Fourier Series

Fourier series Representation of Continuous Time periodic signals

Signals can have dual personalities.

Time domain perpective

(i) Depicts its wave form

(ii) Signal width (duration)

(iii) Rate a which the waveform decays

Frequency Domain perpectives.

(i) in Terms of sinusoidal components

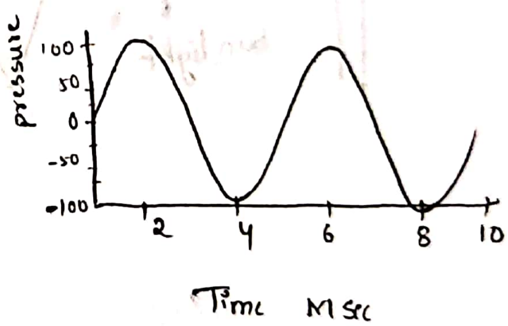
(ii) their relative amplitudes and phases

Time Domain

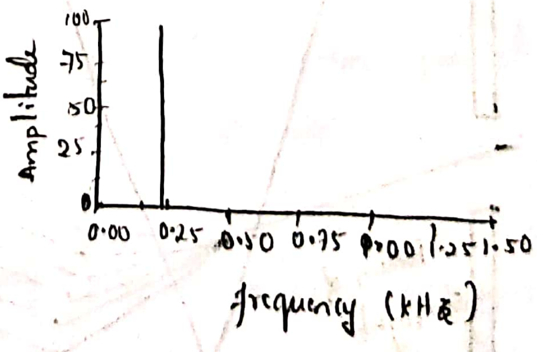
→ Despicts its wave form

Amplitude = 100

frequency = number of cycles in One second = 200Hz



Frequency Domain

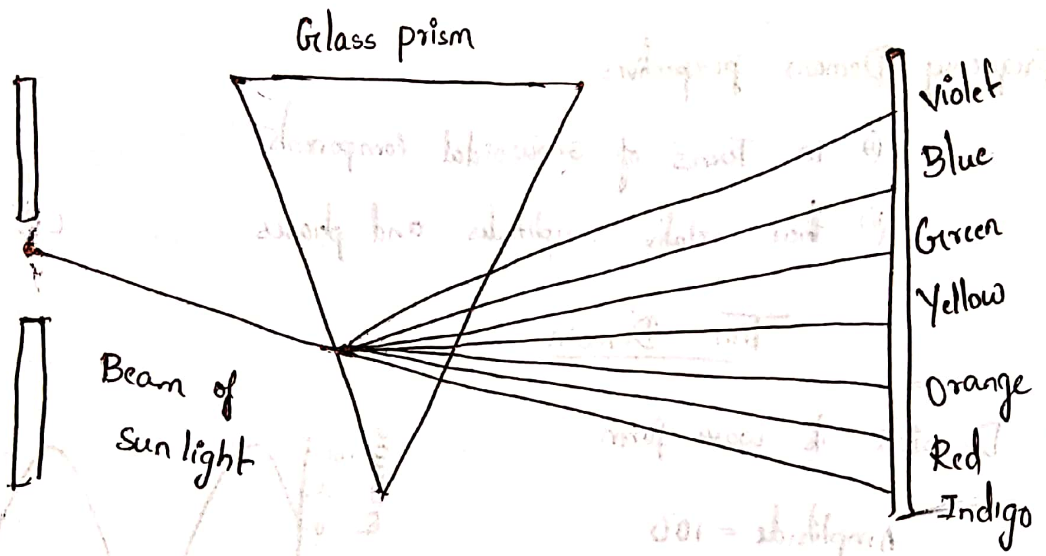


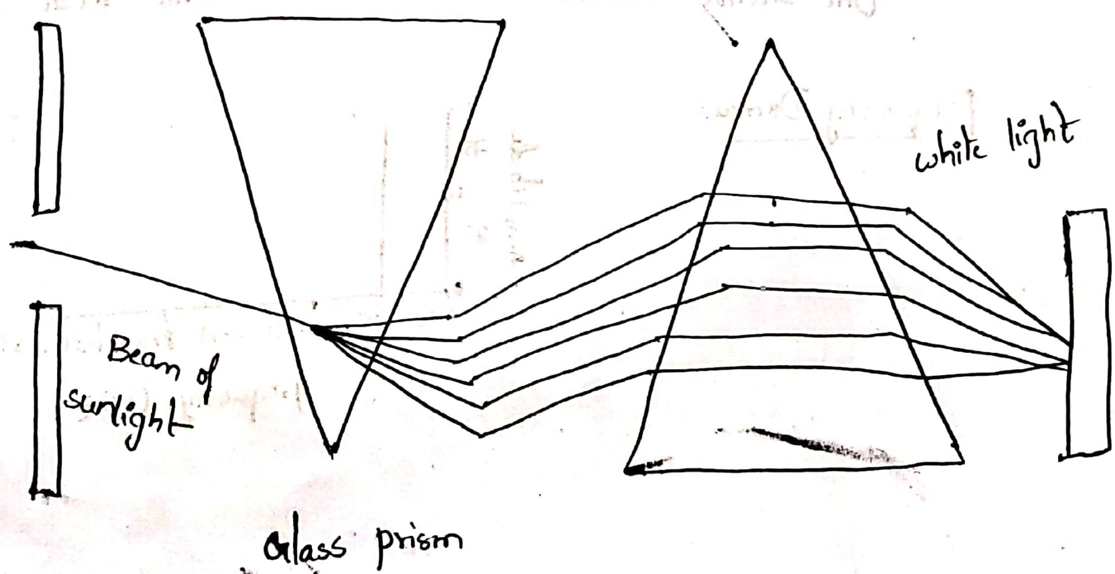
Fourier Series Representation

→ Resolving the signal into its frequency (sinusoids) components is called as frequency Analysis of the signal

→ The Decomposition of the signals in terms of sinusoidal or Complex Exponential components

→ Which such a Decomposition, a signal is said to be represented in the frequency Domain





Fourier Series Representation

→ in Our treatment of frequency analysis, we will develop the proper mathematical tools (prism) for decomposition of the signals (light) into sinusoidal frequency components (colours)

→ The basic motivation for developing the frequency analysis tool is to provide a mathematical & pictorial representation for the frequency components that are contained in any given signal.

Fourier Series

→ The basic mathematical representation of periodic signal is the Fourier series

→ which is a linear weighted sum of harmonically related sinusoids or complex exponentials.

→ The Decomposition of the periodic signals into sinusoids or complex exponentials.

Fourier Transforms

→ Such a Decomposition of the Non-periodic / Energy Signals is called fourier transform representation

Fourier Series Representation of Continuous-time period Signals

→ A signal is periodic, if for some positive value of T, $x(t) = x(t+T)$ for all T

→ The fundamental period of $x(t)$ is the minimum positive, non zero value of T for which the above Equation is satisfied and the value $\omega_0 = 2\pi/T$ is referred to as the fundamental frequency or Angular frequency.

Periodic Signals

Sinusoidal signal

$x(t) = \cos\omega_0 t$

periodic Complex Exponential

$x(t) = e^{j\omega_0 t}$

→ Both of these signals are periodic with fundamental frequency ω_0 & fundamental period $T = \frac{2\pi}{\omega_0}$

Let $\phi_k(t) = e^{j\omega_0 t} = e^{jk(2\pi/T)t}$, $k = 0, \pm 1, \pm 2 \ldots$

is the set of harmonically related complex exponentials

→ Each of these signals has a fundamental frequency that is a multiple of ω_0 and therefore each is periodic with period T.

→ thus, a linear combination of harmonically related Complex Exponentials of the form.

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}$$

is also periodic with period T.

the term for $k=0$ is a constant.

→ The terms for $k = +1$ & $k = -1$ both have fundamental frequency Equal to ω_0 and are collectively referred to as fundamental components or the first harmonic.

→ The two terms for $k = +2$ & $k = -2$ are periodic with half the period (or, Equivalently, twice the frequency) of the fundamental components and are referred to as the Second harmonic

→ More generally, the components for $k = +N$ & $k = -N$ are referred to as the N^{th} harmonic Components

→ thus a linear combination of harmonically related complex exponentials of the form

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}$$

→ The representation of a periodic signal in the form of equation is referred to as the fourier Series representation

Dirichlet's Condition

Condition - 1 $x(t)$ is absolutely integrable over one period

(i.e)

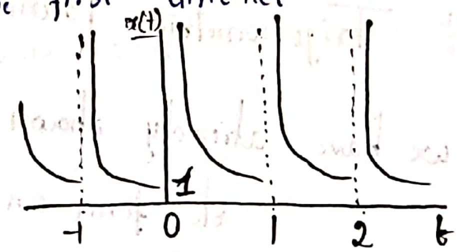
$$\int_T |x(t)|\, dt < \infty$$

Condition - 2 In a finite time interval, $x(t)$ has a finite number of maxima and minima

Condition - 3 In a finite time interval, $x(t)$ has only a finite number of discontinuities

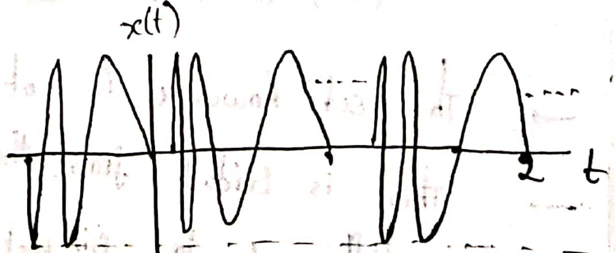
→ A periodic signal that violets the first dirichlet condition is

$$x(t) = \frac{1}{t}, \quad 0 < t \le 1;$$



→ An Example of a function the meets Condition 1 but not Condition 2 is

$$x(t) = \sin\left(\frac{2\pi}{t}\right) \quad 0 < t \le 1$$



an Infinite number of maxima and minima in the interval

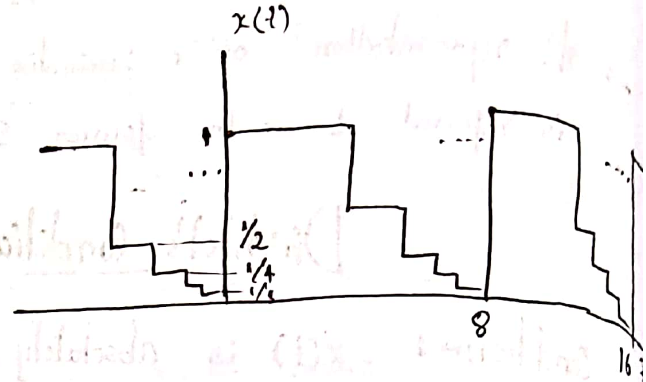
A Signal periodic with period 8 that violates the third Dirichlet condition (for $0 < t < 8$ the value of $x(t)$ decreases by a factor of 2 whenever the distance from t to 8 decreases by a factor 2

$x(t) = 1, \quad 0 \le t < 4$

$x(t) = 1/2 \quad 4 \le t < 6$

$x(t) = 1/4 \quad 6 \le t < 7$

$x(t) = 1/8 \quad 7 \le t < 7.5$



Types of fourier Series Representations

Two Different ways of Expressing the same periodic function they are

(i) Trigonometric fourier Series (TFS) Representation
(ii) Exponential fourier Series (EFS) Representation

Trigonometric fourier Series (TFS) Representation

→ we have already shown that function $\sin \omega_0 t$, $\sin 2\omega_0 t$..... etc. form an Orthogonal set over any interval

$(t_0, \; t_0 + T, \; T = \frac{2\pi}{\omega_0})$

→ this set however is not complete

→ This is Evident from the fact a function $\cos \omega_0 t$ is Orthogonal to $\sin \omega_0 t$ over the same interval.

→ Hence to complete the set, we must include cosine as well as sine functions

→ It can be shown that the composite set of functions consisting a set $\cos n\omega_0 t$ and $\sin n\omega_0 t$ for $(n = 0, 1, 2 \dots)$ forms a complete orthogonal set.

→ Note that for $n = 0$, $\sin n\omega_0 t$ is zero but $\cos n\omega_0 t$ is one. Thus we have a completed orthogonal set represented by functions 1, $\cos \omega_0 t$, $\cos 2\omega_0 t$ ..... $\cos n\omega_0 t$; $\sin \omega_0 t$, $\sin 2\omega_0 t$ ... $\sin n\omega_0 t$ ... etc

→ It therefore follows that any function $f(t)$ can be represented in terms of these functions over any interval $(t_0, t_0+T \quad T = 2\pi/\omega_0)$

thus.

$$f(t) = a_0 + a_1 \cos \omega_0 t + a_2 \cos 2\omega_0 t + \dots + a_n \cos n\omega_0 t + \dots$$
$$+ \; b_1 \sin \omega_0 t + b_2 \sin 2\omega_0 t + \dots + b_n \sin \omega_0 t + \dots$$
$$\left(t_0 < t < t_0 + \frac{2\pi}{\omega_0}\right) \rightarrow ①$$

For convenience we shall denote $\frac{2\pi}{\omega_0}$ by T

the equation can be expressed as

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin \omega_0 t) \qquad (t_0 < t < t_0 + T) \rightarrow ②$$

Equation 2 is the trigonometric fourier series representation of $f(t)$ over an interval $(t_0, t_0 + T)$

$$a_n = \frac{\int_{t_0}^{(t_0+T)} f(t) \cos n\omega_0 t \, dt}{\int_{t_0}^{(t_0+T)} \cos^2 n\omega_0 t \, dt}$$

$$b_n = \frac{\int_{t_0}^{(t_0+T)} f(t) \sin n\omega_0 t \, dt}{\int_{t_0}^{t_0+T} \sin^2 n\omega_0 t \, dt}$$

If we let $n = 0$ in a_n we get

$$a_0 = \frac{1}{T}\int_{t_0}^{(t_0+T)} f(t)\, dt$$

we also have

$$\int_{t_0}^{(t_0+T)} \cos^2 n\omega_0 t\, dt = \int_{t_0}^{(t_0+T)} \sin^2 n\omega_0 t\, dt = \frac{T}{2}$$

therefore

$$a_n = \frac{2}{T}\int_{t_0}^{(t_0+T)} f(t) \cos n\omega_0 t\, dt$$

$$b_n = \frac{2}{T}\int_{t_0}^{(t_0+T)} f(t) \sin n\omega_0 t\, dt$$

the constant term a_0 in the series is given by

$$a_0 = \frac{1}{T}\int_{t_0}^{(t_0+T)} f(t)\, dt$$

the constant term a_0 is the average value of $f(t)$ over the interval (t_0, t_0+T). Thus a_0 is the DC component of $f(t)$ over this interval.

Exponential fourier Series (EFS) Representation

It can be shown easily that a set of exponential function $\{e^{jn\omega_0 t}\}$, $(n = 0, \pm 1, \pm 2 \ldots)$ is orthogonal over an interval $(t_0, t_0 + 2\pi/\omega_0)$ for any value of t_0. Note that this is a set of complex functions we can demonstrate the orthogonality of this set by considering the integral

$$I = \int_{t_0}^{t_0 + \frac{2\pi}{\omega_0}} (e^{jn\omega_0 t})(e^{jm\omega_0 t})^* \, dt$$

$$= \int_{t_0}^{t_0 + \frac{2\pi}{\omega_0}} e^{jn\omega_0 t} e^{-jm\omega_0 t} \, dt$$

If $n = m$ the Integral in the Integral I is given by

$$I = \int_{t_0}^{t_0 + 2\pi/\omega_0} dt = \frac{2\pi}{\omega_0}$$

If $n \neq m$ the Integral I is given by

$$I = \frac{1}{j(n-m)\omega_0} e^{j(n-m)\omega_0 t} \Big|_{t_0}^{t_0 + 2\pi/\omega_0}$$

$$= \frac{1}{j(n-m)\omega_0} e^{j(n-m)\omega_0 t_0} \left[e^{j2\pi(n-m)} - 1 \right]$$

Since both n and m are Integers $e^{j2\pi(n-m)}$ is equal to Unity, and hence the Integral is zero. $I = 0$

Thus

$$\int_{t_0}^{t_0 + \frac{2\pi}{\omega_0}} e^{jn\omega_0 t} (e^{jn\omega_0 t})^* \, dt = \begin{cases} \frac{2\pi}{\omega_0} & m = n \\ 0 & m \neq n \end{cases}$$

$$\frac{2\pi}{\omega_0} = T$$

It is evident from eq that the set of functions

$$\{ e^{jn\omega_0 t} \} \quad (n = 0, \pm 1, \pm 2 \ldots)$$

is orthogonal over the interval $(t_0, t_0 + T)$ where $T = \frac{2\pi}{\omega_0}$. Further, it can be shown that this is a complete set. It is therefore possible to represent an arbitrary function $f(t)$ by a linear combination of exponential functions over an interval $(t_0, t_0 + T)$

This is a complete set. It is therefore possible to represent an arbitrary function $f(t)$ by a linear combination of exponential functions over an interval (t_0, t_0+T):

$$f(t) = F_0 + F_1 e^{j\omega_0 t} + F_2 e^{j2\omega_0 t} + \cdots F_n e^{jn\omega_0 t} \cdots$$
$$F_{-1} e^{-j\omega_0 t} + F_{-2} e^{-j2\omega_0 t} + \cdots F_{-n} e^{-jn\omega_0 t} + \cdots$$

for

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} \qquad (t_0 < t < t_0+T)$$

Known as Exponential fourier series representation of $f(t)$ over the interval (t_0, t_0+T)

The various coefficient in series is given by

$$F_n = \frac{\int_{t_0}^{t_0+T} f(t) \, e^{(jn\omega_0 t)^*} dt}{\int_{t_0}^{t_0+T} e^{jn\omega_0 t} \left(e^{jn\omega_0 t}\right)^* dt}$$

$$= \frac{\int_{t_0}^{t_0+T} f(t) \, e^{-jn\omega_0 t} dt}{\int_{t_0}^{t_0+T} e^{jn\omega_0 t} \cdot e^{-jn\omega_0 t} dt}$$

$$= \frac{1}{T} \int_{t_0}^{t_0+T} f(t) \, e^{-jn\omega_0 t} dt$$

Summarizing the results:- Any given function $f(t)$ may be expressed as a discrete sum of exponential functions $\{e^{jn\omega_0 t}\}$, $(n = 0, \pm 1, \pm 2 \ldots)$ over an interval $t_0 < t < t_0 + T$ $\left(\omega_0 = \frac{2\pi}{T}\right)$

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} \qquad (t_0 < t < t_0 + T)$$

$$F_n = \frac{1}{T} \int_{t_0}^{t_0+T} f(t)\, e^{-jn\omega_0 t}\, dt$$

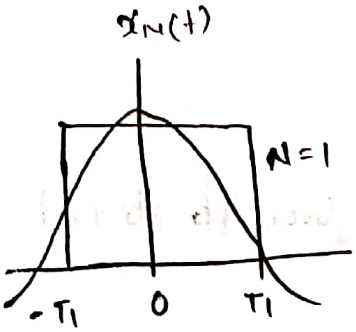
Relationship between TFS & EFS Representations

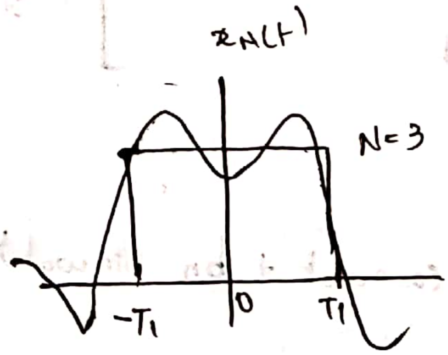
It should be noted that the trigonometric and the exponential Fourier Series are not two different types of Series but two different types of ~~Series~~ ways of expressing the same series. The coefficients of one series can be obtained from those of the other

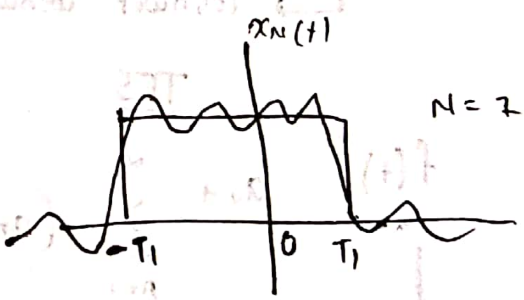
$$a_0 = F_0$$

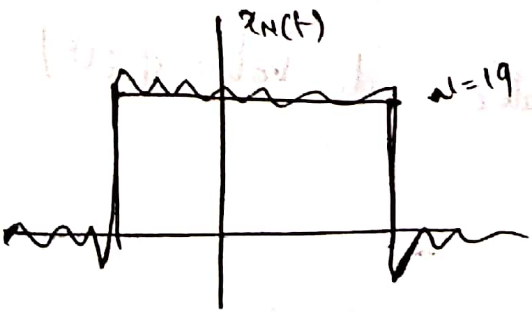
$$a_n = F_n + F_{-n} \qquad \& \qquad F_n = \frac{1}{2}(a_n - j b_n)$$

$$b_n = j(F_n - F_{-n})$$



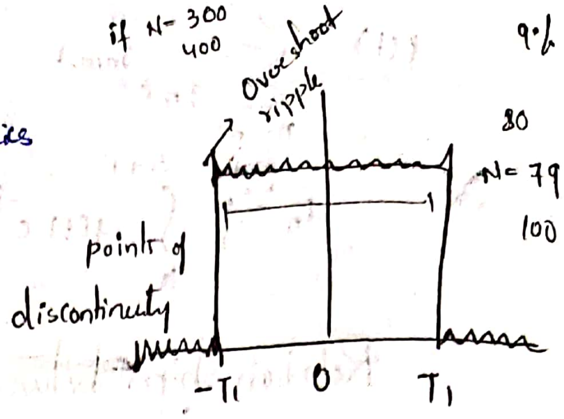






GIBB's phenomenon

Convergence of the fourier Series representation of a Square wave



Fourier Series

Classifications of Fourier Series — TFS / EFS

(i) Set of mutually Orthogonal functions forms a Complete set then any function (periodic / non-periodic) can be Expressed in terms of these Orthogonal functions

→ Sin nω₀t, cos nω₀t → orthogonal
→ sin, cos
→ cos nω₀t cos nω₀t

Only sin ω₀t → not complete
Only cos ×
Sin nω₀t, cos nω₀t — complete

TFS

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t) \quad \text{Over } (t_0, t_0+T)$$

↓

periodic

$$a_0 = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)\,dt \rightarrow \text{avg value} = \text{dc value of } f(t)$$

$$a_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t) \cos n\omega_0 t$$

$$b_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t) \sin n\omega_0 t\, dt$$

EFS

→ Complex exponential function = $e^{j\omega_0 t}$

set of complex exponential functions

$$= e^{j\omega_0 t}, \; e^{j2\omega_0 t} \; \ldots\ldots \; e^{jn\omega_0 t}$$

$\downarrow$ 1st harmonic, $\downarrow$ 2nd, $\downarrow$ 3rd

$$n = [0, \pm 1, \pm 2 \ldots]$$

$$= e^{-j\omega_0 t}, \; e^{-jn\omega_0 t} \; \ldots \; e^{-jn\omega_0 t}$$

TFS

sin cos → form complete

$\dfrac{\sin n\omega_0 t}{\sin n\omega_0 t}$ → set

whether orthogonal or not

$$\phi_K(t) = e^{\pm jn\omega_0 t}$$

→ check for orthogonality complex exponential

$$T = \int_{t_1}^{t_2} f_1(t)\, f_2^*(t)\, dt = 0$$

$$f_1(t) = e^{jn\omega_0 t}$$

$$f_2(t) = e^{jm\omega_0 t} \longrightarrow f_2^*(t) = a \pm ib$$

$$f_2^*(t) = e^{-jm\omega_0 t}$$

$$I = \int_{t_0}^{t_0 + \frac{2\pi}{\omega_0}} e^{jn\omega_0 t} \cdot e^{-jm\omega_0 t}\, dt$$

$$= \int_{t_0}^{t_0 + \frac{2\pi}{\omega_0}} e^{j(n-m)\omega_0 t}\, dt$$

$$\left. \frac{e^{j(n-m)\omega_0 t}}{(n-m)\omega_0} \right|_{t_0}^{t_0 + \frac{2\pi}{\omega_0}}$$

$$= \frac{1}{(n-m)\omega_0}\left[e^{j(n-m)\omega_0 + \frac{2\pi}{\omega_0}} - e^{+j(n-m)\omega_0 t}\right]$$

$$= \frac{1}{(n-m)\omega_0}\left[e^{j(n-m)\omega_0 t_0} \cdot e^{j(n-m)2\pi} - e^{j(n-m)\omega_0 t_0}\right]$$

n, m are Integers

$$I = \frac{1}{(n-m)\omega_0}\left[e^{j(n-m)\omega_0 t_0}\left(\frac{e^{j(n-m)2\pi}}{1} - 1\right)\right]$$

$e^{j2\pi k}$

$\underset{1}{\underline{\cos 2\pi k}} + \underset{0}{\underline{\sin 2\pi k}}$

$$I = \cancel{\frac{1}{(n-m)\omega_0}}\left[\cancel{e^{j(n-m)\omega_0 t_0}}\right.$$

$\boxed{I = 0}$ $\int_{t_1}^{t_2} f_1(t)\, f_2^{*}(t)\, dt = 0$ $\left.\begin{matrix} e^{jn\omega_0 t} \\ e^{jm\omega_0 t}\end{matrix}\right\}$ Orthogonal set

(i) Orthogonal (ii) Complete $\rightarrow e^{jn\omega_0 t} \rightarrow ; n = 0, \pm 1, \pm 2, \pm 3 \ldots$

$e^{jn\omega_0 t} = \cos n\omega_0 t + j \sin n\omega_0 t$ complete set

(iii) Any function can be expressed by these set of mutually orthogonal complex exponential functions $\rightarrow$ EFS

Sinusoids, TFS

$$f(t) \cong F_0 + F_1 e^{j\omega_0 t} + F_2 e^{j2\omega_0 t} + \ldots\ldots F_n e^{+jn\omega_0 t} + \ldots\ldots$$
$$F_{-1} e^{-j\omega_0 t} + F_{-2} e^{-j2\omega_0 t} + \ldots\ldots F_{-n} e^{-jn\omega_0 t}$$

$$F_i(t) = F_0 + \sum_{n=\infty}^{\infty} F_n e^{jn\omega_0 t} \quad (n \neq 0)$$

$$f(t) = \sum_{n=\infty}^{\infty} F_n e^{jn\omega_0 t} \rightarrow \text{EFS of } \left(t_0,\ t_0 + \frac{2\pi}{\omega_0}\right)$$

Over One period

$$F_n = \frac{\int_{t_0}^{t_0 + \frac{2\pi}{\omega_0}} f(t) \cdot e^{-jn\omega_0 t}\, dt}{\int_{t_0}^{t_0 + \frac{2\pi}{\omega_0}} e^{jn\omega_0 t} \cdot e^{-jn\omega_0 t}\, dt}$$

$$C_{12} = \frac{\int f_1(t)\, f_2^*(t)}{f_2(t)\, f_2^*(t)}$$

$$e^0 = \int_T 1\, dt$$

$$\Rightarrow F_n = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)\, e^{-jn\omega_0 t}\, dt \longrightarrow \text{EFS}$$

$$n = 0 \Rightarrow F_0 = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)\, dt \Rightarrow a_0 \ \frac{\text{avg value}}{\text{DC value}}$$

$$f_1 \cdot f(t)$$

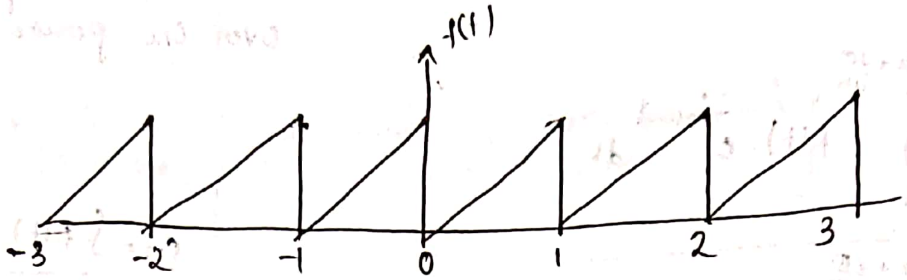
EFS $$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} \longrightarrow$$

where $$F_n = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)\, e^{-jn\omega_0 t}\, dt$$

TFS $f(t) =$ periodic

$$a_0 + \sum_{n=\infty}^{\infty} \left(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t\right)$$

Problem



f(t) → periodic

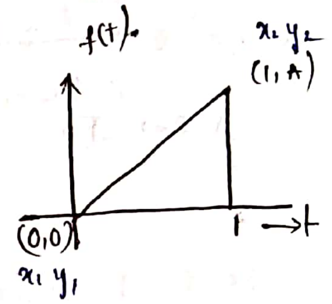
T = 1

0 - 1 = 1
2 - 1 = 1
3 - 1 = 1

Sol

$$TFS = f(t) \approx a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

$$a_0 = \frac{1}{T}\int_0^T f(t)\,dt =$$



$$\overset{f(t)}{y} - y_1 = m(x - x_1)$$

$$m = \frac{y_2 - y_1}{x_2 - x_1} \longrightarrow \text{slope}$$

$$f(t) - 0 = \frac{A - 0}{1 - 0}(t - 0)$$

$$f(t) = At \quad \text{for} \quad (0 \le t \le 1)$$

$$a_0 = \frac{1}{1}\int_0^1 At\,dt$$

$$\left[\frac{A t}{2}\right]$$

$$A\int_0^1 t\,dt$$

$$a_0 = \frac{A}{2}$$

$$A\left[\frac{t^2}{2}\right]_0^1$$

$$\left[\frac{At^2}{2}\right]_0^1$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega_0 t\, dt$$

$T = \frac{2\pi}{\omega_0} \Rightarrow \frac{2\pi}{T} = \omega_0$

$$= \frac{2}{1}\int_0^1 At\cos(n\,2\pi t)\, dt$$

$\int uv = u\int v - \int u'\int v$

$\int uv = u\int v - \int u'\int v$

$$= 2A\left[t\,\frac{\sin 2\pi nt}{2\pi n} - \int 1\cdot\frac{\sin 2\pi nt}{2\pi n}\,dt\right]$$

$$= 2A\left[t\,\frac{\sin 2\pi nt}{2\pi n} + \frac{\cos 2\pi nt}{(2\pi n)^2}\right]_0^1$$

$$a_n = 2A\left[\frac{1}{2\pi n^2} - \frac{1}{2\pi n^2)}\right] = 0$$

$$\boxed{\begin{gathered} 2A\left(\frac{1}{2\pi n} - 0 - 0\right) \\ \frac{A}{\pi n} \\ b_n = \frac{A}{\pi n}\end{gathered}}$$

$\omega_0 = 2\pi$

$T = 1$

$$\underline{b_n} = \frac{2}{T}\int_0^T f(t)\sin(n\omega_0 t)\, dt$$

$$= 2A\left[t\cdot\frac{\cos 2\pi nt}{2\pi n} + \int 1\cdot\frac{\cos 2\pi nt}{(2\pi n)}\,dt\right]$$

$2A\left[\frac{1}{2\pi n} - 0 - 0\right]$

$-\frac{A}{\pi n}$

$b_n = -\frac{A}{\pi n}$

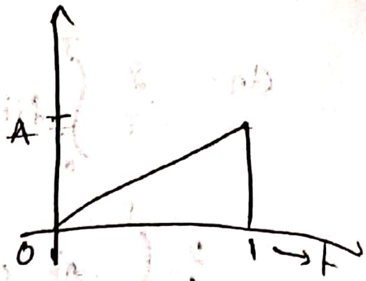
$$= 2A\left[t\cdot\frac{\cos 2\pi nt}{2\pi n} + \frac{\sin 2\pi nt}{(2\pi n)^2}\right]_0^1$$

$a_0 = \frac{A}{2}$ $\quad a_n = 0$ $\quad b_n = -\frac{A}{\pi n}$

$$f(t) = \frac{A}{2} + \sum_{n=1}^{\infty} \frac{A}{\pi n}\sin(2\pi nt) \quad \text{for } (0 \le t \le 1)$$

$$\boxed{f(t) = \frac{A}{2} + -\frac{A}{\pi n}\sum_{n=1}^{\infty} \sin 2\pi nt}$$ TFS

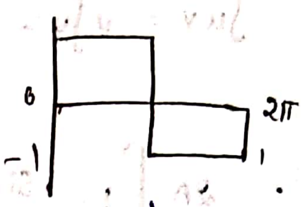
$$F(t) = \frac{A}{2} + \sum_{n=1}^{\infty} \left(\frac{A}{-\pi n}\right) \sin(2\pi n t)$$



period $T = 1$

$$F(t) = \frac{A}{2} - \frac{A}{\pi} \sum_{n=1}^{\infty} \frac{\sin 2\pi n t}{n}$$

$$= \frac{A}{2} - \frac{A}{\pi} \sin(2\pi t) - \frac{A}{2} \sin(4\pi t)$$



$$- \frac{A}{3} \sin(6\pi t) \qquad a_n = 0$$

TFS

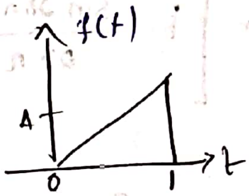
$\sin nt$

$\frac{4}{\pi} \sin t + \frac{4}{3\pi} \sin 3t \ldots$

$\frac{4}{\pi} \sin nt$.

EFS ?

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$$



$F_0 + F_n = ?$

$f(t) = At$ for $0 \le t \le 1$

$T = 1$

$\omega_0 = \frac{2\pi}{T} = 2\pi$

$$F_0 = \frac{1}{T} \int_0^T f(t)\, dt$$

$$= \frac{1}{1} \int_0^1 At\, dt = \frac{A}{2} = a_0$$

$$F_n = \frac{1}{T} \int_0^T f(t) e^{-jn\omega_0 t} dt$$

$$F_n = \frac{1}{1} \int_0^1 At \cdot e^{-j(n 2\pi t)} dt$$

$e^{-j2\pi n}$

$= 1$

$$= A\left[t \cdot \frac{e^{-j2\pi n t}}{-j2\pi n} - \int 1 \cdot \frac{e^{-j2\pi n t}}{(-j2\pi n)} dt\right]$$

$F_n = -\frac{A}{j2\pi n}$

$f(t) = \frac{A}{2} - A \sum_{n=-\infty}^{\infty} \frac{e^{j2\pi n t}}{j2\pi n}$

$$= A\left[\frac{-t \cdot e^{-j2\pi n t}}{2j\pi n} - \frac{e^{-j2\pi n t}}{(-j2\pi n)^2}\right]_0^1$$

$$= A\left[\frac{-1}{j2\pi n} + \frac{1}{(2\pi n)^2} - \frac{0+1}{(2\pi n)^2}\right]$$

$$= A\left[-\frac{1}{j2\pi n} + \frac{\cancel{1}}{2\pi n^2} - \frac{\cancel{1}}{2\pi n^2}\right]$$

$$f(t) = \frac{A}{2} - \frac{A}{j2}\sum_{n=-\alpha}^{\alpha} \frac{e^{-j2\pi nt}}{\pi n} \quad \text{EFS}$$

$$f(t) = \frac{A}{2} + \frac{A}{2}\sum_{n=1}^{\infty} \frac{\sin 2\pi nt}{\pi n} \quad \text{TFS}$$

$$F_n = -\frac{A}{j2\pi n}$$

Relationship between TFS & EFS

$$\text{EFS } f(t) = \sum_{n=-\infty}^{\alpha} F_n e^{jn\omega_0 t} \quad \text{where}$$

$F_n = \alpha_n + j\beta_n$

$F_1 = \alpha_1 + j\beta_1$

$F_{-1} = \alpha_{-1} + j\beta_{-1}$

$$F_n = \frac{1}{T}\int_0^T f(t)\, e^{-jn\omega_0 t}\, dt \longrightarrow (1)$$

$$F_{-n} = \frac{1}{T}\int_0^T f_1(t)\, e^{-jn\omega_0 t}\cdot dt \longrightarrow (2)$$

$$F_n = F_{-n}^*$$

$$\rightarrow f(t) = F_0 + F_1 e^{j\omega_0 t} + F_2 e^{j2\omega_0 t} + \ldots\ldots F_n e^{jn\omega_0 t}$$
$$F_{-1} e^{-j\omega_0 t} + F_{-2} e^{-j2\omega_0 t} + \ldots\ldots F_{-n} e^{-jn\omega_0 t}$$

$F_n = \alpha_n + j\beta_n$

$F_{-n} = \alpha_n - j\beta_n$

$$= F_0 + (\alpha_1 + j\beta_1)e^{j\omega_0 t} + (\alpha_2 + j\beta_2)e^{j2\omega_0 t} + \ldots\ldots (\alpha_n + j\beta_n)e^{jn\omega_0 t}$$
$$+ (\alpha - j\beta_1)e^{-j\omega_0 t} + (\alpha_2 - j\beta_2)e^{-j2\omega_0 t} + \ldots\ldots (\alpha_n - j\beta_n)e^{-jn\omega_0 t}$$

$$= F_0 + \alpha_1(e^{j\omega_0 t} + e^{-j\omega_0 t}) + \alpha_2(e^{j2\omega_0 t} + e^{-j2\omega_0 t}) + \ldots\ldots$$
$$+ j\beta_1(e^{j\omega_0 t} - e^{-j\omega_0 t}) + j\beta_2(e^{j2\omega_0 t} - e^{-j2\omega_0 t}) + \ldots\ldots$$

$= F_0 + \alpha_1 2\cos\omega_0 t + \alpha_2 \cos 2\omega_0 t + \cdots\cdots \alpha_n 2\cos n\omega_0 t + \cdots$

$+ j\beta_1 2j\sin n\omega_0 t + j\beta_2 (\sin 2\omega_0 t) + \cdots (j\beta_n)(j)(\sin n\omega_0 t)$

$$= F_0 + \sum_{n=1}^{\infty} (2\alpha_n \cos n\omega_0 t - 2\beta_n \sin n\omega_0 t) \xrightarrow{(3)} \text{EFS}$$

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t) \longrightarrow (4)$$

$a_0 = F_0$ $\quad a_n = 2\alpha_n$ $\quad b_n = -2\beta_n$ $= j(F_n - F_{-n})$

$F_n = \alpha_n + j\beta_n$

$F_{-n} = \alpha_n - j\beta_n$

$F_n + F_{-n} = 2\alpha_n$

$F_n + F_{-n}, F_0 \longrightarrow$ $a_n = F_n + F_{-n}$

$a_0 = F_0$

$b_n = j(F_n - F_{-n})$

$F_n - F_{-n} = 2j\beta_n$

$F_n = \frac{-A}{j2\pi n}$

$F_n = \frac{-A}{j2\pi n} + \frac{A}{j2\pi n}$

$a_0\ a_n\ b_n \longrightarrow$ $F_n = \frac{1}{2}(a_n - jb_n)$

$F_{-n} = \frac{1}{2}(a_n + jb_n)$

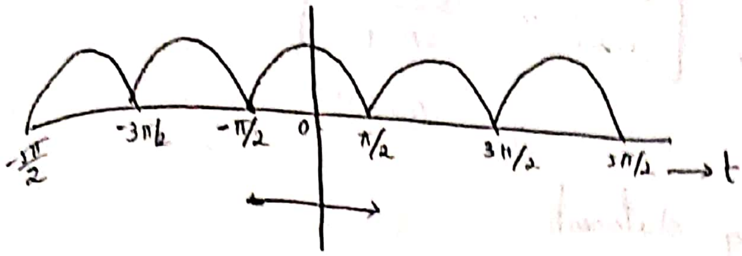
$F_n = \frac{-A}{j2\pi n}$

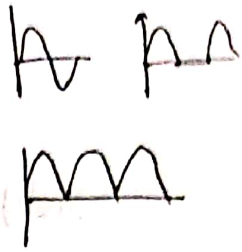
$F_{-n} = \frac{-A}{j2\pi(-n)} = \frac{A}{j2\pi n} \Rightarrow F_n + F_{-n} = 0 = a_n$

$j(F_n - F_{-n}) = j\left(\frac{-A}{j2\pi n} - \frac{A}{j2\pi n}\right)$

$= \cancel{j}\left(\frac{-2A}{\cancel{j}2\pi n}\right) = \frac{-A}{\pi n} = b_n$

Full wave rectified cosine function





$T = \pi$

$$F(t) = a_0 + \sum^{\infty} a_n (\cos n\omega_0 t + b_n \sin n\omega_0 t)$$

$$a_0 = \frac{1}{T}\int_0^{\pi} f(t)\,dt$$

$$= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos t \; dt$$

$$= \frac{1}{\pi} \sin t \Big|_{-\pi/2}^{\pi/2}$$

$$= \frac{2}{\pi}\left(\sin \pi/2 - \sin\left[-\pi/2\right)\right]$$

$$\omega_0 = \frac{2\pi}{T}$$

$$= \frac{2\pi}{\pi} = 2$$

$$a_n = \frac{1}{T}\int_{-\pi/2}^{\pi/2} f(t) \cos n\omega_0 t \; dt$$

$$= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos t \quad \cos(2n\pi)\,dt$$

$$= 2\cos A \cos B = A(A+B) + \cos(A-B)$$

$$\frac{1}{T}\int_{-\pi/2}^{\pi/2} \frac{\cos(1+2n)t + \cos(2n-1)t}{2}\,.\,dt$$

$\cos t \quad \cos 2nt$

$$\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} \cos(1+2n)t\,dt. \quad \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} \cos(2n-1)t\,dt$$

$$a_n = \frac{2}{\pi}\left[\frac{(-1)^n}{2n+1} + \frac{(-1)^{n+1}}{2n-1}\right]$$

$b_n = 0 \rightarrow \qquad \rightarrow \left(-\pi/2 \le t \le \frac{\pi}{2}\right)$

$$f(t) = \frac{2}{\pi} + 2\sum_{n=1}^{\infty}\left[\frac{(-1)^n}{2n+1} + \frac{(-1)^{n+1}}{2n-1}\right] \cos 2nt$$

Justify the following statements

(i) Odd function have only Sine terms. $(a_n = 0,\ b_n \neq 0)$

(ii) Even function have only cosine terms $(a_n \neq 0,\ b_n = 0)$

(iii) function with wave symmetry having only odd harmonics $(a_n \neq 0,\ b_n \neq 0$ but $n = \text{odd})$

Definition $\longrightarrow$ Even function $\longrightarrow$ $f(t) = f(-t)$ cosine

$f(-t) = f(t) \rightarrow$ Symmetrical about Vertical axis

Odd $\rightarrow f(t) \longrightarrow f(-t) = -f(t) \longrightarrow$ Anti Symmetric about Vertical axis

Even x even = even

even x odd = odd

odd x odd = Even

We know that $\left[\begin{array}{l} \cos n\omega_0 t \rightarrow \text{is an Even function} \\ \sin n\omega_0 t \rightarrow \text{is an odd function} \end{array}\right.$

$f(t) \rightarrow$ periodic

$$I = \int_{-T}^{T} f_e(t)\,dt = \int_{-T}^{0} f_e(t)\,dt + \int_{0}^{T} f_e(t)\,dt$$

$t = -t$

$$= \int_0^T f_e(t)\,dt + \int_0^T f_e(t)\,dt = 2\int_0^T f_e(t)\,dt$$

$I =$ $f(t) =$ periodic, $f_o(t)$

$$I = \int_{-T}^{T} f_o(t)\,dt$$

$$I = \int_{-T}^{0} f_o(t)\,dt + \int_0^T f_o(t)\,dt$$

$$I = \int_{-T}^{0} f_o(-t)\,dt + \int_0^T f_o(t)\,dt$$

$$= -\int_0^T f_o(t)\,dt + \int_0^T f_o(t)\,dt$$

$I = 0$ → Even $\quad \int_{-T}^{T} f_e(t)\,dt = 2\int_0^T f_e(t)\,dt$

→ Odd $\quad \int_{-T}^{T} f_o(t)\,dt = 0$

$f(t) \Rightarrow$ Even function − $f_e(t) \longrightarrow$

$$f_e(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t\right) \longrightarrow \text{TFS}$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} \underbrace{f_e(t)}_{\text{Even}} \times \underbrace{\cos n\omega_0 t}_{\text{Even}}\,dt \longrightarrow \text{Even}$$

$$= \frac{2}{T}\int_{-T/2}^{T/2} f(t)\,dt$$

$$a_n = \frac{4}{T}\int_0^{T/2} f_e(t)\cdot \cos n\omega_0 t \cdot dt$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f_e(t)\sin n\omega_0 t \, dt$$

Even x odd → odd

$$I = \int_{-T/2}^{T/2} \text{odd function} = 0$$

$b_n = 0$

(i.e) $f_e(t)$ for Even fn → only cosine Terms are Existing.

$f(t) = f_o(t) \longrightarrow$ periodic → TFS / EFS

$$F_o(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f_o(t)\cos n\omega_0 t \, dt = 0$$

odd x even
odd

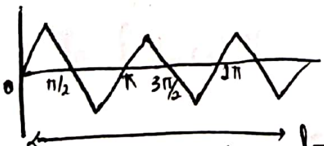
$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f_o(t)\sin n\omega_0 t \, dt$$

odd x odd
even

$$= \frac{4}{T}\int_{0}^{T/2} f_o(t)\sin n\omega_0 t \, dt$$

for odd function have Only Sine Terms, ($b_n \neq 0$, $a_n = 0$)

Functions with half wave Symmetry



half wave Symmetry

even = $f(-t) = f(t)$

odd = $f(-t) = -f(t)$

half wave symmetry

$f(t \pm T/2) = -f(t)$

$f(t) = -f(t \pm T/2)$

$= (T/2 + T/2)$

$$a_n = \frac{2}{T}\int_{0}^{T} f(t)\cos n\omega_0 t \, dt$$

$$a_n = \frac{2}{T}\int_0^{T/2} f(t)\cos n\omega_0 t\, dt + \frac{2}{T}\int_{T/2}^{T} f(t)\cos n\omega_0 t\, dt$$

$t + T/2$

$$= \frac{2}{T}\int_0^{T/2} f(t)\cos n\omega_0 t\, dt + \int_{T/2}^{T} f(t+T/2)\cos n\omega_0 (t+T/2)\, dt$$

$$f(t \pm T/2) = -f(t)$$

$$a_n = \frac{2}{T}\int_0^{T/2} f(t)\cos n\omega_0 t\, dt - \frac{2}{T}\int_0^{T/2} f(t)\cos n\omega_0 t \cos n\pi t\, dt$$

n - even $\quad I = 0$

n - odd

$$= \frac{4}{T}\int_0^{T/2} f(t)\cos n\omega_0 t\, dt$$

odd hamonics

$t = t + T/2$

$T/2 = 0$

$t = t/2$

$\cos n\omega_0 t \cdot \cos \frac{n\omega_0 t}{2}$

$T = \frac{2\pi}{\omega_0} \Rightarrow \omega_0 t = 2\pi$

$\cos n\omega_0 t \cos(\pi n t)$

$b_n = 0$ for n = even

$$= \frac{4}{T}\int_0^{T/2} f(t)\sin\omega_0 t\, dt = \text{odd value of } n$$

half wave symmetry $\rightarrow$ $a_n = 0$ for n = even

$$\frac{4}{T}\int_0^{T/2} f(t)\cos n\omega_0 t\, dt \quad \text{for } n = \text{odd}$$

$b_n = 0$ for n = even

$$= \frac{4}{T}\int_0^{T/2} f(t)\sin n\omega_0 t\, dt \quad \text{for } n = \text{odd}$$

$\rightarrow$ have only odd harmonics

Even $\rightarrow$ cosine terms ($a_n \neq 0$) $\quad b_n = 0$

odd $\rightarrow$ sine terms ($b_n \neq 0$)

EFS co efficients $f(t) = \sin\omega_0 t$

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} \quad (t_0, t_0+T)$$

$$= F_{-2} e^{-j2\omega_0 t} + F_{-1} e^{-j\omega_0 t} + F_0 \rightarrow \text{dc value/avg value}$$

$$F_2 e^{j2\omega_0 t} + F_1 e^{j\omega_0 t}$$

2nd harmonic — 1st harmonic

EFS

TFS → a_0, a_n, b_n co efficients

EFS = F_n co efficients

$F_0 \rightarrow$ dc component

$e^{j\omega_0 t}, e^{-j\omega_0 t} \longrightarrow$ 1st harmonic

$|F_1| \rightarrow$ EFS co-efficient

$|F_n|$

$|F_{-n}|$

$e^{2j\omega_0 t}, e^{-j2\omega_0 t} \longrightarrow$ EFS

$F_2, F_{-2} \rightarrow$ EFS cofficients

same in magnitude diffent phase

$e^{jn\omega_0 t}, e^{-jn\omega_0 t} \longrightarrow$ n^{th} harmonic

F_n, F_{-n}

$$\longrightarrow f(t) = \sin\omega_0 t = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}$$

$$= \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t}$$

$F_1 = \frac{1}{2j}$ $\quad F_0 \rightarrow dc$

$F_{-1} = -\frac{1}{2j}$ $\quad \underbrace{\begin{bmatrix} F_2 & \cdots & F_n \\ F_{-n} & & F_{-n} \end{bmatrix}}_{\text{zero}}$

$|F_1| = |F_{-1}|$

* $f(t) = 1 + \sin\omega_0 t + 2\cos\omega_0 t + \cos(2\omega_0 t + \pi/4)$

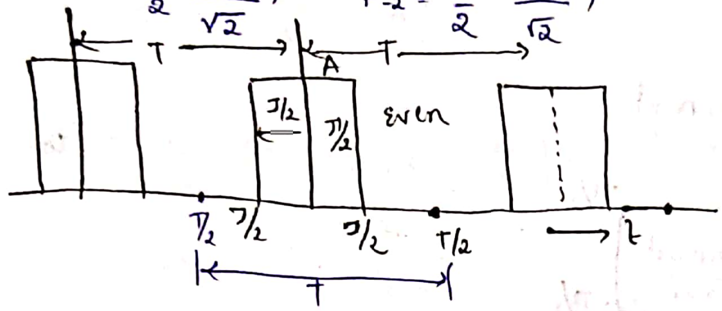
$$1 + \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} + \frac{\cancel{2}(e^{j\omega_0 t} + e^{-j\omega_0 t})}{\cancel{2}} + \frac{e^{(j2\omega_0 t + \pi/4)} + e^{-j2\omega_0 t + \pi/4}}{2}$$

$$1+\left(1+\frac{1}{2j}\right)e^{j\omega_0 t}+\left(1-\frac{1}{2j}\right)e^{-j\omega_0 t}+\frac{1}{2}e^{j2\omega_0 t}\cdot e^{j\pi/4}+\frac{1}{2}e^{-j2\omega_0 t}\cdot e^{j\pi/4}$$

$F_0 = 1$, $F_1 = 1+\frac{1}{2j}$ $\quad F_{-1} = \left(1-\frac{1}{2j}\right)$

$F_2 = \frac{1}{2}e^{j\pi/4}$ $\quad F_{-2} = \frac{1}{2}e^{-j\pi/4}$

$\frac{1}{2}\frac{(1+j)}{\sqrt{2}}$ $\quad F_{-2} = \frac{1}{2}\frac{(1-j)}{\sqrt{2}}$



$a_0 = 0$

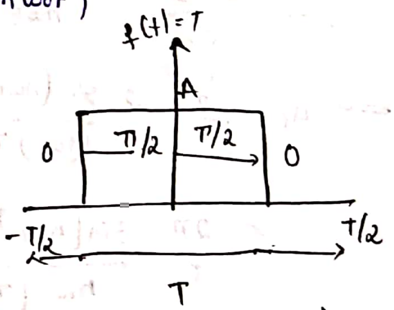
$a_n \neq 0$

$b_n = 0$

$$\int_{-T/2}^{T/2} f(t) = A \quad -\tau/2 < t < \tau/2$$
$$= 0 \quad \text{other}$$

<u>TFS</u>

$$f(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

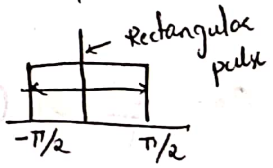


$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\,dt$$

$f(t) = A \quad (-\tau/2 \text{ to } \tau/2)$

$(-\tau/2 < t < \tau/2)$

$0 \quad$ otherwise



$$f(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\,dt$$

$f(t) = A \quad (-\tau/2 \text{ to } \tau/2)$

$-\tau/2 < t < \tau/2$

$0 \quad$ otherwise

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$$a_0 = \frac{1}{T}\int_{-\tau/2}^{\tau/2} A\, dt = \frac{At}{T}\Big]_{-\tau/2}^{\tau/2} = \frac{A\tau}{T} = a_0$$

τ = width
T = period

$$a_n = \frac{2}{T}\int_{-\tau/2}^{\tau/2} f(t)\cos n\omega_0 t\, dt$$

$$\frac{2}{T}\int_{-\tau/2}^{\tau/2} A\cdot\cos n\omega_0 t$$

$$\omega_0 = \frac{2\pi}{T}$$

$$= \frac{2A}{T}\left[\frac{\sin n\omega_0 t}{n\omega_0}\right]_{-\tau/2}^{\tau/2}$$

$$= \frac{2A}{T}\left[\frac{\sin n\omega_0 \tau/2 + \sin n\omega_0 \tau/2}{n\omega_0}\right]$$

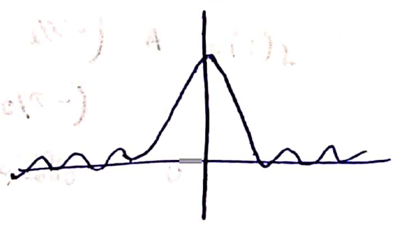
$$= \frac{4A}{T}\,\frac{\sin n\omega_0 \tau/2}{n\omega_0}$$

$$\Rightarrow a_n = \frac{2}{T}\,\frac{2\sin(n\omega_0\tau/2)}{(n\omega_0)\tau/2}\times\tau/2$$

$$Sa(x) = \frac{\sin x}{x}$$

$$= \frac{2\tau}{T}\,\frac{\sin(n\omega_0\tau/2)}{n\omega_0(\tau/2)}$$

$\sin(x)$



$$a_n = \frac{2\tau}{T}\, Sa(n\omega_0\tau/2)$$

$$b_n = \frac{2}{T}\int_{-\tau/2}^{\tau/2} f(t)\sin n\omega_0 t\, dt$$

$$= 0$$

$$f(t) = \frac{A\tau}{T} + \sum_{n=1}^{\infty}\frac{2\tau}{T}\, Sa(n\omega_0\tau/2)\cdot\cos(n\omega_0 t)$$

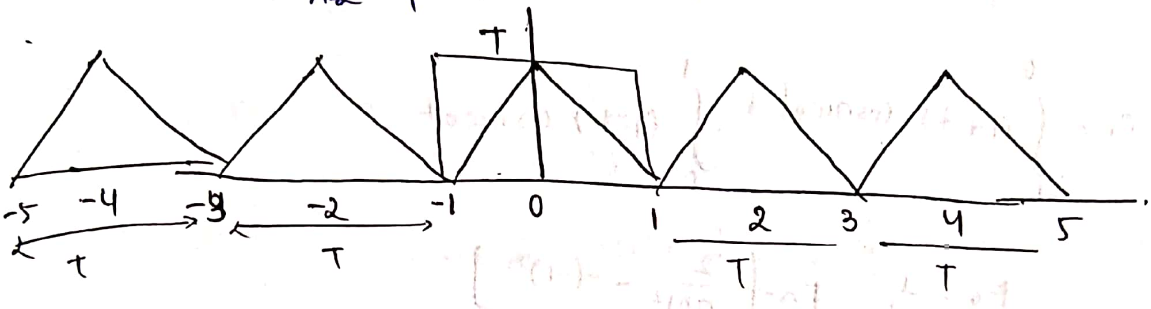
EFS = $f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$

$F_n = \frac{1}{T}\int_0^T \underline{f(t)}\, e^{-jn\omega_0 t}$
$A \quad e^{-jn\omega_0 t}$

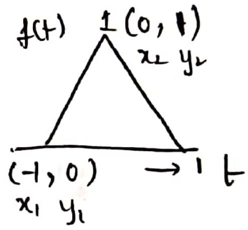
$\frac{\sin x}{x} = sa(x)$

$F_n = \frac{AT}{T}\; sa\left(n\omega_0 T/2\right)$

$f(t) = \sum_{n=-\infty}^{\infty} \frac{AT}{T}\left(sa\left(n\omega_0 T/2\right)\right) e^{jn\omega_0 t}$ EFS



T=2



Equation $f(t) =$

$\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1} \Rightarrow f(t)-0$

$= \frac{1-0}{0+1}$

$f(t) = 1+t \qquad -1 < t < 0$

$\frac{f(t)-1}{t-0} = \frac{0-1}{1-0}$

$f(t)-1 = -t$

$f(t) = 1-t, \quad 0 \le t < 1$

$f(t) = 1-t$ for $0 < t < 1$

$f(t) = 1+t \qquad -1 < t < 0$

$f(t) = 1 - |t|$

$$a_0 = \frac{1}{T}\int_{-1}^{1} f(t)\,dt$$

$|t| = -t$ for $t<0$

$|t| = t$ for $t>0$

$$a_n = \frac{2}{T}\int_{-1}^{1} f(t)\cos n\omega_0 t$$

$$b_n = \frac{2}{T}\int_{-1}^{1} f(t)\sin n\omega_0 t\,dt$$

$$a_n = \int_{-1}^{0} (1+t)\cos n\omega_0 t + \int_{0}^{1} (1-t)\cos n\omega_0 t$$

$F_0 = \frac{1}{2}$ $\quad F_n = \left[\frac{2}{(n\pi)^2} - (-1)^n\right]$

$$F_n = \frac{1}{n^2\pi^2}\left[1-(-1)^n\right]$$

$$f(t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{n^2\pi^2}\left[1-(-1)^n\right] e^{jn\omega_0 t}$$

$\omega_0 = \frac{2\pi}{T}$ $\quad \pi$

$T = 2$

properties of fourier Series

$$f(t) = \sum_{n=\infty} F_n e^{jn\omega_0 t}$$

$f(t) \xleftrightarrow{f.s} F_n$

If $f(t) \rightarrow$ periodic $\rightarrow T$

$$F_n = \frac{1}{T}\int_{0}^{T} f(t) e^{-jn\omega_0 t}\,dt$$

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$$

$f(t) \xleftrightarrow{f.s} F_n$

Linearity property of F.S

Linearity → two properties → additivity
└→ homogenity (or) scaling

$x(t) \xleftrightarrow{F.S} a_n$

$y(t) \xleftrightarrow{F.S} b_n$

$x(t)+y(t) \xleftrightarrow{F.S} \overline{a_n+b_n}$; additivity $Ax(t)+By(t)$
$\rightarrow A \cdot a_n + B b_n$

$x(t) \xleftrightarrow{F.S} a_n$

Scalar 'A' $x(t) \xleftrightarrow[Ax(t)]{F.S} A \cdot a_n$

If $x(t) \xleftrightarrow[T]{F.S} a_n$ & $y(t) \xleftrightarrow[T]{F.S} b_n$ then

$Z(t) = Ax(t) + By(t) \longrightarrow Aa_n + Bb_n$

$f(t) = \sum_{n=-\alpha}^{\alpha} F_n e^{jn\omega_0 t}$. $\quad F_n = \frac{1}{T}\int_0^T f(t) e^{-jn\omega_0 t} dt$

Let $Z(t) = Ax(t) + By(t)$

$Z(t) \xleftrightarrow[F_n]{} C_n$

$\overset{a_n}{C_n} = \frac{1}{T}\int_0^T \underset{y(t)}{\overset{x(t)}{Z(t)}} e^{-jn\omega_0 t} dt$

$C_n = \frac{1}{T}\int_0^T (Ax(t) + By(t)) e^{-jn\omega_0 t} dt$

$C_n = \frac{1}{T}\int_0^T Ax(t) e^{-jn\omega_0 t} dt + \frac{1}{T}\int_0^T By(t) e^{-jn\omega_0 t} dt$

$= A \frac{1}{T}\int_0^T x(t) e^{-jn\omega_0 t} + B\frac{1}{T}\int_0^T y(t) e^{-jn\omega_0 t} dt$

$C_n = Aa_n + Bb_n$

$Z(t) \longleftrightarrow C_n$

$$\boxed{Ax(t) + By(t) \xleftrightarrow{F.S} Aa_n + Bb_n}$$

Linearity property

$x(t) \xleftrightarrow{F.S} a_n$

$y(t) \longleftrightarrow b_n$

$Ax(t) + By(t) \rightarrow Aa_n + Bb_n$

$A = B = 1$

$$x(t) = \sum_{n=-\infty}^{\infty} a_n e^{jn\omega_0 t} \rightarrow ①$$

$$a_n = \frac{1}{T}\int_T x(t) e^{-jn\omega_0 t} \rightarrow ②$$

analysis $\rightarrow x(t) \leftarrow f_1, f_2, f_3, f_4$

synthesis $\rightarrow$ combining all the frequence $x(t)$

Eqn ① synthesis a_1

Eq ② Analysis a_2 a_3

② Time shifting property of F.S

if $x(t) \rightarrow$ periodic with 'T' $\xrightarrow{F.S}$ a_n then if $x(t)$ is ~~scaled~~ shifted by t_0 seconds $\rightarrow x(t-t_0) \rightarrow ?$

$$a_n = \frac{1}{T}\int_T x(t) e^{-jn\omega_0 t} dt$$

$b_n \rightarrow x(t-t_0) \Longrightarrow$

$$b_n = \frac{1}{T}\int_T x(t-t_0) e^{-jn\omega_0 t} \cdot dt$$

Let $t - t_0 = \tau$

$\Rightarrow t = (t_0 + \tau)$

$t = (t_0 + T)$

$dt = dT$

$$b_n = \frac{1}{T}\int_T x(T)\, e^{-jn\omega_0 (t_0 + T)}\, dT$$

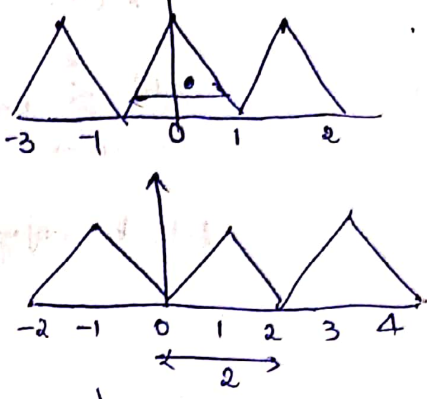
$$b_n = \frac{1}{T}\int_T x(T)\, e^{-jn\omega_0 t_0} \cdot e^{-jn\omega_0 T}\, dT$$

$$= e^{-jn\omega_0 t_0}\, \underbrace{\frac{1}{T}\int x(T)\, e^{-jn\omega_0 T}\, dT}_{a_n}$$

$$b_n = e^{-jn\omega_0 t_0}\, a_n$$

$x(t) \xleftrightarrow{F.S} a_n$

$x(t-t_0) \longrightarrow ?$



$\downarrow$

phas $\rightarrow$

both $x(t)$ & $x(t-t_0) \rightarrow$ same magnitude

but $x(t-t_0)$ phase change $e^{-jn\omega_0 t_0}$

$x(t) \xleftrightarrow{F.S} a_n$

$x(t-t_0) \longrightarrow e^{-jn\omega_0 t_0} \cdot a_n$

mag $x(t-t_0)$ $\quad |e^{-jn\omega_0 t_0}|\, |a_n|$

$|e^{j0}| = \quad a_n = a_n$

mag $|x(t)| = \quad |a_n| =$

2)* Time Reversal

$x(t) \longrightarrow T \longrightarrow a_n$

$x(-t) \longrightarrow T \longrightarrow a_{-n}$

$$x(t) = \sum_{n=-\infty}^{\infty} a_n\, e^{+jn\omega_0 t}$$

let $\boxed{-n = m} \Rightarrow n = -m$

$$x(-t) = \sum_{m=-\infty}^{\infty} a_{-m}\, e^{jm\omega_0 t}$$

$x(-t) \longrightarrow a_{-n}$

If $x(t) =$ even $\Rightarrow x(-t) = x(t)$

$a_{-n} = a_n$

$x(t) =$ odd $\Rightarrow x(-t) = -x(t)$

$a_{-n} = -a_n$

$-a_{-n} = a_n$

3) Conjugation & Congugation Symmetiy

$x(t) \longrightarrow T \rightarrow a_n$ then

$x^*(t) \longrightarrow T \rightarrow ?$

$$x(t) = \sum_{n=-\alpha}^{\alpha} a_n e^{jn\omega_0 t}$$

take the conjugation both sides $x^*(t) = \sum_{n=\alpha}^{\alpha} a_n^* e^{-jn\omega_0 t}$

$$= \sum_{m=\alpha}^{\alpha} a_{-m}^* e^{jm\omega_0 t}$$

$$x^*(t) = \sum_{n=-\alpha}^{\alpha} a_{-n}^* e^{jn\omega_0 t}$$

$(e^{j\theta}) \rightarrow \cos\theta + j\sin\theta$

$(e^{-j\theta}) \rightarrow \cos\theta + j\sin\theta$

$x^*(t) \longrightarrow a_{-n}^*$

$y = b$

$x = a + jb$

$x^* = a - jb$

$y^* = b$

If $x(t) \Rightarrow$ real

both $x(t) = x^*(t)$

$a_n = a_{-n}^* \rightarrow a_n^* = a_{-n}$

if $x(t) \rightarrow$ real $= a_n = a_{-n}^*$ $(x(t) = x^*(t))$

$\Rightarrow \boxed{a_n^* = a_{-n}}$ Conjugate Symmetric

$|a_n| = |a_{-n}^*|$.

If $x(t) \longrightarrow$ real + Even $\Rightarrow x(-t) = x(t)$

$a_{-n} = a_n$

$$\boxed{a_n = a_n^* = a_{-n}}$$

Parseval's Relation of periodic Signals

$x(t) \rightarrow T \rightarrow a_n$

parseval's $\rightarrow$ periodic $\rightarrow$ power signals

avg power $x(t) = \frac{1}{T}\int_0^T |x(t)|^2 dt$

avg power in time domain

periodic $\rightarrow$ F.S $\rightarrow$ a_n

$$P_{avg} = \frac{1}{T}\int_0^T |x(t)|^2 dt$$

$$= \frac{1}{T}\int_0^T x(t)\, x^*(t)\, dt$$

we know that $x(t) = \sum_{n=\alpha} a_n e^{jn\omega_0 t}$

$$x^*(t) = \sum_{n=-\alpha}^{\alpha} a^*(n)\, e^{-jn\omega_0 t}$$

$$P_{avg} = \frac{1}{T}\int_T x(t) \left(\sum_{n=-\alpha}^{\alpha} a_n^* e^{-jn\omega_0 t} \right) dt$$

$$= \sum_{n=-\alpha}^{\alpha} a_n^* \frac{1}{T}\int_T x(t)\, e^{-jn\omega_0 t}\, dt$$

$$= \sum_{n=-\alpha}^{\alpha} a_n^* . a_n = \sum_{n=-\alpha} |a_n|^2 = |a_0|^2 + |a_1|^2 + |a_2|^2 + \cdots |a_n|^2$$

$$\frac{1}{T}\int_0^T |x(t)|^2 dt = \sum_{n=\alpha} |a_n|^2$$

$\rightarrow$ avg power in the periodic signal Equals to

The sum of the avg powers in all of its harmonics components

$$\boxed{\frac{1}{T}\int_0^t |x(t)|^2 dt = \sum_{n=-\alpha}^{\alpha} |a_n|^2}$$

parsevals Relation

Time scaling

No change an

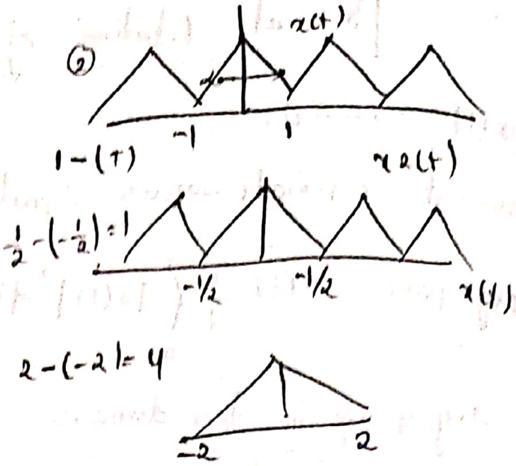
But T - will be changing

$T = \frac{2\pi}{\alpha\omega_0}$

Period - T/α

$\omega_0 \rightarrow \alpha\omega_0$

$T \rightarrow \frac{1}{T/\alpha}$



Parseval's Relation :- Explain the Relation in power signal in the domain & freq Domain (F.S co-efficients)

$x(t) \rightarrow$ periodic $\rightarrow$ power signals $\rightarrow$ power relation

$$x(t) \rightarrow a_n \rightarrow \frac{1}{T}\int_T |x(t)|^2 dt$$

$$= \sum_{n=-\infty}^{\infty} |a_n|^2$$

$$= |a_0|^2 + |a_1|^2 + \cdots |a_n|^2$$

F   1st harmonic   nth freq.

avg power = Sum of the powers of all freq Components

Properties of Fourier Series

$x(t) + y(t) \longrightarrow$ 'T'

1. Linearity property $\rightarrow Ax(t) + By(t) \longrightarrow Aa_n + Bb_n$
2. Time Reversal $\longrightarrow x(t) \rightarrow a_n \Rightarrow a(-t) \rightarrow a_{-n}$
3. Time shifting $\longrightarrow x(t) \rightarrow a_n \Rightarrow x(t-t_0) \rightarrow e^{-jn\omega_0 t_0} \cdot a_n$
4. Time Scaling $\longrightarrow x(t) \rightarrow a_n \Rightarrow x(\alpha t)$

7) Conjugation → $x(t) \to a_n \Rightarrow x^*(t) \to a^*_{-n}$

If $x(t)$ real $\Rightarrow x(t) = x^*(t) \Rightarrow a_n = a^*_{-n} \to a_n = a_n^*$

$x(t)$ = real + even $\Rightarrow x(t) = x(-t)$

$= a_n = a_{-n} = a_n^*$

Gibb's phenomenon

$f(t)/x(t) \longrightarrow f(t) = \sum_{n=-\infty}^{\infty} a_n e^{jn\omega_0 t}$

periodic → approximated by Sinusoids / complex exponential

If the no. of terms (functions) in the approximation ↑

MSE ↓

The occurance of Overshoot / Ripple at the point of discontinuities in the approximation of F.S, it is called as Gibbs phenomenon

Concept of Negative frequency

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$$

$$= \underbrace{F_0}_{DC} + F_1 e^{j\omega_0 t} + F_2 e^{j2\omega_0 t} + \cdots + F_n e^{jn\omega_0 t}$$
$$+ F_{-1} e^{-j\omega_0 t} + F_{-2} e^{-j2\omega_0 t} + \cdots + F_{-n} e^{-jn\omega_0 t}$$

F_1 = F.S coefficient of 1st freq component

F_{-1} = " " " " " "

F_2 = F.S coefficient of 2nd freq component

F_{-2} = " " " " "

F_n / F_{-n} :- F.S co-efficient of n^{th} component

$F_1 = \omega_0$

$F_{-1} = \omega_0$

sunlight V I B G Y O R } F.S

$F_1 = a + jb$

$F_{-1} = a - jb$

$|F_1| = |F_{-1}|$

phasors are diff (opposite) in direction but magnitudes are same

freq = no of cycles/sec → 2

→ Negative freq = 2 but phase opposite

$e^{j\omega_0 t}$ & $e^{-j\omega_0 t}$

positive freq → construct Real signal

In the Analysis → +ve & -ve
mathematical model freq components

$$(e^{j\omega_0 t} + e^{-j\omega_0 t}) = 2\cos\omega_0 t$$

Complex fourier Spectrum

Spectrum → set of frequencies

Fourier → F.S/F.T

Complex → the F.S co-efficients of any periodic signal is Complex

Freq present $0, \omega_0, -\omega_0, 2\omega_0, -2\omega_0 \ldots\ldots n\omega_0, -n\omega_0$

$f(t)$. $\omega_0 = \frac{2\pi}{T}$

$$f(t) = \underbrace{F_0}_{DC} + F_1 e^{j\omega_0 t} + F_2 e^{+j2\omega_0 t} + \cdots F_n e^{jn\omega_0 t} + \cdots$$

$$F_{-1} e^{-j\omega_0 t} + F_{-2} e^{-j2\omega_0 t} + \cdots F_{-n} e^{-jn\omega_0 t} + \cdots$$

dc component freq = 0

1st freq. freq $= \omega_0, -\omega_0 \rightarrow F_1, F_{-1}$

2nd freq freq $= 2\omega_0, -2\omega_0 \rightarrow F_2, F_{-2}$

3rd freq freq $= 3\omega_0, -3\omega_0 \rightarrow F_3, F_{-3}$

nth freq $= n\omega_0, -n\omega_0 \rightarrow F_n, F_{-n}$

$\omega_0 = 0 = F_0$

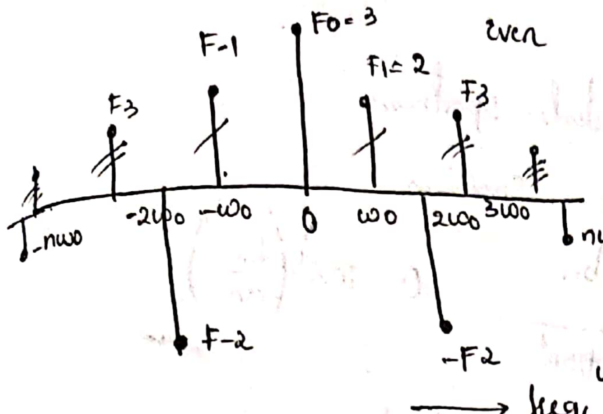
$\omega = \omega_0 = F_1, F_{-1}$

$2\omega_0 = F_2, F_{-2}$

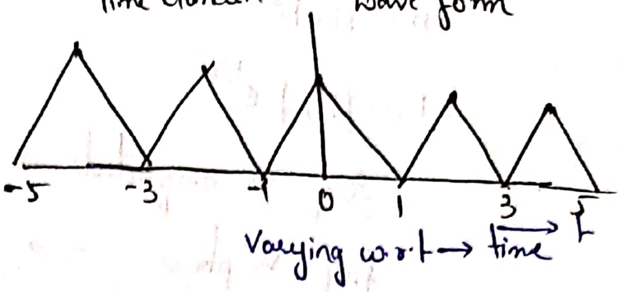
$F_n = F_{-n}$

$f(t) \rightarrow$ periodic $\rightarrow$ time domain

$\quad \hookrightarrow$ Freq Domain representation



Time domain wave form



Varying w.r.t → time

$$F_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, e^{-jn\omega_0 t}\, dt$$

magnitude versus frequency components

$$F_{-n} = \frac{1}{T}\int_{-T/2}^{T/2} |f(t)|\, e^{jn\omega_0 t} \cdot dt$$

freq vs magnitude → spectrum magnitude

$|F_n| = |F_{-n}|$ = same

$\int f(t)\, dt$

complex fourier spectrum

$n = 0, \pm 1, \pm 2, \pm 3 \ldots\ldots \quad n\omega_0$ ✓

Only discrete freq $\omega_0, 2\omega_0, 3\omega_0 \ldots\ldots$ discrete spectrum

line spectrum

magnitude spectrum

$$F_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, e^{-jn\omega_0 t} \cdot dt$$

$$F_{-n} = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, e^{+jn\omega_0 t}\, dt$$

$|F_n| = |F_{-n}| =$ same

$F_n = |F_n| e^{-j\theta}$

$F_{-n} = |F_n| e^{+j\theta}$

$F_n \rightarrow$ complex Quantity

$a_0 = 1, \quad a_1 = \left(1 - \frac{1}{2}j\right) \quad = a + jb$

$a_{-1} \left(1 + \frac{1}{2}j\right)$

$a_2 = \frac{\sqrt{2}}{4}(1 - j)$

$a_{-2} = \frac{\sqrt{2}}{4}(1 + j)$

Complex nature of $F_n \rightarrow$ two Spectrums

$F_n = |F_n| e^{-j\theta}$

$F_{-n} = |F_{-n}| e^{+j\theta}$

freq vs magnitude $\rightarrow$ magnitude spectrum

freq vs phase $\rightarrow$ phase spectrum

$F_n = a_n + j b_n$ $\qquad \theta = \tan^{-1}\left(\frac{b_n}{a_n}\right)$

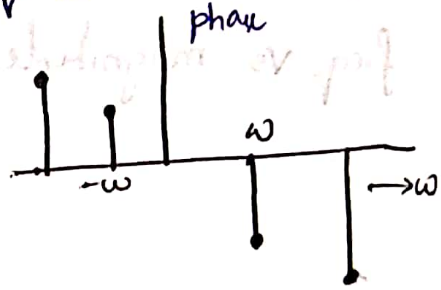
$|F_n| = \sqrt{a_n^2 + j b_n^2}$

magnitude spectrum $\rightarrow$ symmetric about vertical axis

Even function of freq 'ω'

phase spectrum - Anti symmetric

odd function of freq ω.



Summary Fourier Series

$f(t) \rightarrow$ periodic $\rightarrow$ all sum of complex exponentials / sinusoidal
signal

e^{jw_0t} complete

$\sin nw_0t$ & $\cos nw_0t$

Complete set of orthogonal functions

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nw_0t + b_n \sin nw_0t) \rightarrow \text{TFS}$$

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, dt$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t) \cos nw_0t\, dt$$

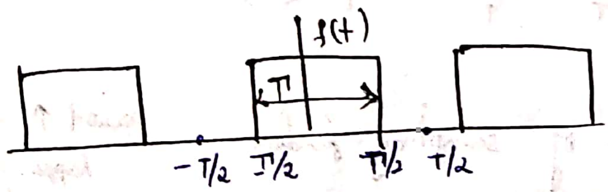
$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t) \sin nw_0t\, dt$$

$$f(t) = \sum_{n=-\infty}^{\infty} F_n\, e^{jnw_0t}$$

$$F_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, e^{-jnw_0t}\, dt$$

EFS

Fourier Transform Representation



$\frac{T}{2} - -\frac{T}{2} = T$

$f(t)$ period function T

Gate function / Rectangular pulse

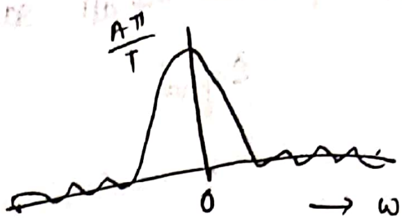
$$\text{EFS} = f(t) = \sum_{n=1} F_n e^{jnw_0t}$$

$$F_n = \frac{AT}{T}\, Sa\left(\frac{nw_0 T}{2}\right)$$

F_0

$$f(t) = \sum_{n=-\infty}^{\infty} \frac{A\tau}{T}\, Sa\left(\frac{n\omega_0 \tau}{2}\right) e^{jn\omega_0 t}$$

$\rightarrow$ $Sa(x) = \frac{\sin x}{x}$



$Sa(0) = 1$

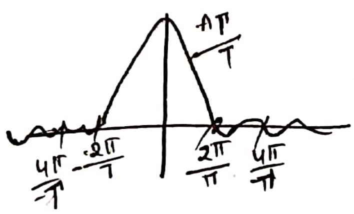
$$\frac{\sin(n\omega_0 \tau/2)}{n\omega_0 \tau/2} = Sa\left(\frac{n\omega_0 \tau}{2}\right)$$

$$\frac{\sin n\omega_0 \tau/2}{n\omega_0 \tau/2} = 0$$

$\Rightarrow$ $\sin \frac{n\omega_0 \tau}{2} = 0$

$= \sin \frac{n\omega_0 \tau}{2} = \sin(n\pi)$

$\frac{n\omega_0 \tau}{2} = \pm n\pi$ $\qquad n\omega_0 \tau = \pm \frac{2n\pi}{\tau}$



$n\omega_0 = \pm \frac{2n\pi}{\tau}$ $\qquad$ T-width

$= \frac{2\pi}{\tau}, \frac{4\pi}{\tau}, \frac{6\pi}{\tau} \dots \frac{2n\pi}{\tau}$ $\qquad$ T-period

$$f(t) = \sum \frac{A\tau}{T}\, Sa\left(\frac{n\omega_0 \tau}{2}\right) e^{jn\omega_0 t}$$

$n=0$ $\quad Sa(0)=1$ $\quad \frac{A\tau}{T}, \frac{A\tau}{T}, Sa\left(\frac{\omega_0 \tau}{2}\right), \frac{A\tau}{T} Sa\left(\frac{2\omega_0 \tau}{2}\right)$

$\frac{A\tau}{T} \rightarrow$ magnitude of sampling function $\qquad$ period $\uparrow$ what happen in frequency domain

$\tau = \frac{1}{20}$ width $\qquad$ $T \rightarrow$ period of pulse

$\tau \rightarrow$ width of pulse

$T = \frac{1}{4}$ period

$\frac{\tau}{T} = \frac{\frac{1}{20}}{\frac{1}{4}} = \frac{1}{5}$ $\qquad \omega_0 = \frac{2\pi}{T}$

$\frac{n\omega_0 \tau}{2} = \pm n\pi$

$n\omega_0 \tau = \pm 2n\pi$

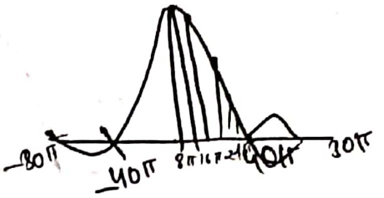
$n\omega_0 = \pm 40n\pi$

$= \pm 40\pi, \pm 80\pi, \pm 120\pi$

Zero's at $\pm \frac{2n\pi}{\tau} = \frac{2n\pi}{\frac{1}{20}}$ $\pm 40n\pi$

$T = \frac{1}{20}$ = width

$T = \frac{1}{4}$



$T = \frac{1}{2}$ zeros $\pm 40\pi$, $\pm 80\pi$ $\pm 120\pi$

$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{1/2} = \cancel{8\pi}\ 4\pi$

$\underbrace{\cancel{4\pi}, 8\pi, 12\pi, 16\pi, 20\pi, 24, 28\pi, 32\pi, 36\pi, 40\pi}$

$T = 1 \Rightarrow T = \frac{1}{20}$ 10 freq components

$T = \alpha$

$\omega_0 = \frac{2\pi}{1} = 2\pi$

$f = \frac{1}{T} = \frac{1}{\alpha} = 0$

$T = 50 \Rightarrow \frac{2\pi}{50}$ $0.02 \times 2\pi$ no spacing b/w freq components

$T \uparrow \Rightarrow$ spectrum

$\boxed{\text{As } T \text{ Increases frequency components } \uparrow \text{ frequency spacing decreases}}$

T = Large   ↑ close to Unity

$\frac{1}{4} = 5$

$T = \frac{1}{f}$

$\omega = \frac{2\pi}{T} = 0$

$\frac{1}{2} = 10$

$1 = 20$

there no spacing between successive freq components

50

100

|

α

freq = continuous

$F.S \Rightarrow f(t) = \sum_{n=-\alpha}^{\alpha} F_n e^{jn\omega_0 t} \rightarrow$ discrete sum of complex Exponential

freq = finite

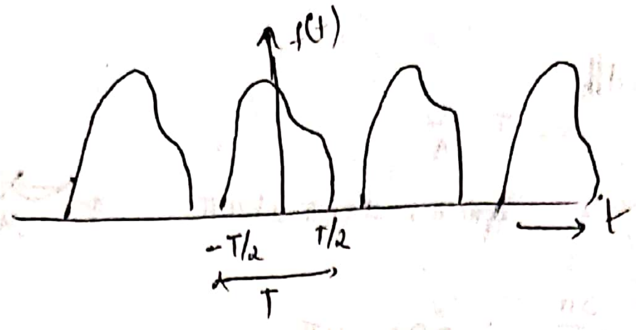
Spectrum $\rightarrow$ discrete in nature

$F.T \Rightarrow$ freq = infinite

Spectrum $\rightarrow$ Continuous sum of Exponentials

Repeating Fn over the Entire Interval (α, α)

$\cancel{\Sigma} \rightarrow$ Replaced with $\int_{-\alpha}^{\alpha}$



at T is Very Large

$T = \alpha$

$f_T(t)$ = periodic $\Rightarrow$ F.S

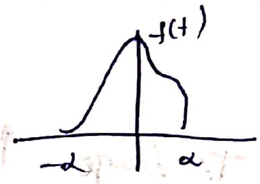
$T\uparrow \rightarrow$ periodic becomes non periodic

$$f(t) = \sum_{n=-\alpha}^{\alpha} F_n e^{jn\omega_0 t}$$

$$F_n = \frac{1}{T}\int_{-T/2}^{T/2} f_T(t) e^{-jn\omega_0 t} dt$$

. periodic $T=\alpha$ means

$$\lim_{T\to\alpha} \underbrace{f_T(t)}_{\text{periodic}} = \underbrace{f(t)}_{\text{Non periodic}}$$



$$f_T(t) = \sum_n F_n e^{jn\omega_0 t}$$

$$F_n = \frac{1}{T}\int_{-T/2}^{T/2} f_T(t) e^{-jn\omega_0 t} dt$$

$$F_n = \int_{-T/2}^{T/2} f_1(t) e^{-jn\omega_0 t} dt$$

$$\boxed{\text{Let } F_n(\omega) = \int_{-\alpha}^{\alpha} f(t) e^{-j\omega_n t} . dt}$$

$\uparrow F_n = F(n\omega_0) \Rightarrow$

$$F_n = \frac{1}{T} F(n\omega_0)$$

$$f_T(t) = \sum_{n=-\alpha}^{\alpha} \frac{1}{T} F(n\omega_0) e^{jn\omega_0 t} dt$$

$$\boxed{T = \frac{2\pi}{\omega_0} \rightarrow T = \alpha}$$

$$\text{Let } f_T(t) = \frac{1}{2\pi} \sum_{n=-\alpha}^{\alpha} F(n\omega_0) e^{jn\omega_0 t} \omega_0$$

$T \rightarrow \alpha$

as $T \to \alpha$ & $\omega = 0$ small $= \Delta\omega$

freq spacing $= 0$ close to each other in the range more

freq

$$f(t) = \frac{1}{2\pi}$$

Let $f_1(t) = \frac{1}{2\pi} \sum_{n=-\alpha}^{\alpha} F(n\Delta\omega_0) e^{j(n\Delta\omega_0)t} \cdot \Delta\omega_0$

non periodic $f(t) = \frac{1}{2\pi} \int_{-\alpha}^{\alpha} F(\omega) e^{j\omega t} d\omega$ $\boxed{\text{IFT of } F(\omega)}$ $F(t)$

$$F(\omega) = \int_{-\alpha}^{\alpha} f(t) e^{j\omega t} dt \rightarrow FT[f(t)] = F(\omega)$$

FS $\to$ periodic

FS $\to$ non periodic

the $F_1[f(t)]$ denoted $F(\omega)$

$$F(\omega) = \int_{-\alpha}^{\alpha} f(t) e^{-j\omega t} \cdot dt \rightarrow (1)$$

$f(t) \to$ periodic FS

$F(t) \to$ non periodic FT

freq Domain

$$IFT[F(\omega)] = f(t) = \frac{1}{2\pi} \int_{-\alpha}^{\alpha} F(\omega) e^{j\omega t} \cdot d\omega \rightarrow (2)$$

$$F(t) \xleftrightarrow{FT} F(\omega)$$

$$F(\omega) = \int_{-\alpha}^{\alpha} f(t) e^{-j\omega t} dt$$

TD FD

$f(t) \longleftrightarrow F(\omega)$

IFT $f(t) = \frac{1}{2\pi} \int_{-\alpha}^{\alpha} F(\omega) e^{j\omega t} \cdot d\omega$