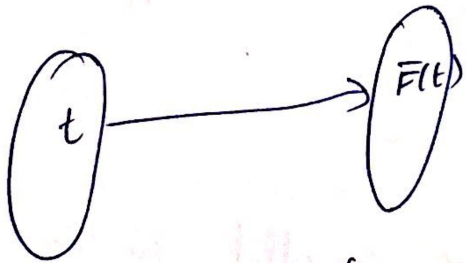
Vector differentiation Unit - 4

we can not express speed, accelaration, Force without explaining direction — vectors

Vector: A quantity which has both magnitude and direction.

ex: speed, accelaration, Force, displacement, momentum etc.

vector function of a scalar variable t is a function $\bar{F} = \bar{F}(t)$ which uniquely associates a vector $\bar{F}$ to each scalar.



A vector function $\bar{F}(t)$ can be expressed interms of unit vectors $\hat{i}, \hat{j}, \hat{k}$ as

$$\bar{F}(t) = f_1(t)\,\hat{i} + f_2(t)\,\hat{j} + f_3(t)\,\hat{k}$$

where $f_1(t)$, $f_2(t)$, $f_3(t)$ are scalar functions of the scalar variable t.

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Scalar point function

If to each point $P(x, y, z)$ of a region R in the space, a unique scalar $\phi(x, y, z)$ is associated, then ϕ is called a scalar point function.

Ex: The temperature $\phi(P)$ at any point P of a body is a scalar point function.

Vector point function:

If to each point $P(x, y, z)$ of a region R in the space a unique vector $\bar{f}(x, y, z)$ is associated, then $\bar{f}$ is said to be a vector point function.

The set of all points of the region R together with the set of all values of the function $\bar{f}$ is said to form a vector field.

Ex: The velocity $\bar{V}$ of a particle moving in a certain region, at any time t, is a vector point function. EX $\bar{F}(t) = t^2\hat{i} + (2t+3)\hat{j} + (t^3+6)\hat{k}$

Note: Any vector $\bar{r}$ can be written as $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Vector Differentiation

Let $\bar{r} = \bar{f}(t)$ be a vector equation of a curve in space.

Let $\bar{r}$, $\bar{r}+\delta\bar{r}$ be the position vectors of two neighbouring pts P and Q on the curve respectively.

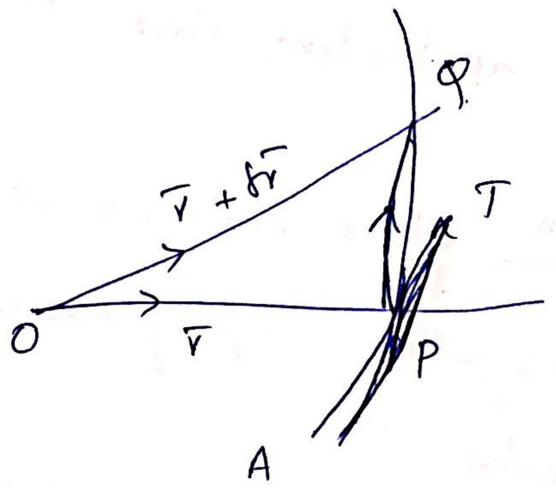
$\overline{OP} = \bar{r} = \bar{f}(t)$

$\overline{OQ} = \bar{r}+\delta\bar{r} = \bar{f}(t+\delta t)$

$\overline{PQ} = \overline{OQ} - \overline{OP}$

$= (\bar{r}+\delta\bar{r}) - \bar{r}$

$= \delta\bar{r}$



Thus $\dfrac{\delta\bar{r}}{\delta t}$ is a vector parallel to the chord PQ.

As $Q \to P$ i.e as $\delta t \to 0$, chord $PQ \to$ tangent at P to the curve.

$$\therefore \lim_{\delta t\to 0} \frac{\delta\bar{r}}{\delta t} = \lim_{\delta t\to 0} \frac{\bar{f}(t+\delta t) - \bar{f}(t)}{\delta t} = \frac{d\bar{r}}{dt}$$ is a vector parallel to the tangent at P to the curve $\bar{r} = \bar{f}(t)$

Derivative of a ~~func~~ vector function $\bar{f}(t)$ is

$$\lim_{\delta t \to 0} \frac{\bar{f}(t+\delta t) - \bar{f}(t)}{\delta t} = \frac{d\bar{f}}{dt} \text{ or } \bar{f}'(t)$$

General rules of differentiation are similar to those of ordinary calculus provided the order of factors in vector products is maintained.

If $\bar{a}, \bar{b}$ and $\bar{c}$ are differentiable vector functions of a scalar variable t and ϕ is a differentiable scalar function of the same variable t, then

1) $\frac{d}{dt}\left(\bar{a} + \bar{b}\right) = \frac{d\bar{a}}{dt} + \frac{d\bar{b}}{dt}$; $\frac{d}{dt}\left(\bar{a} - \bar{b}\right) = \frac{d\bar{a}}{dt} - \frac{d\bar{b}}{dt}$

2) $\frac{d}{dt}\left(\bar{a} \cdot \bar{b}\right) = \bar{a} \cdot \frac{d\bar{b}}{dt} + \frac{d\bar{a}}{dt} \cdot \bar{b}$

3) $\frac{d}{dt}\left(\bar{a} \times \bar{b}\right) = \bar{a} \times \frac{d\bar{b}}{dt} + \frac{d\bar{a}}{dt} \times \bar{b}$

4) $\frac{d}{dt}\left[\phi \bar{a}\right] = \phi \frac{d\bar{a}}{dt} + \frac{d\phi}{dt} \bar{a}$

5) $\frac{d}{dt}\left[\bar{a}\ \bar{b}\ \bar{c}\right] = \left[\frac{d\bar{a}}{dt}\ \bar{b}\ \bar{c}\right] + \left[\bar{a}\ \frac{d\bar{b}}{dt}\ \bar{c}\right] + \left[\bar{a}\ \bar{b}\ \frac{d\bar{c}}{dt}\right]$

6) $\frac{d}{dt}\left(\bar{a} \times (\bar{b} \times \bar{c})\right) = \frac{d\bar{a}}{dt} \times (\bar{b} \times \bar{c}) + \bar{a} \times \left(\frac{d\bar{b}}{dt} \times \bar{c}\right) + \bar{a} \times \left(\bar{b} \times \frac{d\bar{c}}{dt}\right)$

7) Derivative of a constant vector is null vector.

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Problems

(1) If $\bar{r} = (1+t)\hat{i} + (1+t+t^2)\hat{j} + (1+t+t^2+t^3)\hat{k}$

find $\dfrac{d\bar{r}}{dt}$, $\dfrac{d^2\bar{r}}{dt^2}$

Soln $\bar{r} = (1+t)\hat{i} + (1+t+t^2)\hat{j} + (1+t+t^2+t^3)\hat{k}$

$$\frac{d\bar{r}}{dt} = \hat{i} + (1+2t)\hat{j} + (1+2t+3t^2)\hat{k}$$

$$\frac{d^2\bar{r}}{dt^2} = 0 + 2\hat{j} + (2+6t)\hat{k}$$

(2) Find the unit vector tangent to the curve $x=t$, $y=t^2$, $z=t^3$ at $t=1$

Soln $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$= t\hat{i} + t^2\hat{j} + t^3\hat{k}$$

$$\frac{d\bar{r}}{dt} = \hat{i} + 2t\hat{j} + 3t^2\hat{k}$$

at $t=1$ $= \hat{i} + 2\hat{j} + 3\hat{k}$ and $\left|\dfrac{d\bar{r}}{dt}\right|_{t=1} = \sqrt{1+2^2+3^2}$

$= \sqrt{14}$

$\therefore$ Unit tangent vector to the curve $= \dfrac{\frac{d\bar{r}}{dt}}{\left|\frac{d\bar{r}}{dt}\right|} = \dfrac{\hat{i}+2\hat{j}+3\hat{k}}{\sqrt{14}}$

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(3) A particle moves along the curve $x = 2t^2$, $y = t^2 - 4t$, $z = 3t - 5$ where t is the time. Find the component of its velocity and acceleration at time $t = 1$ in the direction $\hat{i} - 3\hat{j} + 2\hat{k}$

$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k} = (2t^2)\hat{i} + (t^2 - 4t)\hat{j} + (3t - 5)\hat{k}$

velocity $\bar{v} = \frac{d\bar{r}}{dt} = 4t\,\hat{i} + (2t - 4)\,\hat{j} + 3\,\hat{k}$

When $t = 1$ i.e $\left(\frac{d\bar{r}}{dt}\right)_{t=1} = 4\hat{i} + (-2)\hat{j} + 3\hat{k}$

acceleration $\bar{a} = \frac{d^2\bar{r}}{dt^2} = 4\hat{i} + 2\hat{j}$

at $t = 1$ $= 4\hat{i} + 2\hat{j}$

Unit vector in the direction of $\hat{i} - 3\hat{j} + 2\hat{k}$ is $\bar{e} = \frac{\hat{i} - 3\hat{j} + 2\hat{k}}{\sqrt{1 + 9 + 4}}$

$\therefore$ Component of velocity in the direction of $\bar{e}$ is $\hat{i} - 3\hat{j} + 2\hat{k}$ is

$$\bar{v} \cdot \bar{e} = (4\hat{i} - 2\hat{j} + 3\hat{k}) \cdot \frac{\hat{i} - 3\hat{j} + 2\hat{k}}{\sqrt{14}} = \frac{4 + 6 + 6}{\sqrt{14}} = \frac{16}{\sqrt{14}}$$

Component of acceleration in the direction of $\hat{i} - 3\hat{j} + 2\hat{k}$ is

$$\bar{a} \cdot \bar{e} = (4\hat{i} + 2\hat{j}) \cdot \frac{\hat{i} - 3\hat{j} + 2\hat{k}}{\sqrt{14}} = \frac{4 - 6}{\sqrt{14}} = \frac{-2}{\sqrt{14}}$$

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(4) A particle moves so that its position vector is given by (7)

$$\bar{r} = (\cos\omega t)\,\hat{i} + (\sin\omega t)\,\hat{j}$$ where ω is a constant.

Show that the velocity $\bar{v}$ of the particle is perpendicular to $\bar{r}$ and $\bar{r}\times\bar{v}$ is a constant vector.

Soln The position vector of the particle is given by

$$\bar{r} = (\cos\omega t)\,\hat{i} + (\sin\omega t)\,\hat{j}$$

The velocity is given by

$$\bar{V} = \frac{d\bar{r}}{dt} = -\omega\sin\omega t\,\hat{i} + \omega\cos\omega t\,\hat{j}$$

Consider $\bar{v}\cdot\bar{r} = -\omega\cos\omega t\sin\omega t + \omega\sin\omega t\cos\omega t = 0$

$\bar{v}\cdot\bar{r} = 0 \Rightarrow \bar{V}$ is $\perp^r$ to $\bar{r}$

Now $$\bar{r}\times\bar{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\omega t & \sin\omega t & 0 \\ -\omega\sin\omega t & \omega\cos\omega t & 0 \end{vmatrix}$$

$$= 0\,\hat{i} + 0\,\hat{j} + (\omega\cos^2\omega t + \omega\sin^2\omega t)\,\hat{k}$$

$$= \omega\hat{k} = \text{Constant vector.}$$

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Vector operator:

The vector operator denoted by ∇ (read as del & nabla) is defined by

$$\nabla \equiv \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$ is called the

vector differential operator.

Gradient:

The Gradient of a scalar point function $\phi(x,y,z)$ denoted by grad ϕ & $\nabla\phi$ and is defined by

$$\text{grad}\,\phi = \nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$\nabla\phi$ is a vector function.

Note: $\nabla\phi$ is a vector normal to the surface $\phi(x,y,z)=C$

Unit normal to the surface $\phi(x,y,z)=C$ $= \dfrac{\nabla\phi}{|\nabla\phi|}$

Theorem: The directional derivative of a scalar point function ϕ at ~~ad~~ a point $P(x, y, z)$ in the direction of a unit vector $\bar{e}$ is given by

$$\frac{d\phi}{ds} = \nabla\phi \cdot \bar{e}$$

If $\bar{r}$ is the position vector of P then $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Then $\frac{d\bar{r}}{ds}$ represents a tangent vector at P

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If θ is the angle made by the vector $\hat{a}$ with direction of $\nabla\phi$, then $\nabla\phi \cdot \hat{a} = |\nabla\phi| \cos\theta$

If $\theta = 0$, then $\cos\theta = 1$ which is the maximum value of $\cos\theta$

$$\nabla\phi \cdot \hat{a} = |\nabla\phi|$$

10 Maximum direction derivative $|\nabla\phi|$

Formulae :

1) The normal vector to the surface $\phi(x, y, z) = c$ is $\nabla\phi$
2) The directional derivative in the direction of $\bar{a}$ is $\nabla\phi \cdot \frac{\bar{a}}{a}$ or $\nabla\phi \cdot \hat{a}$
3) The maximum directional derivative is $|\nabla\phi|$
4) If $\bar{n}_1 = \nabla\phi$ is normal at $P(x_1, y_1, z_1)$ and $\bar{n}_2 = \nabla\phi$ is normal at $Q(x_2, y_2, z_2)$ then the angle θ between the normals is given by $\cos\theta = \frac{\bar{n}_1 \cdot \bar{n}_2}{|\bar{n}_1||\bar{n}_2|}$

SOLVED EXAMPLES – 8.2

EXAMPLE - 1 :

Find the unit normal vector at the point $(1, -1, 2)$ to the surface $x^2y + y^2z + z^2x = 5$.

Solution : Normal vector to the surface $\phi = x^2y + y^2z + z^2x = 5$ is $\nabla\phi$

$$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$= \hat{i}\frac{\partial}{\partial x}(x^2y + y^2z + z^2x) + \hat{j}\frac{\partial}{\partial y}(x^2y + y^2z + z^2x) + \hat{k}\frac{\partial}{\partial z}(x^2y + y^2z + z^2x)$$

$$= (2xy + z^2)\hat{i} + (x^2 + 2yz)\hat{j} + (y^2 + 2zx)\hat{k}$$

At $(1, -1, 2)$, $\nabla\phi = 2\hat{i} - 3\hat{j} + 5\hat{k}$

$$|\nabla\phi| = \sqrt{(2)^2 + (-3)^2 + (5)^2} = \sqrt{38}$$

$$\text{Unit normal vector} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\hat{i} - 3\hat{j} + 5\hat{k}}{\sqrt{38}}$$

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EXAMPLE - 2 :

Find a unit normal vector to the leval surface $x^2y + 2xz = 4$ at the point (2, – 2, 3).

Solution : Let $\phi(x, y, z) = x^2y + 2xz = 4$

Normal to the surface $\phi(x, y, z) = x^2y + 2xz = 4$ is

$$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$= \hat{i}\frac{\partial}{\partial x}(x^2y + 2xz) + \hat{j}\frac{\partial}{\partial y}(x^2y + 2xz) + \hat{k}\frac{\partial}{\partial z}(x^2y + 2xz)$$

$$= (2xy + 2z)\hat{i} + x^2\hat{j} + 2x\hat{k}$$

$\nabla\phi$ at the point (2, – 2, 3) is $-2\hat{i} + 4\hat{j} + 4\hat{k}$

$$\text{Unit normal vector} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{-2\hat{i} + 4\hat{j} + 4\hat{k}}{\sqrt{(-2)^2 + (4)^2 + (4)^2}}$$

$$= \frac{-2\hat{i} + 4\hat{j} + 4\hat{k}}{\sqrt{4 + 16 + 16}} = \frac{-2\hat{i} + 4\hat{j} + 4\hat{k}}{6}$$

EXAMPLE - 3 :

Find the directional derivative of $\phi(x, y, z) = x^2yz + 4xz^2$ at the point (1, – 2, – 1) in the direction of the vector $2\hat{i} - \hat{j} - 2\hat{k}$.

Solution : $\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$

$$= \hat{i}\frac{\partial}{\partial x}(x^2yz + 4xz^2) + \hat{j}\frac{\partial}{\partial y}(x^2yz + 4xz^2) + \hat{k}\frac{\partial}{\partial z}(x^2yz + 4xz^2)$$

$$= (2xyz + 4z^2)\hat{i} + x^2z\hat{j} + (x^2y + 8xz)\hat{k}$$

$(\nabla\phi)$ at the point (1, – 2, – 1) $= 8\hat{i} - \hat{j} - 10\hat{k}$

If $\hat{a}$ is a unit vector in the direction of $2\hat{i} - \hat{j} - 2\hat{k}$

then $\hat{a} = \frac{2\hat{i} - \hat{j} - 2\hat{k}}{3} = \frac{2}{3}\hat{i} - \frac{1}{3}\hat{j} - \frac{2}{3}\hat{k}$

Directional derivative in the direction of $\hat{a} = \nabla\phi \cdot \hat{a}$

$$= \left(8\hat{i} - \hat{j} - 10\hat{k}\right) \cdot \left(\frac{2}{3}\hat{i} - \frac{1}{3}\hat{j} - \frac{2}{3}\hat{k}\right) = \frac{16}{3} + \frac{1}{3} + \frac{20}{3} = \frac{37}{3}$$

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EXAMPLE - 4 :

Find the directional derivative of $\phi(x, y, z) = x^4 + y^4 + z^4$ at the point $(-1, 2, 3)$ in the direction towards the point $(2, -1, -1)$.

Solution : $\phi(x, y, z) = x^4 + y^4 + z^4$

$$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$= \hat{i}\frac{\partial}{\partial x}(x^4 + y^4 + z^4) + \hat{j}\frac{\partial}{\partial y}(x^4 + y^4 + z^4) + \hat{k}\frac{\partial}{\partial z}(x^4 + y^4 + z^4)$$

$$= 4x^3\hat{i} + 4y^3\hat{j} + 4z^3\hat{k}$$

At $(-1, 2, 3)$, $\nabla\phi = -4\hat{i} + 32\hat{j} + 108\hat{k}$

Let $P = (-1, 2, 3)$; $Q = (2, -1, -1)$

$$\bar{a} = \overline{PQ} = \overline{OQ} - \overline{OP} = \left(2\hat{i} - \hat{j} - \hat{k}\right) - \left(-\hat{i} + 2\hat{j} + 3\hat{k}\right)$$

$$= 3\hat{i} - 3\hat{j} - 4\hat{k}$$

$$\hat{a} = \frac{\bar{a}}{a} = \frac{3\hat{i} - 3\hat{j} - 4\hat{k}}{\sqrt{9 + 9 + 16}} = \frac{3\hat{i} - 3\hat{j} - 4\hat{k}}{\sqrt{34}}$$

directional derivative ϕ at $(-1, 2, 3)$ along $\bar{a}$ is given by

$$\nabla\phi \cdot \hat{a} = \left(-4\hat{i} + 32\hat{j} + 108\hat{k}\right) \cdot \left(\frac{3\hat{i} - 3\hat{j} - 4\hat{k}}{\sqrt{34}}\right)$$

$$= \frac{1}{\sqrt{34}}(-12 - 96 - 432) = -\frac{540}{\sqrt{34}}$$

EXAMPLE - 5 :

In what direction from the point $(2, 1, -1)$ is the directional derivative of $\phi = x^2yz^3$ a maximum ? What is its magnitude ?

Solution : $\Delta\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$

$$= \hat{i}\frac{\partial}{\partial x}(x^2yz^3) + \hat{j}\frac{\partial}{\partial y}(x^2yz^3) + \hat{k}\frac{\partial}{\partial z}(x^2yz^3)$$

$$= 2xyz^3\hat{i} + x^2z^3\hat{j} + 3x^2yz^2\hat{k}$$

At $(2, 1, -1)$, $\nabla\phi = -4\hat{i} - 4\hat{j} + 12\hat{k}$

Directional derivative is maximum in the direction of $\nabla\phi$.

Hence Maximum directional derivative is in the direction of $-4\hat{i}-4\hat{j}+12\hat{k}$

Its magnitude $=\sqrt{16+16+144} = 4\sqrt{11}$

EXAMPLE - 6 :

In what direction from the point (1, 1, – 1) is the directional derivative of $\phi = x^2 - 2y^2 + 4z^2$ a maximum. Also find the maximum directional derivative.

Solution : $\phi = x^2 - 2y^2 + 4z^2$

$$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$= 2x\,\hat{i} - 4y\,\hat{j} + 8z\,\hat{k}$$

At (1, 1, – 1), $\nabla\phi = 2\hat{i} - 4\hat{j} + 8\hat{k}$

The directional derivative of ϕ is a maximum in the direction of $\nabla\phi = 2\hat{i} - 4\hat{j} - 8\hat{k}$

The maximum value of the directional derivative $= |\nabla\phi|$

$$= \sqrt{4+16+64} = \sqrt{84} = 2\sqrt{21}$$

EXAMPLE - 7 :

What is the greatest rate of increase of $\phi = xyz^2$ at the point (1, 0, 3).

Solution : $\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$

$$= \hat{i}\frac{\partial}{\partial x}(xyz^2) + \hat{j}\frac{\partial}{\partial y}(xyz^2) + \hat{k}\frac{\partial}{\partial z}(xyz^2)$$

$$= yz^2\,\hat{i} + xz^2\,\hat{j} + 2xyz\,\hat{k}$$

At the point (1, 0, 3), $\nabla\phi = 0\,\hat{i} + 9\,\hat{j} + 0\,\hat{k} = 9\,\hat{j}$

The greatest rate of increase of ϕ at the point (1, 0, 3)

= the maximum value of $\frac{d\phi}{ds}$ at the point (1, 0, 3)

$= |\nabla\phi|$ at the point (1, 0, 3) $= \left|9\,\hat{j}\right| = 9$

EXAMPLE - 8 :

Find the angle between the normals to the surface $xy = z^2$ at (1, 4, 2) and (– 3, – 3, 3).

Solution : $\phi = xy - z^2$

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$$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$= \hat{i}\frac{\partial}{\partial x}(xy - z^2) + \hat{j}\frac{\partial}{\partial y}(xy - z^2) + \hat{k}\frac{\partial}{\partial z}(xy - z^2)$$

$$= y\hat{i} + x\hat{j} - 2z\hat{k}$$

At (1, 4, 2), $\nabla\phi = 4\hat{i} + \hat{j} - 4\hat{k}$, $\bar{n}_1$ say

At (– 3, – 3, 3), $\nabla\phi = -3\hat{i} - 3\hat{j} - 6\hat{k}$, $\bar{n}_2$ say

$$\cos\theta = \frac{\bar{n}_1 \cdot \bar{n}_2}{|\bar{n}_1||\bar{n}_2|} = \frac{\left(4\hat{i} + \hat{j} - 4\hat{k}\right)\cdot\left(-3\hat{i} - 3\hat{j} - 6\hat{k}\right)}{\sqrt{(4)^2 + (1)^2 + (-4)^2}\sqrt{(-3)^2 + (-3)^2 + (-6)^2}} = \frac{1}{\sqrt{22}}$$

$$\theta = \cos^{-1}\frac{1}{\sqrt{22}}$$

EXAMPLE - 9 :

Find the constants a and b such that the surface $3x^2 - 2y^2 - 3z^2 + 8 = 0$ is orthogonal to $ax^2 + y^2 = bz$ at the point (– 1, 2, 1).

Solution : Let $\phi_1(x, y, z) \quad 3x^2 - 2y^2 - 3z^2 + 8$

$$\nabla\phi_1 = \hat{i}\frac{\partial\phi_1}{\partial x} + \hat{j}\frac{\partial\phi_1}{\partial y} + \hat{k}\frac{\partial\phi_1}{\partial z}$$

$$\nabla\phi_1 = 6x\hat{i} - 4y\hat{j} - 6z\hat{k}$$

At the point (– 1, 2, 1), $\nabla\phi_1 = -6\hat{i} - 8\hat{j} - 6\hat{k}$

Unit normal to the point (– 1, 2, 1) = $\hat{n}_1 = \dfrac{-6\hat{i} - 8\hat{j} - 6\hat{k}}{\sqrt{136}}$ for $\phi_1(x, y, z)$

Let $\phi_2(x, y, z) = ax^2 + y^2 - bz$

$$\nabla\phi_2 = \hat{i}\frac{\partial\phi_2}{\partial x} + \hat{j}\frac{\partial\phi_2}{\partial y} + \hat{k}\frac{\partial\phi_2}{\partial z}$$

$$\nabla\phi_2 = 2ax\hat{i} + 2y\hat{j} - b\hat{k}$$

At the point (– 1, 2, 1), $\nabla\phi_2 = -2a\hat{i} + 4\hat{j} - b\hat{k}$

The unit normal at the point (– 1, 2, 1) for $\phi_2(x, y, z) = \hat{n}_2 = \dfrac{-2a\,\hat{i} + 4\,\hat{j} - b\,\hat{k}}{\sqrt{4a^2 + 16 + b^2}}$

Given that the surfaces intersect orthogonally at (– 1, 2, 1)

$$\hat{n}_1 \cdot \hat{n}_2 = 0$$

$$\frac{-6\,\hat{i} - 8\,\hat{j} - 6\,\hat{k}}{\sqrt{136}} \cdot \frac{-2a\,\hat{i} + 4\,\hat{j} - b\,\hat{k}}{\sqrt{4a^2 + 16 + b^2}}$$

$\Rightarrow \quad 12a + 6b - 32 = 0$

$\Rightarrow \quad 6a + 3b - 16 = 0$ .... (1)

Also, the point (– 1, 2, 1) lies on the surface $ax^2 + y^2 = bz$

$\therefore \quad a + 4 = b$

$a - b + 4 = 0$ .... (2)

Solving (1) and (2),

we get $a = \dfrac{4}{9}$ and $b = \dfrac{40}{9}$

EXAMPLE - 10 :

Find the equations of the tangent plane and normal to the surface xyz = 4 at the point (1, 2, 2).

Solution : The equation of the surface is $\phi(x, y, z) = xyz - 4 = 0$

$$\nabla\phi = yz\,\hat{i} + xz\,\hat{j} + xy\,\hat{k}$$

At (1, 2, 2), $\nabla\phi = 4\,\hat{i} + 2\,\hat{j} + 2\,\hat{k}$

Let $\bar{r} = \hat{i} + 2\,\hat{j} + 2\,\hat{k}$, the position vector of the point (1, 2, 2)

$4\,\hat{i} + 2\,\hat{j} + 2\,\hat{k}$ is a vector along the normal to the surface at the point (1, 2, 2)

The equation of the tangent plane is $(\bar{R} - \bar{r}) \cdot \nabla\phi = 0$

Where $\bar{R} = Xi + yj + Zk$ is a point on the plane

i.e. $\left[\left(X\,\hat{i} + Y\,\hat{j} + Z\,\hat{k}\right) - \left(\hat{i} + 2\,\hat{j} + 2\,\hat{k}\right)\right] \cdot (4\,\hat{i} + 2\,\hat{j} + 2\,\hat{k}) = 0$

$$[(X-1)\,\hat{i} + (Y-2)\,\hat{j} + (Z-2)\,\hat{k}] \cdot (4\,\hat{i} + 2\,\hat{j} + 2\,\hat{k}) = 0$$

$$4(X-1) + 2(Y-2) + 2(Z-2) = 0$$

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$$4X + 2Y + 2Z = 12$$

$$2X + Y + Z = 6$$

The equation of the normal to the surface at the point (1, 2, 2) is

$$\frac{X-1}{\frac{\partial f}{\partial x}} = \frac{Y-2}{\frac{\partial f}{\partial y}} = \frac{Z-2}{\frac{\partial f}{\partial z}}$$

i.e. $\frac{X-1}{4} = \frac{Y-2}{2} = \frac{Z-2}{2}$

i.e. $\frac{X-1}{2} = \frac{Y-2}{1} = \frac{Z-2}{1}$

EXAMPLE - 11 :

Find the directional derivative of the scalar point function $\phi(x\,y\,z) = 4xy^2 + 2x^2yz$ at the point A (1, 2, 3) in the direction of the line AB where B = (5, 0, 4).

(JNTU 2007 set 1)

Solution : Given points A (1, 2, 3) and B (5, 0 , 4)

$\overline{OA} = i + 2j + 3k$

$\overline{OB} = 5i + 0j + 4k$

$\overline{AB} = \overline{OB} - \overline{OA} = 5i + 4k - i - 2j - 3k$

$= 4i - 2j + k$

$$\nabla\phi = i\frac{\partial \phi}{\partial x} + j\frac{\partial \phi}{\partial y} + k\frac{\partial \phi}{\partial z}$$

$$= i\frac{\partial}{\partial x}(4xy^2 + 2x^2yz) + j\frac{\partial}{\partial y}(4xy^2 + 2x^2yz) + k\frac{\partial}{\partial z}(4xy^2 + 2x^2yz)$$

$= (4y^2 + 4xyz)\,i + (8xy + 2x^2z)\,j + (2x^2y)\,k$

$(\nabla\phi)$ at the point (1, 2, 3)

$= 40\,i + 22j + 4k$

$\nabla\phi$ in the direction of $4i - 2j + k$

$$= \frac{(40i + 22j + 4k)\cdot(4i - 2j + k)}{\sqrt{21}}$$

$$= \frac{120}{\sqrt{21}}$$

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EXAMPLE - 12 :

Find the angle between the surfaces $x^2+y^2+z^2=9$, $x^2+y^2-z=3$ at $(2,-1,2)$

(JNTU 2006, 2004)

Solution : Let $\phi_1 = x^2+y^2+z^2-9$

$\phi_2 = x^2+y^2-z-3$

Angle between the surfaces ϕ_1 and ϕ_2 is the angle between the normals $\bar{n}_1$ and $\bar{n}_2$ to the surfaces at the point $(2,-1,2)$

$$\bar{n}_1 = \nabla\phi_1 = i\frac{\partial\phi_1}{\partial x} + j\frac{\partial\phi_1}{\partial y} + k\frac{\partial\phi_1}{\partial z}$$

$$= 2xi + 2yj + 2zk$$

$$(\nabla\phi_1)_{(2,-1,2)} = 4i - 2j + 4k$$

$$\bar{n}_2 = \nabla\phi_2 = i\frac{\partial\phi_2}{\partial x} + j\frac{\partial\phi_2}{\partial y} + k\frac{\partial\phi_2}{\partial z}$$

$$= 2xi + 2yj - k$$

$$(\nabla\phi_2)_{(2,-1,2)} = 4i - 2j - k$$

Angle between the normals is

$$\cos\theta = \frac{\bar{n}_1 \cdot \bar{n}_2}{|\bar{n}_1||\bar{n}_2|}$$

$$= \frac{(4i-2j+4k)\cdot(4i-2j-k)}{\sqrt{16+4+16}\sqrt{16+4+1}}$$

$$= \frac{16+4-4}{6\sqrt{21}} = \frac{16}{6\sqrt{21}} = \frac{8}{3\sqrt{21}}$$

$$\cos\theta = \frac{8}{3\sqrt{21}}$$

$$\theta = \cos^{-1}\frac{8}{3\sqrt{21}}$$

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8.12 FORMULAE INVOLVING GRADIENT :

If f and g are scalar point functions, then

1) $\nabla(f+g) = \nabla f + \nabla g$

$\nabla(f-g) = \nabla f - \nabla g$

2) $\nabla f = 0$, f is a constant function

3) $\nabla(fg) = f\nabla g + g\nabla f$

4) $\nabla(cf) = c\nabla f$, c is a constant

5) $\nabla\left(\dfrac{f}{g}\right) = \dfrac{g\nabla f - f\nabla g}{g^2}$

SOLVED EXAMPLES – 8.3

EXAMPLE - 1 :

If $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $r = |\bar{r}|$

Prove that i) $\nabla r = \dfrac{\bar{r}}{r}$

ii) $\nabla f(r) = \dfrac{f'(r)}{r}\bar{r}$

iii) $\nabla\left(\dfrac{1}{r}\right) = -\dfrac{\bar{r}}{r^3}$

Solution : $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$,

$r = |\bar{r}| = \sqrt{x^2 + y^2 + z^2}$

i.e., $r^2 = x^2 + y^2 + z^2$

differentiating with respect to x

$$2r\frac{\partial r}{\partial x} = 2x \quad \Rightarrow \quad \frac{\partial r}{\partial x} = \frac{x}{r}$$

differentiating with respect to y

$$2r\frac{\partial r}{\partial y} = 2y \quad \Rightarrow \quad \frac{\partial r}{\partial y} = \frac{y}{r}$$

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differentiating with respect to z

$$2r\frac{\partial r}{\partial z} = 2z \qquad \Rightarrow \frac{\partial r}{\partial z} = \frac{z}{r}$$

i) $$\nabla r = \hat{i}\frac{\partial r}{\partial x} + \hat{j}\frac{\partial r}{\partial y} + \hat{k}\frac{\partial r}{\partial z}$$

$$= \hat{i}\frac{x}{r} + \hat{j}\frac{y}{r} + \hat{k}\frac{z}{r}$$

$$= \frac{x\hat{i} + y\hat{j} + z\hat{k}}{r} = \frac{\bar{r}}{r}$$

$$\boxed{\nabla r = \frac{\bar{r}}{r}}$$

ii) $$\nabla f(r) = \hat{i}\frac{\partial}{\partial x}f(r) + \hat{j}\frac{\partial}{\partial y}f(r) + \hat{k}\frac{\partial}{\partial z}f(r)$$

$$= f'(r)\,\hat{i}\frac{\partial r}{\partial x} + f'(r)\,\hat{j}\frac{\partial r}{\partial y} + f'(r)\,\hat{k}\frac{\partial r}{\partial z} = f'(r)\nabla r$$

$$= f'(r)\cdot\frac{\bar{r}}{r} \text{ by (i)}$$

$$\boxed{\nabla f(r) = f'(r)\nabla r}$$

iii) $$\nabla\left(\frac{1}{r}\right) = \hat{i}\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + \hat{j}\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + \hat{k}\frac{\partial}{\partial z}\left(\frac{1}{r}\right)$$

$$= \hat{i}\left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial x} + \hat{j}\left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial y} + \hat{k}\left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial z}$$

$$= -\frac{1}{r^2}\left(\hat{i}\frac{x}{r} + \hat{j}\frac{y}{r} + \hat{k}\frac{z}{r}\right)$$

$$= -\frac{1}{r^3}\bar{r} \qquad \nabla\left(\frac{1}{r}\right) = -\frac{1}{r^3}\bar{r}$$

Note : ∇ (scalar function) = (differentiation of function) ∇r

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EXAMPLE - 2 :

Prove that $\nabla (r^n) = n r^{n-2} \bar{r}$

Solution : By Example 1 (ii)

$$\nabla f (r) = f' (r) \nabla r$$

$$\nabla (r^n) = n r^{n-1} \nabla r = n r^{n-1} \frac{\bar{r}}{r} = n r^{n-2} \bar{r}$$

EXAMPLE - 3 :

Prove that $\nabla (\log r) = \frac{\bar{r}}{r^2}$

Solution : By Example 1 (ii)

$$\nabla f (r) = f' (r) \nabla r$$

$$\nabla \log r = \frac{1}{r} \nabla r = \frac{1}{r} \cdot \frac{\bar{r}}{r} = \frac{\bar{r}}{r^2}$$

EXAMPLE - 4 :

If $\phi = x^2y^3z^4$ and $\psi = xy + yz + zx$ evaluate $\nabla (\phi\psi)$.

Solution : $\nabla\phi = \hat{i} \frac{\partial\phi}{\partial x} + \hat{j} \frac{\partial\phi}{\partial y} + \hat{k} \frac{\partial\phi}{\partial z} = (2xy^3z^4) \hat{i} + (3x^2y^2z^4) \hat{j} + (4x^2y^3z^3) \hat{k}$

$$\psi = xy + yz + zx$$

$$\nabla\psi = (y + z) \hat{i} + (z + x) \hat{j} + (x + y) \hat{k}$$

$$\nabla (\phi\psi) = \phi (\nabla\psi) + \psi (\nabla\phi)$$

$$= (x^2y^3z^4) (y + z) \hat{i} + (z + x) \hat{j} + (x + y) \hat{k}$$

$$+ (xy + yz + zx) [(2xy^3z^4) \hat{i} + (3x^2y^2z^4) \hat{j} + (4x^2y^3z^3) \hat{k}]$$

EXAMPLE - 5 :

Show that $\nabla (\bar{r} \cdot \bar{a}) = \bar{a}$

Solution : Let $\bar{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$

$$\bar{r} = x \hat{i} + y \hat{j} + z \hat{k}$$

$$\bar{r} \cdot \bar{a} = (a_1x + a_2y + a_3z)$$

$$\nabla (\bar{r} \cdot \bar{a}) = \nabla (a_1x + a_2y + a_3 z)$$

$$= \hat{i} \frac{\partial}{\partial x} (a_1x + a_2y + a_3z) + \hat{j} \frac{\partial}{\partial y} (a_1x + a_2y + a_3z)$$

$$+ \hat{k} \frac{\partial}{\partial z} (a_1x + a_2y + a_3z)$$

$$= a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k} = \bar{a}$$

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EXAMPLE - 6 :

Show that $(\bar{a}\cdot\nabla)\phi = \bar{a}\cdot\nabla\phi$

Solution : Let $\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$$(\bar{a}\cdot\nabla) = \left(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}\right)\cdot\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)$$

$$= a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial y} + a_3\frac{\partial}{\partial z}$$

$$\boxed{\bar{a}\cdot\nabla = a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial y} + a_3\frac{\partial}{\partial z}}$$

$$(\bar{a}\cdot\nabla)\,\phi = \left(a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial y} + a_3\frac{\partial}{\partial z}\right)\phi$$

$$= a_1\frac{\partial\phi}{\partial x} + a_2\frac{\partial\phi}{\partial y} + a_3\frac{\partial\phi}{\partial z}$$

$$= \left(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}\right)\cdot\left(\hat{i}\frac{\partial\phi}{dx} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right) = \bar{a}\cdot\nabla\phi$$

EXAMPLE - 7 :

Show that $(\bar{a}\cdot\nabla)\bar{r} = \bar{a}$

Solution : $$(\bar{a}\cdot\nabla)\bar{r} = \left(a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial y} + a_3\frac{\partial}{\partial z}\right)\bar{r}$$

$$= \left(a_1\frac{\partial\bar{r}}{\partial x} + a_2\frac{\partial\bar{r}}{\partial y} + a_3\frac{\partial\bar{r}}{\partial z}\right)$$

But $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ $\quad \frac{\partial\bar{r}}{\partial x} = \hat{i},\ \frac{\partial\bar{r}}{\partial y} = \hat{j},\ \frac{\partial\bar{r}}{\partial z} = \hat{k}$

$$(\bar{a}\cdot\nabla)\bar{r} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k} = \bar{a}$$

$$(\bar{a}\cdot\nabla)\bar{r} = \bar{a}$$

EXERCISE 8.2

1. If $\phi(x, y, z) = 3x^2y - y^3z^2$ find $\nabla\phi$ at the point $(1, -2, -1)$.

Ans : $-12\hat{i} - 9\hat{j} - 16\hat{k}$

page 15 Ex 10 not important

Pb) Find the equations of the tangent plane and normal to the surface $xyz = 4$ at the point $(1, 2, 2)$

Soln: The eqn of the surface $\phi(x,y,z) = xyz - 4 = 0$

$$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$= \hat{i}\frac{\partial}{\partial x}(xyz-4) + \hat{j}\frac{\partial}{\partial y}(xyz-4) + \hat{k}\frac{\partial}{\partial z}(xyz-4)$$

$$= yz\,\hat{i} + xz\,\hat{j} + xy\,\hat{k}$$

$$\nabla\phi \text{ at } (1,2,2) = 4\hat{i} + 2\hat{j} + 2\hat{k}$$

This $\nabla\phi$ is normal to the surface $\phi(x,y,z) = 0$ at $(1, 2, 2)$

vector Eqn of a plane passing through the pt $\bar{a}$ and $\perp^r$ to the unit vector $\bar{n}$ is $(\bar{r} - \bar{a}) \cdot \bar{n} = 0$

Let $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ passing thru $\bar{a} = 1\hat{i} + 2\hat{j} + 2\hat{k}$

$\perp^r$ to $\nabla\phi$ i.e $\bar{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{4\hat{i} + 2\hat{j} + 2\hat{k}}{\sqrt{16+4+4}}$

$$\bar{r} - \bar{a} = (x-1)\hat{i} + (y-2)\hat{j} + (z-2)\hat{k}$$

$$(\bar{r} - \bar{a}) \cdot \bar{n} = \frac{4(x-1) + 2(y-2) + 2(z-2)}{\sqrt{24}} = 0$$

$\Rightarrow \quad 4(x-1) + 2(y-2) + 2(z-2) = 0$

This is the eqn of the tangent plane to the given surface at $(1, 2, 2)$.

Eqn of the normal plane to the given surface ϕ at $(1,2,2)$ means $\perp^r$ to the above plane is ~~...~~ passing through $(1,2,2)$ is

$\phi : xyz - 4 = 0$

$$\frac{x-1}{\frac{\partial \phi}{\partial x}} = \frac{y-2}{\frac{\partial \phi}{\partial y}} = \frac{z-2}{\frac{\partial \phi}{\partial z}}$$

$\frac{\partial \phi}{\partial x} = yz \Big|_{(1,2,2)} = 4$

$\frac{\partial \phi}{\partial y} = xz \Big|_{(1,2,2)} = 2$

$\frac{\partial \phi}{\partial z} = xy \Big|_{(1,2,2)} = 2$

$$\Rightarrow \quad \frac{x-1}{4} = \frac{y-2}{2} = \frac{z-2}{2}$$

NOTE: Maximum directional derivative is obtained if it is calculated in the direction normal to that surface i.e in the direction of $\nabla\phi$.

$$\therefore \bar{e} = \frac{\nabla\phi}{|\nabla\phi|} \quad \text{unit vector}$$

directional derivative of a scalar function ϕ in the direction of a unit vector $\bar{e}$ is $\nabla\phi \cdot \bar{e}$

$$= \nabla\phi \cdot \frac{\nabla\phi}{|\nabla\phi|}$$

$$= \frac{|\nabla\phi|^2}{|\nabla\phi|} \quad \because \bar{a}\cdot\bar{a} = |a|^2$$

$$= |\nabla\phi|$$

NOTE: $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$$

$$\frac{\partial r}{\partial x} = \frac{x}{r} \qquad \frac{\partial r}{\partial y} = \frac{y}{r} \qquad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\because r^2 = x^2+y^2+z^2$$

$$2r\frac{\partial r}{\partial x} = 2x \qquad 2r\frac{\partial r}{\partial y} = 2y \qquad 2r\frac{\partial r}{\partial z} = 2z$$

$$\therefore \frac{\partial r}{\partial x} = \frac{\cancel{2}x}{\cancel{2}r} \qquad \frac{\partial r}{\partial y} = \frac{y}{r} \qquad \frac{\partial r}{\partial z} = \frac{z}{r}$$

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(1) $\nabla r = \sum \hat{i} \frac{\partial}{\partial x}(r)$

$= \sum \hat{i} \frac{\partial r}{\partial x}$

$= \sum \hat{i} \frac{x}{r}$ summation is on 3 terms x, y, z

$= \hat{i}\frac{x}{r} + \hat{j}\frac{y}{r} + \hat{k}\frac{z}{r}$

$= \frac{1}{r}(x\hat{i} + y\hat{j} + z\hat{k})$

$= \frac{1}{r}\bar{r}$

(2) $\nabla f(r) = \sum \hat{i} \frac{\partial}{\partial x}[f(r)]$

$= \sum \hat{i} f'(r) \frac{\partial r}{\partial x}$

$= \sum \hat{i} f'(r) \frac{x}{r}$

$= \frac{f'(r)}{r} \sum x\hat{i}$ $\quad \because \sum x\hat{i} = x\hat{i} + y\hat{j} + z\hat{k} = \bar{r}$

$= \frac{f'(r)}{r}[x\hat{i} + y\hat{j} + z\hat{k}] = \frac{f'(r)}{r}\bar{r}$

(3) $\nabla\left(\frac{1}{r}\right) = \frac{-1}{r^2}\frac{1}{r}\bar{r} = \frac{-\bar{r}}{r^3}$ $\quad \because \nabla f(r) = \frac{f'(r)}{r}\bar{r}$

if $f(r) = \frac{1}{r}$ $\quad f'(r) = \frac{-1}{r^2}$

<u>OR</u> $\nabla\left(\frac{1}{r}\right) = \sum \hat{i} \frac{\partial}{\partial x}\left(\frac{1}{r}\right)$

$= \sum \hat{i}\left(\frac{-1}{r^2}\right)\frac{\partial r}{\partial x} = \frac{-1}{r^2}\sum \hat{i}\frac{x}{r} = \frac{-1}{r^3}\sum x\hat{i}$

$= \frac{-1}{r^3}\bar{r}$

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(4) $\nabla(r^n) = n r^{n-2}\,\bar{r}$ $\quad \because \nabla f(r) = \frac{f'(r)}{r}\,\bar{r}$

if $f(r) = r^n$

$f'(r) = n r^{n-1}$

OR

$$\nabla(r^n) = \sum \hat{i}\,\frac{\partial}{\partial x}(r^n)$$

$$= \sum \hat{i}\; n r^{n-1}\,\frac{\partial r}{\partial x}$$

$$= \sum \hat{i}\; n r^{n-1}\,\frac{x}{r}$$

$\because$ Summation is on x, y, z

$$= \frac{n r^{n-1}}{r}\sum x\hat{i}$$

$\sum x\hat{i} = x\hat{i} + y\hat{j} + z\hat{k}$

$= \bar{r}$

$$= n r^{n-2}\,\bar{r}$$

(5) $\nabla(\log r)$

$$\nabla(\log r) = \frac{1}{r}\,\frac{1}{r}\,\bar{r}$$

$\because \nabla f(r) = f'(r)\,\frac{\bar{r}}{r}$

if $f(r) = \log r$ Then $f'(r) = \frac{1}{r}$

$$= \frac{\bar{r}}{r^2}$$

OR

$$\nabla(\log r) = \sum \hat{i}\,\frac{\partial}{\partial x}(\log r)$$

$$= \sum \hat{i}\;\frac{1}{r}\,\frac{\partial r}{\partial x}$$

$$= \sum \hat{i}\;\frac{1}{r}\,\frac{x}{r}$$

$$= \frac{1}{r^2}\sum x\hat{i}$$

$\because \sum x\hat{i} = x\hat{i} + y\hat{j} + z\hat{k}$

$= \bar{r}$

$$= \frac{1}{r^2}\,\bar{r}$$

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Practice Problems

(27)

2. Find the directional derivative of $\phi = xy + yz + zx$ in the direction of vector $\hat{i} + 2\hat{j} + 2\hat{k}$ at (1, 2, 0).

Ans : $\frac{10}{3}$

3. Find the unit vector normal to the surface $x^2 - y^2 + z = 2$ at the point (1, – 1, 2).

Ans : $\frac{1}{3}\left(2\hat{i} + 2\hat{j} + \hat{k}\right)$

4. Find the unit normal vector to the surface $x^3 + y^3 + z^3 = 14 + 3xyz$ at (1, – 1, 2)

Ans : $\frac{1}{\sqrt{35}}\left(3\hat{i} - \hat{j} + 5\hat{k}\right)$

5. Find the direction at (1, 2, 3) in which the directional derivative of $2xz - y^2$ is maximum and find the magnitude.

Ans : $\frac{1}{\sqrt{14}}\left(3\hat{i} - 2\hat{j} + \frac{1}{\sqrt{41}}\hat{k}\right)$; $2\sqrt{14}$

6. Find the rate of change of $\phi = xyz$ in the direction of the normal to the surface $x^2y + y^2z + z^2y = 3$ at (1, 1, 1).

Ans : $\frac{(3yz + 4zx + 2xy)}{\sqrt{29}}$

7. Find the directional derivative of $\phi = xy^2 + yz^2$ at (2, – 1, 1) in the direction of the vector $\hat{i} + 2\hat{j} + 2\hat{k}$.

Ans : – 3

8. Find the greatest value of the directional derivative of the function $2x^2 - y - z^4$ at the point (2, – 1, 1).

Ans : 9

9. Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$; $z = x^2 + y^2 - 3$ at (2, – 1, 2).

(JNTU 2006) **Ans :** $\cos\theta = \frac{8}{3\sqrt{21}}$

10. Find the equation of the tangent plane to the surface $yz - zx + xy + 5 = 0$ at the point (1, – 1, 2).

Ans : $3x - 3y + 2z = 10$

11. Find the constants a and b such that the surface $ax^2 - 2byz = (a + 4)x$ will be orthogonal to the surface $4x^2y + z^2 = 4$ at (1, – 1, 2)

Ans : $a = 5, b = 1$

12. Find the angle between the surfaces $x^2 + y^2 + z^2 = 12$, $x^2 + y^2 - z = 6$ at (2, – 2, 2)

Ans : $\cos\theta = \frac{7}{3\sqrt{11}}$

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13. Prove that $\nabla\phi \cdot d\bar{r} = d\phi$

14. Prove that $\bar{A} \cdot \left(\nabla \frac{1}{r}\right) = -\frac{\bar{A} \cdot \bar{r}}{r^3}$

15. Find the directional derivative of $xyz^2 + xz$ at (1, 1, 1) in the direction of the normal to the surface $3xy^2 + y + z$ at (0, 1, 1). **Ans :** $\frac{4}{\sqrt{11}}$ ***(JNTU 2000)***

16. If ϕ and ψ are scalar point functions, show that the components of the former tangential and normal to the level surface $\psi = 0$ are $\frac{\nabla\psi \times (\phi \times \nabla\psi)}{(\nabla\psi)^2}$ and $\frac{(\phi \cdot \nabla\psi)\,\nabla\psi}{(\nabla\psi)^2}$

17. Find a unit normal to the surface $x^2 + y^2 + 2z^2 = 26$ at the point (2, 1, 3).

Ans : $\frac{4\hat{i} + 4\hat{j} + 12\hat{k}}{\sqrt{176}}$ ***(JNTU 2001)***

18. If $r = |\bar{r}|$ where $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ then prove that $\bar{a}\left[\nabla\left(\frac{1}{r}\right)\right] = -\frac{\bar{a} \cdot \bar{r}}{r^3}$

19. Find the directional derivative of $\phi(x, y, z) = xy^2 + yz^3$ at the point (2, – 1, 1) in the direction of the vector $\hat{i} + 2\hat{j} + 2\hat{k}$. **Ans :** $-\frac{11}{3}$ ***(JNTU 99/S)***

20. If $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and ϕ is a function of x, y, z show that $\frac{d\phi}{ds} = \nabla\phi \cdot \frac{d\bar{r}}{ds}$.

DIVERGENCE AND CURL OF A VECTOR POINT FUNCTION :

8.13 DIVERGENCE :

Definition : The divergence of a continuously differentiable vector point function $\bar{f}$, denoted by div $\bar{f}$ or $\nabla \cdot \bar{f}$ is defined by

$$\text{div } \bar{f} = \nabla \cdot \bar{f} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \bar{f}$$

$$= \hat{i} \cdot \frac{\partial \bar{f}}{\partial x} + \hat{j} \cdot \frac{\partial \bar{f}}{\partial y} + \hat{k} \cdot \frac{\partial \bar{f}}{\partial z}$$

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$$\boxed{\nabla \cdot \bar{f} = \hat{i} \cdot \frac{\partial \bar{f}}{\partial x} + \hat{j} \cdot \frac{\partial \bar{f}}{\partial y} + \hat{k} \cdot \frac{\partial \bar{f}}{\partial z}}$$

If $\bar{f} = f_1 \hat{i} + f_2 \hat{j} + f_3 \hat{k}$ then

$$\nabla \cdot \bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$\nabla \cdot \bar{f}$ is a scalar function.

Note : To find divergence of a vector, differentiate coefficient of $\hat{i}$ with respect to x, coefficient of $\hat{j}$ with respect to y and coefficient of $\hat{k}$ with respect to z.

Solenoidal vector : If $\nabla \cdot \bar{f} = 0$, then $\bar{f}$ is said to be solenoidal vector.

8.14 CURL :

Definition : The curl of a continuously differentiable vector point function $\bar{f}$, denoted by curl $\bar{f}$ or $\nabla \times \bar{f}$ is defined by

$$\text{curl } \bar{f} = \nabla \times \bar{f} = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \times \bar{f}$$

$$= \hat{i} \times \frac{\partial \bar{f}}{\partial x} + \hat{j} \times \frac{\partial \bar{f}}{\partial y} + \hat{k} \times \frac{\partial \bar{f}}{\partial z}$$

$$\boxed{\nabla \times \bar{f} = \hat{i} \times \frac{\partial \bar{f}}{\partial x} + \hat{j} \times \frac{\partial \bar{f}}{\partial y} + \hat{k} \times \frac{\partial \bar{f}}{\partial z}}$$

If $\bar{f} = f_1 \hat{i} + f_2 \hat{j} + f_3 \hat{k}$

$$\nabla \times \bar{f} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \hat{i} \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right) - \hat{j} \left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z} \right) + \hat{k} \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right)$$

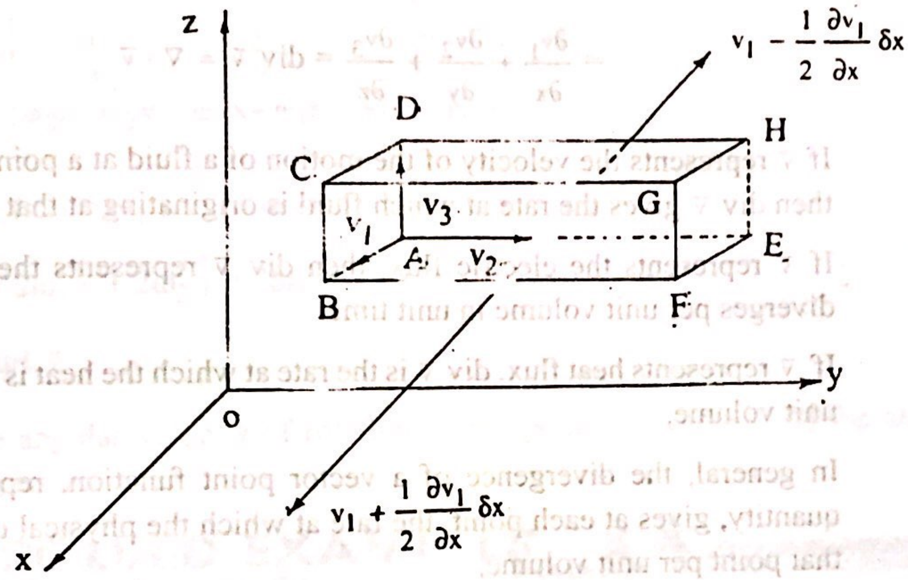
Irrotational vector : If $\nabla \times \bar{f} = 0$, then $\bar{f}$ is said to be irrotational vector.

8.15 PHYSICAL INTERPRETATION OF DIVERGENCE :

Consider a region of space filled with a fluid which moves so that its velocity vector at any point P (x, y, z) is $\bar{v}(x, y, z)$

Let $\bar{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$

where v_1, v_2, v_3 are scalar functions of x, y, z and are the components of velocity parallel to the coordinate axes.



Now, construct a parallelopiped having centre at P (x, y, z), edges parallel to the coordinate axes and having magnitudes $\delta x, \delta y, \delta z$ respectively.

The amount of fluid, leaving the face CBFG in time δt

= velocity component normal to CBFG × area × time

$$= \left(v_1 + \frac{1}{2}\frac{\partial v_1}{\partial x}\delta x\right)\delta y\,\delta z\,\delta t$$

Similarly, the amount of fluid entering the face ADEH

$$= \left(v_1 - \frac{1}{2}\frac{\partial v_1}{\partial x}\delta x\right)\delta y\,\delta z\,\delta t$$

Gain of fluid in the parallelopiped in the direction of the x-axis

$$= \frac{\partial v_1}{\partial x}\delta x\,\delta y\,\delta z\,\delta t$$

Similarly, the gain of fluid in the parallelopiped in the directions of the y and z axes

are $\frac{\partial v_2}{\partial y}\delta x\,\delta y\,\delta z\,\delta t$ and $\frac{\partial v_3}{\partial z}\delta x\,\delta y\,\delta z\,\delta t$

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Total gain of the fluid in the parallelopiped $= \left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}\right) \delta x\, \delta y\, \delta z\, \delta t.$

But $\delta x\, \delta y\, \delta z$ is the volume of the parallelopiped.

$\therefore$ Total gain of fluid in the parallelopiped per unit volume per unit time

$$= \left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}\right)\left(\frac{\delta x\, \delta y\, \delta z}{\delta x\, \delta y\, \delta z}\right)\left(\frac{\delta t}{\delta t}\right)$$

$$= \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} = \text{div}\ \bar{v} = \nabla \cdot \bar{v}$$

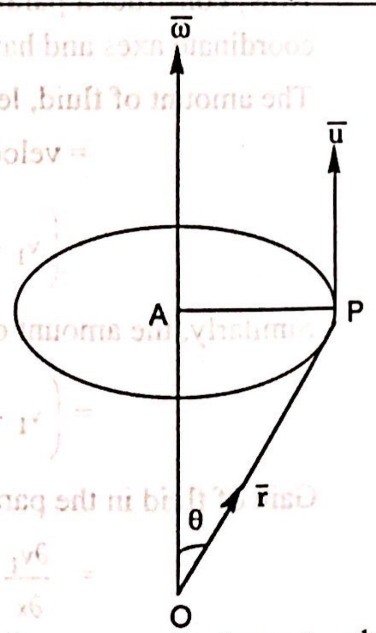
If $\bar{v}$ represents the velocity of the motion of a fluid at a point (x, y, z) then div $\bar{v}$ gives the rate at which fluid is originating at that point per unit volume.

If $\bar{v}$ represents the electric flux, then div $\bar{v}$ represents the amount of flux which diverges per unit volume in unit time.

If $\bar{v}$ represents heat flux, div $\bar{v}$ is the rate at which the heat is issuing from a point per unit volume.

In general, the divergence of a vector point function, representing any physical quantity, gives at each point, the rate at which the physical quantity is issuing from that point per unit volume.

8.16 PHYSICAL INTERPRETATION OF CURL : *(JNTU 2007)*



Let a rigid body be rotating about the axis OA with angular velocity ω radians/sec.

Let P be a point of the body such that

$\overline{OP} = \bar{r}$ and $\angle AOP = \theta$

Let PA ⊥ OA

If $\bar{n}$ is a unit vector perpendicular to $\bar{\omega}$ and $\bar{r}$, then

$$\bar{\omega} \times \bar{r} = \omega \cdot r \cdot \sin\theta\ \hat{n} = (\omega\, PA)\hat{n}$$

$$= (\text{ speed of P })\ \hat{n}$$

= velocity $\bar{v}$ of P in the direction perpendicular to the plane AOP.

If $\bar{\omega} = \omega_1 \hat{i} + \omega_2 \hat{j} + \omega_3 \hat{k}$ and $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

then $\bar{v} = \bar{\omega} \times \bar{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix}$

$$= (\omega_2 z - \omega_3 y)\,\hat{i} + (\omega_3 x - \omega_1 z)\,\hat{j} + (\omega_1 y - \omega_2 x)\,\hat{k}$$

$$\text{Curl } \bar{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \omega_2 z - \omega_3 y & \omega_3 x - \omega_1 z & \omega_1 y - \omega_2 x \end{vmatrix}$$

$$= (\omega_1 + \omega_1)\,\hat{i} + (\omega_2 + \omega_2)\,\hat{j} + (\omega_3 + \omega_3)\,\hat{k}$$

$$= 2\omega_1\,\hat{i} + 2\omega_2\,\hat{j} + 2\omega_3\,\hat{k} = 2\,\bar{\omega}$$

$$\bar{\omega} = \frac{1}{2}\text{ curl } \bar{v}$$

Thus, the angular velocity of rotation at any point is equal to half the curl of the velocity vector.

SOLVED EXAMPLES – 8.4

EXAMPLE - 1 :

If $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$, prove that **i)** div $\bar{r} = 3$ **ii)** curl $\bar{r} = 0$

Solution : **i)** $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

[Formula $\bar{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

$$\nabla \cdot \bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \Big]$$

$$\nabla \cdot \bar{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1 + 1 + 1 = 3$$

ii) Curl $\bar{r} = \nabla \times \bar{r}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \hat{i}\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right) - \hat{j}\left(\frac{\partial z}{\partial x} - \frac{\partial x}{\partial z}\right) + \hat{k}\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right) = 0$$

EXAMPLE - 2 :

If $\bar{f} = x^2y\,\hat{i} - 2xz\,\hat{j} + 2yz\,\hat{k}$ find **i)** div $\bar{f}$ **ii)** curl $\bar{f}$ **iii)** curl curl $\bar{f}$

Solution : $\bar{f} = x^2y\,\hat{i} - 2xz\,\hat{j} + 2yz\,\hat{k}$

i) $$\nabla \cdot \bar{f} = \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial y}(-2xz) + \frac{\partial}{\partial z}(2yz)$$

$$= 2xy + 0 + 2y = 2y(x+1)$$

ii) $$\nabla \times \bar{f} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & -2xz & 2yz \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(2yz) - \frac{\partial}{\partial z}(-2xz)\right]\hat{i} - \left[\frac{\partial}{\partial x}(2yz) - \frac{\partial}{\partial z}(x^2y)\right]\hat{j}$$

$$+ \left[\frac{\partial}{\partial x}(-2xz) - \frac{\partial}{\partial y}(x^2y)\right]\hat{k}$$

$$= (2x + 2z)\,\hat{i} - 0\,\hat{j} + (-2z - x^2)\,\hat{k}$$

$$= (2x + 2z)\,\hat{i} - (x^2 + 2z)\,\hat{k}$$

iii) Curl curl $\bar{f} = \nabla \times (\nabla \times \bar{f})$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x + 2z & 0 & -x^2 - 2z \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}\left(-x^2 - 2z\right) - \frac{\partial}{\partial z}(0)\right]\hat{i} - \left[\frac{\partial}{\partial x}\left(-x^2 - 2z\right) - \frac{\partial}{\partial z}(2x + 2z)\right]\hat{j}$$

$$+ \left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(2x + 2z)\right]\hat{k}$$

$$= (2x + 2)\,\hat{j}$$

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EXAMPLE - 3 :

Show that the vector field given by $\bar{f} = (x + 3y)\,\hat{i} + (y - 3z)\,\hat{j} + (x - 2z)\,\hat{k}$ is solenoidal.

Solution : $\bar{f} = (x + 3y)\,\hat{i} + (y - 3z)\,\hat{j} + (x - 2z)\,\hat{k}$ is solenoidal

if $\nabla \cdot \bar{f} = 0$

$$\nabla \cdot \bar{f} = \frac{\partial}{\partial x}(x + 3y) + \frac{\partial}{\partial y}(y - 3z) + \frac{\partial}{\partial z}(x - 2z) = 1 + 1 - 2 = 0$$

$\bar{f}$ is solenoidal.

EXAMPLE - 4 :

Find the constant a so that

$\bar{f} = y(ax^2 + z)\,\hat{i} + x(y^2 - z^2)\,\hat{j} + 2xy(z - xy)\,\hat{k}$ is solenoidal.

Solution : If $\bar{f}$ is solenoidal, $\nabla \cdot \bar{f} = 0$

$$\nabla \cdot \bar{f} = 0 \Rightarrow \frac{\partial}{\partial x} y[ax^2 + z] + \frac{\partial}{\partial y} x(y^2 - z^2) + \frac{\partial}{\partial z} 2xy(z - xy) = 0$$

$$\Rightarrow 2ayx + 2xy + 2xy = 0$$

$$\Rightarrow 2xy(a + 2) = 0 \qquad \Rightarrow a = -2$$

EXAMPLE - 5 :

Show that $\bar{V} = (\sin y + z)\,\hat{i} + (x \cos y - z)\,\hat{j} + (x - y)\,\hat{k}$ is irrotational.

Solution : A vector $\bar{V}$ is irrotational if $\nabla \times \bar{V} = 0$

$$\nabla \times \bar{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin y + z & x \cos y - z & x - y \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(x - y) - \frac{\partial}{\partial z}(x \cos y - z)\right]\hat{i} - \left[\frac{\partial}{\partial x}(x - y) - \frac{\partial}{\partial z}(\sin y + z)\right]\hat{j}$$

$$+ \left[\frac{\partial}{\partial x}(x \cos y - z) - \frac{\partial}{\partial y}(\sin y + z)\right]\hat{k}$$

$$= (-1 + 1)\,\hat{i} - (1 - 1)\,\hat{j} + (\cos y - \cos y)\,\hat{k} = 0$$

$\bar{V}$ is irrotational.

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EXAMPLE - 6 :

Find the constants a, b, c if the vector

$\bar{f} = (2x + 3y + az)\,\hat{i} + (bx + 2y + 3z)\,\hat{j} + (2x + cy + 3z)\,\hat{k}$ is irrotational.

Solution : If $\bar{f}$ is irrotational, $\nabla \times \bar{f} = 0$

$$\nabla \times \bar{f} = 0 \;\Rightarrow\; \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x + 3y + az & bx + 2y + 3z & 2x + cy + 3z \end{vmatrix} = 0$$

$\Rightarrow (c-3)\,\hat{i} - (2-a)\,\hat{j} + (b-3)\,\hat{k} = 0$

$\Rightarrow 2 - a = 0$ i.e. $a = 2$

$b - 3 = 0$ i.e. $b = 3$

$c - 3 = 0$ i.e. $c = 3$

EXAMPLE - 7 :

If $\bar{f} = (x + y + 1)\,\hat{i} + \hat{j} - (x + y)\,\hat{k}$ prove that $\bar{f} \cdot \text{curl}\,\bar{f} = 0$

Solution : Curl $\bar{f} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x + y + 1 & 1 & -x - y \end{vmatrix}$

$= \hat{i}\,(-1) - \hat{j}\,(-1) + \hat{k}\,(-1) = -\hat{i} + \hat{j} - \hat{k}$

$\bar{f} \cdot \text{curl}\,\bar{f} = \{(x + y + 1)\,\hat{i} + \hat{j} - (x + y)\,\hat{k}] \cdot [-\hat{i} + \hat{j} - \hat{k}]$

$= -(x + y + 1) + 1 + (x + y) = 0$

EXAMPLE - 8 :

If $\bar{a}$ is a constant vector, find **i)** $\nabla \cdot (\bar{r} \times \bar{a})$ **ii)** $\nabla \times (\bar{r} \times \bar{a})$

Solution : Let $\bar{a} = a_1\,\hat{i} + a_2\,\hat{j} + a_3\,\hat{k}$, where a_1, a_2, a_3 are constants.

$$\bar{r} \times \bar{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$= (a_3 y - a_2 z)\,\hat{i} + (a_1 z - a_3 x)\,\hat{j} + (a_2 x - a_3 y)\,\hat{k}$

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i) $$\nabla \cdot (\bar{r} \times \bar{a}) = \frac{\partial}{\partial x}(a_3 y - a_2 z) + \frac{\partial}{\partial y}(a_1 z - a_3 x) + \frac{\partial}{\partial z}(a_2 x - a_3 y)$$

$$= 0 + 0 + 0 = 0$$

ii) $$\nabla \times (\bar{r} \times \bar{a}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_3 y - a_2 z & a_1 z - a_3 x & a_2 x - a_1 y \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(a_2 x - a_1 y) - \frac{\partial}{\partial z}(a_1 z - a_3 x)\right]\hat{i}$$

$$- \left[\frac{\partial}{\partial x}(a_2 x - a_1 y) - \frac{\partial}{\partial z}(a_3 y - a_2 z)\right]\hat{j}$$

$$+ \left[\frac{\partial}{\partial x}(a_1 z - a_3 x) - \frac{\partial}{\partial y}(a_3 y - a_2 z)\right]\hat{k}$$

$$= -2 a_1 \hat{i} - 2a_2 \hat{j} - 2a_3 \hat{k}$$

$$= -2\left(a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}\right) = -2\,\bar{a}$$

EXERCISE 8.3

1. If $\bar{f} = 3x^2\hat{i} + 5xy\hat{j} + xyz^3 k$ find div $\bar{f}$ and curl $\bar{f}$ at (1, 2, 3)

Ans : 65, $27\hat{i} - 18\hat{j} + 10\hat{k}$

2. If $\bar{f} = x^2y\hat{i} - 2xz\hat{j} + 2yz\hat{k}$ show that

div $\bar{f} = 2y(x+1)$, curl $\bar{f} = 2(x+y)\hat{i} - (x^2 + 2z)\hat{k}$

and curl curl $\bar{f} = 2(x+1)\hat{j}$

3. If $\bar{v} = e^{xyz}(\hat{i} + \hat{j} + \hat{k})$ find curl $\bar{v}$

Ans : $e^{xyz}(xz - xy)\hat{i} + e^{xyz}(xy - yz)\hat{j} + e^{xyz}(yz - xz)\hat{k}$

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4. Given $\phi = 2x^3y^2z^4$ find div (grad ϕ)

Ans : $12\,xy^2z^4 + 4x^3z^4 + 24\,x^3y^2z^2$

5. Show that $\bar{F} = (y+z)\,\hat{i} + (z+x)\,\hat{j} + (x+y)\,\hat{k}$ is solenoidal.

6. Determine the constant a so that the vector

$\bar{f} = (x+3y)\,\hat{i} + (y-2z)\,\hat{j} + (x-az)\,\hat{k}$ is a solenoidal.

Ans : $a = 2$

7. Show that the vector $\bar{F} = yz\,\hat{i} + zx\,\hat{j} + xy\,\hat{k}$ is solenoidal.

8. Show that the vector $\bar{f} = \dfrac{x\,\hat{i} + y\,\hat{j} + z\,\hat{k}}{\sqrt{x^2 + y^2 + z^2}}$ is irrotational.

9. Find the constants a, b, c so that the vector

$\bar{F} = (x + 2y + az)\,\hat{i} + (bx - 3y - z)\,\hat{j} + (4x + cy + 2z)\,\hat{k}$ is irrotational.

Ans : $a = 4, b = 2, c = -1$.

10. Show that the vector $\bar{F} = \dfrac{-y\,\hat{i} + x\,\hat{j}}{x^2 + y^2}$ is irrotational.

11. Show that $\dfrac{\bar{r}}{r^3}$ is both solenoidal and irrotational.

12. If $\bar{a}$ and $\bar{b}$ are constant vectors, show that :

i) $\nabla \cdot (\bar{a} \times \bar{b}) = 0$ **ii)** $\nabla \cdot [(\bar{a} \cdot \bar{r})\,\bar{b}] = \bar{a} \cdot \bar{b}$

8.17 IDENTITIES :

I. Del applied twice to point function :

∇f and $\nabla \times \bar{F}$ being vector point functions, their divergence and curl can be formed.
$\nabla \cdot \bar{F}$ being a scalar point function, its gradient can be formed.

1) div grad $f = \nabla^2 f = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2}$

2) div curl $\bar{F}$ $= \nabla \cdot \nabla \times \bar{F} = 0$

3) curl grad f $= \nabla \times \nabla f = 0$

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NOTE

gradiant of a Scalar function ϕ is $\nabla\phi$

$$= \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$ which is a vector

divergence of a vector function $\bar{f}$ is $\nabla\cdot\bar{f}$

$$\nabla\cdot\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$ which is a Scalar

here $\bar{f} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$

Curl of a vector function $\bar{f}$ is $\nabla\times\bar{f}$

$$\nabla\times\bar{f} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} = \hat{i}\left[\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right] - \hat{j}\left[\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right] + \hat{k}\left[\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right]$$

which is a vector

$$\nabla \underset{\text{Scalar}}{\phi} = \text{vector}$$

$$\underset{\text{vector}}{\nabla\cdot\bar{f}} = \text{scalar}$$

$$\nabla\times\underset{\text{vector}}{\bar{f}} = \text{vector}$$

$$\boxed{\nabla \equiv \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}}$$

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UNIT-IV

Vector Differentiation and Vector Operators

INTRODUCTION

In this chapter, vector differential calculus is considered, which extends the basic concepts of differential calculus, such as, continuity and differentiability to vector functions in a simple and natural way. Also, the new concepts of gradient, divergence and curl are introduced.

DIFFERENTIATION OF A VECTOR POINT FUNCITON

Let S be a set of real numbers. Corresponding to each scalar t ε S, let there be associated a unique vector $\bar{f}$. Then $\bar{f}$ is said to be a vector (vector valued) function. S is called the domain of $\bar{f}$. We write $\bar{f} = \bar{f}(t)$.

Let $\bar{i}, \bar{j}, \bar{k}$ be three mutually perpendicular unit vectors in three dimensional space. We can write $\bar{f} = \bar{f}(t) = f_1(t)\bar{i} + f_2(t)\bar{j} + f_3(t)\bar{k}$, where $f_1(t), f_2(t), f_3(t)$ are real valued functions (which are called components of $\bar{f}$). (we shall assume that $\bar{i}, \bar{j}, \bar{k}$ are constant vectors).

1. Derivative:

Let $\bar{f}$ be a vector function on an interval *I* and a ϵ *I*. then $Lt_{t\to a} \dfrac{\bar{f}(t) - \bar{f}(a)}{t-a}$, if exists, is called the derivative of $\bar{f}$ at a and is denoted by $\bar{f}^{1}(a)$ or $\left(\dfrac{d\bar{f}}{dt}\right)$ at t = a. we also say that $\bar{f}$ is differentiable at t =a if $\bar{f}^{1}(a)$ exists.

2. Higher order derivatives

Let $\bar{f}$ be differentiable on an interval *I* and $\bar{f}^{1} = \dfrac{d\bar{f}}{dt}$ be the derivative of $\bar{f}$. $Lt_{t\to a} \dfrac{\bar{f}^{1}(t) - \bar{f}^{1}(a)}{t-a}$ exists for every a Є I_1 . it is denoted by $\bar{f}^{11} = \dfrac{d^2\bar{f}}{dt^2}$.

Similarly we can define $\bar{f}^{111}(t)$ etc.

We now state some properties of differentiable functions (without proof)

(1) Derivative of a constant vector is $\bar{a}$.

If $\bar{a}$ and $\bar{b}$ are differentiable vector functions, then

(2). $\dfrac{d}{dt}(\bar{a} \pm \bar{b}) = \dfrac{d\bar{a}}{dt} \pm \dfrac{d\bar{b}}{dt}$

(3). $\dfrac{d}{dt}(\bar{a}.\bar{b}) = \dfrac{d\bar{a}}{dt}.\bar{b} + \bar{a}.\dfrac{d\bar{b}}{dt}$

(4). $\dfrac{d}{dt}(\bar{a}x\bar{b}) = \dfrac{d\bar{a}}{dt}x\bar{b} + \bar{a}x\dfrac{d\bar{b}}{dt}$

(5). If $\bar{f}$ is a differentiable vector function and ϕ is a scalar differential function, then $\frac{d}{dt}(\phi\,\bar{f}) = \phi\frac{d\bar{f}}{dt} + \frac{d\phi}{dt}\bar{f}$

(6). $\bar{f} = f_1(t)\bar{i} + f_2(t)\,\bar{j} + f_3(t)\bar{k}$, where $f_1(t)$, $f_2(t)$, $f_3(t)$ are Cartesian components of the vector $\bar{f}$, then $\frac{d\bar{f}}{dt} = \frac{df_1}{dt}\bar{i} + \frac{df_2}{dt}\bar{j} + \frac{df_3}{dt}\bar{k}$

(7). The necessary and sufficient condition for $\bar{f}$ (t) to be constant vector function is $\frac{d\bar{f}}{dt}$ = $\bar{0}$.

3. Partial Derivatives

Partial differentiation for vector valued functions can be introduced as was done in the case of functions of real variables. Let $\bar{f}$ be a vector function of scalar variables *p, q, t*. Then we write $\bar{f}$ = $\bar{f}$ (*p,q,t*). Treating *t* as a variable and *p,q* as constants, we define

$$Lt_{\delta t \to 0}\frac{\bar{f}(p,q,t+\delta t) - \bar{f}(p,q,t)}{\delta t}$$

If exists, as partial derivative of $\bar{f}$ w.r.t. *t* and is denot by $\frac{\partial\bar{f}}{\partial t}$

Similarly, we can define $\frac{\partial\bar{f}}{\partial p}, \frac{\partial\bar{f}}{\partial q}$ also. The following are some useful results on partial differentiation.

4. Properties

1) $\frac{\partial}{\partial t}(\phi\bar{a}) = \frac{\partial\phi}{\partial t}\bar{a} + \phi\frac{\partial\bar{a}}{\partial t}$

2). If λ is a constant, then $\frac{\partial}{\partial t}(\lambda\bar{a}) = \lambda\frac{\partial\bar{a}}{\partial t}$

3). If $\bar{c}$ is a constant vector, then $\frac{\partial}{\partial t}(\phi\bar{c}) = \bar{c}\frac{\partial\phi}{\partial t}$

4). $\frac{\partial}{\partial t}(\bar{a} \pm \bar{b}) = \frac{\partial\bar{a}}{\partial t} \pm \frac{\partial\bar{b}}{\partial t}$

5). $\frac{\partial}{\partial t}(\bar{a}.\bar{b}) = \frac{\partial\bar{a}}{\partial t}.\bar{b} + \bar{a}.\frac{\partial\bar{b}}{\partial t}$

6). $\frac{\partial}{\partial t}(\bar{a}x\bar{b}) = \frac{\partial\bar{a}}{\partial t}x\bar{b} + \bar{a}x\frac{\partial\bar{b}}{\partial t}$

7). Let $\bar{f} = f_1\bar{i} + f_2\,\bar{j} + f_3\bar{k}$, where f_1, f_2, $f_3$are differential scalar functions of more then one variable, Then $\frac{\partial\bar{f}}{\partial t} = \bar{i}\frac{\partial f_1}{\partial t} + \bar{j}\frac{\partial f_2}{\partial t} + \bar{k}\frac{\partial f_3}{\partial t}$ (treating $\bar{i}, \bar{j}, \bar{k}$ as fixed directions)

5. Higher order partial derivatives

Let $\bar{f} = \bar{f}(p,q,t)$. Then $\frac{\partial^2 \bar{f}}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial \bar{f}}{\partial t}\right), \frac{\partial^2 \bar{f}}{\partial p \partial t} = \frac{\partial}{\partial p}\left(\frac{\partial \bar{f}}{\partial t}\right) etc.$

6. Scalar and vector point functions: Consider a region in three dimensional space. To each point $p(x,y,z)$, suppose we associate a unique real number (called scalar) say ϕ. This $\phi(x,y,z)$ is called a scalar point function. Scalar point function defined on the region. Similarly if to each point $p(x,y,z)$we associate a unique vector $\bar{f}(x,y,z)$ we associate a unique vector $\bar{f}(x,y,z)$. $\bar{f}$ is called a ***vector point function***.

VECTOR DIFFERENTIAL OPERATOR

Def. The vector differential operator ∇(read as del) is defined as

$\nabla \equiv \bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}$. This operator possesses properties analogous to those of ordinary vectors as well as differentiation operator. We will define now some quantities known as "**gradient", "divergence"** and **"curl"** involving this operator ∇. We must note that this operator has no meaning by itself unless it operates on some function suitably.

GRADIENT OF A SCALAR POINT FUNCTION

Let $\phi(x,y,z)$ be a scalar point function of position defined in some region of space. Then the vector function $\bar{i}\frac{\partial \phi}{\partial x} + \bar{j}\frac{\partial \phi}{\partial y} + \bar{k}\frac{\partial \phi}{\partial z}$ is known as the gradient of ϕ or $\nabla\phi$

$$\nabla\phi = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\phi = \bar{i}\frac{\partial \phi}{\partial x} + \bar{j}\frac{\partial \phi}{\partial y} + \bar{k}\frac{\partial \phi}{\partial z}$$

Properties:

(1) If f and g are two scalar functions then grad(f $\pm$g)= grad f $\pm$ grad g

(2) The necessary and sufficient condition for a scalar point function to be constant is that $\nabla f = \bar{O}$

(3) grad(fg) = f(grad g)+g(grad f)

(4) If c is a constant, grad (cf) = c(grad f)

(5) grad $\left(\frac{f}{g}\right)=\frac{g(grad\ f)-f(grad\ g)}{g^2}$, $(g\neq 0)$

(6) Let $\bar{r}$ = $x\bar{i}+y\bar{j}+z\bar{k}$. Then $d\bar{r}=(dx)\bar{i}+(dy)\bar{j}+(dz)\bar{k}$. if ϕ is any scalar point function, then

$$d\phi=\frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial y}dy+\frac{\partial\phi}{\partial z}dz=\left(\bar{i}\frac{\partial}{\partial x}+\bar{j}\frac{\partial}{\partial y}+\bar{k}\frac{\partial}{\partial z}\right)$$

DIRECTIONAL DERIVATIVE

Let $\phi(x,y,z)$ be a scalar function defined throughout some region of space. Let this function have a value ϕ at a point P whose position vector referred to the origin O is OP = r. let $\phi+\Delta\phi$ be the value of the function at neighbouring point Q. If $\overline{OQ}=\bar{r}+\Delta r$. . Let Δr be the length of $\Delta\bar{r}$.

$\frac{\Delta\phi}{\Delta r}$ gives a measure of the rate at which ϕ change when we move from P to Q. then limiting value $\frac{\Delta\phi}{\Delta r}$ as $\Delta r\to 0$ is called the derivative of ϕ in the direction of PQ or simply directional derivative of ϕ at P and is denoted by $d\phi/dr$.

Theorem 1: The directional derivative of a scalar point function ϕ at a point P(x,y,z) in the direction of a unit vector $\bar{e}$ is equal to e. grad ϕ=e. $\nabla\phi$.

The physical interpretation of $\nabla\phi$

The gradient of a scalar function $\phi(x,y,z)$ at a point $P(x,y,z)$ is a vector along the normal to the level surface $\phi(x,y,z)$ = c at P and is in increasing direction. Its magnitude is equal to the greatest rate of increase of ϕ. Greatest value of directional derivative of **ϕ at a point P = |grad ϕ|** at that point.

SOLVED EXAMPLES

Example 1: If a=x+y+z, b= $x^2+y^2+z^2$, c = xy+yz+zx, prove that [grad a, grad b, grad c] = 0.

Sol:- Given a=x+y+z $\quad \frac{\partial a}{\partial x}=1,\ \frac{\partial a}{\partial y}=1,\ \frac{\partial a}{\partial z}=1$

Grad a = ∇a = $\sum\bar{i}\frac{\partial a}{\partial x}=\bar{i}+\bar{j}+\bar{z}$

Given b= $x^2+y^2+z^2$ $\frac{\partial b}{\partial x}=2x,\ \frac{\partial b}{\partial y}=2y,\ \frac{\partial b}{\partial z}=2z$

Grad b = ∇b = $\bar{i}\frac{\partial b}{\partial x}+\bar{j}\frac{\partial b}{\partial y}+\bar{z}\frac{\partial b}{\partial z}=2x\bar{i}+2y\bar{j}+2z\bar{k}$

Again c = xy+yz+zx $\frac{\partial c}{\partial x} = y+z,\ \frac{\partial c}{\partial y} = z+x, \frac{\partial c}{\partial z} = y+x$

Grad c = $\bar{i}\frac{\partial c}{\partial x} + \bar{j}\frac{\partial c}{\partial y} + \bar{z}\frac{\partial c}{\partial z} = (y+z)\bar{i} + (z+x)\bar{j} + (x+y)\bar{k}$

[grad a, grad b, grad c] = $\begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & z+x & x+y \end{vmatrix} = 0, (on\ simplification)$

Example 2 : Show that ∇[f(r)] = $\frac{f^{i}(r)}{r}\bar{r}$ where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$.

Sol:-since $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$, we have $r^2= x^2+y^2+z^2$

Differentiating w.r.t. 'x' partially, we get

$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}.Similarly\ \frac{\partial r}{\partial y} = \frac{y}{r}\ ,\ \frac{\partial r}{\partial z} = \frac{z}{r}$

$\nabla[f(r)] = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) f(r) = \sum \bar{i} f^{1}(r)\frac{\partial r}{\partial x} = \sum \bar{i} f^{1}(r)\frac{x}{r}$

$= \frac{f^{1}(r)}{r}\sum \bar{i}x = \frac{f^{1}(r)}{r}.\bar{r}$

Note : From the above result, ∇(log r) = $\frac{1}{r^2}r$

Example 3 : Prove that $\nabla(r^n) = nr^{n-2}\bar{r}$.

Sol:- Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ and r = $|\bar{r}|$. Then we have $r^2 = x^2+y^2+z^2$ Differentiating w.r.t. x partially, we have

$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}.Similarly = \frac{\partial r}{\partial y} = \frac{y}{r}\ and\ \frac{\partial r}{\partial z} = \frac{z}{r}$

$\nabla(r^n) = \sum \bar{i}\frac{\partial}{\partial x}(r^n) = \sum \bar{i} n r^{n-1}\frac{\partial r}{\partial x} = \sum \bar{i} n r^{n-1}\frac{x}{r} = n r^{n-2}\sum \bar{i}x = n r^{n-2}(\bar{r})$

Note : From the above result, we can have

(1). $\nabla\left(\frac{1}{r}\right) = -\frac{\bar{r}}{r^3}$, taking n = -1 (2) grad r = $\frac{\bar{r}}{r}$, taking n = 1

Example 4 : Find the directional derivative of $f = xy+yz+zx$ in the direction of vector $\bar{i} + 2\bar{j} + 2\bar{k}$ at the point (1,2,0).

Sol:- Given $f = xy+yz+zx$.

Grad f = $\bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{z}\frac{\partial f}{\partial z} = (y+z)\bar{i} + (z+x)\bar{j} + (x+y)\bar{k}$

If $\bar{e}$ is the unit vector in the direction of the vector $\bar{i} + 2\bar{j} + 2\bar{k}$, then

$$\bar{e} = \frac{\bar{i} + 2\bar{j} + 2\bar{k}}{\sqrt{1^2 + 2^2 + 2^2}} = \frac{1}{3}(\bar{i} + 2\bar{j} + 2\bar{k})$$

Directional derivative of f along the given direction $= \bar{e}.\nabla f$

$$= \frac{1}{3}(\bar{i} + 2\bar{j} + 2\bar{k})\left[(y+2)\bar{i} + (z+x)\bar{j} + (x+y)\bar{k}\right] at\ (1,2,0)$$

$$= \frac{1}{3}\left[(y+z) + 2(z+x) + 2(x+y)\right](1,2,0) = \frac{10}{3}$$

Example 5 : Find the directional derivative of the function $xy^2+yz^2+zx^2$ along the tangent to the curve $x = t$, $y = t^2$, $z = t^3$ at the point (1,1,1).

Sol: - here $f = xy^2+yz^2+zx^2$

$$\nabla f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z} = \text{(y2+2xy)j+(x2+2yz)k}$$

At (1,1,1) , $\nabla f = 3\bar{i} + 3\bar{j} + 3\bar{k}$

Let r be the position vector of any point on the curve $x = t$, $y = t^2$, $z = t^3$. then

$$r = x\bar{i} + y\bar{j} + z\bar{k} = t\bar{i} + t^2\bar{j} + t^3\bar{k}$$

$$\frac{\partial \bar{r}}{\partial t} = \bar{i} + 2t\bar{j} + 3t^2\bar{k} - (\bar{i} + 2\bar{j} + 3\bar{k}) \text{ at } (1,1,1)$$

We know that $\frac{\partial \bar{r}}{\partial t}$ is the vector along the tangent to the curve.

Unit vector along the tangent $= \bar{e} = \frac{\bar{i}+2\bar{j}+3\bar{k}}{\sqrt{1+2^2+3^2}} = \frac{\bar{i}+2\bar{j}+3\bar{k}}{\sqrt{14}}$

Directional derivative along the tangent = $\nabla f.\bar{e}$

$$= \frac{1}{\sqrt{14}}(\bar{i} + 2\bar{j} + 3\bar{k}).3(\bar{i} + \bar{j} + \bar{k})\ \frac{3}{\sqrt{14}}(1+2+3) = \frac{18}{\sqrt{14}}$$

Example 6 : Find the directional derivative of the function $f = x^2-y^2+2z^2$ at the point P =(1,2,3) in the direction of the line $\overline{PQ}$ where Q = (5,0,4).

Sol:- The position vectors of P and Q with respect to the origin are OP = $\bar{i} + 2\bar{j} + 3\bar{k}$ and OQ = $5\bar{i} + 4\bar{k}$

$$\overline{PQ} = \overline{OQ} - \overline{OP} = 4\bar{i} - 2\bar{j} + \bar{k}$$

Let $\bar{e}$ be the unit vector in the direction of PQ. Then $\bar{e} = \frac{4\bar{i} - 2\bar{j} + \bar{k}}{\sqrt{21}}$

grad $f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z} = 2x\bar{i} - 2y\bar{j} + 4z\bar{k}$

The directional derivative of $\bar{f}$ at P (1,2,3) in the direction of PQ = $\bar{e}.\nabla f$

$$= \frac{1}{\sqrt{21}}(4\bar{i}-2\bar{j}+\bar{k}).(2x\bar{i}-2y\bar{j}+4z\bar{k})\ \frac{1}{\sqrt{21}}(8x+4y+4z)_{at(1,2,3)} = \frac{1}{\sqrt{21}}(28)$$

Example 7 : Find the greatest value of the directional derivative of the function f = x^2yz^3 at (2,1,-1).

Sol: we have

grad $f = \bar{i}\frac{\partial f}{\partial x}+\bar{j}\frac{\partial f}{\partial y}+\bar{k}\frac{\partial f}{\partial z} = 2xyz^3\bar{i}+x^2z^3\bar{j}+3x^2yz^2\bar{k} = -4\bar{i}-4\bar{j}+12\bar{k}$ at (2,1,-1).

Greatest value of the directional derivative of $f = |\nabla f| = \sqrt{16+16+144} = 4\sqrt{11}.$

Example 8 : Find the directional derivative of xyz^2+xz at (1, 1 ,1) in a directional of the normal to the surface $3xy^2+y= z$ at (0,1,1).

Sol:- Let $f(x, y, z) \equiv 3xy^2+y- z = 0$

Let us find the unti normal e to this surface at (,1,1). Then

$$\frac{\partial f}{\partial x} = 3y^2,\ \frac{\partial f}{\partial y} = 6xy+1, \frac{\partial f}{\partial z} = -1.$$

∇f = 3y2i+(6xy+1)j-k

$(\nabla f)_{(0,1,1)}$ = 3i+j-k = n

$$\bar{e} = \frac{\bar{n}}{|\bar{n}|} = \frac{3i+j-k}{\sqrt{9+1+1}} = \frac{3i+j-k}{\sqrt{11}}$$

Let $g(x,y,z) = xyz^2+xz$ then

$$\frac{\partial g}{\partial x} = yz^2+z,\ \frac{\partial g}{\partial y} = xz^2, \frac{\partial g}{\partial z} = 2xy+x$$

$\nabla g=(yz^2+z)i+xz^2j+(2xyz+x)k$

And $[\nabla g]_{(1,1,1)}$ = 2i+j+3k

Directional derivative of the given function in the direction of $\bar{e}$ at (1,1,1) = $\nabla g.\bar{e}$

$$=(2i+j+3k).\left(\frac{3i+j-k}{\sqrt{11}}\right) = \frac{6+1-3}{\sqrt{11}} = \frac{4}{\sqrt{11}}$$

Example 9 : Find the directional derivative of $2xy+z^2$ at (1,-1,3) in the direction of $\bar{i}+2\bar{j}+3\bar{k}$.

Sol: Let $f = 2xy+z^2$ $\frac{\partial f}{\partial x} = 2y,\ \frac{\partial f}{\partial y} = 2x, \frac{\partial f}{\partial z} = 2z.$

grad f = $\sum \bar{i}\frac{\partial f}{\partial x} = 2y\bar{i}+2x\bar{j}+2z\bar{k}$ and (grad f)at (1,-1,3)= $-2\bar{i}+2\bar{j}+6\bar{k}$

given vector is $\bar{a} = \bar{i}+2\bar{j}+3\bar{k} \Rightarrow |\bar{a}| = \sqrt{1+4+9} = \sqrt{14}$

directional derivative of f in the direction of $\bar{a}$

$$\frac{\bar{a}.\nabla f}{|\bar{a}|} = \frac{(\bar{i}+2\bar{j}+3\bar{k})(-2\bar{i}+2\bar{j}+6\bar{k}).}{\sqrt{14}} = \frac{-2+4+18}{\sqrt{14}} = \frac{20}{\sqrt{14}}$$

Example 10: Find the directional derivative of $\phi = x^2yz+4xz^2$ at (1,-2,-1) in the direction 2i-j-2k.

Sol:- Given $\phi = x^2yz+4xz^2$

$$\frac{\partial\phi}{\partial x} = 2xyz + 4z^2,\ \frac{\partial\phi}{\partial y} = x^2 z, \frac{\partial\phi}{\partial z} = x^2 y + 8xz.$$

Hence $\nabla\phi = \sum \bar{i}\,\frac{\partial\phi}{\partial x} = \bar{i}(2xyz + 4z^2) + \bar{j}x^2 z + \bar{k}(x^2 y + 8xz)$

$\nabla\phi$ at (1,-2,-1) = i(4+4)+j(-1)+k(-2-8)= 8i-j-10k.

The unit vector in the direction 2i-j-2k is

$$\bar{a} = \frac{2i - j - 2k.}{\sqrt{4+1+4}} = \frac{1}{3}(2i - j - 2k)$$

Required directional derivative along the given direction = $\nabla\phi.\ \bar{a}$

= (8i-j-10k). 1/3 (2i-j-2k)

= 1/3(16+1+20) = 37/3.

Example:11 If the temperature at any point in space is given by t = xy+yz+zx, find the direction in which temperature changes most rapidly with distance from the point (1,1,1) and determine the maximum rate of change.

Sol:- The greatest rate of increase of t at any point is given in magnitude and direction by ∇t.

We have $\nabla t = \left(\bar{i}\,\frac{\partial}{\partial x} + \bar{j}\,\frac{\partial}{\partial y} + \bar{k}\,\frac{\partial}{\partial z}\right)(xy + yz + zx)$

$= \bar{i}(y + z) + \bar{j}(z + x) + \bar{k}(x + y) = 2\bar{i} + 2\bar{j} + 2\bar{k}$ at (1,1,1)

Magnitude of this vector is $\sqrt{2^2 + 2^2 + 2^2} = \sqrt{12} = 2\sqrt{3}$

Hence at the point (1,1,1) the temperature changes most rapidly in the direction given by the vector $2\bar{i} + 2\bar{j} + 2\bar{k}$ and greatest rate of increase = $2\sqrt{3}$.

Example12 : Find the directional derivative of ϕ(x,y,z) = $x^2yz+4xz^2$ at the point (1,-2,-1) in the direction of the normal to the surface *f*(x,y,z) = x log z-y^2 at (-1,2,1).

Sol:- Given $\phi(x,y,z) = x^2yz+4xz^2$ at (1,-2,-1) and $f(x,y,z) = x \log z-y^2$ at (-1,2,1)

Now $\nabla\phi = \frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k}$

$= (2xyz + 4z^2)\bar{i} + (x^2 z)\bar{j} + (x^2 y + 8xz)\bar{k}$

$(\nabla\phi)_{(1,-2,-1)} = [2(1)(-2)(-1) + 4(-1)^2]i + [(1)^2(-1)\bar{j}] + [(1^2)(-2) + 8(-1)]\bar{k} ----(1)$

$= 8\bar{i} - \bar{j} - 10\bar{k}$

Unit normal to the surface

f(x,y,z)= x log z- y^2 is $\dfrac{\nabla f}{|\nabla f|}$

now ∇f = $\bar{i}\dfrac{\partial f}{\partial x}+\bar{j}\dfrac{\partial f}{\partial y}+\bar{k}\dfrac{\partial f}{\partial z}=\log z\,\bar{i}+(-2y)\,\bar{j}+\dfrac{x}{z}\bar{k}$

at (-1,2,1), ∇f = $\log(1)\,\bar{i}-2(2)\,\bar{j}+\dfrac{-1}{1}\bar{k}=-4\bar{j}-\bar{k}$

$\dfrac{\nabla f}{|\nabla f|}=\dfrac{-4\bar{j}-\bar{k}.}{\sqrt{16+1}}=\dfrac{-4\bar{j}-\bar{k}.}{\sqrt{17}}=$

Directional derivative = $\nabla\phi.\dfrac{\nabla f}{|\nabla f|}$

$$=(8\bar{i}-\bar{j}-10\bar{k}).\frac{-4\bar{j}-\bar{k}.}{\sqrt{17}}=\frac{4+10}{\sqrt{17}}=\frac{14}{\sqrt{17}}.$$

Example 13 : Find a unit normal vector to the given surface x^{2}y+2xz = 4 at the point (2,-2,3).

Sol:- Let the given surface be f = x^{2}y+2xz – 4

On differentiating,

$$\frac{\partial f}{\partial x}=2xy+2z,\ \frac{\partial f}{\partial y}=x^2,\frac{\partial f}{\partial z}=2x.$$

grad f = $\sum\bar{i}\dfrac{\partial f}{\partial x}=\bar{i}(2xy+2z)+\bar{j}x^2+2x\bar{k})$

(grad f) at (2,-2,3) = $\bar{i}(-8+6)+4\bar{j}+4\bar{k})=-2\bar{i}+4\bar{j}+4\bar{k}$

grad (f) is the normal vector to the given surface at the given point.

Hence the required unit normal vector $\dfrac{\nabla f}{|\nabla f|}=\dfrac{2(-\bar{i}+2\bar{j}+2\bar{k}).}{2\sqrt{1+2^2+2^2}}=\dfrac{-\bar{i}+2\bar{j}+2\bar{k}}{3}$

Example 14 : Evaluate the angle between the normals to the surface xy= z^2 at the points (4,1,2) and (3,3,-3).

Sol:- given surface is f(x,y,z) = xy= z^2

Let $\bar{n}_1$ and $\bar{n}_2$ be the normals to this surface at (4,1,2) and (3,3,-3) respectively.

Differentiating partially, we get

$$\frac{\partial f}{\partial x}=y,\ \frac{\partial f}{\partial y}=x,\frac{\partial f}{\partial z}=-2z.$$

grad f = $y\bar{i}+x\bar{j}-2z\bar{k}$

$\bar{n}_1$= (grad f) at (4,1,2) $=\bar{i}+4\bar{j}-4\bar{k}$

$\bar{n}_2$ = (grad *f*) at (3,3,-3) = $3\bar{i}+3\bar{j}+6\bar{k}$

Let θ be the angle between the two normals.

$$\cos\theta = \frac{\bar{n}_1\,\bar{n}_2}{|\bar{n}_1\,\bar{n}_2|} = \frac{(i+4j-4k)}{\sqrt{1+16+16}}\cdot\frac{(3i+3j+6k)}{\sqrt{9+9+36}}$$

$$\frac{(3+12-24)}{\sqrt{33}\sqrt{54}} = \frac{-9}{\sqrt{33}\sqrt{54}}$$

Example15 : Find a unit normal vector to the surface $x^2+y^2+2z^2 = 26$ at the point (2, 2 ,3).

Sol:- Let the given surface be $f(x,y,z) \equiv x^2+y^2+2z^2 - 26=0$. Then

$$\frac{\partial f}{\partial x} = 2x,\ \frac{\partial f}{\partial y} = 2y,\ \frac{\partial f}{\partial z} = 4z.$$

grad f = $\sum \bar{i}\,\frac{\partial f}{\partial x} = 2xi+2yj+4zk$

normal vector at(2,2,3) = $[\nabla f\,]_{(2,2,3)}$ = 4I+4J+12K

unit normal vector = $\frac{\nabla f}{|\nabla f|} = \frac{4(\bar{i}+\bar{j}+3\bar{k})}{4\sqrt{11}} = \frac{\bar{i}+\bar{j}+3\bar{k}}{\sqrt{11}}$

Example 16: Find the values of a and b so that the surfaces $ax^2-byz = (a+2)x$ and $4x^2y+z^3= 4$ may intersect orthogonally at the point (1, -1,2).

(or) Find the constants a and b so that surface $ax^2-byz=(a+2)x$ will orthogonal to $4x^2y+z^3=4$ at the point (1,-1,2).

Sol:- let the given surfaces be $f(x,y,z) = ax^2-byz - (a+2)x$-------------(1)

And $g(x,y,z) = 4x^2y+z^3- 4$------------(2)

Given the two surfaces meet at the point (1,-1,2).

Substituting the point in (1), we get

a+2b-(a+2) = 0 ⇒ b=1

now $\frac{\partial f}{\partial x} = 2ax-(a+2),\ \frac{\partial f}{\partial y} = -bz =,\ \frac{\partial f}{\partial z} = -by.$

$\nabla f = \sum \bar{i}\,\frac{\partial f}{\partial x}$ = [(2a-(a+2)]i-2bj+bk = (a-2)i-2bj+bk

= (a-2)i-2j+k = $\bar{n}_1$, normal vector to surface 1.

Also $\frac{\partial g}{\partial x} = 8xy,\ \frac{\partial g}{\partial y} = 4x^2,\ \frac{\partial g}{\partial z} = 3z^2.$

∇g = $\sum \bar{i}\,\dfrac{\partial g}{\partial x} = 8xyi+4x^2j+3z^2k$

$(\nabla g)_{(1,-1,2)}$ = -8i+4j+12k = $\bar{n}_2$, normal vector to surface 2.

Given the surfaces f(x,y,z), g(x,y,z) are orthogonal at the point (1,-1,2).

$[\nabla f].[\nabla g] = 0 \Rightarrow$ ((a-2)i-2j+k). (-8i+4j+12k)=0

$\Rightarrow$-81+16-8+12 $\Rightarrow$ a =5/2

Hence a = 5/2 and b=1.

Example17 : Find a unit normal vector to the surface z= x^2+y^2 at (-1,-2,5)

Sol:- let the given surface be f = x^2+y^2-z

$$\frac{\partial f}{\partial x} = 2x,\ \frac{\partial f}{\partial y} = 2y,\ \frac{\partial f}{\partial z} = -1.$$

grad f = $\nabla f = \sum \bar{i}\,\dfrac{\partial f}{\partial x}$ = 2xi+2yj-k

(∇f) at (-1,-2,5)= -2i-4j-k

∇f is the normal vector to the given surface.

Hence the required unit normal vector = $\dfrac{\nabla f}{|\nabla f|}$ =

$$\frac{-2i-4j-k}{\sqrt{(-2)^2+(-4)^2+(-1)^2}} = \frac{-2i-4j-k}{\sqrt{21}} = -\frac{1}{\sqrt{21}}(2i+4j+k)$$

Example 18: Find the angle of intersection of the spheres $x^2+y^2+z^2$ =29 and $x^2+y^2+z^2$ +4x-6y-8z-47 =0 at the point (4,-3,2).

Sol:- Let f = $x^2+y^2+z^2$ -29 and g = $x^2+y^2+z^2$ +4x-6y-8z-47

Then grad f= $\bar{i}\dfrac{\partial f}{\partial x} + \bar{j}\dfrac{\partial f}{\partial y} + \bar{k}\dfrac{\partial f}{\partial z} = 2x\bar{i} + 2y\bar{j} + 2z\bar{k}$ and

grad g = $(2x+4)\bar{i} + (2y-6)\bar{j} + (2z-8)\bar{k}$

The angle between two surfaces at a point is the angle between the normals to the surfaces at that point.

Let $\bar{n}_1$ = (grad *f*) at (4,-3,2) =$8\bar{i} - 6\bar{j} + 4\bar{k}$

$\bar{n}_2$ = (grad *f*) at (4,-3,2) = $12\bar{i} - 12\bar{j} - 4\bar{k}$

The vectors $\bar{n}_1$ and $\bar{n}_2$ are along the normals to the two surfaces at (4,-3,2). Let θ be the angle between the surfaces. Then

$$\text{Cos } \theta = \frac{\bar{n}_1\,\bar{n}_2}{|\bar{n}_1\,\bar{n}_2|} = .\frac{152}{\sqrt{116}\sqrt{304}} \qquad \therefore \theta = \cos^{-1}\left(\sqrt{\frac{19}{29}}\right)$$

Example19 : Find the angle between the surfaces $x^2+y^2+z^2=9$, and $z = x^2+y^2-3$ at point (2,-1,2).

Sol:- Let $\phi_1 = x^2+y^2+z^2-9=0$ and $\phi_2 = x^2+y^2-z-3=0$ be the given surfaces. Then

$\nabla\phi_1 = 2xi+2yj+2zk$ and $\nabla\phi_2 = 2xi+2yj-k$

Let $\bar{n}_1 = \nabla\phi_1$ at(2,-1,2)= 4i-2j+4k and

$\bar{n}_2 = \nabla\phi_2$ at (2,-1,2) = 4i-2j-k

The vectors $\bar{n}_1$ and $\bar{n}_2$ are along the normals to the two surfaces at the point (2,-1,2). Let θ be the angle between the surfaces. Then

$$\text{Cos } \theta = \frac{\bar{n}_1\,\bar{n}_2}{|\bar{n}_1\,\bar{n}_2|} = \frac{(4i-2j+4k)}{\sqrt{16+4+16}}.\frac{(4i-2j-k)}{\sqrt{16+4+16}} = \frac{16+4-4}{6\sqrt{21}} = \frac{16}{6\sqrt{21}} = \frac{8}{3\sqrt{21}}$$

$$\therefore \theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right).$$

Example : If $\nabla\phi = yz\bar{i} + zx\bar{j} + xy\bar{k}$, find φ.

Sol:- we know that $\nabla\phi = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}$

Given that $\nabla\phi = yz\bar{i} + zx\bar{j} + xy\bar{k}$

Comparing the corresponding coefficients, we have $\frac{\partial\phi}{\partial x} = yz, \frac{\partial\phi}{\partial y} = zx, \frac{\partial\phi}{\partial z} = xy$

Integrating partially w.r.t. x,y,z, respectively, we get

φ= xyz + a constant independent of x.

φ= xyz + a constant independent of y.

φ= xyz + a constant independent of z.

Here a possible form of φ is φ= xyz+a constant.

DIVERGENCE OF A VECTOR

Let $\bar{f}$ be any continuously differentiable vector point function. Then $\bar{i}.\frac{\partial\bar{f}}{\partial x} + \bar{j}.\frac{\partial\bar{f}}{\partial y} + \bar{k}.\frac{\partial\bar{f}}{\partial z}$ is called the divergence of $\bar{f}$ and is written as div $\bar{f}$.

i.e div $\bar{f} = \bar{i}.\frac{\partial \bar{f}}{\partial x} + \bar{j}.\frac{\partial \bar{f}}{\partial y} + \bar{k}.\frac{\partial \bar{f}}{\partial z} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right).\bar{f}$

hence we can write div $\bar{f}$ as

div $\bar{f} = \nabla.\bar{f}$

This is a scalar point function.

Theorem 1: If the vector $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$, then div $\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$

Prof: Given $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$

$$\frac{\partial \bar{f}}{\partial x} = \bar{i}\frac{\partial f_1}{\partial x} + \bar{j}\frac{\partial f_2}{\partial x} + \bar{k}\frac{\partial f_3}{\partial x}$$

Also $\bar{i}.\frac{\partial \bar{f}}{\partial x} = \frac{\partial f_1}{\partial x}$. Similarly $\bar{j}.\frac{\partial \bar{f}}{\partial y} = \frac{\partial f_2}{\partial y}$ and $\bar{k}.\frac{\partial \bar{f}}{\partial z} = \frac{\partial f_3}{\partial z}$

We have div $\bar{f} = \sum \bar{i}.\left(\frac{\partial \bar{f}}{\partial x}\right) = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$

Note : If $\bar{f}$ is a constant vector then $\frac{\partial f_1}{\partial x}, \frac{\partial f_2}{\partial y}, \frac{\partial f_3}{\partial z}$ are zeros.

div $\bar{f}$ =0 for a constant vector $\bar{f}$.

Theorem 2: div ($\bar{f} \pm \bar{g}$) = $div\,\bar{f} \pm div\,\bar{g}$

Proof: div ($\bar{f} \pm \bar{g}$) $= \sum \bar{i}.\frac{\partial}{\partial x}(\bar{f}) \pm \sum \bar{i}.\frac{\partial}{\partial x}(\bar{g}) = div\,\bar{f} \pm div\,\bar{g}$.

Note: If ϕ is a scalar function and $\bar{f}$ is a vector function, then

(i). $(\bar{a}.\nabla)\phi = \left[\bar{a}\left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\right]\phi$

$$= \left[(\bar{a}.\bar{i})\frac{\partial}{\partial x} + (\bar{a}.\bar{j})\frac{\partial}{\partial y} + (\bar{a}.\bar{k})\frac{\partial}{\partial z}\right]\phi$$

$$= \left[(\bar{a}.\bar{i})\frac{\partial \phi}{\partial x} + (\bar{a}.\bar{j})\frac{\partial \phi}{\partial y} + (\bar{a}.\bar{k})\frac{\partial \phi}{\partial z}\right]$$

$= \sum(\bar{a}.\bar{i})\frac{\partial \phi}{\partial x}$. and

(ii). $(\bar{a}.\nabla)\bar{f} = \sum(\bar{a}.\bar{i})\frac{\partial \bar{f}}{\partial x}$.by proceeding as in (i) [simply replace ϕ by $\bar{f}$ in (i)].

SOLENOIDAL VECTOR

A vector point function $\bar{f}$ is said to be $\bar{f}$ solenoidal if div $\bar{f}$ =0.

Physical interpretation of divergence:

Depending upon $\bar{f}$ in a physical problem, we can interpret div $\bar{f}$ (= $\nabla.\bar{f}$).

Suppose $\overline{F}$ (x,y,z,t) is the velocity of a fluid at a point(x,y,z) and time 't'. though time has no role in computing divergence, it is considered here because velocity vector depends on time.

Imagine a small rectangular box within the fluid as shown in the figure. We would like to measure the rate per unit volume at which the fluid flows out at any given time. The divergence of $\overline{F}$ measures the outward flow or expansions of the fluid from their point at any time. This gives a physical interpretation of the divergence.

Similar meanings are to be understood with respect to divergence of vectors $\bar{f}$ from other branches. A detailed elementary interpretation can be seen in standard books on fluid dynamics, electricity and magnetism etc.

SOLVED EXAMPLES

Example 1: If $\bar{f} = xy^2\bar{i} + 2x^2yz\bar{j} - 3yz^2\bar{k}$ find div $\bar{f}$ at(1, -1, 1).

Sol:- $\bar{f} = xy^2\bar{i} + 2x^2yz\bar{j} - 3yz^2\bar{k}$. Then

div $\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(2x^2yz) + \frac{\partial}{\partial z}(-3yz^2) = y^2+2x^2z-6yz$

(div $\bar{f}$) at (1, -1, 1) = 1+2+6 =9

Example 2: find div $\bar{f}$ = grad($x^3+y^3+z^3-3xyz$)

Sol:- Let $\phi = x^3+y^3+z^3-3xyz$. Then

$$\frac{\partial\phi}{\partial x} = 3x^2 - 3yz,\ \frac{\partial\phi}{\partial y} = 3y^2 - 3zx, \frac{\partial\phi}{\partial z} = 3z^2 - 3xy$$

$$\text{grad } \phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z} = 3[(x^2 - yz)\bar{i} + (y^2 - zx)\bar{j} + (z^2 - xy)\bar{k}]$$

$$\text{div } \bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = \frac{\partial}{\partial x}[3(x^2 - yz)].\frac{\partial}{\partial y}[3(y^2 - zx)].\frac{\partial}{\partial z}[3(z^2 - xy)]$$

= 3(2x)+3(2y)+3(2z) = 6(x+y+z)

Example 3: If $\bar{f} = (x+3y)\bar{i} + (y-2z)\bar{j} + (x+pz)\bar{k}$ is solenoidal, find P.

Sol:- Let $\bar{f} = (x+3y)\bar{i} + (y-2z)\bar{j} + (x+pz)\bar{k} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$

We have $\frac{\partial f_1}{\partial x} = 1,\ \frac{\partial f_2}{\partial y} = 1, \frac{\partial f_3}{\partial z} = p$

div $\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$ = 1+1+p =2+p

since $\bar{f}$ is solenoidal, we have div $\bar{f}$ = 0 $\rightarrow$ p = -2

Example 4: Find div $\bar{f} = r^n \bar{r}$. Find n if it is solenoidal?

Sol: Given $\bar{f} = r^n \bar{r}$. where $\bar{f} = \bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ *and* $r = |\bar{r}|$

We have $r^2 = x^2+y^2+z^2$

Differentiating partially w.r.t. x , we get

$$2r\,\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r},$$

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$, *and* $\frac{\partial r}{\partial z} = \frac{z}{r}$

$\bar{f} = r^n\,(x\bar{i} + y\bar{j} + z\bar{k})$

div $\bar{f} = \frac{\partial}{\partial x}(r^n x) + \frac{\partial}{\partial y}(r^n y) + \frac{\partial}{\partial z}(r^n z)$

$= nr^{n-1}\frac{\partial r}{\partial x}x + nr^{n-1}\frac{\partial r}{\partial y}y + nr^{n-1}\frac{\partial r}{\partial z}z + r^n$

$= nr^{n-1}\left[\frac{x^2}{r} + \frac{y^2}{r} + \frac{z^2}{r}\right] + 3r^n = nr^{n-1}\frac{(r^2)}{r} + 3r^n = nr^n + 3r^n = (n+3)r^n$

Let $\bar{f} = r^n \bar{r}$. be solenoidal. Then div $\bar{f}$ = 0

$(n+3)r^n = 0$ ➔ n= -3

Example 5: Evaluate $\nabla.\left(\frac{\bar{r}}{r^3}\right)$ where $\bar{r} = xi + yj + zk$ *and* $r = |\bar{r}|$.

Sol:- We have

r = xi+yj+zk and r = $\sqrt{x^2 + y^2 + z^2}$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}, and\ \frac{\partial r}{\partial z} = \frac{z}{r}$$

$\therefore \frac{\bar{r}}{r^3} = \bar{r}.\ r^{-3} = r^{-3}xi + r^{-3}yj + r^{-3}zk = f_1 i + f_2 j + f_3 k$

Hence $\nabla.\left(\frac{\bar{r}}{r^3}\right) = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$

We have $f_1 = r^{-3}x$ ➔ $\frac{\partial f_1}{\partial x} = r^{-3}.1 + x(-3)r^{-4}.\frac{\partial r}{\partial x}$

$$\nabla\left(\frac{\bar{r}}{r^3}\right) = \sum \bar{i}.\frac{\partial f_1}{\partial x} = 3r^{-3} - 3r^{-5}\sum x^2$$

$= 3r^{-3} - 3r^{-5}r^2 = 3r^{-3} - 3r^{-3} = 0$

Example 6: Find div $\bar{r}$. where $\bar{r}. = x\bar{i} + y\bar{j} + z\bar{k}$

Sol:- We have $\bar{r}. = x\bar{i} + y\bar{j} + z\bar{k} = f_1\bar{i} + f_2\,\bar{j} + f_3\,\bar{k}$

$$\text{div } \bar{r}. = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1+1+1=3$$

CURL OF A VECTOR

Def: Let $\bar{f}$ be any continuously differentiable vector point function. Then the vector function defined by $\bar{i}x\frac{\partial \bar{f}}{\partial x} + \bar{j}x\frac{\partial \bar{f}}{\partial y} + \bar{k}x\frac{\partial \bar{f}}{\partial z}$ is called curl of $\bar{f}$ and is denoted by curl $\bar{f}$ or (∇x $\bar{f}$).

$$\text{Curl } \bar{f} = \bar{i}x\frac{\partial \bar{f}}{\partial x} + \bar{j}x\frac{\partial \bar{f}}{\partial y} + \bar{k}x\frac{\partial \bar{f}}{\partial z} = \sum\left(\bar{i}x\frac{\partial \bar{f}}{\partial x}\right)$$

Theorem 1: If $\bar{f}$ is differentiable vector point function given by $\bar{f} = f_1\bar{i} + f_2\,\bar{j} + f_3\,\bar{k}$ then curl $\bar{f} = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\bar{i} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\bar{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\bar{k}$

Proof : curl $\bar{f} = \sum \bar{i}x\frac{\partial}{\partial x}(\bar{f}) = \sum \bar{i}x\frac{\partial}{\partial x}(f_1\bar{i} + f_2\bar{j} + f_3\bar{k}) = \sum\left(\frac{\partial f_2}{\partial x}\bar{k} - \frac{\partial f_3}{\partial x}\bar{j}\right)$

$$= \left(\frac{\partial f_2}{\partial x}\bar{k} - \frac{\partial f_3}{\partial x}\bar{j}\right) + \left(\frac{\partial f_3}{\partial y}\bar{i} - \frac{\partial f_1}{\partial y}\bar{k}\right) + \left(\frac{\partial f_1}{\partial z}\bar{j} - \frac{\partial f_2}{\partial z}\bar{i}\right)$$

$$= \bar{i}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) + \bar{j}\left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right) + \bar{k}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

Note : (1) The above expression for curl $\bar{f}$ can be remembered easily through the representation.

$$\textbf{curl } \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} = \nabla \textbf{x } \bar{f}$$

note : (2) If $\bar{f}$ is a constant vector then curl $\bar{f} = \bar{o}$.

Theorem 2: curl $\left(\bar{a} \pm \bar{b}\right) = curl\,\bar{a} \pm curl\,\bar{b}$

Proof: curl$\left(\bar{a} \pm \bar{b}\right) = \sum \bar{i}x\frac{\partial}{\partial x}\left(\bar{a} \pm \bar{b}\right)$

$$= \sum ix\left(\frac{\partial \bar{a}}{\partial x} \pm \frac{\partial \bar{b}}{\partial x}\right) = \sum ix\frac{\partial \bar{a}}{\partial x} \pm \sum ix\frac{\partial \bar{b}}{\partial x}$$

$$= curl\,\bar{a} \pm curl\,\bar{b}$$

1. Physical Interpretation of curl

If $\overline{w}$ is the angular velocity of a rigid body rotating about a fixed axis and $\bar{v}$ is the velocity of any point P(x,y,z) on the body, then $\overline{w}$ = ½ curl $\bar{v}$. Thus the angular velocity of rotation at any point is equal to half the curl of velocity vector. This justifies the use of the word "curl of a vector".

2. Irrotational Motion, Irrotational Vector

Any motion in which curl of the velocity vector is a null vector i.e curl $\bar{v} = \bar{0}$ is said to be Irrotational.

Def: A vector $\bar{f}$ is said to be Irrotational if curl $\bar{f} = \bar{0}$.

If $\bar{f}$ is Irrotational, there will always exist a scalar function φ(x,y,z) such that $\bar{f}$ =grad ϕ. This is called scalar potential of $\bar{f}$.

It is easy to prove that, if $\bar{f}$ = grad ϕ, then curl $\bar{f}$ = 0.

Hence ∇x $\bar{f}$ = 0 ⇔ there exists a scalar function ϕ such that $\bar{f} = \nabla\phi$.

This idea is useful when we study the "work done by a force" later.

SOLVED EXAMPLES

Example 1: if $\bar{f} = xy^2\bar{i} + 2x^2yz\,\bar{j} - 3yz^2\,\bar{k}$ find curl $\bar{f}$ at the point (1,-1,1).

Sol:- Let $\bar{f} = xy^2\bar{i} + 2x^2yz\,\bar{j} - 3yz^2\,\bar{k}$. Then

$$\text{curl } \bar{f} = \nabla \times \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & 2x^2yz & -3yz^2 \end{vmatrix}$$

$$= \bar{i}\left(\frac{\partial}{\partial y}(-3yz^2) - \frac{\partial}{\partial z}(2x^2yz)\right) + \bar{j}\left(\frac{\partial}{\partial z}(xy^2) - \frac{\partial}{\partial x}(-3yz^2)\right) + \bar{k}\left(\frac{\partial}{\partial x}(2x^2yz) - \frac{\partial}{\partial y}(xy^2)\right)$$

$$= \bar{i}\left(-3z^2 - 2x^2z\right) + \bar{j}(0-0) + \bar{k}(4xyz - 2xy)$$

= curl $\bar{f}$ = at (1,-1,1) = $-\bar{i} - 2\bar{k}$.

Example 2: Find curl $\bar{f}$ where $\bar{f}$ = grad($x^3+y^3+z^3-3xyz$)

Sol:- Let ϕ= $x^3+y^3+z^3-3xyz$. Then

grad ϕ= $\sum \bar{i}\,\frac{\partial \phi}{\partial x} = 3(x^2 - yz)\bar{i} + 3(y^2 - zx)\,\bar{j} + 3(z^2 - xy)\bar{k}$

curl grad ϕ= ∇x grad ϕ= $3\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - yz & y^2 - zx & z^2 - xy \end{vmatrix}$

$$= 3[\bar{i}(-x+x) - \bar{j}(-y+y) + \bar{k}(-z+z)] = \bar{0}$$

curl $\bar{f} = \bar{0}$.

Note: We can prove in general that curl (grad ϕ)=$\bar{0}$.(i.e) grad ϕ is always irrotational.

Example 3: Prove that if $\bar{r}$ is the position vector of an point in space, then $r^n\bar{r}$ is Irrotational. (or) Show that $(r^n\bar{r}) = 0$

Sol:- Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ and r = $|\bar{r}|$ $\therefore$ $r^2 = x^2+y^2+z^2$.

Differentiating partially w.r.t. ‘x’ partially, we get

$$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r},$$

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}, and \frac{\partial r}{\partial z} = \frac{z}{r}$

$r^n = |\bar{r}|$ $r^n(x\bar{i} + y\bar{j} + z\bar{k})$

x($r^n\bar{r}$)= $\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xr^n & yr^n & zr^n \end{vmatrix}$

$$= \bar{i}\left(\frac{\partial}{\partial y}(r^n z) - \frac{\partial}{\partial z}(r^n y)\right) + \bar{j}\left(\frac{\partial}{\partial z}(r^n x) - \frac{\partial}{\partial x}(r^n z)\right) + \bar{k}\left(\frac{\partial}{\partial x}(r^n y) - \frac{\partial}{\partial y}(r^n x)\right)$$

$$= \sum \bar{i}\left\{znr^{n-1}\frac{\partial r}{\partial y} - ynr^{n-1}\frac{\partial r}{\partial z}\right\} = nr^{n-1}\sum \bar{i}\left\{z\left(\frac{y}{r}\right) - z\left(\frac{z}{r}\right)\right\}$$

$$nr^{n-2}[(zy - yz)\bar{i} + (xz - zx)\bar{j} + (xy - yz)\bar{k}]$$

$nr^{n-2}[0\bar{i} + 0j + 0\bar{k}]$ = nr^{n-2} [$\bar{0}$]=$\bar{0}$

Hence $r^n\bar{r}$ is Irrotational.

Example 4: Prove that curl $\bar{r} = \bar{0}$

Sol:- Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

curl $\bar{r} = \sum \bar{i} x \frac{\partial}{\partial x}(\bar{r}) = \sum(\bar{i}x\bar{i}) = \bar{0} + \bar{0} = \bar{0}$

$\bar{r}$ is Irrotational vector.

Example 5: If $\bar{a}$ is a constant vector, prove that curl $\left(\frac{\bar{a}x\bar{r}}{r^3}\right) = -\frac{\bar{a}}{r^3} + \frac{3\bar{r}}{r^5}(\bar{a}.\bar{r})$.

Sol:- We have $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$$\frac{\partial \bar{r}}{\partial x} = \bar{i}\,,\ \frac{\partial \bar{r}}{\partial y} = \bar{j}\,,\ \frac{\partial \bar{r}}{\partial z} = \bar{k}$$

If $|\bar{r}|$ = r then $r^2 = x^2+y^2+z^2$

$$\frac{\partial r}{\partial x} = \frac{x}{r},\ \frac{\partial r}{\partial y} = \frac{y}{r}, and\ \frac{\partial r}{\partial z} = \frac{z}{r}$$

curl $\left(\frac{\bar{a}x\bar{r}}{r^3}\right) = \sum \bar{i}\, x \frac{\partial}{\partial x}\left(\frac{\bar{a}x\bar{r}}{r^3}\right)$

Now $\frac{\partial}{\partial x}\left(\frac{\bar{a}x\bar{r}}{r^3}\right) = \bar{a}x\frac{\partial}{\partial x}\left(\frac{\bar{r}}{r^3}\right) = \bar{a}x\left[\frac{1}{r^3}\frac{\partial \bar{r}}{\partial x} - \frac{3}{r^4}\frac{\partial r}{\partial x}\bar{r}\right]$

$$= \bar{a}x\left[\frac{1}{r^3}\bar{i} - \frac{3}{r^5}x\bar{r}\right] = \frac{\bar{a}x\bar{i}}{r^3} - \frac{3x(\bar{a}.x\bar{r})}{r^5}.$$

$$\therefore\ ix\frac{\partial}{\partial x}\left(\frac{\bar{a}x\bar{r}}{r^3}\right) = \bar{i}x\left[\frac{\bar{a}x\bar{i}}{r^3} - \frac{3x}{r^5}(\bar{a}x\bar{r})\right] = \frac{\bar{i}x(\bar{a}x\bar{i})}{r^3} - \frac{3x}{r^5}\bar{i}x(\bar{a}x\bar{r})$$

$$= \frac{(\bar{i}.\bar{i})\bar{a} - (\bar{i}.\bar{a})i}{r^3} - \frac{3x}{r^5}[(\bar{i}.\bar{r})\bar{a} - (i.a)\bar{r}]$$

Let $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$. Then $\bar{i}\,.\,\bar{a}$ = a_1 , etc.

$$\therefore\ ix\frac{\partial}{\partial x}\left(\frac{\bar{a}x\bar{r}}{r^3}\right) = \sum\frac{(\bar{a} - a_1 i)}{r^3} - \frac{3x}{r^3}(x\bar{a} - a_1\bar{r})$$

$$\therefore\quad \sum ix\frac{\partial}{\partial x}\left(\frac{\bar{a}x\bar{r}}{r^3}\right) = \sum\frac{\bar{a} - a_1\bar{i}}{r^3} - \frac{3}{r^5}\sum(x^2\bar{a} - a_1x\bar{r})$$

$$= \frac{3\bar{a} - \bar{a}}{r^3} - \frac{3\bar{a}}{r^5}(r^2) + \frac{3\bar{r}}{r^5}(a_1x + a_2y + a_3z)$$

$$= \frac{2\bar{a}}{r^3} - \frac{3\bar{a}}{r^3} + \frac{3\bar{r}}{r^5}(\bar{r}.\bar{a}) = -\frac{\bar{a}}{r^3} + \frac{3\bar{r}}{r^5}(\bar{r}.\bar{a})$$

Example 6: Show that the vector $(x^2 - yz)\bar{i} + (y^2 - zx)\,\bar{j} + (z^2 - xy)\,\bar{k}$ is irrotational and find its scalar potential.

Sol: let $\bar{f} = (x^2 - yz)\bar{i} + (y^2 - zx)\,\bar{j} + (z^2 - xy)\,\bar{k}$

$$\text{Then curl } \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - yz & y^2 - zx & z^2 - xy \end{vmatrix} = \sum i(-x + x) = \bar{0}$$

$\bar{f}$ is Irrotational. Then there exists ϕ such that $\bar{f} = \nabla\phi$.

$\Rightarrow \bar{i}\frac{\partial\phi}{\partial x}+\bar{j}\frac{\partial\phi}{\partial y}+\bar{k}\frac{\partial\phi}{\partial z}=(x^2-yz)\bar{i}+(y^2-zx)\,\bar{j}+(z^2-xy)\,\bar{k}$

Comparing components, we get

$\frac{\partial\phi}{\partial x}=x^2-yz\Rightarrow\phi=\int\left(x^2-yz\right)dx=\frac{x^3}{3}-xyz+f_1(y,z)......(1)$

$\frac{\partial\phi}{\partial y}=y^2-zx\Rightarrow\phi=\frac{y^3}{3}-xyz+f_2(z,x)......(2)$

$\frac{\partial\phi}{\partial z}=z^2-xy\Rightarrow\phi=\frac{z^3}{3}-xyz+f_3(x,y)......(3)$

From (1), (2),(3), $\phi=\frac{x^3+y^3+z^3}{3}-xyz$

$\therefore\quad \phi=\frac{1}{3}(x^3+y^3+z^3)-xyz+cons\tan t$

Which is the required scalar potential.

Example 7: Find constants a,b and c if the vector $\bar{f}=(2x+3y+az)\bar{i}+(bx+2y+3z)\,\bar{j}+(2x+cy+3z)\,\bar{k}$ is Irrotational.

Sol:- Given $\bar{f}=(2x+3y+az)\bar{i}+(bx+2y+3z)\,\bar{j}+(2x+cy+3z)\,\bar{k}$

$$\text{Curl }\bar{f}=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 2x+3y+az & bx+2y+3z & 2x+cy+3z\end{vmatrix}=$$

$(c-3)\bar{i}+(2-a)\,\bar{j}+(b-3)\,\bar{k}$

If the vector is Irrotational then curl $\bar{f}=\bar{0}$

c-3 = 2-a=0, b-3 = 0 $\Rightarrow$c=3, a=2, b=3.

Example 8: If f(r) is differentiable, show that curl { $\bar{r}$ f(r)} = $\bar{0}$ where $\bar{r}=x\bar{i}+y\bar{j}+z\bar{k}$.

Sol: r = $\bar{r}=\sqrt{x^2+y^2+z^2}$ $r^2=x^2+y^2+z^2$

$\Rightarrow 2r\frac{\partial r}{\partial x}=2x\Rightarrow\frac{\partial r}{\partial x}=\frac{x}{r}$, similarly $\frac{\partial r}{\partial y}=\frac{y}{r}, and\ \frac{\partial r}{\partial z}=\frac{z}{r}$

curl{ $\bar{r}$ f(r)}= curl{f(r)($x\bar{i}+y\bar{j}+z\bar{k}$)}= curl $(x.f(r)\bar{i}+y.f(r)\,\bar{j}+z.f(r)\,\bar{k}$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xf(r) & yf(r) & zf(r) \end{vmatrix} = \sum \bar{i}\left[\frac{\partial}{\partial y}[zf(r)] - \frac{\partial}{\partial z}[yf(r)]\right]$$

$$\sum \bar{i}\left[zf^{1}(r)\frac{\partial r}{\partial y} - yf^{1}(r)\frac{\partial r}{\partial z}\right] = \sum \bar{i}\left[zf^{1}(r)\frac{y}{r} - yf^{1}(r)\frac{z}{r}\right]$$

$= \bar{0}$.

Example 9: If $\bar{A}$ is Irrotational vector, evaluate div($\bar{A}$ x $\bar{r}$) where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$.

Sol: we have $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

Given $\bar{A}$ is an irrational vector

∇x$\bar{A} = \bar{0}$

div ($\bar{A}$ x $\bar{r}$) = ∇.($\bar{A}$ x $\bar{r}$)

= $\bar{r}$.(∇x$\bar{A}$)- $\bar{A}$.(∇x$\bar{r}$)

= $\bar{r}$.($\bar{0}$)- $\bar{A}$.(∇x$\bar{r}$) [using (1)]

= - $\bar{A}$.(∇x$\bar{r}$)…..(2)

Now $\nabla \text{x} \bar{r} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \bar{i}\left(\frac{\partial}{\partial y}z - \frac{\partial}{\partial z}y\right) - \bar{j}\left(\frac{\partial}{\partial x}z - \frac{\partial}{\partial z}x\right) + \bar{k}\left(\frac{\partial}{\partial x}y - \frac{\partial}{\partial y}x\right) = \bar{0}$

$\bar{A}$.(∇x$\bar{r}$)=0 …(3)

Hence div ($\bar{A}$ x $\bar{r}$)=0. [using (2) and (3)]

Example 10: Find constants a,b,c so that the vector $\bar{A} =$ $(x+2y+az)\bar{i} + (bx-3y-z)\,\bar{j} + (4x+cy+2z)\,\bar{k}$ is Irrotational. Also find ϕ such that $\bar{A} = \nabla\phi$.

Sol: Given vector is $\bar{A} = (x+2y+az)\bar{i} + (bx-3y-z)\,\bar{j} + (4x+cy+2z)\,\bar{k}$

Vector $\bar{A}$ is Irrotational $\Rightarrow$ curl $\bar{A} = \bar{0}$

$$\Rightarrow \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix} = \bar{0}$$

$\Rightarrow (c+1)\bar{i} + (a-4)\,\bar{j} + (b-2)\,\bar{k} = \bar{0}$

$\Rightarrow (c+1)\bar{i} + (a-4)\,\bar{j} + (b-2)\,\bar{k} = 0\bar{i} + 0\,\bar{j} + 0\,\bar{k}$

Comparing both sides,

c+1=0, a-4=0, b-2=0

c= -1, a=4,b=2

now $\bar{A} = (x+2y+az)\bar{i} + (bx-3y-z)\,\bar{j} + (4x+cy+2z)\,\bar{k}$, on substituting the values of a,b,c

we have $\bar{A} = \nabla\phi$.

$\Rightarrow \bar{A} = (x+2y+az)\bar{i} + (bx-3y-z)\,\bar{j} + (4x+cy+2z)\,\bar{k} = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$

Comparing both sides, we have

$\frac{\partial\phi}{\partial x}$ =x+2y+4z $\Rightarrow\phi$= $x^2/2+2xy+4zx+f_1(y,z)$

$\frac{\partial\phi}{\partial y}$ =2x-3y-z $\Rightarrow\phi$= 2xy-3y2/2-yz+f_2(z,x)

$\frac{\partial\phi}{\partial z}$ =4x-y+2z $\Rightarrow\phi$= 4xz-yz+z2+f_3(x,y)

Hence $\phi = x^2/2 - 3y^2/2 + z^2 + 2xy + 4zx - yz + c$

Example 11: If ω is a constant vector, evaluate curl V where V = ωx$\bar{r}$.

Sol: curl (ωx$\bar{r}$) = $\sum \bar{i}x\frac{\partial}{\partial x}(\varpi x\bar{r}) = \sum \bar{i}x\left[\frac{\partial\varpi}{\partial x}x\bar{r} + \varpi x\frac{\partial\bar{r}}{\partial x}\right]$

$$= \sum \bar{i}x[\bar{0} + \omega x\bar{i}] \quad [\therefore \bar{a}x(\bar{b}x\bar{c}) = (\bar{a}.\bar{c})\bar{b} - (\bar{a}.\bar{b}).\bar{c}]$$

$$= \sum \bar{i}x(\omega x\bar{i}) = \sum[(\bar{i}.\bar{i})\omega - (\bar{i}.\omega)\bar{i}] = \sum\omega - \sum(\bar{i}.\omega)\bar{i} = 3\omega - \omega = 2\omega$$

OPERATORS

Vector differential operator ∇

The operator $\nabla = \bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}$ is defined such that $\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$ where ϕ is a scalar point function.

Note: If ϕ is a scalar point function then $\nabla\phi$= grad ϕ= $\sum i\frac{\partial\phi}{\partial x}$

(2) Scalar differential operator $\bar{a}.\nabla$

The operator $\bar{a}.\nabla = (\bar{a}.\bar{i})\frac{\partial\phi}{\partial x} + (\bar{a}.\bar{j})\frac{\partial\phi}{\partial y} + (\bar{a}.\bar{k})\frac{\partial\phi}{\partial z}$ is defined such that

$(\bar{a}.\nabla)\phi = (\bar{a}.\bar{i})\dfrac{\partial\phi}{\partial x} + (\bar{a}.\bar{j})\dfrac{\partial\phi}{\partial y} + (\bar{a}.\bar{k})\dfrac{\partial\phi}{\partial z}$

And $(\bar{a}.\nabla)\,\bar{f} = (\bar{a}.\bar{i})\dfrac{\partial\bar{f}}{\partial x} + (\bar{a}.\bar{j})\dfrac{\partial\bar{f}}{\partial y} + (\bar{a}.\bar{k})\dfrac{\partial\bar{f}}{\partial z}$

(3). Vector differential operator $\bar{a}\,x\nabla$

The operator $\bar{a}\,x\nabla = (\bar{a}x\bar{i})\dfrac{\partial}{\partial x} + (\bar{a}x\bar{j})\dfrac{\partial}{\partial y} + (\bar{a}x\bar{k})\dfrac{\partial}{\partial z}$ is defined such that

(i). $(\bar{a}\,x\nabla)\phi = (\bar{a}x\bar{i})\dfrac{\partial\phi}{\partial x} + (\bar{a}x\bar{j})\dfrac{\partial\phi}{\partial y} + (\bar{a}x\bar{k})\dfrac{\partial\phi}{\partial z}$

(ii). $(\bar{a}\,x\nabla).\,\bar{f} = (\bar{a}x\bar{i}).\dfrac{\partial\bar{f}}{\partial x} + (\bar{a}x\bar{j}).\dfrac{\partial\bar{f}}{\partial y} + (\bar{a}x\bar{k}).\dfrac{\partial\bar{f}}{\partial z}$

(iii). $(\bar{a}\,x\nabla)x\,\bar{f} = (\bar{a}x\bar{i})x\dfrac{\partial\bar{f}}{\partial x} + (\bar{a}x\bar{j})x\dfrac{\partial\bar{f}}{\partial y} + (\bar{a}x\bar{k})x\dfrac{\partial\bar{f}}{\partial z}$

(4). Scalar differential operator $\nabla.$

The operator $\nabla = \bar{i}.\dfrac{\partial}{\partial x} + \bar{j}.\dfrac{\partial}{\partial y} + \bar{k}.\dfrac{\partial}{\partial z}$ is defined such that $\nabla.\,\bar{f} = \bar{i}.\dfrac{\partial\bar{f}}{\partial x} + \bar{j}.\dfrac{\partial\bar{f}}{\partial y} + \bar{k}.\dfrac{\partial\bar{f}}{\partial z}$

Note: $\nabla.\,\bar{f}$ is defined as div $\bar{f}$ it is a scalar point function.

(5). Vector differential operator ∇ x

The operator ∇ x = $\bar{i}x\dfrac{\partial}{\partial x} + \bar{j}x\dfrac{\partial}{\partial y} + \bar{k}x\dfrac{\partial}{\partial z}$ is defined such that

$$\nabla x\,\bar{f} = \bar{i}x\frac{\partial\bar{f}}{\partial x} + \bar{j}x\frac{\partial\bar{f}}{\partial y} + \bar{k}x\frac{\partial\bar{f}}{\partial z}$$

Note : $\nabla x\,\bar{f}$ is defined as curl $\bar{f}$. It is a vector point function.

(6). Laplacian Operator ∇^2

$$\nabla.\nabla\phi = \sum \bar{i}.\frac{\partial}{\partial x}\left(\bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}\right) = \sum\frac{\partial^2\phi}{\partial x^2} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi = \nabla^2\phi$$

Thus the operator $\nabla^2 \equiv \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$ is called Laplacian operator.

Note : (i). $\nabla^2\phi = \nabla.(\nabla\phi)$ = div(grad ϕ)

(ii). if $\nabla^2\phi = 0$ then ϕ is said to satisfy Laplacian equation. This ϕ is called a harmonic function.

VECTOR IDENTITIES

Theorem 1: If $\bar{a}$ is a differentiable function and ϕ is a differentiable scalar function. Then prove that div($\phi\,\bar{a}$)= (grad ϕ). $\bar{a}$ +ϕ div $\bar{a}$ or $\nabla.(\phi\,\bar{a}) = (\nabla\phi)\,\bar{a} + \phi(\nabla.\,\bar{a})$

Proof: div($\phi\,\bar{a}$)=$\nabla.(\phi\,\bar{a}) = \sum i\dfrac{\partial}{\partial x}(\phi\bar{a})$

$$=\sum \bar{i}.\left(\frac{\partial \phi}{\partial x}\bar{a}+\phi\frac{\partial \bar{a}}{\partial x}\right)=\sum\left(i\frac{\partial \phi}{\partial x}\bar{a}\right)+\sum\left(i\frac{\partial \bar{a}}{\partial x}\right)\phi$$

$$=\sum\left(\bar{i}\frac{\partial \phi}{\partial x}\right).\bar{a}+\left(\sum \bar{i}\frac{\partial a}{\partial x}\right)\phi=(\nabla\phi)\,\bar{a}+\phi(\nabla.\bar{a})$$

Theorem 2: prove that curl ($\phi\,\bar{a}$)= (grad ϕ)x$\bar{a}$ +ϕ curl $\bar{a}$

Proof : curl ($\phi\,\bar{a}$)=∇x($\phi\,\bar{a}$)=$\sum ix\frac{\partial}{\partial x}(\phi\bar{a})$

$$=\sum \bar{i}x\left(\frac{\partial \phi}{\partial x}\bar{a}+\phi\frac{\partial \bar{a}}{\partial x}\right)=\sum\left(i\frac{\partial \phi}{\partial x}\right)x\bar{a}+\sum\left(ix\frac{\partial \bar{a}}{\partial x}\right)\phi$$

$$=\nabla\phi x\,\bar{a}+(\nabla x\,\bar{a})\phi=(\text{grad }\phi)x\,\bar{a}+\phi\text{ curl }\bar{a}$$

Theorem 3: Prove that grad ($\bar{a}\,.\,\bar{b}$)= $(\bar{b}.\nabla)\bar{a}+(\bar{a}.\nabla)\bar{b}+\bar{b}x\,curl\,\bar{a}+\bar{a}x\,curl\,\bar{b}$

Proof: Consider

$$\bar{a}x\,curl(\bar{b})=\bar{a}x(\nabla x\bar{b})=\bar{a}x\sum \bar{i}x\left(\bar{i}x\frac{\partial \bar{b}}{\partial x}\right)$$

$$=\sum \bar{a}x\left(\bar{i}x\frac{\partial \bar{b}}{\partial x}\right)$$

$$=\sum\left\{\left(\bar{a}.\frac{\partial \bar{b}}{\partial x}\right)\bar{i}-(\bar{a}.\bar{i})\frac{\partial \bar{b}}{\partial x}\right\}=\sum \bar{i}\left\{\bar{a}.\frac{\partial \bar{b}}{\partial x}\right\}-\left\{\bar{a}.\sum i\frac{\partial}{\partial x}\right\}\bar{b}$$

$$\therefore \bar{a}x\,curl\,\bar{b}=\sum \bar{i}\left(\bar{a}.\frac{\partial \bar{b}}{\partial x}\right)-(\bar{a}.\nabla)\bar{b}\;\ldots.(1)$$

Similarly , $\bar{b}x\,curl\,\bar{b}=\sum \bar{i}\left(\bar{a}.\frac{\partial \bar{a}}{\partial x}\right)-(\bar{b}.\nabla)\bar{a}\;\ldots.(2)$

(1)+(2) gives

$$\bar{a}x\,curl\,\bar{b}+\bar{b}xcurl\,\bar{a}=\sum \bar{i}\left(\bar{a}.\frac{\partial \bar{b}}{\partial x}\right)-(\bar{a}.\nabla)\bar{b}+\sum \bar{i}\left(\bar{b}.\frac{\partial \bar{a}}{\partial x}\right)-(\bar{b}.\nabla)\bar{a}$$

$$\bar{a}x\,curl\,\bar{b}+\bar{b}xcurl\,\bar{a}+(\bar{a}.\nabla)\bar{b}+(\bar{b}.\nabla)\bar{a}=\sum \bar{i}\left(\bar{a}.\frac{\partial \bar{b}}{\partial x}+\bar{b}.\frac{\partial \bar{a}}{\partial x}\right)$$

$$=\sum \bar{i}\frac{\partial}{\partial x}(\bar{a}.\bar{b})$$

$$= \nabla(\bar{a}\,.\,\bar{b}) = \text{grad}\,(\bar{a}\,.\,\bar{b})$$

Theorem 4: Prove that div ($\bar{a} \times \bar{b}$)= $\bar{b}.curl\,\bar{a} - \bar{a}.curl\,\bar{b}$

Proof: div ($\bar{a} \times \bar{b}$) $= \sum \bar{i}\,\frac{\partial}{\partial x}(\bar{a}x\bar{b}) = \sum \bar{i}.\left(\frac{\partial \bar{a}}{\partial x}x\bar{b} + \bar{a}x\frac{\partial \bar{b}}{\partial x}\right)$

$$= \sum \bar{i}.\left(\frac{\partial \bar{a}}{\partial x}x\bar{b}\right) + \sum \bar{i}.\left(\bar{a}x\frac{\partial \bar{b}}{\partial x}\right) = \sum\left(\bar{i}x\frac{\partial \bar{a}}{\partial x}\right).\bar{b} - \sum\left(\bar{i}x\frac{\partial \bar{b}}{\partial x}\right).\bar{a}$$

$$= (\nabla x\bar{a}).\bar{b} - (\nabla x\bar{b}).\bar{a} = \bar{b}.curl\,\bar{a} - \bar{a}.curl\,\bar{b}$$

Theorem 5 : $curl\,(\bar{a}x\bar{b}) = \bar{a}\,div\,\bar{b} - \bar{b}\,div\,\bar{a} + (\bar{b}.\nabla)\bar{a} - (\bar{a}.\nabla)\bar{b}$

$Proof\,:\, curl\,(\bar{a}x\bar{b}) = \sum \bar{i}x\frac{\partial}{\partial x}(\bar{a}x\bar{b}) = \sum \bar{i}x\left[\frac{\partial \bar{a}}{\partial x}x\bar{b} + \bar{a}x\frac{\partial \bar{b}}{\partial x}\right]$

$$\sum \bar{i}x\left(\frac{\partial \bar{a}}{\partial x}x\bar{b}\right) + \sum \bar{i}x\left(\bar{a}x\frac{\partial \bar{b}}{\partial x}\right)$$

$$= \sum\left\{(\bar{i}.\bar{b})\frac{\partial \bar{a}}{\partial x} - \left(\bar{i}.\frac{\partial \bar{a}}{\partial x}\right)\bar{b}\right\} + \sum\left\{\left(\bar{i}.\frac{\partial \bar{b}}{\partial x}\right)\bar{a} - (\bar{i}.\bar{a})\frac{\partial \bar{b}}{\partial x} - \right\} +$$

$$= \sum(\bar{b}.\bar{i})\frac{\partial \bar{a}}{\partial x} - \sum\left(\bar{i}.\frac{\partial \bar{a}}{\partial x}\right)\bar{b} + \sum\left(\bar{i}.\frac{\partial \bar{b}}{\partial x}\right)\bar{a} - \left(.\bar{a}\sum \bar{i}.\frac{\partial}{\partial x}\right)\bar{b}$$

$$= (\bar{b}.\nabla)\bar{a} - (\bar{a}.\nabla)\bar{b} + (\nabla.\bar{b})\bar{a} - (\bar{a}.\nabla)\bar{b}$$

$$= (\nabla.\bar{b})\bar{a} - (\nabla.\bar{a})\bar{b} + (\bar{b}.\nabla)\bar{a} - (\bar{a}.\nabla)\bar{b}$$

$$= \bar{a}\,div\,\bar{b} - \bar{b}\,div\,\bar{a} + (\bar{b}.\nabla)\bar{a} - (\bar{a}.\nabla)\bar{b}$$

Theorem 6: Prove that curl grad $\phi = 0$.

Proof: Let ϕ be any scalar point function. Then

$$.grad\ \phi = \bar{i}\frac{\partial \phi}{\partial x} + \bar{j}\frac{\partial \phi}{\partial y} + \bar{k}\frac{\partial \phi}{\partial z}$$

$$curl\ (\text{grad}\ \Phi) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ \frac{d\phi}{dx} & \frac{d\phi}{dy} & \frac{d\phi}{dz} \end{vmatrix}$$

$$\bar{i}\left(\frac{\partial^2\phi}{\partial y\partial z}-\frac{\partial^2\phi}{\partial z\partial y}\right)-\bar{j}\left(\frac{\partial^2\phi}{\partial x\partial z}-\frac{\partial^2\phi}{\partial z\partial x}\right)-\bar{k}\left(\frac{\partial^2\phi}{\partial x\partial y}-\frac{\partial^2\phi}{\partial y\partial x}\right)=\bar{0}$$

note: Since crul (grad ϕ)= 0, we have grad ϕ is always Irrotational.

Theorem 7: Prove that div crul *f=0*

$\text{Proof} : Let\ \bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$

$$\therefore curl\ \bar{f}\ \nabla x\bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$=\left(\frac{\partial f_3}{\partial y}-\frac{\partial f_2}{\partial z}\right)\bar{i}-\left(\frac{\partial f_3}{\partial x}-\frac{\partial f_2}{\partial z}\right)\bar{j}+\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial y}\right)\bar{k}$$

$$\therefore \quad div\ \ curl\ \bar{f} = \nabla.(\nabla x\bar{f}) = \frac{\partial}{\partial x}\left(\frac{\partial f_3}{\partial y}-\frac{\partial f_2}{\partial z}\right)-\frac{\partial}{\partial y}\left(\frac{\partial f_3}{\partial x}-\frac{\partial f_2}{\partial z}\right)+\frac{\partial}{\partial z}\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial y}\right)$$

$$=\frac{\partial^2 f_3}{\partial x\partial y}-\frac{\partial^2 f_2}{\partial x\partial z}-\frac{\partial^2 f_3}{\partial y\partial x}+\frac{\partial^2 f_1}{\partial y\partial z}+\frac{\partial^2 f_2}{\partial z\partial x}-\frac{\partial^2 f_1}{\partial z\partial y}=0$$

Theorem 8: If *f* and g are two scalar point functions, prove that *div*($f\nabla$g)= $f\nabla^2$g+∇f. ∇g

Sol: Let *f* and g are two scalar point functions. Then

$$\nabla g = \bar{i}\,\frac{\partial g}{\partial x}+\bar{j}\,\frac{\partial g}{\partial y}+\bar{k}\,\frac{\partial g}{\partial z}$$

Now $f\nabla\text{g} = \bar{i}f\,\frac{\partial g}{\partial x}+\bar{j}f\,\frac{\partial g}{\partial y}+\bar{k}f\,\frac{\partial g}{\partial z}$

$$\nabla.(f\nabla\text{g}) = \frac{\partial}{\partial x}\left(f\,\frac{\partial g}{\partial x}\right)+\frac{\partial}{\partial y}\left(f\,\frac{\partial g}{\partial y}\right)+\frac{\partial}{\partial z}\left(f\,\frac{\partial g}{\partial z}\right)$$

$$= f\left(\frac{\partial^2 g}{\partial x^2}+\frac{\partial^2 g}{\partial y^2}+\frac{\partial^2 g}{\partial x^2}\right)+\left(\frac{\partial f}{\partial x}.\frac{\partial g}{\partial x}+\frac{\partial f}{\partial y}.\frac{\partial g}{\partial y}+\frac{\partial f}{\partial z}.\frac{\partial g}{\partial z}\right)$$

$$=f\nabla^2 g+\left(\bar{i}\frac{\partial f}{\partial x}+\bar{j}\frac{\partial f}{\partial y}+\bar{k}\frac{\partial f}{\partial z}\right).\left(\bar{i}\frac{\partial g}{\partial x}+\bar{j}\frac{\partial g}{\partial y}+\bar{k}\frac{\partial g}{\partial z}\right).$$

$$= f\nabla^2 g+\nabla f.\nabla g$$

Theorem 9: Prove that $\nabla x(\nabla x\,\bar{a}) = \nabla(\nabla.\bar{a})-\nabla^2\bar{a}$.

Proof: $\nabla x(\nabla x\,\bar{a}) = \sum \bar{i}\frac{\partial}{\partial x}(\nabla x\bar{a})$

Now $\bar{i}x\frac{\partial}{\partial x}(\nabla x\bar{a}) = \bar{i}x\frac{\partial}{\partial x}\left(\bar{i}x\frac{\partial\bar{a}}{\partial x}+\bar{j}x\frac{\partial\bar{a}}{\partial y}+\bar{k}x\frac{\partial\bar{a}}{\partial z}\right)$

$$=\bar{i}x\left(\bar{i}x\frac{\partial^2\bar{a}}{\partial x^2}+\bar{j}x\frac{\partial^2\bar{a}}{\partial y^2}+\bar{k}x\frac{\partial^2\bar{a}}{\partial z^2}\right)$$

$$=\bar{i}x\left(\bar{i}x\frac{\partial^2\bar{a}}{\partial x^2}\right)+\bar{i}x\left(\bar{j}x\frac{\partial^2\bar{a}}{\partial x\partial y}\right)+\bar{i}x\left(\bar{k}x\frac{\partial^2\bar{a}}{\partial x\partial z}\right)$$

$$=\left(\bar{i}.\frac{\partial^2\bar{a}}{\partial x^2}\right)\bar{i}-\frac{\partial^2\bar{a}}{\partial x^2}+\left(\bar{i}.\frac{\partial^2\bar{a}}{\partial x\partial y}\right)\bar{j}+\left(\bar{i}.\frac{\partial^2\bar{a}}{\partial x\partial z}\right)\bar{k} \qquad [\because i.i=1, i.j=i.k=0]$$

$$= \bar{i}\frac{\partial}{\partial x}\left(\bar{i}.\frac{\partial\bar{a}}{\partial x}\right)+j\frac{\partial}{\partial y}\left(\bar{i}.\frac{\partial\bar{a}}{\partial y}\right)+k\frac{\partial}{\partial z}\left(\bar{i}.\frac{\partial\bar{a}}{\partial z}\right)-\frac{\partial^2\bar{a}}{\partial x^2}=\nabla\left(\bar{i}.\frac{\partial\bar{a}}{\partial x}\right)-\frac{\partial^2\bar{a}}{\partial x^2}$$

$$=\sum\bar{i}x\frac{\partial}{\partial x}(\nabla x\bar{a})=\sum\bar{i}.\frac{\partial\bar{a}}{\partial x}-\sum\frac{\partial^2\bar{a}}{\partial x^2}=\nabla(\nabla.\bar{a})-\left(\frac{\partial^2\bar{a}}{\partial x^2}+\frac{\partial^2\bar{a}}{\partial y^2}+\frac{\partial^2\bar{a}}{\partial z^2}\right)$$

$$=\nabla x(\nabla x\,\bar{a}) = \nabla(\nabla.\bar{a})-\nabla^2\bar{a}$$

Assignment

Maths unit - 4

* Vector defination

① A particle moves along the curve $\bar{r} = (t^3 - 4t)\hat{i} + (t^2 + 4t)\hat{j} + (8t^2 - 3t^3)k$. where t denotes the time. Find the magnitude of velocity and acceleration at $t = 2$.

sol :-
The position vector of particle is given by,

$$\bar{r} = (t^3 - 4t)\hat{i} + (t^2 + 4t)\hat{j} + (8t^2 - 3t^3)k$$

velocity, $\bar{v} = \frac{d\bar{r}}{dt}$

$$\bar{v} = (3t^2 - 4)\hat{i} + (2t + 4)\hat{j} + (16t - 9t^2)k$$

$$\bar{v}\Big|_{t=2} = 8\hat{i} + 8\hat{j} - 4k$$

$$|\bar{v}| = \sqrt{64 + 64 + 16} = \sqrt{144} = \pm 12 = 12$$

∴ magnitude of velocity is 12 units/sec.

acceleration, $\bar{a} = \frac{d^2\bar{r}}{dt^2} = \frac{d}{dt}(\bar{v})$

$$= 6t\hat{i} + 2\hat{j} + (16 - 18t)k$$

$$\therefore \bar{a}\Big|_{t=2} = 12\hat{i} + 2\hat{j} - 20k$$

$$|\bar{a}| = \sqrt{144 + 4 + 400} = \sqrt{548} = 2\sqrt{137}$$

∴ magnitude of acceleration is $2\sqrt{137}$ units/sec^2.

(2) Find the directional derivative of $f(x,y,z) = xy^2 + yz^3$ at the point $(2,-1,1)$ in the directional vector $\hat{i} + 2\hat{j} + 2\hat{k}$.

Sol :- let, $\phi = f(x,y,z) = f = xy^2 + yz^3$

$$\nabla\phi = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$$

$$= y^2\hat{i} + (2xy + z^3)\hat{j} + 3yz^2 k$$

$\therefore$ $\nabla\phi$ at the point $(2,-1,1)$ is

$$\nabla\phi\Big|_{(2,-1,1)} = \hat{i} - 3\hat{j} - 3k$$

let $\bar{e}$ be the unit vector in the direction of $\hat{i} + 2\hat{j} + 2k$

then $\bar{e} = \frac{1}{3}(\hat{i} + 2j + 2k)$

$\therefore$ The D.D. of f in the direction of $\bar{e}$

$$= \nabla\phi \cdot \bar{a}$$

$$= (i - 3j - 3k) \cdot (i + 2j + 2k)\frac{1}{3}$$

$$= (1 - 6 - 6)\frac{1}{3} = -11/3$$

(3) Find the angle b/w the surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the point $(2,-1,2)$.

Let, $\phi_1 = x^2 + y^2 + z^2 - 9$

$\phi_2 = x^2 + y^2 - z - 3$

Angle b/w the surfaces ϕ_1 and ϕ_2 is the angle b/w the normals $\bar{n}_1$ and $\bar{n}_2$ to the surfaces at the point $(2,-1,2)$.

$$\bar{n}_1 = \nabla\phi_1 = i\frac{\partial\phi_1}{\partial x} + j\frac{\partial\phi_2}{\partial y} + k\frac{\partial\phi_3}{\partial z}$$

$$= 2xi + 2yj + 2zk$$

$$\nabla\phi_1\Big|_{(2,-1,2)} = 4i - 2j + 4k$$

$$n_2 = \nabla\phi_2 = 2xi + 2yj - k$$

$$\nabla\phi_2\Big|_{(2,-1,2)} = 4i - 2j - k$$

Angle b/w the normals is,

$$\cos\theta = \frac{\bar{n}_1 \cdot \bar{n}_2}{|\bar{n}_1||\bar{n}_2|}$$

$$= \frac{(4i - 2j + 4k)\cdot(4i - 2j - k)}{\sqrt{16+4+16}\,\sqrt{16+4+1}}$$

$$= \frac{16+4-4}{\sqrt{36}\,\sqrt{21}}$$

$$= \frac{16}{6\sqrt{21}} = \frac{8}{3\sqrt{21}}$$

$$\theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$$

(4) P.T. $\nabla \times \left(\frac{\bar{a}\times\bar{r}}{r^n}\right) = \left(\frac{2-n}{r^n}\right)\bar{a} + \frac{n(\bar{a}\cdot\bar{r})}{r^{n+2}}\bar{r}$

where $\bar{a}$ is a constant vector.

Sol. $\rightarrow$ let, $\bar{a} = a_1 i + a_2 j + a_3 k$

$\bar{r} = xi + yj + zk$

$\therefore |\bar{r}| = \sqrt{x^2+y^2+z^2}$

$r^2 = x^2+y^2+z^2 \longrightarrow (1)$

$1 = \frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2} \longrightarrow (2)$

$\therefore \frac{\partial x}{\partial r} = \frac{r}{x}, \frac{\partial y}{\partial r} = \frac{r}{y}, \frac{\partial z}{\partial r} = \frac{r}{z}$

consider $= \bar{a}\times\bar{r}$

$$= \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix}$$

$$= i(a_2 z - a_3 y) - j(a_1 z - a_3 x) + k(a_1 y - a_2 x)$$

$$\therefore \nabla \times \left(\frac{\bar{a}\times\bar{r}}{r^n}\right) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{a_2 z - a_3 y}{r^n} & \frac{a_3 x - a_1 z}{r^n} & \frac{a_1 y - a_2 x}{r^n} \end{vmatrix}$$

$$= i\left(\frac{\partial}{\partial y}\left(\frac{a_1 y - a_2 x}{r^n}\right) - \frac{\partial}{\partial z}\left(\frac{a_3 x - a_1 z}{r^n}\right)\right)$$

$$- j\left(\frac{\partial}{\partial x}\left(\frac{a_1 y - a_2 x}{r^n}\right) - \frac{\partial}{\partial z}\left(\frac{a_2 z - a_3 y}{r^n}\right)\right)$$

$$+ k\left(\frac{\partial}{\partial x}\left(\frac{a_3 x - a_1 z}{r^n}\right) - \frac{\partial}{\partial y}\left(\frac{a_2 z - a_3 y}{r^n}\right)\right)$$

$$= i\left(\frac{a_1 r^n - n r^{n-1}(a_1 y - a_2 x)\frac{\partial r}{\partial y}}{r^{2n}} - \frac{-a_1 r^n - n r^{n-1}(a_3 x - a_1 z)\frac{\partial r}{\partial z}}{r^{2n}}\right)$$

$$- j\left(\frac{-a_2 r^n - n r^{n-1}(a_1 y - a_2 x)\frac{\partial r}{\partial x}}{r^{2n}} - \frac{a_2 r^n - n r^{n-1}(a_2 z - a_3 y)\frac{\partial r}{\partial z}}{r^{2n}}\right)$$

$$+ k\left(\frac{a_3 r^n - n r^{n-1}(a_3 x - a_2 z)\frac{\partial r}{\partial x}}{r^{2n}} - \frac{-a_3 r^n - n r^{n-1}(a_2 z - a_3 y)\frac{\partial r}{\partial y}}{r^{2n}}\right)$$

$$= i\left(\frac{a_1 - ny(a_1 y - a_2 x)\frac{1}{r^2} + a_1 + nz(a_3 x - a_1 z)\frac{1}{r^2}}{r^n}\right)$$

$$+ j\left(\frac{a_2 + nx(a_1 y - a_2 x)\frac{1}{r^2} + a_2 - nz(a_2 z - a_3 y)\frac{1}{r^2}}{r^n}\right)$$

$$+ k\left(\frac{a_3 - nx(a_3 x - a_2 z)\frac{1}{r^2} + a_3 + ny(a_2 z - a_3 y)\frac{1}{r^2}}{r^n}\right)$$

$$= \frac{2\bar{a}}{r^n} + \left(\begin{array}{l}(nxya_2 - ny^2 a_1 + nzya_3 - nz^2 a_1)i + \\ (nxya_1 - nx^2 a_2 - nz^2 a_2 + nzya_3)j + \\ (nxza_2 - nx^2 a_3 - ny^2 a_3 + nzya_2)k\end{array} \Big/ r^{n+2}\right)$$

$$[\because r^2 = x^2 + y^2 + z^2]$$

$$= \frac{2\bar{a}}{r^n} + \left(\begin{array}{l}(nxya_2 + nzya_3 + (x^2 - r^2)a_1 n)i + \\ (nxya_1 + nzya_3 + (y^2 - r^2)na_2)j + \\ (nxza_2 + nzya_2 + (z^2 - r^2)na_3)k\end{array} \Big/ r^{n+2}\right)$$

$$= \frac{2\bar{a}}{r^n} + \frac{n(\bar{a}\cdot\bar{r})}{r^{n+2}}\bar{r} - \frac{n\bar{a}}{r^n} + \left(\frac{n}{r^{n+2}}\left(\begin{array}{l}yxa_2 i + zya_3 i + xya_1 j + \\ zya_3 j + xza_2 k + \\ zya_2 k\end{array}\right)\right)$$

$$= \frac{2-n}{r^n}\bar{a} + \frac{n(\bar{a}\cdot\bar{r})}{r^{n+2}}\bar{r} + \bar{0}$$

$$= \frac{2-n}{r^n}\bar{a} + \frac{n(\bar{a}\cdot\bar{r})}{r^{n+2}}\bar{r}.$$

⑤ Find the value of a if

i) E is solenoidal ii) E is irrotational

where $\bar{E} = (ax^2y + yz)\hat{i} + (xy^2 - xz^2)\hat{j} + (2xyz - 2x^2y^2)k$

i) Given E is solenoidal.

$$\therefore \nabla \cdot \bar{E} = 0$$

$$\frac{\partial}{\partial x}(ax^2y + yz) + \frac{\partial}{\partial y}(xy^2 - xz^2) + \frac{\partial}{\partial z}(2xyz - 2x^2y^2) = 0$$

$$2axy + 2xy + 2xy = 0$$

$$2a + 4 = 0$$

$$\boxed{a = -2}$$

ii) Given E is irrotational.

$$\therefore \nabla \times \bar{E} = 0$$

$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ax^2y+yz & xy^2-xz^2 & 2xyz-2x^2y^2 \end{vmatrix} = 0$$

$\Rightarrow \cancel{\hat{i}\left[\frac{\partial}{\partial y}(xy^2 - xz^2)\right]}$

$$\Rightarrow \hat{i}\left[\frac{\partial}{\partial y}(2xyz - 2x^2y^2) - \frac{\partial}{\partial z}(xy^2 - xz^2)\right]$$
$$- \hat{j}\left[\frac{\partial}{\partial x}(2xyz - 2x^2y^2) - \frac{\partial}{\partial z}(ax^2y + yz)\right]$$
$$+ k\left[\frac{\partial}{\partial x}(xy^2 - xz^2) - \frac{\partial}{\partial y}(ax^2y + yz)\right] = 0$$

$$(2xz - 4x^2y + 2xz)\hat{i} + \hat{j}(y - 2yz + 4xy^2) +$$
$$k(y^2 - z^2 - ax^2 + z) = 0$$

$$\therefore \quad 2xz - 4x^2y + 2xz = 0$$
$$4xz - 4x^2y = 0$$
$$x(z - xy) = 0$$
$$x = 0 \mid z = xy$$

$$\therefore \quad y - 2yz + 4xy^2 = 0$$

case(i): $\boxed{z = xy}$

$$y - 2yz + 4zy = 0$$
$$y + 2zy = 0$$
$$y(1 + 2z) = 0$$
$$y = 0 \mid z = -1/2$$

case(ii): $\boxed{x = 0}$

$$y - 2yz + 4xy^2 = 0$$
$$y(1 - 2z) = 0$$
$$y = 0 \mid z = -1/2$$

$$\therefore \quad y^2 - z^2 - ax^2 - z = 0$$

$$\boxed{\cancel{0 - \left(\frac{1}{4}\right) - ax^2 + \frac{1}{2} = 0}}$$

$$y^2 - \frac{1}{4} - ax^2 + \frac{1}{2} = 0$$

$$ax^2 - y^2 = \frac{1}{4}$$

$$a = \left(\frac{1}{4} + y^2\right)\frac{1}{x^2}$$

(6) If $f(x,y,z) = x^2+y^2-z$, then calculate curl(grad f).

<u>Sol.</u> grad $f = \nabla f = \Sigma \frac{\partial f}{\partial x}\hat{i}$

$= 2x\hat{i} + 2y\hat{j} - k$

$\therefore$ curl (grad f)

$= \text{curl}(\nabla f)$

$= \nabla \times \nabla f$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x & 2y & -1 \end{vmatrix}$$

$= \hat{i}(0) - \cancel{(+}\, j(0)) + k(0)$

$= \bar{0}$.

(7) If $\bar{F} = x^2y\hat{i} + y^2z\hat{j} + z^2y\,k$, then find curl (curl $\bar{F}$).

<u>Sol.:-</u> curl $\bar{F} = \nabla \times \bar{F}$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & y^2z & z^2y \end{vmatrix}$$

$= \hat{i}(z^2-y^2) - \hat{j}(0-0) + k(-x^2)$

$= (z^2-y^2)\hat{i} - x^2 k$.

$\therefore$ curl (curl $\bar{F}$) $= \nabla \times (\nabla \times \bar{F})$

$= \nabla \times (\nabla \times \bar{R})$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2-y^2 & 0 & -x^2 \end{vmatrix}$$

$$= i(0) - j(-2x-2z) + k(+2y)$$

$$= (2x+2z)j + 2yk$$

8) $\nabla^2 f(r) = \Sigma \frac{\partial^2 f(r)}{\partial x^2}$

let, $\frac{\partial^2}{\partial x^2}(f(r)) = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x} f(r)\right]$

$$= \frac{\partial}{\partial x}\left[f'(r)\frac{\partial r}{\partial x}\right]$$

$$= \frac{\partial}{\partial x}\left[f'(r)\frac{x}{r}\right]$$

$$= \frac{\partial}{\partial x}\left[\frac{x}{r} \cdot f'(r)\right]$$

$$= \frac{r\left[x \cdot f''(r) \cdot \frac{\partial r}{\partial x} + f'(r)\right] - x f'(r)\frac{\partial r}{\partial x}}{r^2}$$

$$= \frac{x^2 f''(r) + r f'(r) - x^2 f'(r)/r}{r^2}$$

$$\therefore \frac{\partial^2 f(r)}{\partial x^2} = \frac{x^2 \cdot f''(r)}{r^2} + \frac{f'(r)}{r} - \frac{x^2}{r^3} f'(r).$$

$$\therefore \nabla^2 f(r) = \Sigma \frac{\partial^2}{\partial x^2} f(r)$$

$$= \Sigma\left[\frac{x^2 \cdot f''(r)}{r^2} + \frac{1}{r} f'(r) - \frac{1}{r^3} f'(r) x^2\right]$$

$$= \frac{1}{r^2} f''(r) \Sigma x^2 + \frac{3}{r} f'(r) - \frac{1}{r^3} f'(r) \Sigma x^2.$$

$$= f''(r) + \frac{3}{r} f'(r) - \frac{1}{r} f'(r) = f''(r) + \frac{2}{r} f'(r)$$