Distributed Orchestration of Concurrent Federated MARL Tasks over Communication-Constrained O-RAN

Amardip Kumar Singh, Kim Khoa Nguyen

École de Technologie Supérieure, Montreal, Canada

amardip-kumar.singh.1@ens.etsmtl.ca, kimkhoa.nguyen@etsmtl.ca

APPENDIX

We begin by establishing conditions under which the orchestration problem admits optimal solutions.

Lemma 1: [Compactness of Feasible Region]

The feasible region Θ defined by constraints (14)-(23) is compact (closed and bounded) in $\mathbb{R}^n$.

We show that Θ is both closed and bounded, hence compact by the Heine-Borel theorem [1].

Boundedness:

- Binary variables: $X_{t,a}, Y_{t,r,s}, z_t, R_{l,a} \in \{0,1\}$ form a finite set with $2^{|\mathcal{T}|\cdot|\mathcal{A}|+|\mathcal{T}|\cdot|\mathcal{R}|\cdot|\mathcal{S}|+|\mathcal{T}|+|\mathcal{L}|\cdot|\mathcal{A}|}$ possible configurations.
- Continuous allocation ratios: By definition, $u_{r,t} \in [0,1]$ and $v_t \in [0,1]$ for all $r \in \mathcal{R}, t \in \mathcal{T}$.
- Bandwidth variables: From constraint (18), $\phi_{l,t} = f_{l,t} \leq B_l$ for all $l \in \mathcal{L}, t \in \mathcal{T}$. Since available bandwidth $B_l < \infty$ is finite by physical network constraints, all $f_{l,t}$ are bounded.

Therefore, $\Theta \subseteq [0,1]^{n_1} \times [0, B_{\max}]^{n_2}$ for some dimensions n_1, n_2, where $B_{\max} = \max_{l \in \mathcal{L}} B_l < \infty$.

Closedness: All constraints (14)-(23) are defined by:

- Linear inequalities: $\sum_t X_{t,a} \leq 1$, $\sum_r \rho_{r,t} \leq \text{CPU}_{\text{total}}^{\text{DU}}$, etc.
- Linear equalities: $\rho_{r,t} = u_{r,t} \cdot M_a(t) \cdot \sum_{a \in \mathcal{A}_t} (X_{t,a} \cdot O_{r,a})$, etc.
- Implications that can be reformulated as linear constraints using standard techniques from integer programming [2].

Since Θ is the intersection of closed half-spaces and hyperplanes in $\mathbb{R}^n$, it is closed.

By Heine-Borel theorem, a subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded. Thus Θ is compact.

Lemma 2: [Continuity of Performance Functions] *Under the system model assumptions, the following functions are continuous over the feasible region Θ:*

1) Resource utility functions $R_U(\rho_{r,t})$, $R_V(\nu_t)$, $R_F(\phi_{l,t})$ defined in (5)-(7),
2) Policy performance metric $P_{\text{task}}(t)$ in (8),
3) QoS violation function $\text{QoS}_{r,s}^{\text{vio}}(t)$ in (10).

We prove continuity for each function class.

Part (i) - Resource Utilities:

For $R_U(\rho_{r,t}) = k_U \cdot \log(1 + \rho_{r,t}/M_{a,\text{req}})$ defined in (5):

- From (1), $\rho_{r,t} = u_{r,t} \cdot M_a(t) \cdot \sum_{a \in \mathcal{A}_t} (X_{t,a} \cdot O_{r,a})$.

- Since $u_{r,t} \geq 0$, $M_a(t) > 0$, and the sum is non-negative, we have $\rho_{r,t} \geq 0$.
- For any feasible solution with at least one agent assigned ($\sum_a X_{t,a} \geq 1$ when $z_t = 1$ by constraint (16)), we have $\rho_{r,t} > 0$.
- The logarithm function $\log : (0, \infty) \to \mathbb{R}$ is continuous [3].
- The composition $\log(1 + x)$ with the affine function $x = \rho_{r,t}/M_{a,\text{req}}$ is continuous by continuity of function composition.

Similarly, $R_V(\nu_t) = k_V \cdot \log(1 + \nu_t/M_{\text{xApp,req}})$ is continuous since $\nu_t = v_t \cdot M_{\text{xApp}}(t) \geq 0$ from (2), and the logarithm is continuous on $(0, \infty)$.

For $R_F(\phi_{l,t}) = k_F \cdot \sqrt{\phi_{l,t}/(\text{DT}_{\text{req}}/\text{time period})}$ from (7):

- From (3), $\phi_{l,t} = f_{l,t} \cdot B \geq 0$.
- The square root function $\sqrt{\cdot} : [0, \infty) \to [0, \infty)$ is continuous [3].

Part (ii) - Policy Performance:

From (8), $P_{\text{task}}(t) = k_P \cdot R_V(\nu_t) \cdot R_F(\phi_{\text{total},t})$ where $\phi_{\text{total},t} = \sum_{l \in \mathcal{L}} \phi_{l,t}$ from (4).

- The sum $\phi_{\text{total},t}$ is a continuous function (finite sum of continuous functions).
- By Part (i), both R_V and R_F are continuous.
- The product of continuous functions is continuous [3].

Part (iii) - QoS Violation:

From (9), the actual QoS is:

$$\text{QoS}_{r,s}^{\text{act}}(t) = \text{QoS}_{r,s}^{\text{base}} + Y_{t,r,s} \cdot (\omega_{r,s}^{\text{policy}} \cdot P_{\text{task}}(t) \\ + \omega_{r,s}^{\text{res},U} R_U(\rho_{r,t}) + \omega_{r,s}^{\text{res},V} R_V(\nu_t) + \omega_{r,s}^{\text{res},F} R_F(\phi_{\text{total},t}))$$

This is a weighted sum of continuous functions (by Parts (i) and (ii)), multiplied by the binary variable $Y_{t,r,s}$. Since $Y_{t,r,s} \in \{0,1\}$ and the expression is a polynomial in the variables, it is continuous.

For the violation in (10):

$$\text{QoS}_{r,s}^{\text{vio}}(t) = \max(0, \text{QoS}_{r,s}^{\text{req}} - \text{QoS}_{r,s}^{\text{act}}(t))$$

The maximum of two continuous functions is continuous [3]. Therefore, $\text{QoS}_{r,s}^{\text{vio}}(t)$ is continuous.

Now, considering the multi-objective optimization problem (13) subject to constraints (14)-(23). Under the following conditions:

1) The feasible region Θ is non-empty,
2) The cost coefficients satisfy $C_{\text{odu}}^{\text{comp}}, C_{\text{ric}}^{\text{comp}}, C^{\text{comm}}(l) \geq 0$ for all $l \in \mathcal{L}$,
3) The weights satisfy $W_{\text{reward}}, W_{\text{QoS}} \geq 0$,

the optimization problem admits at least one global optimal solution $\theta^* \in \Theta$, and the optimal objective value $J^* = J(\theta^*)$ is finite.

Proof: We apply the Weierstrass Extreme Value Theorem [3], which states that a continuous function on a compact set attains its minimum and maximum.

Step 1: Showing that J is continuous on Θ.
The objective function (13) is:

$$J(\theta) = \sum_{t \in \mathcal{T}} \sum_{r \in \mathcal{R}} \sum_{a \in \mathcal{A}_t} C_{\text{odu}}^{\text{comp}} \cdot \rho_{r,t} + C_{\text{ric}}^{\text{comp}} \cdot \nu_t$$
$$+ \sum_{l \in \mathcal{L}} \sum_{t \in \mathcal{T}} C^{\text{comm}}(l) \cdot \phi_{l,t}$$
$$- W_{\text{reward}} \sum_{t \in \mathcal{T}} R_{\text{global}}(t) \cdot z_t$$
$$+ W_{\text{QoS}} \sum_{r \in \mathcal{R}} \sum_{s \in \mathcal{S}} \sum_{t \in \mathcal{T}} \text{QoS}_{r,s}^{\text{vio}}(t) \cdot Y_{t,r,s}$$

Analyzing each term:

Cost Terms: The first two terms are linear combinations of $\rho_{r,t}$, ν_t, and $\phi_{l,t}$:
- From (1), (2), (3), these are linear or bilinear functions of the decision variables.
- Linear functions are continuous [3].
- Products of continuous functions (like $u_{r,t} \cdot X_{t,a}$) are continuous.
- Finite sums of continuous functions are continuous.

Reward Term: From (11), $R_{\text{global}}(t)$ is defined as:

$$R_{\text{global}}(t) = \lambda_{\text{QoS}} \cdot \sum_{r,s} Y_{t,r,s} \cdot \Delta\text{QoS}_{r,s}(t) - \lambda_{\text{cost}} \cdot \text{TotalCost}(t)$$

where $\Delta\text{QoS}_{r,s}(t)$ is the improvement component of $\text{QoS}_{r,s}^{\text{act}}(t)$ from (9), and $\text{TotalCost}(t)$ from (12) is:

$$\text{TotalCost}(t) = \sum_{r,a} C_{\text{odu}}^{\text{comp}} \cdot \rho_{r,t} + C_{\text{ric}}^{\text{comp}} \cdot \nu_t + \sum_l C^{\text{comm}}(l) \cdot \phi_{l,t}$$

- By Lemma 2, $\Delta\text{QoS}_{r,s}(t)$ is continuous (it's part of $\text{QoS}_{r,s}^{\text{act}}(t)$).
- $\text{TotalCost}(t)$ is a linear function, hence continuous.
- Therefore, $R_{\text{global}}(t)$ is continuous as a linear combination of continuous functions.
- The product $R_{\text{global}}(t) \cdot z_t$ is continuous (polynomial in variables).

QoS Penalty Term: By Lemma 2 Part (iii), $\text{QoS}_{r,s}^{\text{vio}}(t)$ is continuous. The product $\text{QoS}_{r,s}^{\text{vio}}(t) \cdot Y_{t,r,s}$ is continuous, and the finite sum is continuous.

Therefore, J is continuous on Θ as a sum of continuous functions.

Step 2: Applying Weierstrass Theorem.
By Lemma 1, Θ is compact. By Step 1, $J : \Theta \to \mathbb{R}$ is continuous. By the Weierstrass Extreme Value Theorem [3],

J attains its minimum on Θ. Therefore, there exists $\theta^* \in \Theta$ such that:

$$J(\theta^*) = \min_{\theta \in \Theta} J(\theta) =: J^*$$

Step 3: Proving finiteness of J^*.
Upper bound: By condition (a), $\Theta \neq \emptyset$. Let $\theta_0 \in \Theta$ be any feasible point. Then:

$$J^* = J(\theta^*) \leq J(\theta_0) < \infty$$

since all terms in $J(\theta_0)$ are finite (bounded variables, finite coefficients).

Lower bound: We show J is bounded below. Consider the worst-case scenario:

$$J(\theta) \geq -W_{\text{reward}} \sum_{t \in \mathcal{T}} R_{\text{global}}(t) \cdot z_t$$
$$= -W_{\text{reward}} \sum_{t \in \mathcal{T}} z_t \left[\lambda_{\text{QoS}} \sum_{r,s} Y_{t,r,s} \cdot \Delta\text{QoS}_{r,s}(t) - \lambda_{\text{cost}} \cdot \text{TotalCost}(t) \right.$$

Since $\text{TotalCost}(t) \geq 0$ by condition (b), the most negative contribution from the reward term occurs when costs are zero and QoS improvements are maximized. From (9), the QoS improvement is bounded by the utility functions:

$$\Delta\text{QoS}_{r,s}(t) \leq \omega_{r,s}^{\text{policy}} \cdot k_P \cdot M_V \cdot M_F + \omega_{r,s}^{\text{res},U} M_U$$
$$+ \omega_{r,s}^{\text{res},V} M_V + \omega_{r,s}^{\text{res},F} M_F$$

where M_U, M_V, M_F are the upper bounds on the utility functions (which are bounded since logarithm and square root of bounded arguments are bounded).

Therefore:

$$J(\theta) \geq -W_{\text{reward}} \cdot |\mathcal{T}| \cdot |\mathcal{R}| \cdot |\mathcal{S}| \cdot \lambda_{\text{QoS}} \cdot C_{\max} > -\infty$$

for some constant $C_{\max}$ depending on the system parameters. Thus, $-\infty < J^* < \infty$, proving finiteness.

[Performance Estimate State] Let $\mathcal{E}^{(k)} = \{E^{(k)}[R_{\text{global}}(t)], E^{(k)}[\text{Cost}(t)], E^{(k)}[\text{QoS}_{r,s}^{\text{vio}}(t)]\}_{t \in \mathcal{T}, r \in \mathcal{R}, s \in \mathcal{S}}$ denote the vector of estimated performance metrics at iteration k, used as input to the TSA sub-problem in Stage 1.

[Piecewise Linear Approximation Error] Let $R : [a, b] \to \mathbb{R}$ be any of the utility functions R_U, R_V, R_F satisfying: label=()
1) $R \in C^2[a, b]$ (twice continuously differentiable),
2) $|R''(x)| \leq M$ for all $x \in [a, b]$.
If $R^N(x)$ is the N-point piecewise linear approximation with uniform breakpoints $x_j = a + jh$ for $j = 0, 1, \ldots, N$ where $h = (b - a)/N$, then

$$\|R - R^N\|_\infty := \sup_{x \in [a,b]} |R(x) - R^N(x)| \leq \frac{Mh^2}{8} = \frac{M(b-a)^2}{8N^2}.$$

This is a standard result from numerical analysis [4]. We provide the derivation for completeness.

On each subinterval $[x_j, x_{j+1}]$, the piecewise linear approximation is:

$$R^N(x) = R(x_j) + \frac{R(x_{j+1}) - R(x_j)}{h}(x - x_j)$$

By Taylor's theorem with Lagrange remainder [3], for any $x \in [x_j, x_{j+1}]$:

$$R(x) = R(x_j) + R'(x_j)(x - x_j) + \frac{1}{2}R''(\xi_1)(x - x_j)^2$$

for some $\xi_1 \in (x_j, x)$, and:

$$R(x_{j+1}) = R(x_j) + R'(x_j)h + \frac{1}{2}R''(\xi_2)h^2$$

for some $\xi_2 \in (x_j, x_{j+1})$.

From the second equation:

$$\frac{R(x_{j+1}) - R(x_j)}{h} = R'(x_j) + \frac{1}{2}R''(\xi_2)h$$

The approximation error at x is:

$$
\begin{aligned}
|R(x) - R^N(x)| &= \left| R'(x_j)(x - x_j) + \frac{1}{2}R''(\xi_1)(x - x_j)^2 \right. \\
&\quad \left. - \left[R'(x_j) + \frac{1}{2}R''(\xi_2)h \right](x - x_j) \right| \\
&= \left| \frac{1}{2}R''(\xi_1)(x - x_j)^2 - \frac{1}{2}R''(\xi_2)h(x - x_j) \right| \\
&\leq \frac{1}{2}|R''(\xi_1)|(x - x_j)^2 + \frac{1}{2}|R''(\xi_2)|h|x - x_j| \\
&\leq \frac{M}{2}(x - x_j)^2 + \frac{M}{2}h(x - x_j)
\end{aligned}
$$

where we used condition (ii): $|R''(\xi)| \leq M$ for all ξ.

To find the maximum, let $s = x - x_j \in [0, h]$ and define:

$$g(s) = \frac{M}{2}s^2 + \frac{M}{2}hs = \frac{M}{2}s(s + h)$$

Taking the derivative: $g'(s) = M(s + h/2)$, which is always positive for $s \geq 0$. However, this analysis is too coarse. A tighter bound comes from analyzing the error at the midpoint.

Using the classical result [4], the maximum interpolation error for linear interpolation occurs at the midpoint $x = x_j + h/2$:

$$\max_{x \in [x_j, x_{j+1}]} |R(x) - R^N(x)| \leq \frac{Mh^2}{8}$$

Taking the supremum over all subintervals:

$$
\begin{aligned}
\|R - R^N\|_\infty &= \max_{j=0,\ldots,N-1} \max_{x \in [x_j, x_{j+1}]} |R(x) - R^N(x)| \\
&\leq \frac{Mh^2}{8} = \frac{M(b-a)^2}{8N^2}
\end{aligned}
$$

Therefore, the approximation error is $O(N^{-2})$.

For the logarithmic utilities R_U, R_V in (5)-(6), the second derivative is:

$$\frac{d^2}{dx^2}\log(1 + x) = -\frac{1}{(1+x)^2}$$

which is bounded on any compact interval $[a, b]$ with $a > 0$. Similarly, for $R_F(\phi) = k_F\sqrt{\phi}$ in (7), the second derivative $-\frac{1}{4}\phi^{-3/2}$ is bounded away from zero. Thus, Lemma applies to all utility functions in our model.

[Estimate Update Contraction] Suppose the performance estimate update mechanism follows an exponential moving average:

$$\mathcal{E}^{(k+1)} = (1 - \lambda) \cdot \mathcal{E}^{(k)} + \lambda \cdot \mathcal{M}^{(k)}$$

where $\mathcal{M}^{(k)}$ represents the measured performance from executing the FL MARL tasks at iteration k, and $0 < \lambda < 1$ is a learning rate. If the measurement noise is bounded:

$$\|\mathcal{M}^{(k)} - \mathcal{E}^*\| \leq \sigma$$

for all k, where $\mathcal{E}^*$ represents the true equilibrium estimate and $\|\cdot\|$ denotes the Euclidean norm, then the estimate mapping $\mathcal{T} : \mathcal{E}^{(k)} \mapsto \mathcal{E}^{(k+1)}$ is a contraction with:

$$\|\mathcal{E}^{(k+1)} - \mathcal{E}^*\| \leq (1 - \lambda)\|\mathcal{E}^{(k)} - \mathcal{E}^*\| + \lambda\sigma$$

By the definition of the update mechanism:

$$
\begin{aligned}
\mathcal{E}^{(k+1)} - \mathcal{E}^* &= (1 - \lambda)\mathcal{E}^{(k)} + \lambda\mathcal{M}^{(k)} - \mathcal{E}^* \\
&= (1 - \lambda)\mathcal{E}^{(k)} + \lambda\mathcal{M}^{(k)} - [(1 - \lambda) + \lambda]\mathcal{E}^* \\
&= (1 - \lambda)(\mathcal{E}^{(k)} - \mathcal{E}^*) + \lambda(\mathcal{M}^{(k)} - \mathcal{E}^*)
\end{aligned}
$$

Taking norms and applying the triangle inequality [3]:

$$
\begin{aligned}
\|\mathcal{E}^{(k+1)} - \mathcal{E}^*\| &= \|(1 - \lambda)(\mathcal{E}^{(k)} - \mathcal{E}^*) + \lambda(\mathcal{M}^{(k)} - \mathcal{E}^*)\| \\
&\leq \|(1 - \lambda)(\mathcal{E}^{(k)} - \mathcal{E}^*)\| + \|\lambda(\mathcal{M}^{(k)} - \mathcal{E}^*)\| \\
&= (1 - \lambda)\|\mathcal{E}^{(k)} - \mathcal{E}^*\| + \lambda\|\mathcal{M}^{(k)} - \mathcal{E}^*\| \\
&\leq (1 - \lambda)\|\mathcal{E}^{(k)} - \mathcal{E}^*\| + \lambda\sigma
\end{aligned}
$$

where the last inequality follows from the bounded noise assumption.

Setting $\beta = 1 - \lambda \in (0, 1)$, we have:

$$\|\mathcal{E}^{(k+1)} - \mathcal{E}^*\| \leq \beta\|\mathcal{E}^{(k)} - \mathcal{E}^*\| + (1 - \beta)\sigma$$

This is a contraction mapping with contraction constant $\beta < 1$ plus a bounded perturbation term.

The exponential moving average update with bounded noise is a standard approach in federated optimization to handle heterogeneous network conditions and measurement variance [5]. The choice of learning rate λ trades off between fast adaptation (λ close to 1) and steady-state accuracy (small λ reduces sensitivity to noise).

[Fixed-Point Convergence of Iterative Orchestration] Consider the iterative orchestration framework where at each iteration k: label=()

1) TSA (Stage 1) selects assignments $(X_{t,a}^{(k)}, Y_{t,r,s}^{(k)}, z_t^{(k)})$ based on estimates $\mathcal{E}^{(k)}$ using Algorithm 1,
2) RAR (Stage 2) optimizes resources $(u_{r,t}^{(k)}, v_t^{(k)}, f_{l,t}^{(k)}, R_{l,a}^{(k)})$ given fixed assignments using N-point PWL approximation,
3) FL MARL tasks execute with allocated resources and produce measurements $\mathcal{M}^{(k)}$,
4) Estimates update via $\mathcal{E}^{(k+1)} = \mathcal{T}(\mathcal{E}^{(k)}, \mathcal{M}^{(k)})$ as in Lemma .

Under the following conditions: label=()

1) The estimate update mapping $\mathcal{T}$ satisfies the contraction property of Lemma with $\beta = 1 - \lambda < 1$,

achieved through exponential moving average updates as employed in federated optimization for heterogeneous networks [5],
2) The measurement noise is uniformly bounded: $\|\mathcal{M}^{(k)} - \mathcal{E}^*\| \leq \sigma$ for all $k \geq 0$,
3) The TSA assignments are stable under small estimate perturbations: there exists $\epsilon > 0$ such that if $\|\mathcal{E}_1 - \mathcal{E}_2\| < \epsilon$, then Algorithm 1 produces identical assignments,
4) The objective function J is L-Lipschitz continuous in the resource variables for fixed assignments [6]:

$$|J(\xi, \psi_1) - J(\xi, \psi_2)| \leq L\|\psi_1 - \psi_2\|$$

for any fixed assignment ξ and resource allocations ψ_1, ψ_2,

the following hold: label=()
1) **(Unique Fixed Point)** There exists a unique fixed-point estimate $\mathcal{E}^* \in \mathbb{R}^n$ such that $\mathcal{T}(\mathcal{E}^*, \mathcal{E}^*) = \mathcal{E}^*$,
2) **(Geometric Convergence of Estimates)** The estimates converge geometrically to a neighborhood of the fixed point:

$$\|\mathcal{E}^{(k)} - \mathcal{E}^*\| \leq \beta^k\|\mathcal{E}^{(0)} - \mathcal{E}^*\| + \frac{\sigma}{1 - \beta}$$

3) **(Assignment Stabilization)** There exists a finite iteration $K < \infty$ such that the assignments remain constant for all $k \geq K$:

$$(X_{t,a}^{(k)}, Y_{t,r,s}^{(k)}, z_t^{(k)}) = (X_{t,a}^{(K)}, Y_{t,r,s}^{(K)}, z_t^{(K)}) \quad \forall k \geq K$$

4) **(Objective Convergence)** The objective value converges geometrically to a neighborhood of the optimal:

$$J(\theta^{(k)}) - J^* \leq \Delta_0\beta^k + C_\infty$$

where $\Delta_0 = L\|\mathcal{E}^{(0)} - \mathcal{E}^*\|$ and $C_\infty = \frac{L\sigma}{1-\beta} + \epsilon_N$ represents the steady-state approximation error, with $\epsilon_N = O(N^{-2})$ from Lemma .

We prove each part sequentially.

Part (i) - Unique Fixed Point:

By condition (a), the estimate update mapping $\mathcal{T}$ is a contraction. Specifically, from Lemma , for any two estimate vectors $\mathcal{E}_1, \mathcal{E}_2$:

$$\|\mathcal{T}(\mathcal{E}_1, \mathcal{M}) - \mathcal{T}(\mathcal{E}_2, \mathcal{M})\| \leq \beta\|\mathcal{E}_1 - \mathcal{E}_2\|$$

where $\beta < 1$.

Consider the space $(\mathbb{R}^n, \|\cdot\|)$ with $n = |\mathcal{T}| \times (2 + |\mathcal{R}| \times |\mathcal{S}|)$ (the dimension of the estimate vector). This is a complete metric space [3].

By the Banach Fixed-Point Theorem [7], a contraction mapping on a complete metric space has a unique fixed point. Therefore, there exists a unique $\mathcal{E}^* \in \mathbb{R}^n$ such that:

$$\mathcal{E}^* = \mathcal{T}(\mathcal{E}^*, \mathcal{E}^*)$$

This fixed point represents the equilibrium estimate state where the measured performance matches the predicted performance.

Part (ii) - Geometric Convergence of Estimates:

From Lemma , we have:

$$\|\mathcal{E}^{(k+1)} - \mathcal{E}^*\| \leq \beta\|\mathcal{E}^{(k)} - \mathcal{E}^*\| + (1 - \beta)\sigma$$

Applying this inequality recursively:

$$\begin{aligned}
\|\mathcal{E}^{(k)} - \mathcal{E}^*\| &\leq \beta\|\mathcal{E}^{(k-1)} - \mathcal{E}^*\| + (1 - \beta)\sigma \\
&\leq \beta\left[\beta\|\mathcal{E}^{(k-2)} - \mathcal{E}^*\| + (1 - \beta)\sigma\right] + (1 - \beta)\sigma \\
&= \beta^2\|\mathcal{E}^{(k-2)} - \mathcal{E}^*\| + (1 - \beta)\sigma(1 + \beta) \\
&\vdots \\
&\leq \beta^k\|\mathcal{E}^{(0)} - \mathcal{E}^*\| + (1 - \beta)\sigma\sum_{i=0}^{k-1}\beta^i
\end{aligned}$$

The geometric series converges [3]:

$$\sum_{i=0}^{k-1}\beta^i = \frac{1 - \beta^k}{1 - \beta} < \frac{1}{1 - \beta}$$

Therefore:

$$\|\mathcal{E}^{(k)} - \mathcal{E}^*\| \leq \beta^k\|\mathcal{E}^{(0)} - \mathcal{E}^*\| + \frac{\sigma}{1 - \beta}$$

The first term vanishes exponentially fast as $k \to \infty$ since $\beta < 1$. The second term represents the steady-state error induced by measurement noise.

Part (iii) - Assignment Stabilization:

From Part (ii), for any $\delta > 0$, there exists K_δ such that for all $k \geq K_\delta$:

$$\|\mathcal{E}^{(k)} - \mathcal{E}^*\| < \beta^k\|\mathcal{E}^{(0)} - \mathcal{E}^*\| + \frac{\sigma}{1 - \beta} < \frac{\epsilon}{2}$$

where ϵ is the threshold from condition (c), choosing K_δ large enough such that:

$$\beta^{K_\delta}\|\mathcal{E}^{(0)} - \mathcal{E}^*\| < \frac{\epsilon}{2} - \frac{\sigma}{1 - \beta}$$

This is always possible since $\beta^k \to 0$ as $k \to \infty$, provided $\frac{\sigma}{1-\beta} < \frac{\epsilon}{2}$ (which can be ensured by choosing sufficiently small λ or reducing noise σ).

For any two iterations $k_1, k_2 \geq K_\delta$, by the triangle inequality:

$$\begin{aligned}
\|\mathcal{E}^{(k_1)} - \mathcal{E}^{(k_2)}\| &\leq \|\mathcal{E}^{(k_1)} - \mathcal{E}^*\| + \|\mathcal{E}^{(k_2)} - \mathcal{E}^*\| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

By condition (c), Algorithm 1 produces identical assignments when estimate differences are smaller than ϵ. Therefore, for all $k \geq K := K_\delta$:

$$(X_{t,a}^{(k)}, Y_{t,r,s}^{(k)}, z_t^{(k)}) = (X_{t,a}^{(K)}, Y_{t,r,s}^{(K)}, z_t^{(K)})$$

The assignments stabilize in finite time.

Part (iv) - Objective Convergence:

Let $\theta^{(k)} = (\xi^{(k)}, \psi^{(k)})$ denote the complete solution at iteration k, where $\xi^{(k)} = (X^{(k)}, Y^{(k)}, z^{(k)})$ are assignments and $\psi^{(k)} = (u^{(k)}, v^{(k)}, f^{(k)}, R^{(k)})$ are resource allocations.

Decompose the suboptimality:

$$J(\theta^{(k)}) - J^* = J(\theta^{(k)}) - J(\theta^*)$$
$$= \underbrace{[J(\xi^{(k)}, \psi^{(k)}) - J(\xi^{(k)}, \psi^*(\xi^{(k)}))]}_{\text{Term A: RAR suboptimality}}$$
$$+ \underbrace{[J(\xi^{(k)}, \psi^*(\xi^{(k)})) - J(\xi^*, \psi^*(\xi^*))]}_{\text{Term B: TSA suboptimality}} \quad (1)$$

where $\psi^*(\xi)$ denotes the optimal resource allocation for fixed assignment ξ.

Bound Term A (RAR Error):

In Stage 2, the RAR problem uses PWL approximation to linearize the nonlinear utility functions. Let J_2^N denote the linearized objective with N breakpoints, and let $\psi^{(k),N}$ be its optimal solution computed by the MILP solver.

Since the MILP is solved to global optimality for the linearized problem:

$$J_2^N(\xi^{(k)}, \psi^{(k),N}) \le J_2^N(\xi^{(k)}, \psi^*(\xi^{(k)}))$$

By Lemma , each utility function has approximation error $O(N^{-2})$. Since the objective (13) contains a constant number of utility function evaluations (proportional to $|\mathcal{T}| \cdot |\mathcal{R}| \cdot |\mathcal{S}|$), the total approximation error is:

$$|J(\xi, \psi) - J_2^N(\xi, \psi)| \le C \cdot \frac{M(b-a)^2}{8N^2} =: \epsilon_N$$

for some constant C depending on the problem size and weight coefficients.

Therefore:

$$J(\xi^{(k)}, \psi^{(k)}) \le J_2^N(\xi^{(k)}, \psi^{(k)}) + \epsilon_N$$
$$\le J_2^N(\xi^{(k)}, \psi^*(\xi^{(k)})) + \epsilon_N$$
$$\le J(\xi^{(k)}, \psi^*(\xi^{(k)})) + 2\epsilon_N$$

Thus: Term A $\le 2\epsilon_N = O(N^{-2})$.

Bound Term B (TSA Error):

Algorithm 1 (greedy TSA) selects assignments based on estimates $\mathcal{E}^{(k)}$. The assignment quality depends on how close these estimates are to the true equilibrium.

By condition (d), the objective is L-Lipschitz in the resource variables. Since $\psi^*(\xi)$ is the optimal allocation for assignment ξ, we can write:

$$J(\xi^{(k)}, \psi^*(\xi^{(k)})) = \min_{\psi} J(\xi^{(k)}, \psi) =: \tilde{J}(\xi^{(k)})$$

The TSA decisions at iteration k depend on $\mathcal{E}^{(k)}$. By design, Algorithm 1 attempts to minimize the expected objective based on these estimates. While the greedy algorithm does not guarantee global optimality, the quality of its solution improves as estimates improve.

Assume the greedy TSA satisfies a stability property: there exists a constant $\kappa > 0$ such that:

$$\tilde{J}(\xi^{(k)}) - \tilde{J}(\xi^*) \le \kappa \|\mathcal{E}^{(k)} - \mathcal{E}^*\|$$

This assumes that better estimates lead to better assignment decisions, which is reasonable for greedy algorithms that monotonically improve with better information [8].

From Part (ii):

$$\|\mathcal{E}^{(k)} - \mathcal{E}^*\| \le \beta^k \|\mathcal{E}^{(0)} - \mathcal{E}^*\| + \frac{\sigma}{1-\beta}$$

Therefore:

$$\text{Term B} = J(\xi^{(k)}, \psi^*(\xi^{(k)})) - J(\xi^*, \psi^*(\xi^*))$$
$$\le \kappa \|\mathcal{E}^{(k)} - \mathcal{E}^*\|$$
$$\le \kappa \left[\beta^k \|\mathcal{E}^{(0)} - \mathcal{E}^*\| + \frac{\sigma}{1-\beta} \right]$$

Combining Both Terms:

From equation (1):

$$J(\theta^{(k)}) - J^* \le \text{Term A} + \text{Term B}$$
$$\le 2\epsilon_N + \kappa \beta^k \|\mathcal{E}^{(0)} - \mathcal{E}^*\| + \frac{\kappa \sigma}{1-\beta}$$

Setting $\Delta_0 = \kappa \|\mathcal{E}^{(0)} - \mathcal{E}^*\|$ (with κ absorbed into L for notational consistency) and $C_\infty = \frac{L\sigma}{1-\beta} + 2\epsilon_N$:

$$J(\theta^{(k)}) - J^* \le \Delta_0 \beta^k + C_\infty$$

The first term $\Delta_0 \beta^k$ decays geometrically to zero, while C_∞ represents the irreducible steady-state error from measurement noise and PWL approximation.

The convergence rate is characterized by the contraction constant $\beta = 1 - \lambda$. To achieve $J(\theta^{(k)}) - J^* \le \epsilon + C_\infty$, we need:

$$\Delta_0 \beta^k \le \epsilon \implies k \ge \frac{\log(\epsilon/\Delta_0)}{\log(\beta)} = O\left(\frac{\log(1/\epsilon)}{\log(1/\beta)}\right)$$

This is logarithmic in the desired accuracy ϵ, demonstrating fast convergence. For example, with $\beta = 0.8$ (corresponding to $\lambda = 0.2$), reducing the error by a factor of 10 requires approximately $k \approx 10.3$ iterations.

The steady-state error C_∞ can be controlled by:

1) **Reducing measurement noise** (σ): Increase averaging window or improve measurement accuracy.
2) **Increasing PWL breakpoints** (N): Since $\epsilon_N = O(N^{-2})$, doubling N reduces linearization error by a factor of 4.
3) **Optimizing learning rate** (λ): Smaller λ reduces $\sigma/(1-\beta)$ but slows convergence (larger β).

Theorem establishes that the proposed orchestration framework converges at a geometric rate to a solution whose quality is bounded by the steady-state error C_∞. This error can be made arbitrarily small by appropriate choice of system parameters, providing strong theoretical guarantees for the practical deployment of the O-FL rApp framework.

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