Unit – III Random Processes – Temporal Characteristics

The Random Process Concept, Classification of Processes, Deterministic and Nondeterministic Processes, Distribution and Density Functions, concept of Stationarity and Statistical Independence. First-Order Stationary Processes, Second- Order and Wide-Sense Stationarity, (N-Order) and Strict-Sense Stationarity, Time Averages and Ergodicity, Mean-Ergodic Processes, Correlation-Ergodic Processes, Autocorrelation Function and Its Properties, Cross-Correlation Function and Its Properties, Covariance Functions, Gaussian Random Processes, Poisson Random Process. Random Signal Response of Linear Systems: System Response – Convolution, Mean and Mean-squared Value of System Response, autocorrelation Function of Response, Cross-Correlation Functions of Input and Output.

Introduction

A Random Variable 'X' is defined as a function of the possible outcomes of an experiment or whose value is unknown and possibly depends on a set of random events. It is denoted by X(s).

The Concept of Random Process is based on enlarging the random variable concept to include time't' and is denoted by X(t,s) i.e., we assign a time function to every outcome according to some rule.In short, it is represented as X(t). A random process clearly represents a family or ensemble of time functions when t and s are variables. Each member time function is called a sample function or ensemble member.

Depending on time 't' and outcome ' s' fixed or variable,

- A random process represents a single time function when t is a variable and s is fixed at a specific value.
- A random process represents a random variable when t is fixed and s is a variable
- A random process represents a number when t and s are both fixed

Classification of Random Processes

A Random Processes X(t) has been classified in to four types as listed below depending on whether random variable X and time t is continuous or discrete.

1. **Continuous Random Processes**
   If a random variable X is continuous and time t can have any of a continuum of values, then X(t) is called as a continuous random pr
   Example: Thermal Noise

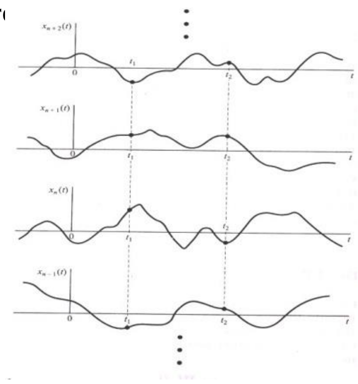


Fig 1: Continuous Random Processes

2. **Discrete Random Processes**
   If a random variable X is discrete and time t is continuous, then X(t) is called as a discrete random process. The sample functions will have only two discrete values.

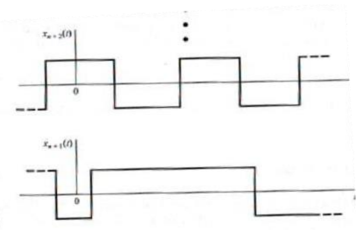


Fig 2: Discrete Random Processes

3. **Continuous random Sequence**

If a random variable X is continuous and time t is discrete, then X(t) is called as a continuous random sequence. Since a continuous random sequence is defined at only discrete times, it is also called as discrete time random process. It can be generated by periodically sampling the ensemble members of continuous random processes. These types of processes are important in the analysis of digital signal processing systems.

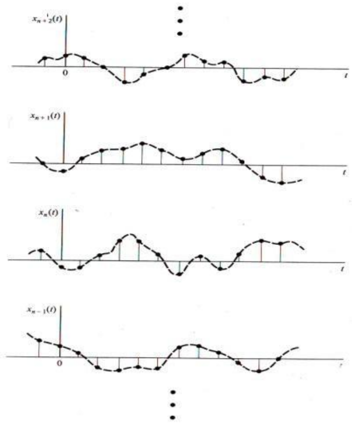


Fig 3: Continuous Random Sequence

4. **Discrete Random Sequence**

If a random variable X and time t areboth discrete, then X(t) is called as a discrete random sequence. It can be generated by sampling the sample functions of discrete random process or rounding off the samples of continuous random sequence.

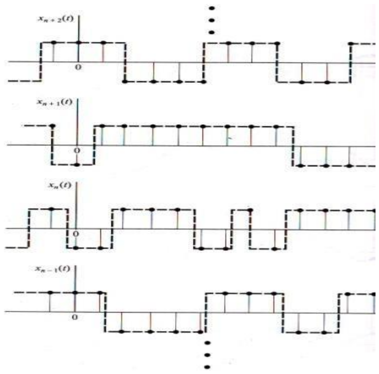


Fig 4: Discrete Random sequence

Deterministic and Non-deterministic processes

In addition to the processes, discussed above a random process can be described by the form of its sample functions.

A Process is said to be deterministic process, if future values of any sample function can be predicted from past values. These are also called as regular signals, which have a particular shape.

Example: X(t) = A Sin (ωt + Θ), A, ω and Θ may be random variables

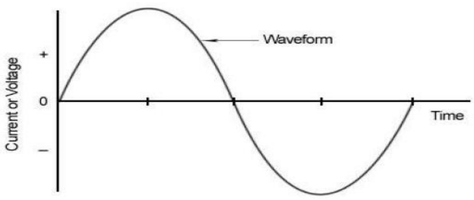


Fig 5: Example of Deterministic Process

A Process is said to be non-deterministic process, if future values of any sample function cannot be predicted from past values.

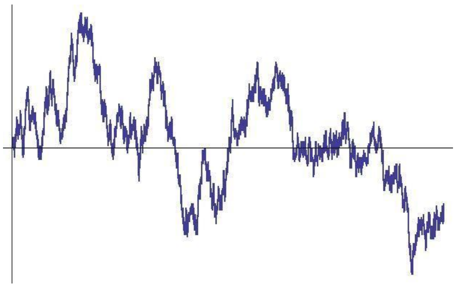


Fig 6: Example of Non-Deterministic Processes

Distribution Function and Density function

Distribution Function:

Probability distribution function (PDF) which is also be called as Cumulative Distribution Function (CDF) of a real valued random variable 'X ' is the probability that X will take value less than or equal to X.
It is given by

$$F_X \quad x \;=\; P\{X \;\leq x\}$$

In case of random process X(t), for a particular time t, the distribution function associated with the random variable X is denoted as

$$F_X \quad x: t \;=\; P\{X(t) \;\leq x\}$$

In case of two random variables, X1 = X(t1) and X2 = X (t2), the second order joint distribution function is two dimensional and given by

$$F_X\, x_{1,}, x_2 : t_{1,}, t_2 = P\{X(t_1) \leq x_{1,}, X(t_2) \leq x_2 \}$$

and can be similarly extended to N random variables, called as Nth order joint distribution function

Density Function:

The probability density function(pdf) in case of random variable is defined as the derivative of the distribution function and is given by

$$f_X \quad x = \frac{dF_X(x)}{dx}$$

In case of random process, density function is given by

$$f_X \quad x: t = \frac{dF_X(x: t)}{dx}$$

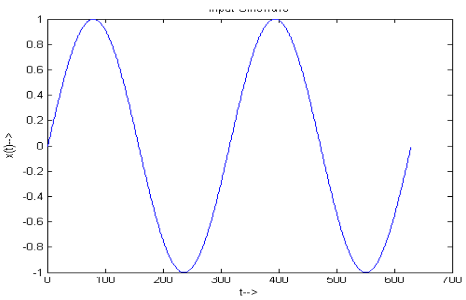
In case of two random functions, two dimensional density function is given by

$$f_X\, x_{1,}, x_2 : t_{1,}, t_2 = \frac{\partial^2 F_X\, x_{1,}, x_2 : t_{1,}, t_2}{\partial x\, \partial x}$$

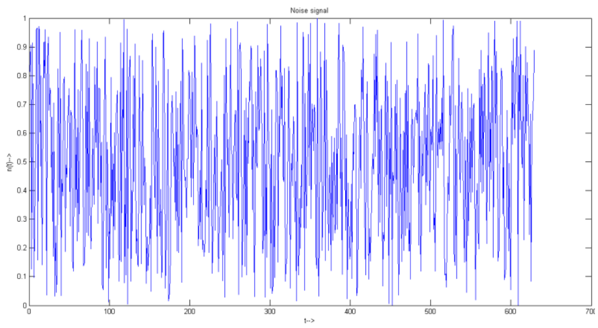
Simulation:

To write a program in MATLAB Program to generate deterministic (Sine wave), random signal (noise) and also to understand the noise effect on signal

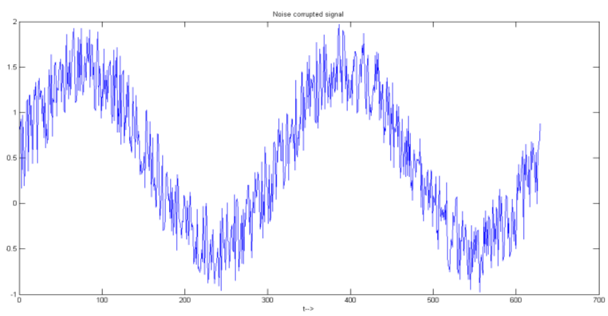
```
% Clearing and Closing Commands
Clc;
Clear
all;
Close
all;
% Generating Sine Wave (Deterministic Signal)
t=0:0.01:2*pi;
x=sin(2*t);
figure,plot(x)
; xlabel('t--
>');
ylabel('x(t)-->')
title('Input Sinewave');
```




```
Generating Noise (Non-Deterministic Signal)
noise=rand(size(x));
figure,plot(noise);
xlabel('t-->');
ylabel('n(t)-->');
title('Noise signal');
```




```
% Generating Noisy Sine Wave (Non-Deterministic Signal)
y=x+noise;
figure,plot(y)
; xlabel('t--
>');
ylabel('y(t)-->');
title('Noise corrupted signal');
```



Independence and Stationarity

Introduction

In probability theory, two events are independent, statistically independent, or stochastically independent if the occurrence of one does not affect the probability of occurrence of other i.e., having the probability of their joint occurrenceequal to the product of their individual probabilities. A random process becomes a random variable when time is fixed at some particular value. The random variable will possess statistical properties such as a mean value, moments, variance etc. that are related to its density function. If two random variables are obtained from the process for two time instants they will have statistical properties related to their joint density function. More generally, N random variables will possess statistical properties related to their N dimensional joint density function.**A random process is said to be stationary if all its statistical properties do not change with time other processes are called non-stationary.** It can also be defined as a stationary time series is one whose statistical properties such as mean, variance, autocorrelation, etc. are all constant over time.

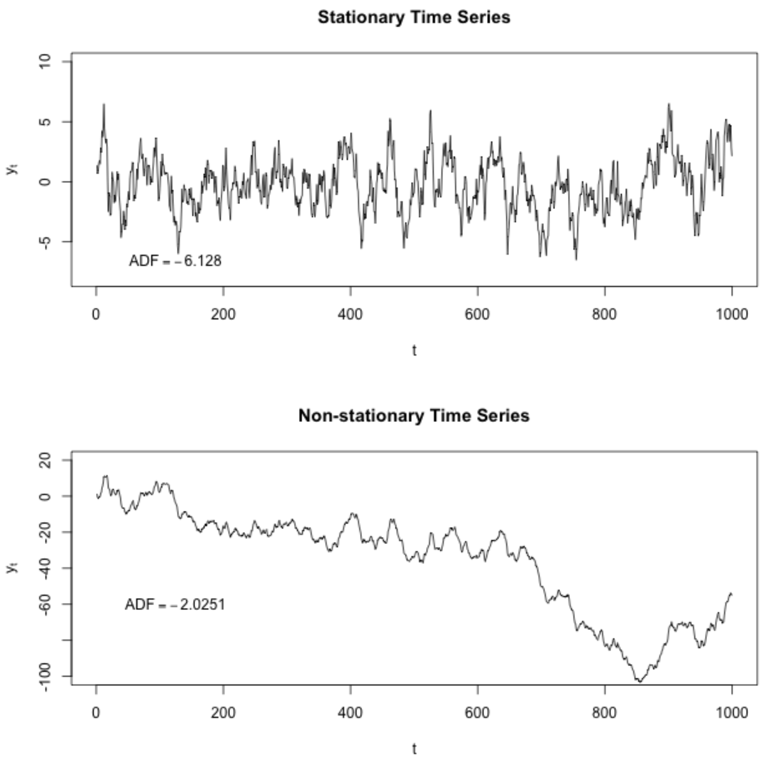


Fig 1: Stationary and Non – Stationary Time Series

As an example, white noise is stationary. The sound of a cymbal clashing, if hit only once, is not stationary because the acoustic power of the clash (and hence its variance) diminishes with time.

Statistical Independence

In case of two random variables X and Y, defined by the events A = {X ≤ x}, B = {Y ≤ y}, to be statistically independent, Consider two processes X(t) and Y(t). The two processes, X(t) and Y(t) are said to be statistically independent, if the random variable group $X(t_1)$, $X(t_2)$, $X(t_3)$ …… $X(t_N)$ is independent of the group $Y(t_1')$, $Y(t_2')$, $Y(t_3')$… $Y(t_M')$. Independence requires that the joint density function be factorable by groups.

Independence requires that the joint density be factorable by groups

$$f_{x,Y}(x_1 ,.... x_N, y_1,......y_N ; t_1,......t_N,t' ,...... t') = f_x(x_1 ,.... x_N; t_1,......t_N) f_y(y_1 ,.... y_N; t' ,...... t')$$

Statistical independence means one event conveys no information about the other; statistical dependence means there is some information

Stationarity

To define Stationarity, Distribution and density functions of random process $X(t)$ must be defined. For a particular time t_1, the distribution function associated with the random variable $X_1=X(t_1)$ will be denoted $F_x(x_1;\, t_1)$. It is defined as

$$F_x(x_1;\, t_1)=P\{X(t_1)\leq x_1\}$$

For any real number x_1.

For two random variables $X_1=X(t_1)$ and $X_2=X(t_2)$, the second order joint distribution function is the two dimensional extension of above equation

$$F_x(x_1, x_2;\, t_1,t_2)=P\{X(t_1)\leq x_1,\, X(t_2)\leq x_2\}$$

In a similar manner, for N random variables $X_i=X(t_i)$, $i=1,2,3,\ldots\ldots\ldots..N$, the Nth order joint distribution function is

$$F_x(x_1,\ldots.\, x_N;\, t_1,\ldots\ldots t_N)=P\{X(t_1)\leq x_1,\ldots\ldots\ldots.\, X(t_N)\leq x_N\}$$

Joint density functions are found from derivatives of the above joint distribution functions

$$f_x(x_1;\, t_1)=d\, F_x(x_1;\, t_1)/dx_1$$

$$f_x(x_1, x_2;\, t_1,t_2)=d^2F_x(x_1, x_2;\, t_1,t_2)/(dx_1dx_2)$$

$$f_x(x_1,\ldots.\, x_N;\, t_1,\ldots\ldots t_N)=d^NF_x(x_1,\ldots.\, x_N;\, t_1,\ldots\ldots t_N)/(dx_1\ldots\ldots\ldots..\, dx_N)$$

First order stationary processes

A Random process is called stationary to order one, if its first order density function does not change with a shift in time origin. In other words

$f_x(x_1;\, t_1) = f_x(x_1;\, t_1+\Delta)$ must be true for any t_1 and any real number Δ if $X(t)$ is to be a first order stationary process.

If $fx(x_1;\, t_1)$ is independent of t_1, the process mean value $E[X(t)]=\overline{x}=$constant.

Second Order and wide sense Stationarity

A process is called stationary to order two if its second order density function satisfies

$$f_x(x_1, x_2;\, t_1,t_2)= f_x(x_1, x_2;\, t_1+\Delta,t_2+\Delta) \text{ for all } t_1,\, t_2 \text{ and } \Delta.$$

The correlation $E[X(t_1)\, X(t_2)]$ of a random process will be a function of t_1 and t_2. Let us denote the function by $R_{XX}(t_1,t_2)$ and call it the autocorrelation function of the random process $X(t)$:

$$R_{XX}(t_1,t_2)= E[X(t_1)\, X(t_2)]$$

The autocorrelation function of second order stationary process is a function only of time differences and not absolute time, that is , if τ=t2-t1, then

$$R_{XX}(t_1,t_1+\tau)= E*X(t_1)\, X(t_1+\tau)= R_{XX}(\tau)$$

A Second order stationary process is wide sense stationary for which the following two conditions are true:

$$E[X(t)]=\overline{x}=\text{constant} \quad \text{and} \quad E[X(t_1)\, X(t_1+\tau)= R_{XX}(\tau)$$

Most of the practical problems, deal with autocorrelation and mean value of a random process. If these quantities are not dependent on absolute time, problem solutions are greatly simplified.

N – Order and Strict Sense Stationary

A Random process is stationary to order N if its Nth order density function is invariant to a time origin shift ; that is, if

$$f_x(x_1, ... x_N; t_1, ...t_N) = f_x(x_1, ... x_N; t_1+\Delta, ... t_N+\Delta), \text{ for all } t_1, ...t_N \text{ and } \Delta.$$

Stationary of order N implies Stationarity to all orders k ≤ N. A process stationary to all orders N=1,2,3,……is called Strict sense Stationary.

Time Averages and Ergodicity

In probability theory, an ergodic system is one that has the same behaviour averaged over time as averaged over the samples. A random process is ergodic if its time average is the same as its average over the probability space

The time average of a quantity is defined as

$$A[.] = \lim_{T\to\infty} 1/2\pi \int_{-}^{T} [.]\,dt$$

Here A is used to denote time average in a manner analogous to E for the statistical average.

Specific averages of interest are the mean value $\bar{x}$= A[$x(t)$] of a sample function and the time autocorrelation function is denoted R_{xx}=A[$x(t)x(t+\tau)$]. These functions are defined by

$$\bar{x} = A\,[x(t)] = \lim_{T\to\infty} 1/2\pi \int_{-T}^{T} [(t)]\,dt$$

$$R_{xx}(\tau) = A[x(t)x(t+\tau)] = \lim_{T\to\infty} 1/2\pi \int_{-T}^{T} [x(t)x(t+\tau)]\,dt$$

By taking the expected value on both sides of the above two equations and assuming the expectation can be brought inside the integrals, then

$$E[\bar{x}] = \bar{X} \quad E[R_{xx}(\tau)] = R_{XX}(\tau)$$

Ergodic Theorem

The Ergodic Theorem states that for any random process *X(t),* all the time averages of the sample functions of the *X(t)* are equal to the corresponding statistical or ensemble averages of *X(t).*

$$\text{i.e., } \bar{x} = \bar{X}$$

$$R_{xx}(\tau) = R_{XX}(\tau)$$

Ergodic Process

Random processes that satisfy the ergodic theorem are called ergodic processes. The analysis of Ergodicity is extremely complex. In most physical applications, it is assumed that all stationary processes are ergodic processes.

Mean Ergodic process

A process *X(t)* with a constant mean value $\bar{X}$ is called mean ergodic or ergodic in the mean, if its statistical average $\bar{X}$ equals the time average $\bar{x}$ of any sample function $x(t)$ with probability 1 for all sample functions; that is, if

$$E[\bar{x}] = \bar{X} = A[x(t)] = \bar{x} \text{ with probability 1 for all } x(t).$$

Correlation Ergodic Processes

Analogous to a mean ergodic process, a stationary continuous process *X(t)* with autocorrelation function $R_{XX}(\tau)$ is called autocorrelation ergodic or ergodic in the autocorrelation if , and only if, for all τ

.

$$\lim_{T\to\infty} 1/2\pi \int_{-T}^{T} [X(t)X(t+\tau)]\,dt = R_{XX}(\tau)\,.$$

Example for the application of concept of Ergodicity

Ergodicity means the ensemble average equals the time average. Each resistor has thermal noise associated with it and it depends on the temperature. Take N resistors (N should be very large) and plot the voltage across those resistors for a long period. For each resistor you will have a waveform. Calculate the average value of that waveform. This gives you the time average. You should also note that you have N waveforms as we have N resistors. These N plots are known as an ensemble. Now take a particular instant of time in all those plots and find the average value of the voltage. That gives you the ensemble average for each plot. If both ensemble average and time average are the same then it is ergodic.

References:

1. Probability, Random Variables and Random Signal Principles, Peyton Z. Peebles Jr. 4th Edition, Tata McGRAW-Hill.
2. Probability Theory and Stochastic Processes, Y. Mallikarjuna Reddy, University Press