Task 1

1) Define coefficient and augmented matrix. Write down the coefficient and augmented matrix for the system

$$x + 2y - 3z + w = 2$$
$$-3x - 8y + 4z - 4w = -5$$
$$-2x - 4y + 5z - 5w = -7$$
$$2x + 4y - 4z - 2w = -10$$

Sol:

~~Coefficient~~ Coefficient matrix is the matrix formed when coefficient of linear system of equation are represented in matrix form. Augmented matrix is the matrix formed when coefficient of linear system along with constant are represented in matrix form.

For the given ~~matrix~~ system of linear equation, coefficient matrix of the system is :-

$$\begin{bmatrix} 1 & 2 & -3 & 1 \\ -3 & -8 & 4 & -4 \\ -2 & -4 & 5 & -5 \\ 2 & 4 & -4 & -2 \end{bmatrix}$$

Augmented matrix of the system is

$$\left[\begin{array}{cccc|c} 1 & 2 & -3 & 1 & 2 \\ -3 & -8 & 4 & -4 & -5 \\ -2 & -4 & 5 & -5 & -7 \\ 2 & 4 & -4 & -2 & -10 \end{array}\right]$$

Page 2

Echelon form of matrix and Gauss Elimination method:-

1) What are the row operations (elementary row operations) on a matrix? What when is the matrix said to be in row echelon and reduced echelon form?

→ Row operations are operations performed to make given matrix into row echelon or reduced echelon form, interchanging rows or by doing arithmetic operations like addition, subtraction, multiplication and division of rows of a matrix.

The matrix is in row echoleon form if it satisfies the following conditions:-

→ left most non-zero entry in each row is 1 (leading 1)

→ All non-zero rows are above any row of all zeros.

→ The leading 1 of a row is always to the right of leading 1 of row above it

→ All entries below a leading 1 are zeroes.

The matrix is in reduced echoleon form if it satisfies the axioms:-

→ All the entries below and above leading 1 are zeros

→ All each columns of augmented matrix, the only non-zero entry is leading 1.

2) Define row echeleon and reduced echoleon form of a matrix with suitable example

ans Examples :-

Row echelon form:-

$$\begin{bmatrix} 0 & 1 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \left[\begin{array}{ccc:c} 1 & -3 & 2 & -1 \\ 0 & 1 & 5 & 7 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Reduced echoleon form:-

$$\left[\begin{array}{ccc:c} 1 & 0 & 0 & -14 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

3) What is Gaussian Elimination method of solving linear equation. ~~with~~ write step by step procedure for gaussian elimination ~~of~~ to solve the system of linear equations.

ans Gaussian Elimination method of solving linear equation is the method in which the system of linear equation in matrix form $Ax = B$ is transformed into the upper triangular system by using row operations method and obtain the solution of system by using back substitution method to upper triangular system.

Examples:-

Solving the system of linear equations by using Gauss Elimination method

$$4x - 11y + 18z = 3$$
$$4x - 12y + 10z = 0$$
$$x - 3y + 2z = -1$$

Step 1: Form Augmented matrix :- $\begin{bmatrix} 4 & -11 & 13 & : & 3 \\ 4 & -12 & 10 & : & 0 \\ 1 & -3 & 2 & : & -1 \end{bmatrix}$

Step 2: Row operation method to transform the augmented matrix into upper triangular matrix.

$R_1 \longrightarrow R_3$ then,

$$\begin{bmatrix} 1 & -3 & 2 & : & -1 \\ 4 & -12 & 10 & : & 0 \\ 4 & -11 & 13 & : & 3 \end{bmatrix}.$$

$R_2 \longrightarrow R_2 - 4R_1$ and $R_3 = R_3 - 4R_1$ then.

$$\begin{bmatrix} 1 & -3 & 2 & : & 1 \\ 0 & 0 & 2 & : & 4 \\ 0 & 1 & 5 & : & 7 \end{bmatrix}$$

$R_2 \longrightarrow R_3$; $R_3 \to 1/2 R_3$

$$\begin{bmatrix} 1 & -3 & 2 & : & -1 \\ 0 & 1 & 5 & : & 7 \\ 0 & 0 & 1 & : & 2 \end{bmatrix}$$

Step 3: Back substitution to upper triangular system, $z = 2$

$y + 5z = 7 \Rightarrow y + 10 = 7 \Rightarrow y = -3$

$x - 3y + 2z = -1 \Rightarrow y + 9 + 4 = -1 \Rightarrow x = -14$

so $x = -14$, $y = -3$, $z = 2$

Now $z=1$

$y+6z=6$ $\quad$ $x-2y+3z=2$

$y=0$ $\quad$ or $x=2-3+2\times 0=-1$

so, $y=0,\ x=-1,\ z=1$.

(ii) $x+2y-3z+w=2$

$-3x-8y+4z-4w=-5$

$-2x-4y+5z-5w=-7$

$2x+4y-4z-2w=-10$

Sol

The augmented matrix of given linear system is

$$\left[\begin{array}{cccc:c} 1 & 2 & -3 & 1 & 2 \\ -3 & -8 & 4 & -4 & -5 \\ -2 & -4 & 5 & -5 & -7 \\ 2 & 4 & -4 & -2 & -10 \end{array}\right]$$

$R_2 \to R_2+3R_1$, $R_3 \to R_3+2R_1$, $R_4 \to R_4-2R_1$

$$\left[\begin{array}{cccc:c} 1 & 2 & -3 & 1 & 2 \\ 0 & -2 & -5 & -1 & 1 \\ 0 & 0 & -1 & -2 & -3 \\ 0 & 0 & 2 & -4 & -14 \end{array}\right]$$

$R_2 \to \dfrac{R_2}{-2}$ $\left[\begin{array}{cccc:c} 1 & 2 & -3 & 1 & 2 \\ 0 & 1 & 2.5 & 0.5 & -0.5 \\ 0 & 0 & -1 & -3 & -3 \\ 0 & 0 & 2 & -4 & -14 \end{array}\right]$

$R_4 \to R_4-2R_3$ $\left[\begin{array}{cccc:c} 1 & 2 & -3 & 1 & 2 \\ 0 & 1 & 2.5 & 0.5 & -0.5 \\ 0 & 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & -10 & -20 \end{array}\right]$

Use gaussian elimination to find the solution of the system using the matrix into row echoleon

1) $3x - 6y + 9z = 6$

$4x - 9y + 6z = 2$

$-2x + 3y - 5z = -3$

sol:

The augmented matrix of the given system of linear equation is

$$\left[\begin{array}{ccc:c} 3 & -6 & 9 & 6 \\ 4 & -9 & 6 & 2 \\ -2 & 3 & -5 & -3 \end{array}\right]$$

$R_1 \to R_1/3$: $$\left[\begin{array}{ccc:c} 1 & -2 & 3 & 2 \\ 4 & -9 & 6 & 2 \\ -2 & 3 & -5 & -3 \end{array}\right]$$

$R_2 \to R_2 - 4R_1$; $R_3 \to R_3 + 2R_1$

$$\left[\begin{array}{ccc:c} 1 & -2 & 3 & 2 \\ 0 & -1 & -6 & -6 \\ 0 & -1 & 1 & 1 \end{array}\right]$$

$R_2 \to R_2 \times (-1)$: $$\left[\begin{array}{ccc:c} 1 & -2 & 3 & 2 \\ 0 & 1 & 6 & 6 \\ 0 & -1 & 1 & 1 \end{array}\right]$$

$R_3 \to R_3 + R_2$: $$\left[\begin{array}{ccc:c} 1 & -2 & 3 & 2 \\ 0 & 1 & 6 & 6 \\ 0 & 0 & 7 & 7 \end{array}\right]$$

$R_3 \to R_3/7$: $$\left[\begin{array}{ccc:c} 1 & -2 & 3 & 2 \\ 0 & 1 & 6 & 6 \\ 0 & 0 & 1 & 7 \end{array}\right] \Rightarrow \text{Row echoleon form}$$

$R_4 \to \frac{R_4}{-10}$: $\begin{bmatrix} 1 & 2 & -3 & 1 & : 2 \\ 0 & 1 & 2.5 & 0.5 & : -0.5 \\ 0 & 0 & 1 & 3 & : 3 \\ 0 & 0 & 0 & 1 & : -2 \end{bmatrix} \Rightarrow$ Row echelon.

Back substitution :-

$w = -2$

$z + 3w = 3 \Rightarrow z = 3 + 6 = 9$

$y + 2.5z + 0.5w = 0.5 \Rightarrow y = 0.5 + 1 - 22.5 = -22$.

$x + 2y - 3z + w = 2$

$\Rightarrow x = 2 + 2 + 27 + 44 = 75$

$-5x + 3y - 2z + 4w = 8$

$6x + 8y + 5z - 3w = 4$

$x + 2y - 7z - 5w = -1$.

$-3x - 7y + 6z + 2w = 2$.

Sol:-

The augmented matrix of linear system of given equation is:-

$\begin{bmatrix} -5 & 3 & -2 & 4 & : 8 \\ 6 & 8 & 5 & -3 & : 4 \\ 1 & 2 & 7 & -5 & : -1 \\ -3 & -7 & 6 & 2 & : 2 \end{bmatrix}$

$R_1 \leftrightarrow R_3$ $\begin{bmatrix} 1 & 2 & -7 & -5 & : -1 \\ 6 & 8 & 5 & -3 & : 4 \\ -5 & 3 & -2 & 4 & : 8 \\ -3 & -7 & 6 & 2 & : 2 \end{bmatrix}$.

$R_2 \to R_2 - 6R_1$, $R_3 \to R_3 + 5R_1$, $R_4 \to R_4 + 3R_1$

$\begin{bmatrix} 1 & 2 & -7 & -5 & : -1 \\ 0 & -4 & 47 & -33 & : 10 \\ 0 & 13 & -37 & -21 & : 3 \\ 0 & -1 & -15 & -13 & : -1 \end{bmatrix}$

$R_2 \to \frac{R_2}{-4}$

$$\left[\begin{array}{cccc|c} 1 & 2 & -7 & -5 & 1 \\ 0 & 1 & -47/4 & 33/4 & -10/4 \\ 0 & -13 & -37 & -21 & 3 \\ 0 & -1 & -15 & -13 & -1 \end{array}\right]$$

$R_3 \to R_3 - 13R_2, \quad R_4 \to R_4 + R_2$

$$\left[\begin{array}{cccc|c} 1 & 2 & -7 & -5 & -1 \\ 0 & 1 & -47/4 & 33/4 & -10/4 \\ 0 & 0 & 463/4 & -513/4 & 71/2 \\ 0 & 0 & -107/4 & -19/4 & -3.5 \end{array}\right]$$

$R_3 \to R_3 \times \left(\frac{4}{463}\right)$

$$\left[\begin{array}{cccc|c} 1 & 2 & -7 & -5 & -1 \\ 0 & 1 & -47/4 & 33/4 & -10/4 \\ 0 & 0 & 1 & -513/463 & 142/463 \\ 0 & 0 & -107/4 & -19/4 & -7/2 \end{array}\right]$$

$R_4 = R_4 + \frac{107}{4} R_3$

$$\left[\begin{array}{cccc|c} 1 & 2 & -7 & -5 & -1 \\ 0 & 1 & -47/4 & 33/4 & -10/4 \\ 0 & 0 & 1 & -513/463 & 142/463 \\ 0 & 0 & 0 & -15922/469 & 2178/463 \end{array}\right]$$

$R_4 \to R_4 \times \frac{-463}{15922}$

$$\left[\begin{array}{cccc|c} 1 & 2 & -7 & -5 & -1 \\ 0 & 1 & -47/4 & 33/4 & -10/4 \\ 0 & 0 & 1 & -513/463 & 142/463 \\ 0 & 0 & 0 & 1 & -1089/7961 \end{array}\right]$$

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Back Substitution :-

$$w = \frac{-1089}{7961} = -0.1368$$

$$z = \frac{513\,w}{463} = \frac{142}{463}$$

$$z = \frac{65}{419} = 0.1551$$

$$y = \frac{-47z}{4} + \frac{33}{4}w = \frac{-10}{4}$$

$$y = \frac{3593}{7961} = 0.4513$$

$$x + 2y - 7z - 5w = -1$$

or $x = -1.5296$ //

Task 3. [Multisystem and low Rank]

) Solve the multi system using gaussian elimination and back addition.

) $x_1 + 2x_2 - 3x_3 = -11$ $\quad$ $x_1 + 2x_2 - 3x_3 = -2.$

$3x_1 + 5x_2 + 3x_3 = 2$ $\quad$ $3x_1 + 5x_2 + 3x_3 = -3$

$-5x_1 - 4x_2 + 8x_3 = 51$ $\quad$ $-5x_1 - 4x_2 + 8x_3 = -16.$

Sol: The augmented matrix for the above multi system is

$$\left[\begin{array}{ccc|cc} 1 & 2 & -3 & -1 & 2 \\ 3 & 5 & 3 & 2 & -3 \\ -5 & -4 & 8 & 51 & -16 \end{array}\right]$$

$R_2 \to R_2 - 3R_1$, $\quad R_3 \to R_3 + 5R_1$

$$\left[\begin{array}{ccc|cc} 1 & 2 & -3 & -1 & 2 \\ 0 & -1 & 2 & 5 & -9 \\ 0 & 6 & -7 & 46 & -6 \end{array}\right]$$

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$R_2 \rightarrow -1 \times R_2$

$$\left[\begin{array}{ccc:cc} 1 & 2 & -3 & -1 & 2 \\ 0 & 1 & -12 & -5 & 9 \\ 0 & 6 & -2 & 46 & -6 \end{array}\right].$$

$R_3 \rightarrow R_3 - 6R_2$

$$\left[\begin{array}{ccc:cc} 1 & 2 & -3 & -1 & 2 \\ 0 & 1 & -12 & -5 & 9 \\ 0 & 0 & 65 & 76 & -60 \end{array}\right]$$

$R_3 \rightarrow R_3/65$

$$\left[\begin{array}{ccc:cc} 1 & 2 & -3 & -1 & 2 \\ 0 & 1 & -12 & -5 & 9 \\ 0 & 0 & 1 & 76/65 & -60/65 \end{array}\right]$$

From back substitution;

$x_3 = \frac{76}{65} = 1.169.$

$x_2 - 12x_3 = 5x-1.$

$x_2 = 9.031$

$x_1 + 2x_2 - 3x_2 = -1$

$x_1 = -15.55$

$x_3 = -0.923$

$x_2 - 12x_3 = 9.$

so, $x_2 = -2.076.$

$x_1 + x_2 - 3x_3 = 2$

or $x_1 = 3.383$

) $x - 2y + 3z = 2$ and $x + 2y + 3z = -2$

$4x - 9y + 6z = 2$ $\quad$ $4x - 9y + 6z = 1$

$-2x + 3y - 5z = -3$ $\quad$ $-2x + 3y - 5z = -1$

Sol:

The augmented matrix is :-

The augmented matrix is :-

$$\begin{bmatrix} 1 & -2 & 3 & : & 2 & -2 \\ 4 & -9 & 6 & : & 2 & 1 \\ -2 & 3 & 3 & : & -3 & -21 \end{bmatrix}$$

$R_2 \to R_2 - 4R_1$, $R_3 \to R_3 + 2R_1$

$$\begin{bmatrix} 1 & -2 & 3 & : & 2 & -2 \\ 0 & -1 & -6 & : & -6 & 9 \\ 0 & -1 & 1 & : & 1 & -5 \end{bmatrix}$$

$R_2 \to -R_2$

$$\begin{bmatrix} 1 & -2 & 3 & : & 2 & -2 \\ 0 & 1 & 6 & : & 6 & -9 \\ 0 & -1 & 1 & : & 1 & -5 \end{bmatrix}.$$

$R_3 \to R_3 + R_2$.

$$\begin{bmatrix} 1 & -2 & 3 & : & 2 & -2 \\ 0 & 1 & 6 & : & 6 & -9 \\ 0 & 0 & 7 & : & 7 & -14 \end{bmatrix}.$$

$R_3 \to R_3/7$

$$\begin{bmatrix} 1 & -2 & 3 & : & 2 & -2 \\ 0 & 1 & 6 & : & 6 & -9 \\ 0 & 0 & 7 & : & 1 & -2 \end{bmatrix}.$$

Back substitution:-

$z = 1$

$y + 6z = 6$ $\quad\quad z = -2$

$y = 0$ $\quad\quad z + 6z = -9$

$x - 2y + 3z = 2$ $\quad\quad y = 3$

so, $x = -1$ $\quad\quad x - 2y + 3z = -2$

so $x = 10$ //

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2 Define row rank of a matrix. Find the rank of coefficient and augmented matrix and check consistency of the given system of equation.

Sol:

The row rank of a matrix is the number of non-zero rows in a row echelon form of the matrix.

[A linear system $Ax = b$ is said to be consistent, if the row rank of the coefficient matrix is equal to the row rank of the augmented matrix).

$$2x - 6y + 8z = 2.$$
$$-4x + 13y + 3z = 6$$
$$-6x + 20y + 14z = 2$$

Sol:

$$\begin{bmatrix} 2 & -6 & 8 & : 2 \\ -4 & 13 & 3 & : 6 \\ +6 & 20 & 14 & : -2 \end{bmatrix}$$

$$R_1/2 \Rightarrow \begin{bmatrix} 1 & -3 & 4 & : 1 \\ -4 & 13 & 3 & : 6 \\ 6 & 20 & 14 & : 2 \end{bmatrix}.$$

$R_2 \to R_2 + R_1$, $R_3 \to R_3 + 6R_1$

$$\begin{bmatrix} 1 & -3 & 4 & : 1 \\ 0 & 1 & 19 & : 10 \\ 0 & 2 & 38 & : 4 \end{bmatrix}.$$

$R_3 \to R_3 - 2R_2$

$$\begin{bmatrix} 1 & -3 & 4 & : 1 \\ 0 & 1 & 19 & : 10 \\ 0 & 0 & 0 & : -16 \end{bmatrix}. \Rightarrow \text{Echoleon matrix form.}$$

Row rank of coefficient matrix = $\begin{bmatrix} 1 & -3 & 4 \\ 0 & 1 & 19 \\ 0 & 0 & 0 \end{bmatrix} = 2$

Row rank of augmented matrix $\left[\begin{array}{ccc:c} 1 & -3 & 4 & 1 \\ 0 & 1 & 19 & 10 \\ 0 & 0 & 0 & 16 \end{array}\right] = 3.$

Since row rank of coefficient matrix $\neq$ row rank of augmented matrix, the given system of linear equation is not consistent.

$2x - 2y + 4z + 6w = 8$

$-4x + 5y - 2z - 7w = -10$

$2x + y + 2z + 21w = 36$

$-3x + 5y - 4z + 21w = 10$

Solⁿ:

$$\left[\begin{array}{cccc:c} 2 & -2 & 4 & 6 & 8 \\ -4 & 5 & -2 & 7 & -10 \\ 2 & 1 & 22 & -21 & 36 \\ -3 & 5 & -42 & 11 & 10 \end{array}\right]$$

$R_1 \to R_1/2$

$$\left[\begin{array}{cccc:c} 1 & 1 & 2 & 3 & 4 \\ -4 & 5 & -2 & -7 & -10 \\ 2 & 1 & 22 & 21 & 36 \\ -3 & 5 & -4 & 11 & 10 \end{array}\right]$$

$R_2 \to R_2 + 4R_1$, $R_3 \to R_3 - 2R_1$, $R_4 \to R_4 + 3R_1$

$$\left[\begin{array}{cccc:c} 1 & -1 & 2 & 2 & 4 \\ 0 & 1 & 6 & 5 & 6 \\ 0 & 3 & 18 & 15 & 28 \\ 0 & 2 & 2 & 20 & 22 \end{array}\right]$$

$R_3 \to R_3 - 3R_2, \quad R_4 \to R_4 - 2R_2$

$$\begin{bmatrix} 1 & -1 & 2 & 5 & : 4 \\ 0 & 1 & 6 & 5 & -6 \\ 0 & 0 & 0 & 0 & : 10 \\ 0 & 0 & -10 & 10 & : 10 \end{bmatrix}$$

Row rank of column matrix = 3

row rank of augmented matrix = 4

Not consistent [row rank of column matrix) $\neq$ row rank of colu matrix.]

Task 4 (Eigen Values and Eigen Vectors).

1) Define Eigen Values, Eigen vectors and characteristics equation.

A number λ is called an eigen value of square matrix A if there exist a non-zero vector x such that $A\vec{x} = \lambda\vec{x}$. If λ is an eigen value of A than any vector $\vec{x} \neq 0$ that satisfies $A\vec{x} = \lambda\vec{x}$ is called eigen vectors of A associated with λ.

Determinant of $(A - \lambda I)$ ie $|A - \lambda I| = 0$ gives an polynomial (in terms of λ) and this eq^n is called characteristic equation of equation. gives the eigen value.

For each of the following problems, compute the characteristics eq^n. Find the eigen value and eigen space for each eigen value of given matrix.

$$\begin{bmatrix} -3 & 5 \\ -10 & 12 \end{bmatrix} = A$$

4

E.F

$$A - \lambda I = \begin{bmatrix} -3-\lambda & 5 \\ -10 & 12-\lambda \end{bmatrix}$$

determinant of $(A - \lambda I) = |A - \lambda I|$

$$= -3(3+\lambda)(12-\lambda) - (5 \times -10)$$

$$= 50 - 36 + 3\lambda - 12\lambda + \lambda^2$$

$$= \lambda^2 - 9\lambda + 14 = 0 \quad \text{is characteristic equation}$$

Eigen value $\lambda = 7, 2$

for eigen value $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Take $\lambda = 7$ $(A - \lambda I)\vec{x} = \vec{0}$

$$\begin{bmatrix} -10 & 5 \\ -10 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -10x_1 + 5x_2 \\ -10x_1 + 5x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

we get, $-10x_1 = -5x_2$

So $x_2 = 2x_1$

Hence, eigen vector $\vec{v_1} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \ldots$

Eigen space $= t\begin{pmatrix} 1 \\ 2 \end{pmatrix}, t \neq 0.$

Take $\lambda = 2$

$$\begin{bmatrix} -5 & 5 \\ -10 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -5x_1 + 5x_2 \\ -10x_1 + 10x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

we get, $-5x_1 = -5x_2$

so $x_1 = x_2$.

Hence eigen ~~value~~ vector,

$\vec{v_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ Eigen space $\vec{v} = t\begin{pmatrix} 1 \\ 1 \end{pmatrix}, t \neq 0$

1 $\begin{bmatrix} -12 & -4 \\ 12 & 2 \end{bmatrix}$

Soln

$$A - \lambda I = \begin{bmatrix} -12-\lambda & -4 \\ 12 & 2-\lambda \end{bmatrix}$$

$$|A - \lambda I| = (-12-\lambda)(2-\lambda) - (12 \cdot -4)$$

$$= -24 + 12\lambda - 2\lambda + \lambda^2 + 48$$

or $\lambda^2 + 10\lambda + 24 = 0$ $\Rightarrow$ characteristic equation

$\lambda = -4, 6 \Rightarrow$ Eigen values

Take $\lambda = -4$

For eagen vector $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

$(A - \lambda I)\vec{x} = \vec{0}$

or $\begin{bmatrix} -12-\lambda & -4 \\ 12 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

or $\begin{bmatrix} -8 & -4 \\ 12 & 6 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

or $\begin{bmatrix} -8x_1 - 4x_2 \\ 12x_1 + 6x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

above,

$-8x_1 - 4x_2 = 0$

So, $x_2 = -2x_1$

Eigen vector $\vec{v_1} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

Eigen space $\vec{v} = t \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad t \neq 0$

Take $\lambda = -6$

For eigen vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$(A - \lambda I)\vec{x} = \vec{0}$

$$\begin{bmatrix} -6 & -4 \\ 12 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$\begin{bmatrix} -6x_1 - 4x_2 \\ 12x_1 + 8x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$\therefore$ Eigen vector $\vec{v_1} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ Eigen space $\vec{v} = t \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ $t \neq 0$

$$\begin{bmatrix} 5 & -3 & -3 \\ -6 & 8 & 6 \\ 12 & -12 & 10 \end{bmatrix}.$$

Sol^n:

$$A - \lambda I = \begin{bmatrix} 5-\lambda & -3 & -3 \\ -6 & 8-\lambda & 6 \\ 12 & -12 & 10-\lambda \end{bmatrix}$$

$|(A - \lambda I)| = (5-\lambda)(-80 + 2\lambda + \lambda^2 + 72) - (-3)(60 + 6\lambda - 72) + (-3)(60 + 6\lambda - 72$
$(72 - 96 + 12\lambda)$ $+(-3)$.

or $0 = 72 - 36\lambda - 36 + 8\lambda + (-40 + 8\lambda + 10\lambda - \lambda^2 + 8\lambda^2 - \lambda^3)$.

or $\lambda^3 + 3\lambda^2 + 0\lambda - 4 = 0$ $\longrightarrow$ characteristic equation.

so, $\lambda = 2, -1 \rightarrow$ Eigen values

take $\lambda = 2$

For eigen vector, $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$(A - \lambda I)\vec{x} = \vec{0}$

or $\begin{bmatrix} 3 & -3 & -3 \\ -6 & 6 & 6 \\ 12 & -12 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

or $\begin{bmatrix} 1 & -1 & -1 \\ -6 & 6 & 6 \\ 12 & 12 & -12 \end{bmatrix} \vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

or $\left[\begin{array}{ccc:c} 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Here x_1 and x_2 are free /non leading variables. let $x_2 = s$, $x_3 = t$. then $x_1 = s+t$

~~Eg~~ Eigen vector $\vec{v_1} = \begin{bmatrix} s+t \\ s \\ t \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$.

Eigen space $\vec{V} = s\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $s \neq 0$, $t \neq 0$

Take $\lambda = -1$

$\begin{bmatrix} 6 & -3 & -3 \\ -6 & 9 & 6 \\ 12 & -12 & -11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\text{or}\quad \begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & 6 & 3 \\ 0 & -6 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{or}\quad \begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & 9 & 6 \\ 0 & -6 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{or}\quad \begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-2x_3 = 0$ so, $x_3 = 0$

$x_2 + 1/2\,x_3 = 0$ so, $x_2 = 0$

$x_1 - 1/2\,x_2 - 1/2\,x_3 = 0$ $\Rightarrow x_1 = 0$

or $x_1 - \frac{1}{2}x_2 - \frac{1}{2}x_3 = 0$

So, Eigen vector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} -4 & 4 & -1 \\ 6 & -2 & 3 \\ 10 & -8 & 3 \end{bmatrix}$$

Sol^n

$$A - \lambda I = \begin{bmatrix} -4-\lambda & -4 & -1 \\ 6 & -2-\lambda & 3 \\ 10 & -8 & 3-\lambda \end{bmatrix}.$$

$|A - \lambda I| = (-4-\lambda)(18-\lambda+\lambda^2) - 4(18-6\lambda-30)(-48+20+10\lambda).$

$= -72 - 18\lambda + 4\lambda + \lambda^2 - 72 + 2\lambda + 120 + 48 - 20 - 16\lambda$

$0 = -\lambda^3 + 3\lambda^2 + 0\lambda + 4.$

$\lambda = 1; 2 \rightarrow$ Eigen values

Take $\lambda = 1$

Eigen vector be $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \vec{x}$

$$\begin{pmatrix} -5 & 4 & -1 \\ 6 & -3 & 3 \\ 10 & -8 & 2 \end{pmatrix} \vec{x} = \vec{0}$$

Augmented matrix

$$\begin{pmatrix} -5 & 4 & -1 & : 0 \\ 6 & -3 & 3 & : 0 \\ 10 & -8 & 2 & : 0 \end{pmatrix}$$

or $\begin{pmatrix} 1 & -4/5 & 1/5 & : 0 \\ 0 & 9/5 & 9/5 & : 0 \\ 0 & 0 & 0 & : 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -4/5 & 1/5 & : 0 \\ 0 & 1 & 1 & : 0 \\ 0 & 0 & 0 & : 0 \end{pmatrix}$

x_3 is a free / non-leading variable. let $x_3 = t$ then,

$x_2 + x_3 = 0$, $x_2 = -x_3$

$x_2 = -t$

$x_1 - 4/5\, x_2 + 1/5\, x_3 = 0$

or $x_1 = \frac{4}{5}(-t) - 1/5\, t =$

so $x_1 = t$.

so eigen vector $\vec{x} = \begin{pmatrix} -t \\ -t \\ t \end{pmatrix}$, t be any arbitrary const.

Eigen space $\vec{v} = t \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$, $t \neq 0$

take $\lambda = -2$

Then, augmented matrix is

$$\begin{bmatrix} -2 & 4 & -1 & : 0 \\ 6 & 0 & 3 & : 0 \\ 10 & -8 & 5 & : 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1/2 & : 0 \\ 0 & 12 & 0 & : 0 \\ 0 & 12 & 0 & : 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1/2 & : 0 \\ 0 & 1 & 0 & : 0 \\ 0 & 0 & 0 & : 0 \end{bmatrix}$$

$\therefore x_3$ is a free variable. let $x_3 = t$

$x_2 = 0$

$x_1 - 2x_2 + 1/2 x_3 = 0$

$x_1 = -t/2$

Eigen vector $= \begin{bmatrix} -t/2 \\ 0 \\ t \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

Eigen space $\vec{v} = t \begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix}, t \neq 0$

$$\begin{bmatrix} -1 & 3 & 1 \\ -1 & 2 & 1 \\ 2 & -9 & -4 \end{bmatrix}$$

Sol^n

$|A - \lambda I| = \begin{vmatrix} -1-\lambda & 3 & 1 \\ -1 & 2-\lambda & 1 \\ 2 & -9 & -4-\lambda \end{vmatrix}$

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$= (-1-\lambda)\{(-8-2\lambda+4\lambda+\lambda^2)+9\} - 3(4+\lambda 2) + 1(9-4+2\lambda)\}$

$= -(1+\lambda)(\lambda^2+2\lambda+1) - 3(\lambda+2) + (5+2\lambda)$

$= -\lambda^2 - 2\lambda - 1 - \lambda^3 - 2\lambda^2 - \lambda - 3\lambda - 6 + 5 + 2\lambda$

$0 = -\lambda^3 - 3\lambda^2 + 4\lambda - 2$

$\therefore$ $\lambda^3 + 3\lambda^2 + 4\lambda + 2 = 0$ is characteristic equation.

so $\lambda = -1$ (rest of values are imaginary).

Take $\lambda = -1$

Now $A - \lambda I = \begin{bmatrix} 0 & 3 & 1 \\ -1 & 3 & 1 \\ 2 & -9 & -3 \end{bmatrix} = \begin{bmatrix} -1 & 3 & 1 \\ 2 & -9 & -3 \\ 0 & 3 & 1 \end{bmatrix}$

let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be eigen vector.

Then, $(A - \lambda I)\vec{x} = \vec{0}$

In augmented matrix form,

$$\left[\begin{array}{ccc:c} -1 & 3 & 1 & 0 \\ 2 & -9 & -3 & 0 \\ 0 & 3 & 1 & 0 \end{array}\right]$$

$$= \left[\begin{array}{ccc:c} 1 & -3 & -1 & 0 \\ 0 & -3 & -1 & 0 \\ 0 & 3 & 1 & 0 \end{array}\right]$$

$$= \left[\begin{array}{ccc:c} 1 & -3 & -1 & 0 \\ 0 & 1 & -1/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

x_3 is free or non leading variable

let $x_3 = t$

Then, $x_2 + 1/3\, x_3 = 0$

$\Rightarrow x_2 = -1/3\, t$

$\Rightarrow x_2 - x_3 = 0$

Again, $x_1 - 3x_2 - x_3 = 0$

$x_1 + t - t = 0$

so $x_1 = 0$

$\therefore$ Eigen vector $\vec{x} = \begin{bmatrix} 0 \\ -1/3\, t \\ t \end{bmatrix} = t\begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$

Eigen space $= \vec{v} = t\begin{bmatrix} 0 \\ -1/3 \\ 1 \end{bmatrix}, t \neq 0.$

Task 5 [Linear Transformation]

Define linear transformation.

If $v_1, v_2, v_3, \ldots v_n$ are vectors in R^n space and $c_1, c_2, c_3 \ldots c_n$ are scalars then the vectors $\vec{v} = c_1\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3} + \ldots c_n v_n$ is linear combination of vectors $v_1, v_2, v_3 \ldots v_n$

i.e. $\sum_{i=1}^{n} c_i v_i = c\vec{v}$

Determine which of the following mappings are linear (i) $T: R^2 \to R^2$ defined by $T(x,y) = (x+y, x-y)$.

Sol:

$\vec{v} = (x,y)$; $\vec{v_1} = (x_1, y_1)$, $\vec{v_2} = (x_2, y_2)$

$\vec{v_1} + \vec{v_2} = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$

$k\vec{v} = k(x, y) = (kx, ky)$

$T(v_1 + v_2) = T(x_1 + x_2, y_1 + y_2)$

$= (x_1 + y_1 + x_2 + y_2, x_1 + x_2 - y_1 - y_2)$

$= \cancel{T(x_1 + y_1} \; T(x_1, y_1) + T(x_2, y_2)$

$T(v_1 + v_2) = T(v_1) + T(v_2)$

Also, $T(kv) = T(kx, ky)$

$= (kx + ky, kx - ky)$

$= k(x+y, x-y)$

$\therefore T(kv) = kT(v)$

As both of them are satisfied. Hence $T(v)$ is linear transformation.

$T: R^2 \to R^3$ defined by $T(x,y) = (x, y, xy)$.

Sol^n

$\vec{V_1} = (x,y)$, $\vec{V_1} = (x_1, y_1)$, $\vec{V_2} = (x_2, y_2)$.

$\vec{V_1} + \vec{V_2} = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$

$T(V_1 + V_2) = T(x_1 + x_2, y_1 + y_2)$.

$= (x_1 + x_2, y_1 + y_2$

$T(V_1) + T(V_2) = T(x_1, y_1) + T(x_2, y_2)$.

$= (x_1, y_1, x_1 y_1) + (x_2, y_2, x_2 y_2)$

$= (x_1 + x_2, y_1 + y_2, x_1 y_1 + x_2 y_2)$

As $T(V_1 + V_2) \neq T(V_1) + T(V_2)$

Hence $T(V)$ is not linear transformation.

28/

$T: R^3 \rightarrow R^2$ defined by $T(x,y,z) = (x+y+z, 0)$

Sol^n

$\vec{V} = (x,y,z)$, $\vec{V_1} = (x_1, y_1, z_1)$, $\vec{V_2} = (x_2, y_2, z_2)$

$K\vec{V} = (Kx, Ky, Kz)$

$\vec{V_1} + \vec{V_2} = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$

$= (x_1 + y_1 + z_1, 0) + (x_2 + y_2 + z_2, 0)$

$K\vec{V} = (Kx, Ky, Kz)$

$\vec{V_1} + \vec{V_2} = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$

$= (x_1 + y_1 + z_1, 0) + (x_2 + y_2 + z_2, 0)$

$\therefore T(V_1 + V_2) = T(V_1) + T(V_2)$

Again,

$T(K\vec{V}) = T(Kx, Ky, Kz)$

$= (Kx + Ky + Kz, 0)$

$= K(x+y+z, 0) = KT(\vec{V})$

Hence, $T(V)$ is a linear transformation

$T: R^2 \rightarrow R$ defined by $T(x,y) = |x+y|$

Sol^n

$\vec{V} = (x,y)$, $\vec{V_1} = (x_1, y_1)$, $\vec{V_2} = (x_2, y_2)$

$\vec{V_1} + \vec{V_2} = (x_1 + x_2, y_1 + y_2)$

Now,

$T(V_1 + V_2) = |x_1 + x_2 + y_1 + y_2| \leq |x_1 + y_1| + |y_1 + y_2|$

$+ T(V_1) + T(V_2)$

So $T(V_1 + V_2) \leq T(V_1) + T(V_2)$

Hence, $T(V)$ is not linear transformation.

ii) $T: R^3 \to R^2$ defined by $T(x,y,z) = (x+z, y-z)$

Sol^n

$\vec{V} = (x, y, z)$; $\vec{V_1} = (x_1, y_1, z_1)$, $\vec{V_2} = (x_2, y_2, z_2)$

$\vec{V_1} + \vec{V_2} = (x_1+x_2, y_1+y_2, z_1+z_2)$

~~$T(\vec{V_1}+\vec{V_2}) =$~~

$T(V_1+V_2) = T(x_1+x_2, y_1+y_2, z_1+z_2)$

$= (x_1+x_2+z_1+z_2, y_1+y_2-z_1-z_2)$

$= (x_1+z_1, y_1-z_1) + (x_2+z_2, y_2-z_2)$

So, $T(V_1+V_2) = T(V_1) + T(V_2)$

$T(K\vec{V}) = T(Kx, Ky, Kz)$

$= (Kx+Kz, ky-Kz)$

$= K(x+z, y-z)$

So, $T(K\vec{V}) = TK(\vec{V})$

Hence, the given transformation is linear transformation.

iii) $T: R^3 \to R^2$ defined by $T(x,y,z) = (x,y)$

Sol^n

$\vec{V} = (x,y,z)$, $\vec{V_1} = (x_1, y_1, z_1)$, $\vec{V_2} = (x_2, y_2, z_2)$

$K\vec{V} = (Kx, Ky, Kz)$

$T(V_1+V_2) = T(x_1+x_2, y_1+y_2, z_1+z_2)$

$= (x_1+x_2, y_1+y_2)$

$= (x_1, y_1) + (x_2, y_2)$

So, $T(V_1+V_2) = T(Kx, Ky, Kz)$

$= (Kx, Ky)$

$= K(x,y)$

So, $T(K\vec{V}) = KT(\vec{V})$

So, $T(V_1+V_2) = T(V_1) + T(V_2)$ and

Since, $T(K\vec{V}) = KT(\vec{V})$, $T(V)$ is linear transformation.

26

$T: R^2 \longrightarrow R^3$ defined by $T(x,y) = (x, x+y, 2y)$

Sol^n,

$\vec{v} = (x, y)$, $\vec{v_1} = (x_1, y_1)$, $\vec{v_2} = (x_2, y_2)$

$\vec{v_1} + \vec{v_2} = (x_1 + x_2, y_1 + y_2)$

$T(V_1 + V_2) = (x_1 + x_2, x_1 + x_2 + y_1 + y_2, 2y_1 + 2y_2)$

$= (x_1, x_1 + y_1, 2y_1) + (x_2, x_2 + y_2, 2y_2)$

So, $T(V_1 + V_2) = T(V_1) + T(V_2)$

$K\vec{v} = (Kx, Ky)$

$T(K\vec{v}) = (Kx, Kx + Ky, 2Ky)$

$= K(x, x+y, 2y)$

So, $T(K\vec{v}) = KT(\vec{v})$

Hence, $T(\vec{v})$ is linear transformation.

Task 1: Vector space.

Linear Combination

1) $\vec{u_1} = (4, -2, 1)$, $\vec{u_2} = (6, 3, 2)$ and $\vec{u_3} = (-2, -1, 3)$ then find $\vec{V} = \vec{u_1} - 3\vec{u_2} + 4\vec{u_3}$

Sol^n.

$\vec{V} = \vec{u_1} - 3\vec{u_2} + 4\vec{u_3}$

$= (4, -2, 1) - 3(6, 3, 2) + 4(-2, -1, 3)$

$= (4, -2, 1) - (18, 9, 6) + (-8, -4, 12)$

$= (4 - 18 - 8, -2 - 9 - 4, 1 - 6 + 12)$

$= (-22, -15, 7)$ //

Express v as a linear combination of u_1 and u_2 where $\vec{V} = (2, 3, -2)$, $\vec{u_1} = (2, 4, -4)$ and $\vec{u_2} = (-2, -5, 6)$

Sol^n,

let $\vec{V} = x\vec{u_1} + y\vec{u_2}$

or, $(2, 3, -2) = x(2, 4, -4) + y(-2, -5, 6)$

or, $(2, 3, -2) = (2x, 4x, -4x) + (-2y, -5y, +6y)$

or, $(2, 3, -2) = (2x - 2y, 4x - 5y, -4x + 6y)$.

So,

$2x - 2y = 2$ — (i)

$4x - 5y = 3$ — (ii)

$-4x + 6y = -2$ — (iii)

Solving (i) and (ii),

$x = 2, y = 1$

$\therefore \vec{V} = 2\vec{u_1} + \vec{u_2}$ //

11 Find a solution of $Ax=b$ and use it to determine a linear combination of the column vectors of A that will be equal to b. (Express b in column space of A).

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 3 & 10 & 5 \\ 2 & 7 & 1 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 6 \\ 4 \\ -2 \end{bmatrix}$$

Sol:

~~$Ax = b$.~~

$$\begin{bmatrix} 1 & 3 & 4 \\ 3 & 10 & 5 \\ 2 & 7 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ -2 \end{bmatrix}$$

$x_1 + 3x_2 + 4x_3$

$3x_1 + 10x_2 + 5x_3$

$2x_1 + 7x_2 + x_3$

Augmented matrix for $A\vec{x} = \vec{B}$

$$\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \left[\begin{array}{ccc|c} 1 & 3 & 4 & 6 \\ 3 & 10 & 5 & 4 \\ 2 & 7 & 1 & -2 \end{array}\right].$$

$$= \left[\begin{array}{ccc|c} 1 & 3 & 4 & 6 \\ 0 & 1 & -7 & -14 \\ 0 & 0 & -2 & 0 \end{array}\right]$$

Back substitution:

$z = 0.$

$y - 7z = -14$

$y = -14.$

Span of vectors :

Express null space of the matrix A as a span of vectors where :

$$A = \begin{bmatrix} 1 & 4 & 2 & -3 \\ -2 & 9 & 8 & 1 \\ 1 & 3 & 14 & -8 \end{bmatrix}$$

let $\vec{x}$ be null space of A.

$$A\vec{x} = 0.$$

$$\begin{bmatrix} 1 & 4 & 2 & -3 & : 0 \\ -2 & 9 & 8 & 1 & : 0 \\ 1 & 3 & 14 & -8 & : 0 \end{bmatrix}.$$

$$\begin{pmatrix} 1 & 4 & 2 & -3 \\ -2 & -9 & 8 & 1 \\ 1 & 3 & 14 & 8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Augumented matrix of given homogenous equation can be written

$$\begin{bmatrix} 1 & 4 & 2 & -3 & : 0 \\ -2 & -9 & 8 & 1 & : 0 \\ 1 & 3 & 14 & -8 & : 0 \\ 0 & 0 & 0 & 0 & : 0 \end{bmatrix}.$$

$$= \begin{bmatrix} 1 & 4 & 2 & -3 & : 0 \\ 0 & 1 & -12 & 5 & : 0 \\ 0 & 0 & 0 & 0 & : 0 \\ 0 & 0 & 0 & 0 & : 0 \end{bmatrix}.$$

x_3 and x_4 are free/ non-leading variables.

$x + 3y - 4z = 6.$

$x = 48.$

$$48\begin{pmatrix}1\\3\\2\end{pmatrix} - 14\begin{bmatrix}3\\10\\7\end{bmatrix} + 0\begin{bmatrix}4\\5\\1\end{bmatrix} = \begin{pmatrix}6\\4\\-2\end{pmatrix}.$$

Let $x_3 = s$ and $x_4 = t$.

$x_2 - 12x_3 + 5x_4 = 0$

$x_2 = 12s - 5t$

$x_1 + 4x_2 + 2x_3 - 3x_4 = 0$

$x_1 = 3t - 2s - 48s + 20t$

$= 23t - 50s.$

Hence, null space of A is span of vector u_1 and u_2 where $u_1 = (23, -5, 0, 1)$ and $u_2 = (-50, 12, 1, 0)$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 23t - 50s \\ 12s - 5t \\ s \\ t \end{pmatrix} = t\begin{pmatrix} 23 \\ -5 \\ 0 \\ 1 \end{pmatrix} + s\begin{pmatrix} -50 \\ 12 \\ 1 \\ 0 \end{pmatrix}.$$

Describe the space of given vectors in the term of coefficient x, y, z of a general vector $b = (x, y, z)$ in the span. Write b as a linear combination of given vector $\vec{u_1} = (3, -6, 3)$, $\vec{u_2} = (-3, 5, 6)$ and $\vec{u_3} = (3, -7, 12)$

Sol:

let c_1, c_2, c_3 be arbitary const such that

$\vec{b} = c_1\vec{u_1} + c_2\vec{u_2} + c_3\vec{u_3}$

The span of set of vector $\{\vec{u_1}, \vec{u_2}, \vec{u_3}\}$ in $\mathbb{R}^3$ is set of all vectors $c_1\vec{u_1} + c_2\vec{u_2} + c_3\vec{u_3}$ that are linear combination of given vector. In short $\vec{b}$ is the span of given set of vector if it is the linear combination of those vectors.

Sol:

$b = (x, y, z)$

to write as a linear combination of $\vec{u_1}, \vec{u_2}$ and $\vec{u_3}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = c_1\begin{pmatrix} 3 \\ -6 \\ 3 \end{pmatrix} + c_2\begin{pmatrix} -3 \\ 5 \\ 6 \end{pmatrix} + c_3\begin{pmatrix} 3 \\ -7 \\ 12 \end{pmatrix}.$$

Basis:

Find the basis ~~for subspace S of~~ R^3 that is the plane with equation $4x - 2y + 7z = 0$.

$$x = 3c_1 - 3c_2 + 3c_3.$$

$$y = -6c_1 + 5c_2 + 7c_3,$$

$$z = 3c_1 + 6c_2 + 12c_3.$$

(x, y, z) can be at any ~~co~~ co-ordinate of R^3. So $\vec{b}$ is the span of u_1, u_2, and u_3 on all of R^3.

Soln

$$\vec{u_1} = (1, 3, -1).$$

$$\vec{u_2} = (1, 3, 0).$$

$$\vec{u_3} = (-2, -7, 3)$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} -2 \\ -7 \\ 3 \end{pmatrix}.$$

$$x = c_1 + c_2 - 2c_3.$$

$$y = 3c_1 + 3c_2 - 7c_3.$$

$$z = -c_1 + 3c_3$$

Determine if the given vector are linearly dependent or independent. If the dependent then find the dependent eq.

$(2,3,-5)$, $(-4,-1,3)$ and $(-8,3,-1)$

Soln.

let c_1, c_2 and c_3 be any arbitrary const.

for dependency check

$$c_1(2,3,-5) + c_2(-4,-1,3) + c_3(-8,3,-1) = (0,0,0).$$

We get.

$$\left[\begin{array}{ccc:c} 2 & -4 & -8 & 0 \\ 3 & -1 & 3 & 0 \\ -5 & 3 & -1 & 0 \end{array}\right].$$

$$\left[\begin{array}{ccc:c} 1 & -2 & -4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

c_3 is free variable / non leading variable, let $c_3 = t$.

then,

$$c_2 = -3t$$

$$c_1 = -6t + 4t$$

$$= -2t.$$

$(c_1, c_2, c_3) = (-2t, -3t, t) = t(-2,-3,1).$

For $t \neq 0$ given vector are linearly independent

Dependency eq for particular solution.

When

$t = 1$; $-2(2,3,-5) - 3(-4,-1,3) + 1(-8,3,-1) = 0.$

$t = 2$; $-4(2,3,-5) - 6(-4,-1,3) + 2(-8,3,-1) = 0$

for $t = 0$, $c_1 = c_2 = c_3 = 0$

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∴ The vectors are linearly independent. But t=0. Hence we conclude that given vector is linearly independent.

b) $(2, 3, -5, -2)$, $(4, -4, -11, -2)$ and $(-4, -2, 6, 5)$

Sol^n

$$c_1 = (2, 3, -5, -2) + c_2 (4, -4, -11, -2) + c_3 (-4, -2, 6, 5) = 0$$

$$\left[\begin{array}{ccc|c} 2 & 4 & -4 & 0 \\ 3 & -4 & -2 & 0 \\ -5 & -11 & 6 & 0 \\ -2 & -2 & 5 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -2 & 0 \\ 0 & 1 & -2/5 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

C_4 is non-leading free variable.

$c_4 = t$.

$c_3 = 0$

$c_2 - 2/5\, c_3 = 0$

$c_2 = 0$.

$c_1 + 2c_2 - 2c_3 = 0$.

$c_1 = 0$.

$c_1 = c_2 = c_3 = 0$

Hence, given vector are linearly independent.

Basis

Find a basis for the subspace S of R^3 that is the plane eq^n $4x - 2y + 3z = 0$

Sol^n

The eq^n of plane $4x - 2y + 3z = 0$ has infinitely many sol^n :-

So, let $z = t$ and $y = s$

Then, $4x = 2s - 3t$

$$x = \frac{2s - 3t}{4}$$

$$\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{2s-3t}{4} \\ s \\ t \end{bmatrix} = s\begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3/4 \\ 0 \\ 1 \end{bmatrix}.$$

$$\vec{v} = s\vec{u_1} + t\vec{u_2}.$$

Hence, $\vec{v}$ is the span of vector $\vec{u_1}$ and $\vec{u_2}$ as

$\vec{v} = s\vec{u_1} + t\vec{u_2}$; $s \& t \neq 0$ be any arbitrary constants.

$$u_1 = \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} \quad u_2 = \begin{bmatrix} -3/4 \\ 0 \\ 1 \end{bmatrix}$$

Now

to show u_1 and u_2 are linearly dependent

$c_1\vec{u_1} + c_2\vec{u_2} = 0$

that gives,

$0.1c_1 - 3/4\, c_2 = 0$

$c_1 = 0$

$c_2 = 0.$

$c_1 = c_2 = 0$. Hence, u_1 and u_2 are linearly dependent.

10 Show that given vector are not basis for R^3?

$(1,4,-2)$, $(-2,-7,4)$ and $(-4,-13,8)$.

Sol

check linearly:

$$c_1\vec{u_1} + c_2\vec{u_2} + c_3\vec{u_3} = \vec{0}$$

$$c_1\begin{pmatrix}1\\4\\-2\end{pmatrix} + c_2\begin{pmatrix}-2\\-7\\4\end{pmatrix} + c_3\begin{pmatrix}-4\\-13\\8\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\left[\begin{array}{ccc:c}1 & -2 & -4 & 0\\4 & -7 & -13 & 0\\-2 & 4 & 8 & 0\end{array}\right]$$

$$\left[\begin{array}{ccc:c}1 & -2 & -4 & 0\\0 & 1 & 3 & 0\\0 & 0 & 0 & 0\end{array}\right]$$

x_3 is non-leading variable.

$x_3 = t$

$x_2 = -3t$

$x_1 = -2t$

For any values to t, x_2 and x_1 are different take

$t = 1$, $x_3 = 1$

$x_2 = -3$

$x_1 = -2$.

Hence, $\vec{u_1}$, $\vec{u_2}$ and $\vec{u_3}$ are linearly dependent [illegible] and [illegible] doesn't satisfy the condition for forming a equation.

Find the rank and nullity of the given matrix A. find null space of A.

$$A = \begin{bmatrix} 5 & 10 & -5 \\ 3 & 7 & -2 \\ 2 & 7 & 1 \end{bmatrix}$$

Arranging in row echelon form.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 7 & -2 \\ 2 & 7 & 1 \end{bmatrix}.$$

$$= \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

row rank = no of non-zero row.
$= 2.$

row rank + nullity = n.

nullity = 3 - 2
$= 1.$

Infinite series

6 Decide the convergence and divergence of the following sequences. If the sequence converges then find its limit.

1) $a_n = \dfrac{n+3}{n^2+5n+6}$

$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \dfrac{n+3}{n^2+5n+6}$

$= \lim_{n \to \infty} \dfrac{\cancel{n^2}(1/n + 3/n^2)}{\cancel{n^2}(1+5/n+6/n^2)}$

$= \dfrac{1/\infty + 3/\infty}{1+5/\infty+6/\infty}$

$= 0$ [Finite value]

~~divergence~~ converges and limit is 0.

(ii) $a_n = \sqrt[n]{3^{2n+1}}$

$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \sqrt[n]{3^{2n+1}}$

$= \lim_{n \to \infty} 3^{\frac{2n+1}{n}}$

$= \lim_{n \to \infty} 3^{2+1/n}$

$= 3^{2+0}$ 3

$= 6 > 3\ 1$

converges.

$a_n = \frac{\sin^2 n}{2^n}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\sin^2 n}{2^n}$

$= \frac{\sin^2 \infty}{2^\infty}$

$= \frac{\text{Finite value}}{\infty}$

$= 0$ ie finite value

$\therefore a_n$ converges and its limit is 0.

(ii) $a_n = \sqrt[n]{3^{2n+1}}$

- Soln

$a_n = \sqrt[n]{3^{2n+1}}$

$= 3^{\frac{2n+1}{n}}$

$= 3^{2+1/n}$

$= 3^2 \cdot 3^{1/n}$

$= 9 \cdot 3^{1/n}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 9 \cdot 3^{1/n}$

$= \lim_{n\to\infty} 9 \cdot 3^{1/n}$

$= 9 \cdot 3^{1/\infty}$

$= 9$ [Finite value]

a_n converges and its value is 9.

$a_n = \frac{3^n}{n^3}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3^n}{n^3}$ $\left[\frac{\infty}{\infty} \text{ form}\right]$

Applying L-Hospital rule;

$a^n \quad a^n \ln a$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3^n \ln 3}{3n^2}$

$= \lim_{n\to\infty} \frac{3^n (\ln 3)^2}{3 \times 2n}$

$= \lim_{n\to\infty} \frac{3^n (\ln 3)^3}{6}$

$= \frac{3^\infty \cdot \ln^3 3}{6}$

$= \infty$ ie infinite value

Thus, a_n diverges.

$3^n \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{}$

$= \frac{\infty}{}$

v $a_n = \left(1 - \frac{1}{n}\right)^n$

$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \left(1 - \frac{1}{n}\right)^n$

$= e^{-1}$

$= \frac{1}{e}$ //

a_n converges and its limit is $1/e$.

7. Find n^{th} partial sum and sum of series for the following

1) $\frac{5}{1 \cdot 2} + \frac{5}{2 \cdot 3} + \frac{5}{3 \cdot 4} + \cdots$

$= 5\left[\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots\right]$

$= 5\left[\frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots\right]$

$= 5 \cdot \left[1 - \frac{1}{n+1}\right]$.

$= 5 \cdot \left(\frac{n+1-1}{n+1}\right)$

$= \frac{5n}{n+1}$

So, n^{th} partial sum $(S_n) = \frac{5n}{n+1}$

For

sum of series,

$S_n = \lim_{n \to \infty} \frac{5n}{n+1}$

$= \lim_{n\to\infty} \frac{5}{1+\frac{1}{n}}$

$= \frac{5}{1+1/\infty}$

$= \frac{5}{1}$

$\therefore$ Sum of series $= 5$

$\sum_{n=0}^{\infty}\left(\frac{5}{2^n} - \frac{1}{3^n}\right)$

or

Given, $\sum_{n=0}^{\infty}\left(\frac{5}{2^n} - \frac{1}{3^n}\right)$.

$= \sum_{n=0}^{\infty} \frac{5}{2^n} - \sum_{n=0}^{\infty} \frac{1}{3^n}$.

$= 5 \sum_{n=0}^{\infty} \frac{1}{2^n} - \sum_{n=0}^{\infty} \frac{1}{3^n}$

$= 5\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{2^2} + \cdots\right)$

which is geometric series

we know, $S_n = \frac{a}{1-r}$ for infinite series and $S_n = \frac{a(1-r^n)}{1-r}$ for g.p.

for partial sum,

$S_n = 5\left[\frac{1(1-(1/2)^n)}{1-1/2}\right] - \left[\frac{1(1-(1/3)^n)}{1-1/3}\right]$

$= 5\frac{(1-1/2^n)}{1/2} - \frac{1-1/3^n}{2/3}$

$= 10\left[1-1/2^n\right] - 3/2\left[1-1/3^n\right]$ is required partial sum.

And for sum of series,

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} 10\left[1-\frac{1}{2^n}\right] - \frac{3}{2}\left[1-\frac{1}{3^n}\right]$

$= 10\left[1-\frac{1}{2^\infty}\right] - \frac{3}{2}\left[1-\frac{1}{3^\infty}\right]$

$= 10[1-0] - 3/2[1-0]$.

$= 10 - 3/2$

$\therefore$ Sum of series $= 17/2$.

Decide whether the following series converges or diverges using.

8) n^{th} term test

i) $\sum_{n=1}^{\infty} \frac{n}{n+1}$

$= \lim_{n \to \infty} \frac{n}{n+1}$

$= \lim_{n \to \infty} \frac{1}{1+\frac{1}{n}}$

$= 1 \div 1/\infty$

$= 1$.

Diverges.

iii) $\sum_{n=1}^{\infty} n \sin \frac{1}{n}$

$= \lim_{n \to \infty} n \sin \frac{1}{n}$

$= \lim_{n \to \infty} \frac{\sin 1/n}{1/n}$ $\quad \infty \cdot \sin 1/\infty$

$= \infty \times \sin 0$

$= \cancel{\sin 0}\ 1$ $\quad = \infty$

Diverges.

ii) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt[3]{n+1}}{\sqrt{n}+1}$

$= \lim_{n \to \infty} (-1)^{n+1} \frac{\sqrt[3]{n+1}}{\sqrt{n}+1}$

$= \lim_{n \to \infty} (-1)^{n} \cdot (-1) \frac{\sqrt[3]{n+1}}{\sqrt{n}+1}$.

Not defined as $(-1)^n$ does not exist.

& so, it diverges

iv) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{10^n}{n^{10}}$

sol^n

$a_n = (-1)^{n+1} \cdot \frac{10^n}{n^{10}}$

$\lim_{n \to \infty} a_n = \lim_{n \to \infty} (-1)^{n} \cdot (-1)^1 \frac{10^n}{n^{10}}$

$= \lim_{n \to \infty} \frac{(-1)\ 10^n}{n^{10}} = -\lim_{n \to \infty} \frac{(-10)^n}{n^{10}}$

a_n fails to exist

so, it diverges.

45

(iii) $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

$f(x) = \frac{x}{x^2+1}$

Now, $\int^{b}$ $\lim_{b \to \infty} \int_1^b \frac{x}{x^2+1}\, dx.$

$= \lim_{b \to \infty} \int_1^b \cancel{\frac{1}{x} + x}$

let $y = x^2+1$

$dy = 2x\, dx.$

Then $\lim_{b \to \infty} \int_1^b \frac{dy/2}{y}$

$= \lim_{b \to \infty} \int_1^b \frac{1}{2} \frac{dy}{y}.$

$= \lim_{b \to \infty} \frac{1}{2} \left[\log y\right]_1^b$

$= \lim_{b \to \infty} \frac{1}{2} \left[\log (x^2+1)\right]_1^b$

$= \cancel{\log} \lim_{b \to \infty} \frac{1}{2} \left[\log (b^2+1) - \log 2\right]$

$= \frac{1}{2} \left[\log \cancel{(\infty+1)} - \log 2\right]$

$= \cancel{\log - \frac{1}{2} \log 2} = \infty.$

= f(x) diverges

so, we conclude that $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ diverges too.

comparison test.

$\left(\frac{n}{3n+1}\right)^n$

$= \left(\frac{n}{3n+1}\right)^n$

$= \frac{n^n}{(3n+1)^n}$

$= \frac{n^n}{n^n}\left(\frac{1}{3+1/n}\right)^n$

$= \frac{1}{(3+1/n)^n}$

$b_n = 1/n$

(ii). $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$

Sol:

$a_n = \frac{1}{n\sqrt{n^2-1}}$

$= \frac{1}{n\, n\sqrt{1+1/n^2}}$ comparing with $1/n^{3/2}$

$= \frac{1}{n^2\sqrt{1+1/n^2}}$

let

$a_n = \frac{1}{2\sqrt{3}} + \frac{1}{3\sqrt{8}} + \frac{1}{4\sqrt{15}} + \dots$

$1/n^{3/2} = 1/2\sqrt{2} + 1/3\sqrt{3} + 1/4\sqrt{4}$

As, $1/n^{3/2} > a_n$

and $\lim_{n\to\infty} \frac{1}{n^{3/2}} > \lim_{n\to\infty} a_n$ and as

$\lim_{n\to\infty} 1/n^{3/2}$ converges so, $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$

$$\sum_{n=1}^{\infty} \frac{n}{n^2+1}$$

$$a_n = \frac{n}{n^2+1}$$

$$= \frac{n}{\cancel{n(1/n)}\, n^2(1+1/n)}$$

$$= \frac{1}{n(1+1/n)}$$

iii) $$\sum_{n=1}^{\infty} \frac{n+2^n}{n^2 2^n}$$

Sol^n

$$a_n = \frac{n+2^n}{n^2 2^n} = \frac{n}{n^2 2^n} + \frac{2^n}{n^2 2^n}$$

$$= \frac{1}{n 2^n} + \frac{1}{n^2}$$

Comparing $\frac{1}{n 2^n}$ with $\frac{1}{2^n}$, $\frac{1}{2^n} > \frac{1}{n2^n}$

ie $$\sum_{n=1}^{\infty} \frac{1}{2^n} > \sum_{n=1}^{\infty} \frac{1}{n 2^n}$$

Thus, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n+2^n}{n^2 2^n}$ converges as

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Limit comparision test

1) $\sum_{n=1}^{\infty} \frac{n+1}{n^2\sqrt{n}}$

$a_n = \sum \frac{n+1}{n^2\sqrt{n}}$

$= \frac{n+1}{n^2\sqrt{n}}$

$= \frac{n(1+1/n)}{n^2\sqrt{n}}$

$= \frac{1+1/n}{n\sqrt{n}} = \frac{1+1/n}{n^{3/2}}$

let $c = \cancel{n^2}\ 1/n^{3/2}$.

$\lim_{n\to\infty} \frac{a_n}{c_n}$

$= \lim_{n\to\infty} \frac{\frac{1+1/n}{n^{3/2}}}{1/n^{3/2}}$

$= 1 + 1/\infty = 1$

Converges

(ii). $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^2}$

let $a_n = \frac{1}{(\ln n)^2}$

$= \frac{1}{\ln n^2}$

$= \frac{2}{\ln n}$.

11) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$

$a_n = \frac{\sqrt{n}}{n^2+1}$

$= \frac{\sqrt{n}}{n^{3/2}(\cdot} \quad \frac{\sqrt{n}}{n^2(1+1/n^2)}$

$= \frac{1}{n^{3/2}(1+1/n^2)}$

let $c_n = 1/n^{3/2}$.

$\lim_{n \to \infty} \frac{a_n}{c_n}$

$= \lim_{n \to \infty} \frac{1/n^{3/2}(1+1/n^2)}{1/n^{3/2}}$

$= 1$

converges,,

12 Raho test :-

1) $\sum_{n=1}^{\infty} \frac{(n+3)!}{3!\,n!\,3^n}$

$a_n = \frac{(n+3)!}{3!\,n!\,3^n}$

$= \frac{(n+3)(n+2)(n+1)\,n!}{3!\,n!\,3^n}$

$= \frac{(n+3)(n+2)(n+1)}{3!\,3^n}$

Then a_n

Ratio test:-

$$\sum_{n=1}^{\infty} \frac{(n+3)!}{3!\,n!\,3^n}$$

$$a_n = \frac{(n+3)!}{3!\,n!\,3^n}$$

$$a_{n+1} = \frac{(n+1+3)!}{3!\,(n+1)!\,3^n} = \frac{(n+4)!}{3!\,(n+1)!\,3^{n+1}}$$

Then,

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

$$= \lim_{n\to\infty} \frac{\dfrac{(n+4)!}{3!\,(n+1)!\,3^{n+1}}}{\dfrac{(n+3)!}{3!\,(n)!\,(3^{n})}}$$

$$= \lim_{n\to\infty} \frac{(n+4)}{(n+1)\,3}$$

$$= \lim_{n\to\infty} \frac{n\,(1+4/n)}{3n\,(1+1/n)}$$

$$= \lim_{n\to\infty} \frac{1+4/n}{3(1+1/n)}$$

$$= \frac{1}{3} < 1$$ ∴ converges

(ii). $$\sum_{n=1}^{\infty} \frac{n\,2^n\,(n+1)!}{3^n\,n!}$$

$$a_n = \frac{n\,2^n\,(n+1)!}{3^n\,n!}$$

$$a_{n+1} = \frac{(n+1)\,2^{n+1}\,(n+2)!}{3^{n+1}\,(n+1)!}$$

Then,

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

$$= \lim_{n\to\infty} \frac{\dfrac{(n+1)\,\cancel{2^n}\cdot 2\,(n+2)\cancel{(n+1)!}}{\cancel{3^n}\cdot 3\,(n+1)\cancel{n!}}}{\dfrac{\cancel{2^n}\,n\,\cancel{(n+1)!}}{\cancel{3^n}\,\cancel{n!}}}$$

$$= \lim_{n\to\infty} \frac{\cancel{(n+1)}\,(n+2)\,2}{3\,\cancel{(n+1)}\,n}$$

$$= \lim_{n\to\infty} \frac{2(n+2)}{3n}$$

$$= \lim_{n\to\infty} \frac{2n\,(1+2/n)}{3n}$$

$$= \lim_{n\to\infty} \frac{2(1+2/n)}{3}$$

$$= \frac{2}{3} \nless 1$$ ~~diverges~~ converges.

(ii) $\sum_{n=1}^{\infty} \frac{(3n)!}{n!(n+1)!(n+2)!}$

$a_n = \frac{(3n)!}{n!(n+1)!(n+2)!}$

$a_{n+1} = \frac{(3n+3)!}{(n+1)!(n+2)!(n+3)!}$

$\lim_{n \to \infty} \frac{a_{n+1}}{a_n}$

$= \lim_{n \to \infty} \frac{\frac{(3n+3)!}{(n+1)!\cancel{(n+2)!}(n+3)!}}{\frac{(3n)!}{n!(n+1)!\cancel{(n+2)!}}}$

$= \lim_{n \to \infty} \frac{\frac{(3n+3)(3n+2)(3n+1)(3n)!}{(n+3)(n+2)(n+1)n!}}{\frac{(3n)!}{n!}}$

$\lim_{n \to \infty} \frac{(3n+3)(3n+2)(3n+1)}{(n+3)(n+2)(n+1)}$

$\lim_{n \to \infty} \frac{n^3(3+3/n)(3+2/n)(3+1/n)}{n^3(1+3/n)(1+2/n)(1+1/n)}$

$\frac{3 \cdot 3 \cdot 3}{1}$

$27 > 1$ diverges

nth root test:

$$\sum_{n=2}^{\infty} \left(\frac{1}{n} - \frac{1}{n^2}\right)^n$$

$$a_n = \left(\frac{1}{n} - \frac{1}{n^2}\right)^n$$

$$= \lim_{n \to \infty} \sqrt[n]{a_n}$$

$$= \lim_{n \to \infty} \sqrt[n]{\left(\frac{1}{n} - \frac{1}{n^2}\right)^n}$$

$$= \lim_{n \to \infty} \frac{1}{n} - \frac{1}{n^2} \quad = \frac{n^2 - n}{n^3}$$

$$= 0 - 0 = \lim_{n \to \infty} \frac{n-1}{n^2}$$

$$= 0 \quad = \lim_{n \to \infty} \frac{1}{2n} = \frac{1}{2} \lim_{n \to \infty} \frac{1}{n} \to 0 < 1$$

convergent.

(ii). $$\sum_{n=2}^{\infty} \frac{n}{(\ln n)^{n/2}}$$

$$a_n = \frac{n}{(\ln n)^{n/2}}$$

$$= \lim_{n \to \infty} \sqrt[n]{\frac{n}{(\ln n)^{n/2}}}$$

$$= \lim_{n \to \infty} \frac{n^{1/n}}{(\ln n)^{1/2}}$$

$$= \lim_{n \to \infty} \frac{1}{(\ln n)^{1/2}}.$$

$$= \frac{1}{(\ln \infty)^{1/2}}$$

$$= 0 < 1$$

As $\rho < 1$ so, $\sum_{n=1}^{\infty} \frac{n}{(\ln n)^{n/2}}$ converges

$$\sum_{n=1}^{\infty} \frac{(n!)^n}{n^{(n^2)}}$$

$$a_n = \frac{(n!)^n}{n^{n^2}}$$

$$\lim_{n \to \infty} \sqrt[n]{a_n}$$

$$\lim_{n \to \infty} \left[\frac{(n!)^n}{n^{n^2}}\right]^{\frac{1}{n}}$$

$$= \lim_{n \to \infty} \frac{n!}{n^{n^2 \times 1/n}}$$

$$= \lim_{n \to \infty} \frac{n!}{n^n}$$

$$= \lim_{n \to \infty} 1/n \cdot 2/n \cdot 3/n \cdot 4/n \cdots n/n$$

$$= 0 < 1$$

$$\sum_{n=1}^{\infty} (-1)^n \frac{(n+1)^n}{(2n)^n}$$

Sol^n

$$a_n = (-1)^n \frac{(n+1)^n}{(2n)^n}$$

The corresponding series of absolute value is

$$\sum |a_n| = \frac{(n+1)^n}{(2n)^n}$$

Applying root test,

$$\lim_{n \to \infty} \sqrt[n]{|a_n|} = \lim_{n \to \infty} \sqrt[n]{\left(\frac{n+1}{2n}\right)^n}$$

$$= \lim_{n \to \infty} \frac{n+1}{2n}$$

$$= \lim_{n \to \infty} \frac{1 + 1/n}{2}$$

$$= \frac{1}{2} < 1$$ [$\because$ $\sum |a_n|$ converges from n^{th} root te

Thus from absolute convergence test, as the corresponding series of absolute value converges so,

$$\sum_{n=1}^{\infty} (-1)^n \frac{(n+1)^n}{(2n)^n}$$ also converges

Alternation series test (Leibniz test).

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\ln n}{n}.$$

Sol^n

$$a_n = (-1)^{n+1} \frac{\ln n}{n}.$$

Absolute convergence test

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(0.1)^n}{n}.$$

Here, $a_n = (-1)^{n+1} \frac{(0.1)^n}{n}$

The corresponding absolute value is,

$$\sum |a_n| = \sum \frac{(0.1)^n}{n}$$

$$= \sum \frac{1}{10^n . n}$$

Comparing above with $\sum \frac{1}{10^n}$,

$$= \sum \frac{1}{10^n} > \sum \frac{1}{10^n n}.$$

and as $\sum 1/10^n$ converges, $\sum 1/10^n n$ also converges

Then, from absolute convergence test,

$\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{(0.1)^n}{n}$ also convergence

as its corresponding series of absolute value converges

ii) $\sum_{n=1}^{\infty} (-1)^n n^2 \left(\frac{2}{3}\right)^n$

Sol:

$$a_n = (-1)^n n^2 \left(\frac{2}{3}\right)^n$$

The corresponding series of absolute value is

$$\sum |a_n| = \sum n^2 \left(\frac{2}{3}\right)^n$$

applying root test,

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \sqrt[n]{n^2 \left(\frac{2}{3}\right)^n}$$

$$= \lim_{n\to\infty} n^{2/n} \left(\frac{2}{3}\right)$$

$$= \lim_{n\to\infty} (n^{1/n})^2 \left(\frac{2}{3}\right).$$

$$= \lim_{n\to\infty} \frac{2}{3}. \quad \left[\lim_{n\to\infty} n^{1/n} = 1\right]$$

$$= \frac{2}{3} < 1$$

∴ Thus, as $S < 1$ so, $\sum |a_n|$ converges

$$\frac{\ln 1}{1} - \frac{\ln 2}{2} + \frac{\ln 3}{3} - \cdots + (-1)^{n+1}\frac{\ln n}{n}$$

Here, $\frac{\ln n}{n} > 0$

we have,

$$\sum_{n=1}^{\infty} (-1)^{n+1}\frac{\ln n}{n} = \frac{\ln 1}{1} - \frac{\ln 2}{2} + \frac{\ln 3}{3} - \frac{\ln 4}{4}$$

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\ln n}{n} = 0 - \frac{\ln 2}{2} + \sum_{n=3}^{\infty} (-1)^{n+1}\frac{\ln n}{n}$$

Here,

$$\frac{\ln(n+1)}{n+1} \leq \frac{\ln n}{n} \quad \text{for } n \geq 3$$

And. $\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{1}{n} = 0$

Thus, $\sum_{n=1}^{\infty} (-1)^{n+1}\frac{\ln n}{n} = -\frac{\ln 2}{2} + \sum_{n=3}^{\infty} (-1)^{n+1}\frac{\ln n}{n}$

satisfies condition of Leibniz test, hence it converges.

$$\sum_{n=1}^{\infty} \frac{6}{(2n-1)(2n+1)}$$

Sol^n

Firstly, using partial fraction

$$\frac{6}{(2n-1)(2n+1)} = \frac{A}{(2n-1)} + \frac{B}{2n+1}$$

or $6 = A(2n+1) + B(2n-1)$

If $n = 1/2$

or $6 = A \times 2 + B \times 0$

or $A = 3$.

If $n = -1/2$

$6 = A \times 0 + B(-2)$

or $B = -2$

Hence, $$\frac{6}{(2n-1)(2n+1)} = \frac{3}{2n-1} + \frac{(-3)}{2n+1}$$

Now for partial sum,

$$S_n = \sum_{n=1}^{n} \frac{6}{(2n-1)(2n+1)}$$

$$= \sum_{n=1}^{n} \frac{3}{2n-1} - \frac{3}{2n+1}$$

$$= 3\left[\sum_{n=1}^{n} \frac{1}{2n-1} - \frac{1}{2n+1}\right]$$

$$= 3\left[\left(\frac{1}{1} - \frac{1}{3} + \frac{1}{3} - \frac{1}{5} + \frac{1}{5} - \frac{1}{7} + \cdots \left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)\right)\right]$$

$$= 3\left[1 - \frac{1}{2n+1}\right]$$

$$= 3\left(\frac{2n+1-1}{2n+1}\right) = \frac{3 \times 2n}{2n+1}$$

$\therefore S_n = \dfrac{6n}{2n+1}$

Now for sum of series,

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{6n}{2n+1}$$

$$= \lim_{n\to\infty} \frac{6}{2+1/n}$$

$$= \frac{6}{2+1/\infty}$$

$$= 6/2 = 3$$

$\therefore$ sum of series $= 3$

iv. $\displaystyle\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}$

Here,

applying partial fraction,

$$\frac{2n+1}{n^2(n+1)^2} = \frac{A}{n} + \frac{B}{(n+1)} + \frac{C}{n^2} + \frac{D}{(n+1)^2}.$$

$2n+1 = An(n+1)^2 + B(n+1)n^2 + C(n+1)^2 + Dn^2$

When $n=0$, $C=1$ --(i)

When $n=-1$, $D=-1$ --(ii)

When $n=1$, $4A+2B=0$

$\Rightarrow 2A+B=0$ --(iii)

When $n=2$, $A=0$, so, $B=0$

Thus,

$$\frac{2n+1}{n^2(n+1)^2} = \frac{1}{n^2} - \frac{1}{(n+1)^2}$$

For n^{th} partial sum,

$$S_n = \sum_{n=1}^{n} \frac{2n+1}{n^2(n+1)^2} = \sum_{n=1}^{n} \frac{1}{n^2} - \frac{1}{(n+1)^2}.$$

$$= \left[\frac{1}{1} - \frac{1}{2^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots \left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right)\right]$$

$$= 1 - \frac{1}{(n+1)^2}.$$

$$= \frac{(n+1)^2 - 1}{(n+1)^2}$$

$$\therefore S_n = \frac{n^2+2n}{(n+1)^2}$$

Now, for sum of series,

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n^2+2n}{(n+1)^2}$$

$$= \lim_{n\to\infty} \frac{1+2/n}{(1+1/n)^2}$$

$$= \frac{1+2/\infty}{(1+1/\infty)^2}$$

$$= 1$$

$\therefore$ sum of series $= 1$

$$\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right)$$

Sol^n

For n^{th} partial sum

$$S_n = \sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right).$$

$$= \left[1/\sqrt{1} - 1/\sqrt{2} + 1/\sqrt{2} - 1/\sqrt{3} + 1/\sqrt{3} - 1/\sqrt{4} + \dots + \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right)\right]$$

$$= 1/\sqrt{1} - \frac{1}{\sqrt{n+1}}$$

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$\therefore S_n = 1 - \frac{1}{\sqrt{n+1}}$

Now, for sum of series

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} 1 - \frac{1}{\sqrt{n+1}}$$

$$= 1 - \frac{1}{\sqrt{\infty+1}}$$

$$= 1 - \frac{1}{\infty}.$$

$\therefore$ Sum of series $= 1$

Q) $\sum_{n=1}^{\infty} n \sin 1/n.$

Here, $a_n = n \sin 1/n$

Then, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} n \sin 1/n.$

$$= \infty \times \sin 1/\infty$$

$$= \infty$$

So, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} n \sin 1/n = \lim_{n\to\infty} \frac{(\sin 1/n)}{1/n}$ $\left[\frac{0}{0} \text{ form}\right]$

Applying L-Hospital rule,

$\lim_{n\to\infty} \cos 1/n \; d$

$a_n = n \sin 1/n$

$= \lim_{n\to\infty} n \sin 1/n$

$= \lim_{n\to\infty} \frac{\sin 1/n}{1/n} = 1$

diverge

Root comparison test

$$\sum_{n=1}^{\infty} \left(\frac{n}{3n+1}\right)^{n}$$

Sol:

We have, $\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^{n} = e^{x} \Rightarrow \lim_{n\to\infty} \left(\frac{n}{n+x}\right)^{n} = e^{-x}$

Now as $a_n = \left(\frac{n}{3n+1}\right)^{n}$.

And $\lim_{n\to\infty} a_n = \left(\frac{n}{3n+1}\right)^{n} < \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{n}$.

ie $\lim_{n\to\infty} \left(\frac{n}{3n+1}\right)^{n} < \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{n}$

$\sum_{n=1}^{\infty} \left(\frac{n}{3n+1}\right)^{n}$ converges as $\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{n}$ converges to e^{-1} //

Alternating series test

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt{n}+1}{n+1}$$

Here, $a_n = (-1)^{n+1} \frac{\sqrt{n}+1}{n+1}$

$$= \frac{\sqrt{1}+1}{1+1} - \frac{\sqrt{2}+1}{2+1} + \frac{\sqrt{3}+1}{3+1} - \cdots$$

Here,

i) $\frac{\sqrt{n}+1}{n+1} > 0$.

ii) $\frac{\sqrt{n+1}+1}{(n+1)+1} \le \frac{\sqrt{n}+1}{n+1}$

iii) $\lim_{n\to\infty} \frac{\sqrt{n}+1}{n+1}$

$\Rightarrow \lim_{n\to\infty} \frac{1+1/\sqrt{n}}{\sqrt{n}+1/\sqrt{n}}$

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$$= \frac{1+0}{\infty+0}$$

$$= \frac{1}{\infty}$$

$$= 0.$$

Thus, $\frac{\sqrt{n+1}}{n+1} \to 0$

Thus, as $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt{n+1}}{n+1}$ satisfies all conditions of Leibniz test, Hence it converges

III $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(n+1)}{n^2}$.

Here,

$$a_n = (-1)^{n+1} \frac{(n+1)}{n^2} = \frac{2}{1^2} - \frac{3}{4} + \frac{4}{9} - \dots$$

Here,

i) $\frac{n+1}{n^2} > 0$

ii) $\frac{(n+1)+1}{(n+1)^2} \le \frac{(n+1)}{n^2}$

iii) $\lim_{n\to\infty} \frac{n+1}{n^2}$

$$= \lim_{n\to\infty} \frac{1+1/n}{n}$$

$$= 1/\infty = 0.$$

Thus, $\lim_{n\to\infty} \frac{n+1}{n^2} = 0$

63

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... $\sum_{n=1} (-1)^{n+1} \frac{n+1}{n^2}$ satisfies all the conditions of Leibnitz test

~~con~~ Hence it converges

...ve that following series converges conditionally.

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n+3}$$

$$= (-1)^n \frac{1}{n+3}$$

...e,

$$\frac{1}{n+3} > 0$$

$$\frac{1}{(n+1)+3} \leq \frac{1}{n+3}$$

$$\lim_{n \to \infty} \frac{1}{n+3} \to 0$$

...hus, from leibniz test $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n+3}$ converges.

..., the corresponding absolute series $\sum_{n=1}^{\infty} \frac{1}{n+3}$. But the

...ies $\sum_{n=1}^{\infty} \frac{1}{n+3}$ diverges. Thus the series $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n+3}$ converges

...ditionally.

64

(ii) $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n - \ln n}$

Here,

$a_n = (-1)^n \frac{\ln n}{n - \ln n}$

Here,

i $\frac{\ln n}{n - \ln n} > 0$

ii) $\frac{\ln(n+1)}{(n+1) - \ln(n+1)} \leq \frac{\ln n}{n - \ln n}$ for only $n \geq 3$.

iii) $\lim_{n\to\infty} \frac{\ln n}{n - \ln n}$ (indeterminate form)

$= \lim_{n\to\infty} \frac{1/n}{1 - 1/n}$

$= \lim_{n\to\infty} \frac{1/n}{(n-1)/n}$

$= \lim_{n\to\infty} \frac{1}{n-1}$

$= 0$

Thus, $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n - \ln n}$ converges from leibniz test

Now corresponding series of absolute sequence is $\sum_{n=1}^{\infty} \frac{\ln n}{n - \ln n}$. Here, let $a_n = \frac{\ln n}{n - \ln n}$ and $b_n = 1/n > 0$

Then $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\ln n}{n - \ln n} \times n$ $\left[\frac{\infty}{\infty} \text{ form}\right]$

By leibniz test the series converges. But the corresponding series of absolute sequence.

$$\sum_{n=1}^{\infty} \frac{\ln n}{n-\ln n} \text{ diverges.}$$

$\therefore$ The given series converge conditionally.

c)

$$\sum_{n=1}^{\infty} (-1)^n (\sqrt{n+1} - \sqrt{n}).$$

Sol^n

i) $\sqrt{n+1} - \sqrt{n} > 0$

ii) $\sqrt{n+2} - \sqrt{n+1} < \sqrt{n+1} - \sqrt{n}$

iii) $\lim_{n \to \infty} (\sqrt{n+1} - \sqrt{n}).$

$$= \lim_{n \to \infty} \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}}$$

$$= \lim_{n \to \infty} \frac{1}{\sqrt{n+1}+\sqrt{n}}$$

$= 0$

Thus, $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n-\ln n}$ converges from leibniz test.

Now, the corresponding series of absolute sequence is,

$$\sum_{n=1}^{\infty} (\sqrt{n+1} - \sqrt{n})$$

Now,

$$a_n = \sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} - \sqrt{n}}.$$

test

$\frac{1}{n} > 0$

let $b_n = \frac{1}{\sqrt{n}}$, then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{\sqrt{1+1/n}+1}$

$= \frac{1}{2}$

Thus, as $1/\sqrt{n}$ diverge so $(\sqrt{n+1} - \sqrt{n})$ also diverges. Hence given ser converges conditionally.

Integral t.

1) $\sum_{n=2}^{\infty} \frac{\ln n}{\sqrt{n}}$

Sol^n

$a_n = \frac{\ln n}{\sqrt{n}}$

then $f(x) = \frac{\ln x}{\sqrt{x}}$ is decreasing for $x \leq 2$

Here $\int_2^{\infty} f(x)\,dx$

$= \lim_{a\to\infty} \int_2^a \frac{\ln x}{\sqrt{x}}\,dx.$

$= \lim_{a\to\infty} \left[\ln x \cdot \frac{x^{1/2}}{1/2} - \int \frac{1}{x} \cdot \frac{x^{1/2}}{1/2}\,dx \right]_2^{\infty}$

$= \lim_{a\to\infty} \left[2\sqrt{x}\ln x - 2\int x^{-1/2}\,dx \right]_2^a$

$= \lim_{a\to\infty} \left[2\sqrt{x}\ln x - 4\sqrt{x} \right]_2^a$

$\left(2\sqrt{a}\ \ln a - 4\sqrt{a}\right) - \left(2\sqrt{2}\ \ln 2 - 4\sqrt{2}\right)$

$\left(2\sqrt{a}\ \ln a - 4\sqrt{a} - 2\sqrt{2}\ \ln 2 + 4\sqrt{2}\right)$.

$\therefore \int_2^\infty f(x)\,dx$ diverges.

$\frac{1}{\sqrt{n}\,(\sqrt{n}+1)}$

Let $f(x) = \frac{1}{\sqrt{x}\,(\sqrt{x}+1)}$ (a function decreasing at $x \geq 1$)

Now, $\int_1^\infty f(x)\,dx$.

$= \lim_{b\to\infty} \int_1^b \frac{1}{\sqrt{x}\,(\sqrt{x}+1)}\,dx$.

$= \lim_{b\to\infty} \int_2^{\sqrt{b}+1} \frac{2\,dz}{z}$.

$= \lim_{b\to\infty} 2\left[\ln z\right]_2^{\sqrt{b}+1}$

$\lim_{b\to\infty} 2\left[\ln(\sqrt{b}+1) - \ln 2\right]$

$= 2\left(\ln\infty - \ln 2\right)$

$= \infty$

68

Since $f(x)$ diverges, the given series also diverges