Unit – III Random Processes – Temporal Characteristics

To analyze the characteristics of a random signal in time domain, one need to estimate certain parameters like Mean, Variance, correlation etc. This particular module deals with Autocorrelation, Cross Correlation, Covariance and its properties. These parameters play a vital role in signal estimation and analysis in various applications like RADR, SONAR, and NAVY etc.

Introduction

Autocorrelation, also known as correlation, is the correlation of a signal with itself at different points in time. Informally, it is the similarity between observations as a function of the time lag between them or is defined as Autocorrelation is a mathematical representation of the degree of similarity between a given time series and a lagged version of itself over successive time intervals. Cross-correlation is a measure of similarity of two different time series as a function of one relative to the other. covariance is a measure of the joint variability of two random variables.

For example in Radar Communications, one transmits a signal and expects to receive the same to estimate any targets / objects within the required range. In this scenario, autocorrelation technique is used to verify whether the same signal what is transmitted is received or not.

Autocorrelation Function and Its Properties

The autocorrelation function of a random process $X(t)$ is the correlation $E[X_1X_2]$ of two random variables $X_1=X(t_1)$ and $X_2=X(t_2)$ defined by the process at times t_1 and t_2 .

$$R_{XX}(t_1,t_2)= E[X(t_1)\, X(t_2)] \quad R_{XX}(t_1,t_1+ \tau)= E[X(t_1)\, X(t_1+ \tau)= R_{XX}(\tau)$$

If $X(t)$ is at least wide sense stationary, such that $R_{XX}(t_1,t_1+ \tau)$ must be a function of only of time difference of τ . Thus, for wide sense stationary processes

$$R_{XX}(\tau) = E[X(t_1)\, X(t_1+ \tau)].$$

Autocorrelation function properties

1. Autocorrelation function is maximum at Origin. It is given by $R_{XX}\ \tau \le R_{XX}(0)$
2. Autocorrelation function exhibits Symmetry property i.e, an even function $R_{XX}(-\tau) = R_{XX}(\tau)$
3. The mean square value of X(t) is equal to the power of the process $X(t) R_{XX}(0)= E[X^2(t)]$
4. If X(t) has a periodic component, then $R_{XX}(\tau)$ will have a periodic component with the same period.
5. If X(t) is ergodic, zero mean, and has no periodic component, then

$$\lim_{\tau \to \propto} R_{XX}(\tau) = 0$$

6. $R_{XX}(\tau)$ cannot have an arbitrary shape.

Cross Correlation Function and Its Properties

The cross correlation function of two random processes *X(t)* and *Y(t)* is defined as

$$R_{XY}(t\,,t+\tau)=E[X(t)Y(t+\tau)]$$

If *X(t)* and *Y(t)* are at least jointly wide sense stationary $R_{XY}(t\,,t+\tau)$ is independent of absolute time and we can write

$$R_{XY}(\tau) = E[X(t)\ Y(t+\tau)]$$

If $R_{XY}(t,\ t+\tau) = 0$ then X *(t)* and *Y (t)* are called orthogonal processes. If the two processes are statistically independent, the cross correlation function becomes

$$R_{XY}(\tau)= E[X(t)]E[Y(t+\tau)]$$

If, in addition to being independent, X *(t)* and *Y (t)* are at least wide sense stationary,

$$R_{XY}(\tau)= \overline{X}\,\overline{Y}$$

Properties of the cross correlation function

1. Cross correlation function is a symmetry function $R_{XY}(-\tau)= R_{YX}(\tau)$
2. Absolute value of the cross correlation function is always less than or equal to geometric mean of the autocorrelation functions of X(t) and Y(t) at $\tau = 0$ and is given by $R_{XY}(\tau)$ $\le \overline{R_{XX}\,0\ R_{YY}(0)}$
3. If X(t) and Y(t) are two random processes with autocorrelation functions $R_{xx}(\tau)$ and $R_{yy}(\tau)$, then cross correlation function satisfies $R_{XY}(\tau) \le \overline{R_{XX}\,0 + R_{YY}(0)}$
4. If two random processes X(t) and Y(t) are statistically independent and are atleast wide sense stationary then $R_{xy}\,\tau = XY$
5. If two random processes X(t) and Y(t) have Zero mean and are jointly wide sense stationary then $\lim_{\tau\to\propto} R_{XY}\,\tau = 0$

Covariance Functions

The concept of the covariance of two random variables can be extended to random processes. The auto covariance function is defined by

$$C_{XX}(t,\ t+\tau) = E[\{X(t) - E[X(t)]\}\{ X(t+\tau)\text{-}E[X(t+\tau)]\}]$$

This can also be put in the form

$$C_{XX}(t,\ t+\tau) = R_{XX}(t,\ t+\tau) - E[X(t)]\ E[X(t+\tau)]$$

The cross-covariance function for two processes *X (t)* and *Y (t)* is defined by

$$C_{XY}(t,\ t+\tau)= E[\{X(t) - E[X(t)]\}\{ Y(t+\tau)\text{-}E[Y(t+\tau)]\}]$$

Or, alternatively,

$$C_{XX}(t,\ t+\tau) = R_{XY}(t,\ t+\tau) - E[X(t)]\ E[Y(t+\tau)]$$

For the processes that are at least jointly wide sense stationary, the above equation reduces to

$$C_{XX}(\tau) = R_{XX}(\tau) - \overline{X}^2$$

$$C_{XY}(\tau) = R_{XY}(\tau) - \overline{X}\,\overline{Y}$$

Program

```
clc;
clear all;
close all;
x=input('enter the first sequence 1') ;             % Input sequence 1
y=input('enter the second sequence')  ;             % Input sequence 2

xx=xcorr(x);                                      % Performing Auto correlation
xy=xcorr(x,y);                                    % Performing Cross correlation

disp('auto correlation of the above sequence is')
disp(xx)
disp('crosscorelation of above sequence is')
disp(xy)

n1=length(xx);
n2=length(xy);

l=max(n1,n2);
n=-(l-1):1:(l-1);
subplot(2,1,1);
stem(xx);
xlabel('number of samples');
ylabel('amplitude'); title('auto
correlation'); subplot(2,1,2);
stem(xy);
xlabel('number of samples');
ylabel('amplitude');
title('Crosscorrelation');
```

References:

1. Probability, Random Variables and Random Signal Principles, Peyton Z. Peebles Jr. 4th Edition, Tata McGRAW-Hill.
2. Probability Theory and Stochastic Processes, Y. Mallikarjuna Reddy, University Press

In probability theory and statistics, a **Gaussian process** is a statistical model where observations occur in a continuous domain, e.g. time or space. In a Gaussian process, every point in some continuous input space is associated with a normally distributed random variable. Moreover, every finite collection of those random variables has a multivariate normal distribution. The distribution of a Gaussian process is the joint distribution of all those (infinitely many) random variables, and as such, it is a distribution over functions with a continuous domain, e.g. time or space.

The concept of Gaussian processes is named after Carl Friedrich Gauss because it is based on the notion of the Gaussian distribution (normal distribution). Gaussian processes can be seen as an infinite-dimensional generalization of multivariate normal distributions.

Gaussian processes are useful in statistical modelling, benefiting from properties inherited from the normal. For example, if a random process is modelled as a Gaussian process, the distributions of various derived quantities can be obtained explicitly. Such quantities include the average value of the process over a range of times and the error in estimating the average using sample values at a small set of times.

The **Poisson process** is one of the most important **random processes** in probability theory. It is widely used to model **random** points in time and space, such as the times of radioactive emissions, the arrival times of customers at a service center, and the positions of flaws in a piece of material.

Gaussian Random Processes

Consider a continuous random process $X(t)$. Let N random variables $X_1=X(t_1)$, $X_2=X(t_2)$, ..., $X_N=X(t_N)$ be defined at time instants $t_1, t_2,$.............. , t_N respectively. If these random variables are jointly Gaussian for any N = 1,2... and at any time instants $t_1, t_2,$................, t_N , then the random process $X(t)$ is called Gaussian random process. The joint density function for a Gaussian random variable is given as

$$f_X(x_1,\ldots x_N; t_1,\ldots t_N) = \frac{exp\,\{(-1/2)\,[x-\bar{X}]^t\,[C_{XX}]^{-1}\,[x-\bar{X}]\}}{\sqrt{2\pi\,[C_{XX}]}}$$

Where $\bar{X} = E[X_i] = E[x(t)]$ and

C_{XX} = covariance matrix and its elements are

$$c_{ik} = c_{X_i X_k} = E[(X_i - \bar{X}_i)(X_i - \bar{X}_k)]$$

$$= E[(x(t_i) - \bar{X}(t_i))(x(t_k) - \bar{X}(t_k))]$$

$$c_{ik} = c_{xx}(t_i, t_k)$$

c_{ik} is the autocovariance of $X(t_i)$ and $X(t_k)$. Also by expanding the above equation, we can get

$$c_{XX}(t, t_k) = R_{XX}(t_i, t_k) - E[X(t_i)]E[X(t_k)]$$

Where $R_{XX}(t, t_k)$ is the autocorrelation function of X.

Poisson Random Processes

The Poisson process *X(t)* is a discrete random process which represents the number of times that some event has occurred as a function of time. *X(t)* has integer valued, non decreasing sample functions, such as check in registers, arrival of a customer, arrival of vehicles at a particular point etc.

In these functions, a single event occurs at a random time. Counting the number of occurrences with time is Poisson process. It is, therefore, also called a counting process.

Probability density function

If the number of occurrences of an event in any finite interval of time, is described by a Poisson distribution with the average rate of occurrences is λ, then the probability of exactly k occurrences over a time interval (0,t) is

$$P[X(t)=k] = \frac{(\lambda t)^k e^{-(\lambda t)}}{k!}, \quad k = 0,1,2,....$$

And the probability density function

$$f_X(x) = \sum_{k=0}^{\infty} \frac{(\lambda t)^k e^{-(\lambda t)}}{k!} \delta(x-k)$$

Mean Value

The mean value of a Poisson density function is

$$E[X(t)] = \int_{-\infty}^{\infty} x \sum_{k=0}^{\infty} \frac{(\lambda t)^k e^{-(\lambda t)}}{k!} \delta(x-k)dx$$

Since $\int_{-\infty}^{\infty} x\,(x-k)dx = k$

$$E[X(t)] = \sum_{k=0}^{\infty} \frac{k(\lambda t)^k e^{-(\lambda t)}}{k!}$$

but we know from the Poisson density function of a random variable that the mean value is λt, so

$$E[X(t)] = \lambda t$$

Therefore,

$$E[X(t)] = \sum_{k=0}^{\infty} \frac{k(\lambda t)^k e^{-(\lambda t)}}{k!} = \lambda t$$

Program

```
clc; clear
all; close
all;

% Generate first set of 5000 samples of Gaussian distributed random numbers x1 =
randn(1,5000);

% Generate second set of 5000 samples of Gaussian distributed random numbers x2 =
randn(1,5000);

%Plot the joint distribution of both the sets using 'dots' so as to get a scatter plot figure,
plot( x1 , x2 , ' . ' );
title('Scatter Plot of Gaussian Distributed Random Numbers');

% Generate first set of 5000 samples of uniformly distributed random numbers x1 =
rand(1,5000);

% Generate second set of 5000 samples of uniformly distributed random numbers x2 =
rand(1,5000);

%Plot the joint distribution of both the sets using 'dots' so as to get a scatter plot figure,
plot( x1 , x2 , ' . ' );
title('Scatter Plot of Uniform Distributed Random Numbers');

%Generate one lakh samples of uniformly distributed random numbers x3 =
rand(1,100000);

% Plot a histogram graph of x3 in the 1st portion of a new figure window figure;
subplot(2,1,1);
hist(x3)
title('Uniform Distribution')

% Generate one lakh samples of Gaussian distributed random numbers y =
randn(1,100000);

%Plot a histogram graph of y in the 2nd portion of the figure window
subplot(2,1,2);
hist(y)
title('Gaussian Distribution')

ymu = mean(y)                     % Find the mean value of y
ymsq = sum(y .^ 2 ) / length(y)   % Find the mean square value of y
ysigma = std(y)                   % Find the standard deviation of y
yvar = var(y)                     % Find the variance value of y
yskew = skewness(y)               %Find the skew value of y
ykurt = kurtosis(y)               %Find the kurtosis value of y
```

Module 26: Analysis of LTI System Response

Objective:

This particular module discusses the methods of describing the out response of a linear time invariant system (LTI) when a continuous random process is applied at the input**.**

Introduction

Response of Linear System to Random Signals

Consider a continuous LTI system with impulse response *h(t).* Assume that the system is always causal and stable. When a continuous time random process *X(t)* with ensemble members $x(t)$is applied on this system, the output response is also a continuous time random process *Y(t)* with ensemble members $y(t)$. If the random *X*and *Y*are discrete time signals, then the linear system is called a discrete time system. In this we concentrate on the statistical and spectral characteristics of the random process *Y(t).*

System Response

Let a random process *X(t)* be applied to a continuous linear time invariant system whose impulse response *h(t)* . Then the output response $y(t)$ is also a random process. It can be expressed by the convolution integral,

*Y(t)=h(t)*X(t)*

That is, the output response is That is, the output response is $Y(t)=\int_{-\infty}^{\infty} h(\lambda)X(t-\lambda)d\lambda$

Mean value of output Response

Consider that the random process *X(t)* is wide sense stationary process.

Mean value of output response = *E[Y(t)]*

$$\text{Then, } E[Y(t)] = E\,[h(t)*X(t)]$$
$$= E\,[\int_{-\infty}^{\infty} h(\lambda)X(t-\lambda)d\lambda]$$
$$=\int_{-\infty}^{\infty} h(\lambda)E[X(t-\lambda)]d\lambda$$

But $E[X(t-\lambda)$= $\bar{X}$= constant , since *X(t)* is wide sense stationarity

Then $E[Y(t)]=\bar{Y} = \bar{X}\int_{-\infty}^{\infty} h(t)dt$

Mean Square value of Output Response

Mean Square Value of output response is

$$E[Y^2(t)] = E\,([h(t)*X(t)]^2]$$
$$=E\,[(h(t)*X(t))(\,h(t)*X(t))]$$
$$=E[\int_{-\infty}^{\infty} h(\tau_1)X(t-\tau_1)d\tau_1 \int_{-\infty}^{\infty} h(\tau_2)X(t-\tau_2)d\tau_2]$$
$$=E[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(t-\tau_1)X(t-\tau_2)h(\tau_1)h(\tau_2)d\tau_1 d\tau_2]$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E[X(t-\tau_1)X(t-\tau_2)]h(\tau_1)h(\tau_2)d\tau_1 d\tau_2$$

Where $\tau_1$and $\tau_2$are the shift in time variables

If the input *X(t)* is a wide sense stationary random process, then

$$E[X(t-\tau_1)X(t-\tau_2)] = R_{XX}(\tau_1-\tau_2)$$

Therefore,

$$E[Y^2(t)]=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(\tau_1-\tau_2)h(\tau_1)h(\tau_2)d\tau_1 d\tau_2$$

The above expression is independent of time *t.* It represents the output power.

Autocorrelation Function of Output Response

The autocorrelation of *Y(t)* is

$$R_{YY}(t_1,t_2)= E[Y(t_1)\,Y(t_2)] = E\,[(h(t_1)*X(t_1))(\,h(t_2)*X(t_2))]$$
$$= E\,[\int_{-\infty}^{\infty} h(\lambda_1)X(t-\lambda_1)d\lambda_1 \int_{-\infty}^{\infty} h(\lambda_2)X(t-\lambda_2)d\lambda_2]$$
$$= E[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(t-\lambda_1)X(t-\lambda_2)h(\lambda_1)h(\lambda_2)d\lambda_1 d\lambda_2]$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E[X(t-\lambda_1)X(t-\lambda_2)]h(\lambda_1)h(\lambda_2)d\lambda_1 d\lambda_2$$

We know that $E[X(t-\tau_1)X(t-\tau_2)]$ = R_{XX} $(t_2-t_1+\tau_1-\tau_2)$
if the input *X(t)* is a wide sense stationary random process, let the time difference
$\tau = t_2 - t_1$ and $t = t_1$
Then $E[X(t-\tau_1)X(t+\tau-\tau_2)]$ = R_{XX} $(\tau+\tau_1-\tau_2)$
It is observed that the output autocorrelation function is a function of only τ. Hence the output random process *Y(t)* is also wide sense stationary process.

Cross Correlation Function of Response

If the input X(t) is a wide sense stationary random process, then the cross correlation function of input X(t) and output Y(t) is

$$R_{XY}(t,t+\tau)= E[X(t)\, Y(t+\tau)]$$

$$R_{XY}(\tau)= E[X(t)\int_{-\infty}^{\infty} h(\tau_1)X(t+\tau-\tau_1)d\tau_1]$$

$$= \int_{-\infty}^{\infty} h(\tau_1)R_{XX}(\tau-\tau_1)d\tau_1$$

$$= h(\tau) * R_{XX}(\tau)$$

Therefore, $R_{YX}(\tau)= h(-\tau) * R_{XX}(\tau)$
This expression show the relationship between the autocorrelation functions and cross correlation functions

% Program to Verify Linearity

```
clc
clear all
close all
x1 = input (' type the samples of  x1 ');     % Input sequence 1
x2 = input (' type the samples of  x2 ');     % Input sequence 2
if (length(x1) ~= length(x2))
   disp(' ERROR : Lengths of x1 & x2 are different ')
   return;
end;
h = input (' type the samples of impulse response ');
N = length(x1) + length(h) -1;
disp('length of the output signal will be ');
disp(N);
a1 = input (' The scale factor a1 is ');                % Scaling factor 1
a2 = input (' The scale factor a2 is ');                % Scaling factor 2
x = a1 * x1 + a2 * x2;
yo1 = conv(x,h);                                        % Response due to combined input
y1 = conv(x1,h);
y1s = a1 * y1;
y2 = conv(x2,h);
y2s = a2 * y2;
yo2 = y1s + y2s;                    % Response due to individual inputs and combining outputs
disp ('Input signal x1 is '); disp(x1);
disp ('Input signal x2 is '); disp(x2);
disp ('Output Sequence yo1 is ');
disp(yo1);
disp ('Output Sequence yo2 is ');
disp(yo2);
if ( yo1 == yo2 )                                   % Verifying Linearity Property
   disp(' yo1 = yo2. Hence the LTI system is LINEAR ')
end;
```

Program for Time-Invariance Verification

```
x = input( ' Type the samples of  signal x(n) ' );          % Input Sequence  x(n)
h = input( ' Type the samples of  signal h(n) ' );          % Input Sequence  h(n)
y = conv(x,h);                                              % Performing convolution
disp( ' Enter a POSITIVE number for delay  ' );
d = input( ' Desired delay of the signal is ' );
xd = [zeros(1,d), x];
nxd = 0 : length(xd)-1;
yd = conv(xd,h);
nyd = 0:length(yd)-1;
disp(' Original Input Signal x(n) is ');
disp(x);
disp(' Original Output Signal y(n) is ');
disp(y);
disp(' Delayed Input Signal xd(n) is ');
disp(xd);
disp(' output yd(n) obtained for Delayed input Signal xd(n) is ');
disp(yd);
xp = [x , zeros(1,d)];
figure
subplot(2,1,1);
stem(nxd,xp); grid;
xlabel( ' Time Index n ' );ylabel( ' x(n) ' );
title( ' Original Input Signal x(n) ' );
subplot(2,1,2);
stem(nxd,xd); grid;
xlabel( ' Time Index n ' );ylabel( ' xd(n) ' );
title( ' Delayed Input Signal xd(n) ' );
yp = [y zeros(1,d)];
figure
subplot(2,1,1);
stem(nyd,yp); grid;
xlabel( ' Time Index n ' );ylabel( ' y(n) ' );
title( ' Original Output Signal y(n) ' );
subplot(2,1,2);
stem(nyd,yd); grid;
xlabel( ' Time Index n ' );ylabel( ' yd(n) ' );
title( ' Delayed Output Signal yd(n) ' );
yo1=[zeros(1,d),y] % delayed original output
yo2=yd                                                      %output obtained for delayed input
if ( yo1 == yo2 )                                           % Verifying Time invariance
   disp(' yo1 = yo2. Hence the system is TIME INVARIANT ')
end;
```

%To calculate Unit Impulse response

```
clc;
clear all;
close all;
% Given system
%y(n)=(3/8)y(n-1)+(2/3)y(n-2)+x(n)+(1/4)x(n-1)
num = input ('type the numerator vector   ');
den = input ('type the denominator vector   ');
N = input ('type the desired length of the output sequence N  ');
n = 0 :  N-1;
h=impz(num,den);
disp('The impulse response of LTI system is'); disp(h(1:N));
stem(n,h(1:N))
xlabel ('time index n');
ylabel ('Amplitude ');
title ('Impulse Response of LTI system');
```

% To calculate Unit Step response

```
% Given system
%y(n)=(3/8)y(n-1)+(2/3)y(n-2)+x(n)+(1/4)x(n-1)
num = input ('type the numerator vector   ');
den = input ('type the denominator vector   ');
N = input ('type the desired length of the output sequence N  ');
n = 0 : 1 : N-1;
s=stepz(num,den);
disp('The step response of LTI system is'); disp(s(1:N));
stem(n,s(1:N))
xlabel ('time index n');
ylabel ('s(n)');
title ('Step Response of LTI system');
```

% To calculate Sinusoidal response

```
% Given system
%y(n)=(3/8)y(n-1)+(2/3)y(n-2)+x(n)+(1/4)x(n-1)
num = input ('type the numerator vector   ');
den = input ('type the denominator vector   ');
n = 0 : 0.1 :2*pi;
in = sin(n);
s = filter ( num, den, in );
subplot(2,1,1);
stem(n,in);
xlabel ('time index n');
ylabel ('(n)');
title('Input sinusoidal sequence');
subplot(2,1,2);
stem(n,s);
xlabel ('time index n');
ylabel ('s(n)');
title ('Sinusoidal Response of LTI system');
```

References:

1. Probability, Random Variables and Random Signal Principles, Peyton Z. Peebles Jr. 4th Edition, Tata McGRAW-Hill.
2. Probability Theory and Stochastic Processes, Y. Mallikarjuna Reddy, University Press