$$f(t) = (e^{-t} - e^{-3t})u(t) + (3e^{-3(t-2)} - e^{-(t-2)})u(t-2) + (2e^{-(t-4)} - 2e^{-3(t-4)})u(t-4)$$

Q) $F(s) = \dfrac{1 - e^{-2s}}{3s^2 + 2s}$

Q) $F(s) = \dfrac{5s + 13}{s(s^2 + 4s + 13)}$

Q) $F(s) = \dfrac{s+3}{s^2+4s+8}$, ~~f(t)~~ find the initial and final value ?

w.k.t., $f(0) = \lim_{t \to 0} f(t) = \lim_{s \to \infty} s\,F(s)$

and $f(\infty) = \lim_{t \to \infty} f(t) = \lim_{s \to 0} s.\,F(s)$

$$f(0) = \lim_{s \to \infty} s\left(\frac{s+3}{s^2+4s+8}\right)$$

$$= \lim_{s \to \infty} \frac{\cancel{s^2}\left(1 + \frac{3}{s}\right)}{\cancel{s^2}\left(1 + \frac{4}{s} + \frac{8}{s^2}\right)} = 1$$

$$f(\infty) = \lim_{s \to 0} \frac{(s^2 + 3s)}{(s^2 + 4s + 8)} = 0$$

Q) $f(t) = 2.e^{-5t}u(t)$, find initial value.

$$F(s) = \frac{2}{s} - \frac{1}{s+5}$$

$$= \frac{2s + 10 - 5}{s(s+5)} = \frac{s+10}{s(s+5)}$$

$$f(0) = \lim_{s \to \infty} \cancel{s}\left(\frac{s+10}{\cancel{s}(s+5)}\right)$$

$$= \lim_{s \to \infty} \frac{\cancel{s}\left(1 + \frac{10}{s}\right)}{\cancel{s}\left(1 + \frac{5}{s}\right)} = 1$$

$f(t) = 5e^{-4t}$, find initial value

$$F(s) = 5\,\frac{1}{s+4}$$

$$f(\infty) = \lim_{s \to 0} s\left(\frac{5}{s+4}\right) = 0.$$

<u>L.T of a periodic function</u> :-

Periodic function $\Rightarrow f(t) = f(t \pm T)$, $T = T_0, 2T_0, 3T_0, \ldots$

$$F(s) = \int_0^{\infty} f(t)e^{-st} = \int_0^{T} f(t)e^{-st}\,dt + \int_T^{2T} f(t)e^{-st}\,dt + \cdots$$

Replace, $t = t + T$ in 2nd term
and, $t = t + 2T$ in 3rd "

$$F(s) = \int_0^T f(t)e^{-st}\,dt + \int_0^T f(t+T)\,e^{-s(t+T)}\,dt + \int_0^T f(t+2T)\,e^{-s(t+2T)}\,dt + \cdots$$

$$= \int_0^T f(t)e^{-st}\,dt + e^{-sT}\int_0^T f(t+T)e^{-st}\,dt + e^{-2sT}\int_0^T f(t+2T)e^{-st}\,dt + \cdots$$

w.k.t., $f(t) = f(t+T) = f(t+2T) = \ldots$

$$\therefore F(s) = \int_0^T f(t)e^{-st}\,dt + e^{-sT}\int_0^T f(t)e^{-st}\,dt + e^{-2sT}\int_0^T f(t)e^{-st}\,dt + \cdots$$

$$= \int_0^T f(t)e^{-st}\,dt\left(1 + e^{-sT} + e^{-2sT} + e^{-3sT} + \cdots + \infty\right)$$

$$= \frac{1}{1 - e^{-sT}}\int_0^T f(t)e^{-st}\,dt = \text{L.T of periodic function.}$$

Z - TRANSFORMS

CTFT → existence conditions → $\int_{-\infty}^{\infty} |f(t)| dt < \infty$ → absolutely integrable

If not absolutely integrable, Laplace transforms are used.

DTFT → existence conditions → $\sum_{n=-\infty}^{\infty} |x(n)| < \infty$ → absolutely summable

If not, z- transforms are used.

∴ z-transforms are used for discrete time signals

If $x[n]$ is a D.T sequence, then $z.T[x(n)] = X(z)$

i.e., $\boxed{z.T[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}}$

where 'z' is known as complex variable which is usually given in polar form as,

$\boxed{z = re^{j\omega}}$

LT → S → rectangular form → $s = \sigma + j\omega$
ZT → Z → polar form → $z = re^{j\omega}$

where, 'r' is called as the magnitude of the complex variable 'z'

'ω' is called as the angle of the complex variable 'z'

W.K.T,

$$DTFT[x(n)] = X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \rightarrow (1)$$

We have, $ZT[x(n)] = X(z) = \sum x(n) \cdot z^{-n}$

Let us replace, $z = re^{j\omega}$, then

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) (re^{j\omega})^{-n}$$

$$\Rightarrow X(z) = \sum_{n=-\infty}^{\infty} x(n) r^{-n} e^{-j\omega n} \rightarrow (2)$$

$$X(z) = \sum f(n) e^{-j\omega n} = DTFT[f(n)] \quad (\text{from } (1))$$

$$X(re^{j\omega}) = \sum x(n) r^{-n} e^{-j\omega n}$$

When $x(n)$ is not absolutely summable, we have r^{-n} which is a real exponential decaying & increasing function.

Now $x(n) r^{-n}$ will be absolutely summable.

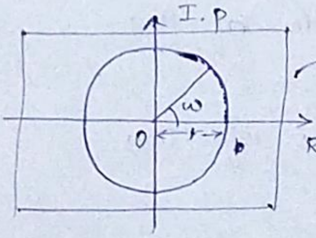
If $\sum_{n=-\infty}^{\infty} |x(n) r^{-n}| < \infty$, = absolutely summable, then the F.T is converging (i.e., exist)

Let, $|z| = \underset{\text{radius}}{r} = 1 =$ unit radius

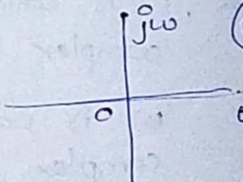
$$\Rightarrow X(e^{j\omega}) = \sum x(n) e^{-j\omega n}$$

$$\boxed{X(z)\Big|_{r=1} = X(e^{j\omega}) = X(\omega)}$$

which is known as DTFT.



z-plane



L.T

→ If $r=1$, then it is known as unit circle.

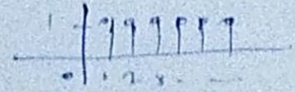
→ Here, unit circle is playing the similar role as $j\omega$ axis in L-Transforms.

① $x(n) = a^n u(n)$, find Z.T of $x(n)$.
R.S. sequence.

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \quad = \text{Bilateral / two sided Z.T}$$

$$= \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n}$$

$$u(n) = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$



$$= \sum_{n=0}^{\infty} (a z^{-1})^n$$

$$X(z) = \frac{1}{1 - a z^{-1}} = \frac{z}{z-a}$$

for absolutely summable, $\sum_n |a z^{-1}| < 1$

and $\left|\frac{a}{z}\right| < 1 \Rightarrow |a| < |z|$

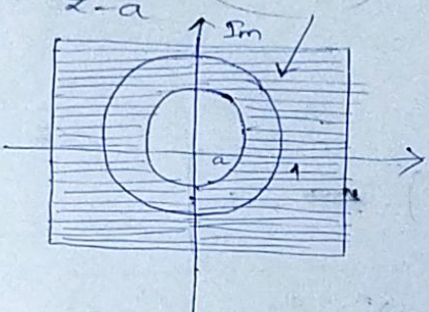
or $|z| > |a|$

$$\therefore X(z) = \frac{1}{1 - a z^{-1}} = \frac{z}{z-a} \quad \text{for } |z| > |a|$$

For $|z| > |a|$, are the range of values for which $X(z)$ is valid (i.e., $X(z)$ is converging)

Region of convergence (ROC) → Range of values for which the system is converging.

$X(z) = \frac{z}{z-a}$, $|z| > |a|$ → circle with radius 'a'.

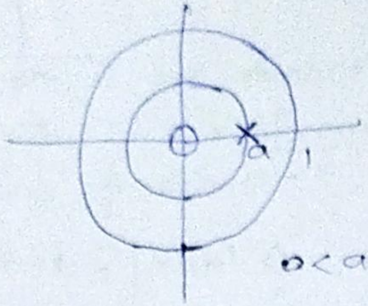


$$x(n) \xleftrightarrow{Z.T} X(z)$$

$$\boxed{a^n u(n) \xleftrightarrow{Z.T} \frac{z}{z-a}}$$

w.r.t., roots of numerator are zeros
and " " denominator are poles.

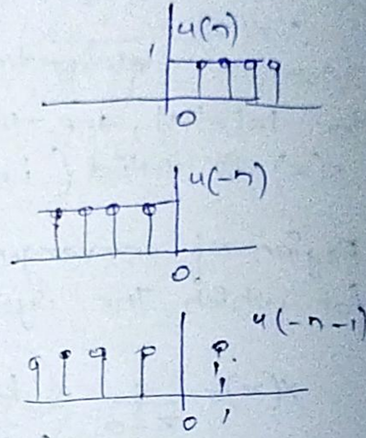
Here, $z=0$ is a root (zero)
and $z=a$ is a pole.



$0<a<1$

Q) $x(n) = -a^n u(-n-1)$

- left sided sequence



$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} -a^n u(-n-1) z^{-n}$$

$$= -\sum_{n=-\infty}^{-1} a^n z^{-n}$$

$$= -\sum_{n=1}^{\infty} a^{-n} z^n$$

$$= -\left[\sum_{n=0}^{\infty} (a^{-1}z^{1})^n - 1\right]$$

$$= 1 - \sum_{n=0}^{\infty} (a^{-1}z^{1})^n$$

$$= 1 - \frac{1}{1-a^{-1}z^{1}}$$

$$= \frac{1-a^{-1}z^{1}-1}{1-a^{-1}z^{1}}$$

$$= \frac{-z/a}{1-z/a}$$

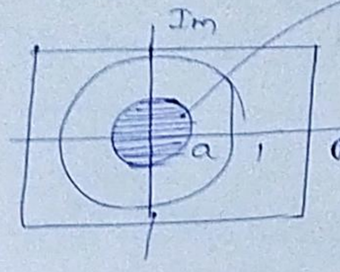
$$= \frac{-z}{a-z}$$

$$X(z) = \frac{z}{z-a}$$

$$-a^n u(-n-1) \xleftrightarrow{Z.T} \frac{z}{z-a}$$

ROC → $\Sigma |a^{-1}z| < 1 \Rightarrow |z| < |a|$

$X(z) = \frac{z}{z-a}$, ROC → $|z| < |a|$



$0 < a < 1$

Q) $x(n) = 7\left(\frac{1}{3}\right)^n u(n) - 6\left(\frac{1}{2}\right)^n u(n)$, $X(z) = ?$

$$X(z) = \frac{7}{1-\frac{1}{3}z^{-1}} - \frac{6}{1-\frac{1}{2}z^{-1}}$$

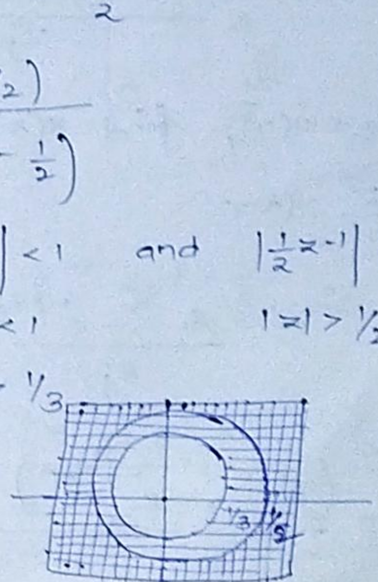
$$= \frac{7z}{z-\frac{1}{3}} - \frac{6z}{z-\frac{1}{2}}$$

$$X(z) = \frac{z(z-3/2)}{\left(z-\frac{1}{3}\right)\left(z-\frac{1}{2}\right)}$$

ROC → $\left|\frac{1}{3}z^{-1}\right| < 1$ and $\left|\frac{1}{2}z^{-1}\right| < 1$

$\left|\frac{1}{3z}\right| < 1$ $|z| > 1/2$

$|z| > 1/3$



∴ ROC = $|z| > 1/2$

(Q) If $x(n) = \{1, 2, 5, 7, 0, 1\}$, $X(z) = ?$

i.e., $x(n) = \begin{cases} 1, & n=0 \\ 2, & n=1 \\ 5, & n=2 \\ 7, & n=3 \\ 0, & n=4 \\ 1, & n=5 \end{cases}$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$= x(0) + x(1) z^{-1} + x(2) z^{-2} + \cdots$$

$$X(z) = 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + 1z^{-5} \text{ //}$$

$\Rightarrow$ If $\boxed{x(n) = \delta(n)}$

$\Rightarrow \boxed{X(z) = 1}$

$a^n u(n) = x(n)$

If $a = 1$,

$\boxed{x(n) = u(n)}$,

and $\boxed{X(z) = \frac{z}{z-1}}$

(Q) $x(n) = \cos\omega_0 n \, u(n) = \frac{e^{j\omega_0 n} + e^{-j\omega_0 n}}{2}$

$x(n) = \sin\omega_0 n \, u(n) = \frac{e^{j\omega_0 n} - e^{-j\omega_0 n}}{2j}$

(Q) $x(n) = \sin\omega_0 n \, u(n)$, find $X(z)$?

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

W.K.T., $\sin\omega_0 n = \frac{e^{j\omega_0 n} - e^{-j\omega_0 n}}{2j}$

$$\therefore X(z) = \sum_{n=0}^{\infty} \frac{\left(e^{j\omega_0 n} - e^{-j\omega_0 n}\right)}{2j} z^{-n}$$

$$= \frac{1}{2j}\left[\sum_{n=0}^{\infty} e^{j\omega_0 n} z^{-n} - \sum_{n=0}^{\infty} e^{-j\omega_0 n} z^{-n}\right]$$

$$= \frac{1}{2j}\left[\sum_{n=0}^{\infty} (e^{j\omega_0} z^{-1})^n - \sum_{n=0}^{\infty} (e^{-j\omega_0} z^{-1})^n\right]$$

$$= \frac{1}{2j}\left[\frac{1}{1-e^{j\omega_0}z^{-1}} - \frac{1}{1-e^{-j\omega_0}z^{-1}}\right] \quad \left(\because \sum_{n=0}^{\infty} r^n = \frac{1}{1-r}\right)$$

$$= \frac{1}{2j}\left[\frac{\cancel{1} - e^{-j\omega_0}z^{-1} - \cancel{1} + e^{j\omega_0}z^{-1}}{(1-e^{j\omega_0}z^{-1})(1-e^{-j\omega_0}z^{-1})}\right]$$

$$= \cancel{\frac{1}{2j}}\left[\frac{\cancel{2j}\sin\omega_0 z^{-1}}{1-e^{j\omega_0}z^{-1} - e^{j\omega_0}z^{-1} + z^{-2}}\right]$$

$$\boxed{X(z) = \frac{\sin\omega_0 z^{-1}}{1-\cos\omega_0 z^{-1} + z^{-2}}}$$

(9) $x(n) = a^n \cos\omega_0 n \, u(n)$, $X(z) = ?$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n \frac{(e^{j\omega_0 n} + e^{-j\omega_0 n})}{2} z^{-n}$$

$$= \frac{1}{2}\left[\sum_{n=0}^{\infty} (a e^{j\omega_0} z^{-1})^n + \sum_{n=0}^{\infty} (a e^{-j\omega_0} z^{-1})^n\right]$$

$$= \frac{1}{2}\left[\frac{1}{1-ae^{j\omega_0}z^{-1}} + \frac{1}{1-ae^{-j\omega_0}z^{-1}}\right]$$

$$= \frac{1}{2}\left[\frac{1-a^{-j\omega_0}z^{-1} + 1 - ae^{-j\omega_0}z^{-1}}{(1-ae^{j\omega_0}z^{-1})(1-ae^{-j\omega_0}z^{-1})}\right]$$

$$\boxed{X(z) = \frac{z^2 - az\cos\omega_0}{z^2 - 2az\cos\omega_0 + a^2}}$$

(g) $x(n) = \sin\omega_0 n\, u(n)$

$$\Rightarrow X(z) = \frac{z\sin\omega_0}{z^2 - 2z\cos\omega_0 + 1}$$

(h) $x(n) = a^n \sin\omega_0 n\, u(n)$

$$\Rightarrow X(z) = \frac{az\sin\omega_0}{z^2 - 2az\cos\omega_0 + a^2}$$

(Q) $x(n) = \left(\frac{1}{3}\right)^n \sin\left(\frac{\pi}{4}n\right) u(n)$

Comparing with $x(n) = a^n \sin\omega_0 n\, u(n)$

$$a = \frac{1}{3}, \quad \omega_0 = \frac{\pi}{4}$$

$$\Rightarrow X(z) = \frac{\frac{1}{3}z\sin\frac{\pi}{4}}{z^2 - \frac{2}{3}z\cos\frac{\pi}{4} + \frac{1}{9}}$$

$$= \frac{3z}{9z^2 - 6z + 1}$$

* Properties of z-transforms :-

W.k.t., $x(n) \xleftrightarrow{Z.T} X(z)$

(i) Linearity property :-

$$x(n) \xleftrightarrow{Z.T} X(z)$$

$x_1(n)\ x_2(n) \ldots x_K(n)$ $\qquad$ $X_1(z)\ X_2(z) \ldots X_K(z)$

Superposition principle must be satisfied for linearity
↓
Combination of both additivity and scaling

STATEMENT :- If $x_1(n) \xleftrightarrow{Z.T} X_1(z)$
and $x_2(n) \xleftrightarrow{Z.T} X_2(z)$

then, $a x_1(n) + b x_2(n) \xleftrightarrow{Z.T} a X_1(z) + b X_2(z)$

PROOF :- W.K.T, $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$$= \sum_n \left(a x_1(n) + b x_2(n)\right) z^{-n}$$

$$= \sum_n a x_1(n) z^{-n} + \sum_n b x_2(n) z^{-n}$$

$$= a X_1(z) + b X_2(z)$$

(ii) Time shifting property :-

STATEMENT :- If $x(n) \longleftrightarrow X(z)$
then, $x(n-n_0) \longleftrightarrow z^{-n_0} X(z)$

PROOF :- $X(z) = \sum_n x(n) z^{-n}$

$$X'(z) = \sum x(n-n_0) z^{-n}$$

let $(n-n_0) = l$ (say)

$\Rightarrow n = l + n_0$

$$X(z) = \sum_l x(l) z^{-(l+n_0)}$$

$$= \sum_l x(l) z^{-l} z^{-n_0}$$

$$= z^{-n_0} X(z)$$

$$\therefore \boxed{x(n-n_0) \longleftrightarrow z^{-n_0} X(z)}$$

i.e., If $\boxed{\begin{array}{l} \delta(n) \longleftrightarrow 1 \\ \delta(n-k) \longleftrightarrow z^{-k} \end{array}}$

(iii) Scaling property :-

STATEMENT :- If $x(n) \longleftrightarrow X(z)$

then $a^n x(n) \longleftrightarrow X\left(\frac{z}{a}\right)$

PROOF : $X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$

$= \sum_n a^n x(n) z^{-n}$

$= \sum_n x(n) (a^{-1} z^{+1})^{-n}$

$= \sum_n x(n) \left(\frac{z}{a}\right)^{-n}$

$= X\left(\frac{z}{a}\right)$

$\therefore \boxed{a^n x(n) \longleftrightarrow X\left(\frac{z}{a}\right)}$

(iv) Time reversal property :-

STATEMENT :- If $x(n) \longleftrightarrow X(z)$

then, $x(-n) \longleftrightarrow X(z^{-1}) = X\left(\frac{1}{z}\right)$

PROOF :- $X(z) = \sum x(n) z^{-n}$

(v) Convolution property :-

STATEMENT : If $x_1(n) \longleftrightarrow X_1(z)$

and $x_2(n) \longleftrightarrow X_2(z)$,

then, $x(n) = x_1(n) * x_2(n) = \sum_k x_1(k)\, x_2(n-k) = X_1(z).X_2(z)$

PROOF :- $X(z) = \sum x(n) z^{-n}$

$$= \sum (x_1(n) * x_2(n)) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left(\sum_{k=0}^{\infty} x_1(k)\, x_2(n-k) \right) z^{-n}$$

$$= \underbrace{\sum_{n=-\infty}^{\infty} x_2(n-k) z^{-n}}\ \sum_{k=-\infty}^{\infty} x_1(k)$$

$$= \sum_k x_1(k) z^{-k}\ X_2(z)$$

$$= X_1(z) X_2(z)$$

$$x_1(n) * x_2(n) \longleftrightarrow X_1(z) \cdot X_2(z).$$

i.e., convolution in t.d = multiplication in z.d.

(vi) Differentiation in z-domain :-

STATEMENT :- If $x(n) \longleftrightarrow X(z)$

then, $n\, x(n) \longleftrightarrow -z \frac{d}{dz} X(z)$

PROOF :- $X(z) = \sum_n x(n) z^{-n}$

$$\frac{d}{dz} X(z) = \sum_n x(n) \frac{d}{dz} (z^{-n})$$

$$= \sum_n x(n) \cdot -n z^{-n-1}$$

$$= z^{-1} \sum n\, x(n) z^{-n} \quad (-1)$$

$$= -\frac{1}{z} \sum n\, x(n) z^{-n}$$

$$-z \frac{d}{dz} X(z) = n\, x(n)$$

i.e., $\boxed{n\,x(n) \longleftrightarrow -z\frac{d}{dz}X(z)}$

(*) Initial and final value theorem :-

→ Initial value theorem :-

For a causal sequence,

$$\boxed{\lim_{n\to 0} x(n) = x(0) = \lim_{z\to\infty} X(z)}$$

PROOF :- $X(z) = \sum_{n=0}^{\infty} x(n)z^{-n}$

$$X(z) = x(0) + x(1)z^{-1} + x(2)z^{-2} + \dots + x(n)z^{-n}$$

$$\lim_{z\to\infty} X(z) = \lim_{z\to\infty} \sum_{n=0}^{\infty} x(n)z^{-n}$$

$$\lim_{z\to\infty} X(z) = \lim_{z\to\infty}\left[x(0) + x(1)z^{-1} + x(2)z^{-2} + \dots + x(n)z^{-n}\right]$$

$$\therefore \boxed{\lim_{z\to\infty} X(z) = x(0)}$$

→ Final value theorem :-

For a causal sequence,

$$\boxed{\lim_{n\to\infty} x(n) = x(\infty) = \lim_{z\to 1}(z-1)X(z)}$$

PROOF :- W.K.T., $X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$

Let us consider, $Z.T\left[x(n) - x(n-1)\right]$

$$X(z) - z^{-1}X(z) = \sum_{n}\left[x(n) - x(n-1)\right]z^{-n}$$

$$= \left[x(0) - x(-1) + x(1) - x(0) + x(2) - x(1) + x(3) - x(2) + \dots\right]z^{-n}$$

$$X(z)\left(1-\frac{1}{z}\right) = \left[\cancel{x(0)} - \cancel{x(-1)} + \cancel{x(1)} - \cancel{x(0)} + \cancel{x(2)} - \cancel{x(1)} + \cancel{x(3)} - \cancel{x(2)} \ldots\right] z^{-n}$$

$$\frac{X(z)(z-1)}{z} = x(\infty)\, z^{-n}$$

$$\lim_{z\to 1} X(z)(z-1) = \lim_{z\to 1} z\, x(\infty)$$

$$\Rightarrow \boxed{x(\infty) = \lim_{z\to 1} (z-1)\, X(z)}$$

(Q) find the Z.T of $\delta(n-k)$

w.k.t, $\delta(n) \xleftrightarrow{Z.T} 1 = X(z)$

$\delta(n-k) \xleftrightarrow{Z.T} z^{-k} \cdot 1$

$\delta(n-n_0) \xrightarrow{Z.T} \left[\because z^{-n_0} X(z)\right]$

(Q) Z.T of $n\left(\frac{1}{4}\right)^n u(n) = ?$

w.k.t, $a^n u(n) \longleftrightarrow \frac{z}{z-a} = X(z)$

then $n a^n u(n) \longleftrightarrow -z \frac{d}{dz}(X(z))$

let, $\left(\frac{1}{4}\right)^n u(n) = x(n)$

Then, $X(z) = \frac{z}{z-\frac{1}{4}}$

then $n\left(\frac{1}{4}\right)^n u(n) = -z \frac{d}{dz}\left(\frac{z}{z-\frac{1}{4}}\right)$

$$= -z\left[\frac{\left(z-\frac{1}{4}\right) - z(1)}{\left(z-\frac{1}{4}\right)^2}\right]$$

$$= -z\left[\frac{(\cancel{4z} - 1 - \cancel{4z})/\cancel{4}}{(4z-1)^2/\cancel{16}_{\,4}}\right]$$

$$X(z) = \frac{+4z}{(4z-1)^2} \quad \text{i.e., } \frac{az}{(az-1)^2}$$

(*) Inverse z-Transform :-

When $X(z)$ = Rational function = $\frac{A(z)}{B(z)}$

We can inverse z-T by various methods →

(i) Partial fractional expansion method.
(ii) Long division method (power series method)
(iii) Residue method

(i) Partial fraction expansion method :-

Let $X(z) = \frac{A(z)}{B(z)}$

Then $X(z) = \frac{A(z)}{B(z)} = \frac{A_1}{z-z_1} + \frac{A_2}{z-z_2} + \cdots + \frac{A_K}{z-z_K}$

where $z_1, z_2, \ldots z_K$ are simple poles.

$IZT[X(z)] = z^{-1}[X(z)] = z^{-1}\left(\frac{A_1}{z-z_1}\right) + z^{-1}\left(\frac{A_2}{z-z_2}\right) + \ldots z^{-1}\left(\frac{A_K}{z-z_K}\right)$

But w.k.t., $a^n u(n) \xleftrightarrow{z.T} \frac{z}{z-a}$ for $|z| > |a|$

(i) for $|z| > |a|$, i.e., for Right hand sequence,

$$z^{-1}\left(\frac{z}{z-a}\right) = a^n u(n)$$

(ii) for $|z| < |a|$ i.e., for LHS sequence,

$$z^{-1}\left(\frac{z}{z-a}\right) = -a^n u(-n-1)$$

(1) Find the IZT of $X(z) = \dfrac{z}{3z^2 - 4z + 1}$

$$X(z) = \frac{z}{3z^2 - 4z + 1}$$

$$\frac{X(z)}{z} = \frac{1}{3\left(z^2 - \frac{4}{3}z + \frac{1}{3}\right)}$$

$$\frac{X(z)}{z} = \frac{1}{3(z-1)\left(z - \frac{1}{3}\right)} = \left(\frac{A_1}{z-1} + \frac{A_2}{z - \frac{1}{3}}\right)\frac{1}{3}.$$

$\Rightarrow A_1 = \frac{1}{2}, \; A_2 = -\frac{1}{2}$ (By partial fraction method

$$\frac{X(z)}{z} = \left(\frac{\frac{1}{2}}{z-1} - \frac{\frac{1}{2}}{z - \frac{1}{3}}\right)\frac{1}{3}.$$

$$X(z) = \frac{1}{6}\frac{z}{z-1} - \frac{1}{6}\frac{z}{z - \frac{1}{3}}.$$

$|z| > 1$ $\qquad$ $|z| > \frac{1}{3}$

$$\therefore IZT(X(z)) = \frac{1}{6}(1)^n u(n) - \frac{1}{6}\left(\frac{1}{3}\right)^n u(n)$$

for RHS $\quad$ ROC is $|z| > 1 \cap |z| > \frac{1}{3} = |z| > 1$

$$\therefore IZT(X(z)) = \frac{1}{6}(1)^n u(n) - \frac{1}{6}\left(\frac{1}{3}\right)^n u(n), \text{ ROC is } |z| > 1$$

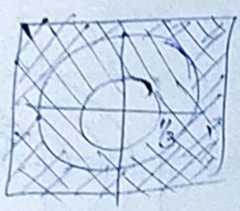
for LHS

$|z| < 1 \cap |z| < \frac{1}{3} = |z| < \frac{1}{3}$

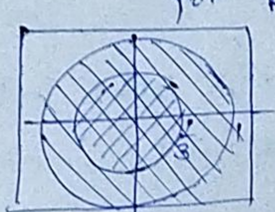
$\therefore$ ROC is $|z| < \frac{1}{3}$.

$$\Rightarrow IZT(X(z)) = x(n) = \frac{1}{6} - (1)^n u(-n-1) + \frac{1}{6}\left(\frac{1}{3}\right)^n u(-n-1)$$

for ROC, $|z| < \frac{1}{3}$

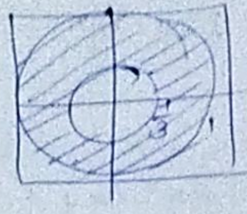


ROC for RHS



ROC for LHS

for $\frac{1}{3} < |z| < 1$



$$x(n) = -\frac{1}{6}(1)^n u(-n-1) - \frac{1}{6}\left(\frac{1}{3}\right)^n u(n)$$

(g) $X(z) = \dfrac{z(z^2-4z+5)}{(z-1)(z-2)(z-3)}$

$$\Rightarrow \frac{X(z)}{z} = \frac{z^2-4z+5}{(z-1)(z-2)(z-3)}$$

$$= \frac{A}{z-1} + \frac{B}{z-2} + \frac{C}{z-3}$$

$$\rightarrow A = \frac{z^2-4z+5}{\cancel{(z-1)}(z-2)(z-3)} \times \cancel{(z-1)}\Bigg|_{z=1} = \frac{2}{2} = 1$$

$$B = \frac{z^2-4z+5}{(z-1)\cancel{(z-2)}(z-3)} \times \cancel{(z-2)}\Bigg|_{z=2} = \frac{4-8+5}{1(-1)} = -1$$

$$C = \frac{z^2-4z+5}{(z-1)(z-2)\cancel{(z-3)}} \times \cancel{(z-3)}\Bigg|_{z=3} = \frac{9-12+5}{2\times 1} = 1$$

$$\Rightarrow \frac{X(z)}{z} = \frac{1}{z-1} - \frac{1}{z-2} + \frac{1}{z-3}$$

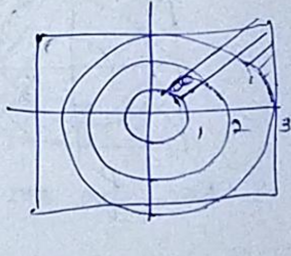
$$X(z) = \frac{z}{z-1} - \frac{z}{z-2} + \frac{z}{z-3}$$

$$\Rightarrow \text{IFT}\,[X(z)] = x(n) = z^{-1}\left(\frac{z}{z-1}\right) - z^{-1}\left(\frac{z}{z-2}\right) + z^{-1}\left(\frac{z}{z-3}\right)$$

(i) for RHS sequence ;

ROC is $|z|>1 \cap |z|>2 \cap |z|>3$

$= |z| > 3$.



$\Rightarrow x(n) = (1)^n u(n) - (2)^n u(n) + (3)^n u(n)$ for $|z| > 3$

(ii) for LHS sequence,

ROC is $|z| < 1 \cap |z| < 2 \cap |z| < 3 = |z| < 1$

$\Rightarrow x(n) = -(1)^n u(-n-1) + (2)^n u(-n-1) - (3)^n u(-n-1)$

for $|z| < 1$

(iii) $1 < |z| < 3$ (i.e., $|z| > 1$ and $|z| < 2$, $|z| < 3$)

$\therefore x(n) = (1)^n u(n) + (2)^n u(-n-1) - (3)^n u(-n-1)$

(Q) $$X(z) = \frac{(z+1)}{3z^2 - 4z + 1}$$

$\times$ & $\div$ by z

$$X(z) = \frac{z(z+1)}{z(3z^2 - 4z + 1)}$$

$$\frac{X(z)}{z} = \frac{z+1}{z(3z^2 - 4z + 1)} = \frac{z+1}{z(z-1)(z - 1/3)}$$

$$= \frac{A}{z} + \frac{B}{z-1} + \frac{C}{z - 1/3}$$

$\Rightarrow A = 1, B = 1, C = -2$

$$\frac{X(z)}{z} = \frac{1}{z} + \frac{1}{z-1} - \frac{2}{z - 1/3}$$

$$X(z) = 1 + \frac{z}{z-1} - 2\frac{z}{z - 1/3}$$

$$IFT\left(X(z)\right) = x(n) = z^{-1}(1) + z^{-1}\left(\frac{z}{z-1}\right) - 2z^{-1}\left(\frac{z}{z - 1/3}\right)$$

(i) for RHS sequence,

$|z| > 1 \cap |z| > 1/3 = |z| > 1$

$x(n) = u(n) + (1)^n u(n) - 2\left(\frac{1}{3}\right)^n u(n)$

(ii) for LHS sequence,

$|z|<1 \cap |z|<\frac{1}{3} = |z|<\frac{1}{3}$

$x(n) = u(-n) - (1)^n u(-n-1) + 2\left(\frac{1}{3}\right)^n u(-n-1), \quad |z|<\frac{1}{3}$

(iii) $\frac{1}{3}<|z|<1$, (i.e., $|z|>\frac{1}{3}$ and $|z|<1$)

$x(n) = u(n) - (1)^n u(-n-1) - 2\left(\frac{1}{3}\right)^n u(n)$, for $\frac{1}{3}<|z|<1$.

* <u>Long division method (power series expansion method)</u>

Let $x(z) = \frac{A(z)}{B(z)} \Rightarrow B(z)\overline{)A(z)}($

$x(z) = \frac{1}{1-az^{-1}}, \quad |z|>a.$

$$1-az^{-1}\overline{)1}\left(1 + az^{-1} + a^2z^{-2} + \dots\right.$$
$$\frac{\cancel{1}-az^{-1}}{az^{-1}}$$
$$\frac{\cancel{az^{-1}}-a^2z^{-2}}{a^2z^{-2}}$$
$$\frac{\cancel{a^2z^{-2}}-a^3z^{-3}}{a^3z^{-3}}$$

$= 1 + az^{-1} + a^2z^{-2} + a^3z^{-3} + \dots$

$$x(z) = \sum_{n=0}^{\infty} x(n)z^{-n}$$

$= x(0) + x(1)z^{-1} + x(2)z^{-2} + \dots$

$\Rightarrow \boxed{x(n) = a^n u(n)}$

If $x(z) = \frac{1}{1-az^{-1}} = \frac{z}{z-a}$

$$z-a\overline{)z}\left(1 + \frac{a}{z} + \frac{a^2}{z^2} + \dots\right.$$
$$\frac{\cancel{z}-a}{a}$$
$$\frac{\cancel{a}-a^2z^{-1}}{a^2z^{-1}}$$
$$\frac{\cancel{a^2z^{-1}}-a^3z^{-2}}{a^3z^{-2}}$$

(i) for RHS sequence, $(|z|>|a|)$

Arrange $x(z)$ in decreasing powers of z (or) increasing powers of z^{-1}.

(ii) for LHS sequence, $(|z|<|a|)$

Arrange $x(z)$ in increasing powers of z (or) decreasing powers of z^{-1}.

Let $x(z) = \frac{1}{1-az^{-1}}, \quad |z|<|a|$

$$-az^{-1}+1 \,)\, 1 \,(\, 1 - a^{-1}z - a^{-2}z^{2} - a^{-3}z^{3}$$

$$1 - a^{-1}z$$

$$a^{-1}z$$

$$a^{-1}z - a^{-2}z^{2}$$

$$a^{-2}z^{2}$$

$$a^{-2}z^{2} - a^{-3}z^{3}$$

$$a^{-3}z^{3}$$

$x(z) = 1 - a^{-1}z - a^{-2}z^{2} - a^{-3}z^{3} - a^{-4}z^{4} - \ldots\ldots$

$\therefore \boxed{x(n) = -a^{n}\, u(-n-1)}$

$x(z) = \frac{z}{z-a}$ $\qquad -a+z \,)\, z \,(\, 1 - a^{-1}z \ldots$

$x(n) = -a^{n}\, u(-n-1)$

(*) Using Contour (or) Residue theorem

$$IZT[X(z)] = x(n) = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$$

$$= \text{sum of residues of } X(z) z^{n-1} \text{ at pole } z = z_i$$

$$x(n) = R_1 + R_2 + R_3 + \cdots$$

Q) Find IZT of $X(z) = \frac{5z}{(z-1)(z-2)}$ using residue theorem.

$x(n) = z^{-1}[X(z)] =$ sum of Residue of $X(z) z^{n-1}$ at $z = z_i$

Residue at $z = z_i = (z - z_i) X(z) z^{n-1} \Big|_{z = z_i}$

$$X(z) = \frac{5z}{(z-1)(z-2)}$$

$$X(z) z^{n-1} = \frac{5z}{(z-1)(z-2)} z^{n-1}$$

$$= \frac{5z^n}{(z-1)(z-2)}$$

poles of $X(z) z^{n-1}$

Roots of denominator polynomial $= (z-1)(z-2) = 0$

$z_1 = 1$ and $z_2 = 2$

$X(z) z^{n-1}$ has two distinct poles.

$x(n) =$ Residue of $X(z) z^{n-1}$ at pole $z = z_1$ + Residue of $X(z) z^{n-1}$ at pole $z = z_2$.

$= R_1 + R_2$

$$\therefore R_1 = (z - z_1) X(z) z^{n-1} \Big|_{z = z_1}$$

$$= \cancel{(z-1)} \frac{5z}{\cancel{(z-1)}(z-2)} z^{n-1} \Big|_{z=1}$$

$R_1 = \frac{5(1)^n}{-1} = (-5)(1)^n$

$R_2 = \cancel{(z-2)} \frac{5z}{(z-1)\cancel{(z-2)}} z^{n-1} \Big|_{z=2}$

$R_2 = 5(2)^n$

$\therefore x(n) = R_1 + R_2$

$$\boxed{x(n) = -5(1)^n + 5(2)^n}$$

(9) Find IZT of $X(z) = \frac{10z}{(z-2)(z-3)}$ using residue theorem

$x(n) = z^{-1}[X(z)]$ = sum of residues of $X(z)z^{n-1}$ at $z = z_i$

$$X(z)z^{n-1} = \frac{10z}{(z-2)(z-3)} z^{n-1}$$

$$= \frac{10z^n}{(z-2)(z-3)}$$

poles $= (z-2)(z-3) = 0$

$\Rightarrow z_1 = 2, \; z_2 = 3$

$\therefore x(n) = R_1 + R_2$

$R_1 = (z-z_1) X(z) z^{n-1} \Big|_{z=2}$

$= \cancel{(z-2)} \frac{10z^n}{\cancel{(z-2)}(z-3)} \Big|_{z=2}$

$R_1 = -10(2)^n$

$R_2 = (z-z_2) X(z) z^{n-1} \Big|_{z=3}$

$= \cancel{(z-3)} \frac{10z^n}{(z-2)\cancel{(z-3)}} \Big|_{z=3}$

$R_2 = 10(3)^n$

$\therefore x(n) = R_1 + R_2$

$$\boxed{x(n) = -10(2)^n + 10(3)^n}$$

(8) $X(z) = \dfrac{6z^3 - 2z^2 - z}{(z+1)(z-1)^2}$, find $x(n)$ using residue theorem

$x(n) = z^{-1}[X(z)]$: Sum of residues of $X(z)z^{n-1}$ at $z = z_i$

$$X(z)z^{n-1} = \frac{(6z^2 - 2z - 1)\cancel{z}}{(z+1)(z-1)^2} z^{n-\cancel{1}}$$

$$= \frac{(6z^2 - 2z - 1)z^n}{(z+1)(z-1)^2}$$

poles $= (z+1)(z-1)^2 = 0$

$z+1 = 0 \Rightarrow z_1 = -1$, which is a simple pole

$(z-1)^2 = 0 \Rightarrow z = 1$, which is a 2nd order pole

i.e, $z = 1$ is occuring two times

$x(n) = R_1 + R_2$

$$R_1 = (z - z_1)\, X(z) z^{n-1} \Big|_{z = z_1}$$

$$= \cancel{(z+1)} \frac{(6z^2 - 2z - 1)z^n}{\cancel{(z+1)}(z-1)^2} \Bigg|_{z = -1}$$

$$= \frac{(6 + 2 - 1)(-1)^n}{4}$$

$$R_1 = (-1)^n \frac{7}{4}$$

$$R_2 = (z - z_2)X(z)\, z^{n-1} \Big|_{z = z_2}$$

But, Here z_2 is a 2nd order pole.

$\therefore$ If there is n^{th} order pole, then residue of $X(z)z^{n-1}$ at m^{th} order poles, is given by,

$$R = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[(z - z_0)^n X(z) z^{n-1} \right] \Bigg|_{\text{at } z = z_0}$$

$$\therefore R_2 = \frac{1}{(2-1)!}\frac{d^{2-1}}{dz^{2-1}}\left[\cancel{(z-1)^2}\frac{(6z^2-2z-1)z^n}{(z+1)^1\cancel{(z-1)^2}}\right]\Bigg|_{at\ z=1}$$

$$= \frac{d}{dz}\left[\frac{6z^{n+2}-2z^{n+1}-z^n}{z+1}\right]\Bigg|_{at\ z=1}$$

$$= \frac{\left[(z+1)\left[6(n+1)z^{n+1}-2(n+1)z^n-nz^{n-1}\right]\right] - \left(6z^{n+2}-2z^{n+1}-z^n\right)1}{(z+1)^2}\Bigg|_{at\ z=1}$$

$$= \frac{\left(2\left[6(n+2)(1)^{n+1}-2(n+1)(1)^n-n(1)^{n-1}\right]\right)-\left(6(1)^{n+2}-2(1)^{n+1}-(1)^n\right)}{4}$$

$$= \frac{\left[2\left[6n+12-2n-2-n\right]\right]-\left[6-2-1\right]}{4}$$

$$= \frac{\left[2(3n+10)\right]-3}{4} \neq \cancel{\frac{6n+8-3}{4}} = \cancel{\frac{6n+5}{4}}$$

$$R_2 = \frac{6n+20-3}{4} = \frac{6n+17}{4}$$

$$\therefore x(n) = R_1 + R_2$$

$$x(n) = (-1)^n\frac{7}{4} + \left(\frac{6n+17}{4}\right)$$