Signals and systems

Communication:- it is the process of Exchanging the information message, data, or any other from One Location to another location called as Transmitter to the Reciever (Source to the destination)

→ To exchange the data/information
- It is the information the drives the Entire world

- In olden days people Used to Communicate with Each other through Speech, gestures, graphical symbols, Drum beats, smoke signals carrier, pigeons and Light beams.
- Now a days electrical signals

Electrical Signals: To Transmit information Over longer distances with very high Speed (3×10^8 m/sec)

→ the Interference of noise is also less.

Signal: signal is a time Varying physical phenomenon/ physical quantity

→ carries or contains some set of information or Data that can be conveyed, displayed or manipulated

→ Mathematically, Signal is represented as a function of an independent Variable t

Usually t represents "time" denoted by $f(t)$ or $x(t)$

Examples of Signals

Speech - Which we Encounter for example in telephony, radio and Everyday life.

Bio medical signal - such as electro cardio gram (ECG - heart)
Electro Encephalogram (EEG - brain)

Sound and Music: Such as reproduced by the Compact disc player

Video and image: which most people watch on the television, and

Radar signals: Which are Used to determine the range of the distant target.

Some more examples of signals include.

- A voltage or current in a electronic circuit
- The position, Velocity, or acceleration of an object
- A force or torque in a mechanical system
- A flow rate of a Liquid or gas in a chemical process
- A digital image, digital Video, Or digital audio.
- A stock market index

→ Signal = function (Independent Variable)

$S = f(x_i) = f(x_1, x_2, x_3 \cdots\cdots x_i)$

$x_i \rightarrow$ no. of independent Variables.

- If $i=1 \rightarrow$ means signal is a function Of One independent variab and it is called as One-dimensional Signal (1-D)

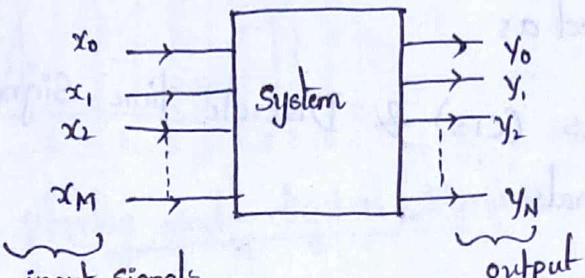
Ex → Speed signal

Whose amplitude Varies with time depending On the Spoken word.

- If $i=2 \rightarrow$ means signal is a function of two independent Variables and it is called as Two dimesional Signal (2-D)

Ex → image/picture Signal with the horizontal and Vertical co-ordinates of the image representing the two dimensions called Spatial Co-ordinates

SYSTEMS



→ It's a mathematical model of a physical process that relates the input and output.

→ A system is a physical device / Hard ware / Software that performs some operations / responds to applied input signals, to produce one or more output signals

→ Input is also called as excitation and output as response

Signal processings

- We need to process the signals by system because
- To modify, analyse the signals.
- To extract additional information
- To remove the noise and interference from the signal
- To obtain the spectrum of the signal
- To transform the signal into more suitable form
- When the signal is passed through the system then it is said to be processed

Example of Systems

→ Filter in a communication system

→ Radar system - knowing the past location and velocity of the target the future location of the moving target is being estimated and tracked

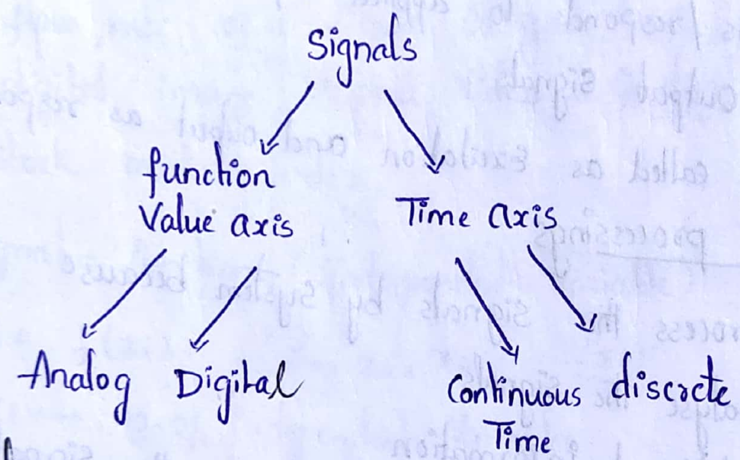
→ Computer or in mbl phone

→ Any GPP like micro processor or specific Digital signal processing Applications

Classifications of signals

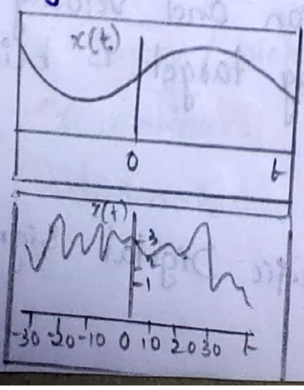
Signals can be classified as

1) Continuous Time Signals (CTS) & Discrete time Signals (DTS)
2) Analog and Digital signals
3) Even and odd signals
4) periodic and non-periodic Signals
5) Energy and power signals
6) Deterministic and Random Signals



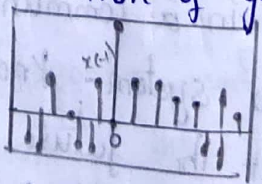
Continuos Time Signals (CTS)

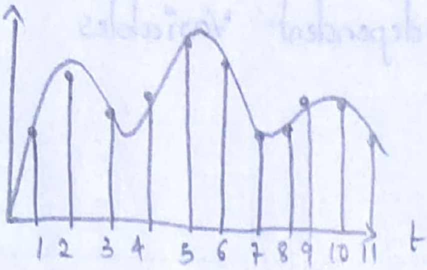
- x(t) Varies Continuosly with respect to independent Variable
- Most of the signals in the physical world are CT Signals
- Voltage & current, presure, temperature Velocity etc



Discrete Time Signals (DTS)

- x(n) n→integer time Varies discretely
- Population of the n^{th} generation of certain Species
- stock market listings, Company production of goods



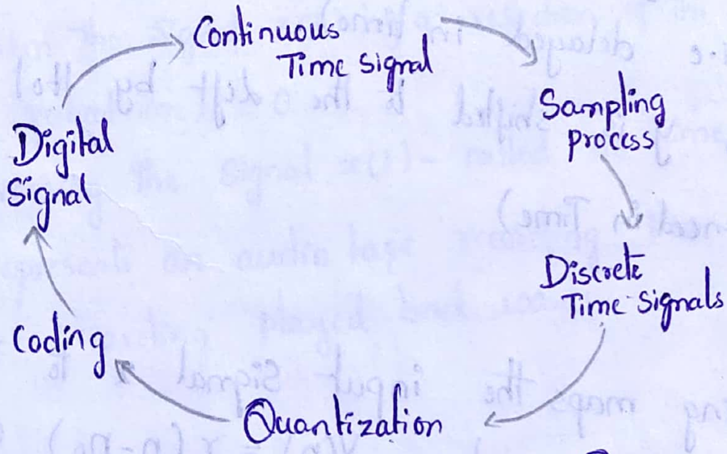


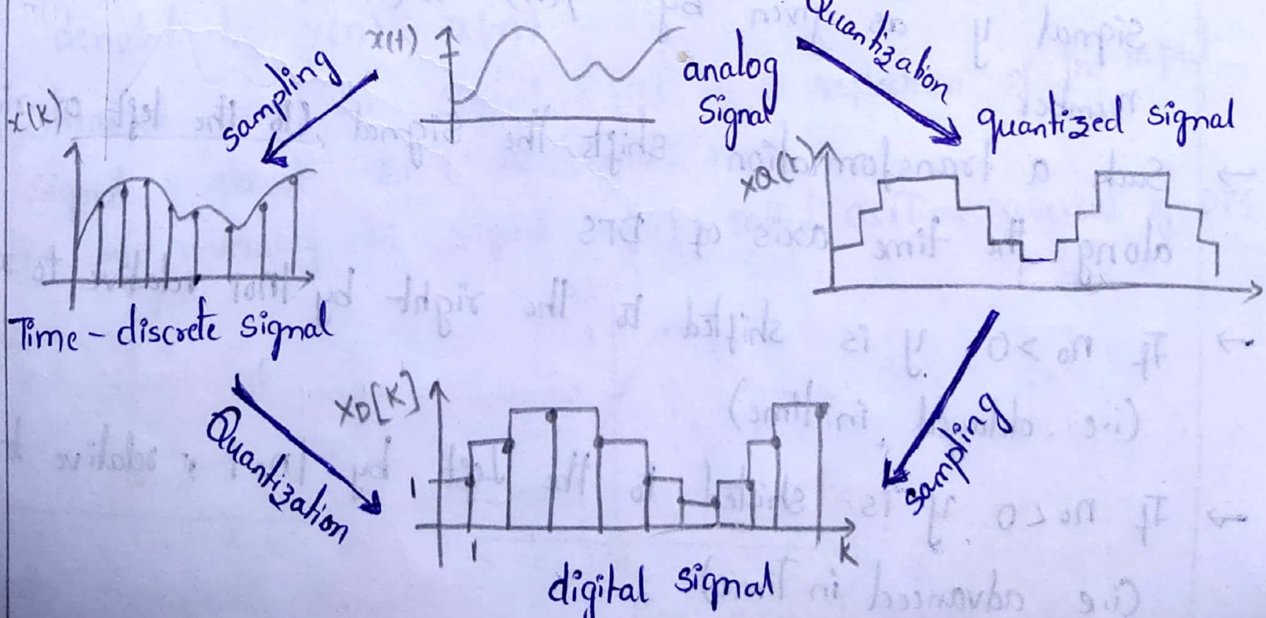
Analog and digital signals

Analog signal If a Continuous time signal $x(t)$ can take on any Value in the Continuous interval $(-\infty$ to $\infty)$

Digital signal: If a discrete time signal $x(n)$ can take Only a finite number of distinct values

Steps to convert Continuous Time Signal to digital Signal





Transformation of the independent Variables

- Time shifting
- Time reversal
- Time scaling

Time shifting:

→ Time shifting (also called translation) maps the input Signal x to the Output Signal y as given by $y(t) = x(t - t_0)$ where t_0 is real number

→ Such a transformation shifts the signals (to the left or right) along the time axis of CTS

→ If $t_0 > 0$ y is shifted to the right by $|t_0|$ relative to x (i.e delayed in time)

→ If $t_0 < 0$ y is shifted to the left by $|t_0|$ relative to x (i.e advanced in Time)

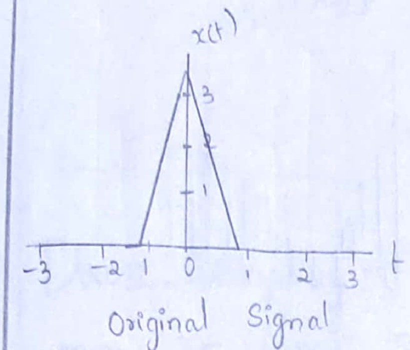
DTS

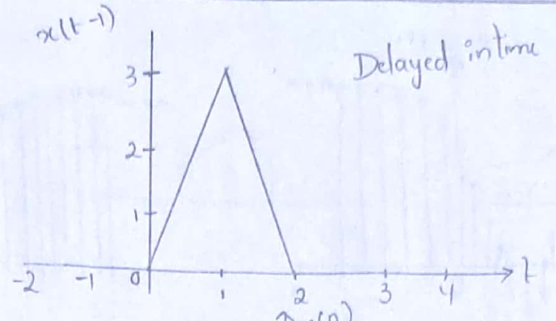
→ Time shifting maps the input Signal x to the output Signal y as given by $y(n) = x(n - n_0)$ where n_0 is real number

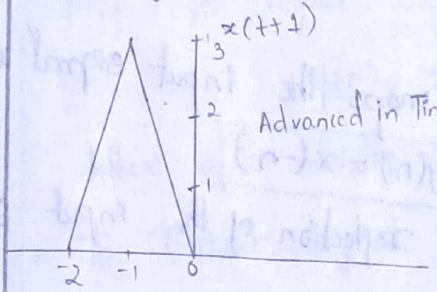
→ Such a transformation shifts the Signal (to the left or right) along the time axis of DTS

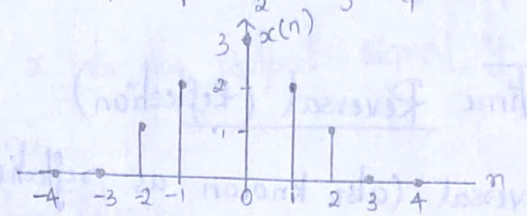
→ If $n_0 > 0$ y is shifted to the right by $|n_0|$ relative to x (i.e delayed in time)

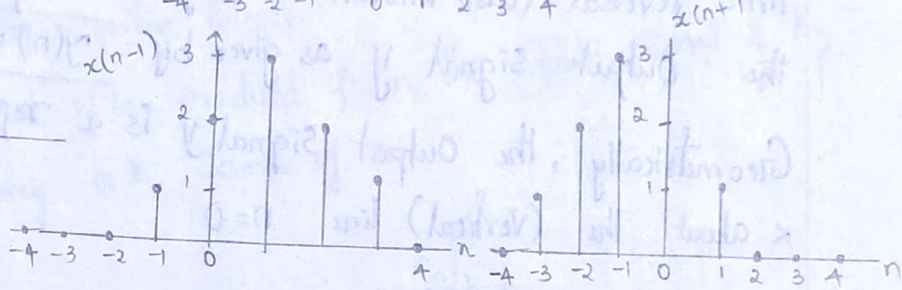
→ If $n_0 < 0$ y is shifted to the left by $|n_0|$, relative to x (i.e advanced in Time)











Time Reversal :-

Time Reversal : denoted by $y(t) = x(-t)$

→ Obtained from the Signal $x(t)$ by a reflection of the input Signal x about the (Vertical line) $t = 0$ called as Time Reversal of CTS

→ i.e by reversing the Signal $x(t)$ - called as Time Reversal of CTS

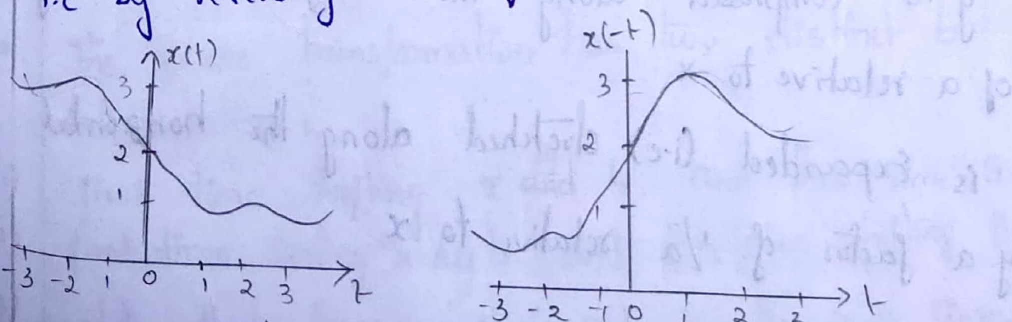
→ If $x(t)$ represents an audio tape recording then $x(-t)$ is the same tape recording played back words

DTS

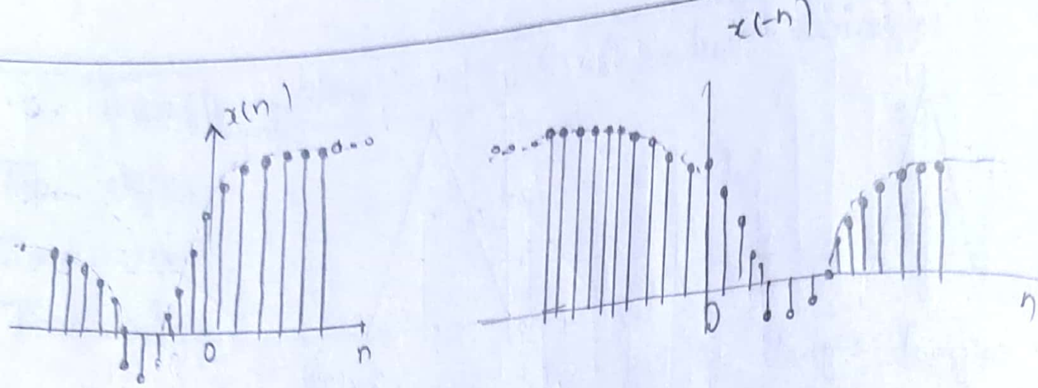
→ denoted by $y(n) = x[-n]$

→ Obtained from the Signal $x[n]$ by a reflection of the input Signal x about the (Vertical line) $n = 0$

→ i.e by reversing the Signal $x[n]$ - called as Time reversal of DTS



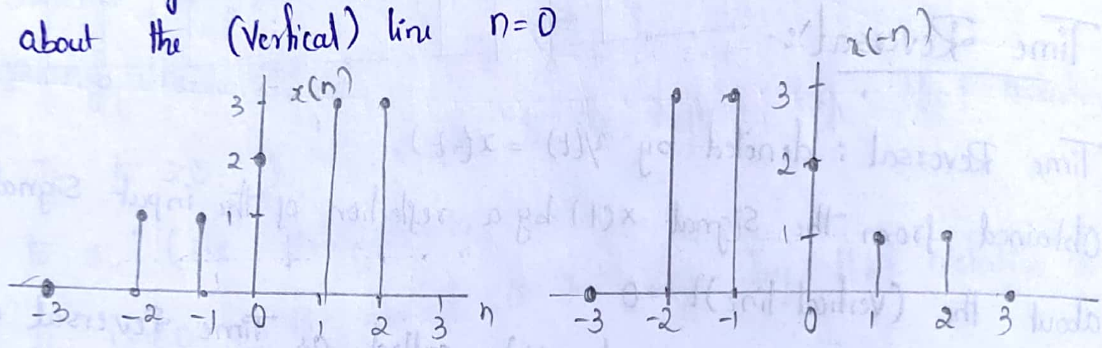
Original Signal   Time reversed signal



Time Reversal (Reflection)

Time reversal (also known as reflection) maps the input signal x to the output signal y as given by $\boxed{y(n) = x(-n)}$

Geometrically, the output signal y is a reflection of the input signal x about the (vertical) line $n=0$



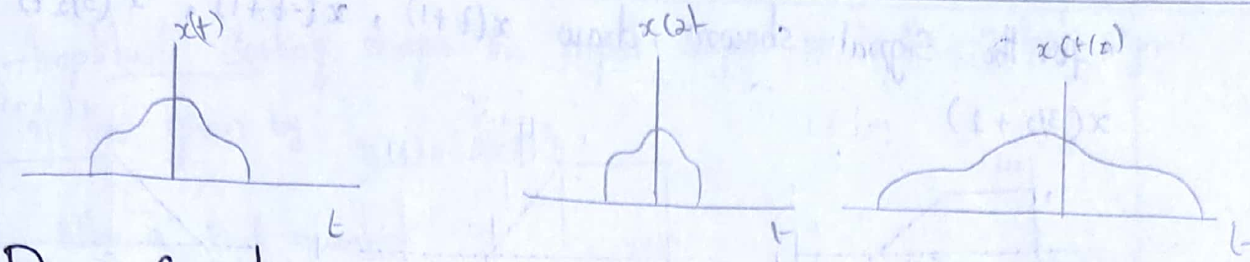
Time scaling

→ Let the three signals $x(t)$, $x(2t)$ and $x(t/2)$ which are linearly related to input signal x to the output signal y as given by $y(t) = x(at)$, where a is a strictly positive real number.

→ Such a transformation is associated with a compression / expansion along the time axis

→ if $a > 1$ y is compressed along the horizontal axis by a factor of a relative to x

→ if $a < 1$ is expanded (i.e) stretched along the horizontal axis by a factor of $1/a$ relative to x



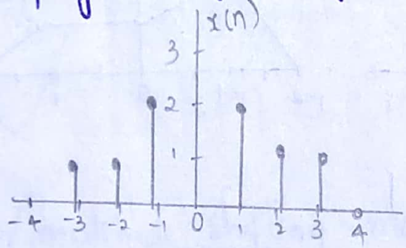
Down Sampling :-

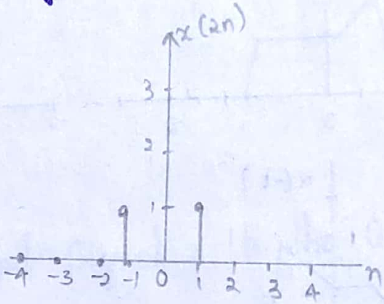
maps the input signal x to the output signal y as given by

$$y(n) = x(an)$$

Where a is strictly positive integer

- The Output Signal y is produced from the input signal x by Keeping Only Every ath Sample of x





Time Scaling

→ Consider a transformation $y(t) = x(at - b)$ Where a and b are real numbers and a not Equal to Zero

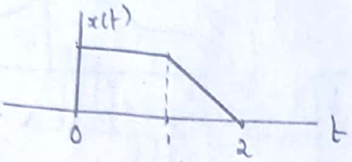
→ the above transformation can be shown to be Combination of a time Scaling Operation and Time shifting Operation

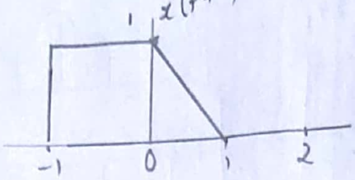
→ Since time Scaling and time shifting do not commute, we must particularly careful about the Order in which these transformations are applied

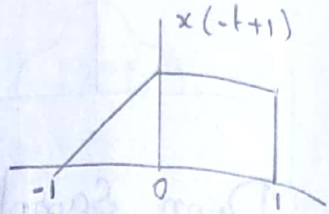
- The above transformation has two distinct but Equivalent interpretations :

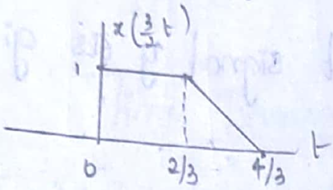
1) first time shifting x by b, and then Time Scaling the result by a
2) first time Scaling x by b and then time shifting the result by b/a
3) Note that time shift is not by the Same amount in both cases

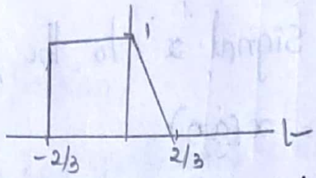
- For the Signal shown, draw $x(t+1)$, $x(-t+1)$, $x(3/2\,t)$ and $x(3/2+1)$



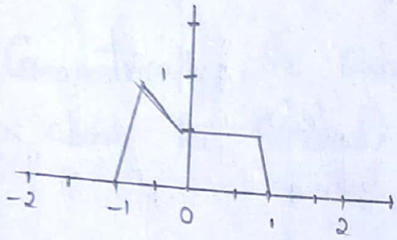


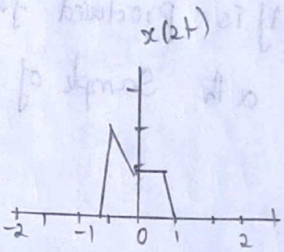


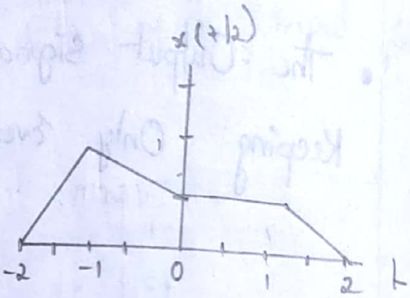


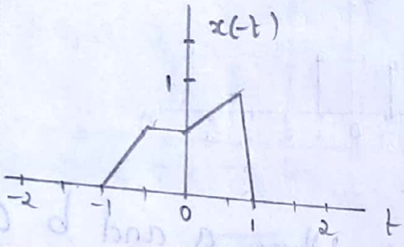


Time Scaling (Dilation/ Reflection Examples)



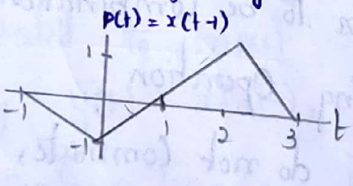


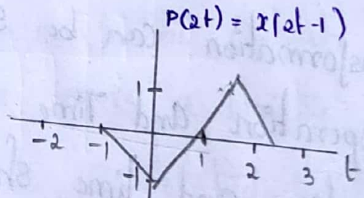




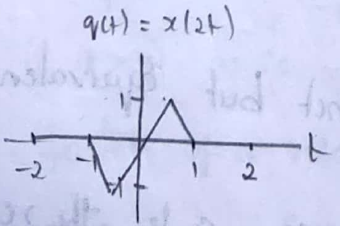
Combined Time scaling and Time shifting Examble

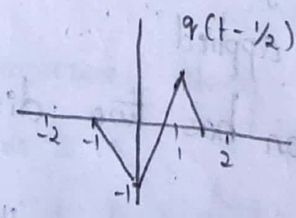
Time shift by 1 and then time Scale 2





Time Scale by 2 and then shift by $\frac{1}{2}$



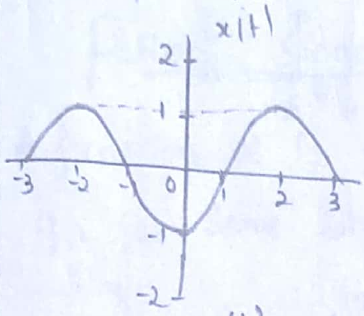


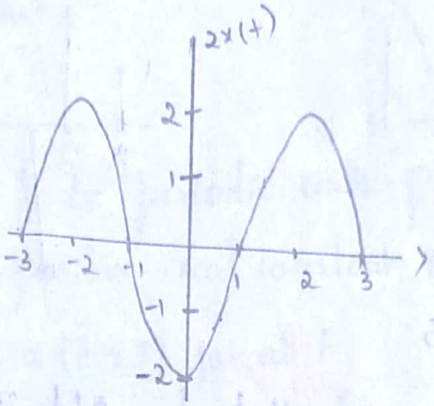
$q(t-\frac{1}{2}) = x(2(t-1)/2)$

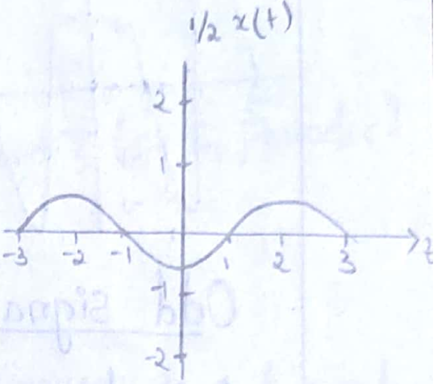
$= x(2t-t)$

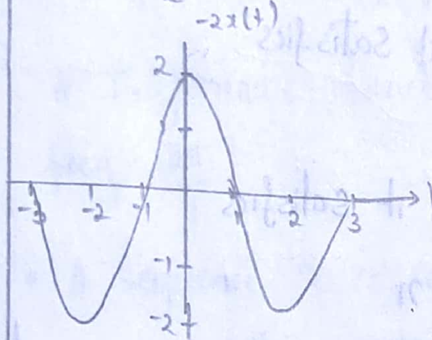
Amplitude Scaling maps the input Signal x to the Output signal y as given by $y(t) = ax(t)$

Where a Real number





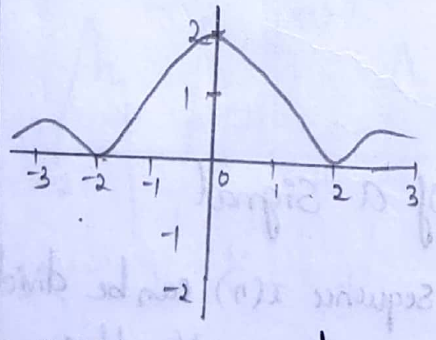


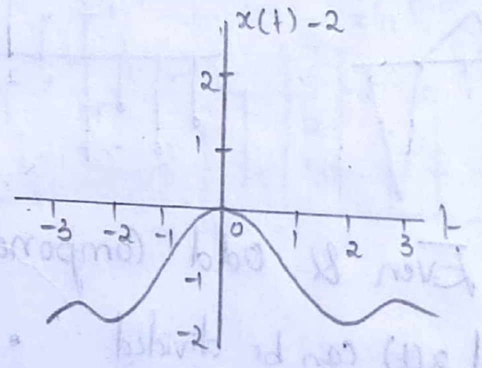


Amplitude shifting : maps the input Signal x to the Output Signal y as given by $y(t) = x(t) + b$

where b is real number

Geometrically amplitude shifting adds Vertical displacement to x





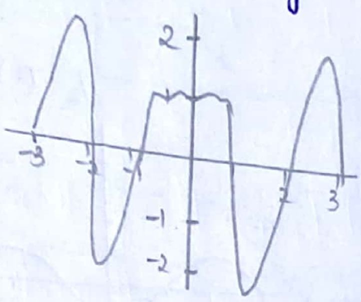
Even Signals

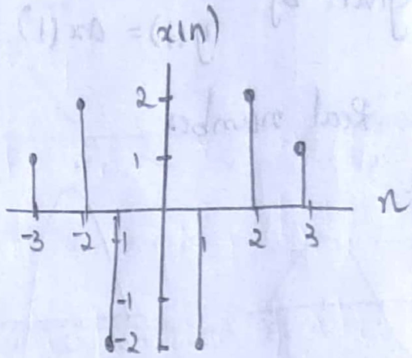
- A function x is Said to be Even if it Satisfies

$$x(t) = x(-t) \text{ for all } t$$

- A sequence x is Said to be Even if it Satisfies

$$x(n) = x(-n) \text{ for all } n$$

- Geometrically the graph of an Even signal is Symmetrical about the origin.
- 



Odd signals

A function x is said to be Odd if it satisfies

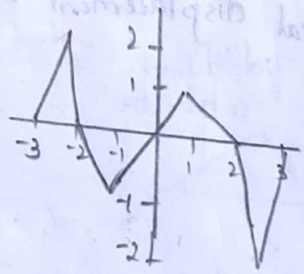
$$x(t) = -x(-t) \quad \text{for all } t$$

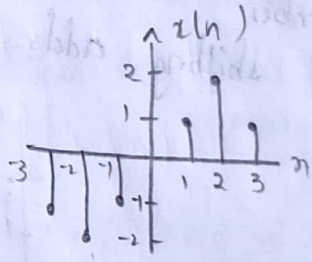
A sequence x is said to be Odd if it satisfies

$$x(n) = -x(-n) \quad \text{for all } n$$

Geometrically the graph of an Odd Signal is anti symmetric about the Origin.

→ An Odd signal x must be such that $x(0) = 0$





Even & Odd Component of a Signal

- Any signal $x(t)$ can be divided into Even component & odd component
- i.e $x(t) = x_e(t) + x_o(t)$ ---- (1)

replacing t by $-t$ in Eqn (1)

$x(-t) = x_e(-t) + x_o(-t)$

$x(-t) = x_e(t) - x_o(t)$ --- (2)

Adding (1) & (2)

$x(t) + \cancel{x}(-t) = 2x_e(t)$

- Any Sequence $x(n)$ can be divided into its Even component & odd Component

i.e $x(n) = x_e(n) + x_o(n)$ -- 1

replacing n by $-n$ in Eq 1

$x(-n) = x_e(-n) + x_o(-n)$

$x(-n) = x_e(n) - x_o(n)$ --- (2)

Adding (1) & (2)

$x(n) + x(-n) = 2x_e(n)$

$x_e(t) = 0.5\,[x(t) + x(-t)]$ = Even component

Similarly (1) - (2)

$x_o(t) = 0.5\,[x(t) - x(-t)]$ odd component

$x_e(n) = 0.5\,[x(n) + x(-n)]$ Even Component

Similarly after (1) - (2)

$x_o(n) = 0.5\,[x(n) - x(-n)]$ odd component

Periodic Signals

A function x is said to be periodic with period T (or T-Periodic) if, for some strictly-positive real constant T

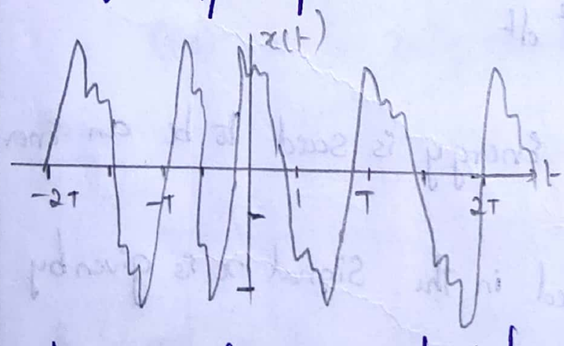
$$x(t) = x(t+T) \text{ for all } t$$

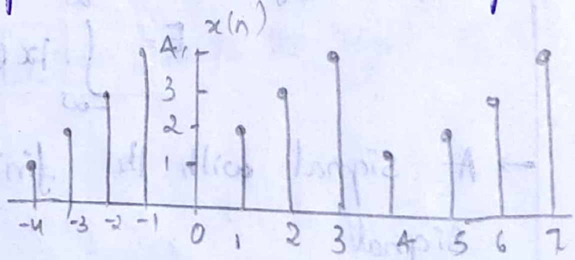
A T-periodic function x is said to have frequency $\frac{1}{T}$ and angular freq $\frac{2\pi}{T}$

- A sequence x is said to be periodic with period N if, for some strictly positive integer constant N

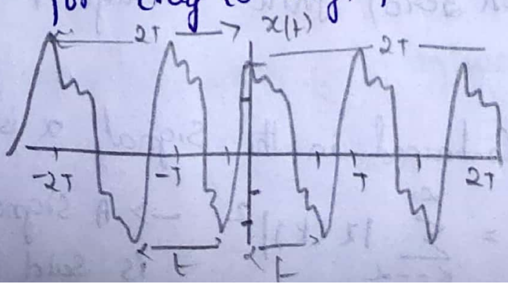
$$x(n) = x(n+N) \text{ for all } n$$

- A N-periodic sequence x is said to have frequency $\frac{1}{N}$ and angular frequency $\frac{2\pi}{n}$
- A function/sequence that is not periodic is said to be aperiodic





The period of a periodic signal is not Unique. That is a signal that is periodic with period T is also periodic with period kT for every (strictly) positive integer k



The smallest period with which a Signal is periodic is called the fundamental period and its corresponding frequency is cally called fundamental frequency.

Sum of period functions: Let x_1 & x_2 be periodic functions with fundamental periods T_1 & T_2 Respectively then the sum $y = x_1 + x_2$ is a periodic function if and only if the ratio T_1/T_2 is a rational number (i.e) the quotient of two integers). Suppose that $T_1/T_2 = q/r$ wher q & r are the integers and co prime (i.e have no common factor) then the fundamental period of y is rT_1 (or Equivalently, qT_2 Since $rT_1 = qT_2$)

Note that rT_1 is simply the Least Common multiple (LCM) of T_1 & T_2

Signal Energy and power

→ The Energy E contained in the signal x is given by

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

→ A Signal with the finite Energy is said to be an Energy signal

→ The Average power P contained in the Signal x is given by

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt$$

→ A signal with (non zero) finite average power is said to be power Signal

→ The Energy E Contained in the Signal x is given by

$$E = \sum_{k=-\infty}^{\infty} |x(k)|^2$$ → A signal with finite Energy is said to be an Energy signal

S.No	Deterministic Signal	Random Signals
1)	Deterministic Signals can be represented or described by a mathematical equation or lookup table	Random signals that cannot be represented or described by a mathematical equation or lookup table
(2)	Deterministic signals are preferable because for analysis & processing of signals we can use mathematical model of the signal	No preferable. The Random signals can be described with the help of their statistical properties
3)	The value of deterministic signal can be evaluated at time without certainly	The value of the random signal cannot be evaluated at any instant of time
Example	Sine or Exponential wave	Noise Signal or Speech Signal

Elementary Signals

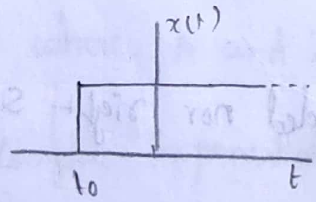
Right Sided signals:-

→ A Signal x is said to be right Sided if for some (finite) real constant t_0, the

$$x(t) = 0 \quad \text{for all } t < t_0$$

(i.e) x is Only potentially non zero to the right of t_0)

Ex:-



A Signal x is said to be Causal if

$$x(t) = 0 \quad \text{for all } t < 0$$

A causal signal is a Special case of right sided signal

A causal signal is not be Confused with a causal System.

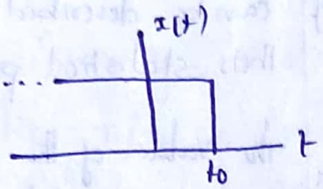
Left Sided Signals

A Signal x is said to be left sided if for some (finite) real constant t_0 the

$$x(t) = 0 \quad \text{for all } t > t_0$$

(i.e. x is only potentially non zero to the left of t_0)

Ex



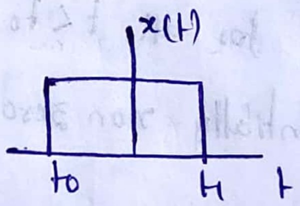
Similarly a signal x is said to be anti causal if

$$x(t) = 0 \quad \text{for all } t > 0$$

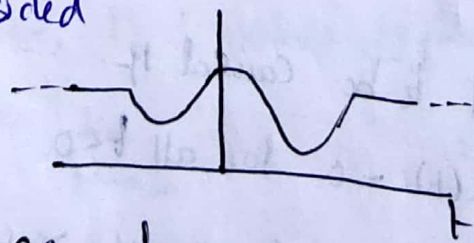
→ An anticausal signal is a special case of a left sided signal

Finite Duration and Two Sided Signals

• A Signal that is both left sided and right sided is said to be finite duration (or time limited)



→ A signal that is neither left sided nor right sided is said to be two sided



Bounded Signals

→ A Signal x is said to be bounded if there exists some (finite) positive real constant A such that

$|x(t)| \leq A$ for all t

(i.e) $x(t)$ is finite for all t)

Ex:- Sine & Cosine wave functions

Non bounded Ex:- tan functions and any non constant polynomial function.

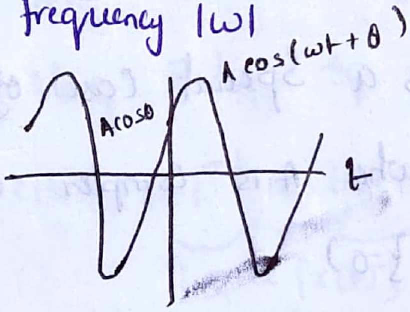
Real Sinosoids

→ A real sinusoid is a function of the form

$$x(t) = A \cos(\omega t + \theta)$$

where A, ω and θ are real constants

→ such a function is periodic with fundamental period $T = \frac{2\pi}{|\omega|}$ & fundament frequency $|\omega|$



→ A (CT) complex exponential is a function of the form

$$x(t) = A e^{\lambda t}$$

where A and λ are complex constants

→ A complex exponential can exhibit one of the number of distinct modes of behaviour depending on the values of its parameters A & λ.

→ For example as special cases complex exponentials include real exponentials and complex sinusoids.

Real Exponentials

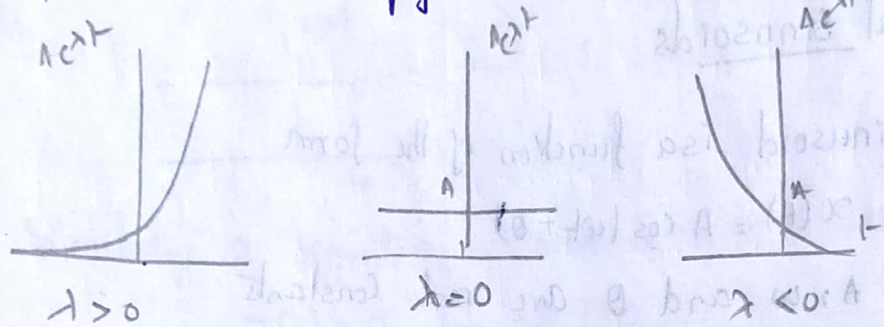
→ A real exponential is a special case of a complex exponential $x(t) = A e^{\lambda t}$ when A & λ are restricted to be real numbers

→ A real exponential can exhibit one of three distinct modes of behaviour depending on the value of λ

If $\lambda > 0$ $x(t)$ increases exponentially as t increases (growing exponential)

If $\lambda < 0$ $x(t)$ decrease exponentially as t increases (decaying)

If $\lambda = 0$ $x(t)$ simply equals the constant A



Complex Sinusoids

A complex sinusoid is a special case of a complex exponential $x(t) = Ae^{\lambda t}$ where A is complex and λ is purely imaginary (i.e) $Re\{\lambda\} = 0$)

→ that is (CT) complex sinusoidal is a function of the form

$$x(t) = Ae^{j\omega t}$$

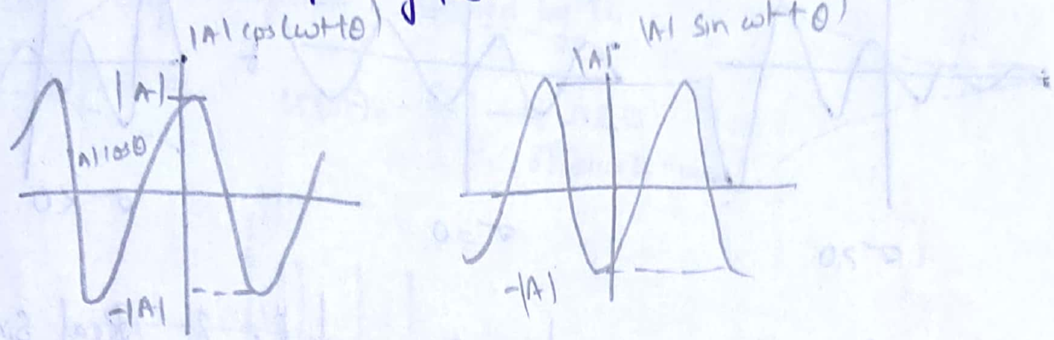
where A is complex & ω is real

→ By expressing A in polar form as $A = |A|e^{j\theta}$ where θ is real and using Euler's relation $x(t)$ as

$$x(t) = \underbrace{|A| \cos(\omega t + \theta)}_{\text{Real }(x(t))} + \underbrace{j|A| \sin(\omega t + \theta)}_{\text{img } x(t)}$$

$Re\{x\}$ & $Im\{x\}$ are same exp except for a time shift

Also x is a periodic with fundamental period $T = \frac{2\pi}{|\omega|}$ & fundamental frequency $|\omega|$



General Complex Exponentials

→ In the most general case of a Complex Exponential $x(t) = Ae^{\lambda t}$ A & λ are both complex

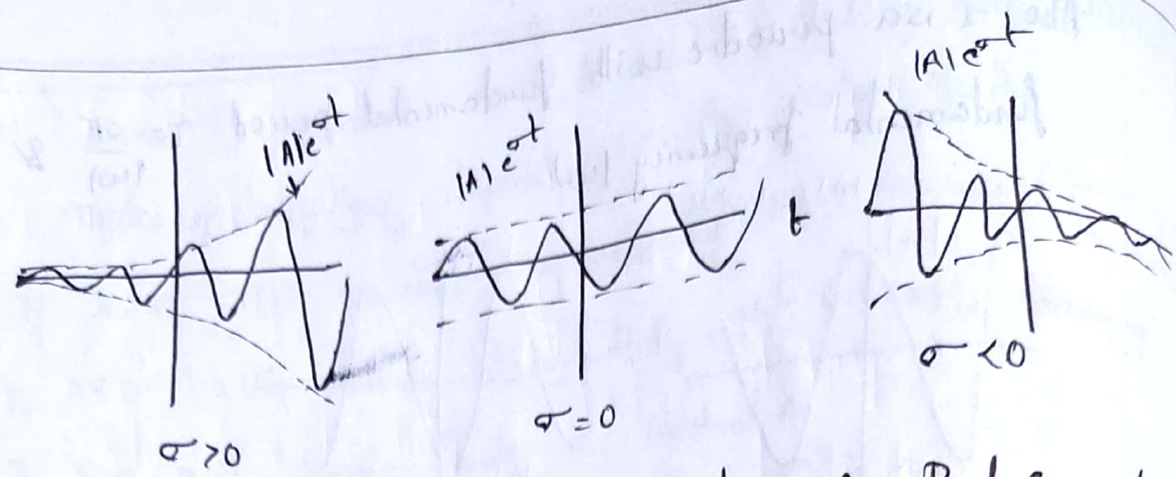
Let $A = |A| e^{j\theta}$ & $\lambda = \sigma + j\omega$ where θ σ and ω are real) & using Euler's eq

$$x(t) = \underbrace{|A|^{\sigma t} \cos(\omega t + \theta)}_{Re[x(t)]} + \underbrace{j|A| e^{\sigma t} \sin(\omega t + \theta)}_{Im\{x(t)\}}$$

→ Thus $Re\{x\}$ and $Im\{x\}$ are each the product of real ~~and~~ exponential & real sinusoidal

→ One of three distinct modes of $x(t)$ depending on σ

- $\sigma = 0 \rightarrow Re\{x\}$ & $Im\{x\}$ Real Sinusoids
- $\sigma > 0 \rightarrow Re\{x\}$ & $Im\{x\}$ are each the product of real sinusoidal and growing real exponential
- $\sigma < 0$ $Re\{x\}$ & $Im\{x\}$ are each the product of a real sinusoid and decaying real exponential



Relation ship between complex Exponentials & Real Sinusoids

form Euler's formula a Complex Sinusoido can be Expressed as the sum of two Sinusoid as

$$A e^{j\omega t} = A\cos \omega t + jA \sin \omega t$$

→ Mover Over a real sinusoidal can be Expressed as the sum of two Complex Sinusoids Using the identities

$$A\cos(\omega t+\theta) = \frac{A}{2}\left[e^{j(\omega t+\theta)} + e^{-j(\omega t+\theta)}\right] \;\&$$

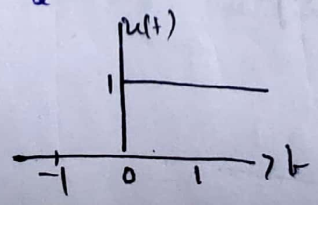
$$A\sin(\omega t+\theta) = \frac{A}{2j}\left[e^{j(\omega t+\theta)} - e^{-j(\omega t+\theta)}\right]$$

Unit step function

A Unit step function (Heaviside function) denoted by u difinded as

$$u(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

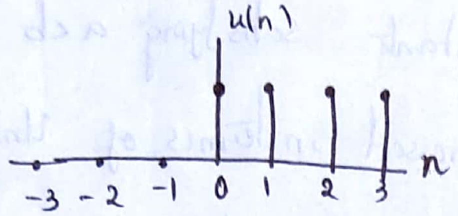
Due to the manner in which u is Used in practice, the actual value of u(0) is Unimportant some time value 0 and $\frac{1}{2}$ are also Used for u(0)



Unit-step sequence

Unit step sequence denoted by u

defined as $u(n) = \begin{cases} 1 & \text{if } n \geq 0 \\ 0 & \text{otherwise} \end{cases}$

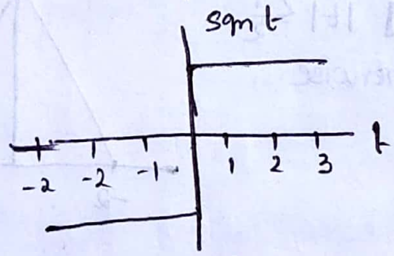


Signum function

Signum function denoted sgn

defined $\operatorname{sgn} t \begin{cases} 1 & \text{if } t > 0 \\ 0 & t = 0 \\ -1 & t < 0 \end{cases}$

from its defination One can see the signum function simply computes the sign of a number

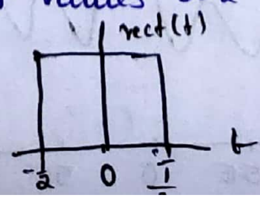


Rectangular function

The Rectangular function (Unit rectangular pulse) denoted rect

$\operatorname{rect}(t) \begin{cases} 1 & \text{if } -\frac{1}{2} \leq t < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$

→ Due to the manner in which the rect function is used in practice the actual value rect(t) at $t = \pm\frac{1}{2}$ is Unimportant Some times different values are Used from Those specified above



Uni Rectangular - Pulses

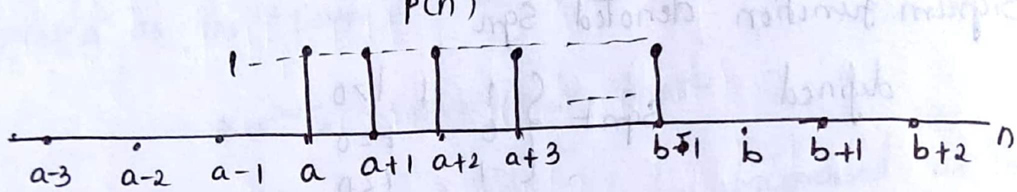
A Uni Rectangular pulse is a Sequence of the form

$$p(n) = \begin{cases} 1 & \text{if } a \le n < b \\ 0 & \text{otherwise} \end{cases}$$

Where a & b Integer constants satisfying a<b

Such a sequence can be Expressed in terms of Unit step sequency as

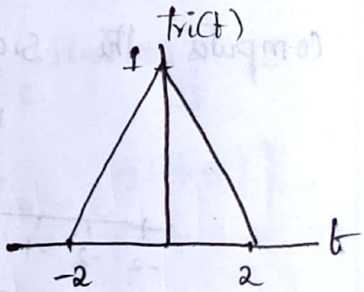
$$p(n) = u(n-a) - u(n-b)$$



Triangular function (Unit- Triangular pulse function)

denoted tri is defined as

$$tri(t) = \begin{cases} 1 & 2|t|\ |t| \le \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$



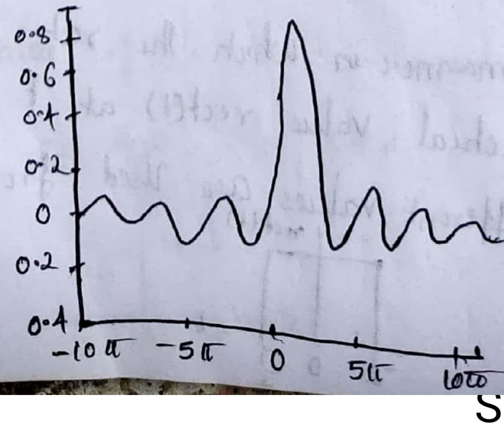
Cardinal Sine function:

The cardinal Sine function denoted sinc is given by

$$\mathrm{sinc}(t) = \frac{\sin t}{t}$$

By l Hopital's rule sinc 0 = 1

Note That Oscillations in sinc(t) do not die out for finite t



Unit impulse function

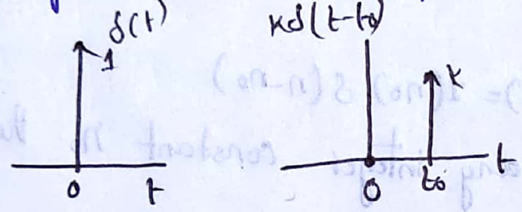
The Unit impulse function (Dirac delta function or delta function) denoted δ is defined by the following properties

$$\delta(t) = 0 \quad \text{for } t \neq 0 \text{ and}$$

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

→ Technically δ is not a function in the Ordinary Sense Rather it is what is known as a generalized function. Consequently, the δ function Sometimes behaves in Unusual ways

→ Graphically the delta function represented as



properties of a Unit Impulse function

Equivalence property: for any continuous function x and any real constant t_0

$$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$$

Shifting property: For any continuous function x and any real constant t_0

$$\int_{-\infty}^{\infty} x(t)\,\delta(t-t_0)\, dt = x(t_0)$$

The δ function also has the following properties

$$\delta(t) = \delta(-t) \text{ and}$$

$$\delta(at) = \frac{1}{|a|}\delta(t)$$

where a is a non zero real constant

Unit Impulse Sequence

→ The Unit Impulse Sequence (delta sequence) denoted by δ is defined as

$$\delta(n) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

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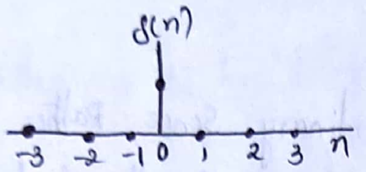
The first order difference of u is δ that is

$$\delta(n) = u(n) - u(n-1)$$

The running sum of δ is u that is

$$u(n) = \sum_{k=-\infty}^{n} \delta(k)$$



properties of the Unit Impulse sequence

→ For any sequence x and any Integer constant n_0 the following identity hold

$$x(n)\,\delta(n-n_0) = x(n_0)\,\delta(n-n_0)$$

→ for any sequence x and any integer constant n_0 the following identity holds

$$\sum_{n=-\infty}^{\infty} x(n)\,\delta(n-n_0) = x(n_0)$$

Trivally the sequence δ is also Even

Representing a Rectangular pulse Using Unit step functions

→ for real constants a and b where $a \le b$ consider a function of the form

$$x(t) = \begin{cases} 1 & \text{if } a \le t < b \\ 0 & \text{otherwise} \end{cases}$$

$x(t)$ is a rectangular pulse of height One with a rising edge at a and falling edge at b)

→ The function x can be Equivalently written as

$$x(t) = u(t-a) - u(t-b)$$

→ Unlike the Original Expression for x this latter Expression for x does not involve multiple cases

→ In Effect by Using Unit step functions we have Collapsed a formula involving multiple cases into a single Expression

Unit I

1) ## Analogy between Vectors and signals

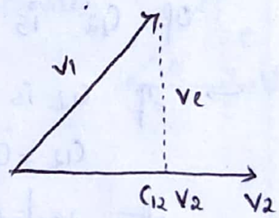
Vector:

1) A Vector is specified by magnitude and direction
2) all Vectors are denoted by boldface type and their magnitudes by Light face type
3) consider two Vectors V_1 and V_2
4) Let the component of V_1 along V_2 be given by $C_{12}V_2$
5) Geometrically the component of Vector V_1 along Vector V_2 is Obtained by drawing a perpendicular from the End of V_1 on the Vector V_2
6) The Vector V_1 can now be expressed in terms of Vector V_2

$$V_1 = C_{12}V_2 + V_e \qquad V_e \rightarrow \text{Error Vector}$$

However this is not the Only way of expressing vector V_1 in terms of Vector V_2

Two infinite alternate possibilities



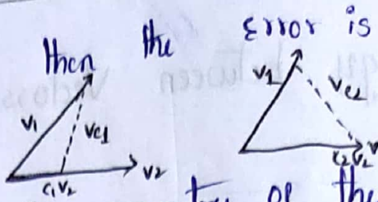
$$V_1 = C_1V_2 + V_{e1}$$
$$V_1 = C_2V_2 + V_{e2}$$

In Each representation V_1 is represented in terms of V_2 plus another vector which will be called the Error Vector

→ If we are asked to approximate the Vector V_1 by a Vector in the direction of V_2 then we represents the error in this approximation

For example: If we approximate the ~~error~~ V_1 by $C_{12}V_2$ then the Error in the approximation is V_e

If V_1 is approximated by $C_1 V_2$ then the error is given by V_{e_1} and so on.



→ It is immedialety evident from the geometry of these figures the error vector is smallest

→ the component of a vector V_1 along the vector V_2 is given by $C_{12} V_2$ where C_{12} is choosen such that the error vector is minimum

→ Let us now interpret physically the component of one vector along another

→ It is clear that larger component of a vector along other vector the more closely do the two vectors resemble each other in their direction and the smaller is the error vector.

→ If the component of a vector V_1 along V_2 is $C_{12} V_2$ then the magnitude of C_{12} is the indication of the similarity of the two vectors

→ If C_{12} is zero then the vector has no component along the other vector and hence the two vectors are mutually perpendicular

$C_{12} = 0$

Such vectors are known as Orthogonal vectors

→ Orthogonal vectors are thus independent vectors If the vectors are orthogonal then the parameter C_{12} is zero

For convienience we define the dot product of two vectors A and B

$$A \cdot B = AB \cos\theta$$

where θ is the angle between vectors A and B as If

It follows from the definition that

$$A \cdot B = B \cdot A$$

The component of A along B = $A \cos\theta = \frac{A \cdot B}{B}$

The component of B along A = $B \cos\theta = \frac{A \cdot B}{A}$

Similarly

the component of V_1 along $V_2 = \frac{V_1 \cdot V_2}{V_2}$

therefore $C_{12} = \frac{V_1 \cdot V_2}{V_2^2} = \frac{V_1 \cdot V_2}{V_2 \cdot V_2}$

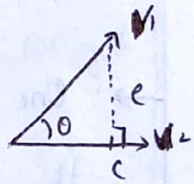
Note that if V_1 & V_2 are orthogonal then

$$V_1 \cdot V_2 = 0$$

$$C_{12} = 0$$

→ Signals can be represented in terms of orthogonal functions. All signals basically vectors

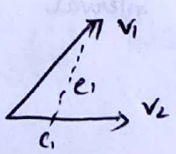
A vector can be represent in terms of its co-ordinates

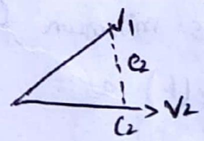


$$V_1 = C V_2 + e$$

V_1 & V_2 are vectors

$C \rightarrow$ component of V_1 along with V_2





$V_1 = C_1 V_2 + e_1$ $\qquad$ $V_1 = C_2 V_2 + e_2$

$$C = \frac{V_1 V_2 \cos\theta}{V_2^2}$$

e_1, e_2 are greater than e

$$C = \frac{V_1 \cdot V_2}{V_2^2} \quad \text{— ④}$$

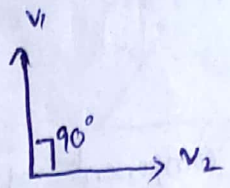
$V_1 \cdot V_2 = |V_1||V_2| \cos\theta \longrightarrow ②$

$C|V_2| = |V_1| \cos\theta$

multiply V_2 on both sides

$C V_2^2 = V_1 V_2 \cos\theta$

If Vectors are perpendicular



$$C = \frac{V_1 V_2 \cos\theta}{V_2^2} = \frac{V_1 V_2 \cos 90^\circ}{V_2^2}$$

$C = 0 \longrightarrow$ no projection

V_1 & V_2 are mutually perpendicular / mutually orthogonal

there is no component

Signal

1) The concept of vector comparison and orthogonality can be extended to signals

Let us consider two signals $f_1(t)$ and $f_2(t)$

Suppose we want to approximate $f_1(t)$ in terms of $f_2(t)$ over a ~~lest~~ certain interval $(t_1 < t < t_2) \rightarrow f_1(t) \cong C_{12} f_2(t)$

$$f_e(t) = f_1(t) - C_{12} f_2(t)$$

→ One possible criterion for minimizing the error $f_e(t)$ over the interval t_1 & t_2 is to minimize the average value of $f_e(t)$ over this interval, that is to minimize $\frac{1}{(t_2 - t_1)} \int_{t_1}^{t_2} [f_1(t) - C_{12} f_2(t)] dt$

→ ~~How~~ we must find C_{12} such that the error between actual function and approximated function is minimum over the interval $(t_1 < t < t_2)$

Let us define error function $f_e(t)$ as

$$f_e(t) = f_1(t) - C_{12} f_2(t)$$

→ However this criterion is inadequate because there can be large positive and negative errors present that may can one another in the process of averaging and give the false indication of error is zero

For Example

→ if we approximate a function Sin t with a null function $f(t)=0$ over an interval 0 to 2π the average Error will be zero. Indicating wrongly that sin t can be approximated to zero over the interval 0 to 2π without any error

→ this situation can be corrected if we choose to minimize the average value (or the mean) of the square of the error instead of the error itself. Let us designate the average of $f_e^2(t)$ by ε

$$\varepsilon = \frac{1}{(t_2-t_1)}\int_{t_1}^{t_2} f_e^2(t)\,dt = \frac{1}{(t_2-t_1)}\int_{t_1}^{t_2}\left[f_1(t) - C_{12} f_2(t)\right]^2 dt$$

→ To find the value of C_{12} which will minimize ε we must have

$$\frac{d\varepsilon}{dC_{12}} = 0$$

that is

$$\frac{d}{dC_{12}}\left\{\frac{1}{(t_2-t_1)}\int_{t_1}^{t_2}\left[f_1(t) - C_{12} f_2(t)\right]^2 dt\right\} = 0$$

changing the order of integration and differentiation we get

$$\frac{1}{t_2-t_1}\left[\int_{t_1}^{t_2}\frac{d}{dC_{12}} f_1^2(t)\,dt - 2\int_{t_1}^{t_2} f_1(t)\, f_2(t)\,dt + 2C_{12}\int_{t_1}^{t_2} f_2^2(t)\,dt\right] = 0$$

→ the first integral is obviously zero

$$C_{12} = \frac{\int_{t_1}^{t_2} f_1(t)\, f_2(t)\,dt}{\int_{t_1}^{t_2} f_2^2(t)\,dt}$$

Observe the similarity between Eqns which express C_{12} for vectors

→ But analogy with vectors we say that $f_1(t)$ has a component of wave form $f_2(t)$ and this component has a magnitude C_{12}. If C_{12} vanishes then the signal $f_1(t)$ contains no component of signal $f_2(t)$

we say that the two functions are orthogonal over the interval (t_1, t_2) It therefore follows that the two functions $f_1(t)$ & $f_2(t)$ are orthogonal over an interval $(t_0, t_0 + 2\pi/\omega_0)$ for integral values of n and m.

consider integral I :

$$I = \int_{t_0}^{t_0+2\pi/\omega_0} \sin n\omega_0 t \; \sin m\omega_0 t \; dt$$

$$= \int_{t_0}^{t_0+2\pi/\omega_0} \frac{1}{2}\left[\cos(n-m)\omega_0 t - \cos(n+m)\omega_0 t\right] dt$$

$$= \frac{1}{2\omega_0}\left[\frac{1}{n-m}\sin (n-m)\omega_0 t - \frac{1}{n+m}\sin(n+m)\omega_0 t\right]_{t_0}^{t_0+2\pi/\omega_0}$$

Since n and m are integers $(n-m)$ & $(n+m)$ are also integer In such case the integral I is zero Hence two functions are orthogonal it can be shown that $\sin n\omega_0 t$ & $\cos m\omega_0 t$ are orthogonal functions and $\cos n\omega_0 t$ $\cos m\omega_0 t$ are also mutually orthogonal

<u>Example : 1.1</u> → A rectangular function $f(t)$ is defined as

$$f(t) = \begin{cases} 1 & (0 < t < \pi) \\ -1 & (\pi < t < 2\pi) \end{cases}$$

→ Approximate this function by a wave form $\sin t$ over the interval $(0, 2\pi)$ such that the mean square error is minimum.

<u>Sol</u> The function $f(t)$ will be approximated over the interval $(0, 2\pi)$ as

$f(t) \simeq c_{12} \sin t$

we shall find the optimum value of c_{12} which will minimize the Mean square Error in this approximation

To minimize the mean square Error

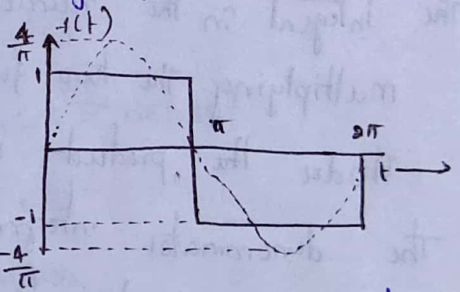
$$c_{12} = \frac{\int_0^{2\pi} f(t) \sin t \, dt}{\int_0^{2\pi} \sin^2 t \, dt}$$

$$= \frac{1}{\pi} \left[\int_0^{\pi} \sin t \, dt + \int_0^{2\pi} -\sin t \, dt \right.$$

$$= \frac{4}{\pi}$$

Thus $f(t) \simeq \frac{4}{\pi} \sin t$

represent the best approximation of $f(t)$ by a function $\sin t$ which will minimize the mean square Error



By analogy with Vectors we may say that the rectangular function $f(t)$ has a component of function $\sin t$ and the magnitude of this component is $\frac{4}{\pi}$.

→ In this case of vectors, Orthogonality implies that one Vector has no component along the other.

→ Similarly a function does not contain any component of the form of the function which is orthogonal to it

→ If we try to approximate a function by its orthogonal function, the Error will be Larger than the original function itself and it is better to approximate a function with a null function $f(t)=0$ rather than with a function Orthogonal to it. Hence the optimum value of $c_{12}=0$ is such a case

Graphical Evaluation of a component of one function in the other.

→ It is possible to evaluate the component of a function in the other function by graphical means,

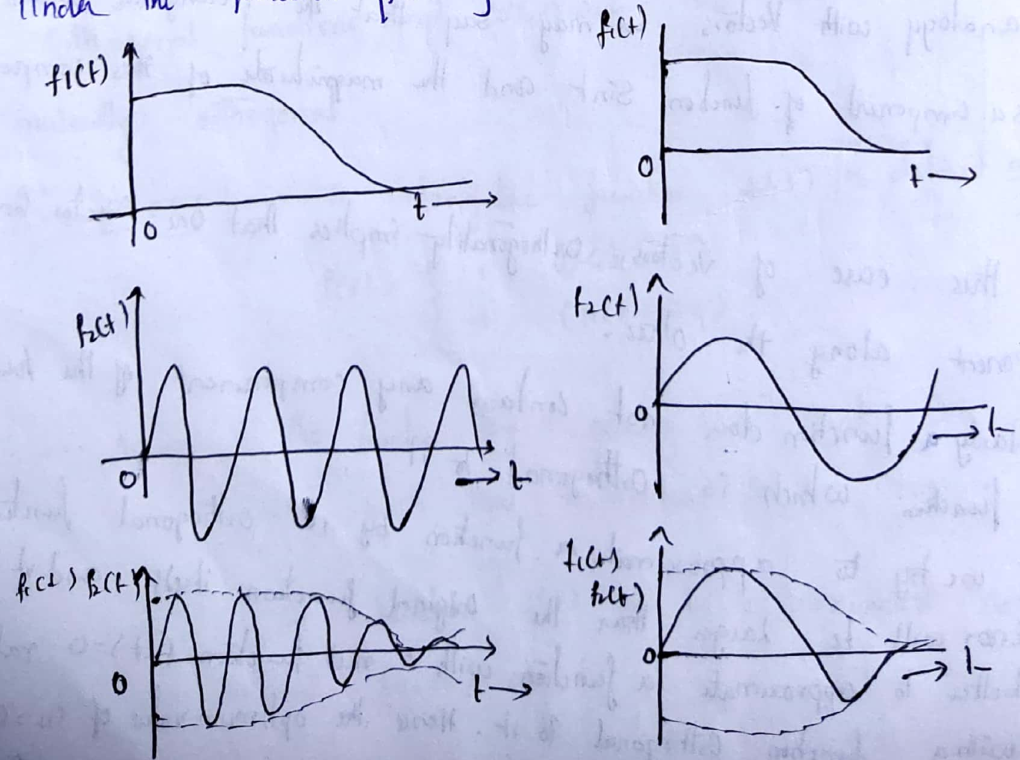
suppose two functions $f_1(t)$ & $f_2(t)$ are known graphically. and it is desired to evaluate the component of wave form ($f_2(t)$) contained in signal $f_1(t)$ over a period $(0, T)$.

We know that this component of function $f_2(t)$ of magnitude C_{12} given by

$$C_{12} = \frac{\int_0^T f_1(t) f_2(t)\, dt}{\int_0^T f_2^2(t)\, dt}$$

The integral in the numerator in this equation can be found by multiplying the two functions and ~~equa~~ Evaluating the area Under the product curve.

The denominator integral can be evaluated by finding the area Under the function $[f_2(t)]^2$ in a similar way.



It is evident that if $f_1(t)$ varies much more slowly than $f_2(t)$ the area under the curve $f_1(t) f_2(t)$ will be very small since the positive and negative area's will be approximately equal and will tend to cancel each other.

→ Hence $f_1(t)$ contains a small component of $f_2(t)$.

→ If however $f_1(t)$ varies at about the same rate as $f_2(t)$, then the area under the product curve $f_1(t) f_2(t)$ will be much larger.

→ hence $f_1(t)$ will contain a large component of function $f_2(t)$.

This result is also intuitively obvious, since if two functions vary at about same rate there must be great deal of similarity between the two functions & hence $f_1(t)$ will contain a large component of the function $f_2(t)$.

→ Consider two signals $f_1(t)$, $f_2(t)$ let $f_1(t)$ can be represented in terms of $f_2(t)$ over the interval $[t_1, t_2]$

$$f_1(t) \cong c_{12} f_2(t)$$

$$f_1(t) = c_{12} f_2(t) + f_e(t)$$

$$f_e(t) = f_1(t) - c_{12} f_2(t) \; ; \quad t_1 < t < t_2$$

minimum $f_e(t)$ gives best approximation

$$\varepsilon = \int_{t_1}^{t_2} f_e^2(t)\, dt$$

mean square value of error

$$\overline{f_e^2(t)} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} f_e^2(t)\, dt$$

$$\varepsilon_r = \int_{t_1}^{t_2} \left[f_1(t) - c_{12} f_2(t) \right]^2 dt$$

The value of c_{12} should be selected in such a way that the error will be min.

This can be obtained by differentiating $f_e(t)$ w.r.t c_{12} and equating to zero

$$\frac{d\varepsilon_r}{dC_{12}} = 0$$

$$\frac{d\varepsilon_r}{dC_{12}} = \frac{d}{C_{12}}\left\{\int_{t_1}^{t_2}\left[f_1(t) - C_{12} f_2(t)\right]^2 dt\right\} = 0$$

$$= \frac{d}{dC_{12}}\int_{t_1}^{t_2} f_1^2(t)\,dt - 2\frac{d}{dC_{12}}\int_{t_1}^{t_2} C_{12} f_1(t) f_2(t)\,dt + \frac{d}{dC_{12}}\int_{t_1}^{t_2} C_{12}^2 f_2^2(t)\,dt$$

$$= 0 - 2\int_{t_1}^{t_2} f_1(t) f_2(t)\,dt + 2C_{12}\int_{t_1}^{t_2} f_2^2(t)\,dt = 0$$

$$= -2\int_{t_1} f_1(t) f_2(t) + 2C_{12}\int_{t_1} f_2(t)\,dt = 0$$

$$2C_{12}\int_{t_1}^{t_2} f_2^2(t)\,dt = 2\int_{t_1}^{t_2} f(t) f_2(t)\,dt$$

$$C_{12} = \frac{\int_{t_1}^{t_2} f_1(t) f_2(t)\,dt}{\int_{t_1}^{t_2} f_2^2(t)\,dt} \longrightarrow (6)$$

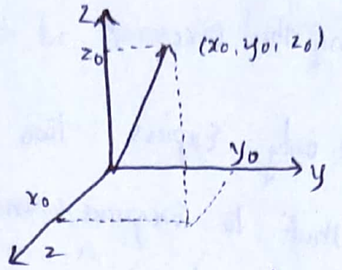
$$C_{12} = \frac{\int_{t_1}^{t_2} f_1(t) f_2(t)\,dt}{\int_{t_1}^{t_2} f_2^2(t)\,dt} \equiv C = \frac{A_1 \cdot A_2}{A_2^2}$$

If $A_1 \cdot A_2 = 0 \rightarrow$ orthogonal vectors

$$\int_{t_1}^{t_2} f_1(t) f_2(t)\,dt = 0 \rightarrow \text{orthogonal functions}$$

Orthogonal Vector Space

→ Let us now consider a three-dimensional vector space described by rectangular co-ordinates.



We shall designate a vector of unit length along the x axis by a_x. Similarly unit vectors along the y and z axes will be designated by a_y and a_z respectively.

→ Since the magnitude of vectors a_x, a_y and a_z is unity. It follows that for any general vector A

The component of A along the x-axis $= A \cdot a_x$

The component of A along the y-axis $= A \cdot a_y$

The component of A along the z-axis $= A \cdot a_z$

A vector A drawn from the origin to a general point (x_0, y_0, z_0) in space has components x_0, y_0 and z_0 along the x, y and z axes respectively.

→ We can express this vector A in terms of its component along the three mutually perpendicular axes

$$A = x_0 a_x + y_0 a_y + z_0 a_z$$

→ Any vector in the space can be expressed in terms of the three vectors a_x, a_y and a_z

Since the three vectors a_x, a_y and a_z are mutually perpendicular it follows

$$a_x \cdot a_y = a_y \cdot a_z = a_z \cdot a_x = 0$$

$$a_x \cdot a_x = a_y \cdot a_y = a_z \cdot a_z = 1$$

The properties of the three vectors as expressed

$$a_m \cdot a_n = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

where m and n can assume any value x, y and z

→ If the co-ordinate system has only two axes x and y then the system is in adequate to express a general vector A in terms of the component along these axes.

→ This system can only express two components of vector A. therefore it is necessary that to express any general vector A in terms of its co-ordinate components. the system of co-ordinates must be complete. In this case there must be three co-ordinate axes.

→ A single straight line represents a one-dimensional space. A single plane represents two dimensional space; and our universe in general has three dimensional space.

→ we may extend our concepts as developed here to a general n-dimensional space. such a physical space of course does not exist in nature.

<u>For Example</u> A linear equation in n independent variable may be viewed as a vector expressed in terms of its components along n mutual perpendicular co-ordinates.

If unit vectors along these n mutually perpendicular co-ordinates are designated as $x_1, x_2 \ldots x_n$ and a general vector A in this n-dimensional space has component $c_1, c_2, c_3 \ldots c_n$ respectively along these n co-ordinates then

$$A = c_1x_1 + c_2x_2 + c_3x_3 + \ldots c_nx_n$$

All the vectors $x_1, x_2 \ldots x_n$ are mutually orthogonal and the set must be complete in order of any general vector A to be represented by

Example :-

The condition of orthogonality implies that the dot product of any two vectors x_n and x_m must be zero and the dot product of any vector with itself must be unity.

$$x_m \cdot x_n = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

The constants $c_1, c_2, c_3 \dots c_n$ in Eq represent the magnitudes of the components of A along the vectors $x_1, x_2, x_3 \dots x_n$ respectively

A follow $\quad c_r = A \cdot x_r$

This result can also be obtained by taking the dot product of both sides with vector x_r we have

$$A \cdot x_r = c_1 x_1 \cdot x_r + c_2 x_2 \cdot x_r + \dots + c_r x_r x_r + \dots \quad c_n x_n x_r$$

it follows that the terms of the form $c_j x_j \cdot x_r \ (j \neq r)$

on the right hand side are zero therefore

$$A \cdot x_r = c_r x_r \cdot x_r = c_r$$

→ we call that the set of vectors $(x_1, x_2 \dots x_n)$ are orthogonal vector space

In general the product $x_m \cdot x_n$ can be some constant k_m instead of unity when k_m is unity the set is called normalized orthogonal set or orthogonal normal vector space therefore in general for orthogonal vector space $\{x_r\}$ ---- $(r = 1, 2, 3 \dots n)$ we have

$$x_m \cdot x_n = \begin{cases} 0 & m \neq n \\ k_m & m = n \end{cases}$$

for orthogonal vector space $\quad A \cdot x_r = c_r x_r \cdot x_r = c_r k_r$

$$c_r = \frac{A \cdot x_r}{k_r}$$

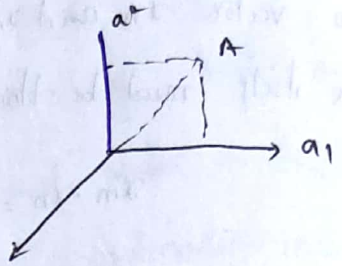
we shall now summarize the results of our discussion for an orthogonal vector space $\{x_r\}$ -- $\{r = 1, 2 \dots n\}$

$$\left(x_m \cdot x_n = \begin{cases} 0 & m \neq n \\ k_m & m = n \end{cases} \right)$$

If this vector space is complete then any vector F can be expressed as

$$F = c_1 x_1 + c_2 x_2 + \cdots \quad c_r x_r + \cdots$$

$$c_r = \frac{F \cdot x_r}{K_r} = \frac{F \cdot x_r}{x_r^2}$$



→ $A = c_1 a_1 + c_2 a_2 + c_3 a_3$

For N dimensional

$A = c_1 a_1 + c_2 a_2 + \cdots \quad c_N a_N$

$a_1, a_2, a_3 \cdots a_n$ are set of orthogonal vectors

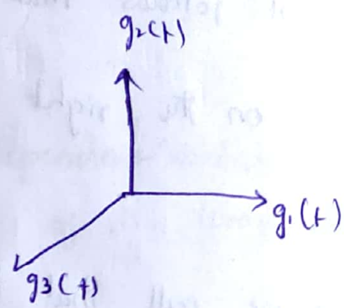
$$a_i \cdot a_j = 0 \quad \text{for } i \neq j$$
$$= K \quad \text{for } i = j$$

a_i & a_j are called orthogonal vectors

If $K = 1$ then a_i, a_j are called orthogonal vectors

$$a_i \cdot a_j = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

Component: $c_i = \frac{A \cdot a_i}{a_i^2} = \frac{A \cdot a_i}{K}$



$$f(t) = c_1 g_1(t) + c_2 g_2(t) + c_3 g_3(t)$$

N – dimensional

$$f(t) = c_1 g_1(t) + c_2 g_2(t) + \cdots \quad c_N g_N(t)$$

Orthogonal Signal Space

→ Any Vector can be expressed as a sum of its components along n mutually Orthogonal Vectors, provided these vectors formed a complete set of co-ordinate system.

→ Similarly it is possible to express any function $f(t)$ as a sum of its components along a set of mutually orthogonal functions if these functions form a complete set

→ called as Orthogonal Signal space

Approximation of a function by a set of mutually Orthogonal functions

Let us consider a set of n functions $g_1(t), g_2(t) \ldots g_n(t)$ which are Orthogonal to One another Over an interval t_1 to t_2 ; that is

$$\int_{t_1}^{t_2} g_j(t)\, g_k(t)\, dt = 0 \qquad j \neq k$$

and let

$$\int_{t_1}^{t_2} g_j^2(t)\, dt = k_j$$

Let an arbitary function $f(t)$ be approximated over an interval (t_1, t_2) by a linear combination of these n mutually Orthogonal functions

$$f(t) \simeq C_1 g_1(t) + C_2 g_2(t) + \cdots\cdots C_k g_k(t) + \cdots\cdots C_n g_n(t)$$

$$\sum_{r=1}^{n} C_r g_r(t)$$

For the best approximation we must find the proper values of constants $C_1, C_2 \ldots, C_n$ Such that ε, the mean square of $f_e(t)$ is minimized by definative

$$f_e(t) = f(t) - \sum_{r=1}^{n} C_r g_r(t)$$

and $\varepsilon = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \left[f(t) - \sum_{r=1}^{n} c_r g_r(t) \right]^2 dt$

It is evident that ε is a function of $c_1, c_2 \ldots c_n$ and to minimize ε we must have

$$\frac{\partial \varepsilon}{\partial c_1} = \frac{\partial \varepsilon}{\partial c_2} = \cdots \frac{\partial \varepsilon}{\partial c_j} = \cdots \frac{\partial \varepsilon}{\partial c_n} = 0$$

Let us consider the equation

$$\frac{\partial \varepsilon}{\partial c_j} = 0$$

Since $(t_2 - t_1)$ is constant eq may be expressed as

$$\frac{\partial}{\partial c_j} \left(\int_{t_1}^{t_2} \left[f(t) - \sum_{r=1}^{n} c_r g_r(t) \right]^2 dt \right\} = 0$$

When we expand the integrand, we note that all the terms arising due to the cross product of the orthogonal functions and zero by a virtue of orthogonality;

→ that is, all the terms of the form $\int g_i(t) g_k(t)\, dt$ are zero as expressed.

→ Similarly the derivative with respect to c_j of all the terms that do not contain c_j is zero; that is

$$\frac{\partial}{\partial c_j} \int_{t_1}^{t_2} f^2(t)dt = \frac{\partial}{\partial c_j} \int_{t_1}^{t_2} c_r^2 g_r^2(t)dt = \frac{\partial}{\partial c_j} \int_{t_1}^{t_2} c_r f(t)\, g_r f(t)\, dt = 0$$

This leaves only two non zero terms in eq

$$\frac{\partial}{\partial c_j} \int_{t_1}^{t_2} \left[-2 c_j f(t)\, g_j(t) + c_j^2 g_j^2(t) \right] dt = 0$$

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changing the Order of differentiation and integration

$$2\int_{t_1}^{t_2} f(t)\,g_j(t)\,dt = 2C_j\int_{t_1}^{t_2} g_j^2(t)\,dt$$

$$C_j = \frac{\int_{t_1}^{t_2} f(t)\,g_j(t)\,dt}{\int_{t_1}^{t_2} g_j^2(t)\,dt}$$

$$= \frac{1}{K}\int_{t_1}^{t_2} f(t)\,g_j(t)\,dt$$

We may summarize this result as follows.

Given set of n function $g_1(t), g_2(t) \ldots g_n(t)$ mutually orthogonal Over the interval (t_1, t_2) it is possible to approximate an arbitary function $f(t)$ Over this interval by a linear combination of these n functions

$$f(t) \simeq C_1 g_1(t) + C_2 g_2(t) + \ldots C_n g_n(t)$$

$$= \sum_{r=1}^{n} C_r g_r(t)$$

For best approximation that is the one that will minimize the mean of the square Error Over the interval we must choose the co-efficients $C_1, C_2 \ldots C_n$ etc.

Evaluation of Mean Square Error

Let us now find the value of ε when optimum values of co-efficients $c_1, c_2, \ldots, c_n$ are choosen

$$\varepsilon = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \left[f(t) - \sum_{r=1}^{n} c_r g_r(t) \right]^2 dt$$

$$\frac{1}{t_2 - t_1} \left[\int_{t_1}^{t_2} f^2(t)\, dt + \sum_{r=1}^{n} c_r^2 \int_{t_1}^{t_2} g_r^2(t)\, dt - 2 \sum_{r=1}^{n} c_r \int_{t_2}^{t_2} f(t)\, g_r(t)\, dt \right.$$

$$\int_{t_1}^{t_2} f(t)\, g_r(t)\, dt = c_r \int_{t_1}^{t_2} g_r^2(t)\, dt = c_r K_r$$

Substituting we get

$$\varepsilon = \frac{1}{(t_2 - t_1)} \left[\int_{t_1}^{t_2} f^2(t)\, dt + \sum_{r=1}^{n} c_r^2 K_r - 2 \sum_{r=1}^{n} c_r^2 K_r \right]$$

$$= \frac{1}{(t_2 - t_1)} \left[\int_{t_1}^{t_2} f^2(t)\, dt - \sum_{r=1}^{n} c_r^2 K_r \right]$$

$$= \frac{1}{(t_2 - t_1)} \left[\int_{t_1}^{t_2} f^2(t)\, dt - (c_1^2 K_1 + c_2^2 K_2 + \cdots + c_n^2 K_n) \right]$$

Representation of a function by a closed or a Complete set of mutually Orthogonal functions.

→ It is evident from eq that if we increase n, that is if we approximate f(t) by a Larger number of Orthogonal functions the Error will become Smaller

→ but by its Very definition ε is a positive quantity

→ hence in the limit as the number of terms is made infinity the sum

$$\sum_{r=1}^{\infty} c_r^2 K_r$$ may converge to the integral

$$\int_{t_1}^{t_2} f^2(t)\,dt$$ and then ε vanishes thus

$$\int_{t_1}^{t_2} f^2(t)\,dt = \sum_{r=1}^{\infty} c_r^2 K_r$$

Under these Conditions f(t) is represented by the infinite Series

$$f(t) = c_1 g_1(t) + c_2 g_2(t) + \cdots + c_r g_r(t) + \cdots$$

the infinite Series on the right hand side of Eq thus converges to f(t) such that the mean square of Error is zero.

→ the Series is said to converge in the mean.

NOTE: that the representation of f(t) is now Exact

→ A set of functions $g_1(t), g_2(t) \ldots g_r(t)$ mutually Orthogonal Over the interval (t_1, t_2) is said to be a complete or closed set if there Exists no function x(t) for which is true that

$$\int_{t_1}^{t_2} x(t)\, g_K(t)\, dt = 0 \qquad \text{for } K = 1, 2 \ldots$$

→ If a function x(t) Could be found such that the above integral is zero the obviously x(t) is orthogonal to Each member of the set {g(t)} and consequently is itself a member of the set

Evidently the set cannot be complete without $x(t)$ being its member.

Let us now summarize the results of this discussion. For a set $\{g_r(t)\}$ ($r = 1, 2, 3 \ldots$) mutually orthogonal over the interval (t_1, t_2)

$$\int_{t_1}^{t_2} g_m(t)\, g_n(t)\, dt = \begin{cases} 0 & \text{if } m \neq n \\ K_m & m = n \end{cases}$$

→ If this function set is complete then any function $f(t)$ can be expressed as

$$f(t) = C_1 g_1(t) + C_2 g_2(t) + \cdots C_r g_r(t)$$

$$C_r = \frac{\int_{t_1}^{t_2} f(t)\, g_r(t)\, dt}{K_r} = \frac{\int_{t_1}^{t_2} f(t)\, g_r(t)\, dt}{\int_{t_1}^{t_2} g_r^2(t)\, dt}$$

Compare equations brings out forcefully the analogy between vectors and signals.

→ Any vector can be expressed as a sum of its components along n mutually orthogonal vectors, provided these vectors form a complete set. Similarly any function $f(t)$ can be expressed as a sum of its components along mutually orthogonal functions provided these functions form a closed or a complete set.

→ In the comparison of vectors and signals the dot product of two vectors is analogous to the integral of the product of two signals that is $A \cdot B \sim \int_{t_1}^{t_2} f_A(t)\, f_B(t)\, dt$

It follows that the square of the length A of a vector A is analogous to the integral of the square of a function that is

$$A \cdot A = A^2 \sim \int_{t_1}^{t_2} f_A^2(t)\,dt$$

→ If a vector is expressed in terms of mutually orthogonal components the square of the length is given by the sum of the squares of the lengths of the component vectors.

→ An analogous result holds true for signals.

→ Since the component functions are not orthonormal the right hand side is $\sum c_r^2 K_r^2$ instead of $\sum c_r^2$ for an ~~orthogonal~~ orthonormal set $K=1$.

→ Thus analogous to the case where a vector is expressed in terms of its components along mutually orthogonal vectors whose length squares are $k_1, k_2, \ldots, k_r$ ... etc.

→ $f(t)$ contains a component of signal $g_r(t)$ and this component has a magnitude c_r.

→ Representation of $f(t)$ by a set of infinite mutually orthogonal functions is called generalized fourier series representation of $f(t)$

Example 1.2

An example we shall consider the rectangular function of Example

this function was approximated by a single function $\sin(t)$

→ we shall now see how the approximation improves when a large number of mutually orthogonal functions are used

→ It was shown previously that functions $\sin n\omega_0 t$ and $\sin m\omega_0 t$ are mutually orthogonal over the interval $(t_0, t_0 + 2\pi/\omega_0)$ for all integral values of n and m

→ Hence it follows that a set of functions $\sin t$, $\sin 2t$, $\sin 3t$ are mutually orthogonal over the interval $(0, 2\pi)$, the rectangular function will now be approximated by a finite series of sinusoidal functions

$$f(t) \simeq C_1 \sin t + C_2 \sin 2t + \cdots C_n \sin nt$$

the constants C_r can be ~~Equo~~ Evaluated by Using

$$C_r = \frac{\int_0^{2\pi} f(t) \sin rt \, dt}{\int_0^{2\pi} \sin^2 rt \, dt}$$

$$= \frac{1}{\pi} \int_0^{\pi} \sin rt \, dt - \int_0^{2\pi} \sin rt \, dt$$

$$= \frac{4}{\pi r} \quad \text{if } r \text{ is odd}$$

$$= 0 \quad \text{if } r \text{ is Even}$$

thus $f(t)$ is approximated by

$$f(t) = \frac{4}{\pi} \left(\sin t + \frac{1}{3} \sin 3t + \frac{1}{5} \sin 5t + \frac{1}{7} \sin 7t + \cdots \right)$$

Figure shows the actual function and approximated function when the function is approximated with One, two, three and four terms respectively.

→ for the given number of terms of the form $\sin rt$ these are the optimum approximations which minimize the mean square Error.

→ As we increase the no. of terms, the approximation improves and the mean square Error dimishes. For infinite terms mean square Error is zero

Let us evaluate the Error ε in these approximations

$$\varepsilon = \frac{1}{t_2 - t_1}\left[\int_{t_2}^{t_1} f^2(t)\,dt - c_1^2 K_1 - c_2^2 K_2 - \cdots\right]$$

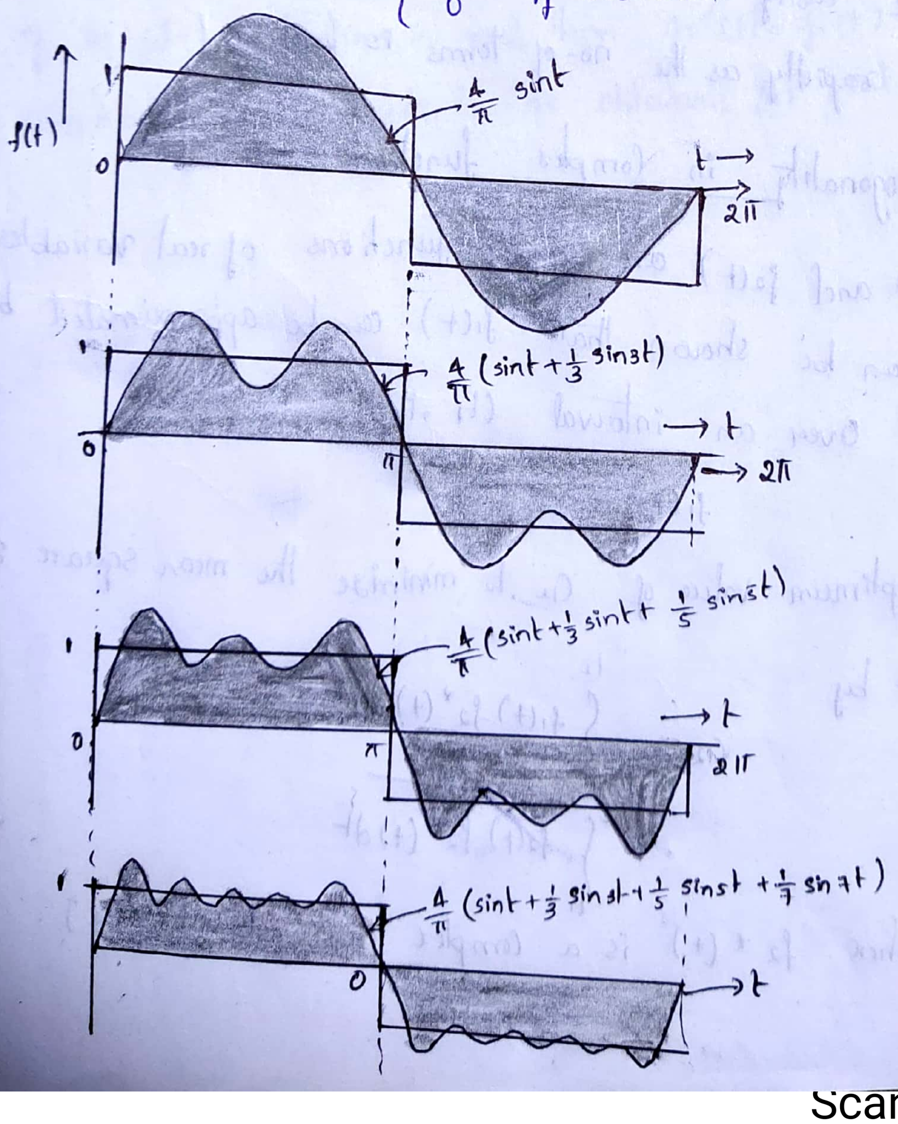
In these case $(t_2 - t_1) = 2\pi$

$$f(t) = \begin{cases} 1 & (0 < t < \pi) \\ -1 & (\pi < t < 2\pi) \end{cases}$$

ther fore

$$\int_0^{2\pi} f^2(t)\,dt = 2\pi$$

$$c_r = \begin{cases} \frac{4}{\pi r} & \text{if } r \text{ is odd} \\ 0 & \text{if } r \text{ is Even} \end{cases}$$



$$K_r = \int_0^{2\pi} \sin^2 rt \; dt = 0$$

therefore for one term approximation

$$\varepsilon_1 = \frac{1}{2\pi}\left[2\pi - \left(\frac{4}{\pi}\right)^2 \pi\right] = 0.19$$

For two terms approximation

$$\varepsilon_2 = \frac{1}{2\pi}\left[2\pi - \left(\frac{4}{\pi}\right)^2 \pi - \left(\frac{4}{3\pi}\right)^2 \pi\right] = 0.1$$

for three terms

$$\varepsilon_3 = \frac{1}{2\pi}\left[2\pi - \left(\frac{4}{\pi}\right)^2 \pi - \left(\frac{4}{3\pi}\right)^2 \pi - \left(\frac{4}{5\pi}\right)^2 \pi\right] = 0.0675$$

and

$$\varepsilon_4 = \frac{1}{2\pi}\left[2\pi - \left(\frac{4}{\pi}\right)^2 \pi - \left(\frac{4}{3\pi}\right)^2 \pi - \left(\frac{4}{5\pi}\right)^2 \pi - \left(\frac{4}{7\pi}\right)^2 \pi\right] = 0.051$$

and so on

It can be easily seen that in these case the mean square error diminishes rapidly as the no. of terms is increased

Orthogonality in Complex functions

→ If $f_1(t)$ and $f_2(t)$ are complex functions of real variables t then it can be shown that $f_1(t)$ can be approximated by $C_{12} f_2(t)$ over an interval (t_1, t_2)

$$f_1(t) \simeq C_{12} f_2(t)$$

The optimum value of C_{12} to minimize the mean square error magnitude by

$$C_{12} = \frac{\int_{t_1}^{t_2} f_1(t) f_2^*(t)\, dt}{\int_{t_1}^{t_2} f_2(t) f_2^*(t)\, dt}$$

where $f_2^*(t)$ is a complex conjugate of $f_2(t)$

The two complex functions $f_1(t)$ and $f_2(t)$ are orthogonal over the interval (t_1, t_2) if

$$\int_{t_1}^{t_2} f_1(t)\, f_2^*(t)\, dt = \int_{t_1}^{t_2} f_1^*(t)\, f_2(t) = 0$$

for a set of complex functions $\{g(t)\}$, $r(1, 2, 3 \ldots)$ mutually orthogonal over the interval (t_1, t_2):

$$\int_{t_1}^{t_2} g_m(t)\, g_n^*(t)\, dt = \begin{cases} 0 & \text{if } m \neq n \\ K_m & m = n \end{cases}$$

If this set of functions is complete then any function $f(t)$ can be expressed as

$$f(t) = C_1 g_1(t) + C_2 g_2(t) + \cdots\cdots C_r g_r(t) + \cdots$$

+ ~~see for instance~~ S. mr

$$C_r = \frac{1}{K_r} \int_{t_1}^{t_2} f(t)\, g_r^*(t)\, dt$$

If the set of functions is real then $g_r^*(t) = g_r(t)$ and all results for complex functions reduced to those obtained for real functions.

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