UNIT V

DIFFERENTIAL EQUATIONS

First Order and First Degree

Differential Equations of First Order and First Degree ➥ Formation ➥ Exact, Linear and Bernoulli ➥ Applications to Newton's law of Cooling ➥ Law of Natural Growth and Decay ➥ Orthogonal Trajectories.

5.1 INTRODUCTION :

i) Definition and Classification :

An equation which involves differentials or differential coefficients with or without the variables from which these differentials or differential coefficients are formed is called a differential equation.

Examples :	Order	Degree	
$\frac{dy}{dx} = e^x$	1	1	(1)
$\left(\frac{dy}{dx}\right)^2 = ax^2 + bx + c$	1	2	(2)
$\frac{d^2y}{dx^2} + a^2x = 0$	2	1	(3)
$\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} = \frac{d^2y}{dx^2}$	2	2	(4)
$\left(\frac{d^3y}{dx^3}\right)^2 = x^2\frac{dy}{dx}$	3	2	(5)
$\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^3 + 2y = 0$	2	1	(6)
$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u$	1	1	(7)
$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$	2	1	(8)

Differential equations are divided into two classes viz. ordinary and partial.

ii) **An ordinary differential equation** is one in which all the differentials (derivatives) involved have reference to a single independent variable. Examples (1) to (6).

iii) **A partial differential equation** is one which contains two or more independent variables. Examples (7) and (8).

iv) **The order** of a differential equation is the order of the highest derivative in the equation.

v) **The degree** of a differential equation is the degree of the highest derivative in the equation after the equation is freed from radicals and fractions in its derivatives.

Examples : The order and degree are indicated against each example above from (1) to (8).

5.2a) ORDINARY DIFFERENTIAL EQUATIONS :

Formation :

i) Consider the example $y = mx$ .... (1)

differentiating (1) with respect to x,

$$\frac{dy}{dx} = m \qquad \text{............... (2)}$$

eliminating m between (1) and (2), we get $y = x\dfrac{dy}{dx}$,

which is the required differential equation.

Note - 1 : If the given relation contains only one arbitrary constant, we have to differentiate one time and eliminate the arbitrary constant. The order of the differential equation is one.

ii) Consider the example $y = a\cos x + b\sin x$ .... (1)

a and b being arbitrary constants.

differentiating (1) with respect to x twice

$$\frac{dy}{dx} = -a\sin x + b\cos x \qquad \text{............ (2)}$$

$$\frac{d^2y}{dx^2} = -a\cos x - b\sin x$$

$$= -(a\cos x + b\sin x) \qquad \ldots\ldots\ldots (3)$$

eliminating a and b from (1) and (3), we get

$$\frac{d^2y}{dx^2} = -y$$

i.e.,
$$\frac{d^2y}{dx^2} + y = 0$$

which is the required differential equation.

Note - 2 : If the given relation contains two arbitrary constants, we have to differentiate two times and eliminate arbitrary constants. The order of the differential equation thus formed is two.

From Note (1) and Note (2), it is clear that a differential equation can be formed by differentiating the number of times contains the arbitrary constants in the given relation.

In general, consider an equation $f(x, y, c_1, c_2, \ldots\ldots c_n) = 0$

Containing n arbitrary constants $c_1, c_2, \ldots\ldots\ldots c_n$.

Differentiate n times the given equation.

Eliminate arbitrary constants from the given equation and the equations obtained by differentiating n times.

Then a differential equation of the n^{th} order can be formed.

5.2 b) SOLUTION OF A DIFFERENTIAL EQUATION :

i) **Solution :** Any relation connecting the variables of an equation and not involving their derivatives, which satisfies the given differential equation is called a solution.

ii) **General Solution :** A solution of a differential equation in which the number of arbitrary constants is equal to the order of the equation is called a general or complete solution or complete primitive of the equation.

iii) **Particular Solution :** The solution obtained by giving particular values to the arbitrary constants of the general solution, is called a particular solution of the equation.

Example : Consider the differential equation $\frac{d^2y}{dx^2} = 0$

The general solution is $y = Ax + B$
The particular solution is $y = 2x + 3$
giving particular values $A = 2, B = 3$.

SOLVED EXAMPLES – 5.1

EXAMPLE - 1 :

Form the differential equation from $y = A\cos(x+\alpha)$ where A and α are arbitrary constants.

Solution : Given equation $y = A\cos(x+\alpha)$ .... (1)

Since, A and α are the arbitrary constants, differentiate the given equation twice with respect to x.

$$\frac{dy}{dx} = -A\sin(x+\alpha) \quad \text{............ (2)}$$

$$\frac{d^2y}{dx^2} = -A\cos(x+\alpha) \quad \text{............ (3)}$$

Substituting (1) in (3)

$$\frac{d^2y}{dx^2} = -y$$

$\therefore$ $\frac{d^2y}{dx^2} + y = 0$, the required differential equation.

EXAMPLE - 2 :

Find the differential equation of the family of curves $y = e^x(a\cos x + b\sin x)$ where a and b are arbitrary constants. **(JNTU 2006)**

Solution : Given equation $y = e^x(a\cos x + b\sin x)$ .... (1)

differentiating (1) twice with respect to x, we get

$$\frac{dy}{dx} = e^x(-a\sin x + b\cos x) + e^x(a\cos x + b\sin x)$$

$$= e^x(-a\sin x + b\cos x) + y \quad \text{............ (2)}$$

$$\frac{d^2y}{dx^2} = e^x(-a\cos x - b\sin x) + e^x(-a\sin x + b\cos x) + \frac{dy}{dx}$$

$$= -e^x(a\cos x + b\sin x) + \left(\frac{dy}{dx} - y\right) + \frac{dy}{dx} \text{ by (2)}$$

$$= -y + \left(\frac{dy}{dx} - y\right) + \frac{dy}{dx} \quad \text{by (1)}$$

$$\therefore \quad \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$$

which is the required differential equation.

EXAMPLE - 3 :

Form the differential equation from $y = a\tan^{-1}x + b$ where a and b are arbitrary constants.

Solution : Given equation $y = a\tan^{-1}x + b$ .... (1)

Differentiating (1) with respect to x twice

$$\frac{dy}{dx} = \frac{a}{1+x^2}$$

$$\therefore \quad (1+x^2)\frac{dy}{dx} = a \quad \text{.................. (2)}$$

$$(1+x^2)\frac{d^2y}{dx^2} + 2x\cdot\frac{dy}{dx} = 0$$

$$(1+x^2)\,y_2 + 2xy_1 = 0$$

which is the required differential equation.

EXAMPLE - 4 :

Form the differential equation from $y = ae^x + be^{-2x}$ where a and b are arbitrary constants.

Solution : Given equation $y = ae^x + be^{-2x}$ .... (1)

differentiating (1) with respect to x twice

$$\frac{dy}{dx} = ae^x - 2be^{-2x} \quad \text{.............. (2)}$$

$$\frac{d^2y}{dx^2} = ae^x + 4be^{-2x} \quad \text{.............. (3)}$$

Now $$\frac{d^2y}{dx^2} - \frac{dy}{dx} = 6be^{-2x}$$

$$\therefore \quad b = \frac{1}{6}\left(\frac{d^2y}{dx^2} - \frac{dy}{dx}\right)e^{2x}$$

From (3), $$\frac{d^2y}{dx^2} = ae^x + \frac{2}{3}\left(\frac{d^2y}{dx^2} - \frac{dy}{dx}\right)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 3ae^x$$

$$\therefore\ a = \frac{1}{3}\left(\frac{d^2y}{dx^2} + 2\frac{dy}{dx}\right)e^{-x}$$

Substituting the values of a and b in (1)

$$y = \frac{1}{3}\left(\frac{d^2y}{dx^2} + 2\frac{dy}{dx}\right) + \frac{1}{6}\left(\frac{d^2y}{dx^2} - \frac{dy}{dx}\right)$$

$$\text{i.e.,}\ 6y = 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx}$$

$$\therefore\ \frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0,\ \text{the required differential equation.}$$

EXAMPLE - 5 :

Form the differential equation of all circles of radius.

Solution : The solution of any circle of radius a is

$$(x-h)^2 + (y-k)^2 = a^2 \quad \ldots\ldots\ldots\ldots\ldots (1)$$

where (h, k) the coordinates of the centre of circle and h, k are arbitrary.

Differentiating (1) with respect to x,

$$2(x-h) + 2(y-k)\cdot\frac{dy}{dx} = 0$$

$$\text{i.e.,}\ (x-h) + (y-k)\frac{dy}{dx} = 0 \quad \ldots\ldots\ldots\ldots\ldots (2)$$

Again differentiating (2) with respect to x

$$1 + (y-k)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots (3)$$

$$\text{From (3), } y-k = -\frac{1+\left(\frac{dy}{dx}\right)^2}{\left(\frac{d^2y}{dx^2}\right)}$$

$$\text{From (2) } x-h = -(y-k)\frac{dy}{dx}$$

$$= \frac{\left(\frac{dy}{dx}\right)\left[1+\left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}}$$

Substituting these values of x – h and y – k in (1), we get

$$\frac{\left(\frac{dy}{dx}\right)^2\left[1+\left(\frac{dy}{dx}\right)^2\right]^2}{\left(\frac{d^2y}{dx^2}\right)^2}+\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^2}{\left(\frac{d^2y}{dx^2}\right)^2}=a^2$$

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^2\left[\left(\frac{dy}{dx}\right)^2+1\right]=a^2\left(\frac{d^2y}{dx^2}\right)^2$$

or $$\left[1+\left(\frac{dy}{dx}\right)^2\right]^3=a^2\left(\frac{d^2y}{dx^2}\right)^2$$

which is the required differential equation.

EXAMPLE - 6 :

Form the differential equation of all circles passing through the origin and having their centres on the x-axis.

Solution : Let the general equation of circle be $x^2+y^2+2gx+2fy+c=0$

Since the circles pass through the origin, $c=0$

Also, the centre $(-g, -f)$ lies on X-axis.

So, the y-coordinate of the centre i.e., $f=0$

The system of circles passing through the origin and having their centres on x-axis is

$$x^2+y^2+2gx=0 \quad \text{.......... (1)} \quad g \text{ is arbitrary}$$

Differentiating (1) with respect to x

$$2x+2y\frac{dy}{dx}+2g=0$$

$$x+y\frac{dy}{dx}+g=0$$

$$\Rightarrow \quad g=-x-y\frac{dy}{dx}$$

Substituting in (1)

$$x^2+y^2+2\left(-x-y\frac{dy}{dx}\right)x=0$$

$$2xy\frac{dy}{dx}+x^2-y^2=0$$

the required differential equation.

EXAMPLE - 7 :

Form the differential equation of all central conics whose axes coincide with the axes of coordinates.

Solution : The equation of all central conics whose axes coincide with the axes is given by

$$ax^2 + by^2 = 1 \qquad \text{............ (1)}$$

Differentiating (1) with respect to x

$$2ax + 2by\frac{dy}{dx} = 0$$

$$ax + by\frac{dy}{dx} = 0 \qquad \text{............ (2)}$$

Differentiating (2) with respect to x

$$a + b\left(y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right) = 0 \qquad \text{............ (3)}$$

Eliminating a, b from (1), (2) and (3)

$$\begin{vmatrix} x^2 & y^2 & 1 \\ x & y\frac{dy}{dx} & 0 \\ 1 & y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 & 0 \end{vmatrix} = 0$$

Expanding $xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\left(\frac{dy}{dx}\right) = 0$ is the required differential equation.

EXAMPLE - 8 :

Find the differential equation of all parabolas having vertex at (α, β) and axes being parallel to X - axis with given latus rectum 4a.

Solution : The equation of the parabola is

$$(y - \beta)^2 = 4a(x - \alpha) \qquad \text{............ (1)}$$

α, β arbitrary constant, 4a is given

Differentiating (1) with respect to x,

$$2(y - \beta)\frac{dy}{dx} = 4a$$

$$(y - \beta)\frac{dy}{dx} = 2a \qquad \text{............ (2)}$$

$$(y - \beta)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0 \qquad \text{............ (3)}$$

From (2) $\quad y - \beta = \dfrac{(2a)}{\left(\dfrac{dy}{dx}\right)}$

Substituting in (3)

$$\frac{2a}{\left(\frac{dy}{dx}\right)} \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$$

$$2a \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = 0$$

EXERCISE 5.1

1. Form the differential equation from $y = (a + bx)e^{-x}$ where a, b are arbitrary constants.

   Ans : $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = 0$

2. Form the differential equation from $4y = c_1x^5 + c_2x + \frac{1}{3}x$

   Ans : $x^2 \dfrac{d^2y}{dx^2} - 5x\dfrac{dy}{dx} + 5y = \dfrac{1}{x}$

3. Find the differential equation from $y = c_1 + \dfrac{c_2}{x}$ Ans : $\dfrac{d^2y}{dx^2} + \dfrac{2}{x}\dfrac{dy}{dx} = 0$

4. Eliminate c_1 and c_2 from $xy = c_1e^x + c_2e^{-x} + x^2$

   Ans : $x\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} - xy + x^2 - 2 = 0$

5. Form the differential equation from $\dfrac{x^2}{a^2 + \lambda} + \dfrac{y^2}{b^2 + \lambda} = 1$, λ being parameter.

   Ans : $(x + yy_1)(xy_1 - y) = (a^2 - b^2)y_1$

   $y_1 = \dfrac{dy}{dx}$

6. Eliminate arbitrary constants A, B, C from $y = Ae^x + Be^{-x} + C$

   Ans : $\dfrac{d^3y}{dx^3} - \dfrac{dy}{dx} = 0$

7. Show that the differential equation corresponding to the family of curves $y = c(x - c)^2$ where c is an arbitrary constant is $\left(\frac{dy}{dx}\right)^3 = 4y\left(x\frac{dy}{dx} - 2y\right)$.

8. Form the differential equation of all parabolas with the origin as focus and axis along x-axis. **Ans :** $2x\frac{dy}{dx} + y\left(\frac{dy}{dx}\right)^2 - y = 0$

9. Form the differential equation of all circles with centre on the line $y = x$ and having radius 1. **Ans :** $\left(1 + \frac{dy}{dx}\right)^2 = (x - y)^2\left[1 + \left(\frac{dy}{dx}\right)^2\right]$

10. Find the differential equation of all parabolas with x-axis as the axis and $(a, 0)$ as focus. **Ans :** $y\frac{dy}{dx} = 2a$

5.3 EXACT DIFFERENTIAL EQUATIONS :

A differential equation which can be obtained from its solution or primitive by direct differentiation is said to be exact without any further transformation such as elimination etc. For example, $x\,dy + y\,dx = 0$ is obtained from its solution $xy = c$ by differentiating directly.

Thus, the differential equation

$$M dx + N dy = 0$$

where M and N are functions of x and y, is exact if it can be obtained directly by differentiation from an equation of the form $f(x, y) = c$.

Theorem : The necessary and sufficient condition that the differential equation $M\,dx + N\,dy = 0$ to be exact is that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Proof : Let the equation be $M\,dx + N\,dy = 0$ .... (1)

Let the solution be $f(x, y) = c$ .... (2)

where c is an arbitrary constant.

Differentiating (2) with respect to x, we have

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0 \qquad \text{........... (3)}$$

As equation (3) is obtained from the solution (2), equation (3) is an exact differential equation.

Comparing (1) and (3), we get

$$\frac{\partial f}{\partial x} = M \qquad \text{.......... (4)}$$

$$\frac{\partial f}{\partial y} = N \qquad \text{.......... (5)}$$

Differentiating (4) partially with y,

$$\frac{\partial^2 f}{\partial y\, \partial x} = \frac{\partial M}{\partial y}$$

Differentiating (5) partially with x,

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial N}{\partial x}$$

Since $\dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{\partial^2 f}{\partial x \partial y}$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Method for solving Exact Differential Equation :

1) Test for the exactness of differential equation $M\,dx + N\,dy = 0$ by $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

2) If the differential equation is exact, integrate terms in M with respect to x keeping y as constant.

3) Integrate the terms in N which are free from x with respect to y.

4) The sum of (2) and (3) equated to an arbitrary constant will be the required solution.

Solution of Exact Differential Equation : $\underset{(\text{y constant})}{\int M\,dx} + \underset{(\text{terms free from x})}{\int N\,dy} = c$

SOLVED EXAMPLES – 5.2

EXAMPLE - 1 :

Solve $y \sin 2x\, dx - (y^2 + \cos^2 x)\, dy = 0$

Solution : Here $M = y \sin 2x$, $N = -(y^2 + \cos^2 x)$

$$\frac{\partial M}{\partial y} = \sin 2x, \quad \frac{\partial N}{\partial x} = 2\cos x \sin x = \sin 2x$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

The equation is an exact.

$$M = y \sin 2x, \quad N = -\left[y^2 + \frac{(1 + \cos 2x)}{2}\right]$$

$\therefore$ terms in N which are free from x are $-y^2 - \frac{1}{2}$

$$\underset{(y\text{ constant})}{\int y \sin 2x \, dx} + \underset{(\text{terms free from } x)}{\int} \left(-y^2 - \frac{1}{2}\right) dy = c$$

$$-\frac{1}{2} y \cos 2x - \frac{y^3}{3} - \frac{1}{2} y = c \quad \text{or} \quad \frac{y \cos 2x}{2} + \frac{y^3}{3} + \frac{y}{2} = c$$

the required solution.

EXAMPLE - 2 :

Solve $\left[y\left(1 + \frac{1}{x}\right) + \cos y\right] dx + [x + \log x - x \sin y]\, dy = 0$

Solution : $M = y\left(1 + \frac{1}{x}\right) + \cos y, \; N = x + \log x - x \sin y$

$$\frac{\partial M}{\partial y} = 1 + \frac{1}{x} - \sin y, \quad \frac{\partial N}{\partial x} = 1 + \frac{1}{x} - \sin y$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

The equation is an exact.

$$\underset{(y\text{ constant})}{\int M\, dx} + \underset{(\text{terms free from } x)}{\int N\, dy} = c$$

$$\underset{(y\text{ constant})}{\int} \left[y\left(1 + \frac{1}{x}\right) + \cos y\right] dx + \underset{(\text{terms free from } x)}{\int 0 \, dy} = C$$

$$y(x + \log x) + x \cos y = c$$

Hence the required solution is $y(x + \log x) + x \cos y = c$

EXAMPLE - 3 :

Solve $[\cos x \tan y + \cos(x+y)]\,dx + [\sin x \sec^2 y + \cos(x+y)]\,dy = 0$

(JNTU 2005)

Solution :

$$M = \cos x \tan y + \cos(x+y)$$

$$\frac{\partial M}{\partial y} = \cos x \sec^2 y - \sin(x+y)$$

$$N = \sin x \sec^2 y + \cos(x+y)$$

$$\frac{\partial N}{\partial x} = \cos x \sec^2 y - \sin(x+y)$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

The equation is an exact.

$$\underset{(y\text{ constant})}{\int M\,dx} + \underset{(\text{terms free from } x)}{\int N\,dy} = c$$

$$\underset{(y\text{ constant})}{\int \left[\cos x \tan y + \cos(x+y)\right] dx} + \underset{(\text{terms free from } x)}{\int 0\,dy} = c$$

$$\sin x \tan y + \sin(x+y) = c$$

Hence, the required solution is

$$\sin x \tan y + \sin(x+y) = c$$

EXAMPLE - 4 :

Solve the equation

$$\left(y^2 e^{xy^2} + 4x^3\right) dx + \left(2xye^{xy^2} - 3y^2\right) dy = 0$$

Solution :

$$M = y^2 e^{xy^2} + 4x^3$$

$$\frac{\partial M}{\partial y} = y^2 \cdot e^{xy^2} \cdot 2xy + 2y\,e^{xy^2} = e^{xy^2}(xy^2+1)\,y$$

$$N = 2xy\,e^{xy^2} - 3y^2$$

$$\frac{\partial N}{\partial x} = 2xy\,e^{xy^2}\,y^2 + 2y\,e^{xy^2} = e^{xy^2}(xy^2+1)\,y$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

The equation is an exact.

$$\underset{(y\ \text{constant})}{\int M\,dx} + \underset{\substack{(\text{terms free} \\ \text{from } x)}}{\int N\,dy} = c$$

$$\underset{(y\ \text{constant})}{\int}\left(y^2 e^{xy^2} + 4x^3\right)dx + \underset{\substack{(\text{terms free} \\ \text{from } x)}}{\int -3y^2\,dy} = c$$

$$e^{xy^2} + x^4 - y^3 = c$$

$$\therefore \quad e^{xy^2} + x^4 - y^3 = c \text{ is the required solution.}$$

EXAMPLE - 5 :

Solve $(1 + e^{x/y})\,dx + \left(1 - \frac{x}{y}\right)e^{x/y}\,dy = 0$ ***(JNTU 2006)***

Solution : $M = 1 + e^{x/y}$

$$\frac{\partial M}{\partial y} = e^{x/y}\left(-\frac{x}{y^2}\right)$$

$$N = \left(1 - \frac{x}{y}\right)e^{x/y}$$

$$\frac{\partial N}{\partial x} = \left(1 - \frac{x}{y}\right)e^{x/y}\cdot\frac{1}{y} + e^{x/y}\left(-\frac{1}{y}\right)$$

$$= e^{x/y}\left(\frac{1}{y} - \frac{x}{y^2} - \frac{1}{y}\right) = e^{x/y}\left(-\frac{x}{y^2}\right)$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \text{ the given equation is exact.}$$

Solution : $$\underset{(y\ \text{constant})}{\int M\,dx} + \underset{\substack{(\text{terms free} \\ \text{from } x)}}{\int N\,dy} = c$$

$$\int\left(1 + e^{x/y}\right)dx + \int 0\cdot dy = c \quad (\text{no term free from } x \text{ in } N)$$

$x + ye^{x/y} = c$, the required general solution.

EXAMPLE - 6 :

Solve $(2y \sin x + \cos y)\,dx = (x \sin y + 2\cos x + \tan y)\,dy$ ***(JNTU 2007)***

Solution : Given differential can be written as $(2y \sin x + \cos y)\,dx$

$$+ (-x \sin y - 2\cos x - \tan y)\,dy = 0$$

Here $M = 2y \sin x + \cos y$, $\quad N = -x \sin y - 2\cos x - \tan y$

$$\frac{\partial M}{\partial y} = 2\sin x - \sin y, \qquad \frac{\partial N}{\partial x} = -\sin y + 2\sin x$$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the given equation is an exact solution

$$\underset{(\text{y constant})}{\int M\,dx} + \underset{(\text{terms free from x})}{\int N\,dy} = C$$

$$\underset{(\text{y constant})}{\int (2y \sin x + \cos y)\,dx} + \underset{(\text{terms free from x})}{\int (-\tan y)\,dy} = C$$

$$-2y\cos x + x\cos y + \log \sec y = C$$

EXAMPLE - 7 :

Solve $(5x^4 + 3x^2y^2 - 2xy^3)\,dx + (2x^3y - 3x^2y^2 - 5y^4)\,dy = 0$ ***(JNTU 2008)***

Solution : Given equation is of the form $Mdx + Ndy = 0$

$M = 5x^4 + 3x^2y^2 - 2xy^3$, $\quad N = 2x^3y - 3x^2y^2 - 5y^4$

$$\frac{\partial M}{\partial y} = 6x^2y - 6xy^2, \qquad \frac{\partial N}{\partial x} = 6x^2y - 6xy^2$$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the given equation is an exact solution.

$$\underset{(\text{y constant})}{\int M\,dx} + \underset{(\text{terms free from x})}{\int N\,dy} = C$$

$$\underset{(\text{y constant})}{\int (5x^4 + 3x^2y^2 - 2xy^3)\,dx} + \underset{(\text{terms free from x})}{\int 0\,dy} = C$$

$$x^5 + x^3y^2 - x^2y^3 = C$$

EXAMPLE - 8 :

Solve $(x^2 - 2xy + 3y^2)\,dx + (y^2 + 6xy - x^2)\,dy = 0$ ***(JNTU 2006)***

Solution : The given equation is of the form $Mdx + Ndy = 0$

$M = x^2 - 2xy + 3y^2$, $\quad N = y^2 + 6xy - x^2$

$$\frac{\partial M}{\partial y} = -2x, \quad \frac{\partial N}{\partial x} = -2x$$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. the given equation is an exact.

The solution is

$$\underset{(y \text{ constant})}{\int M dx} + \underset{(\text{terms free from } x)}{\int N dy} = C$$

$$\underset{(y \text{ constant})}{\int (x^2 - 2xy + 3y^2)\, dx} + \underset{(\text{terms free from } x)}{\int y^2 dx} = C$$

$$\frac{x^3}{3} - x^2 y + 3y^2 x + \frac{y^3}{3} = C$$

EXERCISE 5.2

Solve the following differential equations :

1. $(x + y\cos x)\, dx + \sin x\, dy = 0$ **Ans :** $\frac{x^2}{2} + y \sin x = c$

2. $(2x^2 + 6xy - y^2)\, dx + (3x^2 - 2xy + y^2)\, dy = 0$ **Ans :** $2x^3 + 9x^2 y - 3xy^2 + y^3 = c$

3. $\frac{dy}{dx} + \frac{ax + hy + g}{hx + by + f} = 0$ **Ans :** $ax^2 + 2hxy + by^2 + 2fx + 2fy + c = 0$

4. $(x^2 - ay)\, dx - (ax - y^2)\, dy = 0$ **Ans :** $x^3 - 3axy + y^3 = c$

5. $x\, dx + y\, dy = \frac{a^2 (x\, dy - y\, dx)}{x^2 + y^2}$ **Ans :** $x^2 + y^2 + 2a^2 \tan^{-1} \frac{x}{y}$

6. $(x^2 - 4xy - 2y^2)\, dx + (y^2 - 4xy - 2x^2)\, dx = 0$ **Ans :** $x^3 - 6x^2 y - 6xy^2 + y^3 = c$

7. $(2xy \cos x^2 - 2xy + 1)\, dx + (\sin x^2 - x^2)\, dy = 0$ **Ans :** $y \sin x^2 - x^2 y + x = c$

8. $\frac{dy}{dx} + \frac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0$ **Ans :** $y \sin x + (\sin y + y)\, x = c$

9. $(3x^2 + 6xy^2)\, dx + (6x^2 y + 4y^3)\, dy = 0$ **Ans :** $x^4 + 2x^2 y^2 - 2a^2 x^2 - y^4 - 2b^2 y^2 = c$

10. $(5x^4 + 3x^2 y^2 - 2xy^3)\, dx + (2x^3 y - 3x^2 y^2 + 5y^4)\, dy = 0$ **Ans :** $x^5 + x^3 y^2 - x^2 y^3 + y^5 = c$

5.4 INTEGRATING FACTORS :

If a differential equation multiplied by a factor becomes exact, the factor is called the integrating factor (I.F.).

Integrating Factors by Inspection :

It is necessary to group certain terms in the given equation to make the equation readily integrable. Some of the frequently occuring integrable combinations are given below.

1) $y\,dx + x\,dy = d(xy)$

2) $\dfrac{y\,dx - x\,dy}{y^2} = d\left(\dfrac{x}{y}\right)$

3) $\dfrac{x\,dy - x\,dx}{x^2} = d\left(\dfrac{y}{x}\right)$

4) $\dfrac{x\,dy - y\,dx}{xy} = d\left[\log\left(\dfrac{y}{x}\right)\right]$

5) $\dfrac{x\,dy - y\,dx}{x^2 + y^2} = d\left[\tan^{-1}\dfrac{y}{x}\right]$

SOLVED EXAMPLES – 5.3

EXAMPLE - 1 :

Solve $x\,dy - y\,dx = x\sqrt{x^2 - y^2}\,dx$

Solution : The given equation can be written as

$$x\,dy - y\,dx = x^2\sqrt{1 - \frac{y^2}{x^2}}\,dx$$

i.e., $$\frac{x\,dy - y\,dx}{x^2} = \sqrt{1 - \frac{y^2}{x^2}}\,dx$$

i.e., $$\frac{\left(\dfrac{x\,dy - y\,dx}{x^2}\right)}{\sqrt{1 - \dfrac{y^2}{x^2}}} = dx$$

i.e., $$d\left(\sin^{-1}\frac{y}{x}\right) = dx$$ which is exact (directly integrable)

Integrating $$\sin^{-1}\frac{y}{x} = x + c$$

or $$y = x\sin(x + c)$$

which is the required solution.

EXAMPLE - 2 :

Solve $y(x^2 + y^2 - 1)\,dx + x(x^2 + y^2 + 1)\,dy = 0$

Solution : Grouping the terms,

$$x\,dy - y\,dx + (x^2 + y^2)(y\,dx + x\,dy) = 0$$

i.e., $$\frac{x\,dy - y\,dx}{x^2 + y^2} + (y\,dx + x\,dy) = 0$$

i.e., $$d\left(\tan^{-1}\frac{y}{x}\right) + d(xy) = 0$$ which is exact (directly integrable)

Integrating $\tan^{-1}\dfrac{y}{x} + xy = 0$, the required solution.

EXAMPLE - 3 :

Solve $y(1 + xy)\,dx - x\,dy = 0$

Solution : Given equation can be written as

$$y\,dx - x\,dy + y^2x\,dx = 0$$

Dividing by y^2, $$\frac{y\,dx - x\,dy}{y^2} + x\,dx = 0$$

or $$d\left(\frac{x}{y}\right) + x\,dx = 0$$ which is exact.

Integrating $\dfrac{x}{y} + \dfrac{x^2}{2} = c$, the required solution.

EXAMPLE - 4 :

Solve $y\,dx - x\,dy + 3x^2y^2e^{x^3}\,dx = 0$

Solution : Dividing the equation by y^2,

$$\frac{y\,dx - x\,dy}{y^2} + e^{x^3}3x^2\,dx = 0$$

$$d\left(\frac{x}{y}\right) + d\left(e^{x^3}\right) = 0$$ which is exact and readily integrable

Integrating $\frac{x}{y} + e^{x^3} = c$

EXAMPLE - 5 : **(JNTU 2006)**

Solve $y(xy + e^x)\,dx - e^x\,dy = 0$

Solution : The equation can be written as

$$2xy^2dx + ye^xdx - e^xdy = 0$$

$$2x\,dx + \frac{ye^x dx - e^x dy}{y^2}$$

$$2x\,dx + d\left(\frac{e^x}{y}\right)$$

Integrating, $x^2 + \frac{e^x}{y} = c$ the required solution.

EXAMPLE - 6 : **(JNTU 2008)**

Solve $y(2x^2y + e^x)\,dx = (e^x + y^3)\,dy$

Solution : The given equation can be written as $y(2x^2y + e^x)\,dx = (e^x + y^3)\,dy$

i.e., $2x^2y^2dx + e^xydx - e^xdy - y^3dy = 0$

Dividing the equation by y^2, we get

$$2x^2dx + \frac{e^xy}{y^2}dx - \frac{e^x}{y^2}dy - y\,dy = 0$$

i.e., $$2x^2dx + \frac{y\,e^xdx - e^xdy}{y^2} - ydy = 0$$

i.e., $$2x^2dx + d\left(\frac{e^x}{y}\right) - y\,dy = 0$$ which is exact and readily integrable

Integrating $\frac{2x^3}{3} + \frac{e^x}{y} - \frac{y^2}{2} = C$

5.5 EQUATIONS REDUCIBLE TO EXACT EQUATIONS :

If the given equation $M\,dx + N\,dy = 0$ is not exact, it can be made exact by multiplying it by suitable integrating factor.

Rules to find integrating factors :

Rule - 1 : If M and N are homogeneous functions of the same order and $Mx + Ny \neq 0$, then $\dfrac{1}{Mx + Ny}$ is an integrating factor of $Mdx + Ndy = 0$

Rule - 2 : If $M\,dx + N\,dy = 0$ is of the form $y\,f_1(xy) + x\,f_2(xy) = 0$ and $Mx - Ny \neq 0$, then $\dfrac{1}{Mx - Ny}$ is an integrating factor of $M\,dx + N\,dy = 0$

Rule - 3 : For the equation $M\,dx + N\,dy = 0$

if $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$ is a function of x only, say $f(x)$, then $e^{\int f(x)\,dx}$ is an integrating factor.

Rule - 4 : For the equation $M\,dx + N\,dy = 0$

if $\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}$ is a function of y only, say $g(y)$, then $e^{\int g(y)\,dy}$ is an integrating factor.

Rule - 5 : If the equation $M\,dx + N\,dy = 0$ is of the form $x^a y^b(my\,dx + nx\,dy) + x^c y^d(py\,dx + qx\,dy) = 0$ where a, b, c, d, m, n, q, p are all constants, then $x^h y^k$ is an integrating factor where h, k are so chosen that after multiplication by $x^k y^k$, the equation becomes exact.

SOLVED EXAMPLES – 5.4

EXAMPLE - 1 :

Solve $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0$

Solution :

$M = x^2y - 2xy^2 \qquad N = -x^3 + 3x^2y$

$$\frac{\partial M}{\partial y} = 2xy - 4xy \qquad \frac{\partial N}{\partial x} = -3x^2 + 6xy$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \therefore \text{ The equation is not exact.}$$

To make it exact, we have to multiply by Integrating factor. Since M and N are homogeneous function of same degree the integrating factor is $\dfrac{1}{Mx + Ny}$.

$$\frac{1}{Mx + Ny} = \frac{1}{x^3y - 2x^2y^2 - x^3y + 3x^2y^2} = \frac{1}{x^2y^2}$$

Multiplying the given equation by integrating factor $\frac{1}{x^2y^2}$,

we get $\left(\frac{1}{y} - \frac{2}{x}\right) dx - \left(\frac{x}{y^2} - \frac{3}{y}\right) dy = 0$

This equation is an exact.

Solution :

$$\underset{(y \text{ constant})}{\int} M\,dx + \underset{\substack{(\text{terms free}\\ \text{from } x)}}{\int} N\,dy = c$$

$$\underset{(y \text{ constant})}{\int} \left(\frac{1}{y} - \frac{2}{x}\right) dx + \underset{\substack{(\text{terms free}\\ \text{from } x)}}{\int} \frac{3}{y}\,dy = c$$

$$\frac{x}{y} - 2\log x + 3\log y = c$$

$\frac{x}{y} - \log x^2 + \log y^3 = c$, the required solution.

EXAMPLE - 2 :

Solve $x^2y\,dx - (x^3 + y^3)\,dy = 0$

Solution : Here $M = x^2y$ $\qquad N = -x^3 - y^3$

$$\frac{\partial M}{\partial y} = x^2 \qquad \frac{\partial N}{\partial x} = -3x^2$$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ the equation is not exact.

To make it exact, we have to multiply by integrating factor. Since M and N are homogeneous functions of same degree, the integrating factor is $\frac{1}{Mx + Ny}$.

$$\frac{1}{Mx + Ny} = \frac{1}{x^3y - x^3y - y^4} = -\frac{1}{y^4}$$

Multiply the given equation by $-\frac{1}{y^4}$

$$-\frac{x^2}{y^3}\,dx + \frac{x^3+y^3}{y^4}\,dy = 0$$

Now the equation is exact.

Solution : $\underset{(y\ \text{constant})}{\int} M\,dx + \underset{(\text{terms free from } x)}{\int} N\,dy = c$

$$-\int \frac{x^2}{y^3}\,dx + \int \frac{1}{y}\,dy = c \qquad \Bigg| \qquad N = \frac{x^3+y^3}{y^4} = \frac{1}{y}\ (\text{terms free from } x)$$

$$-\frac{x^3}{3y^3} + \log y = c$$

$$\log y = \frac{x^3}{3y^3} + c, \text{ the required general solution.}$$

EXAMPLE - 3 :

Solve $(xy\sin xy + \cos xy)\,y\,dx + (xy\sin xy - \cos xy)\,x\,dy = 0$

Solution : $M = y(xy\sin xy + \cos xy)$, $N = x(xy\sin xy - \cos xy)$

This equation is of the form $f_1(xy)\,y\,dx + f_2(xy)\,x\,dy = 0$

The integrating factor $= \dfrac{1}{Mx - Ny}$

$$\frac{1}{Mx - Ny} = \frac{1}{2xy\cos xy}$$

Multiply the equation by $\dfrac{1}{2xy\cos xy}$,

$$\frac{1}{2xy\cos xy}(xy\sin xy + \cos xy)\,y\,dx + \frac{1}{2xy\cos xy}(xy\sin xy - \cos xy)\,y\,dy = 0$$

$$= \left(y\tan xy + \frac{1}{x}\right)dx + \left(x\tan xy - \frac{1}{y}\right)dy = 0$$

This is an exact differential equation.

Solution : $\underset{(y\ \text{constant})}{\int} M\,dx + \underset{(\text{terms free from } x)}{\int} N\,dy = c$

$$\underset{(y\ \text{constant})}{\int M\,dx} = \int \left(y\tan xy + \frac{1}{x}\right)dx = y\cdot\frac{1}{y}\log\sec(xy) + \log x$$

$$= \log\sec(xy) + \log x$$

$$\underset{\substack{\text{(terms free}\\ \text{from x)}}}{\int N\,dy} = \int -\frac{1}{y}\,dy = -\log y$$

The required solution $\log \sec(xy) + \log x - \log y = \log c$

$$\frac{x}{y}\sec(xy) = c$$

EXAMPLE - 4 :

Solve $(1 + xy)\,y\,dx + (1 - xy)\,x\,dy = 0$

Solution : The given equation is of the form $f_1(xy)\,y\,dx + f_2(xy)\,x\,dy = 0$

The integrating factor $= \dfrac{1}{Mx - Ny}$

$$= \frac{1}{xy(1+xy) - xy(1-xy)} = \frac{1}{2x^2y^2}$$

Multiply the given equation by $\dfrac{1}{2x^2y^2}$

$$\frac{y(1+xy)}{2x^2y^2}\,dx + \frac{x(1-xy)}{2x^2y^2}\,dy = 0$$

$$\frac{y + xy^2}{2x^2y^2}\,dx + \frac{x - x^2y}{2x^2y^2}\,dy = 0$$

$$\left(\frac{1}{2x^2y} + \frac{1}{2x}\right)dx + \left(\frac{1}{2xy^2} - \frac{1}{2y}\right)dy = 0$$

This is an exact differential equation.

Solution : $\underset{(y\text{ constant})}{\int} M\,dx + \underset{\substack{\text{(terms free}\\ \text{from x)}}}{\int N\,dy} = c$

$$\underset{(y\text{ constant})}{\int}\left(\frac{1}{2x^2y} + \frac{1}{2x}\right)dx + \underset{\substack{\text{(terms free}\\ \text{from x)}}}{\int} -\frac{1}{2y}\,dy = c$$

$$-\frac{1}{2xy} + \frac{1}{2}\log x - \frac{1}{2}\log y = c$$

$$-\frac{1}{xy} + \log\frac{x}{y} = c^1 \text{ where } c^1 = 2c$$

is the required general solution.

EXAMPLE - 5 :

Solve $y\,dx - x\,dy + (1 + x^2)\,dx + x^2 \sin y\,dy = 0$

Solution : $(y + 1 + x^2)\,dx + (x^2 \sin y - x)\,dy = 0$

$M = y + 1 + x^2 \qquad N = x^2 \sin y - x$

$\frac{\partial M}{\partial y} = 1 \qquad \frac{\partial N}{\partial x} = 2x \sin y - 1$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ the equation is not exact.

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{x^2 \sin y - x}(1 - 2x \sin y + 1) = \frac{2(1 - x \sin y)}{-x(1 - x \sin y)} = -\frac{2}{x} = f(x)$$

$\therefore$ Integrating factor $e^{\int f(x)\,dx} = e^{\int -\frac{2}{3}dx} = e^{-2\log x} = e^{\log \frac{1}{x^2}} = \frac{1}{x^2}$

Multiply the equation by the integrating factor $\frac{1}{x^2}$

$$\left(\frac{y + 1 + x^2}{x^2}\right) dx + \left(\frac{x^2 \sin y - x}{x^2}\right) dy = 0$$

$$\left(\frac{y}{x^2} + \frac{1}{x^2} + 1\right) dx + \left(\sin y - \frac{1}{x}\right) dy = 0$$

Solving $\underset{(y\text{ constant})}{\int} M\,dx + \underset{(\text{terms free from } x)}{\int N\,dy} = c$

$$\underset{(y\text{ constant})}{\int}\left(\frac{y}{x^2} + \frac{1}{x^2} + 1\right)dx + \underset{(\text{terms free from } x)}{\int \sin y\,dy} = c$$

$$-\frac{y}{x} - \frac{1}{x} + x - \cos y = c$$

$x^2 - y - 1 - x \cos y = cx$, the required general solution.

EXAMPLE - 6 :

Solve $(y^4 + 2y)\,dx + (xy^3 + 2y^4 - 4x)\,dy = 0$

Solution : $M = y^4 + 2y \qquad N = xy^3 + 2y^4 - 4x$

$\frac{\partial M}{\partial y} = 4y^3 + 2 \qquad \frac{\partial N}{\partial x} = y^3 - 4$

EXAMPLE - 5 :

Solve $y\,dx - x\,dy + (1 + x^2)\,dx + x^2 \sin y\,dy = 0$

Solution : $(y + 1 + x^2)\,dx + (x^2 \sin y - x)\,dy = 0$

$M = y + 1 + x^2 \qquad N = x^2 \sin y - x$

$\dfrac{\partial M}{\partial y} = 1 \qquad \dfrac{\partial N}{\partial x} = 2x \sin y - 1$

$\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$ the equation is not exact.

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{x^2 \sin y - x}(1 - 2x \sin y + 1) = \frac{2(1 - x\sin y)}{-x(1 - x\sin y)} = -\frac{2}{x} = f(x)$$

$\therefore$ Integrating factor $e^{\int f(x)\,dx} = e^{\int -\frac{2}{3}dx} = e^{-2\log x} = e^{\log \frac{1}{x^2}} = \dfrac{1}{x^2}$

Multiply the equation by the integrating factor $\dfrac{1}{x^2}$

$$\left(\frac{y + 1 + x^2}{x^2}\right)dx + \left(\frac{x^2 \sin y - x}{x^2}\right)dy = 0$$

$$\left(\frac{y}{x^2} + \frac{1}{x^2} + 1\right)dx + \left(\sin y - \frac{1}{x}\right)dy = 0$$

Solving $\underset{(y\text{ constant})}{\int} M\,dx + \underset{(\text{terms free from } x)}{\int} N\,dy = c$

$$\underset{(y\text{ constant})}{\int}\left(\frac{y}{x^2} + \frac{1}{x^2} + 1\right)dx + \underset{(\text{terms free from } x)}{\int} \sin y\,dy = c$$

$$-\frac{y}{x} - \frac{1}{x} + x - \cos y = c$$

$x^2 - y - 1 - x\cos y = cx$, the required general solution.

EXAMPLE - 6 :

Solve $(y^4 + 2y)\,dx + (xy^3 + 2y^4 - 4x)\,dy = 0$

Solution : $M = y^4 + 2y \qquad N = xy^3 + 2y^4 - 4x$

$\dfrac{\partial M}{\partial y} = 4y^3 + 2 \qquad \dfrac{\partial N}{\partial x} = y^3 - 4$

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$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. The equation is not exact.

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 3y^3 + 6 = 3(y^3 + 2)$$

Integrating factor $\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{y\left(y^3+2\right)} 3(y^2+2) = \frac{3}{y}$ (a function of y only)

$$\text{I. F.} = e^{-\int \frac{3}{y} dy} = e^{-3 \log y} = e^{\log \frac{1}{y^3}} = \frac{1}{y^3}$$

Multiplying the equation by I.F.,

$$\frac{\left(y^4 + 2y\right) dx}{y^3} + \frac{\left(xy^3 + 2y^4 - 4x\right)}{y^3} dy = 0$$

$$\left(y + \frac{2}{y^2}\right) dx + \left(x + 2y - 4\frac{x}{y^3}\right) dy = 0$$

This is an exact differential equation.

$$\text{Solution} \quad \int_{(y\ \text{constant})} M\,dx + \int_{(\text{terms free from } x)} N\,dy = c$$

$$\int_{(y\ \text{constant})} \left(y + \frac{2}{y^2}\right) dx + \int_{(\text{terms free from } x)} 2y\,dy = c$$

$yx + \frac{2x}{y^2} + y^2 = c$, the required general solution.

EXAMPLE - 7 :

Solve $y(8x - 9y)\,dx + 2x(x - 3y)\,dy = 0$

Solution : $M = 8xy - 9y^2$ $\qquad N = 2x^2 - 6xy$

$$\frac{\partial M}{\partial y} = 8x - 18y \qquad \frac{\partial N}{\partial x} = 4x - 6y$$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ the equation is not an exact.

$$\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=\frac{1}{2x(x-3y)}\,4(x-3y)=\frac{2}{x}\text{ (a function of x only)}$$

$$\text{I. F.}=e^{\int \frac{2}{x}\,dx}=e^{2\log x}=e^{\log x^2}=x^2$$

Multiply the given equation by x^2

$x^2y(8x-9y)\,dx+x^2\cdot 2x\cdot(x-3y)\,dy=0$

$(8x^3y-9x^2y^2)\,dx+(2x^4-6x^3y)\,dy=0$

This is an exact differential equation.

$$\underset{(y\text{ constant})}{\int}\left(8x^3y-9x^2y^2\right)dx+\underset{\substack{\text{(terms free}\\\text{from x)}}}{\int 0\cdot dy}=c$$

$$\frac{8x^4}{4}\cdot y-\frac{9x^3}{3}\cdot y^2=c$$

$2x^4y-3x^3y^2=c$, the required general solution.

EXAMPLE - 8 :

Solve $(3x^2y^4+2xy)\,dx+(2x^3y^3-x^2)\,dy=0$

Solution : $M=3x^2y^4+2xy$ $\quad N=2x^3y^3-x^2$

$$\frac{\partial M}{\partial y}=12x^2y^3+2x \qquad \frac{\partial N}{\partial x}=6x^2y^3-2x$$

$$\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}\text{, the equation is not an exact.}$$

$$\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)=\frac{6x^2y^3-2x-12x^2y^3-2x}{3x^2y^4+2xy}$$

$$=\frac{-2\left(2x+3x^2y^3\right)}{y\left(2x+3x^2y^3\right)}=-\frac{2}{y}$$

$$\text{I. F.}=e^{-\int\frac{2}{y}dy}=\frac{1}{y^2}$$

Multiply the given equation by $\frac{1}{y^2}$, we get

$$\left(\frac{3x^2y^4 + 2xy}{y^2}\right) dx + \frac{\left(2x^3y^3 - x^2\right)}{y^2} dy = 0$$

$$\left(3x^2y^2 + \frac{2x}{y}\right) dx + \left(2x^3y - \frac{x^2}{y^2}\right) dy = 0$$

The equation is an exact.

$$\int_{(y\ \text{constant})} \left(3x^2y^2 + \frac{2x}{y}\right) dx + \int_{\substack{(\text{terms free} \\ \text{from } x)}} 0 \cdot dy = c$$

$x^3y^2 + \frac{x^2}{y} = c$, the required solution.

EXAMPLE - 9 :

Solve $xy^3 (y\, dx + 2x\, dy) + (3y\, dx + 5x\, dy) = 0$

Solution : The given equation is

$$xy^3 (y\, dx + 2x\, dy) + (3y\, dx + 5x\, dy) = 0$$

Comparing with $x^ay^b (my\, dx + nx\, dy) + x^cy^d (py\, dx + qx\, dy) = 0$

$$a = 1, b = 3, c = 0, d = 0$$

$$m = 1, n = 2, p = 3, q = 5$$

Now let x^hy^k be the integrating factor.

Multiplying the given equation by x^hy^k, we get

$$x^{h+1} y^{k+3} (y\, dx + 2x\, dy) + 3\, x^hy^{k+1}\, dx + 5x^{h+1}y^k\, dy = 0$$

This equation is an exact.

$$M = x^{h+1}y^{k+3}y + 3x^h y^{k+1}$$

i.e., $M = x^{h+1} y^{k+4} + 3x^hy^{k+1}$

$$N = x^{h+1} y^{k+3} \cdot 2x + 5x^{h+1} y^k$$

i.e., $N = 2x^{h+2} y^{k+3} + 5x^{h+1} y^k$

Since, the equation is exact, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Multiply the given equation by $\frac{1}{y^2}$, we get

$$\left(\frac{3x^2y^4 + 2xy}{y^2}\right) dx + \frac{\left(2x^3y^3 - x^2\right)}{y^2} dy = 0$$

$$\left(3x^2y^2 + \frac{2x}{y}\right) dx + \left(2x^3y - \frac{x^2}{y^2}\right) dy = 0$$

The equation is an exact.

$$\underset{(y \text{ constant})}{\int} \left(3x^2y^2 + \frac{2x}{y}\right) dx + \underset{\substack{(\text{terms free} \\ \text{from } x)}}{\int} 0 \cdot dy = c$$

$x^3y^2 + \frac{x^2}{y} = c$, the required solution.

EXAMPLE - 9 :

Solve $xy^3 (y\,dx + 2x\,dy) + (3y\,dx + 5x\,dy) = 0$

Solution : The given equation is

$$xy^3 (y\,dx + 2x\,dy) + (3y\,dx + 5x\,dy) = 0$$

Comparing with $x^a y^b (my\,dx + nx\,dy) + x^c y^d (py\,dx + qx\,dy) = 0$

$$a = 1, b = 3, c = 0, d = 0$$

$$m = 1, n = 2, p = 3, q = 5$$

Now let $x^h y^k$ be the integrating factor.

Multiplying the given equation by $x^h y^k$, we get

$$x^{h+1} y^{k+3} (y\,dx + 2x\,dy) + 3\,x^h y^{k+1}\,dx + 5x^{h+1} y^k\,dy = 0$$

This equation is an exact.

$$M = x^{h+1} y^{k+3} y + 3x^h y^{k+1}$$

i.e., $M = x^{h+1} y^{k+4} + 3x^h y^{k+1}$

$$N = x^{h+1} y^{k+3} \cdot 2x + 5x^{h+1} y^k$$

i.e., $N = 2x^{h+2} y^{k+3} + 5x^{h+1} y^k$

Since, the equation is exact, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\therefore \quad (k+h)\,x^{h+1} \cdot y^{k+3} + (k+1)\,3\,x^h y^k$

$= 2(h+2)\,x^{h+1} y^{k+3} + 5(h+1)\,x^h y^k$

Equating the coefficient of $x^{h+1} y^{k+3}$ and $x^h y^k$

$k + h = 2h + h \qquad$ i.e., $2h - k = 0$

$3k + 3 = 5h + 5 \qquad$ i.e., $5h - 3k = 2$

Solving the equation, we get $h = 2, k = 4$

$\therefore \quad$ The I.F. $= x^2y^4$

Multiply the equation by x^2y^4, we get

$x^3y^7 (y\,dx + 2x\,dy) + (3x^2y^5\,dx + 5x^3y^4\,dy) = 0$

i.e. $(x^3y^8 + 3x^2y^5)\,dx + (2x^4y^7 + 5x^3y^4)\,dy = 0$

This is an exact equation.

Solution $\quad \underset{(y \text{ constant})}{\int M\,dx} + \underset{(\text{terms free from } x)}{\int N\,dy} = c$

$$\underset{(y\text{ constant})}{\int} \left(x^3y^8 + 3x^2y^5\right) dx + \underset{(\text{terms free from } x)}{\int 0 \cdot dy} = c$$

$\dfrac{x^4}{4} y^8 + x^3y^5 = c$, the required general solution.

EXAMPLE - 10 :

Solve $xy(y\,dx + x\,dy) + x^2y^2(2y\,dx - x\,dy) = 0$

Solution : Comparing the equation with

$x^ay^b(my\,dx + nx\,dy) + x^cy^d(py\,dx + qx\,dy) = 0$

$$a = 1, b = 1, c = 2, d = 2$$

$$m = 1, n = 1, p = 2, q = -1$$

Let x^hy^k be the integrating factor.

Multiplying the equation by the integrating factor x^hy^k

$x^{h+1}y^{k+1}(y\,dx + x\,dy) + x^{h+2}y^{k+2}(2y\,dx - x\,dy) = 0$

$$(x^{h+1}y^{k+2} + 2x^{h+2}y^{k+3})\,dx + (x^{h+2}y^{k+1} - x^{h+3}y^{k+2})\,dy = 0$$

This is exact differential equation of the form $M\,dx + N\,dy = 0$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$(k+2)x^{h+1}y^{k+1} + 2(k+3)x^{h+2}y^{k+2}$$
$$= (h+2)x^{h+1}y^{k+1} - (h+3)x^{h+2}y^{k+2}$$

Comparing the coefficient of $x^{h+1}y^{k+1}$ and $x^{h+2}y^{k+2}$ on both sides

$k+2 = h+2$ i.e., $h - k = 0$

$2k + 6 = -h - 3$ i.e., $h + 2k = -9$

Solving $h = k = -3$ I.F. $= x^{-3}y^{-3} = \dfrac{1}{x^3y^3}$

Multiply the equation by $\dfrac{1}{x^3y^3}$, we get

$$\frac{xy(y\,dx + x\,dy)}{x^3y^3} + \frac{x^2y^2}{x^3y^3}(2y\,dx - x\,dy) = 0$$

$$\frac{y\,dx + x\,dy}{x^2y^2} + \frac{1}{xy}(2y\,dx - x\,dy) = 0$$

$$\frac{y}{x^2y^2}dx + \frac{x}{x^2y^2}dy + \frac{2y}{xy}dx - \frac{x}{xy}dy = 0$$

$$\left(\frac{1}{x^2y} + \frac{2}{x}\right)dx + \left(\frac{1}{xy^2} - \frac{1}{y}\right)dy = 0$$

The solution $\underset{(y \text{ constant})}{\int M\,dx} + \underset{(\text{terms free from } x)}{\int N\,dy} = c$

$$\underset{(y\text{ constant})}{\int}\left(\frac{1}{x^2y} + \frac{2}{x}\right)dx + \underset{(\text{terms free from } x)}{\int} -\frac{1}{y}\,dy = c$$

$-\dfrac{1}{xy} + 2\log x - \log y = c$ the required general solution.

EXAMPLE - 11 :- (JNTU 2006)

Solve $y(2xy + e^x)dx - e^x\,dy = 0$

Solution : Here $M = y(2xy + e^x)$, $N = -e^x$

$$\frac{\partial M}{\partial y} = 4xy + e^x, \quad \frac{\partial N}{\partial x} = -e^x$$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, the equation is not an exact.

$$\frac{1}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] = \frac{-e^x - 4xy - e^x}{y(2xy + e^x)} = -\frac{2}{y}$$

$$\text{I.F} = e^{-\int \frac{2}{y}dy} = \frac{1}{y^2}$$

Multiply the given equation by $\frac{1}{y^2}$, we get

$$y\left(\frac{2xy + e^x}{y^2}\right)dx - \frac{e^x}{y^2}dy = 0$$

$$2x\,dx + \frac{e^x}{y}dx - \frac{e^x}{y^2}dy = 0$$

$$\left(2x + \frac{e^x}{y}\right)dx - \frac{e^x}{y^2}dy = 0$$

This equation is an exact.

$$\int\left(2x + \frac{e^x}{y}\right)dx + \int 0 \cdot dy = c$$

(y constant) (terms free from x)

$$x^2 + \frac{e^x}{y} = c$$

EXAMPLE - 12 :-

Solve $(y^4 + 2y)\,dx + (xy^3 + 2y^4 - 4x)\,dy = 0$

Solution : Here $M = y^4 + 2y$, $N = xy^3 + 2y^4 - 4x$

$$\frac{\partial M}{\partial y} = 4y^3 + 2, \quad \frac{\partial N}{\partial x} = y^3 - 4$$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, the given equation is not an exact.

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{y^3 - 4 - 4y^3 - 2}{y^4 + 2y} = -\frac{3}{y}$$

$$\text{I.F} = e^{-\int \frac{3}{y}dy} = \frac{1}{y^3}$$

Multiply the given equation by $\frac{1}{y^3}$, we get

$$\left(y + \frac{2}{y^2}\right)dx + \left(x + 2y - \frac{4x}{y^3}\right)dy = 0$$

This is an exact differential equation.

Solution is

$$\underset{(\text{y constant})}{\int M dx} + \underset{(\text{terms free from x})}{\int N\, dy} = c$$

$$\underset{(\text{y constant})}{\int \left(y + \frac{2}{y^2}\right) dx} + \underset{(\text{terms free from x})}{\int 2y\, dy} = c$$

$$xy + \frac{2x}{y^2} + y^2 = c$$

EXERCISE 5.3

Solve the following differential equations.

1. $x\,dy - y\,dx = xy^2 dx$ **Ans : $2x + x^2y = cy$**
2. $x\,dx = y(x^2 + y^2 - 1)\,dy$ **Ans : $\log(x^2 + y^2) = y^2 + c$**
3. $\frac{y\,dx - x\,dy}{(x-y)^2} = \frac{dx}{2\sqrt{1-x^2}}$ **Ans : $\frac{x}{x-y} = \frac{1}{2}\sin^{-1}x + c$**
4. $ye^{x/y}\,dx = (xe^{x/y} + y^2)\,dy$ **Ans : $e^{x/y} = y + c$**
5. $(y^2 + y + x)\,dy - y\,dx = 0$ **Ans : $y + \log y - \frac{x}{y} = c$**
6. $(y^4 + 2y)\,dx + (xy^3 + 2y^4 - 4x)\,dy = 0$ **Ans : $xy + x^3 + \frac{2x}{y^3} = c$**
7. $(x^2 + y^2)\,dx - 2xy\,dy = 0$ **Ans : $x^2 - y^2 = cx$**
8. $(x^2y^2 + xy + 1)\,y\,dx + (x^2y^2 - xy + 1)\,x\,dy = 0$ **Ans : $xy - \log\left(\frac{x}{y}\right) - \frac{1}{xy} = c$**

9. $x^2y\,dx - (x^3 + y^3)\,dy = 0$ **Ans :** $\frac{x}{y} - 2\log x + 3\log y = c$

10. $(xy^2 + 2x^3y^3)\,dx + (x^2y - x^3y^2)\,dy = 0$ **Ans :** $\log\left(\frac{x^2}{y}\right) - \frac{1}{xy} = c$

11. $(6x^2 + 4y^3 + 12y)\,dx + 3x(1 + y^2)\,dy = 0$ **Ans :** $x^6 + x^4y^3 + 3yx^4 = c$

12. $(xy - 3)\,y\,dx + (3xy + 7)\,x\,dy = 0$ **Ans :** $y^7(xy + 5)^8 = cx^3$

13. $(5x^3 + 3xy + 2y^2)\,dx + (x^2 + 2xy)\,dy = 0$ **Ans :** $x^5 + x^3y + x^2y^2 = c$

14. $(x^2 + y^2 + 2x)\,dx + 2y\,dy = 0$ **Ans :** $e^x(x^2 + y^2) = c$

15. $x(3y\,dx + 2x\,dy) + 8y^4(y\,dx + 3x\,dy) = 0$ **Ans :** $x^2y^3(x + 4y^4) = c$

16. $y(y\,dx - x\,dy) + 2x^2(y\,dx + x\,dy) = 0$ **Ans :** $6x^{1/2}y^{1/2} - x^{-3/2}y^{3/2} = c$

17. $(x^2 - 3xy + 2y^2)\,dx + x(3x - 2y)\,dy = 0$ **Ans :** $x^2\log x + 3xy = y^2 + cx^2$

18. $x(4y\,dx + 2x\,dy) + y^3(3y\,dx + 5x\,dy) = 0$ **Ans :** $x^4y^2 + x^3y^5 = c$

5.6 LINEAR EQUATIONS :

An equation of the form $\frac{dy}{dx} + Py = Q$ ....(1)

where P and Q are functions of x alone or constants is called a linear equation of the first order.

Solution : The general solution can be obtain as follows.

Multiply the equation (1) both sides by $e^{\int P dx}$

$$\frac{dy}{dx} e^{\int P dx} + Py e^{\int P dx} = Q e^{\int P dx}$$

i.e., $$\frac{d}{dx}\left(ye^{\int P dx}\right) = Q e^{\int P dx}$$

Integrating, $ye^{\int P dx} = \int Q e^{\int P dx}\,dx + c$ which is the required solution.

Working Rule :

1) Write the given equation in the form $\frac{dy}{dx} + Py = Q$

2) Find the Integration factor (I.F.) $e^{\int P dx}$

3) Solution : $y\,(\text{I.F.}) = \int Q \cdot (\text{I.F.})\,dx + c$

Note : If the given equation is of the form

$$\frac{dx}{dy} + Px = Q$$

then $\text{I.F.} = e^{\int P dy}$

Solution : $x e^{\int P dy} = \int Q e^{\int P dy} \cdot dy + c$

SOLVED EXAMPLES - 5.5

EXAMPLE - 1 :

Solve $x(x-1)\frac{dy}{dx} - y = x^2(x-1)^2$ ***(JNTU 1995)***

Solution : Given equation $x(x-1)\frac{dy}{dx} - y = x^2(x-1)^2$

divide throughout by $x(x-1)$

$$\frac{dy}{dx} - \frac{1}{x(x-1)}y = x(x-1)$$

$$P = -\frac{1}{x(x-1)}, \quad Q = x(x-1)$$

Finding partial fractions

$$\frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$$

$$1 = A(x-1) + Bx$$

Coefficient of x, $A + B = 0$

Constant, $A = 1, B = -1$

$$\frac{1}{x(x-1)} = \frac{1}{x} - \frac{1}{x-1} = \frac{1}{x} + \frac{1}{1-x}$$

$$\text{I.F.} = e^{\int P dx} = e^{\int\left(\frac{1}{x}+\frac{1}{1-x}\right)dx}$$

$$= e^{\log x - \log(1-x)} = e^{\log\frac{x}{1-x}} = \frac{x}{1-x}$$

Solution : $y \cdot \frac{x}{1-x} = \int x(x-1) \cdot \frac{x}{1-x}\,dx + c$

$$= -\int x^2 dx + c = -\frac{x^3}{3} + c$$

$\therefore \frac{yx}{1-x} + \frac{x^3}{3} = c$ is the required solution.

EXAMPLE - 2 :

Solve $\cos x \dfrac{dy}{dx} + y \sin x = 1$

Solution : Dividing the given equation by $\cos x$

$$\frac{dy}{dx} + \frac{\sin x}{\cos x} y = \frac{1}{\cos x}$$

$$\frac{dy}{dx} + (\tan x) y = \sec x$$

Equation (1) is of the form $\dfrac{dy}{dx} + Py = Q$

$$P = \tan x,\ Q = \sec x$$

$$\text{I.F.} = e^{\int \tan x\, dx} = e^{\log \sec x} = \sec x$$

$$\text{Solution : } y \cdot \sec x = \int (\sec x) \sec x\, dx + c$$
$$= \int \sec^2 x\, dx + c = \tan x + c$$

$\therefore$ $y \sec x = \tan x + c$ is the required solution.

EXAMPLE - 3 :

Solve $\dfrac{dy}{dx} + 2xy = e^{-x^2}$

Solution :

$$P = 2x,\ Q = e^{-x^2}$$

$$\text{I.F} = e^{\int 2x\, dx} = e^{x^2}$$

$$y \cdot e^{x^2} = \int e^{-x^2} \cdot e^{x^2} \cdot dx + c$$
$$= \int dx + c = x + c$$

$y \cdot e^{x^2} = x + c$ is the required solution.

EXAMPLE - 4 :

Solve $(1 + y^2)\, dx + \left(x - e^{-\tan^{-1} y}\right) dy = 0$ ***(JNTU 2001 S, 2002,***

Solution : The equation can be written as

$$(1 + y^2) \frac{dx}{dy} = -x + e^{-\tan^{-1} y}$$

$$\frac{dx}{dy} + \frac{1}{1 + y^2} x = \frac{e^{-\tan^{-1} y}}{1 + y^2} \qquad \ldots\ldots\ldots($$

This equation is of the form $\dfrac{dx}{dy} + Px = Q$

$$\text{I.F} = e^{\int P dy} = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}$$

Solution :

$$x \cdot e^{\tan^{-1} y} = \int \frac{1}{1+y^2} e^{\tan^{-1} y} \cdot e^{-\tan^{-1} y} dy + c$$

$$= \int \frac{1}{1+y^2} dy + c$$

$$x e^{\tan^{-1} y} = \tan^{-1} y + c$$ is the required solution.

EXAMPLE - 5 :

Solve $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}} \right) \frac{dx}{dy} = 1$

Solution : The given equation can be written as

$$\frac{dy}{dx} + \frac{y}{\sqrt{x}} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$$

This equation is of the form $\frac{dy}{dx} + Py = Q$

$$P = \frac{1}{\sqrt{x}}, \quad Q = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$$

$$\text{I.F.} = e^{\int P dx} = e^{\int \frac{1}{\sqrt{x}} dx} = e^{2\sqrt{x}}$$

Solution :

$$y \cdot e^{2\sqrt{x}} = \int x^{-1/2} \cdot dx + c = 2\sqrt{x} + c$$

$$y \cdot e^{2\sqrt{x}} = 2\sqrt{x} + c$$

is the required solution.

EXAMPLE - 6 :

Solve $(x^2 - 1) \frac{dy}{dx} + 2xy = 1$ ***(JNTU 1999)***

Solution : $(x^2 - 1) \frac{dy}{dx} + 2xy = 1$

Divide by $(x^2 - 1)$

$$\frac{dy}{dx} + \frac{2x}{x^2 - 1} y = \frac{1}{x^2 - 1}$$

This equation is of the form $\frac{dy}{dx} + Py = Q$

$$P = \frac{2x}{x^2 - 1}, \; Q = \frac{1}{x^2 - 1}$$

$$\text{I.F. } e^{\int P dx} = e^{\int \frac{2x}{x^2-1} dx} = e^{\log(x^2-1)} = x^2 - 1$$

Solution : $y(x^2 - 1) = \int \frac{1}{x^2 - 1}(x^2 - 1)\,dx + c$

$$= \int dx + c = x + c$$

$y(x^2 - 1) = x + c$ is the required general solution.

EXAMPLE - 7 :

Solve $\frac{dy}{dx} + y \cot x = 5e^{\cos x}$ given $y = -4$ when $x = \frac{\pi}{2}$

Solution : $\frac{dy}{dx} + y \cot x = 5e^{\cos x}$

This equation is of the form $\frac{dy}{dx} + Py = Q$ $\quad P = \cot x, \; Q = 5e^{\cos x}$

I.F. $= e^{\int \cot x \, dx} = e^{\log \sin x} = \sin x$

Solution :

$$y \sin x = \int 5e^{\cos x} \cdot \sin x \, dx + c$$

$$y \sin x = -5e^{\cos x} + c$$

Now when $x = \frac{\pi}{2}, y = -4$

$\therefore -4 = -5 + c$ i.e., $c = 1$

The required particular solution is

$+ y \sin x + 5e^{\cos x} = 1$

Put $\cos x = t$

$-\sin x \, dx = dt$

$\int 5e^{+\cos x} \sin x \, dx$

$= \int 5e^{t}(-dt) = -(5e^{t})$

EXAMPLE - 8 :

Solve $\cos^2 x \frac{dy}{dx} + y = \tan x$ **(JNTU 99/S)**

Solution : Given equation $\cos^2 x \frac{dy}{dx} + y = \tan x$

dividing the equation by $\cos^2 x$

$$\frac{dy}{dx} + \frac{1}{\cos^2 x} \cdot y = \frac{\tan x}{\cos^2 x} = \tan x \sec^2 x$$

This equation is of the form $\frac{dy}{dx} + Py = Q$

$$P = \frac{1}{\cos^2 x},\ Q = \tan x \sec^2 x$$

$$\text{I.F. } e^{\int P dx} = e^{\int \frac{1}{\cos^2 x} dx} = e^{\int \sec^2 x\, dx} = e^{\tan x}$$

Solution : $y e^{\tan x} = \int \tan x \sec^2 x \cdot e^{\tan x} \cdot dx + c$

Put $\tan x = t$

$\sec^2 x\, dx = dt$

$$y e^t = \int t e^t \cdot dt + c$$

$$y e^t = t e^t - \int e^t\, dt + c$$

$$= t e^t - e^t + c$$

$$y e^{\tan x} = (\tan x)\, e^{\tan x} - e^{\tan x} + c$$

$$y e^{\tan x} = (\tan x - 1)\, e^{\tan x} + c$$

$$y = \tan x - 1 + c e^{-\tan x}$$

is the required general solution.

EXAMPLE - 9 :

Solve $(x + 2y^2)\frac{dy}{dx} = y$

Solution : Given equation can be written as

$$\frac{dx}{dy} - \frac{x}{y} = 2y^2$$

which is of the form $\frac{dx}{dy} + Px = Q$

$$P = -\frac{1}{y},\ Q = 2y^2$$

$$\text{I.F. } e^{\int -\frac{1}{y} dy} = e^{-\log y} = e^{\log \frac{1}{y}} = \frac{1}{y}$$

Solution

$$x \cdot \frac{1}{y} = \int 2y^2 \cdot \frac{1}{y}\, dy + c$$

$\frac{x}{y} = y + c$ is the required general solution.

EXAMPLE - 10 :

Solve $\frac{dy}{dx} - \frac{xy}{1-x^2} = \frac{1}{1-x^2}$

Solution : Given equation is of the form

$$\frac{dy}{dx} + Py = Q$$

$$P = -\frac{x}{1-x^2}, \quad Q = \frac{1}{1-x^2}$$

$$\text{I.F.} = e^{\int -\frac{x}{1-x^2} dx}$$

$$= e^{\frac{1}{2}\int -\frac{2x}{1-x^2} dx} = e^{\frac{1}{2}\log\left(1-x^2\right)} = e^{\log\sqrt{1-x^2}} = \sqrt{1-x^2}$$

Solution :

$$y\sqrt{1-x^2} = \int \frac{1}{1-x^2} \cdot \sqrt{1-x^2}\, dx + c$$

$$= \int \frac{1}{\sqrt{1-x^2}} dx + c = \sin^{-1} x + c$$

$$y\sqrt{1-x^2} = \sin^{-1} x + c$$

EXAMPLE - 11 :

Solve $(x+y+1)\frac{dy}{dx} = 1$

Solution : $\frac{dx}{dy} = x + y + 1$

$$\frac{dx}{dy} - x = y + 1$$

This equation is of the form $\frac{dx}{dy} + Px = Q$

$$P = -1, Q = y + 1$$

$$\text{I.F.} = e^{\int -dy} = e^{-y}$$

Solution :

$$x \cdot e^{-y} = \int (y+1)e^{-y} dy + c$$

$$xe^{-y} = \int (ye^{-y} + e^{-y}) dy + c$$

$$= \int ye^{-y} dy + \int e^{-y} dy + c$$

$$= y(-e^{-y}) - e^{-y} - e^{-y} + c$$

$$= -e^{-y}(y+2) + c$$

$$xe^{-y} = -e^{-y}(y+2)+c$$
$$(x+y+2)e^{-y} = c \text{ required general solution.}$$

EXAMPLE - 12 :

Solve $\frac{dy}{dx} + 2y\tan x = \sin x$ given $y = 0$ when $x = \frac{\pi}{3}$

Solution : The given equation is of the form $\frac{dy}{dx} + Py = Q$

$P = 2\tan x,\ Q = \sin x$

$$\text{I.F.} = e^{\int 2\tan x} = e^{2\log\sec x} = e^{\log\sec^2 x} = \sec^2 x$$

Solution : $y \cdot \sec^2 x = \int \sin x \cdot \sec^2 x\, dx + c$

$$= \int \sec x \tan x\, dx + c = \sec x + c$$

$$y\sec^2 x = \sec x + c \qquad \ldots\ldots\ldots\ldots (1)$$

Given $y = 0$ when $x = \frac{\pi}{3}$

$$0 = \sec\frac{\pi}{3} + c \qquad \therefore\ c = -2$$

The particular solution is given by substituting c in (1)

$$y\sec^2 x = \sec x - 2$$

EXAMPLE - 13 :

Solve $x^2\frac{dy}{dx} = e^y - x$ ***(JNTU 2007)***

Solution : The given equation can be written as

$$e^{-y}\frac{dy}{dx} = -\frac{dt}{dx} \qquad \ldots\ldots (1)$$

Put $e^{-y} = t,\ e^{-y}\frac{dy}{dx} = -\frac{dt}{dx}$

Equation (1) becomes

$$-\frac{dt}{dx} = \frac{1}{x^2} - \frac{t}{x}$$

$$\text{i.e. } \frac{dt}{dx} - \frac{t}{x} = -\frac{1}{x^2}$$

This equation is of the form $\frac{dy}{dx} + py = Q$

$$P = -\frac{1}{x},\ Q = -\frac{1}{x^2}$$

$$\text{I.F} = e^{\int p\,dx} = e^{\int -\frac{1}{x}dx} = e^{-\log x} = \frac{1}{x}$$

Solution is given by

$$t\frac{1}{x} = \int \left(-\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)dx$$

$$\text{i.e., } \frac{t}{x} = \int \frac{1}{x^3}\,dx$$

$$\text{i.e., } \frac{t}{x} = -\frac{1}{2x^2} + c$$

Substituting $t = e^{-y}$, we get

$$\frac{e^{-y}}{x} = -\frac{1}{2x^2} + c$$

EXAMPLE - 14 :

Solve $\frac{dy}{dx} + (y - 1)\cos x = e^{-\sin x}\cos^2 x$ **(JNTU 2007, 2006, 2006 S)**

Solution : The given differential equation can be written as

$$\frac{dy}{dx} + y\cos x = \cos x + e^{-\sin x}\cos^2 x$$

This equation is of the form $\frac{dy}{dx} + Py = Q$

$P = \cos x,\ Q = \cos x + e^{-\sin x}\cos^2 x$

$\text{I.F} = e^{\int P dx} = e^{\int \cos x\, dx} = e^{\sin x}$

Solution is given by

$$y\,e^{\sin x} = \int e^{\sin x}\left(\cos x + e^{-\sin x}\cos^2 x\right)dx + c$$

$$= \int e^{\sin x}\cos x\,dx + \int \cos^2 x\,dx + c$$

$$= e^{\sin x} + \frac{x}{2} + \frac{\sin 2x}{4} + c$$

EXAMPLE - 15 :

Solve $dr + (2r\cot\theta + \sin 2\theta)\,d\theta = 0$ **(JNTU 2007, 2006S, 2005)**

Solution : The given equation can be written as

$$\frac{dr}{d\theta} + 2r\cot\theta = -\sin 2\theta$$

This is of the form $\frac{dr}{d\theta} + Pr = Q$

$P = 2\cot\theta,\ Q = -\sin 2\theta$

$$\text{I.F} = e^{\int P dx} = e^{\int \frac{2\cos\theta}{\sin\theta} d\theta} = e^{2\log\sin\theta} = \sin^2\theta$$

Solution is given by

$$r\sin^2\theta = -\int \sin^2\theta \cdot 2\sin\theta\ \cos\theta\ d\theta + c$$

$$= -2\int \sin^3\theta\ \cos\theta\ d\theta + c$$

$$= -2\frac{\sin^4\theta}{4} + c$$

$$r = -\frac{1}{2}\sin^2\theta + c\operatorname{cosec}^2\theta$$

EXAMPLE - 16 :

Solve $\dfrac{dy}{dx} + \dfrac{y}{x\log x} = \dfrac{\sin 2x}{\log x}$ *(JNTU 2006 S)*

Solution : The given equation is of the form

$$\frac{dy}{dx} + Py = Q$$

$$P = \frac{1}{x\log x},\ Q = \frac{\sin 2x}{\log x}$$

$$\text{I.F.} = e^{\int P dx} = e^{\int \frac{dx}{x\log x}} = e^{\log\log x} = \log x$$

Solution is given by

$$y\log x = \int \log x \cdot \frac{\sin 2x}{\log x} dx + c$$

$$= \int \sin 2x\, dx + c = -\frac{\cos 2x}{2} + c$$

$$y\log x = -\frac{\cos 2x}{2} + c$$

EXAMPLE - 17 :

Solve $\dfrac{dy}{dx} + \dfrac{2xy}{\left(1+x^2\right)} = \dfrac{1}{\left(1+x^2\right)^2}$

given $y = 0$ when $x = 1$ *(JNTU 2006, 2004)*

Solution : The given equation is of the form $\frac{dy}{dx} + Py = Q$

$$P = \frac{2x}{1+x^2},\ Q = \frac{1}{\left(1+x^2\right)^2}$$

$$\text{I.F.} = e^{\int P dx} = e^{\int \frac{2x}{1+x^2} dx} = e^{\log(1+x^2)} = 1 + x^2$$

The solution is given by

$$y(1+x^2) = \int \frac{1+x^2}{(1+x^2)^2} dx + c$$

$$= \int \frac{1}{1+x^2} dx + c$$

$$= \tan^{-1} x + c$$

$$y(1+x^2) = \tan^{-1} x + c$$

given $y = 0, x = 1, \; c = -\frac{\pi}{4}$

$$y(1+x^2) = \tan^{-1} x - \frac{\pi}{4}$$

EXAMPLE - 18 :

Solve $\cos hx \dfrac{dy}{dx} + y \sin hx = 2 \cosh^2 x \sin hx$ **(JNTU 2005, 2004)**

Solution : $\cos hx \dfrac{dy}{dx} + y \sin hx = 2 \cosh^2 x \sin hx$

Dividing by cos hx, we get

$$\frac{dy}{dx} + y \frac{\sin hx}{\cos hx} = 2 \cos hx \sin hx$$

i.e., $\dfrac{dy}{dx} + \tan hx \cdot y = 2 \cos hx \sin hx$

This is of the form $\dfrac{dy}{dx} + Py = Q$

$P = \tan hx, \; Q = 2 \cos hx \sin x$

$\text{I.F.} = e^{\int P dx} = e^{\int \tan hx\, dx} = e^{\log \cos hx} = \cos hx$

Solution is given by

$$y \cos hx = \int 2 \cosh^2 x \sin hx + c$$

$$y \cos hx = 2 \frac{\cos h^3 x}{3} + c$$

EXERCISE 5.4

Solve the following differential equations.

1. $(x+1)\dfrac{dy}{dx} - y = e^x (x+1)^2$

Ans : $y = (xe^x + c)(x+1)$

2. $\dfrac{dy}{dx} + y \sec x = \tan x$

Ans : $(y-1)(\sec x + \tan x) + x = c$

3. $x \log x \dfrac{dy}{dx} + y = 2 \log x$

Ans : $y \log x = (\log x)^2 + c$

4. $\dfrac{dy}{dx} + \dfrac{3x^2 y}{1+x^3} = \dfrac{1+x^2}{1+x^3}$

Ans : $y(1+x^3) = x + \dfrac{x^3}{3} + c$

5. $\dfrac{dy}{dx} + \dfrac{2x}{1+x^2} y = \dfrac{1}{\left(1+x^2\right)^2}$

Ans : $y(1+x^2) = \tan^{-1}x - \dfrac{\pi}{4}$

given $y = 0$ when $x = 1$

6. $x \cos x \dfrac{dy}{dx} + y(x \sin x + \cos x) = 1$

Ans : $xy \sec x = \tan x + c$

7. $\dfrac{dy}{dx} - y \tan x = 3e^{-\sin x}$

Ans : $y \cos x = 7 - 3e^{-\sin x}$

given $y = 4$ when $x = 0$

8. $\dfrac{dy}{dx} = \dfrac{y}{2y \log y + y - x}$

Ans : $x = \dfrac{c}{y} + y \log y$

9. $(x + 2y^3)\dfrac{dy}{dx} = y$

Ans : $x = y(y^2 + c)$

10. $y e^y dx = (y^2 + 2xe^y) dy$

Ans : $x = y^2(c - e^{-y})$

(JNTU 1998/S)

11. $e^{-y} \sec^2 y \, dy = dx + x \, dy$

Ans : $xe^y = \tan y + c$

12. $x(x-2)\dfrac{dy}{dx} - 2(x-1)y = x^3(x-2)$

given that $y = 9$ when $x = 3$

Ans : $y = x(x-2)[x + \log(x-2)^2]$

13. $\sec x \dfrac{dy}{dx} = y + \sin x$

Ans : $y = ce^{\sin x} - (1 + \sin x)$

14. $y^2\,dy = (x^3 + y^3)\,dx$ *(JNTU 1998/S)*

Ans : $y^3 = -x^3 - x^2 - \frac{2}{3}x - \frac{2}{9} + ce^{3x}$

15. $x \log x \frac{dy}{dx} + y = 2 \log x$

Ans : $y \log x = \frac{(\log x)^2}{2} + c$

16. $x \frac{dy}{dx} + y = \log x$

Ans : $y(\log x - 1) + \frac{c}{x}$ *(JNTU 1996/S)*

17. $\sqrt{1 - y^2}\,dx = (\sin^{-1}y - x)\,dy$ *(JNTU 99/S)*

Ans : $x = \sin^{-1}y - 1 + ce^{\sin^{-1}y}$

18. $(\cos^3 x)\frac{dy}{dx} + y\cos x = \sin x$

Ans : $1 + y = \tan x - ce^{-\tan^{-1}x}$

19. $dr + (2r\cot\theta + \sin 2\theta)\,d\theta = 0$

Ans : $r\sin^2\theta = \frac{-\sin^4\theta}{2} + c$

20. $(x + 1)\frac{dy}{dx} - ny = e^x(x + 1)^{n+1}$

Ans : $y(x + 1)^{-n} = e^x + c$

5.7 BERNOULLI'S EQUATION :

An equation of the form $\frac{dy}{dx} + Py = Qy^n$

where P and Q are functions of x alone, is known as Bernoulli's equation. It can be reduced to linear form as follows.

$$\frac{dy}{dx} + Py = Qy^n \qquad \text{............(1)}$$

Divide both sides by y^n, we get

$$y^{-n}\frac{dy}{dx} + Py^{-n+1} = Q \qquad \text{............(2)}$$

Put $y^{-n+1} = v$

$$(-n + 1)\,y^{-n}\cdot\frac{dy}{dx} = \frac{dv}{dx}$$

$$y^{-n}\frac{dy}{dx} = \frac{1}{1-n}\cdot\frac{dv}{dx}$$

Substituting in (2)

$$\therefore \quad \frac{1}{1-n} \cdot \frac{dv}{dx} + Pv = Q$$

$$\frac{dv}{dx} + (1-n)Pv = (1-n)Q$$

This being linear in v, can be solved as discussed earlier.

SOLVED EXAMPLES – 5.6

EXAMPLE - 1 :

Solve $\frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$

Solution : Given equation is $\frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$ .... (1)

Divide (1) by $\cos^2 y$

$$\frac{1}{\cos^2 y} \cdot \frac{dy}{dx} + x \frac{\sin 2y}{\cos^2 y} = x^3$$

$$\sec^2 y \frac{dy}{dx} + \frac{x\, 2\sin y \cos y}{\cos^2 y} = x^3$$

$$\sec^2 y \frac{dy}{dx} + 2x \tan y = x^3 \qquad \text{........................ (2)}$$

Put $\tan y = v$

$$\sec^2 y \frac{dy}{dx} = \frac{dv}{dx}$$

Substituting in (2)

$$\frac{dv}{dx} + 2x\,v = x^3$$

This is linear equation in v where $P = 2x$, $Q = x^3$

$$\text{I.F.} = e^{\int P dx} = e^{\int 2x\, dx} = e^{x^2}$$

$$\therefore \text{ Solution : } v e^{x^2} = \int x^3 e^{x^2} dx + c \qquad \text{..................... (3)}$$

To evaluate $\int x^3 e^{x^2} dx$, Put $x^2 = t$

$$2x\, dx = dt$$

$$x\, dx = \frac{dt}{2}$$

$$\int x^3 e^{x^2}\, dx = \int x^3 e^{x^2} \cdot x\, dx$$

$$= \int t \cdot e^t \cdot \frac{dt}{2}$$

$$= \frac{1}{2}\left(te^t - e^t \right)$$

Substituting in (3),

$$ve^t = \frac{1}{2}\left(te^t - e^t \right) + c$$

$$v = \frac{1}{2}(t-1)\frac{e^t}{e^t} + \frac{c}{e^t}$$

$$v = \frac{1}{2}(t-1) + ce^{-t}$$

Substituting $t = x^2$ and $v = \tan y$

$$\tan y = \frac{1}{2}\left(x^2 - 1 \right) + ce^{-x^2}$$

EXAMPLE - 2 :

Solve $\dfrac{dy}{dx} + \dfrac{x}{1-x^2}\, y = x\sqrt{y}$

Solution : Divide the equation by $\sqrt{y}$

$$y^{-1/2}\frac{dy}{dx} + \frac{x}{1-x^2}\, y^{1/2} = x \qquad \text{.........(1)}$$

Put $y^{1/2} = v$

$$\frac{1}{2}\, y^{-1/2}\frac{dy}{dx} = \frac{dv}{dx}$$

$$y^{-1/2}\frac{dy}{dx} = 2\frac{dv}{dx}$$

$$2\frac{dv}{dx} + \frac{x}{1-x^2}\, v = x$$

i.e., $$\frac{dv}{dx} + \frac{x}{2\left(1-x^2\right)}\, v = \frac{x}{2} \qquad \text{..................(2)}$$

This is a linear equation in v where $P = \dfrac{x}{2\left(1-x^2\right)}$, $Q = \dfrac{x}{2}$

$$\text{I.F.} = e^{\int P\,dx} = e^{\int \frac{x}{2(1-x^2)}dx} = e^{-\frac{1}{4}\log(1-x^2)} = e^{\log(1-x^2)^{-1/4}} = (1-x^2)^{-1/4}$$

Solution :

$$v \cdot (1-x^2)^{-1/4} = \int \frac{x}{2}\cdot(1-x^2)^{-1/4}\,dx + c \quad \ldots\ldots\ldots\ldots (3)$$

To evaluate $\int \frac{x}{2}\cdot(1-x^2)^{-1/4}\,dx$, Put $x^2 = t$

$2x\,dx = dt$

$x\,dx = \frac{dt}{2}$

$$= \frac{1}{4}\int (1-t)^{-1/4}\cdot dt$$

$$= +\frac{1}{4}\,\frac{(1-t)^{-\frac{1}{4}+1}}{-\frac{1}{4}+1} = +\frac{1}{3}(1-t)^{3/4}$$

Substituting in (3)

$$v(1-t)^{-1/4} = +\frac{1}{3}(1-t)^{3/4} + c$$

Put $v = \sqrt{y}$, $t = x^2$

$$\sqrt{y}\,(1-x^2)^{-1/4} = +\frac{1}{3}(1-x^2)^{3/4} + c$$

$$\sqrt{y} = +\frac{1}{3}(1-x^2) + c(1-x^2)^{1/4}$$

EXAMPLE - 3 :

Solve $xy(1+xy^2)\frac{dy}{dx} = 1$

Solution : The given equation can be written as

$$\frac{dx}{dy} - xy = y^3x^2$$

Divide by x^2, we get

$$x^{-2}\frac{dx}{dy} - yx^{-1} = y^3 \quad \ldots\ldots\ldots\ldots (1)$$

Put $x^{-1} = v$

$$-x^{-2}\frac{dx}{dy} = \frac{dv}{dy} \quad \text{i.e., } x^{-2}\frac{dx}{dy} = -\frac{dv}{dy}$$

Substituting in (1)

$$-\frac{dv}{dy} - yv = y^3$$

$$\frac{dv}{dy} + yv = -y^3$$

This is a linear equation in v where $P = y$, $Q = -y^3$

$$\text{I.F.} = e^{\int y\, dy} = e^{\frac{y^2}{2}}$$

Solution :

$$v \cdot e^{\frac{y^2}{2}} = \int -y^3 \cdot e^{\frac{y^2}{2}} \cdot dy + c \quad \text{.............. (2)}$$

To evaluate $\int -y^3\, e^{\frac{y^2}{2}} \cdot dy$, Put $\frac{y^2}{2} = t$, i.e., $y^2 = 2t$

$y\, dy = dt$

$$= \int -y^2\, e^{\frac{y^2}{2}} \cdot y\, dy$$

$$= -\int 2t\, e^t \cdot dt$$

$$= -2\int te^t \cdot dt$$

$$= -2\,[\,te^t - e^t\,]$$

Substituting this value in (2)

$$ve^t = -2\,(\,te^t - e^t\,) + c$$

Substituting $v = x^{-1}$ and $t = \frac{y^2}{2}$

$$\frac{1}{x} e^{\frac{y^2}{2}} = -2\, e^{\frac{y^2}{2}} \left(\frac{y^2}{2} - 1 \right) + c$$

i.e., $\frac{1}{x} = 2 - y + ce^{-\frac{y^2}{2}}$ the required general solution.

EXAMPLE - 4 :

Solve $x \frac{dy}{dx} + y = y^2 \log x$

Solution : Divide the equation by xy^2

$$y^{-2} \frac{dy}{dx} + \frac{y^{-1}}{x} = \frac{\log x}{x} \quad \text{.............. (1}$$

Put $y^{-1} = v$

$$-y^{-2}\frac{dy}{dx} = \frac{dv}{dx}$$

$$y^{-2}\frac{dy}{dx} = -\frac{dv}{dx}$$

Substituting these in (1), we get

$$-\frac{dv}{dx} + \frac{v}{x} = \frac{\log x}{x}$$

$$\frac{dv}{dx} - \frac{v}{x} = -\frac{\log x}{x}$$

This is a linear equation in v where $P = -\frac{1}{x}$, $Q = -\frac{\log x}{x}$

$$\text{I.F.} = e^{\int P\,dx} = e^{\int -\frac{1}{x}dx} = e^{-\log x} = e^{\log\frac{1}{x}} = \frac{1}{x}$$

Solution :

$$v \cdot \frac{1}{x} = \int -\frac{\log x}{x} \cdot \frac{1}{x}\,dx + c$$

$$= \int\left(-\frac{1}{x^2}\right)\log x\,dx + c$$

$$= \frac{1}{x}\log x - \int \frac{dx}{x^2} + c$$

$$= \left(\frac{1}{x}\right)\log x + \frac{1}{x} + c$$

$$= \left(\frac{1}{x}\right)(\log x + 1) + c$$

$$\therefore \quad \frac{1}{xy} = \frac{1}{x}(\log x + 1) + c$$

$$\frac{1}{y} = \log x + 1 + cx$$

EXAMPLE - 5 :

Solve $x\dfrac{dy}{dx} + y \log y = xye^x$

Solution : $x\dfrac{dy}{dx} + y \log y = xye^x$

Dividing by xy

$$\frac{1}{y}\frac{dy}{dx} + \frac{1}{x}\log y = e^x \qquad \text{..............}(1)$$

Put $\log y = v$

$$\frac{1}{y}\frac{dy}{dx} = \frac{dv}{dx}$$

Equation (1) becomes

$$\frac{dv}{dx} + \frac{v}{x} = e^x, \quad P = \frac{1}{x}, Q = e^x$$

I.F. $= e^{\int \frac{1}{x} dx} = e^{\log x} = x$

Solution :

$$vx = \int xe^x \, dx + c$$

$$vx = xe^x - e^x + c$$

$$x \log y = xe^x - e^x + c$$

EXAMPLE - 6 :

Solve $\tan y \dfrac{dy}{dx} + \tan x = \cos y \cos^2 x$

Solution : $\tan y \dfrac{dy}{dx} + \tan x = \cos y \cos^2 x$

divide by cos y

$$\sec y \tan y \frac{dy}{dx} + \sec y \tan x = \cos^2 x \qquad \text{..............}(1)$$

Put $\sec y = v$

$$\sec y \tan y \frac{dy}{dx} = \frac{dv}{dx}$$

Equation (1) becomes

$$\frac{dv}{dx} + \tan x \, v = \cos^2 x \qquad P = \tan x, Q = \cos^2 x$$

I.F. $e^{\int \tan x \, dx} = e^{\log \sec x} = \sec x$

$$v \sec x = \int \cos^2 x \cdot \sec x \, dx + c$$

$$= \int \cos x \, dx + c = \sin x + c$$

$$\sec y \sec x = \sin x + c$$

$$\sec y = (\sin x + c) \cos x$$

the required general solution.

EXAMPLE - 7 :

Solve $\frac{dy}{dx} + \frac{y}{x} = \frac{y^2}{x^2}$

Solution : $\frac{dy}{dx} + \frac{y}{x} = \frac{y^2}{x^2}$

divide by y^2, we get

$$y^{-2} \frac{dy}{dx} + \frac{y^{-1}}{x} = \frac{1}{x^2} \quad \text{.............. (1)}$$

Put $y^{-1} = v$

$$-y^{-2} \frac{dy}{dx} = \frac{dv}{dx} \quad \text{i.e., } y^{-2} \frac{dy}{dx} = -\frac{dv}{dx}$$

Equation (1) becomes

$$-\frac{dv}{dx} + \frac{v}{x} = \frac{+1}{x^2}$$

$$\frac{dv}{dx} - \frac{v}{x} = \frac{-1}{x^2}$$

This is a linear equation in v, $P = -\frac{1}{x}$, $Q = -\frac{1}{x^2}$

I.F. $e^{\int -\frac{1}{x} dx} = e^{-\log x} = e^{\log \frac{1}{x}} = \frac{1}{x}$

Solution :

$$v \cdot \frac{1}{x} = -\int \frac{1}{x^2} \cdot \frac{1}{x} \, dx + c$$

$$\frac{v}{x} = -\int x^{-3} \, dx + c$$

$$\frac{v}{x} = -\left(\frac{x^{-2}}{-2} \right) + c$$

Substituting $v = \frac{1}{y}$

$$\frac{1}{xy} = \frac{1}{+2x^2} + c$$

is the required general solution.

EXAMPLE - 8 :

Solve $\frac{dy}{dx} + \frac{y}{x-1} = xy^{1/3}$

Solution : Given equation $\frac{dy}{dx} + \frac{y}{x-1} = xy^{1/3}$

Divide by $y^{1/3}$, we get

$$\frac{1}{y^{1/3}} \cdot \frac{dy}{dx} + \frac{1}{x-1} \cdot y^{2/3} = x \quad \text{.............(1)}$$

Put $y^{2/3} = v$

$$\frac{2}{3} y^{-1/3} \frac{dy}{dx} = \frac{dv}{dx}$$

i.e., $\frac{1}{y^{-3}} \cdot \frac{dy}{dx} = \frac{3}{2}\frac{dv}{dx}$

Equation (1) becomes

$$\frac{3}{2}\frac{dv}{dx} + \frac{1}{x-1} v = x$$

i.e. $\frac{dv}{dx} + \frac{2}{3} \cdot \frac{1}{x-1} v = \frac{2}{3} x$

This is a linear equation in v,

where $P = \frac{2}{3} \cdot \frac{1}{x-1}$; $Q = \frac{2}{3} x$

$$\text{I.F.} = e^{\int \frac{2}{3} \cdot \frac{1}{x-1} dx} = e^{\frac{2}{3} \log(x-1)} = e^{\log(x-1)^{2/3}} = (x-1)^{2/3}$$

I.F. $= (x - 1)^{2/3}$

The solution is

$$v (x - 1)^{2/3} = \frac{2}{3} \int x (x - 1)^{2/3} dx + c \quad \text{.............(2)}$$

consider $\frac{2}{3} \int x (x - 1)^{2/3} dx$

Integrating by parts

$$\frac{2}{3}\int x(x-1)^{2/3}\,dx = \frac{2}{3}\left[\frac{3}{5}x(x-1)^{5/3} - \frac{3}{5}\int (x-1)^{5/3}\,dx\right]$$

$$= \frac{2}{3}\left[\frac{3}{5}x(x-1)^{5/3} - \frac{9}{40}(x-1)^{8/3}\right]$$

Substituting in (2)

$$v(x-1)^{2/3} = \frac{2}{3}\left[\frac{3}{5}x(x-1)^{5/3} - \frac{9}{40}(x-1)^{8/3}\right] + c$$

Put $v = y^{2/3}$

$$y^{2/3}(x-1)^{2/3} = \frac{2}{3}\left[\frac{3}{5}x(x-1)^{5/3} - \frac{9}{40}(x-1)^{8/3}\right] + c$$

$$y^{2/3}(x-1)^{2/3} = \frac{2}{5}x(x-1)^{5/3} - \frac{3}{20}(x-1)^{8/3} + c$$

EXAMPLE - 9 :

Solve $x^3 \sec^2 y \dfrac{dy}{dx} + 3x^2 \tan y = \cos x$ **(JNTU 2008)**

Solution : Given equation is of Bernoulli's form

Put $\tan y = v$, $\sec^2 y \dfrac{dy}{dx} = \dfrac{dv}{dx}$

Substituting in the given equation, we get

$$x^3 \frac{dv}{dx} + 3x^2 v = \cos x$$

$$\frac{dv}{dx} + \frac{3x^2}{x^3} v = \frac{\cos x}{x^3}$$

$$\frac{dv}{dx} + \frac{3}{x} v = \frac{\cos x}{x^3}$$

This is a linear equation in v.

$$P = \frac{3}{x},\quad Q = \frac{\cos x}{x^3}$$

$$\text{I.F.} = e^{\int \frac{3}{x}dx} = e^{3\log x} = x^3$$

The solution is given by

$$v(x^3) = \int \frac{\cos x}{x^3}\cdot x^3\,dx + c$$

$$v(x^3) = \sin x + c$$

$$x^3 \tan y = \sin x + c$$

EXAMPLE - 10 :

Solve $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)e^x \sec y$ (JNTU 2007, 2006)

Solution : The given equation is $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)e^x \sec y$

Multiplying by cos y

$$\cos y \frac{dy}{dx} - \frac{\sin y}{1+x} = (1+x)e^x \quad \text{............}(1)$$

Put $\sin y = v$, $\cos y \frac{dy}{dx} = \frac{dv}{dx}$

Substituting in (1), the equation becomes

$$\frac{dv}{dx} - \frac{v}{1+x} = (1+x)e^x$$

This is a linear equation in v.

$$\text{I.F.} = e^{\int P dx} = e^{-\int \frac{1}{1+x} dx} = \frac{1}{1+x}$$

Solution is given by

$$\frac{y}{1+x} = \int \frac{(1+x)e^x}{1+x} dx + c$$

$$y = (1+x)e^x + c(1+x)$$

EXAMPLE - 11 :

Solve $x\frac{dy}{dx} + y = x^3y^6$ (JNTU 2008, 1995)

Solution : The given differential equation is

$$x\frac{dy}{dx} + y = x^3y^6$$

Dividing by xy^6, the equation becomes

$$\frac{1}{y^6}\frac{dy}{dx} + \frac{1}{xy^5} = x^2 \quad \text{.........}(1)$$

Put $\frac{1}{y^5} = v$, $-\frac{5}{y^6}\frac{dy}{dx} = \frac{dv}{dx}$

Equation (1) becomes

$$\frac{dv}{dx} - \frac{5}{x}v = -5x^2$$

This is a linear equation in v.

$$\text{I.F.} = e^{\int -\frac{5}{x}dx} = e^{-5\log x} = \frac{1}{x^5}$$

The solution is given by

$$\frac{v}{x^5} = -5\int \frac{x^2 dx}{x^5} + c = \frac{5}{2x^2} + c$$

$$\frac{1}{y^5 x^5} = \frac{5}{2x^2} + c$$

EXAMPLE - 12 :

Solve $y(2xy + e^x)\,dx - e^x\,dy = 0$ ***(JNTU 2006)***

Solution : The given differential equation is

$(2xy^2 + ye^x)\,dx - e^x\,dy = 0$

$$\frac{dy}{dx} - y = 2xy^2 e^{-x} \qquad \text{.........(1)}$$

This is of Bernoulli's form.

Dividing equation (1) by y^2

$$\frac{1}{y^2}\frac{dy}{dx} - \frac{1}{y} = 2xe^{-x} \qquad \text{.........(2)}$$

$$\text{Put } \frac{1}{y} = v,\ -\frac{1}{y^2}\frac{dy}{dx} = \frac{dv}{dx}$$

The equation (2) reduces to

$$\frac{dv}{dx} + v = -2xe^{-x}$$

This is a linear equation in v.

$\text{I.F} = e^{\int dx} = e^x$

The solution is

$$ve^x = \int -2xe^{-x}\cdot e^x\,dx + c$$

$$= -x^2 + c$$

$$\frac{e^x}{y} = -x^2 + c$$

EXAMPLE - 13 :

Solve $x^2\dfrac{dy}{dx} = e^y - x$ ***(JNTU 2007)***

Solution : The given equation can be written as

$$x^2\frac{dy}{dx} + x = e^y$$

Dividing by $x^2 e^y$, the equation becomes

$$e^{-y}\frac{dy}{dx} + \frac{e^{-y}}{x} = \frac{1}{x^2} \qquad \text{.........(1)}$$

Put $e^{-y} = v$, $-e^{-y}\frac{dy}{dx} = \frac{dv}{dx}$

Substituting in (1), the equation reduces to

$$-\frac{dv}{dx} + \frac{v}{x} = \frac{1}{x^2}$$

i.e., $\frac{dv}{dx} - \frac{v}{x} = -\frac{1}{x^2}$

This is a linear in v.

$$P = -\frac{1}{x},\ Q = -\frac{1}{x^2}$$

$$\text{I.F.} = e^{\int P dx} = e^{-\int \frac{1}{x}dx} = e^{-\log x} = \frac{1}{x}$$

$$\frac{v}{x} = \int -\frac{1}{x^3}\,dx + c$$

$$\frac{v}{x} = \frac{1}{2x^2} + c$$

$$xe^{-y} = \frac{1}{2x^2} + c$$

EXAMPLE - 14 :

Solve $(1 - x^2)\frac{dy}{dx} + xy = y^3 \sin^{-1} x$ **(JNTU 2007)**

Solution : Given differential equation is

$$(1 - x^2)\frac{dy}{dx} + xy = y^3 \sin^{-1} x$$

Dividing by $(1 - x^2)\,y^3$, the equation becomes

$$\frac{1}{y^3}\frac{dy}{dx} + \frac{x}{y^2(1 - x^2)} = \frac{\sin^{-1} x}{(1 - x^2)} \qquad \text{........ (1)}$$

Put $\frac{1}{y^2} = v$, $\frac{-2}{y^3}\frac{dy}{dx} = \frac{dv}{dx}$

Substituting in (1), the equation becomes

$$\frac{dv}{dx} - \frac{2xv}{1 - x^2} = -\frac{2\sin^{-1} x}{1 - x^2}$$

$$\text{I.F.} = e^{\int P dx} = e^{\int -\frac{2x}{1+x^2} dx} = e^{\log(1-x^2)} = 1 - x^2$$

The solution is given by

$$v(1-x^2) = -2\int \sin^{-1} x\, dx + c$$

$$= -2\left[x \sin^{-1} x - \frac{1}{2}\int 2x \cdot \frac{dx}{\sqrt{1-x^2}}\right] + c$$

(Integrating by parts)

$$= -2\left[x \sin^{-1} x - \frac{1}{2}\int \frac{2t}{t} dt\right] + c$$

(put $1 - x^2 = t^2$, $-2x dx = -2t dt$)

$$= -2x \sin^{-1} x - \sqrt{1-x^2} + c$$

$$\frac{1-x^2}{y^2} = -2x \sin^{-1} x - \sqrt{1-x^2} + c$$

EXERCISE 5.5

Solve the following equations :

1. $\frac{dy}{dx} + \frac{y}{x}\log y = \frac{y}{x^2}(\log y)^2$ **Ans :** $x = \log y\left(cx^2 + \frac{1}{2}\right)$

2. $e^y\left(\frac{dy}{dx} + 1\right) = e^x$ **Ans :** $e^{x+y} + \frac{e^{2x}}{2} = c$

3. $\frac{dy}{dx} - 2y \tan x + y^2 \tan^2 x = 0$ **Ans :** $5 \sec^2 x = y(\tan^2 x + c)$

4. $\frac{dy}{dx} + \frac{1}{x}\tan y = \frac{1}{x^2}\tan y \sin y$ **Ans :** $\frac{1}{x \sin y} = \frac{1}{2x^2} + c$

5. $\frac{dy}{dx} + x = xe^{(n-1)y}$ **Ans :** $e^{-(n-1)y} = 1 + ce^{(n-1)x^2/2}$

6. $\frac{dy}{dx} + y \cos x = y^n \sin 2x$ **Ans :** $y^{-n+1} = ce^{(n-1)\sin x} + 2 \sin x + \frac{2}{n-1}$

7. $y(2xy + e^x)dx - e^x dy = 0$ **Ans :** $y^{-1}e^x = c - x^2$

8. $3y^2 \frac{dy}{dx} + 2xy^3 = 4xe^{-x^2}$ **Ans :** $y^3 = (2x^2 + c)\, e^{-x^2}$

9. $x^3 \dfrac{dy}{dx} + y^4 \cos x = x^2 y$ **Ans :** $x^3 = (c + 3 \sin x)\, y^3$

10. $xy^2 dy - (x^3 + y^3)\, dx = 0$ given $y = 0$ when $x = 1$ **Ans :** $y^3 = 3x^3 \log x$

11. $3x(1 - x^2)\, y^2 \dfrac{dy}{dx} + (2x^2 - 1)\, y^3 = ax^3$ **Ans :** $y^3 = ax + cx\sqrt{1 - x^2}$

12. $x \dfrac{dy}{dx} + y = x^3 y^6$ **Ans :** $\dfrac{1}{y^5} = cx^5 + \dfrac{5x^3}{2}$ *(JNTU '95)*

13. $\left(xy^2 - e^{1/x^3}\right) dx - x^2 y\, dy = 0$ **Ans :** $\dfrac{y^2}{x^2} = \dfrac{2}{3} e^{1/x^3} + c$

14. $\dfrac{dy}{dx} + (2x \tan^{-1} y - x^3)(1 + y^2) = 0$ **Ans :** $\tan^{-1} y = \dfrac{x^2 - 1}{2} + c e^{-x^2}$

15. $\dfrac{dy}{dx} - \dfrac{\tan y}{1 + x} = (1 + x)\, e^x \sec y$ **Ans :** $\sin y = (1 + x)(e^x + c)$

5.8 APPLICATIONS :

Newton's Law of Cooling :

Statement : The rate of change of temperature of a body is proportional to the difference between the temperature of the body and that of the surrounding medium.

Let θ be the temperature of the body at time t and θ_0 be the temperature of the surrounding medium (usually air).

By Newton's Law of cooling

$$\frac{d\theta}{dt} \propto \theta - \theta_0$$

$$\frac{d\theta}{dt} = -k(\theta - \theta_0) \qquad \ldots\ldots\ldots\ldots (1)$$

$$\frac{d\theta}{\theta - \theta_0} = -k\, dt$$

$$\log(\theta - \theta_0) = -kt + c \qquad \ldots\ldots\ldots\ldots (2)$$

If initially $\theta = \theta_1$ is the temperature of the body at time $t = 0$.

then (2) gives $c = \log(\theta_1 - \theta_0)$

Substituting this value of c in (2), we get

$$\log\left(\frac{\theta - \theta_0}{\theta_1 - \theta_0}\right) = -kt$$

$$\theta - \theta_0 = (\theta_1 - \theta_0)\, e^{-kt}$$

$$\theta = \theta_0 + (\theta_1 - \theta_0)\, e^{-kt}$$

This formula gives the temperature of the body at anytime t.

SOLVED EXAMPLES - 5.7

EXAMPLE - 1 :

If the air is maintained at 30 °C and the temperature of the body cools from 80° to 60° in 12 minutes, find the temperature of the body after 24 minutes.

Solution : Let θ be the temperature of the body at any time t then, by Newton's law of cooling

$$\frac{d\theta}{dt} = -k(\theta - 30), \quad (\theta_0 = 30°); \quad \frac{d\theta}{\theta - 30} = -k\,dt$$

Integrating $\int \frac{d\theta}{\theta - 30} = -\int k\,dt + k_1$ where k_1 is constant.

$$\log(\theta - 30) = -kt + \log c \qquad \text{where } k_1 = \log c$$

$$\log\left(\frac{\theta - 30}{c}\right) = -kt$$

$$\theta - 30 = ce^{-kt}$$

$$\theta = 30 + ce^{-kt} \qquad \text{.................. (1)}$$

when $t = 0, \ \theta = 80°$

$$80 = 30 + ce^{-k0} \Rightarrow c = 50$$

$$\theta = 30 + 50\, e^{-kt} \qquad \text{............... (2)}$$

when $t = 12, \theta = 60°$

From (2), $60 = 30 + 50\, e^{-k \cdot 12}$

i.e., $50\, e^{-12k} = 30$

$$e^{-12k} = \frac{30}{50} = \frac{3}{5} \quad \Rightarrow e^{12k} = \frac{5}{3} \Rightarrow k = \frac{1}{12}\log\left(\frac{5}{3}\right)$$

Now from (2),

$$\theta = 30 + 50\, e^{-\left(\frac{1}{12}\log\frac{5}{3}\right)t} \qquad \text{............... (3)}$$

Now when t = 24, to find θ

From (3), $\theta = 30 + 50\, e^{-\frac{1}{12}\left(\log\frac{5}{3}\right) \cdot 24}$

$$\theta = 30 + 50e^{-2\log\frac{5}{3}}$$

$$= 30 + 50e^{2\log\left(\frac{3}{5}\right)}$$

$$= 30 + 50e^{\log\left(\frac{3}{5}\right)^2}$$

$$= 30 + 50 \cdot \frac{9}{25} = 48$$

$$\therefore \theta = 48^\circ$$

EXAMPLE - 2 :

The rate of cooling of a body is proportional to the difference between the temperature of the body and the surrounding air. If the air temperature is 20°C and the body cools for 20 minutes from 140°C to 80°C, find when the temperature will be 35°C. **(JNTU 2006)**

Solution : Let θ be the temperature of the body at time t

By Newton's law of cooling

$$\frac{d\theta}{dt} = -k(\theta - 20), \quad \theta_0 = 20$$

$$\frac{d\theta}{\theta - 20} = -k\,dt$$

Integrating $\int \frac{d\theta}{\theta - 20} = -\int k\,dt + k_1$, k_1 is constant.

$$\log(\theta - 20) = -kt + \log c \quad \text{where } k_1 = \log c$$

$$\theta - 20 = ce^{-kt}$$

$$\theta = 20 + ce^{-kt} \quad \text{............ (2)}$$

when $t = 0, \theta = 140$

$$140 = 20 + ce^{-k \cdot 0} \quad \Rightarrow c = 120$$

$\therefore$ (2) becomes

$$\theta = 20 + 120\,e^{-kt} \quad \text{............ (3)}$$

when $t = 20, \ \theta = 80$

from (3), $80 = 20 + 120\,e^{-k20}$

$$60 = 120\,e^{-20k}$$

$$e^{-20k} = \frac{60}{120} \quad \text{i.e., } e^{20k} = \frac{120}{60} = 2$$

$$20k = \log 2$$

$$k = \frac{1}{20}\log 2$$

(3) becomes

$$\theta = 20 + 120\, e^{-\left(\frac{1}{20}\log 2\right)t} \qquad \text{..............} (4)$$

Now to find t when $\theta = 35°C$

From (4),

$$35 = 20 + 120\, e^{-\left(\frac{1}{20}\log 2\right)t}$$

$$120\, e^{-\left(\frac{1}{20}\log 2\right)t} = 35 - 20 = 15$$

$$e^{-\left(\frac{1}{20}\log 2\right)t} = \frac{15}{120} = \frac{1}{8}$$

$$e^{\left(\frac{1}{20}\log 2\right)t} = 8$$

$$\left(\frac{1}{20}\log 2\right)t = \log 8$$

$$t = \frac{20\log 8}{\log 2} = \frac{20\log 2^3}{\log 2} = 60\,\frac{\log 2}{\log 2} = 60 \text{ minutes.}$$

EXAMPLE - 3 :

Suppose that an object is heated to 300°F and allowed to cool in a room whose air temperature is 80°F. If after 10 minutes the temperature of the object is 250°F, what will be its temperature after 20 minutes ? ***(JNTU 2008, 2008 S)***

Solution : Let θ be temperature of the body at any time t, then by Newton's law of cooling

$$\frac{d\theta}{dt} = -k(\theta - \theta_0) \text{ where } \theta_0 \text{ is the temperature of the air. Here } \theta_0 = 80$$

$$\frac{d\theta}{dt} = -k(\theta - 80)$$

$$\frac{d\theta}{\theta - 80} = -k\,dt$$

Integrating

$$\theta - 80 = ce^{-kt}$$

$$\theta = 80 + ce^{-kt} \qquad \text{...............} (1)$$

when $t = 0,\ \theta = 300,\ c = 220$

Equation (1) gives

$$\theta = 80 + 220\, e^{-kt} \qquad \text{...........} (2)$$

when $t = 10, \theta = 250$

$$250 = 80 + 220\, e^{-kt}$$

$$170 = 220\, e^{-kt}$$

$$17 = 22e^{-kt}$$

$\log \frac{17}{22} = -10\,k$

$-k = \frac{1}{10} \log \frac{17}{22}$

Substituting in (2),

$\theta = 80 + 220\, e^{\left(\frac{1}{10} \log \frac{17}{22}\right)t}$ .... (3)

Now to find θ when $t = 20$,

from (3) $\quad \theta = 80 + 220\, e^{\left(\frac{1}{10} \log \frac{17}{22}\right)20}$

$= 80 + 220\, e^{\left(\log \frac{17}{22}\right)^2}$

$= 80 + 131{\cdot}36 \; = 211{\cdot}36$

When $t = 20$, $\theta = 211{\cdot}36F$

EXAMPLE - 4 :

If the temperature of the air is 20°C and the temperature of the body drops from 100°C to 80°C in 10 minutes, what will be its temperature after 20 minutes ? When will be the temperature 40°C ? ***(JNTU 2006)***

Solution : Let θ be the temperature at any time t

By Newton's law of cooling

$\frac{d\theta}{dt} = -k(\theta - \theta_0)$ where θ_0 is the temperature of the air. Here $\theta_0 = 20$

$\frac{d\theta}{dt} = -k(\theta - 20)$

$\frac{d\theta}{\theta - 20} = -kdt$ .... (1)

Integrating $\theta - 20 = ce^{-kt}$

$\theta = 20 + ce^{-kt}$

When $t = 0$, $\theta = 100$, $c = 80$

Equation (1) gives

$\theta = 20 + 80e^{-kt}$ .... (2)

When $t = 10$, $\theta = 80$

$80 = 20 + 80\, e^{-kt}$

$60 = 80\, e^{-kt}$

$80 = 20 + 80\, e^{-10k}$

$60 = 80\, e^{-10k}$

$e^{-10k} = \frac{3}{4}$

When t = 20, to find θ

$\theta = 20 + 80\, e^{-20k} = 20 + 80\,(e^{-10k})^2$

$20 + 80\,\dfrac{9}{16} = 65°$

When θ = 40, to find t

$40 = 20 + 80e^{-kt}$

$\dfrac{1}{4} = \left(\dfrac{3}{4}\right)^{t/10}$

$t = 10\left(\dfrac{\log 4}{\log 4 - \log 3}\right)$

EXAMPLE - 5 :

The temperature of the body drops from 100°C to 75°C in 10 minutes when the surrounding air is 20°C temperature. What will be its temperature after half an hour? When will be the temperature be 25°C? *(JNTU 2007, 2005, 2004)*

Solution : Let θ be the temperature at any time t.

By Newton's law of cooling,

$\dfrac{d\theta}{dt} = -k(\theta - \theta_0)$ where θ_0 is the temperature of the air.

Here $\theta_0 = 20°C$

$\dfrac{d\theta}{dt} = -k(\theta - 20)$

$\dfrac{d\theta}{\theta - 20} = -k\,dt$

Integrating $\theta = 20 + ce^{-kt}$ .... (1)

When t = 0, θ = 100, c = 80

Equation (1) becomes

$\theta = 20 + 80\, e^{-kt}$ .... (2)

When t = 10, θ = 75°C

$75 = 20 + 8\, e^{-10k}$

$55 = 80\, e^{-10k}$

$e^{-10k} = \dfrac{55}{80} = \dfrac{11}{16}$ or $e^{-k} = \left(\dfrac{11}{16}\right)^{1/10}$

When t = 30, to find θ

$\theta = 20 + 80\, e^{-kt}$

$= 20 + 80\,(e^{-30k})$

$= 20 + 80(e^{-10k})^3$

$$= 20 + 80\left(\frac{11}{16}\right)^3$$

$$\approx 46°C$$

When $\theta = 25°C$ to find t

$25 = 20 + 80\, e^{-kt}$

$5 = 80\, e^{-kt}$

$$\frac{5}{80} = e^{-kt}$$

$$e^{-kt} = \frac{1}{16}$$

$$(e^{-k})^t = \frac{1}{16}$$

$$\left(\frac{11}{16}\right)^{t/10} = \frac{1}{16}$$

$$t = 10\left(\frac{\log 16}{\log 16 - \log 11}\right)$$

5.9 LAW OF NATURAL GROWTH OR DECAY :

If the rate of change of a quantity y is proportional to the quantity present at any time t, then

$$\frac{dy}{dt} \propto y$$

$$\frac{dy}{dt} = ky$$

For growth, k is positive $\quad \frac{dy}{dt} = ky, \quad k > 0$

For decay, k is negative $\quad \frac{dy}{dt} = -ky, \quad k > 0$

SOLVED EXAMPLES - 5.8

EXAMPLE - 1 :

The number N of bacteria in a culture grew at a rate proportional to N. The value of N was initially 100 and increased to 332 in one hour. What would be the value of N after $1\frac{1}{2}$ hours ? *(JNTU 2006 S, 2005)*

Solution : Given $\frac{dN}{dt} \propto N$

$$\frac{dN}{dt} = kN$$

$$\frac{dN}{N} = k\,dt$$

Integrating $\log N = kt + k_1$, (k_1 constant)

$$N = e^{kt+k_1} = e^{kt} \cdot e^{k_1} = ce^{kt}, \quad (e^{k_1} = c)$$

$$N = ce^{kt} \qquad \text{.............(1)}$$

The equation (1) gives the number N of bacteria at any time t.

Given when $t = 0$, $N = 100$

$$100 = ce^{k \cdot 0} \Rightarrow c = 100$$

Equation (1) becomes $N = 100\,e^{kt}$ ....(2)

Also given, when $t = 1$, $N = 332$

Equation (2) becomes $332 = 100\,e^{k}$

$$e^k = \left(\frac{332}{100}\right)$$

$$k = \log\left(\frac{332}{100}\right)$$

Now to find N when $t = 1\frac{1}{2} = \frac{3}{2}$

Equation (2) becomes

$$N = 100\,e^{\log\left(\frac{332}{100}\right)^t} = 100 \cdot \left(\frac{332}{100}\right)^t$$

Now to find N, when $t = 1\frac{1}{2} = \frac{3}{2}$

$$N = 100 \cdot \left(\frac{332}{100}\right)^{3/2} = 604.9$$

EXAMPLE - 2 :

A radioactive substance disintegrate at a rate proportional to its mass. When mass is 10 mgm, the rate of disintegration is 0.051 mgm per day. How long will it take for the mass to reduce from 10 to 5 mgm ?

Solution : Given $\frac{dN}{dt} \propto N$

$$\frac{dN}{dt} = -kN \qquad \text{............ (1)}$$

$$\frac{dN}{N} = -k\,dt$$

Integrating $\log N = -kt + k_1$ (k_1 is constant)

$$N = e^{-kt+t_1} = e^{t_1} \cdot e^{-kt} = ce^{-kt}$$

$$N = ce^{-kt} \qquad \text{............ (2)}$$

Equation (1) gives the radioactive substance at any time t

Given $N = 10$ mgm, $\frac{dN}{dt} = 0.051$

from equation (1), $0.051 = -k(10)$

$$k = -\frac{0.051}{10} = -0.0051$$

Equation (2) becomes

$$N = ce^{-(0.0051)t} \qquad \text{............ (3)}$$

Also, given when $t = 0$, $N = 10$

$$10 = ce^{-(0.0051)0} \qquad \Rightarrow c = 10$$

Equation (3) becomes

$$N = 10e^{-(0.0051)t}$$

To find t when N = 5 mgm

$$5 = 10\,e^{-(0.0051)t}$$

$$e^{-(0.0051)t} = \frac{5}{10} = \frac{1}{2} \Rightarrow e^{(0.0051)t} = 2$$

$$(0.0051)\,t = \log 2$$

$$t = \frac{\log 2}{0.0051} = \frac{0.69}{0.0051} = 135 \text{ days approximately.}$$