(\alpha) t^{\alpha} \right], \tag{37}$$

where $y(t)$ is defined by equation (29).

These solutions describe the proposed generalization of the Cagan model, in which we take into account the memory effects and one-paremeter memory fading. Expressions (32)-(34) (and equations (35)-(37) for $\alpha \in (0, 1)$ describe the fractional dynamics of the actual inflation that takes into account memory with power-law fading parameter $\alpha > 0$.

4 Conclusion

In this paper we consider a generalization of the model that is proposed by Phillip D. Cagan. In the standard Cagan model, the memory effects and memory fading are not taken into account. The indicator of nervousness of economic agents, which characterizes the speed of revising the expectations, is represented as a constant parameter in standard model. We assume that the nervousness of economic agents can be caused not only by the current state of the process, but also by the history of its changes. The speed of revising the expectations of inflation can depend on the history of changes in the difference between the real inflation rate and the rate expected by economic agents. The use of the memory function instead of the indicator of nervousness, allows us to take into account the memory effects in the Cagan model. We consider the fractional dynamics of the actual inflation that takes into account memory with power-law fading. The fractional differential equation, which describes the proposed economic model with memory, and the expression of its exact solution are suggested.

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