Random Processes – Spectral Characteristics

Objective: To design any LTI filter which is intended to extract or suppress the signal, it is necessary to understand how the strength of a signal is distributed in the frequency domain, relative to the strengths of other ambient signals. Similar to the deterministic signals, it turns out to be just as true in the case of random signals.

There are two immediate challenges in trying to find an appropriate frequency-domain description for a WSS random process. First, individual sample functions typically don't have transforms that are ordinary, well-behaved functions of frequency; rather, their transforms are only defined in the sense of generalized functions. Second, since the particular sample function is determined as the outcome of a probabilistic experiment, its features will actually be random, and it is to be searched for features of the transforms that are representative of the whole class of sample functions, i.e., of the random process as a whole.

The present module focuses on the expected power in the signal which is a measure of signal strength and will be shown that it meshes nicely with the second moment characterizations of a WSS process. For a process that is second-order ergodic, this will also correspond to the time average power in any realization.

Module Description:

- Ideally, all the sample functions of a random process are assumed to exist over the entire time interval $(-\infty, +\infty)$, and thus, are power signals.
- Thus, the existence of Power spectral density should be enquired.
- The concept of Power spectral density may not appear to be meaningful for a random process for the reasons as follows:
  - It may not be possible to describe a sample function analytically
  - For a given process, every sample function may be different from another one
- Hence, even PSD exists for each sample function, it may be different for different sample functions
- It is possible to define a meaningful PSD for a stationary(at least in the wide sense) random process.
- For non-stationary processes, PSD does not exist.
- For random signals and random Variables, because of the anon availability of the enough information to predict the output with certainty, the respective measures are done in- terms of averages.
- On these lines, the PSD of a random process is defined as a weighted mean of the PSDs of all sample functions, as it is not known exactly which of the sample functions may occur in a given trial.

- **Difficulty in Fourier Representation of a Random Process**
  - The Fourier transform of a WSS $X(t)$ can not be defined by the integral process

$$F[X(t)] = \int_{-\infty}^{\infty} X(t).e^{-j\omega t}\ .dt$$

- The existence of the above integral would have implied the existence the Fourier transform of every realization of $X(t)$.
- But the very notion of stationarity demands that the realization does not decay with time and the first condition of Dirichlet is violated
- This difficulty is avoided by a frequency-domain representation $X(t)$ in of terms of the *power spectral density (PSD)*.
- The power of a WSS process $X(t)$ is a constant and given by $E[X^2(t)]$.
- The PSD denotes the distribution of this power over frequencies.

❖ Defining the Power Spectral Density of a random Process

- Let

$$\begin{aligned} X_T(t) &= X(t) \qquad -T < t < T \\ &= 0 \qquad \text{otherwise} \\ &= X(t)rect(\frac{t}{2T}) \end{aligned}$$

where $rect(\frac{t}{2T})$ is the unity-amplitude rectangular pulse of width $2T$ cantered at origin. As $t \to \infty, X_T(t)$ will represent the random process $X(t)$.

- Define $F[X_T(t)] = X_T(\omega) = \int_{-\infty}^{\infty} X_T(t).e^{-j\omega t}.dt$
- Applying Parseval's theorem to find the energy of the signal
- $.\int_{-T}^{T} X_T{}^2(t).dt = \int_{-\infty}^{\infty} |X_T(\omega)|^2.d\omega$
- Therefore, the power associated with $X_T(t)$ is
- $\frac{1}{2T}\int_{-T}^{T} X_T{}^2(t).dt = \frac{1}{2T}\int_{-\infty}^{\infty} |X_T(\omega)|^2.d\omega$
- The average power is given by

$$\frac{1}{2T}E\left[\int_{-T}^{T} X_T{}^2(t).dt\right] = \frac{1}{2T}E\left[\int_{-\infty}^{\infty} |X_T(\omega)|^2.d\omega\right] = E\left[\int_{-\infty}^{\infty} \frac{|X_T(\omega)|^2}{2T}d\omega\right]$$

- where $E\left[\int_{-\infty}^{\infty} \frac{|X_T(\omega)|^2}{2T}\right]$ is the contribution to the average power at frequency ω and represents the power spectral density for $X_T(t)$. As $T \to \infty$, the left-hand side in the above expression represents the average power of $X(t)$. Therefore, the PSD $S_X(\omega)$ of the process $X(t)$ is defined in the limiting sense by

$$S_{xx}(\omega) = \lim_{T\to\infty} \frac{E[|X_T(\omega)|^2]}{2T}$$

- Thus, The PSD $S_x(\omega)$ of a random process X(t) is defined as the ensemble average of the PSDs of all sample functions Thus,

$$S_{xx}(\omega) = \lim_{T\to\infty} \left[\frac{\overline{|X_T(\omega)|^2}}{T}\right]$$

  Where, $X_T(\omega)$ is theFourier Transform of the truncated random process $X(t).\, rect(\frac{t}{T})$ and the bar represents the ensemble average.
- The ensemble averaging is done before the limiting operation.

Relation Between Power-spectral Density and Autocorrelation function of the Random Process

- We have PSD $S_{xx}(\omega) = \lim_{T\to\infty} \frac{E[|X_T(\omega)|^2]}{2T}$

$$= \lim_{T\to\infty} \frac{E[X^*{}_T(\omega) X_T(\omega).]}{2T}$$

- $X_T(\omega) = \int_{-T}^{T} X(t).\, e^{-j\omega t}\, .\, dt$ and $X^*{}_T(\omega) = \int_{-T}^{T} X(t).\, e^{j\omega t}\, .\, dt$
- $S_{xx}(\omega) = \lim_{T\to\infty} \frac{1}{2T} E\left[\int_{-T}^{T} X(t_1).\, e^{j\omega t_1}\, .\, dt_1.\, \int_{-T}^{T} X(t_2).\, e^{-j\omega t_2}\, .\, dt_2\right]$
- $= \lim_{T\to\infty} \frac{1}{2T} E\left[\int_{-T}^{T} \int_{-T}^{T} X(t_1)\, X(t_2).\, e^{-j\omega(t_2-t_1)}\, .\, dt_1 dt_2.\right]$
- $= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \int_{-T}^{T} E[X(t_1)\, X(t_2)].\, e^{-j\omega(t_2-t_1)}\, .\, dt_1 dt_2$

- $= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \int_{-T}^{T} R_{xx}(t_1, t_2)\, e^{-j\omega(t_2-t_1)} \,.\, dt_1 dt_2$

- Consider the inverse Fourier Transform of PSD i.e. $\frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(\omega)\, e^{j\omega\tau} \,.\, d\omega$

- $F^{-1}[S_{xx}(\omega)] =$
$$\frac{1}{2\pi} \int_{-\infty}^{\infty} [\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \int_{-T}^{T} R_{xx}(t_1, t_2)\, e^{-j\omega(t_2-t_1)} \,.\, dt_1 dt_2]\, e^{j\omega\tau} \,.\, d\omega$$

- $= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \int_{-T}^{T} R_{xx}(t_1, t_2) \frac{1}{2\pi} \,.\, \int_{-\infty}^{\infty} e^{-j\omega(t_2-t_1)} \,.\, d\omega .\, dt_1 dt_2$

- Since, $F[\delta(t)] = 1, \rightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} 1.\, e^{j\omega t}\, d\omega = \delta(t)$

- On similar lines, $\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega(\tau - t_2 + t_1)}\, d\omega = \delta(\tau - t_2 + t_1)$

- $F^{-1}[S_{xx}(\omega)] = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \int_{-T}^{T} R_{xx}(t_1, t_2) \delta(\tau - t_2 + t_1) .\, dt_1 dt_2$

- $since\ \delta(\tau - t_2 + t_1) = 1\ at\ \ \tau - t_2 + t_1 = 0\ i.e.\ t_2 = \tau + t_1$

- $F^{-1}[S_{xx}(\omega)] = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} R_{xx}(t_1, \tau + t_1)\, dt_1$

- Let $t_1 = \tau \rightarrow dt_1 = d\tau$

- Hence, $F^{-1}[S_{xx}(\omega)] = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} R_{xx}(t, t+\tau)\, dt$

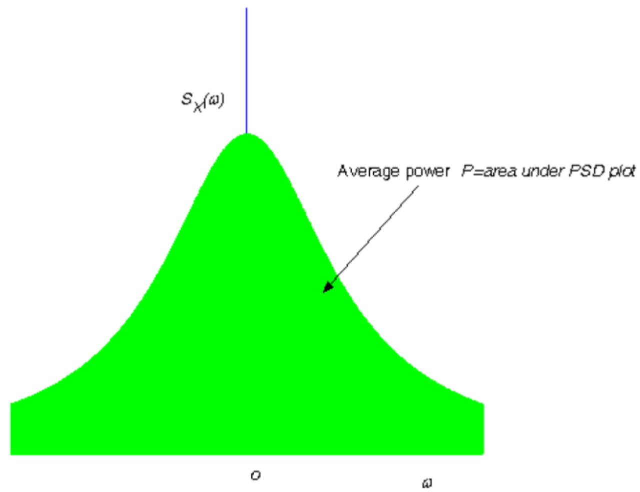
- The RHS of the above eq. is the time average of Auto correlation function.
- Thus, Time average of Autocorrelation function and the PSD form a Fourier Transform Pair.

- If the process is stationary, the time average of $R_{xx}(t, t+\tau)$ will be $R_{xx}(\tau)$, since it is independent of time.
- Thus, for a WSS process, Autocorrelation and Power Spectral Density form a Fourier Transform Pair and this is referred to as Wiener-Khintchine relation.
- $S_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-j\omega\tau}\, d\tau$
- $R_{xx}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)\, e^{j\omega\tau} . d\omega$

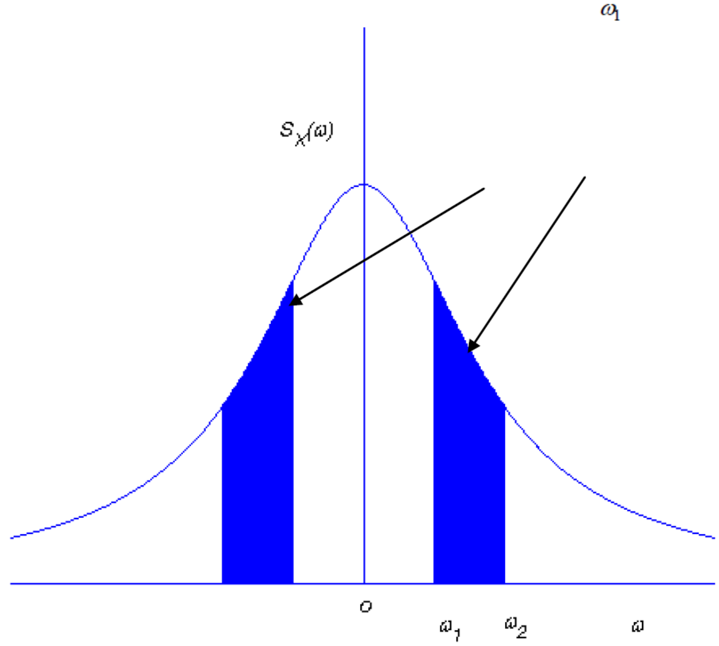
Properties of Power Spectral Density

- The average power of a random process $X(t)$ is

$$E[X^2(t)] = R_{xx}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega) . d\omega$$



The average power in the band $[\omega_1, \omega_2]$ is $2\int_{\omega_1}^{\omega_2} S_X(\omega) d\omega$



- If $\{X(t)\}$ is real, $\mathrm{R}_X(\tau)$ is a real and even function of τ. Therefore,

$$
\begin{aligned}
S_X(\omega) &= \int_{-\infty}^{\infty} R_X(\tau) e^{-j\omega\tau} d\tau \\
&= \int_{-\infty}^{\infty} R_X(\tau)(\cos\omega\tau + j\sin\omega\tau) d\tau \\
&= \int_{-\infty}^{\infty} R_X(\tau)\cos\omega\tau d\tau \\
&= 2\int_{0}^{\infty} R_X(\tau)\cos\omega\tau d\tau
\end{aligned}
$$

Thus $S_X(\omega)$ is a real and even function of ω.

- From the definition $S_X(w) = \lim_{T\to\infty} \dfrac{E\left|X_T(\omega)\right|^2}{2T}$ is always non-negative. Thus $S_X(\omega) \geq 0$.
- If $\{X(t)\}$ has a periodic component, $\mathrm{R}_X(\tau)$ is periodic and so $S_X(\omega)$ will have impulses.

1) The function $S_X(\omega)$ is the PSD of a WSS process $\{X(t)\}$ if and only if $S_X(\omega)$ is a non-negative, real and even function of ω and $\int_{-\infty}^{\infty} S_X(\omega) < \infty$
2) The above condition on $S_X(\omega)$ also ensures that the corresponding autocorrelation function $\mathrm{R}_X(\tau)$ is non-negative definite. Thus the non-negative definite property of an autocorrelation function can be tested through its power spectrum.

ILLUSTRATIVE PROBLEMS:

1.The autocorrelation function of a WSS process $X(t)$ is given by

$$R_X(\tau) = a^2 e^{-b|\tau|} \quad b > 0$$

Find the power spectral density of the process.

Soln:

$$S_X(\omega) = \int_{-\infty}^{\infty} R_x(\tau) e^{-j\omega\tau} d\tau$$

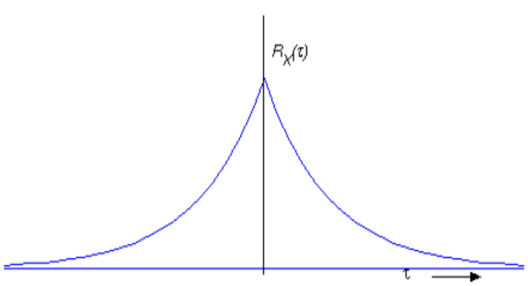
$$= \int_{-\infty}^{\infty} a^2 e^{-b|\tau|} e^{-j\omega\tau} d\tau$$

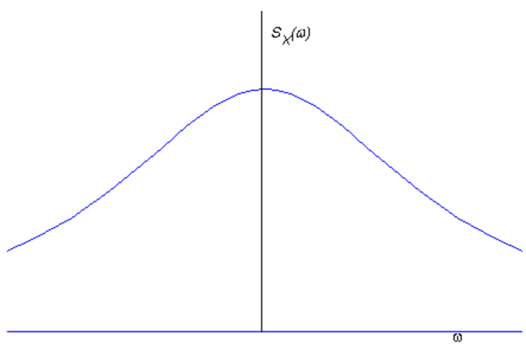
$$= \int_{-\infty}^{0} a^2 e^{b\tau} e^{-j\omega\tau} d\tau + \int_{0}^{\infty} a^2 e^{-b\tau} e^{-j\omega\tau} d\tau$$

$$= \frac{a^2}{b - j\omega} + \frac{a^2}{b + j\omega}$$

$$= \frac{2a^2 b}{b^2 + \omega^2}$$

The autocorrelation function and the PSD are shown in Fig.





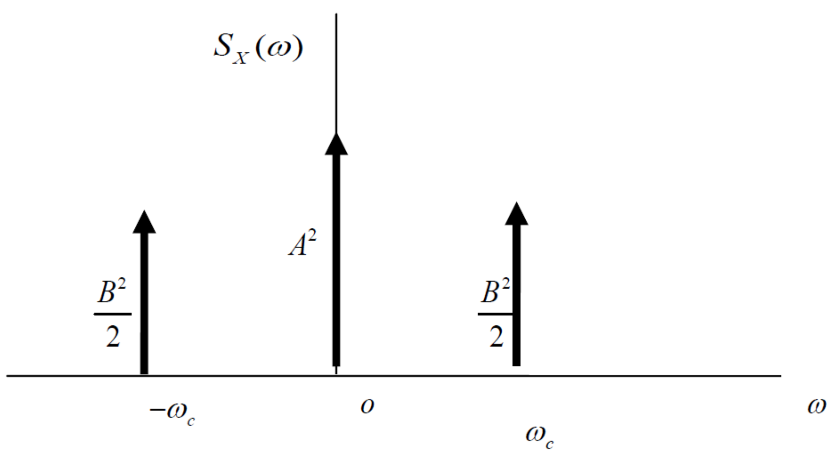
2. Suppose $X(t) = A + B\sin(\omega_c t + \Phi)$ where A is a constant bias and $\Phi \sim U[0, 2\pi]$. Find $R_X(\tau)$ and $S_X(\omega)$..

Soln:

$$R_x(\tau) = EX(t+\tau)X(t)$$
$$= E(A + B\sin(\omega_c(t+\tau) + \Phi))(A + B\sin(\omega_c t + \Phi))$$
$$= A^2 + \frac{B^2}{2}\cos\omega_c\tau$$

$$\therefore\ S_X(\omega) = A^2\delta(\omega) + \frac{B^2}{4}\left(\delta(\omega+\omega_c) + \delta(\omega-\omega_c)\right)$$

where $\delta(\omega)$ is the Dirac Delta function.



3. Find the PSD of the amplitude-modulated random-phase sinusoid

$$X(t) = M(t)\cos\left(\omega_c t + \Phi\right), \qquad \Phi \sim U\left(0, 2\pi\right)$$

where $M(t)$ is a WSS process independent of Φ.

Soln:

$$\Rightarrow R_X(\tau) = E\ M(t+\tau)\ \cos\left(\omega_c(t+\tau) + \Phi\right)\ M(t)\cos\left(\omega_c\tau + \Phi\right)$$
$$= E\ M(t+\tau)M(t)\ E\cos\left(\omega_c(t+\tau) + \Phi\right)\cos\left(\omega_c\tau + \Phi\right)$$

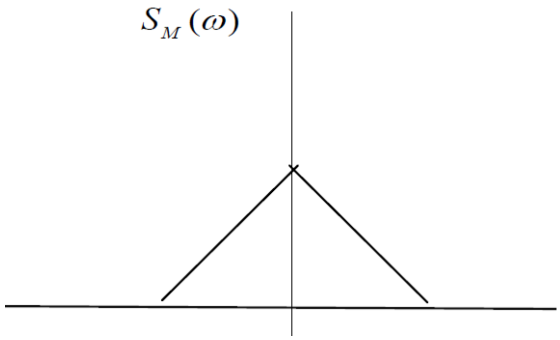
(Using the independence of $M(t)$ and the sinusoid)

$$= R_M(\tau)\frac{A^2}{2}\cos\omega_c\tau$$

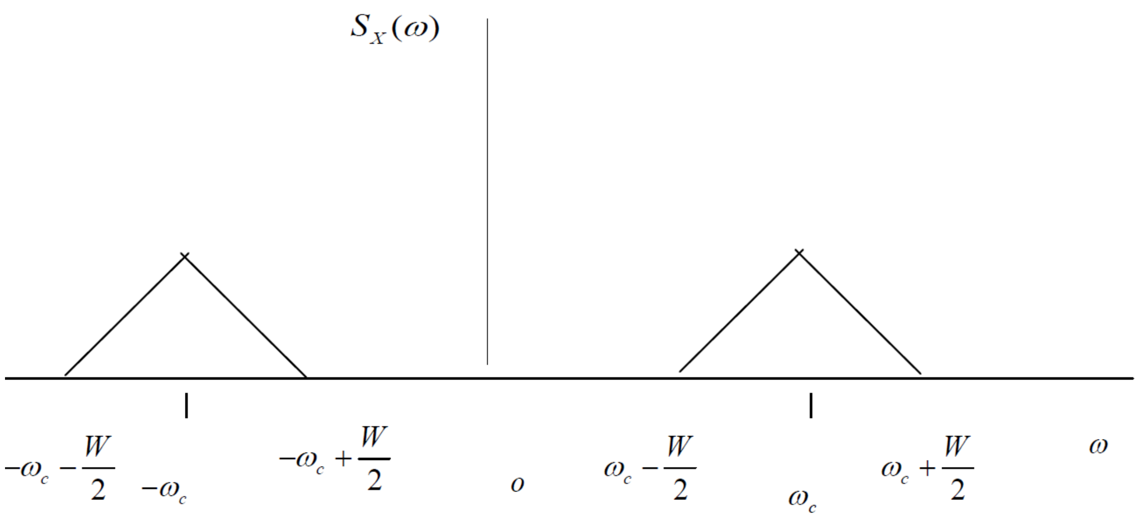
$$\therefore\ S_X(\omega) = \frac{A^2}{4}\left(S_M(\omega+\omega_c) + S_M(\omega-\omega_c)\right)$$

where $S_M(\omega)$ is the PSD of M(t)

Let



Then,



4 . The PSD of a noise process is given by

$$S_N(\omega) = \frac{N_0}{2} \qquad |\omega \pm \omega_c| \leq \frac{W}{2}$$
$$= 0 \qquad \text{Otherwise}$$

Find the autocorrelation of the process.

Soln:

$$\therefore\ R_N(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_N(\omega)\ e^{j\omega\tau}\, d\omega$$

$$= \frac{1}{2\pi} \times 2 \times \int_{\omega_c - \frac{W}{2}}^{\omega_c + \frac{W}{2}} \frac{No}{2} \cos\omega\tau\ d\tau$$

$$= \frac{No}{2\pi} \left[\frac{\sin\left(\omega_c + \frac{W}{2}\right)\tau - \sin\left(\omega_c - \frac{W}{2}\right)\tau}{\tau} \right]$$

$$= \frac{NoW}{2\pi} \frac{\sin\frac{W\tau}{2}}{\frac{W\tau}{2}} \cos\omega_o\tau$$

5.Determine which of the following functions can be valid PSDs.

(a)$\frac{\omega^2}{\omega^6+3\omega^2+3}$ b)$\exp[-(\omega-1)^2]$ c)$\frac{1}{(1+\omega^2)^2}$ d)$\frac{Cos3\omega}{1+\omega^2}$

Soln:

a,c,and d are valid as those are the even functions of frequency and b is invalid

Exercise Problems:

1.Find the Auto correlation function of (i) White Noise (ii) Band Limited white noise (iii) Band pass white noise.

2.Let A and B are the random variables and a random process is defined as X(t)=ACoswt+BSinwt, where W is a real constant. Find the PSD of the process, if A and B are uncorrelated random variables with zero mean and same variance.

3.A random process is defined as Y(t)=X(t)-X(t-a), where X(t) is a WSS process and a>0. Find the PSD of Y(t).

4.Find the PSD of the Random Process X(t)=A.Cos(Bt+Y), where Y is a uniform random variable over(0,2π)

5.Find the Power Spectral density of an ergodic process X(t) whose Autocorrelation function is given by $R_{xx}(\tau) = \begin{matrix} 1 - |\tau| for |\tau| \leq 1 \\ = 0\ otherwise \end{matrix}$

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