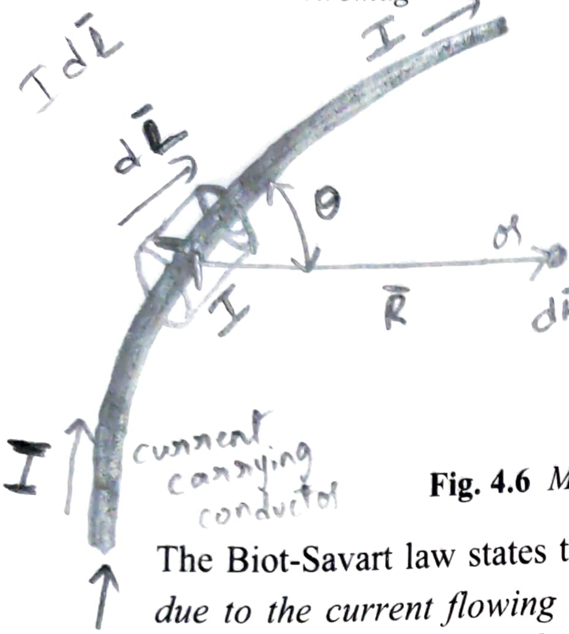
4.4 Biot-Savart Law

Consider a current carrying conductor producing a steady magnetic field around it. Let dl be a differential length on the conductor with current element Idl. Let us consider a point P at a distance R from the current element, as shown in Fig. 4.6.



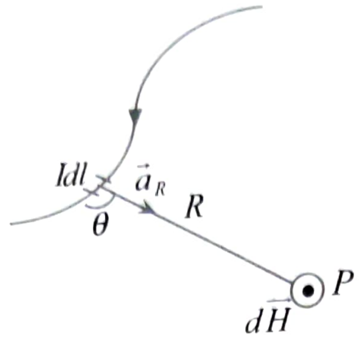


Fig. 4.6 *Magnetic field at point P due to current element*

The Biot-Savart law states that *the magnetic field intensity* $d\vec{H}$ *produced at a point P due to the current flowing through a conductor is proportional to the product of the current I and differential conductor length dl, called current element (Idl), and the sine of the angle between the direction of the element and the line joining point P to the current element, and is inversely proportional to the square of the distance R between the current element and the point P.*

Mathematically, $$d\vec{H} \propto \frac{Id\vec{l}\sin\theta}{R^2} \cdot \hat{a}_R$$

A × B = A . B sinθ

Idl × â_R = Idl . â_R

$K = \frac{1}{4\pi}$ or $$d\vec{H} = \frac{Id\vec{l}\sin\theta}{4\pi R^2} \cdot \hat{a}_R$$

From the definition of the cross product, this can be written as

R = meter

I units Ampere

dl = meter

$\frac{\text{Ampere meter}}{\text{meter}^2}$ (I dL / R^2)

$$d\vec{H} = \frac{Id\vec{l} \times \vec{a}_R}{4\pi R^2} \quad \text{A/m} \tag{4.4}$$

where $1/4\pi$ is the proportionality constant and $\vec{a}_R = \frac{\vec{R}}{|R|}$ is the unit vector along the direction of $\vec{R}$.

Then, $$d\vec{H} = \frac{I\,d\vec{l} \times \vec{R}}{4\pi R^3}. \tag{4.5}$$

∴ The total magnetic field intensity $\vec{H}$ is

$$\vec{H} = \oint \frac{Id\vec{l} \times \vec{a}_R}{4\pi R^2} \tag{4.6}$$

and the magnetic flux density $\vec{B}$ is

$\vec{B} = \mu\vec{H}$

$$\vec{B} = \frac{\mu}{4\pi}\oint \frac{Id\vec{l} \times \vec{a}_R}{R^2}.$$

The direction of the magnetic field can be determined by the right hand thumb rule or the right hand screw rule. Biot-Savart law is also called Ampere's law for current element.

Example 4.3 A steady current element $10^{-3}\ \vec{a}_z$ A-m is located at the origin in free space. (i) What is the magnetic field $\vec{B}$ due to this element at the point (1 m, 0, 0) in rectangular coordinates? (ii) What is the magnetic field at the point (0, 0, 1)?

Solution (i) Given current element, $Id\vec{l} = 10^{-3}\vec{a}_z$.

The magnetic field $d\vec{B}$ at the point (1, 0, 0) due to $Id\vec{l}$ located at origin (0, 0, 0) is,

$$d\vec{B} = \frac{\mu_0 Id\vec{l} \times \vec{R}}{4\pi |R|^3}.$$

The distance vector is $\quad \vec{R} = (1-0)a_x + (0-0)a_y + (0-0)a_z = \vec{a}_x;\ R = 1.$

$$\therefore\ d\vec{B} = \frac{\mu_0}{4\pi}\frac{10^{-3}\left(\vec{a}_z \times \vec{a}_x\right)}{(1)^3}$$

$$= \frac{\mu_0}{4\pi}(10^{-3}\vec{a}_z \times \vec{a}_x) = \frac{4\pi}{4\pi} \times 10^{-7} \times 10^{-3}(\vec{a}_y)$$

$$= 10^{-10}\vec{a}_y \text{ Tesla}$$

(ii) The magnetic field at the point (0, 0, 1) due to current element Idl at (0, 0, 0) is

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{I\vec{dL} \times \vec{R}}{|R|^3}.$$

Now, $\quad \vec{R} = (0-0)\vec{a}_x + (0-0)\vec{a}_y + (1-0)\vec{a}_z = \vec{a}_z;\ R = 1.$

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{10^{-3}a_z \times a_z}{(1)^3} = 0.$$

Note: It is observed that the magnetic field is always perpendicular to the current carrying element.

4.5 Biot-Savart Law for Distributed Currents

Sheet current

Consider a sheet of conductor carrying current I uniformly distributed over the surface on the x–y plane as shown in Fig. 4.7. Let the surface current density be $\vec{K}$ A/m. If we consider a differential surface area dS, the current element with differential length dl having the current density is

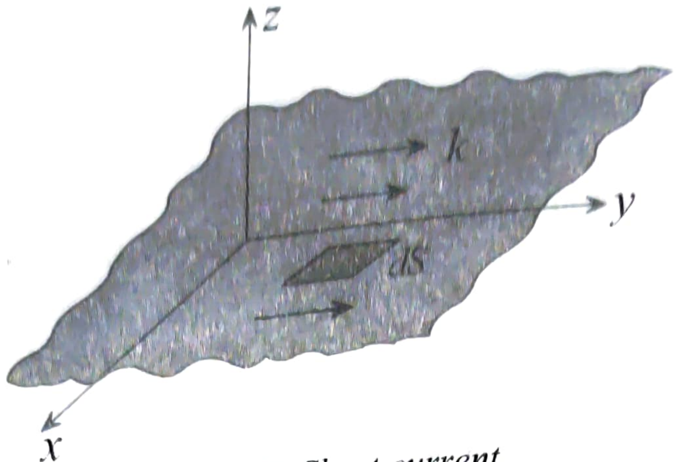


Fig. 4.7 *Sheet current*

$$I d\vec{l} = \vec{K} dS.$$

The Biot-Savart law for sheet current is then

$$\vec{H} = \int_S \frac{\vec{K} dS \times \vec{a}_R}{4\pi R^2} \quad \text{A/m} \tag{4.7}$$

Volume current

Consider a solid conductor carrying current I uniformly distributed over the volume as shown in Fig. 4.8. Let the current density be J A/m^2. If we consider a differential volume dv, the current element with differential length dl having current density is

$$I\, d\vec{l} = \vec{J}\, dv.$$

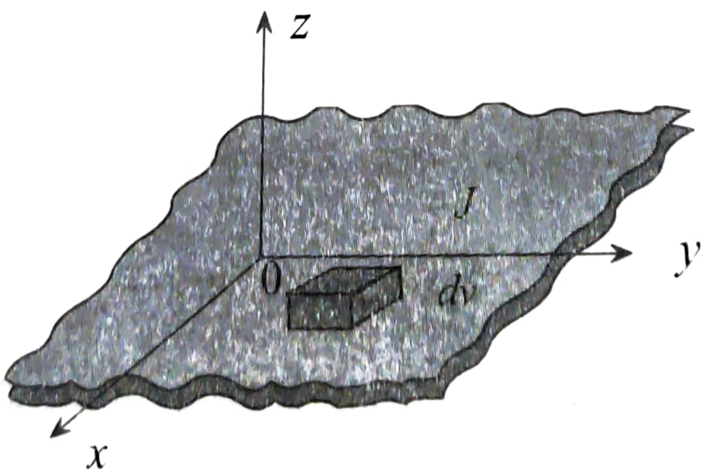


Fig 4.8 *Volume current*

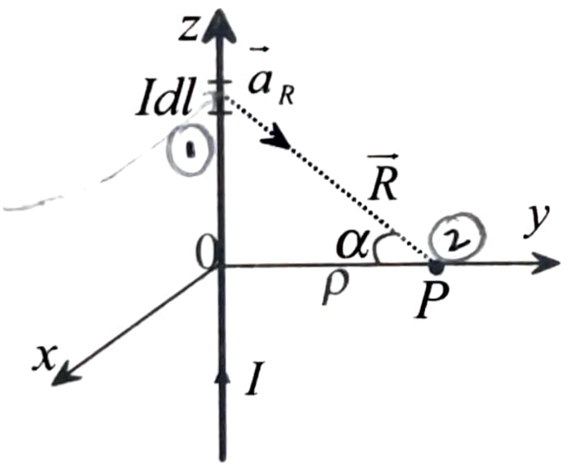
The Biot-Savart law for volume current is then

$$\vec{H} = \int_V \frac{\vec{J} \times \vec{a}_R\, dv}{4\pi R^2} \text{ A/m.} \tag{4.8}$$

4.7 Magnetic Field Intensity Due to Infinitely Long Conductor

Consider an infinitely long conductor carrying current I along the z-axis. Let us consider a current element $I d\vec{l}$ on the conductor at a distance z from the origin. Also let us consider a point P on the y-axis at a distance ρ from the origin, as shown in Fig. 4.10.

$$I d l = I dz\,\hat{a}_z$$



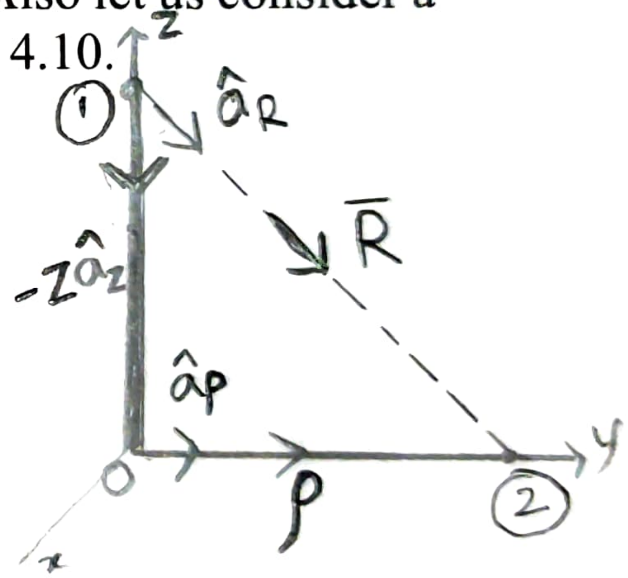


Fig. 4.10 *Magnetic field due to infinitely long conductor*

$$\vec{R} = \rho\hat{a}_\rho - z\hat{a}_z$$

In cylindrical coordinates, the position at the current element is $z\vec{a}_z$ and the position vector at the point P is (ρ , 0, 0).

The distance vector is given by

$$\vec{R} = \rho\vec{a}_\rho - z\vec{a}_z.$$

The unit vector is $\vec{a}_R = \frac{\vec{R}}{|\vec{R}|} = \frac{\rho\vec{a}_\rho - z\vec{a}_z}{\sqrt{\rho^2+z^2}}$ and the current element is $\boxed{Id\vec{l} = Idz\,\vec{a}_z}$

The magnetic field intensity at point P is

$$\vec{H} = \int_L \frac{Id\vec{l}\times\vec{a}_R}{4\pi R^2}.$$

$I\,dz\,\hat{a}_z = I\,d\vec{l}$

Evaluating the cross product,

$$Id\vec{l}\times\vec{a}_R = Idz\,\vec{a}_z \times \frac{(\rho\vec{a}_\rho - z\vec{a}_z)}{\sqrt{r^2+z^2}}$$

$$= \frac{Idz}{\sqrt{\rho^2+z^2}}(\rho\vec{a}_z\times\vec{a}_\rho - z\vec{a}_z\times\vec{a}_z)$$

Since $a_z \times a_\rho = a_\phi$; $\hat{a}_z \times \hat{a}_z = 0$

$$\boxed{Id\vec{l}\times\vec{a}_R = \frac{Idz}{\sqrt{\rho^2+z^2}}\rho\vec{a}_\phi}$$

$\vec{R}^2 = (\rho^2 + z^2)$

$\vec{H} = \int_L \frac{Id\vec{l}\times\hat{a}_R}{4\pi R^2}$ $\therefore$ $$\vec{H} = \int_{-\infty}^{\infty} \frac{I\rho\vec{a}_\phi\,dz}{4\pi(\rho^2+z^2)^{3/2}}$$

To evaluate the integration, let $z = \rho\tan\alpha$. Then $\boxed{dz = \rho\sec^2\alpha\,d\alpha}$, where α is the angle at P due to the current element. $z^2 = \rho^2\tan^2\alpha$

$\boxed{1+\tan^2\alpha = \sec^2\alpha}$

The limits of θ are from $-\pi/2$ to $\pi/2$.

So $$\frac{\rho dz}{(\rho^2+z^2)^{3/2}} = \frac{\rho\rho\sec^2\alpha\,d\alpha}{(\rho^2+\rho^2\tan^2\alpha)^{3/2}} = \frac{\sec^2\alpha\,d\alpha}{\rho\sec^3\alpha} = \frac{\cos\alpha\,d\alpha}{\rho}$$

$\frac{1}{\sec\alpha} = \cos\alpha$

$\frac{\rho^2}{\rho^3}$

$$\therefore \quad H = \int_{-\pi/2}^{\pi/2}\frac{I\vec{a}_\phi}{4\pi\rho}\cos\alpha\,d\alpha = \frac{I\vec{a}_\phi}{4\pi\rho}\int_{-\pi/2}^{\pi/2}\cos\alpha\,d\alpha$$

$$= \frac{I\vec{a}_\phi}{4\pi\rho}[\sin\alpha]_{-\pi/2}^{\pi/2} = \frac{I\vec{a}_\phi}{4\pi\rho}(2) = \frac{I\vec{a}_\phi}{2\pi\rho}\text{ A/m}. \qquad (4.9)$$

If $L = 2\pi\rho$, the length of the magnetic path of radius ρ,

$$\vec{H} = \frac{I}{L}\vec{a}_\rho\text{ A/m}.$$

$\boxed{dz = r\sec^2\theta\,d\theta}$

$z = -\infty$, $\theta = -\pi/2$ and $z = +\infty$, $\theta = +\frac{\pi}{2}$

The magnetic flux density is

$$\vec{B} = \mu\vec{H} = \frac{\mu I \vec{a}_\phi}{2\pi\rho} \quad \text{Wb/m}^2. \tag{4.10}$$

Note:

1. The magnetic field intensity $\vec{H}$ at a point P due to an infinite length conductor is proportional to the current I and inversely proportional to the perpendicular distance ρ.
2. The direction of $\vec{H}$ is along ϕ. It is tangential to the concentric circles around the conductor as shown in Fig. 4.11 and follows the right hand thumb rule.

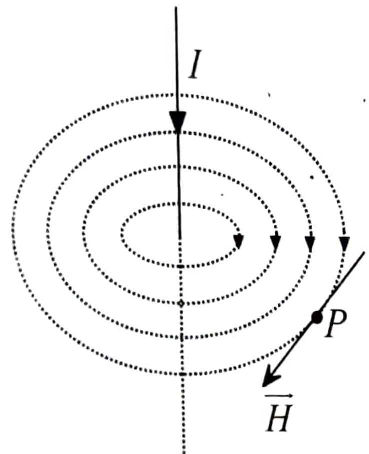


Fig. 4.11 *Direction of the magnetic field due to infinitely long conductor*

For example, a long straight wire carries a current of $I = 1$ A. At a distance of 1 m, around the conductor, the magnetic field produced is $H = \frac{1}{2\pi}$ A/m.

Example 4.4 A current filament, $I = 100$ A along the y-axis, is passing through the point (4,0,–5). Find the magnetic field intensity at the origin.

Solution Given $I = 100$ A and a point on the current element at (4,0,–5).

The magnetic field at the origin is given by

$$\vec{H} = \frac{I}{4\pi\rho}\vec{a}_\phi,$$

where ρ in terms of the radial distance is given by

$$\rho = \sqrt{4^2 + 5^2} = \sqrt{41} = 6.403$$

$$\vec{H} = \frac{100}{4\pi \times 6.403}\vec{a}_\phi = 1.24\,\vec{a}_\phi.$$

OR State Biot-Savart's law.

Ans. : The **Biot-Savart law** states that,

The magnetic field intensity $d\overline{H}$ produced at a point P due to a differential current element IdL is,

1. Proportional to the product of the current I and differential length dL.
2. The sine of the angle between the element and the line joining point P to the element.
3. And inversely proportional to the square of the distance R between point P and the element.

- The current carrying conductor is shown in the Fig. Q.4.1.
- Consider a differential length dL hence the differential current element is IdL.
- The point P is at a distance R from the differential current element.

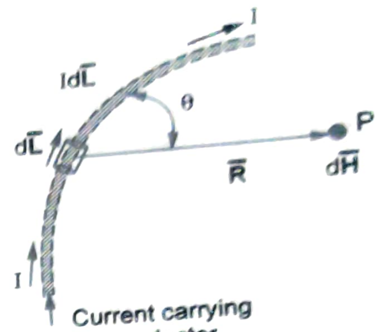


Fig. Q.4.1

- **Mathematically**, the Biot-Savart law can be stated as,

$$d\overline{H} \propto \frac{I\, dL \sin\theta}{R^2} \quad \text{i.e.} \quad \boxed{d\overline{H} = \frac{k\, I\, dL \sin\theta}{R^2}} \qquad \text{... (1)}$$

- In SI units, $k = \frac{1}{4\pi}$ hence $d\overline{H} = \frac{I\, dL \sin\theta}{4\pi R^2}$ ... (2)

Biot-Savart law in vector form :

Let dL = Magnitude of vector length $d\overline{L}$ and

$\bar{a}_R$ = Unit vector in the direction **from differential current element** to P

- Then from rule of cross product,

$$d\overline{L} \times \bar{a}_R = dL\, |\bar{a}_R| \sin\theta = dL \sin\theta \qquad \text{... } |\bar{a}_R| = 1$$

Replacing in equation (2),

$$\boxed{d\overline{H} = \frac{I d\overline{L} \times \bar{a}_R}{4\pi R^2} \ \text{A/m}} \qquad \text{... (3)}$$

But $\bar{a}_R = \frac{\overline{R}}{|\overline{R}|} = \frac{\overline{R}}{R}$ hence

$$\boxed{d\overline{H} = \frac{I d\overline{L} \times \overline{R}}{4\pi R^3} \ \text{A/m}} \qquad \text{... (4)}$$

- The equations (3) and (4) is the vector form of Biot-Savart law.
- The entire conductor is made up of all such differential elements. Hence to obtain total magnetic field intensity $\overline{H}$, the above equation (4) takes the integral form as,

$$\boxed{\overline{H} = \oint \frac{I d\overline{L} \times \bar{a}_R}{4\pi R^2}} \qquad \text{... Integral form of Boat-Savart law}$$

x sheet current

AMPERE'S CIRCUIT LAW:

It states that the line integral of magnetic field Intensity $\bar{H}$ around the closed path is equal to the net current ~~by the~~ enclosed $I_{enclosed}$ by the path

$$\boxed{\oint \bar{H} \cdot \bar{dl} = I_{enclosed}} \quad (11) \text{ (maxwell's 3rd law)}$$

By applying the Stokes theorem we know that

$$\boxed{\oint_L \bar{A} \cdot \bar{dl} = \int_S (\nabla \times \bar{A}) \cdot \bar{ds}}$$

$$\boxed{\oint_L \bar{H} \cdot \bar{dl} = \int_S (\nabla \times \bar{H}) ds} \quad (12)$$

with respect to time

$$\boxed{I_{enclosed} = \int_S \bar{J} \cdot ds} \quad (13)$$

Application of Ampere's Circuital Law: Various application of Ampere's circuital law

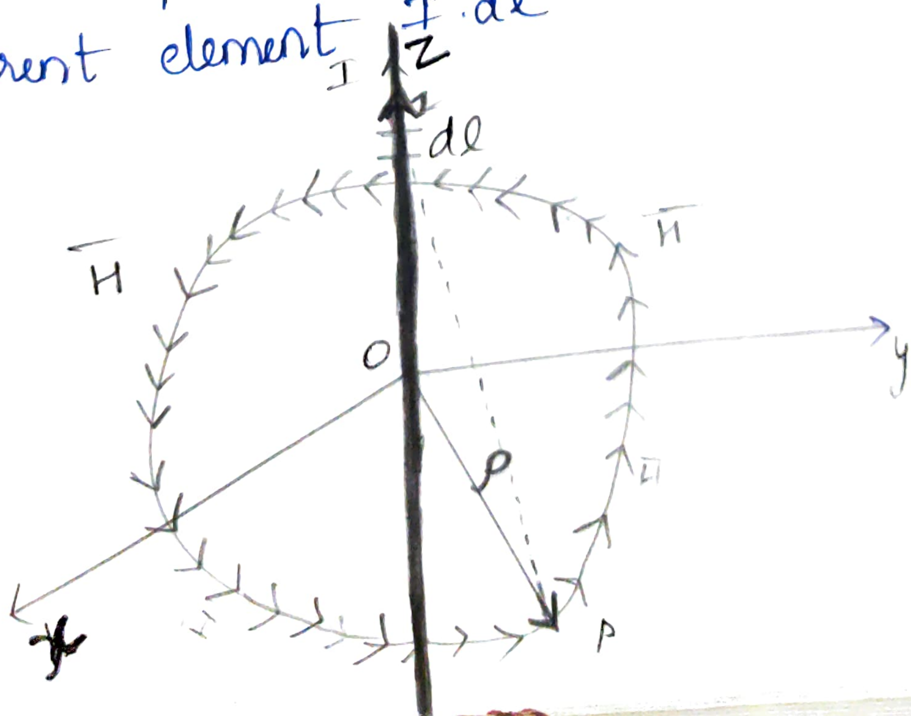
1. calculate magnetic field intensity H due to infinitely long straight conductor.
2. calculate magnetic field intensity H due to co-axial cable
3. calculate magnetic field intensity H due to infinite sheet of current
4. calculate magnetic field intensity H in a circular conductor carrying current I.

from (11), (12), (13)

$$\int_S J \cdot ds = \int_S (\nabla \times \bar{H}) ds$$

$$\boxed{\bar{\nabla} \times \bar{H} = J} \quad (14) \text{ (i.e, Maxwell 3}^{rd}\text{ eq)}$$

Proof: consider a long conductor carrying the current I along the z-axis taken a closed path of radius and called Ampere path around the conductor and as show in a figure the current element $I \cdot dl$



$$\vec{H} = \frac{I}{2\pi\rho}\hat{a}_\phi \text{ and } d\vec{l} = \rho\, d\phi\, \hat{a}_\phi$$

Then $\vec{H}\cdot d\vec{l} = \frac{I}{2\pi\rho}\hat{a}_\phi\, \rho\, d\phi\, \hat{a}_\phi$

line Integral over the closed path

$$\oint \vec{H}\cdot d\vec{l} = \int_0^{2\pi} \frac{I}{2\pi} \times 2\pi$$

If $I = I_{enc}$, the current enclosed by the Path

$$\boxed{\oint H \cdot d\vec{l} = I_{enc}}$$

Note ①:- Ampere's current law is analogous to gaussian law in electrostatics. It can be applied to determine the determine the magnetic field when current distribution is symmetrical. The closed path (amperial path) consideration need not be a circular, it may be any shape.

Note ②:- The closed path on which Ampere's ckt law is to be applied is called Amperian path. It is analogous to gaussian surface for electric field in general gaussian surface can be used for both electrical and magnetical fields

Note ③ :- For a conductor cross section(S) units are m^2 with current density $J(A/m^2)$. Ampere's law can be expressed as

$$I_{enc} = \oint_{l} \vec{H} \cdot d\vec{l} = \int_{S} \vec{J} \cdot d\vec{s}$$

$$I_{enc} = \int_{S} \vec{J} \cdot d\vec{s}$$

10/8/2018

MAGNETIC FLUX

5.3 Magnetic Field Intensity Due to a Solid Conductor

Consider a solid conductor of radius a of infinite length carrying a current of I amperes.

Let a Gaussion surface of radius ρ be considered in the conductor as shown in Fig. 5.2. The magnetic field intensity $\vec{H}$ at various values of radius ρ are given below.

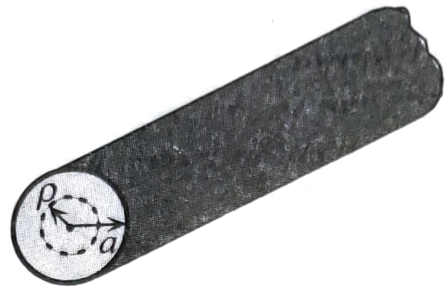


Fig. 5.2 *Solid conductor of infinite length*

Case 1: $\rho < a$

The current enclosed by the circular path of radius ρ is given by

$$I_{\text{encl}} = \frac{\pi\rho^2 I}{\pi a^2} = \frac{\rho^2 I}{a^2}.$$

According to Ampere's circuital law,

$$\oint \vec{H} \cdot d\vec{l} = I_{\text{encl}}$$

$$\int_0^{2\pi} H_\phi \, \rho \, d\phi = I_{\text{encl}}$$

$$\boxed{H_\phi 2\pi\rho = \frac{\rho^2}{a^2} I}$$

$$H_\phi = \frac{\rho I}{2\pi a^2}$$

or

$$\vec{H} = \frac{\rho I}{2\pi a^2} \vec{a}_\phi. \tag{5.2}$$

Case 2: $\rho = a$

$$\vec{H} = \frac{aI}{2\pi a^2} \vec{a}_\phi = \frac{I}{2\pi a} \vec{a}_\phi. \tag{5.3}$$

Case 3: $\rho > a$

The current enclosed by the circular path is

$$I_{\text{encl}} = I$$

$$\therefore \quad H_\phi 2\pi\rho = I$$

$$H_\phi = \frac{I}{2\pi\rho}$$

or

$$\vec{H} = \frac{I}{2\pi\rho}\vec{a}_\phi. \tag{5.4}$$

Hence,

$$\vec{H} = \begin{cases} \dfrac{\rho I}{2\pi a^2}\vec{a}_\phi & \rho < a \\ \dfrac{I}{2\pi a}\vec{a}_\phi & \rho = a \\ \dfrac{I}{2\pi\rho}\vec{a}_\phi & \rho \geq a \end{cases} \tag{5.5}$$

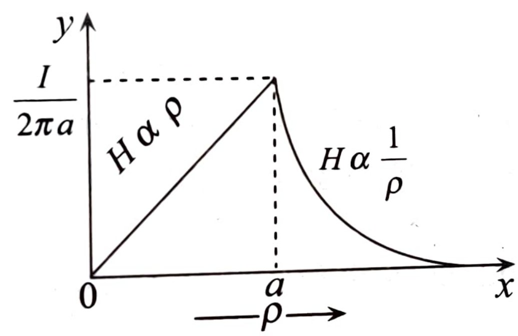


Fig. 5.3 *Variation of $\vec{H}$ along radius in a solid conductor*

The variation of magnetic field intensity $\vec{H}$ with ρ is shown in Fig. 5.3.

Magnetic field between two-wire transmission line

Consider the system as two solid infinite conductors separated by distance d. If the conductors are carrying currents I_1 and I_2 in opposite direction, then the magnite field intensity at point P half way between the two wires is equal to the sum of the fields produced by the two lines at P.

The total magnetic field intensity is

$$\vec{H} = \frac{I_1}{2\pi\rho} + \frac{I_2}{2\pi\rho}.$$

If $\quad I_1 = I_2,$

then $\quad \vec{H} = \dfrac{I}{\pi\rho}\vec{a}_\phi.$

If the point P makes an angle α with the conductors, then

$$\vec{H} = \frac{I\cos^2\alpha}{\pi\rho}\vec{a}_\phi.$$

Ans. : Consider a co-axial cable as shown in the Fig. Q.16.1.

This cable is placed along z axis. The current I is uniformly distributed in the inner conductor. While –I is uniformly distributed in the outer conductor.

Region 1 : Within the inner conductor, $r < a$. Consider a closed path having radius $r < a$. Hence it encloses only part of the conductor as shown in the Fig. Q.16.2.

$r < a$

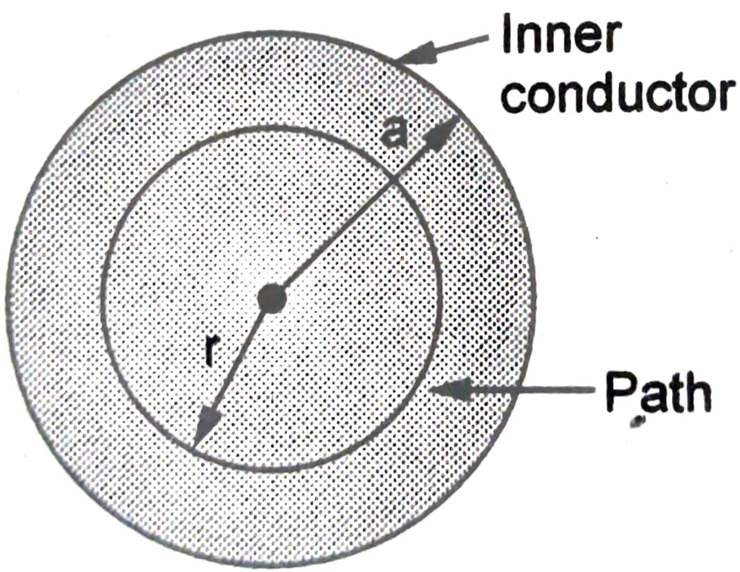


Fig. Q.16.2

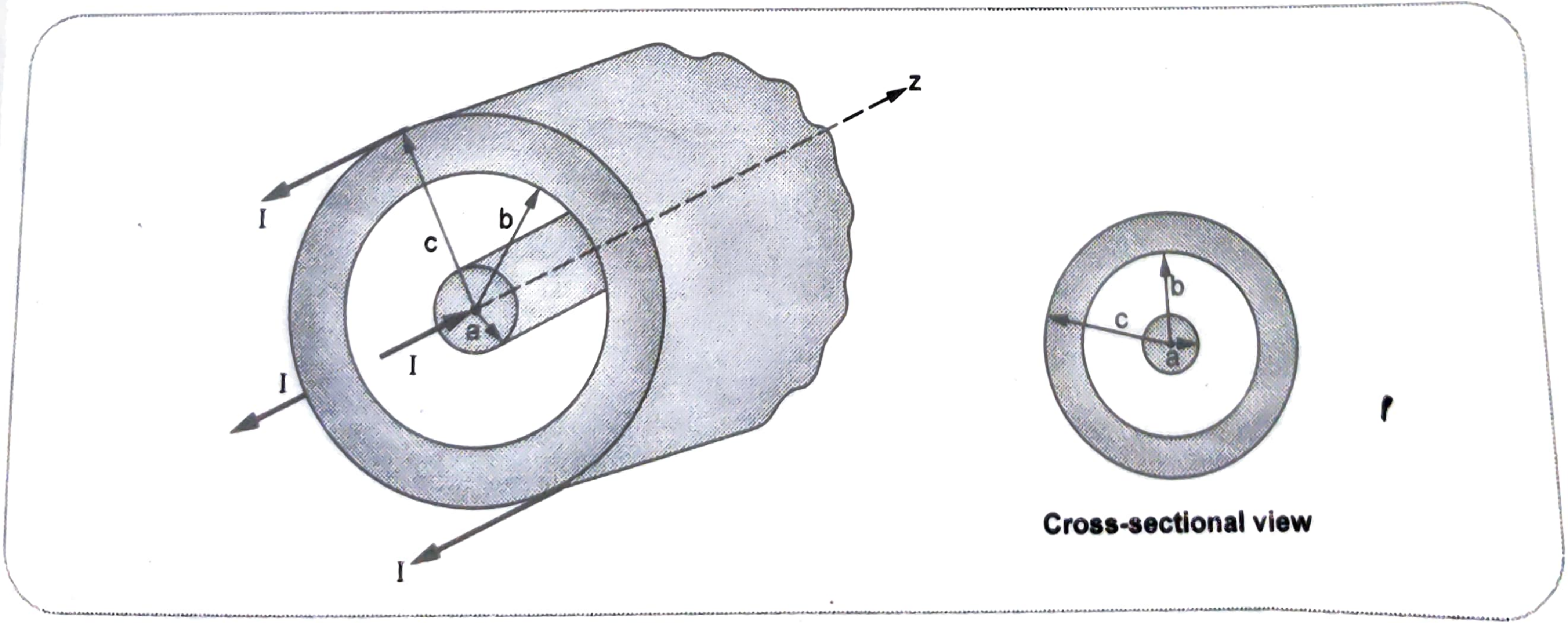


Fig. Q.16.1 Co-axial cable

The area of cross-section enclosed is $\pi r^2\ m^2$.

The total current flowing is I through the area πa^2. Hence the current enclosed by the closed path is,

$$I'_{enc} = \frac{\pi r^2}{\pi a^2} I = \frac{r^2}{a^2} I \qquad \dots (1)$$

The $\bar{H}$ is again only in $\bar{a}_\phi$ direction and depends only on r.

$$\therefore \qquad \bar{H} = H_\phi\, \bar{a}_\phi \text{ and } d\bar{L} = r d\phi \bar{a}_\phi$$

$$\therefore \qquad \bar{H} \bullet d\bar{L} = H_\phi\, \bar{a}_\phi \bullet r d\phi \bar{a}_\phi = H_\phi\ r\, d\phi \qquad \dots (2)$$

According to Ampere's circuital law,

$$\oint \bar{H} \bullet d\bar{L} = I'_{enc}$$

i.e. $$\int_{\phi=0}^{2\pi} H_\phi\, r\, d\phi = \frac{r^2}{a^2} I \qquad \text{... Current enclosed}$$

Solving, $$\boxed{\bar{H} = H_\phi\, \bar{a}_\phi = \frac{I r}{2\pi a^2}\bar{a}_\phi \ \text{A/m}} \qquad \dots r < a$$

Region 2 : Within $a < r < b$ consider a circular path which encloses the inner conductor carrying direct current I. This is the case of infinitely long conductor along z-axis. Hence $\bar{H}$ in this region is,

$\therefore H_\phi 2\pi r = I_{enc}$ $\qquad \vec{H} = \frac{I}{2\pi r}\hat{a}_\phi$

$H_\phi = \frac{I}{2\pi r}$

$$\boxed{\bar{H} = \frac{I}{2\pi r}\bar{a}_\phi \ \text{A/m}} \qquad \dots (a < r < b)$$

Region 3 : Within outer conductor, $b < r < c$

Consider the closed path as shown in the Fig. Q.16.3. The current enclosed by the closed path is only the part of the current $-$ I, in the outer conductor. The total current $-$ I is flowing through the cross section $\pi(c^2 - b^2)$ while the closed path encloses the cross section $\pi(r^2 - b^2)$.

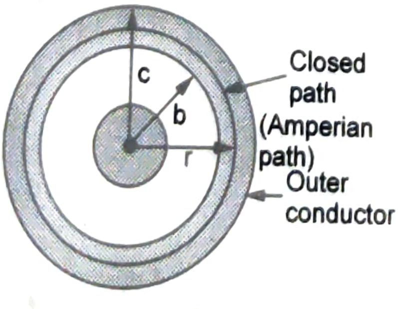


Fig. Q.16.3

Hence the current enclosed by the closed path of **outer conductor** is,

$$I' = \frac{\pi(r^2 - b^2)}{\pi(c^2 - b^2)}(-I) = -\frac{(r^2 - b^2)}{(c^2 - b^2)} I \qquad \dots (3)$$

$\therefore \quad I'' = I =$ Current in inner conductor enclosed

Total current enclosed by the closed path is,

$$\boxed{I_{enc} = I' + I''} = -\frac{(r^2 - b^2)}{(c^2 - b^2)} I + I$$

$$= I\left[1 - \frac{(r^2 - b^2)}{(c^2 - b^2)}\right] = I\left[\frac{c^2 - r^2}{c^2 - b^2}\right] \quad \ldots (5)$$

$$\therefore \; H_\phi = \frac{I}{2\pi r}\left[\frac{c^2 - r^2}{c^2 - b^2}\right]$$

$$\boxed{\bar{H} = H_\phi \bar{a}_\phi = \frac{I}{2\pi r}\left[\frac{c^2 - r^2}{c^2 - b^2}\right] \bar{a}_\phi \text{ A/m}} \quad \ldots b < r < c$$

Region 4 : Outside the cable, r > c.

Consider the closed path with r > c such that encloses both the conductors i.e. both currents + I and – I.

Thus the total current enclosed is,

$$I_{enc} = +I - I = 0 \text{ A}$$

$$\therefore \quad \oint \bar{H} \bullet d\bar{L} = 0 \quad \ldots \text{Ampere's circuital law}$$

$$\therefore \quad \boxed{\bar{H} = 0 \text{ A/m}} \quad \ldots r > c$$

The magnetic field does not exist outside the cable. The variation of $\bar{H}$ against r is shown in the Fig. Q.16.4.

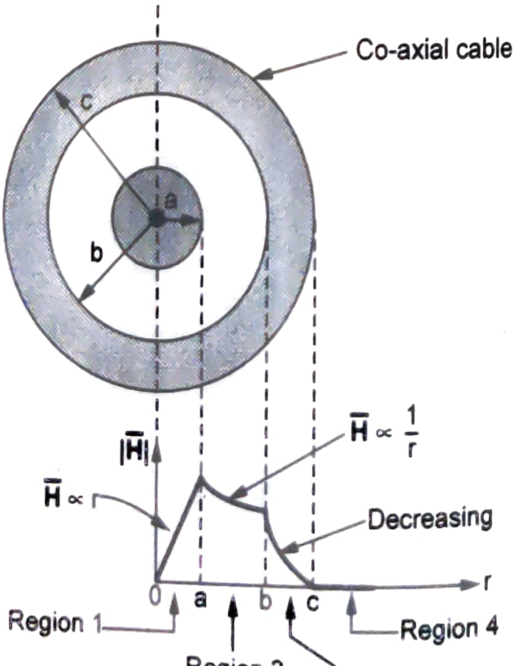


Fig. Q.16.4 Variation of $\bar{H}$ against r in co-axial cable

Magnetic field intensity or magnetic field strength

The magnetic field intensity at any point in a magnetic field is defined as *the magnetic force experienced by a unit pole at that point. It is also called magnetic field strength.*

It is a vector quantity, analogous to electric field intensity and is denoted by $\vec{H}$. Its units are newtons/weber (N/Wb) or ampere/metre (A/m).

When a current is flowing through a conducting wire, a magnetic field is produced around the wire. The field can be represented by concentric circles drawn around the wire with proper direction (flux line) as shown in Fig. 4.5. According the right-hand thumb rule, *I* indicates the current flowing out of the paper; the flux lines are in the anti-clockwise direction. The field strength decreases as the distance from the wire increases.

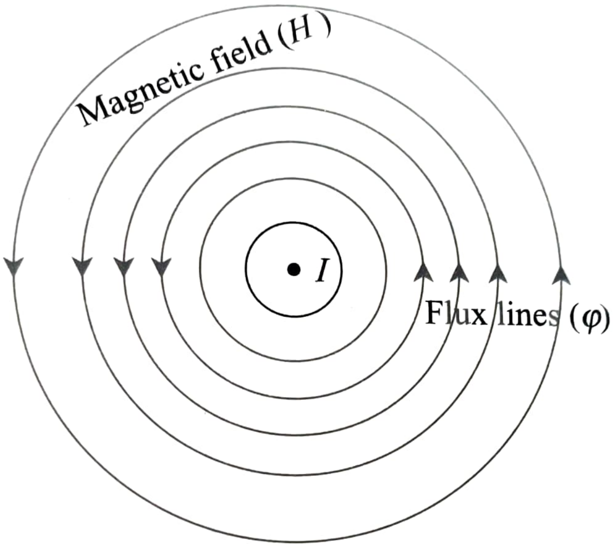


Fig. 4.5 *Flux lines around a current-carrying wire*

The magnetic flux density $\vec{B}$ is related to magnetic field intensity ($\vec{H}$) through the material property called permeability (μ), i.e.,

$$\vec{B} = \mu\vec{H} \tag{4.3}$$

or $\quad \vec{B} = \mu_0\mu_r\vec{H}$

where μ_r = relative permeability and μ_0 = absolute permeability.

For free space, $\mu_r = 1$ and $\mu = \mu_0 = 4\pi \times 10^{-7}$ H/m.

For magnetic materials, μ_r is greater than 1 and for non-magnetic materials, μ_r is equal to 1.

The magnetic field strength is independent of the medium's permeability.

Example 4.2 Given $\vec{H} = 4\vec{a}_x + 2\vec{a}_y - 3\vec{a}_z$ A/m at a point in free space, what is the flux density?

Solution Given $\vec{H} = 4\vec{a}_x + 2\vec{a}_y - 3\vec{a}_z$.

We know that flux density $\vec{B} = \mu_0\vec{H} = 4\pi \times 10^{-7}(4\vec{a}_x + 2\vec{a}_y - 3\vec{a}_z)$

$$= 5.026\vec{a}_x + 2.513\vec{a}_y - 3.767\vec{a}_z\ \mu\text{Wb/m}^2.$$

4.3 Magnetic Flux and Flux Density

Magnetic flux gives the distribution of magnetic field passing through a given surface. In other words, the total number of magnetic lines of force passing through a given surface is called magnetic flux. It is denoted by the symbol φ, and its unit is weber (Wb). In electrostatics, we saw how the electric field could be described on the basis of electric flux density D. In magnetostatics, the corresponding quantity is magnetic flux density, denoted by B, whose units are Wb/m^2, also known as Tesla (T).

Magnetic flux density is defined *as the magnetic flux crossing a unit area normal to the direction of the magnetic flux*.

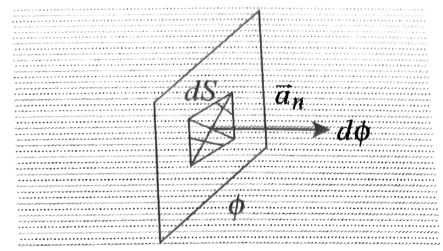


Fig. 4.4 *Magnetic flux density*

If the differential flux $d\varphi$ crosses the differential area dS normal to the plane of the loop as shown in Fig. 4.4, then the magnetic flux density is given by

$$\vec{B} = \frac{d\varphi}{dS}\vec{a}_n \quad \text{Wb/m}^2 \text{ or Tesla.} \tag{4.1}$$

The magnetic flux density is a vector quantity. The direction of B is normal to the plane area which encloses the maximum flux.

The magnetic flux is given by

$$\varphi = \int_s \vec{B}\cdot d\vec{S}. \tag{4.2}$$

Example 4.1 If the magnetic flux density in a medium is given by $\vec{B} = \frac{1}{\rho}\cos\phi\vec{a}_\rho$, find the flux crossing the surface defined by $0 \le z \le 4$ m and $0 < \phi < \pi/4$.

Solution Given $\vec{B} = \frac{1}{\rho}\cos\phi\vec{a}_\rho$. In cylindrical coordinates, $dS = \rho d\phi dz\,\vec{a}_\rho$. The flux is

$$\varphi = \int_s B\cdot dS = \int_0^{\pi/4}\int_0^{4} \frac{1}{\rho}\cos\phi\rho d\phi dz\,\vec{a}_\rho\cdot\vec{a}_\rho$$

$$= \int_0^{\pi/4}\int_0^{4} \cos\phi d\phi dz = 4[\sin\phi]_0^{\pi/4} = 4\times\sin(\pi/4)$$

$$= 4\times\frac{1}{\sqrt{2}} = 2.828 \text{ Wb.}$$

4.14 Maxwell's Second Equation for Magnetic Flux Density

We know from Biot–Savart's law that the magnetic flux density at distance R from a current-carrying conductor is

$$\vec{B} = \frac{\mu_0 I}{4\pi} \oint \frac{d\vec{l} \times \vec{a}_R}{R^2}.$$

Now, $$\nabla \cdot \vec{B} = \nabla \cdot \left[\frac{\mu_0 I}{4\pi} \oint \frac{d\vec{l} \times \vec{a}_R}{R^2} \right]$$

$$= \frac{\mu_0 I}{4\pi} \oint \frac{\nabla \cdot (\overset{A}{d\vec{l}} \times \overset{B}{\vec{a}_R})}{R^2}. \quad (4.36)$$

We know that $\nabla \cdot (\text{A} \times B) = B \cdot \nabla \times \text{A} - \text{A}. \nabla \times B$ (See Chapter 1)

$$\therefore \quad \nabla \cdot \vec{B} = \frac{\mu_0 I}{4\pi} \oint \left[\frac{\vec{a}_R}{R^2} \cdot (\nabla \times d\vec{l}) - d\vec{l} \cdot \left(\nabla \times \frac{\vec{a}_R}{R^2} \right) \right].$$

Since R is only in the radial direction,

$$\frac{\vec{a}_R}{R^2} = \frac{\vec{a}_\rho}{R^2} = -\nabla \left(\frac{1}{R} \right) \vec{a}_\rho.$$

$$\therefore \quad \nabla \cdot \vec{B} = \frac{\mu_0 I}{4\pi} \oint \left[-\nabla \left(\frac{1}{R} \right) \vec{a}_\rho \cdot (\nabla \times \vec{dl}) + d\vec{l} \cdot \left(\nabla \times \nabla \left(\frac{-1}{R} \right) \vec{a}_\rho \right) \right].$$

Let $\frac{1}{R} \vec{a}_\rho = C$ and $d\vec{l} = D$.

Then $$\nabla \cdot \vec{B} = \frac{\mu_0 I}{4\pi} \oint [-\nabla C \cdot (\nabla \times D) + D \cdot (\nabla \times \nabla C)].$$

From vector identity, $\nabla \times \nabla C = 0$ and $\nabla \cdot \nabla \times D = 0$,

$$\nabla \cdot \vec{B} = \frac{\mu_0 I}{4\pi} \oint (-0 + 0) = 0$$

$$\therefore \quad \nabla \cdot \vec{B} = 0. \quad \text{(Proved). (4.37)}$$

Since div $\vec{B} = 0$, the magnetic flux lines are continuous. This equation is known as Maxwell's equation for magnetic flux density.

(V_m) SCALAR MAGNETIC POTENTIAL

(V_m) (units Amperes)

- Scalar magnetic potential denoted by V_m. It unit are Amperes.

- As we know relationship B/w E & V from electrostatic.

$$\vec{E} = -\nabla V$$

$\because \vec{E} \approx \vec{H}$

$\vec{D} \approx \vec{B}$

$\because \nabla \times \vec{E} = 0$

So, in Magneto static, $\boxed{\vec{H} = -\nabla V_m}$

Since, $\nabla \times \vec{H} = \vec{J}$ $\longrightarrow$ Ampere's Law.

$$\nabla \times \vec{H} = \nabla \times (-\nabla V_m)$$

$$\nabla \times \vec{H} = 0$$

$$\therefore \vec{J} = 0$$

Using Vector Identity

$\nabla \times (\nabla V) = 0$

$\nabla \cdot (\nabla \times \vec{A}) = 0$

curl of the gradient of any Scalar is always zero.

Note:- Magnetic Scalar potential V_m is only defined in region where $\vec{J} = 0$

the -Ve gradient of the Scalar magnetic potential is equal to the magnetic field intensity at which the current density $\bar{J} = 0$

i,e $\boxed{\bar{H} = -\nabla V_m \; ; \text{ if } \bar{J} = 0}$

V_m is the Scalar magnetic potential & its units are Amperes.

* The Scalar potential Satisfies or follow Laplace's Equation:

$$\boxed{\nabla^2 V_m = 0 \quad \text{for } \vec{J} = 0}$$

We know that $\nabla \cdot \vec{B} = 0$ → Non-existance of Magnetic monopole

Divergence of Vector is zero Bcz it

$\vec{B} = \mu_0 \vec{H}$

free space.

$\mu_0 \nabla \cdot \vec{H} = 0$

$\mu_0 \nabla \cdot (-\nabla V_m) = 0$

∴ $\boxed{H = -\nabla V_m}$ if $J=0$

$$\boxed{\nabla^2 V_m = 0} \text{ .... only for } \vec{J} = 0$$

laplace's equation in Magnetostatic field.

* The Scalar Magnetic potential is defined as (V_m) the line integral of the magnetic field intensity along a given path, where the current density $(\vec{J})$ is zero.

Mathematically

$$\boxed{V_m = -\int \vec{H} \cdot d\vec{l}}$$

* we Consider the Scalar magnetic potential between two point A & B in a magnetic field is

$$V_m = -\int_A^B \vec{H} \cdot d\vec{l} \text{ ....(Units Ampere)}$$

$\vec{H} = A/m$

$d\vec{l} = m.$

VECTOR MAGNETIC POTENTIAL ($\bar{A}$)

→ Exist where J is present

"Magnetic Vector potential is defined in such a way that its curl gives the magnetic flux density"

$$\boxed{\vec{B} = \nabla \times \vec{A}} \quad \text{—— (A)}$$

where $\vec{A}$ is the Vector magnetic potential.

its units are (Wb/m)

Using Curl because curl of vector ≈ Vector quantity.

―― * ―― * ―― * ―― * ―― * ――

Poisson's equation for Vector Magnetic potential

From Ampere's Circuital law, $\nabla \times \vec{H} = \vec{J}$

free space. $\vec{B} = \mu_0 \vec{H}$

$$\boxed{\vec{H} = \frac{\vec{B}}{\mu_0}}$$

$$\boxed{\nabla \times \vec{B} = \mu_0 \vec{J}}$$

Since we know from Eq. (A) $\boxed{\vec{B} = \nabla \times \vec{A}}$

$$\therefore \nabla \times \nabla \times \vec{A} = \mu_0 \vec{J}$$

from Using Laplacian of Vector idenity

ie $\boxed{\nabla^2 \vec{A} = \nabla(\nabla \cdot \vec{A}) - \nabla \times \nabla \times \vec{A}}$

$$\therefore \nabla(\nabla \cdot \vec{A}) - \nabla^2 A = \mu_0 \vec{J}$$

$$\therefore \quad \nabla(\underbrace{\nabla\cdot\vec{A}}_{=0}) - \nabla^2\vec{A} = \mu_0\vec{J}$$

Since, for D.C current only

$\nabla\cdot\vec{B} = 0$, or $\nabla\cdot\vec{A} = 0$

$$0 - \nabla^2\vec{A} = \mu_0\vec{J}$$

$$-\nabla^2\vec{A} = \mu_0\vec{J}$$

$$\boxed{\nabla^2\vec{A} = -\mu_0\vec{J}}$$

This Equation is know as Vector Poisson's Equation for magnetic field.

If $\vec{J} = 0$ then

$$\nabla^2\vec{A} = 0 \quad \text{(becomes Vector Laplace's Equation)}$$

Properties of $(\vec{A})$ vector magnetic potential

1. the Vector magnetic potential is proportional to the current density flowing through the conductor.
2. the curl of Vector magnetic potential given the magnetic flux density
3. It Satisfies the poisson's equation.

Vector magnetic potential $(\vec{A})$ is used to obtain radiat characteristics of antennas.

The Vector Magnetic potential for Line Current Element.

the Vector magnetic potential at a point P in the magnetic field is proportional to the differential current element $Id\vec{l}$ flowing through the Conductor and inversely proportional to the distance vector from current element to the field point 'P'. It is denoted by $\bar{A}$ and it is Expressed as

$$\vec{A} = \int_L \frac{\mu I d\vec{l}}{4\pi R}$$

Proof: From Bio-Savart law we know that for a differential line current element, $Id\vec{l}$.

$$\vec{H} = \int_L \frac{I d\vec{l} \times \hat{a}_R}{4\pi R^2}$$

where $\vec{R}$ is the distance vector from the current element to the field point P

$$\vec{B} = \mu_0 \vec{H} = \frac{\mu_0}{4\pi} \int_L \frac{I d\vec{l} \times \hat{a}_R}{R^2}$$

Note: to bring the cross product out of the integral, let us do the following

we know that $\boxed{-\nabla\left(\frac{1}{R}\right) = \frac{\hat{a}_R}{R^2}}$ Substituting in above.

$$\boxed{\vec{B} = \frac{-\mu_0}{4\pi} \int_L I d\vec{l} \times \nabla\left(\frac{1}{R}\right)} \quad \text{— B}$$

from Vector Identity

$$\boxed{\nabla \times (f\bar{c}) = f\,\nabla \times \bar{c} + \nabla f \times \vec{c}}$$

$\boxed{f = \frac{1}{R}}$

$\boxed{\vec{c} = I d\vec{l}}$

$$\nabla \times (f\vec{c}) = f\nabla \times \vec{c} + \nabla f \times \vec{c}$$

$$\nabla \times \left(\frac{1}{R}(I d\vec{l})\right) = \frac{1}{R}(\nabla \times I d\vec{l}) + \nabla\left(\frac{1}{R}\right) \times I d\vec{l}$$

(or)

$$\nabla\left(\frac{1}{R}\right) \times I d\vec{l} = \nabla \times \left(\frac{I d\vec{l}}{R}\right) - \frac{1}{R}\nabla \times I d\vec{l}$$

Since $\nabla \times I d\vec{l} = 0$

$$\nabla\left(\frac{1}{R}\right) \times I d\vec{l} = \nabla \times \left(\frac{I d\vec{l}}{R}\right)$$

$\bar{A} \times \bar{B} = -\bar{B} \times \bar{A}$

(or)

$$\boxed{I d\vec{l} \times \nabla\left(\frac{1}{R}\right) = -\nabla \times \left(\frac{I d\vec{l}}{R}\right)}$$

Substitute in eq (B)

$$\vec{B} = \frac{-\mu}{4\pi}\int_L I d\vec{l} \times \nabla\left(\frac{1}{R}\right)$$

$$\vec{B} = \frac{\mu}{4\pi}\int_L \nabla \times \frac{I d\vec{l}}{R}$$

$$\vec{B} = \nabla \times \int_L \frac{\mu I d\vec{l}}{4\pi R}$$

In The term of vector magnetic potential $(\bar{A})$ we know $\vec{B} = \nabla \times \vec{A}$

$$\vec{A} = \int_L \frac{\mu I d\vec{l}}{4\pi R}$$

∴ For a differential current Element $I d\vec{l}$, The vector magnetic potential at a distance R from The current element is

$$\boxed{\vec{A} = \int_L \frac{\mu I d\vec{l}}{4\pi R} \quad wb/m}$$

for differential Surface current Element $\vec{K}.ds$, $\vec{A} = \int_S \frac{\mu \bar{K} ds}{4\pi R}$ wb/m

for differential Volume current Element $\vec{J}.dv$, $\vec{A} = \int_V \frac{\mu \vec{J} dv}{4\pi R}$ wb/m.

Product B/w Two current elements

FORCE DUE TO MAGNETIC Fields

(a) FORCE ON A CHARGE PARTICLE:-

From the Couloumb's Law:-

$\overline{F_E}$ on a stationary or a moving electric charge Q in a electric field intensity $\overline{E}$ is given by —

$$\boxed{F_E = Q\overline{E}} \quad (29)$$

A magnetic field ~~can~~ experience a force only on the moving charge.

From experiments it is found that magnetic force $\vec{F_m}$ experienced by a charge Q moving with a velocity $(\vec{u})$. In a magnetic field density $(\vec{B})$ is $\vec{F_m} = Q * u \times \overline{B}$

$$\boxed{\vec{F_m} = Q\vec{u} \times \vec{B}} \quad (30)$$

It shows ~~ll~~ clearly perpendicular to both $\vec{u}$ and $\vec{B}$

For a moving charge Q in the presence of electric field intensity $\vec{E}$ and magnetic field density $\vec{B}$, the total force on the charge is 'F'

$$\boxed{\vec{F} = \vec{F_m} + \vec{F_E}} \quad (31)$$

$$\vec{F} = Q\vec{u} \times \vec{B} + Q\vec{E}$$

$$\boxed{\vec{F} = Q[\vec{E} + (\vec{u} \times \vec{B})]} \quad (32)$$

This is called ~~kor~~ LORENTZ Force equation

From Newton 2nd law $F = ma$

F = mass × acceleration

acceleration = $\frac{du}{dt}$

$$F = m \cdot \frac{du}{dt}$$

$$\text{Force} \rightarrow \boxed{\vec{F} = m \cdot \frac{du}{dt} = Q[\vec{E} + (\vec{u} \times \vec{B})]} \quad (33)$$

b) FORCE ON A CURRENT ELEMENT :- A force on a current element ($I \cdot dl$) in presence of magnetic field density a $\vec{B}$ is given by

$$\boxed{I = \frac{dQ}{dt}} \qquad I \cdot dl = \frac{dQ}{dt} \cdot dl = dQ \cdot \frac{dl}{dt}$$

since $\boxed{\frac{dl}{dt} = u}$

$$\boxed{I\,dl = dQu} \quad (34)$$

It shows that an elementary charge dQ moving with a velocity u is equivalent to the current element.

→ The equation of the magnetic force $\vec{F_m} = Q\vec{u} \times \vec{B}$ can be written as

$$d\vec{F_m} = d[Q\vec{u} \times \vec{B}]$$

$$d\vec{F_m} = dQ\vec{u} \times \vec{B}$$

$$\boxed{d\vec{F_m} = I d\vec{l} \times \vec{B}} \quad \text{(35)} \qquad (\because dQu = Idl)$$

If the current I is through the closed path L (or) circuit then force on the circuit is given by

$$\boxed{\vec{F_m} = \oint_L I\, dl \times \vec{B}} \quad \text{(36)}$$

$$\boxed{\vec{F_m} = \int_S K \cdot ds \times \vec{B}} \quad \text{(37)}$$ Force extend on Surface current Element

$$\boxed{\vec{F_m} = \int_V \vec{J}\, dV \times \vec{B}} \quad \text{(38)}$$ Force extend on Volome current element

(c) FORCE Between TWO CURRENT ELEMENTS:

~~The force b/w two~~

Consider the force b/w two current elements $I_1 dl_1$ and $I_2 dL_2$. Both current elements produce magnetic field. So we can find the force $d(d\vec{F_1})$

on current element $I_1 dL_1$ due to the field $d\overline{B}_2$ Produced by current element $I_2 dl_2$ as —

$$\boxed{d(d\overline{F}_1) = I_1 d\overline{L}_1 \times d\overline{B}_2} \quad (39)$$

From Bio-savart's law
The magnetic field intensity at $I_2 dl_2$ due to current element $I_1 d_1$ is

$$\boxed{d\overrightarrow{B}_2 = \frac{\mu_0 I_2 dl_2 \times \hat{a}_{R12}}{4\pi R_{12}^2}} \quad (40)$$

$$\boxed{d\overrightarrow{H_2} = \frac{I_1 d\vec{l}_1 \times \vec{a}_{R12}}{4\pi R_{12}}}$$ or $\boxed{d\overrightarrow{B_2} = \mu_0 dH_2}$

Substituting eq (40) in eq (30)

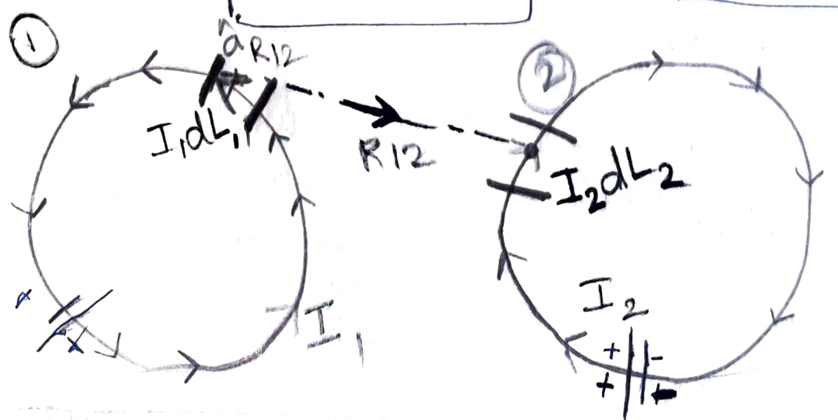
Due to the magnetic field B_2, the differential force on The current element $I_2 dL_2$ is

$$d(d\vec{F}_1) = \frac{I_1 d\overline{L}_1 \times \mu_0 I_2 dl_2 \times \hat{a}_{R12}}{4\pi R_{12}^2}$$

$$= \frac{\mu_0 \; I_1 dL_1 \times (I_2 dl_2 \times \hat{a}_{R12})}{4\pi R_{12}^2}$$

$$\boxed{\vec{F}_1 = \frac{\mu_0 I_1 I_2}{4\pi} \oint_{L_1} \oint_{L_2} \frac{d\overline{L}_1 \times (dL_2 \times \hat{a}_{R12})}{R_{12}^2}}$$

then $\boxed{\vec{F}_1 = -\vec{F}_2}$ (or) $\boxed{\overline{F}_2 = -\overline{F}_1}$



AMPERES FORCE LAW:-

Amperes force law states that force b/w two current carrying ~~conductor~~ elements $I_1 dL_1$ and $I_2 dL_2$ placed at the distance of R_{12} is given as

$$\boxed{\vec{F} = \frac{\mu}{4\pi} \oint_{L_1} \oint_{L_2} \frac{I_2 d\vec{l}_2 \times (I_1 d\vec{l}_1 \times \hat{a}_{R_{12}})}{R_{12}^2}}$$

where $I_1 d\vec{l}_1$ is current element of the first conductor.

$I_2 d\vec{l}_2$ is current element of the second conductor

R_{12} → It is the distance b/w the two current carrying conductor elements.

Ampere's law is used to experience ~~o~~the force on one current carrying conductor direct in terms of a second current carrying conductor without kNeeding to find the magnetic field b/w them.

(Completed (a) part)

5.9 Ampere's Force Law: Force between Two Current Elements

Ampere's force law states that *the force between two current-carrying elements $I_1 dl_1$ and $I_2 dl_2$ placed at a distance R_{12} is given by*

$$\vec{F} = \frac{\mu}{4\pi} \oint_{L_1} \oint_{L_2} \frac{I_2 d\vec{l}_2 \times (I_1 d\vec{l}_1 \times \vec{a}_{R_{12}})}{R_{12}^2}, \tag{5.42}$$

where, $I_1 d\vec{l}_1$ is the current element of the first conductor,

$I_2 d\vec{l}_2$ is the current element of the second conductor, and

R_{12} is the distance between the current elements of the two conductors.

Ampere's law is used to express the force on one current carrying conductor directly in terms of a second current-carrying conductor without needing to find the magnetic field between them.

Proof Consider two conductors, in the form of loop L_1 and loop L_2, carrying currents I_1 and I_2 respectively as shown in Fig. 5.6.

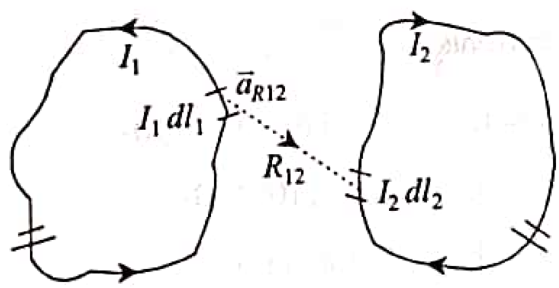


Fig. 5.6 *Force between two current loops*

Let the current element of the first conductor be $I_1 dl_1$ and that of the second conductor be $I_2 d\vec{l}_2$. The current elements are placed at a distance R_{12} from each other; the unit vector $\vec{a}_{R_{12}}$ is as shown in Fig. 5.6.

According to Biot–Savart's law, the magnetic field intensity at $I_2 d\vec{l}_2$ due to the current element $I_1 d\vec{l}_1$ is

$$d\vec{H}_2 = \frac{I_1 d\vec{l}_1 \times \vec{a}_{R_{12}}}{4\pi R_{12}^2},$$

or $$d\vec{B}_2 = \frac{\mu I_1 d\vec{l}_1 \times \vec{a}_{R_{12}}}{4\pi R_{12}^2}. \tag{5.43}$$

Due to the magnetic field B_2, the differential force on the current element $I_2 d\vec{l}_2$ is

$$d\vec{F}_2 = I_2 d\vec{l}_2 \times \vec{B}_2.$$

Taking differential magnetic flux density,

$$d(d\vec{F}_2) = I_2 d\vec{l}_2 \times d\vec{B}_2. \tag{5.44}$$

Substituting Eq. (5.43) in Eq. (5.44), we get

$$d(d\vec{F}_2) = \frac{\mu I_2 d\vec{l}_2 \times I_1 d\vec{l}_1 \times \vec{a}_{R_{12}}}{4\pi R_{12}^2}.$$

Taking double integration on closed paths, the total force $\vec{F}_2$ on current loop L_2 due to current lo
L_1 is

$$\vec{F}_2 = \frac{I_1 I_2 \mu}{4\pi} \oint_{L_1} \oint_{L_2} \frac{d\vec{l}_2 \times d\vec{l}_1 \times \vec{a}_{R_{12}}}{R_{12}^2}$$ (Prove

Note: The force $\vec{F}_1$ on current loop L_1 due to current loop L_2 is

$$\vec{F}_1 = -\vec{F}_2.$$ (5

Sr. No.	Scalar magnetic potential	Vector magnetic potential
1.	It has to satisfy the equation $\nabla \times \nabla V_m = 0$	It has to satisfy the equation $\nabla \bullet (\nabla \times \overline{A}) = 0$
2.	It is defined as $V_{ma,b} = -\int_b^a H \bullet d\overline{L}$	It is defined as $\overline{B} = \nabla \times \overline{A}$
3.	It is defined for the source free region where current density is zero.	It is defined for the region with any finite current density. The current density need not be zero.
4.	It is measured in amperes (A).	It is measured in weber per metre (Wb/m).
5.	It is not a single valued function.	It is single valued function.
6.	It satisfies Laplace's equation for magnetic field.	It satisfies Possion's equation for magnetic field.