SIGNAL ANALYSIS AND FOURIER SERIES

Analogy between Vectors and Signals:

Before the concept, let us know the definction for signals and systems.

Signals: A function of one or more independent variables which contain some information.

(or)

Defined as any physical quantity that varies with time, space or any other independept variables.

Ex: Electric vltg or current. that includes radio sgl, TV sgl, telephone sgl etc.,
Non electric signals such as sound signal, pressure signal etc.,

→ A speech signal can be represented mathematically by acoustic (pertaining to the sense) pressure as function of time.

→ A picture can be represented by brightness as a function of two spatial variables.
(Relating to space)

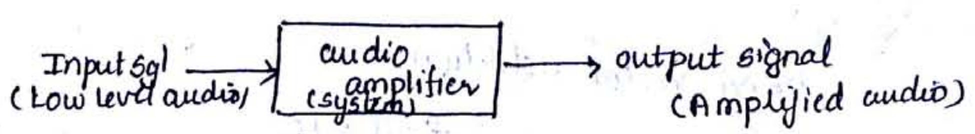
Systems:

It is a set of elements or functional blocks that are connected together and produces an output in response to the input signal.

(or)

An entity that processes a set of input signals to yield another set of output signals.

Ex: An audio amplifier, attenuator, TV set, transmitter, receiver etc., or any engine or machine



Relation b/w signal & systems.

Ex: If we take example of central government, they we can see different sub-systems together considered as big system.

→ Systems is composed of many subsystems like finance, defence, foreign affairs, home culture, social welfare, industries etc.,

→ Inputs to the systems are in the form of revenue, import, complaints, business suggestions, policies for foreign countries through which central government functions.

→ The central government produces output signals in the form of exports, government resolutions, financial aids, welfare programs etc.,

→ This is the concept of signal and systems.

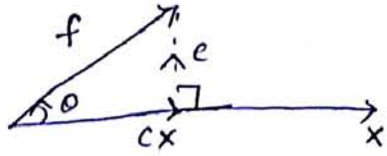
Analogy:

→ Signals are represented in terms of orthogonal functions.

Orthogonality Concept in Vectors:

→ All signals are basically vectors where vector is represented in terms of its co-ordinates.

→ consider a vector f and another vector x then projection of vector along other vector is shown below



The dot product of vectors f and x is given as

$$f \cdot x = |f||x| \cos\theta$$

where θ is the angle between f & x

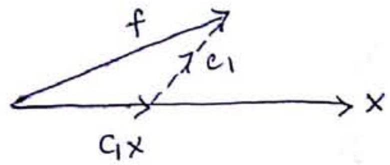
cx is component of vector f along x or projection of f on x.

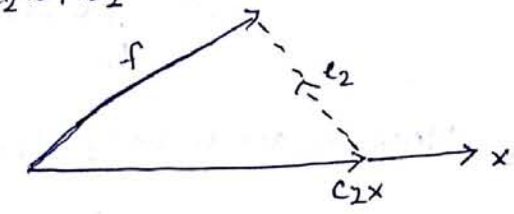
using vector addition

$$f = cx + e$$ → error vector

Note: 'e' is minimum only when it is perpendicular to x. Below are figures in which 'e' is not $\perp^{lar}$

$$f = c_1 x + e_1 = c_2 x + e_2$$ (Here e_1 & e_2 are > than e)





→ The component of f along x is cx which is given as $|f| \cos\theta$

$$\therefore c|x| = |f| \cos\theta$$

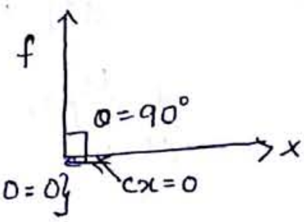
Multiplying both the sides by $|x|$

$$c|x|^2 = \underbrace{|f||x| \cos\theta}_{\text{dot product of vectors } f \text{ \& } x.}$$

$$c|x|^2 = f.x$$

$$c = \frac{f.x}{|x|^2} \Rightarrow \frac{f.x}{x.x}$$ {∵ $x.x$ & $f.x$ are vector product x and x cannot get cancelled}

→ When 'f' is $\perp^{lar}$ to x, 'f' will not have component along x because $\theta = 90°$ as shown in figure



$$f.x = |f||x| \cos\theta$$

$$= |f||x| \cos 90° \quad \{\because \cos 90 = 0\}$$

$$f.x = 0$$

→ The vectors 'f' and 'x' are said to be orthogonal if their dot product is zero. (or) vectors are orthogonal if they are mutually perpendicular.

Orthogonality in Signals:

Consider a signal $f(t)$ to be represented in terms of $x(t)$ over an interval t_1 & t_2.

$$f(t) = c\,x(t) + e(t)$$

$$e(t) = f(t) - c\,x(t) \qquad t_1 \leq t \leq t_2 \longrightarrow (1)$$

Energy of $e(t)$ will be

$$E_e = \int_{t_1}^{t_2} e^2(t)\,dt \rightarrow (2)$$

Mean square value of $e(t)$ will be

$$\overline{e^2(t)} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} e^2(t)\,dt \qquad \{\text{from eq (2)}\}$$

$$\overline{e^2(t)} = \frac{E_e}{t_2 - t_1}$$

From eq (1) we can write eqn (2)

$$E_e = \int_{t_1}^{t_2} [f(t) - c\,x(t)]^2\,dt$$

Here the value of 'c' should be selected such that E_e will be minimum

→ This can be obtained by differentiating E_e w.r. to c and equating it to zero

→ For minimum E_e, $\frac{dE_e}{dc} = 0$

$$\text{i.e } \frac{d}{dc}\left[\int_{t_1}^{t_2} [f(t) - c\,x(t)]^2\right] dt = 0$$

$$\underbrace{\frac{d}{dc}\int_{t_1}^{t_2} f^2(t)\,dt}_{\text{Independent of 'c' so it will be zero}} - \frac{d}{dc}\int_{t_1}^{t_2} 2c\,f(t)\cdot x(t)\,dt + \frac{d}{dc}\int_{t_1}^{t_2} c^2 x^2(t)\,dt = 0$$

$$-2\int_{t_1}^{t_2} f(t)\cdot x(t)\,dt + 2c\int_{t_1}^{t_2} x^2(t)\,dt = 0$$

$$\cancel{2}c\int_{t_1}^{t_2} x^2(t)\,dt = \cancel{2}\int_{t_1}^{t_2} f(t)\cdot x(t)\,dt$$

$$c = \frac{\int_{t_1}^{t_2} f(t)\cdot x(t)\,dt}{\int_{t_1}^{t_2} x^2(t)\,dt}$$

The same expression can be obtained for minimum value of $\overline{e^2(t)}$.
The above equation denominator represents energy of $x(t)$. which cannot be zero. Hence numerator must be zero to make 'c' zero. If 'c' is zero, there will be no component of $f(t)$ along $x(t)$.

→ $f(t)$ and $x(t)$ are said to be orthogonal over an interval $[t_1, t_2]$ i.e

$$\int_{t_1}^{t_2} f(t)x(t)\,dt = 0$$

||ly if $f(t)$ and $x(t)$ are complex signals then they are orthogonal over an interval $[t_1, t_2]$ if

$$\int_{t_1}^{t_2} f(t)\underbrace{x^*(t)}_{\text{complex conjugate of } x(t)}\,dt = 0 \quad \text{or} \quad \int_{t_1}^{t_2} \underbrace{f^*(t)}_{\text{complex conjugate of } f(x)} x(t)\,dt = 0$$

Problems:

(1) Show that the following signals are orthogonal over an interval $[0,1]$

$$f(t) = 1, \; x(t) = \sqrt{3}(1-2t)$$

Sol we know for orthogonal if

$$\int_{t_1}^{t_2} x(t)f(t)\,dt = 0$$

$$\int_{t_1}^{t_2} f(t)x(t)\,dt = \int_0^1 1\cdot(\sqrt{3})(1-2t)\,dt$$

$$= \int_0^1 \sqrt{3}\,dt - \int_0^1 2\sqrt{3}\,t\,dt$$

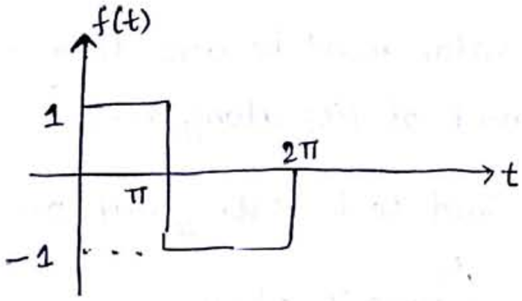
$$= \sqrt{3}\,[t]_0^1 - 2\sqrt{3}\left[\frac{t^2}{2}\right]_0^1$$

$$= \sqrt{3}[1-0] - 2\sqrt{3}\left[\frac{1}{2}\right]$$

$$= \sqrt{3} - \sqrt{3} = 0$$

Two given signals are orthogonal over an interval $[0,1]$

(2) Figure shows a square wave. Represent this signal by sint. plot an error in this representation.



Sol

Square wave be f(t). and sine wave be $x(t) = \sin t$. Then

$$f(t) = C \cdot x(t)$$
$$= C \cdot \sin t$$

value of c given by eqn

$$C = \frac{\int_{t_1}^{t_2} f(t)\, x(t)\, dt}{\int_{t_1}^{t_2} x^2(t)\, dt}$$

$$\int_{t_1}^{t_2} f(t)\, x(t)\, dt = \int_0^{2\pi} f(t) \cdot \sin t \, dt$$

$$= \int_0^{\pi} 1 \cdot \sin t \, dt + \int_{\pi}^{2\pi} (-1) \sin t \, dt$$

$$= \left[-\cos t\right]_0^{\pi} - \left[-\cos t\right]_{\pi}^{2\pi} = \left[-\{\cos \pi - \cos 0\}\right]$$

$$= \left[-(-1-1) + (1-(-1))\right] \qquad + \left[\cos 2\pi - \cos \pi\right]$$

$$= 4.$$

$$\int_{t_1}^{t_2} x^2(t)\, dt = \int_0^{2\pi} \sin^2 t \, dt$$

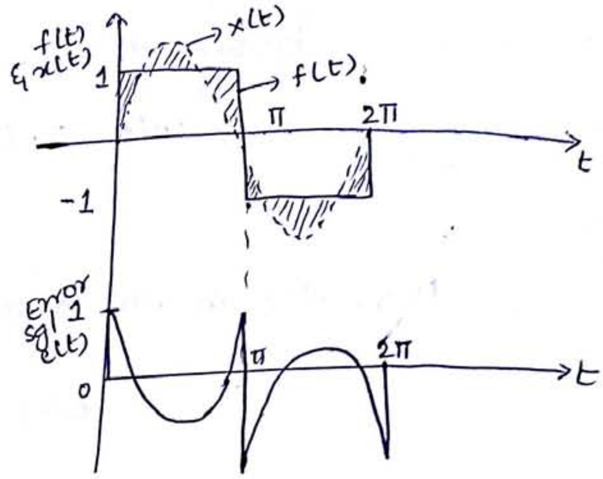
$$= \int_0^{2\pi} \frac{1 - \cos 2t}{2} dt = \frac{1}{2}\int_0^{2\pi} dt - \frac{1}{2}\int_0^{2\pi} \cos 2t \, dt$$

$$= \frac{1}{2}\left[t\right]_0^{2\pi} - \frac{1}{2}\left[\frac{\sin 2t}{2}\right]_0^{2\pi}$$

$$= \frac{1}{2} 2\pi - \frac{1}{4}\left[\sin 4\pi - \sin 0\right]$$

$$= \pi$$

$$\therefore \; C = \frac{\int_{t_1}^{t_2} f(t)\, x(t)\, dt}{\int_{t_1}^{t_2} x^2(t)\, dt}$$



$$C = \frac{4}{\pi}$$

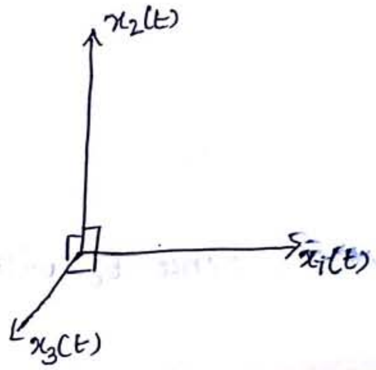
$$\therefore \; f(t) = \frac{4}{\pi} \sin t$$

and error $e(t) = f(t) - c\,x(t)$

Orthogonal Signal Space:

Let $x_1(t)$, $x_2(t)$, $x_3(t)$ be orthogonal to each other i.e '3' signals are mutually $\perp^{lar}$ which forms three dimensional signal space which is also called as orthogonal signal space.

→ which is used to represent any signal lying in that space.



Note:

If there are 'N' such mutually orthogonal signals i.e $x_1(t), x_2(t), x_3(t), x_4(t) \ldots x_N(t)$ then they form N-dimensional orthogonal signal space.

Signal Approximation Using Orthogonal functions:

→ Consider a set of signals which are mutually orthogonal over an interval $[t_1, t_2]$. f(t) can be represented as

$$f(t) \simeq C_1 x_1(t) + C_2 x_2(t) + C_3 x_3(t) + \cdots + C_N x_N(t)$$

$$\simeq \sum_{n=1}^{N} C_n x_n(t)$$

In above eqn any two signals $x_m(t)$ and $x_n(t)$ are orthogonal over an interval $[t_1, t_2]$ i.e

$$\int_{t_1}^{t_2} x_m(t)\, x_n(t)\, dt = \begin{cases} 0 & \text{for } m \neq n \\ E_n & \text{for } m = n \end{cases}$$

because if $m=n$

$$\int_{t_1}^{t_2} x_n(t) \cdot x_n(t)\, dt = \int_{t_1}^{t_2} x_n^2(t)\, dt = E_n = \text{energy of the sgl.}$$

Error $e(t)$ in the approximation of equation is given as

$$e(t) = f(t) - \sum_{n=1}^{N} c_n x_n(t)$$

Hence error energy

$$E_e = \int_{t_1}^{t_2} e^2(t)\, dt = \int_{t_1}^{t_2} \left[f(t) - \sum_{n=1}^{N} c_n x_n(t) \right]^2 dt$$

where E_e is fn of $c_1, c_2 \cdots c_N$.

→ Hence E_e will be minimized w.r.to c_i if

$$\frac{\partial E_e}{\partial c_i} = 0$$

$$\frac{\partial}{\partial c_i} \left\{ \int_{t_1}^{t_2} \left[f(t) - \sum_{n=1}^{N} c_n x_n(t) \right]^2 dt \right\} = 0$$

$$\frac{\partial}{\partial c_i} \left\{ \int_{t_1}^{t_2} f^2(t)\, dt - \int_{t_1}^{t_2} \sum_{n=1}^{N} 2 c_n f(t) x_n(t)\, dt + \int_{t_1}^{t_2} \sum_{n=1}^{N} c_n^2 x_n^2(t)\, dt \right\} = 0$$

→ for $i = 1, 2, 3 \ldots N$ the equation is executed. The first integration term is independent of c_i so its derivative is zero.

So $$\frac{\partial}{\partial c_i} \left[- \int_{t_1}^{t_2} 2 c_i f(t) x_i(t)\, dt + \int_{t_1}^{t_2} c_i^2 x_i^2(t)\, dt \right] = 0$$

$$\therefore\ -2 \int_{t_1}^{t_2} f(t) x_i(t)\, dt + 2 c_i \int_{t_1}^{t_2} x_i^2(t)\, dt = 0$$

$$c_i = \frac{\int_{t_1}^{t_2} f(t) x_i(t)\, dt}{\int_{t_1}^{t_2} x_i^2(t)\, dt} \quad \text{where } i = 1, 2, 3 \ldots N$$

we know that $\int_{t_1}^{t_2} x_i^2(t)\, dt = E_i = \text{energy}$

$$\boxed{c_i = \frac{1}{E_i}\int_{t_1}^{t_2} f(t)\, x_i(t)\, dt}$$

Mean Square Error:

The error energy is given by equation

$$E_e = \int_{t_1}^{t_2}\left[f(t) - \sum_{n=1}^{N} c_n x_n(t)\right]^2 dt$$

$$E_e = \int_{t_1}^{t_2} f^2(t)\, dt - 2\int_{t_1}^{t_2}\sum_{n=1}^{N} c_n f(t)\, x_n(t)\, dt + \int_{t_1}^{t_2}\sum_{n=1}^{N} c_n^2 x_n^2(t)\, dt$$

Integration & Summation order if we interchange

$$E_e = \int_{t_1}^{t_2} f^2(t)dt - 2\sum_{n=1}^{N} c_n \int_{t_1} f(t)\, x_n(t)\, dt + \sum_{n=1}^{N} c_n^2 \int_{t_1} x_n^2(t) dt.$$

$$E_e = \int_{t_1}^{t_2} f^2(t)\, dt - 2\sum_{n=1}^{N} c_n . c_n E_n + \sum_{n=1}^{N} c_n^2 . E_n \quad \left\{ \because \int_{t_1}^{t_2} f(t)\, x_n(t)\, dt = c_n E_n \ \& \ \int_{t_1}^{t_2} x_n^2(t) = E_n \right\}$$

$$= \int_{t_1}^{t_2} f^2(t)\, dt - 2\sum_{n=1}^{N} c_n^2 . E_n + \sum_{n=1}^{N} c_n^2 . E_n$$

$$= \int_{t_1}^{t_2} f^2(t)\, dt - \sum_{n=1}^{N} c_n^2 . E_n .$$

The mean square error and error energy are related as

$$\overline{e^2(t)} = \frac{E_e}{t_2 - t_1} \Rightarrow \frac{1}{t_2 - t_1}\left[\int_{t_1}^{t_2} f^2(t)\, dt - \sum_{n=1}^{N} c_n^2 . E_n\right]$$

$\therefore \sum_{n=1}^{N} c_n^2 . E_n$ is always positive so if $E_e \to 0$ as $N \to \infty$

Closed or Complete set of Orthogonal Functions:

→ Mean square error approaches to zero as number of terms $C_n^2 E_n$ are made infinite. Under this condition,

with $\overline{e^2(t)} = 0$ as $N = \infty$

$$0 = \frac{1}{t_2 - t_1}\left[\int_{t_1}^{t_2} f^2(t)\,dt - \sum_{n=1}^{N} C_n^2 \cdot E_n\right]$$

$$\therefore \int_{t_1}^{t_2} f^2(t)\,dt = \sum_{n=1}^{\infty} C_n^2 \cdot E_n.$$

from

$$f(t) = \sum_{n=1}^{N} C_n x_n(t) \quad \text{when } N \to \infty$$

$$\therefore f(t) = \sum_{n=1}^{\infty} C_n x_n(t). \quad \{\text{Generalized fourier Series}\}$$

→ It is said to be complete or closed set if there exists no function $p(t)$ for which

$$\int_{t_1}^{t_2} p(t)\, x_n(t)\,dt = 0 \quad \text{for } n = 1, 2 \ldots$$

→ If $p(t)$ exists and above integral is zero, then $p(t)$ must be member of set $\{x_n(t)\}$

→ For complete set, function $f(t)$ is expressed as

$$f(t) = C_1 x_1(t) + C_2 x_2(t) + C_3 x_3(t) + \ldots$$

$$C_i = \frac{\int_{t_1}^{t_2} f(t)\, x_i(t)\,dt}{\int_{t_1}^{t_2} x_i^2(t)\,dt} = \frac{1}{E_i}\int_{t_1}^{t_2} f(t)\, x_i(t)$$

Orthogonality in Complex functions:

Let set of signals $x_1(t), x_2(t), x_3(t) \ldots$ are complex. Then those signals are mutually orthogonal if

$$\int_{t_1}^{t_2} x_m(t)\, x_n^*(t)\, dt = \int_{t_1}^{t_2} x_m^*(t)\, x_n(t)\, dt = \begin{cases} 0 & \text{for } m \neq n \\ E_n & \text{for } m = n. \end{cases}$$

Then $f(t)$ can be expressed as

$$f(t) = \sum_{n=1}^{\infty} C_n x_n(t)$$

$$\text{where } C_n = \frac{1}{E_n} \int_{t_1}^{t_2} f(t)\, x_n^*(t)\, dt$$

$$E_n = \int_{t_1}^{t_2} x_n(t) \cdot x_n^*(t)\, dt.$$

Problems:

(1) Show that the signal set $\{1, \cos w_0t, \cos 2w_0t, \ldots \cos nw_0t, \ldots \sin w_0t, \sin 2w_0t \ldots \sin nw_0t\}$ are orthogonal over an interval $T_0 = \frac{2\pi}{w_0}$.

Sol (i) To check orthogonality of cosine waves:

Consider the orthogonality of $\cos nw_0t$ and $\cos mw_0t$ i.e

$$\int_{t}^{t+T_0} \cos nw_0t \cos mw_0t\, dt$$

$$\left\{ \because \cos x \cos y = \frac{1}{2}\left[\cos(x-y) + \cos(x+y)\right] \right.$$

$$\int_{t}^{t+T_0} \cos nw_0t \cos mw_0t\, dt = \frac{1}{2}\int_{t}^{t+T_0} \cos(n-m)w_0t\, dt + \frac{1}{2}\int_{t}^{t+T_0} \cos(n+m)w_0t\, dt.$$

For $n=m$, $\cos(n-m)w_0t = 1$ but for $n \neq m$, the integration of $(n-m)$ full cycles of cosine wave is taken over one period. Hence integration is zero. Similarly for integration of $(n+m)$ full cycles

$\therefore$ Equation becomes.

$$\int_t^{t+T_0} \cos n\omega_0 t \cdot \cos m\omega_0 t \, dt = \frac{1}{2}\int_t^{t+T_0} 1 \, dt$$

$$= \frac{1}{2}[t]_t^{t+T_0} = \frac{T_0}{2}.$$

$$\therefore \int_t^{t+T_0} \cos n\omega_0 t \cos m\omega_0 t \, dt = \begin{cases} 0 & \text{for } n \neq m \\ \frac{T_0}{2} & \text{for } n = m. \end{cases}$$ where it shows that two cosine waves of given set are orthogonal over one period.

(ii) To check orthogonality of sine waves.

$$\int_t^{t+T_0} \sin n\omega_0 t \sin m\omega_0 t \, dt = \frac{1}{2}\int_t^{t+T_0} \cos(n-m)\omega_0 t \, dt + \frac{1}{2}\int_t^{t+T_0} \cos(n+m)\omega_0 t \, dt$$

$$\left\{ \because \sin x \sin y = \frac{1}{2}\{\cos(x-y) - \cos(x+y)\} \right\}$$

same as above explanation

$$\therefore \int_t^{t+T_0} \sin n\omega_0 t \sin m\omega_0 t \, dt = \frac{1}{2}\int_t^{t+T_0} 1 \, dt = \frac{T_0}{2}.$$

$$\int_t^{t+T_0} \sin n\omega_0 t \sin m\omega_0 t \, dt = \begin{cases} 0 & \text{for } n \neq m \\ \frac{T_0}{2} & \text{for } n = m. \end{cases}$$

(iii) To check orthogonality of $\sin n\omega_0 t$ and $\cos m\omega_0 t$

$$\int_t^{t+T_0} \sin n\omega_0 t \cos m\omega_0 t \, dt = \frac{1}{2}\int_t^{t+T_0} \sin(n-m)\omega_0 t \, dt + \frac{1}{2}\int_t^{t+T_0} \sin(n+m)\omega_0 t \, dt$$

$$\left\{ \because \sin x \cos y = \frac{1}{2}[\sin(x-y) + \sin(x+y)] \right\}$$

Integration of $(n-m)$ or $(n+m)$ full cycles of sine wave over a period will be zero. hence both the above integrals are zero.

$$\therefore \int_t^{t+T_0} \sin n\omega_0 t \cos m\omega_0 t \, dt = 0$$ for all values of n & m.

$\therefore$ Thus sine & cosine waves of given set are orthogonal over one period.

② Prove that set of exponentials $1, e^{\pm j\omega_0 t}, e^{\pm 2j\omega_0 t}, e^{\pm j3\omega_0 t} \ldots$ is orthogonal over any interval T_0.

Sol Here we have to check orthogonality of complex function. It is given as

$$\int_{t_1}^{t_2} x_m(t)\, x_n^*(t)\, dt = \begin{cases} 0 & \text{for } m \neq n \\ E_n & \text{for } m = n \end{cases}$$

For $x_m(t) = e^{jm\omega_0 t}$, $x_n(t) = e^{jn\omega_0 t}$

$$\int_t^{t+T_0} e^{jm\omega_0 t} [e^{jn\omega_0 t}]^* dt = \int_t^{t+T_0} e^{jm\omega_0 t} \cdot e^{-jn\omega_0 t} dt$$

$$= \int_t^{t+T_0} e^{j(m-n)\omega_0 t} dt \qquad \left(\because \int e^x \Rightarrow \frac{1}{x} \cdot e^x \right)$$

$$= \frac{1}{j(m-n)\omega_0} \left[e^{j(m-n)\omega_0 (t+T_0)} - e^{+j(m-n)\omega_0 t} \right]$$

$$= \frac{1}{j(m-n)\omega_0} \left[e^{j(m-n)\omega_0 t} \cdot e^{j(m-n)\omega_0 T_0} - e^{j(m-n)\omega_0 t} \right]$$

$$= \frac{1}{j(m-n)\omega_0} e^{j(m-n)\omega_0 t} \left[e^{j(m-n)\omega_0 T_0} - 1 \right]$$

Here $\omega_0 = \frac{2\pi}{T_0}$ $\therefore \omega_0 T_0 = 2\pi$

$$\int_t^{t+T_0} e^{jm\omega_0 t} [e^{jn\omega_0 t}]^* dt = \frac{1}{j(m-n)\omega_0} e^{j(m-n)\omega_0 t} [1-1]$$

$$= 0 \qquad \left\{ \because e^{j(m-n) \cdot 2\pi} = 1 \text{ always} \right\}$$

Thus complex exponentials are orthogonal over any time period T_0.

Now when $n = m$ i.e

$$\int_t^{t+T_0} e^{jm\omega_0 t} [e^{jn\omega_0 t}]^* dt = \int_t^{t+T_0} e^{j(m-m)\omega_0 t} dt = \int_t^{t+T_0} 1\, dt$$

$$= [t]_t^{t+T_0} = T_0$$

$$\int_{t}^{t+T_0} e^{jm\omega_0 t} e^{-jn\omega_0 t}\, dt = \begin{cases} 0 & \text{for } m \neq n \\ T_0 & \text{for } m = n \end{cases}$$

$\downarrow$

Energy of complex exponential fn.

(3) If $x(t)$ and $y(t)$ are orthogonal then show that the energy of the signal $x(t)+y(t)$ is identical to the energy of the signal $x(t)$ plus energy of the signal $y(t)$.

Sol Let energy of $x(t)$ be E_x & energy of $y(t)$ be E_y i.e

$$E_x = \int_{-\infty}^{\infty} x^2(t)\, dt \quad ; \quad E_y = \int_{-\infty}^{\infty} y^2(t)\, dt.$$

Energy of sum of signal i.e $x(t)$ and $y(t)$ will be

$$\int_{-\infty}^{\infty} [x(t)+y(t)]^2\, dt = \int_{-\infty}^{\infty} [x^2(t)+y^2(t)+2x(t)y(t)]\, dt$$

$$= \int_{-\infty}^{\infty} x^2(t)\,dt + \int_{-\infty}^{\infty} y^2(t)\, dt + 2\int_{-\infty}^{\infty} x(t)\,y(t)\,dt$$

Since $x(t)$ & $y(t)$ are orthogonal, third integration term in above eqn will be zero

$$\int_{-\infty}^{\infty} [x(t)+y(t)]^2\, dt = \int_{-\infty}^{\infty} x^2(t)\, dt + \int_{-\infty}^{\infty} y^2(t)\, dt$$

$$= E_x + E_y$$

$\therefore$ Sum of energies of orthogonal is equal to energy of the total sum of signals.

(4) Show that over the period of interval '0' to 2π, a rectangular function is orthogonal to signals $\cos t, \cos 2t \ldots \cos nt$ for all integers values of n.

Sol Rectangular function over the period 0 to 2π

$$f(t) = \begin{cases} 1 & \text{for } 0 \le t \le 2\pi \\ 0 & \text{for others} \end{cases}$$

$$x(t) = \cos nt$$

$$\int_t^{t+T_0} f(t)\, x(t)\, dt = \int_t^{t+T_0} 1 \cdot \cos nt\, dt$$

$$= \int_t^{2\pi} \cos nt\, dt \Rightarrow \left[\frac{\sin nt}{n}\right]_0^{2\pi} \Rightarrow \frac{1}{n}\left[\sin 2\pi t - \sin 0\right] = 0$$

$\therefore$ $\cos nt$ and rectangular fn are orthogonal over an interval 0 to 2π

(5) Show that the sequence $e^{j2\pi kn/N}$ is an orthogonal sequence, periodic in N.

Sol Given sequence $x(n) = e^{j2\pi kn/N}$

It will be periodic if $x(n+N) = x(n)$

$$x(n+N) = e^{j2\pi k(n+N)/N} = e^{j2\pi kn/N} \cdot e^{j2\pi k}$$

Here $e^{j2\pi k} = \cos 2\pi k + j\sin 2\pi k$

$= 1 + 0 = 1$ always

$$x(n+N) = e^{j2\pi kn/N} = x(n)$$

$\therefore$ $x(n)$ is periodic with period N.

Let two sequences be $x_k(n) = e^{j2\pi kn/N}$ and $x_l(n) = e^{j2\pi ln/N}$

Orthogonality of discrete time sequences can be checked over one period

$$\sum_{n=0}^{N-1} x_k(n)\, x_l^*(n) = \sum_{n=0}^{N-1} e^{j2\pi kn/N} \left[e^{j2\pi ln/N}\right]^*$$

$$= \sum_{n=0}^{N-1} e^{j2\pi kn/N} \cdot e^{-j2\pi ln/N} = \sum_{n=0}^{N-1} e^{j2\pi \left(\frac{k-l}{N}\right)n}.$$

Standard series formula

$$\sum_{n=N_1}^{N_2} a^n = \frac{a^{N_1} - a^{N_2+1}}{1-a} \quad \{N_2 > N_1\}$$

Here $a = e^{j2\pi\left(\frac{K-l}{N}\right)}$

$$\sum_{n=0}^{N-1} x_K(n)\, x_l^*(n) = \frac{\left[e^{j2\pi \frac{K-l}{N}}\right]^0 - \left[e^{j2\pi\left(\frac{K-l}{N}\right)}\right]^N}{1-e^{j2\pi\frac{K-l}{N}}} = \frac{1-e^{j2\pi(K-l)}}{1-e^{j2\pi\frac{K-l}{N}}}.$$

K & l are integers, so K-l will also be an integer. Therefore $e^{j2\pi(K-l)} = 1$ always

$$\sum_{n=0}^{N-1} x_K(n)\, x_l^*(n) = \frac{1-1}{1-e^{j2\pi\frac{K-l}{N}}} = 0$$

⑥ Find if following signals are orthogonal $x_1(n) = e^{jK\left(\frac{\pi}{8}\right)n}$ and $x_2(n) = e^{jm\left(\frac{\pi}{8}\right)n}$

Sol $x_1(n) = e^{jK\left(\frac{\pi}{16}\right)n}$ & $x_2(n) = e^{jm\left(\frac{\pi}{16}\right)n}$

$$\sum_{n=0}^{N-1} x_1(n)\, x_2^*(n) = \sum_{n=0}^{N-1} e^{j2\pi Kn/16} \cdot e^{-j2\pi mn/16} = \sum_{n=0}^{N-1} e^{j2\pi(K-m)n/16}$$

$$= \frac{\left[e^{j2\pi(K-m)/16}\right]^0 - \left[e^{j2\pi(K-m)/16}\right]^N}{1-e^{j2\pi(K-m)/16}}$$

Here $N = 16$

$$= \frac{1-e^{j2\pi(K-m)}}{1-e^{j2\pi(K-m)/16}}.$$

but $e^{j2\pi(K-m)} = 1$ always so

$$\sum_{n=0}^{N-1} x_1(n)\, x_2^*(n) = 0$$

∴ two sgls are orthogonal.

Classification of Signals:

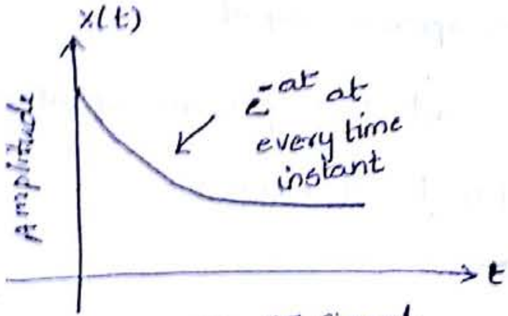
→ Signals classified into two types depending on independent variable time

a) Continuous Time (CT) signals

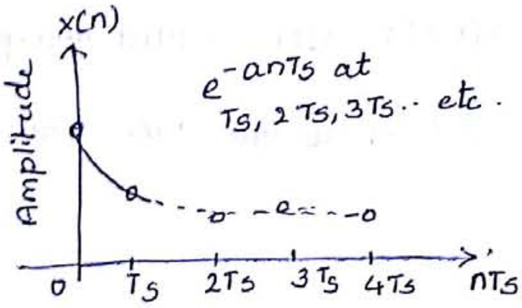
b) Discrete Time (DT) signals.

CT signals & DT signals:

It is defined continuously with respect to time. A DT signal is defined only at specific or regular time instants.



(a) CT signal



(b) DT signal.

Significance:

(i) Analog circuits process CT signals. Such circuits are op-amps, filters, amplifier etc.,

(ii) Digital circuits process DT signals. Such circuits are microprocessors, counters, flipflops etc..

→ When amplitude of CT signal varies continuously it is called analog sgl. In other words amplitude and time are continuous for analog signal.

→ When amplitude of DT signal takes only finite varies it is digital signal.

Periodic and Non-Periodic Signals:

→ A signal is said to be periodic if it repeats at regular intervals.
(or)

→ A signal which repeats after every time interval T is called periodic signal.

· $x(t)$ is called periodic if and only if

$$\underset{\text{time}}{x(t+T)} = x(t) \text{ for all } t$$

constant

→ The smallest value of T that satisfies this condition is called fundamental period or simple period of $x(t)$.

→ The reciporal of fundamental period T is called fundamental frequency f of $x(t)$

$$f = \frac{1}{T}$$

$$\text{Angular frequency} = \omega = 2\pi f = \frac{2\pi}{T}$$

$$T = \frac{2\pi}{\omega}$$

→ A signal $x(t)$ for which there is no value of T satisfying the condition $x(t+T) = x(t)$ is called non-periodic or aperiodic signal.

$\parallel^{ly}$ for discrete-time signals, $x[n]$ is said to be periodic if it satisfies

$$x[n+\underset{\substack{\downarrow \\ \text{+ve integer}}}{N}] = x[n] \text{ for all integers 'n'}$$

$$\text{Angular frequency } \Omega = \frac{2\pi}{N}$$

$$N = \frac{2\pi}{\Omega}$$

Note :

(1) Sum of two continuous-time periodic signals may not be periodic

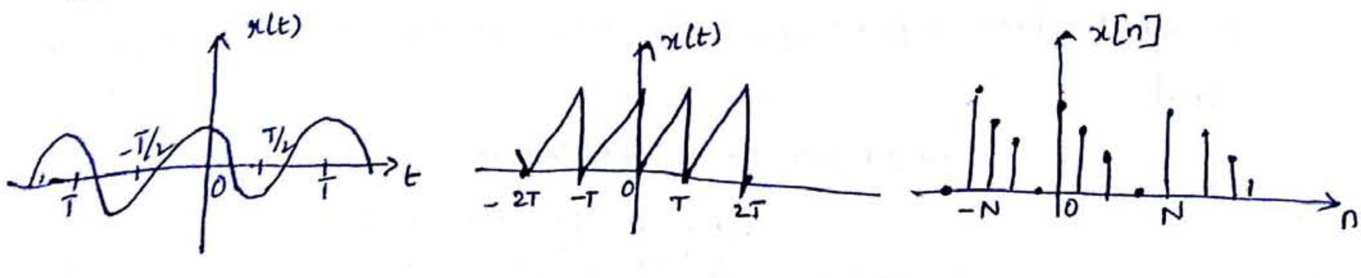
(2) Sum of two periodic sequences is always periodic.

(3) Sum of two periodic signals is periodic only if the ratio of their respective periods is a rational number.

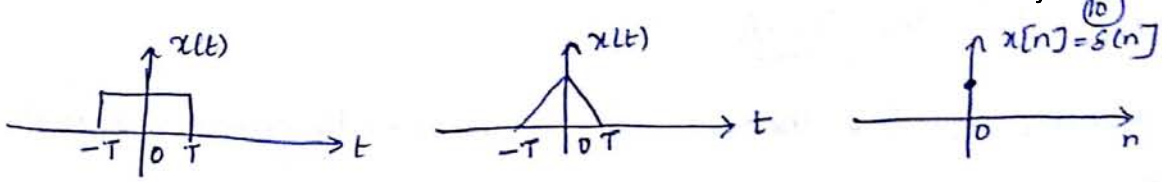
$$\frac{T_1}{T_2} = \text{rational number.}$$

(4) Fundamental period is the LCM of T_1 & T_2.

(5) If the ratio T_1/T_2 is an irrational number, then signals $x_1(t)$ & $x_2(t)$ do not have a common period and $x(t)$ cannot be periodic.



(a) periodic



(b) Non-periodic

Even and Odd Signals:

→ A signal $x(t)$ or $x[n]$ is said to be an even sgl if it satisfies the condition

$$x(-t) = x(t) \text{ for all } t,$$

$$x[-n] = x[n] \text{ for all } n.$$

→ A signal $x(t)$ or $x[n]$ is said to be an odd sgl if it satisfies the condition

$$x(-t) = -x(t) \text{ for all 't'}$$

$$x[-n] = -x[n] \text{ for all 'n'}.$$

→ Even signals (cosine) are symmetrical abt vertical axis or time origin whereas odd signals (sine) are asymmetric.

→ A signal $x(t)$ or $x[n]$ can be expressed as sum of two signals i.e one odd & one even.

$$x(t) = x_e(t) + x_o(t)$$

$$x[n] = x_e[n] + x_o[n]$$

where

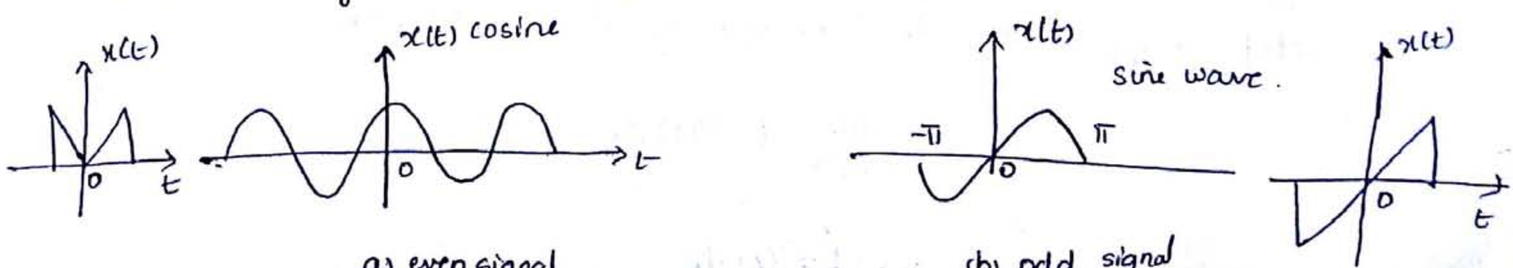
$$x_e[n] = \frac{1}{2}\{x(n) + x(-n)\}, \text{ even part}$$

$$x_o[n] = \frac{1}{2}[x(n) - x(-n)]$$

$$x_e(t) = \frac{1}{2}\{x(t) + x(-t)\}$$

$$x_o(t) = \frac{1}{2}\{x(t) - x(-t)\}$$

→ The product of two even or odd signals is an even signal & product of an even signal and odd signal is an odd signal.



a) even signal (b) odd signal

Deterministic and Random Signals

→ A deterministic signal is the one where no uncertainity occurs w.r.to its value at any time.

$$x(t) = 100 \sin 50t \text{ (Continuous)}$$

$$x[n] = 100 \sin 50n \text{ (Discrete)}$$

→ A random signal is the one about which there is some degree of uncertainity before it actually occurs.

For example : the o/p of TV/radio receiver when tuned to frequency where there is no broadcast.

Real And Complex Signals:

→ $x(t)$ is real signal if its value is a real number and is a complex signal if its value is a complex number.

$$\text{Eg: } x(t) = x_1(t) + jx_2(t)$$

real signals and $j = \sqrt{-1}$

Energy and Power Signals:

→ In electrical systems, signals may represent current or vltg.

Consider a voltage signal $v(t)$ across resistor 'R' producing current $i(t)$

Then power dissipated in resistor is given by

$$p(t) = \frac{v^2(t)}{R} = i^2(t).R.$$

when $R = 1\Omega$

$$p(t) = v^2(t) = i^2(t).$$

In general $x(t)$ whether it is vltg or current sgl we get power given by

$$p(t) = x^2(t).$$

Total energy or normalized energy E of sgl $x(t)$ is defined by

$$E = \lim_{T \to \infty} \int_{-T/2}^{T/2} x^2(t)\, dt.$$

$$\therefore E = \int_{-\infty}^{\infty} x^2(t)\, dt$$

The average power or normalized average power P of the signal x(t) is

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t)\, dt$$

→ In case of discrete-time signal $x[n]$, integrals replaced by summation

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

Note:

→ The signals for which total energy is finite $(0 < E < \infty)$ are called energy signals. They have zero average power.

Ex: deterministic & non-periodic sgls.

→ The signals for which the average power is finite $(0 < P < \infty)$ are called power signals. They have infinite energy Ex: Random, periodic sgl.

→ Both energy & power signals are mutually exclusive.

Elementary Signals:

Unit step function:

↓

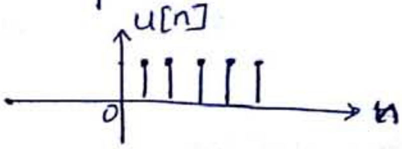
Important signal used in many cases. Ex: When we apply brake to an automobile we are applying constant force.

→ If a step function has unity magnitude then it is called unit step fn.

It is defined as

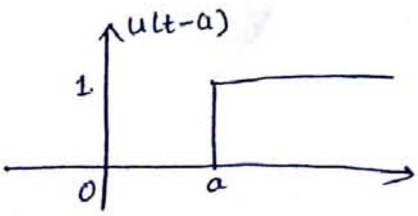
$$u(t) = 1 \quad \text{for } t \geq 0$$
$$= 0 \quad \text{for } t < 0$$

discrete $u[n] = 1$ for $n \geq 0$
0 for $n < 0$.



$||^{ly}$ for shifted unit step fn $u(t-a)$ is zero if $t-a<0$ or $t<a$. and $t-a>0$ or $t>a$.

$$\therefore u(t-a) = 1 \quad \text{for } t>a$$
$$= 0 \quad \text{for } t<a$$



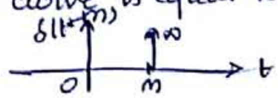
Impulse function (Dirac Delta fn):

It is defined as

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1 \text{ at } t=0 \text{ and } \delta(t) = 0 \text{ for } t \neq 0$$

i.e The function has zero amplitude everywhere except at $t=0$. At $t=0$ amplitude is infinity such that area under the curve is equal to one.

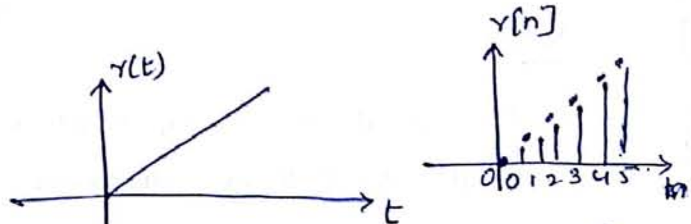
$$\delta(t-m) = \infty \;;\; t=m \quad \& \quad \int_{-\infty}^{\infty} \delta(t-m)\, dt = 1$$
$$= 0 \;\;\; t \neq m$$



Unit Ramp Function:

Unit ramp is defined as

$$r(t) = t \quad \text{for } t \geq 0$$
$$= 0 \quad \text{for } t < 0$$



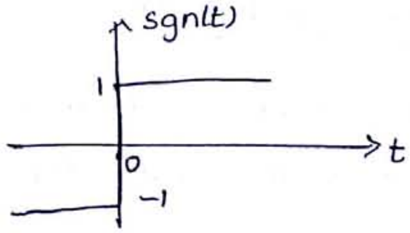
→ Ramp fn can be obtained by applying unit step fn to integrator

$$r(t) = \int u(t)\, dt = \int dt = t \quad (\text{in interval } t>0).$$

Signum function:

It is defined by

$$sgn(t) = \begin{cases} 1, & t>0 \\ -1, & t<0 \\ 0, & t=0 \end{cases}$$



→ This fn in terms of unit step fn

$$sgn(t) = 2u(t) - 1$$

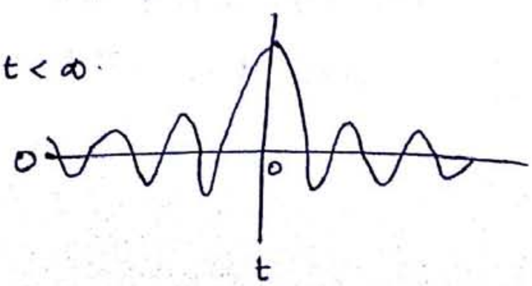
For $t>0$ $\quad 2u(t) = 2$

for $t<0$ $\quad 2u(t) = 0$.

Sinc function:

The sinc fn defined by expression

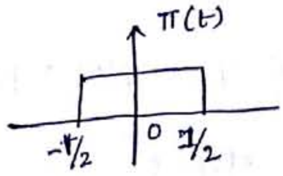
$$sinc(t) = \frac{\sin t}{t} \quad -\infty < t < \infty.$$



Rectangular pulse function:

$\Pi(t) = 1$ for $|t| \leq \frac{1}{2}$

$= 0$ otherwise

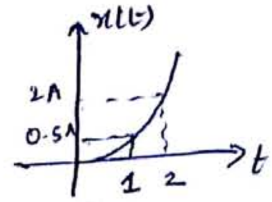


Parabolic signal:

defined as

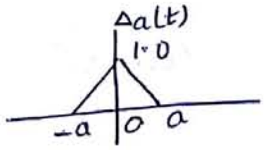
$x(t) = \frac{At^2}{2}$ for $t \geq 0$

$= 0 \; ; t < 0$



Triangular pulse function:

$$\Delta_a(t) = \begin{cases} 1 - \frac{|t|}{a} & |t| \leq a \\ 0 & |t| > a \end{cases}$$

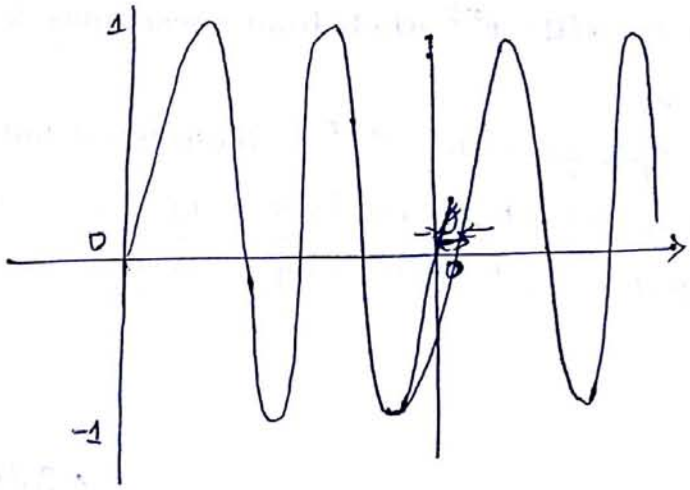


Sinusoidal signal: / cosinusoidal signal $x(t) = A\cos(\Omega t + \phi)$

A continuous-time sinusoidal signal is given by

$x(t) = A\sin(\Omega t + \theta)$ → phase angle in radians.

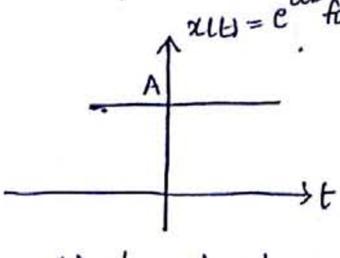
A → amplitude; Ω → frequency in radians per second.



$A = 1$, $\theta = -\frac{\pi}{3}$, time period of $\frac{2\pi}{\Omega}$.

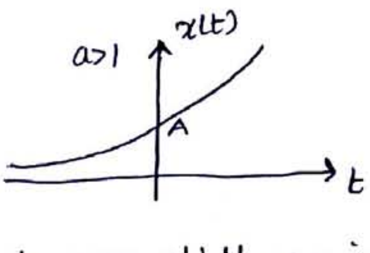
Real Exponential signals:

$x(t) = e^{at}$ for $a = 0$



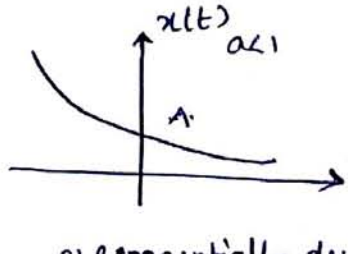
a) dc signal

'a' is positive



b) exponentially growing

'a' is negative



c) exponentially decaying

A real exponential signal is defined as

$$x(t) = Ae^{at}$$

where A, a are real.

Complex exponential signal:

The most general form of complex exponential is given by

$$x(t) = e^{st}$$

where s = complex variable = $\sigma + j\Omega = s$

$$x(t) = e^{st} = e^{(\sigma + j\Omega)t} = e^{\sigma t} \cdot e^{j\Omega t} \rightarrow ①$$

Using Euler's identity

$$e^{j\Omega t} = \cos\Omega t + j\sin\Omega t \rightarrow ②$$

Substitute ② in ①

$$x(t) = e^{\sigma t}[\cos\Omega t + j\sin\Omega t]$$

→ Depending on the values of σ and Ω we get different signals.

1. If $\sigma = 0$, $\Omega = 0$; $x(t) = 1$; pure DC signal
2. If $\Omega = 0$ then $s = \sigma$, $x(t) = e^{\sigma t}$ which decay exponentially for $\sigma < 0$ & grows exponentialy for $\sigma > 0$.
3. If $\sigma = 0$ then $s = \pm j\Omega$ gives $x(t) = e^{j\Omega t}$ a sinusoidal sgl with $\phi = 0$.
4. If $\sigma < 0$ then finite Ω we get exponentially decaying sinusoidal signal.
5. If $\sigma > 0$ with finite Ω, we get exponentially growing sinusoidal signal.

Gaussian Signal

It is defined as

$$x(t) = g_a(t) = e^{-a^2 t^2} ; -\infty < t < \infty$$