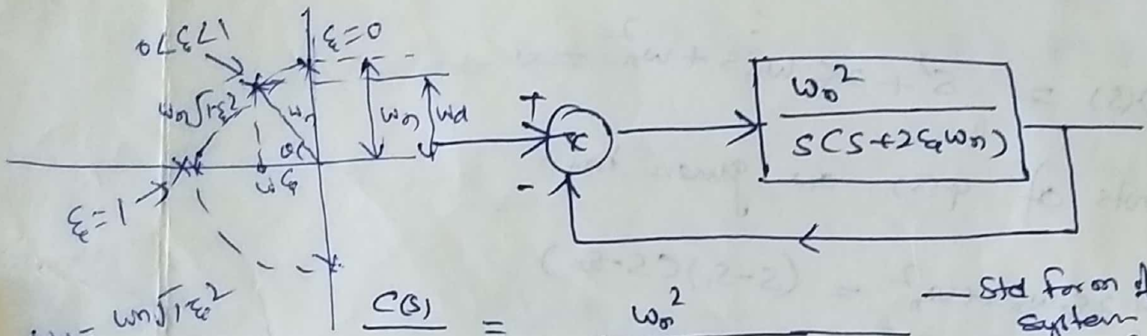
Second order System

Response of second order System to the unit Step

Consider the second order System



$\omega_d = \omega_n\sqrt{1-\xi^2}$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$ — Std form of second order System

$\xi \rightarrow$ (Zeta) damping ratio

$\omega_n \rightarrow$ undamped natural freq

$s^2+2\xi\omega_n s+\omega_n^2 = 0 = (s-s_1)(s-s_2)$

$s_1, s_2 = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$

→ If $\xi = 0$, Poles are purely imaginary and lies on $j\omega$ axis and system is called ~~under damped~~ undamped. It Purely oscillatory system. → Transient response does not die out.

→ If $0 < \xi < 1$, the closed loop poles are Complex conjugates and lies in the LHS of S-Plane, the system is called under damped and transient response is oscillatory.

→ If $\xi = 1$, the poles are real, negative, and equal. the system is called critically damped. The response rises slowly and reaches the final value.

→ If $\xi > 1$, the poles are real, negative and unequal. the system is called over damped. The o/p rises ~~slow~~ towards its final value slowly.

Critical and over damped system do not exhibit any over shoot.

This time response of any System is characterised by Poles of transfer function. In fact root of denominator

$$q(s) = 0$$

is Called characteristics eqn.

$$q(s) = s^2 + 2\xi w_n s + w_n^2 = 0$$

The roots of $q(s)$ are given by

$$s^2 + 2\xi w_n s + w_n^2 = (s-s_1)(s-s_2)$$

for $\xi < 1$,

$$s_1, s_2 = -\xi w_n \pm j w_n \sqrt{1-\xi^2}$$

$$\boxed{s_1, s_2 = -\xi w_n \pm j w_d}$$

$\therefore w_d = w_n\sqrt{1-\xi^2}$ is called damped natural freq

⇒ most of the Control System with exception of robotic Control are design with $\xi < 1$ to have high ~~time~~ response speed.

Response of an under damped System $(0 < \xi < 1)$.

In This Case $\frac{C(s)}{R(s)}$ Can be written as

$$\frac{C(s)}{R(s)} = \frac{w_n^2}{s^2 + 2\xi w_n s + w_n^2} = \frac{w_n^2}{(s + \xi w_n + j w_d)(s + \xi w_n - j w_d)} \quad \text{—(1)}$$

For unit step $R(s) = 1/s$ there fore eqn (1) Can be writt

$$\frac{C(s)}{R(s)} = \frac{w_n^2}{s(s^2 + 2\xi w_n s + w_n^2)}$$

$$= \frac{1}{s} - \frac{s + 2\xi w_n}{(s^2 + 2\xi w_n s + w_n^2)}$$

$$= \frac{1}{s} - \frac{s + 2\xi w_n}{\left[(s + \xi w_n)^2 + w_n^2 - w_n^2 \xi^2\right]}$$

$$= \frac{1}{s} - \frac{s+\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2} - \frac{\xi\omega_n}{\omega_d}\cdot\frac{\omega_d}{(s+\xi\omega_n)^2+\omega_d^2}$$

$$= \frac{1}{s} - \frac{s+\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2} - \frac{\xi}{\sqrt{1-\xi^2}}\cdot\frac{\omega_d}{(s+\xi\omega_n)^2+\omega_d^2} \quad \text{--- (2)}$$

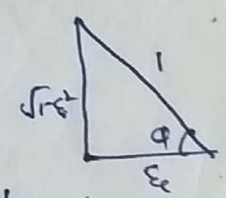
$\left\{ \because \text{Substitute } \omega_d = \omega_n\sqrt{1-\xi^2} \right.$

taking inverse Laplace transform of (2) we get

$$c(t) = 1 - e^{-\xi\omega_n t}\cos\omega_d t - \frac{\xi e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin\omega_d t \quad \text{--- (3)}$$

$$= 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left(\sqrt{1-\xi^2}\cos\omega_d t + \xi\sin\omega_d t\right)$$

$$= 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left(\sin\theta\cos\omega_d t + \cos\theta\sin\omega_d t\right)$$



$$= 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t + \theta) \quad \text{--- (4)}$$

$$= 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin\left[\omega_n\sqrt{1-\xi^2}\,t + \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}\right) \text{ for } t \geq 0 \quad \text{--- (5)}$$

The error signal for this system is $r(t) - c(t)$, $r(t) = 1$ g.h (3)

$$e(t) = r(t) - c(t)$$
$$= 1 - \left(1 - e^{-\xi\omega_n t}\cos\omega_d t - \frac{\xi e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin\omega_d t\right)$$

$$\therefore e(t) = e^{-\xi\omega_n t}\left(\cos\omega_d t + \frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_d t\right), \text{ for } t \geq 0 \quad \text{--- (6)}$$

The error signal exhibits a damped sinusoidal osc. At steady state, or $t \to \infty$, no error exists between i/p & o/p

for $\xi = 0$, the response become undamped and oscillation is continue indefinitely, the response $c(t)$ for $\xi = 0$, is

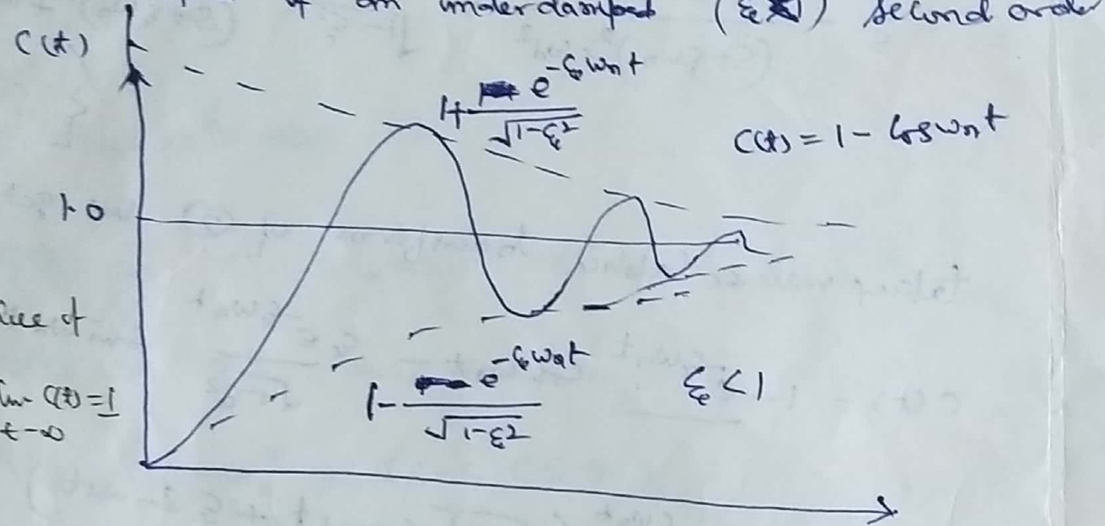
$$c(t) = 1 - \cos\omega_n t \quad \text{for } t \geq 0$$

Thus ω_n represents the undamped natural freq

$\omega_d = \omega_n$ for $\xi = 0$,

$\omega_d < \omega_n$ for $\xi > 0$,

The time response of an underdamped ($\xi < 1$) second order



The steady value of

$C(t) = C_{ss} = \lim_{t \to \infty} C(t) = 1$

$\boxed{e_{ss} = 0}$

Fig 1: unit step response of a second order System

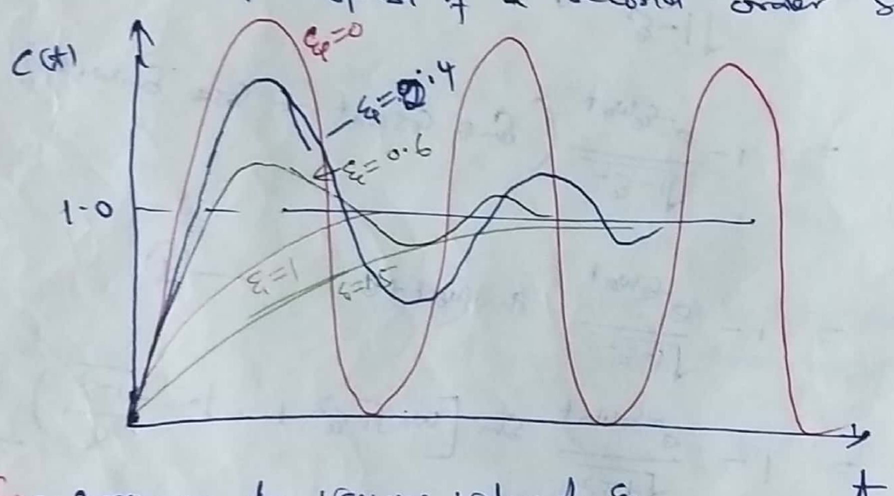


Fig 2: response for various value of ξ.

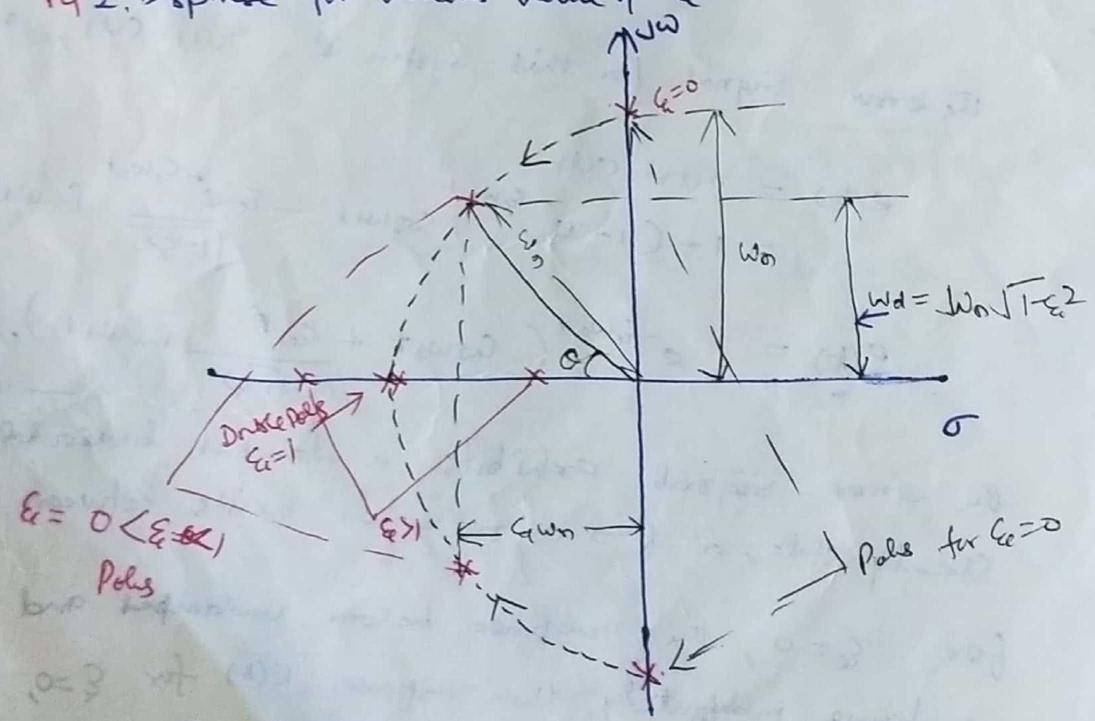
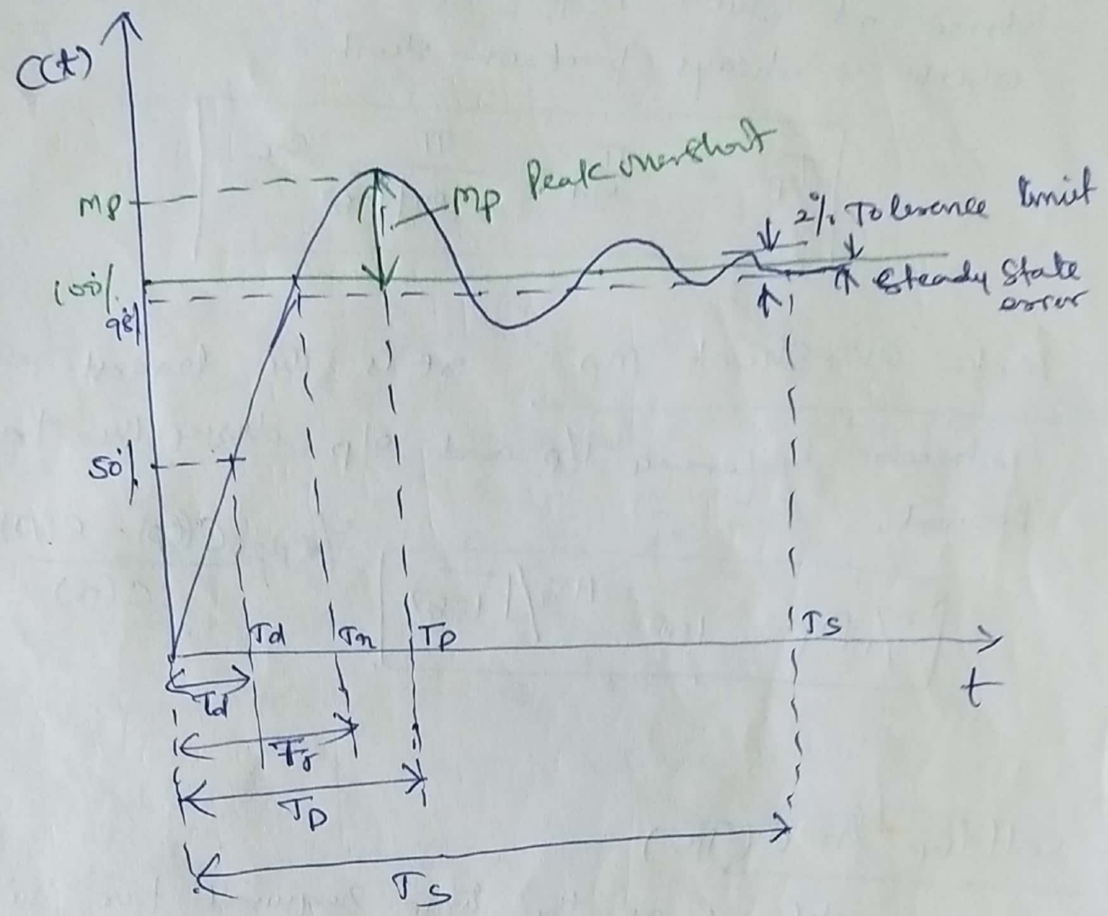


Fig 3: Poles location for second order System

Transient response Specifications



Delay time (T_d) It is time required for o/p to reach 50% of final value in first attempt. It is given as

$$\boxed{T_d = \frac{1+0.7\xi}{\omega_n}}$$

Rise time (T_r) It is time required by the response to rise 10% to 90% of final value for over damped and systems and 0 to 100% of final value for under damped system. It is reciprocal of the slope of the response at the instant, the response equal to 50% of final value. It is given as

$$\boxed{T_r = \frac{\pi - \theta}{\omega_d} \text{ Sec}}$$

Peak time (T_p) :-

It is time required for the response to reach its peak value. It is also defined as the time at which response undergoes the first overshoot which is always peak overshoot

$$\boxed{T_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} \text{ sec}}$$

Peak overshoot M_p

It is the largest error between reference i/p and o/p during the transient period.

$$\%M_p = \frac{C(t_p) - C(\infty)}{C(\infty)} \times 100$$

$$\boxed{\%M_p = 100e^{-\pi\xi/\sqrt{1-\xi^2}}}$$

Settling time (T_s)

This defined as the time required for the response to decrease and stay within specified % of its final value (within tolerance band)

$$\text{Time constant of system } T = \frac{1}{\xi\omega_n}$$

$$T_s = 4T = \frac{4}{\xi\omega_n} \quad \pm 2\%$$

(Time constant T is the time taken by system o/p to reach 63.2% of final value.

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Calculation of rise time

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \theta)$$

at rise time t_r $\quad c(t_r) = 1$, then

$$c(t_r) = 1 - \frac{e^{-\xi\omega_n t_r}}{\sqrt{1-\xi^2}} \sin(\omega_d t_r + \theta) = 1$$

then

$$\frac{e^{-\xi\omega_n t_r}}{\sqrt{1-\xi^2}} \sin(\omega_d t_r + \theta) = 0$$

Here 1st terms is not equal to zero, then second term

$$\sin(\omega_d t_r + \theta) = 0 \equiv \cancel{\sin n\pi}$$

to satisfy the eqn.

Trigonometrically it is true only

$$(\omega_d t_r + \theta) = n\pi \quad \text{where } n = 1, 2, \ldots$$

As we are interested in first attempt use $n = 1$

$$\omega_d t_r + \theta = \pi$$

$$\boxed{t_r = \frac{\pi - \theta}{\omega_d}}$$

$$\omega_d = \omega_n\sqrt{1-\xi^2}$$

$$\theta = \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}$$

or,

$$\boxed{t_r = \frac{\pi - \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}}{\omega_n\sqrt{1-\xi^2}}}$$

Derivation for T_p (Peak time)

We know

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \theta)$$

As at $t = t_p$, $c(t)$ will attain maximum value, According to maxima theorem

$$\left.\frac{dc(t)}{dt}\right|_{t=t_p} = 0.$$

$$\left.\frac{dc(t)}{dt}\right|_{t=t_p} = -\frac{e^{-\xi\omega_n t}(-\xi\omega_n)\sin(\omega_d t+\theta)}{\sqrt{1-\xi^2}} - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\,\omega_d\cos(\omega_d t+\theta) = 0$$

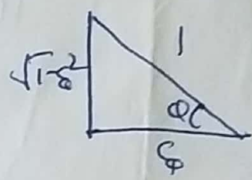
$$= \frac{\xi\omega_n e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t+\theta) + \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\cdot\omega_n\sqrt{1-\xi^2}\cos(\omega_d t+\theta) = 0$$

$\therefore\ \omega_d = \omega_n\sqrt{1-\xi^2}$

or, $$= \frac{\omega_n e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left(\xi\sin(\omega_d t+\theta) + \sqrt{1-\xi^2}\cos(\omega_d t+\theta)\right] = 0$$

or, $$\xi\sin(\omega_d t+\theta) - \sqrt{1-\xi^2}\cos(\omega_d t+\theta) = 0$$

or, $$\frac{\sin(\omega_d t+\theta)}{\cos(\omega_d t+\theta)} = \frac{\sqrt{1-\xi^2}}{\xi} = \tan\theta$$



or, $$\tan(\omega_d t+\theta) = \tan\theta$$

or, for trignometrical formula

$$\tan(n\pi+\theta) = \tan\theta$$

So, $$n\pi = \omega_d T_p \qquad (\because t = T_p)$$

or $$t_p = \frac{n\pi}{\omega_d} \qquad \because\ n=1 \text{ 1st peak}$$

$$\boxed{T_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}\ \text{sec}}$$

Derivation for m_p

$$m_p = c(T_p) - 1$$

$$m_p = \left\{1 - \frac{e^{-\xi\omega_n T_p}}{\sqrt{1-\xi^2}}\sin(\omega_d T_p+\theta)\right\} - 1$$

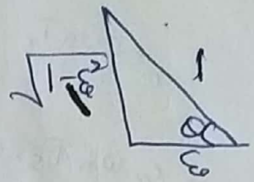
$$= \frac{e^{-\xi\omega_n T_p}}{\sqrt{1-\xi^2}}\sin(\omega_d T_p+\theta)$$

$T_p = \pi/\omega_d$ Substitute than

$$M_p = -\frac{e^{-\xi\omega_n T_p}}{\sqrt{1-\xi^2}} \cdot \sin(\pi + \theta)$$

Now $\sin(\pi+\theta) = -\sin\theta$, then

$$M_p = \frac{e^{-\xi\omega_n T_p}}{\sqrt{1-\xi^2}} \sin\theta$$



$$= \frac{e^{-\xi\omega_n T_p}}{\sqrt{1-\xi^2}} \cdot \sqrt{1-\xi^2} \qquad \therefore \sin\theta = \sqrt{1-\xi^2}$$

$$\boxed{M_p = e^{-\xi\omega_n T_p}} \qquad T_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}$$

or

$$M_p = e^{-\frac{\xi\pi\cancel{\omega_n}}{\cancel{\omega_n}\sqrt{1-\xi^2}}}$$

$$\boxed{\therefore M_p = e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}}}$$

$$\boxed{\% M_p = 100\, e^{-\frac{\pi\xi}{\sqrt{1-\xi^2}}}}$$

Derivation for T_s (Settling time)

The settling time T_s is the time required by the o/p to settle down within $\pm 2\%$ of tolerance band. So T_s is the time when o/p becomes 98% of its final value

$$c(t)\Big|_{t=T_s} = 0.98$$

Now at $t = T_s$ the transient oscillating terms completely vanishes. The term which controls the amplitude of o/p within $\pm 2\%$ is $e^{-\xi\omega_n t}$. Hence value of T_s is obtained considering only expontial term, neglecting all other terms

$$\therefore c(t)\Big|_{t=T_s} = 1 - \frac{e^{-\xi\omega_n T_s}}{\sqrt{1-\xi^2}} = .98$$

or) $e^{-\xi\omega_n T_s} = .02\sqrt{1-\xi^2}$

For small value of ξ

$$e^{-\xi\omega_n T_s} = 0.02$$

$$-\xi\omega_n T_s = \log_e(.02)$$

$$-\xi\omega_n T_s = -3.912$$

$$\boxed{T_s = \frac{3.912}{\xi\omega_n} \simeq \frac{4}{\xi\omega_n}}, \quad \left\{ T = \frac{1}{\xi\omega_n} \right. = \text{Time constant of system}$$

$$\boxed{T_s = 4T}$$

For 5%

$$c(t)\big|_{t=T_s} = 0.95$$

$$\boxed{T_s = \frac{2.995}{\xi\omega_n} \simeq \frac{3}{\xi\omega_n}}$$

Ramp $R(s) = 1/s^2$

$$c(t) = L^{-1}\left[\frac{\omega_n^2}{s^2(s^2+2\xi\omega_n s+\omega_n^2)}\right]$$

$$= t - \frac{2\xi}{\omega_n} + \frac{e^{-\xi\omega_n t}}{\omega_n\sqrt{1-\xi^2}}\sin(\omega_d t$$

$\xi < 1$

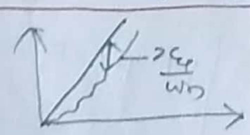
$$\lim_{t\to\infty} e_{ss}[t - c(t)] = \frac{2\xi}{\omega_n} = 1/K_v$$

(5) Steady state error e_{ss} (unit step)

$$e_{ss} = \lim_{t\to\infty}[1 - c(t)] = 0$$

unit ramp $R(s) = 1/s^2$

$$\boxed{e_{ss} = \frac{2\xi}{\omega_n} = 1/K_v}$$



Summary

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t + \theta)$$, for under damped unit step i/p

$$\omega_d = \omega_n\sqrt{1-\xi^2},$$

$$\theta = \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}$$

$$T_p = \frac{\pi}{\omega_d} \text{ sec},$$

$$T_r = \frac{\pi-\theta}{\omega_d} \text{ sec}$$

$$\%M_p = 100\, e^{-\frac{\pi\xi}{\sqrt{1-\xi^2}}}$$

$$T_s(2\%) = \frac{4}{\xi\omega_n}, \quad T_s(5\%) = \frac{3}{\xi\omega_n}$$

$$T_d = \frac{1+0.7\xi}{\omega_n} \text{ Sec}$$

Step: Poles location for

Steady State errors (Second order System)

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

For unit step input $R(s) = 1/s$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s(s^2+2\xi\omega_n s+\omega_n^2)}$$ we know that

$$c(t) \quad 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin \omega_d t + \theta$$

$$e_{ss} = \lim_{t \to \infty} \left(1 - \left(1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin \omega_d t + \theta\right)\right)$$

$$= \underline{\underline{0}}$$

or, $\boxed{c_{ss} = 1}$

$$\lim s\left(\frac{1}{s} - \frac{\omega_n^2}{s(s^2+2\xi\omega_n s+\omega_n^2)}\right)$$

$$\cancel{s}\frac{(s^2+2\xi\omega_n s+\cancel{\omega_n^2}-\omega_n^2)}{\cancel{s}(s^2+2\xi\omega_n s+\omega_n^2)}$$

$$\underline{\underline{0}}$$

For unit ramp I/P $\quad R(s) = 1/s^2$

$$C(s) = \frac{\omega_n^2}{s^2(s^2+2\xi\omega_n s+\omega_n^2)}$$

$$E(s) = [R(s) - C(s)]$$

$$= \frac{1}{s^2} - \frac{\omega_n^2}{s^2(s^2+2\xi\omega_n s+\omega_n^2)}$$

$$E(s) = \frac{s^2+2\xi\omega_n s+\cancel{\omega_n^2}-\cancel{\omega_n^2}}{s^2(s^2+2\xi\omega_n s+\omega_n^2)}$$

$$e_{ss} = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{\cancel{s}.(s^2+2\xi\omega_n s)}{s^{\cancel{2}}(s^2+2\xi\omega_n s+\omega_n^2)}$$

$$= \lim_{s \to 0} \frac{s+2\xi\omega_n}{s^2+2\xi\omega_n s+\omega_n^2} = \frac{2\xi\omega_n}{\omega_n^2} = \frac{2\xi}{\omega_n}$$

$$\boxed{e_{ss} = \frac{2\xi}{\omega_n}} \boxed{= 1/K_v}$$ for unit ramp

$$\therefore \omega_n = \sqrt{\frac{K_v}{\tau}} = \sqrt{\frac{K}{J}}$$

$$\xi = \frac{1}{2\sqrt{K_v\tau}}, \quad t_s = 4/\xi\omega_n$$

Design Specification of Second order System

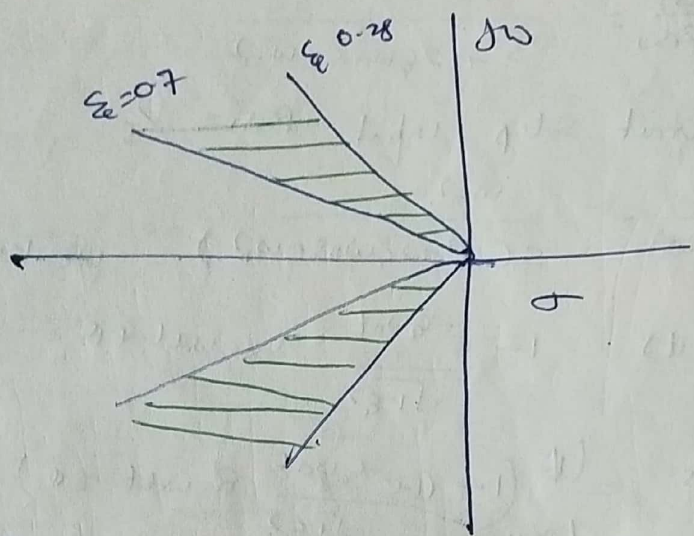


fig Desirable region of Poles locations for a Second order System

Type-1 second order ess, ω_n, ξ & t_s

$$\xi = \frac{1}{2\sqrt{K_v \tau}}, \quad \omega_n = \sqrt{K_v/\tau}$$

$$t_s = 4/\xi\omega_n,$$

$$e_{ss} = 2\xi/\omega_n = 1/K_v$$

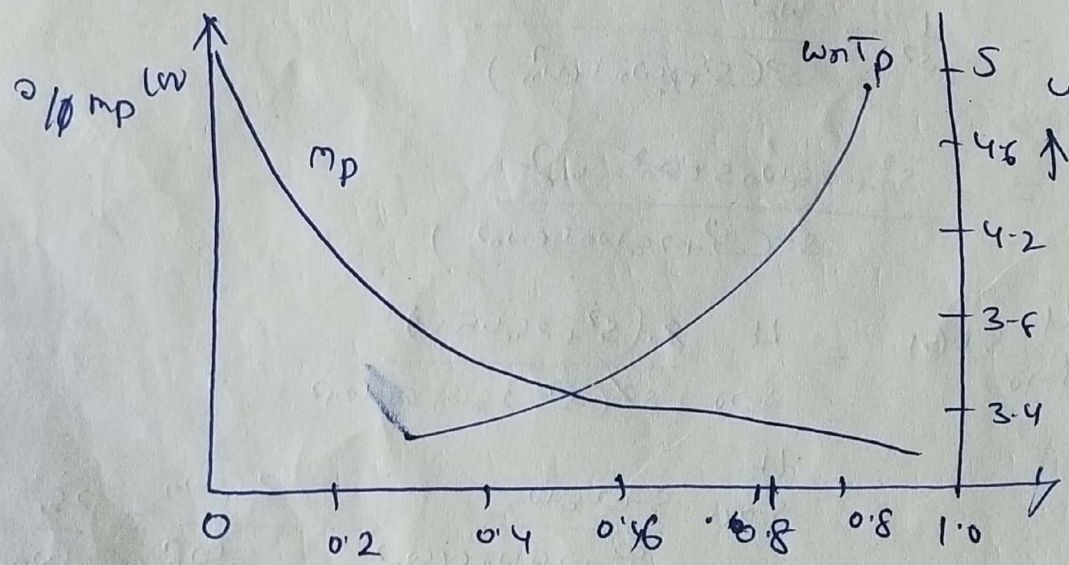


fig mp & $\omega_n T_p$ Vs ξ for Second order System