<u>UNIT-II</u>

① The probability density of uniform random variable is

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & a \le x \le b \\ 0 & \text{elsewhere} \end{cases}$$

<u>mean:-</u>

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx.$$

$$= \int_{a}^{b} x \cdot \frac{1}{b-a}\, dx. = \frac{1}{b-a} \int_a^b x\, dx.$$

$$= \frac{1}{b-a} \left[\frac{x^2}{2}\right]_a^b = \frac{1}{b-a} \left(\frac{b^2-a^2}{2}\right).$$

$$= \frac{1}{\cancel{b-a}} \frac{(b+a)\cancel{(b-a)}}{2}$$

$$\boxed{E[X] = \frac{b+a}{2}}$$

<u>Variance:-</u>

$$\sigma_X^2 = m_2 - m_1^2$$

$$\sigma_X^2 = E[X^2] - \overline{X}^2$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx. = \int_a^b x^2 \cdot \frac{1}{b-a}\, dx.$$

$$= \frac{1}{b-a} \left[\frac{x^3}{3}\right]_a^b = \frac{\cancel{(b-a)}(a^2+b^2+ab)}{\cancel{(b-a)}\, 3} = \frac{a^2+b^2+ab}{3}$$

$$(\overline{X})^2 = (E[X])^2 = \left(\frac{a+b}{2}\right)^2.$$

$$\sigma_x^2 = \frac{a^2+b^2+ab}{3} - \frac{(a+b)^2}{4}$$

$$= \frac{4a^2+4b^2+4ab - 3a^2-3b^2-6ab}{12}$$

$$= \frac{a^2+b^2-2ab}{12}$$

$$\boxed{\sigma_x^2 = \frac{(a-b)^2}{12}}$$

② properties of Variance:-

(i) The variance of a constant is zero.

$$\mu_2 = E[X^2] - (E[X])^2 \quad \cancel{[X=a]} \text{ [constant} = a]$$

$$= E[a^2] - (E[a])^2$$

$$= a^2 - a^2 = 0.$$

(ii) Variance of kx = k^2 variance of (X)

$$Var(kx) = k^2 var(x).$$

$$Var(kx) = E[k^2x^2] - (E[kx])^2$$

$$= k^2 E[x^2] - k^2 E[x]^2$$

$$= k^2 (E[x^2] - E[x]^2)$$

$$= k^2 var(x).$$

(iii) $var(ax+b) = a^2 var(x) + 0.$

(iv) The positive square root of variance is called standard deviation and it is a measure of spreading the function $f_X(x)$ about the mean.

③ * let X be a random variable and ω be a real ~~fund~~ number then the characteristic function of random variable x is given by-

$$\phi_X(\omega) = E\left[e^{j\omega x}\right] \quad -\infty < \omega < \infty$$

* The characteristic function can be obtained as

$$\phi_X(\omega) = \int_{-\infty}^{\infty} f_X(x) e^{-j\omega x} \, dx.$$

Properties of characteristic function

① The characteristic function of a random variable X at $\omega = 0$ is unity

i.e, $\phi_X(\omega)\big/_{\omega=0} = \phi_X(0) = 1.$

② The amplitude of characteristic function is maximum at $\omega = 0$ and it is unity.

i.e, $|\phi_X(\omega)| \leq \phi_X(0)$
≤ 1

③ The characteristic function of $Y = aX + b$ is

$\phi_Y(\omega) = e^{j\omega b} \phi_X(a\omega).$

④ If X_1 and X_2 are 2 independent random variables then

$$\phi_{(X_1+X_2)}(\omega) = \phi_{X_1}(\omega) \cdot \phi_{X_2}(\omega).$$

Q. The binomial density function is given by

$$f_X(x) = \sum_{r=0}^{n} {}^nC_r p^r q^{n-r} \delta(x-r).$$

$$E[X] = \sum_{x=0}^{n} x\, (P_X = x).$$

$$P(X=r) = {}^nC_r p^r q^{n-r} \delta(r-r).$$

$$P(X=r) = {}^nC_r p^r q^{n-r}.$$

$$P(X=x) = {}^nC_x p^x q^{n-x}.$$

$$E[X] = \sum_{x=0}^{n} x\, {}^nC_x p^x q^{n-x}$$

$$= \sum_{x=0}^{n} \frac{n!}{(n-x)!\,(x-1)!} p^x q^{n-x}.$$

$$= \sum_{x=0}^{n} \frac{n\,(n-1)!}{[(n-1)-(x-1)]!\,(x-1)!} p.p^{x-1} q^{(n-1)-(x-1)}$$

$$= np \sum_{x=0}^{n} {}^{(n-1)}C_{(x-1)} p^{(x-1)} q^{(n-1)-(x-1)} \rightarrow (1)$$

Binomial expansion of

$$(p+q)^n = \sum_{x=0}^{n} {}^nC_x p^x q^{n-x}.$$

$$= np(p+q)^{n-1} \quad (p+q=1)$$

$$= np$$

$$\therefore \boxed{E[X] = np} \rightarrow \text{mean.}$$

$\rightarrow \mu_2 = E[x^2] - (E[x])^2$

$= E[x(x-1)] + E[x] - E[x]^2$

$$E[x(x-1)] = \sum_{x=0}^{n} x(x-1)\, {}^nC_x\, p^x (1-p)^{n-x}$$

$$= \sum_{x=0}^{n} x(x-1) \frac{n!}{(n-x)!\, x!} p^x q^{n-x}$$

$$= \sum_{x=2}^{n} \frac{n!}{(n-x)!\,(x-2)!} p^x q^{n-x}$$

$$= \sum_{x=2}^{n} \frac{n(n-1)(n-2)!}{(n-x)!\,(x-2)!} p^2 p^{x-2} q^{n-x}$$

$$= n(n-1)p^2 \sum_{x=2}^{n} \frac{(n-2)!}{(n-x)!\,(x-2)!} p^{x-2} q^{n-x}$$

$$= n(n-1)p^2 \sum_{x=2}^{n} \frac{(n-2)!}{[n-2-(x-2)]!\,(x-2)!} p^{x-2} q^{n-2-(x-2)}$$

$$= n(n-1)p^2 \sum_{x=2}^{n} {}^{n-2}C_{x-2}\, p^{x-2} q^{n-2-(x-2)}$$

$$= n(n-1)p^2 [p+q]^{n-2}$$

$$= n(n-1)p^2 [\cancel{p} + 1 - \cancel{p}]^{n-2}$$

$$= n(n-1)p^2$$

$$\mu_2 = n(n-1)p^2 + np - (np)^2$$

$$= \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2} \qquad (q = 1-p)$$

$$= np(1-p) = \boxed{npq}$$

⑮ Joint distribution function:-

→ For any ordered pair of random Variable (x, y). let A and B are the 2 events defined as $A = \{X \le x\}$ and $B = \{Y \le y\}$

→ The event $A \cap B$ defined on a sample space i.e, $\{X \le x, Y \le y\}$ is known as joint event

→ The distribution function of this joint event is known as joint distribution function and it is defined as

$$F_{XY}(x, y) = P\{X \le x, Y \le y\}$$

Properties of Joint distribution function:-

① $F_{XY}(-\infty, -\infty) = 0$; $F_{XY}(-\infty, y) = 0$; $F_{XY}(x, -\infty) = 0$

② $F_{XY}(\infty, \infty) = 1$

③ $0 \le F_{XY}(x, y) \le 1$

④ $F_{XY}(x, y)$ is a monotonic non decreasing function.

⑤ $P\{x_1 < X \le x_2, y_1 < Y \le y_2\} = F_{XY}(x_2, y_2) + F_{XY}(x_1, y_1) - F_{XY}(x_2, y_1) - F_{XY}(x_1, y_2)$.

⑥ $F_{XY}(x, \infty) = F_X(x)$, $F_{XY}(\infty, y) = F_Y(y)$.

→ $F_{XY}(x, y) = \sum_{n=1}^{N} \sum_{m=1}^{M} P(x_n, y_m)\, u(x - x_n)\, u(y - y_m)$

→ for n-random variables:-

$$F_{X_1, X_2 \ldots X_n}(x_1, x_2 \ldots x_N) = P\{X_1 \le x_1, X_2 \le x_2 \ldots X_n \le x_n\}.$$

les,

EXAMPLE 4.5-2. The joint density of two random variables X and Y is

$$f_{X,Y}(x, y) = \frac{1}{12} u(x)u(y)e^{-(x/4)-(y/3)}$$

We determine if X and Y are statistically independent. From (4.3-5g) and (4.3-5h)

$$f_X(x) = \int_0^{\infty} (1/12)u(x)e^{-x/4}e^{-y/3}\, dy = (1/4)u(x)e^{-x/4}$$

$$f_Y(y) = \int_0^{\infty} (1/12)u(y)e^{-y/3}e^{-x/4}\, dx = (1/3)u(y)e^{-y/3}$$

Since $f_X(x)f_Y(y) = f_{X,Y}(x, y)$, then X and Y are independent.

⑦ $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{xy}(x,y)\,dx\,dy = 1.$$

$$\int_{y=0}^{b}\int_{x=0}^{1} 3xy\,dx\,dy = 1.$$

$$\int_{y=0}^{b}\left(\frac{3x^2}{2}\right)_0^1 y\,dy = 1$$

$$\frac{3}{2}\int_{y=0}^{b} y\,dy = 1$$

$$\frac{3}{2}\left[\frac{y^2}{2}\right]_0^b = 1$$

$$\frac{3}{2}\left[\frac{b^2}{2}\right] = 1$$

$$3b^2 = 4$$

$$b^2 = \frac{4}{3}$$

$$\boxed{b = \frac{2}{\sqrt{3}}}$$

(9) $\bar{X}=0, \bar{Y}=1, E[X^2]=2, E[Y^2]=4,$

$R_{XY}=-2.$

$W = 2X+Y$

$V = -X-3Y$

$R_{XY} = E[XY] = -2.$

$\sigma_X^2 = E[X^2] - (E[X])^2$

$= 2 - 0$

$\boxed{\sigma_X^2 = 2}$

$\sigma_Y^2 = E[Y^2] - (E[Y])^2.$

$= 4 - 1$

$\boxed{\sigma_Y^2 = 3}$

$E[W] = E[2X+Y] = 2E[X] + E[Y]$

$= 2(0) + 1 = 1$

$E[V] = E[-X-3Y] = -E[X] - 3E[Y]$

$= -0 - 3(1) = -\cancel{4}\,3.$

$\bar{W}=1, \bar{V} = -\cancel{4}\,3.$

$R_{WV} = E[WV]$

$= E[(2X+Y)(-X-3Y)]$

$= E[-2X^2 - 6XY - XY - 3Y^2]$

$= -2E[X^2] - 7E[XY] - 3E[Y^2]$

$= -2(2) - 7(-2) - 3(4)$

$= -4 + 14 - 12$

$= -16 + 14.$

(10). The joint characteristic function of 2 R.V X & Y is defined as

$$\phi_{X,Y}(\omega_1, \omega_2) = E\left[e^{j\omega_1 X + j\omega_2 Y}\right].$$

where ω_1, ω_2 are real number.

$$\phi_{X,Y}(\omega_1, \omega_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\, e^{j\omega_1 x + j\omega_2 y}\, dx dy.$$

Prop'r

(1) by putting either $\omega_1 = 0$ or $\omega_2 = 0$ the characteristic functions of X or Y are obtained. They are called marginal characteristic functions.

$$\phi_X(\omega_1) = \phi_{XY}(\omega_1, 0)$$

$$\phi_X(\omega_2) = \phi_{XY}(0, \omega_2).$$

(2) Joint moments m_{nk} can be found from the Joint characteristic function as follows:

$$m_{nk} = (-j)^{n+k} \left. \frac{\partial^{n+k} \phi_{X,Y}(\omega_1, \omega_2)}{\partial \omega_1^n \, \partial \omega_2^k} \right|_{\omega_1 = 0, \omega_2 = 0}$$

(3) This expression is recognised as 2-D. fourier transform the joint density function.

EXAMPLE 5.2-1. Two random variables X and Y have the joint characteristic function

$$\Phi_{X,Y}(\omega_1, \omega_2) = \exp(-2\omega_1^2 - 8\omega_2^2)$$

We show that X and Y are both zero-mean random variables and that they are uncorrelated.

The means derive from (5.2-6):

$$\bar{X} = E[X] = m_{10} = -j\frac{\partial\Phi_{X,Y}(\omega_1, \omega_2)}{\partial\omega_1}\bigg|_{\omega_1=0,\omega_2=0}$$

$$= -j(-4\omega_1)\exp(-2\omega_1^2 - 8\omega_2^2)\bigg|_{\omega_1=0,\omega_2=0} = 0$$

$$\bar{Y} = E[Y] = m_{01} = -j(-16\omega_2)\exp(-2\omega_1^2 - 8\omega_2^2)\bigg|_{\omega_1=0,\omega_2=0} = 0$$

Also from (5.2-6);

$$R_{XY} = E[XY] = m_{11} = (-j)^2\frac{\partial^2}{\partial\omega_1\partial\omega_2}[\exp(-2\omega_1^2 - 8\omega_2^2)]\bigg|_{\omega_1=0,\omega_2=0}$$

$$= -(-4\omega_1)(-16\omega_2)\exp(-2\omega_1^2 - 8\omega_2^2)\bigg|_{\omega_1=0,\omega_2=0} = 0$$

Since means are zero, $C_{XY} = R_{XY}$ from (5.1-14). Therefore, $C_{XY} = 0$ and X and Y are uncorrelated.

$= \frac{7}{81} + \frac{9}{81} + \frac{11}{81} + \frac{13}{81} + \frac{15}{81} + \frac{17}{81}$

$= \frac{72}{81}$.

(8) $\bar{x} = 0 \quad \sigma_x^2 = 9$.

$Y = 5x^2$

$E[Y] = 5E[x^2]$

$\sigma_x^2 = E[x^2] - (E[x])^2$

$9 = E[x^2] - 0$.

$E[x^2] = 9$.

$E[Y] = 5 \times 9 = 45$.

(9) $m_{10} = 2, \; m_{20} = 14, \; m_{02} = 12$.

$m_{11} = -6$. $\quad \mu_{22}$ & m_{22}

$m_{10} = E[x] = 2$

$m_{20} = E[x^2] = 14$.

$m_{02} = E[Y^2] = 12$

$m_{11} = E[XY] = -6$.

$E[XY] = E[X]E[Y]$

$-6 = 2 \times E[Y]$

$E[Y] = -3$.

$\bar{Y} = -3$.

$m_{22} = E[x^2 y^2]$.

$= E[x^2]\,E[y^2]$ as they are independent

$= 14 \times 12$.

$= \underline{168}$

$\mu_{22} = E[(x-\bar{x})^2 (y-\bar{y})^2]$.

$= E[(x-\bar{x})^2]\,E[(y-\bar{y})^2]$.

$= \sigma_x^2 \sigma_y^2$.

$= \mu_{20}\,\mu_{02}$.

$\mu_{20} = \sigma_x^2 = E[x^2] - (E[x])^2$

$= 14 - 4$

$= 10$.

$\mu_{02} = \sigma_y^2 = E[y^2] - (E[y])^2$

$= 12 - 9$

$= 3$.

$\mu_{22} = 10 \times 3$

$= 30$.