30/3/2019 Saturday

UNIT-5

Vector Integral Calculus

* **Line integral:**

→ $\bar{F}$ is a vector pt fn

An integral which is evaluated along a curve C is called a line integral along C.

* **Workdone by a force vector**

→ If $\vec{F}$ represents force vector acting on a particle moving along an arc AB ($\overset{\frown}{AB}$), then the workdone during a small displacement is given by

$$\bar{F} \cdot \delta\bar{r}$$

Then the total workdone by $\bar{F}$ during displacement A to B is given by

$$\int_A^B \bar{F} \cdot d\bar{r}.$$

$$= \int_A^B (F_1 dx + F_2 dy + F_3 dz)$$

* **Circulation:**

→ If $\vec{F}$ represents velocity of a fluid particle and c is a closed curve then the integral $\oint_c \vec{F} \cdot d\bar{r}$ is known as circulation of $\vec{F}$ round the curve C.

(*) <u>Note</u> :

- If $\oint_C \vec{F}\cdot\vec{dr} = 0$ then the field $\vec{F}$ is said to be conservative since there is no workdone i.e., energy is conserved.
- If the circulation of $\vec{F}$ round every close curve in a region D vanishes then $\vec{F}$ is said to be irrotational in D.
- $\vec{F}$ is a conservative field if and only if it is an exact differential of a scalar fn. i.e., $\vec{F} = \nabla\phi$. where ϕ is scalar potential.
- $\vec{F}$ is a conservative fied if and only if the workdone is independent on the path joining.
- $\vec{F}$ is a conservative field if and only if it is irrotational.
- Area A of a regular region B bounded by the curve 'C' is given by $A = \frac{1}{2}\oint_C (x\,dy - y\,dx)$.

Q. Evaluate $\int_C \vec{F}\cdot\vec{dr}$ where $\vec{F} = 3xy\bar{i} - y^2\bar{j}$ and the curve C is $y = 2x^2$ in xy plane from O(0,0) to A(1,2)

Sol. Let $\vec{F} = 3xy\bar{i} - y^2\bar{j}$

$C : y = 2x^2$ in xy plane from $(0,0) \to (1,2)$

In xy-plane,

$d\bar{r} = dx\bar{i} + dy\bar{j} + 0\bar{k}$

$= dx\bar{i} + dy\bar{j}$

Let $\vec{F} \cdot d\bar{r} = (3xy\bar{i} - y^2\bar{j}) \cdot (dx\bar{i} + dy\bar{j})$

$= 3xy\,dx - y^2 dy$

$= 3x(2x^2)dx - y^2 dy$

$= 6x^3 dx - y^2 dy$ —— (1)

$x : 0$ to 1 ; $y : 0$ to 2

Let $\int_C \vec{F} \cdot d\vec{r} = \int_C (6x^3 dx - y^2 dy)$

$= \int_0^1 6x^3 dx - \int_0^2 y^2 dy$

$= 6\left(\frac{x^4}{4}\right)_0^1 - \left[\frac{y^3}{3}\right]_0^2$

$= -\frac{7}{6}.$

Q. Find the workdone by $\vec{F}$ which $= 3x^2\bar{i} + (2xz - y)\bar{j} + z\bar{k}$ along (i) the st line from the point $(0,0,0)$ to $(2,1,3)$

(ii) the curve $x^2 = 4y$, $3x^3 = 8z$ from $x = 0$ to $x = 2$.

Sol: $\vec{F} \cdot d\bar{r} = \int_C (3x^2 dx + (2xz - y)\bar{j} + z\bar{k})$

$= \int_C (3x^2 dx + (2xz - y)\,dy + z\,dz)$

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1. O to A $\Rightarrow$ Eqn of the line AB in parametric form

$$= \frac{x-0}{2-0} = \frac{y-0}{1-0} = \frac{z-0}{3-0} = t$$

$\Rightarrow x = 2t \,;\, y = t \,;\, z = 3t$

$\Rightarrow dx = 2dt \,;\, dy = dt \,;\, dz = 3dt.$

t : 0 to 1.

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \left(3(2t)^2(2dt) + (2(2t)(3t) - t)\,dt + (3t)(3dt)\right)$$

$$= \int_0^1 (12t^2dt + 12t^2dt - tdt + 9tdt)$$

$= 16.$

7/4/2019

Monday

Q. Find the circulation of $\vec{F}$ where
$\vec{F} = (2x-y+2z)\bar{i} + (x+y-z)\bar{j} + (3x-2y-5z)\bar{k}$
along the circle $x^2+y^2=4$ in xy-plane.

Sol: Circulation $= \oint_C \vec{F}\cdot\vec{dr}$ ——(1)

In the xy-plane,

$d\bar{r} = dx\bar{i} + dy\bar{j}$

$\Rightarrow \vec{F}\cdot\vec{dr} = (2x-y+0)dx\,(x+y)dy \quad (\because z=0)$

sub in (1)

Circulation $= \oint((2x-y)dx + (x+y)dy)$ →(2)

$x = r\cos\theta$, $y = r\sin\theta$, $dxdy = r\,dr\,d\theta$.

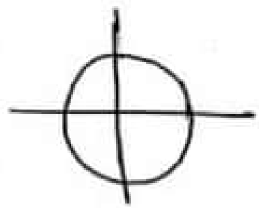
$x = 2\cos\theta$, $y = 2\sin\theta$

$dx = -2\sin\theta d\theta$, $dy = 2\cos\theta d\theta$.

$x^2+y^2 = 4 \Rightarrow r^2 = 4 \Rightarrow r = 2$

$r : 0$ to 2

$\theta : 0$ to 2π



Circulation: $\int_0^{2\pi}\{(4\cos\theta - 2\sin\theta)(-2\sin\theta d\theta) + (2\cos\theta + 2\sin\theta)(2\cos\theta d\theta)\}$

$= \int_0^{2\pi}(-4\sin\theta\cos\theta\, d\theta) + \int_0^{2\pi} 4$

$= 4(\cos\theta)_0^{2\pi} + 4(2\pi) = 8\pi$

Q. If $\vec{F} = (4xy - 3x^2z^2)\bar{i} + (2x^2)\bar{j} - 2x^3z\bar{k}$ then P.T

$\int \vec{F}\cdot\vec{dr}$ i.e, the workdone is independent of the curve joining 2 points

To prove that the workdone is independent of the path joining we need to prove that $\vec{F}$ is irrotational

$$\text{curl } \vec{F} = 0$$

$$\text{curl } \vec{F} = \nabla \times \vec{F}$$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4xy-3x^2z^2 & 2x^2 & -2x^3z \end{vmatrix}$$

$$= \bar{i}(0) + j(0) + \bar{k}(0) = \bar{0}.$$

Hence it is irrotational. Therefore, $\vec{F}$ is the workdone is independent of the curve joining 2 points.

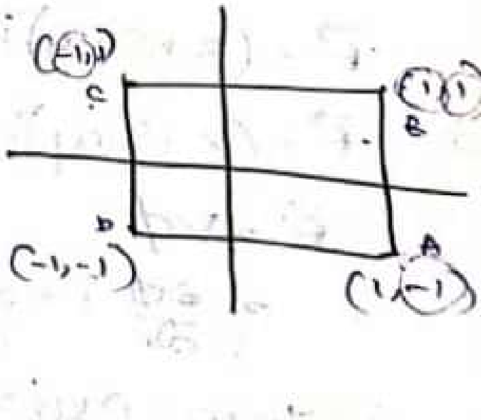
$\therefore \vec{F}$ is a conservative field.

Q. If $\vec{F} = x(x+y)\bar{i} + (x^2+y^2)\bar{j}$ evaluate $\int_C \vec{F}\cdot d\vec{r}$ where c is a square in xy plane bounded by lines $x = \pm 1, y = \pm 1$

Sol: $\int_C \vec{F}\cdot d\vec{r} = \int_C (x^2+xy)dx + (x^2+y^2)dy$ ——— (1)

If c is a square,

$\int_C = \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA}$ ——— (2)



On AB: $x = 1 \Rightarrow dx = 0$

$y = -1$ to 1

from (1)

$$\int_{AB} \vec{F}\cdot\vec{dr} = \int_{-1}^{1} (1^2+y^2)dy = \left[y + \frac{y^3}{3} \right]_{-1}^{1} = \frac{8}{3} \longrightarrow AB$$

On BC: $y = 1 \Rightarrow dy = 0$

$x : 1$ to -1

$$\int_{BC} \vec{F}\cdot\vec{dr} = \int_{1}^{-1} (x^2+x)dx$$

$$= \left(\frac{x^3}{3} + \frac{x^2}{2} \right)_{1}^{-1}$$

$$= -\frac{2}{3} \longrightarrow BC$$

On CD: $x = -1 \rightarrow dx = 0$

$y : 1$ to -1

$$\int_{CD} \vec{F}\cdot\vec{dr} = \int_{1}^{-1} (1+y^2)dy = \frac{-8}{3} \longrightarrow CD$$

On DA: $y = -1 \Rightarrow dy = 0$

$x : -1$ to 1

$$\int_{DA} (x^2-x)dx = \left(\frac{x^3}{3} - \frac{x^2}{2} \right)_{-1}^{1} = \frac{2}{3}$$

$$\int_{C} \vec{F}\cdot\vec{dr} = \frac{8}{3} - \frac{2}{3} - \frac{8}{3} + \frac{2}{3} = 0.$$

Q. Find the scalar potential of $\vec{F}$ where

$\vec{F} = (z+\sin y)\bar{i} + (-z + x\cos y)\bar{j} + (x-y)\bar{k}$.

Sol: $\vec{F} = (x+\sin y)\bar{i} + (-z+x\cos y)\bar{j} + (x-y)\bar{k}$

$\vec{F} = \nabla\phi$

$= \frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k}$

from (1) & (2)

$\frac{\partial\phi}{\partial x} = z + \sin y \Rightarrow \phi = x(z + \sin y) + C_1$

$\frac{\partial\phi}{\partial y} = -z + x\cos y \Rightarrow \phi = -zy + x\sin y + C_2$

$\frac{\partial\phi}{\partial z} = x - y \Rightarrow \phi = (x - y)z + C_3$

$\therefore \phi = xz - zy + x\sin y + C$

Q. Prove that ~~the~~ force field $\vec{F} = 2xyz^3\bar{i} + x^2z^3\bar{j} + 3x^2yz^2\bar{k}$ is conservative. Find the work done by moving a particle from $P(1,-1,2)$ to $Q(3,2,-1)$ in this force fixed. Find the scalar potential of $\vec{F}$

Sol: Curl $\vec{F} = \nabla \wedge \phi$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz^3 & x^2z^3 & 3x^2yz^2 \end{vmatrix}$$

$$= \bar{i}(3x^2z^2 - 3z^2x^2) - \bar{j}(6xyz^2 - 6xyz^2) + \bar{k}(2xz^3 - 2xz^3)$$

$= 0$

Hence it is irrotational. Thereby it is conservative

Line PQ $\Rightarrow$ ~~$x = -3t$; $y = -3t + 2$~~

$x = 2t + 1$; $y = 3t - 1$; $z = -3t + 2$

$dx = 2dt$; $dy = 3dt$; $dz = -3dt$.

$\int \vec{F} \cdot d\vec{r} = \int 2(2t+1)(3t-1)$

$\vec{F}\cdot\vec{dr} = 2xyz^3dx + x^2z^3dy + 3x^2yz^2dz$

$= d(x^2yz^3)$

Workdone $= \int_P^Q d(x^2yz^3) = [x^2yz^3]_P^Q$

$= -18+8 = -10.$

2/4/2019

Tuesday

Evaluate line integral $\int_{AB}(x+y)dx + xdy$ for the following parts.

(a) AB → A(0,0) to B(1,1)

(b) AB → $y = x^2$ from A(0,0) to B(1,1)

(c) along the st. line AC & CB where A(0,0), C(1,0), B(1,1).

(i) $\frac{x-0}{1-0} = \frac{y-0}{1-0} = t$

$x = t$; $y = t$.

$dx = dt$; $dy = dt$

$\int_{AB}(x+y)dx + xdy = \int_{AB} 3x\,dx = \left[\frac{3x^2}{2}\right]_0^1 = \frac{3}{2}$

(b) $\int \vec{F}\cdot d\vec{r} = \int_{AB} (x+y)\,dx + x\,dy$

$y = x^2$ } x : 0 to 1
$dy = 2x\,dx$

$$\int \vec{F}\cdot d\vec{r} = \int_0^1 (x+x^2)\,dx + x(2x\,dx)$$

$$= \int_0^1 x\,dx + x^2\,dx + 2x^2\,dx$$

$$= \left[\frac{x^2}{2} + \frac{x^3}{3} + \frac{2x^3}{3}\right]_0^1$$

$$= \frac{1}{2} + \frac{1}{3} + \frac{2}{3} = \frac{3}{2}.$$

(c) $\int \vec{F}\cdot d\vec{r} = \int_{AC} + \int_{CB}$

$$= \int_0^1 x\,dx + \int_0^1 dy = \left[\frac{x^2}{2}\right]_0^1 + [y]_0^1$$

$$= \frac{3}{2}.$$

Along the path A, B, C, the workdone is $\frac{3}{2}$.
∴ it is independent of path joining.

Surface integral:

- It is a generalization of double integral.
- In a surface integral, the integrand is integrated over a curve surface.
- If $\vec{F}$ is any vector pt fn taken over a surface S then surface integral of $\vec{F}$ is defined as

$\int_S \vec{F}\cdot\vec{ds} = \iint_S \vec{F}\cdot\bar{n}\, ds$ where $\bar{n}$ is unit outward drawn normal to the given surfa

$$\bar{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

Evaluation of surface integral!

It is evaluated by reducing it to a double integral i.e., by projecting the surface s onto one of the ~~into~~ coordinate planes.

Case-1:

Let R be the projection of s onto xy-plane then $ds = \frac{dx\,dy}{|\bar{n}\cdot\bar{k}|}$

$$\iint_S \vec{F}\cdot\bar{n}\, ds = \iint_R \vec{F}\cdot\bar{n}\,\frac{dx\,dy}{|\bar{n}\cdot\bar{k}|}$$

Case-2:

Let R be the projection of s onto yz plane then $ds = \frac{dy\,dz}{|\bar{n}\cdot\bar{i}|}$

$$\iint_S \vec{F}\cdot\bar{n}\, ds = \iint_R \vec{F}\cdot\bar{n}\,\frac{dy\,dz}{|\bar{n}\cdot\bar{i}|}$$

Case-3:

Let R be the projection of s onto zx plane then $ds = \frac{dz\,dx}{|\bar{n}\cdot\bar{j}|}$

* Flux of a vector field:

If $\bar{v}$ represents velocity of the fluid particle then total outward flux of $\bar{v}$ along a closed surface is

$$\text{flux} = \int_{S} \bar{F} \cdot \bar{n}\, ds \quad \bar{v}$$

* Note:

→ When flux of $\bar{v}$ across every close surface S in a region E vanishes then $\bar{v}$ is said to be a solenoidal vector pt fn in E.

→ Surface Area of S is given by

$$S.A = \int_S ds = \iint_S ds = \iint_R \frac{dx\,dy}{\bar{n}\cdot\bar{k}}$$

$$= \iint_R \frac{\sqrt{F_x^2 + F_y^2 + F_z^2}}{|F_z|}\, dx\,dy.$$

→ Surface integral in component form is given by

$$\int_S \bar{F}\cdot\bar{n}\, ds = \int_S \left(F_1\, dy\,dz + F_2\, dx\,dz + F_3\, dx\,dy\right)$$

Q. Evaluate $\int_S \vec{F}\cdot\bar{n}\, ds$ where $\vec{F} = 18z\bar{i} - 12\bar{j} + 3y\bar{k}$ and S is the part of the surface of the plane $2x + 3y + 6z = 12$ located in the first oct

Sol: $S = \phi = 2x + 3y + 6z - 12 = 0$

$$\bar{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$\bar{n} = \frac{2\bar{i} + 3\bar{j} + 6\bar{k}}{\sqrt{49}}$$

$$= \frac{2\bar{i} + 3\bar{j} + 6\bar{k}}{7} \quad \text{——} (1)$$

Let R be the projection of S onto xy-plane.

$$\int_S \bar{F}\cdot\bar{n}\,ds = \iint_R \bar{F}\cdot\bar{n}\frac{dx\,dy}{|\bar{n}\cdot\bar{k}|} \quad \text{——} (2)$$

where $R: 2x + 3y = 12$

$x : 0$ to $\frac{12-3y}{2}$

$y : 0$ to 4

$$|\bar{n}\cdot\bar{k}| = \frac{6}{7} \quad \text{——} (3)$$

Sub (1) & (3) in (2)

$$\int_S \bar{F}\cdot\bar{n}\,ds = \int_0^4 \int_0^{\frac{12-3y}{2}} \frac{6}{7}(6-2x)\frac{dx\,dy}{\frac{6}{7}}$$

$$= \int_0^4 \int_0^{12-3y/2} (6-2x)\,dx\,dy$$

$$= \int_0^4 \left[6x - \frac{\cancel{2}x^2}{\cancel{2}}\right]_0^{\frac{12-3y}{2}} dy$$

$$= \int_0^4 \left[6\left(\frac{12-3y}{2}\right) - \left(\frac{12-3y}{2}\right)^2\right] dy.$$

$$= \int_0^4 \left[3(12-3y) - \frac{1}{4}\left(144 + 9y^2 - 72y\right)\right] dy$$

$$= \int_0^4 \left[36 - 9y - \frac{1}{4}\left(144 + 9y^2 - 72y\right)\right] dy$$

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$$= \left[36y - \frac{9y^2}{2} - \frac{\cancel{144}}{\cancel{4}}y - \frac{\cancel{9}}{4}\frac{y^3}{\cancel{3}} + \frac{\cancel{72}}{4}\frac{y^2}{\cancel{2}}\right]_0^4$$

$$= \left[36y - \frac{9y^2}{2} - 36y - \frac{3}{4}y^3 + 9y^2\right]_0^4$$

$$= 36(4) - \frac{9}{2}(16) - 36(4) - \frac{3}{\cancel{4}}(\cancel{64}) + 9(16)$$

$$= \cancel{\frac{9}{2}(16)} - 48 = 72 - 48$$

$$= 24.$$

$$\begin{array}{r} \cancel{6}\cancel{12}^{12} \\ 48 \\ \hline 24 \end{array}$$

4/4/2019 Thursday

Evaluate $\int_S \bar{F}\cdot\bar{n}\,ds$ where $\bar{F} = z\bar{i} + x\bar{j} - 3y^2 z\bar{k}$

and S is a surface $x^2+y^2=16$ included in first octant b/w $z=0$ to $z=5$.

$\vec{F} = z\bar{i} + x\bar{j} - 3y^2 z\bar{k}$

$S: \phi = x^2+y^2-16=0$ b/w $z=0$ to $z=5$.

Let $\bar{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\bar{i} + 2y\bar{j}}{\sqrt{4x^2+4y^2}}$

$\bar{n} = \frac{1}{4}(x\bar{i} + y\bar{j})$

$\bar{n}\cdot i = \frac{x}{4}$.

Let R be the projection S onto yz-plane.

$ds = \frac{dy\,dz}{|\bar{n}\cdot\bar{i}|} = \frac{dy\,dz}{x/4}$

Let $z : 0$ to 5
$y : 0$ to 4

Let $\int_S \bar{F}\cdot\bar{n}\,ds = \iint_R \bar{F}\cdot\bar{n}\,\dfrac{dy\,dz}{|\bar{n}\cdot\bar{i}|}$

$= \int_0^5\int_0^4 \frac{1}{4}\left[zx + xy\right]\dfrac{dy\,dz}{x/4}$

$= \int_0^5 \left[3y + \frac{y^2}{2}\right]_0^4 dz.$

$= \int_0^5 (4z+8)\,dz = 90.$

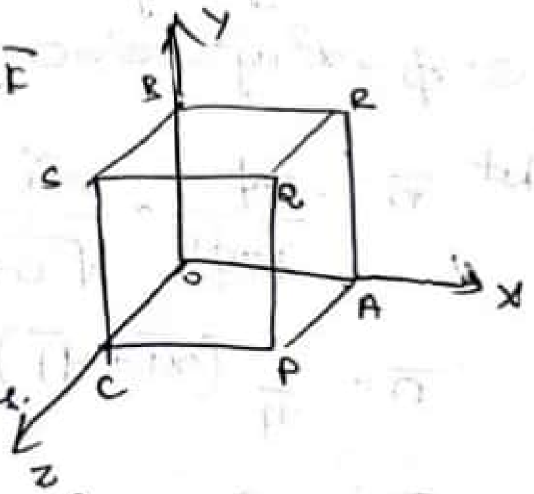
If: $\bar{F} = 4xz\bar{i} - y^2\bar{j} + yz\bar{k}$ then evaluate $\int_S \bar{F}\cdot\bar{n}\,ds$ where S is the surface of the cube $x=0, y=0, z=0$; $x=a$, $y=a$ & $z=a$.

Given $\bar{F} = 4xz\bar{i} - y^2\bar{j} + yz\bar{k}$

S: cube :

$x=0$; $y=0$, $z=0$

$x=a$, $y=a$, $z=a$.



A cube has 6 surfaces.

$\therefore \int_S \vec{F}\cdot\bar{n}\,ds = \int_{S_1} + \int_{S_2} + \cdots \int_{S_6}$

S_1: (ARQP) Let R_1 be the projection of S onto yz plane.

$x=a$; $y: 0$ to a ; $z: 0$ to a ; $\bar{n} = \bar{i}$

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$$ds = \frac{dydz}{|\bar{n}\cdot\bar{i}|} = dydz.$$

$$\int_{S_1} \vec{F}\cdot\bar{n}\,ds = \int_0^a\int_0^a (\bar{F}\cdot\bar{i})\,dy\,dz$$

$$= \int_0^a\int_0^a 4az\,dy\,dz = 4a\cdot\frac{a^2}{2}\cdot a = 2a^4 \hookrightarrow (S_1)$$

S_2 (OBSC) : $\rightarrow$ yz plane

$x = 0$, $\bar{n} = -\bar{i}$

$$ds = \frac{dydz}{|\bar{n}\cdot\bar{i}|} = dydz.$$

$$\int_{S_2} \vec{F}\cdot\bar{n}\,ds = \iint_R \bar{F}\cdot(-\bar{i})\,dydz = \iint_R 0\,dydz = 0 \hookrightarrow (S_2)$$

S_3 (BSQR) $\rightarrow y = a$, $\bar{n} = \bar{j} \Rightarrow zx$ plane.

$$ds = \frac{dzdx}{|\bar{n}\cdot\bar{j}|} = dzdx.$$

$$\int_{S_3} \vec{F}\cdot\bar{n}\,ds = \iint (-y^2)\,dz\,dx.$$

$$= \int_0^a\int_0^a -a^2\,dzdx = -a^2(a)(a) = -a^4 \hookrightarrow (S_3)$$

S_4 (OCPA) $\Rightarrow y = 0$, $\bar{n} = -\bar{j} \longrightarrow zx$ plane.

$ds = dzdx.$

$$\int_{S_4} y^2\,dzdx = 0.$$

S_5 (CPQS) $\rightarrow z = a$, $\bar{n} = \bar{k} \rightarrow xy$

$ds : dxdy$

$\int_{S_5}$

Q. Evaluate $\iint_S (yz\,dy\,dz + zx\,dz\,dx + xy\,dx\,dy)$ over the surface of sphere $x^2+y^2+z^2=1$ in the positive octant.

8/4/2019

Monday

* Volume integral: A volume integral is simply an integral evaluated over a volume.

Let V be the region in space enclosed by closed surface and $\bar{F}$ is any differentiable vector pt. fn then $\iiint \bar{F}\,dv$ is known as volume integral or space integral

$$\iiint_V \bar{F}\,dv = \int_V \bar{F}\,dv = \iiint_V \bar{F}\,dx\,dy\,dz$$

(:) Note: In component form, a volume integral can be expressed as $\iiint_V \bar{F} dv = \bar{i} \iiint_V f_1 dx dy dz$ $+ \bar{j} \iiint_V \bar{F_2} dx dy dz + \bar{k} \iiint_V F_3 dx dy dz$.

Q. If $\bar{F} = 2xz\bar{i} - x\bar{j} + y^2\bar{k}$ then evaluate $\iiint_V \bar{F} dv$ where V is the region bounded by the surfaces $V: x=0, x=2, y=0, y=6, z=x^2, z=4$.

Sol: Let $\bar{F} = 2xz\bar{i} - x\bar{j} + y^2\bar{k}$

$V: z \to x^2$ to 4

$y \to 0$ to 6 $\quad x: 0$ to 2.

$$\int_V \bar{F} dv = \int_0^2 \int_0^6 \int_{x^2}^4 [2xz\bar{i} - x\bar{j} + y^2\bar{k}] dz\, dy\, dx.$$

$$= \int_0^2 \int_0^6 ((16x - x^5)\bar{i} - (4x - x^3)\bar{j} + (4y^2 - x^2y^2)\bar{k}) dy\, dx$$

$$= \int_0^2 (96x - 6x^5)\bar{i} + (6x^3 - 24x)\bar{j} + (288 - 72x^2)\bar{k}.$$

$$= (192 - 64)\bar{i} + (24 - 48)\bar{j} + (576 - 192)\bar{k}$$

$$= 128\bar{i} - 24\bar{j} + 384\bar{k}.$$

Q. If $\bar{F} = (2x^2 - 3z)\bar{i} - 2xy\bar{j} - 4x\bar{k}$ then evaluate a) $\int_V \nabla \cdot \bar{F} dv$ b) $\int_V \nabla \times \bar{F} dv$. where $V: x = y = z = 0$

$2x + 2y + z = 4$

Sol: $\nabla \cdot \bar{F} = 4x - 2x = 2x$

$$\nabla \times \bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x^2 - 3z & -2xy & -4x \end{vmatrix}$$

$$= \bar{i}(0-0) - \bar{j}(-4+3) + \bar{k}(-2y-0)$$

$$= \bar{j} - 2y\bar{k}$$

<u>Limits</u>: $z : 0$ to $4-2x-2y$

$y : 0$ to ~~$4-2x$~~ $\cdot$ 0 to $2-x$

$x : 0$ to 2

$$\int_V \nabla\cdot\bar{F} = \int_0^2 \int_0^{2-x} \int_0^{4-2x-2y} 2x \, dz\, dy\, dx$$

$$= \int_0^2 \int_0^{2-x} 2x(4-2x-2y)\, dy\, dx$$

$$= \int_0^2 8x(2-x) - 4x^2(2-x) - 2x(2-x)^2 \, dx$$

$$= \int_0^2 (16x - 8x^2 - 8x^2 + 4x^3 - 4x + 2x^2)\, dx.$$

$$= \int_0^2 (4x^3 - 14x^2 + 12x)\, dx = 16 - \frac{14\times 8}{3} + 24$$

$$= 40 - \frac{112}{3} = \frac{120-112}{3}$$

$$\int_0^2 \int_0^{2-x} \int_0^{4-2x-2y} \bar{j} - 2y\bar{k} \, dz\, dy\, dx$$

$$= \int_0^2 \int_0^{2-x} \left[(z\bar{j}) - (2yz)\bar{k}\right]_0^{4-2x-4y} dy\, dx$$

$$= \int_0^2 \int_0^{2-x} (4-2x-2y)\bar{j} - 2y(4-2x-2y)\, dy\, dx$$

$$= \int_0^2 (x^2-4x+4)\bar{j} - \left(2x^3 + \frac{8}{3}x^3 + 12x^2 + 44x - 16 + \frac{64}{3}\right)\bar{k}\, dx$$

$$= \frac{8}{3}\bar{j} - \frac{8}{3}\bar{k}.$$

$\int$, ... $)dv$ where v is closed bounded by $v: z=4-x^2, z=0, x=0, y=0$
$x=y=2$.

* **Green's theorem:-**

It is a transformation between line & double integral.

(•) **Statement:** If R is a closed region in the xy plane bounded by a simple closed curve C and $M(x,y), N(x,y)$ are continuous fns of x and y having continuous derivations in R, then

$$\oint_C (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy \quad \text{——— (1)}$$

where C is traversed in the '+ve' direction (anticlockwise)

(•) **Note-1:** Vector notation of green's theorem

$$\bar{F} = M\bar{i} + N\bar{j}, \; \bar{dr} = dx\,\bar{i} + dy\,\bar{j}$$

$$\Rightarrow \vec{F}\cdot\vec{dr} = M\,dx + N\,dy \longrightarrow (a)$$

Let curl $\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & 0 \end{vmatrix}$

$$= \bar{k}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{y}\right)$$

Let $\text{curl}\,\bar{F}\cdot\bar{k} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$ — ⑥

$\therefore \oint_C \bar{F}\cdot d\bar{r} = \iint_R (\text{curl}\,\bar{F}\cdot\bar{k})\,dx\,dy$ in vector form

2. Area as a line integral

Let $M = -y$ and $N = x$ in eq ①

then we have ① $\Rightarrow \oint_C (-y\,dx + x\,dy)$

$= \iint_R (1+1)\,dx\,dy = 2\iint_R dx\,dy$

$= 2A$.

$\Rightarrow A = \frac{1}{2}\oint_C (x\,dy - y\,dx)$

3. Area in polar form $A = \frac{1}{2}\oint_C r^2\,dt$

If $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$ then by greens theorem,

$\oint_C \bar{F}\cdot d\bar{r} = 0$ (or) $\oint_C M\,dx + N\,dy = 0$

Stokes theorem:

It is a transformation b/w line & surface integral.

statement:

Let S be an open surface bounded by a closed non-intersecting curve C. If $\bar{F}$ is any differentiable vector pt. fn then $\oint_C \bar{F}\cdot d\bar{r}$

$= \iint_S \text{curl}\,\bar{F}\cdot\bar{n}\,ds$ — ② where $\bar{n}$ = unit outward normal at any point on the surface and

C is tranversed in the positive direction.

* Note:

It is applicable for any open surfaces.

(c) Note:

Green's theorem is a special case of stoke's theorem. Let R be the projection of surface S onto x-y plane such that normal to the surface S lies along z-axis.

i.e., $\bar{n} \cdot \bar{k}$ then surface integral in eq^n-(2) can be reduced to

$$\iint_S \text{curl}\,\bar{F}\cdot\bar{n}\,ds = \iint_R (\text{Curl}\,\bar{F}\cdot\bar{k})\,\frac{dx\,dy}{|\bar{n}\cdot\bar{k}|}$$

$$= \iint_R (\text{curl}\,\bar{F}\cdot\bar{k})\,dx\,dy$$

Substituting (a) in (2)

$\oint_C \bar{F}\cdot d\bar{r} = \iint_R (\text{curl}\,\bar{F}\cdot\bar{k})\,dx\,dy$ which is same a green's theorem in Vector form!

i.e., green's theorem is deduced from stoke's theorem.

* Gauss divergence theorem:

→ It is a transformation between surface & volume integrals.

(i) Statement:

Let S be a closed surface enclosing a volume 'V'.

If $\bar{F}$ is a continuously differentiable vector pt. fn then $\int_V \text{div}\,\bar{F}\,dv = \int_S \bar{F}\cdot\bar{n}\,ds$ — (3)

(or)

$$\iiint_V (\nabla\cdot\bar{F})\,dx\,dy\,dz = \iint_S \bar{F}\cdot\bar{n}\,ds$$

where $\bar{n}$ = unit outward normal.

* Note:

Gauss divergence theorem in component form is given by $\iiint_V \left[\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right] dx\,dy\,dz$

$= \iint_S f_1\,dy\,dz + f_2\,dz\,dx + f_3\,dx\,dy.$

Q. Verify green's theorem in plane for $\oint_C (3x^2-8y^2)dx + (4y-6xy)dy$ where C is the region bounded by $C: y=\sqrt{x}$ and $y=x^2$.

Sol Green's theorem states that

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy \quad \text{— (A)}$$

from equation (1)

$M = 3x^2 - 8y^2$ and $N = 4y - 6xy$

$R: y = \sqrt{x}$ and $y = x^2$

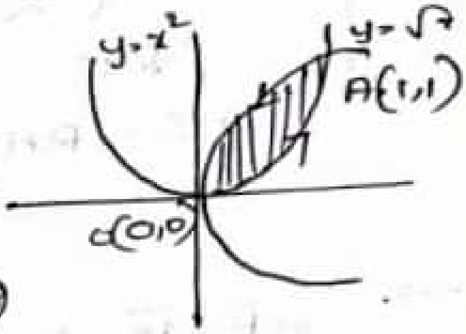
$C: OAO = OA + AO$

LHS: $\oint_C (M\,dx + N\,dy) = \int_{OA} + \int_{AO}$ — (a).

on OA :- $y = x^2$ $dy = 2x\,dx$

$x : 0$ to 1

$$\int_{OA} M\,dx + N\,dy = \int_0^1 (3x^2 - 8(x^2)^2]\,dx + [4x^2 - 6x^3](2x\,dx]$$



$$= \int_0^1 [3x^2 - 8x^4 + 8x^3 - 12x^4]\,dx$$

$$= 3x^2 + 8x^3 - 20x^4$$

$$= [x^3 + 2x^4 - 4x^5]_0^1 = -1 \longrightarrow \text{OA}$$

on AO : $y = \sqrt{x} \Rightarrow x = y^2$ $dx = 2y\,dy$, $y : 1$ to 0

$$\int_{AO} M\,dx + N\,dy = \int_1^0 (3(y)^4 - 8y^2)\,2y\,dy + (4y - 6y^3)\,dy$$

$$= 6y^5 - 16y^3 + 4y - 6y^3$$

$$= 4y - 22y^3 + 6y^5.$$

$$[2y^2 - 11y^4 + y^6]_1^0 = -2 + \frac{11}{2} - 1 = 5/2 \longrightarrow \text{AO}$$

$$\oint_C M\,dx + N\,dy = -1 + 5/2 = 3/2 \longrightarrow \text{LHS.}$$

RHS : $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

$M = 3x^2 - 8y^2$; $N = 4y - 6xy$

$\frac{\partial M}{\partial y} = -16y$; $\frac{\partial N}{\partial x} = -6y$

R : y(C.L) : 0 to 1

: x(V.L) : y^2 to $\sqrt{y}$

$$\int_0^1 \int_{y^2}^{\sqrt{y}} (-6y + 16y)\,dx\,dy = \int_0^1 \int_{y^2}^{\sqrt{y}} (10y)\,dx\,dy$$

$$= \int_0^1 10y(\sqrt{y} - y^2)\,dy = \int_0^1 (10\, y^{3/2} - y^3)\,dy$$

$$= 10\left[\frac{y^{5/2}}{5/2} - \frac{y^4}{4}\right]_0^1 = 10\left[\frac{2}{5} - \frac{1}{4}\right]$$

$$= 10\left(\frac{8-5}{20}\right) = 10\left(\frac{3}{20}\right) = \frac{3}{2}$$

$= RHS$

$LHS = RHS$

Hence verified.

→ $\oint_C \bar{F}\cdot\bar{dr} = 0 \Rightarrow \bar{F}$ is irrotational.

→ $\iint_S \bar{F}\cdot\bar{n}\,ds = 0 \Rightarrow \bar{F}$ is solenoidal.

* S.chand

Q. Show that in an irrotational field, the value of a line integral between 2 points A and B will be independent of the path of integration and be equal to their potential difference.

Sol: Let $\bar{F}$ determine an irrotational field. We know that $\bar{F} = \nabla\phi$ where ϕ is scalar fn and is called scalar potential

Now,

$$\bar{F}\cdot\bar{dr} = \nabla\phi\cdot\bar{dr} = \left[\bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}\right]\cdot(dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k})$$

$$= \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = d\phi$$

$$\int_A^B \bar{F}\cdot\bar{dr} = \int_A^B d\phi = \left[\phi(x,y,z)\right]_A^B = \phi_B - \phi_A$$

Verify stoke's theorem for $\bar{F} = (2x-y)\bar{i} - yz^2\bar{j} - y^2z\bar{k}$ over the upper half surface of the sphere $x^2+y^2+z^2 = 1$ bounded by the projection of xy plane.

Given

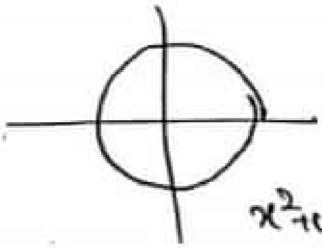
$$\bar{F} = (2x-y)\bar{i} - yz^2\bar{j} - y^2z\bar{k}$$

Acc Stoke's theorem, line $\Leftrightarrow$ Surface

$$\oint_C \vec{F}\cdot d\bar{r} = \iint_S \text{curl}\,\bar{F}\cdot\bar{n}\,ds. \quad \text{—— (1)}$$

L.H.S:

$$\oint_C \vec{F}\cdot d\vec{r} \cdot$$



$x^2+y^2=1$

On xy-plane, $z=0 \Rightarrow dz=0$

$$\vec{F}\cdot d\bar{r} = (2x-y)dx - y(0)dy - y^2z(0)$$

$$= (2x-y)dx.$$

Let $x = \cos\theta$, $y = \sin\theta$.

$$\vec{F}\cdot d\bar{r} = (2\cos\theta - \sin\theta)(-\sin\theta)\,d\theta.$$

$$= -\sin 2\theta + \left(\frac{1-\cos 2\theta}{2}\right) d\theta.$$

$$\oint_C \bar{F}\cdot d\bar{r} = \int_0^{2\pi} \left[-\sin 2\theta - \frac{1}{2} + \frac{\cos 2\theta}{2}\right]$$

$$= \left[\frac{\cos 2\theta}{2} - \frac{1}{2}\theta + \frac{\sin 2\theta}{4}\right]_0^{2\pi}$$

$$= \left[\frac{1}{2} + \pi - 0 - \frac{1}{2}\right] = \pi \quad \text{—— (2)}$$

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R.H.S

$$\text{Curl } \vec{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix}$$

$$= \bar{i}(-2yz+2yz) - \bar{j}(0-0) + \bar{k}(0+1)$$

$$= \bar{k}.$$

$$\bar{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$= \frac{2x\bar{i}+2y\bar{j}+2z\bar{k}}{\sqrt{4(x^2+y^2+z^2)}} = x\bar{i}+y\bar{j}+z\bar{k}.$$

$$\iint_S \text{Curl } \bar{F}\cdot\bar{n}\,ds = \iint_R \bar{k}\cdot(x\bar{i}+y\bar{j}+z\bar{k})\frac{dx\,dy}{|\bar{n}\cdot\bar{k}|}$$

$$= \iint_R z\,\frac{dx\,dy}{z}$$

$$= \iint_R dx\,dy.$$

where R be the projection of 's' onto xy plane

x : V.L : $-\sqrt{1-y^2}$ to $\sqrt{1-y^2}$

y : H.L : -1 to 1

$$\iint_S \text{curl } \bar{F}\cdot\bar{n}\,ds = \int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dx\,dy$$

$$= \int_{-1}^{1} 2\sqrt{1-y^2}\,dy.$$

$$= 2\left[\sqrt{1-y^2}\left(\frac{y}{2}\right) + \frac{1}{2}\sin^{-1}\left(\frac{y}{1}\right)\right]_{-1}^{1} \longrightarrow (3)$$

$= \pi$.

from (1), (2), (3)

Stoke's theorem is verified.

Q. Verify Gauss-Divergence theorem for $\bar{F} = (x^3 - yz)\bar{i} - 2x^2y\bar{j} + z\bar{k}$ taken over the surface of the cube bounded by the planes $x = y = z = 0$ & $x = y = z = a$.

Sol: Gauss divergence theorem:

$$\iiint_V \operatorname{div} \vec{F}\, dv = \iint_S \bar{F}\cdot\bar{n}\, ds$$

$$\operatorname{div}\vec{F} = \nabla \vec{F}$$

$$= \cancel{3x^2 - y + 1}$$

$$= 3x^2 - 2x^2 + 1$$

$$= x^2 + 1$$

$$\iiint_V \operatorname{div}\vec{F}\, dv = \int_0^a\int_0^a\int_0^a (x^2+1)\, dx\, dy\, dz$$

$$= \int_0^a\int_0^a \left(\frac{x^3}{3} + x\right)_0^a dy\, dz$$

$$= \left(\frac{a^3}{3} + a\right)(a)(a)$$

$$= \frac{a^5}{3} + a^3$$

Cube has 6 faces

$$\int_S \bar{F}\cdot\bar{n}\,ds = \int_{S_1} + \int_{S_2} + \cdots \int_{S_6}$$

S_1 : (ARQP) Let R_1 be the projection of s onto yz plane

$x = a$; y : 0 to a ; z : 0 to a ; $\bar{n} = \bar{i}$

$$ds = \frac{dy\,dz}{|\bar{n}\cdot\bar{i}|} = dy\,dz.$$

$$\int_{S_1} \bar{F}\cdot\bar{n}\,ds = \int_0^a\int_0^a (\bar{F}\cdot i)\,dy\,dz.$$

$$= \int_0^a\int_0^a (x^3 - yz)\,dy\,dz.$$

$$= \int_0^a \left[x^3y - \frac{y^2z}{2}\right]_0^a dz.$$

$$= \int_0^a \left[ax^3 - \frac{a^2z}{2}\right] dz.$$

$$= \int_0^a \left[a^4 - \frac{a^2z}{2}\right] dz.$$

$$= a^5 - \frac{a^4}{4}$$

S_2 : (OBSC) $\rightarrow$ yz plane.

$x = 0$, $\bar{n} = -i$

$$\int_{S_2} \bar{F}\cdot\bar{n}\,ds = \iint_R (yz - x^3)\,dy\,dz.$$

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$$\int_{S_2} \bar{F}\cdot\bar{n}\,ds = \int_0^a\int_0^a yz\,dy\,dz$$

$$= \frac{a^4}{4}.$$

$$\int_{S_3} \vec{F}\cdot\bar{n}\,ds = \int_0^a\int_0^a (-2x^2a)\,dx\,dz.$$

$\bar{n} = \bar{j}$ $\quad = -\frac{2}{3}a^5$

S_4: OCPA : $y = 0$

$$\int_{S_4} \bar{F}\cdot(-\bar{j})\,ds = 0.$$

S_5(CPQS) : $z = a$, $\bar{n} = \bar{k}$, x, y : 0 to a

$$\int_{S_5} \bar{F}\cdot\bar{k}\,ds = \int_0^a\int_0^a a\,dx\,dy = a^3.$$

S_6 (OARB) : $z = 0$

$$\int_{S_6} \vec{F}\cdot(-\bar{k})\,ds = 0$$

$$\int_S \bar{F}\cdot\bar{n}\,ds = \int_{S_1} + \int_{S_2} + \cdots \int_{S_6}$$

$$= \frac{a^5}{3} + a^3.$$

12/4/2019 Friday

Q. Evaluate by green's theorem

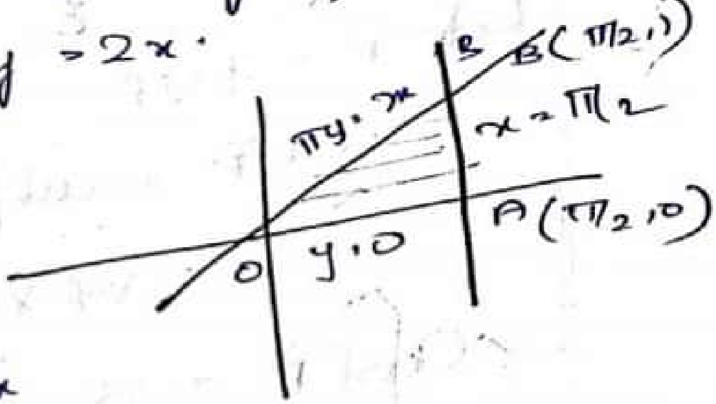
$\oint_C (y-\sin x)\,dx + \cos x\,dy$ where C is a Δ^{le} enclosed by $y = 0$, $x = \pi/2$, $\pi y = 2x$.

Sol: $$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$\int_C (y - \sin x)\,dx + \cos x\, dy$

C: $y=0$, $x=\pi/2$ & $\pi y = 2x$.

C: ΔOAB



y : C.L : 0 to 1

x : V.L : $\frac{\pi y}{2}$ to $\frac{\pi}{2}$.

$M = y - \sin x$, $N = \cos x$

$\frac{\partial M}{\partial y} = 1$; $\frac{\partial N}{\partial x} = -\sin x$.

$$= \int_0^1 \int_{\pi y/2}^{\pi/2} (-\sin x - 1)\,dx\,dy.$$

$$= -\int_0^1 \int_{\pi y/2}^{\pi/2} (1 + \sin x)\,dx\,dy.$$

$$= -\int_0^1 \left[x - \cos x\right]_{\pi y/2}^{\pi/2} dy.$$

$$= -\int_0^1 \left[\frac{\pi}{2} - (0) - \left(\frac{\pi y}{2} - \cos\frac{\pi y}{2}\right)\right] dy.$$

$$= -\int_0^1 \left[\frac{\pi}{2}\left[1 - y\right] + \cos\frac{\pi y}{2}\right] dy.$$

$$= -\left[\frac{\pi}{2}\int_0^1 (1-y)\,dy + \int_0^1 \cos\frac{\pi y}{2}\,dy\right]$$

$$= -\left[\frac{\pi}{2}\left[y - \frac{y^2}{2}\right]_0^1 + \left[\sin\frac{\pi y}{2} \times \frac{2}{\pi}\right]_0^1\right]$$

$$= -\left[\frac{\pi}{4} + \frac{2}{\pi}\right] = -\frac{\pi}{4} - \frac{2}{\pi}$$

Q. Prove that $\oint_C f\nabla f \cdot d\bar{r} = 0$

Sol: Let $\bar{F} = f\nabla f$

$\text{Curl}\,\bar{F} = \text{curl}(f\nabla f)$

$= \nabla f \times \nabla f + f(\text{curl}(\nabla f))]$

$\text{Curl}\,\bar{F} = \bar{0} + f(\bar{0})$ $[\because \text{Curl}(\nabla\phi) = \bar{0}]$

$\text{Curl}(f\nabla f) = \bar{0} \longrightarrow (1)$

By green's theorem:

$$\oint_c f\nabla f \cdot d\bar{r} = \iint_R \text{curl}\,\bar{F} \cdot \bar{E}\, dR$$

$$= \iint_R 0 \cdot E\, dR = 0.$$

Q. Prove that $\oint_C f\nabla g \cdot d\bar{r} = \int(\nabla f \times \nabla g) \cdot \bar{n}\, ds.$

Sol: Let $\bar{F} = f\nabla g$

$\text{Curl}\,\bar{F} = \text{Curl}(f\nabla g)$

$= \nabla f \times \nabla g + f\,\text{curl}(\nabla g)$

$\text{Curl}\,\bar{F} = \nabla f \times \nabla g + f(\bar{0})$ $(\because \text{curl}(\nabla\phi) = \bar{0})$

$\text{Curl}(f\nabla g) = \nabla f \times \nabla g \longrightarrow (1)$

By stoke's theorem,

$$\oint_C f\nabla f \cdot d\bar{r} = \iint_S \text{curl}\,\bar{F} \cdot \bar{n}\, ds$$

$$= \iint_S (\nabla f \times \nabla g) \cdot \bar{n}\, ds$$

Q. Evaluate $\int \bar{r}\cdot\bar{n}\,ds$ by gauss divergence theorem

A: G.D theorem states that

$$\int_V \text{div}\,\bar{F}\,dv = \int_S \bar{F}\cdot\bar{n}\,ds \quad \text{--- (1)}$$

$\bar{F} = \bar{r} \Rightarrow \text{div}\,\bar{r} = 3$

sub. in ①

$$\Rightarrow \int_S \bar{r}\cdot\bar{n}\,ds = \int_V 3\,dv = 3V.$$

Q. If $\bar{F} = y\bar{i} + (x-2xz)\bar{j} - xy\bar{k}$ then evaluate $\int_S (\nabla\times\bar{F})\cdot\bar{n}\,ds$ where 's' is surface of sphere $x^2+y^2+z^2=a^2$ above the xy plane.

A: Stoke's theorem states that

$$\int_S \text{curl}\,\bar{F}\cdot\bar{n}\,ds = \int_C \bar{F}\cdot\bar{dr} \quad \text{--- (1)}$$

On xy-plane, $x^2+y^2=a^2$

$x = a\cos\theta$; $y = a\sin\theta$ $\quad \theta: 0$ to 2π

$\bar{F}\cdot\bar{dr} = (y\bar{i} + (x-0)\bar{j} - xy\bar{k})(dx\bar{i} + dy\bar{j})$

$= y\,dx + x\,dy.$ --- ②

from ① & ②,

$$\int_S (\nabla\times\bar{F})\cdot\bar{n}\,ds = \oint_C (y\,dx + x\,dy)$$

$= \cancel{2(\text{Area})}\ \oint_C d(xy)$

$= \cancel{\frac{2\pi a^2}{xy}}.$

$= (a\cos\theta)(a\sin\theta)$

$= \left[a^2\cos\theta\sin\theta\right]_0^{2\pi}$

$= \left[\frac{a^2}{2}\sin 2\theta\right]_0^{2\pi} = 0$

Compute $\int (ax^2+by^2+cz^2)\,ds$ over the surface of the sphere $x^2+y^2+z^2=1$

Given $\phi = ax^2+by^2+cz^2$.

$$\nabla\phi = 2ax\,\bar{i} + 2yb\,\bar{j} + 2zc\,\bar{k}$$

$$|\nabla\phi| = \sqrt{(2ax)^2+(2yb)^2+(2zc)^2}$$

$$= \sqrt{4a^2x^2+4y^2b^2+2z^2c^2}$$

$$= 2\sqrt{a^2x^2+y^2b^2+z^2c^2}$$

$$\bar{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{ax\,\bar{i} + yb\,\bar{j} + zc\,\bar{k}}{\sqrt{a^2x^2+y^2b^2+z^2c^2}}$$

$$\bar{n} = \frac{\nabla\phi}{|\nabla\phi|} = x\bar{i} + y\bar{j} + z\bar{k} = \bar{r} \text{ —— (1)}$$

Let $\bar{F}\cdot\bar{n} = ax^2+by^2+cz^2$.

$$(F_1\bar{i}+F_2\bar{j}+F_3\bar{k})\cdot(x\bar{i}+y\bar{j}+z\bar{k}) = ax^2+by^2+cz^2$$

$$\Rightarrow F_1x+F_2y+F_3z = ax^2+by^2+cz^2$$

$$F_1 = ax,\ F_2 = by,\ F_3 = cz$$

$$\bar{F} = ax\bar{i}+by\bar{j}+cz\bar{k} \text{ —— (2)}$$

$$\operatorname{div}\bar{F} = \frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}$$

$$= a+b+c \text{ —— (3)}$$

By G.D theorem,

$$\int_S(\bar{F}\cdot\bar{n})\,ds = \int_V \operatorname{div}\bar{F}\,dv$$

$$= \int(a+b+c)\,dv = (a+b+c)V$$

$$= \frac{4}{3}\pi(1)^3(a+b+c).$$

Q: By transforming into $\iiint$ evaluate $\iint \{x^3 dy dz + x^2 y\, dz dx + x^2 z\, dx dy\}$ $S : x^2 + y^2 = a^2$, $z = 0$ & $z = b$.

A: Given $\iint \{x^3 dy dz + x^2 y\, dz dx + x^2 z\, dx dy\}$

$F_1 = x^3$, $F_2 = x^2 y$, $F_3 = x^2 z$

$\frac{\partial F_1}{\partial x} =$ $\operatorname{div} \bar{F} = 3x^2 + x^2 + x^2 = 5x^2$.

By gauss. D th,

$$\int_S \bar{F} \cdot \bar{n}\, ds = \int_V \operatorname{div} \bar{F}\, dv = \int_V 5x^2\, dv$$

$$= \int_0^b \int_{-a}^{a} \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} 5x^2\, dx\, dy\, dz.$$

$$= 5 \int_0^b \int_{-a}^{a} \left[\frac{x^3}{3}\right]_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} dy\, dz.$$

$$= \frac{5}{3} \int_0^b \int_{-a}^{a} 2(a^2-y^2)^{3/2}\, dy\, dz. \qquad y = a\sin$$

$$= \frac{10}{3} \int_0^b \int_{-a}^{a} (a^2-y^2)^{3/2}\, dy\, dz.$$

$$= \frac{10a^3}{3} \int_0^b \int_{-\pi/2}^{\pi/2} \cos^3\theta\, dy\, dz$$

$\frac{\sqrt{\pi}}{2} \Gamma\left(\frac{n+1}{2}\right)$

$$= \frac{20}{3} a^3 \int_0^b \int_0^{\pi/2} \cos^3\theta\, dy\, dz.$$

$=$

Using green's theorem, evaluate $\oint_C (2x^2-y^2)dx + (x^2+y^2)dy$ where C is the boundary in x-y plane, C: $x^2+y^2=a^2$ in the upper half plane (semicircle), x-axis.

$$\oint_C (Mdx+Ndy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy.$$

$M = 2x^2-y^2$, $N = x^2+y^2$

$\frac{\partial M}{\partial y} = -2y$; $\frac{\partial N}{\partial x} = 2x$.

y : CL : 0 to a

x : VL : $-\sqrt{a^2-y^2}$ to $\sqrt{a^2-y^2}$

Monday

Green's Identities:

If f and g are 2 continuous, differentiable scalar pt fns over the region enclosed by the surface S then (a) $\int_V [f\nabla^2 g + \nabla f \cdot \nabla g]\,dv$
$= \int_S (f\nabla g)\cdot\bar{n}\,ds$

(b) $\int_V [f\nabla^2 g - g\nabla^2 f]\,dv = \int_S (f\nabla g - g\nabla f)\cdot\bar{n}\,ds$

Proofs:

(a) Gauss. D th states that

$$\int_V \operatorname{div}\bar{F}\,dv = \int_S \bar{F}\cdot\bar{n}\,ds \quad \text{—— (1)}$$

w.k.t

$$\operatorname{div}(\phi\bar{f}) = \nabla\phi\cdot\bar{f} + \phi\operatorname{div}\bar{F} \quad \text{—— (2)}$$

$\cdot \bar{F} = f\nabla g$

$\operatorname{div}\bar{F} = \operatorname{div}(f\nabla g) = \nabla f\cdot\nabla g + f\nabla^2 g$

sub in R.H.S in (1)

$$\int_S (f\nabla g)\cdot\bar{n}\,ds = \int_V (\nabla f\cdot\nabla g + f\nabla^2 g)\,dv$$

(b) $\bar{F} = f\nabla g - g\nabla f$

$\operatorname{div}\bar{F} = \operatorname{div}(f\nabla g - g\nabla f)$

$= \operatorname{div}(f\nabla g) - \operatorname{div}(g\nabla f)$

$= \{\nabla f\cdot\nabla g + f\nabla^2 g\} - \{\nabla g\cdot\nabla f + g\nabla^2 f\}$

$= f\nabla^2 g - g\nabla^2 f$ sub in (1)

Hence proved.

Find the area of the circle with radius 'a' by using green's theorem.

By green's theorem,

$$A = \frac{1}{2}\oint_C (x\,dy - y\,dx) \quad \text{——} (1)$$

Radius of the circle is 'a'.

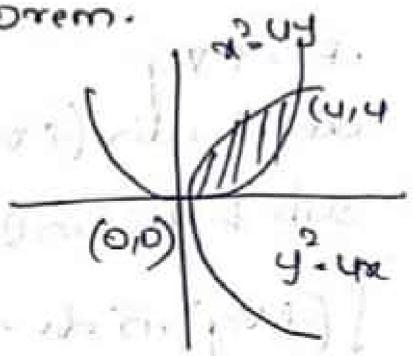
$x = a\cos\theta$, $y = a\sin\theta$, $\theta : 0$ to 2π

$dx = -a\sin\theta\, d\theta$, $dy = a\cos\theta\, d\theta$

Area of the circle $= \frac{1}{2}\int_0^{2\pi}[a^2\cos^2\theta + a^2\sin^2\theta]\,d\theta$

$= \pi a^2$.

Find the area of the region b/w the parabolas $y^2 = 4x$ & $x^2 = 4y$ using green's theorem.



Revision

Tuesday

1. $\int_0^3 \int_0^1 (x^2 + 3y^2)\,dy\,dx$

2. $\int_{-1}^{1} \int_0^2 (1 - 6x^2y)\,dx\,dy$

$= \int_{-1}^{1} \left[x - \frac{\cancel{6}^{2}x^3}{\cancel{3}}y\right]_0^2 dy = \int_{-1}^{1} (2 - 16y)\,dy$

$= \left[2y - 8y^2\right]_{-1}^{1} = 2 - 8 - [-2 - 8]$

$= -6 + 2 + 8 = 4$

3. $\int_0^1 \int_0^x e^{y/x}\,dy\,dx = \int_0^1 \left[e^{y/x} x\right]_0^x dx$

$= \int_0^1 (xe - x)\,dx = \frac{e-1}{2}$

4. $\int_0^1 \int_0^y \underset{u}{xy}\, \underset{v}{e^{x-2}}\,dx\,dy.$

DATE

$\int_0^1 y\left[x e^{x-2} - \int e^{x-2}dx\right]_0^y dy.$

$= \int_0^1 y\left[e^{x-2}(x-1)\right]_0^y dy$

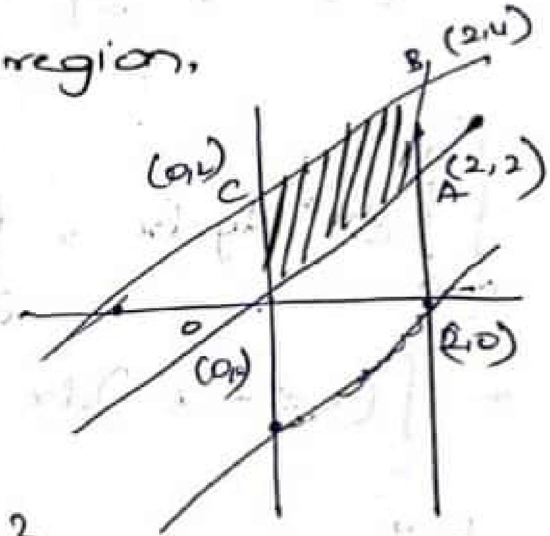
$= \int_0^1 y\left[e^{y-2}(y-1) + e^{-2}\right] dy.$

$= \int_0^1 \left[(y^2 - y)e^{y-2} + e^{-2}y\right] dy$

$$\int_0^{\pi/2}\int_0^{2a\cos\theta} r^2 \sin\theta \, d\theta \, dr$$

$$\int_0^{\pi/4}\int_0^{1} r \, dr \, d\theta.$$

Evaluate $\iint_R (x+y)\,dy\,dx$ where R is the region bounded by $x=0, x=2, y=x, y=x+2$. Also find the area of the region.



R : OABC

x : (C.L) : 0 to 2

y : (V.L) : x to $x+2$

$$\int_0^2\int_x^{x+2} (x+y)\,dy\,dx$$

$$\int_0^2 \left[xy + \frac{y^2}{2}\right]_x^{x+2} dx = \int_0^2 \left[x(x+2) + \frac{(x+2)^2}{2}\right] dx$$

$$= \int_0^2 \left[x^2+2x+\frac{1}{2}(x^2+4+4x)\right] dx = 12.$$

Changing the order:

$$\iint_R (x+y)\,dy\,dx = \iint_{R_1} + \iint_{R_2} \quad \text{——} \; (2)$$

R_1 : (OACO) y : C.L : 0 to 2
x : V.L : 0 to y

R_2 : (ABCA) : y : 2 to 4
x : $y-2$ to 2

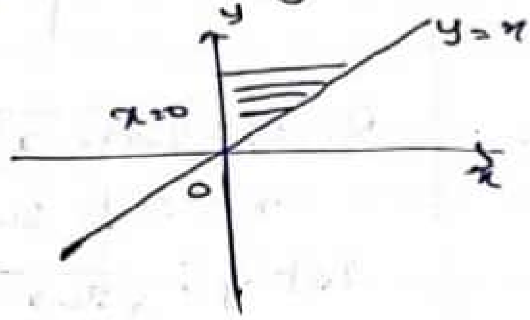
$$\iint_R dx\,dy = \int_0^2 \int_x^{x+2} dy\,dx = 4$$

8. $$\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dy\,dx$$

Q: We can't ~~evaluate~~ evaluate this in the given order. So we need to change the order of integration.

Let y : x to ∞

x : 0 to ∞



y : C.L : 0 to ∞

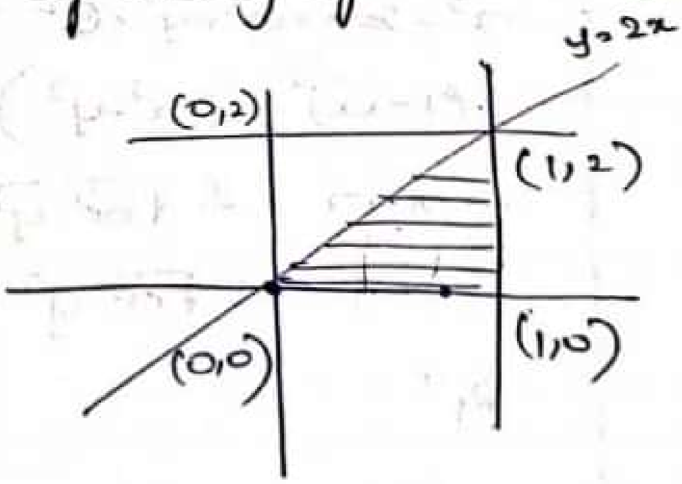
x : V.L : 0 to y

$$I = \int_0^\infty \int_0^y \frac{e^{-y}}{y}\,dx\,dy = \int_0^\infty \cancel{y}\frac{e^{-y}}{\cancel{y}}\,dy$$

$$= \left[-e^{-y}\right]_0^\infty = -e^{-\infty} + e^{-0}$$

$$= -\frac{1}{e^\infty} + 1 = 1 - \frac{1}{\infty} = 1$$

9. Evaluate $\int_0^2 \int_{y/2}^1 e^{x^2}\,dx\,dy$ by changing the order of integration.



x : $y/2$ to 1

y : 0 to 2

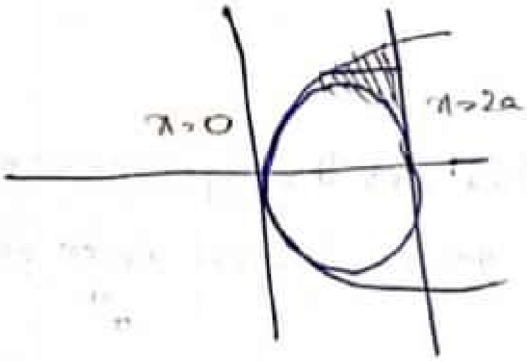
x : (C.L) : 0 to 1

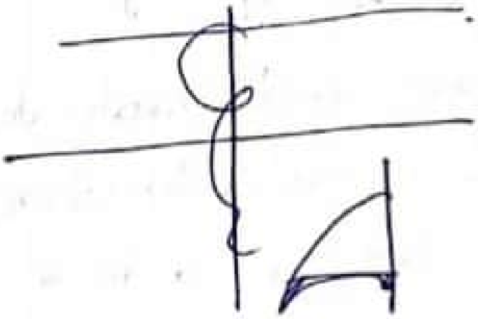
y : (V.L) : 0 to $2x$

$$\int_0^1 \int_0^{2x} e^{x^2}\,dx\,dy = \int_0^1 2xe^{x^2}\,dx = 2\int_0^1 xe^{x^2}\,dx$$

$= e-1$

Change the order of integration $\int_0^{2a}\int_{\sqrt{2ax-x^2}}^{\sqrt{2ax}} f(x,y)\,dx$





R: $y : \sqrt{2ax-x^2}$ to $\sqrt{2ax}$

$x : 0$ to $2a$

Let $y = \sqrt{2ax-x^2}$

$\Rightarrow x^2+y^2-2ax = 0$

which circle centered at $(a,0)$

$y = \sqrt{2ax} \Rightarrow y^2 = 2ax \Rightarrow$ parabola

$I = \iint_{R_1} + \iint_{R_2} + \iint_{R_3}$

$x^2+y^2-2ax = 0$

$x^2-2ax+a^2+y^2 = a^2$

$(x-a)^2 = (a^2-y^2)$

$x-a = \pm\sqrt{a^2-y^2}$

$x = a \pm \sqrt{a^2-y^2}$

R_1 :

$\int \cos^n x\, dx =$

$$\int_0^{\pi/2} \sin^n x\, dx = \int_0^{\pi/2} \cos^n x\, dx = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)} \text{ for } n>-1$$

$$\beta(m,n) = \frac{\Gamma m \Gamma n}{\Gamma(m+n)} ; \quad \beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx .$$

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt. ;$$

Q. find the area of the folium of descartes $x^3+y^3 = 3axy$ $(a>0)$ using green's theorem.

Sol: from green's theorem, we have

$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$$

By green's theorem, $A = \frac{1}{2}\oint_C x dy - y dx$

considering the loop of folium descartes $(a>0)$

Let $x = \frac{3at}{1+t^3}$ $\quad y = \frac{3at^2}{1+t^3}$

$$dx = \left[\frac{d}{dt}\left(\frac{3at}{1+t^3}\right)\right] dt \text{ and } dy = \left[\frac{d}{dt}\left(\frac{3at^2}{1+t^3}\right)\right] dt$$

the pt of intersection of loop is $\left(\frac{3a}{2}, \frac{3a}{2}\right)$

$\Rightarrow t = 1$

Along OA, $t \to 0$ to 1

$$\frac{1}{2}\oint x dy - y dx = \frac{1}{2}\int_0^1 \left[\left(\frac{3at}{1+t^3}\right)\frac{d}{dt}\left(\frac{3at^2}{1+t^3}\right)\right] dt$$

$$-\left(\frac{3at^2}{1+t^3}\right)\left[\frac{d}{dt}\left(\frac{3at}{1+t^3}\right)\right] dt$$

$$= \frac{1}{2}\int_0^1 \frac{3at}{1+t^3}\left[\frac{3at(2-t^3)}{(1+t^3)^2}\right] - \frac{3at^2}{1+t^3}\left(\frac{3a(1-2t^3)}{(1+t^3)^2}\right)\Bigg] dt$$

$$= \frac{9a^2}{2}\int_0^1 \left[\frac{t^2(2-t^3)}{(1+t^3)^3} - \frac{t^2(1-2t^3)}{(1+t^3)^3}\right] dt$$

$$= \frac{9a^2}{2}\int_0^1 \frac{2t^2-t^5-t^2+2t^5}{(1+t^3)^3}\, dt.$$

$$= \frac{9a^2}{2}\int_0^1 \frac{t^2+t^5}{(1+t^3)^3}\, dt = \frac{9a^2}{2}\int_0^1 \frac{t^2(1+t^3)}{(1+t^3)^3}\, dt.$$

$$\cancel{= \frac{9a^2}{2}\int_0^1 \frac{t^2+t^5}{1+t}}$$

$$= \frac{9a^2}{2}\int_0^1 \frac{t^2}{(1+t^3)^2}$$

Put $\Rightarrow 1+t^3 = x$

$\Rightarrow 3t^2 dt = dx.$

$$= \frac{9a^2}{2}\int_1^2 \frac{t^2}{x^2}\cdot\frac{dx}{3t^2}.$$

$$= \frac{9a^2}{6}\int_1^2 dx \frac{1}{x^2} = \frac{3a^2}{4} \text{ sq. units}$$

Using double integration determine the area of the region bounded by the curves $y^2 = 4ax$, $x+y = 3a$, $y = 0$

Given curves are

$y^2 = 4ax$ —— (1)

$x+y = 3a$ —— (2)

$y = 0$ —— (3)

To find the pts of intersection of the 2 curves $y^2 = 4ax$, $x+y = 3a$ solve (1) & (2)

Sub the value of y from eq(2) in eq(1)

$(3a-x)^2 = 4ax$

$x^2 - 10ax + 9a^2 = 0$

$(x-a)(x-9a) = 0$

$x = a, x = 9a$

Sub. $x = a$ in (2) we get $y = 2a$

The curves (1) & (2) intersect at pt $B(a, 2a)$

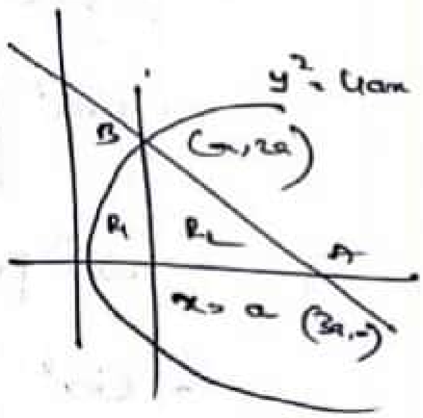
(2) & (3) at $A(3a, 0)$

$A = \iint_R dxdy$

$= \iint_{R_1} dxdy + \iint_{R_2} dxdy$

$= \int_0^a \int_0^{\sqrt{4ax}} dxdy + \int_a^{3a} \int_0^{3a-x} dydx$

$= \frac{10a^2}{3}$ sq.units.



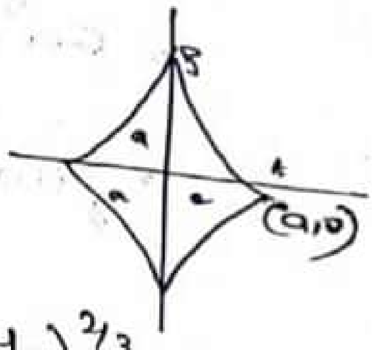
Q. Find by D.I, the area enclosed by the curve $x^{2/3} + y^{2/3} = a^{2/3}$.

Sol: Curve symmetrical abt both axes.

curve cuts x-axis in $(\pm a, 0)$

y-axis in $(0, \pm a)$

This is astroid or star.



Astroid also given by

$$\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$$

y (V.L) :- 0 to $(a^{2/3}-x^{2/3})^{3/2}$

x (C.L) :- 0 to a.

Required Area $= 4$ (area OAB)

$$= 4\int_0^a \int_0^{(a^{2/3}-x^{2/3})^{3/2}} dy\,dx$$

$$= 4\int_0^a (a^{2/3}-x^{2/3})^{3/2}\,dx$$

put $x = at^{3/2} \rightarrow dx = a\frac{3}{2}t^{1/2}\,dt$

When $x=0, t=0$ and when $x=a, t=1$

$$\text{Area} = 4\int_0^1 (a^{2/3}-a^{2/3}t)^{3/2}\cdot\frac{3a}{2}t^{1/2}\,dt.$$

$$= 6a^2\int_0^1 t^{1/2}(1-t)^{3/2}\,dt.$$

$$= 6a^2\int_0^1 t^{(3/2-1)}(1-t)^{(5/2-1)}\,dt.$$

$$= 6a^2\,\beta\left(\frac{3}{2},\frac{5}{2}\right) \qquad \left[\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx\right]$$

$$= 6a^2\,\frac{\Gamma\frac{3}{2}\,\Gamma\frac{5}{2}}{\Gamma\left(\frac{3}{2}+\frac{5}{2}\right)} \qquad \left[\beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma m+n}\right]$$

$$= 6a^2\,\frac{\frac{1}{2}\Gamma\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\,\Gamma\frac{1}{2}}{\Gamma 4} \qquad \left[\because \Gamma n = (n-1)\Gamma n-1\right]$$

$$= 6a^2\cdot\frac{3}{8\times 6}(\sqrt{\pi})^2 \qquad \left[\Gamma n+1 = n!,\ \Gamma\frac{1}{2} = \sqrt{\pi}\right]$$

$$= \frac{3}{8}\pi a^2$$

$$(1-x)^{-1} = 1+x+x^2+x^3+ \cdots$$

$$(1-x)^{-2} = 1+2x+3x^2+4x^3+ \cdots$$

$$(1-x)^{-3} = 1+3x+6x^2+10x^3+ \cdots$$

$$(1+x)^{-1} = 1-x+x^2-x^3+ \cdots$$

$$(1+x)^{n} = 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+ \cdots$$

$$(1-x)^{-n} = 1+nx+\frac{n(n+1)}{2!}x^2+\frac{n(n+1)(n+2)}{3!}x^3+ \cdots$$

$$(1+x)^{-n} = 1-nx+\frac{n(n+1)}{2!}x^2-\frac{n(n+1)(n+2)}{3!}x^3+ \cdots$$

$$(1-x)^{n} = 1-nx+\frac{n(n-1)}{2!}x^2-\frac{n(n-1)(n-2)}{3!}x^3+ \cdots$$

UNIT-V
VECTOR INTEGRATION

1. Line integral:- (i) $\int_c \bar{F}.d\bar{r}$ is called Line integral of $\bar{F}$ along c

Note : Work done by $\bar{F}$ along a curve c is $\int_c \bar{F}.d\bar{r}$

Example : If $\bar{F}$ $(x^2-27)\,\bar{i}$ $-6yz\,\bar{j}+8xz^2\,\bar{k}$, evaluate $\int \bar{F}\,.\,d\bar{r}$ from the point (0,0,0) to the point (1,1,1) along the Straight line from (0,0,0) to (1,0,0), (1,0,0) to (1,1,0) and (1,1,0) to (1,1,1).

Solution : Given $\bar{F} = (x^2-27)\,\bar{i}$ $-6yz\,\bar{j}+8xz^2\,\bar{k}$

Now $\bar{r} = x\bar{i}+y\bar{j}+z\bar{k} \Rightarrow d\bar{r}\ dx\bar{i}+dy\bar{j}+dz\bar{k}$

$\therefore$ $\bar{F}\,.\,d\bar{r} = (x^2-27)dx - (6yz)dy + 8xz^2dz$

(i) Along the straight line from O = (0,0,0) to A = (1,0,0)

Here y =0 =z and dy=dz=0. Also x changes from 0 to 1.

$$\int_{OA} \bar{F}\,.\,d\bar{r} = \int_o^1 (x^2-27)dx = \left[\frac{x^3}{3}-27x\right]_0^1 = \frac{1}{3}-27 = \frac{-80}{3}$$

(ii) Along the straight line from A = (1,0,0) to B = (1,1,0)

Here x =1, z=0 $\Rightarrow$ dx=0, dz=0. y changes from 0 to 1.

$$\int_{AB} \bar{F}\,.\,d\bar{r} = \int_{y=0}^{1}(-6yz)dy = 0$$

(iii) Along the straight line from B = (1,1,0) to C = (1,1,1)

x =1 =y $\Rightarrow$ dx=dy=0 and z changes from 0 to 1.

$$\int_{BC} \bar{F}\,.\,d\bar{r} = \int_{z=0}^{1} 8xz^2dz = \int_{z=0}^{1} 8xz^2dz = \left[\frac{8z^3}{3}-\right]_0^1 = \frac{8}{3}$$

$$(i)+(ii)+(iii) \Rightarrow \int_C \bar{F}\,.\,d\bar{r} = \frac{88}{3}$$

Example : If $\bar{F}$ (5xy-6x^2) $\bar{i}$ (2y-4x) $\bar{j}$, evaluate $\int_C \bar{F} . d\bar{r}$ along the curve C in xy plane y=x^{3}from (1,1) to (2,8).

Solution : Given $\bar{F}$ (5xy-6x^2) $\bar{i}$ (2y-4x) $\bar{j}$,

Along the curve y=x^2, dy =3x^2 dx

$\therefore$ $\bar{F}$ (5x^4-6x^2) $\bar{i}$ +(2x^3-4x) $\bar{j}$, [Putting y=x^3 in (1)]

$d\bar{r} = dx\bar{i} + dy\bar{j} + dx\bar{i}$ +3x^{2}dx

$\therefore$ $\bar{F} . d\bar{r}$ = [(5x^4-6x^2) $\bar{i}$ +(2x^3-4x) $\bar{j}$]. dx$\bar{i}$ +3x^{2}dx $\bar{j}$

= (5x^4 – 6x^2) dx+(2x^3 – 4x)3x^{3}dx

= (6x^5+5x^4-12x^3 -6x^2)dx

Hence $\int_{y=x^3} \bar{F} . d\bar{r} = \int_1^2 (6x^5 + 5x^4 - 12x^3 - 6x^2)dx$

$$= \left(6.\frac{x^6}{6} + 5.\frac{x^3}{5} - 12.\frac{x^4}{4} - 6.\frac{x^3}{4}\right) = \left(x^6 + x^3 - 3x^4 - 2x3\right)_1^2$$

= 16(4+2-31) – (1+1-3-2) = 32+3 = 35

Example : Find the work done by the force $\bar{F}$ = $z\bar{i} + x\bar{j} + y\bar{k}$, when it moves a particle along the arc of the curve $\bar{r}$ = cost $\bar{i}$ + sint $\bar{j}$ -t $\bar{k}$ from t = 0 to t = 2π

Solution : Given force $\bar{F}$ = zi = xj +yk and the arc is $\bar{r}$ = cost $\bar{i}$ + sin t $\bar{j}$ -t

i.e., x = cost, y= sin t, z = -t

$d\bar{r}$ = (-sin t $\bar{i}$ +cost $\bar{j}$ - $\bar{k}$)dt

$\bar{F} . d\bar{r}$ = (-t $\bar{i}$ +cost $\bar{j}$ +sin t $\bar{k}$). (-sin t $\bar{i}$ + cost $\bar{j}$ - $\bar{k}$)dt = (t sin t + cos^2 t – sin t)dt

Hence work done = $\int_0^{2\pi} \bar{F} . d\bar{r}$ = $\int_0^{2\pi}$ (t sin t + cos^2 t – sin t) dt

$$= [t(-\cos t)]_0^{2\pi} - \int_0^{2\pi} (-\sin t)dt +) + \int_0^{2\pi} \frac{1+\cos 2t}{2} dt - \int_0^{2\pi} \sin t \, dt$$

$$= -2\pi - (\cos t)_0^{2\pi} + \frac{1}{2}\left(t + \frac{\sin 2t}{2}\right)_0^{2\pi} + (\cos t)_0^{2\pi}$$

$$= -2\pi - (1-1) + \frac{1}{2}(2\pi) + (1-1) = -2\pi + \pi = -\pi$$

Surface integral: $\int_c \bar{F}.\bar{n}\,ds$ is called surface integral

Problem 1 : Evaluate $\int \bar{F}.\text{ndS}$ where $\bar{F}$ = zi + xj − 3y^{2}zk and S is the surface $x^2 + y^2 = 16$ included in the first octant between z = 0 and z = 5.

Sol. The surface S is $x^2 + y^2 = 16$ included in the first octant between z = 0 and z = 5.

Let $\phi = x^2 + y^2 = 16$

Then $\nabla\phi = i\dfrac{\partial\phi}{\partial x} + j\dfrac{\partial\phi}{\partial x} + k\dfrac{\partial\phi}{\partial x} = 2xi + 2yj$

$\therefore$ unit normal $\bar{n} = \dfrac{\nabla\phi}{|\nabla\varphi|} = \dfrac{xi + yj}{4}$ $(\because\ x^2 + y^2 = 16)$

Let R be the projection of S on yz plane

Then $\int_S \bar{F}.\text{ndS} = \iint_R \bar{F}.\bar{n}\dfrac{dydz}{|\bar{n}\ .\ \bar{i}|}$ ……………. *

Given $\bar{F}$ = zi + xj − 3y^{2}zk

$\therefore$ $\bar{F}\ .\ \bar{n} = \dfrac{1}{4}(xz + xy)$

and $\bar{n}\ .\ \bar{i} = \dfrac{x}{4}$

In yz plane, x = 0, y = 4

In first octant, y varies from 0 to 4 and z varies from 0 to 5.

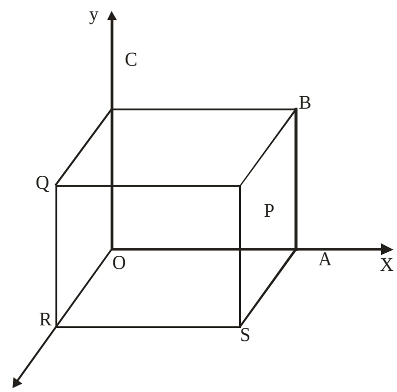
$$\int_S \bar{F}.\text{ndS} = \int_{y=0}^{4}\int_{z=0}^{5}\left(\frac{xz + xy}{4}\right)\frac{dydz}{\left|\frac{x}{4}\right|}$$

$$= \int_{y=0}^{4}\left(\int_{z=0}^{5}(y + dz)dz\right)dy$$

$= 90.$

Problem 2 : If $\bar{F}$ = zi + xj − 3y^{2}zk, evaluate $\int_S \bar{F}.\text{ndS}$ where S is the surface of the cube bounded by x = 0, x = a, y = 0, y= a, z = 0, z = a.

Sol. Given that S is the surface of the $x = 0$, $x = a$, $y = 0$, $y = a$, $z = 0$, $z = a$, and $\overline{F} = zi + xj - 3y^2zk$ we need to evaluate $\int_S \overline{F}.ndS$.



(i) For OABC

Eqn is $z = 0$ and $dS = dxdy$

$$\overline{n} = -\overline{k}$$

$$\int_{S_1} \overline{F}.ndS = -\int_{x=0}^{a} -\int_{y=0}^{a} (yz)\, dxdy = 0$$

(ii) For PQRS

Eqn is $z = a$ and $dS = dxdy$

$$\overline{n} = \overline{k}$$

$$\int_{S_2} \overline{F}.ndS = \int_{x=0}^{a}\left(\int_{y=0}^{a} y(a)dy\right) dx = \frac{a^4}{2}$$

(iii) For OCQR

Eqn is $x = 0$, and $\overline{n} = -\overline{i}$, $dS = dydz$

$$\int_{S_3} \overline{F}.ndS = \int_{y=0}^{a} \int_{z=0}^{a} 4xzdydz = 0$$

(iv) For ABPS

Eqn is $x = a$, and $\overline{n} = -\overline{i}$, $dS = dydz$

$$\int_{S_3} \overline{F}.\overline{n}dS = \int_{y=0}^{a}\left(\int_{z=0}^{a} 4azdz\right)dy = 2a^4$$

(v) For OASR

Eqn is $y = 0$, and $\overline{n} = -\overline{j}$, $dS = dxdz$

$$\int_{S_5} \overline{F}.ndS = \int_{y=0}^{a} \int_{z=0}^{a} y^2dzdx = 0$$

(vi) For PBCQ

Eqn is $y = a$, and $\overline{n} = -\overline{j}$, $dS = dxdz$

$$\int_{S_6} \overline{F}.\overline{n}dS = -\int_{y=0}^{a} \int_{z=0}^{a} y^2dzdx = 0$$

From (i) – (vi) we get

$$\int_{S_6} \overline{F}.\overline{n}dS = 0 + \frac{a^4}{2} + 0 + 2a^4 + 0 - a4 = \frac{3a^4}{2}$$

3._VOLUME INTEGRALS

Let V be the volume bounded by a surface $\bar{r} = \bar{f}$ (u,v). Let $\bar{F}(\bar{r})$ be a vector point function define over V. Divide V into m sub-regions of volumes $\delta V_1, \delta V_2, ...\delta V_p....\delta V_m$

Let P$_i$ ($\bar{r}_i$) be a point in δV_r then form the sum I$_m$ = $\sum_{i=1}^{m} \bar{F}(r1)\delta V_1$. Let m $\to \infty$ in such a way that δV_1

shrinks to a point,. The limit of I$_m$ if it exists, is called the volume integral of $\bar{F}(\bar{r})$ in the region V is denoted by $\int_V \bar{F}(\bar{r})\,dv$ or $\int_V \bar{F}dv$.

Cartesian form : Let $\bar{F} = (r)i = F_1\bar{i} + F_2\bar{i} + F_3\bar{k}$ where F_1, F_2, F_3 are functions of x,y,z. We know that dv = dx dy dz. The volume integral given by

$\int_v \bar{F}dv = \iiint F_1 \ (F_1\bar{i} + F_2\bar{i} + F_3\bar{k})$ dx dy dz = $\bar{i} = \iiint F_1$ dxdydz + $\bar{j} = \iiint F_2$ dxdydz + $\bar{k} = \iiint F_3$

Example 2 : If $\overline{F} = (2x^2 - 3z)\overline{i} - 2xy\overline{j} - 4x\overline{k}$ then evaluate (*i*) $\int_V \nabla . \overline{F}\, dv$ and (*ii*) $\int_V \nabla \times \overline{F} dv$

V is the closed region bounded by $x = 0$, $y = 0$, $z = 0$, $2x + 2y + z = 4$.

Solution : (*i*) $\nabla \cdot \overline{F} = \overline{i} \cdot \frac{\partial \overline{F}}{\partial x} + \overline{j} \cdot \frac{\partial \overline{F}}{\partial y} + \overline{k} \cdot \frac{\partial \overline{F}}{\partial z} = 4x - 2x = 2x.$

The limits are : $z = 0$ to $z = 4 - 2x - 2y$, $y = 0$ to $\frac{4-2x}{2}$ (*i.e.*) $2 - x$ and $x = 0$ to $\frac{4}{2}$ (*i.e.*) 2

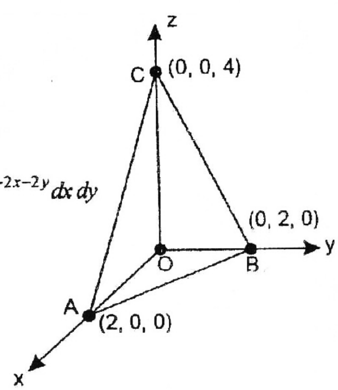
$$\therefore \int_V \nabla . \overline{F} dv = \int_{x=0}^{2} \int_{y=0}^{2-x} \int_{z=0}^{4-2x-2y} 2x\, dx\, dy\, dz = \int_{x=0}^{2} \int_{y=0}^{2-x} (2x)(z)_0^{4-2x-2y} dx\, dy$$

$$= \int_{x=0}^{2} \int_{y=0}^{2-x} 2x(4 - 2x - 2y) dx\, dy = 4 \int_{x=0}^{2} \int_{y=0}^{2-x} (2x - x^2 - xy) dx\, dy$$

$$= 4 \int_0^2 \left(2xy - x^2 y - \frac{xy^2}{2}\right)_0^{2-x} dx = 4 \int_0^2 \left[(2x - x^2)(2 - x) - \frac{x}{2}(2 - x)^2\right] dx$$

$$= \int_0^2 (2x^3 - 8x^2 + 8x) dx = \left[\frac{x^4}{2} - \frac{8x^3}{2} + 4x^2\right]_0^2 = \frac{8}{3}$$

$$(ii) \quad \nabla \times \overline{F} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x^2 - 3z & -2xy & -4x \end{vmatrix} = \overline{j} - 2y\overline{k}$$



$$\therefore \int_V \nabla \times \overline{F}\, dv = \iiint_V (\overline{j} - 2y\overline{k}) dx\, dy\, dz = \int_{x=0}^{2} \int_{y=0}^{2-x} (\overline{j} - 2y\overline{k})(z)_0^{4-2x-2y} dx\, dy$$

$$= \int_{x=0}^{2} \int_{y=0}^{2-x} (\overline{j} - 2y\overline{k})(4 - 2x - 2y) dx\, dy$$

$$= \int_{x=0}^{2} \int_{y=0}^{2-x} \left\{\overline{j}\left[(4 - 2x) - 2y\right] - \overline{k}\left[(4 - 2x) \cdot 2y - 4y^2\right]\right\} dx\, dy$$

$$= \int_{x=0}^{2} \overline{j}\left[(4 - 2x)y - y^2\right]_0^{2-x} dx - \overline{k} \int_{x=0}^{2} \left[(4 - 2x)y^2 - \frac{4y^3}{3}\right]_0^{2-x} dx$$

$$= \bar{j}\int_0^2 (2-x)^2 dx - \bar{k}\int_0^2 \frac{2}{3}(2-x)^3 dx$$

$$= \bar{j}\left[\frac{(2-x)^3}{-3}\right]_0^2 - \frac{2\bar{k}}{3}\left[\frac{(2-x)^4}{-4}\right]_0^2 = \frac{8}{3}(\bar{j}-\bar{k})$$

EXERCISE 12.3

(1) Evaluate $\iiint (2x+y)dv$ where V is the closed region bounded by the cylinder $z=4-x^2$, and planes $x=0,\ y=0,\ y=2$, and $z=0$.

(2) If $\phi = 45x^2y$ evaluate $\iiint_V \phi\, dv$ where V is the closed region bounded by the planes $4x+2y+z=8, y=0, z=0$.

(3) Evaluate $\int_V \bar{F} dv$ when $\bar{F} = x\bar{i} + y\bar{j} + z\bar{k}$ and V is the region bounded by $x=0, y=0, y=6, z=4, z=x^2$.

ANSWERS

(1) $\frac{80}{3}$ (2) 128 (3) $24\bar{i} + 96\bar{j} + \frac{384}{5}\bar{k}$

2. Vector Integral Theorems

Introduction

In this chapter we discuss three important vector integral theorems: (i) Gauss divergence theorem, (ii) Green's theorem in plane and (iii) Stokes theorem. These theorems deal with conversion of

(i) $\int_S \bar{F}.\bar{n}$ ds into a volume integral where S is a closed surface.

(ii) $\int_C \bar{F}.d\bar{r}$ into a double integral over a region in a plane when C is a closed curve in the plane and.

(iii) $\int_S (\nabla x \bar{A}).\bar{n}$ ds into a line integral around the boundary of an open two sided surface.

In solid mechanics, fluid mechanics, quantum mechanics, electrical engineering and various other fields, these theorems will be of great use. Evaluation of an integral of one type may be difficult and using one of the appropriate theorems we may be able to evaluate to the equivalent integral easily. Hence readers are advised to grasp the significance in each case.

I. GAUSS'S DIVERGENCE THEOREM

(Transformation between surface integral and volume integral)

Let S be a closed surface enclosing a volume v. if $\bar{F}$ is a continuously differentiable vector point function, then

$$\int_V div\, \bar{F} dv = \int_s \bar{F}.\bar{n}\ \mathrm{d}S$$

When $\bar{n}$ is the outward drawn normal vector at any point of S.

Example : Verify Gauss Divergence theorem for $\overline{F} = (x^3 - yz)\overline{i} - 2x^2y\overline{j} + z\overline{k}$ taken over the surface of the cube bounded by the planes x = y = z = a and coordinate planes.

Sol: By Gauss Divergence theorem we have

$$\int \overline{F}\,.\,\overline{n}dS = \int_v div\ \overline{F}\ dv$$

$$RHS = \int_0^a\int_0^a\int_0^a (3x^2 - 2x^2 + 1)dx\ dy\ dz = \int_0^a\int_0^a\int_0^a (x^2+1)\,dx\ dy\ dz = \int_0^a\int_0^a \left(\frac{x^3}{3}+x\right)_0^a dy\ dz$$

$$= \int_0^a\int_0^a \left[\frac{a^3}{3}+a\right] dy\ dz = \int_0^a \left[\frac{a^3}{3}+a\right](y)_0^a\ dz = \left(\frac{a^3}{3}+a\right)a\int_0^a dz = \left(\frac{a^3}{3}+a\right)(a^2)$$

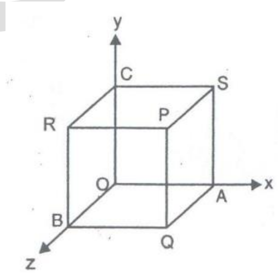
$$= \frac{a^3}{3} + a^3 \ldots (1)$$

Verification: We will calculate the value of $\int_S F.\overline{n}dS$ *over the six faces of the cube.*

(i) For S_1 = PQAS; unit outward drawn normal $\overline{n} = \overline{i}$

x=a; ds=dy dz; $0 \le y \le a$, $0 \le z \le a$

$\overline{F}.\overline{n} = x^3 - yz = a^3 - yz$ since $x = a$



$$\int_{S_1}\int \overline{F}.\overline{n}dS = \int_{z=0}^a\int_{y=0}^a (a^3 - yz)dy\,dz$$

$$= \int_{z=0}^a \left[a^3y - \frac{y^2}{2}z\right]_{y=0}^a dz$$

$$= \int_{z=0}^a \left(a^4 - \frac{a^2}{2}z\right)dz$$

$$= a^5 - \frac{a^4}{4} \ldots (2)$$

(ii) For S_2 = OCRB; unit outward drawn normal $\overline{n} = -\overline{i}$

x=a; ds=dy dz; $0 \le y \le a$, $y \le z \le a$

$\bar{F}.\bar{n} = -(x^3 - yz) = yz \; since \; x = 0$

$$\int_{S_2}\int \bar{F}.\bar{n}dS = \int_{z=0}^{a}\int_{y=0}^{a} yz\,dy\,dz = \int_{z=0}^{a}\left[\frac{y^2}{2}\right]_{y=0}^{a} zdz$$

$$= \frac{a^2}{2}\int_{z=0}^{a} zdz = \frac{a^4}{4} \dots (3)$$

(iii) For S_3 = RBQP; Z = a; ds = dxdy; $\bar{n} = \bar{k}$

0≤x≤a, 0≤y≤a

$\bar{F}.\bar{n} = z = a \;\; since \; z = a$

$$\int_{S_3}\int \bar{F}.\bar{n}dS = \int_{y=0}^{a}\int_{x=0}^{a} a\,dx\,dy = a^3 \dots (4)$$

(iv) For S_4 = OASC; z = 0; $\bar{n} = -\bar{k}$, ds = dxdy;

0≤x≤a, 0≤y≤a

$\bar{F}.\bar{n} = -z = 0 \;\; since \; z = 0$

$$\int_{S_4}\int \bar{F}.\bar{n}dS = 0 \dots (5)$$

(v) For S_5 = PSCR; y = a; $\bar{n} = \bar{j}$, ds = dzdx;

0≤x≤a, 0≤z≤a

$\bar{F}.\bar{n} = -2x^2y = -2ax^2 \;\; since \; y = a$

$$\int_{S_5}\int \bar{F}.\bar{n}dS = \int_{x=0}^{a}\int_{z=0}^{a}(-2ax^2)dzdx$$

$$= \int_{x=0}^{a}(-2ax^2)_{z=0}^{a}\,dx$$

$$= -2a^2\left(\frac{x^3}{3}\right)_0^a = \frac{-2a^5}{3} \dots (6)$$

(vi) For S_6 = OBQA; y = 0; $\bar{n} = -\bar{j}$, ds = dzdx;

0≤x≤a, 0≤y≤a

$\bar{F}.\bar{n} = 2x^2y = 0 \; since \; y = 0$

$$\int_{S_6}\int \bar{F}.\bar{n}dS = 0$$

$$\int_S \int \bar{F}.\bar{n}dS = \int_{S_1}\int + \int_{S_2}\int + \int_{S_3}\int + \int_{S_4}\int + \int_{S_5}\int + \int_{S_6}\int$$

$$= a^5 - \frac{a^4}{4} - \frac{a^4}{4} + a^3 + 0 - \frac{2a^5}{3} + 0$$

$$= \frac{a^5}{3} + a^3 = \int\int_V\int \bar{\nabla}.\bar{F}\, dv \text{ using } (1)$$

Hence Gauss Divergence theorem is verified

Example : Compute $\int (ax^2 + by^2 + cz^2) dS$ over the surface of the sphere $x^2+y^2+z^2 = 1$

Sol: By divergence theorem $\int \bar{F}.\bar{n}dS = \int_V \bar{\nabla}.\bar{F}\ dv$

$Given\ \bar{F}.\bar{n} = ax^2 + by^2 + cz^2. Let\ \phi = x^2 + y^2 + z^2 - 1$

Normal vector $\bar{n}$ *to the surface* ϕ *is*

$$\bar{\nabla}\phi = \left(\bar{\imath}\frac{\delta}{\delta x} + \bar{\jmath}\frac{\delta}{\delta y} + \bar{k}\frac{\delta}{\delta y}\right)(x^2 + y^2 + z^2 - 1) = 2(x\bar{\imath} + y\bar{\jmath} + z\bar{k})$$

$$Unit\ normal\ vector = \bar{n} = \frac{2(x\bar{\imath} + y\bar{\jmath} + z\bar{k})}{2\sqrt{x^2 + y^2 + z^2}} = x\bar{\imath} + y\bar{\jmath} + z\bar{k} \text{ since } x^2 + y^2 + z^2 = 1$$

$$\bar{F}.\bar{n} = \bar{F}.(x\bar{\imath} + y\bar{\jmath} + z\bar{k}) = (ax^2 + by^2 + cz^2) = (ax\bar{\imath} + by\bar{\jmath} + cz\bar{k}).(x\bar{\imath} + y\bar{\jmath} + z\bar{k})$$

$$i.e.\ ax\bar{\imath} + by\bar{\jmath} + cz\bar{k} \qquad \bar{\nabla}.\bar{F} = a + b + c$$

Hence by Gauss Divergence theorem,

$$\int_S (ax^2 + by^2 + cz^2)dS = \int_V (a + b + c)dv = (a + b + c)V = \frac{4\pi}{3}(a + b + c)$$

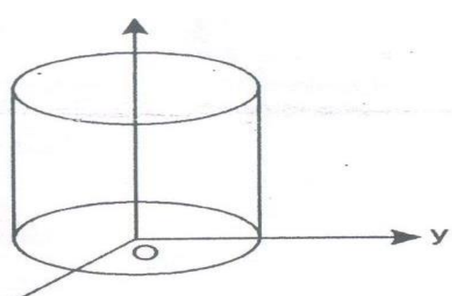
$$\left[SInce\ V = \frac{4\pi}{3}\ is\ the\ volume\ of\ the\ sphere\ of\ unit\ radius\right]$$

Example : By transforming into triple integral, evaluate $\int\int x^3\ dy\ dz + x^2 y\ dz\ dx + x^2 dx\ dy$ where S is the closed surface consisting of the cylinder $x^2+y^2 = a^2$ and the circular discs z= 0 , z= b.

Sol: Here $F_1 = x^3, F_2 = x^2 y, F_3 = x^2 z\ and\ \bar{F} = F_1\bar{\imath} + F_2\bar{\jmath} + F_3\bar{k}$

$$\frac{\delta F_1}{\delta x} = 3x^2, \frac{\delta F_2}{\delta y} = x^2, \frac{\delta F_3}{\delta z} = x^2$$

$$\bar{\nabla}.\bar{F} = \frac{\delta F_1}{\delta x} + \frac{\delta F_2}{\delta y} + \frac{\delta F_3}{\delta z} = 3x^2 + x^2 + x^2 = 5x^2$$



z

x

By Gauss Divergence theorem,

$$\int\int F_1\, dy\, dz + F_2\, dz\, dx + F_3\, dx\, dy = \int\int\int \left(\frac{\delta F_1}{\delta x} + \frac{\delta F_2}{\delta y} + \frac{\delta F_3}{\delta z}\right) dx\, dy\, dz$$

$$\int\int_S (x^3\, dy\, dz + x^2 y\, dz\, dx + x^2 dx\, dy) = \int\int\int 5x^2\, dx\, dy\, dz$$

$$= 5 \int_{-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{z=0}^{b} x^2 dx\, dy\, dz$$

$$= 20 \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_{z=0}^{\sqrt{a^2-x^2}} x^2\, dx\, dy\, dz\ [Integrand\ is\ even\ function]$$

$$= 20 \int_0^a \int_0^{\sqrt{a^2-x^2}} x^2 (z)_0^b dxdy = 20b \int_{x=0}^{a} \int_0^{\sqrt{a^2-x^2}} x^2 dxdy$$

$$= 20b \int_{x=0}^{a} x^2 (y)_0^{\sqrt{a^2-x^2}}\, dx = 20b \int_0^a x^2 \sqrt{a^2 - x^2}\ dx$$

$$= 20b \int_0^{\frac{\pi}{2}} a^2 \sin^2\theta \sqrt{a^2 - a^2 \sin^2\theta}\ (a\cos\theta d\theta)$$

$$\left[put\ x = a\sin\theta ==> dx = a\cos\theta\ d\theta\ when\ x = a ==> \theta = \frac{\pi}{2}\ and\ x = 0 ==> \theta = 0\right]$$

$$= 20a^4 b \int_0^{\frac{\pi}{2}} \sin^2\theta\ \cos^2\theta d\theta = 5a^4 b \int_0^{\frac{\pi}{2}} (2\sin\theta\cos\theta)^2\ d\theta\ = 5a^4 b\ \int_0^{\frac{\pi}{2}} \frac{1-\cos 4\theta}{2}\, d\theta$$

$$= \frac{5a^4 b}{2}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\pi/2} = \frac{5a^4 b}{2}\left[\frac{\pi}{2}\right] = \frac{5}{4}\pi a^4 b$$

Example : Show that $\int_S\ (ax\bar{\imath} + by\bar{\jmath} + cz\bar{k}).\bar{n}dS = \frac{4\pi}{3}(a + b + c)$, where S is the surface of the sphere $x^2+y^2+z^2=1$.

Sol: Take $\bar{F} = ax\bar{\imath} + by\bar{\jmath} + cz\bar{k}$

$$div\ \bar{F} = \frac{\delta F_1}{\delta x} + \frac{\delta F_2}{\delta y} + \frac{\delta F_3}{\delta z} = a + b + c$$

By Gauss divergence theorem, $\int_S \bar{F}.\bar{n}dS = \int_V \bar{V}.\bar{F}\,dV = (a+b+c)\int_V dV = (a+b+c)V$

$We\ have\ V = \frac{4}{3}\pi r^3\ for\ the\ sphere. Here\ r = 1$

$$\int_S \bar{F}.\bar{n}dS = (a+b+c)\,\frac{4\pi}{3}$$

Example : Apply divergence theorem to evaluate

$\int\int_S (x+z)dy\,dz + (y+z_{dz}dx + (x+y)dxdy\ where$ **S is the surface of the sphere $x^2+y^2+z^2=4$**

Sol: Given $\int\int_S (x+z)dy\,dz + (y+z_{dz}dx + (x+y)dxdy$

Here $F_1 = x+z$, $F_2 = y+z$, $F_3 = x+y$

$$\frac{\delta F_1}{\delta x} = 1, \frac{\delta F_2}{\delta y} = 1, \frac{\delta F_3}{\delta z} = 0\ and\ \frac{\delta F_1}{\delta x} + \frac{\delta F_2}{\delta y} + \frac{\delta F_3}{\delta z} = 1 + 1 + 0 = 2$$

By Gauss Divrgence theorem,

$$\int\int_S F_1dydz + F_2dzdx + F_3dxdy = \int\int\int_V \left(\frac{\delta F_1}{\delta x} + \frac{\delta F_2}{\delta y} + \frac{\delta F_3}{\delta z}\right)dxdydz$$

$$= \int\int\int 2dxdydz = 2\int_V dv = 2V$$

$$= 2\left[\frac{4}{3}\pi(2)^3\right] = \frac{64\pi}{3}\ [for\ the\ sphere, radius = 2]$$

Example : Evaluate $\int_S \bar{F}.\bar{n}ds, if\ F = xy\bar{\imath} + z^2\bar{\jmath} + 2yz\bar{k}$ over the tetrahedron bounded by x=0, y=0, z=0 and the plane x+y+z=1.

Sol: Given F = $xy\bar{\imath} + z^2\bar{\jmath} + 2yz\bar{k}$, then div. F = y+2y = 3y

$$\int \bar{F}.\bar{n}dS = \int_v div\ \bar{F}\ dv = \int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y} 3y\ dx\ dy\ dz$$

$$= 3\int_{x=0}^{1}\int_{y=0}^{1-x} y[z]_0^{1-x-y}dxdy = 3\int_{x=0}^{1}\int_{y=0}^{1-x} y(1-x-y)dx\ dy$$

$$= 3\int_{x=0}^{1}\left[\frac{y^2}{2}-\frac{xy^2}{2}-\frac{y^3}{3}\right]_0^{1-x} dx = 3\int_0^1\left[\frac{(1-x)^2}{2}-\frac{x(1-x)^2}{2}-\frac{(1-x)^3}{3}\right]dx$$

$$= 3\int_0^1\left[\frac{(1-x)^3}{2}-\frac{(1-x)^3}{3}\right]dx = 3\int_0^1\frac{(1-x)^3}{6}dx = \frac{3}{6}\left[\frac{-(1-x)^4}{4}\right]_0^1 = \frac{1}{8}$$

Example : Use divergence theorem to evaluate $\int_S\int \bar{F}.d\bar{S}$ where $\bar{F} = x^3i+y^3j+z^3k$ and S is the surface of the sphere $x^2+y^2+z^2 = r^2$

Sol: We have

$$\bar{V}.\bar{F} = \frac{\delta}{\delta x}(x^3)+\frac{\delta}{\delta y}(y^3)+\frac{\delta}{\delta z}(z^3) = 3(x^2+y^2+z^2)$$

By divergence theorem,

$$\bar{V}.\bar{F}\,dV = \int\int_V\int \bar{V}.\bar{F}\,dV = \int\int_V\int 3(x^2+y^2+z^2)dx\,dy\,dz$$

$$= 3\int_{r=0}^{a}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} r^2(r^2\sin\theta\,dr\,d\theta\,d\phi)$$

$[Changing\ into\ spherical\ polar\ coordinates\ x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta]$

$$\int_S\int \bar{F}.dS = 3\int_{r=0}^{a}\int_{\theta=0}^{\pi} r^4\sin\theta\left[\int_{\phi=0}^{2\pi}d\phi\right]dr\,d\theta$$

$$3\int_{r=0}^{a}\int_{\theta=0}^{\pi} r^4\sin\theta\,(2\pi-0)\,dr\,d\theta = 6\pi\int_{r=0}^{a} r^4\left[\int_0^{\pi}\sin\theta\,d\theta\right]dr$$

$$= 6\pi\int_{r=0}^{a} r^4(-\cos\theta)_0^{\pi}\,dr = -6\pi\int_0^a r^4(\cos\pi-\cos 0)\,dr$$

$$= 12\pi\int_0^a r^4 dr = 12\pi\left[\frac{r^5}{5}\right]_0^a = \frac{12\pi a^5}{5}$$

Example : Use Gauss Divergence theorem to evaluate $\int\int_S (yz^2\bar{\imath}+zx^2\bar{\jmath}+2z^2\bar{k}).ds$, where S is the closed surface bounded by the xy plane and the upper half of te sphere $x^2+y^2+z^2=a^2$

Sol: Divergence theorem states that

$$\int_S\int \bar{F}.ds = \int\int_V\int \bar{V}.\bar{F}\,dv$$

$$Here\ \bar{V}.\bar{F} = \frac{\delta}{\delta x}(yz^2)+\frac{\delta}{\delta y}(zx^2)+\frac{\delta}{\delta z}(2z^2) = 4z$$

$$\iint_S \bar{F}.ds = \iiint_V 4z\, dx\, dy\, dz$$

Introducing spheical polar coordinates $x = r\sin\theta\cos\phi\,,\ y = r\sin\theta\sin\phi\,,\ z = r\cos\theta$ *then* $dx\,dy\,dz = r^2\,dr\,d\theta\,d\phi$

$$\iint_S \bar{F}.ds = 4\int_{r=0}^{a}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}(r\cos\theta)\,(r^2\sin\theta\,dr\,d\theta\,d\phi)$$

$$= 4\int_{r=0}^{a}\int_{\theta=0}^{\pi} r^3\sin\theta\cos\theta\left[\int_{\phi=0}^{2\pi}d\phi\right]dr\,d\theta$$

$$= 4.\int_{r=0}^{a}\int_{\theta=0}^{\pi} r^3\sin\theta\cos\theta\ (2\pi-0)dr\,d\theta$$

$$= 4\pi\int_{r=0}^{a} r^3\left[\int_0^{\pi}\sin 2\theta\,d\theta\right]dr = 4\pi\int_{r=0}^{a} r^3\left(-\frac{\cos 2\theta}{2}\right)_0^{\pi}dr$$

$$= (-2\pi)\int_0^a r^3(1-1)dr = 0$$

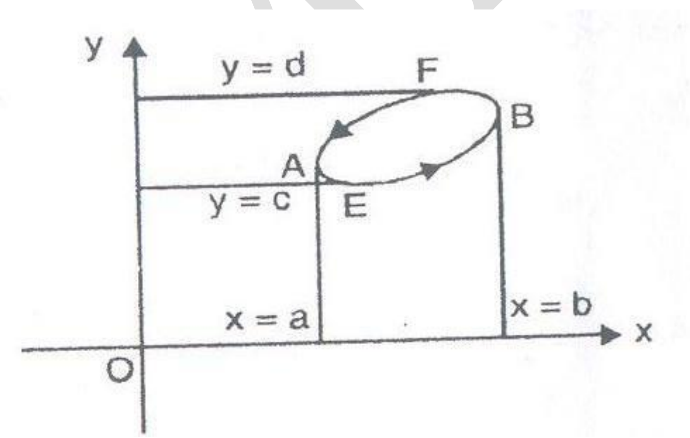
<u>II. GREEN'S THEOREM IN A PLANE</u>

(Transformation Between Line Integral and Surface Integral) [JNTU 2001S].

If S is Closed region in xy plane bounded by a simple closed curve C and if M and N are continuous functions of x and y having continuous derivatives in R, then

$$\oint_c M dx + N dy = \iint_s \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx dy,$$

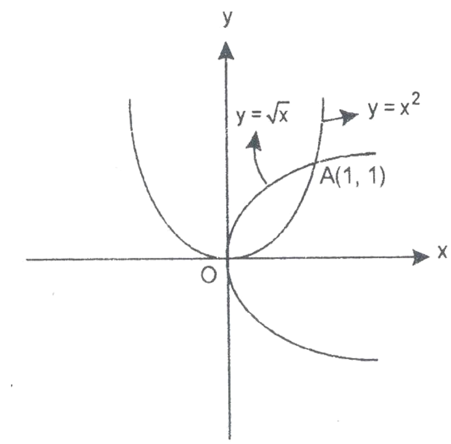
Where C is traversed in the positive(anti clock-wise) direction



1.Verify Green's theorem in plane for $\oint(3x^2 - 8y^2)dx + (4y - 6xy)dy$ where C is the region bounded by y=$\sqrt{x}$ and y=x^2 .

Solution: Let M=3x^2-8y^2 and N=4y-6xy. Then

$\frac{\partial M}{\partial y} = -16y, \frac{\partial N}{\partial x} = -6y$



We have by Green's theorem,

$\oint_c Mdx + Ndy = \iint_s \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$

Now $\int_s \int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \iint_s (16y - 6y) dxdy$

$$=10\iint_s y\,dxdy = 10 \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} y\,dydx = 10 \int_{x=0}^{1} \left[\frac{y^2}{2}\right]_{x^2}^{\sqrt{x}} dx$$

$$=5 \int_0^1 (x - x^4)dx = 5\left(\frac{x^2}{2} - \frac{x^5}{5}\right)_0^1 = 5\left(\frac{1}{2} - \frac{1}{5}\right) = \frac{3}{2} \qquad \ldots.(1)$$

Verification:

We can write the line integral along c

=[line integral along y=x^2(from O to A) + [line integral along y^2=x(from A to O)]

=I_1+I_2(say)

Now $I_1 = \int_{x=0}^{1} \{[3x^2 - 8(x^2)^2]dx + [4x^2 - 6x(x^2)]2xdx\} \left[\because y = x^2 \Rightarrow \frac{dy}{dx} = 2x\right]$

$= \int_0^1 (3x^3 + 8x^3 - 20x^4)dx = -1$

And $I_2 = \int_1^0 \left[(3x^2 - 8x)dx + \left(4\sqrt{x} - 6x^{3/2}\right)\frac{1}{2\sqrt{x}} = dx\right] = \int_1^0 (3x^2 - 11x + 2)dx = \frac{5}{2}$

$\therefore\ I_1 + I_{2=-1+5/2=3/2}.$

From(1) and (2), we have $\oint_c Mdx + Ndy = \iint_s \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$

Hence the verification of the Green's theorem.

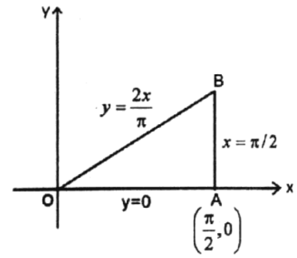
2.Evaluate by Green's theorem $\int_c (y - \sin x)dx + \cos x\,dy$ where C is the triangle enclosed by the lines y=0, x=$\frac{\pi}{2}$, $\pi y = 2x$. [JNTU 1993, 1995 S, 2003 S, 2007, (H) June 2010(Set No.2)]

Solution : Let M=y-$\sin x$ $and\ N = \cos x$ Then

$\frac{\partial M}{\partial y}$=1 and $\frac{\partial N}{\partial x}$=-$\sin x$

By Green's theorem $\oint_c Mdx + Ndy = \int_s \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$

$$\Rightarrow \int_c (y - \sin x)dx + \cos x\, dy = \iint_s (-1 - \sin x)dx($$



$$= -\int_{x=0}^{\pi/2} \int_{y=0}^{\frac{2x}{\pi}} (1 + \sin x)\, dxdy$$

$$= \int_{x=0}^{\pi/2} (\sin x + 1)\, [y]_0^{2x/\pi}\, dx$$

$$= \frac{-2}{\pi} \int_{x=0}^{\pi/2} x(\sin x + 1)dx$$

$$= \frac{-2}{\pi} [(-\cos x + x)]_0^{\frac{\pi}{2}} - \int_0^{\pi/2} 1(-\cos x + x)dx$$

$$= \frac{-2}{\pi} \left[x(-\cos x + x) + \sin x - \frac{x^2}{2}\right]_0^{\pi/2}$$

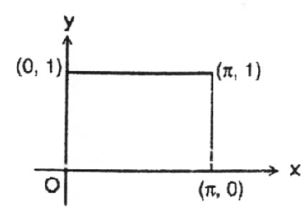
$$= \frac{-2}{\pi} \left[-x\cos x + \frac{x^2}{2} + \sin x\right]_0^{\pi/2} = \frac{-2}{\pi}\left[\frac{\pi^2}{8} + 1\right] = -\left(\frac{\pi}{4} + \frac{2}{\pi}\right)$$

Example 3: Evaluate by Green's theorem for $\oint_c (x^2 - \cosh y)dx + (y + \sin x)dy$ where C is the rectangle with vertices $(0,0), (\pi, 0), (\pi, 1), (0,1)$.

Solution: Let M=$x^2 - \cosh y$, $N = y + \sin x$ $\therefore \frac{\partial M}{\partial y} = -\sinh y\ and\ \frac{\partial N}{\partial x} = \cos x$

By Green's theorem, $\oint_c Mdx + Ndy = \int_s \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$

$$\Rightarrow \oint_c (x^2 - \cosh y)dx + (y + \sin x)dy = \int_s \int (\cos x + \sinh$$



$$= \int_{x=0}^{\pi} \int_{y=0}^{1} (\cos x + \sinh y)dydx = \int_{x=0}^{\pi} (y\cos x + \cosh y)_0^1 dx$$

$$= \int_{x=0}^{\pi} [\cos x + \cosh 1 - 1]dx$$

$$= \pi(\cosh 1 - 1)$$

Example 4: A Vector field is given by $\overline{F} = (\sin y)\boxed{i} + x(1 + \cos y)\boxed{j}$.

Evaluate the line integral over the circular path x^2+$y^2 = a^2$, z=0

(i) Directly (ii) By using Green's theorem

Solution : (i) Using the line integral [JNTU 96, (A) June 2011 (Set No.4)

$$\oint_c \overline{F}.d\overline{r} = \oint_c F_1 dx + F_2 dy = \oint_c \sin y dx + x(1 + \cos y)dy$$

$$= \oint_c \sin y dx + x\ \cos y\, dy = xdy = \oint_c d(x \sin y) + xdy$$

Given Circle is x^2+$y^2 = a^2$. Take x=a $\cos\theta$ and y=a $\sin\theta$ so that dx=-a $\sin\theta\, d\theta$ and dy=a $\cos\theta \, \mathrm{d}\theta$ and $\theta = 0 \rightarrow 2\pi$

$\therefore \;\; \oint \overline{F}.d\overline{r} = \int_0^{2\pi} d[a \;\cos\theta \sin(a \;\sin\theta)] + \int_0^{2\pi} a(\cos\theta)a \;\cos\theta \, d\theta$

$=[a \;\cos\theta \sin(a \;\sin\theta)]_0^{2x} + 4a^2 \;\int_0^{\pi/2} cos^2\,\theta \, d\theta$

$=0+4a^2.\frac{1}{2}.\frac{\pi}{2} - \pi a^2$

(ii)Using Green's theorem

Let M=$\sin y$ and N=x$(1+\cos y)$. Then

$\frac{\partial M}{\partial y}$=$\cos y$ and $\frac{\partial N}{\partial x}$=-$(1+\cos y)$

By Green's theorem,

$\oint_c Mdx + Ndy = \iint_s \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$

$\therefore \; \oint_c \; \sin y \, dx + (1+\cos y)dy = \;\int_s \int \; (-\cos y + 1\cos y)dxdy == \iint dxdy$

$= \;\; \int_s \int dA = A = \pi a^2 \; (\because area \; of \; circle = \pi a^2)$

We observe that the values obtained in (i) and (ii) are same to that Green's theorem is verified.

Example 4: Show that area bounded by a simple closed curve C is given by $\frac{1}{2}\oint xdy - ydx$ and hence find the area of

(i)The ellipse x=$acos\theta \;\; ⟦\theta, y = bsin\theta \;\; ⟦\theta \;\; (i.e) \; \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1⟧ \; ⟧$

(II)The Circle x=$a\cos\theta, y = a\sin\theta \;\; (i.e)x^2 + y^2 = a^2$

Solution: We have by Green's theorem $\oint_c Mdx + Ndy = \int_s \int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \; dxdy$

Here M=-y and N=x so that $\frac{\partial M}{\partial y} = -1 \; and \frac{\partial N}{\partial x} = 1$

$\oint_c xdy - ydx = 2 \; \int_s dxdy = 2A$ where A is the area of the surface.

$\therefore \frac{1}{2}\int xdy - ydx = A$

(i)For the ellipse x=$a\cos\theta$ and y=$b\sin\theta$ and $\theta = 0 \rightarrow 2\pi$

$\therefore \; Area, A=\frac{1}{2}\oint xdy - ydx = \frac{1}{2}\int_0^{2\pi}[(a \;\cos\theta)(b\cos\theta) - (b \;\sin\theta \;(-a \;\sin\theta))]d\,\theta$

$=\frac{1}{2}ab \int_0^{2\pi}(cos^2\theta + sin^2\theta) \, d\theta = \frac{1}{2}ab(\theta)_0^{2\pi} = \frac{ab}{2}(2\pi - 0) = \pi ab$

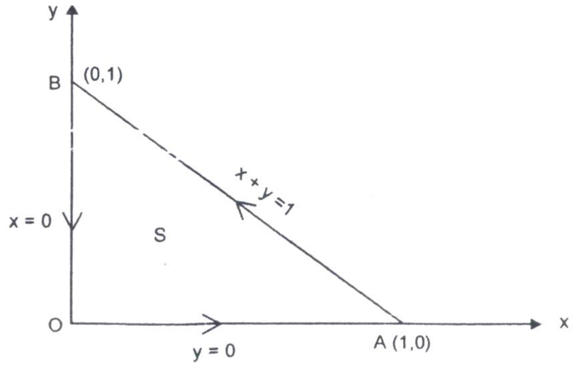
(ii)Put a=b to get area of the circle A=πa^2

Example 5: Verify Green's theorem for $\int_c[(3x^2 - 8y^2)dx + (4y - 6xy)dy]$ where c is the region bounded by x=0, y=0 and x+y=1. [JNTU 2003S, 2007S(Set No.3)

Solution : By Green's theorem, we have

$\int_C Mdx + Ndy = \iint_s \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$

Here M=$3x^2 - 8y^2$ and N=4y-6xy



$\therefore \frac{\partial M}{\partial y} = 16y \;\; and \; \frac{\partial N}{\partial x} = 6y$

Now $\int_C Mdx + Ndy = + \int_{OA} Mdx + Ndy + \int_{AB} Mdx + Ndy + \int_{BC} Mdx + Ndy$ ….(1)

Along OA, y=0 $\therefore dy = 0$

$\int_{OA} Mdx + Ndy = \int_0^1 3x^2 dx = \left(\frac{x^3}{3}\right)_0^1 = 1$

Along AB, x+y=1 $\therefore dy = -dx$ and x=1-y and varies from 0 to 1.

$\int_{AB} Mdx + Ndy = \int_0^1 [3(y-1)^2 - 8y^2](-dy) + [4y + 6y(y-a)]dy$

$= \int_0^1 (-5y^2 - 6y + 3)(-dy) + (6y^2 - 2y)dy$

$= \int_0^1 (11y^2 + 4y - 3)dy = \left(11\frac{y^3}{3} + 4\frac{y^2}{2} - 3y\right)_0^1$

$= \frac{11}{3} + 2 - 3 = \frac{8}{3}$

Along BO, x=0 $\therefore dx = 0$ and limits of y are from 1 to 0

. $\int_{BO} Mdx + Ndy = \int_1^0 4ydy = \left(4\frac{y^2}{2}\right)_1^0 = (2y^2)_0^1 = -2$

from (1), we have $\int_c Mdx + Ndy = 1 + \frac{8}{3} - 2 = \frac{5}{3}$

Now $\iint_s \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \int_{x=0}^1 \int_{y=0}^{1-x} (-6y + 16y)dxdy$

$= 10 \int_{x=0}^1 \left[\int_{y=0}^{1-x} ydy\right] dx = 10 \int_0^1 \left(\frac{y^2}{2}\right)_0^{1-x} dx$

$= 5 \int_0^1 (1-x)^2 dx = 5\left[\frac{(1-x)^3}{-3}\right]_0^1$

$= -\frac{5}{3}[(1-1)^3 - (1-0)^3] = \frac{5}{3}$

From (2) and (3), we have $\int_C Mdx + Ndy = \iint_s \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$

Hence the verification of the Green's Theorem.

Example 11: Verify Green's theorem in the plane for $\int_C (x^2 - xy^3)\, dx + (y^2 - 2xy) dy$
Where C is square with vertices (0,0), (2,0), (2,2), (0,2).

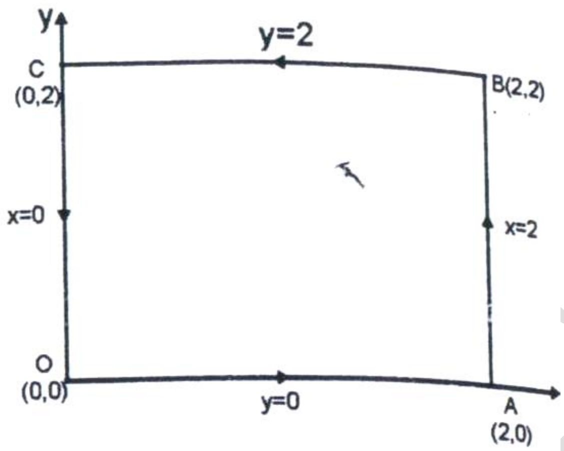
[JNTU Aug, 2008S, (H)June2009,(K) May2010(Set No.2)]

<u>Solution:</u> The Cartesian form of Green's theorem in the plane is

$$\int_C M dx + N dy = \iint_s \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx dy$$

Here M=$x^2 - xy^3$ and N=$y^2 - 2xy$

$\therefore\ \frac{\partial M}{\partial y}$ =3 xy^2 and $\frac{\partial N}{\partial x} = -2y$



Evaluation of $\int_C (\boldsymbol{M dx + N dy})$

To Evaluate $\int_C (x^2 - xy^3)\, dx + (y^2 - 2xy) dy$, we shall take C in four different segments viz (i) along OA(y=0) (ii) along AB(x=2) (iii) along BC(y=2) (iv) along CO(x=0).

(i)Along OA(y=0)

$$\int_C (x^2 - xy^3)\, dx + (y^2 - 2xy) dy = \int_0^2 x^2 dx = \left(\frac{x^3}{3}\right)_0^2 = \frac{8}{3} \quad \text{.....(1)}$$

(ii)Along AB(x=2)

$$\int_C (x^2 - xy^3)\, dx + (y^2 - 2xy) dy = \int_0^2 (y^2 - 4y)\, dy \quad [\because x = 2, dx = 0]$$

$$= \left(\frac{y^3}{3} - 2y^2\right)_0^2 = \left(\frac{8}{3} - 8\right) = 8\left(-\frac{2}{3}\right) = -\frac{16}{3} \quad \text{....(2)}$$

(iii)Along BC(y=2)

$$\int_C (x^2 - xy^3)\, dx + (y^2 - 2xy) dy = \int_2^0 (x^2 - 8x)\, dx \quad [\because y = 2, dy = 0]$$

$$= \left(\frac{x^3}{3} - 4x^2\right)_2^0 = \left(\frac{8}{3} - 16\right) = \frac{40}{3} \quad \text{...(3)}$$

(iv)Along CO(x=0)

$$\int_C (x^2 - xy^3)\, dx + (y^2 - 2xy) dy = \int_2^0 y^2\, dx \quad [\because x = 0, dx = 0] = \left(\frac{y^3}{3}\right)_2^0 = -\frac{8}{3} \quad \text{.....(4)}$$

Adding(1),(2),(3) and (4), we get

$$\int_c (x^2 - xy^3)\,dx + (y^2 - 2xy)dy = \frac{8}{3} - \frac{16}{3} + \frac{40}{4} - \frac{8}{3} = \frac{24}{3} = 8 \qquad \ldots(5)$$

Evaluation of $\iint_s \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \boldsymbol{dxdy}$

Here x ranges from 0 to 2 and y ranges from 0 to 2.

$$\iint_s \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \int_0^2 \int_0^2 (-2y + 3xy^2)dxdy$$

$$= \int_0^2 \left(-2xy + \frac{3x^2}{2} y^2\right)_0^2 dy$$

$$= \int_0^2 (-4y + 6y^2)\,dy = (-2y^2 + 2y^3)_0^2$$

$$= -8+16=8 \qquad \ldots(6)$$

From (5) and (6), we have

$$\int_C M dx + N dy = \iint_s \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

Hence the Green's theorem is verified.

II. <u>STOKE'S THEOREM</u>

(Transformation between Line Integral and Surface Integral) [JNTU 2000]

Let S be a open surface bounded by a closed, non intersecting curve G. if $\bar{F}$ is any differentieable vector point function then $\oint_C \bar{F}.d\,\bar{r} = \int_S curl\ \ \bar{F}.\bar{n}\ ds$ *where c is traversed in the positive* direction and $\bar{n}$ *is unit outward drawn normal at any point of the surface.*

1. Verify Stokes theorem for $\bar{F} = -y^3\bar{\imath} + x^3\bar{\jmath}$, Where S is the circular disc $x^2 + y^2 \leq 1, z = 0$. [JNTU 99,2007,2008S(Set No.4)]

<u>Solution:</u> Given that $\bar{F} = -y^3\bar{\imath} + x^3\bar{\jmath}$. The boundary of C of S is a circle in xy plane. $x^2 + y^2 \leq 1, z = 0$. We use the parametric co-ordinates x=cosθ, $y = sin\theta, z = 0, 0 \leq \theta \leq 2\pi$;

dx=-sinθ $d\theta$ and dy =cosθ $d\theta$

$$\therefore \oint_c \bar{F}.dr = \int_c F_1 dx + F_2 dy + F_3 dz = \int_c -y^3 dx + x^3 dy$$

$$= \int_0^{2\pi} [-sin^3\theta(-sin\theta) + cos^3\theta cos\theta]d\theta = \int_0^{2\pi} (cos^4\theta + sin^4\theta)d\theta$$

$$= \int_0^{2\pi} (1 - 2sin^2\theta\ cos^2\theta)d\theta = \int_0^{2\pi} d\theta - \frac{1}{2}\int_0^{2\pi} (2sin\theta\ cos\theta)^2\, d\theta$$

$$= \int_0^{2\pi} d\theta - \frac{1}{2}\int_0^{2\pi} sin^2 2d\theta = (2\pi - 0) - \frac{1}{4}\int_0^{2\pi} (1 - cos4\theta)d\theta$$

$$= 2\pi + \left[-\frac{1}{4}\theta + \frac{1}{16} sin4\theta\right]_0^{2\pi} = 2\pi - \frac{2\pi}{4} = \frac{6\pi}{4} = \frac{3\pi}{2}$$

Now$\nabla \times \bar{F} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^3 & x^3 & 0 \end{vmatrix} = \bar{k}(3x^2 + 3y^2)$

$\therefore \int_s (\nabla \times \bar{F}).\bar{n}ds = 3\int_s (x^2 + y^2)\bar{k}.\bar{n}ds$

We have $(\bar{k}.\bar{n})ds - dx\, dy$ *and R is the region on* $xy - plane$

$\therefore \iint_s (\nabla \times \bar{F}).\bar{n}ds = 3\iint_R (x^2 + y^2)\, dx\, dy$

Put x=r cos$\emptyset, y = r\, sin\emptyset \therefore dxdy = rdr\, d\emptyset$

R is varying from 0 to 1 and 0$\leq \emptyset \leq 2\pi$.

$\int (\nabla \times \bar{F}).\, \bar{n}ds = 3\int_{\emptyset=0}^{2\pi}\int_{r=0}^{1} r^2$.rdr d$\emptyset = \frac{3\pi}{2}$

L.H.S=R.H.S.Hence the theorem is verified.

Example2: If $\bar{F} = y\bar{\imath} + (x - 2xz)\bar{\jmath} - xy\bar{k}$, *evaluate* $\int_s (\nabla \times \bar{F}).\bar{n}ds$ *Where* Sis the surface of sphere $x^2 + y^2 + z^2 = a^2$, *above the* $xy - plane$.

Solution: Given$\bar{F} = y\bar{\imath} + (x - 2xz)\bar{\jmath} - xy\bar{k}$.

By Stokes Theorem,

$\int_s (\nabla \times \bar{F}).\bar{n}ds = \int_c F_1 dx + F_2 dy + F_3 dz = \int_c ydx + (x - 2xz)dy - xydz$

Above the xy plane the sphere is $x^2 + y^2 += a^2, z = 0$

$\therefore \int_c \bar{F}.d\bar{r} = \int_c ydx + xdy.$

Put x=a cos θ,y=asinθ *so that* $dx = -a\, sin\theta d\theta, dy = acos\theta d\theta$ *and* $\theta = 0 \to 2\pi$

$\int_c \bar{F}.d\bar{r} = \int_0^{2\pi} (a\, sin\theta)(-a\, sin\theta)\, d\theta + (acos\theta)(acos\theta)d\theta$

$= a^2 \int_0^{2\pi} cos2\theta\, d\theta = a^2 \left[\frac{sin2\theta}{2}\right]_0^{2\pi} = \frac{a^2}{2}(0) = 0$

Example 3: Verify Stokes theorem for $\bar{F} = (2x - y)\bar{\imath} - \dot{y}z^2\bar{\jmath} - y^2 z\bar{k}$ over the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$bounded by the projection of the xy-plane.

[JNTU2006,2007,2007S,2008,JNTU(A) June2009(Set No.2)]

Solution: The boundary C of S is a circle in xy plane i.e $x^2 + y^2$=1, z=0

The parametric equations are x=$cos\theta, y = sin\theta, \theta = 0 \to 2\pi$

$\therefore \quad dx = -sin\theta\, d\theta, dy = cos\theta\, d\theta$

$\int_C \bar{F}.d\bar{r} = \int_C \bar{F}.d\bar{r} = \int_c \bar{F}_1.dx + \bar{F}_2.dy + \bar{F}_3.dz = \int_c (2x - y)dx - yz^2dy - y^2zdz$

$= \int_c (2x - y)dx(since\ z = 0\ and\ dz = 0)$

$$=\int_c^{2\pi}(2\cos\theta - \sin\theta)\sin\theta\, d\theta = \int_c^{2\pi}\sin^2\theta\, d\theta = \int_c^{2\pi}\sin 2\, d\theta$$

$$=\int_{\theta=0}^{2\pi}\frac{1-\cos 2\theta}{2}\,d\theta - \int_0^{2\pi}\sin 2\theta\, d\theta = \left[\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta + \frac{1}{2}.\cos 2\theta\right]_0^{2\pi}$$

$$=\frac{1}{2}(2\pi - 0) + 0 + \frac{1}{2}.(\cos 4\pi - \cos 0) = \pi$$

Again $\nabla \times \bar{F} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2 z \end{vmatrix} = \bar{\imath}(-2yz + 2yz) - \bar{\jmath}(0-0) + \bar{k}(0+1) = \bar{k}$

$\therefore \int_S (\nabla \times \bar{F}).\bar{n}ds = \int_S \bar{k}.\bar{n}ds = \int_R \int dxdy$

Where R is the projection of S on xy plane and $\bar{k}.\bar{n}ds = dxdy$

Now $\int\int_R dxdy = 4\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} dydx = 4\int_{x=0}^{1}\sqrt{1-x^2}\,dx = 4\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x\right]_0^1$

$$=4\left[\frac{1}{2}\sin^{-1}1\right] = 2\frac{\pi}{2} = \pi$$

$\therefore$ *The* Stokes theorem is verified.

Example 4: Verify Stokes theorem for the function $\bar{F} = x^2\,\bar{\imath} + xy\,\bar{\jmath}$ integrated round the square in the plan z=0 whose sides are along the lines x=0, y=0, x=a, y=a.

Solution: Given $\bar{F} = x^2\,\bar{\imath} + xy\,\bar{\jmath}$

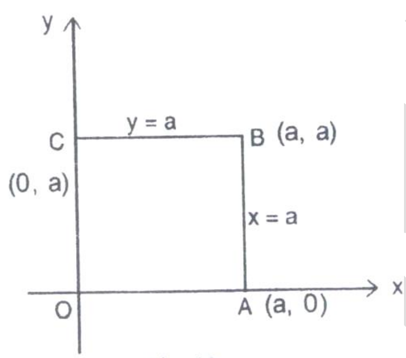


Fig. 13

By Stokes Theorem, $\int_S (\nabla \times \bar{F}).\bar{n}ds = \int_S \bar{F}.d\bar{r}$

Now $\nabla \times \bar{F} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & xy & 0 \end{vmatrix} = \bar{k}$y

L.H.S. , $\int_S (\nabla \times \bar{F}).\bar{n}ds = \int_S y(\bar{n}.\bar{k})\,ds = \int_S ydxdy$

$\bar{n}.\bar{k}.ds = dxdy$ and R is the region bounded for the square.

$\therefore \int_S (\nabla \times \bar{F}).\bar{n}ds = \int_0^a \int_0^a ydydx = \frac{a^3}{2}$

R.H.S. $=\int_c \bar{F}.d\bar{r} = \int_c (x^2dx + xydy)$

But $\int \bar{F}.d\bar{r} = \int_{OA}\bar{F}.d\bar{r} + \int_{AB}\bar{F}.d\bar{r} + \int_{BC}\bar{F}.d\bar{r} + \int_{CO}\bar{F}.d\bar{r}$

(i)Along OA: y=0, z=0, dy=0, dx=0

$\therefore \int_{OA} \bar{F}\cdot d\bar{r} = \int_0^a x^2 dx = \frac{a^3}{3}$

(ii)Along AB:x=a, z=0,dx=0,dz=0

$\int_{AB} \bar{F}\cdot d\bar{r} = \int_0^a ay dx = \frac{1}{2}a^3$

(iii)Along BC: y=a,z=0,dy=0,dz=0

$\therefore \int_{BC} \bar{F}\cdot d\bar{r} = \int_a^0 0 dx = \frac{1}{3}a^3$

(iv)Along CO: x=0, z=0, dx=0, dz=0

$\therefore \int_{CO} \bar{F}\cdot d\bar{r} = \int_a^0 0 dy = 0$

Adding $\int_c \bar{F}\cdot d\bar{r} = \frac{1}{3}a^3 + \frac{1}{2}a^3 + \frac{1}{3}a^3 + 0 = \frac{1}{2}a^3$

Hence the verification.

$\oint_c (x^2 + y - 4)\, dx + 3xydy + (2xz + z^2)dz$

ASSIGNMENT QUESTIONS

UNIT-I ASSIGNMENT QUESTIONS

SET – I

1. Solve $(x^2 - 2xy + 3y^2)dx + (y^2+6xy- x^2)dy = 0$.
2. Solve $(3y+2x+4)dx -(4x + 6y + 5) dy = 0$.

SET – II

1. Solve $(x+y)(dx-dy) = dx+dy$.
2. Solve $(2x - 4y + 5)y^1 + (x-2y+3) = 0$.

SET – III

1. The number N of bacteria in a culture grew at a rate proportional to N. The value of N was initially 100 and increased to 332 in one hour what was the value of N after 1½ hours.
2. Uranium disintegrates at a rate proportional to the amount present at any instant. If m_1 and m_2 are grams of uranium that are present at times T_1 and T_2 respectively, find the half – life of uranium.

SET – IV

1. Solve $(x^2 - 2xy + 3y^2)dx + (y^2+6xy- x^2)dy = 0$.
2. Solve $(2x - 4y + 5)y^1 + (x-2y+3) = 0$.

SET – V

1. Solve $(x+y)(dx-dy) = dx+dy$.
2. Uranium disintegrates at a rate proportional to the amount present at any instant. If m_1 and m_2 are grams of uranium that are present at times T_1 and T_2 respectively, find the half – life of uranium.

SET - VI

1. The number N of bacteria in a culture grew at a rate proportional to N. The value of N was initially 100 and increased to 332 in one hour what was the value of N after 1½ hours.

2. Solve $(3y+2x+4)dx -(4x + 6y + 5) dy = 0$.

UNIT-II ASSIGNMENT QUESTIONS

SET – I

1. Solve $y^{11} + y^{1} - 2y = 0$ y(0) y(0) = 4, $y^{1}(0)$-1.
2. Solve $(D^3 + 1)y = \cos(2x-1)$.

SET – II

1. Solve by method of variation of parameters. $(D^2 + 4)y = \tan 2x$.
2. Solve $(D^3 + 1)y = 3 + 5e^x$.

SET – III

1. $(D^3 + 2D^2 + D)y = e^{2x} + x^2 + x + \sin 2x$
2. $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = 0$

SET – IV

1. Solve $y^{11} + y^{1} - 2y = 0$ y(0) y(0) = 4, $y^{1}(0)$-1.
2. Solve by method of variation of parameters. $(D^2 + 4)y = \tan 2x$.

SET – V

1. $(D^3 + 2D^2 + D)y = e^{2x} + x^2 + x + \sin 2x$
2. Solve $(D^3 + 1)y = 3 + 5e^x$.

SET - VI

1. Solve $(D^3 + 1)y = \cos(2x-1)$.
2. Solve $y^{11} + y^{1} - 2y = 0$ y(0) y(0) = 4, $y^{1}(0)$-1.

Maths Assignment

unit 5 : Vector Integral Calculus

(1) If a force $\bar{F} = 2x^2y\,i + 3xy\,j$ displaces a particle in the xy-plane from (0,0) to (1,4) along a curve $y = 4x^2$. Find the work done.

$$\bar{F} = 2x^2y\,i + 3xy\,j$$

(0,0) to (1,4) in $y = 4x^2$

let, $dr = dx\,i + dy\,j + dz\,k$

$\therefore$ work done

$$\int_C F \cdot dr = \int_C \left[(2x^2y)\,dx + (3xy)\,dy\right]$$

$$= \int_0^1 2x^2(4x^2)\,dx + \int_0^4 \frac{3}{2} y\sqrt{y}\,dy$$

$$= 8\left(\frac{x^5}{5}\right)_0^1 + \frac{3}{2}\times\frac{2}{5}\left(y^{5/2}\right)_0^4$$

$$= \frac{8}{5} + \frac{3}{5}(2^5) = \frac{8+3(32)}{5} = \frac{104}{5}.$$

(2) If $\bar{A} = (3x^2+6y)i - 14zj + 20xz^2k$. evaluate the line integral $\int_C A \cdot dr$ from (0,0,0) to (1,1,1) along the curve C.

$$\bar{A} = (3x^2+6y)i - 14z\,j + 20xz^2\,k.$$

(0,0,0) to (1,1,1) along curve.

let, $x = t$, $y = t^2$, $z = t^3$

$dx = dt$, $dy = 2t\,dt$, $dz = 3t^2$.

$t : 0$ to 1. $\quad dr = dx\,i + dy\,j + dz\,k$

$$\therefore \int_c A \cdot dr = \int (3x^2+6y)\,dx - 14z\,dy + 20xz^2\,dz$$

$$= \int_0^1 (3t^2 + 6t^2)\,dt - 14t^3(2t)\,dt + 20t^7(3t^2\,dt)$$

$$= \int_0^1 \left(9t^2 - 28t^4 + 60t^9\right) dt$$

$$= \left(\frac{9t^3}{3} - 28\frac{t^5}{5} + \frac{60}{10}t^{10}\right)_0^1$$

$$= \left(\frac{9}{3} - \frac{28}{5} + 6\right) = 9 - \frac{28}{5} = \frac{45-28}{5} = \frac{17}{5}$$

(3) Evaluate $\int_s (\bar{f} \cdot \bar{n})\,ds$ where $\bar{f} = 18z\hat{i} - 12\hat{j} + 3y k$ and s is the surface of the place $2x + 3y + 6z = 12$ located in the first octant.

$$\int_s (\bar{f} \cdot \bar{n})\,ds$$

$$\therefore \bar{f} = 18z\hat{i} - 12\hat{j} + 3yk$$

$$\phi = 2x + 3y + 6z - 12$$

$$\nabla\phi = 2\hat{i} + 3\hat{j} + 6k$$

$$\bar{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\hat{i}+3\hat{j}+6k}{\sqrt{49}} = \frac{2\hat{i}+3\hat{j}+6k}{7}$$

let, R be the projection of S onto xy-plane.

$z = 0$

$y : 0$ to $\frac{12-2x}{3}$

$x : 0$ to $12/2 \Rightarrow 0$ to 6.

$$\iint_S (\bar{f}\cdot\bar{n})\,ds = \iint_R \bar{f}\cdot\bar{n}\,\frac{dx\,dy}{|\bar{n}\cdot\bar{k}|}$$

$$= \int_0^6\int_0^{\frac{12-2x}{3}} \frac{-36+18y}{7}\,\frac{dy\,dx}{\left(\frac{6}{7}\right)}$$

$$= \int_0^6\int_0^{\frac{12-2x}{3}} (3y-6)\,dy\,dx$$

$$= \int_0^6 \left(\frac{3y^2}{2}-6y\right)_0^{\frac{12-2x}{3}} dx$$

$$= \int_0^6 \frac{3}{2}\left(\frac{12-2x}{3}\right)^2 - 6\left(\frac{12-2x}{3}\right) dx$$

$$= \int_0^6 \frac{3}{2}\left(\frac{4x^2+144-48x}{9}\right) - 2(12-2x)\,dx$$

$$= \int_0^6 \frac{1}{6}(4x^2+144-48x) - 24+4x\,dx$$

$$= \int_0^6 \left(\frac{2x^2}{3}+24-8x-24+4x\right)dx$$

$$= \left(\frac{2x^3}{3\times3} + \cancel{24x} - 4x^2 - \cancel{24x} + 2x^2\right)_0^6$$

$$= \left(\frac{2x^3}{9}-2x^2\right)_0^6 = \frac{36\times4}{3} - 36\times2$$

$$= 12\times4 - 36\times2$$

$$= 48-72$$

$$= -24$$

(4) If $\bar{f} = (2x^2 - 3z)\hat{i} - 2xy\hat{j} - 4x\hat{k}$, then evaluate $\int_V \operatorname{div} \bar{f}\, dv$, where V is bounded by the plane $x=0, y=0, z=0$ and $2x+2y+z=4$.

$$\bar{f} = (2x^2 - 3z)\hat{i} - (2xy)\hat{j} - 4x\hat{k}$$

$$\int \operatorname{div} \bar{f}\, dv = \int \nabla\cdot\bar{f}\, dv$$

$$\therefore \nabla\cdot\bar{f} = 4x - 2x = 2x$$

$\therefore z : 0$ to $4-2x-2y$

$y : 0$ to $\frac{4-2x}{2} = 2-x$

$x : 0$ to 2

$$\therefore \int_V \operatorname{div}\bar{f}\, dv = \int_{x=0}^{2}\int_{y=0}^{2-x}\int_{z=0}^{4-2x-2y} 2x\, dz\, dy\, dx$$

$$= \int_0^2\left(\int_0^{2-x} 2x(z)_0^{4-2x-2y}\, dy\, dx\right)$$

$$= \int_0^2\int_0^{2-x} 2x(4-2x-2y)\, dy\, dx$$

$$\cancel{= \int_0^2\int_0^{2-x} (8x - 4x^2 - 4xy)\, dy\, dx}$$

$$= \int_0^2\int_0^{2-x} (8x - 4x^2 - 4xy)\, dy\, dx$$

$$= \int_0^2 \left(8xy - 4x^2y - 2xy^2\right)_0^{2-x} dx$$

$$= \int_0^2 [8x(2-x) - 4x^2(2-x) - 2x(x^2+4-4x)]\,dx$$

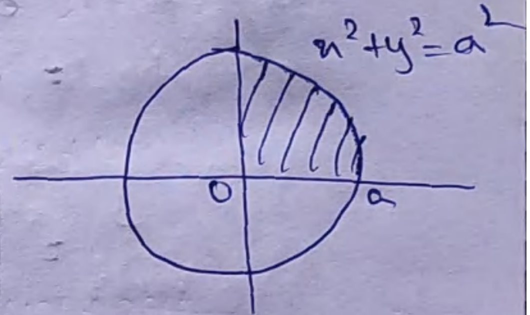
$$= \int_0^2 (16x - 8x^2 - 8x^2 + 4x^3 - 2x^3 - 8x + 8x^2)\,dx$$

$$= \int_0^2 (2x^3 - 8x^2 + 8x)\,dx$$

$$= \left(\frac{x^4}{2} - \frac{8x^3}{3} + 4x^2\right)_0^2$$

$$= 8 - \frac{64}{3} + 16 = 24 - \frac{64}{3} = \frac{72-64}{3} = \frac{8}{3}.$$

(5) Apply Green's theorem to evaluate the line integral $\oint_c \bar{f}\cdot\bar{dr}$ where c is the circular path given by $x^2+y^2=a^2$ and the vector field of $\bar{f}$ is given by $\bar{f} = \sin y\, \hat{i} + x(1+\cos y)\hat{j}$.



$x^2+y^2=a^2$

$\bar{f} = \sin y\, \hat{i} + x(1+\cos y)\hat{j}$

$dr = dx\hat{i} + dy\hat{j} + dz\hat{k}$

$\therefore$ Green's theorem,

$$\oint_c \bar{f}\cdot dr = \oint_c (\sin y\, dx + (x + x\cos y)\,dy)$$

it is in the form $M\,dx + N\,dy$.

$\therefore$ $M = \sin y$ $\qquad$ $N = x + x\cos y$

$\frac{\partial M}{\partial y} = \cos y$ $\qquad$ $\frac{\partial N}{\partial x} = 1 + \cos y$

$$\therefore \oint_c \bar{f}\cdot \bar{dr} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$= \iint_R (1 + \cos y - \cos y)\, dx\, dy$$

$$= \iint_R dx\, dy$$

$\therefore$ R is the circular path

$\therefore$ $y : 0$ to a

$x : 0$ to $\sqrt{a^2 - y^2}$

$$\therefore \oint_C f \cdot dr = \int_{y=0}^{a} \int_{x=0}^{\sqrt{a^2-y^2}} dx\, dy$$

$$\int_0^a (x)_0^{\sqrt{a^2-y^2}} dy$$

$$= \int_0^a \sqrt{a^2-y^2}\, dy$$

$$= \left[\frac{y}{2}\sqrt{a^2-y^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{y}{a}\right) \right]_0^a$$

$$= \frac{a^2}{2}\sin^{-1}\left(\frac{a}{a}\right)$$

$$= \frac{a^2}{2} \times \frac{\pi}{2} = \frac{\pi a^2}{4}.$$

(6) Use the Stoke's theorem to evaluate

$$\int_C \left[(x+2y)\,dx + (x-z)\,dy + (y-z)\,dz \right]$$

where C is the boundry of the triangle with vertices $(2,0,0)$, $(0,3,0)$ and $(0,0,6)$

oriented in the anti-clockwise direction.

$$\int_C [(x+2y)dx + (x-z)dy + (y-z)dz]$$

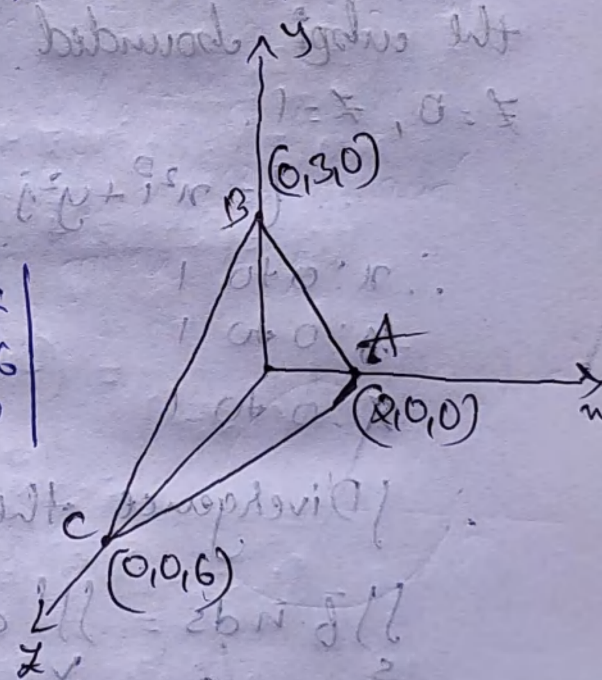
Stokes theorem,

$$\oint_C \bar{F}\cdot d\bar{r} = \iint_S \operatorname{curl} \bar{F} \cdot \bar{n}\, ds$$

$$\therefore \operatorname{curl} \bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y & x-z & y-z \end{vmatrix}$$

$$= i(1+1) - j(0-0) + k(1-2)$$

$$= 2i - k$$



$$\therefore \bar{AB} = (-2,3,0)$$
$$\bar{CA} = (2,0,-6)$$

$$\bar{CA} \times (\bar{AB}) = \begin{vmatrix} i & j & k \\ 2 & 0 & -6 \\ -2 & 3 & 0 \end{vmatrix}$$

$$= 18i + 12j + 6k$$

$$\bar{n} = \frac{\bar{CA}\times\bar{AB}}{|\bar{CA}\times\bar{AB}|}$$

$$= \frac{3i+2j+k}{\sqrt{14}}$$

$$\therefore \iint ds = \frac{1}{2}|\bar{CA}\times\bar{AB}| = \frac{1}{2}(6\sqrt{14}) = 3\sqrt{14}$$

$$\therefore \operatorname{curl} \bar{F}\cdot\bar{n} = \frac{6-1}{\sqrt{14}} = \frac{5}{\sqrt{14}}$$

$$\therefore \iint_S \text{curl}\,\bar{F}\cdot\bar{n}\,ds = \iint_S \frac{5}{\sqrt{14}}\,ds$$

$$= \frac{5}{\sqrt{14}} \iint_S ds$$

$$= \frac{5}{\sqrt{14}}(3\sqrt{14}) = 15$$

(7) Using divergence theorem, evaluate surface integral $\int_S (\bar{F}\cdot\bar{n})\,ds$, where $\bar{F} = x^2\hat{i} + y^2\hat{j} + z^2\hat{k}$ and S is the surface of the cube bounded by $x=0,\ x=1,\ y=0,\ y=1,\ z=0,\ z=1$.

$$\therefore \bar{F} = x^2\hat{i} + y^2 j + z^2 k$$

$\therefore x : 0$ to 1

$y : 0$ to 1

$z : 0$ to 1

$\therefore$ Divergence theorem,

$$\iint_S \bar{F}\cdot\bar{n}\,ds = \iiint_V \text{div}\,\bar{F}\,dv$$

$$\therefore \text{div}.\bar{F} = 2x + 2y + 2z$$

$$\therefore \iint_S \bar{F}\cdot\bar{n}\,ds = \iiint_V \text{div}\,\bar{F}\,dv$$

$$= \int_0^1\int_0^1\int_0^1 (2x+2y+2z)\,dx\,dy\,dz$$

$$= \int_0^1\int_0^1 \left(x^2 + 2xy + 2xz\right)_0^1 dy\,dz$$

$$= \int_0^1\int_0^1 (1+2y+2z)\,dy\,dz$$

$$= \int_0^1 (y + y^2 + 2yz)_0^1 \, dz$$

$$= \int_0^1 (1 + 1 + 2z) \, dz$$

$$\iint_S \bar{f} \cdot \bar{n} \, ds = (2z + z^2)_0^1 = 2 + 1 = 3$$

(8) If $f = \tan^{-1}(y/x)$, then find div(grad f).

$f = \tan^{-1}(y/x)$

div(grad f)

$$= \operatorname{div}\left(\sum \frac{\partial f}{\partial x} \hat{i}\right)$$

$$= \operatorname{div}\left(\frac{y \log x}{1 + \left(\frac{y}{x}\right)^2} \hat{i} + \frac{1}{x} \frac{1}{1 + \left(\frac{y}{x}\right)^2} \hat{j} \right)$$

$$= \operatorname{div}\left(\frac{x^2 y \log x}{x^2 + y^2} \hat{i} + \frac{x}{x^2 + y^2} \hat{j} \right)$$

$$= \frac{(x^2+y^2)(2xy \log x + xy) - x^2 y \log x (2x)}{(x^2+y^2)^2}$$

$$+ \frac{(x^2+y^2)(0) - x(2y)}{(x^2+y^2)^2}$$

$$= \frac{\cancel{2x^3 y \log x} + x^2 y + y^2 (2xy \log x + xy) - \cancel{2x^3 y \log x} - 2xy}{(x^2+y^2)^2}$$

$$= \frac{2xy(y^2 \log x - 1) + xy(x^2 + y^2)}{(x^2+y^2)^2}$$

$$= \frac{xy}{x^2+y^2} + \frac{2xy(y^2 \log x - 1)}{(x^2+y^2)^2}$$

(9) Find the value of curl(grad f), if $f = 2x^2 - 3y^2 + 4z^2$.

$$\text{grad } f = 4x\hat{i} - 6y\hat{j} + 8z k$$

$$\text{curl (grad f)} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4x & -6y & 8z \end{vmatrix}$$

$$= 0i + 0j + 0k = \bar{0}.$$

(10) Find the directional derivative of $\phi(x,y,z) = x^2yz + 4xz^2$ at the point $(1,2,-1)$ in the direction PQ where $P = (1,2,-1)$ and $Q = (-1,2,3)$

$$\phi(x,y,z) = x^2yz + 4xz^2$$

$$\therefore \nabla\phi = [(2xyz) + (4z^2)]\hat{i} + x^2z\hat{j} + (x^2y + 8xz)k$$

$$\nabla\phi \Big|_{(1,2,-1)} = (-4+4)\hat{i} + (-\hat{j}) + (2-8)k$$

$$= -j - 6k$$

$$P = (1,2,-1) \quad Q = (-1,2,3)$$

let, $\bar{a} = \bar{PQ} = (\bar{OQ} - \bar{OP})$

$$= -2\hat{i} + 4k.$$

$$\therefore \hat{a} = \frac{\bar{a}}{|\bar{a}|} = \frac{-2\hat{i}+4k}{\sqrt{4+16}} = \frac{-\hat{i}+2k}{\sqrt{5}}$$

$\therefore$ D.D. of ϕ at $(1,2,-1)$ along PQ

$$= \nabla\phi \cdot \hat{a}$$

$$= (-\hat{j} - 6k) \cdot \left(\frac{-i+2k}{\sqrt{5}}\right) = \frac{-12}{\sqrt{5}}$$

(11) Evaluate $\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)\, dx\, dy$

$$\int_{x=0}^{1} \int_{y=x}^{\sqrt{x}} (x^2+y^2)\, dy\, dx$$

$$= \int_0^1 \left(x^2 y + \frac{y^3}{3}\right)_x^{\sqrt{x}} dx$$

$$= \int_0^1 \left(x^{5/2} + \frac{x^{3/2}}{3} - x^3 - \frac{x^3}{3}\right) dx$$

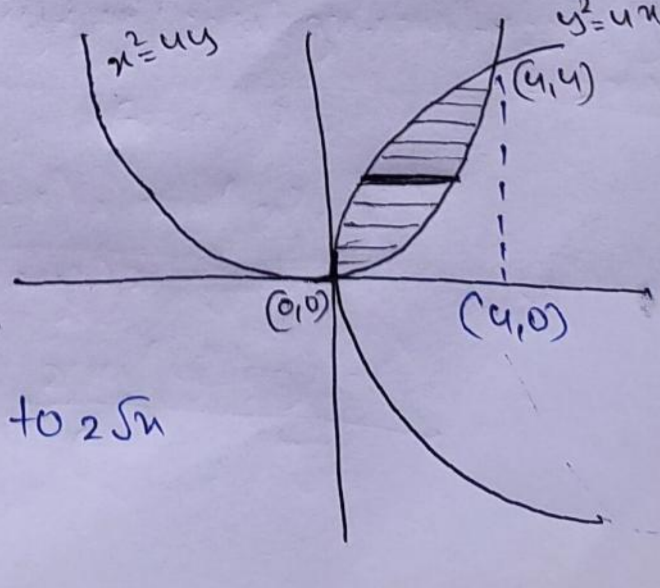
$$= \left(\frac{2}{7} x^{7/2} + \frac{2}{15} x^{5/2} - \frac{x^4}{4} - \frac{x^4}{12}\right)_0^1$$

$$= \frac{2}{7} + \frac{2}{15} - \frac{3}{12} - \frac{1}{12}$$

$$= \frac{30+14}{15\times 9} - \frac{1}{3} = \frac{44}{3\times 35} - \frac{35}{35\times 3}$$

$$= \frac{9}{3\times 35} = \frac{3}{35}$$

(12) evaluate $\iint y\,dx\,dy$, over the region bounded by the parabola $y^2 = 4x$ and $x^2 = 4y$.



$\therefore$ $y^2 = 4x$

$x^2 = 4y$

$\therefore \iint y\,dx\,dy$

$\therefore$ x : 0 to 4

y : $\cancel{2\sqrt{x}\ \text{to}}$ $\frac{x^2}{4}$ to $2\sqrt{x}$

$$\int_{x=0}^{4}\int_{y=x^2/4}^{2\sqrt{x}} y\,dy\,dx$$

$$= \int_0^4 \left(\frac{y^2}{2}\right)_{x^2/4}^{2\sqrt{x}} dx$$

$$= \int_0^4 \left(\frac{4x^1}{2} - \frac{x^4}{32}\right) dx$$

$$= \int_0^4 \left(2x - \frac{x^4}{32}\right) dx$$

$$= \left(x^2 - \frac{x^5}{32\times 5}\right)_0^4$$

$$= \left(16 - \frac{(4\times 4\times 2)\times 2\times 16}{2^5\times 5}\right)$$

$$= 16\left(1-\frac{2}{5}\right)$$

$$= 16\left(\frac{3}{5}\right) = \frac{48}{5}$$