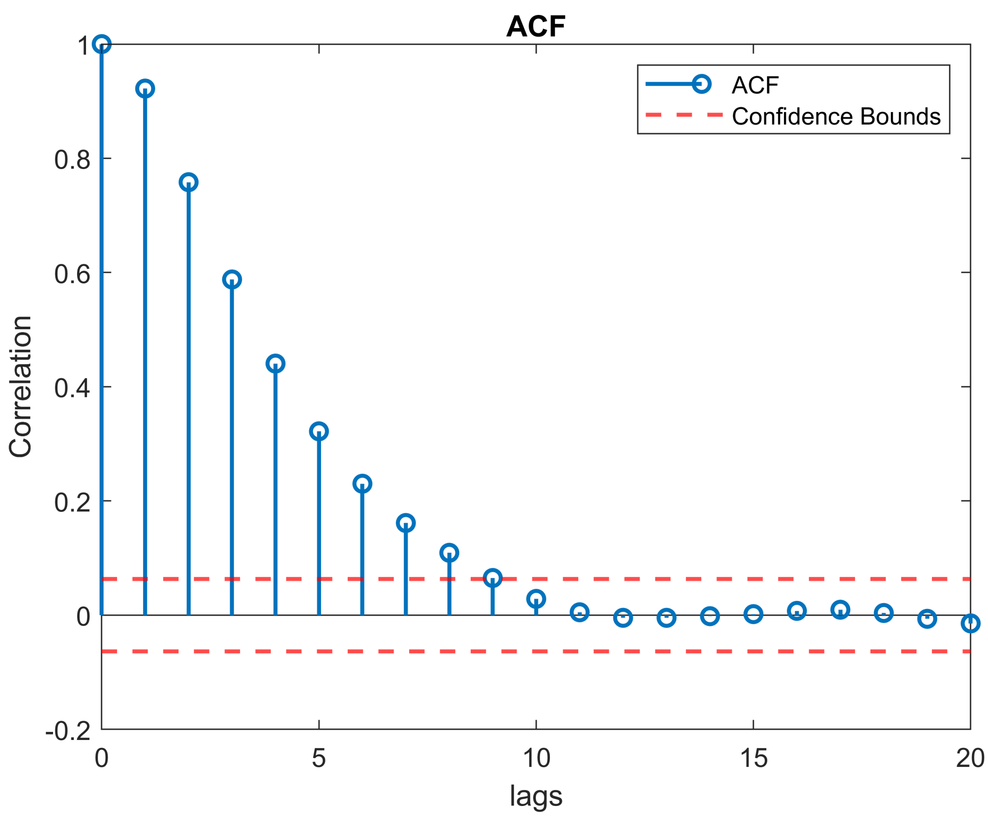
Question 1

(a)

AUTOCORRELATION FUNCTION

```
A=vk;
[acf,lags,abounds]=autocorr(A);
figure;
ACF=[acf';lags']';
stem(lags,acf,LineWidth=1.5);hold on;
yline(abounds(1),'--r',LineWidth=1.5)
yline(abounds(2),'--r',LineWidth=1.5)
title('ACF')
xlabel('lags')
ylabel('Correlation')
legend('ACF','Confidence Bounds')
```

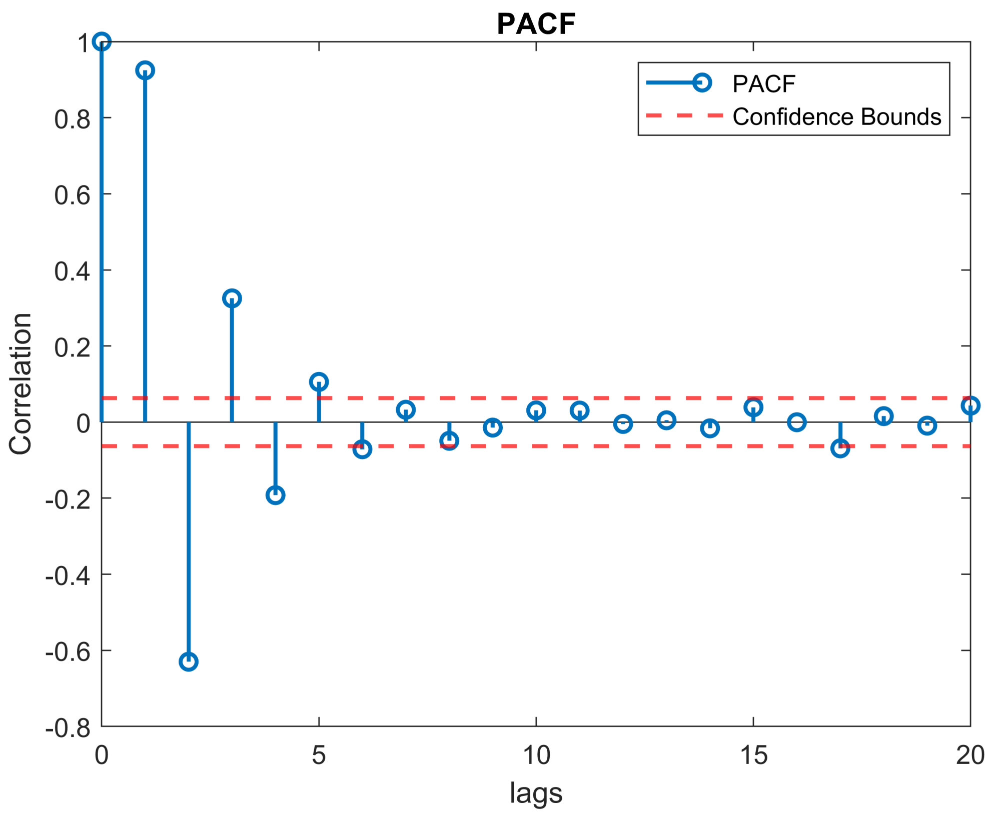


The ACF of the time series data seems to be **exponentially decaying**.

This is the property of an **AR model.**

```
[pacf,plags,bounds]=parcorr(A);
PACF=[pacf';plags']';
figure;
stem(lags,pacf,LineWidth=1.5);hold on;
yline(bounds(1),'--r',LineWidth=1.5)
yline(bounds(2),'--r',LineWidth=1.5)
title('PACF')
```

```
xlabel('lags')
ylabel('Correlation')
legend('PACF','Confidence Bounds')
```



(b) CHOOSING MODEL

From PACF plot, the values are going in bounds after the **6th lag.**

So, till 6 th lag, there is good correlation with the past values.

```
ar(A,6)
```

```
ans =
Discrete-time AR model: A(z)y(t) = e(t)
  A(z) = 1 - 1.803 z^-1 + 1.432 z^-2 - 0.8537 z^-3 + 0.4804 z^-4 - 0.2322 z^-5 + 0.06844 z^-6

Sample time: 1 seconds

Parameterization:
   Polynomial orders:   na=6
   Number of free coefficients: 6
   Use "polydata", "getpvec", "getcov" for parameters and their uncertainties.

Status:
Estimated using AR ('fb/now') on time domain data "A".
Fit to estimation data: 72.8%
FPE: 0.9906, MSE: 0.9788
```

After comparing the above AR model for different polynomial orders 4,5 etc.. and also comparing with ARMA models, this AR model is giving the best Fit. So, AR model with polynomial order 6 is chosen as the best model.

CH5230 ASSIGNMENT - 4.

Amarchand.C
ED21S005.

② → a $y[k] = \frac{b_2^0 q^{-2}}{1+f_1^0 q^{-1}} u[k] + e[k]$.

Considering $b_2^0 = b_2$ (a constant). & $f_1^0 = f_1$ (constant)

$$(1+f_1 q^{-1}) y[k] = b_2 q^{-2} u[k] + (1+f_1 q^{-1}) e[k]$$

$$\boxed{y[k] + f_1 y[k-1] = b_2 u[k-2] + e[k] + f_1 e[k-1]} \quad — ①.$$

$u[k]$ & $e[k]$ have WN properties.

$\Rightarrow E(y[k]) = f_1 E(y[k-1]) = \mu_y$.

$\mu_y + f_1(\mu_y) = 0 \quad \Rightarrow \underline{\underline{\mu_y = 0}}$

Taking variance on both sides,

$$var(y[k] + f_1 y[k-1]) = var(b_2 u[k-2] + e[k] + f_1 e[k-1]).$$

$$\Rightarrow var(y[k]) + f_1^2 var(y[k-1]) + f_1 \cdot cov(y[k]\, y[k-1])$$

$$= b_2^2 var(u[k-2]) + var(e[k] + f_1 e[k-1]) + cov(u[k-2+ \ldots$$

$$\Rightarrow \boxed{(1+f_1^2)\sigma_y^2 + f_1 \sigma_{yy}[1] = b_2^2 \sigma_u^2 + (1+f_1^2)\sigma_e^2.} \quad — ②$$

Multiplying both sides with $u[k-2]$ and taking expectation.

$$E(y[k]u[k-2] + f_1 E(y[k-1]u[k-2])) = b_2 E(u[k-2]u[k-2]).$$
$$+ E(e[k]u[k-2].)$$
$$+ f_1 E(e[k-1]u[k-2])$$

$$\sigma_{yu}[\ell] = E((y[k]-\mu_y.)(y[k]-\mu_u)) \quad \because (\mu_u = \mu_y = 0.)$$

$$\sigma_{uu}[R] = E(y[k]u[k-\ell]) \quad \text{or} \quad E(y[k-\ell]u[k]).$$

$$\Rightarrow \sigma_{yu}[2] + f_1 \sigma_{yu}[1] = b_2 \sigma_u^2 + 0. + 0.$$

$$\boxed{\sigma_{yu}[2] + f_1 \sigma_{yu}[1] = b_2 \sigma_u^2.} \quad \text{—— (3).}$$

Multiplying (1) with $y[k]$ on both sides and taking expectation.

$$E(y[k]y[k]) + f_1 E(y[k-1]y[k]) = b_2 E(y[k]u[k-2]) + 0 + 0.$$

$$\Rightarrow \boxed{\sigma_y^2 + f_1 \sigma_{yy}[1] = b_2 \sigma_{yu}[2].} \quad \text{—— (4)}$$

Multiplying (1). with $y[k-1]$ and taking expectation,

$$\boxed{\sigma_{yy}[1] + f_1 \sigma_y^2 = b_2 \sigma_{yu}[1]} \quad \text{—— (5).}$$

Solving ②, ③, ④ & ⑤,   [Mathematica was used].

$$\boxed{\sigma_y^2 = \frac{\sigma_e^2 + \sigma_e^2 f_1^2 + b_2^2 \sigma_u^2}{1 + f_1 + f_1^2}}$$

$$\boxed{\sigma_{yy}[1] = \frac{\sigma_e^2 + \sigma_e^2 f_1^2 + b_2^2 \sigma_u^2}{1 + f_1 + f_1^2}}$$

$$\boxed{\sigma_{yu}[1] = \frac{-\sigma_e^2(1 + f_1 + f_1^2 + f_1^3) + f^2 b_2^2 \sigma_u^2}{f(1 + f + f^2) b_2}}$$

$$\boxed{\sigma_{yu}[2] = \frac{(1+f_1)(\sigma_e^2 + \sigma_e^2 f^2 + b_2^2 \sigma_u^2}{(1 + f_1 + f_1^2) b_2}}$$

② → b

$$e[k] \longrightarrow \boxed{H(q^{-1})} \longrightarrow v[k]$$

$$\text{ACVF}(e[k]) \longrightarrow \boxed{H(q^{-1})} \longrightarrow \text{CCVF}(v[k], e[k])$$

$$\sigma_{ve}[l] = \sum_{n=0}^{\infty} h[n]\, \sigma_{ee}[l-n] = H(q^{-1})\, \sigma_{ee}[l]$$

$$e[k] \rightarrow WN \rightarrow \sigma_{ee}[l] = \begin{cases} \frac{\sigma_e^2}{2\pi}, & l = 0 \\ 0 & l \neq 0 \end{cases}$$

$\sigma_{ee}[l+n] = \{$

$$\sigma_{ve}[l] = \sum_{n=0}^{\infty} h[n]\, \sigma_{ee}[l-n].$$

$$\sigma_{ve}[l] = h[l] \cdot \frac{\sigma_e^2}{2\pi}.$$

$$\boxed{h[l] = \frac{\sigma_{ve}[l]}{\frac{\sigma_e^2}{2\pi}}}$$ → IR coefficients.

If $e[k]$ is coloured,

$$\sum_{n=0}^{\infty} \sigma_{ee}[l-n]\, h[n] = \sigma_{ve}[l].$$

$n=0 \Rightarrow \sigma_{ve}[l] = h[0]\, \sigma_{ee}[l].$

$l=0 \Rightarrow h[0]\sigma_{ee}[0] + h[1]\sigma_{ee}[-1] + \dots = \sigma_{ve}[0]$

$l=1 \Rightarrow h[0]\sigma_{ee}[1] + h[1]\sigma_{ee}[0] + \dots = \sigma_{ve}[1].$

$l=2 \Rightarrow h[0]\sigma_{ee}[2] + h[1]\sigma_{ee}[1] + \dots = \sigma_{ve}[2].$

These equations can be solved to get the IR coefficients.

Delay Estimation

Let $y[k] = A u[k-D] + e[k].$

multiplying $u[k-l]$ on both sides and taking expectation,

$$E(y[k]u[k-l]) = AE(u[k-D]\, u[k-l]) + E(e[k]u[k-l]).$$

Assuming open loop conditions where noise doesn't affect the input then,

$u[k]$ and $e[k]$ are uncorrelated (Irrespective of $e[k]$ being coloured or not).

$$\Rightarrow \sigma_{yu}[l] = A\,\sigma_{uu}[l-D] + \sigma_{ve}[l].$$

$$(\sigma_{ve}[l] = 0)$$

$$\boxed{\sigma_{yu}[l] = A\,\sigma_{uu}[l-D].}$$

as $\rho_{yu}[l]$ and $\sigma_{yu}[l]$ are functions of $\sigma_{uu}[l-D]$, when $l=D$, $\rho_{yu}[l]$ and $\rho_{uu}[l-D]$ peak at the same time.

$\boxed{\text{Thus, we can find the delay for both coloured and white noise using this method.}}$

③ (a) Given $\gamma_{vv}(f) = \dfrac{1.68}{2.5625 - 3.24\cos(\pi f) + 0.7\cos(2\pi f)}$

$$= \frac{1.68}{2.5625 - 3.24\cos\left(\frac{w}{2}\right) + 0.7\cos(w)}.$$

$$\gamma_{vv}(w) = |H(e^{-jw})|^2 \gamma_{ee}(w).$$

$$\gamma_{vv}(w) = |H(e^{-jw})|^2 \times \frac{\sigma_e^2}{2\pi}. \qquad (\sigma_e^2 = 1.)$$

$$\gamma_{vv}(w) \times 2\pi = |H(e^{-jw})|^2.$$

$$\Rightarrow |H(e^{-jw})| = \sqrt{\frac{1.68 \times 2\pi}{(2.5625 - 3.24\cos\left(\frac{w}{2}\right) + 0.7\cos w)}}$$

$$|a + b_1 e^{iw} + b_2 e^{iw/2}| = \sqrt{a^2 + b_1^2 + b_2^2 + 2ab_1\cos w + 2b_2(a+b_1)\cos\left(\frac{w}{2}\right)}$$

$$\sqrt{2.5625 - 3.24\cos(w/2) + 0.7\cos w} = \sqrt{(a + b_1^2 + b_2^2) + 2ab_1\cos w + 2b_2\cos\left(\frac{w}{2}\right)(a+b_1)}$$

Equating coefficients,

$a = 1.14612$. or $a = -1.14612$

$b_1 = 0.305$ $\qquad b_1 = -0.305$

$b_2 = -1.11609.$ $\qquad b_2 = 1.11609.$

Taking first case,

$$H(e^{-jw}) = \frac{\sqrt{1.68 \times 2\pi}}{1.14612 + 0.305e^{-jw} - 1.11609\, e^{-jw/2}}$$

Scanned with CamScanner

$$H(e^{-j\omega}) = \frac{3.248}{1.14612 + 0.305e^{-j\omega} - (1.255e^{-j\omega})^{1/2}}$$

$$\boxed{H(q^{-1}) = \frac{3.248}{1.14612 + 0.305q^{-1} - (1.255q^{-1})^{1/2}}}$$

$$\Rightarrow \quad v[k] = \frac{3.248}{1.15 + 0.31q^{-1} - (1.25q^{-1})^{1/2}} \cdot e[k]$$

3(b)

$\sigma_{vv}[l]$ = ACVF of the stationary process.

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \gamma_{vv}(\omega)\, e^{j\omega l} \cdot dl$$

$$\sigma_{vv}[l] = \frac{1.68}{2\pi}\int_{-\pi}^{\pi} \frac{(\cos \omega l + i \sin \omega l)\, d\omega}{2.5625 - 3.24\cos(\omega/2) + 0.7\cos\omega}$$

$$= \frac{1.68}{2\pi}\int_{-\pi}^{\pi} \frac{\cos(\omega l)\, d\omega}{2.5625 - 3.24\cos(\frac{\omega}{2}) + 0.7\cos(\omega)}$$

$$+ i \cdot \frac{1.68}{2\pi}\int_{-\pi} \frac{\sin(\omega l)\, .\, d\omega}{2.5625 - 3.24\cos\frac{\omega}{2} + 0.7\cos\omega}$$

Scanned with CamScanner

4) a

$$v[k] = \frac{1}{1+d_1 q^{-1}} e[k].$$

$$v[k] + d_1 v[k-1] = e[k].$$

$$v[k] = -d_1 v[k-1] + e[k].$$

l-step ahead prediction

$$v[k] = \sum_{n=0}^{l-1} h[n]\, e[k-n] + \sum_{n=l}^{\infty} h[n] . e[k-n].$$

$$H(q^{-1}) = \frac{1}{1+d_1 q^{-1}} = (1+d_1 q^{-1})^{-1}$$

$h[0] = 1$

$h[1] = -d_1$

$h[2] = d_1^2$

l-step ~~p~~ ahead prediction.

$$v[k] = \sum_{n=0}^{l-1} h[n]\, e[k-n].$$

$$+ \sum_{n=l}^{\infty} h[n]\, e(k-n].$$

$$H(q^{-1}) = \frac{1}{1+d_1 q^{-1}} = (1+d_1 q^{-1})^{-1}.$$

$$y = [k/k-l] = \left(1 + H_l' H^{-1}(q^{-1})\right) G(q^{-1}) u[k].$$

$$+ H_l'(q^{-1}) H^{-1}(q^{-1}) y[k].$$

$$H(q^{-1}) = \bar{H}_l(q^{-1}) + H_l'(q^{-1}) + H_l'(q^{-1}) H^{-1}(q^{-1})$$

$$1 - H_l'(q^{-1}) H^{-1}(q^{-1}) = \bar{H}_l(q^{-1}) H^{-1}(q^{-1})$$

$$\bar{H}_l = \sum_{n=0}^{2} h[n]\, e[k-n].$$

$$\bar{H}_e = (1 - d_1 q^{-1} + d_1^2 q^{-2})$$

$$\hat{y}[k/k-3] = \bar{H}_e(q^{-1}) H(q^{-1}) G(q^{-1}) u[k].$$
$$+ 1 - (1 - d_1 q^{-1} + d_1^2 q^{-2})(1 + d_1 q^{-1}) y[k]$$

$$\bar{H}_e(q^{-1}) H^{-1}(q^{-1}) = (1 + d_1 q^{-1} - d_1 q^{-1} - d_1^2 q^{-2} + d_1^2 q^{-2} + d_1^3 q^{-3})$$
$$\Rightarrow \boxed{1 + d_1^3 q^{-3} = \bar{H}_e(q^{-1}) H^{-1}(q^{-1})}$$

$$\hat{y}[k|k-3] = (1 + d_1^3 q^{-3}) \{ G(q^{-1}) u[k] - d_1^3 q^{-3} y[k].$$

Prediction Error:

$$y[k] - \hat{y}[k|k-3].$$
$$\Rightarrow y[k] - (((1 + d_1^3 q^{-3}) G(q^{-1}) u[k]$$
$$- d_1^3 y[k-3])).$$

$$PE = y[k]. + d_1^3 y[k-3] - (1 + d_1^3 q^{-3}) G(q^{-1}) u[k].$$

$$\mathrm{var}(PE) = \mathrm{var}[y[k] + d_1^3 y[k-3] - (1 + d_1^3 q^{-3}) G(q^{-1}) u[k]].$$

$$y[k] = G(q^{-1}) u[k] + H(q^{-1}) e[k].$$

$$\mathrm{var}(PE) = \mathrm{var}[d_1^3 y[k-3] - d_1^3 q^{-3} G(q^{-1}) u[k]]$$
$$+ \mathrm{var}[H(q^{-1}) e[k].]$$

as $e[k]$ is uncorrelated with $u[k]$ and $y[k]$.

Assuming $G(q^{-1}) = 0$.

$$var(PE) = var\left[d_1^3 y[k-3]\right] + (1+d_1)^{-1} \sigma_e^2.$$

$$\boxed{var(PE) = d_1^6 \sigma_y^2 + (1+d_1)^{-1} \sigma_e^2}$$

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(4) b.

$$\hat{y}[k|k-2] = L_1(q^{-1})u[k] + L_2(q^{-1})y[k].$$

$$L_1(q^{-1}) = \bar{H}_l(q^{-1})G(q^{-1}) \ ; \ L_2(q^{-1}) = 1 - \bar{H}_l(q^{-1})H^{-1}(q^{-1})$$

$$\bar{H}_l(q^{-1}) = \sum_{n=0}^{l-1} h[n]e[k-n] = h[0] + h[1]q^{-1}$$

$$\boxed{\bar{H}_l(q^{-1}) = 1 + h[1]q^{-1}}$$

$$L_2(q^{-1}) = 1 - \left(1 + h[1]q^{-1}\right)\left(H^{-1}(q^{-1})\right).$$

$$\Rightarrow \cancel{H^{-1}(q^{-1}) = (1 - L_2 q^{-1})(H^{-1}(q^{-}}$$

$$H^{-1}(q^{-1}) = \frac{1 - L_2(q^{-1})}{1 + h[1]q^{-1}}$$

$$H(q^{-1}) = \frac{1 + h[1]q^{-1}}{1 - L_2(q^{-1})}$$

$$\boxed{G(q^{-1}) = \frac{L_1(q^{-1})}{1 - L_2(q^{-1})}} \quad \boxed{H(q^{-1}) = \frac{1 + h[1]q^{-1}}{1 - L_2(q^{-1})}}$$