Unit – I

Probability & Random Variable

Probability & Random Variable: Probability introduced through Sets and Relative Frequency: Experiments and Sample Spaces, Discrete and Continuous Sample Spaces, Events, Probability Definitions and Axioms, Joint Probability, Conditional Probability, Total Probability, Bay's Theorem, Independent Events, Random Variable- Definition, Conditions for a Function to be a Random Variable, Discrete, Continuous and Mixed Random Variable, Distribution and Density functions, Properties, Binomial, Poisson, Uniform, Gaussian, Exponential, Rayleigh, Methods of defining Conditioning Event, Conditional Distribution, Conditional Density and their Properties.

Introduction:

The Primary aim of this course is to introduce the principles of random signals and to provide tools where by one can deal with systems involving such signals. A random signal is a time waveform that can be characterized only in some probabilistic manner. In general it can be desired or undesired waveform. For example one can hear background hiss while listening to an ordinary broadcast Radio receiver. It is undesirable, since it interferes with our ability to hear the radio program and is called noise. One example for desirable random signal is that the output voltage of a wind powered generator would be random because wind speed fluctuates randomly.

There are actually two things to be considered in characterizing random signals.

1. How to describe any one of a variety of random signals (Random variables)
2. How to bring time into problem so as to create the random signal of interest (Random Process)

Several approaches exist for the definition and discussion of Probability. Among them two are only worthy of modern day consideration while others are mainly of historical interest.

- First approach is regarding relative frequency definition of Probability, which gives a degree of physical insight.
- Second approach to probability uses Axiomatic definition of probability, which is mathematically sound of all approaches and as per as this course is concerned we deal with this second approach.

Hence prior to introduction of the axioms of probability, it is necessary to recollect some concepts of set theory.

Introduction to Sets:

A set is defined as a collection of objects. The objects are called elements of the set. A set is usually represented by a capital letter while an element is represented by a lower case letter.

A set is said to be **countable** if its elements can be put in one to one correspondence. If a set is not countable it is said to be **uncountable.**

A set is said to be **empty** if it has no elements and it is often called **Null set**.

A **Finite** set is one that is either empty or has finite number of elements that can be counted. If a set is not finite, it is said to be **Infinite** Set.

Consider two sets namely A and B. If every element of set A is also an element in set B then A is known as **subset** of B and at least one element exists in set B which is not in A, then A is a **proper subset** of B.

Null set is clearly a subset of all other sets.

Two sets are called **disjoint or mutually** exclusive if they have no common elements.

The largest or all encompassing set of objects under discussion is said to be a **Universal set** and denote d by S. All sets are subsets of Universal Set.

Equality and Difference of Sets:

Two sets A and B are equal if all elements in A are present in B and all elements in B are present in A. For equal sets it can be written as A=B.

The difference of two sets A and B is denoted as A-B, is the set containing all elements of A that are not present in B.

Union and Intersection of Sets:

The union of two sets A and B is written as C $=A \cup B$. It is the set of all elements of A or B or both. The union is sometimes called sum of two sets.

The intersection of two sets A and B is written as D $=A \cap B$. It is the set of all elements common to both A and B. Intersection is some times called Product of two Sets.

For Mutually exclusive sets A and B, $A \cap B = \emptyset$.

Complement:

The Complement of set A is denoted as $\bar{A}$, i.e., the set of all elements not in A.

Algebra of Sets:

All subsets of the universal set form an algebraic system for which a number of theorems may be stated. Three of the most important laws involving union and intersections are

1. Commutative Law: For any two sets A and B,

$$A \cap B = B \cap A$$
$$A \cup B = B \cup A$$

2. Distributive Law: For any three sets A, B and C

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

3. Associative Law: For any three sets A, B and C

$$(A \cup B) \cup C = A \cup (B \cup C) = A \cup B \cup C$$
$$(A \cap B) \cap C = A \cap (B \cap C) = A \cap B \cap C$$

De Morgan's Laws:

De Morgan's Laws state that the complement of a union of two sets A and B equals the intersection of the complements $\bar{A}$ and $\bar{B}$. Thus

$$(\overline{A \cup B}) = \bar{A} \cap \bar{B}$$
$$(\overline{A \cap B}) = \bar{A} \cup \bar{B}$$

Definition of a Sample Space and its Types

Sample Space: It is defined as a set of all possible outcomes in any given experiment is called the Sample Space and it is given by the symbol S.

Example: Consider an experiment of throwing a single die and observing the number on die.

Die has six faces and the number on each face will be 1,2,3,4,5,6 respectively. On throwing a die, the possible output may be any of the six numbers. Hence the all possible outcomes of this experiment will be {1,2,3,4,5,6} and it is considered as Sample Space denoted by S.

S = {1,2,3,4,56}

1. Sample space is a universal set for the given experiment as it contains all possible outcomes of any given experiment.
2. Sample Space may be different for different experiments but all the experiments are governed by sample space.

Types of Sample Space:

1. Discrete Sample Space
2. Continuous Sample Space

Discrete Sample Space: If a sample space consists of finite set of elements, then it is said to be discrete sample space.

Example: In tossing a single die, the sample space is given by {1,2,3,4,5,6}, which consists of finite no. of samples and discrete.

In some cases the sample space can be discrete and infinite. For example in an experiment, of "choosing randomly a positive integer", the sample space will be {1,2,3,4,5..........}, which consists of infinite no.of samples and countable.

Continuous Sample Space: If a sample space has uncountably infinite no. of elements, then it is said to be Continuous sample space.
Example: Obtain a number by spinning the pointer on a wheel of chance numbered from 0 to 12. Here any number s from o to 12 can result and $S = [0 < s \leq 12\}$. Such a sample space is called continuous.

Definition of an Event and its Types

Event: An Event is defined as a subset of the sample space.
Example: Consider an Experiment of "drawing a spade" from a deck of 52 cards. Here drawing a spade is an Event.

- If two events have no common outcomes they are said to be **Mutually Exclusive Events.**

Types of Events:

1. Discrete Event
2. Continuous Event

Discrete Event: If an Event consists of distinct and finite set of elements, then it is said to be a discrete Event.
Example: List multiples of 2 in tossing a single die.
Here Sample space is S = {1,2,3,4,5,6} and the outcome of an event " list multiples of 2" is {2,4,6}. Here the outcome of an event is discrete and finite.

Continuous Event: Events defined on continuous sample spaces are usually continuous Events.

Definition of Probability and its Axioms

Probability: To each event defined on a sample space S, there shall be assigned a non negative number called probability. Hence therefore probability is therefore a function of events defined. Probability of an Event A is denoted as P(A).

Example: Consider an experiment of rolling a Single Die. All possible outcomes (Sample Space) of this experiment is given by S = {1,2,3,4,5,6}.
Now, an Event is defined as "getting a number 2 on the face of the dice". The corresponding probability of this event is 1/6.
If an event is defined as "getting an even number" the elements of an Event will be {2,4,6} and the probability of this Event is 3/6.

Axioms of Probability:

1. The Probability of any event A is defined as $P(A) \geq 0$.
   It defines that probability of a certain event will never be negative.

2. The Probability of Sample Space S is P(S) = 1.
   Sample space itself is an Event and it is the all-encompassing event, it should have the highest probability, which is selected as unity.

3. Consider N events A_n, n = 1,2,3......,N, where N may possibly be infinite, defined on sample space S and having the property $A_m \cap A_n = \emptyset$ for all $m \neq n$. Then,

$$\boldsymbol{P\left(\cup_{n=1}^{N} A_n\right) = \sum_{n=1}^{N} P(A_n) \text{ if } A_m \cap A_n = \emptyset}$$

For all $m \neq n$ = 1,2,3.....,N with N Possibly infinite. This states that the probability of the event equal to the union of any number of mutually exclusive events is equal to the sum of the individual event probabilities.

Mathematical Model of an Experiment

Sample space, **Events** and **Axioms of probability** are the three elements required to completely define the mathematical model of an Experiment.

Given some real physical experiment,

- First sample space has to be defined to mathematically represent the physical outcomes.
- Secondly to recognize certain characteristics of the outcomes in the real experiment of interest events were defined to mathematically represent these characteristics.
- Finally, probabilities were assigned to the defined events to mathematically account for the random nature of the experiment.

Hence a real experiment is defined mathematically by three things

1. Assignment of sample space
2. Definition of Events of interest
3. Making probability assignments to the events to mathematically account for the random nature of the experiment.

Probability as a Relative Frequency

The use of common sense and engineering and scientific observations leads to a definition of probability as a relative frequency of occurrence of some event.

For example consider an experiment of flipping coin **n times** and out of that **n_H is the heads** shown up out of n flips, then

$$\lim_{n \to \infty} (n_H / n) = P(H)$$

Where P(H) is interpreted as the probability of the event heads. The ratio n_H/n is the relative frequency for this event.

Joint, Conditional and Total Probability

Joint Probability: Consider any two Events A and B. The Probability of $P(A \cap B)$, i.e, the probability of common elements of two events is called the Joint Probability for the two events A and B which intersect in the sample space.

$$P(A \cap B) = P(A) + P(B) - P(A \cup B)$$

Equivalently $$P(A \cup B) = P(A) + P(B) - P(A \cap B) \leq P(A) + P(B)$$

In other words, the probability of the union of two events never exceeds the sum of the event Probabilities. The equality holds only for mutually exclusive events because $A \cap B = \emptyset$, therefore $P(A \cap B) = P(\emptyset) = 0$.

Conditional Probability:

Given some Event B with non-zero probability $P(B) > 0$, we define the conditional probability of an event A, given B by

$$P(A/B) = P(A \cap B) / P(B)$$

The Probability of P(A/B) simply reflects the fact that the probability of an event A may depend on a second Event B. If A and B are mutually exclusive, $A \cap B = \emptyset$ and $P(A/B) = 0$

Conditional Probability is a defined quantity and cannot be proven. However as a probability it must satisfy the three axioms as discussed earlier.

The conditional probabilities are some times called as transition probabilities in Communications context.

Total Probability:

The Probability P(A) of any Event A defined on a sample space S can be expressed in terms of conditional probabilities. Suppose we are given N mutually exclusive events $B_n =$ 1,2,.....N, whose union equals Sample space S. These events satisfy

$$B_m \cap B_n = \emptyset, \; m \neq n = 1,2, \; .... \; N$$

$$\bigcup_{n=1}^{N} B_n = S$$

and hence therefore the total probability of event A is given by

$$P(A) = \sum_{n=1}^{N} P(A/B_n) \; P(B_n)$$

Proof:

Consider and Event A, and since we know $A \cap S = A$, using the equations

$$\bigcup_{n=1}^{N} B_n = S \text{ ----- (1)}$$

and Commutative Property of sets A,B and C

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \text{ ---- (2)}$$

We can write

$$A \cap S = A \cap \left(\bigcup_{n=1}^{N} B_n \right) = \bigcup_{n=1}^{N} (A \cap B_n)$$

Now the events $A \cap B_n$ becomes mutually exclusive and by applying Axiom 3 of probability,

$$P(A) = P(A \cap S) = P[\cup_{n=1}^{N} (A \cap B_n)] = \sum_{n=1}^{N} P(A \cap B_n) \text{ ---- (3)}$$

and finally substituting Conditional Probability $P(A/B) = P(A \cap B) / P(B)$ in equation (3), we obtain

$$P(A) = \sum_{n=1}^{N} P(A/B_n)\ P(B_n) \text{ --------(4)}$$

Bayes Theorem

The definition of conditional Probability $P(A/B) = P(A \cap B) / P(B)$, applies to any two events.

In Particular let B_n be one of the events defined above in the subsection on total probability.

Then the equation for conditional Probability can be written as

$$P(B_n \backslash A) = \frac{P(B_n \cap A)}{P(A)} \text{ if } P(A) \neq 0 \text{ ---- (1)}$$

or alternatively,

$$P(A \backslash Bn) = \frac{P(A \cap B_n)}{P(B_n)} \text{ if } P(B_n) \neq 0 \text{ ----- (2)}$$

One form of Bayes theorem can be obtained by equating the above two expressions

$$P(B_n \backslash A) = \frac{P(A \backslash B_n)\ P(Bn)}{P(A)} \text{ ---- (3)}$$

Another form of Bayes theorem derives from a substitution of $P(A) = \sum_{n=1}^{N} P(A/B_n)\ P(B_n)$ in the equation (3)

$$P(B_n \backslash A) = \frac{P(A \backslash B_n)\ P(B_n)}{P(A \backslash B_1) P(B_1) + \ldots \ldots . . + P(A \backslash B_N) P(B_N)}$$

In Bayes theorem, the probabilities $P(B_n)$ and $P(A/B_n)$ are usually referred as a Priori probabilities since they known before conducting the event. Probabilities $P(B_n/A)$ are known as posterior probabilities since they apply after the experiments performance when some event A is obtained.

Independent Events

Two Events:

Let two events A and B have nonzero probabilities of occurrence, that is $P(A) \neq 0$ and $P(B) \neq 0$. These two events are said to statistically independent if the probability of occurrence of one event is not affected by the occurrence of the other event.

Mathematically, this statement is equivalent to requiring

$$P(A\backslash B) = P(A) \text{ ---- } (1)$$

or

$$P(B\backslash A) = P(B)$$

for statistically independent events.

Substituting the above eq(1) in the expression for conditional probability $P(A/B) = P(A \cap B) / P(B)$, we have

$$P(A \cap B) = P(A)\, P(B) \quad \text{------ } (2)$$

i.e., independence also means that the probability of the joint occurrence of two events must equal to the product of the two event probabilities.

Generally, when events are considered as independent, probability problems are greatly simplified.

Remind that, in case of two mutually exclusive events A and B, joint probability $P(A \cap B) = 0$ but in case of independent events, the joint probability is given by $P(A \cap B) = P(A)\, P(B)$. Hence it can be stated that two events cannot be both mutually exclusive and statistically independent.

If more than two events are considered and those events satisfying the above criteria (1) & (2) are said to be statistically independent by pairs.

Multiple Events:

When more than two events are considered, independence by pairs is not sufficient to establish the events as statistically independent, even if every pair satisfies.

In the case of three events A_1, A_2, and A_3, they are said to be independent if and only if they are independent by all pairs and are also independent as a triple. i.e., they must satisfy the four equations

$$P(A_1 \cap A_2) = P(A_1)\, P(A_2)$$
$$P(A_1 \cap A_3) = P(A_1)\, P(A_3)$$
$$P(A_2 \cap A_3) = P(A_2)\, P(A_3)$$
$$P(A_1 \cap A_2 \cap A_3) = P(A_1)\, P(A_2)\, P(A_3)$$

and the same can be extended for N events