Laplace Mechanism Accuracy

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Definition 1. Let z be the true value of the statistic and let X be the random variable the noisy release is drawn from. Let α be the statistical significance level, and let $Y = |X - z|$. Then, accuracy a for a given $\alpha \in [0, 1]$ is the a s.t.

$$\alpha = \Pr[Y > a].$$

Theorem 1. *The accuracy of an ϵ-differentially private release from the Laplace mechanism on a function with ℓ_1-sensitivity Δ_1, at statistical significance level α is*

$$a = \frac{\Delta_1}{\epsilon} \ln(1/\alpha).$$

Proof. Recall the definition of the Laplace mechanism, which adds Laplace noise proportional to Δ_1/ϵ to the true query responses [DMNS06]. The probability density function f of the Laplace distribution, for $x \sim X$ with location parameter μ and scaling parameter λ is defined to be

$$f(x) = \frac{1}{2\lambda} e^{\frac{-|x-\mu|}{\lambda}}$$

Then, since the pdf g of Y is the same as the folded pdf of X, shifted over by μ and doubled,

$$g(y) = \frac{1}{\lambda} e^{-y/\lambda}.$$

Then,

$$\begin{aligned}
\alpha &= \Pr[Y > a] \\
&= 1 - \Pr[Y \leq a] \\
&= 1 - \int_{-\infty}^{a} g(y)dy \\
&= 1 - \int_{0}^{a} g(y)dy \\
&= 1 - (1 - e^{-a/\lambda})
\end{aligned}$$

Solving for a gives $a = \lambda \ln(1/\alpha)$. Then, since

$$\lambda = \frac{\Delta_1}{\epsilon},$$

$$a = \frac{\Delta_1}{\epsilon} \ln(1/\alpha).$$

$\square$

References

[DMNS06] Cynthia Dwork, Frank McSherry, Kobbi Nissim, and Adam Smith. Calibrating noise to sensitivity in private data analysis. In *Theory of cryptography conference*, pages 265–284. Springer, 2006.