# LECTURE NOTES

# ON

# SIGNALS AND SYSTEMS
# (15A04303)

## II B.TECH – I SEMESTER ECE
## (JNTUA – R15)

## PREPARED BY


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**JAWAHARLAL NEHRU TECHNOLOGICAL UNIVERSITY ANANTAPUR**

**II B.Tech I-Sem (E.C.E)**

**(15A04303) SIGNALS AND SYSTEMS**

***Course objectives***:

- To study about signals and systems.
- To do analysis of signals & systems (continuous and discrete) using time domain & frequency domain methods.
- To understand the stability of systems through the concept of ROC.
- To know various transform techniques in the analysis of signals and systems.

***Learning Outcomes***:

- For integro-differential equations, the students will have the knowledge to make use of Laplace transforms.
- For continuous time signals the students will make use of Fourier transform and Fourier series.
- For discrete time signals the students will make use of Z transforms.
- The concept of convolution is useful for analysis in the areas of linear systems and communication theory.

**UNIT I**

**SIGNALS & SYSTEMS:** Definition and classification of Signal and Systems (Continuous time and Discrete time), Elementary signals such as Dirac delta, unit step, ramp, sinusoidal and exponential and operations on signals. Analogy between vectors and signals-orthogonality-Mean Square error-Fourier series: Trigonometric & Exponential and concept of discrete spectrum

**UNIT II**

**CONTINUOUS TIME FOURIER TRANSFORM:** Definition, Computation and properties of Fourier Transform for different types of signals. Statement and proof of sampling theorem of low pass signals

**UNIT III**

**SIGNAL TRANSMISSION THROUGH LINEAR SYSTEMS:** Linear system, impulse response, Response of a linear system, linear time-invariant (LTI) system, linear time variant (LTV) system, Transfer function of a LTI system. Filter characteristics of linear systems. Distortion less transmission through a system, Signal bandwidth, system bandwidth, Ideal LPF, HPF and BPF characteristics, Causality and Poly-Wiener criterion for physical realization, Relationship between bandwidth and rise time. Energy and Power Spectral Densities

**UNIT IV**

**DISCRETE TIME FOURIER TRANSFORM:** Definition, Computation and properties of Fourier Transform for different types of signals.

**UNIT V**

**LAPLACE TRANSFORM:** Definition-ROC-Properties-Inverse Laplace transforms-the S-plane and BIBO stability-Transfer functions-System Response to standard signals-Solution of differential equations with initial conditions.

**The Z–TRANSFORM:** Derivation and definition-ROC-Properties-Linearity, time shifting, change of scale, Z-domain differentiation, differencing, accumulation, convolution in discrete time, initial and final value theorems-Poles and Zeros in Z -plane-The inverse Z-Transform-System analysis-Transfer function-BIBO stability-System Response to standard signals-Solution of difference equations with initial conditions. .

**TEXT BOOKS:**

1. B. P. Lathi, "Linear Systems and Signals", Second Edition, Oxford University press,
2. A.V. Oppenheim, A.S. Willsky and S.H. Nawab, "Signals and Systems", Pearson, 2nd Edn.
3. A. Ramakrishna Rao,"Signals and Systems", 2008, TMH.

**REFERENCES:**

1. Simon Haykin and Van Veen, "Signals & Systems", Wiley, 2nd Edition.
2. B.P. Lathi, "Signals, Systems & Communications", 2009,BS Publications.
3. Michel J. Robert, "Fundamentals of Signals and Systems", MGH International Edition, 2008.
4. C. L. Philips, J. M. Parr and Eve A. Riskin, "Signals, Systems and Transforms", Pearson education.3rd

# UNIT-I

# SIGNALS & SYSTEMS

# UNIT-I

# SIGNALS & SYSTEMS

**Signal :** A signal is defined as a time varying physical phenomenon which is intended to convey information. (or) Signal is a function of time. (or) Signal is a function of one or more independent variables, which contain some information.

**Example:** voice signal, video signal, signals on telephone wires , EEG, ECG etc.

Signals may be of continuous time or discrete time signals.

**System :** System is a device or combination of devices, which can operate on signals and produces corresponding response. Input to a system is called as excitation and output from it is called as response.

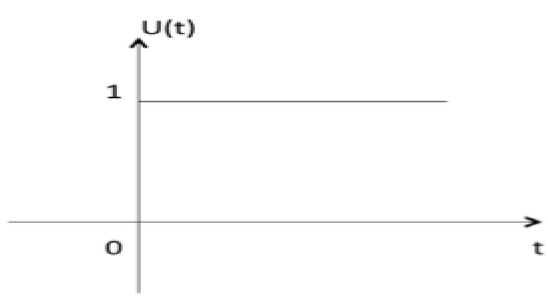
For one or more inputs, the system can have one or more outputs.

**Example:** Communication System


## Elementary Signals or Basic Signals:

### Unit Step Function

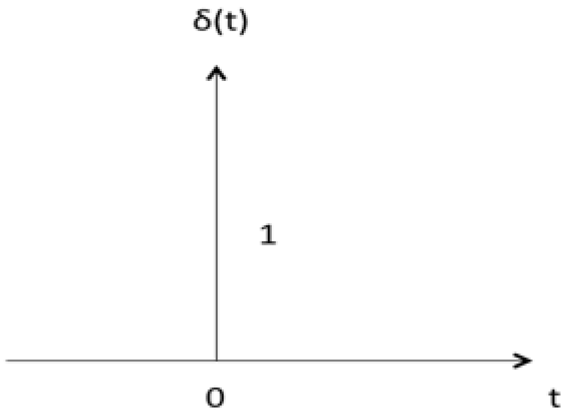
Unit step function is denoted by u(t). It is defined as u(t) = 1 when $t \geq 0$ and
0 when $t < 0$




- It is used as best test signal.
- Area under unit step function is unity.

### Unit Impulse Function

Impulse function is denoted by δ(t). and it is defined as δ(t) = $\begin{cases} 1 & t = 0 \\ 0 & t \neq 0 \end{cases}$

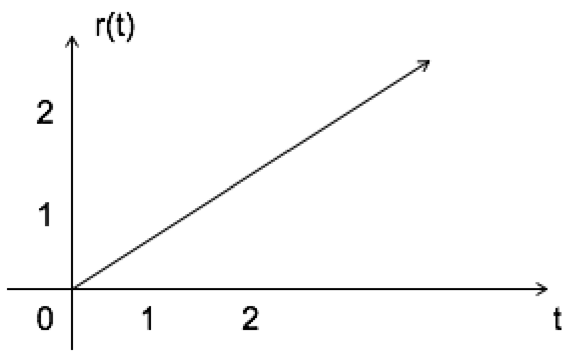



$$\int_{-\infty}^{\infty} \delta(t)dt = u(t)$$

$$\delta(t) = \frac{du(t)}{dt}$$

### Ramp Signal

Ramp signal is denoted by r(t), and it is defined as r(t) = $\begin{cases} t & t \geqslant 0 \\ 0 & t < 0 \end{cases}$



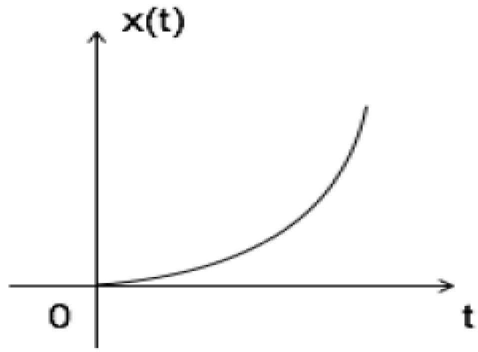

$$\int u(t) = \int 1 = t = r(t)$$

$$u(t) = \frac{dr(t)}{dt}$$

Area under unit ramp is unity.

### Parabolic Signal

Parabolic signal can be defined as x(t) = $\begin{cases} t^2/2 & t \geqslant 0 \\ 0 & t < 0 \end{cases}$



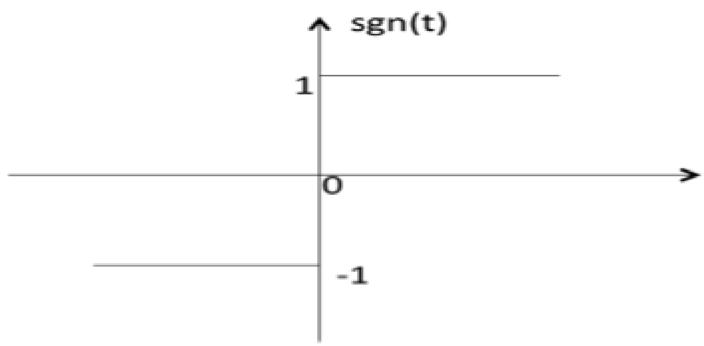

$$\iint u(t)dt = \int r(t)dt = \int tdt = \frac{t^2}{2} = parabolicsignal$$

$$\Rightarrow u(t) = \frac{d^2x(t)}{dt^2}$$

$$\Rightarrow r(t) = \frac{dx(t)}{dt}$$

**Signum Function**

Signum function is denoted as sgn(t). It is defined as sgn(t) = $\begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0 \end{cases}$



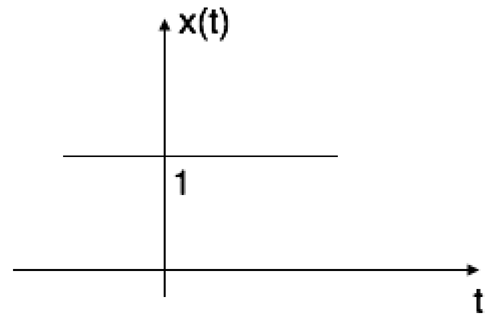

sgn(t) = 2u(t) – 1

**Exponential Signal**

Exponential signal is in the form of x(t) = $e^{\alpha t}$

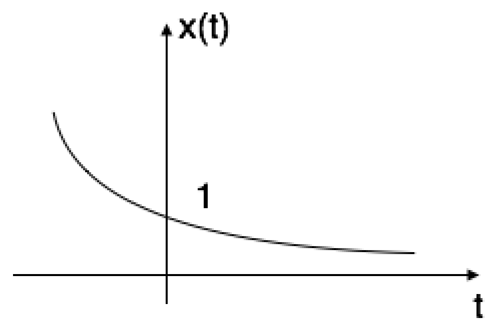
.The shape of exponential can be defined by $\alpha$

**Case i:** if $\alpha = 0 \longrightarrow$ x(t) = $e^0 = 1$



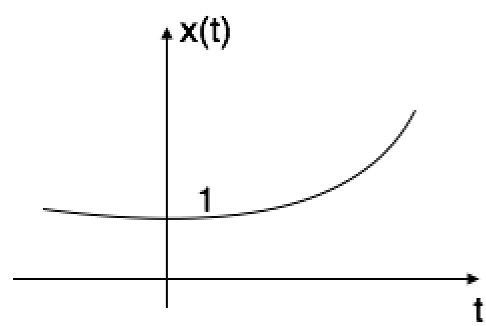

**Case ii:** if $\alpha < 0$ i.e. -ve then x(t) = $e^{-\alpha t}$

. The shape is called decaying exponential.




**Case iii:** if $\alpha > 0$ i.e. +ve then x(t) = $e^{\alpha t}$

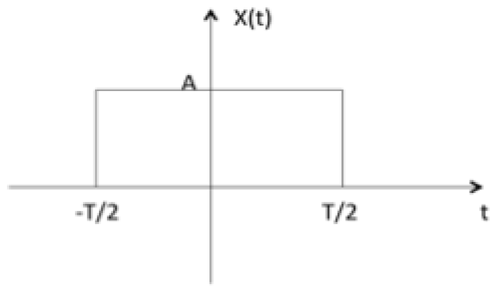
. The shape is called raising exponential.

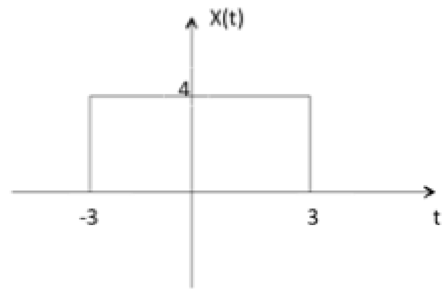


## Rectangular Signal

Let it be denoted as x(t) and it is defined as

$$x(t) = A\,rect\left[\frac{r}{T}\right] \qquad \text{ex}: 4\,rect\left[\frac{r}{6}\right]$$

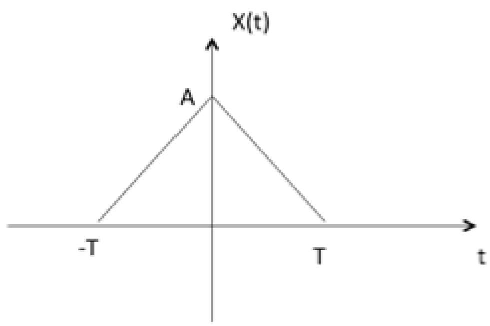


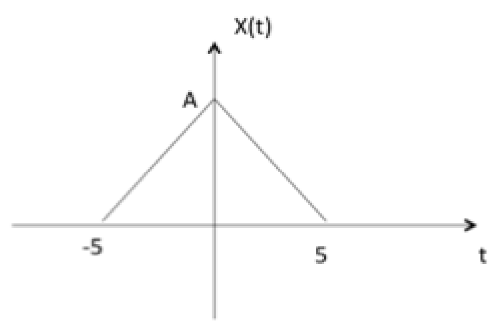




## Triangular Signal

Let it be denoted as x(t)

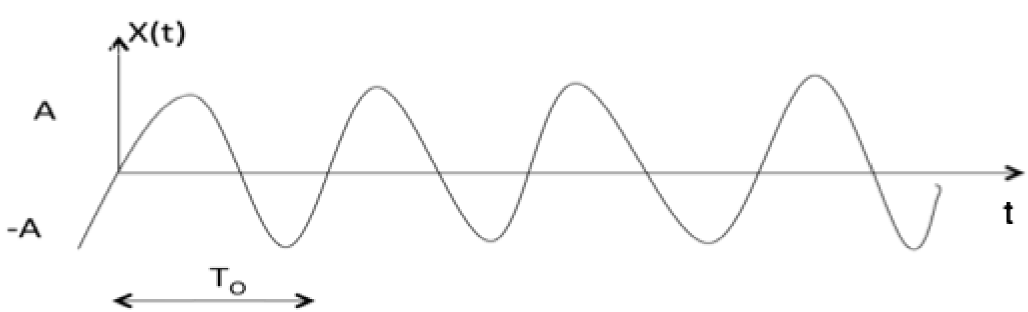
$$x(t) = A\left[1 - \frac{|t|}{T}\right] \qquad \text{ex}: x(t) = A\left[1 - \frac{|t|}{5}\right]$$




## Sinusoidal Signal

Sinusoidal signal is in the form of x(t) = A cos($w_0 \pm \phi$) or A sin($w_0 \pm \phi$)




Where $T_0 = 2\pi/w_0$

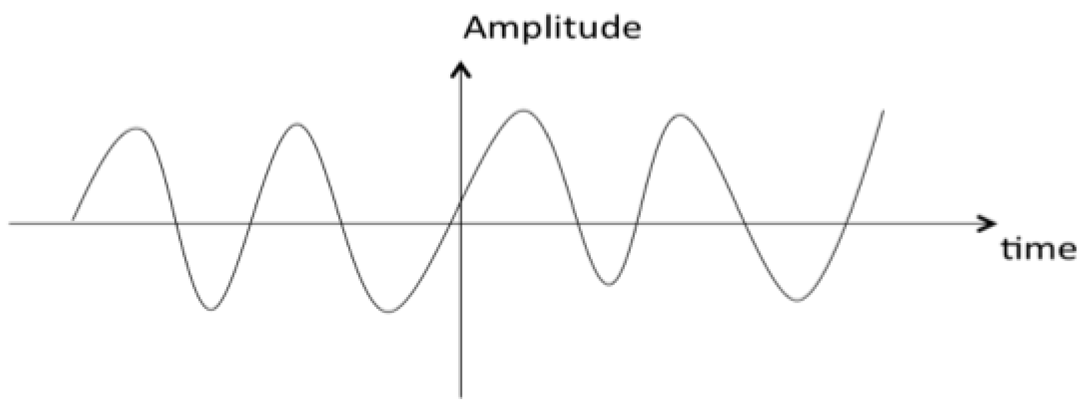
## Classification of Signals:

Signals are classified into the following categories:

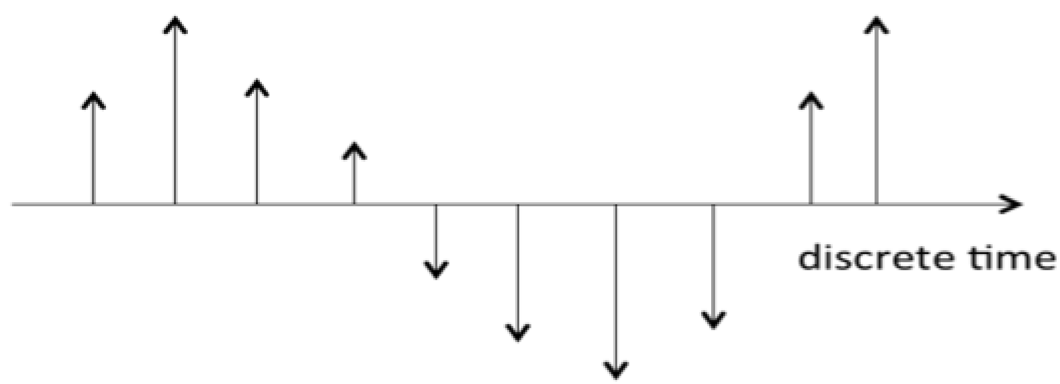
- Continuous Time and Discrete Time Signals
- Deterministic and Non-deterministic Signals
- Even and Odd Signals
- Periodic and Aperiodic Signals
- Energy and Power Signals
- Real and Imaginary Signals

### Continuous Time and Discrete Time Signals

A signal is said to be continuous when it is defined for all instants of time.

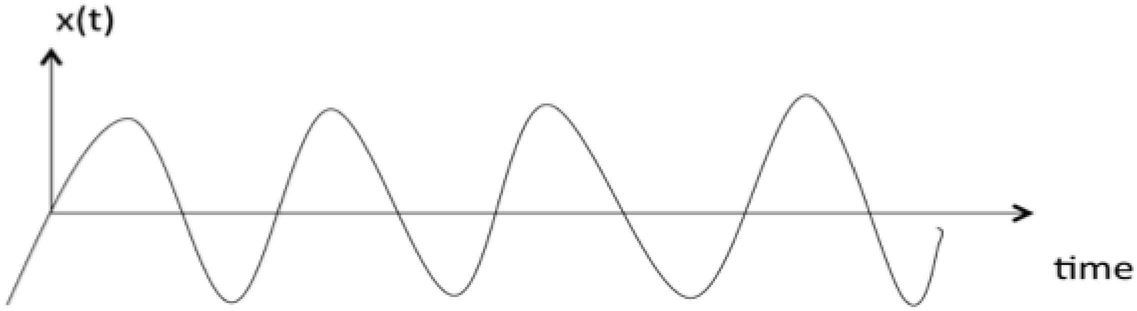



A signal is said to be discrete when it is defined at only discrete instants of time/

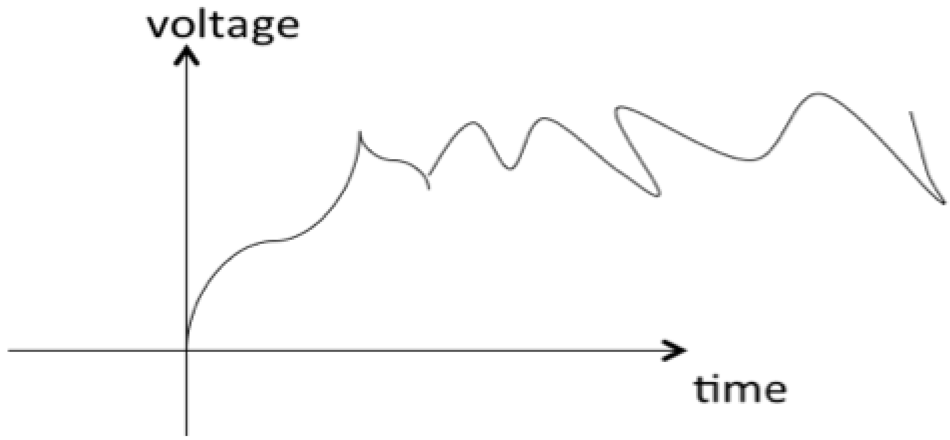



### Deterministic and Non-deterministic Signals

A signal is said to be deterministic if there is no uncertainty with respect to its value at any instant of time. Or, signals which can be defined exactly by a mathematical formula are known as deterministic signals.




A signal is said to be non-deterministic if there is uncertainty with respect to its value at some instant of time. Non-deterministic signals are random in nature hence they are called random signals. Random signals cannot be described by a mathematical equation. They are modelled in probabilistic terms.




## Even and Odd Signals

A signal is said to be even when it satisfies the condition x(t) = x(-t)

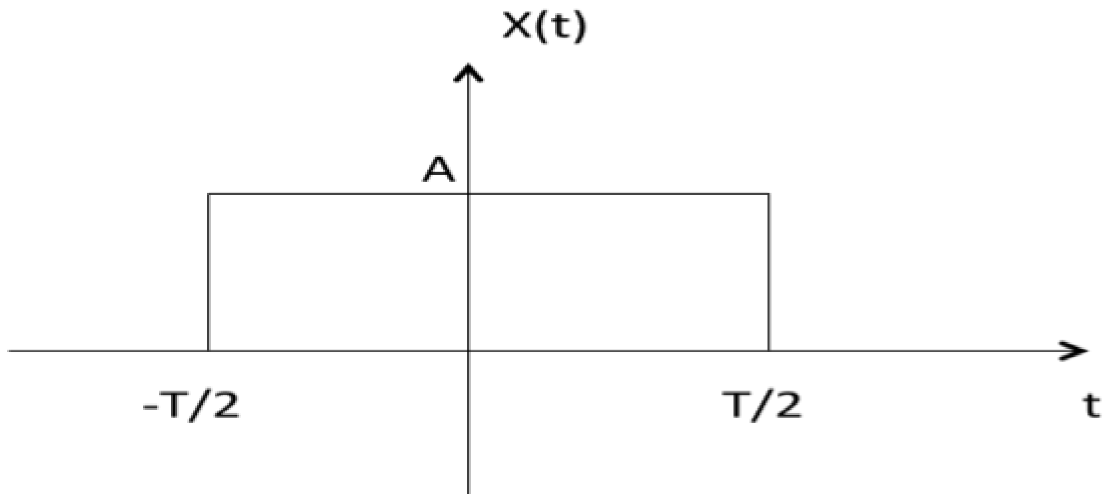
**Example 1:** $t^2$, $t^4$… cost etc.

Let $x(t) = t^2$

$x(-t) = (-t)^2 = t^2 = x(t)$

$\therefore t^2$ is even function

**Example 2:** As shown in the following diagram, rectangle function x(t) = x(-t) so it is also even function.




A signal is said to be odd when it satisfies the condition x(t) = -x(-t)

**Example:** t, $t^3$ ... And sin t

Let x(t) = sin t

x(-t) = sin(-t) = -sin t = -x(t)

∴ sin t is odd function.

Any function $f(t)$ can be expressed as the sum of its even function $f_e(t)$ and odd function $f_o(t)$.

$f(t) = f_e(t) + f_0(t)$

where

$f_e(t) = ½[f(t) + f(-t)]$

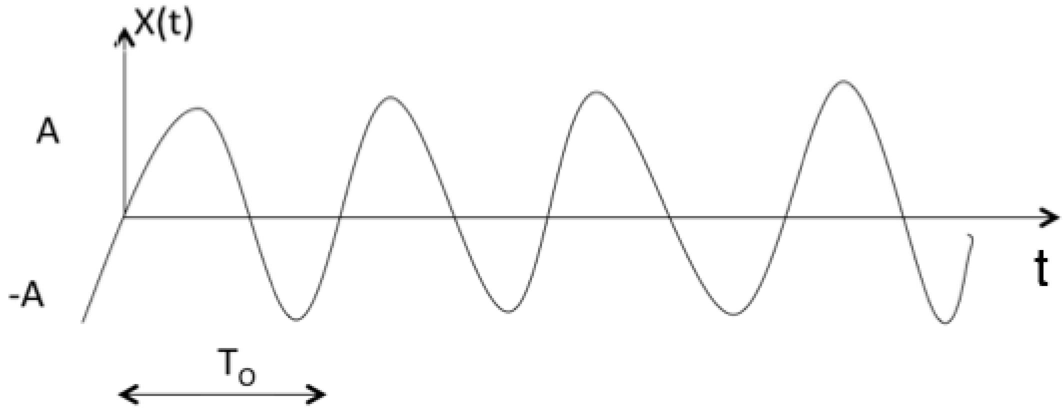
## Periodic and Aperiodic Signals

A signal is said to be periodic if it satisfies the condition x(t) = x(t + T) or x(n) = x(n + N).

Where

T = fundamental time period,

1/T = f = fundamental frequency.




The above signal will repeat for every time interval $T_0$ hence it is periodic with period $T_0$.

## Energy and Power Signals

A signal is said to be energy signal when it has finite energy.

$$\text{Energy } E = \int_{-\infty}^{\infty} x^2\,(t)dt$$

A signal is said to be power signal when it has finite power.

$$\text{Power } P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x^2(t)dt$$

NOTE:A signal cannot be both, energy and power simultaneously. Also, a signal may be neither energy nor power signal.

Power of energy signal = 0
Energy of power signal = ∞

## Real and Imaginary Signals

A signal is said to be real when it satisfies the condition x(t) = x*(t)
A signal is said to be odd when it satisfies the condition x(t) = -x*(t)
Example:

If x(t)= 3 then x*(t)=3*=3 here x(t) is a real signal.
If x(t)= 3j then x*(t)=3j* = -3j = -x(t) hence x(t) is a odd signal.

**Note:** For a real signal, imaginary part should be zero. Similarly for an imaginary signal, real part should be zero.

## Basic operations on Signals:

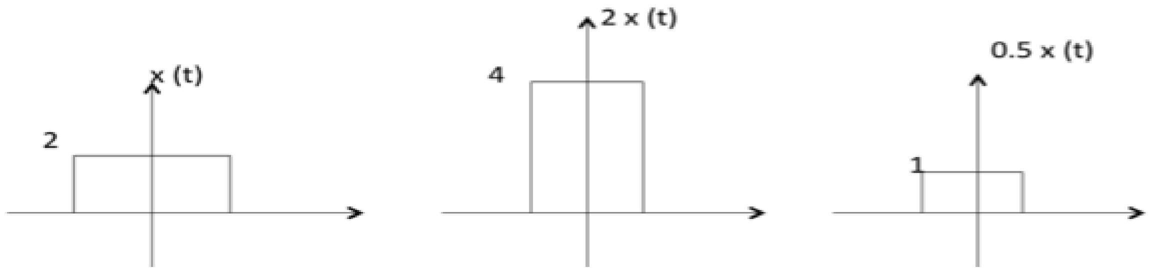
There are two variable parameters in general:

1. Amplitude
2. Time

**(1) The following operation can be performed with amplitude:**

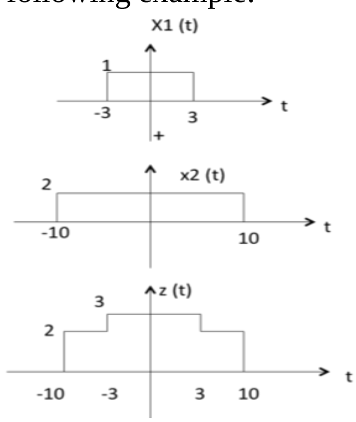
## Amplitude Scaling

C x(t) is a amplitude scaled version of x(t) whose amplitude is scaled by a factor C.




## Addition

Addition of two signals is nothing but addition of their corresponding amplitudes. This can be best explained by using the following example:




As seen from the previous diagram,

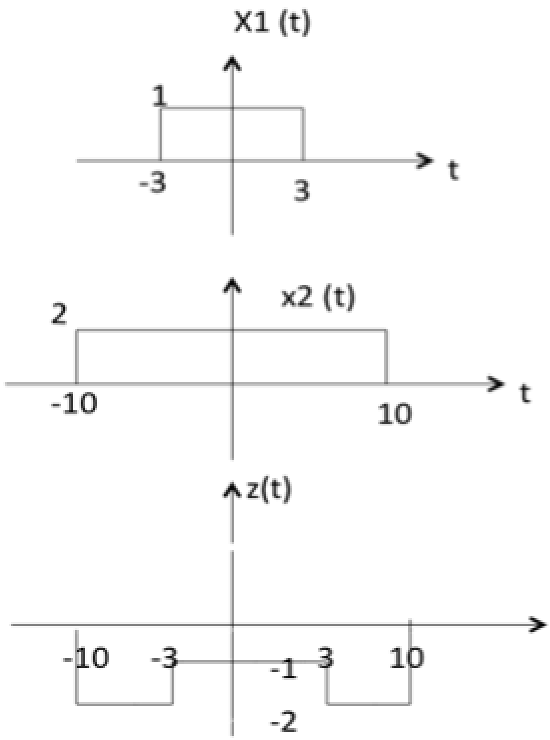
$$-10 < t < -3 \text{ amplitude of } z(t) = x_1(t) + x_2(t) = 0 + 2 = 2$$

$$-3 < t < 3 \text{ amplitude of } z(t) = x_1(t) + x_2(t) = 1 + 2 = 3$$

$$3 < t < 10 \text{ amplitude of } z(t) = x_1(t) + x_2(t) = 0 + 2 = 2$$

## Subtraction

subtraction of two signals is nothing but subtraction of their corresponding amplitudes. This can be best explained by the following example:




As seen from the diagram above,

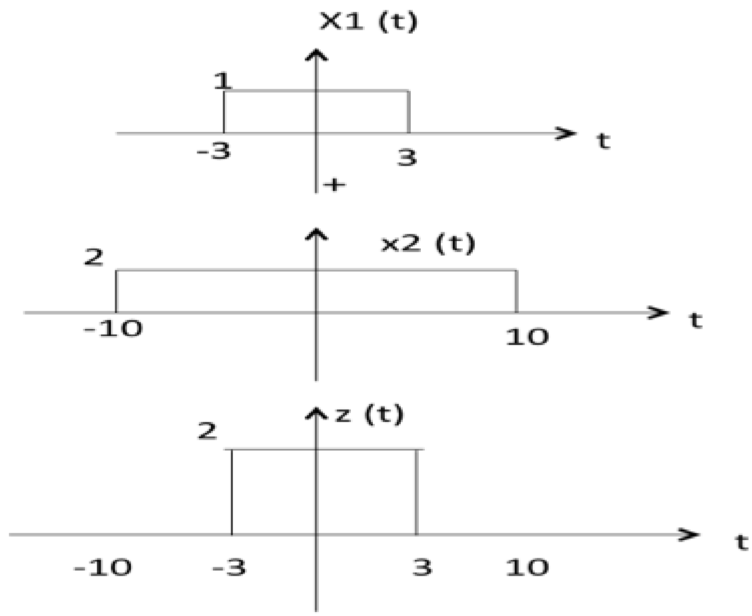
$-10 < t < -3$ amplitude of z (t) = $x_1(t) - x_2(t) = 0 - 2 = -2$

$-3 < t < 3$ amplitude of z (t) = $x_1(t) - x_2(t) = 1 - 2 = -1$

$3 < t < 10$ amplitude of z (t) = $x_1(t) - x_2(t) = 0 - 2 = -2$

## Multiplication

Multiplication of two signals is nothing but multiplication of their corresponding amplitudes. This can be best explained by the following example:




As seen from the diagram above,

$-10 < t < -3$ amplitude of z (t) = $x_1(t) \times x_2(t) = 0 \times 2 = 0$
$-3 < t < 3$ amplitude of z (t) = $x_1(t) - x_2(t) = 1 \times 2 = 2$
$3 < t < 10$ amplitude of z (t) = $x_1(t) - x_2(t) = 0 \times 2 = 0$

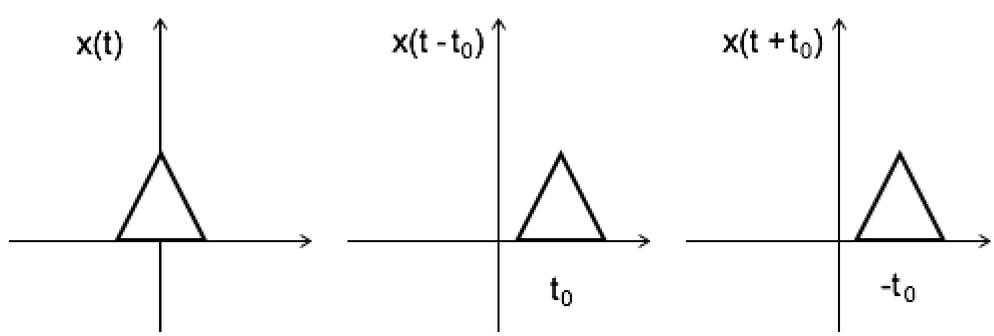
## (2)The following operations can be performed with time:

### Time Shifting

$x(t \pm t_0)$ is time shifted version of the signal x(t).

$x(t + t_0) \longrightarrow$negative shift

$x(t - t_0) \longrightarrow$positive shift

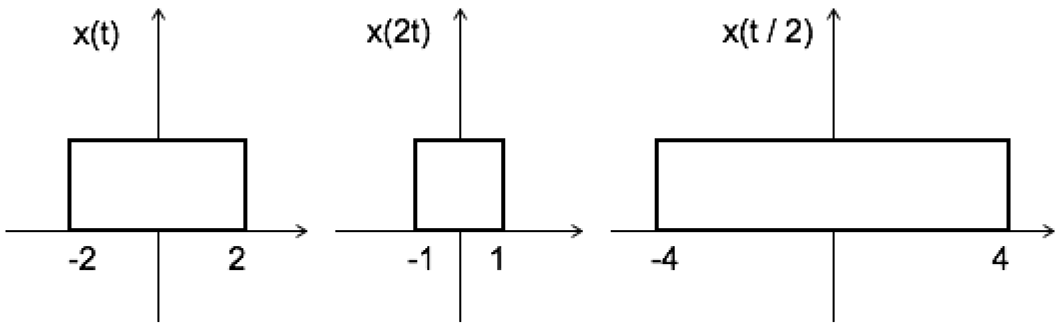



### Time Scaling

x(At) is time scaled version of the signal x(t). where A is always positive.

$|A| > 1 \longrightarrow$ Compression of the signal

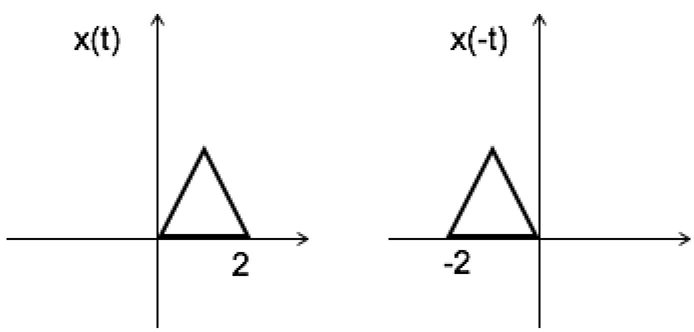
$|A| < 1 \longrightarrow$ Expansion of the signal




Note: u(at) = u(t) time scaling is not applicable for unit step function.

## Time Reversal

x(-t) is the time reversal of the signal x(t).




## Classification of Systems:

Systems are classified into the following categories:

- Liner and Non-liner Systems
- Time Variant and Time Invariant Systems
- Liner Time variant and Liner Time invariant systems
- Static and Dynamic Systems
- Causal and Non-causal Systems
- Invertible and Non-Invertible Systems
- Stable and Unstable Systems

## Linear and Non-linear Systems

A system is said to be linear when it satisfies superposition and homogenate principles. Consider two systems with inputs as $x_1(t)$, $x_2(t)$, and outputs as $y_1(t)$, $y_2(t)$ respectively. Then, according to the superposition and homogenate principles,

$$T\,[a_1\, x_1(t) + a_2\, x_2(t)] = a_1\, T[x_1(t)] + a_2\, T[x_2(t)]$$

$$\therefore\ T\,[a_1\, x_1(t) + a_2\, x_2(t)] = a_1\, y_1(t) + a_2\, y_2(t)$$

From the above expression, is clear that response of overall system is equal to response of individual system.

**Example:**

$y(t) = x^2(t)$

Solution:

$y_1\,(t) = T[x_1(t)] = x_1^2(t)$

$y_2\,(t) = T[x_2(t)] = x_2^2(t)$

$T\,[a_1\, x_1(t) + a_2\, x_2(t)] = [\, a_1\, x_1(t) + a_2\, x_2(t)]^2$

Which is not equal to $a_1\, y_1(t) + a_2\, y_2(t)$. Hence the system is said to be non linear.

## Time Variant and Time Invariant Systems

A system is said to be time variant if its input and output characteristics vary with time. Otherwise, the system is considered as time invariant.

The condition for time invariant system is:

$$y\,(n\,,\,t) = y(n-t)$$

The condition for time variant system is:

$$y\,(n\,,\,t) \neq y(n-t)$$

Where $y\,(n\,,\,t) = T[x(n-t)]$ = input change

$y\,(n-t)$ = output change

**Example:**

$y(n) = x(-n)$

$y(n, t) = T[x(n-t)] = x(-n-t)$

$y(n-t) = x(-(n-t)) = x(-n + t)$

$\therefore$ $y(n, t) \neq y(n-t)$. Hence, the system is time variant.

## Liner Time variant (LTV) and Liner Time Invariant (LTI) Systems

If a system is both liner and time variant, then it is called liner time variant (LTV) system.

If a system is both liner and time Invariant then that system is called liner time invariant (LTI) system.

## Static and Dynamic Systems

Static system is memory-less whereas dynamic system is a memory system.

**Example 1:** $y(t) = 2\,x(t)$

For present value t=0, the system output is $y(0) = 2x(0)$. Here, the output is only dependent upon present input. Hence the system is memory less or static.

**Example 2:** $y(t) = 2\,x(t) + 3\,x(t-3)$

For present value t=0, the system output is $y(0) = 2x(0) + 3x(-3)$.

Here $x(-3)$ is past value for the present input for which the system requires memory to get this output. Hence, the system is a dynamic system.

## Causal and Non-Causal Systems

A system is said to be causal if its output depends upon present and past inputs, and does not depend upon future input.

For non causal system, the output depends upon future inputs also.

**Example 1:** $y(n) = 2\,x(t) + 3\,x(t-3)$

For present value t=1, the system output is $y(1) = 2x(1) + 3x(-2)$.

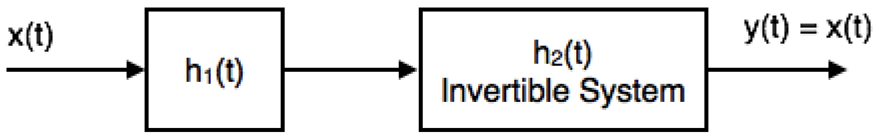
Here, the system output only depends upon present and past inputs. Hence, the system is causal.

**Example 2:** y(n) = 2 x(t) + 3 x(t-3) + 6x(t + 3)

For present value t=1, the system output is y(1) = 2x(1) + 3x(-2) + 6x(4) Here, the system output depends upon future input. Hence the system is non-causal system.

## Invertible and Non-Invertible systems

A system is said to invertible if the input of the system appears at the output.




$$Y(S) = X(S)\ H1(S)\ H2(S)$$

$$= X(S)\ H1(S) \cdot 1(H1(S))$$

*Since H2(S) = 1/( H1(S) )*

$$\therefore Y(S) = X(S)$$

$$\rightarrow y(t) = x(t)$$

Hence, the system is invertible.

If $y(t) \neq x(t)$, then the system is said to be non-invertible.

## Stable and Unstable Systems

The system is said to be stable only when the output is bounded for bounded input. For a bounded input, if the output is unbounded in the system then it is said to be unstable.

**Note:** For a bounded signal, amplitude is finite.

**Example 1:** y (t) = $x^2$(t)

Let the input is u(t) (unit step bounded input) then the output y(t) = u2(t) = u(t) = bounded output.

Hence, the system is stable.

**Example 2:** y (t) = $\int x(t)dt$

Let the input is u (t) (unit step bounded input) then the output y(t) = $\int u(t)dt$ = ramp signal (unbounded because amplitude of ramp is not finite it goes to infinite when t $\longrightarrow$ infinite).

Hence, the system is unstable.

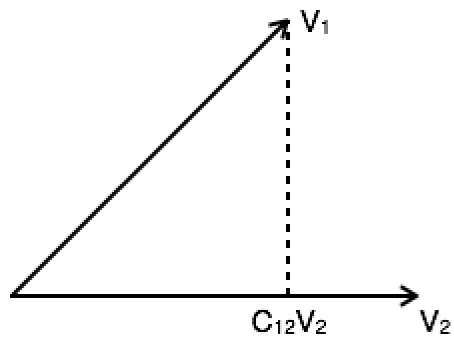
## Analogy Between Vectors and Signals:

There is a perfect analogy between vectors and signals.

### Vector

A vector contains magnitude and direction. The name of the vector is denoted by bold face type and their magnitude is denoted by light face type.

**Example:** V is a vector with magnitude V. Consider two vectors $V_1$ and $V_2$ as shown in the following diagram. Let the component of $V_1$ along with $V_2$ is given by $C_{12}V_2$. The component of a vector $V_1$ along with the vector $V_2$ can obtained by taking a perpendicular from the end of $V_1$ to the vector $V_2$ as shown in diagram:



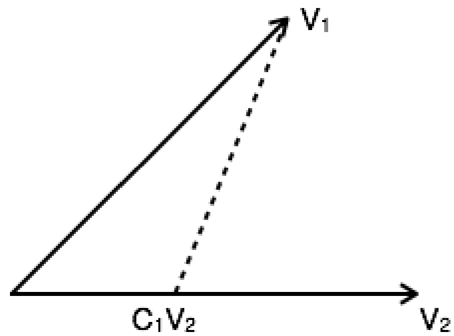

The vector $V_1$ can be expressed in terms of vector $V_2$

$$V_1 = C_{12}V_2 + V_e$$

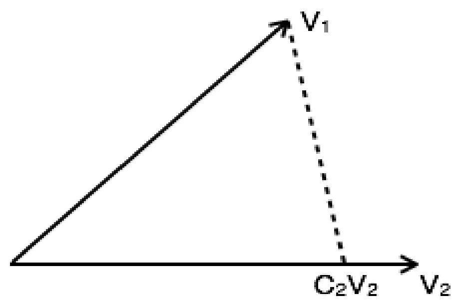
Where Ve is the error vector.

But this is not the only way of expressing vector $V_1$ in terms of $V_2$. The alternate possibilities are:

$V_1=C_1V_2+V_{e1}$




$V_2 = C_2 V_2 + V_{e2}$




The error signal is minimum for large component value. If $C_{12}=0$, then two signals are said to be orthogonal.

Dot Product of Two Vectors

$V_1 \, . \, V_2 = V_1.V_2 \cos\theta$

$\theta$ = Angle between V1 and V2

$V_1. \, V_2 = V_2.V_1$

From the diagram, components of $V_1$ a long $V_2 = C_{12} V_2$

$$\frac{V_1 \, . \, V_2}{V_2 = C_1 2 \, V_2}$$

$$\Rightarrow C_{12} = \frac{V_1 \, . \, V_2}{V_2}$$

**Signal**

The concept of orthogonality can be applied to signals. Let us consider two signals $f_1(t)$ and $f_2(t)$. Similar to vectors, you can approximate $f_1(t)$ in terms of $f_2(t)$ as

$f_1(t) = C_{12} f_2(t) + f_e(t)$ for $(t_1 < t < t_2)$

$\Rightarrow f_e(t) = f_1(t) - C_{12} f_2(t)$

One possible way of minimizing the error is integrating over the interval $t_1$ to $t_2$.

$$\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} [f_e(t)]dt$$

$$\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} [f_1(t) - C_{12} f_2(t)]dt$$

However, this step also does not reduce the error to appreciable extent. This can be corrected by taking the square of error function.

$$\varepsilon = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} [f_e(t)]^2 dt$$

$$\Rightarrow \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} [f_e(t) - C_{12} f_2]^2 dt$$

Where ε is the mean square value of error signal. The value of $C_{12}$ which minimizes the error, you need to calculate $d\varepsilon/dC12=0$

$$\Rightarrow \frac{d}{dC_{12}} [\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} [f_1(t) - C_{12} f_2(t)]^2 dt] = 0$$

$$\Rightarrow \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} [\frac{d}{dC_{12}} f_1^2(t) - \frac{d}{dC_{12}} 2 f_1(t) C_{12} f_2(t) + \frac{d}{dC_{12}} f_2^2(t) C_{12}^2] dt = 0$$

Derivative of the terms which do not have C12 term are zero.

$$\Rightarrow \int_{t_1}^{t_2} -2 f_1(t) f_2(t) dt + 2 C_{12} \int_{t_1}^{t_2} [f_2^2(t)] dt = 0$$

If $C_{12} = \frac{\int_{t_1}^{t_2} f_1(t) f_2(t) dt}{\int_{t_1}^{t_2} f_2^2(t) dt}$ component is zero, then two signals are said to be orthogonal.

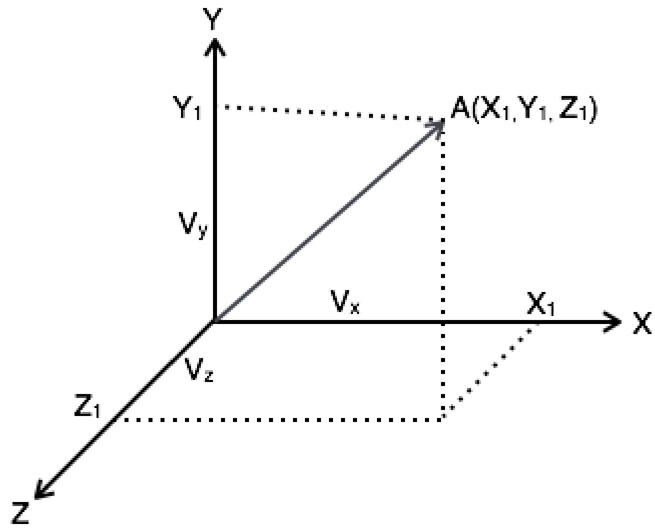
Put $C_{12} = 0$ to get condition for orthogonality.

$$0 = \frac{\int_{t_1}^{t_2} f_1(t) f_2(t) dt}{\int_{t_1}^{t_2} f_2^2(t) dt}$$

$$\int_{t_1}^{t_2} f_1(t) f_2(t) dt = 0$$

## Orthogonal Vector Space

A complete set of orthogonal vectors is referred to as orthogonal vector space. Consider a three dimensional vector space as shown below:




Consider a vector A at a point ($X_1$, $Y_1$, $Z_1$). Consider three unit vectors ($V_X$, $V_Y$, $V_Z$) in the direction of X, Y, Z axis respectively. Since these unit vectors are mutually orthogonal, it satisfies that

$$V_X . V_X = V_Y . V_Y = V_Z . V_Z = 1$$
$$V_X . V_Y = V_Y . V_Z = V_Z . V_X = 0$$

We can write above conditions as

$$V_a \,.\, V_b = \begin{cases} 1 & a = b \\ 0 & a \neq b \end{cases}$$

The vector A can be represented in terms of its components and unit vectors as

$$A = X_1 V_X + Y_1 V_Y + Z_1 V_Z \dots\dots\dots\dots\dots\dots (1)$$

Any vectors in this three dimensional space can be represented in terms of these three unit vectors only.

If you consider n dimensional space, then any vector A in that space can be represented as

$$A = X_1 V_X + Y_1 V_Y + Z_1 V_Z + \dots + N_1 V_N \dots\dots (2)$$

As the magnitude of unit vectors is unity for any vector A

The component of A along x axis = $A.V_X$
The component of A along Y axis = $A.V_Y$
The component of A along Z axis = $A.V_Z$

Similarly, for n dimensional space, the component of A along some G axis

$$= A.V_G \dots\dots\dots\dots\dots (3)$$

Substitute equation 2 in equation 3.

$$\Rightarrow CG = (X_1 V_X + Y_1 V_Y + Z_1 V_Z + \dots + G_1 V_G \dots + N_1 V_N) V_G$$

$$= X_1 V_X V_G + Y_1 V_Y V_G + Z_1 V_Z V_G + \dots + G_1 V_G V_G \dots + N_1 V_N V_G$$

$$= G_1 \quad \text{since } V_G V_G = 1$$

$$If V_G V_G \neq 1 \text{ i.e.} V_G V_G = k$$

$$A V_G = G_1 V_G V_G = G_1 K$$

$$G_1 = \frac{(A V_G)}{K}$$

## Orthogonal Signal Space

Let us consider a set of n mutually orthogonal functions $x_1(t)$, $x_2(t)$... $x_n(t)$ over the interval $t_1$ to $t_2$. As these functions are orthogonal to each other, any two signals $x_j(t)$, $x_k(t)$ have to satisfy the orthogonality condition. i.e.

$$\int_{t_1}^{t_2} x_j(t)x_k(t)dt = 0 \text{ where } j \neq k$$

$$\text{Let } \int_{t_1}^{t_2} x_k^2(t)dt = k_k$$

Let a function f(t), it can be approximated with this orthogonal signal space by adding the components along mutually orthogonal signals i.e.

$$f(t) = C_1x_1(t) + C_2x_2(t) + \ldots + C_nx_n(t) + f_e(t)$$

$$= \Sigma_{r=1}^{n} C_r x_r(t)$$

$$f(t) = f(t) - \Sigma_{r=1}^{n} C_r x_r(t)$$

Mean sqaure error $\varepsilon = \frac{1}{t_2 - t_2} \int_{t_1}^{t_2} [f_e(t)]^2 dt$

$$= \frac{1}{t_2 - t_2} \int_{t_1}^{t_2} [f[t] - \sum_{r=1}^{n} C_r x_r(t)]^2 dt$$

The component which minimizes the mean square error can be found by

$$\frac{d\varepsilon}{dC_1} = \frac{d\varepsilon}{dC_2} = \ldots = \frac{d\varepsilon}{dC_k} = 0$$

Let us consider $\frac{d\varepsilon}{dC_k} = 0$

$$\frac{d}{dC_k}[\frac{1}{t_2 - t_1}\int_{t_1}^{t_2}[f(t) - \Sigma_{r=1}^{n} C_r x_r(t)]^2 dt] = 0$$

All terms that do not contain $C_k$ is zero. i.e. in summation, r=k term remains and all other terms are zero.

$$\int_{t_1}^{t_2} -2f(t)x_k(t)dt + 2C_k\int_{t_1}^{t_2}[x_k^2(t)]dt = 0$$

$$\Rightarrow C_k = \frac{\int_{t_1}^{t_2} f(t)x_k(t)dt}{int_{t_1}^{t_2} x_k^2(t)dt}$$

$$\Rightarrow \int_{t_1}^{t_2} f(t)x_k(t)dt = C_k K_k$$

## Mean Square Error:

The average of square of error function $f_e(t)$ is called as mean square error. It is denoted by ε (epsilon).

$$\varepsilon = \frac{1}{t_2-t_1}\int_{t_1}^{t_2}[f_e(t)]^2 dt$$

$$= \frac{1}{t_2-t_1}\int_{t_1}^{t_2}[f_e(t) - \Sigma_{r=1}^{n} C_r x_r(t)]^2 dt$$

$$= \frac{1}{t_2-t_1}[\int_{t_1}^{t_2}[f_e^2(t)]dt + \Sigma_{r=1}^{n} C_r^2 \int_{t_1}^{t_2} x_r^2(t)dt - 2\Sigma_{r=1}^{n} C_r \int_{t_1}^{t_2} x_r(t)f(t)dt$$

You know that $C_r^2\int_{t_1}^{t_2} x_r^2(t)dt = C_r\int_{t_1}^{t_2} x_r(t)f(d)dt = C_r^2 K_r$

$$\varepsilon = \frac{1}{t_2-t_1}[\int_{t_1}^{t_2}[f^2(t)]dt + \Sigma_{r=1}^{n} C_r^2 K_r - 2\Sigma_{r=1}^{n} C_r^2 K_r]$$

$$= \frac{1}{t_2-t_1}[\int_{t_1}^{t_2}[f^2(t)]dt - \Sigma_{r=1}^{n} C_r^2 K_r]$$

$$\therefore \varepsilon = \frac{1}{t_2-t_1}[\int_{t_1}^{t_2}[f^2(t)]dt + (C_1^2 K_1 + C_2^2 K_2 + \ldots + C_n^2 K_n)]$$

The above equation is used to evaluate the mean square error.

## Closed and Complete Set of Orthogonal Functions:

Let us consider a set of n mutually orthogonal functions $x_1(t)$, $x_2(t)...x_n(t)$ over the interval $t_1$ to $t_2$. This is called as closed and complete set when there exist no function f(t) satisfying the condition

$$\int_{t_1}^{t_2} f(t)x_k(t)dt = 0$$

If this function is satisfying the equation

$$\int_{t_1}^{t_2} f(t)x_k(t)dt = 0$$

For $k$=1,2,.. then f(t) is said to be orthogonal to each and every function of orthogonal set. This set is incomplete without f(t). It becomes closed and complete set when f(t) is included.

f(t) can be approximated with this orthogonal set by adding the components along mutually orthogonal signals i.e.

$$f(t) = C_1x_1(t) + C_2x_2(t) + \ldots + C_nx_n(t) + f_e(t)$$

If the infinite series $C_1x_1(t) + C_2x_2(t) + \ldots + C_nx_n(t)$ converges to $ft$ then mean square error is zero.

## Orthogonality in Complex Functions:

If $f_1(t)$ and $f_2(t)$ are two complex functions, then $f_1(t)$ can be expressed in terms of $f_2(t)$ as

$f_1(t) = C_{12}f_2(t)$.. with negligible error

$$\text{Where } C_{12} = \frac{\int_{t_1}^{t_2} f_1(t)f_2^*(t)dt}{\int_{t_1}^{t_2} |f_2(t)|^2 dt}$$

Where $f_2^*(t)$ is the complex conjugate of $f_2(t)$

If $f_1(t)$ and $f_2(t)$ are orthogonal then $C_{12} = 0$

$$\frac{\int_{t_1}^{t_2} f_1(t)f_2^*(t)dt}{\int_{t_1}^{t_2} |f_2(t)|^2 dt} = 0$$

$$\Rightarrow \int_{t_1}^{t_2} f_1(t)f_2^*(dt) = 0$$

The above equation represents orthogonality condition in complex functions.

## Fourier series:

To represent any periodic signal x(t), Fourier developed an expression called Fourier series. This is in terms of an infinite sum of sines and cosines or exponentials. Fourier series uses orthoganality condition.

**Jean Baptiste Joseph Fourier,**a French mathematician and a physicist; was born in Auxerre, France. He initialized Fourier series, Fourier transforms and their applications to problems of heat transfer and vibrations. The Fourier series, Fourier transforms and Fourier's Law are named in his honour.

### Fourier Series Representation of Continuous Time Periodic Signals

A signal is said to be periodic if it satisfies the condition x (t) = x (t + T) or x (n) = x (n + N).

Where T = fundamental time period,

$\omega_0$= fundamental frequency = $2\pi/T$

There are two basic periodic signals:

$x(t)=\cos\omega_0 t$(sinusoidal) &

$x(t)=e^{j\omega_0 t}$(complex exponential)

These two signals are periodic with period $T=2\pi/\omega_0$

. A set of harmonically related complex exponentials can be represented as {$\phi_k(t)$}

$$\phi_k(t) = \{e^{jk\omega_0 t}\} = \{e^{jk(\frac{2\pi}{T})t}\}\text{where}\, k = 0 \pm 1, \pm 2..n \;\; .....(1)$$

All these signals are periodic with period T

According to orthogonal signal space approximation of a function x (t) with n, mutually orthogonal functions is given by

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \ldots\ldots (2)$$

$$= \sum_{k=-\infty}^{\infty} a_k k e^{jk\omega_0 t}$$

Where $a_k$ = Fourier coefficient = coefficient of approximation.

This signal x(t) is also periodic with period T.

Equation 2 represents Fourier series representation of periodic signal x(t).

The term k = 0 is constant.

The term $k=\pm 1$ having fundamental frequency $\omega_0$ , is called as $1^{st}$ harmonics.

The term $k=\pm 2$ having fundamental frequency $2\omega_0$ , is called as $2^{nd}$ harmonics, and so on...

The term $k=\pm n$ having fundamental frequency $n\omega_0$, is called as $n^{th}$ harmonics.

### Deriving Fourier Coefficient

We know that

$$x(t) = \Sigma_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \ldots\ldots (1)$$

Multiply $e^{-jn\omega_0 t}$ on both sides. Then

$$x(t)e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} . e^{-jn\omega_0 t}$$

Consider integral on both sides.

$$\int_0^T x(t)e^{jk\omega_0 t}\,dt = \int_0^T \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}\,.\,e^{-jn\omega_0 t}\,dt$$

$$= \int_0^T \sum_{k=-\infty}^{\infty} a_k e^{j(k-n)\omega_0 t}\,.\,dt$$

$$\int_0^T x(t)e^{jk\omega_0 t}\,dt = \sum_{k=-\infty}^{\infty} a_k \int_0^T e^{j(k-n)\omega_0 t}\,dt.\ \ldots\ldots(2)$$

by Euler's formula,

$$\int_0^T e^{j(k-n)\omega_0 t}dt. = \int_0^T \cos(k-n)\omega_0 dt + j\int_0^T \sin(k-n)\omega_0 t\,dt$$

$$\int_0^T e^{j(k-n)\omega_0 t}\,dt. = \begin{cases} T & k=n \\ 0 & k\neq n \end{cases}$$

Hence in equation 2, the integral is zero for all values of k except at k = n. Put k = n in equation 2.

$$\Rightarrow \int_0^T x(t)e^{-jn\omega_0 t}\,dt = a_n T$$

$$\Rightarrow a_n = \frac{1}{T}\int_0^T e^{-jn\omega_0 t}\,dt$$

Replace n by k

$$\Rightarrow a_k = \frac{1}{T}\int_0^T e^{-jk\omega_0 t}\,dt$$

$$\therefore x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j(k-n)\omega_0 t}$$

$$\text{where} a_k = \frac{1}{T}\int_0^T e^{-jk\omega_0 t}\,dt$$

**Properties of Fourier series:**

**Linearity Property**

If $x(t) \xleftrightarrow{fourier\ series\ \ coefficient} f_{xn}$ & $y(t) \xleftrightarrow{fourier\ series\ \ coefficient} f_{yn}$

then linearity property states that

$\mathrm{a}\,x(t) + \mathrm{b}\,y(t) \xleftrightarrow{fourier\ series\ \ coefficient} \mathrm{a}\,f_{xn} + \mathrm{b}\,f_{yn}$

**Time Shifting Property**

If $x(t) \xleftrightarrow{fourier\ series\ \ coefficient} f_{xn}$

then time shifting property states that

$x(t-t_0) \xleftrightarrow{fourier\ series\ \ coefficient} e^{-jn\omega_0 t_0}\, f_{xn}$

**Frequency Shifting Property**

If $x(t) \xleftrightarrow{fourier\ series\quad coefficient} f_{xn}$

then frequency shifting property states that

$e^{jn\omega_0 t_0} \,.\, x(t) \xleftrightarrow{fourier\ series\quad coefficient} f_{x(n-n_0)}$

**Time Reversal Property**

If $x(t) \xleftrightarrow{fourier\ series\quad coefficient} f_{xn}$

then time reversal property states that

If $x(-t) \xleftrightarrow{fourier\ series\quad coefficient} f_{-xn}$

**Time Scaling Property**

If $x(t) \xleftrightarrow{fourier\ series\quad coefficient} f_{xn}$

then time scaling property states that

If $x(at) \xleftrightarrow{fourier\ series\quad coefficient} f_{xn}$

Time scaling property changes frequency components from $\omega_0$ to $a\omega_0$.

**Differentiation and Integration Properties**

If $x(t) \xleftrightarrow{fourier\ series\quad coefficient} f_{xn}$

then differentiation property states that

If $\frac{dx(t)}{dt} \xleftrightarrow{fourier\ series\quad coefficient} jn\omega_0 . f_{xn}$

& integration property states that

If $\int x(t)dt \xleftrightarrow{fourier\ series\quad coefficient} \frac{f_{xn}}{jn\omega_0}$

**Multiplication and Convolution Properties**

f $x(t) \xleftrightarrow{fourier\ series\quad coefficient} f_{xn}$ & $y(t) \xleftrightarrow{fourier\ series\quad coefficient} f_{yn}$

:hen multiplication property states that

$x(t). y(t) \xleftrightarrow{fourier\ series\quad coefficient} T f_{xn} * f_{yn}$

& convolution property states that

$x(t) * y(t) \xleftrightarrow{fourier\ series\quad coefficient} T f_{xn} . f_{yn}$

**Conjugate and Conjugate Symmetry Properties**

If $x(t) \xleftrightarrow{fourier\ series\quad coefficient} f_{xn}$

Then conjugate property states that

$x * (t) \xleftrightarrow{fourier\ series\quad coefficient} f *_{xn}$

Conjugate symmetry property for real valued time signal states that

$$f *_{xn} = f_{-xn}$$

& Conjugate symmetry property for imaginary valued time signal states that

$$f *_{xn} = -f_{-xn}$$

## Trigonometric Fourier Series (TFS)

$\sin n\omega_0 t$ and $\sin m\omega_0 t$ are orthogonal over the interval $(t_0, t_0+2\pi\omega_0)$. So $\sin\omega_0 t, \sin 2\omega_0 t$ forms an orthogonal set. This set is not complete without $\{\cos n\omega_0 t\}$ because this cosine set is also orthogonal to sine set. So to complete this set we must include both cosine and sine terms. Now the complete orthogonal set contains all cosine and sine terms i.e. $\{\sin n\omega_0 t, \cos n\omega_0 t\}$ where n=0, 1, 2...

$\therefore$ Any function $xt$ in the interval $(t_0, t_0 + \frac{2\pi}{\omega_0})$ can be represented as

$$\begin{aligned} x(t) &= a_0 \cos 0\omega_0 t + a_1 \cos 1\omega_0 t + a_2 \cos 2\omega_0 t + \ldots + a_n \cos n\omega_0 t + \ldots \\ &\quad + b_0 \sin 0\omega_0 t + b_1 \sin 1\omega_0 t + \ldots + b_n \sin n\omega_0 t + \ldots \\ &= a_0 + a_1 \cos 1\omega_0 t + a_2 \cos 2\omega_0 t + \ldots + a_n \cos n\omega_0 t + \ldots \\ &\quad + b_1 \sin 1\omega_0 t + \ldots + b_n \sin n\omega_0 t + \ldots \end{aligned}$$

$$\boxed{\therefore x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t) \quad (t_0 < t < t_0 + T)}$$

The above equation represents trigonometric Fourier series representation of x(t).

$$\text{Where } a_0 = \frac{\int_{t_0}^{t_0+T} x(t) \cdot 1 dt}{\int_{t_0}^{t_0+T} 1^2 dt} = \frac{1}{T} \cdot \int_{t_0}^{t_0+T} x(t) dt$$

$$a_n = \frac{\int_{t_0}^{t_0+T} x(t) \cdot \cos n\omega_0 t\, dt}{\int_{t_0}^{t_0+T} \cos^2 n\omega_0 t\, dt}$$

$$b_n = \frac{\int_{t_0}^{t_0+T} x(t) \cdot \sin n\omega_0 t\, dt}{\int_{t_0}^{t_0+T} \sin^2 n\omega_0 t\, dt}$$

$$\text{Here } \int_{t_0}^{t_0+T} \cos^2 n\omega_0 t\, dt = \int_{t_0}^{t_0+T} \sin^2 n\omega_0 t\, dt = \frac{T}{2}$$

$$\therefore a_n = \frac{2}{T} \cdot \int_{t_0}^{t_0+T} x(t) \cdot \cos n\omega_0 t\, dt$$

$$b_n = \frac{2}{T} \cdot \int_{t_0}^{t_0+T} x(t) \cdot \sin n\omega_0 t\, dt$$

## Exponential Fourier Series (EFS):

Consider a set of complex exponential functions $\left\{e^{jn\omega_0 t}\right\} (n = 0, \pm 1, \pm 2...)$ which is orthogonal over the interval ($t_0$,$t_0$+$T$). Where $T$=2$\pi$/$\omega_0$ . This is a complete set so it is possible to represent any function f(t) as shown below

$$f(t) = F_0 + F_1 e^{j\omega_0 t} + F_2 e^{j2\omega_0 t} + \ldots + F_n e^{jn\omega_0 t} + \ldots$$

$$F_{-1} e^{-j\omega_0 t} + F_{-2} e^{-j2\omega_0 t} + \ldots + F_{-n} e^{-jn\omega_0 t} + \ldots$$

$$\therefore f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} \qquad (t_0 < t < t_0 + T) \ldots\ldots(1)$$

Equation 1 represents exponential Fourier series representation of a signal f(t) over the interval ($t_0$, $t_0$+T). The Fourier coefficient is given as

$$F_n = \frac{\int_{t_0}^{t_0+T} f(t)(e^{jn\omega_0 t})^* dt}{\int_{t_0}^{t_0+T} e^{jn\omega_0 t}(e^{jn\omega_0 t})^* dt}$$

$$= \frac{\int_{t_0}^{t_0+T} f(t) e^{-jn\omega_0 t} dt}{\int_{t_0}^{t_0+T} e^{-jn\omega_0 t} e^{jn\omega_0 t} dt}$$

$$= \frac{\int_{t_0}^{t_0+T} f(t) e^{-jn\omega_0 t} dt}{\int_{t_0}^{t_0+T} 1\, dt} = \frac{1}{T} \int_{t_0}^{t_0+T} f(t) e^{-jn\omega_0 t} dt$$

$$\therefore F_n = \frac{1}{T} \int_{t_0}^{t_0+T} f(t) e^{-jn\omega_0 t} dt$$

## Relation Between Trigonometric and Exponential Fourier Series:

Consider a periodic signal x(t), the TFS & EFS representations are given below respectively

$$x(t) = a_0 + \Sigma_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)\ldots\ldots(1)$$

$$x(t) = \Sigma_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$$

$$= F_0 + F_1 e^{j\omega_0 t} + F_2 e^{j2\omega_0 t} + \ldots + F_n e^{jn\omega_0 t} + \ldots$$

$$F_{-1} e^{-j\omega_0 t} + F_{-2} e^{-j2\omega_0 t} + \ldots + F_{-n} e^{-jn\omega_0 t} + \ldots$$

$$= F_0 + F_1(\cos\omega_0 t + j\sin\omega_0 t) + F_2(cos2\omega_0 t + j\sin 2\omega_0 t) + \ldots + F_n(\cos n\omega_0 t + j\sin n\omega_0 t) + \ldots$$

$$\ldots + F_{-1}(\cos\omega_0 t - j\sin\omega_0 t) + F_{-2}(\cos 2\omega_0 t - j\sin 2\omega_0 t) + \ldots + F_{-n}(\cos n\omega_0 t - j\sin n\omega_0 t) + \ldots$$

$$= F_0 + (F_1 + F_{-1})\cos\omega_0 t + (F_2 + F_{-2})\cos 2\omega_0 t + \ldots + j(F_1 - F_{-1})\sin\omega_0 t + j(F_2 - F_{-2})\sin 2\omega_0 t + \ldots$$

$$\therefore x(t) = F_0 + \Sigma_{n=1}^{\infty}((F_n + F_{-n})\cos n\omega_0 t + j(F_n - F_{-n})\sin n\omega_0 t)\ldots\ldots(2)$$

Compare equation 1 and 2.

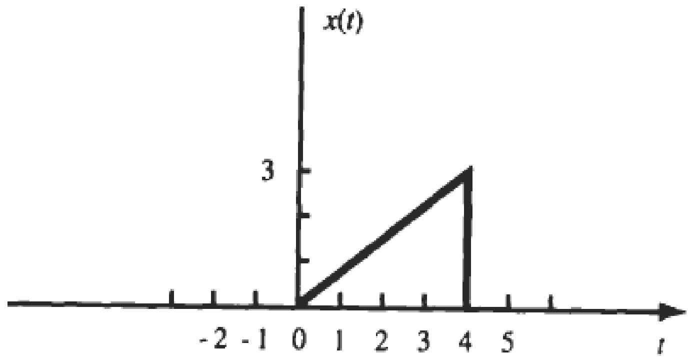
$$a_0 = F_0$$
$$a_n = F_n + F_{-n}$$
$$b_n = j(F_n - F_{-n})$$

Similarly,

$$F_n = \frac{1}{2}(a_n - jb_n)$$
$$F_{-n} = \frac{1}{2}(a_n + jb_n)$$

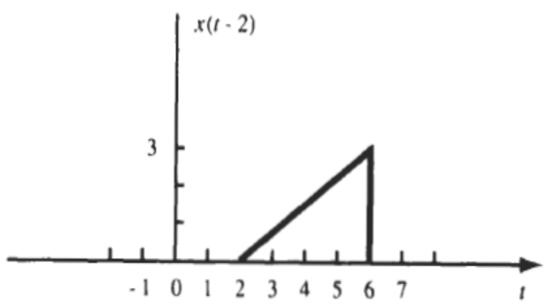
# Problems

1. **A continuous-time signal *x ( t )* is shown in the following figure. Sketch and label each of the following signals.**

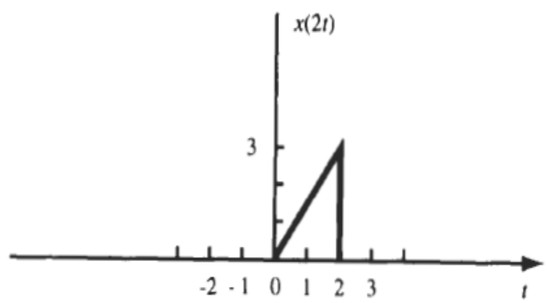



***( a ) x(t - 2); ( b ) x(2t); ( c ) x(t/2); (d) x ( - t )***

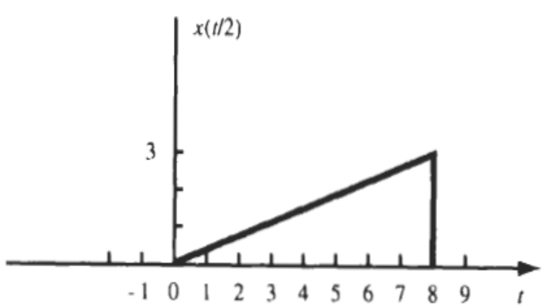
***Sol:***



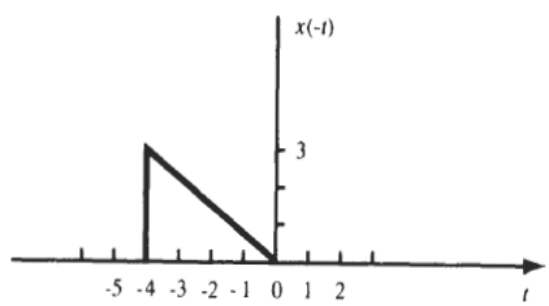

(*a*)




(*b*)




(*c*)




(*d*)

**2. Determine whether the following signals are energy signals, power signals, or neither.**

(a) $x(t) = e^{-at}u(t), \quad a > 0$ (b) $x(t) = A\cos(\omega_0 t + \theta)$

(c) $x(t) = tu(t)$ (d) $x[n] = (-0.5)^n u[n]$

(e) $x[n] = u[n]$ (f) $x[n] = 2e^{j3n}$

(a) $E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_0^{\infty} e^{-2at} dt = \frac{1}{2a} < \infty$

Thus, $x(t)$ is an energy signal.

(b) The sinusoidal signal $x(t)$ is periodic with $T_0 = 2\pi/\omega_0$. Then by the result from Prob. 1.18, the average power of $x(t)$ is

$$P = \frac{1}{T_0}\int_0^{T_0}[x(t)]^2 dt = \frac{\omega_0}{2\pi}\int_0^{2\pi/\omega_0} A^2\cos^2(\omega_0 t + \theta)\, dt$$

$$= \frac{A^2\omega_0}{2\pi}\int_0^{2\pi/\omega_0}\frac{1}{2}[1 + \cos(2\omega_0 t + 2\theta)]\, dt = \frac{A^2}{2} < \infty$$

Thus, $x(t)$ is a power signal. Note that periodic signals are, in general, power signals.

(c) $E = \lim_{T\to\infty}\int_{-T/2}^{T/2}|x(t)|^2 dt = \lim_{T\to\infty}\int_0^{T/2} t^2\, dt = \lim_{T\to\infty}\frac{(T/2)^3}{3} = \infty$

$P = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}|x(t)|^2 dt = \lim_{T\to\infty}\frac{1}{T}\int_0^{T/2} t^2\, dt = \lim_{T\to\infty}\frac{1}{T}\frac{(T/2)^3}{3} = \infty$

Thus, $x(t)$ is neither an energy signal nor a power signal.

(d) we know that energy of a signal is

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

And by using

$$\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha} \qquad |\alpha| < 1$$

we obtain

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \sum_{n=0}^{\infty} 0.25^n = \frac{1}{1-0.25} = \frac{4}{3} < \infty$$

Thus, ***x [ n ]*** is a power signal.

(e) By the definition of power of signal

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{N} 1^2 = \lim_{N\to\infty} \frac{1}{2N+1}(N+1) = \frac{1}{2} < \infty$$

Thus, $x[n]$ is a power signal.

(*f*) Since $|x[n]| = |2e^{j3n}| = 2|e^{j3n}| = 2$,

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} 2^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} 4(2N+1) = 4 < \infty$$

Thus, $x[n]$ is a power signal.

3. **Determine whether or not each of the following signals is periodic. If a signal is periodic, determine its fundamental period.**

(*a*) $x(t) = \cos\left(t + \frac{\pi}{4}\right)$ (*b*) $x(t) = \sin\frac{2\pi}{3}t$

(*c*) $x(t) = \cos\frac{\pi}{3}t + \sin\frac{\pi}{4}t$ (*d*) $x(t) = \cos t + \sin\sqrt{2}\,t$

(*e*) $x(t) = \sin^2 t$ (*f*) $x(t) = e^{j[(\pi/2)t - 1]}$

(*g*) $x[n] = e^{j(\pi/4)n}$ (*h*) $x[n] = \cos\frac{1}{4}n$

(*i*) $x[n] = \cos\frac{\pi}{3}n + \sin\frac{\pi}{4}n$ (*j*) $x[n] = \cos^2\frac{\pi}{8}n$

**Sol:**

(*a*) $x(t) = \cos\left(t + \frac{\pi}{4}\right) = \cos\left(\omega_0 t + \frac{\pi}{4}\right) \longrightarrow \omega_0 = 1$

$x(t)$ is periodic with fundamental period $T_0 = 2\pi/\omega_0 = 2\pi$.

(*b*) $x(t) = \sin\frac{2\pi}{3}t \longrightarrow \omega_0 = \frac{2\pi}{3}$

$x(t)$ is periodic with fundamental period $T_0 = 2\pi/\omega_0 = 3$.

(*c*) $x(t) = \cos\frac{\pi}{3}t + \sin\frac{\pi}{4}t = x_1(t) + x_2(t)$

where $x_1(t) = \cos(\pi/3)t = \cos\omega_1 t$ is periodic with $T_1 = 2\pi/\omega_1 = 6$ and $x_2(t) = \sin(\pi/4)t = \sin\omega_2 t$ is periodic with $T_2 = 2\pi/\omega_2 = 8$. Since $T_1/T_2 = \frac{6}{8} = \frac{3}{4}$ is a rational number, $x(t)$ is periodic with fundamental period $T_0 = 4T_1 = 3T_2 = 24$.

(*d*) $x(t) = \cos t + \sin\sqrt{2}\,t = x_1(t) + x_2(t)$

where $x_1(t) = \cos t = \cos \omega_1 t$ is periodic with $T_1 = 2\pi/\omega_1 = 2\pi$ and $x_2(t) = \sin\sqrt{2}\,t = \sin \omega_2 t$ is periodic with $T_2 = 2\pi/\omega_2 = \sqrt{2}\,\pi$. Since $T_1/T_2 = \sqrt{2}$ is an irrational number, $x(t)$ is nonperiodic.

(*e*) Using the trigonometric identity $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$, we can write

$$x(t) = \sin^2 t = \tfrac{1}{2} - \tfrac{1}{2}\cos 2t = x_1(t) + x_2(t)$$

where $x_1(t) = \frac{1}{2}$ is a dc signal with an arbitrary period and $x_2(t) = -\frac{1}{2}\cos 2t = -\frac{1}{2}\cos\omega_2 t$ is periodic with $T_2 = 2\pi/\omega_2 = \pi$. Thus, $x(t)$ is periodic with fundamental period $T_0 = \pi$.

(*f*) $x(t) = e^{j[(\pi/2)t-1]} = e^{-j}e^{j(\pi/2)t} = e^{-j}e^{j\omega_0 t} \longrightarrow \omega_0 = \dfrac{\pi}{2}$

$x(t)$ is periodic with fundamental period $T_0 = 2\pi/\omega_0 = 4$.

(*g*) $x[n] = e^{j(\pi/4)n} = e^{j\Omega_0 n} \longrightarrow \Omega_0 = \dfrac{\pi}{4}$

Since $\Omega_0/2\pi = \frac{1}{8}$ is a rational number, $x[n]$ is periodic, and by Eq. (*1.55*) the fundamental period is $N_0 = 8$.

(*h*) $x[n] = \cos\frac{1}{4}n = \cos\Omega_0 n \longrightarrow \Omega_0 = \frac{1}{4}$

Since $\Omega_0/2\pi = 1/8\pi$ is not a rational number, $x[n]$ is nonperiodic.

(*i*) $x[n] = \cos\dfrac{\pi}{3}n + \sin\dfrac{\pi}{4}n = x_1[n] + x_2[n]$

where

$$x_1[n] = \cos\frac{\pi}{3}n = \cos\Omega_1 n \longrightarrow \Omega_1 = \frac{\pi}{3}$$

$$x_2[n] = \sin\frac{\pi}{4}n = \cos\Omega_2 n \longrightarrow \Omega_2 = \frac{\pi}{4}$$

Since $\Omega_1/2\pi = \frac{1}{6}$ (= rational number), $x_1[n]$ is periodic with fundamental period $N_1 = 6$, and since $\Omega_2/2\pi = \frac{1}{8}$ (= rational number), $x_2[n]$ is periodic with fundamental period $N_2 = 8$. Thus, from the result of Prob. 1.15, $x[n]$ is periodic and its fundamental period is given by the least common multiple of 6 and 8, that is, $N_0 = 24$.

(*j*) Using the trigonometric identity $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$, we can write

$$x[n] = \cos^2\frac{\pi}{8}n = \frac{1}{2} + \frac{1}{2}\cos\frac{\pi}{4}n = x_1[n] + x_2[n]$$

where $x_1[n] = \frac{1}{2} = \frac{1}{2}(1)^n$ is periodic with fundamental period $N_1 = 1$ and $x_2[n] = \frac{1}{2}\cos(\pi/4)n = \frac{1}{2}\cos\Omega_2 n \longrightarrow \Omega_2 = \pi/4$. Since $\Omega_2/2\pi = \frac{1}{8}$ (= rational number), $x_2[n]$ is periodic with fundamental period $N_2 = 8$. Thus, $x[n]$ is periodic with fundamental period $N_0 = 8$ (the least common multiple of $N_1$ and $N_2$).

**5. Determine the even and odd components of the following signals:**

(a) $x(t) = u(t)$

(b) $x(t) = \sin\left(\omega_0 t + \dfrac{\pi}{4}\right)$

(c) $x[n] = e^{j(\Omega_0 n + \pi/2)}$

(d) $x[n] = \delta[n]$

*Ans.* (a) $x_e(t) = \frac{1}{2}, x_o(t) = \frac{1}{2}\operatorname{sgn} t$

(b) $x_e(t) = \dfrac{1}{\sqrt{2}}\cos\omega_0 t, x_o(t) = \dfrac{1}{\sqrt{2}}\sin\omega_0 t$

(c) $x_e[n] = j\cos\Omega_0 n, x_o[n] = -\sin\Omega_0 n$

(d) $x_e[n] = \delta[n], x_o[n] = 0$

# UNIT – II

# CONTINUOUS TIME FOURIER TRANSFORM

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# CONTINUOUS TIME FOURIER TRANSFORM

## INTRODUCTION:

The main drawback of Fourier series is, it is only applicable to periodic signals. There are some naturally produced signals such as nonperiodic or aperiodic, which we cannot represent using Fourier series. To overcome this shortcoming, Fourier developed a mathematical model to transform signals between time (or spatial) domain to frequency domain & vice versa, which is called 'Fourier transform'.

Fourier transform has many applications in physics and engineering such as analysis of LTI systems, RADAR, astronomy, signal processing etc.

## Deriving Fourier transform from Fourier series:

Consider a periodic signal f(t) with period T. The complex Fourier series representation of f(t) is given as

$$f(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$= \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T_0}kt} \ldots\ldots(1)$$

Let $\frac{1}{T_0} = \Delta f$, then equation 1 becomes

$$f(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k\Delta ft} \ldots\ldots(2)$$

but you know that

$$a_k = \frac{1}{T_0} \int_{t_0}^{t_0+T} f(t) e^{-jk\omega_0 t}\, dt$$

Substitute in equation 2.

$$2 \Rightarrow f(t) = \Sigma_{k=-\infty}^{\infty} \frac{1}{T_0} \int_{t_0}^{t_0+T} f(t)e^{-jk\omega_0 t}\, dt\, e^{j2\pi k\Delta ft}$$

Let $t_0 = \frac{T}{2}$

$$= \Sigma_{k=-\infty}^{\infty} [\int_{\frac{-T}{2}}^{\frac{T}{2}} f(t)e^{-j2\pi k\Delta ft}\, dt]\, e^{j2\pi k\Delta ft} . \Delta f$$

In the limit as T→∞,Δf approaches differential df, kΔf becomes a continuous variable f, and summation becomes integration

$$f(t) = lim_{T\to\infty} \left\{ \Sigma_{k=-\infty}^{\infty} [\int_{\frac{-T}{2}}^{\frac{T}{2}} f(t)e^{-j2\pi k\Delta ft}\, dt]\, e^{j2\pi k\Delta ft} . \Delta f \right\}$$

$$= \int_{-\infty}^{\infty} [\int_{-\infty}^{\infty} f(t)e^{-j2\pi ft}\, dt] e^{j2\pi ft}\, df$$

$$f(t) = \int_{-\infty}^{\infty} F[\omega]e^{j\omega t}\, d\omega$$

$$\text{Where } F[\omega] = [\int_{-\infty}^{\infty} f(t)e^{-j2\pi ft}\, dt]$$

Fourier transform of a signal

$$f(t) = F[\omega] = [\int_{-\infty}^{\infty} f(t)e^{-j\omega t}\, dt]$$

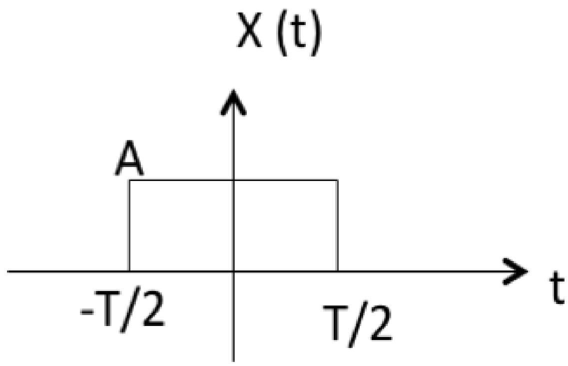
Inverse Fourier Transform is

$$f(t) = \int_{-\infty}^{\infty} F[\omega]e^{j\omega t}\, d\omega$$

## Fourier Transform of Basic Functions:

Let us go through Fourier Transform of basic functions:

**FT of GATE Function**




$$\boxed{F[\omega] = ATSa(\frac{\omega T}{2})}$$

**FT of Impulse Function:**

$$FT[\omega(t)] = [\int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt]$$

$$= e^{-j\omega t} \mid t = 0$$

$$= e^0 = 1$$

$$\boxed{\therefore \delta(\omega) = 1}$$

**FT of Unit Step Function:**

$$U(\omega) = \pi\delta(\omega) + 1/j\omega$$

**FT of Exponentials:**

$$e^{-at}u(t) \xleftrightarrow{\text{F.T}} 1/(a + j\omega)$$

$$e^{-at}u(t) \xleftrightarrow{\text{F.T}} 1/(a + j\omega)$$

$$e^{-a\,|\,t\,|} \overset{\text{F.T}}{\longleftrightarrow} \frac{2a}{a^2+\omega^2}$$

$$e^{j\omega_0 t} \overset{\text{F.T}}{\longleftrightarrow} \delta(\omega-\omega_0)$$

**FT of Signum Function :**

$$sgn(t) \overset{\text{F.T}}{\longleftrightarrow} \frac{2}{j\omega}$$

## Conditions for Existence of Fourier Transform:

Any function f(t) can be represented by using Fourier transform only when the function satisfies Dirichlet's conditions. i.e.

- The function f(t) has finite number of maxima and minima.
- There must be finite number of discontinuities in the signal f(t),in the given interval of time.
- It must be absolutely integrable in the given interval of time i.e.

$$\int_{-\infty}^{\infty} |\,f(t)|\,dt < \infty$$

## Properties of Fourier Transform:

Here are the properties of Fourier Transform:

### Linearity Property:

$$\text{If } x(t) \overset{\text{F.T}}{\longleftrightarrow} X(\omega)$$

$$\&\ y(t) \overset{\text{F.T}}{\longleftrightarrow} Y(\omega)$$

Then linearity property states that

$$ax(t)+by(t) \overset{\text{F.T}}{\longleftrightarrow} aX(\omega)+bY(\omega)$$

**Time Shifting Property:**

$$\text{If } x(t) \overset{\text{F.T}}{\longleftrightarrow} X(\omega)$$

Then Time shifting property states that

$$x(t-t_0) \overset{\text{F.T}}{\longleftrightarrow} e^{-j\omega t_0} X(\omega)$$

**Frequency Shifting Property:**

$$\text{If } x(t) \overset{\text{F.T}}{\longleftrightarrow} X(\omega)$$

Then frequency shifting property states that

$$e^{j\omega_0 t}.\, x(t) \overset{\text{F.T}}{\longleftrightarrow} X(\omega-\omega_0)$$

**Time Reversal Property:**

$$\text{If } x(t) \overset{\text{F.T}}{\longleftrightarrow} X(\omega)$$

Then Time reversal property states that

$$x(-t) \overset{\text{F.T}}{\longleftrightarrow} X(-\omega)$$

**Time Scaling Property:**

$$\text{If } x(t) \overset{\text{F.T}}{\longleftrightarrow} X(\omega)$$

Then Time scaling property states that

$$x(at) \frac{1}{\mid a \mid} X \frac{\omega}{a}$$

**Differentiation and Integration Properties:**

$$\text{If } x(t) \xleftrightarrow{\text{F.T}} X(\omega)$$

Then Differentiation property states that

$$\frac{dx(t)}{dt} \xleftrightarrow{\text{F.T}} j\omega.\, X(\omega)$$

$$\frac{d^n x(t)}{dt^n} \xleftrightarrow{\text{F.T}} (j\omega)^n.\, X(\omega)$$

and integration property states that

$$\int x(t)\, dt \xleftrightarrow{\text{F.T}} \frac{1}{j\omega} X(\omega)$$

$$\iiint \ldots \int x(t)\, dt \xleftrightarrow{\text{F.T}} \frac{1}{(j\omega)^n} X(\omega)$$

**Multiplication and Convolution Properties:**

$$\text{If } x(t) \xleftrightarrow{\text{F.T}} X(\omega)$$

$$\&\ y(t) \xleftrightarrow{\text{F.T}} Y(\omega)$$

Then multiplication property states that

$$x(t).\, y(t) \xleftrightarrow{\text{F.T}} X(\omega) * Y(\omega)$$

and convolution property states that

$$x(t) * y(t) \xleftrightarrow{\text{F.T}} \frac{1}{2\pi} X(\omega).\, Y(\omega)$$

**Sampling Theorem and its Importance:**

**Statement of Sampling Theorem:**

A band limited signal can be reconstructed exactly if it is sampled at a rate atleast twice the maximum frequency component in it."

The following figure shows a signal g(t) that is bandlimited.

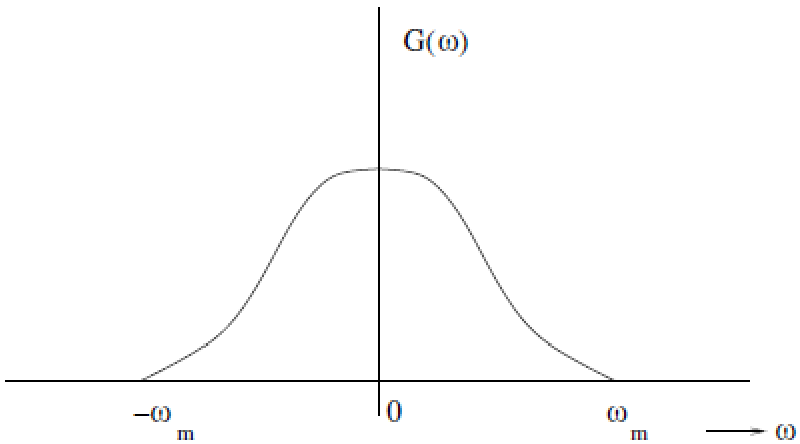



Figure1: Spectrum of band limited signal g(t)

The maximum frequency component of g(t) is fm. To recover the signal g(t) exactly from its samples it has to be sampled at a rate fs ≥ 2fm.

The minimum required sampling rate fs = 2fm is called "Nyquist rate".

**Proof:**

Let g(t) be a bandlimited signal whose bandwidth is $f_m$ ($\omega_m = 2\pi f_m$).

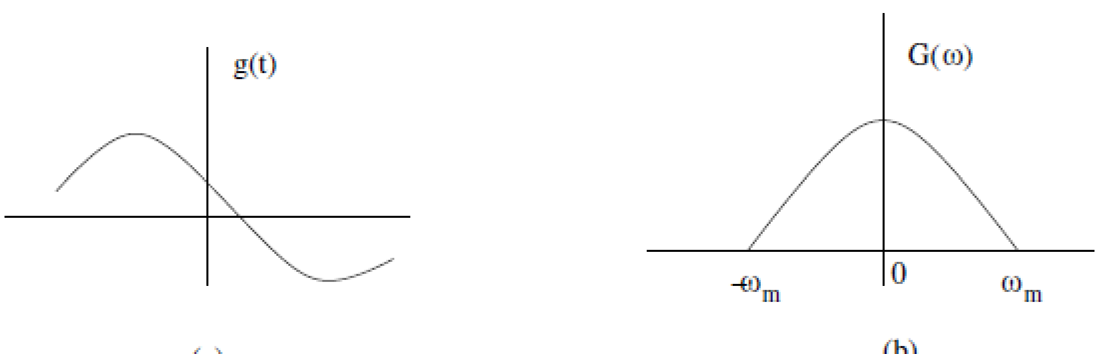



Figure 2: (a) Original signal g(t) (b) Spectrum G(ω)

$\delta_T(t)$ is the sampling signal with $f_s = 1/T > 2f_m$.

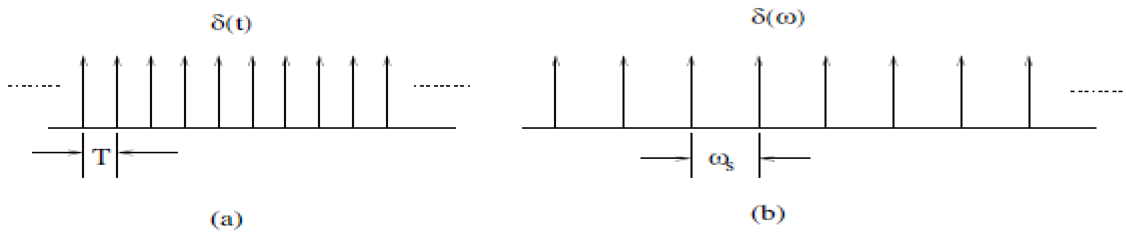



Figure 3: (a) sampling signal $\delta_T(t)$ (b) Spectrum $\delta_T(\omega)$

Let $g_s(t)$ be the sampled signal. Its Fourier Transform $Gs(\omega)$ is given by

$$\begin{aligned}
\mathcal{F}(g_s(t)) &= \mathcal{F}\left[g(t)\delta_T(t)\right] \\
&= \mathcal{F}\left[g(t)\sum_{n=-\infty}^{+\infty}\delta(t-nT)\right] \\
&= \frac{1}{2\pi}\left[G(\omega) * \omega_0 \sum_{n=-\infty}^{+\infty}\delta(\omega-n\omega_0)\right] \\
G_s(\omega) &= \frac{1}{T}\sum_{n=-\infty}^{+\infty} G(\omega) * \delta(\omega-n\omega_0) \\
G_s(\omega) &= \mathcal{F}\left[g(t)+2g(t)\cos(\omega_0 t)+2g(t)\cos(2\omega_0 t)+\cdots\right] \\
G_s(\omega) &= \frac{1}{T}\sum_{n=-\infty}^{+\infty} G(\omega-n\omega_0)
\end{aligned}$$

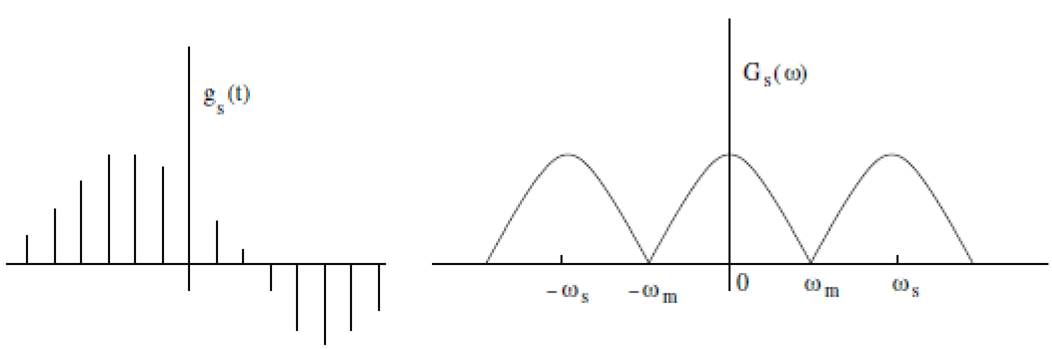



Figure 4: (a) sampled signal $g_s(t)$ (b) Spectrum $G_s(\omega)$

If $\omega_s = 2\omega_m$, i.e., $T = 1/2f_m$. Therefore, $G_s(\omega)$ is given by

$$G_s(\omega) = \frac{1}{T}\sum_{n=-\infty}^{+\infty} G(\omega - n\omega_m)$$

To recover the original signal $G(\omega)$:

1. Filter with a Gate function, $H_{2\omega_m}(\omega)$ of width $2\omega_m$.
2. Scale it by $T$.

$$G(\omega) = TG_s(\omega)H_{2\omega_m}(\omega).$$

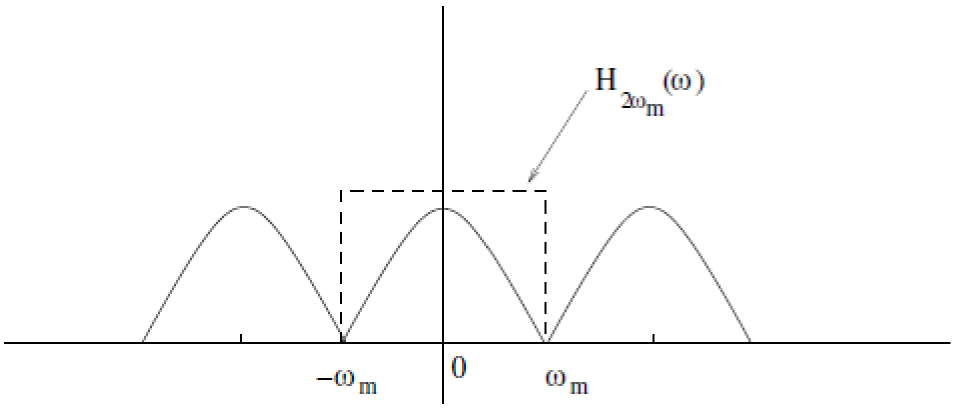



Figure 5: Recovery of signal by filtering with a filter of width $2\omega_m$

**Aliasing:**

Aliasing is a phenomenon where the high frequency components of the sampled signal interfere with each other because of inadequate sampling $\omega_s < \omega_m$

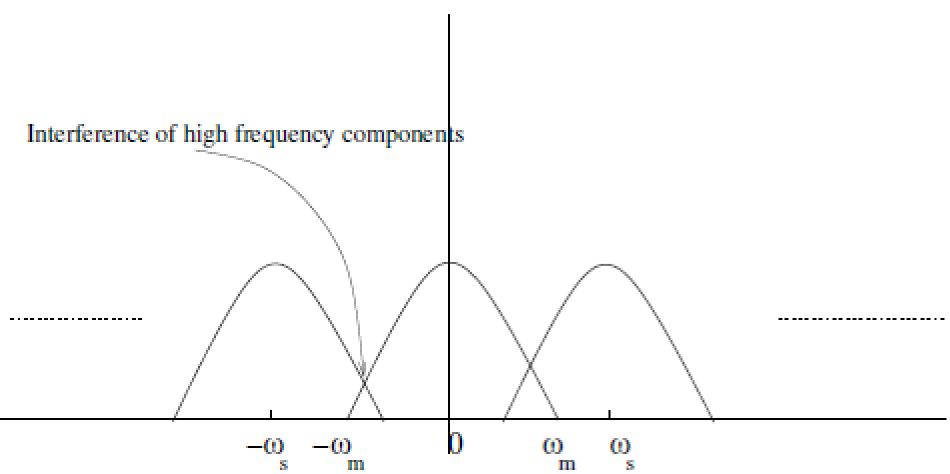



Figure 6: Aliasing due to inadequate sampling

Aliasing leads to distortion in recovered signal. This is the reason why sampling frequency should be atleast twice the bandwidth of the signal.

**Oversampling:**

In practice signal are oversampled, where fs is signi_cantly higher than Nyquist rate to avoid aliasing.

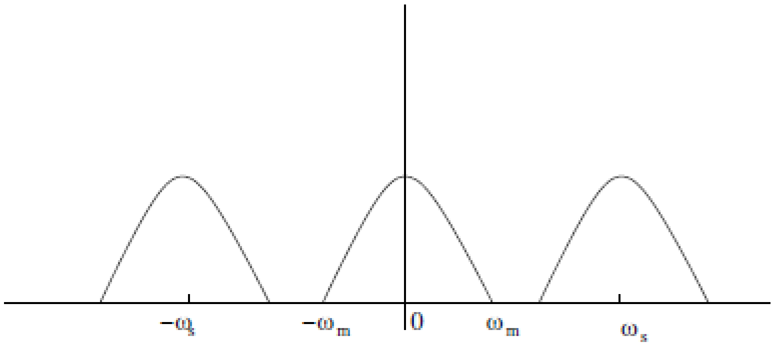
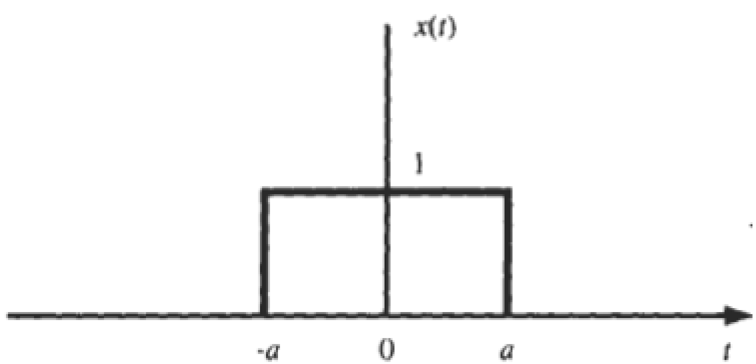



Figure 7: Oversampled signal-avoids aliasing

## Problems

**1. Find the Fourier transform of the rectangular pulse signal *x(t)* defined by**

$$x(t) = p_a(t) = \begin{cases} 1 & |t| < a \\ 0 & |t| > a \end{cases}$$




**Sol:** By definition of Fourier transform

$$X(\omega) = \int_{-\infty}^{\infty} p_a(t)\, e^{-j\omega t}\, dt = \int_{-a}^{a} e^{-j\omega t}\, dt$$

$$= \frac{1}{j\omega}(e^{j\omega a} - e^{-j\omega a}) = 2\frac{\sin \omega a}{\omega} = 2a\frac{\sin \omega a}{\omega a}$$

Hence we obtain

$$p_a(t) \longleftrightarrow 2\frac{\sin \omega a}{\omega} = 2a\frac{\sin \omega a}{\omega a}$$

The following figure shows the Fourier transform of the given signal x(t)

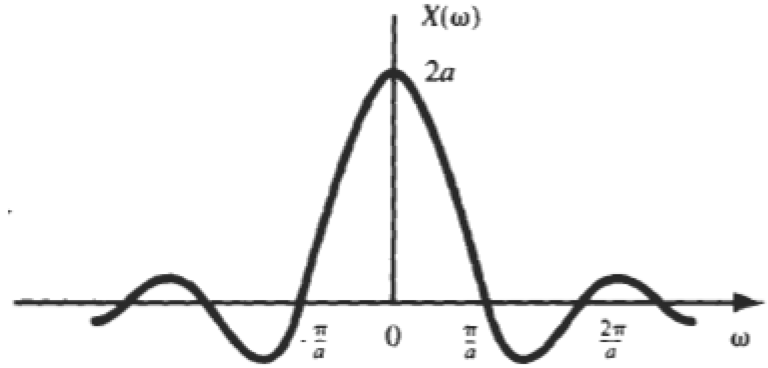



Figure: Fourier transform of the given signal

**2. Find the Fourier transform of the following signal x(t)**

$$x(t) = e^{-a|t|} \qquad a > 0$$

**Sol:** Signal x(t) can be rewritten as

$$x(t) = e^{-a|t|} = \begin{cases} e^{-at} & t > 0 \\ e^{at} & t < 0 \end{cases}$$

Then
$$X(\omega) = \int_{-\infty}^{0} e^{at}e^{-j\omega t}\,dt + \int_{0}^{\infty} e^{-at}e^{-j\omega t}\,dt$$

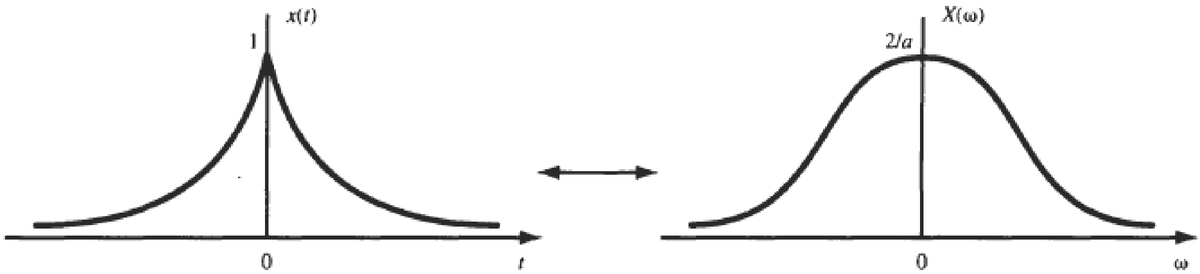
$$= \int_{-\infty}^{0} e^{(a-j\omega)t}\,dt + \int_{0}^{\infty} e^{-(a+j\omega)t}\,dt$$

$$= \frac{1}{a - j\omega} + \frac{1}{a + j\omega} = \frac{2a}{a^2 + \omega^2}$$

**Hence, we get**

$$e^{-a|t|} \longleftrightarrow \frac{2a}{a^2 + \omega^2}$$

The Fourier transform *X(w)* of *x(t)* is shown in the following figures




(a) (b)

Fig: (a) Signal x(t) (b) Fourier transform *X(w)* of *x(t)*

4. **Find the Fourier transform of the periodic impulse train**

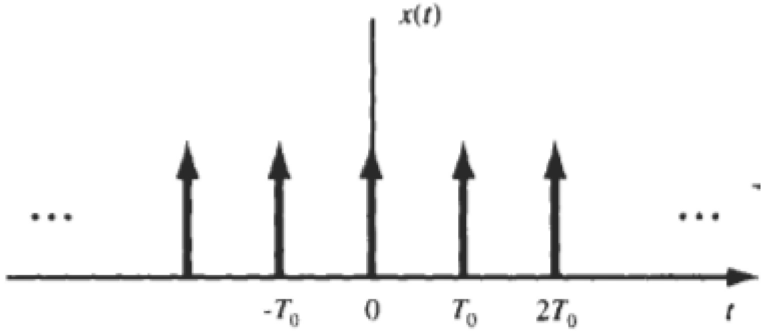



**Fig: Train of impulses**

**Sol:** Given signal can be written as

$$\delta_{T_0}(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_0)$$

the complex exponential Fourier series of $\delta_{T_0}(t)$ is given by

$$\delta_{T_0}(t) = \frac{1}{T_0} \sum_{k=-\infty}^{\infty} e^{jk\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T_0}$$

$$\mathscr{F}\left[\delta_{T_0}(t)\right] = \frac{2\pi}{T_0} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0)$$

$$= \omega_0 \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0) = \omega_0 \delta_{\omega_0}(\omega)$$

$$\sum_{k=-\infty}^{\infty} \delta(t - kT_0) \longleftrightarrow \omega_0 \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0)$$

Thus, the Fourier transform of a unit impulse train is also a similar impulse train. The following figure shows the Fourier transform of a unit impulse train

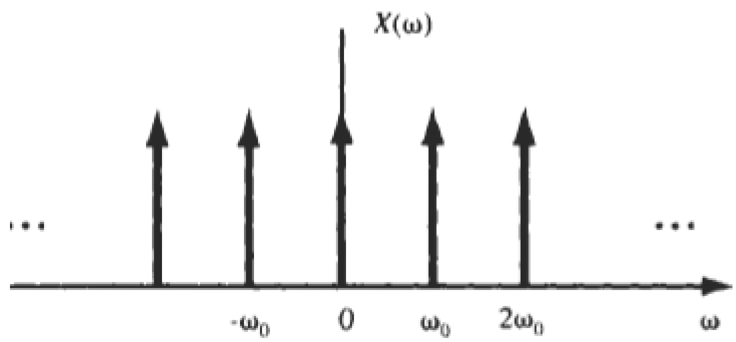



Figure: Fourier transform of the given signal

**5.Find the Fourier transform of the *signum* function**

**Sol:** Signum function is defined as

$$\mathrm{sgn}(t) = \begin{cases} 1 & t > 0 \\ -1 & t < 0 \end{cases}$$

The signum function, sgn(t), can be expressed as

**Sgn(t)= 2u(t)-1**

We know that

$$\frac{d}{dt}\,\mathrm{sgn}(t) = 2\delta(t)$$

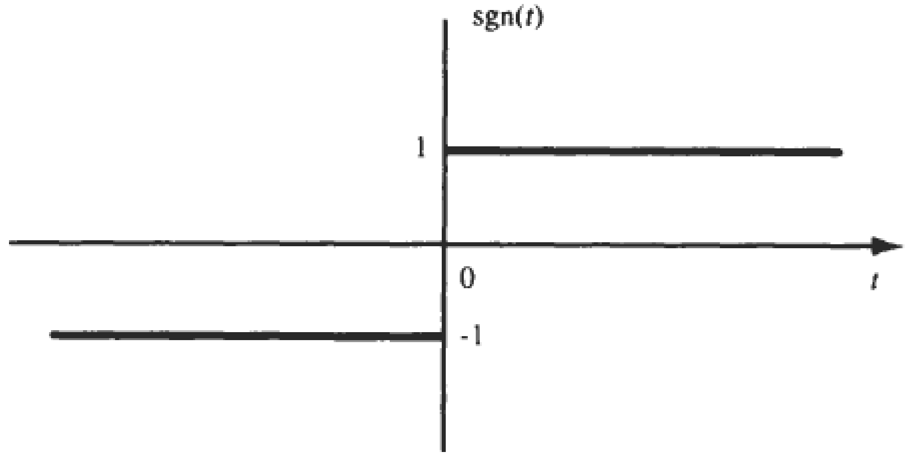



Fig: Signum function

Let

$$\mathrm{sgn}(t) \longleftrightarrow X(\omega)$$

Then applying the differentiation property , we have

$$j\omega X(\omega) = \mathscr{F}[2\delta(t)] = 2 \longrightarrow X(\omega) = \frac{2}{j\omega}$$

$$\mathrm{sgn}(t) \longleftrightarrow \frac{2}{j\omega}$$

**Note that sgn(t) is an odd function, and therefore its Fourier transform is a pure imaginary function of w**

## Properties of Fourier Transform:

| Property | Aperiodic signal | Fourier transform |
|---|---|---|
| | $x(t)$<br>$y(t)$ | $X(j\omega)$<br>$Y(j\omega)$ |
| Linearity | $ax(t) + by(t)$ | $aX(j\omega) + bY(j\omega)$ |
| Time Shifting | $x(t - t_0)$ | $e^{-j\omega t_0}X(j\omega)$ |
| Frequency Shifting | $e^{j\omega_0 t}x(t)$ | $X(j(\omega - \omega_0))$ |
| Conjugation | $x^*(t)$ | $X^*(-j\omega)$ |
| Time Reversal | $x(-t)$ | $X(-j\omega)$ |
| Time and Frequency Scaling | $x(at)$ | $\frac{1}{\|a\|}X\left(\frac{j\omega}{a}\right)$ |
| Convolution | $x(t) * y(t)$ | $X(j\omega)Y(j\omega)$ |
| Multiplication | $x(t)y(t)$ | $\frac{1}{2\pi}\int_{-\infty}^{+\infty}X(j\theta)Y(j(\omega-\theta))d\theta$ |
| Differentiation in Time | $\frac{d}{dt}x(t)$ | $j\omega X(j\omega)$ |
| Integration | $\int_{-\infty}^{t} x(t)dt$ | $\frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$ |
| Differentiation in Frequency | $tx(t)$ | $j\frac{d}{d\omega}X(j\omega)$ |
| Conjugate Symmetry for Real Signals | $x(t)$ real | $\begin{cases} X(j\omega) = X^*(-j\omega) \\ \mathcal{R}e\{X(j\omega)\} = \mathcal{R}e\{X(-j\omega)\} \\ \mathcal{I}m\{X(j\omega)\} = -\mathcal{I}m\{X(-j\omega)\} \\ \|X(j\omega)\| = \|X(-j\omega)\| \\ \sphericalangle X(j\omega) = -\sphericalangle X(-j\omega) \end{cases}$ |
| Symmetry for Real and Even Signals | $x(t)$ real and even | $X(j\omega)$ real and even |
| Symmetry for Real and Odd Signals | $x(t)$ real and odd | $X(j\omega)$ purely imaginary and odd |
| Even-Odd Decomposition for Real Signals | $x_e(t) = \mathcal{E}v\{x(t)\}$ [$x(t)$ real]<br>$x_o(t) = \mathcal{O}d\{x(t)\}$ [$x(t)$ real] | $\mathcal{R}e\{X(j\omega)\}$<br>$j\mathcal{I}m\{X(j\omega)\}$ |

Parseval's Relation for Aperiodic Signals

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega$$

## Fourier Transform of Basic Functions:

| Signal | Fourier transform | Fourier series coefficients (if periodic) |
|---|---|---|
| $\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$ | $2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$ | $a_k$ |
| $e^{j\omega_0 t}$ | $2\pi\delta(\omega - \omega_0)$ | $a_1 = 1$<br>$a_k = 0$, otherwise |
| $\cos\omega_0 t$ | $\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$ | $a_1 = a_{-1} = \frac{1}{2}$<br>$a_k = 0$, otherwise |
| $\sin\omega_0 t$ | $\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$ | $a_1 = -a_{-1} = \frac{1}{2j}$<br>$a_k = 0$, otherwise |
| $x(t) = 1$ | $2\pi\delta(\omega)$ | $a_0 = 1, \quad a_k = 0, \ k \neq 0$<br>(this is the Fourier series representation for any choice of $T > 0$) |
| Periodic square wave<br>$x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & T_1 < \lvert t\rvert \le \frac{T}{2} \end{cases}$<br>and<br>$x(t+T) = x(t)$ | $\sum_{k=-\infty}^{+\infty} \frac{2\sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0)$ | $\frac{\omega_0 T_1}{\pi} \operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$ |
| $\sum_{n=-\infty}^{+\infty} \delta(t - nT)$ | $\frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$ | $a_k = \frac{1}{T}$ for all $k$ |
| $x(t)\begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & \lvert t\rvert > T_1 \end{cases}$ | $\frac{2\sin\omega T_1}{\omega}$ | — |
| $\frac{\sin Wt}{\pi t}$ | $X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$ | — |
| $\delta(t)$ | $1$ | — |
| $u(t)$ | $\frac{1}{j\omega} + \pi\delta(\omega)$ | — |
| $\delta(t - t_0)$ | $e^{-j\omega t_0}$ | — |
| $e^{-at}u(t), \mathcal{R}e\{a\} > 0$ | $\frac{1}{a + j\omega}$ | — |
| $te^{-at}u(t), \mathcal{R}e\{a\} > 0$ | $\frac{1}{(a + j\omega)^2}$ | — |
| $\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)$,<br>$\mathcal{R}e\{a\} > 0$ | $\frac{1}{(a + j\omega)^n}$ | — |

# UNIT – III

## SIGNAL TRANSMISSION THROUGH LINEAR SYSTEMS

# UNIT – III

## SIGNAL TRANSMISSION THROUGH LINEAR SYSTEMS

## Linear Systems:

A system is said to be linear when it satisfies superposition and homogenate principles. Consider two systems with inputs as $x_1(t)$, $x_2(t)$, and outputs as $y_1(t)$, $y_2(t)$ respectively. Then, according to the superposition and homogenate principles,

$$T [a_1 x_1(t) + a_2 x_2(t)] = a_1 T[x_1(t)] + a_2 T[x_2(t)]$$

$$\therefore T [a_1 x_1(t) + a_2 x_2(t)] = a_1 y_1(t) + a_2 y_2(t)$$

From the above expression, is clear that response of overall system is equal to response of individual system.

**Example:**

$y(t) = 2x(t)$

Solution:

$y_1 (t) = T[x_1(t)] = 2x_1(t)$

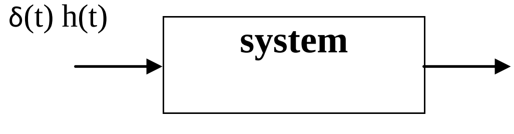
$y_2 (t) = T[x_2(t)] = 2x_2(t)$

$T [a_1 x_1(t) + a_2 x_2(t)] = 2[ a_1 x_1(t) + a_2 x_2(t)]$

Which is equal to $a_1y_1(t) + a_2 y_2(t)$. Hence the system is said to be linear.

## Impulse Response:

The impulse response of a system is its response to the input δ(t) when the system is initially at rest. The impulse response is usually denoted h(t). In other words, if the input to an initially at rest system is δ(t) then the output is named h(t).



## Liner Time variant (LTV) and Liner Time Invariant (LTI) Systems

If a system is both liner and time variant, then it is called liner time variant (LTV) system.

If a system is both liner and time Invariant then that system is called liner time invariant (LTI) system.

**Response of a continuous-time LTI system and the convolution integral**

**(i)Impulse Response:**

The ***impulse response*** h(t) of a continuous-time LTI system (represented by **T)** is defined to be the response of the system when the input is δ(t), that is,

$$h(t)= T\{ \delta(t)\} \quad \text{--------(1)}$$

**(ii)Response to an Arbitrary Input:**

The input ***x(*** t) can be expressed as

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau \qquad \text{--------(2)}$$

Since the system is linear, the response ***y(*** t of the system to an arbitrary input x( t ) can be expressed as

$$y(t) = \mathbf{T}\{x(t)\} = \mathbf{T}\left\{\int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau\right\}$$

$$= \int_{-\infty}^{\infty} x(\tau)\mathbf{T}\{\delta(t-\tau)\}\,d\tau \qquad \text{--------(3)}$$

Since the system is time-invariant, we have

$$h(t-\tau) = \mathbf{T}\{\delta(t-\tau)\} \qquad \text{--------(4)}$$

Substituting **Eq.** (4) into **Eq.** (3), we obtain

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau \qquad \text{-------(5)}$$

Equation *(5)* indicates that a continuous-time LTI system is completely characterized by its impulse response *h( t)*.

**(iii)Convolution Integral:**

Equation (5) defines the convolution of two continuous-time signals x ( t ) and h(t) denoted by

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau \qquad \text{-------(6)}$$

Equation (6) is commonly called the convolution integral. Thus, we have the fundamental result that the output of any continuous-time LTI system is the convolution of the input x ( t ) with the impulse response h(t) of the system. The following figure illustrates the definition of the impulse response h(t) and the relationship of Eq. (6).


Fig. : Continuous-time LTl system.

**(iv) Properties of the Convolution Integral:**

The convolution integral has the following properties.

1. ***Commutative:***

$$x(t) * h(t) = h(t) * x(t)$$

2. ***Associative:***

$$\{x(t) * h_1(t)\} * h_2(t) = x(t) * \{h_1(t) * h_2(t)\}$$

3. ***Distributive:***

$$x(t) * \{h_1(t)\} + h_2(t)\} = x(t) * h_1(t) + x(t) * h_2(t)$$

**(v) Step Response:**

The step response s(t) of a continuous-time LTI system (represented by **T)** is defined to be the response of the system when the input is u(t); that is,

**S(t)= T{u(t)}**

In many applications, the step response s(t) is also a useful characterization of the system.

The step response s(t) can be easily determined by,

$$s(t) = h(t) * u(t) = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)\,d\tau = \int_{-\infty}^{t} h(\tau)\,d\tau$$

Thus, the step response s(t) can be obtained by integrating the impulse response h(t).
Differentiating the above equation with respect to t, we get

$$h(t) = s'(t) = \frac{ds(t)}{dt}$$

Thus, the impulse response h(t) can be determined by differentiating the step response s(t).

**Distortion less transmission through a system:**

Transmission is said to be distortion-less if the input and output have identical wave shapes. i.e., in distortion-less transmission, the input x(t) and output y(t) satisfy the condition:

$$y(t) = Kx(t - t_d)$$

Where $t_d$ = delay time and

k = constant.

Take Fourier transform on both sides

$$FT[y(t)] = FT[Kx(t - t_d)]$$

$$= K\,FT[x(t - t_d)]$$

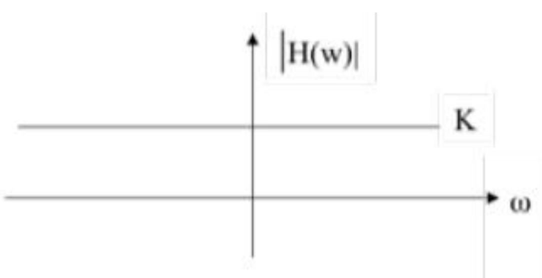
According to time shifting property,

$$Y(w) = KX(w)e^{-j\omega t_d}$$

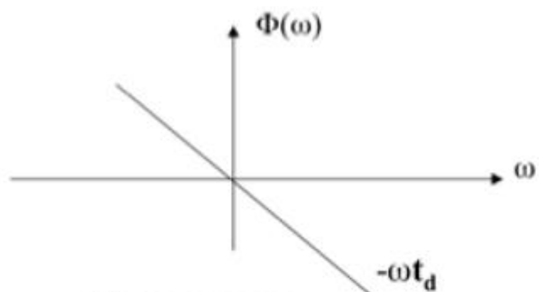
Thus, distortion less transmission of a signal x(t) through a system with impulse response h(t) is achieved when

**$|H(\omega)|=K$ and** (amplitude response)

$$\Phi(\omega) = -\omega t_d = -2\pi f t_d \quad phaseresponse$$

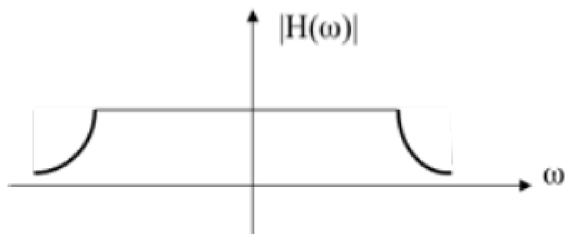


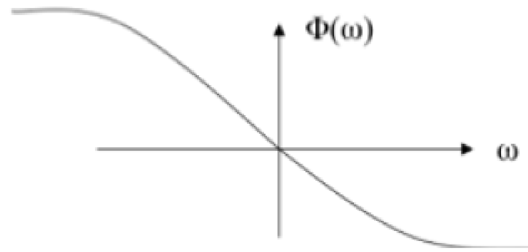

**Amplitude response**




**Phase response**

A physical transmission system may have amplitude and phase responses as shown below:




**FILTERING**

One of the most basic operations in any signal processing system is filtering. Filtering is the process by which the relative amplitudes of the frequency components in a signal are changed or perhaps some frequency components are suppressed. As we saw in the preceding section, for continuous-time LTI systems, the spectrum of the output is that of the input multiplied by the frequency response of the system. Therefore, an LTI system acts as a filter on the input signal. Here the word "filter" is used to denote a system that exhibits some sort of frequency-selective behavior.

**A. Ideal Frequency-Selective Filters:**

An ideal frequency-selective filter is one that exactly passes signals at one set of frequencies and completely rejects the rest. The band of frequencies passed by the filter is referred to as the pass band, and the band of frequencies rejected by the filter is called the stop band.

The most common types of ideal frequency-selective filters are the following.

***1. Ideal Low-Pass Filter:***

An ideal low-pass filter (LPF) is specified by

$$|H(\omega)| = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & |\omega| > \omega_c \end{cases}$$

The frequency $w_c$ is called the cutoff frequency.

***2. Ideal High-Pass Filter:***

An ideal high-pass filter (HPF) is specified by

$$|H(\omega)| = \begin{cases} 0 & |\omega| < \omega_c \\ 1 & |\omega| > \omega_c \end{cases}$$

***3. Ideal Bandpass Filter:***

An ideal bandpass filter (BPF) is specified by

$$|H(\omega)| = \begin{cases} 1 & \omega_1 < |\omega| < \omega_2 \\ 0 & \text{otherwise} \end{cases}$$

***4. Ideal Bandstop Filter:***

An ideal bandstop filter (BSF) is specified by

$$|H(\omega)| = \begin{cases} 0 & \omega_1 < |\omega| < \omega_2 \\ 1 & \text{otherwise} \end{cases}$$

The following figures shows the magnitude responses of ideal filters

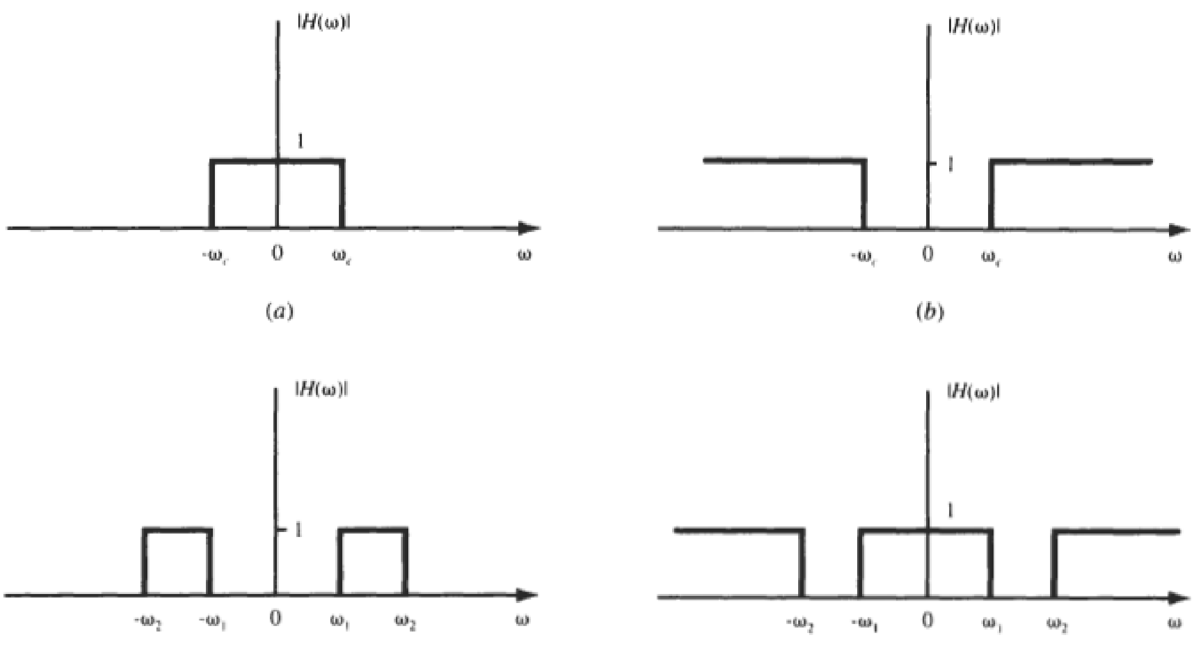



Fig: Magnitude responses of ideal filters (a) Ideal Low-Pass Filter (b)Ideal High-Pass Filter

© Ideal Bandpass Filter (d) Ideal Bandstop Filter

# UNIT – IV

# DISCRETE TIME FOURIER TRANSFORM

# UNIT-IV

## DISCRETE TIME FOURIER TRANSFORM

**Discrete Time Fourier Transforms (DTFT)**

Here we take the exponential signals to be $\{e^{j\omega n}\}$ where $\omega$ is a real number.The representation is motivated by the Harmonic analysis, but instead of following the historical development of the representation we give directly the defining equation. Let $\{x[n]\}$ be discrete time signal such that $\sum_{n=-\infty}^{\infty} |x[n]| < \infty$ that is $\{x[n]\}$ sequence is absolutely summable.
The sequence $\{x[n]\}$ can be represented by a Fourier integral of the form.

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\,d\omega \quad \text{-----------------(1)}$$

Where

$$x(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \quad \text{------------------(2)}$$

Equation (1) and (2) give the Fourier representation of the signal. Equation (1) is referred as synthesis equation or the inverse discrete time Fourier transform (IDTFT) and equation (2)is Fourier transform in the analysis equation. Fourier transform of a signal in general is a complex valued function, we can write

$$X(e^{j\omega}) = X_R(e^{j\omega}) + jX_I(e^{j\omega})$$

where $|X(e^{j\omega})|$is magnitude and $\angle X(e^{j\omega})$is the phase of. We also use the term Fourier spectrum or simply, the spectrum to refer to. Thus $|X(e^{j\omega})|$is called the magnitude spectrum and $\angle X(e^{j\omega})$is called the phase spectrum. From equation (2) we can see that $X(e^{j\omega})$is a periodic function with period $2\pi$ i.e.. We can interpret (1) as Fourier coefficients in the representation of a periodic function. In the Fourier series analysis our attention is on the periodic function, here we are concerned with the representation of the signal. So the roles of the two equation are interchanged compared to the Fourier series analysis of periodic signals.

Now we show that if we put equation (2) in equation (1) we indeed get the signal. Let

$$\hat{x}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{m=-\infty}^{\infty} x[m]e^{-j\omega m}\right)e^{+j\omega n}\,d\omega$$

where we have substituted $X(e^{j\omega})$from (2) into equation (1) and called the result as. Since we have used n as index on the left hand side we have used m as the index variable for the sum defining the Fourier transform. Under our assumption that $\{x[n]\}$ sequence is absolutely summable we can interchange the order of integration and summation. Thus

$$\hat{x}[n] = \sum_{m=-\infty}^{\infty} x[m] \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{+j\omega(n-m)} d\omega \right)$$

**Example:** Let

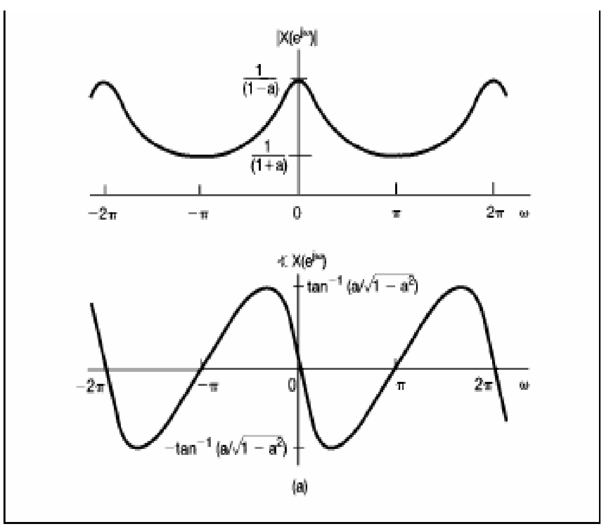
$$\{x[n]\} = \{a^n u[n]\}$$

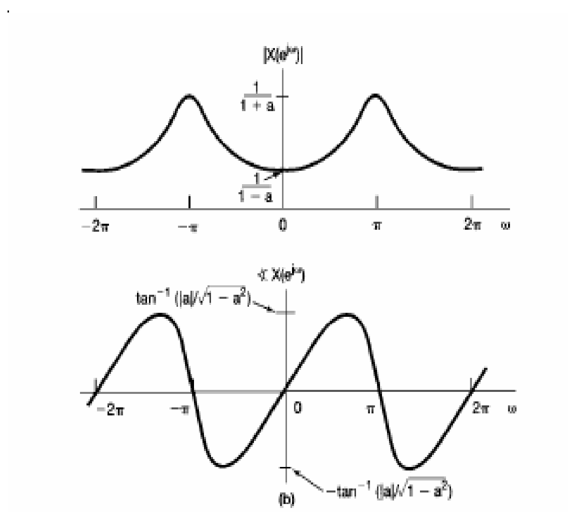
Fourier transform of this sequence will exist if it is absolutely summable. We have

$$\sum_{n=-\infty}^{\infty} |x[n]| = \sum_{n=0}^{\infty} |a|^n$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} a^n u[n]e^{-j\omega n} = \sum_{n=0}^{\infty} \left(ae^{-j\omega}\right)^{-n} = \frac{1}{1-ae^{-j\omega}}.$$

The magnitude and phase for this example are show in the figure below, where $a > 0$ and $a < 0$ are shown in (a) and (b).



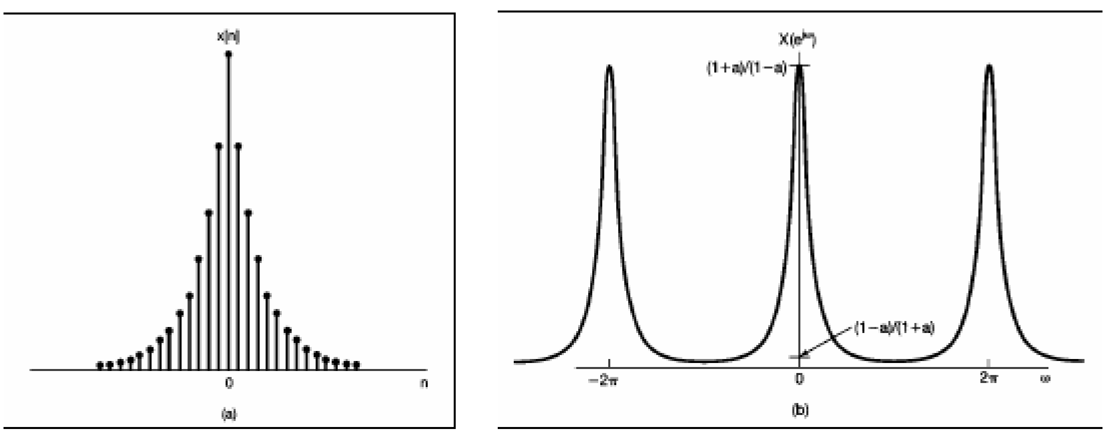
**Example**: $x[n] = a^{|n|}$, $|a| < 1$.

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^{|n|} u[n] e^{-j\omega n} = \sum_{n=-\infty}^{-1} a^{-n} e^{-j\omega n} + \sum_{n=0}^{\infty} a^{n} e^{-j\omega n}$$

Let $m = -n$ in the first summation, we obtain

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^{|n|} u[n] e^{-j\omega n} = \sum_{m=1}^{\infty} a^{m} e^{j\omega m} + \sum_{n=0}^{\infty} a^{n} e^{-j\omega n}$$

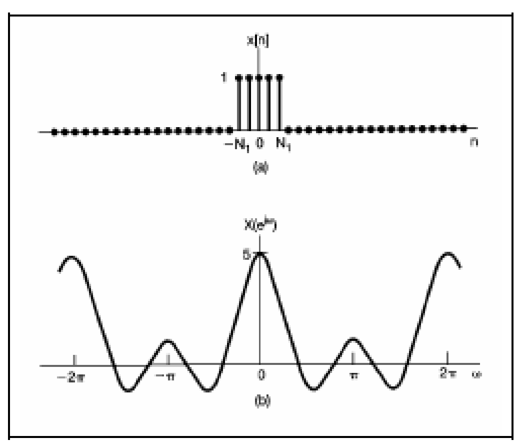
$$= \frac{a e^{j\omega}}{1 - a e^{j\omega}} + \frac{1}{1 - a e^{-j\omega}} = \frac{1 - a^2}{1 - 2a\cos\omega + a^2}$$




**Example**: Consider the rectangular pulse

$$x[n] = \begin{cases} 1, & |n| \le 2 \\ 0, & |n| > 2 \end{cases}, \qquad (5.14)$$

$$X(j\omega) = \sum_{n=-2}^{2} e^{-j\omega n} = \frac{\sin\omega(N_1 + 1/2)}{\sin(\omega/2)}. \qquad (5.15)$$




This function is the discrete counterpart of the sic function, which appears in the Fourier transform of the continuous-time pulse.

The difference between these two functions is that the discrete one is periodic (see figure) with period of $2\pi$, whereas the sinc function is aperiodic.

**Fourier transform of Periodic Signals**

For a periodic discrete-time signal,

$$x[n] = e^{j\omega_0 n},$$

its Fourier transform of this signal is periodic in *w* with period 2∏ , and is given

$$X(e^{j\omega}) = \sum_{l=-\infty}^{+\infty} 2\pi\delta(\omega - \omega_0 - 2\pi l).$$

Now consider a periodic sequence $x[n]$ with period $N$ and with the Fourier series representation

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}.$$

The Fourier transform is

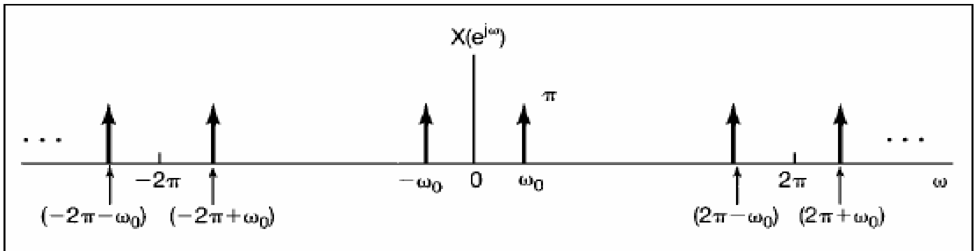
$$X(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta\left(\omega - \frac{2\pi k}{N}\right).$$

**Example**: The Fourier transform of the periodic signal

$$x[n] = \cos\omega_0 n = \frac{1}{2}e^{j\omega_0 n} + \frac{1}{2}e^{-j\omega_0 n}, \text{ with } \omega_0 = \frac{2\pi}{3},$$

is given as

$$X(e^{j\omega}) = \pi\delta\left(\omega - \frac{2\pi}{3}\right) + \pi\delta\left(\omega + \frac{2\pi}{3}\right), \qquad -\pi \le \omega < \pi.$$




Discrete-time Fourier transform of $x[n] = \cos\omega_0 n$.

**Example**: The periodic impulse train

$$x[n]=\sum_{k=-\infty}^{+\infty}\delta[n-kN].$$

The Fourier series coefficients for this signal can be calculated

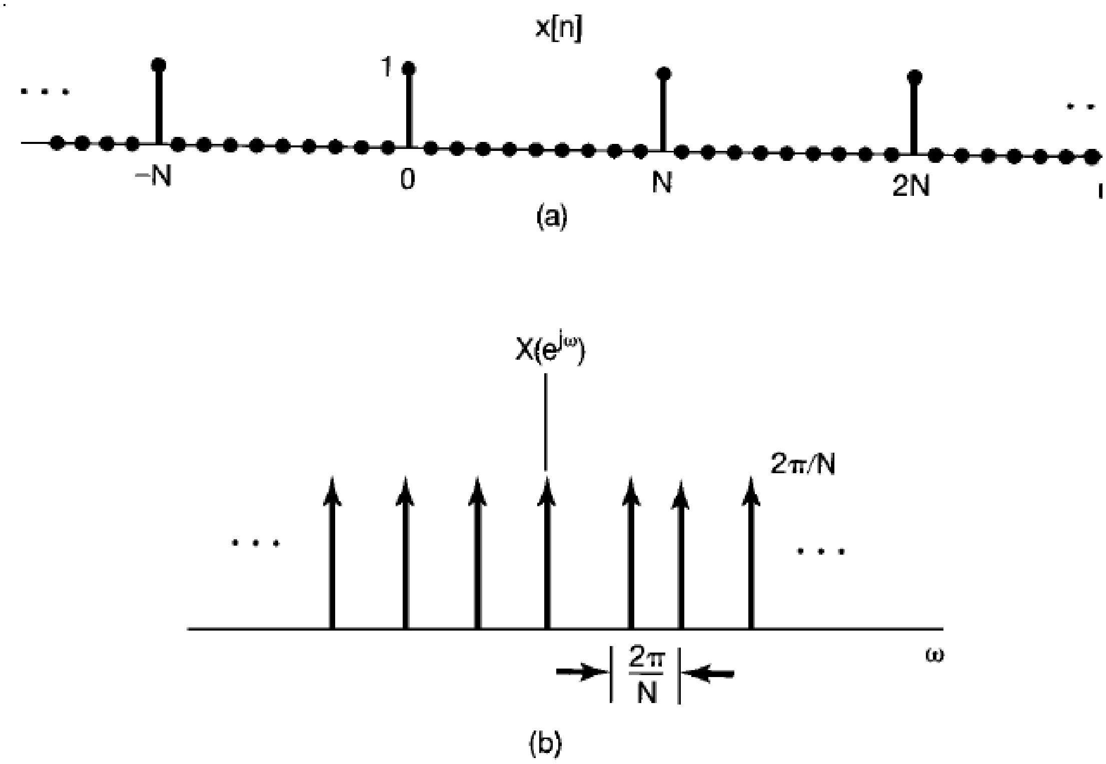
$$a_k=\sum_{n=\langle N\rangle}x[n]e^{-jk(2\pi/N)n}.$$

Choosing the interval of summation as $0\le n\le N-1$, we have

$$a_k=\frac{1}{N}.$$

The Fourier transform is

$$X(e^{j\omega})=\frac{2\pi}{N}\sum_{k=-\infty}^{\infty}\delta\left(\omega-\frac{2\pi k}{N}\right).$$



(a) Discrete-time periodic impulse train; (b) its Fourier transform.

**Properties of the Discrete Time Fourier Transform:**

Let $\{x[n]\}$ and $\{y[n]\}$be two signal, then their DTFT is denoted by $X(e^{j\omega})$and. The notation

$$\{x[n]\} \leftrightarrow X(e^{j\omega})$$

is used to say that left hand side is the signal x[n] whose DTFT is $X(e^{j\omega})$is given at right hand side.

**1. Periodicity of the DTFT:**

The discrete-time Fourier transform is always periodic in $\omega$ with period $2\pi$, i.e.,

$X\left(e^{j(\omega+2\pi)}\right) = X\left(e^{j\omega}\right)$.

**2. Linearity of the DTFT:**

If $x_1[n] \xleftrightarrow{F} X_1(e^{j\omega})$, and $x_2[n] \xleftrightarrow{F} X_2(e^{j\omega})$,

then

$$\boxed{ax_1[n] + bx_2[n] \xleftrightarrow{F} aX_1(e^{j\omega}) + bX_2(e^{j\omega})}$$

3. **Time Shifting and Frequency Shifting:**

If $x[n] \xleftrightarrow{F} X(e^{j\omega})$,

then

$$\boxed{x[n-n_0] \xleftrightarrow{F} e^{-j\omega n_0} X(e^{j\omega})}$$

and

$$\boxed{e^{j\omega_0 n} x[n] \xleftrightarrow{F} X(e^{j(\omega-\omega_0)})}$$

4. **Conjugation and Conjugate Symmetry:**

If $x[n] \xleftrightarrow{F} X(e^{j\omega})$,

then

$$\boxed{x^*[n] \xleftrightarrow{F} X^*(e^{-j\omega})}$$

If $x[n]$ is real valued, its transform $X(e^{j\omega})$ is conjugate symmetric. That is

$$\boxed{X(e^{j\omega}) = X^*(e^{-j\omega})}$$

From this, it follows that $\mathrm{Re}\left\{X(e^{jw})\right\}$ is an even function of $w$ and $\mathrm{Im}\left\{X(e^{jw})\right\}$ is an odd function of $w$. Similarly, the **magnitude** of $X(e^{jw})$ is an even function and the phase angle is an odd function. Furthermore,

$$Ev\{x[n]\} \xleftrightarrow{F} \mathrm{Re}\{X(e^{j\omega})\},$$

and

$$Od\{x[n]\} \xleftrightarrow{F} j\,\mathrm{Im}\{X(e^{j\omega})\}.$$

5. **Differencing and Accumulation**

If $x[n] \xleftrightarrow{F} X(e^{j\omega})$,

then

$$\boxed{x[n] - x[n-1] \xleftrightarrow{F} \left(1 - e^{-j\omega}\right) X(e^{j\omega})}.$$

For signal

$$y[n] = \sum_{m=-\infty}^{n} x[m],$$

its Fourier transform is given as

$$\boxed{\sum_{m=-\infty}^{n} x[m] \xleftrightarrow{F} \frac{1}{1-e^{-j\omega}} X(e^{j\omega}) + \pi X(e^{j0}) \sum_{m=-\infty}^{+\infty} \delta(\omega - 2\pi k)}.$$

The impulse train on the right-hand side reflects the dc or average value that can result from summation.

**For example**, the Fourier transform of the unit step $x[n] = u[n]$ can be obtained by using the accumulation property.

We know $g[n] = \delta[n] \xleftrightarrow{F} G(e^{j\omega}) = 1$, so

$$x[n] = \sum_{m=-\infty}^{n} g[m] \xleftrightarrow{F} \frac{1}{(1-e^{-j\omega})} G(e^{j\omega}) + \pi G(e^{j0}) \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k) = \frac{1}{(1-e^{-j\omega})} + \pi \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k).$$

6. **Time Reversal**

If $x[n] \xleftrightarrow{F} X(e^{j\omega})$,

then

$$\boxed{x[-n] \xleftrightarrow{F} X(-e^{j\omega})}.$$

***7.* Time Expansion**

For continuous-time signal, we have

$$x(at) \xleftrightarrow{F} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right).$$

For discrete-time signals, however, *a* should be an integer. Let us define a signal with *k* a positive integer,

$$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n \text{ is a multiple of } k \\ 0, & \text{if } n \text{ is not a multiple of } k \end{cases}.$$

$x_{(k)}[n]$ is obtained from $x[n]$ by placing $k-1$ zeros between successive values of the original signal.

The Fourier transform of $x_{(k)}[n]$ is given by

$$X_{(k)}(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x_{(k)}[n]e^{-j\omega n} = \sum_{r=-\infty}^{+\infty} x_{(k)}[rk]e^{-j\omega rk} = \sum_{r=-\infty}^{+\infty} x[r]e^{-j(k\omega)r} = X(e^{jk\omega}).$$

That is,

$$\boxed{x_{(k)}[n] \overset{F}{\longleftrightarrow} X(e^{jk\omega})}.$$

For $k > 1$, the signal is spread out and slowed down in time, while its Fourier transform is compressed.

**Example**: Consider the sequence $x[n]$ displayed in the figure (a) below. This sequence can be related to the simpler sequence $y[n]$ as shown in (b).

$$x[n] = y_{(2)}[n] + 2y_{(2)}[n-1],$$

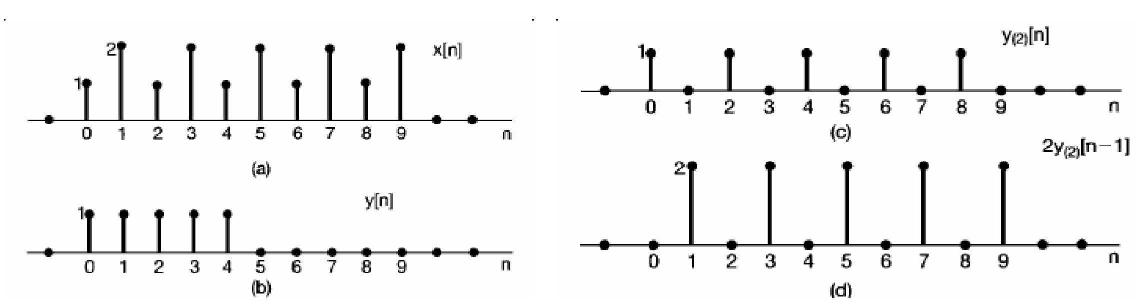
where

$$y_2[n] = \begin{cases} y[n/2], & \textit{if } n \textit{ is even} \\ 0, & \textit{if } n \textit{ is odd} \end{cases}$$

The signals $y_{(2)}[n]$ and $2y_{(2)}[n-1]$ are depicted in (c) and (d).

As can be seen from the figure below, $y[n]$ is a rectangular pulse with 2 $_1 N =$ , its Fourier transform is given by

$$Y(e^{j\omega}) = e^{-j2\omega}\frac{\sin(5\omega/2)}{\sin(\omega/2)}.$$



Using the time-expansion property, we then obtain

$$y_{(2)}[n] \xleftrightarrow{F} e^{-j4\omega}\frac{\sin(5\omega)}{\sin(\omega)}$$

$$2y_{(2)}[n-1] \xleftrightarrow{F} 2e^{-j5\omega}\frac{\sin(5\omega)}{\sin(\omega)}$$

Combining the two, we have

$$X(e^{j\omega}) = e^{-j4\omega}(1+2e^{-j\omega})\left(\frac{\sin(5\omega)}{\sin(\omega)}\right).$$

8. **Differentiation in Frequency**

If $x[n] \xleftrightarrow{F} X(e^{j\omega})$,

Differentiate both sides of the analysis equation $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$

$$\frac{dX(e^{j\omega})}{d\omega} = \sum_{n=-\infty}^{+\infty} -jnx[n]e^{-j\omega n}.$$

The right-hand side of the above equation is the Fourier transform of $-jnx[n]$ . Therefore, multiplying both sides by $j$ , we see that

$$\boxed{nx[n] \xleftrightarrow{F} j\frac{dX(e^{j\omega})}{d\omega}}.$$

**9. Parseval's Relation**

If $x[n] \xleftrightarrow{F} X(e^{j\omega})$, then we have

$$\boxed{\sum_{n=-\infty}^{+\infty}\left|x[n]\right|^2 = \frac{1}{2\pi}\int_{2\pi}\left|X(e^{j\omega})\right|^2 d\omega}$$

**Properties of the Discrete Time Fourier Transform:**

| Property | Aperiodic Signal | Fourier Transform |
|---|---|---|
| | $x[n]$<br>$y[n]$ | $\left.\begin{array}{l}X(e^{j\omega})\\Y(e^{j\omega})\end{array}\right\}$ periodic with period $2\pi$ |
| Linearity | $ax[n] + by[n]$ | $aX(e^{j\omega}) + bY(e^{j\omega})$ |
| Time Shifting | $x[n - n_0]$ | $e^{-j\omega n_0}X(e^{j\omega})$ |
| Frequency Shifting | $e^{j\omega_0 n}x[n]$ | $X(e^{j(\omega-\omega_0)})$ |
| Conjugation | $x^*[n]$ | $X^*(e^{-j\omega})$ |
| Time Reversal | $x[-n]$ | $X(e^{-j\omega})$ |
| Time Expansion | $x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n = \text{multiple of } k \\ 0, & \text{if } n \neq \text{multiple of } k \end{cases}$ | $X(e^{jk\omega})$ |
| Convolution | $x[n] * y[n]$ | $X(e^{j\omega})Y(e^{j\omega})$ |
| Multiplication | $x[n]y[n]$ | $\frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$ |
| Differencing in Time | $x[n] - x[n-1]$ | $(1 - e^{-j\omega})X(e^{j\omega})$ |
| Accumulation | $\sum_{k=-\infty}^{n} x[k]$ | $\frac{1}{1-e^{-j\omega}}X(e^{j\omega}) + \pi X(e^{j0})\sum_{k=-\infty}^{+\infty}\delta(\omega - 2\pi k)$ |
| Differentiation in Frequency | $nx[n]$ | $j\frac{dX(e^{j\omega})}{d\omega}$ |
| Conjugate Symmetry for Real Signals | $x[n]$ real | $\begin{cases} X(e^{j\omega}) = X^*(e^{-j\omega}) \\ \Re e\{X(e^{j\omega})\} = \Re e\{X(e^{-j\omega})\} \\ \Im m\{X(e^{j\omega})\} = -\Im m\{X(e^{-j\omega})\} \\ \lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert \\ \sphericalangle X(e^{j\omega}) = -\sphericalangle X(e^{-j\omega}) \end{cases}$ |
| Symmetry for Real, Even Signals | $x[n]$ real an even | $X(e^{j\omega})$ real and even |
| Symmetry for Real, Odd Signals | $x[n]$ real and odd | $X(e^{j\omega})$ purely imaginary and odd |
| Even-odd Decomposition of Real Signals | $x_e[n] = \mathcal{E}v\{x[n]\}$ $[x[n]$ real$]$<br>$x_o[n] = \mathcal{O}d\{x[n]\}$ $[x[n]$ real$]$ | $\Re e\{X(e^{j\omega})\}$<br>$j\Im m\{X(e^{j\omega})\}$ |

Parseval's Relation for Aperiodic Signals

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega$$

**Basic Discrete Time Fourier Transform Pairs:**

| Signal | Fourier Transform |
|---|---|
| $\sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$ | $2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$ |
| $e^{j\omega_0 n}$ | $2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - \omega_0 - 2\pi l)$ |
| $\cos \omega_0 n$ | $\pi \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) + \delta(\omega + \omega_0 - 2\pi l)\}$ |
| $\sin \omega_0 n$ | $\frac{\pi}{j} \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) - \delta(\omega + \omega_0 - 2\pi l)\}$ |
| $x[n] = 1$ | $2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - 2\pi l)$ |
| Periodic square wave $x[n] = \begin{cases} 1, & \lvert n\rvert \le N_1 \\ 0, & N_1 < \lvert n\rvert \le N/2 \end{cases}$ and $x[n+N] = x[n]$ | $2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$ |
| $\sum_{k=-\infty}^{+\infty} \delta[n - kN]$ | $\frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$ |
| $a^n u[n], \quad \lvert a\rvert < 1$ | $\frac{1}{1 - ae^{-j\omega}}$ |
| $x[n] \begin{cases} 1, & \lvert n\rvert \le N_1 \\ 0, & \lvert n\rvert > N_1 \end{cases}$ | $\frac{\sin[\omega(N_1 + \frac{1}{2})]}{\sin(\omega/2)}$ |
| $\frac{\sin Wn}{\pi n} = \frac{W}{\pi} \operatorname{sinc}\left(\frac{Wn}{\pi}\right)$ $0 < W < \pi$ | $X(\omega) = \begin{cases} 1, & 0 \le \lvert\omega\rvert \le W \\ 0, & W < \lvert\omega\rvert \le \pi \end{cases}$ $X(\omega)$ periodic with period $2\pi$ |
| $\delta[n]$ | $1$ |
| $u[n]$ | $\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{+\infty} \pi\delta(\omega - 2\pi k)$ |
| $\delta[n - n_0]$ | $e^{-j\omega n_0}$ |
| $(n+1)a^n u[n], \quad \lvert a\rvert < 1$ | $\frac{1}{(1 - ae^{-j\omega})^2}$ |
| $\frac{(n+r-1)!}{n!(r-1)!} a^n u[n], \quad \lvert a\rvert < 1$ | $\frac{1}{(1 - ae^{-j\omega})^r}$ |

# UNIT – V

# LAPLACE & Z TRANSFORM

# UNIT – V

## LAPLACE TRANSFORM AND Z–TRANSFORM

### THE LAPLACE TRANSFORM:

we know that for a continuous-time LTI system with impulse response h(t), the output **y(t)** of the system to the complex exponential input of the form $e^{st}$ is

$$y(t) = T\{e^{st}\} = H(s)e^{st}$$

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}\,dt$$

**A. Definition:**

The function H(s) is referred to as the Laplace transform of h(t). For a general continuous-time signal x(t), the Laplace transform X(s) is defined as

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$$

The variable s is generally complex-valued and is expressed as

$$s = \sigma + j\omega$$

**Relation between Laplace and Fourier transforms:**

Laplace transform of x(t)

$$X(S) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$$

Substitute s= σ + jω in above equation.

$$\rightarrow X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t)e^{-(\sigma+j\omega)t}\,dt$$

$$= \int_{-\infty}^{\infty} [x(t)e^{-\sigma t}]e^{-j\omega t}\,dt$$

$$\therefore X(S) = F.T[x(t)e^{-\sigma t}]$$

$$X(S) = X(\omega) \qquad for\ s = j\omega$$

**Inverse Laplace Transform:**

We know that

$$X(S) = F.T[x(t)e^{-\sigma t}]$$

$$\rightarrow x(t)e^{-\sigma t} = F.T^{-1}[X(S)] = F.T^{-1}[X(\sigma + j\omega)]$$

$$= \frac{1}{2}\pi \int_{-\infty}^{\infty} X(\sigma + j\omega)e^{j\omega t} d\omega$$

$$x(t) = e^{\sigma t} \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\sigma + j\omega)e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\sigma + j\omega)e^{(\sigma + j\omega)t} d\omega$$

Here, $\sigma + j\omega = s$

$$jd\omega = ds \rightarrow d\omega = ds/j$$

$$\therefore x(t) = \frac{1}{2\pi j} \int_{-\infty}^{\infty} X(s)e^{st} ds \ldots\ldots$$

**Conditions for Existence of Laplace Transform:**

Dirichlet's conditions are used to define the existence of Laplace transform. i.e.

- The function f has finite number of maxima and minima.
- There must be finite number of discontinuities in the signal f ,in the given interval of time.
- It must be absolutely integrable in the given interval of time. i.e.

$$\int_{-\infty}^{\infty} |\, f(t)|\, dt < \infty$$

**Initial and Final Value Theorems**

If the Laplace transform of an unknown function x(t) is known, then it is possible to determine the initial and the final values of that unknown signal i.e. x(t) at $t=0^+$ and $t=\infty$.

**Initial Value Theorem**

**Statement:** If x(t) and its 1st derivative is Laplace transformable, then the initial value of x(t) is given by

$$x(0^+) = \lim_{s\to\infty} SX(S)$$

**Final Value Theorem**

**Statement:** If x(t) and its 1st derivative is Laplace transformable, then the final value of x(t) is given by

$$x(\infty) = \lim_{s\to\infty} SX(S)$$

**<u>Properties of Laplace transform:</u>**

The properties of Laplace transform are:

**Linearity Property**

If $x(t) \stackrel{\mathrm{L.T}}{\longleftrightarrow} X(s)$

& $y(t) \stackrel{\mathrm{L.T}}{\longleftrightarrow} Y(s)$

Then linearity property states that

$$ax(t) + by(t) \stackrel{\mathrm{L.T}}{\longleftrightarrow} aX(s) + bY(s)$$

**Time Shifting Property**

If $x(t) \stackrel{\mathrm{L.T}}{\longleftrightarrow} X(s)$

Then time shifting property states that

$$x(t-t_0) \stackrel{\mathrm{L.T}}{\longleftrightarrow} e^{-st_0} X(s)$$

**Frequency Shifting Property**

If $x(t) \xleftrightarrow{\text{L.T}} X(s)$

Then frequency shifting property states that

$$e^{s_0 t}.\, x(t) \xleftrightarrow{\text{L.T}} X(s - s_0)$$

**Time Reversal Property**

If $x(t) \xleftrightarrow{\text{L.T}} X(s)$

Then time reversal property states that

$$x(-t) \xleftrightarrow{\text{L.T}} X(-s)$$

**Time Scaling Property**

If $x(t) \xleftrightarrow{\text{L.T}} X(s)$

Then time scaling property states that

$$x(at) \xleftrightarrow{\text{L.T}} \frac{1}{|a|} X(\frac{s}{a})$$

**Differentiation and Integration Properties**

If $x(t) \xleftrightarrow{\text{L.T}} X(s)$

Then differentiation property states that

$$\frac{dx(t)}{dt} \xleftrightarrow{\text{L.T}} s.\, X(s)$$

$$\frac{d^n x(t)}{dt^n} \xleftrightarrow{\text{L.T}} (s)^n.\, X(s)$$

The integration property states that

$$\int x(t)dt \xleftrightarrow{\text{L.T}} \frac{1}{s}X(s)$$

$$\iiint \dots \int x(t)dt \xleftrightarrow{\text{L.T}} \frac{1}{s^n}X(s)$$

**Multiplication and Convolution Properties**

If $x(t) \xleftrightarrow{\text{L.T}} X(s)$

and $y(t) \xleftrightarrow{\text{L.T}} Y(s)$

Then multiplication property states that

$$x(t).\, y(t) \xleftrightarrow{\text{L.T}} \frac{1}{2\pi j}X(s) * Y(s)$$

The convolution property states that

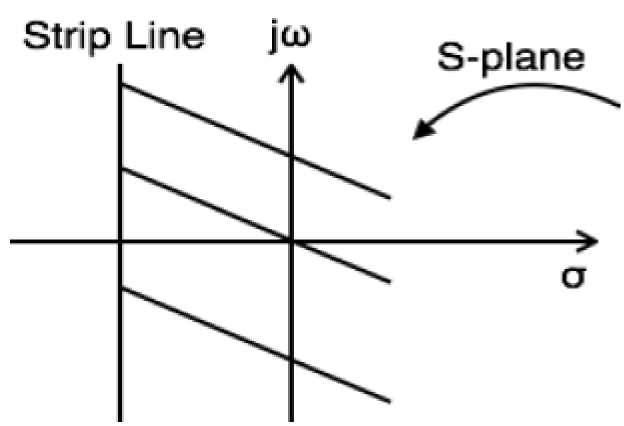
$$x(t) * y(t) \xleftrightarrow{\text{L.T}} X(s).\, Y(s)$$

<u>**Region of convergence**</u>.

The range variation of σ for which the Laplace transform converges is called region of convergence.

**Properties of ROC of Laplace Transform**

- ROC contains strip lines parallel to jω axis in s-plane.



- If x(t) is absolutely integral and it is of finite duration, then ROC is entire s-plane.
- If x(t) is a right sided sequence then ROC : Re{s} > $\sigma_o$.
- If x(t) is a left sided sequence then ROC : Re{s} < $\sigma_o$.
- If x(t) is a two sided sequence then ROC is the combination of two regions.

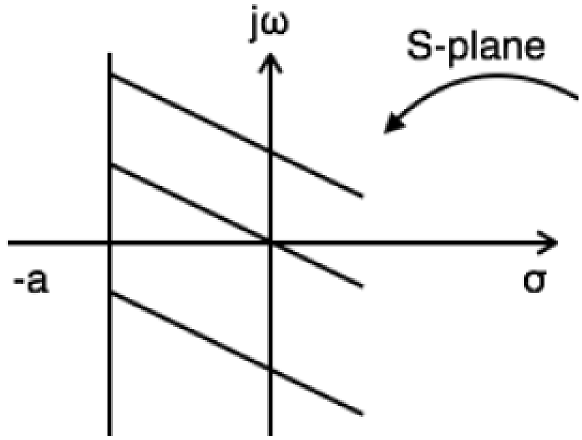
ROC can be explained by making use of examples given below:

**Example 1: Find the Laplace transform and ROC of x(t)=e− at u(t) x(t)=e$^{-at}$u(t)**

$$L.T[x(t)] = L.T[e -^{at} u(t)] = \frac{1}{S+a}$$

$$Re > -a$$

$$ROC : Res >> -a$$

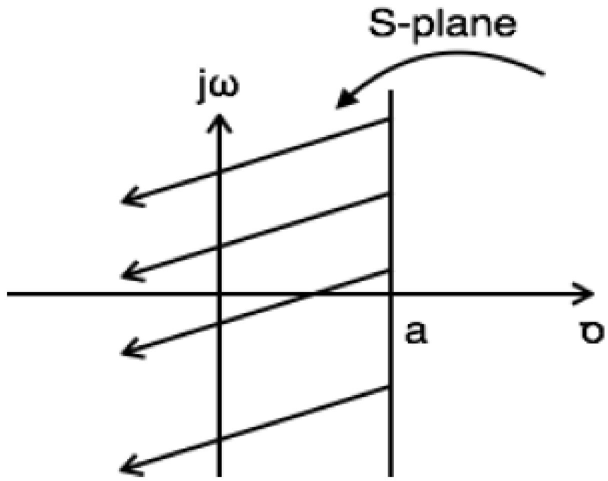



**Example 2: Find the Laplace transform and ROC of x(t)=e at u(−t) x(t)=e$^{at}$u(−t)**

$$L.T[x(t)] = L.T[e^{at}u(t)] = \frac{1}{S-a}$$

$$Res < a$$

$$ROC : Res < a$$

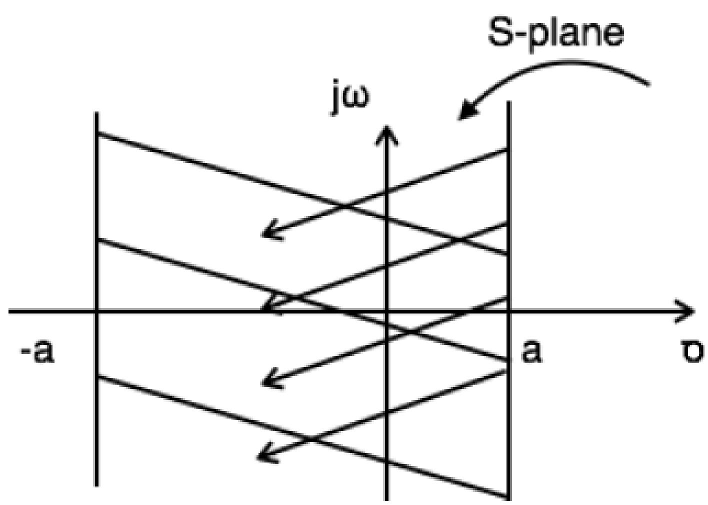



**Example 3: Find the Laplace transform and ROC of x(t)=e −at u(t)+e at u(−t)**
**$x(t)=e^{-at}u(t)+e^{at}u(-t)$**

$$L.T[x(t)] = L.T[e^{-at}u(t) + e^{at}u(-t)] = \frac{1}{S+a} + \frac{1}{S-a}$$

For $\frac{1}{S+a} Re\{s\} > -a$

For $\frac{1}{S-a} Re\{s\} < a$

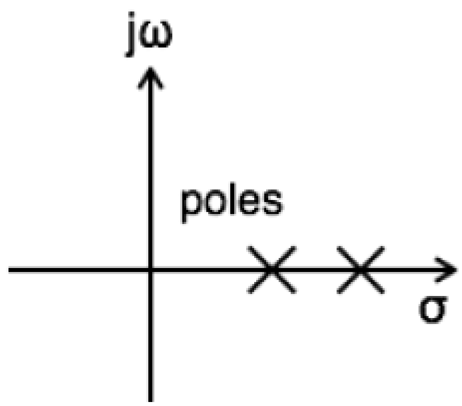



Referring to the above diagram, combination region lies from –a to a. Hence,

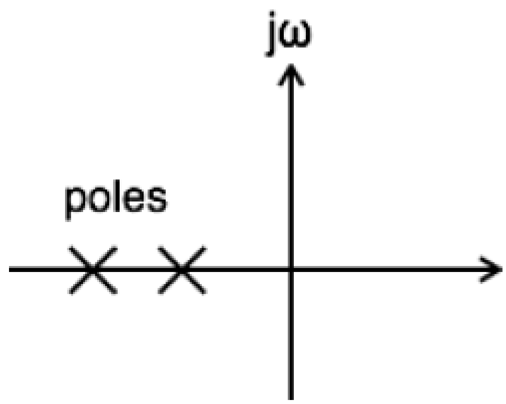
ROC: $-a < Res < a$

**Causality and Stability**

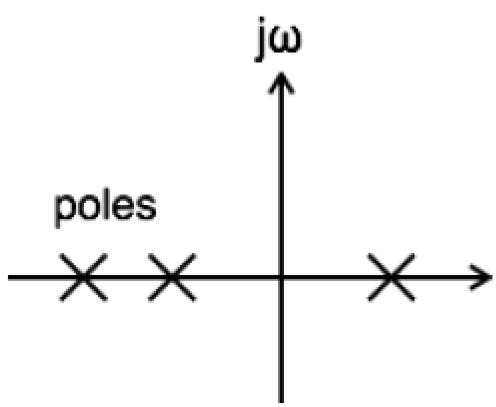
- For a system to be causal, all poles of its transfer function must be right half of s-plane.



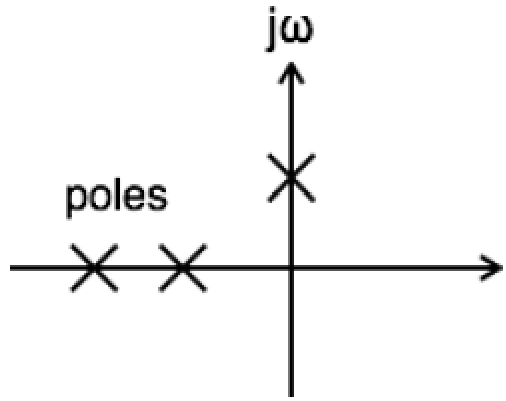

- A system is said to be stable when all poles of its transfer function lay on the left half of s-plane.




- A system is said to be unstable when at least one pole of its transfer function is shifted to the right half of s-plane.



- A system is said to be marginally stable when at least one pole of its transfer function lies on the jω axis of s-plane




**LAPLACE TRANSFORMS OF SOME COMMON SIGNALS**

**A. Unit Impulse Function** δ(***t*** ):

$$\mathscr{L}[\delta(t)] = \int_{-\infty}^{\infty} \delta(t) e^{-st}\, dt = 1 \qquad \text{all } s$$

**B. Unit Step Function *u(t* ):**

$$\mathscr{L}[u(t)] = \int_{-\infty}^{\infty} u(t) e^{-st}\, dt = \int_{0^+}^{\infty} e^{-st}\, dt$$

$$= -\frac{1}{s} e^{-st}\Big|_{0^+}^{\infty} = \frac{1}{s} \qquad \operatorname{Re}(s) > 0$$

where $0^+ = \lim_{\varepsilon \to 0}(0 + \varepsilon)$.

**Some Laplace Transforms Pairs:**

| $x(t)$ | $X(s)$ | ROC |
|---|---|---|
| $\delta(t)$ | $1$ | All $s$ |
| $u(t)$ | $\frac{1}{s}$ | $\mathrm{Re}(s) > 0$ |
| $-u(-t)$ | $\frac{1}{s}$ | $\mathrm{Re}(s) < 0$ |
| $tu(t)$ | $\frac{1}{s^2}$ | $\mathrm{Re}(s) > 0$ |
| $t^k u(t)$ | $\frac{k!}{s^{k+1}}$ | $\mathrm{Re}(s) > 0$ |
| $e^{-at}u(t)$ | $\frac{1}{s+a}$ | $\mathrm{Re}(s) > -\mathrm{Re}(a)$ |
| $-e^{-at}u(-t)$ | $\frac{1}{s+a}$ | $\mathrm{Re}(s) < -\mathrm{Re}(a)$ |
| $te^{-at}u(t)$ | $\frac{1}{(s+a)^2}$ | $\mathrm{Re}(s) > -\mathrm{Re}(a)$ |
| $-te^{-at}u(-t)$ | $\frac{1}{(s+a)^2}$ | $\mathrm{Re}(s) < -\mathrm{Re}(a)$ |
| $\cos \omega_0 t u(t)$ | $\frac{s}{s^2+\omega_0^2}$ | $\mathrm{Re}(s) > 0$ |
| $\sin \omega_0 t u(t)$ | $\frac{\omega_0}{s^2+\omega_0^2}$ | $\mathrm{Re}(s) > 0$ |
| $e^{-at}\cos \omega_0 t u(t)$ | $\frac{s+a}{(s+a)^2+\omega_0^2}$ | $\mathrm{Re}(s) > -\mathrm{Re}(a)$ |
| $e^{-at}\sin \omega_0 t u(t)$ | $\frac{\omega_0}{(s+a)^2+\omega_0^2}$ | $\mathrm{Re}(s) > -\mathrm{Re}(a)$ |

## Z-Transform

Analysis of continuous time LTI systems can be done using z-transforms. It is a powerful mathematical tool to convert differential equations into algebraic equations.

The bilateral (two sided) z-transform of a discrete time signal x(n) is given as

$$Z.T[x(n)] = X(Z) = \Sigma_{n=-\infty}^{\infty} x(n) z^{-n}$$

The unilateral (one sided) z-transform of a discrete time signal x(n) is given as

$$Z.T[x(n)] = X(Z) = \Sigma_{n=0}^{\infty} x(n) z^{-n}$$

Z-transform may exist for some signals for which Discrete Time Fourier Transform (DTFT) does not exist.

**Concept of Z-Transform and Inverse Z-Transform**

Z-transform of a discrete time signal x(n) can be represented with X(Z), and it is defined as

$$X(Z) = \Sigma_{n=-\infty}^{\infty} x(n) z^{-n} \ldots\ldots (1)$$

If $Z = re^{j\omega}$ then equation 1 becomes

$$X(re^{j\omega}) = \Sigma_{n=-\infty}^{\infty} x(n) [re^{j\omega}]^{-n}$$

$$= \Sigma_{n=-\infty}^{\infty} x(n) [r^{-n}] e^{-j\omega n}$$

$$X(re^{j\omega}) = X(Z) = F.T[x(n) r^{-n}] \ldots\ldots (2)$$

The above equation represents the relation between Fourier transform and Z-transform

$$X(Z)|_{z=e^{j\omega}} = F.T[x(n)].$$

**Inverse Z-transform:**

$$X(re^{j\omega}) = F.T[x(n)r^{-n}]$$

$$x(n)r^{-n} = F.T^{-1}[X(re^{j\omega}]$$

$$x(n) = r^n F.T^{-1}[X(re^{j\omega})]$$

$$= r^n \tfrac{1}{2\pi} \int X(re^j\omega)e^{j\omega n} d\omega$$

$$= \tfrac{1}{2\pi} \int X(re^j\omega)[re^{j\omega}]^n d\omega \ldots \ldots (3)$$

Substitute $re^{j\omega} = z$.

$$dz = jre^{j\omega} d\omega = jz d\omega$$

$$d\omega = \tfrac{1}{j} z^{-1} dz$$

Substitute in equation 3.

$$3 \rightarrow x(n) = \tfrac{1}{2\pi} \int X(z) z^n \tfrac{1}{j} z^{-1} dz = \tfrac{1}{2\pi j} \int X(z) z^{n-1} dz$$

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$x(n) = \frac{1}{2\pi j} \int X(z) z^{n-1} dz$$

**Z-Transform Properties:**

Z-Transform has following properties:

**Linearity Property:**

If $x(n) \overset{\text{Z.T}}{\longleftrightarrow} X(Z)$

and $y(n) \overset{\text{Z.T}}{\longleftrightarrow} Y(Z)$

Then linearity property states that

$$a\,x(n) + b\,y(n) \overset{\text{Z.T}}{\longleftrightarrow} a\,X(Z) + b\,Y(Z)$$

**Time Shifting Property:**

If $x(n) \overset{\text{Z.T}}{\longleftrightarrow} X(Z)$

Then Time shifting property states that

$$x(n-m) \overset{\text{Z.T}}{\longleftrightarrow} z^{-m}X(Z)$$

**Multiplication by Exponential Sequence Property**

If $x(n) \overset{\text{Z.T}}{\longleftrightarrow} X(Z)$

Then multiplication by an exponential sequence property states that

$$a^n\,.\,x(n) \overset{\text{Z.T}}{\longleftrightarrow} X(Z/a)$$

**Time Reversal Property**

If $x(n) \overset{\text{Z.T}}{\longleftrightarrow} X(Z)$

Then time reversal property states that

$$x(-n) \overset{\text{Z.T}}{\longleftrightarrow} X(1/Z)$$

**Differentiation in Z-Domain OR Multiplication by n Property**

If $x(n) \xleftrightarrow{\text{Z.T}} X(Z)$

Then multiplication by n or differentiation in z-domain property states that

$$n^k x(n) \xleftrightarrow{\text{Z.T}} [-1]^k z^k \frac{d^k X(Z)}{dZ^K}$$

**Convolution Property**

If $x(n) \xleftrightarrow{\text{Z.T}} X(Z)$

and $y(n) \xleftrightarrow{\text{Z.T}} Y(Z)$

Then convolution property states that

$$x(n) * y(n) \xleftrightarrow{\text{Z.T}} X(Z).Y(Z)$$

**Correlation Property**

If $x(n) \xleftrightarrow{\text{Z.T}} X(Z)$

and $y(n) \xleftrightarrow{\text{Z.T}} Y(Z)$

Then correlation property states that

$$x(n) \otimes y(n) \xleftrightarrow{\text{Z.T}} X(Z).Y(Z^{-1})$$

**Initial Value and Final Value Theorems**

Initial value and final value theorems of z-transform are defined for causal signal.

**Initial Value Theorem**

For a causal signal x(n), the initial value theorem states that

$$x(0) = \lim_{z \to \infty} X(z)$$

This is used to find the initial value of the signal without taking inverse z-transform

**Final Value Theorem**

For a causal signal x(n), the final value theorem states that

$$x(\infty) = \lim_{z\to 1}[z-1]X(z)$$

This is used to find the final value of the signal without taking inverse z-transform

**Region of Convergence (ROC) of Z-Transform**

The range of variation of z for which z-transform converges is called region of convergence of z-transform.

**Properties of ROC of Z-Transforms**

- ROC of z-transform is indicated with circle in z-plane.
- ROC does not contain any poles.
- If x(n) is a finite duration causal sequence or right sided sequence, then the ROC is entire z-plane except at z = 0.
- If x(n) is a finite duration anti-causal sequence or left sided sequence, then the ROC is entire z-plane except at z = ∞.
- If x(n) is a infinite duration causal sequence, ROC is exterior of the circle with radius a. i.e. |z| > a.
- If x(n) is a infinite duration anti-causal sequence, ROC is interior of the circle with radius a. i.e. |z| < a.
- If x(n) is a finite duration two sided sequence, then the ROC is entire z-plane except at z = 0 & z = ∞.

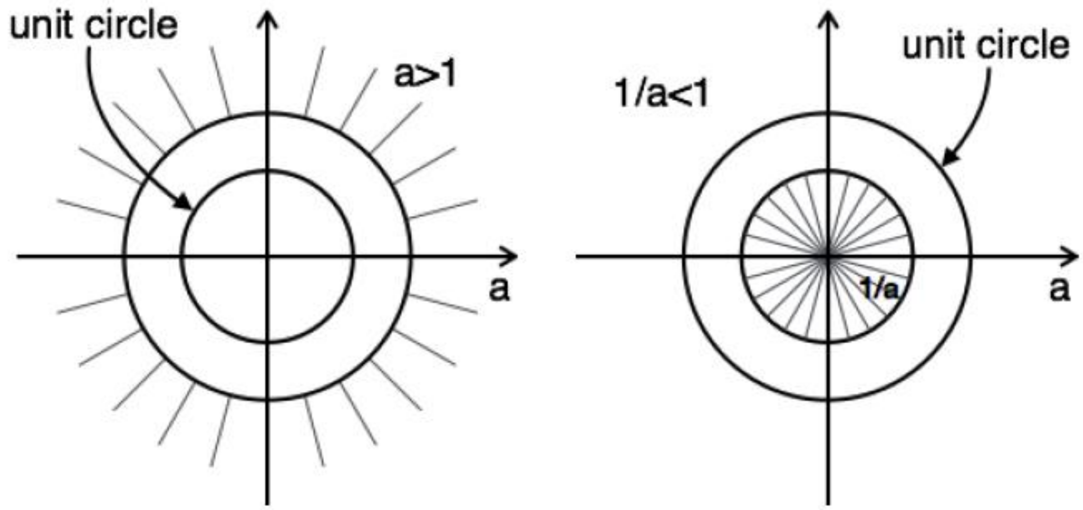
The concept of ROC can be explained by the following example:

**Example 1:** Find z-transform and ROC of $a^{n}u[n]+a^{-n}u[-n-1]$ $a^{n}u[n]+a^{-n}u[-n-1]$

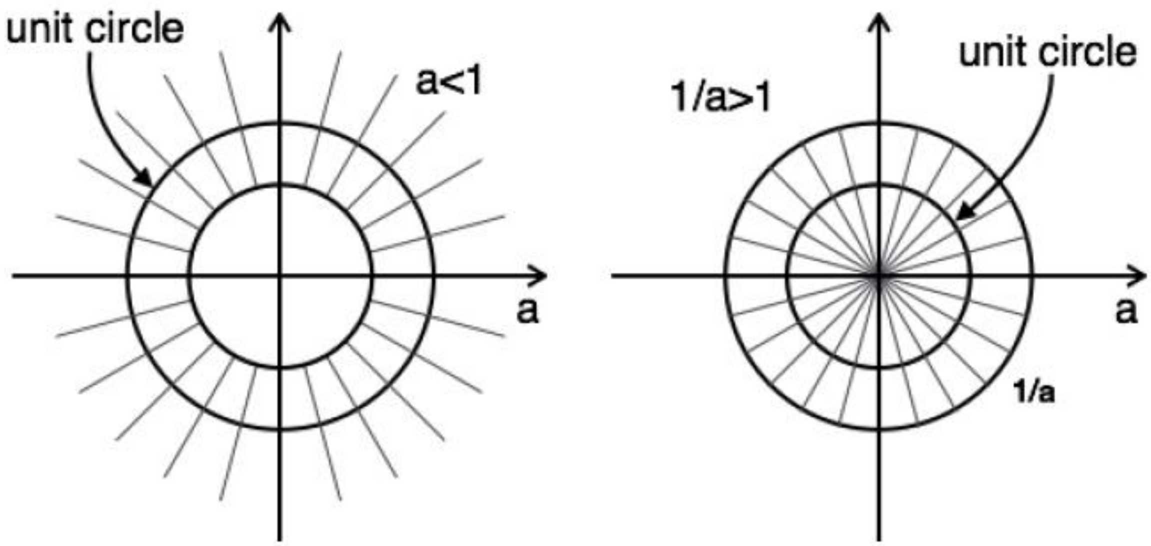
$$Z.T[a^n u[n]] + Z.T[a^{-n}u[-n-1]] = \frac{Z}{Z-a} + \frac{Z}{Z\frac{-1}{a}}$$

$$ROC: |z| > a \qquad ROC: |z| < \frac{1}{a}$$

The plot of ROC has two conditions as a > 1 and a < 1, as we do not know a.




In this case, there is no combination ROC.




Here, the combination of ROC is from a<|z|<1/a

Hence for this problem, z-transform is possible when a < 1.

**Causality and Stability**

**Causality condition for discrete time LTI systems is as follows:**

A discrete time LTI system is causal when

- ROC is outside the outermost pole.
- In The transfer function H[Z], the order of numerator cannot be grater than the order of denominator.

**Stability Condition for Discrete Time LTI Systems**

A discrete time LTI system is stable when

- its system function H[Z] include unit circle |z|=1.
- all poles of the transfer function lay inside the unit circle |z|=1.

**Z-Transform of Basic Signals**

| $x[n]$ | $X(z)$ | ROC |
|---|---|---|
| $\delta[n]$ | $1$ | All $z$ |
| $u[n]$ | $\frac{1}{1-z^{-1}}, \frac{z}{z-1}$ | $\lvert z\rvert > 1$ |
| $-u[-n-1]$ | $\frac{1}{1-z^{-1}}, \frac{z}{z-1}$ | $\lvert z\rvert < 1$ |
| $\delta[n-m]$ | $z^{-m}$ | All $z$ except 0 if $(m>0)$ or $\infty$ if $(m<0)$ |
| $a^n u[n]$ | $\frac{1}{1-az^{-1}}, \frac{z}{z-a}$ | $\lvert z\rvert > \lvert a\rvert$ |
| $-a^n u[-n-1]$ | $\frac{1}{1-az^{-1}}, \frac{z}{z-a}$ | $\lvert z < \lvert a\rvert$ |
| $na^n u[n]$ | $\frac{az^{-1}}{(1-az^{-1})^2}, \frac{az}{(z-a)^2}$ | $\lvert z\rvert > \lvert a\rvert$ |
| $-na^n u[-n-1]$ | $\frac{az^{-1}}{(1-az^{-1})^2}, \frac{az}{(z-a)^2}$ | $\lvert z\rvert < \lvert a\rvert$ |
| $(n+1)a^n u[n]$ | $\frac{1}{(1-az^{-1})^2}, \left[\frac{z}{z-a}\right]^2$ | $\lvert z\rvert > \lvert a\rvert$ |
| $(\cos \Omega_0 n)u[n]$ | $\frac{z^2-(\cos\Omega_0)z}{z^2-(2\cos\Omega_0)z+1}$ | $\lvert z\rvert > 1$ |
| $(\sin \Omega_0 n)u[n]$ | $\frac{(\sin\Omega_0)z}{z^2-(2\cos\Omega_0)z+1}$ | $\lvert z\rvert > 1$ |
| $(r^n \cos \Omega_0 n)u[n]$ | $\frac{z^2-(r\cos\Omega_0)z}{z^2-(2r\cos\Omega_0)z+r^2}$ | $\lvert z\rvert > r$ |
| $(r^n \sin \Omega_0 n)u[n]$ | $\frac{(r\sin\Omega_0)z}{z^2-(2r\cos\Omega_0)z+r^2}$ | $\lvert z\rvert > r$ |
| $\begin{cases} a^n & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$ | $\frac{1-a^N z^{-N}}{1-az^{-1}}$ | $\lvert z\rvert > 0$ |

**Some Properties of the Z- Transform:**

| Property | Sequence | Transform | ROC |
|---|---|---|---|
| | $x[n]$ | $X(z)$ | $R$ |
| | $x_1[n]$ | $X_1(z)$ | $R_1$ |
| | $x_2[n]$ | $X_2(z)$ | $R_2$ |
| Linearity | $a_1x_1[n]+a_2x_2[n]$ | $a_1X_1(z)+a_2X_2(z)$ | $R' \supset R_1 \cap R_2$ |
| Time shifting | $x[n-n_0]$ | $z^{-n_0}X(z)$ | $R' \supset R \cap \{0<\lvert z\rvert<\infty\}$ |
| Multiplication by $z_0^n$ | $z_0^n x[n]$ | $X\left(\frac{z}{z_0}\right)$ | $R' = \lvert z_0\rvert R$ |
| Multiplication by $e^{j\Omega_0 n}$ | $e^{j\Omega_0 n}x[n]$ | $X(e^{-j\Omega_0}z)$ | $R' = R$ |
| Time reversal | $x[-n]$ | $X\left(\frac{1}{z}\right)$ | $R' = \frac{1}{R}$ |
| Multiplication by $n$ | $nx[n]$ | $-z\frac{dX(z)}{dz}$ | $R' = R$ |
| Accumulation | $\sum_{k=-\infty}^{n} x[n]$ | $\frac{1}{1-z^{-1}}X(z)$ | $R' \supset R \cap \{\lvert z\rvert>1\}$ |
| Convolution | $x_1[n]*x_2[n]$ | $X_1(z)X_2(z)$ | $R' \supset R_1 \cap R_2$ |

## Inverse Z transform:

Three different methods are:

1. Partial fraction method
2. Power series method
3. Long division method

**Partial fraction method:**

- In case of LTI systems, commonly encountered form of $z$-transform is

$$X(z) = \frac{B(z)}{A(z)}$$

$$X(z) = \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \ldots + a_N z^{-N}}$$

Usually $M < N$

- If $M > N$ then use long division method and express $X(z)$ in the form

$$X(z) = \sum_{k=0}^{M-N} f_k z^{-k} + \frac{\tilde{B}(z)}{A(z)}$$

where $B(z)$ now has the order one less than the denominator polynomial and use partial fraction method to find $z$-transform

- The inverse $z$-transform of the terms in the summation are obtained from the transform pair and time shift property

$$1 \xleftrightarrow{z} \delta[n]$$

$$z^{-n_o} \xleftrightarrow{z} \delta[n - n_o]$$

- If $X(z)$ is expressed as ratio of polynomials in $z$ instead of $z^{-1}$ then convert into the polynomial of $z^{-1}$
- Convert the denominator into product of first-order terms

$$X(z) = \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{a_0 \prod_{k=1}^{N}(1 - d_k z^{-1})}$$

where $d_k$ are the poles of $X(z)$

**For distinct poles**

- For all distinct poles, the $X(z)$ can be written as

$$X(z) = \sum_{k=1}^{N} \frac{A_k}{(1 - d_k z^{-1})}$$

- Depending on ROC, the inverse $z$-transform associated with each term is then determined by using the appropriate transform pair
- We get

$$A_k (d_k)^n u[n] \xleftrightarrow{z} \frac{A_k}{1 - d_k z^{-1}},$$

with ROC $z > d_k$ OR

$$-A_k (d_k)^n u[-n-1] \xleftrightarrow{z} \frac{A_k}{1 - d_k z^{-1}},$$

with ROC $z < d_k$

- For each term the relationship between the ROC associated with $X(z)$ and each pole determines whether the right-sided or left sided inverse transform is selected

**For Repeated poles**

- If pole $d_i$ is repeated $r$ times, then there are $r$ terms in the partial-fraction expansion associated with that pole

$$\frac{A_{i_1}}{1-d_i z^{-1}}, \frac{A_{i_2}}{(1-d_i z^{-1})^2}, \ldots, \frac{A_{i_r}}{(1-d_i z^{-1})^r}$$

- Here also, the ROC of $X(z)$ determines whether the right or left sided inverse transform is chosen.

$$A\frac{(n+1)\ldots(n+m-1)}{(m-1)!}(d_i)^n u[n] \stackrel{z}{\longleftrightarrow} \frac{A}{(1-d_i z^{-1})^m}, \qquad \text{with ROC} |z| > d_i$$

- If the ROC is of the form $|z| < d_i$, the left-sided inverse $z$-transform is chosen, ie.

$$-A\frac{(n+1)\ldots(n+m-1)}{(m-1)!}(d_i)^n u[-n-1] \stackrel{z}{\longleftrightarrow} \frac{A}{(1-d_i z^{-1})^m}, \quad \text{with ROC} |z| < d_i$$

**Deciding ROC**

- The ROC of $X(z)$ is the intersection of the ROCs associated with the individual terms in the partial fraction expansion.
- In order to chose the correct inverse $z$-transform, we must infer the ROC of each term from the ROC of $X(z)$.
- By comparing the location of each pole with the ROC of $X(z)$.
- Chose the right sided inverse transform: if the ROC of $X(z)$ has the radius greater than that of the pole associated with the given term
- Chose the left sided inverse transform: if the ROC of $X(z)$ has the radius less than that of the pole associated with the given term

## Partial fraction method

- It can be applied to complex valued poles
- Generally the expansion coefficients are complex valued

- If the coefficients in $X(z)$ are real valued, then the expansion coefficients corresponding to complex conjugate poles will be complex conjugate of each other

- Here we use information other than ROC to get unique inverse transform
- We can use causality, stability and existence of DTFT
- If the signal is known to be causal then right sided inverse transform is chosen
- If the signal is stable, then t is absolutely summable and has DTFT
- Stability is equivalent to existence of DTFT, the ROC includes the unit circle in the $z$-plane, ie. $|z| = 1$
- The inverse $z$-transform is determined by comparing the poles and the unit circle
- If the pole is inside the unit circle then the right-sided inverse $z$-transform is chosen
- If the pole is outside the unit circle then the left-sided inverse $z$-transform is chosen

**Power series expansion method**

- Express $X(z)$ as a power series in $z^{-1}$ or $z$ as given in $z$-transform equation
- The values of the signal $x[n]$ are then given by coefficient associated with $z^{-n}$
- Main disadvantage: limited to one sided signals

- Signals with ROCs of the form $|z| > a$ or $|z| < a$
- If the ROC is $|z| > a$, then express $X(z)$ as a power series in $z^{-1}$ and we get right sided signal
- If the ROC is $|z| < a$, then express $X(z)$ as a power series in $z$ and we get left sided signal

**Long division method:**

- Find the $z$-transform of

$$X(z) = \frac{2+z^{-1}}{1-\frac{1}{2}z^{-1}}, \text{with ROC} \quad |z| > \frac{1}{2}$$

- Solution is: use long division method to write $X(z)$ as a power series in $z^{-1}$, since ROC indicates that $x[n]$ is right sided sequence
- We get

$$X(z) = 2 + 2z^{-1} + z^{-2} + \frac{1}{2}z^{-3} + \ldots$$

- Compare with $z$-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

- We get

$$x[n] = 2\delta[n] + 2\delta[n-1] + \delta[n-2] + \frac{1}{2}\delta[n-3] + \ldots$$

- If we change the ROC to $|z| < \frac{1}{2}$, then expand $X(z)$ as a power series in $z$ using long division method
- We get

$$X(z) = -2 - 8z - 16z^2 - 32z^3 + \ldots$$

- We can write $x[n]$ as

$$x[n] = -2\delta[n] - 8\delta[n+1] - 16\delta[n+2] - 32\delta[n+3] + \ldots$$

- Find the $z$-transform of

$$X(z) = e^{z^2}, \text{with ROC all } z \text{ except } |z| = \infty$$

- Solution is: use power series expansion for $e^a$ and is given by

$$e^a = \sum_{k=0}^{\infty} \frac{a^k}{k!}$$

- We can write $X(z)$ as

$$X(z) = \sum_{k=0}^{\infty} \frac{(z^2)^k}{k!}$$

$$X(z) = \sum_{k=0}^{\infty} \frac{z^{2k}}{k!}$$

- We can write $x[n]$ as

$$x[n] = \begin{cases} 0 & n > 0 \text{ or } n \text{ is odd} \\ \frac{1}{(\frac{-n}{2})!}, & \text{otherwise} \end{cases}$$

**Example: A finite sequence *x [ n ]* is defined as**

$$x[n] = \{5, 3, -\underset{\uparrow}{2}, 0, 4, -3\}$$

**Find X(z) and its ROC.**

Sol: We know that

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=-2}^{3} x[n] z^{-n}$$

$$= x[-2]z^2 + x[-1]z + x[0] + x[1]z^{-1} + x[2]z^{-2} + x[3]z^{-3}$$
$$= 5z^2 + 3z - 2 + 4z^{-2} - 3z^{-3}$$

For $z$ not equal to zero or infinity, each term in $X(z)$ will be finite and consequently $X(z)$ will converge. Note that $X(z)$ includes both positive powers of $z$ and negative powers of $z$. Thus, from the result we conclude that the ROC of $X(z)$ is $0 < |z| < \infty$.

**Example: Consider the sequence**

$$x[n] = \begin{cases} a^n & 0 \le n \le N-1,\ a > 0 \\ 0 & \text{otherwise} \end{cases}$$

**Find** $X(z)$ **and plot the poles and zeros of X(z) .**

**Sol:**

$$X(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} (az^{-1})^n = \frac{1-(az^{-1})^N}{1-az^{-1}} = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z-a}$$

From the above equation we see that there is a pole of $(N-1)^{th}$ order at $z = 0$ and a pole at $z = a$. Since x[n] is a finite sequence and is zero for $n < 0$, the ROC is $|z| > 0$. The $N$ roots of the numerator polynomial are at

$$z_k = ae^{j(2\pi k/N)} \qquad k = 0, 1, \ldots, N-1$$

The root at k = **0** cancels the pole at $z = a$. The remaining zeros of $X(z)$ are at

$$z_k = ae^{j(2\pi k/N)} \qquad k = 1, \ldots, N-1$$

The pole-zero plot is shown in the following figure with N=8