8.2 Reflection and Refraction of a Uniform Plane Wave

The propagation of an electromagnetic wave through different materials (media) having different constant parameters such as ε, μ, σ, η, etc., undergoes partial reflections and refractions at the boundaries of the media. Some part of the incident wave is transmitted and the other part is reflected back at the boundary depending on the angle of incidence.

For example, when a plane wave in air is incident normally on the surface of a perfect conductor, the wave completely reflects back. The combination of incident (forward) wave and reflected (backward) wave result in standing waves.

Depending upon the direction of incident wave, two cases can be considered.

(1) Normal incidence: The incident wave is normal to the plane of the boundary surface.

(2) Oblique incidence: The incident wave makes an angle θ with the normal to the plane of the boundary surface.

When the wave is incident obliquely at the boundary between two materials, the following terms are considered (Fig. 8.1).

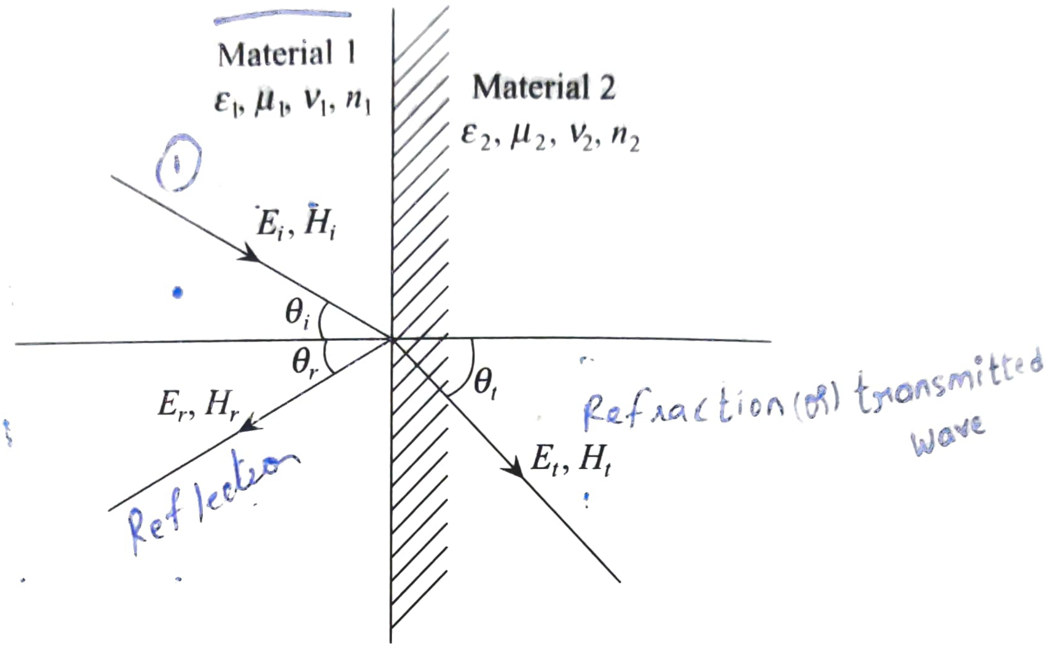


Fig. 8.1 *Reflection and refraction of a plane wave*

Incidence angle θ_i The angle at which the incident wave makes with the normal to the interface (boundary) is called the incidence angle θ_i.

Reflection angle θ_r The angle at which the reflected wave makes with the normal to the interface (boundary) is called the reflected angle θ_r.

Transmission angle θ_t The angle at which the transmitted (refracted) wave makes with the normal to the interface (boundary) is called the transmitted angle or the refracted angle θ_t.

Snell's law (law of refraction) The law of refraction is also known as Snell's law. It is used to determine the direction of waves travelling through refractive media with varying indices of refraction.

Snell's law states that the ratio of the sines of the angles of incidence and refraction is equivalent to the ratio of phase velocities in the two media, or equivalent to the reciprocal of the ratio of the refractive indices. That is

$$\frac{\sin\theta_i}{\sin\theta_t} = \frac{v_1}{v_2} = \frac{n_2}{n_1}, \tag{8.1}$$

where v_1 and v_2 are the velocities of the wave in medium 1 and medium 2, respectively and n_1 and n_2 are the refractive indices of medium 1 and medium 2, respectively.
For dielectric media,

$$\mu_1 = \mu_2 = \mu_0.$$

$$\therefore \quad v_1 = \frac{1}{\sqrt{\varepsilon_1 \mu_0}} \quad \text{and} \quad v_2 = \frac{1}{\sqrt{\varepsilon_2 \mu_0}}.$$

$$\therefore \quad \frac{\sin\theta_i}{\sin\theta_t} = \frac{\dfrac{1}{\sqrt{\varepsilon_1\mu_0}}}{\dfrac{1}{\sqrt{\varepsilon_2\mu_0}}} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} = \frac{n_2}{n_1}$$

or $\quad \sqrt{\varepsilon_1}\sin\theta_i = \sqrt{\varepsilon_2}\sin\theta_t$

and $\quad \dfrac{n_2}{n_1} = \sqrt{\dfrac{\varepsilon_2}{\varepsilon_1}}. \qquad (8.2)$

Law of reflection The law of reflection states that a wave incident upon a reflective surface will be reflected at an angle equal to the incident angle

i.e., incidence angle = reflected angle, $\theta_i = \theta_r$.

Reflection coefficient Γ It is the ratio of the amplitudes of reflected and incident waves. It gives the fraction of the incident wave that is reflected back from the interface. It is a dimensionless quantity and exists in the range of -1 to 1.

It can be expressed as $\Gamma = \dfrac{E_r}{E_i}$ or $\Gamma = \dfrac{H_r}{H_i}$,

where E_i and H_i are incident wave fields and E_r and H_r are reflected wave fields, respectively.

Transmission coefficient T It is the ratio of the amplitudes of refracted and incident waves. It gives the fraction of the incident wave that is transmitted into the second material from the interface. It is a dimensionless quantity and exists in the range of 0 to 2. It can be expressed as $T = \dfrac{E_t}{E_i}$ or $T = \dfrac{H_t}{H_i}$, where E_t, H_t are refracted fields in the second material.

The study of reflection and transmission coefficients of a uniform plane wave is important in various applications where signal transmission is made through transmission lines such as optical fibre systems, radar systems, remote sensing, missile tracking and guidance, stealth operations and medical imaging. For example, in optical fibre communications, light wave propagates due to total internal reflection and in satellite communications, the wave is refracted through the ionosphere.

8.3 Reflection by a Perfect Conductor with Normal Incidence

Consider a plane wave in a homogeneous medium incident normally on a perfect conductor at $z = 0$ plane as shown in Fig. 8.2. Assume that the electric field is in the x direction and the magnetic field is in the y direction. Since no EM fields exist within a perfect conductor, none of the energy is transmitted and absorbed. The incident wave is completely reflected back.

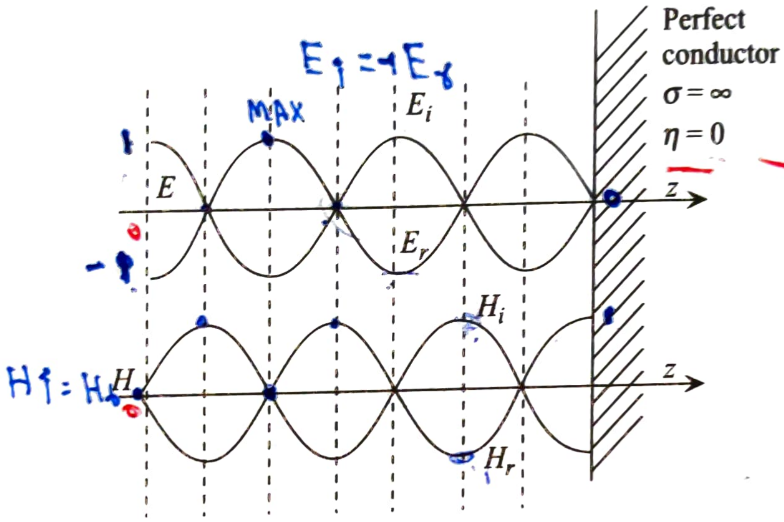
The field vectors of the incidence wave in phasor notation are

$$\boxed{\vec{E}_i = E_i e^{-j\beta z}\vec{a}_x;} \quad \boxed{\vec{H}_i = H_i e^{-j\beta z}\vec{a}_y}$$

and that of the reflected wave,

$$\vec{E}_r = E_r e^{j\beta z}\vec{a}_x;\quad \vec{H}_r = H_r e^{j\beta z}\vec{a}_y, \tag{8.3}$$

where E_i, H_i and E_r, H_r are the field amplitudes of the incident and reflected waves at $z = 0$ respectively.



E
Y
x H

$\hat{a}_x \rightarrow \vec{E} \rightarrow Z$

a

Fig. 8.2 *Standing waves of $\vec{E}$ and $\vec{H}$*

We know that the tangential component of the electric field vector is continuous at the boundary surface. Therefore the fields at the perfect conductor are

$$E_i + E_r = E_t = 0$$

or $$E_i = -E_r. \tag{8.4}$$

Thus the electric field vector reflects with 180° phase shift at the boundary surface. The superposition of incident and reflected waves travelling in opposite directions produces a standing wave.

The electric field strength of the standing wave is

$E_r = -E_i$

$E_i e^{-j\beta z} - E_i e^{+j\beta z} = E(z)$

$$E(z) = E_i e^{-j\beta z} + E_r e^{+j\beta z}$$

$$= E_i(e^{-j\beta z} - e^{+j\beta z}) = -2j\frac{(e^{+j\beta z} - e^{-j\beta z})}{2j} \cdot E_i$$

$$\frac{e^{+j\beta z} - e^{-j\beta z}}{2j} = \sin(\beta z)$$

-ve Common

or $$\vec{E}(z) = -2jE_i\sin(\beta z)\vec{a}_x. \tag{8.5}$$

The field in time domain is

$$\vec{E}(z,t) = \Re e[-2jE_i\sin(\beta z)e^{j\omega t}\vec{a}_x]. \tag{8.6}$$

Since E_i is a real quantity,

(x, t)

$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$

$$\vec{E}(z,t) = 2E_i\sin(\beta z)\sin(\omega t)\vec{a}_x. \tag{8.7}$$

Thus the magnitude of the electric field vector of a standing wave varies sinusoidaly with distance from the boundary. It is zero at the boundary surface. The maximum amplitude $2E_i$ occurs at odd multiples of $\lambda/4$ from the surface as shown in Fig. 8.2.

Similarly, the magnetic field vector, since it is normal to the electric field, reflects without phase reversal at the boundary surface. So at the boundary,

$$\boxed{H_r = H_i.}$$

The magnetic field vector of the standing wave is

$$H(z) = H_i e^{-j\beta z} + H_r e^{j\beta z} = 2H_i \frac{(e^{j\beta z} + e^{-j\beta z})}{2}$$

or $$\vec{H}(z) = 2H_i \cos(\beta z)\,\vec{a}_y. \tag{8.8}$$

$$\boxed{\eta = \frac{\vec{E}}{\vec{H}}}$$

The field in time domain is

$$\vec{H}(z,t) = \Re e\{2H_i \cos(\beta z)\, e^{j\omega t}\}\,\vec{a}_y. \tag{8.9}$$

Since H_i is a real quantity,

$$\vec{H}(z,t) = 2H_i \cos(\beta z)\cos(\omega t)\,\vec{a}_y.$$

$$\vec{H}(z,t) = \frac{2E_i}{\eta}\cos(\beta z)\cos(\omega t)\,\vec{a}_y. \tag{8.10}$$

$$\boxed{\eta = \frac{\vec{E_i}}{H_i}}$$

Thus the magnitude of the magnetic field vector of a standing wave varies sinusoidaly with distance from the boundary. The standing wave distribution of the magnetic field has maximum amplitude, $2H_i$ at the surface of the boundary and at even multiples of $\lambda/4$ from the surface.

The reflection coefficient for the electric field is

$$\Gamma = \frac{E_r}{E_i} = -1.$$

The transmission coefficient is $T = 0$.

Note: From the boundary conditions, the magnetic field strength $\vec{H}$ is equal to the surface current density $\vec{K}$ on the surface.

$$\vec{H} = \vec{K}.$$

8.4 Reflection by a Perfect Dielectric—Normal Incidence

Consider a uniform plane wave travelling along the z direction and incident normally on the surface of a perfect dielectric media. At the interface, the wave is partly reflected and partly transmitted. Assume that the electric field is in the x direction and magnetic field is in the y direction.

Let medium 1 and 2 have the properties ε_1, μ_1, v_1, η_1 and ε_2, μ_2, v_2, η_2 respectively. A perfect dielectric (lossless) has zero conductivity so that there is no loss of power in propagation.

Let E_i and H_i be the incident wave field strengths, E_r and H_r, the reflected wave field strengths, and E_t and H_t, the transmitted wave field strengths, respectively.

If the electric field vector is oriented upwards parallel to the plane of the paper, then the magnetic field vectors for incident and transmitted waves are oriented outward to the plane of the paper, indicated as symbol "$\odot H$", and for the reflected wave, it is oriented into the plane of the paper indicated as symbol "$\otimes H$" as shown in Fig. 8.3.

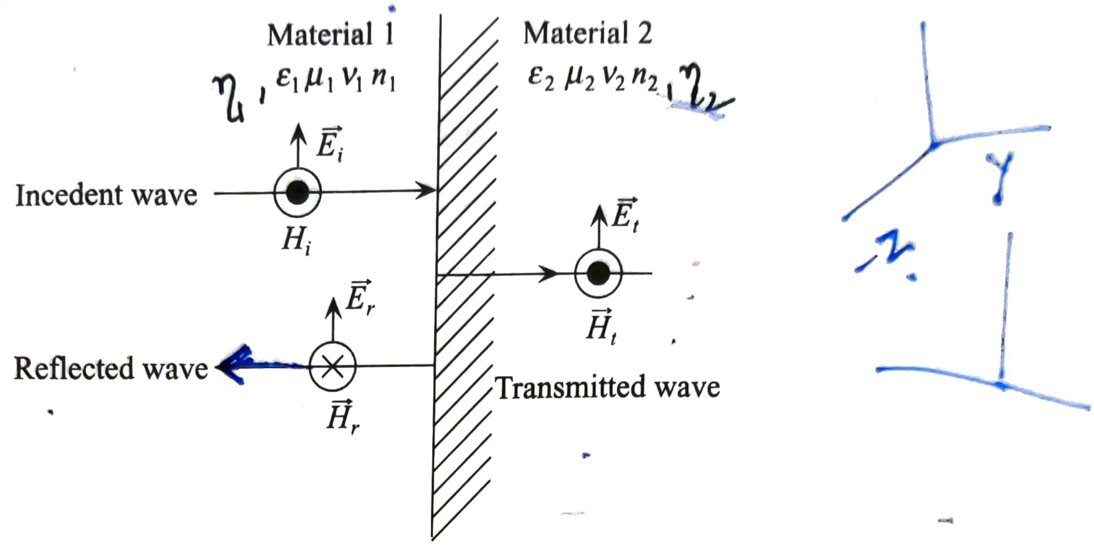


Fig. 8.3 *Normal incidence on a perfect dielectric*

The field vectors for the incident wave in phasor notation are

$$\boxed{\vec{E}_i = E_i e^{-j\beta z} \vec{a}_x,} \quad \boxed{\vec{H}_i = H_i e^{-j\beta z} \vec{a}_y,}$$

and for the reflected wave travelling in the $-\vec{a}_z$ direction,

$$\boxed{\vec{E}_r = E_r e^{j\beta z} \vec{a}_x} \quad \boxed{\vec{H}_r = -H_r e^{j\beta z} \vec{a}_y}$$

and for the transmitted wave

$$\boxed{\vec{E}_t = E_t e^{-j\beta z} \vec{a}_x,} \quad \boxed{\vec{H}_t = H_t e^{-j\beta z} \vec{a}_y,} \tag{8.11}$$

where E_i, H_i, E_r, H_r and E_t, H_t are the field amplitudes of incident, reflected and transmitted waves at $z = 0$ respectively.

We know that the relationships between electric and magnetic fields are

$H_i = \frac{E_i}{\eta}$

$$\boxed{E_i = \eta_1 H_i;} \quad E_r = -\eta_1 H_r; \quad E_t = \eta_2 H_t, \tag{8.12}$$

where η_1 and η_2 are the intrinsic impedances of medium 1 and medium 2 respectively.

At the boundary, the tangential components of the field are continuous, i.e.,

$\eta_1 = \frac{E_i}{H_i}$

$$E_i + E_r = E_t;$$

$$H_i + H_r = H_t. \tag{8.13}$$

$\boxed{H_r = -\frac{E_r}{\eta_1}}$

$\boxed{H_i = \frac{E_i}{\eta_1}}$

Substituting Eq. (8.12) for electric fields,

$$\frac{E_i}{\eta_1} - \frac{E_r}{\eta_1} = \frac{E_t}{\eta_2}.$$

$$\frac{E_i}{\eta_1} - \frac{E_r}{\eta_1} = \frac{(E_i + E_r)}{\eta_2}.$$

$$\eta_2 E_i - \eta_2 E_r = \eta_1 E_i + \eta_1 E_r.$$

$$E_i(\eta_2 - \eta_1) = E_r(\eta_1 + \eta_2).$$

$\therefore$ The reflection coefficient for the electric field is

$$\Gamma = \frac{E_r}{E_i} = \frac{\eta_2 - \eta_1}{\eta_1 + \eta_2}. \tag{8.14}$$

$\Gamma = \frac{E_r}{E_i}$

T

The transmission coefficient for the electric field is

$$T = \frac{E_t}{E_i} = \frac{E_i + E_r}{E_i} = 1 + \frac{E_r}{E_i} = 1 + \Gamma.$$

$$T = 1 + \frac{\eta_2 - \eta_1}{\eta_1 + \eta_2} = \frac{2\eta_2}{\eta_1 + \eta_2}.$$

$\Rightarrow$ $\frac{\cancel{\eta_1} + \eta_2 + \eta_2 - \cancel{\eta_1}}{\eta_1 + \eta_2} = \frac{2\eta_2}{\eta_1 + \eta_2}$

$$T = \frac{E_t}{E_i} = \frac{2\eta_2}{\eta_1 + \eta_2}. \tag{8.15}$$

The reflection coefficient for the magnetic field is

$\eta_1 = \frac{E_i}{H_i}$

$$\Gamma_2 \quad \frac{H_r}{H_i} = \frac{-E_r/\cancel{\eta_1}}{E_i/\cancel{\eta_1}} = \frac{-E_r}{E_i} = \frac{\eta_1 - \eta_2}{\eta_1 + \eta_2} \tag{8.16}$$

$\boxed{H_i = \frac{E_i}{\eta_1}}$

and the transmission coefficient for the magnetic field is

$$T = \frac{H_t}{H_i} = \frac{E_t/\eta_2}{E_i/\eta_1} = \frac{\eta_1}{\eta_2} \times \frac{E_t}{E_i} = \frac{2\eta_1}{\eta_1 + \eta_2}. \tag{8.17}$$

$H_r = \frac{E_r}{\eta_1}$

Reflection and transmission coefficients in terms of dielectric constants

For free space and dielectrics, $\mu_1 = \mu_2 = \mu_0$.
Also, the intrinsic impedances are

$$\eta_1 = \sqrt{\frac{\mu_0}{\varepsilon_1}} \quad \text{and} \quad \eta_2 = \sqrt{\frac{\mu_0}{\varepsilon_2}}.$$

T

Substituting in Eq. (8.14), the reflection coefficient is

$$\frac{E_r}{E_i} = \frac{\sqrt{\frac{\mu_0}{\varepsilon_2}} - \sqrt{\frac{\mu_0}{\varepsilon_1}}}{\sqrt{\frac{\mu_0}{\varepsilon_2}} + \sqrt{\frac{\mu_0}{\varepsilon_1}}} = \frac{\frac{1}{\sqrt{\varepsilon_2}} - \frac{1}{\sqrt{\varepsilon_1}}}{\frac{1}{\sqrt{\varepsilon_2}} + \frac{1}{\sqrt{\varepsilon_1}}}.$$

$$\Gamma = \frac{E_r}{E_i} = \frac{\sqrt{\varepsilon_1} - \sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2}} \tag{8.18}$$

and $$\frac{H_r}{H_i} = \frac{-E_r}{E_i} = \frac{\sqrt{\varepsilon_2} - \sqrt{\varepsilon_1}}{\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2}}. \tag{8.19}$$

The transmission coefficient is

$$T = \frac{E_t}{E_i} = \frac{2\sqrt{\dfrac{\mu_0}{\varepsilon_2}}}{\sqrt{\dfrac{\mu_0}{\varepsilon_1}} + \sqrt{\dfrac{\mu_0}{\varepsilon_2}}} = \frac{2\sqrt{\varepsilon_1}}{\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2}} \tag{8.20}$$

and
$$\frac{H_t}{H_i} = \frac{2\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2}}. \tag{8.21}$$

Example 8.1 A plane wave travelling in air is normally incident on a block of paraffin with $\varepsilon_r = 2.2$. Find the reflection coefficient.

Solution Given: For air, $\varepsilon_{r_1} = 1,\ \mu_{r_1} = \mu_{r_2} = 1,\ \varepsilon_{r_2} = 2.2.$

Intrinsic impedance of air is $\eta_1 = 120\pi\ \Omega$.

Intrinsic impedance of paraffin is

$$\eta_2 = \sqrt{\frac{\mu_0 \mu_r}{\varepsilon_0 \varepsilon_r}} = 120\pi \sqrt{\frac{\mu_r}{\varepsilon_r}}.$$

$$= 120\pi\sqrt{\frac{1}{2.2}} = \frac{120\pi}{\sqrt{2.2}} = 254\ \ \Omega.$$

$$\text{Reflection coefficient} = \frac{\eta_1 - \eta_2}{\eta_1 + \eta_2} = \frac{377 - 254}{377 + 254} = 0.195.$$

Topic Name: Reflection and Refraction of plane wave-

Oblique Incident for both Perfect Conductor and Perfect Dielectrics.

Introduction:

- *Reflection of Plane Wave -Oblique Incidence.*
- *Horizontal Polarization-Oblique Incidence.*
- *Vertical Polarization-Oblique Incidence.*

Topic :

Set 1

- Refection of [redacted] **Horizontal Polarized** Wave by a **Perfect Conductor**- Oblique Incidence.
- Refection of [redacted] **Vertical Polarized** Wave by a **Perfect Conductor**- Oblique Incidence.

Topic :

Set 2

Refection of [redacted] **Horizontal Polarized** Wave by a **Perfect Dielectric**- Oblique Incidence.

- Refection of [redacted] **Vertical Polarized** Wave by a **Perfect Dielectric**- Oblique Incidence.

Introduction

8.5 Reflection of a Plane Wave—Oblique Incidence

H

V

Consider a plane wave travelling in a lossless homogeneous medium. It is incident obliquely on the surface of another medium. At the interface, some part of the wave is reflected and the other part is refracted (transmitted) through the medium. The fields and direction of propagation follow the right-hand screw law. There are two situations which occur depending on the orientation of the electric field vector.

(1) Horizontal polarization (or perpendicular polarization)

(2) Vertical polarization (or parallel polarization)

Horizontal polarization In horizontal polarization the electric field vector of the wave is perpendicular to the plane incidence. It is also called perpendicular polarization. If the interfacing surface is normal to the paper surface, the plane of incidence is parallel to the plane of the paper. So the electric vector is normal to the plane of the paper. As shown in Fig. 8.4, if the electric vector is oriented outward to the plane of the paper, indicated as '$\odot E$', the magnetic vector is perpendicular to E_i and parallel to the plane of the paper.

Horizontally polarized waves can be generated from a horizontal antenna.

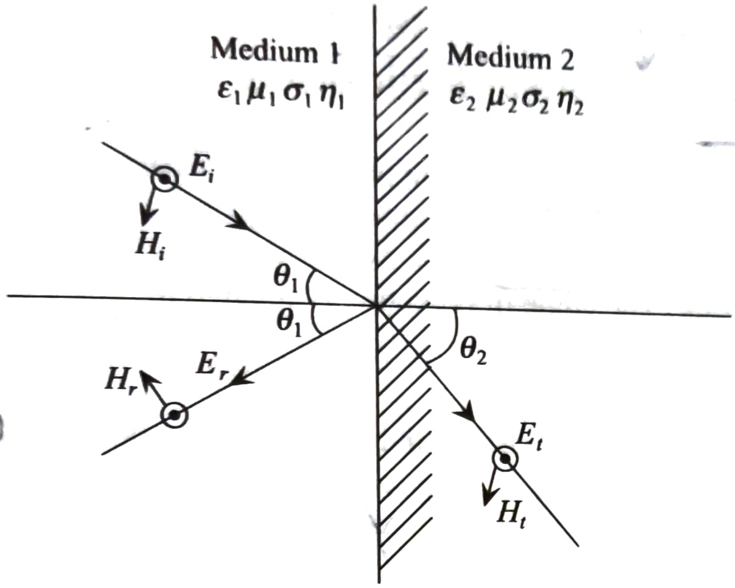


Fig. 8.4 *Reflection and refraction of a horizontally polarized wave*

Vertical polarization In vertical polarization, the electric field vector is parallel to the plane of incidence. It is also called parallel polarization. If the interfacing surface is normal to the paper surface, the plane of incidence is parallel to the plane of the paper. So the electric vector is parallel to the plane of the paper.

As shown in Fig. 8.5, if the electric vector is oriented parallel to the plane of the paper surface, the magnetic vector is oriented outward to the plane of the paper, indicated as "$\odot H$".

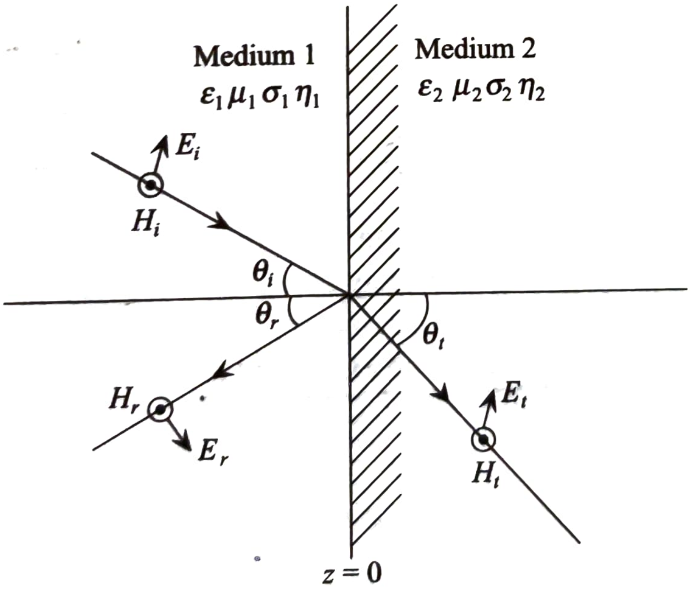


Fig. 8.5 *Reflection and refraction of a vertically polarized wave*

Vertically polarized waves can be generated from a vertical antenna. It is to be noted that the electric field vector in vertical polarization is not completely vertical—it may have some horizontal components.

4.2 Magnetic Field due to a Current-Carrying Conductor

When a conductor carries a current, it produces a magnetic field around it along the conductor. The direction of the magnetic flux lines (magnetic field) is given by the right-hand thumb rule (see Fig. 4.1), which is stated as follows.

When the right-hand thumb is opened and the fingers are closed, if the direction of the current is parallel to the thumb, then the direction of the fingers indicate the direction of magnetic field in a closed loop path.

Alternatively, the right-hand screw rule (Fig. 4.2) stated below also gives the direction of the magnetic field:

When a right-handed screw is turned, if the screw moves in the direction of the current flow, then the direction of the rotation of the screw gives the direction of magnetic field.

Fig. 4.1 *Right-hand thumb rule*

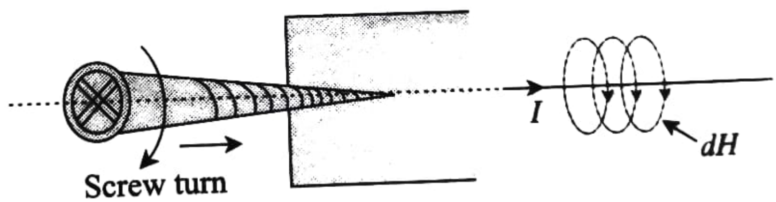
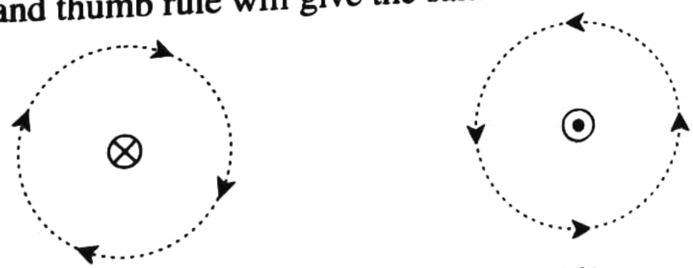


Fig. 4.2 *Right-hand screw rule*

In other words, we can say that if the current flows into the plane of paper (symbol ⊗), then the direction of the magnetic field is in the clockwise direction, and if the current flows outwards (symbol ⊙), the direction of the magnetic field is anti-clockwise (see Fig. 4.3). Both the right-hand screw rule and the right-hand thumb rule will give the same direction for magnetic field.



(*a*) (*b*)

Fig. 4.3 (*a*) *Current is flowing inwards into the paper and the flux direction is clockwise.*
(*b*) *Current is flowing outwards from the paper and the flux direction is anti-clockwise.*

8.6 Reflection of a Horizontally Polarized Wave by a Perfect Conductor—Oblique Incidence

Consider a horizontally polarized wave incident obliquely on a perfect conductor lying on the $z = 0$ plane. Let the electric field vector be perpendicular to the plane of incidence ($x - z$ plane) and H_i is in the downward direction. Since no EM fields exist within a perfect conductor, the incident wave is completely reflected back. The angle of reflection is equal to the angle of incidence,

i.e., $\theta_r = \theta_i = \theta,$

where θ_i and θ_r are the angle of incidence and angle of reflection of the wave with the z-axis respectively, as shown in Fig. 8.6.

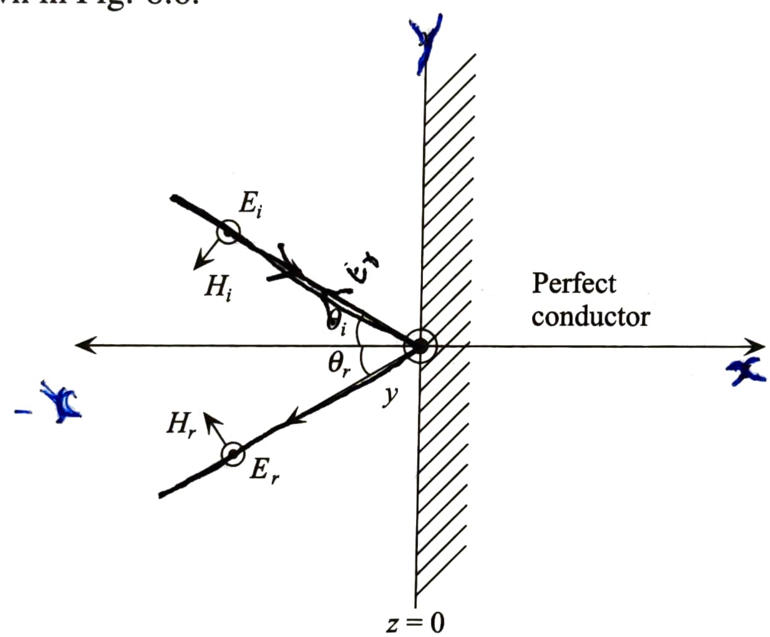


Fig. 8.6 *Reflection of a horizontally polarized wave by a perfect conductor*

The incident and reflected waves form a standing wave distribution.

Assume that the wave is propagating obliquely along any arbitrary direction with unit vector $\vec{a}_r$. Let a position vector $\vec{r}$ be such that $\vec{r}.\vec{a}_r =$ constant.

The incident wave in phasor notation can be expressed as

$$\vec{E}_i = E_i e^{-j\beta(\vec{r}.\vec{a}_r)}\,\vec{a}_r. \tag{8.22}$$

Using direction cosines,

$$\vec{r}.\vec{a}_r = (\cos\theta_x \vec{a}_x + \cos\theta_y \vec{a}_y + \cos\theta_z \vec{a}_z).(x\vec{a}_x + y\vec{a}_y + z\vec{a}_z).$$

$$\vec{r}.\vec{a}_r = x\cos\theta_x + y\cos\theta_y + z\cos\theta_z, \tag{8.23}$$

where θ_x, θ_y and θ_z are the incident angles between the direction of propagation and respective axes.

For horizontal polarization, the electric vector is along the y direction, i.e., normal to the plane of incidence ($x - z$ plane) and direction of propagation.

If θ is the incident angle with the z-axis, then

$$\theta_z = \theta,\ \ \theta_y = \pi/2,\ \ \text{and}\ \ \theta_x = \theta - \pi/2.$$

$x \sin\theta + z\cos\theta.$

Hence, $\vec{r}.\vec{a}_r = x\cos(\theta-\pi/2) + y\cos(\pi/2) + z\cos\theta.$

$\cos(\theta - \pi/2) = \sin\theta$

$\cos \pi/2 = 0$

$$\boxed{\vec{r}\cdot\vec{a}_r = x\sin\theta + z\cos\theta}$$

$$\therefore \boxed{\vec{E}_i = E_i e^{-j\beta(x\sin\theta + z\cos\theta)}\,\vec{a}_y.} \longrightarrow A \qquad (8.24)$$

Similarly, for a reflected wave in the $(-z)$ direction,

$$\vec{r}.\vec{a}_r = x\sin\theta - z\cos\theta.$$

$$\therefore \vec{E}_r = E_r e^{-j\beta(x\sin\theta - z\cos\theta)}\,\vec{a}_y. \longrightarrow B \qquad (8.25)$$

The electric field vector for a standing wave is

$$E = E_i + E_r = E_i e^{-j\beta(x\sin\theta + z\cos\theta)} + E_r e^{-j\beta(x\sin\theta - z\cos\theta)}.$$

$\boxed{E_i = -E_r}$

Since $\boxed{E_i = -E_r,}$

$$E = E_i[e^{-j\beta(x\sin\theta + z\cos\theta)} - e^{-j\beta(x\sin\theta - z\cos\theta)}]$$

$$= E_i e^{-j\beta x\sin\theta}[e^{-j\beta(z\cos\theta)} - e^{j\beta(z\cos\theta)}]$$

$$= -2jE_i e^{-j\beta x\sin\theta}\left(\frac{e^{j\beta z\cos\theta} - e^{-j\beta z\cos\theta}}{2j}\right).$$

$\beta\cos\theta = \beta$

$$\boxed{E = -2jE_i\sin(z\beta\cos\theta)e^{-jx\beta\sin\theta}} \qquad (8.26)$$

If $\boxed{\beta = \frac{2\pi}{\lambda}}$, the phase shift constant along the direction of propagation, $\boxed{\beta_z = \beta\cos\theta}$ is the phase shift constant along the z direction and $\boxed{\beta_x = \beta\sin\theta}$ is the phase shift constant along the x direction.

The wavelengths and velocities of the wave along the axes are:

wavelengths: $$\lambda_z = \frac{2\pi}{\beta_z} = \frac{2\pi}{\beta\cos\theta} = \frac{\lambda}{\cos\theta},$$

$$\lambda_x = \frac{2\pi}{\beta_x} = \frac{2\pi}{\beta\sin\theta} = \frac{\lambda}{\sin\theta} \qquad (8.27)$$

velocities: $$v_z = \frac{v}{\cos\theta}, \quad v_x = \frac{v}{\sin\theta}.$$

Therefore the electric field vector for a standing wave distribution is given by

$$\boxed{\vec{E} = -2jE_i\sin(z\beta_z)e^{-jx\beta_x}\,\vec{a}_y.} \qquad (8.28)$$

$\eta = \frac{\vec{E}}{\vec{H}}$

The magnetic field vector is

$$\vec{H} = \frac{1}{\eta}\left(\vec{a}_r \times \vec{E}\right).$$

Since the magnetic field vector is parallel to the $x-z$ plane, only x- and z- field components exist.

$$\vec{H} = \frac{1}{\eta}(H_x\vec{a}_x + H_z\vec{a}_z)$$

10

$$\therefore \quad \boxed{\vec{H} = H_x \vec{a}_x + H_z \vec{a}_z}$$

The magnetic field along the x-axis is

$$H_x = -2H_i \sin\theta \, \sin(z\beta_z) e^{-jx\beta_x}$$

and the magnetic field along the z-axis is

$$H_z = -2jH_i \cos\theta \, \cos(z\beta_z) e^{-jx\beta_x},$$

where $H_i = E_i/\eta$.

$$\therefore \quad \vec{H} = \frac{-2jE_i}{\eta} [\sin\theta \sin(z\beta_z)\vec{a}_x + \cos\theta \cos(z\beta_z)\vec{a}_z] e^{-jx\beta_x} \tag{8.29}$$

Note: In horizontal polarization, since the electric field vector is normal to the plane of incidence, only E_y, H_x and H_z components of the standing wave exist.

8.7 Reflection of a Vertically Polarized Wave by a Perfect Conductor—Oblique Incidence

Consider a vertically polarized wave incident obliquely on a perfect conductor lying on the $z = 0$ plane. Let the electric field vector is parallel to the plane of incidence ($x - z$ plane) and E_i is in the upward direction. Since no EM fields exist within a perfect conductor, the incident wave is completely reflected back. The angle of reflection is equal to the angle of incidence.

i.e., $\theta_r = \theta_i = \theta$,

where θ_i and θ_r are the angle of incidence and angle of reflection of the wave with the z-axis respectively as shown in Fig. 8.7.

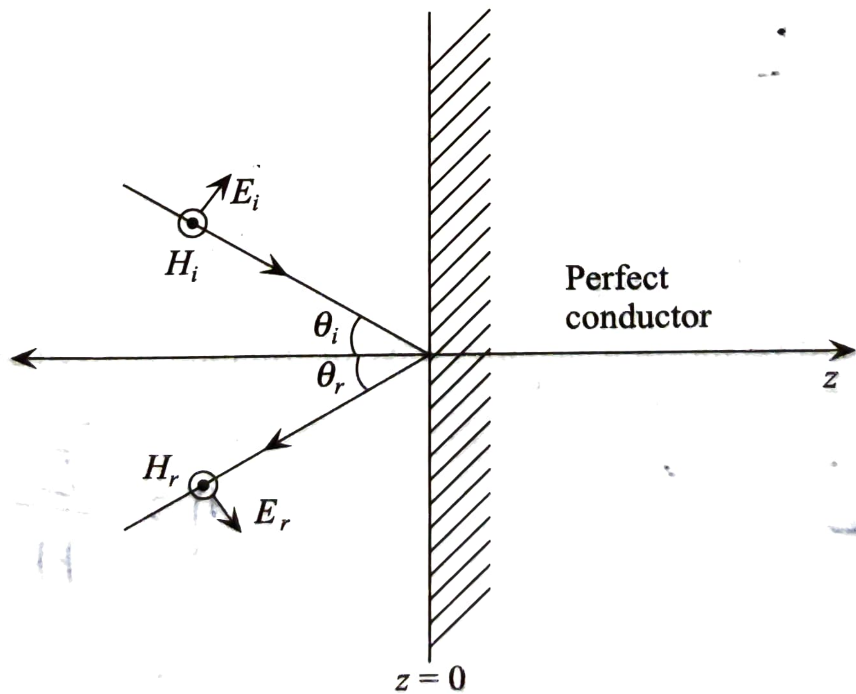


Fig. 8.7 *Reflection of a vertically polarized wave by a perfect conductor*

The incident and reflected waves form a standing wave distribution.

Since the magnetic field vector is parallel to the boundary, it will be reflected without phase reversal.

That is, $\boxed{H_i = H_r}$ and $\dfrac{E_i}{H_i} = \dfrac{E_r}{H_r} = \eta.$

Assume that the wave is propagating obliquely along any arbitrary direction with unit vector $\vec{a}_r$. Let a position vector $\vec{r}$ be such that $\vec{r}.\vec{a}_r =$ constant.

The incident wave in phasor notation can be expressed as

$$\vec{H}_i = H_i e^{-j\beta(\vec{r}.\vec{a}_r)}\vec{a}_r. \tag{8.30}$$

Using direction cosines,

$$\begin{aligned}\vec{r}.\vec{a}_r &= (\cos\theta_x\vec{a}_x + \cos\theta_y\vec{a}_y + \cos\theta_z\vec{a}_z).(x\vec{a}_x + y\vec{a}_y + z\vec{a}_z)\\ &= x\cos\theta_x + y\cos\theta_y + z\cos\theta_z,\end{aligned} \tag{8.31}$$

where θ_x, θ_y and θ_z are the incident angles between direction of propagation and respective axes.

For vertical polarization, the magnetic vector is along the y direction, i.e., normal to the plane of incidence ($x-z$ plane) and direction of propagation. If θ is the incident angle with the z-axis, then

$$\theta_z = \theta,\ \theta_y = \pi/2, \text{ and } \theta_x = \theta - \pi/2.$$

$$\begin{aligned}\vec{r}.\vec{a}_r &= x\cos(\theta - \pi/2) + y\cos(\pi/2) + z\cos\theta\\ &= x\sin\theta + z\cos\theta.\end{aligned}$$

$$\therefore \quad \vec{H}_i = H_i e^{-j\beta(x\sin\theta + z\cos\theta)}\,\vec{a}_y. \tag{8.32}$$

Similarly, for the reflected wave in the $(-z)$ direction,

$$\begin{aligned}\vec{r}\cdot\vec{a}_r &= x\sin\theta - z\cos\theta.\\ \vec{H}_r &= H_r e^{-j\beta(x\sin\theta - z\cos\theta)}\,\vec{a}_y.\end{aligned} \tag{8.33}$$

The total magnetic field vector of the standing wave is

$$\begin{aligned}\vec{H} &= \vec{H}_i + \vec{H}_r\\ &= H_i e^{-j\beta(x\sin\theta + z\cos\theta)}\,\vec{a}_y + H_r e^{-j\beta(x\cos\theta - z\cos\theta)}\,\vec{a}_y.\end{aligned}$$

$$\frac{e^{-x} + e^{x}}{2} = \cos$$

Since $H_i = H_r$,

$$\begin{aligned}\vec{H} &= H_i e^{-jx\beta\sin\theta}(e^{-jz\beta\cos\theta} + e^{jz\beta\cos\theta})\,\vec{a}_y.\\ &\boxed{\vec{H} = 2H_i\cos(z\beta\cos\theta)e^{-jx\beta\sin\theta}\,\vec{a}_y.}\end{aligned} \tag{8.34}$$

Since $\beta = \dfrac{2\pi}{\lambda}$, the phase shift constant along the direction of propagation, $\beta_z = \beta\cos\theta$ and $\beta_x = \beta\sin\theta$ are phase shift constants along the z and x directions, respectively. (8.29)

The velocity and wavelength of the wave along the x-axis is

$$v_x = \frac{v}{\sin\theta} \quad \text{and} \quad \lambda_x = \frac{\lambda}{\sin\theta}. \tag{8.35}$$

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Therefore the magnetic field vector along the y-axis for a standing wave distribution is given by

$$\vec{H} = 2H_i \cos(z\beta_z) e^{-jx\beta_x} \vec{a}_y. \tag{8.36}$$

Since the electric field vector is parallel to the $x - z$ plane, only the x and z field components exist.

$$\vec{E} = E_x \vec{a}_x + E_z \vec{a}_z.$$

The electric field vector along the x-axis is

$$E_x = -2jE_i \cos\theta \sin(z\beta_z) e^{-jx\beta_x}.$$

The electric field vector along the z-axis is

$$E_z = -2jE_i \sin\theta \cos(z\beta_z) e^{-jx\beta_x}$$

where $E_i = \eta H_i$

$$\vec{E} = -2j\eta H_i [\cos\theta \sin(z\beta_z)\vec{a}_x + \sin\theta \cos(z\beta_z)\vec{a}_z] e^{-jx\beta_x} \tag{8.37}$$

Note: In vertical polarization, since the electric field vector is parallel to the plane of incidence, only H_y, E_x and E_z components of the standing wave exist.

8.8 Reflection by a Dielectric or Perfect Insulator—Oblique Incidence

Consider a plane wave incident obliquely on the surface of a dielectric medium or a perfect insulator. Some part of the wave is reflected and the other part is refracted through the medium at the boundary as shown in Fig. 8.8.

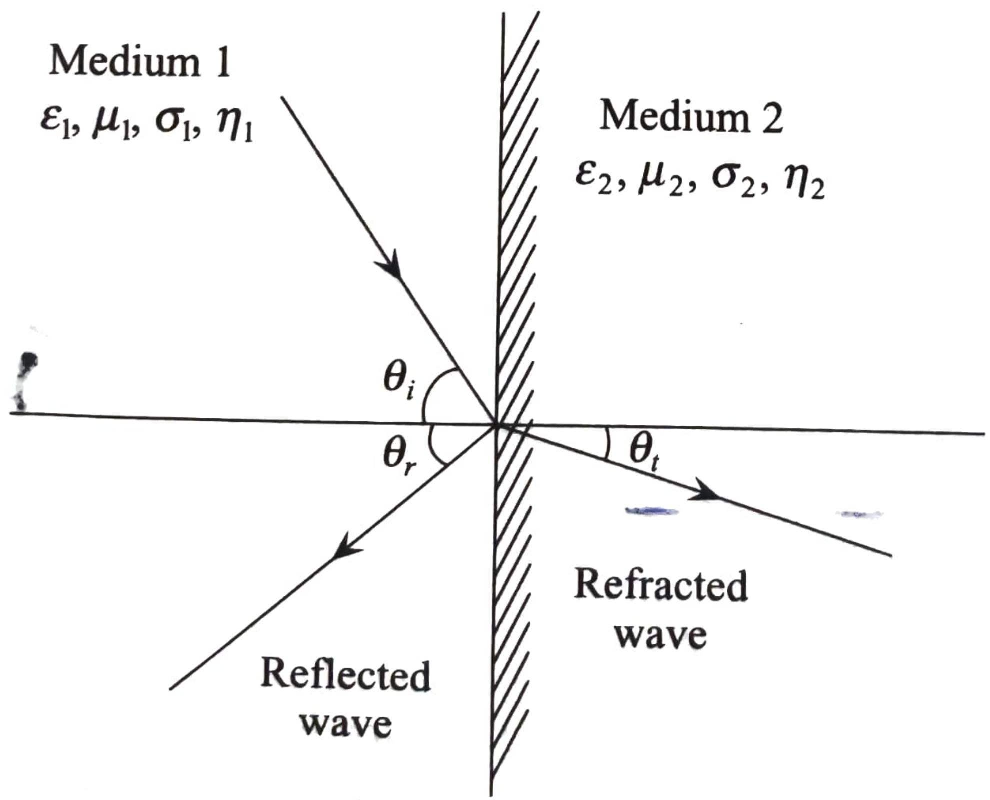


Fig. 8.8 *Oblique incidence of a plane wave on a dielectric medium*

Let θ_i = incident angle, θ_r = reflected angle and θ_t = refracted angle. $\theta_i = \theta_r = \theta_1$, $\theta_t = \theta_2$..

Also let medium 1 and 2 have the properties ε_1, η_1, β_1 and ε_2, η_2, β_2 respectively.

From Snell's law for dielectric media,

$$\sqrt{\varepsilon_1}\sin\theta_1 = \sqrt{\varepsilon_2}\sin\theta_2. \tag{8.38}$$

The field vectors can be obtained from the distribution of powers at the boundary surface.

Assume that the power absorbed by the material is neglected. We know that the wave power density is

$$P = |\vec{E}\times\vec{H}| = \frac{E^2}{\eta}. \tag{8.39}$$

Incident wave power density is

$$P_i = \frac{E_i^2}{\eta_1}\cos\theta_1. \tag{8.40}$$

Similarly, the reflected power density is

$$P_r = \frac{E_r^2}{\eta_1}\cos\theta_1 \tag{8.41}$$

and the transmitted wave power density is

$$P_t = \frac{E_t^2}{\eta_2}\cos\theta_2. \tag{8.42}$$

From the principle of conservation of energy, we know that

Incident power density = reflected power density + refracted power density

i.e., $P_i = P_r + P_t$.

$$\frac{E_i^2}{\eta_1}\cos\theta_1 = \frac{E_r^2}{\eta_1}\cos\theta_1 + \frac{E_t^2}{\eta_2}\cos\theta_2. \longrightarrow$$

Divide. by $\cos\theta_1$ & multiply by η_1 on. B Side

$$1 = \frac{E_r^2}{E_i^2} + \frac{E_t^2}{E_i^2}\left(\frac{\eta_1}{\eta_2}\right)\frac{\cos\theta_2}{\cos\theta_1}.$$

We know that $\dfrac{\eta_1}{\eta_2} = \dfrac{\sqrt{\mu_0/\varepsilon_1}}{\sqrt{\mu_0/\varepsilon_2}} = \sqrt{\dfrac{\varepsilon_2}{\varepsilon_1}}.$

$$\therefore \quad \frac{E_r^2}{E_i^2} = 1 - \frac{E_t^2}{E_i^2}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\left(\frac{\cos\theta_2}{\cos\theta_1}\right), \tag{8.43}$$

where $\dfrac{E_r}{E_i} = \Gamma$, is the reflection coefficient and $\dfrac{E_r}{E_i} = T$, is the transmission coefficient.

To find the reflection and transmission coefficients, the wave polarizations should be considered.

Oblique Incidence

Perfect Dielectric

dielectric

Case 1: Horizontal (perpendicular) polarization

Assume that the electric vector is perpendicular to the plane of incidence. Let the electric field vector E_i of the obliquely incident wave be oriented outward (positive y direction) from the plane paper as shown in Fig. 8.9.

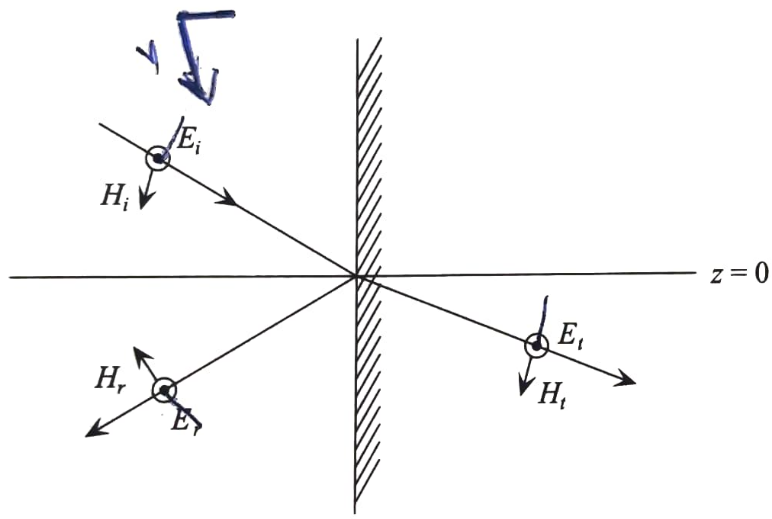


Fig. 8.9 *Reflection and refraction of a perpendicularly polarized wave*

E_i, E_r and E_t are in the positive y direction. Applying boundary conditions, the tangential components are continuous.

i.e., $\boxed{E_i + E_r = E_t}$ or $\dfrac{E_t}{E_i} = 1 + \dfrac{E_r}{E_i}$. (8.44)

Substituting this in Eq. (8.43),

$$\frac{E_r^2}{E_i^2} = 1 - \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\left(1 + \frac{E_r}{E_i}\right)^2 \frac{\cos\theta_2}{\cos\theta_1}.$$

$$1 - \left(\frac{E_r}{E_i}\right)^2 = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\left(1 + \frac{E_r}{E_i}\right)^2 \frac{\cos\theta_2}{\cos\theta_1}.$$

$$\left(1 + \frac{E_r}{E_i}\right)\left(1 - \frac{E_r}{E_i}\right) = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\left(1 + \frac{E_r}{E_i}\right)^2 \frac{\cos\theta_2}{\cos\theta_1}.$$

$$1 - \frac{E_r}{E_i} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\left(1 + \frac{E_r}{E_i}\right) \frac{\cos\theta_2}{\cos\theta_1}.$$

$$1 - \frac{E_r}{E_i} = \frac{\sqrt{\varepsilon_2}\cos\theta_2}{\sqrt{\varepsilon_1}\cos\theta_1} + \frac{E_r}{E_i}\frac{\sqrt{\varepsilon_2}\cos\theta_2}{\sqrt{\varepsilon_1}\cos\theta_1}.$$

$$\frac{E_r}{E_i}\left(\frac{\sqrt{\varepsilon_2}\cos\theta_2}{\sqrt{\varepsilon_1}\cos\theta_1} + 1\right) = 1 - \frac{\sqrt{\varepsilon_2}\cos\theta_2}{\sqrt{\varepsilon_1}\cos\theta_1}.$$

$$\frac{E_r}{E_i}\frac{\left[\sqrt{\varepsilon_2}\cos\theta_2+\sqrt{\varepsilon_1}\cos\theta_1\right]}{\sqrt{\varepsilon_1}\cos\theta_1}=\frac{\sqrt{\varepsilon_1}\cos\theta_1-\sqrt{\varepsilon_2}\cos\theta_2}{\sqrt{\varepsilon_1}\cos\theta_1}.$$

The reflection coefficient $\Gamma = \frac{E_r}{E_i} = \frac{\sqrt{\varepsilon_1}\cos\theta_1-\sqrt{\varepsilon_2}\cos\theta_2}{\sqrt{\varepsilon_1}\cos\theta_1+\sqrt{\varepsilon_2}\cos\theta_2}$. (8.45)

Similarly, the transmission coefficient is

$$T=\frac{E_t}{E_i}=1+\frac{E_r}{E_i}=1+\frac{\sqrt{\varepsilon_1}\cos\theta_1-\sqrt{\varepsilon_2}\cos\theta_2}{\sqrt{\varepsilon_1}\cos\theta_1+\sqrt{\varepsilon_2}\cos\theta_2} \tag{8.46}$$

$$=\frac{2\sqrt{\varepsilon_1}\cos\theta_1}{\sqrt{\varepsilon_1}\cos\theta_1+\sqrt{\varepsilon_2}\cos\theta_2}.$$

θ_2 can be obtained from Snell's law,

$$\sin\theta_2=\frac{\sqrt{\varepsilon_1}\sin\theta_1}{\sqrt{\varepsilon_2}}. \tag{8.47}$$

x

The fields can be expressed in phasor notation as

Incident wave:

$$\vec{E}_i=E_ie^{-j\beta_1(x\sin\theta_1+z\cos\theta_1)}\vec{a}_y;$$
$$\vec{H}_i=H_i(-\cos\theta_1\,\vec{a}_x+\sin\theta_1\,\vec{a}_z)e^{-j\beta_1(x\sin\theta_1+z\cos\theta_1)}. \tag{8.48}$$

Reflected wave:

$$\vec{E}_r=E_re^{-j\beta_1(x\sin\theta_1-z\cos\theta_1)}\vec{a}_y;$$
$$\vec{H}_r=H_r(\cos\theta_1\,\vec{a}_x+\sin\theta_1\,\vec{a}_z)e^{-j\beta_1(x\sin\theta_1-z\cos\theta_1)}. \tag{8.49}$$

Transmitted wave:

$$\vec{E}_t=H_te^{-j\beta_2(x\sin\theta_2+z\cos\theta_2)}\vec{a}_y;$$
$$\vec{H}_t=H_t(-\cos\theta_2\,\vec{a}_x+\sin\theta_2\,\vec{a}_z)e^{-j\beta_2(x\sin\theta_2+z\cos\theta_2)}. \tag{8.50}$$

Perfect Dielectric – oblique Incidence

Case 2: Vertical (parallel) polarization

Assume that the electric vector is parallel to the plane of incidence. Let the electric field vector E_i of the obliquely incident wave be upward and the magnetic vector be outward (positive y direction) from the plane paper as shown is Fig. 8.10.

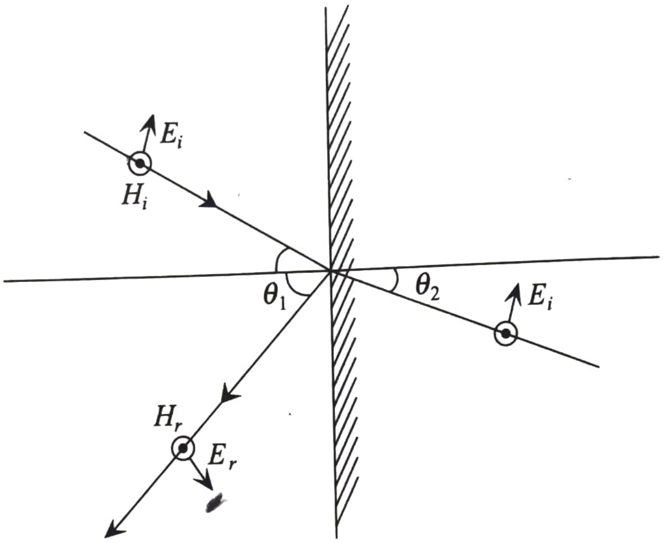


Fig. 8.10 *Obliquely incidence of a vertically polarized wave*

Applying boundary conditions, the tangential component of E is continuous at the boundary.
Since E_i and E_r make an angle θ_1 with the axis and E_t makes an angle θ_2 with the axis,

$$E_i \cos\theta_1 - E_r \cos\theta_1 = E_t \cos\theta_2.$$

$E_t = E_i + E_r$

$$1 - \frac{E_r}{E_i} = \frac{E_t \cos\theta_2}{E_i \cos\theta_1}.$$

$\cos\theta_2 E_t = \cos\theta_1 E_i - \cos\theta_1 E_r$

$$\therefore \quad \frac{E_t}{E_i} = \left(1 - \frac{E_r}{E_i}\right)\frac{\cos\theta_1}{\cos\theta_2}. \tag{8.51}$$

Substituting this in Eq. (8.43),

$$\left(\frac{E_r}{E_i}\right)^2 = 1 - \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\left(\frac{E_t}{E_i}\right)^2 \frac{\cos\theta_2}{\cos\theta_1}$$

or $$\left(\frac{E_r}{E_i}\right)^2 = 1 - \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\left(1 - \frac{E_r}{E_i}\right)^2 \left(\frac{\cos^2\theta_1}{\cos^2\theta_2}\right)\frac{\cos\theta_2}{\cos\theta_1}.$$

$$1 - \left(\frac{E_r}{E_i}\right)^2 = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\left(1 - \frac{E_r}{E_i}\right)^2 \frac{\cos\theta_1}{\cos\theta_2}.$$

$$\left(1 - \frac{E_r}{E_i}\right)\left(1 + \frac{E_r}{E_i}\right) = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\left(1 - \frac{E_r}{E_i}\right)^2 \frac{\cos\theta_1}{\cos\theta_2}.$$

$$1 + \frac{E_r}{E_i} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\left(1 - \frac{E_r}{E_i}\right)\frac{\cos\theta_1}{\cos\theta_2}.$$

$$1 + \frac{E_r}{E_i} = \frac{\sqrt{\varepsilon_2}\cos\theta_1}{\sqrt{\varepsilon_1}\cos\theta_2} - \left(\frac{E_r}{E_i}\right)\frac{\sqrt{\varepsilon_2}\cos\theta_1}{\sqrt{\varepsilon_1}\cos\theta_2}.$$

$$\frac{E_r}{E_i}\left(1+\frac{\sqrt{\varepsilon_2}\cos\theta_1}{\sqrt{\varepsilon_1}\cos\theta_2}\right)=\frac{\sqrt{\varepsilon_2}\cos\theta_1}{\sqrt{\varepsilon_1}\cos\theta_2}-1.$$

The reflection coefficient $\Gamma=\frac{E_r}{E_i}=\frac{\sqrt{\varepsilon_2}\cos\theta_1-\sqrt{\varepsilon_1}\cos\theta_2}{\sqrt{\varepsilon_2}\cos\theta_1+\sqrt{\varepsilon_1}\cos\theta_2}$ (8.52)

and the transmission coefficient is

$$T=\frac{E_t}{E_i}=1-\left(\frac{E_r}{E_i}\right)\frac{\cos\theta_1}{\cos\theta_2}$$

$$=1-\left(\frac{\sqrt{\varepsilon_2}\cos\theta_1-\sqrt{\varepsilon_1}\cos\theta_2}{\sqrt{\varepsilon_2}\cos\theta_1+\sqrt{\varepsilon_1}\cos\theta_2}\right)\left(\frac{\cos\theta_1}{\cos\theta_2}\right).$$

$$\frac{E_t}{E_i}=\frac{\sqrt{\varepsilon_2}\cos\theta_1\cos\theta_2+\sqrt{\varepsilon_1}\cos^2\theta_2-\sqrt{\varepsilon_2}\cos^2\theta_1+\sqrt{\varepsilon_1}\cos\theta_1\cos\theta_2}{\sqrt{\varepsilon_2}\cos\theta_1\cos\theta_2+\sqrt{\varepsilon_1}\cos^2\theta_2}.$$

$$T=\frac{E_t}{E_i}=\frac{\sqrt{\varepsilon_2}\cos\theta_1(\cos\theta_2-\cos\theta_1)+\sqrt{\varepsilon_1}\cos\theta_2(\cos\theta_1+\cos\theta_2)}{(\sqrt{\varepsilon_2}\cos\theta_1+\sqrt{\varepsilon_1}\cos\theta_2)\cos\theta_2}. \tag{8.53}$$

The refracted angle can be obtained from Snell's law

$$\sin\theta_2=\sqrt{\frac{\varepsilon_1}{\varepsilon_2}}\sin\theta_1.$$

The fields can be expressed in phasor notation as

Incident wave:

$$\vec{E}_i=E_i(\cos\theta_1\vec{a}_x-\sin\theta_1\vec{a}_z)e^{-j\beta_1(x\sin\theta_1+z\cos\theta_1)};$$

$$\vec{H}_i=H_i e^{-j\beta_1(x\sin\theta_1+z\cos\theta_1)}\vec{a}_y. \tag{8.54}$$

Reflected wave:

$$\vec{E}_r=E_r(\cos\theta_1\vec{a}_x+\sin\theta_1\vec{a}_z)e^{-j\beta_1(x\sin\theta_1-z\cos\theta_1)};$$

$$\vec{H}_r=-H_r e^{-j\beta_1(x\sin\theta_1-z\cos\theta_1)}\vec{a}_y. \tag{8.55}$$

Transmitted wave:

$$\vec{E}_t=E_t(\cos\theta_2\vec{a}_x-\sin\theta_2\vec{a}_z)e^{-j\beta_2(x\sin\theta_2+z\cos\theta_2)};$$

$$\vec{H}_t=H_t e^{-j\beta_2(x\sin\theta_2+z\cos\theta_2)}\vec{a}_y. \tag{8.56}$$

8.9 Brewster Angle

When a wave is incident obliquely on the surface of a dielectric medium, the angle of incidence at which no reflection takes place is called the Brewster angle θ_B. It exists only for vertically polarized waves.

It can be expressed as $\boxed{\theta_B = \tan^{-1}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}}$. (8.57)

Proof Consider a vertically polarized wave incident obliquely on the surface of a dielectric medium.

The Brewster angle can be obtained from Eq. (8.52) by substituting the reflection coefficient as zero,

i.e.,
$$\frac{E_r}{E_i} = \frac{\sqrt{\varepsilon_2}\cos\theta_1 - \sqrt{\varepsilon_1}\cos\theta_2}{\sqrt{\varepsilon_2}\cos\theta_1 + \sqrt{\varepsilon_1}\cos\theta_2}.$$

At $\theta_1 = \theta_B$, reflection coefficient is zero,

$$\frac{\sqrt{\varepsilon_2}\cos\theta_B - \sqrt{\varepsilon_1}\cos\theta_2}{\sqrt{\varepsilon_2}\cos\theta_B + \sqrt{\varepsilon_1}\cos\theta_2} = 0$$

or $$\sqrt{\varepsilon_2}\cos\theta_B - \sqrt{\varepsilon_1}\cos\theta_2 = 0.$$

$$\sqrt{\varepsilon_2}\cos\theta_B = \sqrt{\varepsilon_1}\cos\theta_2$$

or $\quad \varepsilon_2 \cos^2 \theta_B = \varepsilon_1 \cos^2 \theta_2.$ (8.58)

We know from Snell's law that

$$\sin \theta_2 = \sqrt{\frac{\varepsilon_1}{\varepsilon_2}} \sin \theta_B.$$

Since $\quad \cos^2 \theta_2 = 1 - \sin^2 \theta_2,$

$$\boxed{\cos^2 \theta_2 = 1 - \frac{\varepsilon_1}{\varepsilon_2} \sin^2 \theta_B.}$$

Substituting the value in Eq. (8.58),

$$\varepsilon_2 \cos^2 \theta_B = \varepsilon_1 \left(1 - \frac{\varepsilon_1}{\varepsilon_2} \sin^2 \theta_B\right).$$

$$\frac{\varepsilon_2}{\varepsilon_1} \cos^2 \theta_B = 1 - \frac{\varepsilon_1}{\varepsilon_2} \sin^2 \theta_B.$$

$\cos^2 \theta_B + \sin^2 \theta_B = 1$

$$\frac{\varepsilon_2}{\varepsilon_1} \cos^2 \theta_B = \cos^2 \theta_B + \sin^2 \theta_B - \frac{\varepsilon_1}{\varepsilon_2} \sin^2 \theta_B.$$

$$\cos^2 \theta_B \left(\frac{\varepsilon_2}{\varepsilon_1} - 1\right) = \sin^2 \theta_B \left(1 - \frac{\varepsilon_1}{\varepsilon_2}\right).$$

$$\cos^2 \theta_B \left(\frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_1}\right) = \sin^2 \theta_B \left(\frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2}\right).$$

$$\varepsilon_2 \cos^2 \theta_B = \varepsilon_1 \sin^2 \theta_B.$$

$$\frac{\sin^2 \theta_B}{\cos^2 \theta_B} = \frac{\varepsilon_2}{\varepsilon_1}.$$

$$\tan^2 \theta_B = \frac{\varepsilon_2}{\varepsilon_1}.$$

$$\tan \theta_B = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}.$$

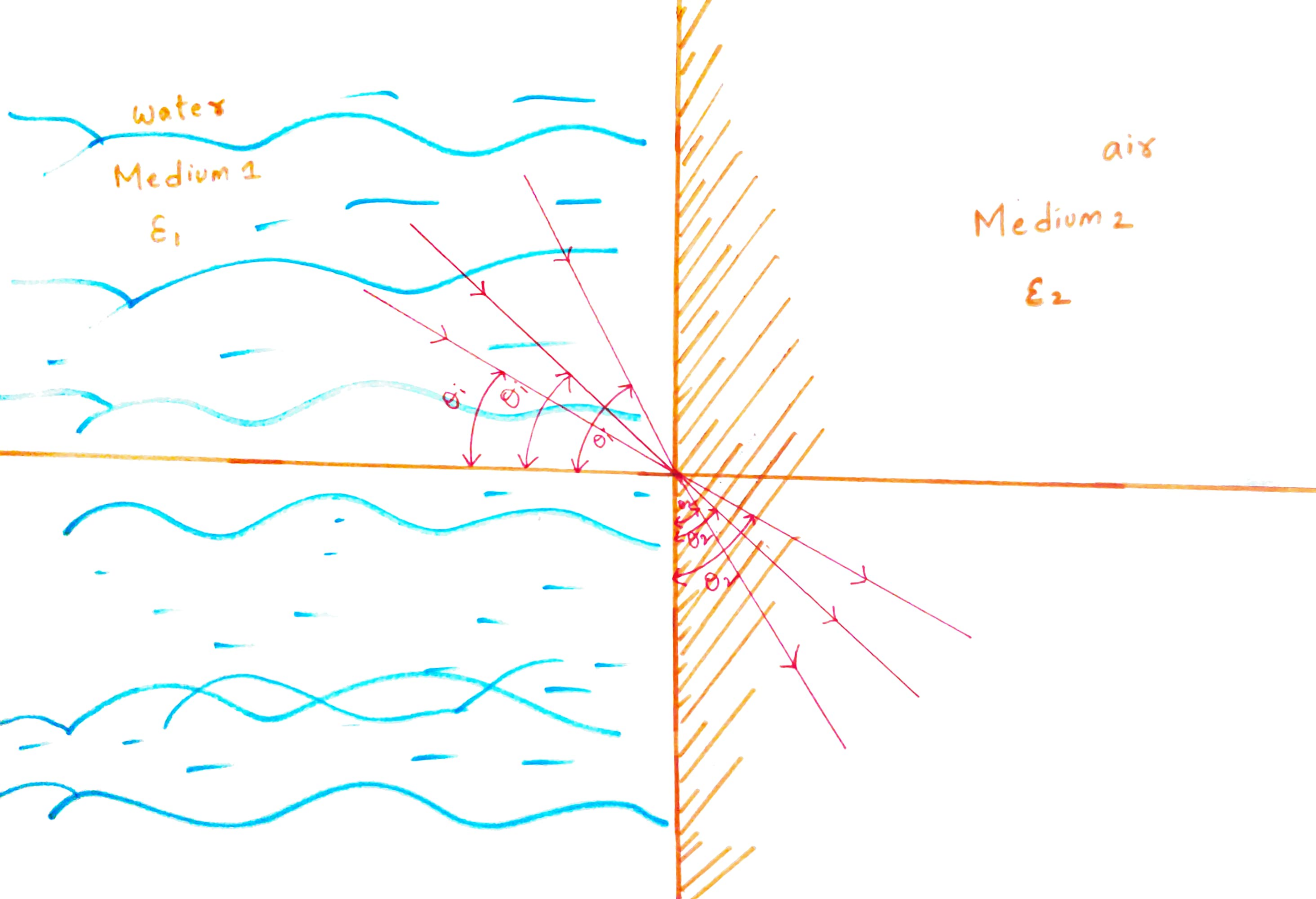
Brewster angle is $\theta_B = \tan^{-1} \sqrt{\dfrac{\varepsilon_2}{\varepsilon_1}}$ (Proved).

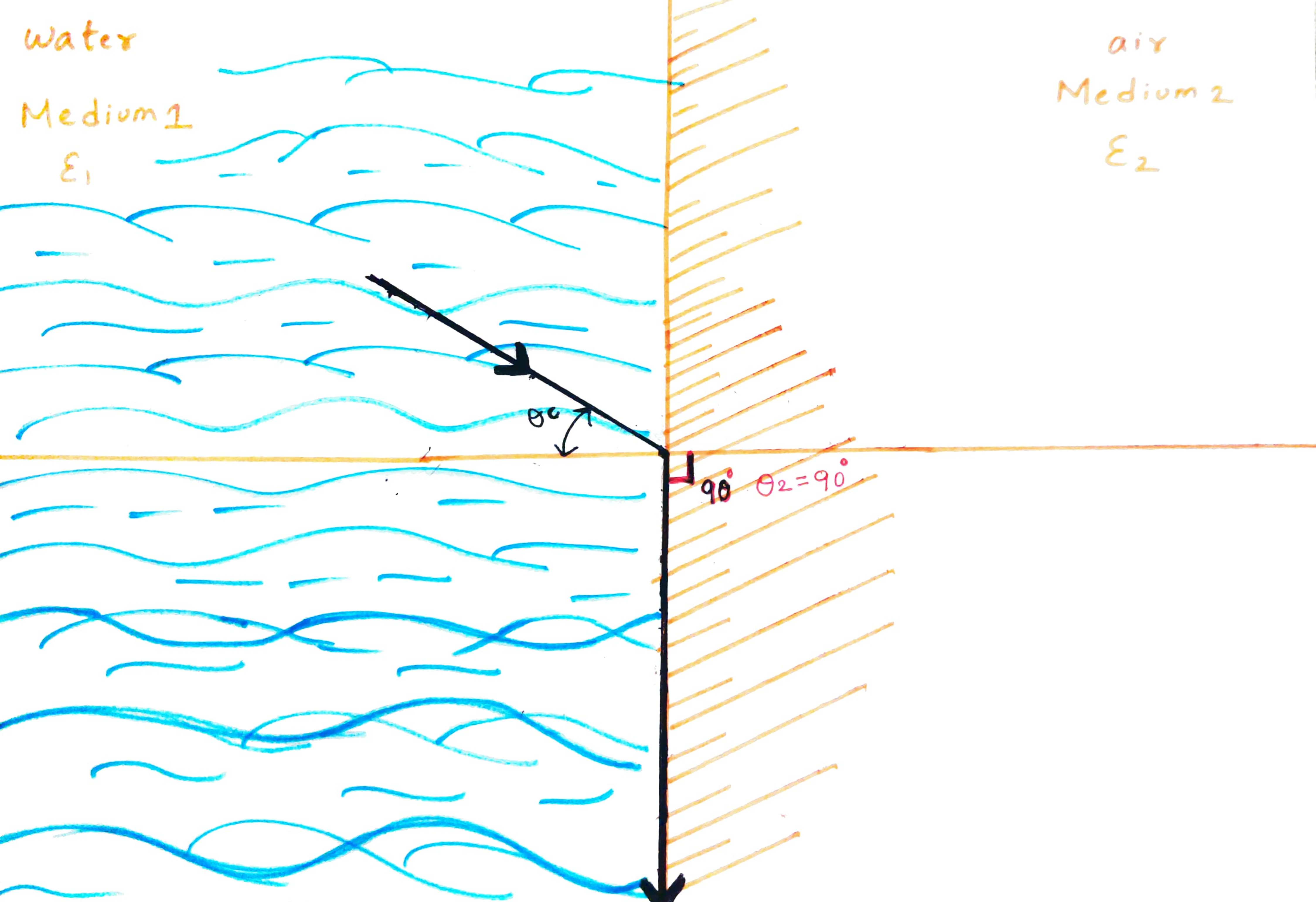
Therefore, when the wave is incident at the Brewster angle θ_B with parallel polarization, no reflections take place. The wave is completely transmitted through the media. Thus the Brewster angle is also called the angle of total transmission.

This phenomenon can be used when we want to transmit a signal from one medium to another medium without loss of power.

For example, when a laser beam is launched into an optical fibre, the incidence angle at the fibre surface is kept at the Brewster angle in order to minimise reflection losses.

When the wave is perpendicularly polarized, no Brewster angle exists. Therefore, when an unpolarized wave is incident obliquely on the dielectric surface at the Brewster angle θ_B, only the field components with perpendicular polarization will be reflected; those with parallel polarization will be transmitted. This method is used to separate the polarization components from the unpolarized wave.


water
Medium 1
ε_1
air
Medium 2
ε_2
θ_i
θ_i
θ_i
θ_2
θ_2
θ_2



water
Medium 1
ε_1
air
Medium 2
ε_2
θ_c
90°
$\theta_2 = 90^\circ$

8.10 Total Internal Reflection

Consider a plane wave incident obliquely on the surface between two media. Let ε_1 of medium 1 be greater than ε_2 of medium 2. If θ_1 and θ_2 are the incident and refracted angles, then from Snell's law, we know that

$$\frac{\sin\theta_1}{\sin\theta_2} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}. \tag{8.59}$$

When $\theta_2 = \pi/2$, the transmitted wave travels along the boundary surface as shown in Fig. 8.11.

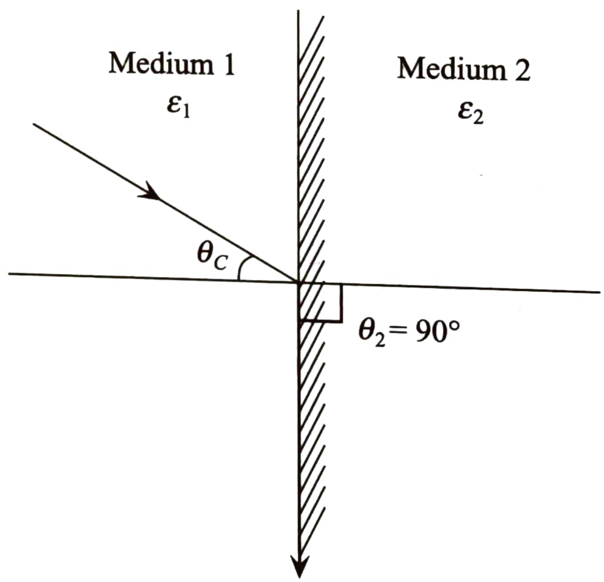


Fig. 8.11 *Total internal reflection*

The angle of incidence at which the refracted angle becomes $\pi/2$ is called the critical angle θ_c, i.e., if $\theta_1 = \theta_c$, then $\theta_2 = \pi/2$.

From Snell's law,

$$\frac{\sin\theta_c}{\sin\pi/2} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$$

sin 90° = 1

or the critical angle is $\theta_c = \sin^{-1}\left(\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\right)$. (8.60)

If the incidence angle $\theta_1 > \theta_c$, no transmission takes place. The wave completely reflects back. This phenomenon is called total internal reflection.

Let n_1 be the refractive index of medium 1, and n_2, the refractive index of medium 2.

For lossless media, we know that $n_1 = \sqrt{\varepsilon_{r1}\mu_{r1}}$ and $n_2 = \sqrt{\varepsilon_{r2}\mu_{r2}}$. Also since $\mu_{r1} = \mu_{r2} = 1$.

$$\sin\theta_c = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} = \frac{n_2}{n_1}$$

$$\theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right). \tag{8.61}$$

Total internal reflection takes place if $n_1 > n_2$, i.e., the wave travels from denser medium to a lighter medium.

For example, if the wave travels from fresh water ($\varepsilon_{r_1} = 81$) to air ($\varepsilon_{r_2} = 1$), the critical angle is

$$\theta_c = \sin^{-1}\left(\sqrt{\frac{\varepsilon_0\varepsilon_{r2}}{\varepsilon_0\varepsilon_{r1}}}\right) = \sin^{-1}\sqrt{\frac{1}{81}} = 6.38°.$$

That is, if the incidence angle of the wave is greater than 6.38°, the entire wave is reflected back into the water.

Example 8.3 A wave propagates from a dielectric medium to the interface with free space. If the angle of incidence is the critical angle of $\theta_c = 40°$, find the relative permittivity.

Solution Given: $\theta_c = 40°$

From Snell's law,

$$\sin\theta_c = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} = \sqrt{\frac{\varepsilon_{r_2}}{\varepsilon_{r_1}}}.$$

Given $\varepsilon_{r_2} = 1$ for free space.

$$\sin\theta_c = \frac{1}{\sqrt{\varepsilon_{r_1}}} \quad \text{or} \quad \sqrt{\varepsilon_{r_1}} = \frac{1}{\sin 40°} = 1.5557$$

or $\quad \varepsilon_{r_1} = 2.42.$

8.11 Surface Impedance

When an electromagnetic wave travels along the conductor at a very high frequency, because of the tangential electric field, the total field current is confined only to the thin sheet of the surface.

Surface impedance is defined as *the ratio of the tangential component of the electric field to the surface current density at the surface*. It can be expressed as

$$Z_s = \frac{E_{\tan}}{K} = \frac{E_{\tan}}{J_s} = \frac{\gamma}{\sigma}\,\Omega, \tag{8.62}$$

where $E_{\tan}$ = tangential component of the electric field vector V/m,

$K = J_s$ = total surface current density A/m.

γ = propagation constant and σ = conductivity of the sheet.

Proof Consider an electromagnetic wave travelling along a conducting plate of thickness t lying at $z=0$ plane as shown in Fig. 8.12.

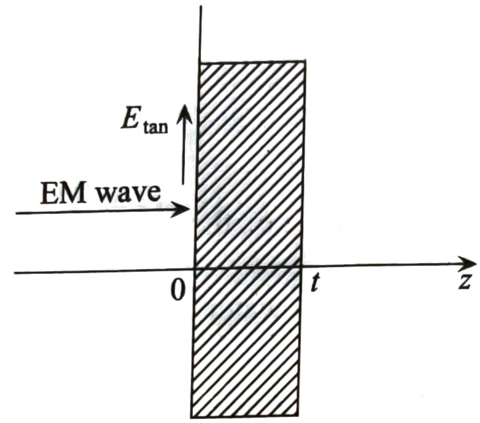


Fig. 8.12 *Conducting plate*

The current flows along the surface of the conductor due to the tangential component of the electric field vector.

Assume that the depth of penetration of the current into the plate along the z direction is less than the plate thickness t.

Let J be the current density. It can be expressed as

$$J = J_0 e^{-\gamma z} \tag{8.63}$$

where J_0 = current density in A/m^2 at the surface at $z=0$ and
γ = propagation constant.

The variation of current density inside the plate is shown in Fig. 8.13.

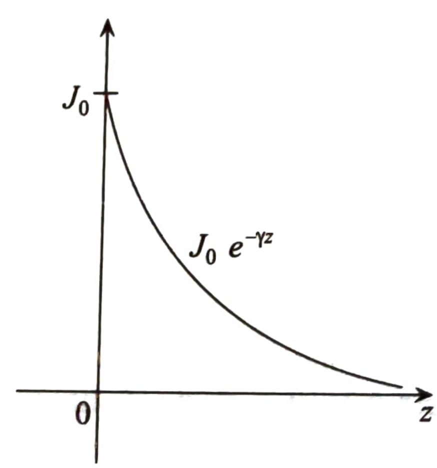


Fig. 8.13 *Variation of current density along the thickness of a conductor.*

If the current is distributed along the conductor surface, the linear current density or surface current density is

$$K = J_s = \int_0^\infty J dz, \quad \text{or} \quad J_s = \int_0^\infty J_0 e^{-\gamma z} dz.$$

$$J_s = J_0\left[\frac{e^{-\gamma z}}{-\gamma}\right]_0^\infty \quad \text{or} \quad J_s = \frac{J_0}{\gamma} \text{ A/m.} \tag{8.64}$$

If σ is the conductivity of the conductor, the current density on the surface is

$$J_0 = \sigma E_{\tan}.$$

$$\therefore \quad J_s = \frac{\sigma E_{\tan}}{\gamma}.$$

The surface impedance is

$$Z_s = \frac{E_{\tan}}{J_s} = \frac{\gamma}{\sigma}\ \Omega. \qquad \text{(Proved)}$$

We know that the propagation constant is $\gamma = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}$.

$$\therefore \quad Z_s = \frac{\sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}}{\sigma}\ \Omega. \tag{8.65}$$

Thus the surface impedance depends on the conductor properties, ε, μ, σ at a given frequency.

Note: For a good conductor, $\sigma >> \omega\varepsilon$.

$$\therefore \quad \gamma = \sqrt{j\omega\mu\sigma}.$$

Surface impedance is $Z_s = \dfrac{\sqrt{j\omega\mu\sigma}}{\sigma} = \sqrt{\dfrac{\omega\mu}{\sigma}}\angle 45^\circ$.

We know that the characteristic impedance $\eta = \sqrt{\dfrac{\omega\mu}{\sigma}}\angle 45^\circ$. (8.66)

Therefore, for a good conductor, the surface impedance is equal to the characteristic impedance.

8.13 Poynting Vector and Poynting Theorem

Poynting vector The Poynting vector is defined as *the cross product of the electric field vector $\vec{E}$ and the magnetic field vector of an electromagnetic wave.*

Thus $\vec{P} = \vec{E} \times \vec{H}$ watts/m^2,

$$(8.72)$$

where $\vec{P}$ = Poynting vector in watts/m^2;
$\vec{E}$ = electric field vector in V/m;
$\vec{H}$ = magnetic field vector in A/m.

The Poynting vector gives instantaneous power density of the wave. The direction of the Poynting vector is perpendicular to the plane of the field and follows the right-hand screw law.

Consider an EM wave which propagates in free space along the z direction. The electric and magnetic fields are

$$\vec{E} = E_0 \cos(\omega t - \beta z)\,\vec{a}_x \quad \text{and} \quad \vec{H} = H_0 \cos(\omega t - \beta z)\,\vec{a}_y. \tag{8.73}$$

In free space, the intrinsic impedance is $\eta_0 = \dfrac{E_0}{H_0} = 377\Omega.$

The Poynting vector is

$$\vec{P} = \vec{E} \times \vec{H}$$

$$= E_0 \cos(\omega t - \beta z)\vec{a}_x \times H_0 \cos(\omega t - \beta z)\vec{a}_y$$

$$= E_0 H_0 \cos^2(\omega t - \beta z)\vec{a}_z.$$

$$\vec{P} = \frac{E_0^2}{\eta_0} \cos^2(\omega t - \beta z)\vec{a}_z \ \text{W/m}^2. \tag{8.74}$$

The Poynting vector is along the direction of wave propagation.

If S is the area of cross-section of power flow, then the instantaneous power is given by

$$\vec{p} = \int_S \vec{P} \cdot d\vec{S} \text{ watts.} \tag{8.75}$$

Average Poynting vector or average power density The average Poynting vector is the average of the instantaneous Poynting vector over one period.

$$\vec{P}_{av} = \frac{1}{T} \int_0^T \vec{P}\, dt.$$

$$\therefore \quad P_{av} = \frac{1}{T} \int_0^T E_0 H_0 \cos^2(\omega t - \beta z)dt.$$

$$= \frac{E_0 H_0}{T} \int_0^T \left(\frac{1}{2} + \frac{\cos 2(\omega t - \beta z)}{2} \right) dt.$$

$\left[\because \cos^2\theta = \frac{1 + \cos 2\theta}{2} \right.$

$\left. \frac{1}{2} + \frac{\cos 2\theta}{2} \right]$

$$= \frac{E_0 H_0}{T} \int_0^T \frac{1}{2} dt + 0.$$

$\#\ . \ 0$

$$\vec{P}_{av} = \frac{E_0 H_0}{2T}(T) = \frac{E_0 H_0}{2} \vec{a}_z \quad \text{watts/m}^2. \tag{8.76}$$

In free space, $\frac{E_0}{H_0} = \eta_0$. The average power density is

$$\vec{P}_{av} = \frac{E_0^2}{2\eta_0} \vec{a}_z = \frac{1}{2}\eta_0 H_0^2 a_z \quad \text{watts/m}^2. \tag{8.77}$$

In a medium, if the impedance is $\eta = |\eta|\angle\theta$, the electric and magnetic fields are

$$\vec{E} = E_0 e^{-\alpha z} \cos(\omega t - \beta z)\vec{a}_x,$$

$$\vec{H} = H_0 e^{-\alpha z} \cos(\omega t - \beta z)\vec{a}_y = \frac{E_0}{|\eta|} e^{-\alpha z} \cos(\omega t - \beta z - \theta)\vec{a}_y$$

and the Poynting vector is

$2\cos A \cos B$
$= \cos(A-B) + \cos(A+B)$

$$\vec{P} = \frac{E_0^2}{|\eta|} e^{-2\alpha z} \cos(\underbrace{\omega t - \beta z}_{A}) \cos(\underbrace{\omega t - \beta z - \theta}_{B}) \vec{a}_z$$

$$\vec{P} = \frac{E_0^2}{|\eta|} e^{-2\alpha z} [\cos\theta + \cos(2\omega t - 2\beta z - \theta)] \vec{a}_z. \quad \frac{1}{2}$$

Then the avarage power density is

$$\vec{P}_{\text{av}} = \frac{1}{T} \int_0^T \vec{P} dt = \frac{E_0^2}{2|\eta|} e^{-2\alpha z} \cos\theta \vec{a}_z.$$

Also, $\vec{P}_{\text{av}} = \frac{1}{2} H_0^2 |\eta| e^{-2\alpha z} \cos\theta\, \vec{a}_z$ watts/m^2.

Average power $\quad p_{av} = |\vec{P}_{av}| S$ watts. (8.78)

Complex Poynting vector The complex Poynting vector is defined as

$$\vec{P}_{\text{com}} = \frac{1}{2} (\vec{E} \times \vec{H}^*).$$

$$\vec{P}_{\text{com}} = \vec{P}_{\text{av}} + \vec{P}_{\text{react}}, \tag{8.79}$$

where $\vec{P}_{\text{av}}$ = average Poynting vector;

$\vec{P}_{\text{react}}$ = reactive Poynting vector.

In complex notation, average Poynting vector is

$$\vec{P}_{\text{av}} = \frac{1}{2} \Re e[E \times H^*]$$

and reactive Poynting vector is

$$\vec{P}_{\text{react}} = \frac{1}{2} \text{Im}[E \times H^*]. \tag{8.80}$$

8.14 Poynting Theorem

The Poynting theorem is one of the most important results in EM theory. It is a statement of energy conservation for electric and magnetic fields and tells us about the power flowing in an EM field. The Poynting theorem states that *the vector product* $\vec{P} = \vec{E} \times \vec{H}$ *at any point is a measure of the rate of energy flow per unit area at that point. The direction of energy flow is perpendicular to the E and H planes.*

It can be expressed as

$$\boxed{\int_v \vec{E}.\vec{J}dv = \frac{-\partial}{\partial t}\int_v \left(\frac{\mu H^2}{2} + \frac{\varepsilon E^2}{2}\right) dv - \oint_S (\vec{E} \times \vec{H}) dS.} \qquad (8.81)$$

← we have to Prove.

From the perspective of conservation of energy, it can be stated that *the rate of energy dissipation in the volume v must be equal to the rate at which the stored energy in volume v is decreasing plus the rate at which energy is entering the volume v from outside.*

Proof Consider an EM wave travelling through a medium. From Maxwell's equation, we know that

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

or $\boxed{\vec{J} = \nabla \times \vec{H} - \varepsilon \frac{\partial \vec{E}}{\partial t}}$ — 1 $[\because D = \varepsilon \vec{E}]$ — 1

Taking dot product with $\vec{E}$. on Both side.

$$\boxed{\vec{E} \cdot \vec{J} = \vec{E} \cdot (\nabla \times \vec{H}) - \varepsilon \vec{E} \cdot \frac{\partial \vec{E}}{\partial t}} \quad \text{—— 2} \qquad (8.82)$$

We know that the vector identity

$$\boxed{\nabla \cdot (\vec{E} \times \vec{H}) = \vec{H} \cdot (\nabla \times \vec{E}) - \vec{E}(\nabla \times \vec{H})}$$

Vector Identity.

or $\boxed{\vec{E} \cdot (\nabla \times \vec{H}) = \vec{H} \cdot (\nabla \times \vec{E}) - \nabla \cdot (\vec{E} \times \vec{H}).}$

Substituting in Eq. (8.83),

$$\boxed{\vec{E} \cdot \vec{J} = \vec{H} \cdot (\nabla \times \vec{E}) - \nabla \cdot (\vec{E} \times \vec{H}) - \varepsilon \vec{E} . \frac{\partial \vec{E}}{\partial t}} \quad \text{—— 3} \qquad (8.83)$$

From Maxwell's equation, we know that

$$\boxed{\nabla \times \vec{E} = \frac{-\partial \vec{B}}{\partial t} = -\frac{\mu \partial \vec{H}}{\partial t}}$$

Maxwell's equation

$\boxed{\vec{B} = \mu \vec{H}}$

Substituting this in Eq. (8.83), 3

$$\boxed{\vec{E} \cdot \vec{J} = \vec{H} \cdot \left(-\mu \frac{\partial \vec{H}}{\partial t} \right) - \nabla \cdot (\vec{E} \times \vec{H}) - \varepsilon \vec{E} \cdot \frac{\partial \vec{E}}{\partial t}}$$

or $\vec{E} \cdot \vec{J} = -\mu \vec{H} \cdot \frac{\partial \vec{H}}{\partial t} - \varepsilon \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} - \nabla \cdot (\vec{E} \times \vec{H}).$

We know that,

$$\vec{H} \cdot \frac{\partial \vec{H}}{\partial t} = \frac{1}{2} \frac{\partial H^2}{\partial t} \text{ and } \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} = \frac{1}{2} \frac{\partial E^2}{\partial t}. \qquad (8.84)$$

$$\therefore \quad \vec{E} \cdot \vec{J} = \frac{-\mu}{2} \frac{\partial H^2}{\partial t} - \frac{\varepsilon}{2} \frac{\partial E^2}{\partial t} - \nabla \cdot (\vec{E} \times \vec{H}).$$

Integrating over the $\boxed{\text{volume,}}$

$$\int_v (\vec{E}\cdot\vec{J})dv = \frac{-\partial}{\partial t}\int_v \left(\frac{\mu H^2}{2}+\frac{\varepsilon E^2}{2}\right)dv - \int_v \nabla\cdot(\vec{E}\times\vec{H})dv. \tag{8.85}$$

Using $\boxed{\text{divergence theorem}}$ we know that,

$$\boxed{\int_v \nabla\cdot(\vec{E}\times\vec{H})dv = \oint_S (\vec{E}\times\vec{H})d\vec{S}.}$$ Divergence Theorem.

Substituting this in Eq. (8.85), the final expression is

$$\underbrace{\int_v (\vec{E}\cdot\vec{J})dv}_{\text{1st term}} = \underbrace{\frac{-\partial}{\partial t}\int_v \left(\frac{\mu H^2}{2}+\frac{\varepsilon E^2}{2}\right)dv}_{\text{2nd term}} - \underbrace{\oint_S (\vec{E}\times\vec{H})\cdot d\vec{S}}_{\text{3rd term}} \qquad \text{(Proved).}$$

The physical interpretation of poynting theorem is given below.

The first term $\int_v (\vec{E}\cdot\vec{J})dv$ = Total power dissipated in a volume when the electric field vector produces a current density J in the conducting medium. It represents the power dissipated due to ohmic ($I^2 R$) loss of the conductor.

$\vec{E}\cdot\vec{J}$ represents power density in watts /m^3.

The second term $\frac{-\partial}{\partial t}\int_v \left(\frac{\mu H^2}{2}+\frac{\varepsilon E^2}{2}\right)dv$ = rate at which the stored energy in the volume is decreasing.

where, $\frac{\mu H^2}{2}$ = energy density stored in the magnetic field.

and $\frac{\varepsilon E^2}{2}$ = energy density stored in the electric field.

The third term $-\oint_S (\vec{E}\times\vec{H})\cdot d\vec{S}$ = The rate of flow of energy inward through the surface of the volume.

where $\vec{E}\times\vec{H} = \vec{P}$ is a Poynting vector in watts/m^2.