UNIT - VII

Random processes - Spectral characteristics.

① The power spectrum :

properties

② ~~Test~~ Relationship between power spectrum and auto correlation function

③ The cross-power density spectrum

④ properties

⑤ Relationship between cross-power spectrum and cross correlation function

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Introduction :

In the previous unit random variable is a function of time, that is random process is studied, specifically in time domain. In this unit, its spectral characteristics i.e. its frequency domain description will be studied.

§ power spectrum or power density spectrum or power spectral density.

(Revision spectral analysis)

A random variable varying with respect to time is a random process and if the variation is known variation or deterministic variation, the corresponding process is a deterministic process or a signal.

For a periodic signal, fourier series is used for the study of spectral behaviour.

If it is non-periodic, fourier transform is used for the study of spectral behaviour. It is given as

$$X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} \cdot dt, \quad \text{where}$$

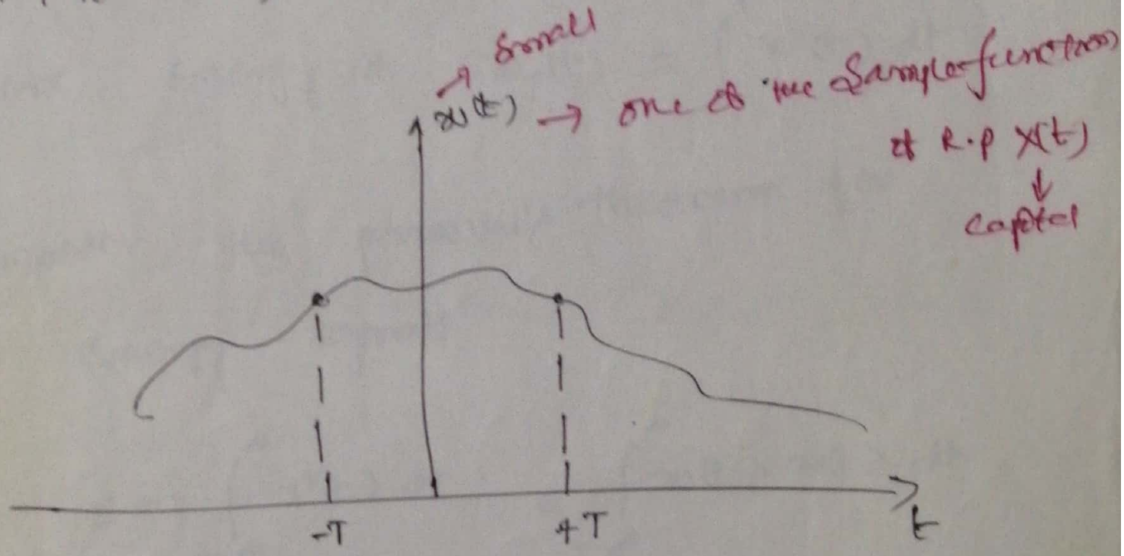
$x(t)$ is the deterministic process.

$\boxed{\text{IFT is } x(t) = \frac{1}{2\pi}\int X(\omega) e^{j\omega t} d\omega}$

If the variation of the r.v. with respect to time is unknown or non-deterministic, the ~~present~~ Result of the variation is a stochastic process. Now, we will consider the spectral analysis of such a process.

A random process or a stochastic process $X(t)$ is a collection of number of sample functions. Let $x(t)$ be a one sample function of Ensemble.

Let $x(t)$ be truncated i.e select a portion from $x(t)$, extending from $t = -T$ to $+T$, and remaining is neglected.



Let $x_T(t) = x(t)$ for $-T < t < T$

$= 0$, elsewhere

Now, $x_T(t)$ is of finite duration and we will apply fourier transform for that.

$$\therefore \quad F\left[x_T(t)\right] = x_T(\omega) = \int_{-T}^{T} x_T(t) e^{-j\omega t} \, dt$$

Since over $(-T, +T)$, $x_T(t) = x(t)$

$$\therefore \quad X_T(\omega) = \int_{-T}^{+T} x(t) \cdot e^{-j\omega t} \cdot dt$$

Energy of a signal is defined as

$$E = \int_{-\infty}^{\infty} x^2(t) \cdot dt$$

Here, Energy of $x_T(t) = \int_{-T}^{T} x^2(t) \cdot dt$

consider the parseval's theorem for

Energy signals

$$E = \int_{-\infty}^{\infty} x^2(t) \, dt = \int_{-\infty}^{\infty} x(t) \cdot x(t) \cdot dt$$

Using Inverse fourier transform,

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \cdot e^{+j\omega t} \cdot d\omega$$

$$\therefore \quad E = \int_{-\infty}^{\infty} x(t) \cdot \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \, e^{j\omega t} \cdot d\omega \right] dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \cdot \left[\underbrace{\int_{-\infty}^{\infty} x(t) \, e^{j\omega t} \cdot dt}_{X(-\omega)} \right] d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \cdot X(-\omega) \cdot d\omega$$

$$\because X(-\omega) = X^*(\omega)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \cdot X^*(\omega) \cdot d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \cdot d\omega.$$

Using this relation for $x_T(t)$, its Energy

$$= \int_{-T}^{T} x^2(t)\,dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X_T(\omega)|^2 \cdot d\omega$$

Since, power is time average of Energy, power available over the interval $(-T, T)$ is

$$\frac{1}{2T} \times \cancel{\frac{1}{2T}} \cdot \int_{-T}^{T} x^2(t)\,dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|X_T(\omega)|^2}{2T}\,d\omega$$

Thus $\frac{|X_T(\omega)|^2}{2T}$ is referred to as power density spectrum or power spectral density

The above power computed is the power available in the interval $(-T, T)$ only, but not in the entire $x(t)$, which is of very large duration. So, to get the expression for the power in $x(t)$, consider the limits $T \to \infty$

But, this is also the power contained in one sample function $x(t)$ of the process $X(t)$. To find the power of the process $X(t)$, its average, i.e. expected value is considered. So the average power of the random process $X(t)$ is

$$P_{avg} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} E[x^2(t)]\, dt$$

~~avalable~~ avg ✓

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{\lim_{T \to \infty} \frac{E[|X_T(\omega)|^2]}{2T}}_{S_{XX}(\omega)} . d\omega$$

Thus, the average power of a random process is obtained by ~~the~~ time averaging its mean square value.

$$\cancel{\text{If} \lim_{T \to \infty} \frac{E[|X_T(\omega)|^2]}{2T}}$$

If $\lim_{T \to \infty} \frac{E[|X_T(\omega)|^2]}{2T}$ is named power spectral density, $S_{XX}(\omega)$.

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then the average power is given as

$$P_{avg} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\cdot d\omega.$$ ie the area

Enclosed by power spectral density curve is

the power.

§ properties of PSD :-

① $S_{XX}(\omega) \geq 0$

② $S_{XX}(-\omega) = S_{XX}(\omega)$, $x(t)$ real

③ $S_{XX}(\omega)$ is real

④ $$\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\, d\omega = A\left\{E\left[x^2(t)\right]\right\}$$

⑤ $$S_{\dot{X}\dot{X}}(\omega) = \omega^2 \cdot S_{XX}(\omega)$$

⑥ $$\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\, d\omega = A\left[R_{XX}(t, t+\tau)\right]$$

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} A\left[R_{XX}(t, t+\tau)\right] e^{-j\omega\tau}\, d\tau$$

It states that the power density spectrum and the time average of the auto correlation function form a fourier transform pair.

If $x(t)$ is at least WSS,

then $A[R_{XX}(t, t+\tau)] = R_{XX}(\tau)$, then

the (6) property indicates that the power spectrum and the auto correlation function form a fourier transform pair.

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)\, e^{-j\omega\tau} . d\tau$$

$$R_{XX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) . e^{j\omega\tau} . d\omega$$

for a W.S.S.

Relationship between power-spectrum and auto-correlation function :-

Relationship :- Time-average of auto correlation function and the power spectral density form a Fourier transform pair. ie

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)\, e^{-j\omega\tau} \cdot d\tau$$

$$\Rightarrow \quad R_{XX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) \cdot e^{j\omega\tau} \cdot d\omega \cdot$$

i.e $R_{XX}(\tau) \overset{F.T}{\longleftrightarrow} S_{XX}(\omega)$ [called wiener-khintchine relation]

proof :

we have P_S^{SD},

$$S_{XX}(\omega) = \lim_{T\to\infty} \frac{E\left[|X_T(\omega)|^2\right]}{2T}$$

$$= \lim_{T\to\infty} \frac{E\left[X_T^*(\omega) \cdot X_T(\omega)\right]}{2T}$$

But, we have

$$X_T(\omega) = \int_{-T}^{+T} x(t)\cdot e^{-j\omega t}\cdot dt$$

replace t by t_2

$$\Rightarrow \quad X_T^*(\omega) = \int_{-T}^{+T} x(t)\cdot e^{j\omega t}\cdot dt$$

replace t by t_1

$$\therefore \quad S_{XX}(\omega) = \lim_{T\to\infty} \frac{1}{2T}\cdot E\left[\int_{-T}^{T} x(t_1)\cdot e^{j\omega t_1}\cdot dt_1 \cdot \int_{-T}^{T} x(t_2) e^{-j\omega t_2}\cdot dt_2\right]$$

$$= \lim_{T\to\infty} \frac{1}{2T}\cdot E\left[\int_{-T}^{+T}\int_{-T}^{+T} x(t_1)\cdot x(t_2)\cdot e^{-j\omega(t_2-t_1)}\cdot dt_1\cdot dt_2\right]$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T}\int_{-T}^{+T} E\left[x(t_1)\cdot x(t_2)\right]\cdot e^{-j\omega(t_2-t_1)}\cdot dt_1\cdot dt_2$$

$$S_{XX}(\omega) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T}\int_{-T}^{+T} R_{XX}(t_1,t_2)\cdot e^{-j\omega(t_2-t_1)}\cdot dt_1\cdot dt_2$$

Consider the Expression, $\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\cdot e^{j\omega\tau}\cdot d\omega$

which is the Inverse-fourier transform of $S_{XX}(\omega)$

$$F^{-1}[S_{XX}(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{+T}\int_{-T}^{+T} R_{XX}(t_1,t_2)\cdot e^{-j\omega(t_2-t_1)}\cdot dt_1\cdot dt_2\right]\cdot e^{j\omega\tau}\cdot d\omega$$

$$= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{+T}\int_{-T}^{+T} R_{XX}(t_1,t_2)\cdot \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega(\tau-t_2+t_1)}\cdot d\omega}\cdot dt_1\cdot dt_2$$

we have $F[\delta(t)] = 1$

$$\Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} 1\cdot e^{j\omega t}\cdot d\omega = \delta(t)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega(\tau-t_2+t_1)}\cdot d\omega = \delta(\tau-t_2+t_1)$$

$$= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{+T}\int_{-T}^{+T} R_{XX}(t_1,t_2)\cdot\delta(\tau-t_2+t_1)\cdot dt_1\cdot dt_2$$

$\int \delta(t) = 1$

$\int_{-T}^{T} \delta(\tau-t_2+t_1) = 1$

$\delta(\tau-t_2+t_1) = 1$ at $\tau-t_2+t_1=0$

ie $t_2=\tau+t_1$.

$$\therefore F^{-1}[S_{XX}(\omega)] = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{+T} R_{XX}(t_1,\tau+t_1)\cdot dt_1$$

let $t_1 = t \Rightarrow dt_1 = dt$

$$\therefore \quad F^{-1}[S_{XX}(\omega)] = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} \underbrace{R_{XX}(t, t+\tau)}_{a.c.f} \, dt$$

The right-hand side of the above equation is the time average of auto correlation function.

Thus time average of auto correlation function and PSD form a Fourier transform pair.

If the process is stationary, the time average of auto correlation $R_{XX}(t, t+\tau)$ will be $R_{XX}(\tau)$, since auto correlation function of stationary process is independent of time t.

$$\therefore \quad F^{-1}[S_{XX}(\omega)] = R_{XX}(\tau)$$

Thus, for a WSS random process, auto correlation and power spectral density form a fourier transform pair. i.e

$$R_{XX}(\tau) \xleftrightarrow{F.T} S_{XX}(\omega)$$

This is called wiener-Khintchine relation.

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau) \cdot e^{-j\omega\tau} \cdot d\tau$$

$$R_{xx}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(\omega)\, e^{j\omega\tau} \cdot d\omega .$$

Bandwidth of the power density spectrum:

The rms bandwidth $\overline{w_{rms}}$ is given by

$$\omega_{rms}^2 = \frac{\int_{-\infty}^{\infty} \omega^2 S_{XX}(\omega)}{\int_{-\infty}^{\infty} S_{XX}(\omega)}$$

$$W_{rms} = \sqrt{\frac{\int_{-\infty}^{\infty} \omega^2 S_{XX}(\omega)}{\int_{-\infty}^{\infty} S_{XX}(\omega)}}.$$

§ Cross-power Density Spectrum :-

or

Cross- spectral Density.

Let $X(t)$ and $Y(t)$ be two random processes and let $x(t)$ and $y(t)$ be their sample functions respectively.

Let $x_T(t) = x(t), \quad -T < t < +T$

$= 0, \quad$ elsewhere

Small letters

and let $y_T(t) = y(t), \quad -T < t < +T$

$= 0, \quad$ elsewhere

The individual fourier transform of $x_T(t)$ and $y_T(t)$ are

$$X_T(\omega) = \int_{-T}^{T} x_T(t) \cdot e^{-j\omega t} \, dt$$

$$= \int_{-T}^{T} x(t) \cdot e^{-j\omega t} \, . dt$$

$$Y_T(\omega) = \int_{-T}^{T} y_T(t) \cdot e^{-j\omega t} \, . dt$$

$$= \int_{-T}^{T} y(t) \cdot e^{-j\omega t} \, . dt$$

Consider $\int_{-\infty}^{\infty} x(t) \cdot y(t) \cdot dt$

we have $y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Y(\omega) \cdot e^{+j\omega t} \cdot d\omega$

$\therefore \int_{-\infty}^{\infty} x(t) \cdot y(t) \cdot dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} x(t) \left[\int_{-\infty}^{\infty} Y(\omega) e^{j\omega t} \cdot d\omega \right] dt$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} Y(\omega) \cdot \Big[\underbrace{\int_{-\infty}^{\infty} x(t) \cdot e^{j\omega t} \cdot dt}_{X(-\omega)} \Big] d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(-\omega) \cdot Y(\omega) \cdot d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega) \cdot Y(\omega) \cdot d\omega$$

$$\therefore \int_{-\infty}^{\infty} x(t) \cdot y(t) \cdot dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega) \cdot Y(\omega) \cdot d\omega$$

$$= \boxed{\frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \cdot Y^*(\omega) \cdot d\omega}$$

the ~~average~~ Cross power $P_{xy} = \frac{1}{2T} \int_{-T}^{T} x_T(t) \cdot y_T(t) \cdot dt$

$$P_{xy} = \frac{1}{2T} \int_{-T}^{T} x(t) \cdot y(t) \cdot dt$$

By using the parseval's relation

$$\frac{1}{2T}\int_{-T}^{T} x(t)\,y(t)\,dt = P_{xy} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{X_T^*(\omega)\cdot Y_T(\omega)}{2T}\cdot d\omega$$

Using the similar argument as made in the case of PSD and $T \to \infty$

Total avg crosspower by taking expectation

$$P_{xy}(avg) = \underset{T\to\infty}{Lt}\ \frac{1}{2T}\int_{-T}^{T} E\left[x(t)\cdot y(t)\right]\cdot dt$$

or

$\bar{P}_{XY}(T)$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \underset{T\to\infty}{Lt}\ \frac{E\left[X_T^*(\omega)\cdot Y_T(\omega)\right]}{2T}\cdot d\omega$$

If we define $\underset{T\to\infty}{Lt}\ \dfrac{E\left[X_T^*(\omega)\,Y_T(\omega)\right]}{2T} = S_{xy}(\omega)$.

i.e. the cross-spectral density.

The average power

$$P_{xy}(avg) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xy}(\omega)\cdot d\omega$$

Similarly, ~~$S_{xy}(\omega)$~~ =

$$P_{xy}(avg) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{yx}(\omega)\cdot d\omega$$

§ properties :-

① $S_{xy}(\omega) = S_{yx}(-\omega) = S^*_{yx}(\omega)$

② $Re[S_{xy}(\omega)]$ and $Re[S_{yx}(\omega)]$ are

even functions of ω.

③ $Im[S_{xy}(\omega)]$ and $Im[S_{yx}(\omega)]$ are

odd functions of ω.

④ $S_{xy}(\omega) = 0$ and $S_{yx}(\omega) = 0$, if $x(t)$ and $y(t)$ are orthogonal.

⑤ If $x(t)$ and $y(t)$ are uncorrelated and have constant means $\bar{x}$ and $\bar{y}$

$$S_{xy}(\omega) = S_{yx}(\omega) = 2\pi \bar{x}\bar{y}\delta(\omega)$$

⑥ $A[R_{xy}(t, t+\tau)] \leftrightarrow S_{xy}(\omega)$

$A[R_{yx}(t, t+\tau)] \Leftrightarrow S_{yx}(\omega)$

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Note,

$$S_{xy}(\omega) = \int_{-\infty}^{\infty} R_{xy}(\tau)\, e^{-j\omega\tau} \cdot d\tau$$

$$S_{yx}(\omega) = \int_{-\infty}^{\infty} R_{yx}(\tau)\, e^{-j\omega\tau} \cdot d\tau$$

$$R_{xy}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xy}(\omega)\, e^{j\omega\tau} \cdot d\omega$$

$$R_{yx}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{yx}(\omega)\, e^{j\omega\tau} \cdot d\omega$$

§ Relationship between cross correlation function and cross spectral density function (or cross power spectrum) form a Fourier transform pair.

Relationship :- Time average of cross correlation function and the cross spectral density function form a F.T. pair.

$$F^{-1}[S_{XY}(\omega)] = R_{XY}(\tau)$$

or $R_{XY}(\tau) \xleftrightarrow{FT} S_{XY}(\omega)$

proof :- we have cross spectral density

$$S_{XY}(\omega) = \lim_{T\to\infty} \frac{E[X_T^*(\omega) \cdot Y_T(\omega)]}{2T}$$

consider $x(t_1)$ and $y(t_2)$

$$X_T^*(\omega) = \int_{-T}^{+T} x(t_1) \cdot e^{j\omega t_1} \cdot dt_1$$

$$Y_T(\omega) = \int_{-T}^{+T} y(t_2) \cdot e^{-j\omega t_2} \cdot dt_2$$

$$\therefore S_{XY}(\omega) = \lim_{T\to\infty} \frac{1}{2T} \cdot E\left[\int_{-T}^{+T} x(t_1) e^{j\omega t_1} \cdot dt_1 \cdot \int_{-T}^{+T} y(t_2) \cdot e^{-j\omega t_2} \cdot dt_2\right]$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T}\int_{-T}^{+T} E[x(t_1) \cdot y(t_2)] \, e^{-j\omega(t_2-t_1)} \cdot dt_1 \cdot dt_2$$

$$S_{XY}(\omega) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T}\int_{-T}^{+T} R_{XY}(t_1,t_2) \, e^{-j\omega(t_2-t_1)} \, dt_1 \cdot dt_2$$

Consider the Expression $\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xy}(\omega) \cdot e^{j\omega\tau} \cdot d\omega$.

which is the I.F.T of $S_{xy}(\omega)$.

$$F^{-1}[S_{xy}(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{+T}\int_{-T}^{+T} R_{xy}(t_1,t_2) \cdot e^{-j\omega(t_2-t_1)} \cdot dt_1 \cdot dt_2\right] e^{j\omega\tau} \cdot d\omega$$

$$= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{+T}\int_{-T}^{+T} R_{xy}(t_1,t_2) \cdot \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega(\tau-t_2+t_1)} d\omega \cdot dt_1 \cdot dt_2$$

Since $\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega(\tau-t_2+t_1)} d\omega = \delta(\tau-t_2+t_1)$

$$F^{-1}[S_{xy}(\omega)] = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{+T}\int_{-T}^{+T} R_{xy}(t_1,t_2) \cdot \delta(\tau-t_2+t_1)\, dt_1 \cdot dt_2$$

Since $\delta(\tau-t_2+t_1) = 1$ at $\tau-t_2+t_1 = 0$

$$\Rightarrow t_2 = \tau+t_1$$

$$F^{-1}[S_{xy}(\omega)] = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{+T} R_{xy}(t_1,t_1+\tau) \cdot dt_1$$

Let $t_1 = t \Rightarrow dt_1 = dt$

$$\therefore F^{-1}[S_{xy}(\omega)] = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{+T} R_{xy}(t,t+\tau) \cdot dt$$

The right hand side of the above equation is the time average of cross correlation function.

It implies that time average of cross correlation function and cross spectral density form a F.T. pair.

If the processes $X(t)$ and $Y(t)$ are jointly WSS processes, the time average of cross correlation function $R_{XY}(t, t+\tau)$ will be $R_{XY}(\tau)$.

$$\therefore \quad F^{-1}[S_{XY}(\omega)] = R_{XY}(\tau)$$

$$\Rightarrow \quad R_{XY}(\tau) \xleftrightarrow{F.T} S_{XY}(\omega)$$