VEHICLE DYNAMICS-ASSIGNMENT 4

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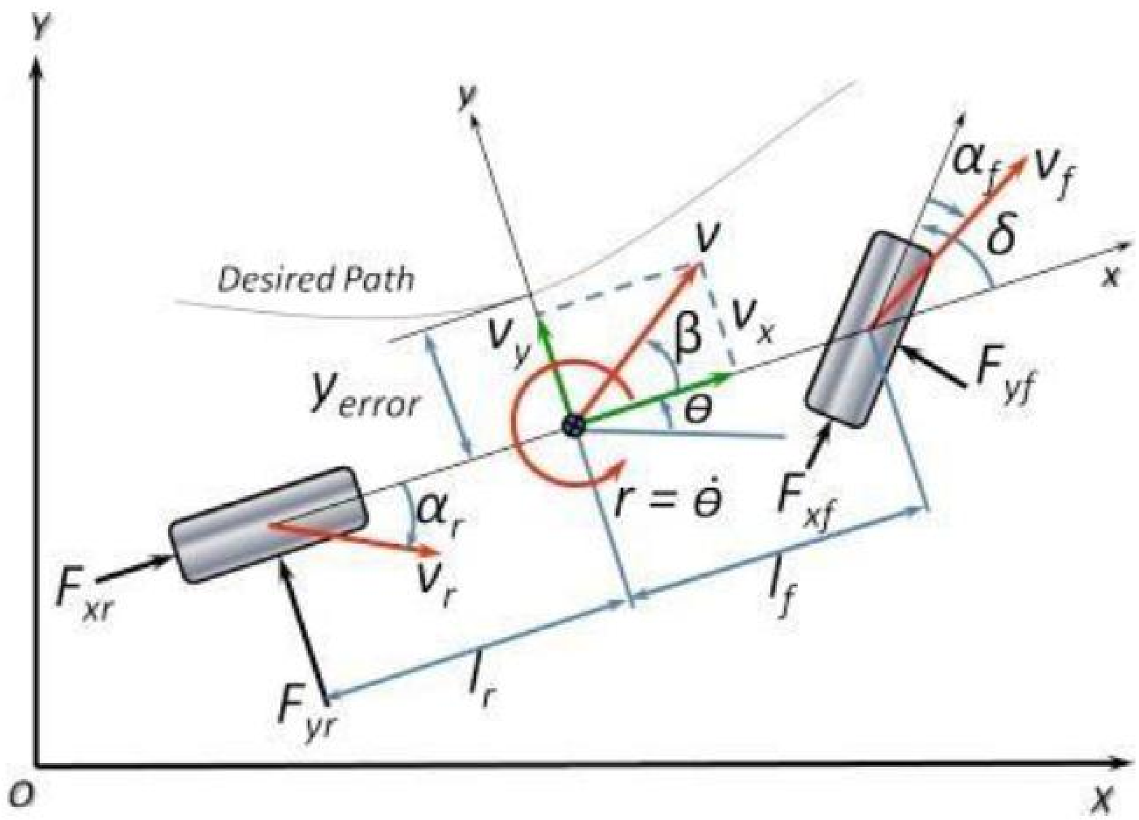


Figure 1 – Bicycle Vehicle Model

Given below are the nomenclature used for the expressions mentioned in figure 1:-

α_f is the front angle	; R is the turning radius
α_r is the rear slip angle	; β is the actual moving angle of vehicle
δ is the steering angle	; l is the wheelbase
v_y is the lateral velocity	; F_{yf} is the force on the front axle along y
v_x is the longitudinal velocity	; F_{yr} is the force on the rear axle along y
l_f is distance of front axle center from vehicle CG	; r is the yaw rate
l_r is distance of rear axle center from vehicle CG	; θ is the heading angle

From figure,

$$m(\dot{v}_y + r v_x) = F_{yf} + F_{yr}.$$

$$I_z \dot{r} = l_f \cdot F_{yf} - l_r \cdot F_{yr}.$$

Also, we can see from figure,

$$\tan(\alpha_r) = \frac{-(l_r \cdot r - v_y)}{v_x}.$$

$$\beta = \frac{v_y}{v_x}.$$

let α_r be very small,

$$\tan(\alpha_r) \approx \alpha_r = -\frac{l_r \cdot r}{v_x} + \frac{v_y}{v_x}.$$

$$\boxed{\alpha_r = \beta - \frac{l_r}{R}.}$$

Similarly,

$$\tan(\alpha_f) \approx \alpha_f = -\left(\delta - \frac{l_f r + v_y}{v_x}\right)$$

α_f becomes negative because as it is in the negative z-direction.

$$\alpha_f = -\left(\delta - \frac{l_f}{R} - \beta\right)$$

$$\delta_f = \frac{l_f}{R} + \alpha_r + \frac{l_r}{R} - \alpha_f$$

$$\boxed{\delta = \frac{l_f + l_r}{R} + \alpha_r - \alpha_f.}$$

Taking a linear tire model,

$$F_y = -C \cdot \alpha .$$

(negative α produces a +ve F_y.)

$$F_{yf} = -C_f \cdot \alpha_f . \qquad \& \quad F_{yr} = -C_r \cdot \alpha_r .$$

$$\therefore \quad m\left(\dot{V_y} + rV_x\right) = F_{yf} + F_{yr}.$$

$$m\left(\dot{V_y} + rV_x\right) = -C_f \alpha_f - C_r \cdot \alpha_r .$$

$$m\left(\dot{V_y} + rV_x\right) = -C_f\left(\frac{l_f + l_r}{R} - \delta + \beta - \frac{l_r}{R}\right) - C_r\left(\beta - \frac{l_r}{R}\right).$$

$$m\left(\dot{V_y} + rV_x\right) = \left(-\frac{C_f l_f}{V_x} + \frac{C_r l_r}{V_x}\right)r - \left(C_f + C_r\right)\beta + C_f \cdot \delta$$

$$m\dot{V_y} + mrV_x + \left(\frac{C_f l_f}{V_x} - \frac{C_r l_r}{V_x}\right)r + \left(C_f + C_r\right)\beta = C_f \cdot \delta$$

$$\therefore \boxed{m\dot{V_y} + \left(\frac{C_f + C_r}{V_x}\right)V_y + \left(mV_x + \frac{C_f l_f - C_r l_r}{V_x}\right)r = C_f \delta .}$$

$\Rightarrow$ lateral Motion of the Vehicle.

$$\therefore \boxed{\dot{V_y} = -\left(\frac{C_f + C_r}{mV_x}\right)V_y + \left(-\frac{C_f l_f + C_r l_r}{mV_x} - V_x\right)r + \left(\frac{C_f}{m}\right)\delta .}$$

From the torque equation,

$$I_z \dot{r} = l_f \cdot F_{yf} - l_r \cdot F_{yr}.$$

$$I_z \cdot \dot{r} = -l_f C_f \alpha_f + l_r C_r \alpha_r.$$

$$I_z \cdot \dot{r} = -l_f C_f \left(\frac{l_f + l_r}{R} - \delta + \beta - \frac{l_r}{R} \right) + l_r C_r \left(\beta - \frac{l_r}{R} \right).$$

$$I_z \dot{r} = \frac{-l_f^2 C_f r}{V_x} + l_f C_f \delta - \frac{l_f V_y C_f}{V_x} + \frac{l_r V_y \cdot C_r}{V_x} - \frac{l_r^2 C_r \cdot r}{V_x}.$$

$\therefore$

$$I_z \dot{r} + \left(\frac{C_f l_f - C_r l_r}{V_x} \right) V_y + \left(\frac{C_f l_f^2 + C_r l_r^2}{V_x} \right) r = C_f l_f \delta.$$

That is,

$$\boxed{\dot{r} = -\left(\frac{C_f l_f - C_r l_r}{I_z \cdot V_x} \right) V_y - \left(\frac{C_f l_f^2 + C_r l_r^2}{I_z \cdot V_x} \right) r + \left(\frac{C_f l_f}{I_z} \right) \delta.}$$

Writing equations for $\dot{V}_y$ and $\dot{r}$ in state-space form,

(A)

$$\begin{bmatrix} \dot{V}_y \\ \dot{r} \end{bmatrix} = \begin{bmatrix} -\dfrac{(C_f + C_r)}{m v_x} & -\dfrac{l_f C_f + l_r C_r}{m V_x} - V_x \\ -\dfrac{(l_f \cdot C_f - l_r C_r)}{I_z V_x} & -\dfrac{(l_f^2 C_f + l_r^2 C_r)}{I_z V_x} \end{bmatrix} \begin{bmatrix} V_y \\ r \end{bmatrix}$$

$$+ \ (B) \begin{bmatrix} C_f/m \\ \dfrac{l_f C_f}{I_z} \end{bmatrix} \delta.$$

If output required is V_y, $C = [1 \ \ 0]$.

If output required is r, $C = [0 \ \ 1]$.

$[D] = 0$.

Parameters Given,

$C_f \Rightarrow C_{af} = 60000 \, N/rad.$

$C_r \Rightarrow C_{ar} = 60000 \, N/rad.$

$m \Rightarrow m = 1000 \, kg.$

$I_z = 1650 \, kg m^2.$

$l_f = a = 1 m.$

$l_r = b = 1.5 m$

$v_x = 16 \, m/s.$

Calculating Matrices A, B, C and D. using the above parameters,

$$A = \begin{bmatrix} -7.5 & -14.125 \\ 1.13636 & -7.38636 \end{bmatrix}$$

$$B = \begin{bmatrix} 60 \\ 36.3636 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$D = 0.$$

Since $C = [0 \;\; 1]$, output is r.

$\therefore$ Converting it into Transfer function. form,

$$\frac{r(s)}{\delta(s)} = C.(sI - A)^{-1} B + D.$$

For getting both responses (Lateral velocity and Yaw rate), $C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Mathematica was ~~solve~~ used to solve the matrix equations and get Transfer Functions for both the outputs.

/Mathematica/

Steering angle input (step) = 8°

$$8^\circ \rightarrow 8 \times \frac{\pi}{180} = \frac{2\pi}{45} \text{ rad.}$$

$$\delta(s) = \frac{2\pi}{45 s}.$$

Now,

$$\boxed{\begin{aligned} \frac{r(s)}{\delta(s)} &= \frac{36.3636s + 340.909}{s^2 + 14.8864s + 71.4488} \\ \frac{V_y(s)}{\delta(s)} &= \frac{60s + -70.4543}{s^2 + 14.8864s + 71.4488} \end{aligned}}$$

Considering $\delta(s) = \frac{2\pi}{45 s}$,

$$r(s) = \frac{5.0774s + 47.6}{s(s^2 + 14.8864 + 71.4487)}.$$

$$v_y(s) = \frac{8.37758s - 9.83727.}{s(s^2 + 14.8864 + 71.4487)}$$

Taking partial Fractions,

$$r(s) = \frac{0.666209}{s} + \frac{-4.84011 - 0.666209 s}{s^2 + 14.8864 s + 71.4488}$$

$$v_y(s) = \frac{-0.13768}{s} + \frac{10.4272 + 0.137683 s}{s^2 + 14.8864 s + 71.4488}.$$

Taking Inverse Laplace Transform,

$$r(t) = 0.666209 + \left[0.02965 \sin(4.006t) - 0.666209 \cos(4.006t)\right] \cdot e^{-(7.443t)}$$

$$v_y(t) = \left[2.3471 \sin(4.006t) + 0.137683 \cos(4.006t)\right] \cdot e^{(-7.443t)} - 0.137683.$$

⇒ The above two are the ss-Model Analytical solutions.

2. (c).

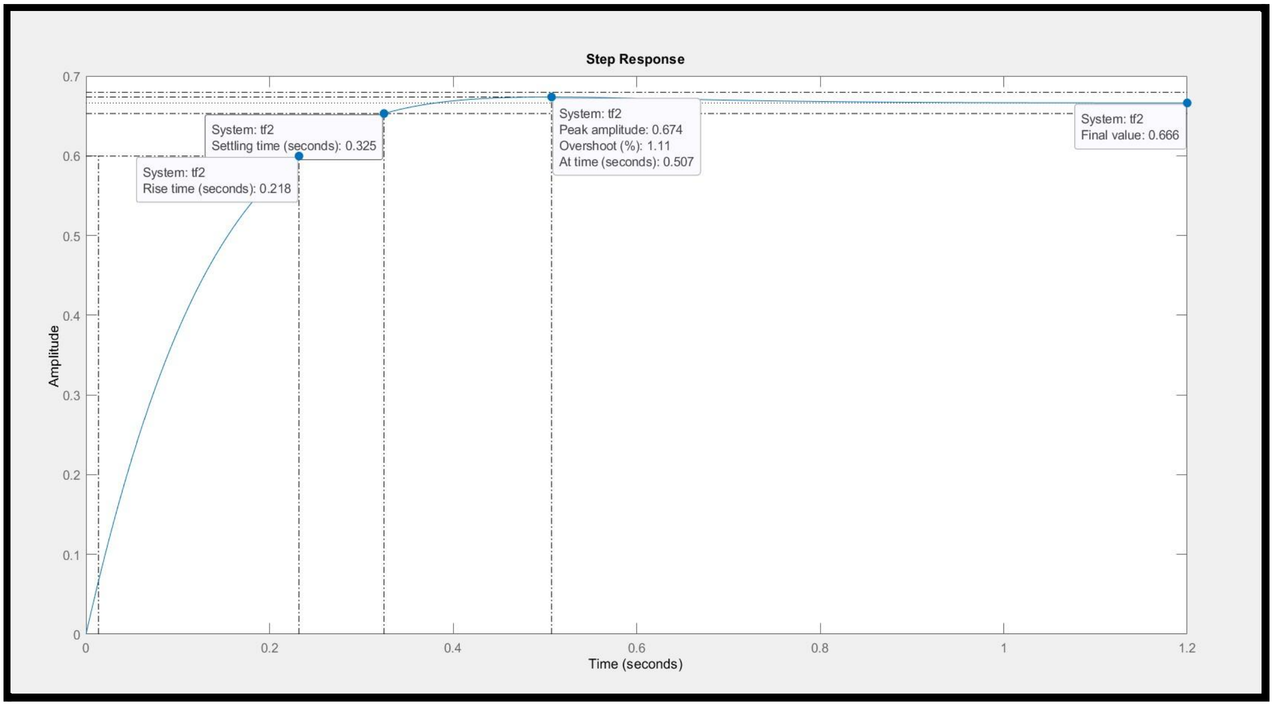
(i) Peak value = 0.674 rad/s.

→It overshoots by 1.1 %.

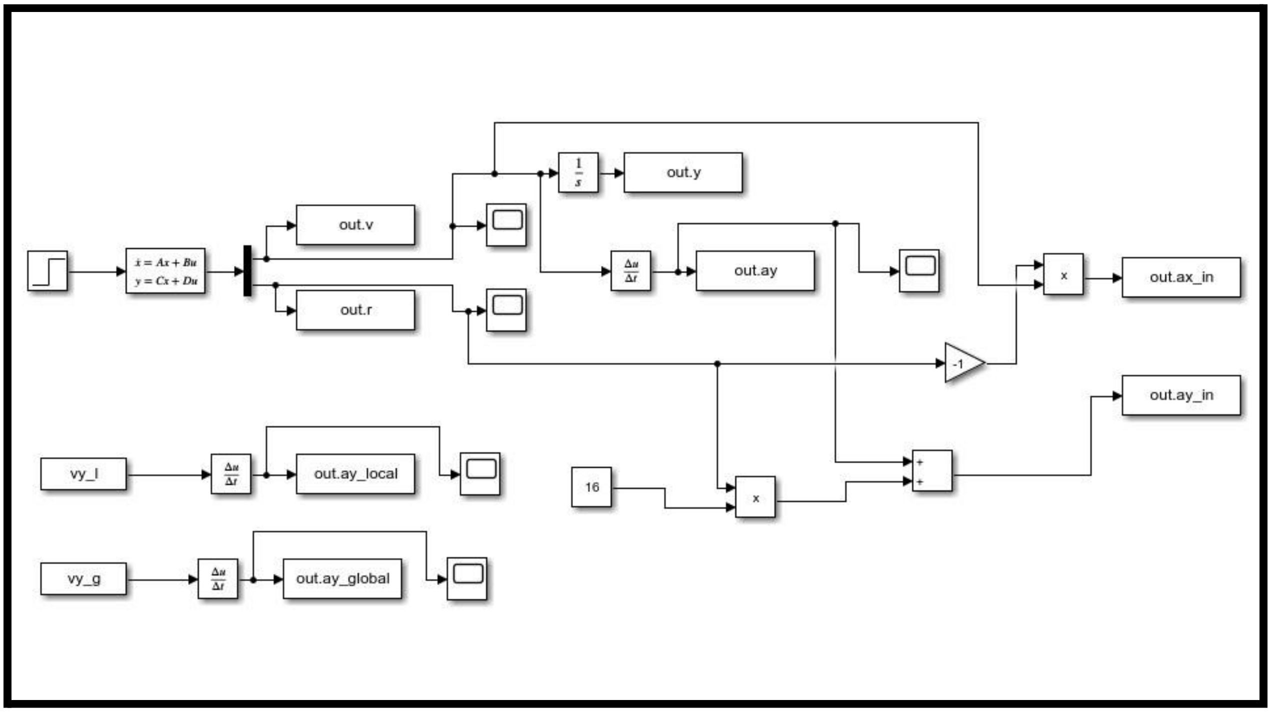
(ii) Steady state value = 0.666 rad/s.

(iii) Settling time = 0.325 seconds

(iv) Peak response time = 0.501 seconds.



Simulink Model



In[1]:=
```
A = {{-7.5, -14.125}, {1.13636, -7.38636}};
B = {{60}, {36.3636}};
C2 = {0, 1};
C1 = {1, 0};
i = IdentityMatrix[2];
```

In[6]:=
```
Mat1 = C1.Inverse[S.i - A].B;
Mat2 = C2.Inverse[S.i - A].B;
```

In[8]:=
```
Mat = {{s + 7.5, 14.125}, {-1.13636, s + 7.38636}};
```

In[9]:=
```
Inmat = Inverse[Mat];
```

In[ ]:=
```
Tf2 = C2.Inmat.B // Simplify
```

Out[ ]=
$$\left\{0. + \frac{340.909}{71.4488 + 14.8864\, s + s^2} + \frac{36.3636\, s}{71.4488 + 14.8864\, s + s^2}\right\}$$

In[ ]:=
```
Tf1 = C1.Inmat.B // Simplify
```

Out[ ]=
$$\left\{0. - \frac{70.4543}{71.4488 + 14.8864\, s + s^2} + \frac{60.\, s}{71.4488 + 14.8864\, s + s^2}\right\}$$

(b) Analytical Solution From the State Space Model and Transfer Function

In simulink, step input was given at 3 seconds and simulation was run for 12 seconds.

For better interpretability, xlim has been used in the plots.

```
%For Lateral Speed (analytical)
t=0:0.001:12;
v=(2.3471*sin(4.006*t)+0.137683*cos(4.006*t)).*exp(-7.443*t)-0.137683;
figure;
plot(t,v,LineWidth=2) ;hold on;

%For Lateral Speed (Transfer Functon)
tf1=tf([60*(2*pi)/45,-70.4543*(2*pi)/45],[1,14.8864,71.4488])
```

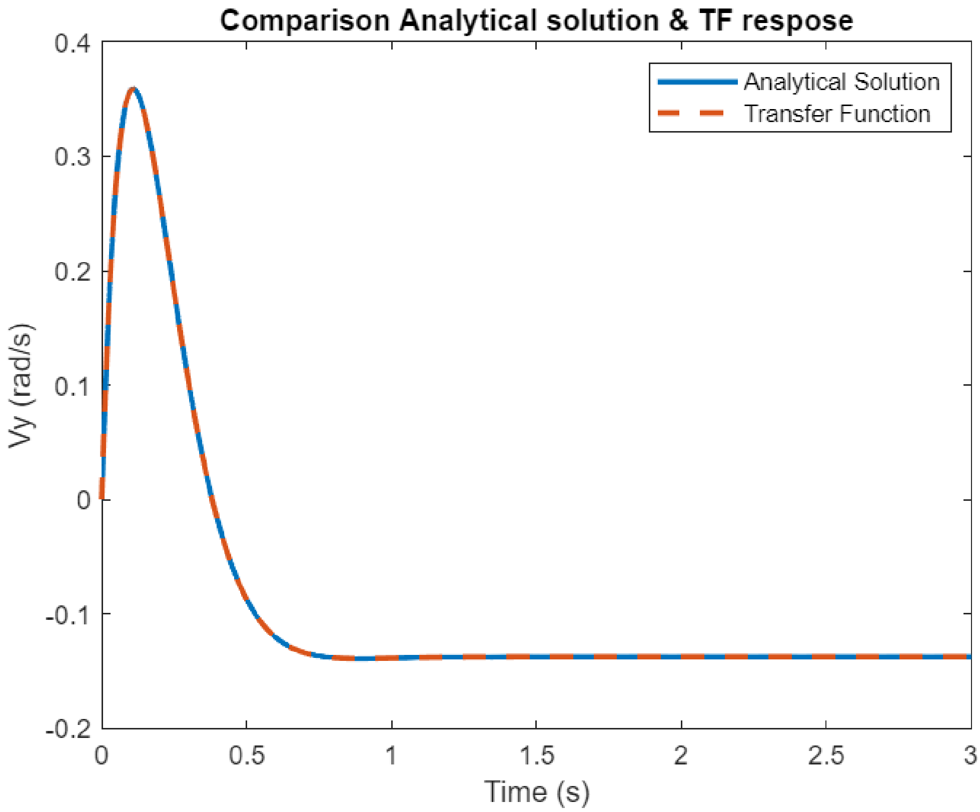
```
tf1 =

    8.378 s - 9.837
  ---------------------
  s^2 + 14.89 s + 71.45

Continuous-time transfer function.
```

```
stepres=step(tf1,t);
plot(t,stepres,'LineStyle','--',LineWidth=2)

title('Comparison Analytical solution & TF respose')
xlabel('Time (s)')
ylabel('Vy (rad/s)')
xlim([0,3])
legend('Analytical Solution','Transfer Function')
```




```
%For Yaw Rate (analytical)

r=(0.02965*sin(4.006*t)-0.666212*cos(4.006*t)).*exp(-7.443*t)+0.666212;
figure;
plot(t,r,LineWidth=2); hold on;

%For yaw rate (Transfer Functon)
tf2=tf([36.3636*(2*pi)/45,340.909*(2*pi)/45],[1,14.8864,71.4488])
```

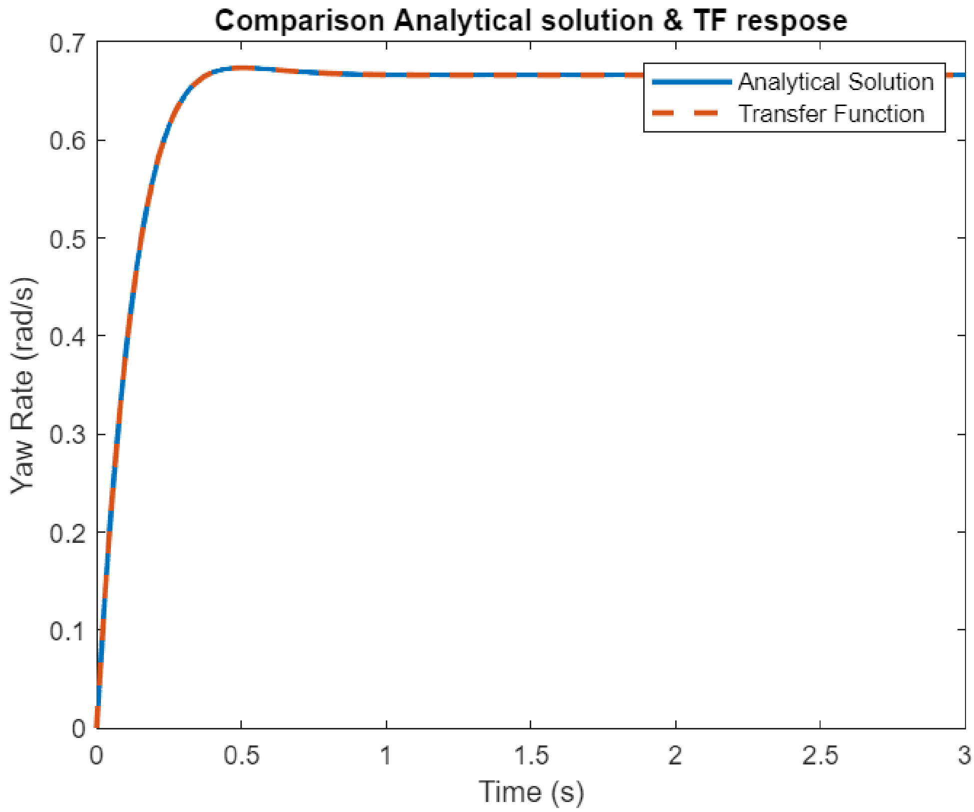
```
tf2 =

     5.077 s + 47.6
  ---------------------
  s^2 + 14.89 s + 71.45

Continuous-time transfer function.
```

```
stepres2=step(tf2,t);
plot(t,stepres2,'LineStyle','--',LineWidth=2)

title('Comparison Analytical solution & TF respose')
xlabel('Time (s)')
ylabel('Yaw Rate (rad/s)')
xlim([0,3])
legend('Analytical Solution','Transfer Function')
```



Frame of Reference changing

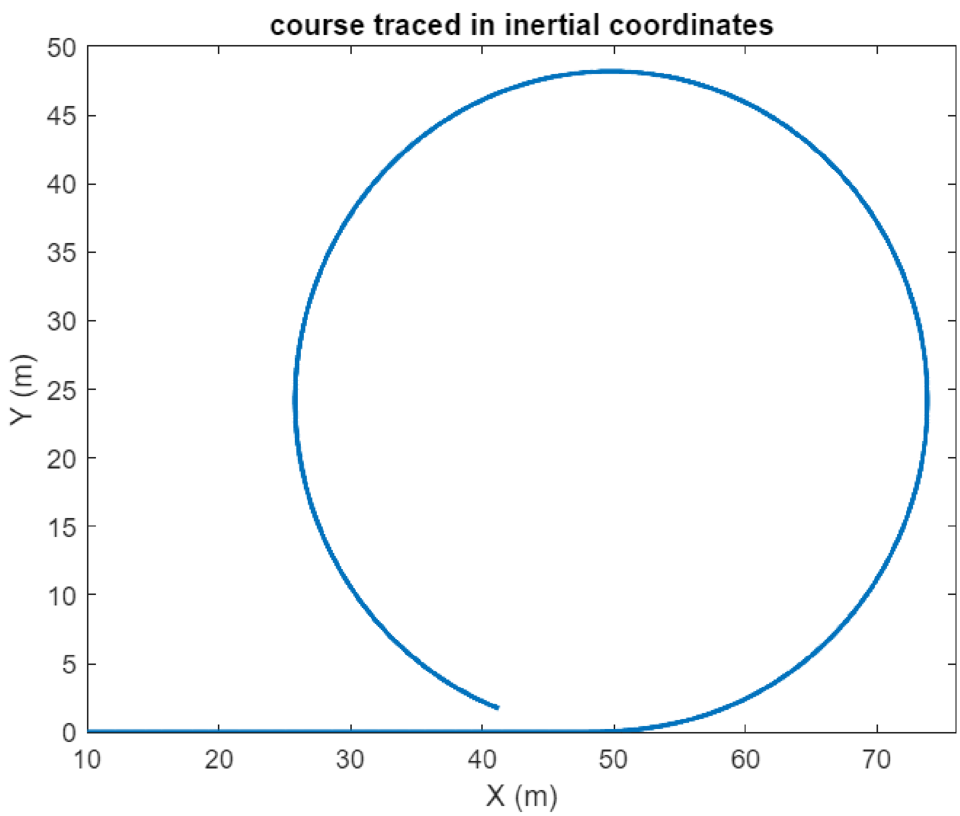
```
vx=16;
yaw_rate =out.r.Data;        %y(:,1)';
vy_local = out.v.Data ;   %y(:,2)';
vx_global = zeros(1,length(t));
vy_global = zeros(1,length(t));
yaw = cumtrapz(t,yaw_rate);
for k = 1:length(t)
    R = [cos(yaw(k)),-sin(yaw(k));...
        sin(yaw(k)),cos(yaw(k))];
    temp = R\[vy_local(k); vx];
    vy_global(k) = temp(1,:);
    vx_global(k) = temp(2,:);
end
vy_g=[t;vy_global]';
vy_l=[t;vy_local']';
x_g=cumtrapz(t,vx_global);
y_g=cumtrapz(t,vy_global);
```

(d)Course traced

The Longitudinal velocity of the vehicle is 16 m/s.

Before giving the step input, with 3 seconds, it runs for 48 meters in the x direction and this is visible in the plot.

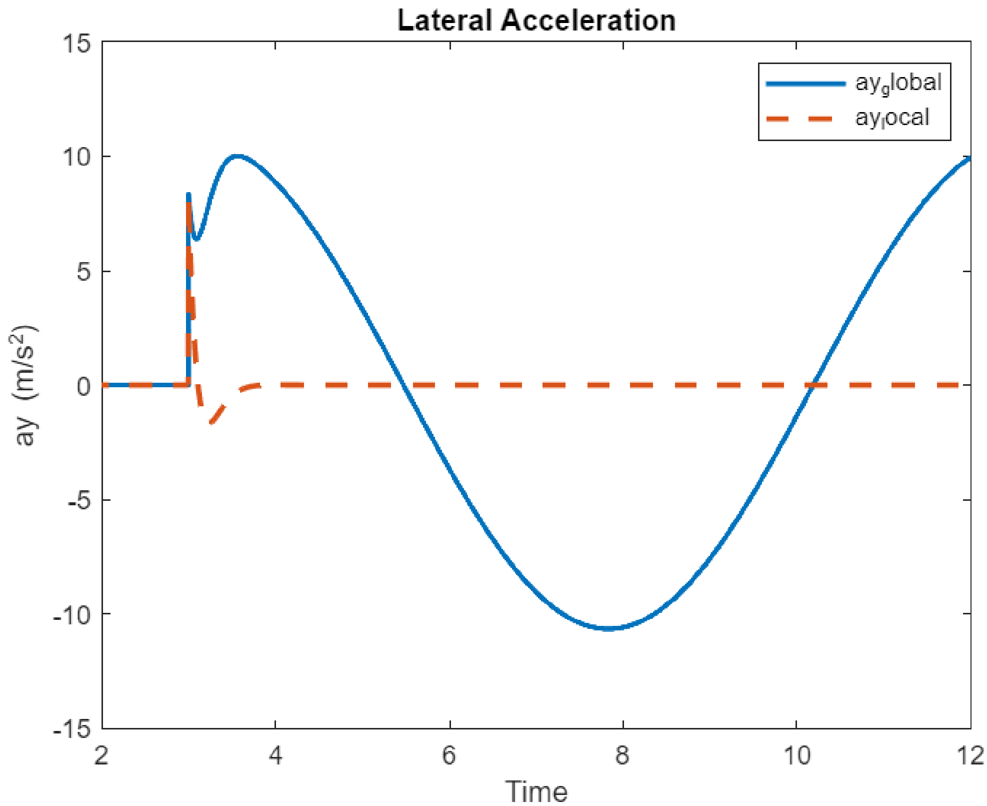
```
% for plotting the course of the vehicle
figure;
plot(x_g,y_g,LineWidth=2)
xlim([10,76])
title('Course traced in Inertial Coordinates')
xlabel('X (m)')
ylabel('Y (m)')
```



(e) Accelerations

```
figure;
plot(t,out.ay_global.Data(:,1),LineWidth=2);hold on;
plot(t,out.ay_local.Data,'linestyle','--',LineWidth=2)
title('Lateral Acceleration')

xlim([2,12])
xlabel('Time')
ylabel('ay (m/s^2)')
legend('ay_global','ay_local')
```


Lateral Acceleration
ay_global
ay_local
ay (m/s^2)
Time
15
10
5
0
-5
-10
-15
2
4
6
8
10
12