UNIT-I
FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS

ORDINARY DIFFERENTIAL EQUATION'S OF FIRST ORDER & FIRST DEGREE

Definition: An equation which involves differentials is called a Differential equation.

Ordinary differential equation: An equation is said to be ordinary if the derivatives have reference to only one independent variable.

Ex . (1) $\frac{dy}{dx}+7xy=x^2$ (2) $\frac{d^2y}{dx^2}+3\frac{dy}{dx}+2y=e^x$

(1) **Partial Differential equation:** A Differential equation is said to be partial if the derivatives in the equation have reference to two or more independent variables.

E.g: 1. $\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2=4z$ 2. $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=2z$

Order of a D.E equation: A Differential equation is said to be of order 'n' if the n^{th} derivative is the highest derivative in that equation.

E.g : (1). $(x^2+1)\cdot\frac{dy}{dx}+2xy=4x^2$

Order of this Differential equation is 1.

(2). $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0.$ Order is 2.

Degree of a Differential equation: Degree of a D .Equation is the degree of the highest derivative in the equation after the equation is made free from radicals and fractions in its derivations.

E.g : 1) $y=x\cdot\frac{dy}{dx}+\sqrt{1+\left(\frac{dy}{dx}\right)^2}$ on solving we get

$$(1-x^2)\left(\frac{dy}{dx}\right)^2+2xy\cdot\frac{dy}{dx}+(1-y^2)=0.$$ Degree = 2

Formation of Differential Equation : In general an O.D Equation is Obtained by eliminating the arbitrary constants c1,c2,c3--------cn from a relation like $\emptyset(x,y,c1,c2,------cn)=0.$ ------ (1). Where c1,c2,c3,---------cn are constants.

Differentiating (1) successively w.r.t x n- times and eliminating the n-arbitrary constant c1,c 2,----cn from the above (n+1) equations, we obtain the differential equation f(x , y, $y_1,y_2,\_\_\_\_\_$) =0.

Exact Differential Equations:

Def: Let M(x,y)dx +N(x,y) dy =0 be a first order and first degree Differential Equation where M & N are real valued functions of x,y . Then the equation Mdx + Ndy =0 is said to be an exact Differential equation if ∃ a function f ∋.

$$d[f(x,y)]=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy$$

Condition for Exactness: If M(x,y) & N (x,y) are two real functions which have continuous partial derivatives then the necessary and sufficient condition for the Differential equation Mdx+ Ndy =0 is to be exact is that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Hence solution of the exact equation M(x,y)dx +N(x,y) dy =0. Is

$$\int M dx+\int N dy=c.$$

(y constant) (terms free from x).

PROBLEMS:

1) Solve $(1+e^{\frac{x}{4}})$ **dx +** $e^{\frac{x}{y}}(1-\frac{x}{y})$ **.dy =0**

Sol: Hence $M = 1+e^{\frac{x}{y}}$ & $N = e^{\frac{x}{y}}(1-\frac{x}{y})$

$$\frac{\partial M}{\partial y} = e^{\frac{x}{y}}\left(\frac{-x}{y^2}\right) \quad \& \quad \frac{\partial N}{\partial x} = e^{\frac{x}{y}}\left(\frac{-1}{y}\right) + \left(1-\frac{x}{y}\right)e^{\frac{x}{y}}\left(\frac{1}{y}\right)$$

$$\frac{\partial M}{\partial y} = e^{\frac{x}{y}}\left(\frac{-x}{y^2}\right) \quad \& \quad \frac{\partial N}{\partial x} = e^{\frac{x}{y}}\left(\frac{-x}{y^2}\right)$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$ equation is exact

General solution is

$$\int M dx \quad + \quad \int N dy \quad = c.$$

(y constant) (terms free from x)

$$\int(1+e^{\frac{x}{y}})\,dx \quad + \quad \int 0\,dy \quad = c.$$

$$\Rightarrow x + \frac{e^{\frac{x}{y}}}{\frac{1}{y}} = c$$

$$\Rightarrow x + y\,e^{\frac{x}{y}} = C$$

2. **Solve [y(**$1+\frac{1}{x}$**) +cos y] dx+ [x +logx –xsiny]dy =0.**

Sol: hence $M = y(1+\frac{1}{x}) + \cos y$ $N = x + \log x - x\sin y$.

$$\frac{\partial M}{\partial y} = 1+\frac{1}{x} - \sin y \qquad \frac{\partial N}{\partial x} = 1+\frac{1}{x} - \sin y$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$ so the equation is exact

General sol $\int M dx \quad + \quad \int N dy \quad = c.$

(y constant) (terms free from x)

$$\int[y+\frac{y}{x}+cosy]dx \quad + \quad \int o\,.dy \quad = c.$$

$\Rightarrow$ Y(x+ logx) +x cosy = c.

3. (cosx-xcosy)dy – (siny+(ysinx))dx =0

Sol: N = cosx-x cosy & M = -siny-ysinx

$$\frac{\partial N}{\partial x} = -\sin x - \cos y \qquad \frac{\partial M}{\partial y} = -\cos y - \sin x$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$$ the equation is exact.

General sol $\int M dx \quad + \quad \int N dy \quad = c.$

(y constant) (terms free from x)

$$\Rightarrow \int(-siny - ysinx).dx \quad + \quad \int o\,.dy \quad = c$$

=> -xsiny+ ycos x =c

=> ycosx – xsiny =c.

REDUCTION OF NON-EXACT DIFFERENTIAL EQUATIONS TO EXACT USING INTEGRATING FACTORS

Definition: If the Differential Equation M(x,y) dx + N (x,y) dy = 0. Can be made exact by multiplying with a suitable function u (x,y) ≠ 0. Then this function is called an Integrating factor(I.F).

Note: there may exits several integrating factors.

Some methods to find an I.F to a non-exact Differential Equation Mdx+N dy =0

Case -1: Integrating factor by inspection/ (Grouping of terms).

Some useful exact differentials

1. d (xy) = xdy +y dx
2. $d\left(\frac{x}{y}\right) = \frac{ydx-xdy}{y^2}$
3. $d\left(\frac{y}{x}\right) = \frac{xdy-ydx}{x^2}$
4. $d\left(\frac{x^2+y^2}{2}\right) = x\,dx + y\,dy$
5. $d\left(\log\left(\frac{y}{x}\right)\right) = \frac{xdy-ydx}{xy}$
6. $d\left(\log\left(\frac{x}{y}\right)\right) = \frac{ydx-xdy}{xy}$
7. $d\left(\tan^{-1}\left(\frac{x}{y}\right)\right) = \frac{ydx-xdy}{x^2+y^2}$
8. $d\left(\tan^{-1}\left(\frac{y}{x}\right)\right) = \frac{xdy-ydx}{x^2+y^2}$
9. $d(\log(xy)) = \frac{xdy+ydx}{xy}$
10. $d(\log(x^2+y^2)) = \frac{2(xdx+ydy)}{x^2+y^2}$
11. $d\left(\frac{e^x}{y}\right) = \frac{ye^x\,dx-e^x\,dy}{y^2}$

PROBLEMS:

1. **Solve ydx –x dy = a (x^2+y^2) dx**

Ans: $\frac{ydx - x\,dy}{(x^2+y^2)} = a\,dx$

$d\left(\tan^{-1}\frac{y}{x}\right) = a\,dx$

integrating on $\tan^{-1}\frac{y}{x} = ax + c$

2.Solve xdx +y dy + $\frac{xdy-ydx}{x^2+y^2}$ = 0.

Sol: Given equationx dx + y dy + $\frac{xdy-ydx}{x^2+y^2} = 0$

$d\left(\frac{x^2+y^2}{2}\right) + d\left(\tan^{-1}\left(\frac{y}{x}\right)\right) = 0$

on Integrating

$\frac{x^2+y^2}{2} + \tan^{-1}\left(\frac{y}{x}\right) = c.$

3 . **Solve y(x^3. $e^{xy} - y$) dx + x (y + x^3. e^{xy}) dy = 0.**

Sol: Given equation is on Regrouping

We get $yx^3 e^{xy}\ dx - y^2 dx + x^2 y\, dy + x^4 e^{xy}\, dy = 0.$

$X^3 e^{xy}(ydx + xdy) + y\,(x\,dy - ydx) = 0$

Dividing by x^3

$e^{xy}\,(ydx + xdy) + \left(\frac{y}{x}\right)\ .\ \left(\frac{xdy - ydx}{x^2}\right) = 0$

$d\,(e^{xy}) + \left(\frac{y}{x}\right)\ .d + \left(\frac{y}{x}\right)\ = 0$

on Integrating

$e^{xy} + ½\left(\frac{y}{x}\right)^2 = C$ is required G.S.

4. **(1+xy) x dy + (1- yx) y dx = 0**

Sol: given equation is $(1+xy)\,x\,dy + (1-yx)\,y\,dx = 0$.

$(xdy + y\,dx) + xy\,(xdy - y\,dx) = 0.$

Divided by x^2y^2 => $\left(\frac{xdy + ydx}{x^2y^2}\right) + \left(\frac{xdy - ydx}{xy}\right) = 0$

$\left(\frac{d(xy)}{x^2y^2}\right) + \frac{1}{y}\,dy - \frac{1}{x}\,dx = 0.$

On integrating => $\frac{1}{xy} + \log y - \log x = \log c$

$-\frac{1}{xy} - \log x + \log y = \log c.$

Method -2: If M(x,y) dx + N (x,y) dy =0 is a homogeneous differential equation and and

Mx +Ny $\neq$ 0, then $\frac{1}{Mx + Ny}$ **is an integrating factor of Mdx+ Ndy =0.**

1 . **Solve x^{2}y dx – (x^3+ y^3) dy = 0**

Sol : $x^2y\,dx - (x^3 + y^3)\,dy = 0$-----------------(1)

Where $M = x^2y$ & $N = (-x^3 - y^3)$

Consider $\frac{\partial M}{\partial y} = x^2$ & $\frac{\partial N}{\partial x} = -3x^2$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ equation is not exact .

But given equation(1) is homogeneous D.Equation then

So $Mx + Ny = x(x^2y) - y\,(x^3 + y^3) = -y^4 \neq 0.$

$$\text{I.F} = \frac{1}{Mx + Ny} = \frac{-1}{y^4}$$

Multiplying equation (1) by $\frac{-1}{y4}$

$$=> \frac{x^2y}{-y^4}\ dx - \frac{x^3 + y^3}{-y^4}\,dy = 0 \qquad \text{----------------------(2)}$$

$$=> -\frac{x2}{y3}\ dx - \frac{x3 + y3}{-y4}\,dy = 0$$

This is of the form $M_1dx + N_1dy = 0$

For $M_1 = \frac{-x2}{y3}$ & $N_1 = \frac{x^3 + y^3}{-y^4}$

$=> \quad \frac{\partial M1}{\partial y} = \frac{3x^2}{y4}$ & $\frac{\partial N1}{\partial x} = \frac{3x^2}{-y^4}$

$=> \quad \frac{\partial M1}{\partial y} = \frac{\partial N1}{\partial x}$ equation (2) is an exact D.equation.

General sol $\int M dx \quad + \int N dy \quad = c.$

(y constant) (terms free from x in N)

$=> \quad \int \frac{-x^2}{y^3} dx \quad + \int \frac{1}{y} dy \quad = c.$

$=> \quad \frac{-x^3}{3y^3}$ + log |y| = c.//

3. **Solve y($y^2 - 2\ x^2$) dx + x (2 $y^2 - \ x^2$) dy =0** -----------(1)

Sol: it is the form Mdx +Ndy =0

Where M = y($y^2 - 2\ x^2$) N= x (2 $y^2 - \ x^2$)

Consider $\frac{\partial M}{\partial y} = 3y^2\text{-}2x^2$ & $\frac{\partial N}{\partial x} = 2y^2\text{-}3x^2$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ equation is not exact .

Since equation(1) is homogeneous D.Equation then

Consider Mx+ N y= x[y($y^2 - 2\ x^2$)] +y [x (2 $y^2 - \ x^2$)]

= 3xy ($y^2 - \ x^2$) ≠ 0.

$=> \quad$ I.F. $= \frac{1}{3xy\ (\ y^2 - x^2)}$

Multiplying equation (1) by $\frac{1}{3xy\ (\ y^2 - x^2)}$ we get

$=> \frac{y\ (\ y^2 - x^2)}{3xy\ (\ y^2 - x^2)}\ dx + = \frac{x(\ y^2 - x^2)}{3xy\ (\ y^2 - x^2)}$ dy =0

=> now it is exact (check)

$\frac{(\ y^2 - x^2) - x^2}{3xy\ (\ y^2 - x^2)}$ dx + $\frac{y^2 + (\ y^2 - x^2)}{3xy\ (\ y^2 - x^2)}$ dy =0.

$\frac{dx}{x} - \frac{xdx}{y^2 - x^2} + \frac{ydy}{y^2 - x^2} + \frac{dy}{y} = 0.$

$(\frac{dx}{x} + \frac{dy}{y}) + \frac{2ydy}{2(y^2 - x^2)} \frac{2xdx}{2(y^2 - x^2)} = 0$

Log x +log y +$\frac{1}{2}$ log $(y^2 - x^2)$ - $\frac{1}{2}$ log $(y^2 - x^2) = c \quad \Rightarrow \quad$ xy = c

Method- 3: If the equation Mdx + N dy =0 is of the form y. f (xy) .dx + x . g (xy) dy = 0 &

Mx- Ny $\neq 0$ then $\frac{1}{\text{Mx}-\text{Ny}}$ is an integrating factor of Mdx+ Ndy =0.

Problems:

1 . solve (xy sinxy +cosxy) ydx + (xy sinxy –cosxy)x dy =0.

Sol: (xy sinxy +cosxy) ydx + (xy sinxy –cosxy)x dy =0 -------(1).

=> this is the form y. f(xy) .dx + x . g (xy) dy =0.

=> consider Mx-Ny

Here M =(xy sinxy + cos xy) y

N= (xy sinxy- cos xy) x

Consider Mx-Ny =2xycosxy

Integrating factor $= \frac{1}{2xycosxy}$

So equation (1) x I.F

$$\Rightarrow \frac{(xy\sin xy+\cos xy)x}{2xy\ \cos xy}dx + \frac{(xy\sin xy+\cos xy)y}{2xy\ \cos xy}dy = 0.$$

$$\Rightarrow (y\tan xy + \frac{1}{x})\,dx + (y\tan xy - \frac{1}{y})\,dy = 0$$

$$\Rightarrow M_1\,dx + N_1\,dx = 0$$

Now the equation is exact.

General sol $\int M_1\,dx + \int N_1\,dy = c.$

(y constant) (terms free from x in N_1)

$$\Rightarrow \int(y\tan xy + \frac{1}{x})dx + \int \frac{-1}{y}dy = c.$$

$$\Rightarrow \frac{y.\log|\sec xy|}{y} + \log x + (-\log y) = \log c$$

$$\Rightarrow \log|\sec(xy)| + \log\frac{x}{y} = \log c.$$

$$\Rightarrow \frac{x}{y}.\sec xy = c.$$

2. Solve (1+xy) y dx + (1-xy) x dy =0

Sol : I.F $=\frac{1}{2x^2y^2}$

$$\Rightarrow \int \frac{1}{2x^2y} + \frac{1}{2x})dx + \int \frac{-1}{2y}dy = c$$

$$\Rightarrow \frac{-1}{2xy} + \frac{1}{2}\log x - \frac{1}{2}\log y = c.$$

$$\Rightarrow \frac{-1}{xy} + \log(\frac{x}{y}) = c^1 \quad \text{where } c^1 = 2c.$$

Method -4: If there exists a single variable function $\int f(x)\,dx$ such that $\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}$ $=f(x)$, then I.F. of Mdx + N dy =0 is e $e^{\int f(x)dx}$

PROBLEMS:

1 . Solve (3xy – 2ay^2) dx + ($x^2 - 2axy$) dy =0

Sol: given equation is the form Mdx+ Ndy = 0

$\Rightarrow$ M = 3xy – 2ay^2 & N = $x^2 - 2axy$

$$\frac{\partial M}{\partial y} = 3x-4ay \ \& \ \frac{\partial N}{\partial x} = 2x-2ay$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \quad \text{equation not exact .}$$

Now consider $\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N} = \frac{(3x-4ay)-(2x-2ay)}{(2x-2ay)}$

$$\Rightarrow \frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N} = \frac{1}{x} = f(x).$$

$\Rightarrow e^{\int\frac{1}{x}dx} = x$ is an Integrating factor of (1)

=> equation (1) x I.F = equation (1) X x

$$\Rightarrow \frac{(3xy-2ay^2)}{1}\ x\,dx + \frac{(x^2-2axy)}{1}\ x\,dy = 0$$

$$\Rightarrow (3x^2y - 2ay^2x)\,dx + (x^3 - 2ax^2y)\,dy = 0$$

It is the form $M_1dx + N_1dy = 0$

General sol $\int M_1dx + \int N_1dy = c.$

$= > \int (3x^2 - 2ay^2x)dx + \int o d\, y = c$

$= > x^3y - ax^2y^2 = c$ .//

2 . **Solve ydx-xdy+(1+x^2)$dx + x^2 \sin y\, dy = 0$**

Sol : given equation is (y+1+x^2) $dx + (x^2 \, siny - x)$ dy =0.

M= y+1+x^2 & N =$x^2 \, siny - x$

$\frac{\partial M}{\partial y} = 1$ $\frac{\partial N}{\partial x} = 2x \sin y - 1$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ = > the equation is not exact.

So consider $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \frac{(1-2x\, siny - x)}{x^2 siny - x} = \frac{-2x\, siny - x)}{x^2 siny - x} = \frac{-2}{x}$

I.F = $e^{\int g(y)dy} = e^{-2\int \frac{1}{x}dx} = e^{-2\log x} = \frac{1}{x^2}$

Equation (1) X I.F => $\frac{y+1+x^2}{x^2}\, dx + \frac{x^2\, siny - x}{x^2}$ dy =0

It is the form of M1dx+ N1 dy =0.

Gen soln =>$\int (\frac{y}{x^2} + \frac{1}{x^2} + 1)dx + \int siny dy$ =0

=>$\frac{-y}{x} - \frac{1}{x}$ +x- cosy =c.

=> $x^2 - y - 1 - xcosy = cx$.//

Method -5: For the equation Mdx + N dy = 0 if $\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}$ = g(y) (is a function of y alone) then $e^{\int g(y)dy}$ is the Integrating factor of M dx + N dy =0.

Problems:

1 . **Solve $(3x^2y^4+2xy)dx + (2x^3y^3-x^2)\, dy = 0$**

Sol: $(3x^2y^4+2xy)dx + (2x^3y^3-x^2)$ dy =0 -----------------(1).

Here M dx + N dy =0.

Where M =$3x^2y^4+2xy$ & N = $2x^3y^3-x^2$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ equation (1) not exact.

So consider $\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{-2}{y}$ = g(y)

I.F = $e^{\int g(y)dy} = e^{-2\int \frac{1}{y}dy} = e^{-2\log y} = \frac{1}{y^2}$.

Equation (1) x I.F => $\left(\frac{3x2y4+2xy}{y^2}\right) dx + (\frac{2x3y3-x2}{y^2})$ dy =0

It is the form M1dx + N1 dy =0

General sol $\int M1dx$ + $\int N1dy$ = c.

(y constant) (terms free from x in N1)

=> $\int (3x^2y^2 + \frac{2x}{y})dx + \int o\, dy$ =c.

=> $\frac{3x^3y^2}{3} + \frac{2x^2}{2y}$ =c.

=> $x^3y^2 + \frac{x^2}{y}$ =c.//

2 . **Solve $(xy^3+y)\, dx + 2(x^2y^2+x+y^4)\, dy = 0$**

Sol: $\frac{\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)}{M} = \frac{(4xy^2+2)-(3xy^2+1)}{xy^3+y} = \frac{1}{y}$ = g(y).

I.F = $e^{\int g(y)dy} = e^{\int \frac{1}{y}dy}$ =y.

Gen sol: $\int(\text{xy4} + \text{y2})dx + \int(\text{2y5})dy = c$

$\frac{x^2y^4}{2} + y^2$ x $+\frac{2y^6}{6}$ =c .

3 . solve $(y^4+2y)dx + (xy^3 + 2y^4 - 4x)dy = 0$

Sol: $\frac{\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)}{M} = \frac{(y^3-4)-(4y^3+2)}{y^4+2y} = \frac{-3}{y}$ = g(y).

I.F = $e^{\int g(y)dy} = e^{-3\int \frac{1}{y}dy} = \frac{1}{y^3}$

Gen soln : $\int\left(y+\frac{2}{y^2}\right)dx + \int 2y\,dy = c$.

$\left(y+\frac{2}{y^2}\right)x + y^2 = c.$

LINEAR DIFFERENTIAL EQUATION'S OF FIRST ORDER:

Def: An equation of the form $\frac{dy}{dx} + P(x).y = Q(x)$ is called a linear differential equation of first order in y.

Working Rule: To solve the liner equation $\frac{dy}{dx} + P(x).y = Q(x)$

first find the Integrating factor I.F $=e^{\int p(x)dx}$

General solution is y x I.F = $\int Q(x)\times I.F.dx + c$

Note: An equation of the form $\frac{dx}{dy} + p(y).x =$ Q(y) called a linear Differential equation of first order in x.

Then Integrating factor $=e^{\int p(y)dy}$

Gen soln is = x X I.F = $\int Q(y)\times I.F.dy + c$

PROBLEMS:

1 . Solve $(1+y^2)$ dx=(tan^{-1}y –x) dy

Sol: $(1+y^2)\frac{dx}{dy} = (tan^{-1}y - x)$

$\frac{dx}{dy} + \left(\frac{1}{1+y^2}\right) . x = \frac{tan^{-1}}{1+y^2}$

It is the form of $\frac{dx}{dy}$ + p(y).x = Q(y)

I.F $=e^{\int p(x)dx} = e^{\int \frac{1}{1+y^2}dy} = e^{tan^{-1}y}$

=> Gen sol is x. $e^{tan^{-1}y} = \int \frac{tan^{-1}}{1+y^2} . e^{tan^{-1}y}\,dy$+ c.

= > x. $e^{tan^{-1}y} = \int t . e^t\,dt + c$

[put $tan^{-1} y = t$

$\Rightarrow \frac{1}{1+y^2} dy = dt$]

$\Rightarrow$x. $e^{tan^{-1}y}$ = t . e^t -e^t +c

=> x. $e^{tan^{-1}y}$ = $tan^{-1}y. e^{tan^{-1}y}$ -$e^{tan^{-1}y}$ +c

=> x = $tan^{-1}y - 1$ + c/ $e^{tan^{-1}y}$ is the required solution

2. Solve (x+y+1) $\frac{dy}{dx}$ = 1.

Sol: g iven equation is (x+y+1) $\frac{dy}{dx}$ = 1.

=> $\frac{dx}{dy} - x$ = y+1.

It is of the form $\frac{dx}{dy}$ + p(y).x = Q(y)

Where p(y) = -1 ; Q(y) = 1+y

=> I.F =$e^{\int p(y)dy}$ = $e^{-\int dy}$ = e^{-y}

Gen soln = x X I.F = $\int Q(y) \times I.F.dy + c$

=> x . $e^{-y} = \int (1+y)\, e^{-y} dy + c$

=> x . $e^{-y} = \int e^{-y} dy + \int y e^{-y} dy + c$

=> xe^{-y} = - $e^{-y} -$ y xe^{-y} - e^{-y} +c

=> xe^{-y} = - $e^{-y}(2+y)$ +c .//

3. Solve $y^1 + y = e^{e^x}$

Sol: this is of the form $\frac{dy}{dx} + p(x).y =$ Q(x)

Where p(x) =1 Q(x) = e^{e^x}

=> I.F = $e^{\int p(x)dx}$ = $e^{\int dx}$ =e^x

Gen soln is is y x I.F = $\int Q(x) \times I.F.dx + c$

=> y. $e^x = \int e^{e^x} e^x dx + c$

=> y. $e^x = \int e^t t\, dt + c$ put $e^x = t$

=> y. e^x = t $e^t -$ e^t +c $e^x dx = dt$

=> y. e^x = $e^{e^x}(e^x - 1) + c.$

4. Solve x. $\frac{dy}{dx}$ + y =log x

Sol : this is of the form $\frac{dy}{dx}$ +p(x)y = $\emptyset(x)$.

Where $p(x) = \frac{1}{x}$ & $\emptyset(x) = \frac{\log x}{x}$

i.e , $\frac{dy}{dx} + \frac{1}{x} . y = \frac{\log x}{x}$

=> I.F = $e^{\int p(x)dx} = e^{\int \frac{1}{x}dx} = e^{\log x} = x.$

Gen soln is is y x I.F = $\int Q(y) \times I.F.dy + c$

=> y.x = $\int \frac{\log x}{x}\, x\, dx + c$

=> y .x = x (logx-1) +c.//

5 . Solve $(1+y^2) + (x- e^{\tan^{-1} y})\frac{dy}{dx} = 0.$

Sol : Given equation is $\frac{dx}{dy} + \frac{x}{1+y2} = \frac{e^{\tan^{-1} y}}{1+y2}$

It is of the form $\frac{dx}{dy}$ + p(y) .x = Q(y)

Where p (y) = $\frac{1}{1+y2}$ Q(y) $= \frac{e^{\tan^{-1} y}}{1+y2}.$

I.F $= e^{\int p(y)dy} = e^{\int \frac{1}{1+y2}dy} = e^{\tan^{-1} y}.$

General solution is is x x I.F = $\int Q(y) \times I.F.dy + c$.

$=> x . e^{\tan^{-1} y}. = \int \frac{e^{\tan^{-1} y}}{1+y2} e^{\tan^{-1} y}.$ dy +c

$=> x . e^{\tan^{-1} y} = \int e^t e^t.$ dt +c

[**Note: put** $\tan^{-1} y$ = t

$=> \frac{1}{1+y2} dy$ = dt]

$=> x . e^{\tan^{-1} y} = \int e^{2t}$. dt +c

$=> x . e^{\tan^{-1} y} = \frac{e^{2t}}{2}$+ c

$=> x . e^{\tan^{-1} y} = \frac{e^{2\tan^{-1} y}}{2}$+ c //

6. Solve $\frac{dy}{dx} - \frac{\tan y}{1+x}$ = (1+x) e^x . sec y

Sol : the above equation can be written as

Divided by sec y => cos y $\frac{dy}{dx} - \frac{\sin y}{1+x}$ = (1+x) e^x ------------(1)

Put sin y = u

$=> \cos y \frac{dy}{dx} = \frac{du}{dx}$

D. Equation (1) is $\frac{du}{dx} - \frac{1}{1+x} . u = (1+x)\ e^x$

this is of the form $\frac{du}{dx} + p(x).u =$ Q(x)

Where $p(x) = \frac{-1}{1+x}$ $\quad Q(x) = (1+x)\ e^x$

=> $\quad$ I.F = $e^{\int p(x)dx} = e^{\int \frac{-1}{1+x}dx} = e^{-\log(1+x)} = \frac{1}{1+x}$

Gen soln is is u x I.F = $\int Q(y) \times I.F.dy + c$

=> $\quad u.\frac{1}{1+x} = \int (1+x)\ e^x \frac{1}{1+x} dx + c$

=> $\quad u.\frac{1}{1+x} = \int e^x dx + c$

=> $(\sin y)\frac{1}{1+x} = e^x + c$

(Or)

= > $\sin y = (1+x)\ e^x + c . (1+x)$ is required solution.

BERNOULLI'S EQUATION :

(EQUATION'S REDUCIBLE TO LINEAR EQUATION)

Def: An equation of the form $\frac{dy}{dx} + p(x) .y = Q(x)\ y^n$ ----- --------**(1)**

Is called Bernoulli's Equation, where p & Q are function of x and n is a real constant.

Working Rule:

Case -1 : if n=1 then the above equation becomes $\frac{dy}{dx} + p. y = Q.$

=> Gen soln of $\frac{dy}{dx} + (p-Q)y = 0$ is

$\int \frac{dy}{dx} + \int (p-Q)dx = c$ by variable separation method.

Case -2: if $n \neq 1$ then divide the given equation (1) by y^n

$\Rightarrow \quad y^{-n} . \frac{dy}{dx} + p(x) . y^{1-n} = Q$ ---------(2)

Then take $y^{1-n} = u$

$(1-n)\ y^{-n}.\frac{dy}{dx} = \frac{du}{dx}$

$\Rightarrow \quad y^{-n}. \frac{dy}{dx} = \frac{1}{1-n} \frac{du}{dx}$

Then equation (2) becomes

$\frac{1}{1-n} \frac{du}{dx} + p(x) . u = Q$

$\frac{du}{dx}$ + (1-n) p.u = (1-n)Q which is linear and hence we can solve it.

Problems:

1 . Solve $x\frac{dy}{dx} + y = x^3 y^6$

Sol: given equation can be written as $\frac{dy}{dx} + \left(\frac{1}{x}\right) y = x^2+y^6$

Which is of the form $\frac{dy}{dx}$ + p(x).y =Q y^n

Where p(x) = $\frac{1}{x}$ Q(x) = x^2 & n=6

Divided by y^2 => $\frac{1}{y^6} \cdot \frac{dy}{dx} + \frac{1}{x}\frac{1}{y^5} = x^2$ ---------------(2)

Take $\frac{1}{y^5}$ = u

$\Rightarrow \frac{-5}{y^6}\frac{dy}{dx} = \frac{du}{dx}$ }----------------(3)

$\Rightarrow \frac{1}{y^6}\frac{dy}{dx} = \frac{-1}{5}\frac{du}{dx}$ }----------------(3)

(3) in (2) => $\frac{du}{dx} - \frac{5}{x}u = -5x^2$

Which is a L.D equation in u

I.F = $e^{\int p(x)dx} = e^{-5\int\frac{1}{x}dx} = e^{-5\log x} = \frac{1}{x^5}$

Gensol $\Rightarrow$ u .I.F = $\int Q(y)\times I.F.dy + c$

$u.\frac{1}{x^5} = \int -5x2.\frac{1}{x^5}\,dx + c$

$\frac{1}{y^5 x^5} = \frac{5}{2x^2} + c$ (or) $\frac{1}{y^5} = \frac{5x^3}{2} + cx^5$

2. Solve $\frac{dy}{dx}(x^2y^3 + xy) = 1$

Sol: $\frac{dx}{dy} - x.y = x^2y^3$ => $\frac{1}{x^2}\cdot\frac{dx}{dy} - \frac{1}{x}.y = y^3$ ---------------------(1)

Put $\frac{1}{x}$ = u

$\Rightarrow \frac{-1}{x^2}\cdot\frac{dx}{dy} = \frac{du}{dx}$ ---------------(2).

(2) in (1) $\Rightarrow -\frac{du}{dx} - u.y = y^3$

(Or) $\frac{du}{dx} + u.y = -y^3$.

Is a L.D Equation in ‘ u’

I.F = $e^{\int P(y)dy} = e^{\int y\,dy} = e^{-\frac{y^2}{2}}$

Gensol $\Rightarrow$ u .I.F $= \int Q(y) \times I.F.dy + c$

$\Rightarrow$ u . $e^{-\frac{y^2}{2}} = \int y^3 . e^{-\frac{y^2}{2}}\,dy$ + c

$\Rightarrow \dfrac{e^{-\frac{y^2}{2}}}{x} = -2(\frac{y^2}{2} - 1)\ . e^{-\frac{y^2}{2}}$ +c

(or)

X(2-y^2)+ cx$e^{-\frac{y^2}{2}}$ =1.

3. $(1-x^2)\frac{dy}{dx}$ + xy = $y^3\ sin^{-1}$x

Sol: given equation can be written as

$\frac{dy}{dx} + \frac{x}{1-x^2}$ y = $\frac{y^3}{1-x^2} sin^{-1}$x

Which is a Bernoulli's equation in ' y '

Divided by $y^3 \Rightarrow \frac{1}{y^3} . \frac{dy}{dx} + \frac{1}{y^2}\ \frac{x}{1-x^2} = \frac{sin^{-1}x}{1-x^2}$ -----------(1).

Let $\frac{1}{y^2}$ = u

$\Rightarrow \frac{-2}{y^3}\frac{dy}{dx} = \frac{du}{dx}$ => $\frac{1}{y^3}\frac{dy}{dx} = -\frac{1}{2}\frac{du}{dx}$-------------(2)

(2) in (1) $\Rightarrow -\frac{1}{2}\frac{du}{dx} + \frac{x}{1-x^2} . u = \frac{sin^{-1}x}{1-x^2}$

Which is a L.D equation in u

$\Rightarrow$ I.F = $e^{\int p(x)dx} = e^{-\int \frac{2x}{1-x^2}dx} = e^{\log(1-x^2)} = (1-x^2)$

Gensol $\Rightarrow$ u .I.F $= \int Q(x) \times I.F.dx + c$

$\Rightarrow \frac{1}{y^2}\ (1-x^2) = -\int \frac{2sin^{-1}x}{1-x^2}(1-x^2)dx + c$

=> $\frac{(1-x^2)}{y^2}$ = -2 [xsin^{-1}x +$\sqrt{1-x^2}$] +c

NEWTON'S LAW OF COOLING

STATEMENT: The rate of change of the temp of a body is proportional to the difference of the temp of the body and that of the surroundings medium.

Let 'θ' be the temp of the body at time 't' and θo be the temp of its surroundings medium(usually air). By the Newton's low of cooling , we have

$\frac{d\theta}{dt} \propto (\theta - \theta o) \Rightarrow -\frac{d\theta}{dt} = k(\theta - \theta o)$ k is +ve constant

$\Rightarrow \int \frac{d\theta}{(\theta - \theta o)} . = -k \int dt$

$\Rightarrow$ log $(\theta - \theta o)$ = -kt +c.

If initially $\theta = \theta 1$ is the temp of the body at time t=0 then

$$c = \log(\theta 1 - \theta o) \Rightarrow \log(\theta - \theta o) = -kt + \log(\theta 1 - \theta o)$$

$$\Rightarrow \log\left(\frac{(\theta - \theta o)}{(\theta 1 - \theta o)}\right) = -kt.$$

$$\Rightarrow \frac{(\theta - \theta o)}{(\theta 1 - \theta o)} = e^{-kt}$$

$$\theta = \theta o + (\theta 1 - \theta o).\, e^{-kt}$$

Which gives the temp of the body at time 't' .

Problems:

1 A body is originally at 80^0 and cools dowm to 60^0 c in 20 min . if the temp of the air is 40^0 c. Find the temp of body after 40 min.

Sol: By Newton's low of cooling we have

$$\frac{d\theta}{dt} = k(\theta - \theta o) \qquad \theta o \text{ is the temp of the air.}$$

$$\Rightarrow \int \frac{d\theta}{(\theta - 40)} . = -k \int dt \qquad \theta o = 40^0 \text{ c}$$

$$\Rightarrow \log(\theta - 40) = -kt + \log c$$

$$\Rightarrow \log\left(\frac{\theta - 40}{c}\right) = -kt$$

$$\Rightarrow \frac{\theta - 40}{c} = e^{-kt}$$

$$\Rightarrow \theta = 40 + c\, e^{-kt} \text{ --------------(1)}$$

When t=0 , $\theta = 80^0$ c $\Rightarrow$ 80 = 40 +c ------------(2).

When t=20 , $\theta = 60^0$ c $\Rightarrow$ 60 = 40 +ce^{-20k} ------------(3).

Solving (2) & (3) $\Rightarrow$ c$e^{-20k} = 20$

C=40 $\Rightarrow$ 40e^{-2k} =20

=> k = $\frac{1}{20}$ log2

When t= 40^0 c => equation (1) is $\theta = 40 + 40\, e^{-(\frac{1}{20}\log 2)40}$

$= 40 + 40\, e^{-2\log 2}$

$= 40 + (\ 40 \times \frac{1}{4}\ )$

$\Rightarrow \theta = 50^0$ c

2 . An object when temp is 75^0c cools in an atmosphere of constant temp. 25^0 c, at the rate k θ ,θ being the excess temp of the body over that of the temp. If after 10min , the temp of the object falls to 66^0 c , find its temp after 20 min . also find the time required to cool down to 55^0c .

Sol : we will take one as unit of time.

It is given that $\frac{d\theta}{dt}$ = - kθ

$\Rightarrow$ sol is θ = c e^{-kt} ------------(1).

Initially when t=0 $\Rightarrow \theta = 75^0 - 25^0 = 50^0$

$\Rightarrow$ c= 50^0 -----------------(2)

When t= 10 min $\Rightarrow \theta = 65^0 - 25^0 = 40^0$

$\Rightarrow$ 40= 50 e^{-10k}

$\Rightarrow e^{-10k} = \frac{4}{5}$ ---------------------(3).

The value of θ when t=20 $\Rightarrow \theta$ = c e^{-kt}

$$\theta = 50e^{-20k}$$
$$\theta = 50(e^{-10k})^2$$
$$\theta = 50(\tfrac{4}{5})^2$$

when t=20 $\Rightarrow$ $\theta = 32^0$ c.

3. A body kept in air with temp 25^0c cools from 140^0 c to 80^0 in 20 min. Find when the body cools down in 35^0.

Sol : here θo $=25^0c$ $\Rightarrow$ $\frac{d\theta}{(\theta-25)}$ = -k dt

$\Rightarrow$ log $(\theta - 25)$ = -kt +c -------------(1).

When t=0 , $\theta = 140^0$ c $\Rightarrow$ log (115) =c

$\Rightarrow$ c =log (115).

$\Rightarrow$ kt = - log $(\theta - 25)$+ log 115--------(2)

When t=20 , $\theta = 80^0$ c

$\Rightarrow$ log(80^0 c) = -20k + log 115

$\Rightarrow$ 20 k =log (115) - log(55) ---------(3)

(2)/ (3) => $\frac{kt}{20k} = \frac{\log 115 - \log(\theta-25)}{\log 115 - \log 55}$

$$\frac{t}{20} = \frac{\log 115 - \log(\theta-25)}{\log 115 - \log 55}$$

When $\theta = 35^0$ c $\Rightarrow$ $\frac{t}{20} = \frac{\log 115 - \log(10)}{\log 115 - \log 55}$

$$\Rightarrow \frac{t}{20} = \frac{\log(11.5)}{\log(\frac{23}{11})} = 3.31$$

$\Rightarrow$ t = 20 $\times$ 3.31 = 66.2

The temp will be 35^0 c after 66.2 min.

LAW OF NATURAL GROWTH OR DECAY

(STATEMENT: Let x(t) or x be the amount of a substance at time ‘ t’ and let the substance be getting converted chemically . A law of chemical conversion states that the rate of change of amount x(t) of a chemically changed substance is proportional to the amount of the substance available at that time

$$\frac{dx}{dt} \propto x \quad \text{(or)} \quad \frac{dx}{dt} = -kt \ ; \ (k>0)$$

Where k is a constant of proportionality

Note: In case of Natural growth we take

$$\frac{dx}{dt} = k.x\ )$$

PROBLEMS

1 The number N of bacteria in a culture grew at a rate proportional to N . The value of N was initially 100 and increased to 332 in one hour. What was the value of N after 1hrs

Sol: The D. Equation to be solved is $\frac{dN}{dt}$ = kN

$$\Rightarrow \frac{dN}{N} = k\,dt$$

$$\Rightarrow \int \frac{dN}{N}. = \int k dt$$

$\Rightarrow$ log N = kt + log e

$\Rightarrow$ N = c e^{kt} ------------(1).

When t= 0sec , N =100 $\Rightarrow$ 100 = c $\Rightarrow$ c =100

When t =3600 sec , N =332 $\Rightarrow$ 332 =100 e^{3600k}

$\Rightarrow$ $e^{3600k} = \frac{332}{100}$

Now when t =$\frac{3}{2}$ hors = 5400 sec then N=?

$\Rightarrow$ N =100 e^{5400k}

$\Rightarrow$ N =100[e^{3600k}]$^{\frac{3}{2}}$

$\Rightarrow$ N = 100 $[\frac{332}{100}]^{\frac{3}{2}}$ = 605.

$\Rightarrow$ N = 605.

2.. The temp of cup of coffie is 92^0c . in which freshly period the room temp being 24^0c . in one min it was cooled to 80^0c . how long a period must elspse , before the temp of the cup becomes 65^0c .

Sol: : By Newton's Law of Cooling,

$\frac{d\theta}{dt}$ = -k($\theta - \theta o$) ; k>0

θo = 24^0c $\Rightarrow$ log ($\theta - 24$) = -kt + log c--------------(1).

When t=0 ; θ = 92 $\Rightarrow$ c =68

When t =1 ; $\theta = 80^0$c $\Rightarrow$ $e^k = \frac{68}{56}$

$\Rightarrow$ k = log ($\frac{68}{56}$).

When $\theta = 65^0$c , t =?

Ans: t = $\frac{41}{56}$ min.

EQUATIONS OF THE FIRST ORDER AND HIGHER DEGREE

Case. I. Equation solvable for p. A differential equation of the first order but of the nth degree is of the form

$$p^n + P_1 p^{n-1} + P_2 p^{n-2} + \dots + P_n = 0 \quad \dots(1)$$

where $P_1, P_2, \dots, P_n$ are functions of x and y.

Splitting up the left hand side of (1) into n linear factors, we have

$$[p - f_1(x, y)]\ [p - f_2(x, y)] \dots [p - f_n(x, y)] = 0.$$

Equating each of the factors to zero,

$$p = f_1(x, y), p = f_2(x, y), \dots, p = f_n(x, y)$$

Solving each of these equations of the first order and first degree, we get the solutions

$$F_1(x, y, c) = 0, F_2(x, y, c) = 0, \dots, F_n(x, y, c) = 0.$$

These n solutions constitute the general solution of (1).

Otherwise, the general solution of (1) may be written as

$$F_1(x, y, c) \cdot F_2(x, y, c) \dots\dots F_n(x, y, c) = 0.$$

Solve $\frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - \frac{y}{x}$.

Solution. Given equation is $p - \frac{1}{p} = \frac{x}{y} - \frac{y}{x}$ where $p = \frac{dy}{dx}$ or $p^2 + p\left(\frac{y}{x} - \frac{x}{y}\right) - 1 = 0$.

Factorising $(p + y/x)(p - x/y) = 0$.

Thus we have $p + y/x = 0$ ...(i) and $p - x/y = 0$

From (i), $\frac{dy}{dx} + \frac{y}{x} = 0$ or $xdy + ydx = 0$

$d(xy) = 0$. Integrating, $xy = c$.

From (ii), $\frac{dy}{dx} - \frac{x}{y} = 0$ or $xdx - ydx = 0$

Integrating, $x^2 - y^2 = c$. Thus $xy = c$ or $x^2 - y^2 = c$, constitute the required solution.

Otherwise, combining these into one, the required solution can be written as

$$(xy - c)(x^2 - y^2 - c) = 0.$$

Solve $p^2 + 2py \cot x = y^2$.

Solution. We have $p^2 + 2py \cot x + (y \cot x)^2 = y^2 + y^2 \cot^2 x$

$$p + y \cot x = \pm y \operatorname{cosec} x$$

$$p = y(-\cot x + \operatorname{cosec} x) \quad ...(i)$$

$$p = y(-\cot x - \operatorname{cosec} x) \quad ...(ii)$$

From (i), $\frac{dy}{dx} = y(-\cot x + \operatorname{cosec} x)$ or $\frac{dy}{y} = (\operatorname{cosec} x - \cot x)\, dx$

Integrating, $\log y = \log \tan \frac{x}{2} - \log \sin x + \log c = \log \frac{c \tan x/2}{\sin x}$

$$y = \frac{c}{2 \cos x^2/2} \text{ or } y(1 + \cos x) = c \quad ...(iii)$$

From (ii), $\frac{dy}{dx} = -y(\cot x + \operatorname{cosec} x)$ or $\frac{dy}{y} = -(\cot x + \operatorname{cosec} x)\, dx$

Integrating, $\log y = -\log \sin x - \log \tan \frac{x}{2} + \log c = \log \frac{c}{\sin x \tan \frac{x}{2}}$

$$y = \frac{c}{2 \sin^2 \frac{x}{2}} \quad \text{or} \quad y(1 - \cos x) = c \quad ...(iv)$$

Thus combining (iii) and (iv), the required general solution is

$$y(1 \pm \cos x) = c.$$

Case II. Equations solvable for y. If the given equation, on solving for y, takes the form

$$y = f(x, p). \quad ...(1)$$

then differentiation with respect to x gives an equation of the form

$$p = \frac{dy}{dx} = \phi\left(x, p, \frac{dp}{dx}\right).$$

Now it may be possible to solve this new differential equation in x and p.

Let its solution be $F(x, p, c) = 0$. ...(2)

The ellimination of p from (1) and (2) gives the required solution.

In case elimination of p is not possible, then we may solve (1) and (2) for x and y and obtain

$$x = F_1(p, c), y = F_2(p, c)$$

as the required solution, where p is the parameter.

Solve $y - 2px = \tan^{-1}(xp^2)$.

Solution. Given equation is $y = 2px + \tan^{-1}(xp^2)$ ...(*i*)

Differentiating both sides with respect to x, $\dfrac{dy}{dx} = p = 2\left(p + x\dfrac{dp}{dx}\right) + \dfrac{p^2 + 2xp\dfrac{dp}{dx}}{1 + x^2p^4}$

$$p + 2x\frac{dp}{dx} + \left(p + 2x\frac{dp}{dx}\right)\cdot\frac{p}{1+x^2p^4} = 0 \quad \text{or} \quad \left(p + 2x\frac{dp}{dx}\right)\left(1 + \frac{p}{1+x^2p^4}\right) = 0$$

This gives $p + 2x\, dp/dx = 0$.

Separating the variables and integrating, we have $\displaystyle\int\frac{dx}{x} + 2\int\frac{dp}{p}$ = a constant

$$\log x + 2\log p = \log c \quad \text{or} \quad \log xp^2 = \log c$$

ce $\quad xp^2 = c \quad$ or $\quad p = \sqrt{(c/x)}$ ...(*ii*)

Eliminating p from (*i*) and (*ii*), we get $y = 2\sqrt{(c/x)}\,x + \tan^{-1}c$

$y = 2\sqrt{(cx)} + \tan^{-1}c$ which is the general solution of (*i*)

Solve $y = 2px + p^n$.

Solution. Given equation is $\quad y = 2px + p^n$

Differentiating it with respect to x, we get

$$\frac{dy}{dx} = p = 2p + 2x\frac{dp}{dx} + np^{n-1}\frac{dp}{dx} \quad \text{or} \quad p\frac{dx}{dp} + 2x = -np^{n-1}$$

$$\frac{dx}{dp} + \frac{2x}{p} = -np^{n-2}$$

This is Leibnitz's linear equation in x and p. Here I.F. $= e^{\int\frac{2}{p}dp} = e^{\log p^2} = p^2$

$\therefore$ the solution of (*ii*) is

$$x(\text{I.F.}) = \int(-np^{n-2})\,.\,(\text{I.F.})\,dp + c \quad \text{or} \quad xp^2 = -n\int p^n\,dp + c = -\frac{np^{n+1}}{n+1} + c$$

$$x = cp^{-2} - \frac{np^{n-1}}{n+1} \qquad \text{...(iii)}$$

Substituting this value of x in (*i*), we get $y = \dfrac{2c}{p} + \dfrac{1-n}{1+n}p^n$ ...(*iv*)

The equations (*iii*) and (*iv*) taken together, with parameter p, constitute the general solution (*i*).

Case III. **Equations solvable for x.** If the given equation on solving for x, takes the form

$$x = f(y, p) \qquad \text{...(1)}$$

then differentiation with respect to y gives an equation of the form

$$\frac{1}{p} = \frac{dx}{dy} = \phi\left(y, p, \frac{dp}{dy}\right)$$

Now it may be possible to solve the new differential equation in y and p. Let its solution be $F(y, p, c) = 0$.

The elimination of p from (1) and (2) gives the required solution. In case the elimination is not feasible, (1) and (2) may be expressed in terms of p and p may be regarded as a parameter.

Solve $y = 2px + y^2p^3$.

Solution. Given equation, on solving for x, takes the form $x = \dfrac{y - y^2p^3}{2p}$

Differentiating with respect to y, $\dfrac{dx}{dy}\left(=\dfrac{1}{p}\right) = \dfrac{1}{2}\cdot\dfrac{p\left(1 - 2y\,.\,p^3 - y^2 3p^2\dfrac{dp}{dy}\right) - (y - y^2p^3)\dfrac{dp}{dy}}{p^2}$

or $$2p = p - 2yp^4 - 3y^2p^3\frac{dp}{dy} - y\frac{dp}{dy} + y^2p^3\frac{dp}{dy}$$

or $$p + 2yp^4 + 2y^2p^3\frac{dp}{dy} + y\frac{dp}{dy} = 0 \text{ or } p\,(1 + 2yp^3) + y\frac{dp}{dy}(1 + 2yp^3) = 0.$$

or $$\left(p + y\frac{dp}{dy}\right)(1 + 2py^3) = 0 \text{ This gives } p + y\frac{dp}{dy} = 0. \text{ or } \frac{d}{dy}(py) = 0.$$

Integrating $\quad py = c.$ ...(*i*)

Thus eliminating from the given equation and (*i*), we get $y = 2\,\dfrac{c}{y}x + \dfrac{c^3}{y^3}y^2$ or $y^2 = 2cx + c^3$

which is the required solution.

CLAIRAUT'S EQUATION*

An equation of the form $y = px + f(p)$ is known as Clairaut's equation ...(1)

Differentiating with respect to x, we have $p = p + x\,\dfrac{dp}{dx} + f'(p)\,\dfrac{dp}{dx}$

or $$[x + f'(p)]\,\frac{dp}{dx} = 0 \qquad \therefore\ \frac{dp}{dx} = 0, \text{ or } x + f'(p) = 0$$

$$\frac{dp}{dx} = 0, \text{ gives } p = c \qquad \text{...(2)}$$

Thus eliminating p from (1) and (2), we get $y = cx + f(c)$ ...(3)

as the general solution of (1).

Hence the solution of the Clairaut's equation is obtained on replacing p by c.

Solve $p = \sin(y - xp)$. Also find its singular solutions.

Solution. Given equation can be written as

$\sin^{-1} p = y - xp$ or $y = px + \sin^{-1} p$ which is the *Clairaut's equation.*

$\therefore$ its solution is $y = cx + \sin^{-1} c$.

To find the singular solution, differentiate (*i*) w.r.t. c giving

$$0 = x + \frac{1}{\sqrt{1 - c^2}} \qquad \text{...(ii)}$$

To eliminate c from (*i*) and (*ii*), we rewrite (*ii*) as

$$c = N(x^2 - 1)/x$$

Now substituting this value of c in (*i*), we get

$$y = N(x^2 - 1) + \sin^{-1}\{N(x^2 - 1)/x\}$$

which is the desired singular solution.

Solve $(px - y)(py + x) = a^2 p$. (

$x^2 = u$ and $y^2 = v$ so that $2xdx = du$ and $2ydy = dv$

$$p = \frac{dy}{dx} = \frac{dv}{y} \Big/ \frac{du}{x} = \frac{x}{y} P, \text{ where } P = \frac{dv}{du}$$

Then the given equation becomes $\left(\frac{xP}{y} . x - y \right)\left(\frac{xP}{y} . y + x \right) = a^2 \frac{xP}{y}$

$$(uP - v)(P + 1) = a^2 P \text{ or } uP - v = \frac{a^2 P}{P + 1}$$

$v = uP - a^2P/(P + 1)$, which is Clairaut's form.

$\therefore$ its solution is $v = uc - a^2c/(c + 1)$, *i.e.*, $y^2 = cx^2 - a^2c/(c + 1)$.

Ex:Solve the equation $ydx - xdy = 0, \quad x, y > 0.$

Solution: It can be easily verified that the given equation is not exact.

Multiplying by $\frac{1}{xy}$, the equation reduces to

$$\frac{1}{xy} ydx - \frac{1}{xy} xdy = 0, \quad \text{or equivalently} \quad d(\ln x - \ln y) = 0.$$

Thus, by definition, $\frac{1}{xy}$ is an integrating factor. Hence, a general solution of the given equation is $G(x, y) = \frac{1}{xy} = c,$ for some constant $c \in \mathbb{R}.$ That is,

$$y = cx, \quad \text{for some constant } c \in \mathbb{R}.$$

UNIT-II

LINEAR DIFFERENTIAL EQUATIONS OF HIGHER ORDER

1. Definitions. **2.** Complete solution. **3.** Operator D. **4.** Rules for finding the Complementary function. **5.** Inverse operator. **6.** Rules for finding the particular integral. **7.** Working procedure. **8.** Two other methods of finding P.I.—Method of variation of parameters ; Method of undetermined coefficients. **9.** Cauchy's and Legendre's linear equations. **10.** Linear dependence of solutions. **11.** Simultaneous linear equations with constant coefficients. **12.** Objective Type of Questions.

2.1 DEFINITIONS Second Order Differential Equations with Constant Coefficients

Linear differential equations *are those in which the dependent variable and its derivatives occur only in the first degree and are not multiplied together.* Thus the general linear differential equation of the nth order is of the form

$$\frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + p_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + p_n y = X,$$

where $p_1, p_2, \dots, p_n$ and X are functions of x only.

Linear differential equations with constant co-efficients are of the form

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + k_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + k_n y = X$$

where $k_1, k_2, \dots, k_n$ are constants. Such equations are most important in the study of electro-mechanical vibrations and other engineering problems.

THEOREM

If y_1, y_2 are only two solutions of the equation

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + k_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + k_n y = 0 \qquad \dots(1)$$

then $c_1 y_1 + c_2 y_2 \,(= u)$ is also its solution.

Since $y = y_1$ and $y = y_2$ are solutions of (1).

$$\therefore \quad \frac{d^n y_1}{dx^n} + k_1 \frac{d^{n-1} y_1}{dx^{n-1}} + k_2 \frac{d^{n-2} y_1}{dx^{n-2}} + \dots + k_n y_1 = 0 \qquad \dots(2)$$

and

$$\frac{d^n y_2}{dx^n} + k_1 \frac{d^{n-1} y_2}{dx^{n-1}} + k_2 \frac{d^{n-2} y_2}{dx^{n-2}} + \dots + k_n y_2 = 0 \qquad \dots(3)$$

If c_1, c_2 be two arbitrary constants, then

$$\frac{d^n (c_1 y_1 + c_2 y_2)}{dx^n} + k_1 \frac{d^{n-1} (c_1 y_1 + c_2 y_2)}{dx^{n-1}} + \dots + k_n (c_1 y_1 + c_2 y_2)$$

$$= c_1\left(\frac{d^n y_1}{dx^n} + k_1\frac{d^{n-1}y_1}{dx^{n-1}} + \ldots + k_n y_1\right) + c_2\left(\frac{d^n y_2}{dx^n} + k_1\frac{d^{n-1}y_2}{dx^{n-1}} + \ldots + k_n y_2\right)$$

$$= c_1(0) + c_2(0) = 0 \qquad \text{[By (2) and (3)]}$$

i.e.,
$$\frac{d^n u}{dx^n} + k_1\frac{d^{n-1}u}{dx^{n-1}} + \ldots + k_n y = 0 \qquad \ldots(4)$$

This proves the theorem.

(2) Since the general solution of a differential equation of the nth order contains n arbitrary constants, it follows, from above, that if $y_1, y_2, y_3, \ldots, y_n$, are n independent solutions of (1), then $c_1y_1 + c_2y_2 + \ldots + c_ny_n (= u)$ is its complete solution.

(3) If $y = v$ be any particular solution of

$$\frac{d^n y}{dx^n} + k_1\frac{d^{n-1}y}{dx^{n-1}} + \ldots + k_n y = X \qquad \ldots(5)$$

then
$$\frac{d^n v}{dx^n} + k_1\frac{d^{n-1}v}{dx^{n-1}} + \ldots + k_n v = X \qquad \ldots(6)$$

Adding (4) and (6), we have $\dfrac{d^n(u+v)}{dx^n} + k_1\dfrac{d^{n-1}(u+v)}{dx^{n-1}} + \ldots + k_n(u+v) = X$

This shows that $y = u + v$ is the complete solution of (5).

The part u is called the **complementary function (C.F.)** *and the part v is called the* **particular integral (P.I.)** *of (5).*

$\therefore$ the complete solution (C.S.) of (5) is **y = C.F. + P.I.**

Thus in order to solve the question (5), we have to first find the C.F., i.e., the complete solution of (1), and then the P.I., i.e. a particular solution of (5).

OPERATOR D

Denoting $\dfrac{d}{dx}, \dfrac{d^2}{dx^2}, \dfrac{d^3}{dx^3}$ etc. by D, D^2, D^3 etc., so that

$\dfrac{dy}{dx} = Dy$, $\dfrac{d^2y}{dx^2} = D^2y$, $\dfrac{d^3x}{dx^3} = D^3y$ etc., the equation (5) above can be written in the *symbolic form* $(D^n + k_1D^{n-1} + \ldots + k_n)y = X$, *i.e.*, $f(D)y = X$,
where $f(D) = D^n + k_1D^{n-1} + \ldots + k_n$, *i.e.*, a polynomial in D.

Thus *the symbol D stands for the operation of differentiation and can be treated much the same as an algebraic quantity i.e., $f(D)$ can be factorised by ordinary rules of algebra and the factors may be taken in any order.* For instance

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = (D^2 + 2D - 3)\,y = (D+3)(D-1)y \text{ or } (D-1)(D+3)\,y.$$

RULES FOR FINDING THE COMPLEMENTARY FUNCTION

To solve the equation $\dfrac{d^n y}{dx^n} + k_1\dfrac{d^{n-1}y}{dx^{n-1}} + k_2\dfrac{d^{n-2}y}{dx^{n-2}} + \ldots + k_n y = 0 \qquad \ldots(1)$

where k's are constants.

The equation (1) in *symbolic form is*

$$(D^n + k_1D^{n-1} + k_2D^{n-2} + \ldots + k_n)y = 0 \qquad \ldots(2)$$

Its symbolic co-efficient equated to zero *i.e.*

$$D^n + k_1D^{n-1} + k_2D^{n-2} + \ldots + k_n = 0$$

is called the *auxiliary equation* (A.E.). Let $m_1, m_2, \ldots, m_n$ be its roots.

Case I. *If all the roots be real and different*, then (2) is equivalent to

$$(D - m_1)(D - m_2)\ldots(D - m_n)y = 0 \qquad \ldots(3)$$

Now (3) will be satisfied by the solution of $(D - m_n)y = 0$, *i.e.*, by $\frac{dy}{dx} - m_n y = 0$.

This is a Leibnitz's linear and I.F. $= e^{-m_n x}$

$\therefore$ its solution is $y e^{-m_n x} = c_n$, *i.e.*, $y = c_n e^{m_n x}$

Similarly, since the factors in (3) can be taken in any order, it will be satisfied by the solutions of $(D - m_1) y = 0$, $(D - m_2)y = 0$ etc. *i.e.*, by $y = c_1 e^{m_1 x}$, $y = c_2 e^{m_2 x}$ etc.

Thus the complete solution of (1) is $y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots + c_n e^{m_n x}$ ...(4)

Case II. *If two roots are equal (i.e., $m_1 = m_2$)*, then (4) becomes

$$y = (c_1 + c_2) e^{m_1 x} + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}$$

$$y = C e^{m_1 x} + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x} \qquad [\because \; c_1 + c_2 = \text{one arbitrary constant } C]$$

It has only $n - 1$ arbitrary constants and is, therefore, not the complete solution of (1). In this case, we proceed as follows :

The part of the complete solution corresponding to the repeated root is the complete solution of $(D - m_1)(D - m_1)y = 0$

Putting $(D - m_1)y = z$, it becomes $(D - m_1) z = 0$ or $\frac{dz}{dx} - m_1 z = 0$

This is a Leibnitz's linear in z and I.F. $= e^{-m_1 x}$. $\therefore$ its solution is $ze^{-m_1 x} = c_1$ or $z = c_1 e^{m_1 x}$

Thus $\qquad (D - m_1)y = z = c_1 e^{m_1 x}$ or $\frac{dy}{dx} - m_1 y = c_1 e^{m_1 x}$ ...(5)

Its I.F. being $e^{-m_1 x}$, the solution of (5) is

$$ye^{-m_1 x} = \int c_1 e^{m_1 x} \, dx + c_2 = c_1 x + c_2 \quad \text{or} \quad y = (c_1 x + c_2) e^{m_1 x}$$

Thus the complete solution of (1) is $y = (c_1 x + c_2) e^{m_1 x} + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}$

If, however, the A.E. has three equal roots (*i.e.*, $m_1 = m_2 = m_3$), then the complete solution is

$$y = (c_1 x^2 + c_2 x + c_3)\, e^{m_1 x} + c_4 e^{m_4 x} + \ldots + c_n e^{m_n x}$$

Case III. *If one pair of roots be imaginary, i.e.*, $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$, then the complete solution is

$$y = c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x} + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}$$

$$= e^{\alpha x}(c_1 e^{i\beta x} + c_2 e^{-i\beta x}) + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}$$

$$= e^{\alpha x}[c_1(\cos \beta x + i \sin \beta x) + c_2 (\cos \beta x - i \sin \beta x)] + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}$$

[$\because$ by *Euler's Theorem*, $e^{i\theta} = \cos \theta + i \sin \theta$]

$$= e^{ax}(C_1 \cos \beta x + C_2 \sin \beta x) + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}$$

where $C_1 = c_1 + c_2$ and $C_2 = i(c_1 - c_2)$.

Case IV. *If two points of imaginary roots be equal i.e.*, $m_1 = m_2 = \alpha + i\beta$, $m_3 = m_4 = \alpha - i\beta$, then by case II, the complete solution is

$$y = e^{ax}[(c_1 x + c_2) \cos \beta x + (c_3 x + c_4) \sin \beta x] + \ldots + c_n e^{m_n x}.$$

Example 13.1. *Solve* $\frac{d^2 x}{dt^2} + 5\frac{dx}{dt} + 6x = 0$, *given* $x(0) = 0$, $\frac{dx}{dt}(0) = 15$. *(V.T.U., 2010)*

Solution. Given equation in *symbolic form is* $(D^2 + 5D + 6)\, x = 0$.

Its A.E. is $D^2 + 5D + 6 = 0$, *i.e.*, $(D + 2)(D + 3) = 0$ whence $D = -2, -3$.

$\therefore$ $\qquad$ C.S. is $x = c_1 e^{-2t} + c_2 e^{-3t}$ and $\frac{dx}{dt} = -2ae^{-2t} - 3c_2 e^{-3t}$

When $t = 0$, $x = 0$. $\quad \therefore \quad 0 = c_1 + c_2$ (*i*)

When $t = 0$, $dx/dt = 15$ $\quad \therefore \quad 15 = -2c_1 - 3c_2$ ...(*ii*)

Solving (i) and (ii), $c_1 = 15, c_2 = -15$.
Hence the required solution is $x = 15\,(e^{-2t} - e^{-3t})$.

Example 13.2. *Solve* $\dfrac{d^2x}{dt^2} + 6\dfrac{dx}{dt} + 9x = 0.$

Solution. Given equation in *symbolic form is* $(D^2 + 6D + 9) = 0$

$\therefore$ A.E. is $D^2 + 6D + 9 = 0$, *i.e.*, $(D + 3)^2 = 0$ whence $D = -3, -3$.

Hence the C.S. is $x = (c_1 + c_2t)\,e^{-3t}$.

Example 13.3. *Solve* $(D^3 + D^2 + 4D + 4) = 0.$

Solution. Here the A.E. is $D^3 + D^2 + 4D + 4 = 0$ *i.e.*, $(D^2 + 4)(D + 1) = 0$ $\therefore$ $D = -1, \pm 2i$.

Hence the C.S. is $y = c_1e^{-x} + e^{0x}(c_2 \cos 2x + c_3 \sin 2x)$

i.e., $y = c_1e^{-x} + c_2 \cos 2x + c_3 \sin 2x$.

Example 13.4. *Solve (i)* $(D^4 - 4D + 4)\,y = 0$ *(Bhopal, 2008)*

(ii) $(D^2 + 1)^3\,y = 0$ *where* $D \equiv d/dx$.

Solution. (i) The A.E. equation is $D^4 - 4D^2 + 4 = 0$ or $(D^2 - 2)^2 = 0$

$\therefore$ $D^2 = 2, 2$ *i.e.*, $D = \pm\sqrt{2}, \pm\sqrt{2}$.

Hence the C.S. is $((c_1 + c_2x)\,e^{\sqrt{2}x} + (c_3 + c_4x)\,e^{-\sqrt{2}x})$ [Roots being repeated]

(ii) The A.E. equation is $(D^2 + 1)^3 = 0$

$\therefore$ $D = \pm i, \pm i, \pm i$.

Hence the C.S. is $y = e^{0x}\,[(c_1 + c_2x + c_3x^2)\cos x + (c_4 + c_5x + c_6x^2)\sin x]$

i.e., $y = (c_1 + c_2 + c_3x^2)\cos x + (c_4 + c_5\,x + c_6x^2)\sin x$.

Example 13.5. *Solve* $\dfrac{d^4x}{dt^4} + 4x = 0.$

Solution. Given equation in *symbolic form is* $(D^4 + 4)\,x = 0$

$\therefore$ A.E. is $D^4 + 4 = 0$ or $(D^4 + 4D^2 + 4) - 4D^2 = 0$ or $(D^2 + 2)^2 - (2D)^2 = 0$

or $(D^2 + 2D + 2)(D^2 - 2D + 2) = 0$

$\therefore$ either $D^2 + 2D + 2 = 0$ or $D^2 - 2D + 2 = 0$

whence $D = \dfrac{-2 \pm \sqrt{(-4)}}{2}$ and $\dfrac{2 \pm \sqrt{(-4)}}{2}$ *i.e.*, $D = -1 \pm i$ and $1 \pm i$.

Hence the required solution is $x = e^{-t}(c_1 \cos t + c_2 \sin t) + e^t(c_3 \cos t + c_4 \sin t)$.

13.5 INVERSE OPERATOR

(1) Definition. $\frac{1}{f(D)}X$ is that function of x, not containing arbitrary constants which when operated upon by $f(D)$ gives X.

i.e.,
$$f(D)\left\{\frac{1}{f(D)}X\right\} = X$$

Thus $\frac{1}{f(D)}X$ *satisfies the equation* $f(D)y = X$ *and is, therefore, its particular integral.*

Obviously, $f(D)$ and $1/f(D)$ are inverse operators.

(2) $$\frac{\mathbf{1}}{\mathbf{D}}\mathbf{X} = \int \mathbf{X}\,\mathbf{dx}$$

Let $$\frac{1}{D}X = y \qquad \text{...}(i)$$

Operating by D, $$D\frac{1}{D}X = Dy \quad i.e., \; X = \frac{dy}{dx}$$

Integrating both sides w.r.t. x, $y = \int X\,dx$, no constant being added as (i) does not contain any constant.

Thus $$\frac{1}{D}X = \int X\,dx.$$

(3) $$\frac{\mathbf{1}}{\mathbf{D-a}}\mathbf{X} = \mathbf{e^{ax}}\int \mathbf{Xe^{-ax}dx}.$$

Let $$\frac{1}{D-a}X = y \qquad \text{...}(ii)$$

Operating by $D - a$, $(D-a)\,.\,\frac{1}{D-a}X = (D-a)y.$

or $$X = \frac{dy}{dx} - ay\,,\; i.e.,\; \frac{dy}{dx} - ay = X$$ which is a Leibnitz's linear equation.

$\therefore$ I.F. being e^{-ax}, its solution is

$$ye^{-ax} = \int Xe^{-ax}dx,$$ no constant being added as (ii) doesn't contain any constant.

Thus $$\frac{1}{D-a}X = y = e^{ax}\int Xe^{-ax}dx.$$

13.6 RULES FOR FINDING THE PARTICULAR INTEGRAL $\mathbf{e^{ax}}$

Consider the equation $\frac{d^n y}{dx^n} + k_1\frac{d^{n-1}y}{dx^{n-1}} + k_2\frac{d^{n-2}y}{dx^{n-2}} + \ldots + k_n y = X$

which is *symbolic form of* $(D^n + k_1D^{n-1} + k_2D^{n-2} + \ldots + k_n)y = X.$

$$\therefore \qquad \text{P.I.} = \frac{1}{D^n + k_1D^{n-1} + k_2D^{n-2} + \ldots + k_n}X.$$

Case I. When $\mathbf{X = e^{ax}}$

Since
$$De^{ax} = ae^{ax}$$
$$D^2e^{ax} = a^2e^{ax}$$
....
....
$$D^ne^{ax} = a^ne^{ax}$$

$$\therefore \quad (D^n + k_1D^{n-1} + \ldots + k_n)e^{ax} = (a^n + k_1a^{n-1} + \ldots + k_n)e^{ax}, \quad i.e., \quad f(D)e^{ax} = f(a)e^{ax}$$

Operating on both sides by $\frac{1}{f(D)}, \frac{1}{f(D)} f(D) e^{ax} = \frac{1}{f(D)} f(a) e^{ax}$ or $e^{ax} = f(a) \frac{1}{f(D)} e^{ax}$

$\therefore$ dividing by $f(a)$,

$$\mathbf{\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}} \quad \textit{provided } f(a) \neq 0 \qquad ...(1)$$

If f(a) = 0, *the above rule fails and we proceed further.*

Since a is a root of A.E. $f(D) = D^n + k_1 D^{n-1} + ... + k_n = 0$.

$\therefore$ $D - a$ is a factor of $f(D)$. Suppose $f(D) = (D - a)\,\phi(D)$, where $\phi(a) \neq 0$. Then

$$\frac{1}{f(D)} e^{ax} = \frac{1}{D-a} \cdot \frac{1}{\phi(D)} e^{ax} = \frac{1}{D-a} \cdot \frac{1}{\phi(a)} e^{ax} \qquad \text{[By (1)]}$$

$$= \frac{1}{\phi(a)} \cdot \frac{1}{D-a} e^{ax} = \frac{1}{\phi(a)} \cdot e^{ax} \int e^{ax} . e^{-ax} dx \qquad \text{[By §13.5 (3)]}$$

$$= \frac{1}{\phi(a)} e^{ax} \int dx = x \frac{1}{\phi(a)} e^{ax} \quad i.e., \quad \mathbf{\frac{1}{f(D)} e^{ax} = x \frac{1}{f'(a)} e^{ax}} \qquad ...(2)$$

$$\left[\begin{array}{l} \because \ f'(D) = (D-a)\phi'(D) + 1 . \phi(D) \\ \therefore \ f'(a) = 0 \times \phi'(a) + \phi(a) \end{array} \right.$$

If $f'(a) = 0$, then applying (2) again, we get $\mathbf{\frac{1}{f(D)} e^{ax} = x^2 \frac{1}{f''(a)} e^{ax}}$, *provided* $f''(a) \neq 0$ ...(3)

and so on.

Example 13.6. *Find the P.I. of $(D^2 + 5D + 6)y = e^x$.*

Solution. P.I. $= \frac{1}{D^2 + 5D + 6} e^x$ [Put $D = 1$] $= \frac{1}{1^2 + 5.1 + 6} e^x = \frac{e^x}{12}$.

Example 13.7. *Find the P.I. of $(D + 2)(D - 1)^2 y = e^{-2x} + 2 \sinh x$.*

Solution. P.I. $= \frac{1}{(D+2)(D-1)^2}[e^{-2x} + 2\sinh x] = \frac{1}{(D+2)(D-1)^2}[e^{-2x} + e^x - e^{-x}]$

Let us evaluate each of these terms separately.

$$\frac{1}{(D+2)(D-1)^2} e^{-2x} = \frac{1}{D+2} \cdot \left[\frac{1}{(D-1)^2} e^{-2x} \right]$$

$$= \frac{1}{D+2} \cdot \frac{1}{(-2-1)^2} e^{-2x} = \frac{1}{9} \cdot \frac{1}{D+2} e^{-2x}$$

$$= \frac{1}{9} . x . \frac{1}{1} e^{-2x} = \frac{x}{9} e^{-2x} \qquad \left[\because \ \frac{d}{dD}(D+2) = 1 \right]$$

$$\frac{1}{(D+2)(D-1)^2} e^x = \frac{1}{1+2} \cdot \frac{1}{(D-1)^2} e^x = \frac{1}{3} . x^2 . \frac{1}{2} e^x = \frac{x^2}{6} e^x \qquad \left[\because \ \frac{d^2}{dD^2}(D-1)^2 = 2 \right]$$

and $$\frac{1}{(D+2)(D-1)^2} e^{-x} = \frac{1}{(-1+2)(-1-1)^2} e^{-x} = \frac{e^{-x}}{4}$$

Hence, P.I. $= \frac{x}{9} e^{-2x} + \frac{x^2}{6} e^x + \frac{1}{4} e^{-x}$.

Case II. When X = sin (ax + b) or cos (ax + b).

Since
$$D \sin(ax + b) = a \cos(ax + b)$$
$$D^2 \sin(ax + b) = -a^2 \sin(ax + b)$$
$$D^3 \sin(ax + b) = -a^3 \cos(ax + b)$$

$$D^4 \sin(ax+b) = a^4 \sin(ax+b)$$

i.e.,
$$D^2 \sin(ax+b) = (-a^2)\sin(ax+b)$$

$$(D^2)^2 \sin(ax+b) = (-a^2)^2 \sin(ax+b)$$

In general $\quad (D^2)^r \sin(ax+b) = (-a^2)^r \sin(ax+b)$

$\therefore \quad f(D^2)\sin(ax+b) = f(-a^2)\sin(ax+b)$

Operating on both sides $1/f(D^2)$,

$$\frac{1}{f(D^2)} \cdot f(D^2)\ \sin(ax+b) = \frac{1}{f(D^2)} f(-a^2)\sin(ax+b)$$

or
$$\sin(ax+b) = f(-a^2)\frac{1}{f(D^2)}\sin(ax+b)$$

$\therefore$ Dividing by $f(-a^2)$. $\mathbf{\dfrac{1}{f(D^2)}\ \sin(ax+b) = \dfrac{1}{f(-a^2)}\ \sin(ax+b)}$ *provided* $f(-a^2) \neq 0$ ...(4)

If $\mathbf{f(-a^2) = 0}$, *the above rule fails and we proceed further.*

Since $\cos(ax+b) + i\sin(ax+b) = e^{i(ax+b)}$ [Euler's theorem]

$\therefore \quad \dfrac{1}{f(D^2)}\sin(ax+b) = \text{I.P. of } \dfrac{1}{f(D^2)} e^{i(ax+b)}$ [Since $f(-a^2) = 0 \quad \therefore$ by (2)]

$= \text{I.P. of } x\dfrac{1}{f'(D^2)} e^{i(ax+b)}$ where $D^2 = -a^2$

$\therefore \quad \mathbf{\dfrac{1}{f(D^2)}\sin(ax+b) = x\ \dfrac{1}{f'(-a^2)}\ \sin(ax+b)}$ *provided* $f'(-a^2) \neq 0$ ...(5)

If $f'(-a^2) = 0$, $\mathbf{\dfrac{1}{f(D^2)} \cdot \sin(ax+b) = x^2\ \dfrac{1}{f''(-a^2)}\ \sin(ax+b)}$, *provided* $f''(-a^2) \neq 0$, and so on.

Similarly, $\quad \mathbf{\dfrac{1}{f(D^2)}\ \cos(ax+b) = \dfrac{1}{f(-a^2)}\ \cos(ax+b)}$, *provided* $f(-a^2) \neq 0$

If $f(-a^2) = 0$, $\quad \mathbf{\dfrac{1}{f(D^2)}\cos(ax+b) = x \cdot \dfrac{1}{f'(-a^2)}\cos(ax+b)}$, *provided* $f'(-a^2) \neq 0$.

If $f'(-a^2) = 0$, $\quad \mathbf{\dfrac{1}{f(D^2)}\cos(ax+b) = x^2\ \dfrac{1}{f''(-a^2)}\cos(ax+b)}$, *provided* $f''(-a^2) \neq 0$ and so on.

Example 13.8. *Find the P.I. of $(D^3+1)y = \cos(2x-1)$.*

Solution. P.I. $= \dfrac{1}{D^3+1}\cos(2x-1)$ [Put $D^2 = -2^2 = -4$]

$= \dfrac{1}{D(-4)+1}\ \cos(2x-1)$ [Multiply and divide by $1 + 4D$]

$= \dfrac{(1+4D)}{(1-4D)(1+4D)}\cos(2x-1) = (1+4D) \cdot \dfrac{1}{1-16D^2}\cos(2x-1)$ [Put $D^2 = -2^2 = -4$]

$= (1+4D)\dfrac{1}{1-16(-4)}\cos(2x-1) = \dfrac{1}{65}[\cos(2x-1) + 4D\cos(2x-1)]$

$= \dfrac{1}{65}[\cos(2x-1) - 8\sin(2x-1)]$.

Example 13.9. *Find the P.I. of $\dfrac{d^3y}{dx^3} + 4\dfrac{dy}{dx} = \sin 2x$.*

Solution. Given equation in *symbolic form* is $(D^3+4D)y = \sin 2x$

$\therefore$ P.I. $= \dfrac{1}{D(D^2+4)}\sin 2x$ [$\because$ $D^2 + 4 = 0$ for $D^2 = -2^2$, $\therefore$ Apply (5) 477]

$= x\dfrac{1}{3D^2+4}\sin 2x$ $\left[\because \dfrac{d}{dD}[D^3+4D] = 3D^2+4\right]$

[Put $D^2 = -2^2 = -4$]

$= x\dfrac{1}{3(-4)+4}\sin 2x = -\dfrac{x}{8}\sin 2x.$

Case III. When X = $\mathbf{x^m}$.

Here P.I. $= \dfrac{1}{f(D)}x^m = [f(D)]^{-1}x^m.$

Expand $[f(D)]^{-1}$ in ascending powers of D as far as the term in D^m and operate on x^m term by term. Since the $(m + 1)$th and higher derivatives of x^m are zero, we need not consider terms beyond D^m.

Example 13.10. *Find the P.I. of* $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = x^2 + 2x + 4.$

Solution. Given equation in *symbolic form* is $(D^2 + D)y = x^2 + 2x + 4$.

$\therefore$ P.I. $= \dfrac{1}{D(D+1)}(x^2+2x+4) = \dfrac{1}{D}(1+D)^{-1}(x^2+2x+4)$

$= \dfrac{1}{D}(1-D+D^2-\ldots)(x^2+2x+4) = \dfrac{1}{D}[x^2+2x+4-(2x+2)+2]$

$= \int(x^2+4)dx = \dfrac{x^3}{3} + 4x.$

Case IV. When X = $\mathbf{e^{ax}}$ V, *V being a function of x.*

If u is a function of x, then

$$D(e^{ax}u) = e^{ax}Du + ae^{ax}u + e^{ax}(D+a)u$$

$$D^2(e^{ax}u) = a^{ax}D^2u + 2ae^{ax}Du + a^2e^{ax}u = e^{ax}(D+a)^2u$$

and in general, $D^n(e^{ax}u) = e^{ax}(D+a)^nu$

$\therefore$ $f(D)(e^{ax}u) = e^{ax}f(D+a)u$

Operating both sides by $1/f(D)$,

$$\frac{1}{f(D)}\cdot f(D)(e^{ax}u) = \frac{1}{f(D)}[e^{ax}f(D+a)u]$$

$$e^{ax}u = \frac{1}{f(D)}[e^{ax}f(D+a)u]$$

Now put $f(D+a)u = V$, *i.e.*, $u = \dfrac{1}{f(D+a)}V$, so that $e^{ax}\dfrac{1}{f(D+a)}V = \dfrac{1}{f(D)}(e^{ax}V)$

i.e., $$\frac{1}{\mathbf{f(D)}}(\mathbf{e^{ax}V}) = \mathbf{e^{ax}}\frac{1}{\mathbf{f(D+a)}}\mathbf{V}. \quad \ldots(6)$$

Example 13.11. *Find P.I. of* $(D^2 - 2D + 4)y = e^x \cos x.$

Solution. P.I. $= \dfrac{1}{D^2-2D+4}e^x\cos x$ [Replace D by $D + 1$]

$= e^x\dfrac{1}{(D+1)^2-2(D+1)+4}\cos x = e^x\dfrac{1}{D^2+3}\cos x$ [Put $D^2 = -1^2 = -1$]

$= e^x\dfrac{1}{-1+3}\cos x = \dfrac{1}{2}e^x\cos x.$

Case V. When X is any other function of x.

Here $\quad \text{P.I.} = \dfrac{1}{f(D)}X.$

If $\quad f(D) = (D - m_1)(D - m_2) \ldots (D - m_n)$, resolving into partial fractions,

$$\frac{1}{f(D)} = \frac{A_1}{D - m_1} + \frac{A_2}{D - m_2} + \ldots + \frac{A_n}{D - m_n}$$

$\therefore$

$$\text{P.I.} = \left[\frac{A_1}{D - m_1} + \frac{A_2}{D - m_2} + \ldots + \frac{A_n}{D - m_n}\right]X$$

$$= A_1 \frac{1}{D - m_1}X + A_2 \frac{1}{D - m_2}X + \ldots + A_n \frac{1}{D - m_n}X$$

$$= A_1 . e^{m_1 x}\int Xe^{-m_1 x}dx + A_2 . e^{m_2 x}\int Xe^{-m_2 x}dx + \ldots + A_n . e^{m_n x}\int Xe^{-m_n x}dx \qquad \text{[By §13.5 ...(3)]}$$

Obs. This method is a general one and can, therefore, be employed to obtain a particular integral in any given case.

13.7 WORKING PROCEDURE TO SOLVE THE EQUATION

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + \ldots + k_{n-1}\frac{dy}{dx} + k_n y = X$$

of which the *symbolic form* is

$$(D^n + k_1 D^{n-1} + \ldots + k_{n-1}D + k_n)y = X .$$

Step I. To find the complementary function

(*i*) *Write the A.E.*

i.e., $\quad D^n + k_1 D^{n-1} + \ldots + k_{n-1}D + k_n = 0$ and solve it for D.

(*ii*) Write the *C.F.* as follows :

Roots of A.E.	*C.F.*
1. $m_1, m_2, m_3 \ldots$ (real and different roots)	$c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \ldots$
2. $m_1, m_1, m_3 \ldots$ (two real and equal roots)	$(c_1 + c_2 x)e^{m_1 x} + c_3 e^{m_3 x} + \ldots$
3. $m_1, m_1, m_1, m_4 \ldots$ (three real and equal roots)	$(c_1 + c_2 x + c_3 x^2)e^{m_1 x} + c_4 e^{m_4 x} + \ldots$
4. $\alpha + i\beta, \alpha - i\beta, m_3 \ldots$ (a pair of imaginary roots)	$e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x) + c_3 e^{m_3 x} + \ldots$
5. $\alpha \pm i\beta, \alpha \pm i\beta, m_5 \ldots$ (2 pairs of equal imaginary roots)	$e^{\alpha x}[(c_1 + c_2 x)\cos \beta x + (c_3 + c_4 x)\sin \beta x] + c_5 e^{m_5 x} + \ldots$

Step II. To find the particular integral

From symbolic form P.I. $= \dfrac{1}{D^n + k_1 D^{n-1} + \ldots + k_{n-1}D + k_n} X = \dfrac{1}{f(D)}$ or $\dfrac{1}{\phi(D^2)}X$

(*i*) *When* $X = e^{ax}$

$$\text{P.I.} = \frac{1}{f(D)}e^{ax}, \text{ put } D = a, \qquad [f(a) \neq 0]$$

$$= x\frac{1}{f'(D)}e^{ax}, \text{ put } D = a, \qquad [f(a) = 0, f'(a) \neq 0]$$

$$= x^2 \frac{1}{f''(D)}e^{ax}, \text{ put } D = a, \qquad [f'(a) = 0, f''(a) \neq 0]$$

and so on.

where $\quad f'(D)$ = diff. coeff. of $f(D)$ w.r.t. D

$f''(D)$ = diff. coeff. of $f'(D)$ w.r.t. D, etc.

(ii) When $X = \sin(ax + b)$ or $\cos(ax + b)$.

$$\text{P.I.} = \frac{1}{\phi(D^2)} \sin(ax+b) \text{ [or } \cos(ax+b)], \text{ put } D^2 = -a^2 \qquad [\phi(-a^2) \neq 0]$$

$$= x \frac{1}{\phi'(D^2)} \sin(ax+b) \text{ [or } \cos(ax+b)], \text{ put } D^2 = -a^2 \qquad [\phi(-a^2) = 0, \phi'(-a^2) \neq 0]$$

$$= x^2 \frac{1}{\phi''(D^2)} \sin(ax+b) \text{ [or } \cos(ax+b)], \text{ put } D^2 = -a^2 \qquad [\phi'(-a^2) \neq 0, \phi''(-a^2) \neq 0]$$

and so on.

where $\phi'(D^2)$ = diff. coeff. of $\phi(D^2)$ w.r.t. D,

$\phi''(D^2)$ = diff. coeff. of $\phi'(D^2)$ w.r.t. D, etc.

(iii) When $X = x^m$, *m* being a positive integer.

$$\text{P.I.} = \frac{1}{f(D)} x^m = [f(D)]^{-1} x^m$$

To evaluate it, expand $[f(D)]^{-1}$ in ascending powers of D by Binomial theorem as far as D^m and operate on x^m term by term.

(iv) When $X = e^{ax}V$, where V is a function of x.

$$\text{P.I.} = \frac{1}{f(D)} e^{ax} V = e^{ax} \frac{1}{f(D+a)} V$$

and then evaluate $\frac{1}{f(D+a)} V$ as in (*i*), (*ii*), and (*iii*).

(v) When X is any function of x.

$$\text{P.I.} = \frac{1}{f(D)} X$$

Resolve $\frac{1}{f(D)}$ into partial fractions and operate each partial fraction on X remembering that

$$\frac{1}{D-a} X = e^{ax} \int X e^{-ax} \, dx.$$

Step III. To find the complete solution

Then the C.S. is y = C.F. + P.I.

Example 13.12. *Solve* $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = (1 - e^x)^2$.

Solution. Given equation in *symbolic form* is $(D^2 + D + 1)y = (1 - e^x)^2$

(i) To find C.F.

Its A.E. is $D^2 + D + 1 = 0$, $\therefore \quad D = \frac{1}{2}(-1 + \sqrt{3}i)$

Thus C.F. $= e^{-x/2}\left(c_1 \cos \frac{\sqrt{3}}{2} x + c_2 \sin \frac{\sqrt{3}}{2} x\right)$

(ii) To find P.I.

$$\text{P.I.} = \frac{1}{D^2+D+1}(1 - 2e^x + e^{2x}) = \frac{1}{D^2+D+1}(e^{0x} - 2e^x + e^{2x})$$

$$= \frac{1}{0^2+0+1} e^{0x} - 2 \cdot \frac{1}{1^2+1+1} e^x + \frac{1}{2^2+2+1} e^{2x} = 1 - \frac{2}{3} e^x + \frac{e^{2x}}{7}$$

(iii) Hence the C.S. is $y = e^{-x/2}\left(c_1 \cos \frac{\sqrt{3}}{2} x + c_2 \sin \frac{\sqrt{3}}{2} x\right) + 1 - \frac{2}{3} e^x + \frac{e^{2x}}{7}$.

Example 13.13. *Solve $y'' + 4y' + 4y = 3 \sin x + 4 \cos x$, $y(0) = 1$ and $y'(0) = 0$.* (*J.N.T.U., 2003*)

Solution. Given equation in *symbolic form* is $(D^2 + 4D + 4)y = 3 \sin x + 4 \cos x$

(*i*) *To find C.F.*

Its A.E. is $(D + 2)^2 = 0$ where $D = -2, -2$ $\therefore$ C.F. $= (c_1 + c_2x)e^{-2x}$.

(*ii*) *To find P.I.*

$$\text{P.I.} = \frac{1}{D^2 + 4D + 4}(3 \sin x + 4 \cos x) = \frac{1}{-1 + 4D + 4}(3 \sin x + 4 \cos x)$$

$$= \frac{4D - 3}{16D^2 - 9}(3 \sin x + 4 \cos x) = \frac{(4D - 3)}{-16 - 9}(3 \sin x + 4 \cos x)$$

$$= \frac{-1}{25}\{3(4 \cos x - 3 \sin x) + 4(-4 \sin x - 3 \cos x)\} = \sin x$$

(*iii*) C.S. is $y = (c_1 + c_2) e^{-2x} + \sin x$

When $x = 0, y = 1,$ $\therefore$ $1 = c_1$

Also $y' = c_2e^{-2x} + (c_1 + c_2x)(-2)e^{-2x} + \cos x$.

When $x = 0, y' = 0,$ $\therefore$ $0 = c_2 - 2c_1 + 1$, *i.e.*, $c_2 = 1$.

Hence the required solution is $y = (1 + x) e^{-2x} + \sin x$.

Example 13.14. *Solve $(D - 2)^2 = 8(e^{2x} + \sin 2x + x^2)$.*

Solution. (*i*) *To find C.F.*

Its A.E. is $(D - 2)^2 = 0$, $\therefore$ $D = 2, 2$.

Thus C.F. $= (c_1 + c_2x) e^{2x}$.

(*ii*) *To find P.I.*

$$\text{P.I.} = 8\left[\frac{1}{(D-2)^2} e^{2x} + \frac{1}{(D-2)^2} \sin 2x + \frac{1}{(D-2)^2} x^2\right]$$

Now $\dfrac{1}{(D-2)^2} e^{2x} = x^2 \dfrac{1}{2(1)} e^{2x}$ [$\because$ by putting $D = 2$, $(D - 2)^2 = 0$, $2(D - 2) = 0$]

$$= \frac{x^2e^{2x}}{2}$$

$$\frac{1}{(D-2)^2} \sin 2x = \frac{1}{D^2 - 4D + 4} \sin 2x = \frac{1}{(-2^2) - 4D + 4} \sin 2x$$

$$= -\frac{1}{4}\int \sin 2x \, dx = -\frac{1}{4}\left(\frac{-\cos 2x}{2}\right) = \frac{1}{8} \cos 2x$$

and $$\frac{1}{(D-2)^2} x^2 = \frac{1}{4}\left(1 - \frac{D}{2}\right)^{-2} . x^2 = \frac{1}{4}\left[1 + (-2)\left(\frac{D}{2}\right) + \frac{(-2)(-3)}{2!}\left(-\frac{D}{2}\right)^2 + \ldots\right] x^2$$

$$= \frac{1}{4}\left(1 + D + \frac{3D^2}{4} + \ldots\right) x^2 = \frac{1}{4}\left(x^2 + 2x + \frac{3}{2}\right)$$

Thus P.I. $= 4x^2e^{2x} + \cos 2x + 2x^2 + 4x + 3$.

(*iii*) Hence the C.S. is $y = (c_1 + c_2x) e^{2x} + 4x^2e^{2x} + \cos 2x + 2x^2 + 4x + 3$.

Example 13.15. *Find the complete solution of $y'' - 2y' + 2y = x + e^x \cos x$.* (*U.P.T.U., 2002*)

Solution. Given equation in *symbolic form* is $(D^2 - 2D + 2)y = x + e^x \cos x$

(*i*) *To find C.F.*

Its A.E. is $D^2 - 2D + 2 = 0$ $\therefore$ $D = \dfrac{2 \pm \sqrt{(4-8)}}{2} = 1 \pm i$.

Thus C.F. $= e^x (c_1 \cos x + c_2 \sin x)$

(*ii*) *To find P.I.*

$$\text{P.I.} = \frac{1}{D^2 - 2D + 2}(x) + \frac{1}{D^2 - 2D + 2}(e^x \cos x)$$

$$= \frac{1}{2}\left[1 - \left(D - \frac{D^2}{2}\right)\right]^{-1}(x) + e^x \frac{1}{(D+1)^2 - 2(D+1) + 2}(\cos x)$$

$$= \frac{1}{2}\left(1 + D - \frac{D^2}{2}\right)x + e^x \frac{1}{D^2 + 1}\cos x \qquad \text{[Case of failure]}$$

$$= \frac{1}{2}(x + 1 - 0) + e^x \cdot x \frac{1}{2D}\cos x = \frac{1}{2}(x+1) + \frac{xe^x}{2}\int \cos x\, dx = \frac{1}{2}(x+1) + \frac{xe^x}{2}\sin x$$

(*iii*) Hence the C.S. is $y = e^x(c_1 \cos x + c_2 \sin x) + \frac{1}{2}(x+1) + \frac{xe^x}{2}\sin x$.

Example 13.16. *Solve* $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = xe^{3x} + \sin 2x$.

(V.T.U., 2008 ; Kottayam, 2005 ; U.P.T.U., 2003)

Solution. Given equation in *symbolic form* is $(D^2 - 3D + 2)y = xe^{3x} + \sin 2x$

(*i*) *To find C.F.*

Its A.E. is $D^2 - 3D + 2 = 0$ or $(D-2)(D-1) = 0$ whence $D = 1, 2$.

Thus C.F. $= c_1e^x + c_2e^{2x}$

(*ii*) *To find P.I.*

$$\text{P.I.} = \frac{1}{D^2 - 3D + 2}(xe^{3x} + \sin 2x) = \frac{1}{D^2 - 3D + 2}(e^{3x} \cdot x) + \frac{1}{D^2 - 3D + 2}(\sin 2x)$$

$$= e^{3x} \cdot \frac{1}{(D+3)^2 - 3(D+3) + 2}(x) + \frac{1}{-4 - 3D + 2}(\sin 2x)$$

$$= e^{3x} \cdot \frac{1}{D^2 + 3D + 2}(x) - \frac{3D - 2}{9D^2 - 4}(\sin 2x) = \frac{e^{3x}}{2} \cdot \left[1 + \left\{\frac{3D + D^2}{2}\right\}\right]^{-1} x - \frac{(3D-2)}{9(-4) - 4}(\sin 2x)$$

$$= \frac{e^{3x}}{2}\left(1 - \frac{3D}{2} \ldots\right)x + \frac{1}{40}(6\cos 2x - 2\sin 2x) = \frac{e^{3x}}{2}\left(x - \frac{3}{2}\right) + \frac{1}{20}(3\cos 2x - \sin 2x)$$

(*iii*) Hence the C.S. is $y = c_1e^x + c_2e^{2x} + e^{3x}\left(\frac{x}{2} - \frac{3}{4}\right) + \frac{1}{20}(3\cos 2x - \sin 2x)$.

Example 13.17. *Solve* $\frac{d^2y}{dx^2} - 4y = x \sinh x$. *(Madras, 2000 S)*

Solution. Given equation in *symbolic form* is $(D^2 - 4)y = x \sinh x$.

(*i*) *To find C.F.*

Its A.E. is $D^2 - 4 = 0$, whence $D = \pm 2$.

Thus C.F. $= c_1e^{2x} + c_2e^{-2x}$

(*ii*) *To find P.I.*

$$\text{P.I.} = \frac{1}{D^2 - 4}x \sinh x = \frac{1}{D^2 - 4}x\left(\frac{e^x - e^{-x}}{2}\right) = \frac{1}{2}\left[\frac{1}{D^2 - 4}e^x \cdot x - \frac{1}{D^2 - 4}e^{-x} \cdot x\right]$$

$$= \frac{1}{2}\left[e^x \frac{1}{(D+1)^2 - 4}x - e^{-x}\frac{1}{(D-1)^2 - 4}x\right] = \frac{1}{2}\left[e^x \frac{1}{D^2 + 2D - 3}x - e^{-x}\frac{1}{D^2 - 2D - 3}x\right]$$

$$= \frac{1}{2}\left[\frac{e^x}{-3}\left\{1-\left(\frac{2D}{3}+\frac{D^2}{3}\right)\right\}^{-1} . x - \frac{e^{-x}}{-3}\left\{1+\left(\frac{2D}{3}-\frac{D^2}{3}\right)\right\}^{-1} \cdot x\right]$$

$$= -\frac{1}{6}\left[e^x\left(1+\frac{2D}{3}+...\right)x - e^{-x}\left(1-\frac{2D}{3}+...\right)x\right] = -\frac{1}{6}\left[e^x\left(x+\frac{2}{3}\right)-e^{-x}\left(x-\frac{2}{3}\right)\right]$$

$$= -\frac{x}{3}\left(\frac{e^x-e^{-x}}{2}\right)-\frac{2}{9}\left(\frac{e^x+e^{-x}}{2}\right) = -\frac{x}{3}\sinh x - \frac{2}{9}\cosh x .$$

(*iii*) Hence the C.S. is $y = c_1e^{2x} + c_2e^{-2x} - \frac{x}{3}\sinh x - \frac{2}{9}\cosh x$.

Example 13.18. *Solve* $(D^2 - 1)y = x \sin 3x + \cos x.$

Solution. (*i*) *To find C.F.*

Its A.E. is $D^2 - 1 = 0$, whence $D = \pm 1$. $\therefore$ C.F. $= c_1e^x + c_2e^{-x}$

(*ii*) *To find P.I.*

$$\text{P.I.} = \frac{1}{D^2-1}(x\sin 3x + \cos x) = \frac{1}{D^2-1}x\,(\text{I.P. of } e^{3ix}) + \frac{1}{D^2-1}\cos x$$

$$= \text{I.P. of } \frac{1}{D^2-1}e^{3ix} . x + \frac{1}{(-1)^2-1}\cos x = \text{I.P. of }\left[e^{3ix}\frac{1}{(D+3i)^2-1}x\right] - \frac{\cos x}{2}$$

[Replacing D by $D + 3i$]

$$= \text{I.P. of }\left[e^{3ix}\frac{1}{D^2+6iD-10}x\right] - \frac{\cos x}{2}$$

$$= \text{I.P. of }\left[e^{3ix} . \frac{1}{-10}\left(1-\frac{3iD}{5}-\frac{D^2}{10}\right)^{-1}x\right] - \frac{\cos x}{2}$$ [Expand by Binomial theorem]

$$= \text{I.P. of }\left[e^{3ix} . \frac{1}{-10}\left(1+\frac{3iD}{5}+......\right)x\right] - \frac{\cos x}{2} = \text{I.P. of }\left[-\frac{e^{3ix}}{10}\left(x+\frac{3i}{5}\right)\right] - \frac{\cos x}{2}$$

$$= \text{I.P. of }\left[\frac{-1}{10}(\cos 3x + i\sin 3x)\left(x+\frac{3i}{5}\right)\right] - \frac{\cos x}{2}$$

$$= -\frac{1}{10}\text{ I.P. of }\left[\left(x\cos 3x - \frac{3\sin 3x}{5}\right) + i\left(x\sin 3x + \frac{3}{5}\cos 3x\right)\right] - \frac{\cos x}{2}$$

$$= -\frac{1}{10}\left(x\sin 3x + \frac{3}{5}\cos 3x\right) - \frac{\cos x}{2} .$$

(*iii*) Hence the C.S. is $y = c_1e^x + c_2e^{-x} - \frac{1}{50}(5x\sin 3x + 3\cos 3x + 25\cos x)$.

Example 13.19. *Solve* $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x \sin x.$ *(S.V.T.U., 2007 ; J.N.T.U., 2006 ; U.P.T.U., 2005)*

Solution. Given equation in *symbolic form* is $(D^2 - 2D + 1)\,y = xe^x \sin x$

(*i*) *To find C.F.*

Its A.E. is $D^2 - 2D + 1 = 0$, *i.e.*, $(D - 1)^2 = 0$

$\therefore$ $D = 1, 1$. Thus C.F. $= (c_1 + c_2x)e^x$

(*ii*) *To find P.I.*

$$\text{P.I.} = \frac{1}{(D-1)^2}e^x \,.\, x\sin x = e^x \,.\, \frac{1}{(D+1-1)^2}\, x\sin x$$

$$= e^x\frac{1}{D^2}\, x\sin x = e^x\frac{1}{D}\int x\sin x\,dx \qquad \textbf{[Integrate by parts]}$$

$$= e^x\frac{1}{D}\left[x(-\cos x) - \int 1\,.\,(-\cos x)\,dx\right] = e^x\int[-x\cos x + \sin x]\,dx$$

$$= e^x\left[-\left\{x\sin x - \int 1\,.\,\sin x\,dx\right\} - \cos x\right] = e^x[-x\sin x - \cos x - \cos x]$$

$$= -e^x(x\sin x + 2\cos x).$$

(*iii*) Hence the C.S. is $y = (c_1 + c_2x)\,e^x - e^x(x\sin x + 2\cos x)$.

Example 13.20. *Solve* $(D^4 + 2D^2 + 1)\,y = x^2\cos x$. *(Nagarjuna, 2008 ; Rajasthan, 2005)*

Solution. (*i*) *To find C.F.*

Its A.E. is $(D^2+1)^2 = 0$ whose roots are $D = \pm i, \pm i$

$\therefore$ C.F. $= (c_1 + c_2x)\cos x + (c_3 + c_4x)\sin x$

(*ii*) *To find P.I.*

$$\text{P.I.} = \frac{1}{(D^2+1)^2}\,x^2\cos x = \frac{1}{(D^2+1)^2}\,x^2(\text{Re.P. of } e^{ix})$$

$$= \text{Re.P. of}\left\{\frac{1}{(D^2+1)^2}e^{ix}\,.\,x^2\right\} = \text{Re.P. of}\left\{e^{ix}\frac{1}{[(D+i)^2+1]^2}\,x^2\right\}$$

$$= \text{Re.P. of}\left\{e^{ix}\frac{1}{(D^2+2iD)^2}\,x^2\right\} = \text{Re.P. of}\left[e^{ix}\left\{-\frac{1}{4D^2}\left(1-\frac{i}{2}D\right)^{-2}x^2\right\}\right]$$

$$= \text{Re.P. of}\left[-\frac{1}{4}e^{ix}\,.\,\frac{1}{D^2}\left\{1 + 2\frac{iD}{2} + 3\left(\frac{iD}{2}\right)^2 + \ldots\right\}x^2\right]$$

$$= \text{Re.P. of}\left\{-\frac{1}{4}e^{ix}\,.\,\frac{1}{D^2}\left(x^2 + 2ix - \frac{3}{2}\right)\right\} = \text{Re.P. of}\left\{-\frac{1}{4}e^{ix}\,.\,\frac{1}{D}\left(\frac{x^3}{3} + ix^2 - \frac{3}{2}x\right)\right\}$$

$$= -\frac{1}{4}\,\text{Re.P. of}\left\{e^{ix}\left(\frac{x^4}{12} + \frac{ix^3}{3} - \frac{3}{4}x^2\right)\right\} = \frac{-1}{48}\ \text{Re.P. of }\{(\cos x + i\sin x)(x^4 + 4ix^3 - 9x^2)\}$$

$$= -\frac{1}{48}\,[(x^4 - 9x^2)\cos x - 4x^3\sin x]$$

(*iii*) Hence the C.S. is $y = (c_1 + c_2x)\cos x + (c_3 + c_4x)\sin x + \dfrac{1}{48}\ [4x^3\sin x - x^2(x^2-9)\cos x]$.

Example 13.21. *Solve* $(D^2 - 4D + 4)y = 8x^2e^{2x}\sin 2x$. *(J.N.T.U., 2006 ; U.P.T.U., 2004)*

Solution. (*i*) *To find C.F.*

Its A.E. is $D^2 - 4D + 4 = 0$ *i.e.*, $(D-2)^2 = 0$. $\therefore$ $D = 2, 2$

$\therefore$ C.F. $= (c_1 + c_2x)\,e^{2x}$

(*ii*) *To find P.I.*

$$\text{P.I.} = \frac{1}{(D-2)^2}\,(8x^2e^{2x}\sin 2x) = 8e^{2x}\,\frac{1}{(D+2-2)^2}\,(x^2\sin 2x)$$

$$= 8e^{2x}\,\frac{1}{D^2}\,(x^2\sin 2x) = 8e^{2x}\,.\,\frac{1}{D}\int x^2\sin 2x\,dx$$

$$= 8e^{2x} \cdot \frac{1}{D}\left\{x^2\left(\frac{-\cos 2x}{2}\right) - \int 2x\left(\frac{-\cos 2x}{2}\right)dx\right\}$$

$$= 8e^{2x}\frac{1}{D}\left\{-\frac{x^2}{2}\cos 2x + x\frac{\sin 2x}{2} - \int 1 . \frac{\sin 2x}{2}\,dx\right\}$$

$$= 8e^{2x}\int\left\{-\frac{x^2}{2}\cos 2x + \frac{x}{2}\sin 2x + \frac{\cos 2x}{4}\right\}dx$$

$$= 8e^{2x}\left[\left\{\frac{-x^2}{2}\frac{\sin 2x}{2} - \int(-x)\frac{\sin 2x}{2}dx\right\} + \left\{\int\frac{x}{2}\sin 2x\,dx\right\} + \frac{\sin 2x}{8}\right]$$

$$= 8e^{2x}\left[\left(\frac{-x^2}{4} + \frac{1}{8}\right)\sin 2x + \int x\sin 2x\,dx\right]$$

$$= 8e^{2x}\left[\left(\frac{1}{8} - \frac{x^2}{4}\right)\sin 2x + x\left(\frac{-\cos 2x}{2}\right) - \int 1 . \left(\frac{-\cos 2x}{2}\right)dx\right]$$

$$= 8e^{2x}\left[\left(\frac{1}{8} - \frac{x^2}{4}\right)\sin 2x - \frac{x\cos 2x}{2} + \frac{\sin 2x}{4}\right]$$

$$= e^{2x}[(3 - 2x^2)\sin 2x - 4x\cos 2x]$$

(*iii*) Hence the C.S. is $y = e^{2x}[c_1 + c_2x + (3 - 2x^2)\sin 2x - 4x\cos 2x]$.

Example 13.22. *Solve* $\dfrac{d^2y}{dx^2} + a^2y = \sec ax.$

Solution. Given equation in *symbolic form* is $(D^2 + a^2)y = \sec ax$.

(*i*) *To find C.F.*

Its A.E. is $D^2 + a^2 = 0 \quad \therefore \quad D = \pm ia.$

Thus C.F. $= c_1\cos ax + c_2\sin ax.$

(*ii*) *To find P.I.*

$$\text{P.I.} = \frac{1}{D^2 + a^2}\sec ax = \frac{1}{(D + ia)(D - ia)}\sec ax \qquad \text{[Resolving into partial fractions]}$$

$$= \frac{1}{2ia}\left[\frac{1}{D - ia} - \frac{1}{D + ia}\right]\sec ax = \frac{1}{2ia}\left[\frac{1}{D - ia}\sec ax - \frac{1}{D + ia}\sec ax\right]$$

Now $\dfrac{1}{D - ia}\sec ax = e^{iax}\displaystyle\int \sec ax \,.\, e^{-iax}\,dx \qquad \left[\because \dfrac{1}{D - a}X = e^{ax}\displaystyle\int Xe^{-ax}\,dx\right]$

$$= e^{iax}\int\frac{\cos ax - i\sin ax}{\cos ax}dx = e^{iax}\int(1 - i\tan ax)dx = e^{iax}\left(x + \frac{i}{a}\log\cos ax\right)$$

Changing i to $-i$, we have

$$\frac{1}{D + ia}\sec ax = e^{-iax}\left\{x - \frac{i}{a}\log\cos ax\right\}$$

Thus $\text{P.I.} = \dfrac{1}{2ia}\left[e^{iax}\left\{x + \dfrac{i}{a}\log\cos ax\right\} - e^{-iax}\left\{x - \dfrac{i}{a}\log\cos ax\right\}\right]$

$$= \frac{x}{a}\frac{e^{iax} - e^{-iax}}{2i} + \frac{1}{a^2}\log\cos ax \,.\, \frac{e^{iax} + e^{-iax}}{2} = \frac{x}{a}\sin ax + \frac{1}{a^2}\log\cos ax \,.\, \cos ax\,.$$

(*iii*) Hence the C.S. is

$$y = c_1\cos ax + c_2\sin ax + (1/a)x\sin ax + (1/a^2)\cos ax\log\cos ax.$$

Method of Variation of parameters:

This method is applicable to equations with constant coefficients but the C.F. must be known before the method is applied.

Procedure to solve $y^{11}+py^1+qy=X$ where p,q are constants and X is function of x

Find C.F. $c_1y_1+c_2y_2$

Assume P.I. as uy_1+vy_2 so that $u^1y_1^1+v^1y_2^1=X$

To find u,v as functions of x assume that $u^1y_1+v^1y_2=0$

Compute $u=-\int \frac{y_2X}{W}dx$ and $v = \int \frac{y_1X}{W}dx$ where $W=\begin{vmatrix} y_1 & y_2 \\ y_1^1 & y_2^1 \end{vmatrix}$ is called the **Wronskian** of y_1, y_2

The general solution is $y = c_1y_1+c_2y_2+uy_1+vy_2$.

Ex: Find the general solution of

$$y'' + y = \frac{1}{2+\sin x}, \quad x \geq 0.$$

Solution: The general solution of the corresponding homogeneous equation $y''+y=0$ is given by

$$y_h = c_1 \cos x + c_2 \sin x.$$

Here, the solutions $y_1 = \sin x$ and $y_2 = \cos x$ are linearly independent over $I = [0, \infty)$ and $W = W(\sin x, \cos x) = 1.$ Therefore, a particular solution, y_h, by Theorem is

$$\begin{aligned} y_p &= -y_1 \int \frac{y_2}{2+\sin x}dx + y_2 \int \frac{y_1}{2+\sin x}dx \\ &= \sin x \int \frac{\cos x}{2+\sin x}dx + \cos x \int \frac{\sin x}{2+\sin x}dx \\ &= -sinx \ln(2+\sin x) + \cos x \left(x - 2\int \frac{1}{2+\sin x}dx\right). \end{aligned}$$

So, the required general solution is

$$y = c_1 \cos x + c_2 \sin x + y_p$$

1. Solve $\frac{d^2y}{dx^2}$ +y = cosec **x** by using method of variation of parameters.

Solution:

Given: $(D^2 + 1)\, y = \text{cosec } x$

The auxiliary equation is $(m^2 + 1) = 0$

$m = \pm i$

$\therefore$ C. F. $= c_1 \cos x + c_2 \sin x$

$= c_1 f_1 + c_2 f_2$

Here, $f_1 = \cos x$

$f_1' = -\sin x$

$f_2 = \sin x$

$f_2' = \cos x$

$X = \text{cosec } x$

$f_1 f_2' - f_1' f_2 = \cos^2 x + \sin^2 x = 1$

$P = -\int \frac{f_2 X}{f_1 f_2' - f_1' f_2}\, dx$

$= -\int \frac{\sin x \text{ cosec } x}{1}\, dx$

$= -x$

$Q = \int \frac{f_1 X}{f_1 f_2' - f_1' f_2}\, dx$

$= \int \frac{\cos x \text{ cosec } x}{1}\, dx$

$= \log(\sin x)$

P.I. $= Pf_1 + Qf_2$

$= -x\cos x + \log(\sin x)$

$\therefore y = C.F + P.I.$

$y = c_1\cos x + c_2\sin x - x\cos x + \log(\sin x)$

2. Solve $(D^2+1)y = \operatorname{cosec} x \cot x$ by the method of variation of parameters.

Solution:

Given: $(D^2+1)y = \operatorname{cosec} x$

The auxiliary equation is $(m^2+1) = 0$

$m = \pm i$

$\therefore$ C. F. $= C_1\cos x + C_2\sin x$

$= c_1f_1 + c_2f_2$

Here, $f_1 = \cos x$

$f_1' = -\sin x$

$f_2 = \sin x$

$f_2' = \cos x$

$X = \operatorname{cosec} x \cot x$

$f_1f_2' - f_1'f_2 = \cos^2 x + \sin^2 x = 1$

$$P = -\int \frac{f_2X}{f_1f_2' - f_1'f_2}dx$$

$$= -\int \frac{\sin x \operatorname{cosec} x \cot x}{1}dx$$

$$= -\int \cot x\, dx$$

$$= -\log(\sin x)$$

$$Q = \int \frac{f_1X}{f_1f_2' - f_1'f_2}dx$$

$$= \int \frac{\cos x \operatorname{cosec} x \cot x}{1}dx$$

$$= \int \frac{\cos^2 x}{\sin^2 x}dx$$

$$= \int \frac{(1-\sin^2 x)}{\sin^2 x}dx$$

$$= \int (\operatorname{cosec}^2 x - 1)dx$$

$$= -\cot x - x$$

P.I. $= Pf_1 + Qf_2$

$= -\log(\sin x)\cos x + [-\cot x - x]\sin x$

$= -\cos x\log(\sin x) - [\cot x + x]\sin x$

$y = C.F. + P.I$

$y = C_1\cos x + C_2\sin x - \cos x\log(\sin x) - [\cot x + x]\sin x$

Example 13.23. *Using the method of variation of parameters, solve*

$$\frac{d^2y}{dx^2} + 4y = \tan 2x.$$ (*V.T.U., 2008 ; Bhopal, 2007 ; S.V.T.U., 2006 S*)

Solution. Given equation in *symbolic form* is $(D^2 + 4)y = \tan 2x$.

(*i*) *To find C.F.*

Its A.E. is $D^2 + 4 = 0$, $\therefore$ $D = \pm 2i$

Thus C.F. is $y = c_1 \cos 2x + c_2 \sin 2x$.

(*ii*) *To find P.I.*

Here $y_1 = \cos 2x$, $y_2 = \sin 2x$ and $X = \tan 2x$

$$\therefore \quad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2$$

Thus,
$$\text{P.I.} = -y_1 \int \frac{y_2 X}{W} dx + y_2 \int \frac{y_1 X}{W} dx$$
$$= -\cos 2x \int \frac{\sin 2x \tan 2x}{2} dx + \sin 2x \int \frac{\cos 2x \tan 2x}{2} dx$$
$$= -\frac{1}{2} \cos 2x \int (\sec 2x - \cos 2x) dx + \frac{1}{2} \sin 2x \int \sin 2x \, dx$$
$$= -\frac{1}{4} \cos 2x [\log(\sec 2x + \tan 2x) - \sin 2x] - \frac{1}{4} \sin 2x \cos 2x$$
$$= -\frac{1}{4} \cos 2x \log(\sec 2x + \tan 2x)$$

Hence the C.S. is $y = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{4} \cos 2x \log(\sec 2x + \tan 2x)$.

Example 13.24. *Solve, by the method of variation of parameters,* $d^2y/dx^2 - y = 2/(1 + e^x)$.

(*V.T.U., 2005 ; Hissar, 2005*)

Solution. Given equation is $D^2 - 1 = 2/(1 + e^x)$

A.E. is $D^2 - 1 = 0$, $D = \pm 1$, $\therefore$ C.F. $= c_1 e^x + c_2 e^{-x}$

Here $y_1 = e^x$, $y_2 = e^{-x}$ and $X = 2/(1 + e^x)$

$$\therefore \quad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -e^x e^{-x} - e^x e^{-x} = -2.$$

Thus
$$\text{P.I.} = -y_1 \int \frac{y_2 X}{W} dx + y_2 \int \frac{y_1 X}{W} dx = -e^x \int \frac{e^{-x}}{-2} \cdot \frac{2}{1+e^x} dx + e^{-x} \int \frac{e^x}{-2} \cdot \frac{2}{1+e^x} dx$$
$$= e^x \int \left(\frac{1}{e^x} - \frac{1}{1+e^x} \right) dx - e^{-x} \log(1+e^x) = e^x \left[e^{-x} - \int \frac{e^{-x}}{e^{-x}+1} dx \right] - e^{-x} \log(1+e^x)$$
$$= e^x [-e^{-x} + \log(e^{-x} + 1)] - e^{-x} \log(1 + e^x) = -1 + e^x \log(e^{-x} + 1) - e^{-x} \log(e^x + 1)$$

Hence C.S. is $y = c_1 e^x + c_2 e^{-x} - 1 + e^x \log(e^{-x} + 1) - e^{-x} \log(e^x + 1)$.

Example 13.25. *Solve by the method of variation of parameters* $y'' - 6y' + 9y = e^{3x}/x^2$.
(Nagpur, 2009 ; S.V.T.U., 2009)

Solution. Given equation is $(D^2 - 6D + 9)y = e^{3x}/x^2$

A.E. is $D^2 - 6D + 9 = 0$ *i.e.* $(D-3)^2 = 0$ $\therefore$ C.F. $= (c_1 + c_2x)e^{3x}$

Here $y_1 = e^{3x}, y_2 = xe^{3x}$ and $X = e^{3x}/x^2$

$$\therefore \qquad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & e^{3x} + 3xe^{3x} \end{vmatrix} = e^{6x}.$$

Thus
$$\text{P.I.} = -y_1 \int \frac{y_2X}{W}\,dx + y_2 \int \frac{y_1X}{W}\,dx = -e^{3x} \int \frac{xe^{3x}}{e^{6x}} \frac{e^{3x}}{x^2}\,dx + xe^{3x} \int \frac{e^{3x}}{e^{6x}} \cdot \frac{e^{3x}}{x^2}\,dx$$

$$= -e^{3x} \int \frac{dx}{x} + xe^{3x} \int x^{-2}\,dx = -e^{3x}(\log x + 1)$$

Hence C.S. is $\quad y = (c_1 + c_2x)e^{3x} - e^{3x}(\log x + 1)$.

Example 13.26. *Solve, by the method of variation of parameters,* $y'' - 2y' + y = e^x \log x$.
(V.T.U., 2006 ; Kurukshetra, 2005 ; Madras, 2003)

Solution. Given equation in symbolic form is $(D^2 - 2D + 1)\,y = e^x \log x$

(*i*) *To find C.F.*

Its A.E. is $(D-1)^2 = 0$, $\therefore$ $D = 1, 1$

Thus C.F. is $y = (c_1 + c_2x)e^x$

(*ii*) *To find P.I.*

Here $y_1 = e^x$, $y_2 = xe^x$ and $X = e^x \log x$

$$\therefore \qquad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & xe^x \\ e^x & (1+x)\,e^x \end{vmatrix} = e^{2x}$$

Thus
$$\text{P.I.} = -y_1 \int \frac{y_2X}{W}\,dx + y_2 \int \frac{y_1X}{W}\,dx$$

$$= -e^x \int \frac{xe^x \,.\, e^x \log x}{e^{2x}}\,dx + xe^x \int \frac{e^x \,.\, e^x \log x}{e^{2x}}\,dx = -e^x \int x \log x\,dx + xe^x \int \log x\,dx$$

$$= -e^x \left(\frac{x^2}{2}\log x - \int \frac{1}{x} \cdot \frac{x^2}{2}\,dx\right) + x \,.\, e^x \left(x \log x - \int \frac{1}{x} \,.\, x\,dx\right)$$

$$= -e^x \left(\frac{x^2}{2}\log x - \frac{x^2}{4}\right) + x \;\; e^x (x \log x - x) = \frac{1}{4}\, x^2 e^x (2 \log x - 3)$$

Hence C.S. is $y = (c_1 + c_2x)\,e^x + \dfrac{1}{4}\,x^2e^x(2 \log x - 3)$.

I. Cauchy's homogeneous linear equation*. An equation of the form

$$x^n \frac{d^n y}{dx^n} + k_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + k_{n-1} x \frac{dy}{dx} + k_n y = X \qquad \dots(1)$$

where X is a function of x, is called *Cauchy's homogeneous linear equation.*

Such equations can be reduced to linear differential equations with constant coefficients, by putting

$$x = e^t \quad \text{or} \quad t = \log x. \text{ Then if } D = \frac{d}{dt}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt} \cdot \frac{1}{x}, \quad i.e., \quad x\frac{dy}{dx} = Dy.$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right) = -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d}{dt}\left(\frac{dy}{dt}\right)\frac{dt}{dx} = -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d^2y}{dt^2}\frac{dt}{dx} = \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

i.e., $\quad x^2 \dfrac{d^2y}{dx^2} = D(D-1)\,y$. Similarly, $x^3 \dfrac{d^3y}{dx^3} = D(D-1)(D-2)\,y$ and so on.

After making these substitutions in (1), there results a linear equation with constant coefficients, which can be solved as before.

Example 13.30. *Solve* $x^2 \dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + y = \log x.$ *(V.T.U., 2010)*

Solution. This is a Cauchy's homogeneous linear.

Put $x = e^t$, *i.e.*, $t = \log x$, so that $x\dfrac{dy}{dx} = Dy$, $x^2\dfrac{d^2y}{dx^2} = D(D-1)y$ where $D = \dfrac{d}{dt}$

Then the given equation becomes $[D(D-1) - D + 1]\,y = t$ or $(D-1)^2 y = t$ ...(*i*)

which is a linear equation with constant coefficients.

Its A.E. is $(D-1)^2 = 0$ whence $D = 1, 1$.

$\therefore$ C.F. $= (c_1 + c_2 t)e^t$ and P.I. $= \dfrac{1}{(D-1)^2}\,t = (1-D)^{-2}\,t = (1 + 2D + 3D^2 + \ldots)\,t = t + 2.$

Hence the solution of (*i*) is $y = (c_1 + c_2 t)e^t + t + 2$ or, putting $t = \log x$ and $e^t = x$, we get

$y = (c_1 + c_2 \log x)\,x + \log x + 2$ as the required solution of (*i*).

Example 13.31. *Solve* $x^2\dfrac{d^2y}{dx^2} + 3x\dfrac{dy}{dx} + y = \dfrac{1}{(1-x)^2}$. (*P.T.U., 2003*)

Solution. Put $x = e^t$ *i.e.*, $t = \log x$ so that $x\,dy/dx = Dy$, $x^2\dfrac{d^2y}{dx^2} = D(D-1)\,y$

Then the given equation becomes

$$[D(D-1) + 3D + 1]\,y = \frac{1}{(1-e^t)^2} \quad \text{or} \quad (D^2 + 2D + 1)\,y = \frac{1}{(1-e^t)^2}$$

Its A.E. is $D^2 + 2D + 1 = 0$ or $(D+1)^2 = 0$ *i.e.*, $D = -1, -1$.

$$\therefore \quad \text{C.F.} = (c_1 + c_2 x)\,e^{-t} = (c_1 + c_2 \log x)\,\frac{1}{x}$$

$$\text{P.I.} = \frac{1}{(D+1)^2}\,\frac{1}{(1-e^t)^2} = \frac{1}{D+1}\,u, \text{ where } u = \frac{1}{D+1}\cdot\frac{1}{(1-e^t)^2} \text{ i.e. } \frac{du}{dt} + u = (1-e^t)^{-2}$$

which is Leibnitz's linear equation having I.F. $= e^t$

$$\therefore \text{ its solution is} \quad ue^t = \int \frac{e^t}{(1-e^t)^2}\,dt = \frac{1}{1-e^t} \quad \text{or} \quad u = \frac{e^{-t}}{1-e^t}$$

$$\therefore \quad \text{P.I.} = \frac{1}{D+1}\left(\frac{e^{-t}}{1-e^t}\right) = e^{-t}\int \frac{1}{1-e^t}\,dt = \frac{1}{x}\int \frac{dx}{x(1-x)}$$

$$= \frac{1}{x}\int\left(\frac{1}{x} + \frac{1}{1-x}\right)dx = \frac{1}{x}\,[\log x - \log(1-x)] = \frac{1}{x}\log\frac{x}{x-1}$$

Hence the solution is $$y = \left\{c_1 + c_2 \log x + \log\frac{x}{x-1}\right\}\frac{1}{x}.$$

Example 13.32. *Solve* $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = \log x \sin(\log x)$.

(*Kurukshetra, 2006 ; Madras, 2006 ; Kerala, 2005*)

Solution. Putting $x = e^t$ *i.e.* $t = \log x$, the given equation becomes

$$[D(D-1) + D + 1]\,y = t \sin t \quad \text{i.e.} \quad (D^2+1)\,y = t\sin t \qquad \ldots(i)$$

Its A.E. is $D^2 + 1 = 0$ *i.e.* $D = \pm i$.

$$\therefore \quad \text{C.F.} = c_1 \cos t + c_2 \sin t$$

and $$P.I. = \frac{1}{D^2+1}\,t\sin t = \frac{1}{D^2+1}\,t\,(\text{I.P. of } e^{it})$$

$$= \text{I.P. of } e^{it}\,\frac{1}{(D+i)^2+1}\,t = \text{I.P. of } e^{it}\,.\,\frac{1}{D^2+2iD}\,t$$

$$= \text{I.P. of } e^{it} \frac{1}{2iD(1 + D/2i)} t = \text{I.P. of } \frac{1}{2i} e^{it} \frac{1}{D}\left(1 - \frac{iD}{2}\right)^{-1} t$$

$$= \text{I.P. of } \frac{1}{2i} e^{it} \frac{1}{D}\left(1 + \frac{iD}{2} + \ldots\right) t = \text{I.P. of } \frac{1}{2i} e^{it} \frac{1}{D}\left(t + \frac{i}{2}\right)$$

$$= \text{I.P. of } \frac{e^{it}}{2i} \int\left(t + \frac{i}{2}\right) dt = \text{I.P. of } \frac{e^{it}}{2i}\left(\frac{t^2}{2} + \frac{it}{2}\right)$$

$$= \text{I.P. of } e^{it}\left(-\frac{i}{4}t^2 + \frac{t}{4}\right) = \text{I.P. of } (\cos t + i \sin t)\left(-\frac{it^2}{4} + \frac{t}{4}\right) = -\frac{t^2}{4}\cos t + \frac{t}{4}\sin t$$

Hence the C.S. of (i) is $y = c_1 \cos t + c_2 \sin t - \dfrac{t^2}{4} \cos t + \dfrac{t}{4} \sin t$

or $\quad y = c_1 \cos(\log x) + c_2 \sin(\log x) - \dfrac{1}{4}(\log x)^2 \cos(\log x) + \dfrac{1}{4}\log(\log x)\sin(\log x)$

which is the required solution.

Example 13.33. *Solve* $x^2 \dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + y = \log x \dfrac{\sin(\log x) + 1}{x}$. *(I.S.M., 2001)*

Solution. Put $x = e^t$, *i.e.*, $t = \log x$ so that $x\dfrac{dy}{dx} = Dy,\ x^2\dfrac{d^2y}{dx^2} = D(D-1)y$

Then the given equation becomes

$$\{D(D-1) - 3D + 1\}y = t\,\frac{\sin t + 1}{e^t} \quad \text{or} \quad (D^2 - 4D + 1)y = e^{-t}t(\sin t + 1)$$

which is a linear equation with constant coefficients.

Its A.E. is $D^2 - 4D + 1 = 0$ whence $D = 2 \pm \sqrt{3}$

$\therefore$ $$\text{C.F.} = c_1e^{(2+\sqrt{3})t} + c_2e^{(2-\sqrt{3})t} = e^{2t}(c_1e^{\sqrt{3}t} + c_2e^{-\sqrt{3}t})$$

and $$\text{P.I.} = \frac{1}{D^2 - 4D + 1} e^{-t}t(\sin t + 1) = e^{-t}\frac{1}{(D-1)^2 - 4(D-1) + 1} t(\sin t + 1)$$

$$= e^{-t}\left\{\frac{1}{D^2 - 6D + 6}t + \frac{1}{D^2 - 6D + 6}t\sin t\right\}$$

$$\frac{1}{D^2 - 6D + 6}t = \frac{1}{6}\left(1 - \frac{6D - D^2}{6}\right)^{-1} t = \frac{1}{6}(1 + D)t = \frac{1}{6}(t + 1)$$

$$\frac{1}{D^2 - 6D + 6}t\sin t = \text{I.P. of } \frac{1}{D^2 - 6D + 6}e^{it} \cdot t$$

$$= \text{I.P. of } e^{it}\frac{1}{(D+i)^2 - 6(D+i) + 6}t = \text{I.P. of } e^{it}\frac{1}{D^2 + (2i-6)D + (5-6i)}t$$

$$= \text{I.P. of } \frac{e^{it}}{5-6i}\left\{1 + \frac{(2i-6)D + D^2}{5-6i}\right\}^{-1} t = \text{I.P. } \frac{e^{it}}{5-6i}\left(1 - \frac{2i-6}{5-6i}D\right)t$$

$$= \text{I.P. of } \frac{(5+6i)}{61}(\cos t + i\sin t)\left(t - \frac{2i-6}{5-6i}\right)$$

$$= \text{I.P. of } \frac{1}{61}\{(5\cos t - 6\sin t) + i(5\sin t + 6\cos t)\}\left(t + \frac{42 + 26i}{61}\right)$$

$$= \frac{26}{3721}(5\cos t - 6\sin t) + \frac{1}{61}(5\sin t + \cos t)\left(t + \frac{42}{61}\right)$$

$$= \frac{t}{61}(5\sin t + \cos t) + \frac{2}{3721}(27\sin t + 191\cos t)$$

$\therefore$
$$\text{P.I.} = e^{-t}\left[\frac{1}{6}(t+1) + \frac{1}{61}(5\sin t + 6\cos t) + \frac{2}{3721}(27\sin t + 191\cos t)\right]$$

Hence
$$y = e^{2t}(c_1 e^{\sqrt{3}t} + c_2 e^{-\sqrt{3}t}) + e^{-t}\left[\frac{1}{6}(t+1) + \frac{t}{61}(5\sin t + 6\cos t) + \frac{2}{3721}(27\sin t + 191\cos t)\right]$$

or
$$y = x^2(c_1 x^{\sqrt{3}} + c_2 x^{-\sqrt{3}}) + \frac{1}{x}\left[\frac{1}{6}(\log x + 1) + \frac{\log x}{61}\{5\sin(\log x) + 6\cos(\log x)\} + \frac{2}{3721}\{27\sin(\log x) + 191\cos(\log x)\}\right].$$

Example 13.34. *Solve* $x^2\dfrac{d^2y}{dx^2} + 4x\dfrac{dy}{dx} + 2y = e^{x^x}$. *(Kurukshetra, 2005 ; U.P.T.U., 2005)*

Solution. Putting $x = e^t$, *i.e.*, $t = \log x$, the given equation becomes

$[D(D-1) + 4D + 2]\, y = e^{e^t}$ *i.e.*, $(D^2 + 3D + 2)\, y = e^{e^t}$

Its A.E. is $D^2 + 3D + 2 = 0$ whence $D = -1, -2$.

$\therefore$ C.F. $= c_1 e^{-t} + c_2 e^{-2t} = c_1 x^{-1} + c_2 x^{-2}$

and
$$\text{P.I.} = \frac{1}{(D^2 + 3D + 2)} e^{e^t} = \frac{1}{(D+1)(D+2)} e^{e^t} = \left(\frac{1}{D+1} - \frac{1}{D+2}\right) e^{e^t}.$$

Now
$$\frac{1}{D+1} e^{e^t} = \frac{1}{D+1} e^{-t} \cdot e^t e^{e^t} = e^{-t}\frac{1}{(D-1)+1} e^t e^{e^t}$$
$$= e^{-t}\frac{1}{D} e^t e^{e^t} = e^{-t}\int e^{e^t} d(e^t) = x^{-1}\int e^x dx = x^{-1}e^x$$
$$\frac{1}{D+2} e^{e^t} = \frac{1}{D+2} e^{-2t} \cdot e^{2t} e^{e^t} = e^{-2t}\frac{1}{(D-2)+2} e^{2t} e^{e^t}$$
$$= e^{-2t}\frac{1}{D} e^{e^t} e^{2t} = e^{-2t}\int e^{e^t} e^t\, d(e^t)$$
$$= x^{-2}\int e^x\, x\, dx \qquad [\because\ e^t = x]$$
$$= x^{-2}(xe^x - e^x) \qquad \text{[Integrating by parts]}$$

$\therefore$
$$\text{P.I.} = x^{-1}e^x - x^{-2}(xe^x - e^x) = x^{-2}e^x$$

Hence the required solution is $y = c_1 x^{-1} + x^{-2}(c_2 + e^x)$.

II. Legendre's linear equation*. An equation of the form

$$(ax+b)^n \frac{d^n y}{dx^n} + k_1 (ax+b)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \ldots + k_n y = X \qquad \ldots(2)$$

where k's are constants and X is a function of x, is called *Legendre's linear equation.*

Such equations can be reduced to linear equations with constant coefficients by the substitution $ax + b = e^t$, *i.e.*, $t = \log(ax+b)$.

Then, if
$$D = \frac{d}{dt}, \frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{a}{ax+b}\cdot\frac{dy}{dt} \quad i.e.\ (ax+b)\frac{dy}{dx} = a\,Dy$$
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{a}{ax+b}\frac{dy}{dt}\right) = \frac{-a^2}{(ax+b)^2}\frac{dy}{dt} + \frac{a}{ax+b}\frac{d}{dt}\left(\frac{dy}{dt}\right)\frac{dt}{dx} = \frac{a^2}{(ax+b)^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

i.e., $(ax+b)^2 \dfrac{d^2y}{dx^2} = a^2 D(D-1)\,y$. Similarly, $(ax+b)^3 \dfrac{d^3y}{dx^3}\; a^3D(D-1)(D-2)\,y$ and so on.

After making these replacements in (2), there results a linear equation with constant coefficients.

Example 13.35. *Solve* $(1+x)^2 \dfrac{d^2y}{dx^2} + (1+x)\dfrac{dy}{dx} + y = 2\sin[\log(1+x)]$. ...(*i*)

(V.T.U., 2009 ; J.N.T.U., 2005 ; Kerala, 2005)

Solution. This is a Legendre's linear equation.

$\therefore$ put $1+x = e^t$, *i.e.*, $t = \log(1+x)$, so that $(1+x)\dfrac{dy}{dx} = Dy$

and
$$(1+x)^2 \frac{d^2y}{dx^2} = D(D-1)\,y, \text{ where } D = \frac{d}{dt}$$

Then (*i*) becomes $D(D-1)\,y + Dy + y = 2\sin t$

or
$$(D^2+1)\,y = 2\sin t \qquad ...(ii)$$

This is a linear equation with constant co-efficients

Its A.E. is $D^2+1=0$, whence $D = \pm i$ $\therefore$ C.F. $= c_1\cos t + c_2\sin t$

and
$$\text{P.I.} = 2\frac{1}{D^2+1}\sin t = 2t\,.\,\frac{1}{2D}\sin t$$

$$= t\int \sin t\,dt = -t\cos t \qquad [\because \text{ on replacing } D^2 \text{ by } -1^2,\ D^2+1=0]$$

Hence the solution of (*ii*) is $y = c_1\cos t + c_2\sin t - t\cos t$ and on replacing t by $\log(1+x)$, we get $y = c_1\cos[\log(1+x)] + c_2\sin[\log(1+x)] - \log(1+x)\cos[\log(1+x)]$ as the required solution.

Example 13.36. *Solve* $(2x-1)^2 \dfrac{d^2y}{dx^2} + (2x-1)\dfrac{dy}{dx} - 2y = 8x^2 - 2x + 3$. (V.T.U., 2006)

Solution. This is a Legendre's linear equation.

$\therefore$ put $2x-1 = e^t$ *i.e.*, $t = \log(2x-1)$ so that $(2x-1)\dfrac{dy}{dx} = 2Dy$

and
$$(2x-1)^2 \frac{d^2y}{dx^2} = 4D(D-1)\,y, \text{ where } D = \frac{d}{dt}.$$

Then the given equation becomes

$$4D(D-1)\,y + 2Dy - 2y = 8\left(\frac{1+e^t}{2}\right)^2 - 2\left(\frac{1+e^t}{2}\right) + 3$$

or
$$2D^2y - Dy - y = e^{2t} + \frac{3}{2}e^t + 2 \qquad ...(i)$$

This is a linear equation with constant coefficients.

Its A.E. is $2D^2 - D - 1 = 0$ whence $D = 1, -1/2$.

$\therefore$ C.F. $= c_1e^t + c_2e^{-t/2}$

and
$$\text{P.I.} = \frac{1}{2D^2-D-1}\left(e^{2t} + \frac{3}{2}e^t + 2\right) = \frac{1}{2.4-2-1}e^{2t} + \frac{3}{2}\,\frac{t}{4D-1}e^t + 2\,.\,\frac{1}{2.0^2-0-1}e^{0t}$$

$[\because$ on putting $t = 1$, $2D^2 - D - 1 = 0$

$$= \frac{1}{5}e^{2t} + \frac{3t}{2}\,.\,\frac{1}{4-1}e^t - 2 = \frac{1}{5}e^{2t} + \frac{t}{2}e^t - 2$$

Hence the solution of (*i*) is

$$y = c_1e^t + c_2e^{-t/2} + \frac{1}{5}e^{2t} + \frac{1}{2}te^t - 2 \text{ and on replacing } t \text{ by } \log(2x-1),$$

$$y = c_1(2x-1) + c_2(2x-1)^{-1/2} + \frac{1}{5}(2x-1)^2 + \frac{1}{2}(2x-1)\log(2x-1) - 2.$$

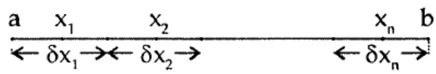
which is the required solution.

UNIT-III

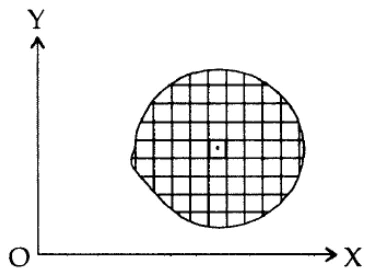
MULTIVARIABLE CALCULUS (MULTIPLE INTEGRALS)

Introduction: The process of integration can be extended to functions of more than one variable. This leads us to two generalizations of the definite integral, namely multiple integrals and repeated integrals. Multiple integrals are definite integrals of functions of several variables. Double and triple integrals arise while evaluating quantities such as area, volume, mass, moments, centroids and moment of inertia find many applications in science and engineering problems.

Double Integrals: The definite integral $\int_a^b f(x)dx$ is defined as the limits of the sum $f(x_1)\,\delta x_1 + f(x_2)\,\delta x_2 + f(x_3)\,\delta x_3 + \ldots\ldots\ldots\ldots + f(x_n)\,\delta x_n$ when $n \to \propto$ and each of the length $\delta x_1, \delta x_2, \delta x_3 \ldots\ldots\ldots\ldots\ldots \delta x_n$ tends to zero. Here $\delta x_1, \delta x_2, \delta x_3 \ldots\ldots\ldots\ldots\ldots \delta x_n$ are n sub intervals into which the range b – a has been divided and $x_1, x_2 \ldots\ldots\ldots\ldots x_n$ are values of x lying respectively in the first, second, third,…………nth sub-interval.



A double integral is its counterpart of two dimensions. Let a single valued and bounded function f(x, y) of two independent variables x, y are defined in a closed region R of the xy – plane.
Divide the region R into subregions by drawing lines parallel to Co-ordinate axes. Number of rectangles which lie entirely inside the region R, from 1 to n. Let (x_r, y_r) be any point inside the rth rectangle whose area is δA_r.



Consider the sum
$f(x_1, y_1)\,\delta A + f(x_2, y_2)\,\delta A_2 + \ldots\ldots\ldots\ldots + f(x_n, y_n)\,\delta A_n$

$$= \sum_{r=1}^{n} f(x_r, y_r)\delta A_r \qquad \text{(i)}$$

Let the number of these sub-regions increase indefinitely, such that the largest linear dimension (i.e. diagonal) of δA_r approaches zero. The limit of the sum (i), if it exists, irrespective of the mode of subdivision is called the double integral of f(x, y) over the region R and is denoted by

$$\iint_R f(x,y)\,dA$$

In other words

$$\lim_{\substack{n\to\infty \\ \delta A_r \to 0}} \sum_{r=1}^{n} f(x_r, y_r)\delta A_r = \iint_R f(x,y)\,dA$$

which is also expressed as

$$\iint_R f(x,y)\,dx\,dy \text{ or } \iint_R f(x,y)\,dy\,dx$$

Example 1: Evaluate $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dydx}{1+x^2+y^2}$

(P.T.U. 2006)

Solution: Since the limits of y are functions of x, the integration will first be performed w.r.t y (treating x as a constant). Thus

$$\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dydx}{1+x^2+y^2} = \int_0^1 \left(\int_0^{\sqrt{1+x^2}} \frac{dy}{1+x^2+y^2} \right) dx$$

$$= \int_0^1 \left[\frac{1}{\sqrt{1+x^2}} \tan^{-1}\left(\frac{y}{\sqrt{1+x^2}} \right) \right]_0^{\sqrt{1+x^2}} dx$$

$$= \int_0^1 \left\{ \frac{1}{\sqrt{1+x^2}} \tan^{-1}(1) \right\} dx$$

$$= \int_0^{\pi/2} \frac{1}{\sqrt{1+x^2}} \tan^{-1} \tan\frac{\pi}{4} dx$$

$$= \frac{\pi}{4} \int_0^1 \frac{dx}{\sqrt{1+x^2}}$$

$$= \frac{\pi}{4} \left[\log\left\{ x + \sqrt{(1+x^2)} \right\} \right]_0^1 = \frac{\pi}{4} \log\left(1+\sqrt{2}\right) \qquad \text{Answer}$$

Example 2: Evaluate $\int_0^1 \int_0^{y^2} e^{x/y} dydx$

Solution: $\int_0^1 \int_0^{y^2} e^{x/y} dydx = \int_0^1 dy \int_0^{y^2} e^{x/y} dx$

$$= \int_0^1 dy \left\{ y e^{x/y} \right\}_0^{y^2}$$

Let $\frac{x}{y} = t$

$\Rightarrow dx = y\,dt$

$$= \int_0^1 y dy \left\{ e^{y^2/y} - e^0 \right\}$$

$$= \int_0^1 y dy \left(e^y - 1 \right)$$

$$= \int_0^1 y \left(e^y - 1 \right) dy$$

$$= \left\{ y\left(e^y - y \right) \right\}_0^1 - \int_0^1 1.\left(e^y - y \right) dy$$

$$= \left\{ y\left(e^y - y \right) \right\}_0^1 - \left\{ \left(e^y - \frac{y^2}{2} \right) \right\}_0^1$$

$$= \left\{ 1\left(e^1 - 1 \right) - 0 \right\} - \left\{ e^1 - \frac{1}{2} \right\} - \left\{ e^0 - 0 \right\}$$

$$= e - 1 - e + \frac{1}{2} + 1$$

$$= \frac{1}{2} \qquad \text{Answer.}$$

Example 3: Evaluate $\int_0^a \int_0^{\sqrt{(a^2-y^2)}} \sqrt{a^2 - x^2 - y^2}\, dy dx$

Solution: Let the given integral be denoted by I

$$\therefore I = \int_0^a \int_0^{\sqrt{(a^2-y^2)}} \sqrt{(a^2 - x^2 - y^2)}\, dy\, dx$$

$$= \int_0^a \left[\int_0^{\sqrt{(a^2-y^2)}} \sqrt{\left\{ (a^2 - y^2) - x^2 \right\}}\, dx \right] dy$$

$$= \int_0^a \left[\frac{x}{2} \sqrt{\left\{ (a^2 - y^2) - x^2 \right\}} + \frac{1}{2}(a^2 - y^2) \sin^{-1} \frac{x}{\sqrt{(a^2 - y^2)}} \right]_0^{\sqrt{a^2-y^2}} dy$$

$$= \int_0^a \left[\frac{1}{2} \sqrt{(a^2 - y^2)} \sqrt{\left\{ (a^2 - y^2) - (a^2 - y^2) \right\}} + \frac{1}{2}(a^2 - y^2) \sin^{-1} 1 \right] dy$$

$$= \int_0^a \frac{1}{2}\left(a^2 - y^2\right)\frac{\pi}{2}dy$$

$$= \frac{\pi}{4}\int_0^a \left(a^2 - y^2\right)dy$$

$$= \frac{\pi}{4}\left[a^2y - \frac{y^3}{3}\right]_0^a$$

$$= \frac{\pi}{4}\left[a^3 - \frac{a^3}{3}\right]$$

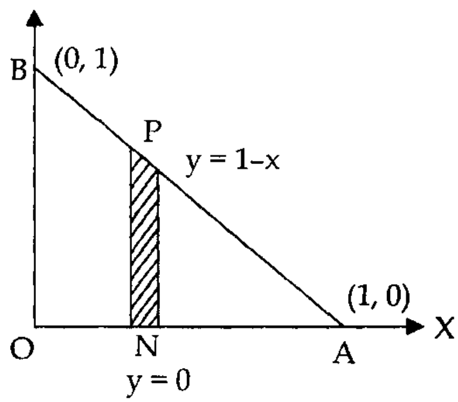
$$= \frac{\pi a^3}{6}$$ Answer.

Example 4: Evaluate $\iint\left(x^2 + y^2\right)dxdy$ over the region in the positive quadrant for which $x + y \leq 1$

OR

Evaluate $\iint_A \left(x^2 + y^2\right)dxdy$, where A is the region bounded by $x = 0$, $y = 0$, $x + y = 1$.

Solution: The region of integration is the triangle OAB, for this region x varies from 0 to A i.e. from $x = 0$ to $x = 1$ and for any intermediary value of x at N. say y, varies from the x axis to P on the line AB given by $x + y = 1$ i.e. y varies from $y = 0$ to $y = 1 - x$



Hence the given integral

$$= \int_0^1\int_0^{1-x}\left(x^2 + y^2\right)dx\,dy$$

$$= \int_0^1\left(x^2y + \frac{y^3}{3}\right)_0^{1-x} dx$$

Multiple Integrals

$$= \int_0^1 \left[x^2 (1-x) + \frac{1}{2}(1-x)^3 \right] dx$$

$$= \int_0^1 (1-x)\left\{ x^2 + \frac{1}{3}(1-x)^2 \right\} dx$$

$$= \int_0^1 \frac{1}{3}\left(1 - 3x + 6x^2 - 4x^3\right) dx$$

$$= \frac{1}{2}\left[x - \frac{3}{2}x^2 + 2x^2 - x^4 \right]_0^1$$

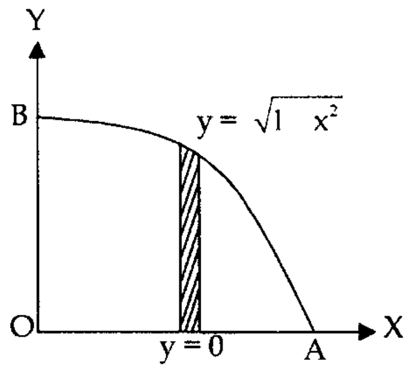
$$= \frac{1}{3}\left[1 - \frac{3}{2} + 2 - 1 \right]$$

$$= \frac{1}{6} \quad \text{Answer.}$$

Example 5: Evaluate $\iint \frac{xy}{\sqrt{(1-y^2)}} dx\,dy$ over the positive quadrant of the circle $x^2 + y^2 = 1$

Solution: The region of integration here is the quadrant OABO of the circle as shown in figure.

Here, the Co-ordinates of A and B are (1, 0) and (0, 1) respectively as the radius of the given circle is 1.



Here the given integral

$$= \int_0^1 \int_{y=0}^{\sqrt{1-x^2}} \frac{xy}{\sqrt{1-y^2}} dx\,dy$$

$$= \int_0^1 x\left[-\sqrt{(1-y^2)}\right]_0^{\sqrt{(1-x^2)}} dx$$

$$= -\int_0^1 x\left[\sqrt{\{1-(1-x^2)\}} - \sqrt{(1-0)}\right]dx$$

$$= -\int_0^1 x(x-1)dx$$

$$= -\int_0^1 (x^2 - x)dx$$

$$= -\left[\frac{x^3}{3} - \frac{x^2}{2}\right]_0^1$$

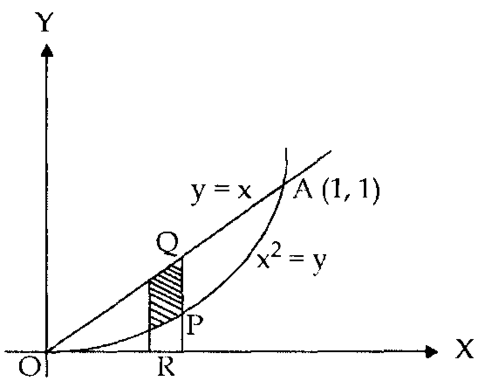
$$= -\left[\frac{1}{3} - \frac{1}{2}\right]$$

$$= \frac{1}{6} \quad \text{Answer.}$$

Example 6: Evaluate $\iint xy(x+y)dx\,dy$ over the area between $y = x^2$ and $y = x$

(B.P.S.C. 2005)

Solution: Here $x^2 = y$ represents a parabola whose vertex is the origin and axis is the axis of y. The equation $y = x$ is a line through origin making an angle of 45^0 with x axis solving $y = x^2$ and $y = x$, we find that the parabola $y = x^2$ and the line $y = x$ intersect in the point (0, 0) and (1, 1)



Required value $= \int_{x=0}^{1} \int_{y=x}^{x^2} xy(x+y)dx\,dy$

Multiple Integrals

$$= \int_{x=0}^{1}\int_{y=x}^{x^2}\left(x^2y + xy^2\right)dx\,dy$$

$$= \int_{x=0}^{1}\left[\frac{1}{2}x^2y^2 + \frac{1}{3}xy^3\right]_x^{x^2} dx$$

$$= \int_0^1\left[\left(\frac{1}{2}x^6 + \frac{1}{3}x^7\right) - \left(\frac{1}{2}x^4 + \frac{1}{3}x^4\right)\right]dx$$

$$= \int_0^1\left(\frac{1}{2}x^6 + \frac{1}{3}x^7 - \frac{3}{6}x^4\right)dx$$

$$= \left[\left(\frac{1}{14}\right)x^7 + \frac{1}{24}x^8 - \frac{1}{6}x^5\right]_0^1$$

$$= \frac{1}{14} + \frac{1}{24} - \frac{1}{6}$$

$$= \frac{3}{56}$$ Numerically Answer.

Example 7: Evaluate $\iint x^2y^2\,dx\,dy$ over the region $x^2 + y^2 \le 1$

Solution:

$$I = \int_{-1}^{1}dx\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x^2y^2\,dy$$

$$= \int_{-1}^{1}x^2\left(\frac{1}{3}y^3\right)_{-\sqrt{(1-x^2)}}^{\sqrt{(1-x^2)}}$$

$$= \frac{1}{3}\int_{-1}^{1}x^2\left[\left(1-x^2\right)^{3/2} - \left\{-\left(1-x^2\right)^{3/2}\right]dx$$

$$= \frac{2}{3}\int_{-1}^{1}x^2\left(1-x^2\right)^{3/2}dx$$

$$= \frac{4}{3}\int_0^1 x^2\left(1-x^2\right)^{3/2}dx$$

$$= \frac{4}{3}\int_0^{\pi/2}\sin^2\theta\cos^3\theta\,d\theta$$

putting $x = \sin\theta$

$$= \frac{4}{3}\int_0^{\pi/2}\sin^2\theta\cos^4\theta\,d\theta$$

$$= \frac{4}{3}\frac{\overline{\left|\left(\frac{3}{2}\right)\right.}\overline{\left|\left(\frac{5}{2}\right)\right.}}{2\overline{|4}}$$

$= \dfrac{\pi}{24}$ Answer.

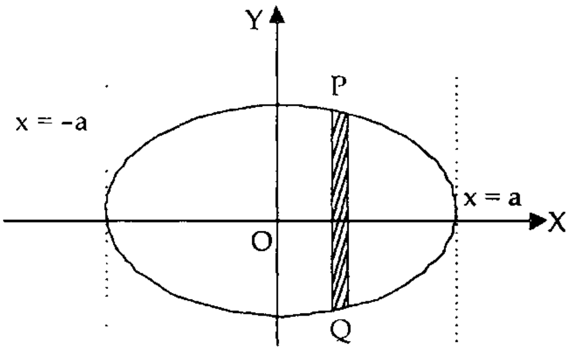
Example 8: Evaluate $\iint (x+y)^2 \, dx\,dy$ over the area bounded by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

Solution: For the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, we have

$$y = \pm \frac{b}{a}\sqrt{(a^2 - x^2)}$$

$\therefore$ Region of integration is for x from –a to + a and for y from $-\dfrac{b}{a}\sqrt{(a^2 - x^2)}$ to $+\dfrac{b}{a}\sqrt{(a^2 - x^2)}$

$\therefore$ the given integral $\displaystyle\int_{x=-a}^{a} \int_{y=-\frac{b}{a}\sqrt{(a^2-x^2)}}^{\frac{b}{a}\sqrt{(a^2-x^2)}} (x+y)^2 \, dx\,dy$



$$= \int_{x=-a}^{a} \int_{y=-\frac{b}{a}\sqrt{(a^2-x^2)}}^{\frac{b}{a}\sqrt{(a^2x^2)}} \left(x^2 + y^2 + 2xy\right) dx\,dy$$

$$= 4\int_{x=0}^{a} \int_{y=0}^{\frac{b}{a}\sqrt{(a^2-x^2)}} \left(x^2 + y^2\right) dx\ dy$$

(The third integral vanishing as 2xy is an odd function of y)

<u>*Multiple Integrals*</u>

$$= 4\int_{x=0}^{a}\left(x^2y + \frac{1}{3}y^3\right)_0^{\frac{b}{a}\sqrt{(a^2-x^2)}} dx$$

$$= 4\int_0^a \frac{b}{a}\sqrt{(a^2-x^2)}\left[x^2 + \frac{1}{3}\frac{b^2(a^2-x^2)}{a}\right] dx$$

$$= \frac{4b}{3a^3}\int_0^a \sqrt{(a^2-x^2)}\left[3a^2x^2 + b^2a^2 - b^2x^2\right]dx$$

$$= \frac{4b}{3a^3}\int_{\theta=0}^{\pi/2} a\,\text{coa}\theta\left[3a^4\sin^2\theta + b^2a^2 - b^2a^2\sin^2\theta\right]a\cos\theta\, d\theta$$

putting $x = a\sin\theta$

$$= \frac{4}{3}ba\int_0^{\pi/2}\left[3a^2\sin^2\theta + b^2 - b^2\sin^2\theta\right]\cos^2\theta\, d\theta$$

$$= \frac{4}{3}ba\left[(3a^2-b^2)\int_0^{\pi/2}\sin^2\theta\cos^2\theta\, d\theta + b^2\int_0^{\pi/2}\cos^2\theta\, d\theta\right]$$

$$= \frac{4}{3}ab\left[(3a^2-b^2)\frac{\Gamma\frac{3}{2}\,\Gamma\frac{3}{2}}{2\Gamma(3)} + b^2.\frac{1}{2}.\frac{\pi}{2}\right]$$

$$= \frac{4}{3}ab\left[(3a^2-b^2)\frac{\frac{1}{2}\sqrt{\pi}.\frac{1}{2}\sqrt{\pi}}{2.2} + \frac{b^2\pi}{4}\right]$$

$$= \frac{\pi ab}{3}\left[\frac{1}{4}(3a^2-b^2) + b^2\right]$$

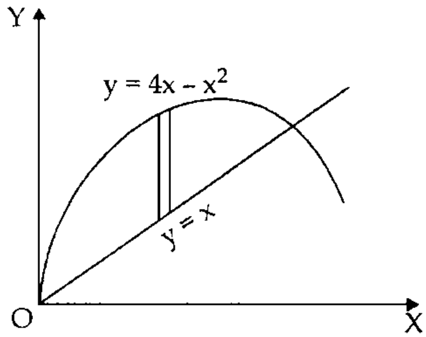
$$= \frac{1}{4}\pi ab(a^2+b^2) \qquad \text{Answer.}$$

Example 9: Find the area lying between the parabola $y = 4x - x^2$ and the line $y = x$.

Solution: The two curves intersect at points whose abscissa are given by

$4x - x^2 = x$

i.e. $x = 0$ or 3



The area can be considered as lying between the curve y =x, y = 4x -x², x =0 and x =3. So, integrating along a vertical strip first, we see that the required area

$$= \int_0^3 \int_x^{4x-x^2} dx\,dy = \int_0^3 [y]_x^{4x-x^2} dx$$

$$= \int_0^3 (4x - x^2 - x)dx$$

$$= \int_0^3 (3x - x^2)dx$$

$$= \left[\frac{3}{2}x^2 - \frac{1}{3}x^3\right]_0^3$$

$$= \frac{27}{2} - 9 = \frac{9}{2} \quad \text{Answer.}$$

Change to Polar Co-ordinates:

We have x = r cosθ, y = r sinθ

Therefore

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r$$

$$\Rightarrow \iint_R f[x,y]\,dx\,dy = \iint_{R'} f[r\cos\theta, r\sin\theta]\,J\,d\theta\,dr$$

$$= \iint_{R'} f[r\cos\theta, r\sin\theta]\,r\,d\theta\,dr$$

Note: In polar dx dy is to be replaced by r dθ dr.

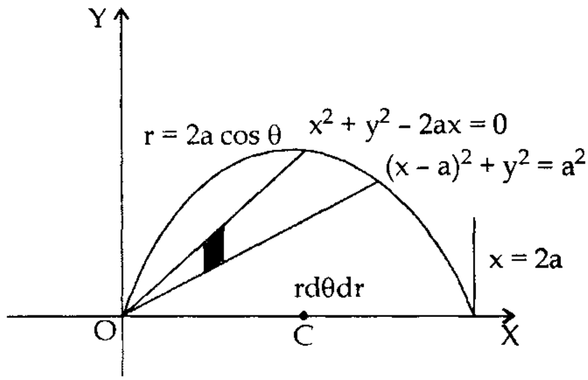
Multiple Integrals

Example 10: Evaluate $\int_0^{2a}\int_0^{\sqrt{2ax-x^2}}\left(x^2+y^2\right)dy\,dx$ by changing to polar Co-ordinates.

Solution: Let I = $\int_{x=}^{2a}\int_{y=0}^{\sqrt{2ax-x^2}}\left(x^2+y^2\right)dy\,dx$, upper limit of y is

$x^2 + y^2 - 2ax = 0$

$(x - a)^2 + y^2 = a^2$ (i)



This equation represent a circle whose centre is (a, 0) and radius a. Region of integration is upper half circle. Let us convert the equation into polar Co-ordinates by putting

$x = r\cos\theta$ and $y = r\sin\theta$

$\Rightarrow r^2 - 2a\,r\cos\theta = 0$

$\Rightarrow r = 2a\cos\theta$ (ii)

$$\therefore I=\int_0^{2a}\int_0^{\sqrt{(2ax-x^2)}}\left(x^2+y^2\right)dy\,dx$$

$$=\int_{\theta=0}^{\pi/2}\int_{r=0}^{2a\cos\theta} r^2\left(r\,dr\,d\theta\right)$$

$$=4a^4\int_0^{\pi/2}\cos^4\theta\,d\theta$$

$$=\frac{3\pi a^4}{4}$$ Answer.

Example 11: Transform the integral $\int_0^a\int_0^{\sqrt{(a^2-x^2)}} y^2\sqrt{(x^2+y^2)}\,dx\,dy$ by changing to polar Co-ordinates and hence evaluate it.

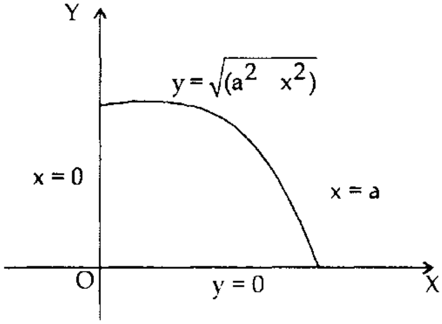
Solution: The given limits of integration show that the region of integration lies between the curves

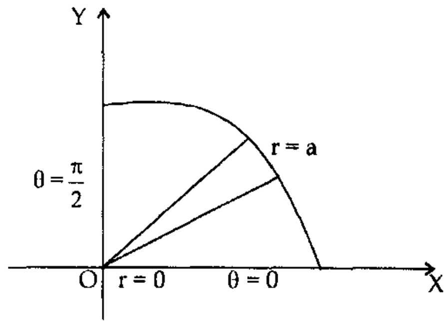
$y = 0,\ y=\sqrt{(a^2-x^2)}$, $x = 0$, $x = a$

Thus the region of integration is the part of the circle $x^2 + y^2 = a^2$ in the first quadrant. In polar Co-ordinates, the equation of the circle is

$r^2 \cos^2\theta + r^2 \sin^2\theta = a^2$

i.e. $r = a$.





Hence in polar Co-ordinates, the region of integration is bounded by the curves $r = 0$, $r = a$, $\theta = 0$, $\theta = \pi/2$

Therefore,

$$\int_0^a \int_0^{\sqrt{(a^2-x^2)}} y^2 \sqrt{(x^2+y^2)}\, dx\, dy = \int_0^{\pi/2} \int_0^a r^2 \sin^2\theta . r . r\, d\theta\, dr$$

$$= \int_0^{\pi/2} \sin^2\theta \left[\frac{r^5}{5}\right]_0^a d\theta$$

$$= \frac{a^5}{5} \int_0^{\pi/2} \sin^2\theta\, d\theta$$

$$= \frac{a^5}{5} \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma(2)}$$

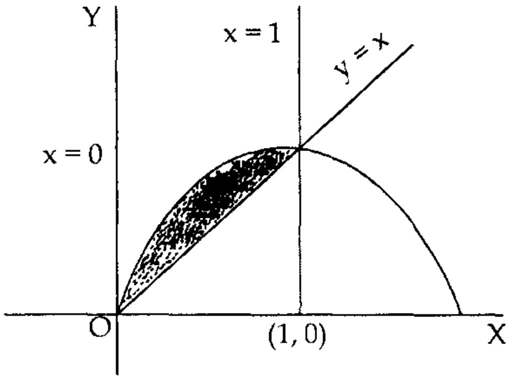
$$= \frac{a^5}{5} \frac{1}{2} \frac{\sqrt{\pi}.\sqrt{\pi}}{2.1} = \frac{1}{20}\pi a^5 \qquad \text{Answer}$$

Example 12: Evaluate $\int_0^1 \int_x^{\sqrt{(2x-x^2)}} (x^2+y^2)\, dx\, dy$ by changing to polars.

Solution: The region of integration is given by $y = x$, $y = \sqrt{(2x - x^2)}$, $x = 0$, $x = 1$.

Thus, the region of integration lies between line $y = x$, a part of circle $(x-1)^2 + y^2 = 1$, $x = 0$ and $x = 1$.

Multiple Integrals



The diameter of the circle is 2 with its end at (0, 0) and (0, 2). Its equation is r = 2 cosθ and θ varies from π/4 to $\frac{\pi}{2}$ (y =x to x =0)

Now the given integral in polar Co-ordinates takes form

$$\int_{\pi/4}^{\pi/2}\int_0^{2\cos\theta} r^2 r\,d\theta\,dr = \frac{1}{4}\int_{\pi/4}^{\pi/2}\left[r^4\right]_0^{2\cos\theta} d\theta$$

$$= \frac{1}{4}\int_{\pi/4}^{\pi/2} 2^4\cos^4\theta\,d\theta$$

$$= 4\int_{\pi/4}^{\pi/2}\cos^4\theta\,d\theta$$

$$= \int_{\pi/4}^{\pi/2}(1+\cos 2\theta)^2\,d\theta$$

$$= \int_{\pi/4}^{\pi/2}(1+2\cos 2\theta+\cos^2 2\theta)\,d\theta$$

$$= \int_{\pi/4}^{\pi/2}\left[1+2\cos 2\theta+\frac{1}{2}(1+\cos 4\theta)\right]d\theta$$

$$= \int_{\pi/4}^{\pi/2}\left[\frac{3}{2}+2\cos 2\theta+\frac{1}{2}\cos 4\theta\right]d\theta$$

$$= \left[\frac{3}{2}\theta+\sin 2\theta+\frac{1}{8}\cos 4\theta\right]_{\pi/4}^{\pi/2}$$

$$= \frac{3}{2}\left(\frac{\pi}{2}-\frac{\pi}{4}\right)+(0-1)+\frac{1}{8}(0-0)$$

$$= \frac{3\pi}{8}-1 \qquad \text{Answer.}$$

Example 13: Evaluate $\int_0^2\int_0^{\sqrt{2x-x^2}}\frac{x\,dy\,dx}{\sqrt{x^2+y^2}}$ by changing to polar Co-ordinates.

Solution: In the given integral, y varies from 0 to $\sqrt{(2x-x^2)}$ and x varies from 0 to 2.

$y = \sqrt{2x - x^2}$

$\Rightarrow y^2 = 2x - x^2$

$\Rightarrow x^2 + y^2 = 2x$

In polar Co-ordinates, we have

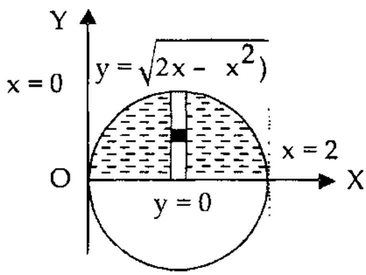
$r^2 = 2r\cos\theta$

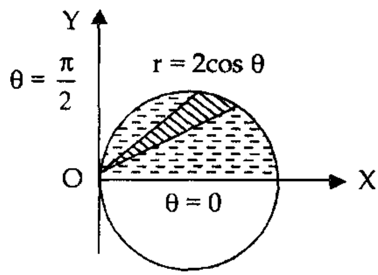
$\Rightarrow r = 2\cos\theta$

$\therefore$ For the region of integration r varies from 0 to 2 cosθ and θ varies from 0 to $\frac{\pi}{2}$.

In the given integral replacing x by r cosθ, y by r sinθ, dy dx by r dr dθ, we have

$$I = \int_0^{\pi/2}\int_0^{2\cos\theta} \frac{r\cos\theta . r\,dr\,d\theta}{r}$$





$$= \int_0^{\pi/2}\int_0^{2\cos\theta} r\cos\theta\,dr\,d\theta$$

$$= \int_0^{\pi/2} \cos\theta \left[\frac{r^2}{2}\right]_0^{2\cos\theta} d\theta$$

$$= \int_0^{\pi/2} 2\cos^3\theta\,d\theta = 2.\frac{2}{3} = \frac{4}{3} \qquad \text{Answer}$$

Example 14: Evaluate $\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}dx\,dy$ by changing to polar Co-ordinates. Hence show that $\int_0^{\infty} e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$

(U.P.T.U. 2002)

Solution: Given that

$$I = \int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\,dx\,dy$$

Here we see that the integration is along a vertical strip extending from y =0 to y = ∞ and this strip slides from x =0 to x = ∞.

Multiple Integrals

Hence, the region of integration is in the first quadrant, as shown in figure (i)

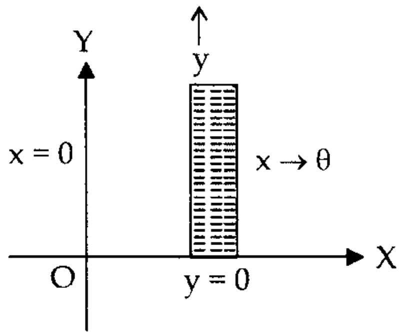
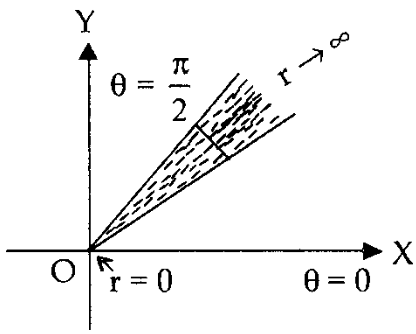


Fig. (i)

The region is covered by the radius strip from r =0 to r =∞ and it starts from θ =0 to θ = π/2 as shown in figure (ii). Thus



$$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\,dy = \int_0^{\pi/2} \int_0^\infty e^{-r^2} r\,d\theta\,dr$$

$$= -\frac{1}{2}\int_0^{\pi/2}\int_0^\infty (-2r)e^{-r^2} dr\,d\theta$$

$$= -\frac{1}{2}\int_0^{\pi/2}\left[e^{-r^2}\right]_0^\infty d\theta$$

$$= -\frac{1}{2}\int_0^{\pi/2}(0-1)\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} 1\,d\theta = \frac{\pi}{4} \qquad \text{Answer}$$

Now, let

$$I = \int_0^\infty e^{-x^2} dx \qquad (1)$$

Also $I = \int_0^\infty e^{-y^2} dy \qquad (2)$

(by property of definite integrals)
Multiplying (1) and (2) we get

$$I^2 = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\,dy = \frac{\pi}{4}$$

$$\Rightarrow I = \sqrt{\left(\frac{\pi}{4}\right)} \text{ as obtained above}$$

$$\Rightarrow I = \frac{\sqrt{\pi}}{2} \qquad \text{Proved.}$$

CHANGE OF ORDER OF INTEGRATION

Introduction: We have seen that $\int_a^b \int_c^d f(x,y)\,dx\,dy = \int_c^d \int_a^b f(x,y)\,dy\,dx$, provided a, b, c, d are constants.

Here we see that the limit of x and y remain the same whatever order of integrations are performed. In case, the limits are not constant the limits of x and y in both the repeated integrals will not be the same.

$$\int_a^b \int_c^d f(x,y)dx\,dy$$

Means integrate f(x,y) first w.r.t y from y = c to y = d treating x as constant and then integrate the result obtained w.r.t x from x =a to x =b.

Sometime, the evaluation of an integrated integral can be simplified by reversing the order of integration. In such cases, the limits of integration are changed if they are variable. A rough sketch of the region of integration helps in fixing the new limits of integration.

Note: In some books particularly American $\int_{x_1}^{x_2} \int_{y_1}^{y_2} f(x,y)dy\,dx$ is written instead of $\int_{x_1}^{x_2} \int_{y_1}^{y_2} f(x,y)\,dx\,dy$ where the first integration performed w.r.t y and then after w.r.t.x.

However in this book we shall generally use the notation given in the beginning of the introduction.

Example 15: Change the order of integration in the integral

$$\int_0^{a\cos\alpha} \int_{x\tan\alpha}^{\sqrt{(a^2-x^2)}} f(x,y)dx\,dy$$

Solution: The given integral $\int_0^{a\cos\alpha} \int_{x\tan\alpha}^{\sqrt{(a^2-x^2)}} f(x,y)dx\,dy$

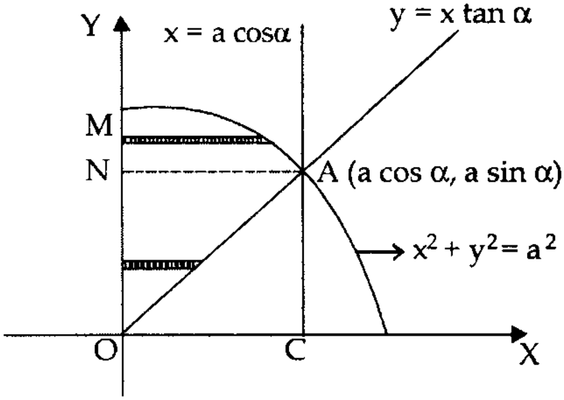
Here the limits are given by x =0, x = a cosα; y = x tanα, $y = \sqrt{(a^2-x^2)}$,

$y = \sqrt{(a^2-x^2)}$ gives $x^2 + y^2 = a^2$ i.e. circle with centre at origin.

To find intersection point of

Multiple Integrals

$x^2 + y^2 = a^2$ (i)
and
$y = x \tan\alpha$ (ii)
Now from (i) and (ii) $\Rightarrow x^2 + x^2 \tan^2 a = a^2$
$\Rightarrow x = a\cos\alpha$
put this in (ii), $y = a\cos\alpha \tan\alpha$
$= a\sin\alpha$
$\therefore$ $(a\sin\alpha, a\cos\alpha)$ is the intersection point A of (i) and (ii)



Through A draw a line AN parallel to x-axis. Evidently the region of integration is OAMO.
In order to change the order of integration let us take elementary strips parallel to x-axis. Such type of strips change their character at the point A. Hence the region of integration is divided into two parts ONAO, NAMN.
In the region ONAO, the strip has its extermities on the lines x =0 and $y = x\tan\alpha$.
$\therefore$ limits of x in term of y are from x =0 to $\dfrac{y}{\tan\alpha}$ (or $y\cot\alpha$) limits of y are form 0 to $a\sin\alpha$.
In the region NAMN, the strip has its extremities on the line x = 0 and the circle $x_2 + y^2 = a^2$
$\therefore$ limits of x in term of y are from 0 to $\sqrt{(a^2 - x^2)}$ and limits of y are from $a\sin\alpha$ to a.

$$\therefore \int_0^{a\cos\alpha}\int_{x\tan\alpha}^{\sqrt{(a^2-x^2)}} f(x,y)\,dx\,dy = \int_0^{a\sin\alpha}\int_0^{y\cot\alpha} f(x,y)\,dy\,dx + \int_{a\sin\alpha}^{a}\int_0^{\sqrt{(a^2-y^2)}} f(x,y)\,dy\,dx$$

Answer.
Example 16: Change the order of integration in the following integrals.

(i) $\int_0^{2a}\int_{x^2/4a}^{3a-x}\phi(x,y)\,dx\,dy$

(ii) $\int_0^{a}\int_0^{\sqrt{(a^2-x^2)}} f(x,y)\,dx\,dy$.

Solution: (i) We denote the given integral by I. Here the limits are given by $x = 0$, $x = 2a$; $y = 3a - x$

or $\dfrac{x}{3a}+\dfrac{y}{3a}=1$, $y=\dfrac{x^2}{4a}$ or $x^2 = 4ay$

For intersection point of

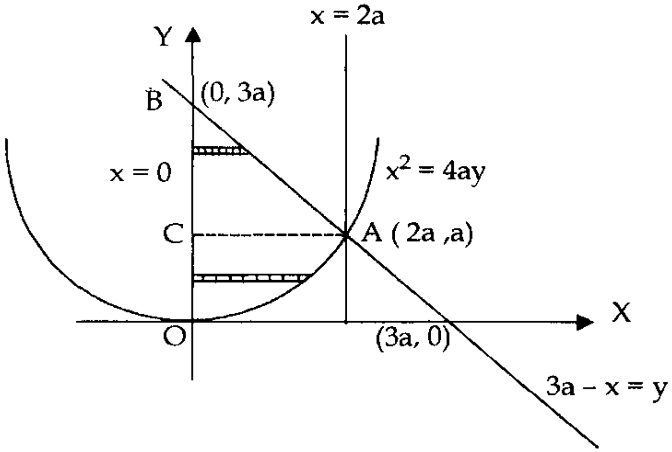
$y = 3a - x$ and $x^2 = 4ay$ we have $x^2 = 4a(3a - x)$

or $(x + 6a)(x - 2a) = 0$

or $x = 2a, -6a$

Put in $y = 3a - x$ we get $y = a, 9a$

$\therefore$ $(-6a, 9a)$ $(2a, a)$



Range of integration is OABO. In order to change the order of integration, we take elementary strip parallel to x axis, such type of strips change their character at A. Hence we draw line CA parallel to x axis. Thus the range is divided into two parts OACO and CABC.

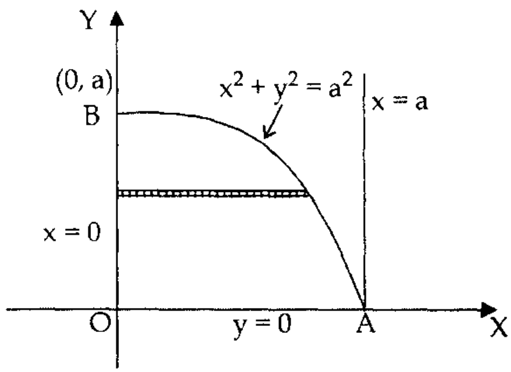
In the region OACO, any strip parallel to x axis has its extremities on $x = 0$ and $x^2 = 4ay$. $\therefore$ For such strips, limits of x in term of y are from 0 to $\sqrt{(4ay)}$ and limits of y are from o to a.

In region CABC, any strip parallel to x axis extermities on $x = 0$ and $x = 3a - y$ and y varies from $y = a$ to $y = 3a$.

$$I=\int_0^{a}\int_0^{\sqrt{(4ay)}}\phi(x,y)\,dy\,dx+\int_0^{3a}\int_0^{3a-y}\phi(x,y)\,dy\,dx \qquad \text{Answer.}$$

Multiple Integrals

(ii) Limits x =0, x =a; y =0, $y=\sqrt{(a^2-x^2)}$ or $x^2 + y^2 = a^2$



Range is OABO.

∴ on changing the order of integration, we get

$I=\int_0^a\int_0^{\sqrt{(a^2-y^2)}} f(x,y)\,dy\,dx$ Answer.

Example 17: Change the order of integration of

$\int_0^{2a}\int_{\sqrt{(2ax-x^2)}}^{\sqrt{2ax}} f(x,y)\,dx\,dy$

Solution: Limits of integration are x =0, x =2a, $y=\sqrt{(2ax-x^2)}$, $y=\sqrt{(2ax)}$

$y=\sqrt{2ax} \Rightarrow y^2 = 2ax$, parabola

$y=\sqrt{2ax-x^2} \Rightarrow x^2 + y^2 - 2ax = 0$

$\Rightarrow (x - a)^2 + (y - 0)^2 = a^2$, is circle with centre at (a, 0) and the radius a

$\Rightarrow x = \pm a \pm \sqrt{(a^2-y^2)}$

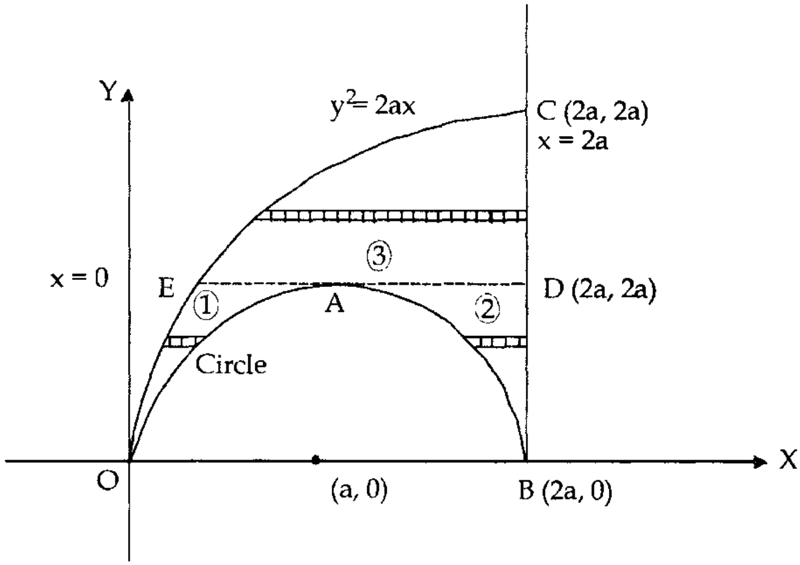
intersection of x = 2a and $y^2 = 2ax$ is C (2a, 2a).

Range of integration is OABCO. Through A draw a line ED parallel to x axis. Thus the range is divided in three parts namely.

(1) OAEO

(2) ABDA

(3) EDCE



Range No (1) for the region OAEO strip parallel to x axis lies its one end on $x = y^2/2a$ and the other end on $x = a - \sqrt{(a^2 - y^2)}$. For this strip y varies from y =0 to y =a.

Range No (2) for the region ABDA, the limits for x are from $a + \sqrt{(a^2 - y^2)}$ to 2a and that for y are from y =0 to y =a.

Range No (3), for the region EDCE, the limits for x are from $y^2/2a$ (from parabola) to 2a and that for y are from y =a to y = 2a.

Hence the transformed integral is given by

$$\int_0^{2a}\int_{\sqrt{(2ax-x^2)}}^{\sqrt{(2ax)}} f(xy)\,dx\,dy = \int_0^{a}\int_{y^2/2a}^{a-\sqrt{(a^2-y^2)}} f\,dy\,dx + \int_0^{a}\int_{a+\sqrt{(a^2-y^2)}}^{2a} f\,dy\,dx + \int_0^{2a}\int_{y^2/2a}^{2a} f\,dy\,dx$$

Answer.

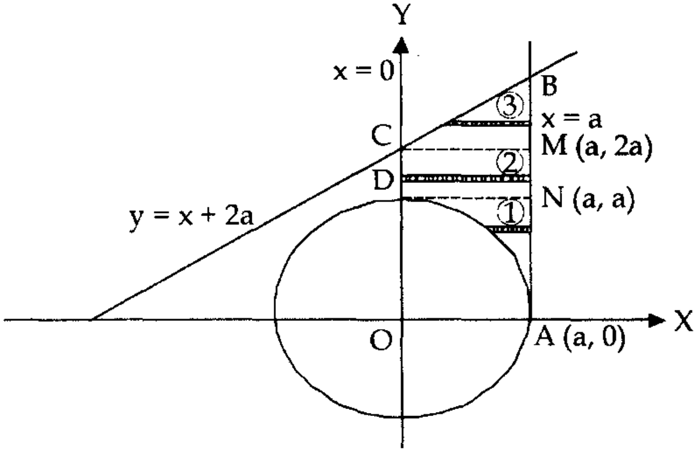
Example 18: Change the order of integration in

$$\int_0^{a}\int_{\sqrt{(a^2-x^2)}}^{x+2a} \phi(x,y)\,dx\,dy$$

Solution: We denote the given integral by I. Limits of integration are x =0, x =a; $y = \sqrt{(a^2 - x^2)}$, y =x + 2a, $y = \sqrt{(a^2 - x^2)}$ gives $x^2 + y^2 = a^2$ circle. y = x + 2a is expressible as

$$\frac{x}{(-2a)} + \frac{y}{2a} = 1$$

<u>*Multiple Integrals*</u>



The range is ABCDA. Any strip parallel to x-axis change its character at D and C both. Hence we draw two parallel lines DN and CM. The range is divided in three parts namely (1) DAND (2) DNMCD, (3) CMBC

Range (1) one end of the strip lies on $x = \sqrt{(a^2 - y^2)}$ and the other end on x = a. For this strip y varies from y =0 to y =a. Similar calculations are done for range (2) and (3)

$$\therefore I = \int_0^a \int_{\sqrt{(a^2-y^2)}}^a \phi\, dy\, dx + \int_a^{2a} \int_0^a \phi\, dy\, dx + \int_{2a}^{3a} \int_{y-2a}^a \phi\, dy\, dx \quad \text{Answer.}$$

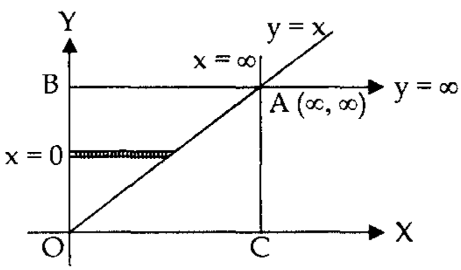
Example 19: Change the order of integration in $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} dx\, dy$ and hence find its value.

(I.A.S. 2006)

Solution: Let $I = \int_0^\infty \int_x^\infty \frac{e^{-y}}{y} dx\, dy$

Here the limits are

x =0, x = ∞; y =x, y = ∞



The range of integration is OABO. In order to change the order of integration, let us take an elementary strip parallel to x axis. One end of this strip is on x =0 and the other on x =y. For this strip y varies from y = 0 to y = ∞

$$\therefore \int_0^\infty \int_x^\infty \frac{e^{-y}}{y} dx\,dy = \int_0^\infty \int_0^y \frac{e^{-y}}{y} dy\,dx$$

$$= \int_0^\infty \frac{e^{-y}}{y} dy \int_0^y dx$$

$$= \int_0^\infty \frac{e^{-y}}{y} dy \left[x\right]_0^y$$

$$= \int_0^\infty \frac{e^{-y}}{y} dy . y$$

$$= \int_0^\infty e^{-y} dy$$

$$= \left[-e^{-y}\right]_{y=0}^\infty$$

$= 1-0 = 1$ Answer.

Example 20: Change the order of integration in $\int_0^{a/2} \int_{x^2/a}^{x-x^2/a} f(x,y)\,dx\,dy$

Solution: The limits of integration are given by the parabolas i.e. $x^2/a = y$; i.e. $x^2 = ay$; $x - x^2/a = y$ i.e. $ax - x^2 = ay$ and the lines $x = 0$; $x = \frac{a}{2}$.

Also the equation of parabola $ax - x^2 = ay$ may be written as $\left(x - \frac{a}{2}\right)^2 = -a\left(y - \frac{a}{4}\right)$ i.e. this parabola has the vertex as the point $\left(\frac{a}{2}, \frac{a}{4}\right)$ and its concavity is downwards.

The points of intersection of two parabolas are given as follows $ax - x^2 = x^2$ or $x = 0, \frac{a}{2}$ and hence from $x^2 = ay$, we get $y = 0$ at $x = 0$ and $y = \frac{a}{4}$ at $x = \frac{a}{2}$.

Hence the points of intersection of the two are (0,0) $\left(\frac{a}{2}, \frac{a}{4}\right)$.

Draw the two parabolas $x^2 = ay$ and $ax = x^2 - ay$ intersecting at the point O (0,0) and P$\left(\frac{a}{2}, \frac{a}{4}\right)$.

Now draw the lines $x = 0$ and $x = \frac{a}{2}$. Clearly the integral extends to the area ONPLO.

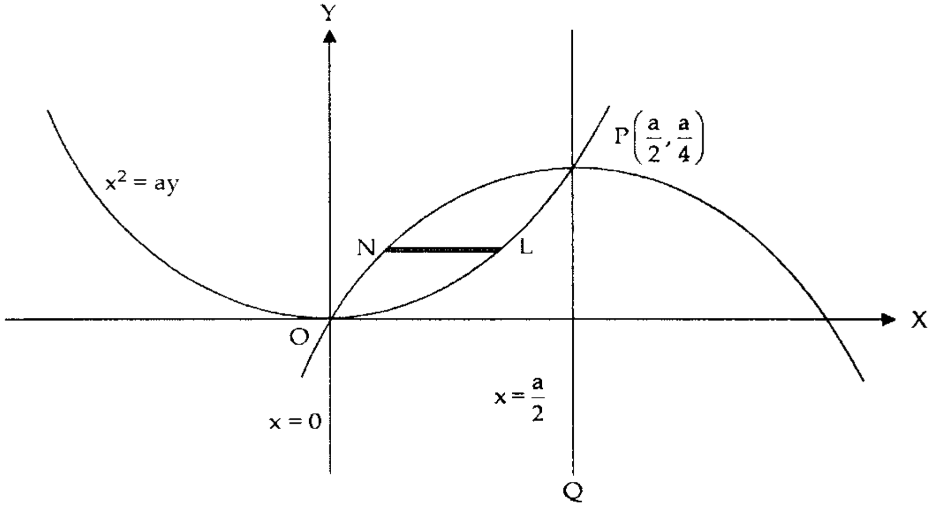
Multiple Integrals

Now take strips of the type NL parallel to the x axis.
Solving ay = ax – x² i.e x² – ax + xy =0 for x, we get

$$x = \frac{1}{2}\left[a \pm \sqrt{(a^2 - 4ay)}\right]$$

$$= \frac{1}{2}\left[a - \sqrt{(a^2 - 4ay)}\right] \qquad (1)$$

rejecting the positive sign before square root, Since x is not greater than $\frac{a}{2}$ for the region of integration.



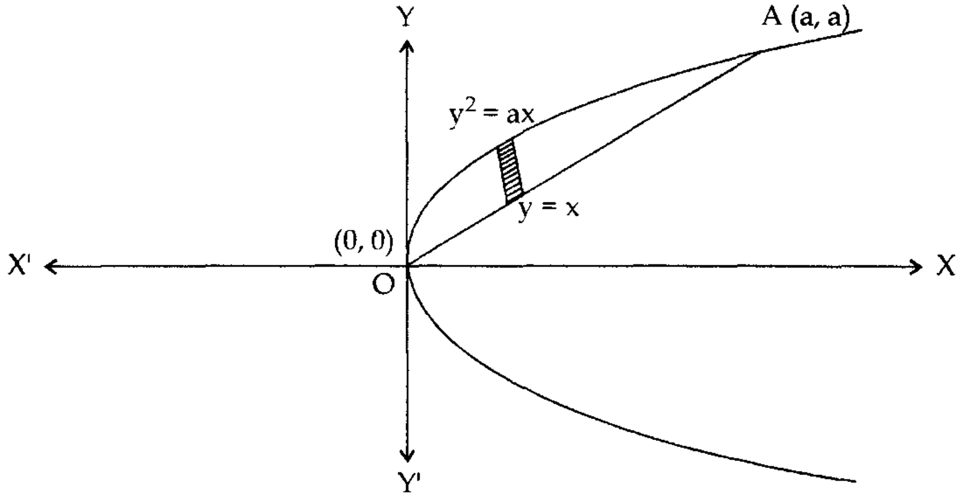
Again the region ONPLO, the elementary strip NL has the extermities N and L on ax – x = ay and x² = ay. Thus the limits of x are from $\frac{1}{2}\left[a - \sqrt{(a^2 - 4ay)}\right]$ to $\sqrt{(ay)}$. For limits of y, at 0, y = 0 and at P, $y = \frac{a}{4}$. Hence changing the order of integration, we have $\int_0^{a/2}\int_{x^2/a}^{x-x^2/a} f(x,y)\,dx\,dy = \int_0^{a/4}\int_{\frac{1}{2}\left[a-\sqrt{(a^2-4ay)}\right]}^{\sqrt{(ay)}} f(x,y)\,dy\,dx$ Answer.

Example 21: By changing the order of integration, evaluate

$$\int_0^a \int_{y^2/a}^{y} \frac{y}{(a-x)\sqrt{ax - y^2}}\,dx\,dy$$

(I.A.S. 2003, U.P.T.U. 2002)

Solution: The given limits show that the area of integration lies between x = y²/a, x =y, y =0, y =a since x = y²/a, y² = ax (a parabola) and $y = x$ is a straight line, these two intersect each other in the point 0 (0, 0) and A (a, a). The area of integration is the shaded portion in the figure.



We can consider it as lying between $y = x$, $y = \sqrt{ax}$; $x = 0$, $x = a$.

Therefore by changing the order of integration we have

$$\int_0^a \int_{y^2/a}^{y} \frac{y}{(a-x)\sqrt{ax-y^2}}\,dx\,dy = \int_0^a \int_x^{\sqrt{ax}} \frac{y\,dy\,dx}{(a-x)\sqrt{ax-y^2}}$$

$$= \int_0^a \left[-\frac{\left(ax-y^2\right)^{1/2}}{a-x} \right]_x^{\sqrt{ax}} dx$$

$$= \int_0^a \frac{\left(ax-x^2\right)^{1/2}}{a-x} dx$$

$$= \int_0^a \left(\frac{x}{a-x} \right)^{1/2} dx$$

put $x = a\sin^2\theta$

$\therefore$ $dx = 2a \sin\theta \cos\theta \, d\theta$

$$= \int_0^{\pi/2} \left(\frac{a\sin^2\theta}{a\cos^2\theta} \right)^{1/2} . \, 2a \sin\theta \cos\theta \, d\theta$$

$$= \int_0^{\pi/2} 2a\sin^2\theta\, d\theta = 2a.\frac{1}{2}.\frac{\pi}{2}$$

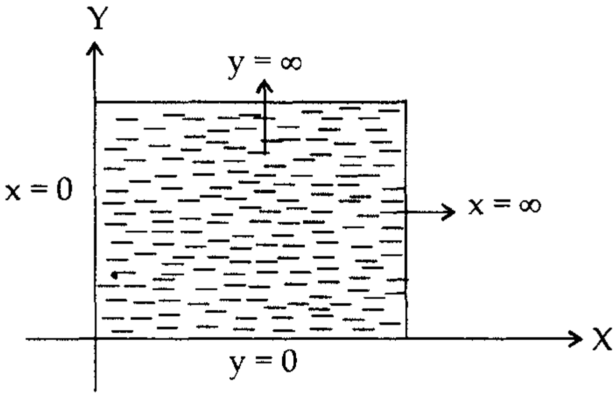
$$= \frac{\pi a}{2}$$ Answer.

Multiple Integrals

Example 22: Changing the order of integration of $\int_0^\infty\int_0^\infty e^{-xy}\sin nx\,dx\,dy$ show that $\int_0^\infty \frac{\sin nx}{x}dx = \frac{\pi}{2}$

(U.P.T.U 2003, 2009)

Solution: The region of integration is bounded by x =0, x = ∞, y = 0, y = ∞.
i.e., the first quadrant, as shown in figure.



Thus

$$\int_0^\infty\int_0^\infty e^{-xy}\sin nx\,dx\,dy = \int_0^\infty dy\int_0^\infty e^{-xy}\sin nx\,dx$$

$$= \int_0^\infty\left[\frac{e^{-xy}}{n^2+y^2}\{-y\sin nx - n\cos nx\}\right]_0^\infty dy$$

$$= \int_0^\infty\left[0+\frac{n}{n^2+y^2}\right]dy$$

$$= \left[\tan^{-1}\frac{y}{n}\right]_0^\infty$$

$$= \frac{\pi}{2} \qquad \text{(i)}$$

on changing the order of integration, we get

$$\int_0^\infty\int_0^\infty e^{-xy}\sin nx\,dx\,dy = \int_0^\infty \sin nx\,dx\int_0^\infty e^{-xy}dy$$

$$= \int_0^\infty \sin nx\, dx \left[\frac{e^{-xy}}{-x} \right]_0^\infty$$

$$= \int_0^\infty \frac{\sin nx}{x} dx \left[-\frac{1}{e^{xy}} \right]_0^\infty$$

$$= \int_0^\infty \frac{\sin nx}{x} dx [-0+1]$$

$$= \int_0^\infty \frac{\sin nx}{x} dx \qquad \text{(ii)}$$

Hence from equations (i) and (ii) we have

$$\int_0^\infty \frac{\sin nx}{x} = \frac{\pi}{2}$$

Example 23: Find the area enclosed between the parabola $y = 4x - x^2$ and the line $y = x$.

(U.P.T.U. 2008)

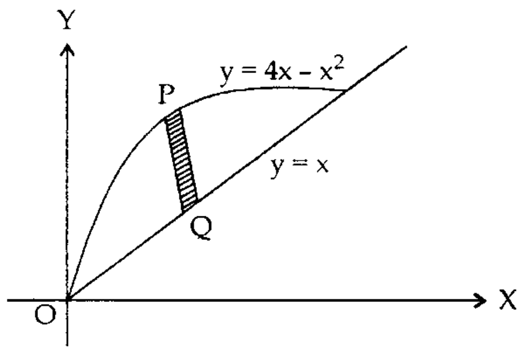
Solution: The given curves intersect at the points whose abscissas are given by $y = 4x - x^2$ and $y = x$, Therefore

$x = 4x - x^2$

or $3x - x^2 = 0$

$\Rightarrow x(3 - x) = 0$

$\Rightarrow x = 0, 3$



The area under consideration lies betv[illegible] curves $y = x$, $y = 4x - x^2$, $x = 0$ and $x = 3$.

Hence, integrating along the vertical strip PQ first, we get the required area as

$$\text{Area} = \int_0^3 \int_{y=x}^{y=4x-x^2} dy\, dx$$

$= \int_0^3 [y]_x^{4x-x^2} dx$

$= \int_0^3 \left(4x - x^2 - x\right) dx$

$= \int_0^3 \left(3x - x^2\right) dx$

$= \left[\frac{3}{2}x^2 - \frac{x^3}{3}\right]_0^3$

$= \left[\frac{3}{2}(3)^2 - \frac{(3)^3}{2}\right]$

$= \frac{27}{2} - 9 = \frac{9}{2}$ Answer.

Example 24: By double integration, find the whole area of the curve $a^2 x^2 = y^3 (2a - y)$

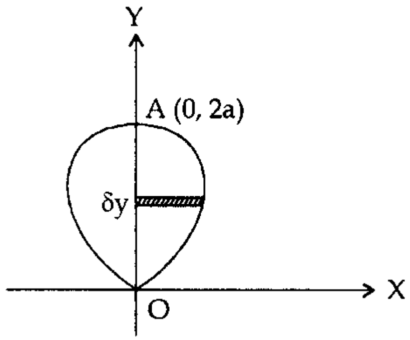
(U.P.T.U. 2001)

Solution: The region of integration is shown in figure. Here, we have

Area = 2 × area of the region OAB

$= 2\int_{y=0}^{2a} \int_{x=0}^{f(y)} dx\, dy$

where $f(y) = \frac{y^{3/2}\sqrt{(2a-y)}}{a}$



Consider the horizontal strip PQ with a small area, we get

$\text{Area} = = 2\int_0^{2a} \int_0^{y^{3/2} \frac{\sqrt{2a-y}}{a}} dx\, dy$

$= 2\int_0^{2a} [x]_0^{y^{3/2} \frac{\sqrt{(2a-y)}}{a}} dy$

$= \frac{2}{a}\int_0^{2a} y^{3/2} \sqrt{2a-y}\, dy$

putting $y = 2a\sin^2\theta$

i.e. $dy = 4a\sin\theta\cos\theta\, d\theta$ we get

$$\text{Area} = \frac{2}{a}\int_0^{\pi/2}\left(2a\sin^2\theta\right)^{3/2}\sqrt{\left(2a - 2a\sin^2\theta\right)}\; 4a\sin\theta\cos\theta\, d\theta$$

$$= 32a^2\int_0^{\pi/2}\sin^4\theta\cos^2\theta\, d\theta$$

$$= 32a^2\frac{\left|\frac{5}{2}\right.\left|\frac{3}{2}\right.}{2\left|4\right.}$$

$$= 16a^2\frac{\frac{3}{2}\times\frac{1}{2}\sqrt{\pi}\times\frac{1}{2}\sqrt{\pi}}{6}$$

$= \pi a^2$ Answer.

EXERCISE

1. Evaluate the integral by changing the order of integration

$$\int_0^\infty\int_0^x xe^{-x^2/y}\,dy\,dx$$

(U.P.T.U. 2006)

Ans. $\frac{1}{2}$

2. Evaluate

$$\int_0^1 dx\int_0^x e^{y/x}\,dy$$

Ans. $\frac{1}{2}$ (e - 1).

3. Evaluate $\iint_R xy\,dx\,dy$ where R is the quadrant of the circle $x^2 + y^2 = a^2$ where $x \geq 0$ and $y \geq 0$

Ans. $\frac{a^4}{8}$

4. Evaluate the following integral by changing the order of integration

$$\int_0^1\int_{e^x}^{e}\frac{dy\,dx}{\log y}$$

Ans. e^{-1}

5. Evaluate by changing the order of integration

$$\int_0^1\int_{2y}^{2} e^{x^2}\,dx\,dy$$

Ans. $\frac{e^4 - 1}{4}$

Multiple Integrals

6. Evaluate $\int_1^e \int_1^{\log y} \int_1^{e^x} \log z \, dz \, dx \, dy$

Ans. $\frac{1}{4}(e^2 - 8e + 13)$

7. Change the order of integration in

$$\int_0^a \int_x^{a^2/x} (x+y)\,dx\,dy$$

and find its value.

Ans. $\int_0^a \int_x^{a^2/x} (x+y)\,dx\,dy = \int_0^a \int_0^y (x+y)\,dy\,dx + \int_a^\infty \int_0^{a^2/y} (x+y)\,dy\,dx$ and its value is ∞

8. Change the order of integration of the integral

$$\int_0^a \int_0^{b/(b+x)} f(x,y)\,dx\,dy$$

Ans. $\int_0^{b/a+b} \int_0^a f(x,y)\,dy\,dx + \int_{b/a+b}^1 \int_0^{b(1-y)/y} f(x,y)\,dy\,dx$

9. Transform $\int_0^{\pi/2} \int_0^{\pi/2} \sqrt{\frac{\sin\phi}{\sin\theta}}\,d\phi\,d\theta$ by the substitution $x = \sin\phi\cos\theta$, $y = \sin\phi\sin\theta$ and show that its value is π.

(U.P.T.U. 2001)

10. Let D be the region in the first quadrant bounded by $x = 0$, $y = 0$ and $x + y = 1$, change the variables x, y to u,v where $x + y = u$, $y = uv$ and evaluate $\iint_D xy(1-x-y)^{1/2}\,dx\,dy$

(U.P.T.U. 2002)

Ans. $\frac{16}{945}$

11. Determine the area of the region bounded by the curves $xy = 2$, $4y = x^2$, $y = 4$.

Ans. $\frac{28}{3} - 4\log 2$

12. Find the volume of the cylindrical column standing on the area common to the parabolas $x = y^2$, $y = x^2$ as base and cut off by the surface $z = 12 + y - x^2$

Ans. $\frac{569}{140}$

OBJECTIVE PROBLEMS

Four alternative answers are given for each question, only one of them is correct. Tick mark the correct answer.

1. $\int_0^2 \int_4^6 (xy + e^x) dy\, dx$ is equal to

(i) $5 + e^2$ (ii) $2(9 + e^2)$

(iii) $2(7 + e^2)$ (iv) None of these

Ans. (ii)

2. $\int_0^1 \int_0^{\sqrt{y}} (x^2 + y^2) dy\, dx$ is equal to

(i) $\frac{7}{65}$ (ii) $\frac{44}{105}$

(iii) $\frac{64}{105}$ (iv) None of these

Ans. (ii)

3. If R s the region bounded by x =0, y =0, x + y =1, then $\iint_R (x^2 + y^2) dx\, dy$ is equal to

(i) $\frac{1}{3}$ (ii) $\frac{1}{5}$

(iii) $\frac{1}{6}$ (iv) $\frac{1}{12}$

Ans. (iii)

4. The area bounded by the parabola $y^2 = 4\,ax$, x-axis and the ordinates x =1, x =2 is given by

(i) $\frac{2}{3}\sqrt{a}\left(\sqrt{2} - 1\right)$

(ii) $\frac{4}{3}\sqrt{a}\left(2\sqrt{2} - 1\right)$

(iii) $\frac{2}{3}a\left(2\sqrt{2} + 1\right)$

(iv) None of these

Ans. (ii)

5. The area above the x - axis bounded by the curves y 2ax and $y^2 = ax$ is given by

<u>*Multiple Integrals*</u>

(i) $a^2\left(\frac{\pi}{2}-\frac{2}{3}\right)$ (ii) $a^2\left(\frac{\pi}{4}-\frac{3}{2}\right)$

(iii) $a^2\left(\frac{\pi}{4}-\frac{2}{3}\right)$ (iv) None of these

Ans. (iii)

6. The area bounded by the curev xy = 4, y axis and the lines y =1 to y =4 is given by

(i) 2 log 2 (ii) 4 log 2

(iii) 8 log 2 (iv) None of these

Ans. (iii)

7. The volume of the area intercepted between the plane x + y + z = 1 and the Co-ordinate planes is

(i) $\frac{1}{2}$ (ii) $\frac{1}{3}$

(iii) $\frac{1}{6}$ (iv) None of these

Ans. (iii)

8. $\int_0^1\int_1^{x^2}\int_{2y}^{x+y} x\,dz\,dy\,dx$ is equal to

(i) $\frac{1}{6}$ (ii) $\frac{1}{10}$

(iii) $\frac{1}{30}$ (iv) $\frac{1}{15}$

Ans. (iii)

9. The volume of the tetrahedron bounded by the Co-ordinate planes and the plane x + y + z = 4 is equal to

(i) $\frac{32}{3}$ (ii) $\frac{16}{3}$ (iii) $\frac{4}{3}$ (iv) $\frac{128}{3}$

Ans. (i)

10. The volume of

$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is

(i) $\frac{8}{3}\pi\,abc$ (ii) $\frac{4}{3}\pi\,abc$

(iii) $\frac{6}{5}\pi\,abc$ (iv) $\frac{4}{5}\pi\,abc$

Ans. (ii)

11. $\int_0^1\int_0^1\int_0^1 e^{x+y+z}\,dx\,dy\,dz$ is equal to

Evaluation of Triple integrals:

Let f(x,y,z) be a function which is defined at all points in a finite region v in space. Let δx, δy, δzbe an elementary volume v enclosing of the point (x,y,z) thus the triple summation.

$$\lim_{\substack{\delta x \to 0 \\ \delta y \to 0 \\ \delta z \to 0}} \sum\sum\sum f(x,y,z)\,\delta x, \delta y, \delta z$$

If it exists is written as $\iiint f(x,y,z)$ dx dydz which is called the triple integral of f(x,y,z) over the region v.

If the region v is bounded by the surfaces $x=x_1$, $x=x_2$, $y=y_1$, $y=y_2$, $z=z_1$, $z=z_2$ then

$$\iiint f(x,y,z)\text{dxdydz} = \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f(x,y,z)dx\,dy\,dz$$

Note:

(i) If x_1, x_2; y_1, y_2; z_1, z_2 are all constants then the order of integration is immaterial provide the limits of integration are changed accordingly.

i.e.

$$= \int_{z_1}^{z_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x,y,z)dx\,dy\,dz$$

$$= \int_{y_1}^{y_2}\int_{z_1}^{z_2}\int_{x_1}^{x_2} f(x,y,z)dx\,dz\,dy$$

$$= \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f(x,y,z)dz\,dy\,dx$$

(ii) If, however z_1, z_2 are functions of x and y and y_1, y_2 are functions of x while $x_1$and x_2 are constants

(iii) then the integration must be performed first w.r.to 'z' then w.r.to 'y' and finally w.r.to 'x'.

i.e.

$$\iiint f(x,y,z)\,dxdydz =$$

$$= \int_{x=a}^{x=b}\int_{y=\emptyset_1(x)}^{y=\emptyset_2(x)}\int_{z=\delta_1(x,y)}^{z=(x,y)} f(x,y,z)dz\,dy\,dx$$

10). Evaluate the following integrals:

(i) $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} xyz\ dz\,dy\,dx$

Sol $= \int_{x=0}^{1}\left\{\int_{y=0}^{\sqrt{1-x^2}}\left[\int_{z=0}^{\sqrt{1-x^2-y^2}} xyz\ dz\right] dy\right\} dx$

$$= \int_{x=0}^{1}\left\{\int_{y=0}^{\sqrt{1-x^2}}\left[xy\frac{z^2}{2}\right]_0^{\sqrt{1-x^2-y^2}}\right\} dy\,dx$$

$$= \int_{x=0}^{1}\left\{\int_{y=0}^{\sqrt{1-x^2}}\left[\frac{xy\,(1-x^2-y^2)}{2}\right]\right\} dy\,dx$$

$$= \frac{1}{2}\int_{x=0}^{1}\left\{\int_{y=0}^{\sqrt{1-x^2}}[\,xy\ - x^3y - xy^3]\right\} dy\,dx$$

$$= \frac{1}{2}\int_{x=0}^{1}\left[\frac{xy^2}{2} - \frac{x^3y^2}{2} - \frac{xy^4}{4}\right]_0^{\sqrt{1-x^2}} dx$$

$$= \frac{1}{2}\int_{x=0}^{1}\frac{x(1-x^2)}{2} - \frac{x^3(1-x^2)}{2} - \frac{x(1-x^2)^2}{4}\,dx$$

$$= \frac{1}{2}\int_{0}^{1}\frac{x-x^3-x^3+x^5)}{2} - \frac{x(1-2x^2+x^4)}{4}\,dx$$

$$=\frac{1}{2}\int_0^1 \frac{x-2x^3+x^5)}{2} - \frac{2x^3-x-x^5)}{4}\,dx$$

$$=\frac{1}{8}\int_0^1 x - 2x^3 + x^5 dx$$

$$=\frac{1}{8}\left[\frac{x^2}{2} - \frac{2x^4}{4} - \frac{x^6}{6}\right]_0^1$$

$$=\frac{1}{8}\left[\frac{1}{2} - \frac{1}{2} + \frac{1}{6}\right] = \frac{1}{48}$$

$$\therefore \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz\ dz\ dy\ dx = \frac{1}{48}$$

(ii) $\int_1^e \int_1^{logy} \int_1^{e^x} \log y\ dz\ dx\ dy$

Sol. $I = \int_{y=1}^{e} \int_{x=1}^{logy} \int_{z=1}^{e^x} \log z\ dz\ dx\ dy$

Consider $\int_{z=1}^{e^x} \log z\ dz = [\mathrm{zlogz} - \mathrm{z}]_1^{e^x}$

$= e^x \log e^x{}_e - e^x + 1$

$= xe^x - e^x + 1$

$= e^x(x-1) + 1$

$I = \int_{y=1}^{e} \int_{x=1}^{logy} \{(x-1)\,e^x + 1\}\ dx$

Consider $\int_{x=1}^{x=logy} \{(x)\,e^x - e^x + 1\}\ dx$

$$= [xe^x - e^x - e^x + 1]_{x=1}^{logy}$$

$$= [xe^x - 2e^x + 1]_1^{logy}$$

$= [\mathrm{ylogy} - 2\mathrm{y} + \mathrm{logy}] - [e - 2e + 1]$

$$= (y + 1)\log y - 2y + (e - 1)$$

$$\therefore I = \int_{y=1}^{e} y\log y + \log y - 2y + (e - 1)dy$$

$$= \left[\frac{y^2}{2}\log y \ - \frac{y^2}{4} + \ y\log y - y - y^2 + (e - 1)y\right]_1^e$$

$$= [\frac{e^2}{2}\log e \ - \frac{e^2}{4} + \ e\log e - e - e^2 + (e - 1)e] -$$

$$- \ [\frac{1}{2}\log 1 \ - \frac{1}{4} + \ \log 1 - 1 - 1 + e - 1]$$

$$= (\frac{e^2}{2} \ - \frac{e^2}{4} + e - 2e) - (-\frac{1}{4} - 3 - e)$$

$$= \frac{2e^2 - e^2 - 8e + 1 + 12}{4} = \frac{1}{4}[e^2 \ -8e+13]$$

$$\int_1^e \int_1^{\log y} \int_1^{e^x} \log y \, dz \, dx \, dy = \frac{1}{4}[e^2 \ -8e+13]$$

5. *Evaluate:* $\int_{-c}^{c}\int_{-b}^{b}\int_{-a}^{a}\left(x^2+y^2+z^2\right)dz\,dy\,dx.$

Solution
$$I = \int_{x=-c}^{c}\int_{y=-b}^{b}\int_{z=-a}^{a}\left(x^2+y^2+z^2\right)dz\,dy\,dx$$

Integrating w.r.t. z, x and y – constant.

$$= \int_{x=-c}^{c}\int_{y=-b}^{b}\left[x^2z+y^2z+\frac{z^3}{3}\right]_{z=-a}^{a} dy\,dx$$

$$= \int_{x=-c}^{c}\int_{y=-b}^{b}\left[x^2(a+a)+y^2(a+a)+\left(\frac{a^3}{3}+\frac{a^3}{3}\right)\right]dy\,dx$$

$$= \int_{x=-c}^{c}\int_{y=-b}^{b}\left(2ax^2+2ay^2+\frac{2a^3}{3}\right)dy\,dx$$

Integrating w.r.t. y, x – constant.

$$= \int_{x=-c}^{c}\left[2ax^2y+\frac{2ay^3}{3}+\frac{2a^3}{3}y\right]_{y=-b}^{b} dx$$

$$= \int_{x=-c}^{c}\left[2ax^2(b+b)+\frac{2a}{3}\left(b^3+b^3\right)+\frac{2a^3}{3}(b+b)\right]dx$$

$$= \int_{x=-c}^{c}\left[4ax^2b+\frac{4ab^3}{3}+\frac{4a^3b}{3}\right]dx$$

$$= \left[4ab\left(\frac{x^3}{3}\right)+\frac{4ab^3}{3}(x)+\frac{4a^3b}{3}(x)\right]_{-c}^{c}$$

$$= 4ab\left(\frac{2c^3}{3}\right)+\frac{4ab^3}{3}\cdot(2c)+\frac{4a^3b}{3}(2c)$$

$$= \frac{8abc^3}{3}+\frac{8ab^3c}{3}+\frac{8a^3bc}{3}$$

$$I = \frac{8abc}{3}\left(a^2+b^2+c^2\right)\cdot$$

UNIT-IV

Vector Differentiation and Vector Operators

INTRODUCTION

In this chapter, vector differential calculus is considered, which extends the basic concepts of differential calculus, such as, continuity and differentiability to vector functions in a simple and natural way. Also, the new concepts of gradient, divergence and curl are introduced.

DIFFERENTIATION OF A VECTOR POINT FUNCITON

Let S be a set of real numbers. Corresponding to each scalar t ε S, let there be associated a unique vector $\bar{f}$. Then $\bar{f}$ is said to be a vector (vector valued) function. S is called the domain of $\bar{f}$. We write $\bar{f} = \bar{f}(t)$.

Let $\bar{i}, \bar{j}, \bar{k}$ be three mutually perpendicular unit vectors in three dimensional space. We can write $\bar{f} = \bar{f}(t) = f_1(t)\bar{i} + f_2(t)\bar{j} + f_3(t)\bar{k}$, where $f_1(t)$, $f_2(t)$, $f_3(t)$ are real valued functions (which are called components of $\bar{f}$). (we shall assume that $\bar{i}, \bar{j}, \bar{k}$ are constant vectors).

1. Derivative:

Let $\bar{f}$ be a vector function on an interval *I* and a ϵ *I*. then $Lt_{t\to a} \dfrac{\bar{f}(t) - \bar{f}(a)}{t-a}$, if exists, is called the derivative of $\bar{f}$ at a and is denoted by $\bar{f}^{1}(a)$ or $\left(\dfrac{d\bar{f}}{dt}\right)$ at t = a. we also say that $\bar{f}$ is differentiable at t =a if $\bar{f}^{1}(a)$ exists.

2. Higher order derivatives

Let $\bar{f}$ be differentiable on an interval *I* and $\bar{f}^{1} = \dfrac{d\bar{f}}{dt}$ be the derivative of $\bar{f}$. $Lt_{t\to a} \dfrac{\bar{f}^{1}(t) - \bar{f}^{1}(a)}{t-a}$ exists for every a Є I_1 . it is denoted by $\bar{f}^{11} = \dfrac{d^2\bar{f}}{dt^2}$.

Similarly we can define $\bar{f}^{111}(t)$ etc.

We now state some properties of differentiable functions (without proof)

(1) Derivative of a constant vector is $\bar{a}$.

If $\bar{a}$ and $\bar{b}$ are differentiable vector functions, then

(2). $\dfrac{d}{dt}(\bar{a} \pm \bar{b}) = \dfrac{d\bar{a}}{dt} \pm \dfrac{d\bar{b}}{dt}$

(3). $\dfrac{d}{dt}(\bar{a}.\bar{b}) = \dfrac{d\bar{a}}{dt}.\bar{b} + \bar{a}.\dfrac{d\bar{b}}{dt}$

(4). $\dfrac{d}{dt}(\bar{a} x \bar{b}) = \dfrac{d\bar{a}}{dt} x\bar{b} + \bar{a} x \dfrac{d\bar{b}}{dt}$

(5). If $\bar{f}$ is a differentiable vector function and ϕ is a scalar differential function, then $\frac{d}{dt}(\phi\,\bar{f}) = \phi\frac{d\bar{f}}{dt} + \frac{d\phi}{dt}\,\bar{f}$

(6). $\bar{f} = f_1(t)\bar{i} + f_2(t)\,\bar{j} + f_3(t)\,\bar{k}$, where $f_1(t)$, $f_2(t)$, $f_3(t)$ are Cartesian components of the vector $\bar{f}$, then $\frac{d\bar{f}}{dt} = \frac{df_1}{dt}\bar{i} + \frac{df_2}{dt}\,\bar{j} + \frac{df_3}{dt}\bar{k}$

(7). The necessary and sufficient condition for $\bar{f}$ (t) to be constant vector function is $\frac{d\bar{f}}{dt}$ = $\bar{0}$.

3. Partial Derivatives

Partial differentiation for vector valued functions can be introduced as was done in the case of functions of real variables. Let $\bar{f}$ be a vector function of scalar variables *p, q, t*. Then we write $\bar{f}$ = $\bar{f}$ (*p,q,t*). Treating *t* as a variable and *p,q* as constants, we define

$$Lt_{\delta t\to 0}\,\frac{\bar{f}(p,q,t+\delta t) - \bar{f}(p,q,t)}{\delta t}$$

If exists, as partial derivative of $\bar{f}$ w.r.t. *t* and is denot by $\frac{\partial\bar{f}}{\partial t}$

Similarly, we can define $\frac{\partial\bar{f}}{\partial p}, \frac{\partial\bar{f}}{\partial q}$ also. The following are some useful results on partial differentiation.

4. Properties

1) $\frac{\partial}{\partial t}(\phi\bar{a}) = \frac{\partial\phi}{\partial t}\bar{a} + \phi\frac{\partial\bar{a}}{\partial t}$

2). If λ is a constant, then $\frac{\partial}{\partial t}(\lambda\bar{a}) = \lambda\frac{\partial\bar{a}}{\partial t}$

3). If $\bar{c}$ is a constant vector, then $\frac{\partial}{\partial t}(\phi\bar{c}) = \bar{c}\frac{\partial\phi}{\partial t}$

4). $\frac{\partial}{\partial t}(\bar{a} \pm \bar{b}) = \frac{\partial\bar{a}}{\partial t} \pm \frac{\partial\bar{b}}{\partial t}$

5). $\frac{\partial}{\partial t}(\bar{a}.\bar{b}) = \frac{\partial\bar{a}}{\partial t}.\bar{b} + \bar{a}.\frac{\partial\bar{b}}{\partial t}$

6). $\frac{\partial}{\partial t}(\bar{a}x\bar{b}) = \frac{\partial\bar{a}}{\partial t}x\bar{b} + \bar{a}x\frac{\partial\bar{b}}{\partial t}$

7). Let $\bar{f} = f_1\bar{i} + f_2\,\bar{j} + f_3\,\bar{k}$, where f_1, f_2, $f_3$are differential scalar functions of more then one variable, Then $\frac{\partial\bar{f}}{\partial t} = \bar{i}\,\frac{\partial f_1}{\partial t} + \bar{j}\,\frac{\partial f_2}{\partial t} + \bar{k}\,\frac{\partial f_3}{\partial t}$(treating $\bar{i}, \bar{j}, \bar{k}$ as fixed directions)

5. Higher order partial derivatives

Let $\bar{f} = \bar{f}(p,q,t)$. Then $\frac{\partial^2 \bar{f}}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial \bar{f}}{\partial t}\right), \frac{\partial^2 \bar{f}}{\partial p \partial t} = \frac{\partial}{\partial p}\left(\frac{\partial \bar{f}}{\partial t}\right) etc.$

6. Scalar and vector point functions: Consider a region in three dimensional space. To each point $p(x,y,z)$, suppose we associate a unique real number (called scalar) say ϕ. This $\phi(x,y,z)$ is called a scalar point function. Scalar point function defined on the region. Similarly if to each point $p(x,y,z)$we associate a unique vector $\bar{f}(x,y,z)$ we associate a unique vector $\bar{f}(x,y,z)$. $\bar{f}$ is called a ***vector point function***.

VECTOR DIFFERENTIAL OPERATOR

Def. The vector differential operator ∇(read as del) is defined as

$\nabla \equiv \bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}$. This operator possesses properties analogous to those of ordinary vectors as well as differentiation operator. We will define now some quantities known as "**gradient", "divergence"** and **"curl"** involving this operator ∇. We must note that this operator has no meaning by itself unless it operates on some function suitably.

GRADIENT OF A SCALAR POINT FUNCTION

Let $\phi(x,y,z)$ be a scalar point function of position defined in some region of space. Then the vector function $\bar{i}\frac{\partial \phi}{\partial x} + \bar{j}\frac{\partial \phi}{\partial y} + \bar{k}\frac{\partial \phi}{\partial z}$ is known as the gradient of ϕ or $\nabla\phi$

$$\nabla\phi = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\phi = \bar{i}\frac{\partial \phi}{\partial x} + \bar{j}\frac{\partial \phi}{\partial y} + \bar{k}\frac{\partial \phi}{\partial z}$$

Properties:

(1) If f and g are two scalar functions then grad(f $\pm$g)= grad f $\pm$ grad g

(2) The necessary and sufficient condition for a scalar point function to be constant is that $\nabla f = \bar{O}$

(3) grad(fg) = f(grad g)+g(grad f)

(4) If c is a constant, grad (cf) = c(grad f)

(5) grad $\left(\frac{f}{g}\right)=\frac{g(grad\ f)-f(grad\ g)}{g^2}$, $(g\neq 0)$

(6) Let $\bar{r}$ = xi+yj+zk. Then $d\bar{r}$= (dx)i+(dy) j+(dz)k. if ϕ is any scalar point function, then

$$d\phi=\frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial y}dy+\frac{\partial\phi}{\partial z}dz=\left(\bar{i}\frac{\partial}{\partial x}+\bar{j}\frac{\partial}{\partial y}+\bar{k}\frac{\partial}{\partial z}\right)$$

DIRECTIONAL DERIVATIVE

Let ϕ(x,y,z) be a scalar function defined throughout some region of space. Let this function have a value ϕ at a point P whose position vector referred to the origin O is OP = r. let ϕ+Δϕ be the value of the function at neighbouring point Q. If $\overline{OQ}=\bar{r}+\Delta$r. . Let Δr be the length of $\Delta\bar{r}$.

$\frac{\Delta\phi}{\Delta r}$ gives a measure of the rate at which ϕ change when we move from P to Q. then limiting value $\frac{\Delta\phi}{\Delta r}$ *as* $\Delta r\to 0$ is called the derivative of ϕ in the direction of PQ or simply directional derivative of ϕ at P and is denoted by dϕ/dr.

Theorem 1: The directional derivative of a scalar point function ϕ at a point P(x,y,z) in the direction of a unit vector e is equal to e. grad ϕ=e. ∇ϕ.

The physical interpretation of ∇ϕ

The gradient of a scalar function ϕ(x,y,z) at a point *P*(x,y,z) is a vector along the normal to the level surface ϕ(x,y,z) = c at *P* and is in increasing direction. Its magnitude is equal to the greatest rate of increase of **ϕ**. Greatest value of directional derivative of **ϕ at a point P = |grad ϕ|** at that point.

SOLVED EXAMPLES

Example 1: If a=x+y+z, b= $x^2+y^2+z^2$, c = xy+yz+zx, prove that [grad a, grad b, grad c] = 0.

Sol:- Given a=x+y+z $\quad \frac{\partial a}{\partial x}=1,\ \frac{\partial a}{\partial y}=1,\ \frac{\partial a}{\partial z}=1$

Grad a = ∇a = $\sum\bar{i}\frac{\partial a}{\partial x}=\bar{i}+\bar{j}+\bar{z}$

Given b= $x^2+y^2+z^2$ $\frac{\partial b}{\partial x}=2x,\ \frac{\partial b}{\partial y}=2y,\ \frac{\partial b}{\partial z}=2z$

Grad b = ∇b = $\bar{i}\frac{\partial b}{\partial x}+\bar{j}\frac{\partial b}{\partial y}+\bar{z}\frac{\partial b}{\partial z}=2x\bar{i}+2y\bar{j}+2z\bar{k}$

Again c = xy+yz+zx $\frac{\partial c}{\partial x} = y + z,\ \frac{\partial c}{\partial y} = z + x, \frac{\partial c}{\partial z} = y + x$

Grad c = $\bar{i}\,\frac{\partial c}{\partial x} + \bar{j}\,\frac{\partial c}{\partial y} + \bar{z}\,\frac{\partial c}{\partial z} = (y + z)\bar{i} + (z + x)\,\bar{j} + (x + y)\bar{k}$

[grad a, grad b, grad c] = $\begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y + z & z + x & x + y \end{vmatrix} = 0, (on\,simplification)$

Example 2 : Show that ∇[f(r)] = $\frac{f^{i}(r)}{r}\bar{r}$ where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$.

Sol:-since $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$, we have $r^2 = x^2+y^2+z^2$

Differentiating w.r.t. 'x' partially, we get

$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}.Similarly\ \frac{\partial r}{\partial y} = \frac{y}{r}\ ,\ \frac{\partial r}{\partial z} = \frac{z}{r}$

$\nabla[f(r)] = \left(\bar{i}\,\frac{\partial}{\partial x} + \bar{j}\,\frac{\partial}{\partial y} + \bar{k}\,\frac{\partial}{\partial z}\right) f(r) = \sum \bar{i} f^{1}(r)\frac{\partial r}{\partial x} = \sum \bar{i} f^{1}(r)\frac{x}{r}$

$= \frac{f^{1}(r)}{r}\sum \bar{i}x = \frac{f^{1}(r)}{r}.\bar{r}$

Note : From the above result, ∇(log r) = $\frac{1}{r^{2}} r$

Example 3 : Prove that $\nabla(r^n) = nr^{n-2}\bar{r}$.

Sol:- Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ and r = $|\bar{r}|$. Then we have $r^2 = x^2+y^2+z^2$ Differentiating w.r.t. x partially, we have

$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}.Similarly = \frac{\partial r}{\partial y} = \frac{y}{r}\,and\,\frac{\partial r}{\partial z} = \frac{z}{r}$

$\nabla(r^n) = \sum \bar{i}\,\frac{\partial}{\partial x}(r^{n}) = \sum \bar{i} nr^{n-1}\frac{\partial r}{\partial x} = \sum \bar{i} nr^{n-1}\frac{x}{r} = n\,r^{n-2}\sum \bar{i}x = n\,r^{n-2}(\bar{r})$

Note : From the above result, we can have

(1). $\nabla\left(\frac{1}{r}\right) = -\frac{\bar{r}}{r^{3}}$, taking n = -1 (2) grad r = $\frac{\bar{r}}{r}$, taking n = 1

Example 4 : Find the directional derivative of $f = xy+yz+zx$ in the direction of vector $\bar{i} + 2\bar{j} + 2\bar{k}$ at the point (1,2,0).

Sol:- Given $f = xy+yz+zx$.

Grad f = $\bar{i}\,\frac{\partial f}{\partial x} + \bar{j}\,\frac{\partial f}{\partial y} + \bar{z}\,\frac{\partial f}{\partial z} = (y + z)\bar{i} + (z + x)\,\bar{j} + (x + y)\bar{k}$

If $\bar{e}$ is the unit vector in the direction of the vector $\bar{i} + 2\bar{j} + 2\bar{k}$, then

$$\bar{e} = \frac{\bar{i} + 2\bar{j} + 2\bar{k}}{\sqrt{1^2 + 2^2 + 2^2}} = \frac{1}{3}(\bar{i} + 2\bar{j} + 2\bar{k})$$

Directional derivative of f along the given direction $= \bar{e}.\nabla f$

$$= \frac{1}{3}(\bar{i} + 2\bar{j} + 2\bar{k})[(y+2)\bar{i} + (z+x)\bar{j} + (x+y)\bar{k}] at\ (1,2,0)$$

$$= \frac{1}{3}[(y+z) + 2(z+x) + 2(x+y)](1,2,0) = \frac{10}{3}$$

Example 5 : Find the directional derivative of the function $xy^2+yz^2+zx^2$ along the tangent to the curve $x = t$, $y = t^2$, $z = t^3$ at the point (1,1,1).

Sol: - here $f = xy^2+yz^2+zx^2$

$$\nabla f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z} = \text{(y2+2xy)j+(x2+2yz)k}$$

At (1,1,1) , $\nabla f = 3\bar{i} + 3\bar{j} + 3\bar{k}$

Let r be the position vector of any point on the curve $x = t$, $y = t^2$, $z = t^3$. then

r = $x\bar{i} + y\bar{j} + z\bar{k} = t\bar{i} + t^2\bar{j} + t^3\bar{k}$

$$\frac{\partial \bar{r}}{\partial t} = \bar{i} + 2t\bar{j} + 3t^2\bar{k} - (\bar{i} + 2\bar{j} + 3\bar{k}) \text{ at } (1,1,1)$$

We know that $\frac{\partial \bar{r}}{\partial t}$ is the vector along the tangent to the curve.

Unit vector along the tangent $= \bar{e} = \frac{\bar{i}+2\bar{j}+3\bar{k}}{\sqrt{1+2^2+3^2}} = \frac{\bar{i}+2\bar{j}+3\bar{k}}{\sqrt{14}}$

Directional derivative along the tangent = $\nabla f.\bar{e}$

$$= \frac{1}{\sqrt{14}}(\bar{i} + 2\bar{j} + 3\bar{k}).3(\bar{i} + \bar{j} + \bar{k})\ \frac{3}{\sqrt{14}}(1+2+3) = \frac{18}{\sqrt{14}}$$

Example 6 : Find the directional derivative of the function $f = x^2-y^2+2z^2$ at the point P =(1,2,3) in the direction of the line $\overline{PQ}$ where Q = (5,0,4).

Sol:- The position vectors of P and Q with respect to the origin are OP = $\bar{i} + 2\bar{j} + 3\bar{k}$ and OQ = $5\bar{i} + 4\bar{k}$

$$\overline{PQ} = \overline{OQ} - \overline{OP} = 4\bar{i} - 2\bar{j} + \bar{k}$$

Let $\bar{e}$ be the unit vector in the direction of PQ. Then $\bar{e} = \frac{4\bar{i} - 2\bar{j} + \bar{k}}{\sqrt{21}}$

grad $f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z} = 2x\bar{i} - 2y\bar{j} + 4z\bar{k}$

The directional derivative of $\bar{f}$ at P (1,2,3) in the direction of PQ = $\bar{e}.\nabla f$

$$= \frac{1}{\sqrt{21}}(4\bar{i} - 2\bar{j} + \bar{k}).(2x\bar{i} - 2y\bar{j} + 4z\bar{k})\ \frac{1}{\sqrt{21}}(8x + 4y + 4z)_{at(1,2,3)} = \frac{1}{\sqrt{21}}(28)$$

Example 7 : Find the greatest value of the directional derivative of the function f = x^2yz^3 at (2,1,-1).

Sol: we have

grad $f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z} = 2xyz^3\bar{i} + x^2z^3\bar{j} + 3x^2yz^2\bar{k} = -4\bar{i} - 4\bar{j} + 12\bar{k}$ at (2,1,-1).

Greatest value of the directional derivative of $f = |\nabla f| = \sqrt{16+16+144} = 4\sqrt{11}.$

Example 8 : Find the directional derivative of xyz^2+xz at (1, 1 ,1) in a directional of the normal to the surface $3xy^2+y= z$ at (0,1,1).

Sol:- Let $f(x, y, z) \equiv 3xy^2+y- z = 0$

Let us find the unti normal e to this surface at (,1,1). Then

$$\frac{\partial f}{\partial x} = 3y^2,\ \frac{\partial f}{\partial y} = 6xy + 1, \frac{\partial f}{\partial z} = -1.$$

∇f = 3y2i+(6xy+1)j-k

$(\nabla f)_{(0,1,1)}$ = 3i+j-k = n

$$\bar{e} = \frac{\bar{n}}{|\bar{n}|} = \frac{3i + j - k}{\sqrt{9+1+1}} = \frac{3i + j - k}{\sqrt{11}}$$

Let $g(x,y,z) = xyz^2+xz$ then

$$\frac{\partial g}{\partial x} = yz^2 + z,\ \frac{\partial g}{\partial y} = xz^2, \frac{\partial g}{\partial z} = 2xy + x$$

$\nabla g=(yz^2+z)i+xz^2j+(2xyz+x)k$

And $[\nabla g]_{(1,1,1)}$ = 2i+j+3k

Directional derivative of the given function in the direction of $\bar{e}$ at (1,1,1) = $\nabla g.\bar{e}$

$$=(2i+j+3k).\left(\frac{3i + j - k}{\sqrt{11}}\right) = \frac{6+1-3}{\sqrt{11}} = \frac{4}{\sqrt{11}}$$

Example 9 : Find the directional derivative of $2xy+z^2$ at (1,-1,3) in the direction of $\bar{i} + 2\bar{j} + 3\bar{k}$.

Sol: Let $f = 2xy+z^2\ \frac{\partial f}{\partial x} = 2y,\ \frac{\partial f}{\partial y} = 2x, \frac{\partial f}{\partial z} = 2z.$

grad f = $\sum \bar{i}\frac{\partial f}{\partial x} = 2y\bar{i} + 2x\bar{j} + 2z\bar{k}$ and (grad f)at (1,-1,3)= $-2\bar{i} + 2\bar{j} + 6\bar{k}$

given vector is $\bar{a} = \bar{i} + 2\bar{j} + 3\bar{k} \Rightarrow |\bar{a}| = \sqrt{1+4+9} = \sqrt{14}$

directional derivative of f in the direction of $\bar{a}$

$$\frac{\bar{a}.\nabla f}{|\bar{a}|} = \frac{(\bar{i} + 2\bar{j} + 3\bar{k})(-2\bar{i} + 2\bar{j} + 6\bar{k}).}{\sqrt{14}} = \frac{-2+4+18}{\sqrt{14}} = \frac{20}{\sqrt{14}}$$

Example 10: Find the directional derivative of $\phi = x^2yz+4xz^2$ at (1,-2,-1) in the direction 2i-j-2k.

Sol:- Given $\phi = x^2yz+4xz^2$

$$\frac{\partial\phi}{\partial x} = 2xyz + 4z^2, \; \frac{\partial\phi}{\partial y} = x^2 z, \frac{\partial\phi}{\partial z} = x^2 y + 8xz.$$

Hence $\nabla\phi = \sum \bar{i}\,\frac{\partial\phi}{\partial x} = \bar{i}(2xyz + 4z^2) + \bar{j}x^2 z + \bar{k}(x^2 y + 8xz)$

$\nabla\phi$ at (1,-2,-1) = i(4+4)+j(-1)+k(-2-8)= 8i-j-10k.

The unit vector in the direction 2i-j-2k is

$$\bar{a} = \frac{2i - j - 2k.}{\sqrt{4+1+4}} = \frac{1}{3}(2i - j - 2k)$$

Required directional derivative along the given direction = $\nabla\phi.\ \bar{a}$

= (8i-j-10k). 1/3 (2i-j-2k)

= 1/3(16+1+20) = 37/3.

Example:11 If the temperature at any point in space is given by t = xy+yz+zx, find the direction in which temperature changes most rapidly with distance from the point (1,1,1) and determine the maximum rate of change.

Sol:- The greatest rate of increase of t at any point is given in magnitude and direction by ∇t.

We have $\nabla t = \left(\bar{i}\,\frac{\partial}{\partial x} + \bar{j}\,\frac{\partial}{\partial y} + \bar{k}\,\frac{\partial}{\partial z}\right)(xy + yz + zx)$

$= \bar{i}(y+z) + \bar{j}(z+x) + \bar{k}(x+y) = 2\bar{i} + 2\bar{j} + 2\bar{k}$ at (1,1,1)

Magnitude of this vector is $\sqrt{2^2 + 2^2 + 2^2} = \sqrt{12} = 2\sqrt{3}$

Hence at the point (1,1,1) the temperature changes most rapidly in the direction given by the vector $2\bar{i} + 2\bar{j} + 2\bar{k}$ and greatest rate of increase = $2\sqrt{3}$.

Example12 : Find the directional derivative of $\phi(x,y,z) = x^2yz+4xz^2$ at the point (1,-2,-1) in the direction of the normal to the surface *f*(x,y,z) = x log z-y^2 at (-1,2,1).

Sol:- Given $\phi(x,y,z) = x^2yz+4xz^2$ at (1,-2,-1) and $f(x,y,z) = x \log z - y^2$ at (-1,2,1)

Now $\nabla\phi = \frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k}$

$= (2xyz + 4z^2)\bar{i} + (x^2 z)\bar{j} + (x^2 y + 8xz)\bar{k}$

$(\nabla\phi)_{(1,-2,-1)} = [2(1)(-2)(-1) + 4(-1)^2]\bar{i} + [(1)^2(-1)\bar{j}] + [(1^2)(-2) + 8(-1)]\bar{k} \text{ ----(1)}$

$= 8\bar{i} - \bar{j} - 10\bar{k}$

Unit normal to the surface

f(x,y,z)= x log z- y^2 is $\frac{\nabla f}{|\nabla f|}$

now ∇f = $\bar{i}\frac{\partial f}{\partial x}+\bar{j}\frac{\partial f}{\partial y}+\bar{k}\frac{\partial f}{\partial z} = \log z\,\bar{i}+(-2y)\,\bar{j}+\frac{x}{z}\bar{k}$

at (-1,2,1), ∇f = $\log(1)\bar{i}-2(2)\bar{j}+\frac{-1}{1}\bar{k} = -4\bar{j}-\bar{k}$

$\frac{\nabla f}{|\nabla f|} = \frac{-4\bar{j}-\bar{k}.}{\sqrt{16+1}} = \frac{-4\bar{j}-\bar{k}.}{\sqrt{17}} =$

Directional derivative = $\nabla\phi.\frac{\nabla f}{|\nabla f|}$

$$= (8\bar{i}-\bar{j}-10\bar{k}).\frac{-4\bar{j}-\bar{k}.}{\sqrt{17}} = \frac{4+10}{\sqrt{17}} = \frac{14}{\sqrt{17}}.$$

Example 13 : Find a unit normal vector to the given surface $x^2y+2xz = 4$ at the point (2,-2,3).

Sol:- Let the given surface be $f = x^2y+2xz-4$

On differentiating,

$$\frac{\partial f}{\partial x} = 2xy+2z,\ \frac{\partial f}{\partial y} = x^2,\frac{\partial f}{\partial z} = 2x.$$

grad f = $\sum\bar{i}\frac{\partial f}{\partial x} = \bar{i}(2xy+2z)+\bar{j}x^2+2x\bar{k})$

(grad f) at (2,-2,3) = $\bar{i}(-8+6)+4\bar{j}+4\bar{k}) = -2\bar{i}+4\bar{j}+4\bar{k}$

grad (f) is the normal vector to the given surface at the given point.

Hence the required unit normal vector $\frac{\nabla f}{|\nabla f|} = \frac{2(-\bar{i}+2\bar{j}+2\bar{k}).}{2\sqrt{1+2^2+2^2}} = \frac{-\bar{i}+2\bar{j}+2\bar{k}}{3}$

Example 14 : Evaluate the angle between the normals to the surface $xy= z^2$ at the points (4,1,2) and (3,3,-3).

Sol:- given surface is f(x,y,z) = $xy= z^2$

Let $\bar{n}_1$ and $\bar{n}_2$ be the normals to this surface at (4,1,2) and (3,3,-3) respectively.

Differentiating partially, we get

$$\frac{\partial f}{\partial x} = y,\ \frac{\partial f}{\partial y} = x,\frac{\partial f}{\partial z} = -2z.$$

grad f = $y\bar{i}+x\bar{j}-2z\bar{k}$

$\bar{n}_1$= (grad f) at (4,1,2) $=\bar{i}+4\bar{j}-4\bar{k}$

$\bar{n}_2$ = (grad f) at (3,3,-3) = $3\bar{i}+3\bar{j}+6\bar{k}$

Let θ be the angle between the two normals.

$$\cos\theta = \frac{\bar{n}_1\,\bar{n}_2}{|\bar{n}_1\,\bar{n}_2|} = \frac{(i+4j-4k)}{\sqrt{1+16+16}}\cdot\frac{(3i+3j+6k)}{\sqrt{9+9+36}}$$

$$\frac{(3+12-24)}{\sqrt{33}\sqrt{54}} = \frac{-9}{\sqrt{33}\sqrt{54}}$$

Example15 : Find a unit normal vector to the surface $x^2+y^2+2z^2 = 26$ at the point (2, 2 ,3).

Sol:- Let the given surface be $f(x,y,z) \equiv x^2+y^2+2z^2 - 26=0$. Then

$$\frac{\partial f}{\partial x} = 2x,\ \frac{\partial f}{\partial y} = 2y,\frac{\partial f}{\partial z} = 4z.$$

grad f = $\sum \bar{i}\,\frac{\partial f}{\partial x} =$ 2xi+2yj+4zk

normal vector at(2,2,3) = $[\nabla f\,]_{(2,2,3)}$ = 4I+4J+12K

unit normal vector = $\frac{\nabla f}{|\nabla f|} = \frac{4(\bar{i}+\bar{j}+3\bar{k})}{4\sqrt{11}} = \frac{\bar{i}+\bar{j}+3\bar{k}}{\sqrt{11}}$

Example 16: Find the values of a and b so that the surfaces ax^2-byz = (a+2)x and $4x^2y+z^3$= 4 may intersect orthogonally at the point (1, -1,2).

(or) Find the constants a and b so that surface ax^2-byz=(a+2)x will orthogonal to $4x^2y+z^3$=4 at the point (1,-1,2).

Sol:- let the given surfaces be f(x,y,z) = ax^2-byz - (a+2)x-------------(1)

And g(x,y,z) = $4x^2y+z^3$- 4------------(2)

Given the two surfaces meet at the point (1,-1,2).

Substituting the point in (1), we get

a+2b-(a+2) = 0 ⇒ b=1

now $\frac{\partial f}{\partial x} = 2ax-(a+2),\ \frac{\partial f}{\partial y} = -bz =,\frac{\partial f}{\partial z} = -by.$

∇f = $\sum \bar{i}\,\frac{\partial f}{\partial x} =$ [(2a-(a+2)]i-2bj+bk = (a-2)i-2bj+bk

= (a-2)i-2j+k = $\bar{n}_1$, normal vector to surface 1.

Also $\frac{\partial g}{\partial x} = 8xy,\ \frac{\partial g}{\partial y} = 4x^2,\frac{\partial g}{\partial z} = 3z^2.$

∇g = $\sum \bar{i}\frac{\partial g}{\partial x}$ = 8xyi+4x²j+3z²k

$(\nabla g)_{(1,-1,2)}$ = -8i+4j+12k = $\bar{n}_2$, normal vector to surface 2.

Given the surfaces f(x,y,z), g(x,y,z) are orthogonal at the point (1,-1,2).

$[\nabla f].[\nabla g] = 0 \Rightarrow$ ((a-2)i-2j+k). (-8i+4j+12k)=0

$\Rightarrow$-81+16-8+12 $\Rightarrow$ a =5/2

Hence a = 5/2 and b=1.

Example17 : Find a unit normal vector to the surface z= x^2+y^2 at (-1,-2,5)

Sol:- let the given surface be f = x^2+y^2-z

$$\frac{\partial f}{\partial x} = 2x, \frac{\partial f}{\partial y} = 2y, \frac{\partial f}{\partial z} = -1.$$

grad f = $\nabla f = \sum \bar{i}\frac{\partial f}{\partial x}$ = 2xi+2yj-k

(∇f) at (-1,-2,5)= -2i-4j-k

∇f is the normal vector to the given surface.

Hence the required unit normal vector = $\frac{\nabla f}{|\nabla f|}$ =

$$\frac{-2i-4j-k}{\sqrt{(-2)^2+(-4)^2+(-1)^2}} = \frac{-2i-4j-k}{\sqrt{21}} = -\frac{1}{\sqrt{21}}(2i+4j+k)$$

Example 18: Find the angle of intersection of the spheres $x^2+y^2+z^2$ =29 and $x^2+y^2+z^2$ +4x-6y-8z-47 =0 at the point (4,-3,2).

Sol:- Let f = $x^2+y^2+z^2$ -29 and g = $x^2+y^2+z^2$ +4x-6y-8z-47

Then grad f= $\bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z} = 2x\bar{i} + 2y\bar{j} + 2z\bar{k}$ and

grad g = $(2x+4)\bar{i} + (2y-6)\bar{j} + (2z-8)\bar{k}$

The angle between two surfaces at a point is the angle between the normals to the surfaces at that point.

Let $\bar{n}_1$= (grad *f*) at (4,-3,2) =$8\bar{i} - 6\bar{j} + 4\bar{k}$

$\bar{n}_2$= (grad *f*) at (4,-3,2) = $12\bar{i} - 12\bar{j} - 4\bar{k}$

The vectors $\bar{n}_1$ and $\bar{n}_2$ are along the normals to the two surfaces at (4,-3,2). Let θ be the angle between the surfaces. Then

$$\text{Cos }\theta = \frac{\bar{n}_1\,\bar{n}_2}{|\bar{n}_1\,\bar{n}_2|} = .\frac{152}{\sqrt{116}\sqrt{304}} \qquad \therefore \theta = \cos^{-1}\left(\sqrt{\frac{19}{29}}\right)$$

Example19 : Find the angle between the surfaces $x^2+y^2+z^2=9$, and $z = x^2+y^2-3$ at point (2,-1,2).

Sol:- Let $\phi_1 = x^2+y^2+z^2-9=0$ and $\phi_2 = x^2+y^2-z-3=0$ be the given surfaces. Then

$\nabla\phi_1 = 2xi+2yj+2zk$ and $\nabla\phi_2 = 2xi+2yj-k$

Let $\bar{n}_1 = \nabla\phi_1$ at(2,-1,2)= 4i-2j+4k and

$\bar{n}_2 = \nabla\phi_2$ at (2,-1,2) = 4i-2j-k

The vectors $\bar{n}_1$ and $\bar{n}_2$ are along the normals to the two surfaces at the point (2,-1,2). Let θ be the angle between the surfaces. Then

$$\text{Cos }\theta = \frac{\bar{n}_1\,\bar{n}_2}{|\bar{n}_1\,\bar{n}_2|} = \frac{(4i-2j+4k)}{\sqrt{16+4+16}}.\frac{(4i-2j-k)}{\sqrt{16+4+16}} = \frac{16+4-4}{6\sqrt{21}} = \frac{16}{6\sqrt{21}} = \frac{8}{3\sqrt{21}}$$

$$\therefore \theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right).$$

Example : If $\nabla\phi = yz\bar{i} + zx\bar{j} + xy\bar{k}$, find ϕ.

Sol:- we know that $\nabla\phi = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}$

Given that $\nabla\phi = yz\bar{i} + zx\bar{j} + xy\bar{k}$

Comparing the corresponding coefficients, we have $\frac{\partial \phi}{\partial x} = yz, \frac{\partial \phi}{\partial y} = zx, \frac{\partial \phi}{\partial z} = xy$

Integrating partially w.r.t. x,y,z, respectively, we get

ϕ= xyz + a constant independent of x.

ϕ= xyz + a constant independent of y.

ϕ= xyz + a constant independent of z.

Here a possible form of ϕ is ϕ= xyz+a constant.

DIVERGENCE OF A VECTOR

Let $\bar{f}$ be any continuously differentiable vector point function. Then $\bar{i}.\frac{\partial \bar{f}}{\partial x} + \bar{j}.\frac{\partial \bar{f}}{\partial y} + \bar{k}.\frac{\partial \bar{f}}{\partial z}$ is called the divergence of $\bar{f}$ and is written as div $\bar{f}$.

i.e div $\bar{f} = \bar{i}.\frac{\partial \bar{f}}{\partial x} + \bar{j}.\frac{\partial \bar{f}}{\partial y} + \bar{k}.\frac{\partial \bar{f}}{\partial z} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right).\bar{f}$

hence we can write div $\bar{f}$ as

div $\bar{f} = \nabla.\bar{f}$

This is a scalar point function.

Theorem 1: If the vector $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$, then div $\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$

Prof: Given $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$

$$\frac{\partial \bar{f}}{\partial x} = \bar{i}\frac{\partial f_1}{\partial x} + \bar{j}\frac{\partial f_2}{\partial x} + \bar{k}\frac{\partial f_3}{\partial x}$$

Also $\bar{i}.\frac{\partial \bar{f}}{\partial x} = \frac{\partial f_1}{\partial x}$. Similarly $\bar{j}.\frac{\partial \bar{f}}{\partial y} = \frac{\partial f_2}{\partial y}$ and $\bar{k}.\frac{\partial \bar{f}}{\partial z} = \frac{\partial f_3}{\partial z}$

We have div $\bar{f} = \sum \bar{i}.\left(\frac{\partial \bar{f}}{\partial x}\right) = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$

Note : If $\bar{f}$ is a constant vector then $\frac{\partial f_1}{\partial x}, \frac{\partial f_2}{\partial y}, \frac{\partial f_3}{\partial z}$ are zeros.

div $\bar{f}$ =0 for a constant vector $\bar{f}$.

Theorem 2: div ($\bar{f} \pm \bar{g}$) = $div\,\bar{f} \pm div\,\bar{g}$

Proof: div ($\bar{f} \pm \bar{g}$) $= \sum \bar{i}.\frac{\partial}{\partial x}(\bar{f}) \pm \sum \bar{i}.\frac{\partial}{\partial x}(\bar{g}) = div\,\bar{f} \pm div\,\bar{g}$.

Note: If ϕ is a scalar function and $\bar{f}$ is a vector function, then

(i). $(\bar{a}.\nabla)\phi = \left[\bar{a}\left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\right]\phi$

$$= \left[(\bar{a}.\bar{i})\frac{\partial}{\partial x} + (\bar{a}.\bar{j})\frac{\partial}{\partial y} + (\bar{a}.\bar{k})\frac{\partial}{\partial z}\right]\phi$$

$$= \left[(\bar{a}.\bar{i})\frac{\partial \phi}{\partial x} + (\bar{a}.\bar{j})\frac{\partial \phi}{\partial y} + (\bar{a}.\bar{k})\frac{\partial \phi}{\partial z}\right]$$

$= \sum (\bar{a}.\bar{i})\frac{\partial \phi}{\partial x}$. and

(ii). $(\bar{a}.\nabla)\bar{f} = \sum (\bar{a}.\bar{i})\frac{\partial \bar{f}}{\partial x}$.by proceeding as in (i) [simply replace ϕ by $\bar{f}$ in (i)].

SOLENOIDAL VECTOR

A vector point function $\bar{f}$ is said to be $\bar{f}$ solenoidal if div $\bar{f}$ =0.

Physical interpretation of divergence:

Depending upon $\bar{f}$ in a physical problem, we can interpret div $\bar{f}$ (= $\nabla.\bar{f}$).

Suppose $\overline{F}$ (x,y,z,t) is the velocity of a fluid at a point(x,y,z) and time 't'. though time has no role in computing divergence, it is considered here because velocity vector depends on time.

Imagine a small rectangular box within the fluid as shown in the figure. We would like to measure the rate per unit volume at which the fluid flows out at any given time. The divergence of $\overline{F}$ measures the outward flow or expansions of the fluid from their point at any time. This gives a physical interpretation of the divergence.

Similar meanings are to be understood with respect to divergence of vectors $\bar{f}$ from other branches. A detailed elementary interpretation can be seen in standard books on fluid dynamics, electricity and magnetism etc.

SOLVED EXAMPLES

Example 1: If $\bar{f} = xy^2\bar{i} + 2x^2yz\bar{j} - 3yz^2\bar{k}$ find div $\bar{f}$ at(1, -1, 1).

Sol:- $\bar{f} = xy^2\bar{i} + 2x^2yz\bar{j} - 3yz^2\bar{k}$. Then

div $\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(2x^2yz) + \frac{\partial}{\partial z}(-3yz^2) = y^2+2x^2z-6yz$

(div $\bar{f}$) at (1, -1, 1) = 1+2+6 =9

Example 2: find div $\bar{f}$ = grad($x^3+y^3+z^3-3xyz$)

Sol:- Let $\phi = x^3+y^3+z^3-3xyz$. Then

$$\frac{\partial \phi}{\partial x} = 3x^2 - 3yz,\ \frac{\partial \phi}{\partial y} = 3y^2 - 3zx, \frac{\partial \phi}{\partial z} = 3z^2 - 3xy$$

grad $\phi = \bar{i}\frac{\partial \phi}{\partial x} + \bar{j}\frac{\partial \phi}{\partial y} + \bar{k}\frac{\partial \phi}{\partial z} = 3[(x^2 - yz)\bar{i} + (y^2 - zx)\bar{j} + (z^2 - xy)\bar{k}]$

div $\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = \frac{\partial}{\partial x}[3(x^2 - yz)].\frac{\partial}{\partial y}[3(y^2 - zx)].\frac{\partial}{\partial z}[3(z^2 - xy)]$

= 3(2x)+3(2y)+3(2z) = 6(x+y+z)

Example 3: If $\bar{f} = (x+3y)\bar{i} + (y-2z)\bar{j} + (x+pz)\bar{k}$ is solenoidal, find P.

Sol:- Let $\bar{f} = (x+3y)\bar{i} + (y-2z)\bar{j} + (x+pz)\bar{k} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$

We have $\frac{\partial f_1}{\partial x} = 1,\ \frac{\partial f_2}{\partial y} = 1, \frac{\partial f_3}{\partial z} = p$

$$\text{div } \bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = 1+1+p = 2+p$$

since $\bar{f}$ is solenoidal, we have div $\bar{f}$ = 0 $\rightarrow$ p = -2

Example 4: Find div $\bar{f} = r^n \bar{r}$. Find n if it is solenoidal?

Sol: Given $\bar{f} = r^n \bar{r}$. where $\bar{f} = \bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ *and* $r = |\bar{r}|$

We have $r^2 = x^2+y^2+z^2$

Differentiating partially w.r.t. x , we get

$$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r},$$

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}, and\ \frac{\partial r}{\partial z} = \frac{z}{r}$

$\bar{f} = r^n (x\bar{i} + y\bar{j} + z\bar{k})$

$$\text{div } \bar{f} = \frac{\partial}{\partial x}(r^n x) + \frac{\partial}{\partial y}(r^n y) + \frac{\partial}{\partial z}(r^n z)$$

$$= nr^{n-1}\frac{\partial r}{\partial x}x + nr^{n-1}\frac{\partial r}{\partial y}y + nr^{n-1}\frac{\partial r}{\partial z}z + r^n$$

$$= nr^{n-1}\left[\frac{x^2}{r} + \frac{y^2}{r} + \frac{z^2}{r}\right] + 3r^n = nr^{n-1}\frac{(r^2)}{r} + 3r^n = nr^n + 3r^n = (n+3)r^n$$

Let $\bar{f} = r^n \bar{r}$. be solenoidal. Then div $\bar{f}$ = 0

$(n+3)r^n = 0 \Rightarrow n = -3$

Example 5: Evaluate $\nabla.\left(\frac{\bar{r}}{r^3}\right)$ where $\bar{r} = xi + yj + zk$ *and* $r = |\bar{r}|$.

Sol:- We have

r = xi+yj+zk and r = $\sqrt{x^2 + y^2 + z^2}$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}, and\ \frac{\partial r}{\partial z} = \frac{z}{r}$$

$\therefore \frac{\bar{r}}{r^3} = \bar{r}.\ r^{-3} = r^{-3}xi + r^{-3}yj + r^{-3}zk = f_1 i + f_2 j + f_3 k$

Hence $\nabla.\left(\frac{\bar{r}}{r^3}\right) = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$

We have $f_1 = r^{-3} x \Rightarrow \frac{\partial f_1}{\partial x} = r^{-3}.1 + x(-3)r^{-4}.\frac{\partial r}{\partial x}$

$$\nabla\left(\frac{\bar{r}}{r^3}\right) = \sum \bar{i}.\frac{\partial f_1}{\partial x} = 3r^{-3} - 3r^{-5}\sum x^2$$

$= 3r^{-3} - 3r^{-5} r^2 = 3r^{-3} - 3r^{-3} = 0$

Example 6: Find div $\bar{r}$. where $\bar{r}. = x\bar{i} + y\bar{j} + z\bar{k}$

Sol:- We have $\bar{r}. = x\bar{i} + y\bar{j} + z\bar{k} = f_1\bar{i} + f_2\,\bar{j} + f_3\,\bar{k}$

$$\text{div } \bar{r}. = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1+1+1 = 3$$

CURL OF A VECTOR

Def: Let $\bar{f}$ be any continuously differentiable vector point function. Then the vector function defined by $\bar{i}x\frac{\partial \bar{f}}{\partial x} + \bar{j}x\frac{\partial \bar{f}}{\partial y} + \bar{k}x\frac{\partial \bar{f}}{\partial z}$ is called curl of $\bar{f}$ and is denoted by curl $\bar{f}$ or (∇x $\bar{f}$).

$$\text{Curl } \bar{f} = \bar{i}x\frac{\partial \bar{f}}{\partial x} + \bar{j}x\frac{\partial \bar{f}}{\partial y} + \bar{k}x\frac{\partial \bar{f}}{\partial z} = \sum\left(\bar{i}x\frac{\partial \bar{f}}{\partial x}\right)$$

Theorem 1: If $\bar{f}$ is differentiable vector point function given by $\bar{f} = f_1\bar{i} + f_2\,\bar{j} + f_3\,\bar{k}$ then curl

$$\bar{f} = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\bar{i} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\bar{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\bar{k}$$

Proof : curl $\bar{f} = \sum \bar{i}x\frac{\partial}{\partial x}(\bar{f}) = \sum \bar{i}x\frac{\partial}{\partial x}(f_1\bar{i} + f_2\bar{j} + f_3\bar{k}) = \sum\left(\frac{\partial f_2}{\partial x}\bar{k} - \frac{\partial f_3}{\partial x}\bar{j}\right)$

$$= \left(\frac{\partial f_2}{\partial x}\bar{k} - \frac{\partial f_3}{\partial x}\bar{j}\right) + \left(\frac{\partial f_3}{\partial y}\bar{i} - \frac{\partial f_1}{\partial y}\bar{k}\right) + \left(\frac{\partial f_1}{\partial z}\bar{j} - \frac{\partial f_2}{\partial z}\bar{i}\right)$$

$$= \bar{i}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) + \bar{j}\left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right) + \bar{k}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

Note : (1) The above expression for curl $\bar{f}$ can be remembered easily through the representation.

$$\textbf{curl } \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} = \nabla \mathbf{x}\, \bar{f}$$

note : (2) If $\bar{f}$ is a constant vector then curl $\bar{f} = \bar{o}$.

Theorem 2: curl $(\bar{a} \pm \bar{b}) = curl\,\bar{a} \pm curl\,\bar{b}$

Proof: curl$(\bar{a} \pm \bar{b}) = \sum \bar{i}x\frac{\partial}{\partial x}(\bar{a} \pm \bar{b})$

$$= \sum ix\left(\frac{\partial \bar{a}}{\partial x} \pm \frac{\partial \bar{b}}{\partial x}\right) = \sum ix\frac{\partial \bar{a}}{\partial x} \pm \sum ix\frac{\partial \bar{b}}{\partial x}$$

$$= curl\,\bar{a} \pm curl\,\bar{b}$$

1. Physical Interpretation of curl

If $\overline{w}$ is the angular velocity of a rigid body rotating about a fixed axis and $\bar{v}$ is the velocity of any point P(x,y,z) on the body, then $\overline{w}$ = ½ curl $\bar{v}$. Thus the angular velocity of rotation at any point is equal to half the curl of velocity vector. This justifies the use of the word "curl of a vector".

2. Irrotational Motion, Irrotational Vector

Any motion in which curl of the velocity vector is a null vector i.e curl $\bar{v} = \bar{0}$ is said to be Irrotational.

Def: A vector $\bar{f}$ is said to be Irrotational if curl $\bar{f} = \bar{0}$.

If $\bar{f}$ is Irrotational, there will always exist a scalar function φ(x,y,z) such that $\bar{f}$ =grad ϕ. This is called scalar potential of $\bar{f}$.

It is easy to prove that, if $\bar{f}$ = grad ϕ, then curl $\bar{f}$ = 0.

Hence ∇x $\bar{f}$ = 0 ⇔ there exists a scalar function ϕ such that $\bar{f}$ = ∇ϕ.

This idea is useful when we study the "work done by a force" later.

SOLVED EXAMPLES

Example 1: if $\bar{f} = xy^2\bar{i} + 2x^2yz\,\bar{j} - 3yz^2\,\bar{k}$ find curl $\bar{f}$ at the point (1,-1,1).

Sol:- Let $\bar{f} = xy^2\bar{i} + 2x^2yz\,\bar{j} - 3yz^2\,\bar{k}$. Then

$$\text{curl } \bar{f} = \nabla\text{x } \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & 2x^2yz & -3yz^2 \end{vmatrix}$$

$$= \bar{i}\left(\frac{\partial}{\partial y}(-3yz^2) - \frac{\partial}{\partial z}(2x^2yz)\right) + \bar{j}\left(\frac{\partial}{\partial z}(xy^2) - \frac{\partial}{\partial x}(-3yz^2)\right) + \bar{k}\left(\frac{\partial}{\partial x}(2x^2yz) - \frac{\partial}{\partial y}(xy^2)\right)$$

$$= \bar{i}\left(-3z^2 - 2x^2z\right) + \bar{j}(0-0) + \bar{k}(4xyz - 2xy)$$

= curl $\bar{f}$ = at (1,-1,1) = $-\bar{i} - 2\bar{k}$.

Example 2: Find curl $\bar{f}$ where $\bar{f}$ = grad($x^3+y^3+z^3-3xyz$)

Sol:- Let ϕ= $x^3+y^3+z^3-3xyz$. Then

grad ϕ= $\sum .\bar{i}\,\frac{\partial \phi}{\partial x} = 3(x^2 - yz)\bar{i} + 3(y^2 - zx)\,\bar{j} + 3(z^2 - xy)\bar{k}$

curl grad ϕ= ∇x grad ϕ= $3\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - yz & y^2 - zx & z^2 - xy \end{vmatrix}$

$$= 3[\bar{i}(-x+x) - \bar{j}(-y+y) + \bar{k}(-z+z)] = \bar{0}$$

curl $\bar{f} = \bar{0}$.

Note: We can prove in general that curl (grad ϕ)=$\bar{0}$.(i.e) grad ϕ is always irrotational.

Example 3: Prove that if $\bar{r}$ is the position vector of an point in space, then $r^n\bar{r}$ is Irrotational. (or) Show that $(r^n\bar{r}) = 0$

Sol:- Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ and r = $|\bar{r}|$ $\therefore$ $r^2 = x^2+y^2+z^2$.

Differentiating partially w.r.t. 'x' partially, we get

$$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r},$$

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}, and \frac{\partial r}{\partial z} = \frac{z}{r}$

$r^n = |\bar{r}|$ $r^n(x\bar{i} + y\bar{j} + z\bar{k})$

x($r^n\bar{r}$)= $\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xr^n & yr^n & zr^n \end{vmatrix}$

$$= \bar{i}\left(\frac{\partial}{\partial y}(r^n z) - \frac{\partial}{\partial z}(r^n y)\right) + \bar{j}\left(\frac{\partial}{\partial z}(r^n x) - \frac{\partial}{\partial x}(r^n z)\right) + \bar{k}\left(\frac{\partial}{\partial x}(r^n y) - \frac{\partial}{\partial y}(r^n x)\right)$$

$$= \sum \bar{i}\left\{znr^{n-1}\frac{\partial r}{\partial y} - ynr^{n-1}\frac{\partial r}{\partial z}\right\} = nr^{n-1}\sum \bar{i}\left\{z\left(\frac{y}{r}\right) - z\left(\frac{z}{r}\right)\right\}$$

$$nr^{n-2}[(zy - yz)\bar{i} + (xz - zx)\bar{j} + (xy - yz)\bar{k}]$$

$nr^{n-2}[0\bar{i} + 0j + 0\bar{k}]$ = nr^{n-2} [$\bar{0}$]=$\bar{0}$

Hence $r^n\bar{r}$ is Irrotational.

Example 4: Prove that curl $\bar{r} = \bar{0}$

Sol:- Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

curl $\bar{r} = \sum \bar{i} x \frac{\partial}{\partial x}(\bar{r}) = \sum(\bar{i} x \bar{i}) = \bar{0} + \bar{0} = \bar{0}$

$\bar{r}$ is Irrotational vector.

Example 5: If $\bar{a}$ is a constant vector, prove that curl $\left(\frac{\bar{a}x\bar{r}}{r^3}\right) = -\frac{\bar{a}}{r^3} + \frac{3\bar{r}}{r^5}(\bar{a}.\bar{r})$.

Sol:- We have $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$$\frac{\partial \bar{r}}{\partial x} = \bar{i}\,,\ \frac{\partial \bar{r}}{\partial y} = \bar{j}\,,\ \frac{\partial \bar{r}}{\partial z} = \bar{k}$$

If $|\bar{r}|$ = r then $r^2 = x^2+y^2+z^2$

$$\frac{\partial r}{\partial x} = \frac{x}{r},\ \frac{\partial r}{\partial y} = \frac{y}{r},\ and\ \frac{\partial r}{\partial z} = \frac{z}{r}$$

curl $\left(\frac{\bar{a}x\bar{r}}{r^3}\right) = \sum \bar{i}\, x \frac{\partial}{\partial x}\left(\frac{\bar{a}x\bar{r}}{r^3}\right)$

Now $\frac{\partial}{\partial x}\left(\frac{\bar{a}x\bar{r}}{r^3}\right) = \bar{a}x\frac{\partial}{\partial x}\left(\frac{\bar{r}}{r^3}\right) = \bar{a}x\left[\frac{1}{r^3}\frac{\partial \bar{r}}{\partial x} - \frac{3}{r^4}\frac{\partial r}{\partial x}\bar{r}\right]$

$$= \bar{a}x\left[\frac{1}{r^3}\bar{i} - \frac{3}{r^5}x\bar{r}\right] = \frac{\bar{a}x\bar{i}}{r^3} - \frac{3x(\bar{a}.x\bar{r})}{r^5}.$$

$$\therefore\ ix\frac{\partial}{\partial x}\left(\frac{\bar{a}x\bar{r}}{r^3}\right) = \bar{i}x\left[\frac{\bar{a}x\bar{i}}{r^3} - \frac{3x}{r^5}(\bar{a}x\bar{r})\right] = \frac{\bar{i}x(\bar{a}x\bar{i})}{r^3} - \frac{3x}{r^5}\bar{i}x(\bar{a}x\bar{r})$$

$$= \frac{(\bar{i}.\bar{i})\bar{a} - (\bar{i}.\bar{a})i}{r^3} - \frac{3x}{r^5}[(\bar{i}.\bar{r})\bar{a} - (i.a)\bar{r}]$$

Let $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$. Then $\bar{i}\,.\,\bar{a}$ = a_1 , etc.

$$\therefore\ ix\frac{\partial}{\partial x}\left(\frac{\bar{a}x\bar{r}}{r^3}\right) = \sum\frac{(\bar{a} - a_1 i)}{r^3} - \frac{3x}{r^3}(x\bar{a} - a_1\bar{r})$$

$$\therefore\quad \sum ix\frac{\partial}{\partial x}\left(\frac{\bar{a}x\bar{r}}{r^3}\right) = \sum\frac{\bar{a} - a_1\bar{i}}{r^3} - \frac{3}{r^5}\sum(x^2\bar{a} - a_1 x\bar{r})$$

$$= \frac{3\bar{a} - \bar{a}}{r^3} - \frac{3\bar{a}}{r^5}(r^2) + \frac{3\bar{r}}{r^5}(a_1x + a_2y + a_3z)$$

$$= \frac{2\bar{a}}{r^3} - \frac{3\bar{a}}{r^3} + \frac{3\bar{r}}{r^5}(\bar{r}.\bar{a}) = -\frac{\bar{a}}{r^3} + \frac{3\bar{r}}{r^5}(\bar{r}.\bar{a})$$

Example 6: Show that the vector $(x^2 - yz)\bar{i} + (y^2 - zx)\bar{j} + (z^2 - xy)\bar{k}$ is irrotational and find its scalar potential.

Sol: let $\bar{f} = (x^2 - yz)\bar{i} + (y^2 - zx)\bar{j} + (z^2 - xy)\bar{k}$

$$\text{Then curl } \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - yz & y^2 - zx & z^2 - xy \end{vmatrix} = \sum i(-x + x) = \bar{0}$$

$\bar{f}$ is Irrotational. Then there exists ϕ such that $\bar{f} = \nabla\phi$.

$$\Rightarrow \bar{i}\frac{\partial\phi}{\partial x}+\bar{j}\frac{\partial\phi}{\partial y}+\bar{k}\frac{\partial\phi}{\partial z}= (x^2-yz)\bar{i}+(y^2-zx)\,\bar{j}+(z^2-xy)\,\bar{k}$$

Comparing components, we get

$$\frac{\partial\phi}{\partial x}=x^2-yz\Rightarrow\phi=\int\left(x^2-yz\right)dx=\frac{x^3}{3}-xyz+\ f_1(y,z)......(1)$$

$$\frac{\partial\phi}{\partial y}=y^2-zx\Rightarrow\phi=\frac{y^3}{3}-xyz+\ f_2(z,x)......(2)$$

$$\frac{\partial\phi}{\partial z}=z^2-xy\Rightarrow\phi=\frac{z^3}{3}-xyz+\ f_3(x,y)......(3)$$

From (1), (2),(3), $\phi=\dfrac{x^3+y^3+z^3}{3}-xyz$

$$\therefore\quad \phi=\frac{1}{3}(x^3+y^3+z^3)-xyz+cons\tan t$$

Which is the required scalar potential.

Example 7: Find constants a,b and c if the vector $\bar{f}$ = $(2x+3y+az)\bar{i}+(bx+2y+3z)\,\bar{j}+(2x+cy+3z)\,\bar{k}$ is Irrotational.

Sol:- Given $\bar{f}=(2x+3y+az)\bar{i}+(bx+2y+3z)\,\bar{j}+(2x+cy+3z)\,\bar{k}$

$$\text{Curl }\bar{f}=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ 2x+3y+az & bx+2y+3z & 2x+cy+3z\end{vmatrix}=$$

$$(c-3)\bar{i}+(2-a)\,\bar{j}+(b-3)\,\bar{k}$$

If the vector is Irrotational then curl $\bar{f}=\bar{0}$

c-3 = 2-a=0, b-3 = 0 $\Rightarrow$c=3, a=2, b=3.

Example 8: If f(r) is differentiable, show that curl { $\bar{r}$ f(r)} = $\bar{0}$ where $\bar{r} = x\bar{i}+y\bar{j}+z\bar{k}$.

Sol: r = $\bar{r}=\sqrt{x^2+y^2+z^2}$ $r^2 = x^2+y^2+z^2$

$$\Rightarrow 2r\,\frac{\partial r}{\partial x}=2x\Rightarrow\frac{\partial r}{\partial x}=\frac{x}{r},\ \text{similarly }\frac{\partial r}{\partial y}=\frac{y}{r},and\,\frac{\partial r}{\partial z}=\frac{z}{r}$$

curl{ $\bar{r}$ f(r)}= curl{f(r)($x\bar{i}+y\bar{j}+z\bar{k}$)}= curl $(x.f(r)\bar{i}+y.f(r)\,\bar{j}+z.f(r)\,\bar{k}$

$$=\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xf(r) & yf(r) & zf(r) \end{vmatrix} = \sum \bar{i}\left[\frac{\partial}{\partial y}[zf(r)] - \frac{\partial}{\partial z}[yf(r)]\right]$$

$$\sum \bar{i}\left[zf^1(r)\frac{\partial r}{\partial y} - yf^1(r)\frac{\partial r}{\partial z}\right] = \sum \bar{i}\left[zf^1(r)\frac{y}{r} - yf^1(r)\frac{z}{r}\right]$$

$= \bar{0}$.

Example 9: If $\bar{A}$ is Irrotational vector, evaluate div($\bar{A}$ x $\bar{r}$) where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$.

Sol: we have $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

Given $\bar{A}$ is an irrational vector

∇x$\bar{A} = \bar{0}$

div ($\bar{A}$ x $\bar{r}$) = ∇.($\bar{A}$ x $\bar{r}$)

= $\bar{r}$.(∇x$\bar{A}$)- $\bar{A}$.(∇x$\bar{r}$)

= $\bar{r}$.($\bar{0}$)- $\bar{A}$.(∇x$\bar{r}$) [using (1)]

= - $\bar{A}$.(∇x$\bar{r}$)…..(2)

Now $\nabla \text{x} \bar{r} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \bar{i}\left(\frac{\partial}{\partial y}z - \frac{\partial}{\partial z}y\right) - \bar{j}\left(\frac{\partial}{\partial x}z - \frac{\partial}{\partial z}x\right) + \bar{k}\left(\frac{\partial}{\partial x}y - \frac{\partial}{\partial y}x\right) = \bar{0}$

$\bar{A}$.(∇x$\bar{r}$)=0 …(3)

Hence div ($\bar{A}$ x $\bar{r}$)=0. [using (2) and (3)]

Example 10: Find constants a,b,c so that the vector $\bar{A}$ = $(x+2y+az)\bar{i} + (bx-3y-z)\,\bar{j} + (4x+cy+2z)\,\bar{k}$ is Irrotational. Also find ϕ such that $\bar{A} = \nabla\phi$.

Sol: Given vector is $\bar{A} = (x+2y+az)\bar{i} + (bx-3y-z)\,\bar{j} + (4x+cy+2z)\,\bar{k}$

Vector $\bar{A}$ is Irrotational $\Rightarrow$ curl $\bar{A} = \bar{0}$

$$\Rightarrow \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix} = \bar{0}$$

$\Rightarrow (c+1)\bar{i} + (a-4)\,\bar{j} + (b-2)\,\bar{k} = \bar{0}$

$\Rightarrow (c+1)\bar{i} + (a-4)\,\bar{j} + (b-2)\,\bar{k} = 0\bar{i} + 0\,\bar{j} + 0\,\bar{k}$

Comparing both sides,

c+1=0, a-4=0, b-2=0

c= -1, a=4,b=2

now $\bar{A} = (x+2y+az)\bar{i} + (bx-3y-z)\,\bar{j} + (4x+cy+2z)\,\bar{k}$, on substituting the values of a,b,c

we have $\bar{A} = \nabla\phi$.

$\Rightarrow \bar{A} = (x+2y+az)\bar{i} + (bx-3y-z)\,\bar{j} + (4x+cy+2z)\,\bar{k} = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$

Comparing both sides, we have

$\frac{\partial\phi}{\partial x}$ = x+2y+4z $\Rightarrow\phi$= $x^2/2+2xy+4zx+f_1(y,z)$

$\frac{\partial\phi}{\partial y}$ = 2x-3y-z $\Rightarrow\phi$= 2xy-3y2/2-yz+$f_2(z,x)$

$\frac{\partial\phi}{\partial z}$ = 4x-y+2z $\Rightarrow\phi$= 4xz-yz+z2+$f_3(x,y)$

Hence $\phi= x^2/2 - 3y^2/2 + z^2 + 2xy + 4zx - yz + c$

Example 11: If ω is a constant vector, evaluate curl V where V = ωx$\bar{r}$.

Sol: curl (ωx$\bar{r}$) = $\sum \bar{i}x\frac{\partial}{\partial x}(\varpi x\bar{r}) = \sum \bar{i}x\left[\frac{\partial\varpi}{\partial x}x\bar{r} + \varpi x\frac{\partial\bar{r}}{\partial x}\right]$

$$= \sum \bar{i}x[\bar{0} + \omega x\bar{i}] \quad [\therefore \bar{a}x(\bar{b}x\bar{c}) = (\bar{a}.\bar{c})\bar{b} - (\bar{a}.\bar{b}).\bar{c}]$$
$$= \sum \bar{i}x(\omega x\bar{i}) = \sum[(\bar{i}.\bar{i})\omega - (\bar{i}.\omega)\bar{i}] = \sum\omega - \sum(\bar{i}.\omega)\bar{i} = 3\omega - \omega = 2\omega$$

OPERATORS

Vector differential operator ∇

The operator ∇ = $\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}$ is defined such that ∇ϕ= $\bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$ where ϕ is a scalar point function.

Note: If ϕ is a scalar point function then ∇ϕ= grad ϕ= $\sum i\frac{\partial\phi}{\partial x}$

(2) Scalar differential operator $\bar{a}\,.\nabla$

The operator $\bar{a}\,.\nabla$ = $(\bar{a}.\bar{i})\frac{\partial\phi}{\partial x} + (\bar{a}.\bar{j})\frac{\partial\phi}{\partial y} + (\bar{a}.\bar{k})\frac{\partial\phi}{\partial z}$ is defined such that

$(\bar{a}.\nabla)\phi = (\bar{a}.\bar{i})\frac{\partial\phi}{\partial x} + (\bar{a}.\bar{j})\frac{\partial\phi}{\partial y} + (\bar{a}.\bar{k})\frac{\partial\phi}{\partial z}$

And $(\bar{a}.\nabla)\bar{f} = (\bar{a}.\bar{i})\frac{\partial\bar{f}}{\partial x} + (\bar{a}.\bar{j})\frac{\partial\bar{f}}{\partial y} + (\bar{a}.\bar{k})\frac{\partial\bar{f}}{\partial z}$

(3). Vector differential operator $\bar{a}\,x\nabla$

The operator $\bar{a}\,x\nabla = (\bar{a}x\bar{i})\frac{\partial}{\partial x} + (\bar{a}x\bar{j})\frac{\partial}{\partial y} + (\bar{a}x\bar{k})\frac{\partial}{\partial z}$ is defined such that

(i). $(\bar{a}\,x\nabla)\phi = (\bar{a}x\bar{i})\frac{\partial\phi}{\partial x} + (\bar{a}x\bar{j})\frac{\partial\phi}{\partial y} + (\bar{a}x\bar{k})\frac{\partial\phi}{\partial z}$

(ii). $(\bar{a}\,x\nabla).\bar{f} = (\bar{a}x\bar{i}).\frac{\partial\bar{f}}{\partial x} + (\bar{a}x\bar{j}).\frac{\partial\bar{f}}{\partial y} + (\bar{a}x\bar{k}).\frac{\partial\bar{f}}{\partial z}$

(iii). $(\bar{a}\,x\nabla)x\,\bar{f} = (\bar{a}x\bar{i})x\frac{\partial\bar{f}}{\partial x} + (\bar{a}x\bar{j})x\frac{\partial\bar{f}}{\partial y} + (\bar{a}x\bar{k})x\frac{\partial\bar{f}}{\partial z}$

(4). Scalar differential operator ∇.

The operator $\nabla = \bar{i}.\frac{\partial}{\partial x} + \bar{j}.\frac{\partial}{\partial y} + \bar{k}.\frac{\partial}{\partial z}$ is defined such that $\nabla.\bar{f} = \bar{i}.\frac{\partial\bar{f}}{\partial x} + \bar{j}.\frac{\partial\bar{f}}{\partial y} + \bar{k}.\frac{\partial\bar{f}}{\partial z}$

Note: $\nabla.\bar{f}$ is defined as div $\bar{f}$ it is a scalar point function.

(5). Vector differential operator ∇ x

The operator ∇ x = $\bar{i}x\frac{\partial}{\partial x} + \bar{j}x\frac{\partial}{\partial y} + \bar{k}x\frac{\partial}{\partial z}$ is defined such that

$$\nabla x\,\bar{f} = \bar{i}x\frac{\partial\bar{f}}{\partial x} + \bar{j}x\frac{\partial\bar{f}}{\partial y} + \bar{k}x\frac{\partial\bar{f}}{\partial z}$$

Note : $\nabla x\,\bar{f}$ is defined as curl $\bar{f}$. It is a vector point function.

(6). Laplacian Operator ∇^2

$$\nabla.\nabla\phi = \sum\bar{i}.\frac{\partial}{\partial x}\left(\bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}\right) = \sum\frac{\partial^2\phi}{\partial x^2} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi = \nabla^2\phi$$

Thus the operator $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is called Laplacian operator.

Note : (i). $\nabla^2\phi = \nabla.(\nabla\phi)$ = div(grad ϕ)

(ii). if $\nabla^2\phi=0$ then ϕ is said to satisfy Laplacian equation. This ϕ is called a harmonic function.

VECTOR IDENTITIES

Theorem 1: If $\bar{a}$ is a differentiable function and ϕ is a differentiable scalar function. Then prove that div($\phi\bar{a}$)= (grad ϕ).$\bar{a}$ +ϕ div $\bar{a}$ or $\nabla.(\phi\bar{a}) = (\nabla\phi)\bar{a} + \phi(\nabla.\bar{a})$

Proof: div($\phi\bar{a}$)=$\nabla.(\phi\bar{a}) = \sum i\frac{\partial}{\partial x}(\phi\bar{a})$

$$=\sum \bar{i}.\left(\frac{\partial \phi}{\partial x}\bar{a}+\phi\frac{\partial \bar{a}}{\partial x}\right)=\sum\left(i\frac{\partial \phi}{\partial x}\bar{a}\right)+\sum\left(i\frac{\partial \bar{a}}{\partial x}\right)\phi$$

$$=\sum\left(\bar{i}\frac{\partial \phi}{\partial x}\right).\bar{a}+\left(\sum \bar{i}\frac{\partial a}{\partial x}\right)\phi=(\nabla\phi)\,\bar{a}+\phi(\nabla.\bar{a})$$

Theorem 2: prove that curl $(\phi\,\bar{a})=(\text{grad }\phi)\text{x}\,\bar{a}+\phi$ curl $\bar{a}$

Proof : curl $(\phi\,\bar{a})=\nabla\text{x}(\phi\,\bar{a})=\sum ix\frac{\partial}{\partial x}(\phi\bar{a})$

$$=\sum \bar{i}x\left(\frac{\partial \phi}{\partial x}\bar{a}+\phi\frac{\partial \bar{a}}{\partial x}\right)=\sum\left(i\frac{\partial \phi}{\partial x}\right)x\bar{a}+\sum\left(ix\frac{\partial \bar{a}}{\partial x}\right)\phi$$

$$=\nabla\phi\text{x}\,\bar{a}+(\nabla\text{x}\,\bar{a})\phi=(\text{grad }\phi)\text{x}\,\bar{a}+\phi\text{ curl }\bar{a}$$

Theorem 3: Prove that grad $(\bar{a}\,.\,\bar{b})=(\bar{b}.\nabla)\bar{a}+(\bar{a}.\nabla)\bar{b}+\bar{b}x\,curl\,\bar{a}+\bar{a}x\,curl\,\bar{b}$

Proof: Consider

$$\bar{a}x\,curl(\bar{b})=\bar{a}x(\nabla x\bar{b})=\bar{a}x\sum \bar{i}x\left(\bar{i}x\frac{\partial \bar{b}}{\partial x}\right)$$

$$=\sum \bar{a}x\left(\bar{i}x\frac{\partial \bar{b}}{\partial x}\right)$$

$$=\sum\left\{\left(\bar{a}.\frac{\partial \bar{b}}{\partial x}\right)\bar{i}-(\bar{a}.\bar{i})\frac{\partial \bar{b}}{\partial x}\right\}=\sum \bar{i}\left\{\bar{a}.\frac{\partial \bar{b}}{\partial x}\right\}-\left\{\bar{a}.\sum i\frac{\partial}{\partial x}\right\}\bar{b}$$

$$\therefore \bar{a}x\,curl\,\bar{b}=\sum \bar{i}\left(\bar{a}.\frac{\partial \bar{b}}{\partial x}\right)-(\bar{a}.\nabla)\bar{b}\ \ldots.(1)$$

Similarly , $\bar{b}x\,curl\,\bar{b}=\sum \bar{i}\left(\bar{a}.\frac{\partial \bar{a}}{\partial x}\right)-(\bar{b}.\nabla)\bar{a}\ \ldots.(2)$

(1)+(2) gives

$$\bar{a}x\,curl\,\bar{b}+\bar{b}xcurl\,\bar{a}=\sum \bar{i}\left(\bar{a}.\frac{\partial \bar{b}}{\partial x}\right)-(\bar{a}.\nabla)\bar{b}+\sum \bar{i}\left(\bar{b}.\frac{\partial \bar{a}}{\partial x}\right)-(\bar{b}.\nabla)\bar{a}$$

$$\bar{a}x\,curl\,\bar{b}+\bar{b}xcurl\,\bar{a}+(\bar{a}.\nabla)\bar{b}+(\bar{b}.\nabla)\bar{a}=\sum \bar{i}\left(\bar{a}.\frac{\partial \bar{b}}{\partial x}+\bar{b}.\frac{\partial \bar{a}}{\partial x}\right)$$

$$=\sum \bar{i}\frac{\partial}{\partial x}(\bar{a}.\bar{b})$$

$$= \nabla(\bar{a}\,.\bar{b}) = \text{grad}\,(\bar{a}\,.\bar{b})$$

Theorem 4: Prove that div ($\bar{a} \times \bar{b}$)= $\bar{b}.curl\,\bar{a} - \bar{a}.curl\,\bar{b}$

Proof: div ($\bar{a} \times \bar{b}$) $= \sum \bar{i}\,\frac{\partial}{\partial x}(\bar{a}x\bar{b}) = \sum \bar{i}.\left(\frac{\partial \bar{a}}{\partial x}x\bar{b} + \bar{a}x\frac{\partial \bar{b}}{\partial x}\right)$

$$= \sum \bar{i}.\left(\frac{\partial \bar{a}}{\partial x}x\bar{b}\right) + \sum \bar{i}.\left(\bar{a}x\frac{\partial \bar{b}}{\partial x}\right) = \sum\left(\bar{i}x\frac{\partial \bar{a}}{\partial x}\right).\bar{b} - \sum\left(\bar{i}x\frac{\partial \bar{b}}{\partial x}\right).\bar{a}$$

$$= (\nabla x\bar{a}).\bar{b} - (\nabla x\bar{b}).\bar{a} = \bar{b}.curl\,\bar{a} - \bar{a}.curl\,\bar{b}$$

Theorem 5 : $curl\,(\bar{a}x\bar{b}) = \bar{a}\,div\,\bar{b} - \bar{b}\,div\,\bar{a} + (\bar{b}.\nabla)\bar{a} - (\bar{a}.\nabla)\bar{b}$

$Proof\,:\,curl\,(\bar{a}x\bar{b}) = \sum \bar{i}x\frac{\partial}{\partial x}(\bar{a}x\bar{b}) = \sum \bar{i}x\left[\frac{\partial \bar{a}}{\partial x}x\bar{b} + \bar{a}x\frac{\partial \bar{b}}{\partial x}\right]$

$$\sum \bar{i}x\left(\frac{\partial \bar{a}}{\partial x}x\bar{b}\right) + \sum \bar{i}x\left(\bar{a}x\frac{\partial \bar{b}}{\partial x}\right)$$

$$= \sum\left\{(\bar{i}.\bar{b})\frac{\partial \bar{a}}{\partial x} - \left(\bar{i}.\frac{\partial \bar{a}}{\partial x}\right)\bar{b}\right\} + \sum\left\{\left(\bar{i}.\frac{\partial \bar{b}}{\partial x}\right)\bar{a} - (\bar{i}.\bar{a})\frac{\partial \bar{b}}{\partial x} - \right\} +$$

$$= \sum(\bar{b}.\bar{i})\frac{\partial \bar{a}}{\partial x} - \sum\left(\bar{i}.\frac{\partial \bar{a}}{\partial x}\right)\bar{b} + \sum\left(\bar{i}.\frac{\partial \bar{b}}{\partial x}\right)\bar{a} - \left(.\bar{a}\sum \bar{i}.\frac{\partial}{\partial x}\right)\bar{b}$$

$$= (\bar{b}.\nabla)\bar{a} - (\bar{a}.\nabla)\bar{b} + (\nabla.\bar{b})\bar{a} - (\bar{a}.\nabla)\bar{b}$$

$$= (\nabla.\bar{b})\bar{a} - (\nabla.\bar{a})\bar{b} + (\bar{b}.\nabla)\bar{a} - (\bar{a}.\nabla)\bar{b}$$

$$= \bar{a}\,div\,\bar{b} - \bar{b}\,div\,\bar{a} + (\bar{b}.\nabla)\bar{a} - (\bar{a}.\nabla)\bar{b}$$

Theorem 6: Prove that curl grad $\phi = 0$.

Proof: Let ϕ be any scalar point function. Then

$$.grad\;\phi = \bar{i}\frac{\partial \phi}{\partial x} + \bar{j}\frac{\partial \phi}{\partial y} + \bar{k}\frac{\partial \phi}{\partial z}$$

$$curl\;(\text{grad}\;\Phi) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ \frac{d\phi}{dx} & \frac{d\phi}{dy} & \frac{d\phi}{dz} \end{vmatrix}$$

$$\bar{i}\left(\frac{\partial^2\phi}{\partial y\partial z}-\frac{\partial^2\phi}{\partial z\partial y}\right)-\bar{j}\left(\frac{\partial^2\phi}{\partial x\partial z}-\frac{\partial^2\phi}{\partial z\partial x}\right)-\bar{k}\left(\frac{\partial^2\phi}{\partial x\partial y}-\frac{\partial^2\phi}{\partial y\partial x}\right)=\bar{0}$$

note: Since crul (grad ϕ)= 0, we have grad ϕ is always Irrotational.

Theorem 7: Prove that div crul *f=0*

$\Pr oof\ : Let\ \bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$

$$\therefore\ curl\ \bar{f}\ \nabla x\bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$=\left(\frac{\partial f_3}{\partial y}-\frac{\partial f_2}{\partial z}\right)\bar{i}-\left(\frac{\partial f_3}{\partial x}-\frac{\partial f_2}{\partial z}\right)\bar{j}+\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial y}\right)\bar{k}$$

$$\therefore\quad div\ \ curl\ \bar{f} = \nabla.(\nabla x\bar{f}) = \frac{\partial}{\partial x}\left(\frac{\partial f_3}{\partial y}-\frac{\partial f_2}{\partial z}\right)-\frac{\partial}{\partial y}\left(\frac{\partial f_3}{\partial x}-\frac{\partial f_2}{\partial z}\right)+\frac{\partial}{\partial z}\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial y}\right)$$

$$=\frac{\partial^2 f_3}{\partial x\partial y}-\frac{\partial^2 f_2}{\partial x\partial z}-\frac{\partial^2 f_3}{\partial y\partial x}+\frac{\partial^2 f_1}{\partial y\partial z}+\frac{\partial^2 f_2}{\partial z\partial x}-\frac{\partial^2 f_1}{\partial z\partial y}=0$$

Theorem 8: If *f* and g are two scalar point functions, prove that *div*(*f*∇g)= *f*∇^2g+∇*f*. ∇g

Sol: Let *f* and g are two scalar point functions. Then

$$\nabla g = \bar{i}\,\frac{\partial g}{\partial x} + \bar{j}\,\frac{\partial g}{\partial y} + \bar{k}\,\frac{\partial g}{\partial z}$$

Now $$f\nabla \mathrm{g} = \bar{i}f\,\frac{\partial g}{\partial x} + \bar{j}f\,\frac{\partial g}{\partial y} + \bar{k}f\,\frac{\partial g}{\partial z}$$

$$\nabla.(f\nabla \mathrm{g}) = \frac{\partial}{\partial x}\left(f\,\frac{\partial g}{\partial x}\right)+\frac{\partial}{\partial y}\left(f\,\frac{\partial g}{\partial y}\right)+\frac{\partial}{\partial z}\left(f\,\frac{\partial g}{\partial z}\right)$$

$$= f\left(\frac{\partial^2 g}{\partial x^2}+\frac{\partial^2 g}{\partial y^2}+\frac{\partial^2 g}{\partial x^2}\right)+\left(\frac{\partial f}{\partial x}.\frac{\partial g}{\partial x}+\frac{\partial f}{\partial y}.\frac{\partial g}{\partial y}+\frac{\partial f}{\partial z}.\frac{\partial g}{\partial z}\right)$$

$$=f\nabla^2 g+\left(\bar{i}\frac{\partial f}{\partial x}+\bar{j}\frac{\partial f}{\partial y}+\bar{k}\frac{\partial f}{\partial z}\right).\left(\bar{i}\frac{\partial g}{\partial x}+\bar{j}\frac{\partial g}{\partial y}+\bar{k}\frac{\partial g}{\partial z}\right).$$

$$= f\nabla^2 g+\nabla f.\nabla g$$

Theorem 9: Prove that $\nabla x(\nabla x\,\bar{a}) = \nabla(\nabla.\bar{a}) - \nabla^2\bar{a}$.

Proof: $\nabla x(\nabla x\,\bar{a}) = \sum \bar{i}\frac{\partial}{\partial x}(\nabla x\bar{a})$

Now $\bar{i}x\frac{\partial}{\partial x}(\nabla x\bar{a}) = \bar{i}x\frac{\partial}{\partial x}\left(\bar{i}x\frac{\partial\bar{a}}{\partial x}+\bar{j}x\frac{\partial\bar{a}}{\partial y}+\bar{k}x\frac{\partial\bar{a}}{\partial z}\right)$

$$=\bar{i}x\left(\bar{i}x\frac{\partial^2\bar{a}}{\partial x^2}+\bar{j}x\frac{\partial^2\bar{a}}{\partial y^2}+\bar{k}x\frac{\partial^2\bar{a}}{\partial z^2}\right)$$

$$=\bar{i}x\left(\bar{i}x\frac{\partial^2\bar{a}}{\partial x^2}\right)+\bar{i}x\left(\bar{j}x\frac{\partial^2\bar{a}}{\partial x\partial y}\right)+\bar{i}x\left(\bar{k}x\frac{\partial^2\bar{a}}{\partial x\partial z}\right)$$

$$=\left(\bar{i}.\frac{\partial^2\bar{a}}{\partial x^2}\right)\bar{i}-\frac{\partial^2\bar{a}}{\partial x^2}+\left(\bar{i}.\frac{\partial^2\bar{a}}{\partial x\partial y}\right)\bar{j}+\left(\bar{i}.\frac{\partial^2\bar{a}}{\partial x\partial z}\right)\bar{k} \qquad [\because i.i=1, i.j=i.k=0]$$

$$= \bar{i}\frac{\partial}{\partial x}\left(\bar{i}.\frac{\partial\bar{a}}{\partial x}\right)+j\frac{\partial}{\partial y}\left(\bar{i}.\frac{\partial\bar{a}}{\partial y}\right)+k\frac{\partial}{\partial z}\left(\bar{i}.\frac{\partial\bar{a}}{\partial z}\right)-\frac{\partial^2\bar{a}}{\partial x^2}=\nabla\left(\bar{i}.\frac{\partial\bar{a}}{\partial x}\right)-\frac{\partial^2\bar{a}}{\partial x^2}$$

$$=\sum\bar{i}x\frac{\partial}{\partial x}(\nabla x\bar{a})=\sum\bar{i}.\frac{\partial\bar{a}}{\partial x}-\sum\frac{\partial^2\bar{a}}{\partial x^2}=\nabla(\nabla.\bar{a})-\left(\frac{\partial^2\bar{a}}{\partial x^2}+\frac{\partial^2\bar{a}}{\partial y^2}+\frac{\partial^2\bar{a}}{\partial z^2}\right)$$

$$=\nabla x(\nabla x\,\bar{a}) = \nabla(\nabla.\bar{a}) - \nabla^2\bar{a}$$

UNIT-V
VECTOR INTEGRATION

1. Line integral:- (i) $\int_c \bar{F}.d\bar{r}$ is called Line integral of $\bar{F}$ along c

Note : Work done by $\bar{F}$ along a curve c is $\int_c \bar{F}.d\bar{r}$

Example : If $\bar{F}$ $(x^2-27)\bar{i}$ $-6yz\bar{j}+8xz^2\bar{k}$, evaluate $\int \bar{F}.d\bar{r}$ from the point (0,0,0) to the point (1,1,1) along the Straight line from (0,0,0) to (1,0,0), (1,0,0) to (1,1,0) and (1,1,0) to (1,1,1).

Solution : Given $\bar{F} = (x^2-27)\bar{i}$ $-6yz\bar{j}+8xz^2\bar{k}$

Now $\bar{r} = x\bar{i}+y\bar{j}+z\bar{k} \Rightarrow d\bar{r}\ dx\bar{i}+dy\bar{j}+dz\bar{k}$

$\therefore \quad \bar{F}.d\bar{r} = (x^2-27)dx - (6yz)dy + 8xz^2dz$

(i) Along the straight line from O = (0,0,0) to A = (1,0,0)

Here y =0 =z and dy=dz=0. Also x changes from 0 to 1.

$$\int_{OA} \bar{F}.d\bar{r} = \int_o^1 (x^2-27)dx = \left[\frac{x^3}{3}-27x\right]_0^1 = \frac{1}{3}-27 = \frac{-80}{3}$$

(ii) Along the straight line from A = (1,0,0) to B = (1,1,0)

Here x =1, z=0 $\Rightarrow$ dx=0, dz=0. y changes from 0 to 1.

$$\int_{AB} \bar{F}.d\bar{r} = \int_{y=0}^1 (-6yz)dy = 0$$

(iii) Along the straight line from B = (1,1,0) to C = (1,1,1)

x =1 =y $\Rightarrow$ dx=dy=0 and z changes from 0 to 1.

$$\int_{BC} \bar{F}.d\bar{r} = \int_{z=0}^1 8xz^2dz = \int_{z=0}^1 8xz^2dz = \left[\frac{8z^3}{3}-\right]_0^1 = \frac{8}{3}$$

$$(i)+(ii)+(iii) \Rightarrow \int_C \bar{F}.d\bar{r} = \frac{88}{3}$$

Example : If $\bar{F}$ (5xy-6x^2) $\bar{i}$ (2y-4x) $\bar{j}$, evaluate $\int_C \bar{F}.d\bar{r}$ along the curve C in xy plane y=x^{3}from (1,1) to (2,8).

Solution : Given $\bar{F}$ (5xy-6x^2) $\bar{i}$ (2y-4x) $\bar{j}$,

Along the curve y=x^2, dy =3x^2 dx

$\therefore$ $\bar{F}$ (5x^4-6x^2) $\bar{i}$ +(2x^3-4x) $\bar{j}$, [Putting y=x^3 in (1)]

$d\bar{r} =$ dx$\bar{i}$ + dy$\bar{j}$ + dx$\bar{i}$ +3x^{2}dx

$\therefore$ $\bar{F}.d\bar{r}$ = [(5x^4-6x^2) $\bar{i}$ +(2x^3-4x) $\bar{j}$]. dx$\bar{i}$ +3x^{2}dx $\bar{j}$

= (5x^4 – 6x^2) dx+(2x^3 – 4x)3x^{3}dx

= (6x^5+5x^4-12x^3 -6x^2)dx

Hence $\int_{y=x^3} \bar{F}.d\bar{r} = \int_1^2 (6x^5+5x^4-12x^3-6x^2)dx$

$$= \left(6.\frac{x^6}{6}+5.\frac{x^3}{5}-12.\frac{x^4}{4}-6.\frac{x^3}{4}\right) = \left(x^6+x^3-3x^4-2x3\right)_1^2$$

= 16(4+2-31) – (1+1-3-2) = 32+3 = 35

Example : Find the work done by the force $\bar{F} = z\bar{i}+x\bar{j}+y\bar{k}$, when it moves a particle along the arc of the curve $\bar{r}$ = cost $\bar{i}$ + sint $\bar{j}$ -t $\bar{k}$ from t = 0 to t = 2π

Solution : Given force $\bar{F}$ = zi = xj +yk and the arc is $\bar{r}$ = cost $\bar{i}$ + sin t $\bar{j}$ -t

i.e., x = cost, y= sin t, z = -t

$d\bar{r}$ = (-sin t $\bar{i}$ +cost $\bar{j}$ - $\bar{k}$)dt

$\bar{F}.d\bar{r}$ = (-t $\bar{i}$ +cost $\bar{j}$ +sin t $\bar{k}$). (-sin t $\bar{i}$ + cost $\bar{j}$ - $\bar{k}$)dt = (t sin t + cos^2 t – sin t)dt

Hence work done = $\int_0^{2\pi} \bar{F}.d\bar{r} = \int_0^{2\pi}$ (t sin t + cos^2 t – sin t) dt

$$= [t(-\cos t)]_0^{2\pi} - \int_0^{2\pi}(-\sin t)dt +) + \int_0^{2\pi}\frac{1+\cos 2t}{2}dt - \int_0^{2\pi}\sin t\,dt$$

$$= -2\pi-(\cos t)_0^{2\pi}+\frac{1}{2}\left(t+\frac{\sin 2t}{2}\right)_0^{2\pi}+(\cos t)_0^{2\pi}$$

$$= -2\pi-(1-1)+\frac{1}{2}(2\pi)+(1-1) = -2\pi+\pi = -\pi$$

Surface integral: $\int_c \bar{F}.\bar{n}\,ds$ is called surface integral

Problem 1 : Evaluate $\int \bar{F}.ndS$ where $\bar{F}$ = zi + xj − 3y²zk and S is the surface $x^2 + y^2 = 16$ included in the first octant between z = 0 and z = 5.

Sol. The surface S is $x^2 + y^2 = 16$ included in the first octant between z = 0 and z = 5.

Let □ = $x^2 + y^2 = 16$

Then $\nabla\phi = i\dfrac{\partial\phi}{\partial x} + j\dfrac{\partial\phi}{\partial x} + k\dfrac{\partial\phi}{\partial x} = 2xi + 2yj$

□ unit normal $\bar{n} = \dfrac{\nabla\phi}{|\nabla\varphi|} = \dfrac{xi + yj}{4}$ $(\because\ x^2 + y^2 = 16)$

Let R be the projection of S on yz plane

Then $\int_S \bar{F}.ndS = \iint_R \bar{F}.\bar{n}\dfrac{dydz}{|\bar{n}\ .\ \bar{i}|}$ ……………. *

Given $\bar{F}$ = zi + xj − $3y^2zk$

□ $\bar{F}\ .\ \bar{n} = \dfrac{1}{4}(xz + xy)$

and $\bar{n}\ .\ \bar{i} = \dfrac{x}{4}$

In yz plane, x = 0, y = 4

In first octant, y varies from 0 to 4 and z varies from 0 to 5.

$$\int_S \bar{F}.ndS = \int_{y=0}^{4}\int_{z=0}^{5}\left(\frac{xz + xy}{4}\right)\frac{dydz}{\left|\frac{x}{4}\right|}$$

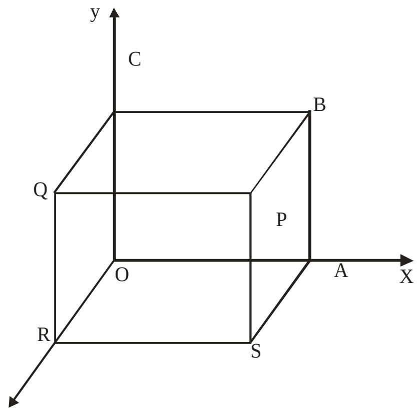
$$= \int_{y=0}^{4}\left(\int_{z=0}^{5}(y + dz)dz\right)dy$$

$$= 90.$$

Problem 2 : If $\bar{F}$ = zi + xj − $3y^2zk$, evaluate $\int_S \bar{F}.ndS$ where S is the surface of the cube bounded by x = 0, x = a, y = 0, y= a, z = 0, z = a.

Sol. Given that S is the surface of the $x = 0$, $x = a$, $y = 0$, $y = a$, $z = 0$, $z = a$, and $\bar{F} = zi + xj - 3y^2zk$

we need to evaluate $\int_S \bar{F}.ndS$.



(i) For OABC

Eqn is $z = 0$ and $dS = dxdy$

$$\bar{n} = -\bar{k}$$

$$\int_{S_1} \bar{F}.ndS = -\int_{x=0}^{a} -\int_{y=0}^{a} (yz)\,dxdy = 0$$

(ii) For PQRS

Eqn is $z = a$ and $dS = dxdy$

$$\bar{n} = \bar{k}$$

$$\int_{S_2} \bar{F}.ndS = \int_{x=0}^{a} \left(\int_{y=0}^{a} y(a)dy \right) dx = \frac{a^4}{2}$$

(iii) For OCQR

Eqn is $x = 0$, and $\bar{n} = -\bar{i}$, $dS = dydz$

$$\int_{S_3} \bar{F}.ndS = \int_{y=0}^{a} \int_{z=0}^{a} 4xzdydz = 0$$

(iv) For ABPS

Eqn is $x = a$, and $\bar{n} = -\bar{i}$, $dS = dydz$

$$\int_{S_3} \bar{F}.\bar{n}dS = \int_{y=0}^{a} \left(\int_{z=0}^{a} 4azdz \right) dy = 2a^4$$

(v) For OASR

Eqn is $y = 0$, and $\bar{n} = -\bar{j}$, $dS = dxdz$

$$\int_{S_5} \bar{F}.ndS = \int_{y=0}^{a} \int_{z=0}^{a} y^2dzdx = 0$$

(vi) For PBCQ

Eqn is $y = a$, and $\bar{n} = -\bar{j}$, $dS = dxdz$

$$\int_{S_6} \bar{F}.\bar{n}dS = -\int_{y=0}^{a} \int_{z=0}^{a} y^2dzdx = 0$$

From (i) – (vi) we get

$$\int_{S_6} \bar{F}.\bar{n}dS = 0 + \frac{a^4}{2} + 0 + 2a^4 + 0 - a4 = \frac{3a^4}{2}$$

3. VOLUME INTEGRALS

Let V be the volume bounded by a surface $\bar{r} = \bar{f}$ (u,v). Let $\bar{F}(\bar{r})$ be a vector point function define over V. Divide V into m sub-regions of volumes $\delta V_1, \delta V_2, \ldots.\delta V_p \ldots.\delta V_m$

Let P_i ($\bar{r}_i$) be a point in δV_r then form the sum $I_m = \sum_{i=1}^{m} \bar{F}(r1)\delta V_1$. Let m $\to\infty$ in such a way that δV_1

shrinks to a point,. The limit of I_m if it exists, is called the volume integral of $\bar{F}(\bar{r})$ in the region V is

denoted by $\int_V \bar{F}(\bar{r})\,dv$ or $\int_V \bar{F}dv$.

Cartesian form : Let $\bar{F} = (r)i = F_1\bar{i} + F_2\bar{i} + F_3\bar{k}$ where F_1, F_2, F_3 are functions of x,y,z. We know that dv = dx dy dz. The volume integral given by

$$\int_v \bar{F}dv = \iiint F_1 \ (F_1\bar{i} + F_2\bar{i} + F_3\bar{k})\text{ dx dy dz} = \bar{i} = \iiint F_1\text{dxdydz} + \bar{j} = \iiint F_2\text{dxdydz} + \bar{k} = \iiint F_3$$

Example 2 : If $\overline{F} = (2x^2 - 3z)\overline{i} - 2xy\overline{j} - 4x\overline{k}$ then evaluate (*i*) $\int_V \nabla.\overline{F}\,dv$ and (*ii*) $\int_V \nabla \times \overline{F}dv$

; V is the closed region bounded by $x = 0$, $y = 0$, $z = 0$, $2x + 2y + z = 4$.

Solution : (*i*) $\nabla \cdot \overline{F} = \overline{i} \cdot \frac{\partial \overline{F}}{\partial x} + \overline{j} \cdot \frac{\partial \overline{F}}{\partial y} + \overline{k} \cdot \frac{\partial \overline{F}}{\partial z} = 4x - 2x = 2x.$

The limits are : $z = 0$ to $z = 4 - 2x - 2y$, $y = 0$ to $\frac{4-2x}{2}$ (*i.e.*) $2 - x$ and $x = 0$ to $\frac{4}{2}$ (*i.e.*) 2

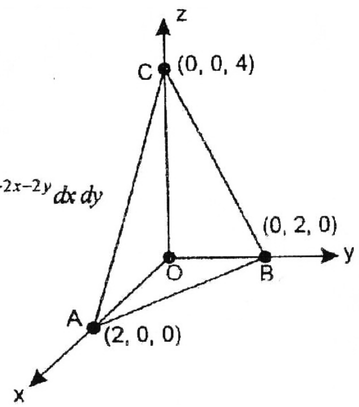
$$\therefore \int_V \nabla.\overline{F}dv = \int_{x=0}^{2}\int_{y=0}^{2-x}\int_{z=0}^{4-2x-2y} 2x\,dx\,dy\,dz = \int_{x=0}^{2}\int_{y=0}^{2-x}(2x)(z)_0^{4-2x-2y}dx\,dy$$

$$= \int_{x=0}^{2}\int_{y=0}^{2-x} 2x(4-2x-2y)dx\,dy = 4\int_{x=0}^{2}\int_{y=0}^{2-x}(2x - x^2 - xy)dx\,dy$$

$$= 4\int_0^2\left(2xy - x^2y - \frac{xy^2}{2}\right)_0^{2-x} dx = 4\int_0^2\left[(2x-x^2)(2-x) - \frac{x}{2}(2-x)^2\right]dx$$

$$= \int_0^2 (2x^3 - 8x^2 + 8x)dx = \left[\frac{x^4}{2} - \frac{8x^3}{2} + 4x^2\right]_0^2 = \frac{8}{3}$$

$$(ii)\quad \nabla \times \overline{F} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x^2 - 3z & -2xy & -4x \end{vmatrix} = \overline{j} - 2y\overline{k}$$



$$\therefore \int_V \nabla \times \overline{F}\,dv = \iiint_V (\overline{j} - 2y\overline{k})dx\,dy\,dz = \int_{x=0}^{2}\int_{y=0}^{2-x}(\overline{j} - 2y\overline{k})(z)_0^{4-2x-2y}dx\,dy$$

$$= \int_{x=0}^{2}\int_{y=0}^{2-x}(\overline{j} - 2y\overline{k})(4 - 2x - 2y)dx\,dy$$

$$= \int_{x=0}^{2}\int_{y=0}^{2-x}\left\{\overline{j}\left[(4-2x) - 2y\right] - \overline{k}\left[(4-2x)\cdot 2y - 4y^2\right]\right\}dx\,dy$$

$$= \int_{x=0}^{2}\overline{j}\left[(4-2x)y - y^2\right]_0^{2-x}dx - \overline{k}\int_{x=0}^{2}\left[(4-2x)y^2 - \frac{4y^3}{3}\right]_0^{2-x}dx$$

$$= \bar{j}\int_0^2 (2-x)^2 dx - \bar{k}\int_0^2 \frac{2}{3}(2-x)^3 dx$$

$$= \bar{j}\left[\frac{(2-x)^3}{-3}\right]_0^2 - \frac{2\bar{k}}{3}\left[\frac{(2-x)^4}{-4}\right]_0^2 = \frac{8}{3}(\bar{j}-\bar{k})$$

EXERCISE 12.3

(1) Evaluate $\iiint (2x+y)dv$ where V is the closed region bounded by the cylinder $z=4-x^2$, and planes $x=0$, $y=0$, $y=2$, and $z=0$.

(2) If $\phi = 45x^2y$ evaluate $\iiint_V \phi dv$ where V is the closed region bounded by the planes $4x+2y+z=8, y=0, z=0$.

(3) Evaluate $\int_V \bar{F} dv$ when $\bar{F} = x\bar{i} + y\bar{j} + z\bar{k}$ and V is the region bounded by $x=0, y=0, y=6, z=4, z=x^2$.

ANSWERS

(1) $\frac{80}{3}$ (2) 128 (3) $24\bar{i} + 96\bar{j} + \frac{384}{5}\bar{k}$

2. Vector Integral Theorems

Introduction

In this chapter we discuss three important vector integral theorems: (i) Gauss divergence theorem, (ii) Green's theorem in plane and (iii) Stokes theorem. These theorems deal with conversion of

(i) $\int_S \bar{F}.\bar{n}$ ds into a volume integral where S is a closed surface.

(ii) $\int_C \bar{F}.d\bar{r}$ into a double integral over a region in a plane when C is a closed curve in the plane and.

(iii) $\int_S (\nabla x \bar{A}).\bar{n}$ ds into a line integral around the boundary of an open two sided surface.

In solid mechanics, fluid mechanics, quantum mechanics, electrical engineering and various other fields, these theorems will be of great use. Evaluation of an integral of one type may be difficult and using one of the appropriate theorems we may be able to evaluate to the equivalent integral easily. Hence readers are advised to grasp the significance in each case.

I. GAUSS'S DIVERGENCE THEOREM

(Transformation between surface integral and volume integral)

Let S be a closed surface enclosing a volume v. if $\bar{F}$ is a continuously differentiable vector point function, then

$$\int_V div\,\bar{F} dv = \int_s \bar{F}.\bar{n}\ dS$$

When $\bar{n}$ is the outward drawn normal vector at any point of S.

Example : Verify Gauss Divergence theorem for $\bar{F} = (x^3 - yz)\bar{\imath} - 2x^2y\bar{\jmath} + z\bar{k}$ taken over the surface of the cube bounded by the planes x = y = z = a and coordinate planes.

Sol: By Gauss Divergence theorem we have

$$\int \bar{F}\,.\,\bar{n}dS = \int_v div\ \bar{F}\ dv$$

$$RHS = \int_0^a\int_0^a\int_0^a (3x^2 - 2x^2 + 1)dx\ dy\ dz = \int_0^a\int_0^a\int_0^a (x^2+1)\,dx\ dy\ dz = \int_0^a\int_0^a \left(\frac{x^3}{3} + x\right)_0^a dy\ dz$$

$$= \int_0^a\int_0^a \left[\frac{a^3}{3} + a\right] dy\ dz = \int_0^a \left[\frac{a^3}{3} + a\right] (y)_0^a\ dz = \left(\frac{a^3}{3} + a\right) a \int_0^a dz = \left(\frac{a^3}{3} + a\right)(a^2)$$

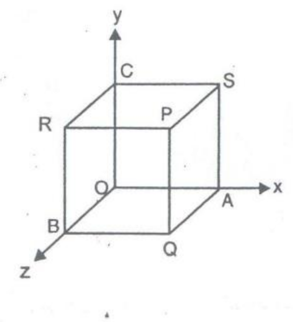
$$= \frac{a^3}{3} + a^3 \ldots (1)$$

Verification: We will calculate the value of $\int_S F.\bar{n}dS$ *over the six faces of the cube.*

(i) For S_1 = PQAS; unit outward drawn normal $\bar{n} = \bar{\imath}$

x=a; ds=dy dz; 0≤y≤a, 0≤z≤a

$\bar{F}.\bar{n} = x^3 - yz = a^3 - yz$ since $x = a$



$$\int_{S_1}\int \bar{F}.\bar{n}dS = \int_{z=0}^{a}\int_{y=0}^{a} (a^3 - yz)dy\,dz$$

$$= \int_{z=0}^{a} \left[a^3y - \frac{y^2}{2}z\right]_{y=0}^{a} dz$$

$$= \int_{z=0}^{a} \left(a^4 - \frac{a^2}{2}z\right) dz$$

$$= a^5 - \frac{a^4}{4} \ldots (2)$$

(ii) For S_2 = OCRB; unit outward drawn normal $\bar{n} = -\bar{\imath}$

x=a; ds=dy dz; 0≤y≤a, y≤z≤a

$\bar{F}.\bar{n} = -(x^3 - yz) = yz \ since\ x = 0$

$$\int\int_{S_2} \bar{F}.\bar{n}dS = \int_{z=0}^{a}\int_{y=0}^{a} yz\,dy\,dz = \int_{z=0}^{a}\left[\frac{y^2}{2}\right]_{y=0}^{a} zdz$$

$$= \frac{a^2}{2}\int_{z=0}^{a} zdz = \frac{a^4}{4} \dots (3)$$

(iii) For S_3 = RBQP; Z = a; ds = dxdy; $\bar{n} = \bar{k}$

0≤x≤a, 0≤y≤a

$\bar{F}.\bar{n} = z = a \ \ since\ z = a$

$$\int\int_{S_3} \bar{F}.\bar{n}dS = \int_{y=0}^{a}\int_{x=0}^{a} a\,dx\,dy = a^3 \dots (4)$$

(iv) For S_4 = OASC; z = 0; $\bar{n} = -\bar{k}$, ds = dxdy;

0≤x≤a, 0≤y≤a

$\bar{F}.\bar{n} = -z = 0 \ \ since\ z = 0$

$$\int\int_{S_4} \bar{F}.\bar{n}dS = 0 \dots (5)$$

(v) For S_5 = PSCR; y = a; $\bar{n} = \bar{j}$, ds = dzdx;

0≤x≤a, 0≤z≤a

$\bar{F}.\bar{n} = -2x^2y = -2ax^2 \ \ since\ y = a$

$$\int\int_{S_5} \bar{F}.\bar{n}dS = \int_{x=0}^{a}\int_{z=0}^{a} (-2ax^2)dzdx$$

$$= \int_{x=0}^{a} (-2ax^2)_{z=0}^{a}\,dx$$

$$= -2a^2\left(\frac{x^3}{3}\right)_0^a = \frac{-2a^5}{3} \dots (6)$$

(vi) For S_6 = OBQA; y = 0; $\bar{n} = -\bar{j}$, ds = dzdx;

0≤x≤a, 0≤y≤a

$\bar{F}.\bar{n} = 2x^2y = 0 \ since\ y = 0$

$$\int\int_{S_6} \bar{F}.\bar{n}dS = 0$$

$$\int_S\int \bar{F}.\bar{n}dS = \int_{S_1}\int + \int_{S_2}\int + \int_{S_3}\int + \int_{S_4}\int + \int_{S_5}\int + \int_{S_6}\int$$

$$= a^5 - \frac{a^4}{4} - \frac{a^4}{4} + a^3 + 0 - \frac{2a^5}{3} + 0$$

$$= \frac{a^5}{3} + a^3 = \int\int_V\int \bar{V}.\bar{F}\, dv\; using\; (1)$$

Hence Gauss Divergence theorem is verified

Example : Compute $\int(ax^2 + by^2 + cz^2)dS$ over the surface of the sphere $x^2+y^2+z^2 = 1$

Sol: By divergence theorem $\int \bar{F}.\bar{n}dS = \int_V \bar{V}.\bar{F}\; dv$

$Given\ \bar{F}.\bar{n} = ax^2 + by^2 + cz^2. Let\ \phi = x^2 + y^2 + z^2 - 1$

$Normal\ vector\ \bar{n}\ to\ the\ surface\ \phi\ is$

$$\bar{V}\phi = \left(\bar{\imath}\frac{\delta}{\delta x} + \bar{\jmath}\frac{\delta}{\delta y} + \bar{k}\frac{\delta}{\delta y}\right)(x^2 + y^2 + z^2 - 1) = 2(x\bar{\imath} + y\bar{\jmath} + z\bar{k})$$

$$Unit\ normal\ vector = \bar{n} = \frac{2(x\bar{\imath} + y\bar{\jmath} + z\bar{k})}{2\sqrt{x^2 + y^2 + z^2}} = x\bar{\imath} + y\bar{\jmath} + z\bar{k}\ \text{since}\ x^2 + y^2 + z^2 = 1$$

$$\bar{F}.\bar{n} = \bar{F}.(x\bar{\imath} + y\bar{\jmath} + z\bar{k}) = (ax^2 + by^2 + cz^2) = (ax\bar{\imath} + by\bar{\jmath} + cz\bar{k}).(x\bar{\imath} + y\bar{\jmath} + z\bar{k})$$

$i.e.\ ax\bar{\imath} + by\bar{\jmath} + cz\bar{k} \qquad \bar{V}.\bar{F} = a + b + c$

Hence by Gauss Divergence theorem,

$$\int_S (ax^2 + by^2 + cz^2)dS = \int_V (a + b + c)dv = (a + b + c)V = \frac{4\pi}{3}(a + b + c)$$

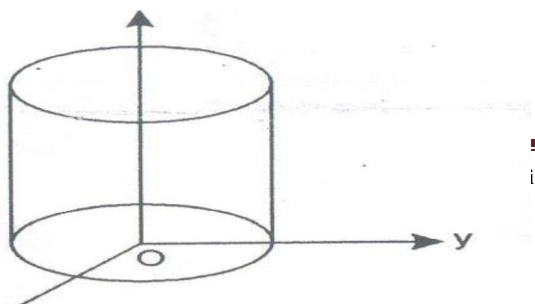
$$\left[SInce\ V = \frac{4\pi}{3}\ is\ the\ volume\ of\ the\ sphere\ of\ unit\ radius\right]$$

Example : By transforming into triple integral, evaluate $\int\int x^3\ dy\ dz + x^2 y\ dz\ dx + x^2 dx\ dy$ where S is the closed surface consisting of the cylinder $x^2+y^2 = a^2$ and the circular discs z= 0 , z= b.

Sol: Here $F_1 = x^3, F_2 = x^2y, F_3 = x^2z\ and\ \bar{F} = F_1\bar{\imath} + F_2\bar{\jmath} + F_3\bar{k}$

$$\frac{\delta F_1}{\delta x} = 3x^2, \frac{\delta F_2}{\delta y} = x^2, \frac{\delta F_3}{\delta z} = x^2$$

$$\bar{V}.\bar{F} = \frac{\delta F_1}{\delta x} + \frac{\delta F_2}{\delta y} + \frac{\delta F_3}{\delta z} = 3x^2 + x^2 + x^2 = 5x^2$$



z

x

By Gauss Divergence theorem,

$$\int\int F_1\, dy\, dz + F_2\, dz\, dx + F_3\, dx\, dy = \int\int\int \left(\frac{\delta F_1}{\delta x} + \frac{\delta F_2}{\delta y} + \frac{\delta F_3}{\delta z}\right) dx\, dy\, dz$$

$$\int\int_S (x^3\, dy\, dz + x^2 y\, dz\, dx + x^2 dx\, dy) = \int\int\int 5x^2\, dx\, dy\, dz$$

$$= 5 \int_{-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{z=0}^{b} x^2 dx\, dy\, dz$$

$$= 20 \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_{z=0}^{\sqrt{a^2-x^2}} x^2\, dx\, dy\, dz \; [Integrand\ is\ even\ function]$$

$$= 20 \int_0^a \int_0^{\sqrt{a^2-x^2}} x^2 (z)_0^b dxdy = 20b \int_{x=0}^{a} \int_0^{\sqrt{a^2-x^2}} x^2 dxdy$$

$$= 20b \int_{x=0}^{a} x^2 (y)_0^{\sqrt{a^2-x^2}}\, dx = 20b \int_0^a x^2 \sqrt{a^2 - x^2}\, dx$$

$$= 20b \int_0^{\frac{\pi}{2}} a^2 \sin^2\theta \sqrt{a^2 - a^2 \sin^2\theta}\, (a\cos\theta d\theta)$$

$$\left[put\ x = a\sin\theta ==> dx = a\cos\theta\ d\theta\ when\ x = a ==>\ \theta = \frac{\pi}{2}\ and\ x = 0 ==>\ \theta = 0\right]$$

$$= 20a^4 b \int_0^{\frac{\pi}{2}} \sin^2\theta\ \cos^2\theta d\theta = 5a^4 b \int_0^{\frac{\pi}{2}} (2\sin\theta\cos\theta)^2\ d\theta = 5a^4 b \int_0^{\frac{\pi}{2}} \frac{1-\cos 4\theta}{2} d\theta$$

$$= \frac{5a^4 b}{2}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\pi/2} = \frac{5a^4 b}{2}\left[\frac{\pi}{2}\right] = \frac{5}{4}\pi a^4 b$$

Example : Show that $\int_S (ax\bar{\imath} + by\bar{\jmath} + cz\bar{k}).\bar{n}dS = \frac{4\pi}{3}(a+b+c)$, where S is the surface of the sphere $x^2+y^2+z^2=1$.

Sol: Take $\bar{F} = ax\bar{\imath} + by\bar{\jmath} + cz\bar{k}$

$$div\ \bar{F} = \frac{\delta F_1}{\delta x} + \frac{\delta F_2}{\delta y} + \frac{\delta F_3}{\delta z} = a + b + c$$

By Gauss divergence theorem, $\int_S \bar{F}.\bar{n}dS = \int_V \bar{V}.\bar{F}dV = (a+b+c)\int_V dV = (a+b+c)V$

$$We\ have\ V = \frac{4}{3}\pi r^3\ for\ the\ sphere. Here\ r = 1$$

$$\int_S \bar{F}.\bar{n}dS = (a+b+c)\ \frac{4\pi}{3}$$

Example : Apply divergence theorem to evaluate

$\int\int_S (x+z)dy\,dz + (y+z_{dz}dx + (x+y)dxdy\ where$ **S is the surface of the sphere $x^2+y^2+z^2=4$**

Sol: Given $\int\int_S (x+z)dy\,dz + (y+z_{dz}dx + (x+y)dxdy$

Here $F_1 = x+z$, $F_2 = y+z$, $F_3 = x+y$

$$\frac{\delta F_1}{\delta x} = 1, \frac{\delta F_2}{\delta y} = 1, \frac{\delta F_3}{\delta z} = 0\ and\ \frac{\delta F_1}{\delta x} + \frac{\delta F_2}{\delta y} + \frac{\delta F_3}{\delta z} = 1 + 1 + 0 = 2$$

By Gauss Divrgence theorem,

$$\int\int_S F_1dydz + F_2dzdx + F_3dxdy = \int\int\int_V \left(\frac{\delta F_1}{\delta x} + \frac{\delta F_2}{\delta y} + \frac{\delta F_3}{\delta z}\right)dxdydz$$

$$= \int\int\int 2dxdydz = 2\int_V dv = 2V$$

$$= 2\left[\frac{4}{3}\pi(2)^3\right] = \frac{64\pi}{3}\ [for\ the\ sphere, radius = 2]$$

Example : Evaluate $\int_S \bar{F}.\bar{n}ds, if\ F = xy\bar{\imath} + z^2\bar{\jmath} + 2yz\bar{k}$ over the tetrahedron bounded by x=0, y=0, z=0 and the plane x+y+z=1.

Sol: Given F = $xy\bar{\imath} + z^2\bar{\jmath} + 2yz\bar{k}$, then div. F = y+2y = 3y

$$\int \bar{F}.\bar{n}dS = \int_v div\ \bar{F}\ dv = \int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y} 3y\ dx\ dy\ dz$$

$$= 3\int_{x=0}^{1}\int_{y=0}^{1-x} y[z]_0^{1-x-y}dxdy = 3\int_{x=0}^{1}\int_{y=0}^{1-x} y(1-x-y)dx\ dy$$

$$= 3\int_{x=0}^{1}\left[\frac{y^2}{2}-\frac{xy^2}{2}-\frac{y^3}{3}\right]_0^{1-x}dx = 3\int_0^1\left[\frac{(1-x)^2}{2}-\frac{x(1-x)^2}{2}-\frac{(1-x)^3}{3}\right]dx$$

$$= 3\int_0^1\left[\frac{(1-x)^3}{2}-\frac{(1-x)^3}{3}\right]dx = 3\int_0^1\frac{(1-x)^3}{6}dx = \frac{3}{6}\left[\frac{-(1-x)^4}{4}\right]_0^1 = \frac{1}{8}$$

Example : Use divergence theorem to evaluate $\int_S\int \bar{F}.d\bar{S}$ *where* $\bar{F}$ =$x^3i+y^3j+z^3k$ and S is the surface of the sphere $x^2+y^2+z^2 = r^2$

Sol: We have

$$\bar{V}.\bar{F} = \frac{\delta}{\delta x}(x^3)+\frac{\delta}{\delta y}(y^3)+\frac{\delta}{\delta z}(z^3) = 3(x^2+y^2+z^2)$$

By divergence theorem,

$$\bar{V}.\bar{F}\,dV = \int\int_V \int \bar{V}.\bar{F}\,dV = \int\int_V \int 3(x^2+y^2+z^2)dx\,dy\,dz$$

$$= 3\int_{r=0}^{a}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} r^2(r^2\sin\theta\,dr\,d\theta\,d\phi)$$

$[Changing\ into\ spherical\ polar\ coordinates\ x = r\sin\theta\cos\phi\,,y = r\sin\theta\sin\phi\,,z = r\cos\theta]$

$$\int_S\int \bar{F}.dS = 3\int_{r=0}^{a}\int_{\theta=0}^{\pi} r^4\sin\theta\left[\int_{\phi=0}^{2\pi}d\phi\right]dr\,d\theta$$

$$3\int_{r=0}^{a}\int_{\theta=0}^{\pi} r^4\sin\theta\ (2\pi-0)\ dr\,d\theta = 6\pi\int_{r=0}^{a} r^4\left[\int_0^{\pi}\sin\theta\ d\theta\right]dr$$

$$= 6\pi\int_{r=0}^{a} r^4(-\cos\theta)_0^{\pi}\ dr = -6\pi\int_0^a r^4(\cos\pi-\cos 0)\ dr$$

$$= 12\pi\int_0^a r^4dr = 12\pi\left[\frac{r^5}{5}\right]_0^a = \frac{12\pi a^5}{5}$$

Example : Use Gauss Divergence theorem to evaluate $\int\int_S\ (yz^2\bar{\imath}+zx^2\bar{\jmath}+2z^2\bar{k}).ds,$ *where* S is the closed surface bounded by the xy plane and the upper half of te sphere $x^2+y^2+z^2=a^2$

Sol: Divergence theorem states that

$$\int_S\int \bar{F}.ds = \int\int_V\int \bar{V}.\bar{F}\ dv$$

$$Here\ \bar{V}.\bar{F} = \frac{\delta}{\delta x}(yz^2)+\frac{\delta}{\delta y}(zx^2)+\frac{\delta}{\delta z}(2z^2) = 4z$$

$$\int\int_S \bar{F}.ds = \int\int_V\int 4z\,dx\,dy\,dz$$

Introducing spheical polar coordinates $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ *then* $dx\,dy\,dz = r^2\,dr\,d\theta\,d\phi$

$$\int\int_S \bar{F}.ds = 4\int_{r=0}^{a}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}(r\cos\theta)\,(r^2\sin\theta\,dr\,d\theta\,d\phi)$$

$$= 4\int_{r=0}^{a}\int_{\theta=0}^{\pi} r^3\sin\theta\cos\theta\left[\int_{\phi=0}^{2\pi}d\phi\right]dr\,d\theta$$

$$= 4.\int_{r=0}^{a}\int_{\theta=0}^{\pi} r^3\sin\theta\cos\theta\,(2\pi-0)dr\,d\theta$$

$$= 4\pi\int_{r=0}^{a} r^3\left[\int_0^{\pi}\sin 2\theta\,d\theta\right]dr = 4\pi\int_{r=0}^{a} r^3\left(-\frac{\cos 2\theta}{2}\right)_0^{\pi} dr$$

$$= (-2\pi)\int_0^a r^3(1-1)dr = 0$$

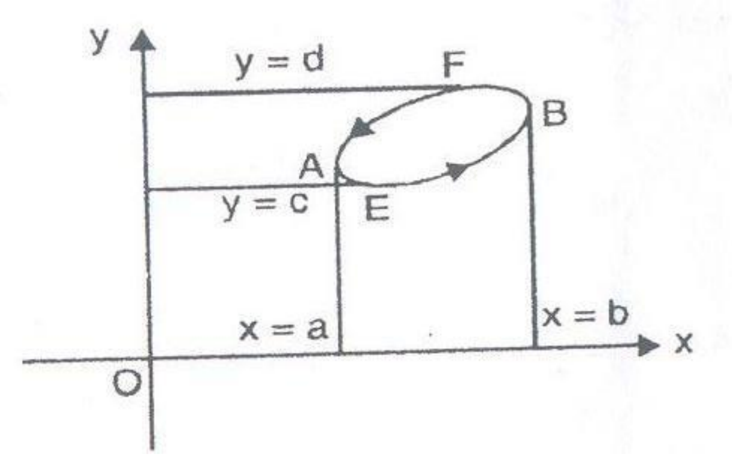
II. GREEN'S THEOREM IN A PLANE

(Transformation Between Line Integral and Surface Integral) [JNTU 2001S].

If S is Closed region in xy plane bounded by a simple closed curve C and if M and N are continuous functions of x and y having continuous derivatives in R, then

$$\oint_c M dx + N dy = \iint_s\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy,$$

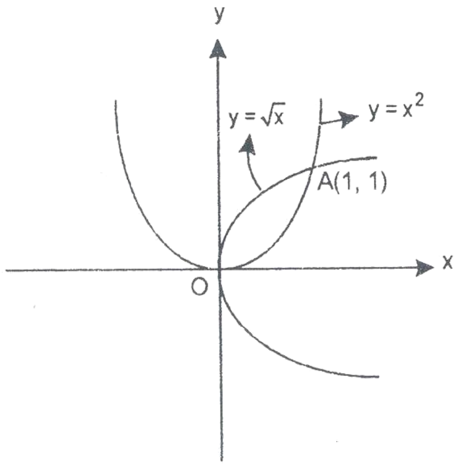
Where C is traversed in the positive(anti clock-wise) direction



1.Verify Green's theorem in plane for $\oint(3x^2 - 8y^2)dx + (4y - 6xy)dy$ where C is the region bounded by y=$\sqrt{x}$ and y=x^2 .

Solution: Let M=3x^2-8y^2 and N=4y-6xy. Then

$\frac{\partial M}{\partial y} = -16y, \frac{\partial N}{\partial x} = -6y$



We have by Green's theorem,

$\oint_c Mdx + Ndy = \iint_s \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$

Now $\int_s \int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \iint_s (16y - 6y) dxdy$

$=10\iint_s y\, dxdy = 10 \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} y\, dydx = 10 \int_{x=0}^{1} \left[\frac{y^2}{2}\right]_{x^2}^{\sqrt{x}} dx$

$=5 \int_0^1 (x - x^4) dx = 5\left(\frac{x^2}{2} - \frac{x^5}{5}\right)_0^1 = 5\left(\frac{1}{2} - \frac{1}{5}\right) = \frac{3}{2}$ ….(1)

Verification:

We can write the line integral along c

=[line integral along y=x^2(from O to A) + [line integral along y^2=x(from A to O)]

=I_1+I_2(say)

Now $I_1 = \int_{x=0}^{1} \{[3x^2 - 8(x^2)^2]dx + [4x^2 - 6x(x^2)]2xdx\} \left[\because y = x^2 \Rightarrow \frac{dy}{dx} = 2x\right]$

$= \int_0^1 (3x^3 + 8x^3 - 20x^4) dx = -1$

And $I_2 = \int_1^0 \left[(3x^2 - 8x)dx + \left(4\sqrt{x} - 6x^{3/2}\right)\frac{1}{2\sqrt{x}} = dx\right] = \int_1^0 (3x^2 - 11x + 2)dx = \frac{5}{2}$

$\therefore I_1 + I_{2=-1+5/2=3/2}$.

From(1) and (2), we have $\oint_c Mdx + Ndy = \iint_s \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$

Hence the verification of the Green's theorem.

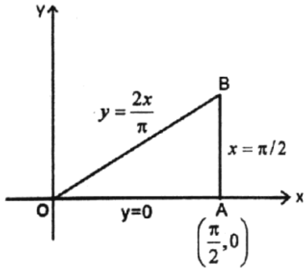
2.Evaluate by Green's theorem $\int_c (y - \sin x)dx + \cos x\, dy$ where C is the triangle enclosed by the lines y=0, x=$\frac{\pi}{2}$, $\pi y = 2x$. [JNTU 1993, 1995 S, 2003 S, 2007, (H) June 2010(Set No.2)]

Solution : Let M=y-$\sin x$ $and\ N = \cos x$ Then

$\frac{\partial M}{\partial y}$=1 and $\frac{\partial N}{\partial x}$=-$\sin x$

By Green's theorem $\oint_c Mdx + Ndy = \int_s \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$

$$\Rightarrow \int_c (y - \sin x)dx + \cos x\, dy = \iint_s (-1 - \sin x)dxdy$$



$$= -\int_{x=0}^{\pi/2} \int_{y=0}^{\frac{2x}{\pi}} (1 + \sin x)\, dxdy$$

$$= \int_{x=0}^{\pi/2} (\sin x + 1)\, [y]_0^{2x/\pi}\, dx$$

$$= \frac{-2}{\pi} \int_{x=0}^{\pi/2} x(\sin x + 1)dx$$

$$= \frac{-2}{\pi} [(-\cos x + x)]_0^{\frac{\pi}{2}} - \int_0^{\pi/2} 1(-\cos x + x)dx$$

$$= \frac{-2}{\pi} \left[x(-\cos x + x) + \sin x - \frac{x^2}{2}\right]_0^{\pi/2}$$

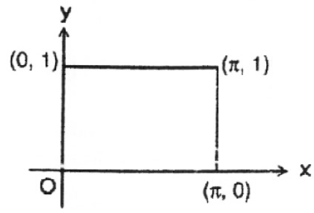
$$= \frac{-2}{\pi} \left[-x\cos x + \frac{x^2}{2} + \sin x\right]_0^{\pi/2} = \frac{-2}{\pi}\left[\frac{\pi^2}{8} + 1\right] = -\left(\frac{\pi}{4} + \frac{2}{\pi}\right)$$

Example 3: Evaluate by Green's theorem for $\oint_c (x^2 - \cosh y)dx + (y + \sin x)dy$ where C is the rectangle with vertices $(0,0), (\pi, 0), (\pi, 1), (0,1)$.

Solution: Let M=$x^2 - \cosh y$,$N = y + \sin x$ $\therefore \frac{\partial M}{\partial y} = -\sinh y\ and\ \frac{\partial N}{\partial x} = \cos x$

By Green's theorem, $\oint_c Mdx + Ndy = \int_s \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$

$$\Rightarrow \oint_c (x^2 - \cosh y)dx + (y + \sin x)dy = \int_s \int (\cos x + \sinh$$



$$= \int_{x=0}^{\pi} \int_{y=0}^{1} (\cos x + \sinh y)dydx = \int_{x=0}^{\pi} (y\cos x + \cosh y)_0^1\, dx$$

$$= \int_{x=0}^{\pi} [\cos x + \cosh 1 - 1]dx$$

$$= \pi(\cosh 1 - 1)$$

Example 4: A Vector field is given by $\overline{F} = (\sin y)\boxed{i} + x(1 + \cos y)\boxed{j}$.

Evaluate the line integral over the circular path $x^2 + y^2 = a^2$, z=0

(i) Directly (ii) By using Green's theorem

Solution : (i) Using the line integral [JNTU 96, (A) June 2011 (Set No.4)

$$\oint_c \overline{F}.d\overline{r} = \oint_c F_1 dx + F_2 dy = \oint_c \sin y dx + x(1 + \cos y)dy$$

$$= \oint_c \sin y dx + x\ \cos y\, dy = xdy = \oint_c d(x\sin y) + xdy$$

Given Circle is $x^2 + y^2 = a^2$. Take x=a $\cos\theta$ and y=a $\sin\theta$ so that dx=-a $\sin\theta\, d\theta$ and dy=a $\cos\theta\, d\theta$ and $\theta = 0 \rightarrow 2\pi$

$\therefore \;\oint \overline{F}.d\overline{r} = \int_0^{2\pi} d[a\ \cos\theta \sin(a\ \sin\theta)] + \int_0^{2\pi} a(\cos\theta)a\ \cos\theta\, d\theta$

$=[a\ \cos\theta \sin(a\ \sin\theta)]_0^{2\pi} + 4a^2\ \int_0^{\pi/2} \cos^2\theta\, d\theta$

$=0+4a^2.\frac{1}{2}.\frac{\pi}{2} - \pi a^2$

(ii)Using Green's theorem

Let M=$\sin y$ and N=x$(1+\cos y)$. Then

$\frac{\partial M}{\partial y}$=$\cos y$ and $\frac{\partial N}{\partial x}$=-$(1+\cos y)$

By Green's theorem,

$\oint_c Mdx + Ndy = \iint_s \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$

$\therefore\ \oint_c \sin y\, dx + (1+\cos y)dy = \int_s \int (-\cos y + 1\cos y)dxdy == \iint dxdy$

$= \int_s \int dA = A = \pi a^2\ (\because area\ of\ circle = \pi a^2)$

We observe that the values obtained in (i) and (ii) are same to that Green's theorem is verified.

Example 4: Show that area bounded by a simple closed curve C is given by $\frac{1}{2}\oint xdy - ydx$ and hence find the area of

(i)The ellipse x=$acos\theta$ ⟦$\theta, y = bsin\theta$ ⟦θ $(i.e)\ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$⟧ ⟧

(II)The Circle x=$a\cos\theta$, $y = a\sin\theta$ $(i.e)x^2 + y^2 = a^2$

Solution: We have by Green's theorem $\oint_c Mdx + Ndy = \int_s \int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$

Here M=-y and N=x so that $\frac{\partial M}{\partial y} = -1\ and \frac{\partial N}{\partial x} = 1$

$\oint_c xdy - ydx = 2\ \int_s dxdy = 2A$ where A is the area of the surface.

$\therefore \frac{1}{2}\int xdy - ydx = A$

(i)For the ellipse x=$a\cos\theta$ and y=$b\sin\theta$ and $\theta = 0 \rightarrow 2\pi$

$\therefore Area, A$=$\frac{1}{2}\oint xdy - ydx = \frac{1}{2}\int_0^{2\pi} [(a\ \cos\theta)(b\cos\theta) - (b\ \sin\theta\ (-a\ \sin\theta))]d\,\theta$

$=\frac{1}{2}ab\int_0^{2\pi}(\cos^2\theta + \sin^2\theta)\, d\theta = \frac{1}{2}ab(\theta)_0^{2\pi} = \frac{ab}{2}(2\pi - 0) = \pi ab$

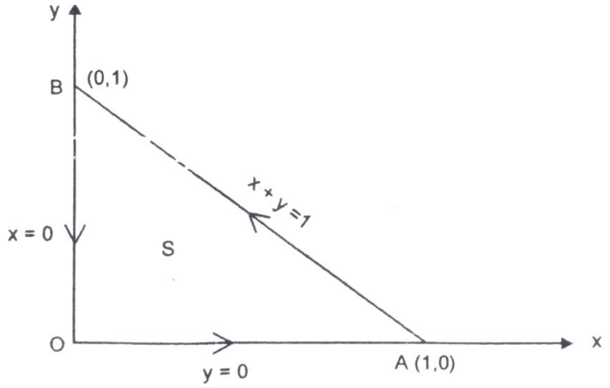
(ii)Put a=b to get area of the circle A=πa^2

Example 5: Verify Green's theorem for $\int_c[(3x^2 - 8y^2)dx + (4y - 6xy)dy]$ where c is the region bounded by x=0, y=0 and x+y=1. [JNTU 2003S, 2007S(Set No.3)]

Solution : By Green's theorem, we have

$\int_C Mdx + Ndy = \iint_s \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$

Here M=$3x^2 - 8y^2$ and N=4y-6xy



$\therefore \frac{\partial M}{\partial y} = 16y \ \ and \ \frac{\partial N}{\partial x} = 6y$

Now $\int_C Mdx + Ndy = + \int_{OA} Mdx + Ndy + \int_{AB} Mdx + Ndy + \int_{BC} Mdx + Ndy$ ....(1)

Along OA, y=0 $\therefore dy = 0$

$\int_{OA} Mdx + Ndy = \int_0^1 3x^2 dx = \left(\frac{x^3}{3}\right)_0^1 = 1$

Along AB, x+y=1 $\therefore dy = -dx$ and x=1-y and varies from 0 to 1.

$\int_{AB} Mdx + Ndy = \int_0^1 [3(y-1)^2 - 8y^2](-dy) + [4y + 6y(y-a)]dy$

$= \int_0^1 (-5y^2 - 6y + 3)(-dy) + (6y^2 - 2y)dy$

$= \int_0^1 (11y^2 + 4y - 3)dy = \left(11\frac{y^3}{3} + 4\frac{y^2}{2} - 3y\right)_0^1$

$= \frac{11}{3} + 2 - 3 = \frac{8}{3}$

Along BO, x=0 $\therefore dx = 0$ and limits of y are from 1 to 0

. $\int_{BO} Mdx + Ndy = \int_1^0 4ydy = \left(4\frac{y^2}{2}\right)_1^0 = (2y^2)_0^1 = -2$

from (1), we have $\int_c Mdx + Ndy = 1 + \frac{8}{3} - 2 = \frac{5}{3}$

Now $\iint_s \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \int_{x=0}^1 \int_{y=0}^{1-x} (-6y + 16y)dxdy$

$= 10 \int_{x=0}^1 \left[\int_{y=0}^{1-x} ydy\right] dx = 10 \int_0^1 \left(\frac{y^2}{2}\right)_0^{1-x} dx$

$= 5 \int_0^1 (1-x)^2 dx = 5\left[\frac{(1-x)^3}{-3}\right]_0^1$

$= -\frac{5}{3}[(1-1)^3 - (1-0)^3] = \frac{5}{3}$

From (2) and (3), we have $\int_C Mdx + Ndy = \iint_s \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$

Hence the verification of the Green's Theorem.

Example 11: Verify Green's theorem in the plane for $\int_C (x^2 - xy^3)\,dx + (y^2 - 2xy)dy$

Where C is square with vertices (0,0), (2,0), (2,2), (0,2).

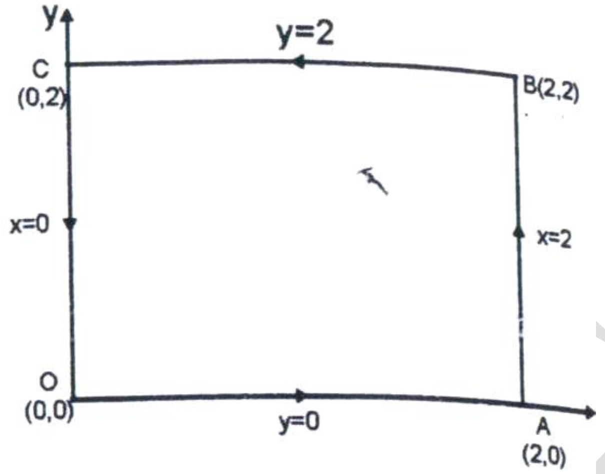
[JNTU Aug, 2008S, (H)June2009,(K) May2010(Set No.2)]

Solution: The Cartesian form of Green's theorem in the plane is

$$\int_C M\,dx + N\,dy = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Here M=$x^2 - xy^3$ and N=$y^2 - 2xy$

$\therefore$ $\frac{\partial M}{\partial y}$ =3 xy^2 and $\frac{\partial N}{\partial x} = -2y$



Evaluation of $\int_C (M\,dx + N\,dy)$

To Evaluate $\int_C (x^2 - xy^3)\,dx + (y^2 - 2xy)dy$, we shall take C in four different segments viz (i) along OA(y=0) (ii) along AB(x=2) (iii) along BC(y=2) (iv) along CO(x=0).

(i)Along OA(y=0)

$$\int_C (x^2 - xy^3)\,dx + (y^2 - 2xy)dy = \int_0^2 x^2 dx = \left(\frac{x^3}{3}\right)_0^2 = \frac{8}{3} \quad \text{.....(1)}$$

(ii)Along AB(x=2)

$$\int_C (x^2 - xy^3)\,dx + (y^2 - 2xy)dy = \int_0^2 (y^2 - 4y)\,dy \quad [\because x = 2, dx = 0]$$

$$= \left(\frac{y^3}{3} - 2y^2\right)_0^2 = \left(\frac{8}{3} - 8\right) = 8\left(-\frac{2}{3}\right) = -\frac{16}{3} \quad \text{....(2)}$$

(iii)Along BC(y=2)

$$\int_C (x^2 - xy^3)\,dx + (y^2 - 2xy)dy = \int_2^0 (x^2 - 8x)\,dx \quad [\because y = 2, dy = 0]$$

$$= \left(\frac{x^3}{3} - 4x^2\right)_2^0 = \left(\frac{8}{3} - 16\right) = \frac{40}{3} \quad \text{...(3)}$$

(iv)Along CO(x=0)

$$\int_C (x^2 - xy^3)\,dx + (y^2 - 2xy)dy = \int_2^0 y^2\,dx \quad [\because x = 0, dx = 0] = \left(\frac{y^3}{3}\right)_2^0 = -\frac{8}{3} \quad \text{.....(4)}$$

Adding(1),(2),(3) and (4), we get

$\int_C (x^2 - xy^3)\, dx + (y^2 - 2xy) dy = \frac{8}{3} - \frac{16}{3} + \frac{40}{4} - \frac{8}{3} = \frac{24}{3} = 8$ …(5)

Evaluation of $\iint_s \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$

Here x ranges from 0 to 2 and y ranges from 0 to 2.

$\iint_s \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \int_0^2 \int_0^2 (-2y + 3xy^2) dxdy$

$= \int_0^2 \left(-2xy + \frac{3x^2}{2} y^2\right)_0^2 dy$

$= \int_0^2 (-4y + 6y^2)\, dy = (-2y^2 + 2y^3)_0^2$

=-8+16=8 …(6)

From (5) and (6), we have

$\int_C M dx + N dy = \iint_s \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$

Hence the Green's theorem is verified.

II. <u>STOKE'S THEOREM</u>

(Transformation between Line Integral and Surface Integral) [JNTU 2000]

Let S be a open surface bounded by a closed, non intersecting curve G. if $\bar{F}$ is any differentieable vector point function then $\oint_C \bar{F}.d\,\bar{r} = \int_S curl\ \bar{F}.\bar{n}\, ds$ *where c is traversed in the positive* direction and $\bar{n}$ *is unit outward drawn normal at any point of the surface.*

1. Verify Stokes theorem for $\bar{F} = -y^3\bar{\imath} + x^3\bar{\jmath}$, Where S is the circular disc $x^2 + y^2 \le 1, z = 0$. [JNTU 99,2007,2008S(Set No.4)]

<u>Solution:</u> Given that $\bar{F} = -y^3\bar{\imath} + x^3\bar{\jmath}$. The boundary of C of S is a circle in xy plane. $x^2 + y^2 \le 1, z = 0$. We use the parametric co-ordinates x=cos$\theta, y = \sin\theta, z = 0, 0 \le \theta \le 2\pi$; dx=-sin$\theta\, d\theta$ and dy =cos$\theta\, d\theta$

$\therefore \oint_c \bar{F}.dr = \int_c F_1 dx + F_2 dy + F_3 dz = \int_c -y^3 dx + x^3 dy$

$= \int_0^{2\pi} [-\sin^3\theta(-\sin\theta) + \cos^3\theta\cos\theta] d\theta = \int_0^{2\pi} (\cos^4\theta + \sin^4\theta) d\theta$

$= \int_0^{2\pi} (1 - 2\sin^2\theta\, \cos^2\theta) d\theta = \int_0^{2\pi} d\theta - \frac{1}{2}\int_0^{2\pi} (2\sin\theta\, \cos\theta)^2\, d\theta$

$= \int_0^{2\pi} d\theta - \frac{1}{2}\int_0^{2\pi} \sin^2 2d\theta = (2\pi - 0) - \frac{1}{4}\int_0^{2\pi} (1 - \cos 4\theta) d\theta$

$= 2\pi + \left[-\frac{1}{4}\theta + \frac{1}{16}\sin 4\theta\right]_0^{2\pi} = 2\pi - \frac{2\pi}{4} = \frac{6\pi}{4} = \frac{3\pi}{2}$

Now $\nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^3 & x^3 & 0 \end{vmatrix} = \bar{k}(3x^2 + 3y^2)$

$\therefore \int_s (\nabla \times \bar{F}).\bar{n}ds = 3\int_s (x^2 + y^2)\bar{k}.\bar{n}ds$

We have $(\bar{k}.\bar{n})ds - dx\ dy\ and\ R\ is\ the\ region\ on\ xy - plane$

$\therefore \iint_s (\nabla \times \bar{F}).\bar{n}ds = 3\iint_R (x^2 + y^2)\ dx\ dy$

Put x=r cos$\emptyset, y = r\ sin\emptyset \therefore dxdy = rdr\ d\emptyset$

R is varying from 0 to 1 and $0 \leq \emptyset \leq 2\pi$.

$\int(\nabla \times \bar{F}).\ \bar{n}ds = 3\int_{\emptyset=0}^{2\pi}\int_{r=0}^{1} r^2$.rdr d$\emptyset = \frac{3\pi}{2}$

L.H.S=R.H.S.Hence the theorem is verified.

Example2: If $\bar{F} = y\bar{i} + (x - 2xz)\bar{j} - xy\bar{k}, evaluate\ \int_s (\nabla \times \bar{F}).\bar{n}ds\ Where$ Sis the surface of sphere $x^2 + y^2 + z^2 = a^2, above\ the\ xy - plane.$

Solution: Given$\bar{F} = y\bar{i} + (x - 2xz)\bar{j} - xy\bar{k}.$

By Stokes Theorem,

$\int_s (\nabla \times \bar{F}).\bar{n}ds = \int_c F_1 dx + F_2 dy + F_3 dz = \int_c ydx + (x - 2xz)dy - xydz$

Above the xy plane the sphere is $x^2 + y^2 += a^2, z = 0$

$\therefore \int_c \bar{F}.d\bar{r} = \int_c ydx + xdy.$

Put x=a cos θ,y=asinθ $so\ that\ dx = -a\ sin\theta d\theta, dy = acos\theta d\theta\ and \theta = 0 \to 2\pi$

$\int_c \bar{F}.d\bar{r} = \int_0^{2\pi}(a\ sin\theta)(-a\ sin\theta)\ d\theta + (acos\theta)(acos\theta)d\theta$

$$=a^2\int_0^{2\pi} cos2\theta\ d\theta = a^2\left[\frac{sin2\theta}{2}\right]_0^{2\pi} = \frac{a^2}{2}(0) = 0$$

Example 3: Verify Stokes theorem for $\bar{F} = (2x - y)\bar{i} - \dot{y}z^2\bar{j} - y^2z\bar{k}$ over the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$bounded by the projection of the xy-plane.

[JNTU2006,2007,2007S,2008,JNTU(A) June2009(Set No.2)]

Solution: The boundary C of S is a circle in xy plane i.e $x^2 + y^2$=1, z=0

The parametric equations are x=$cos\theta, y = sin\theta, \theta = 0 \to 2\pi$

$\therefore\ dx = -sin\theta\ d\theta, dy = cos\theta\ d\theta$

$\int_C \bar{F}.d\bar{r} = \int_C \bar{F}.d\bar{r} = \int_c \bar{F}_1.dx + \bar{F}_2.dy + \bar{F}_3.dz = \int_c (2x - y)dx - yz^2dy - y^2zdz$

$=\int_c (2x - y)dx(since\ z = 0\ and\ dz = 0)$

$=\int_c^{2\pi}(2\cos\theta-\sin\theta)\sin\theta\ d\theta=\int_c^{2\pi}\sin^2\theta\ d\theta=\int_c^{2\pi}\sin 2\ d\theta$

$=\int_{\theta=0}^{2\pi}\frac{1-\cos 2\theta}{2}d\theta-\int_0^{2\pi}\sin 2\theta\ d\theta=\left[\frac{1}{2}\theta-\frac{1}{4}\sin 2\theta+\frac{1}{2}.\cos 2\theta\right]_0^{2\pi}$

$=\frac{1}{2}(2\pi-0)+0+\frac{1}{2}.(\cos 4\pi-\cos 0)=\pi$

Again $\nabla\times\bar{F}=\begin{vmatrix}\bar{\imath} & \bar{\jmath} & \bar{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 2x-y & -yz^2 & -y^2z\end{vmatrix}=\bar{\imath}(-2yz+2yz)-\bar{\jmath}(0-0)+\bar{k}(0+1)=\bar{k}$

$\therefore \int_S(\nabla\times\bar{F}).\bar{n}ds=\int_S\bar{k}.\bar{n}ds=\int_R\int dxdy$

Where R is the projection of S on xy plane and $\bar{k}.\bar{n}ds=dxdy$

Now $\int\int_R dxdy=4\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}}dydx=4\int_{x=0}^{1}\sqrt{1-x^2}\,dx=4\left[\frac{x}{2}\sqrt{1-x^2}+\frac{1}{2}\sin^{-1}x\right]_0^1$

$=4\left[\frac{1}{2}\sin^{-1}1\right]=2\frac{\pi}{2}=\pi$

$\therefore$ *The* Stokes theorem is verified.

Example 4: Verify Stokes theorem for the function $\bar{F}=x^2\,\bar{\imath}+xy\,\bar{\jmath}$ integrated round the square in the plan z=0 whose sides are along the lines x=0, y=0, x=a, y=a.

Solution: Given $\bar{F}=x^2\,\bar{\imath}+xy\,\bar{\jmath}$

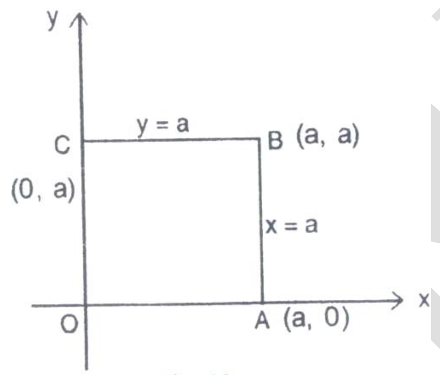


Fig. 13

By Stokes Theorem, $\int_S(\nabla\times\bar{F}).\bar{n}ds=\int_S\bar{F}.d\bar{r}$

Now $\nabla\times\bar{F}=\begin{vmatrix}\bar{\imath} & \bar{\jmath} & \bar{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x^2 & xy & 0\end{vmatrix}=\bar{k}$y

L.H.S. , $\int_S(\nabla\times\bar{F}).\bar{n}ds=\int_S y(\bar{n}.\bar{k})\,ds=\int_S ydxdy$

$\bar{n}.\bar{k}.ds=dxdy$ and R is the region bounded for the square.

$\therefore \int_S(\nabla\times\bar{F}).\bar{n}ds=\int_0^a\int_0^a ydydx=\frac{a^3}{2}$

R.H.S. $=\int_c\bar{F}.d\bar{r}=\int_c(x^2dx+xydy)$

But $\int\bar{F}.d\bar{r}=\int_{OA}\bar{F}.d\bar{r}+\int_{AB}\bar{F}.d\bar{r}+\int_{BC}\bar{F}.d\bar{r}+\int_{CO}\bar{F}.d\bar{r}$

(i)Along OA: y=0, z=0, dy=0, dx=0

$\therefore \int_{OA} \bar{F}\cdot d\bar{r} = \int_0^a x^2 dx = \frac{a^3}{3}$

(ii)Along AB:x=a, z=0,dx=0,dz=0

$\int_{AB} \bar{F}\cdot d\bar{r} = \int_0^a ay dx = \frac{1}{2}a^3$

(iii)Along BC: y=a,z=0,dy=0,dz=0

$\therefore \int_{BC} \bar{F}\cdot d\bar{r} = \int_a^0 0 dx = \frac{1}{3}a^3$

(iv)Along CO: x=0, z=0, dx=0, dz=0

$\therefore \int_{CO} \bar{F}\cdot d\bar{r} = \int_a^0 0 dy = 0$

Adding $\int_c \bar{F}\cdot d\bar{r} = \frac{1}{3}a^3 + \frac{1}{2}a^3 + \frac{1}{3}a^3 + 0 = \frac{1}{2}a^3$

Hence the verification.

$\oint_c (x^2 + y - 4)\, dx + 3xydy + (2xz + z^2) dz$

ASSIGNMENT QUESTIONS

UNIT-I ASSIGNMENT QUESTIONS

SET – I

1. Solve $(x^2 - 2xy + 3y^2)dx + (y^2+6xy- x^2)dy = 0$.
2. Solve $(3y+2x+4)dx –(4x + 6y + 5) dy = 0$.

SET – II

1. Solve $(x+y) (dx-dy) = dx+dy$.
2. Solve $(2x – 4y + 5)y^1 + (x-2y+3) = 0$.

SET – III

1. The number N of bacteria in a culture grew at a rate proportional to N. The value of N was initially 100 and increased to 332 in one hour what was the value of N after 1½ hours.
2. Uranium disintegrates at a rate proportional to the amount present at any instant. If m_1 and m_2 are grams of uranium that are present at times T_1 and T_2 respectively, find the half – life of uranium.

SET – IV

1. Solve $(x^2 - 2xy + 3y^2)dx + (y^2+6xy- x^2)dy = 0$.
2. Solve $(2x – 4y + 5)y^1 + (x-2y+3) = 0$.

SET – V

1. Solve $(x+y) (dx-dy) = dx+dy$.
2. Uranium disintegrates at a rate proportional to the amount present at any instant. If m_1 and m_2 are grams of uranium that are present at times T_1 and T_2 respectively, find the half – life of uranium.

SET - VI

1. The number N of bacteria in a culture grew at a rate proportional to N. The value of N was initially 100 and increased to 332 in one hour what was the value of N after 1½ hours.

2. Solve $(3y+2x+4)dx –(4x + 6y + 5) dy = 0$.

UNIT-II ASSIGNMENT QUESTIONS

SET – I

1. Solve $y^{11} + y^{1} - 2y = 0$ y(0) y(0) = 4, $y^{1}(0)$-1.
2. Solve $(D^3 + 1)y = \cos(2x-1)$.

SET – II

1. Solve by method of variation of parameters. $(D^2 + 4)y = \tan 2x$.
2. Solve $(D^3 + 1)y = 3 + 5e^x$.

SET – III

1. $\left(D^3 + 2D^2 + D\right)y = e^{2x} + x^2 + x + \sin 2x$
2. $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = 0$

SET – IV

1. Solve $y^{11} + y^{1} - 2y = 0$ y(0) y(0) = 4, $y^{1}(0)$-1.
2. Solve by method of variation of parameters. $(D^2 + 4)y = \tan 2x$.

SET – V

1. $\left(D^3 + 2D^2 + D\right)y = e^{2x} + x^2 + x + \sin 2x$
2. Solve $(D^3 + 1)y = 3 + 5e^x$.

SET - VI

1. Solve $(D^3 + 1)y = \cos(2x-1)$.
2. Solve $y^{11} + y^{1} - 2y = 0$ y(0) y(0) = 4, $y^{1}(0)$-1.

UNIT-III ASSIGNMENT QUESTIONS

SET-I

1. Evaluate $\int_0^{\pi}\int_0^{x^2} x(x^2+y^2)$dx dy
2. Evaluate $\int_0^{2}\int_0^{x} e^{x+y}$ dx dy.
3. Evaluate $\int_0^{\frac{\pi}{4}}\int_0^{a\sin\Theta} \frac{r}{\sqrt{a^2-r^2}}$dr dΘ

SET-II

1. Evaluate $\iint (x^2+y^2)dxdy$ in positive quadrant for which x+y< 1
2. Evaluate $\int_0^{\pi}\int_0^{a\sin\Theta} r\,dr\,d\Theta$
3. ..Evaluate $\int_0^1\int_1^2\int_2^3 xyz\ dx\,dy\,dz.$

SET-III

1. Find $\iint (x+y)^2 dxdy$ over the area bounded by the ellipse $x^2/a^2+y^2/b^2=1$
2. Evaluate $\iint r^3 drd\Theta$ over the area included between the circles r=2 sinɵ and r=4 sinɵ.(Dec 2010)
3. Find the area of the loop of the curve r=a(1+cosɵ)

SET-IV

1. Find $\iint (x+y)^2 dxdy$ over the area bounded by the ellipse $x^2/a^2+y^2/b^2=1$
2. Evaluate $\iint r^3 drd\Theta$ over the area included between the circles r=2 sinɵ and r=4 sinɵ.(Dec 2010)
3. Find the area of the loop of the curve r=a(1+cosɵ)

SET-V

1. Find the volume common to the cylinder $x^2+y^2=a^2$ and $x^2+z^2=a^2$
2. Find volume bounded by the cylinder $x^2+y^2=4$, y+z=4 and z=0
3. **3.** Evaluate $\int\int y\,dx\,dy$ where R is the region bounded by the parabolas y^2 = 4x and x^2 = 4y

SET-VI

1. Evaluate $\int_0^{\pi}\int_0^{a\sin\Theta} r\,dr\,d\Theta$

2. $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2}-y^2} xyz\ dz\,dy\,dx$

$$\int_1^e\int_1^{\log y}\int_1^{e^x} \log y\ dz\,dx\,dy$$

UNIT-IV ASSIGNMENT QUESTIONS

SET-I

1. Find a unit normal vector to the given surface $x^2y+2xz = 4$ at the point (2,-2,3).
2. Evaluate the angle between the normals to the surface $xy= z^2$ at the points (4,1,2) and (3,3,-3).
3. If $a=x+y+z$, $b= x^2+y^2+z^2$, $c = xy+yz+zx$, prove that [grad a, grad b, grad c] = 0.

SET-II

1. find div $\bar{f}$ = grad$(x^3+y^3+z^3-3xyz)$
2. Find constants a,b and c if the vector $\bar{f} = (2x+3y+az)\bar{i}+(bx+2y+3z)\,\bar{j}+(2x+cy+3z)\,\bar{k}$ is Irrotational
3. prove that curl $(\phi\,\bar{a})$= (grad ϕ)x $\bar{a}$ +ϕ curl $\bar{a}$

SET-III

1. Show that $\nabla[f(r)] = \dfrac{f^{i}(r)}{r}\bar{r}$ where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$.
2. Find the directional derivative of the function $f = x^2-y^2+2z^2$ at the point P =(1,2,3) in the direction of the line $\overline{PQ}$ where Q = (5,0,4
3. Find the directional derivative of $\phi = x^2yz+4xz^2$ at (1,-2,-1) in the direction 2i-j-2k.

SET-IV

1. Find curl $\bar{f}$ where $\bar{f}$ = grad$(x^3+y^3+z^3-3xyz)$
2. Find constants a,b,c so that the vector $\bar{A} = (x+2y+az)\bar{i}+(bx-3y-z)\,\bar{j}+(4x+cy+2z)\,\bar{k}$ is Irrotational
3. Prove that div.(grad r^m)= $m(m+1)r^{m-2}$

SET-V

1. Prove that div ($\bar{a}$ x $\bar{b}$)= $\bar{b}.curl\,\bar{a}-\bar{a}.curl\,\bar{b}$
2. Show that $\nabla^2[f(r)]= \dfrac{d^2f}{dr^2}+\dfrac{2}{r}\dfrac{df}{dr} = f^{11}(r)+\dfrac{2}{r}f^{1}(r)$
3. Find div $\bar{r}$. where $\bar{r}$. = $x\bar{i}+y\bar{j}+z\bar{k}$

SET-VI

1. Prove that div$(\phi\,\bar{a})$= (grad ϕ). $\bar{a}$ +ϕ div $\bar{a}$ or $\nabla.(\phi\,\bar{a})= (\nabla\phi)\,\bar{a}+\phi(\nabla.\,\bar{a})$
2. Find a unit normal vector to the surface $z= x^2+y^2$ at (-1,-2,5)
3. Prove that $\nabla(r^n)= nr^{n-2}\,\bar{r}$.

UNIT-V ASSIGNMENT QUESTIONS

SET-I

1. If $\bar{F}$ (5xy-6x^2) $\bar{i}$ (2y-4x) $\bar{j}$, evaluate $\int_C \bar{F}.d\bar{r}$ along the curve C in xy plane y=x^{3}from (1,1) to (2,8).
2. If $\phi = x^2yz^3, evaluate \int_c \phi d\bar{r}$ along with curve x= t, y =2t, z=3t from t = 0 to t=1.

3.Compute $\int(ax^2 + by^2 + cz^2)dS$ over the surface of the sphere x^2+y^2+z^2 = 1

1. .

SET-II

1. Show that $\int_S (ax\bar{i} + by\bar{j} + cz\bar{k}).\bar{n}dS = \frac{4\pi}{3}(a+b+c)$, where S is the surface of the sphere x^2+y^2+z^2=1
2. Evaluate $\int_S \bar{F}.\bar{n}ds, if\ F = xy\bar{i} + z^2\bar{j} + 2yz\bar{k}$ over the tetrahedron bounded by x=0, y=0, z=0 and the plane x+y+z=1.
3. Evaluate $\int_S \int \bar{F}.\bar{n}ds,\ where\ \bar{F} = 2x^2yi - y^2j + 4xz^2k$ and S is the closed surface of the region in the octant bounded by the cylinder y^2+z^2 = 9 and the planes x=0, x=2, y=0, z=0

SET-III

1. Evaluate $\iint_s xdy\ dz + y\ dz\ dx + zdxdy$ over $x^2 + y^2 + z^2$=1
2. Compute $\iint (a^2x^2 + b^2y^2 + c^2z^2)^{\frac{1}{2}}\ dS$ over the ellipsoid $a^2x^2 + b^2y^2 + c^2z^2$=1
3. Find $\int_s (4.x\bar{i} - 2y^2\bar{j} + z^2k).\bar{n}\,dS$ Where S Is the region bounded by $x^2 + y^2$ =4, z=0 and z=3.

SET-IV

1. Evaluate by Green's theorem $\int_c (y - \sin x)dx + \cos x\,dy$ where C is the triangle enclosed by the lines y=0, x=$\frac{\pi}{2}$, $\pi y = 2x$.
2. Evaluate by Green's theorem for $\oint_c (x^2 - \cosh y)dx + (y + \sin x)dy$ where C is the rectangle with vertices $(0,0), (\pi,0), (\pi,1), (0,1)$.
3. A Vector field is given by $\bar{F} = (\sin y)\boxed{i} + x(1 + \cos y)\boxed{j}$.Evaluate the line integral over the circular path $x^2+y^2 = a^2$, z=0

SET-V

1. Evaluate $\oint_c (3x + 4y)dx + (2x - 3y)\,dy$ where c is the circle $x^2 + y^2 = 4$
2. Find the area bounded by one arc of the cycloid x=a $(\theta - sin\theta), y = a(1 - cos\theta), a > 0\ and\ the\ x - axis.$
3. Find the area bounded by the hypocycloid $x^{2/3} + y^{2/3} = a^{2/3}, a > 0.$

SET-VI

1. Evaluate $\iint_s (\nabla \times \bar{F}).\bar{n}dS$, where $\bar{F}=(x^2+y-4)\bar{i} + 3xy\bar{j}+(2xz+z^2)\bar{k}$ and S is the surface of, The hemisphere $x^2+y^2+z^2=16$ above the xy-plane ,

2. Find $\int_c \bar{F}.d\bar{r}$ where $\bar{F} = x^2y^2\bar{i}+y\bar{i}$ and the curve $y^2=4x$ in the xy-plane from (0,0) to (4,4).

3. If $\bar{F}=y\bar{i}+z\bar{j}+x\bar{k}$, find the circulation of $\bar{F}$ round the curve c where c is the circule $x^2+y^2=1$, $z=0$

TUTORIAL QUESTIONS

UNIT-I TUTORIAL QUESTIONS

1. Solve y-x dy/dx = a(y^2 + dy/dx).
2. Solve x dy/dx +coty = 0 if y = π/4 when x = √2.
3. The rate at which bacteria multiply is proportional to the instantaneous number present. If the original number doubles in 2 hours, in how many hours will it be triple?
4. A body is originally at 80°C and cools down to 60°C in 20 minutes. If the temperature of the air is 40°C, find the temperature of the body after 40 minutes
5. Solve (x^2 + y^2)dx = 2xy dy.
6. Solve xdy – ydx = $\sqrt{(x^2 + y^2)}$ dx given that y = 1 when x = √3.

UNIT-II TUTORIAL QUESTIONS

1. Solve y^{11}-4 y^1+3y = $4e^{3x}$, y(0) = 1, $y^1(0)$ = 3.
2. Solve y^{11}+4 y^1+4y = 4cosx+3sinx, y(0) = 3.
3. Solve ($D^3 + 2D^2 + D$)y = $e^{2x} + x^2 + x$ + sin2x.
4. (D^2 +1)y = 1/e^x-1.
5. ($D^2 + n^2$)y = Sec nx.
6. (D^2 + 3D+2)y = $e^{2x}+x^2$
7. Solve 4 y^{111} + 4 y^{11} + y^1 = 0.
8. Solve ($D^4 + 18D^2$ + 81)y = 0.
9. Without using variation of parameters solve (D^2+a^2)y = tan ax.

UNIT-III TUTORIAL QUESTIONS

1.Evaluate $\int_0^{\pi}\int_0^{x^2} x(x^2 + y^2)$dx dy.

2.Evaluate $\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}$ dx dy.

3.Evaluate $\int_0^1\int_1^2\int_2^3 xyz\ dx\ dy\ dz.$

4.Evaluate $\int_0^2\int_0^x e^{x+y}$ dx dy.

5.Evaluate $\iiint xyz$ dx dy dz where V is the domain bounded by the coordinate planes and the plane x+y+z=1(Dec 2000)

6.Evaluate $\iiint xyz$ dx dy dz,where the domain V is bounded by the plane x+y+z=a and the Coordinate planes.(sep 2006)

7. Evaluate $\int\int y\ dx\ dy$ where R is the region bounded by the parabolas y^2 = 4x and x^2 = 4y

8. Evaluate $\int\int y\ dx\ dy$ where R is the region bounded by the parabolas y^2 = 4x and x^2 = 4y

9. Evaluate $\int\int x^2 + y^2$ dx dy in positive quadrant for which x+y≤1.

10.Show by double integration, the area between the parabolas y^2 = 4ax and x^2 = 4ay is $\frac{16}{3}a^2$

UNIT-IV TUTORIAL QUESTIONS

1.If the temperature at any point in space is given by t = xy+yz+zx, find the direction in which temperature changes most rapidly with distance from the point (1,1,1) and determine the maximum rate of change

2. Evaluate the angle between the normals to the surface xy= z^2 at the points (4,1,2) and (3,3,-3).

3. Find the values of a and b so that the surfaces ax^2-byz = (a+2)x and $4x^2y+z^3$= 4 may intersect orthogonally at the point (1, -1,2).

4. Find the constants a and b so that surface ax^2-byz=(a+2)x will orthogonal to $4x^2y+z^3$=4 at the point (1,-1,2).

5. Find the angle of intersection of the spheres $x^2+y^2+z^2$ =29 and $x^2+y^2+z^2$ +4x-6y-8z-47 =0 at the point (4,-3,2).

6. Find the angle between the surfaces $x^2+y^2+z^2$ =9, and z = x^2+y^2- 3 at point (2,-1,2).

7. Prove that if $\bar{r}$ is the position vector of an point in space, then $r^n\bar{r}$ is Irrotational. (or) Show that $(r^n\bar{r}) = 0$

8. Show that ∇[f(r)] = $\frac{f^i(r)}{r}\bar{r}$ where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$.

9. Find the directional derivative of $f = xy+yz+zx$ in the direction of vector $\bar{i} + 2\bar{j} + 2\bar{k}$ at the point (1,2,0).

10. Find the directional derivative of xyz^2+xz at (1, 1 ,1) in a directional of the normal to the surface $3xy^2$+y= z at (0,1,1).

UNIT-V TUTORIAL QUESTIONS

1. Find $\int_c \bar{F}.d\bar{r}$ where $\bar{F} = x^2y^2\bar{i} + y\bar{i}$ and the curve y^2=4x in the xy-plane from (0,0) to (4,4).
2. If $\bar{F}$ =3xy$\bar{i}$ -5z$\bar{j}$ +10x$\bar{k}$ evaluate $\int_C \bar{F}.d\bar{r}$ along the curve x=t^2+1,y=2 t^2, z = t^3 from t = 1 to t= 2.
3. If $\bar{F}$ =y$\bar{i}$ +z$\bar{j}$ +x$\bar{k}$, find the circulation of $\bar{F}$ round the curve c where c is the circle $x^2 + y^2$ =1, z=0.
4. 1.Verify Gauss Divergence theorem for $\bar{F} = (x^3 - yz)\bar{\imath} - 2x^2y\bar{\jmath} + z\bar{k}$ taken over the surface of the cube bounded by the planes x = y = z = a and coordinate planes
5. 2. Compute $\int(ax^2 + by^2 + cz^2)dS$ over the surface of the sphere $x^2+y^2+z^2$ = 1
6. Apply divergence theorem to evaluate $\int\int_S (x+z)dy\, dz + (y + z_{dz}dx + (x+y)dxdy$ where S is the surface of the sphere $x^2+y^2+z^2$=4

7. Evaluate $\int_S \bar{F}.\bar{n}ds, if\ F = xy\bar{\imath} + z^2\bar{\jmath} + 2yz\bar{k}$ over the tetrahedron bounded by x=0, y=0, z=0 and the plane x+y+z=1.
8. Use divergence theorem to evaluate $\int_S \int \bar{F}.d\bar{S}\ where\ \bar{F} =$ $x^3i+y^3j+z^3k$ and S is the surface of the sphere $x^2+y^2+z^2 = r^2$.
9. Evaluate by Green's theorem $\int_c (y - \sin x)dx + \cos x\, dy$ where C is the triangle enclosed by the lines y=0, $x=\frac{\pi}{2}$, $\pi y = 2x$.
10. Evaluate by Green's theorem for $\oint_c (x^2 - \cosh y)dx + (y + \sin x)dy$ where C is the rectangle with vertices $(0,0), (\pi, 0), (\pi, 1), (0,1)$.

IMPORTANT QUESTIONS

UNIT- I DIFFERENTIAL EQUATIONS

Exact Differential Equations

1. $(hx+by+f)dy+(ax+hy+g)dx=0$ 2. $\frac{(y\cos x+\sin y+y)}{(\sin x+x\cos y+x)}+\frac{dy}{dx}=0$

3. $(2y-x-1)dy+(2x-y+1)dx=0$ 4. $\left(e^{y}+1\right)\cos x dx+e^{y}\sin x dy=0$

5. $(2y\sin x+\cos y)dx+(x\sin y+2\cos x+\tan y)dy=0$ 6. $(e^{\frac{x}{y}}+1)dx+(1-\frac{x}{y})e^{\frac{x}{y}}dy=0$

Integrating factors

$(1+xy)ydx+(1-xy)xdy=0$ 2. $(y-x^{2})dx+(x^{2}\cot y-x)dy=0$ 3. $xdx+ydy=\frac{a\left(xdy-ydx\right)}{x^{2}+y^{2}}$

4. $\frac{y(xy+e^{x})dx-e^{x}dy}{y^{2}}=0$ 5. $ydx-xdy+3x^{2}y^{2}e^{x^{3}}dx=0$ 6. $xdx+ydy=\frac{xdy-ydx}{x^{2}+y^{2}}$

7. $ydx-xdy+3x^{2}y^{2}e^{x^{3}}dx=0$

Homogeneous exact equation

1. $(x^{2}y)dx=(x^{3}+y^{3})dy$ 2. $(x^{2}-xy-y^{2})dy+x^{2}dx$ 3. $(y^{2}-2x^{2})ydx+x(2y^{2}-x^{2})dy$

Non Homogeneous exact equation

1. $(x^{2}y^{2}+2)ydx+(2-2x^{2}y^{2})xdy=0$ 2. $(xy\sin xy+\cos xy)ydx+(xy\sin xy-\cos xy)xdy=0$

3. $(1+xy)ydx+(1-xy)xdy=0$

Integrating factors $e^{\int pdx}$

1. $(x^{2}+y^{2}+1)dx=2xydy$ 2. $(2x^{2}-xy+1)ydx+(x-y)dy=0$ 3. $(4xy+3y^{2}-x)dx+x(x+2y)dy=0$

Integrating factors $e^{\int qdy}$

1. $\left(y^{4}+2y\right)dx+\left(xy^{3}+2y^{4}-4x\right)\ dy=0$ $\left(xy^{3}+y\right)dx+2\left(x^{2}y^{2}+x+y4\right)\ dy=0$

3. $\left(2xy+e^{x}\right)\ ydx-e^{x}dy\ \ dy=0$

Newton'S Cooling Laws

1. a body is originality at 80and cools down to 60 in 20 minutes if the temperature of the air is 40 find the of the body after 40 minutes

2. If the air is maintained at 15^{0}c and the temperature of the body cools from 70^{0}C to 40^{0}c in 12 minutes ,find the temperature of the body after 30 minutes

3.A copper ball is heated to a temperature of $80^{o}c$ and time t=0, then it is placed in water which is maintained at $30^{o}c$ .if at t=3 minutes the temperature of the ball is reduced to $50^{o}c$. Find the time at which the temperature of the ball is $40^{o}c$

4.The temperature of a body dropped from 200to 100 for the first hour. Determine how many degrees the body cooled in one hour more if the environment temperature is 0

Natural Growth And Decay

1. .The number N of bacteria in a culture grew at a rate a proportional to N. The value of N was initially 100 and increased to 332 in 1 hour , what was the value of N after $1\frac{1}{2}$ hour

2. .The rate at which the bacteria multiply is preoperational to the instantaneous number present. if the original number doubles in 2 hours , when it will be triple.
3. .In chemical reaction a given substance is converted in to another at a rate proportional to the amount of the substance unconverted . If 1/5 th of the original amount has been transformed in 4 minutes , how much time will be required to transform ½
4.The number of bacteria in a liquid culture is observed to grow at a rate proportional to the number of cells present. At the beginning of the experiment there are 10,000 cells and after three hours there are 500,000. How many will there be after one day of growth if this unlimited growth continues? What is the **doubling time** of the bacteria?
5.Carbon-14 is a radioactive isotope of carbon that has a **half life** of 5600 years. It is used extensively in dating organic material that is tens of thousands of years old. What fraction of the original amount of Carbon-14 in a sample would be present after 10,000 years?
6.A certain type of bacteria, given a favorable growth medium, doubles in population every 6.5 hours. Given that there were approximately 100 bacteria to start with, how many bacteria will there be in a day and a half?

UNIT-II

HIGHER ORDER LINEAR DIFFERENTIAL EQUATION IMPORTANT QUESTIONS
I solve the following Differential equation

1) $\frac{d^3y}{dx^3}-9\frac{d^2y}{dx^2}+23\frac{dy}{dx}15y=0$ 2) $\left(D^4-2D^3-3D^2+4D+4\right)y=0$ 3) $\frac{d^2y}{dx^2}+\frac{dy}{dx}+y=0$

4) $\frac{d^2y}{dx^2}-4\frac{dy}{dx}+5y=0$ 5) $\frac{d^2y}{dx^2}-(a+b)\frac{dy}{dx}+aby=0$ 6) $\left(D^4+18D^2+81\right)y=0$

II solve the following Differential equation(e^{ax} **)**

1) $\left(D^2-3D+2\right)y=\cosh x$ 2) $\left(4D^2-4D+1\right)y=100$ 3) $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+3y=e^{2x}$

4) $\left(D^3-6D^2+11D-6\right)y=e^{-2x}+e^{-3x}$ 5) $\left(D^3-5D^2+7D-3\right)y=e^{2x}\cosh x$

6) $4\frac{d^2y}{dx^2}+4\frac{dy}{dx}-3y=e^{2x}$ 7) $(D-1)^3(D+2)y=e^x$ 8) $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+5y=-2\cosh x$

9) $\left(D^2+4D+4\right)y=18Coshx$ 10) $y^{11}-4y^1+3y=4e^{3x}$

III solve the following Differential equation $(\sin bx \ \& \cos bx)$

1) $\left(D^2+3D+2\right)y=\sin 3x$ 2) $\left(D^2-4D+3\right)y=\cos 3x$ 3) $\left(D^2-4\right)y=2\cos^2 x$

4) $\left(D^2+4\right)y=e^x+\sin 2x+\cos 2x$ 5) $\left(D^2-4D+3\right)y=\cos 5x\cos 3x$

6) $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+4y=3\sin 2x+4\cos 2x$ 7) $\frac{d^3y}{dx^3}+4\frac{dy}{dx}=\sin 2x$ 8) $\left(D^2+D+1\right)y=(1+\sin x)^2$

9) $\left(D^4-5D^2+4\right)y=10Cosx$

IV solve the following Differential equation (x^k)

1) $\frac{d^2y}{dx^2}+\frac{dy}{dx}=x^2+2x+4$ 2) $\left(D^2+D+1\right)y=x^3$ 3) $\left(D^3+2D^2+D\right)y=e^{2x}+x^2+x+\sin 2x$

4) $\frac{d^3y}{dx^3}+2\frac{d^2y}{dx^2}-\frac{dy}{dx}-2y=1-4x^3$ 5) $\left(D^2+5D\ \ +4\right)y=x^2$

V solve the following Differential equation ($e^{ax}v$)

1) $\frac{d^2y}{dx^2}+y=e^{-x}+x^3+e^x\sin x$ 2) $\frac{d^2y}{dx^2}+y=\sin x\sin 2x+e^x x^2$ 3) $(D^4-1)y=e^x\cos x$

4) $\frac{d^3y}{dx^3}-4\frac{d^2y}{dx^2}-\frac{dy}{dx}+4y=e^{3x}\cos 2x$ 5) $\frac{d^2y}{dx^2}-6\frac{dy}{dx}+13y=8e^{3x}\sin 3x$

6) $\frac{d^2y}{dx^2}-4\frac{dy}{dx}+4y=e^{2x}+\cos 2x+e^x\sin 2x$ 7) $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+3y=e^2\cos x-\cos 3x+e^x\cos x$

8) $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+4y=e^{-x}\sin 2x$ 9) $\left(D^2-3D+2\right)y=xe^{2x}+\sin 2x$ 10) $\left(D^3-7D^2+14D-8\right)y=e^xCos2x$

VI solve the following Differential equation (xv)

1) $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=x\cos x$ 2) $\frac{d^2y}{dx^2}+3\frac{dy}{dx}+2y=xe^x\cos x$ 3) $\frac{d^2y}{dx^2}-4\frac{dy}{dx}+4y=x^2\sin 2x+e^{2x}+3$

4) $\frac{d^2y}{dx^2}-4\frac{dy}{dx}+4y=8x^2e^{2x}\sin 2x$ 5) $\left(D^3-1\right)y=e^x+Sin^3x+2$ 6) $\left(D^2+2D+1\right)y=xCosx$ 7) $\left(D^2-2D+1\right)y=xe^xSinx$

VII) **Method Of Variation Of Parameter** 1. solve $\frac{d^2y}{dx^2}+a^2y=\sec ax$ 2. solve $\frac{d^2y}{dx^2}+a^2y=\cos ecax$ 3. solve $\frac{d^2y}{dx^2}+a^2y=\tan ax$ 4. solve $\frac{d^2y}{dx^2}+a^2y=\cot ax$ 5. solve $\frac{d^2y}{dx^2}+y=x\cos x$ 6. 7. solve $(D^2-2D+2)y=e^x\tan x$

UNIT - III

1. Evaluate $\int_0^{\pi}\int_0^{a\sin\Theta} r\,dr\,d\Theta$.
2. Evaluate $\int_0^{\pi}\int_0^{x^2} x(x^2+y^2)$dx dy.
3. Evaluate $\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}$ dx dy.
4. Evaluate $\int_0^1\int_1^2\int_2^3 xyz\ dx\,dy\,dz.$
5. Evaluate $\int_0^2\int_0^x e^{x+y}$dx dy.
6. Evaluate $\iint(x^2+y^2)$ dx dy in the positive quarant for which x+y≤1.
7. Evaluate $\int_0^{\frac{\pi}{4}}\int_0^{a\sin\Theta}\frac{r}{\sqrt{a^2-r^2}}$dr dΘ.
8. Find $\iint(x+y)^2dxdy$ over the area bounded by the ellipse $x^2/a^2+y^2/b^2=1$.(sept 2006)
9. Evaluate $\iint(x^2+y^2)dxdy$ in positive quadrant for which x+y< 1.(may 2006)
10. Evaluate $\iint(x^2+y^2)dxdy$ ove the area bounded by the ellipse $x^2/a^2+y^2/b^2=1$.(.(Dec 2010)
11. Evaluate $\iint r^3 dr d\Theta$ over the area included between the circles r=2 sinɵ and r=4 sinɵ.(Dec 2010)
12. Evaluate the triple integral $\iiint xy^2$z dx dy dz taken through the positive octant of the sphere $x^2+y^2+z^2=a^2$. (Dec 2010)
13. Evaluate $\iiint z^2$ dx dy dz taken over the volume bounded by the surfaces $x^2+y^2=a^2$,$x^2+y^2=z$ and Z=0.(may 1999)
14. Evaluate $\iiint xyz$ dx dy dz where V is the domain bounded by the coordinate planes and the plane x+y+z=1(Dec 2000)
15 .Evaluate $\iiint xyz$ dx dy dz,where the domain V is bounded by the plane x+y+z=a and the Coordinate planes.(sep 2006)
16. Find the area of the loop of the curve r=a(1+cosɵ).(sep 2007)
17. Find the volume common to the cylinder $x^2+y^2=a^2$ and $x^2+z^2=a^2$.(Dec 2000)
18. Find volume bounded by the cylinder $x^2+y^2=4$,y+z=4 and z=0.(sep 2000)
19. Find the volume of the solid generated by the revolution of the cardioid r=a(1-cosɵ).(may 2006)

20 .Find the volume of the region bounded by $z=x^2+a^2, z=0, x=-a, x=a, y=-a, y=a$.(sep 2008)

21. Find the volume of the solid generated by the revolution of the cardioid $r=a(1-\cos\theta)$ about its axis.(may 2007)

22. Find by double integral ,the volume of the solid bounded by $z=0, x^2+y^2=1$ and $x+y+z=3$.(may 2010)

23 Evaluate $\int\int y\,dx\,dy$ where R is the region bounded by the parabolas $y^2 = 4x$ and $x^2 = 4y$

24Evaluate $\int\int y\,dx\,dy$ where R is the region bounded by the parabolas $y^2 = 4x$ and $x^2 = 4y$

25 Evaluate $\int\int x^2 + y^2$ dx dy in positive quadrant for which $x+y\leq 1$.

26. Evaluate the following integrals by changing the order of integration

$$\int_0^1 \int_0^{\sqrt{1-x^2}} y^2\,dx\,dy$$

27. Evaluate the following integrals by changing the order of integration $\int_0^a \int_y^a \frac{x}{x^2+y^2}$dy dx

28. Evaluate the following integrals by changing the order of integration

$$\int_0^a \int_{\frac{x}{a}}^{\sqrt{\frac{x}{a}}} (x^2 + y^2)\,dx\,dy$$

29. Evaluate the following integrals by changing to polar co-ordinates. $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}$dxdy

30. Show by double integration, the area between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ is $\frac{16}{3}a^2$

UNIT - IV

1. If $a=x+y+z$, $b= x^2+y^2+z^2$, $c = xy+yz+zx$, prove that [grad a, grad b, grad c] = 0.
2. Show that $\nabla[f(r)] = \frac{f^i(r)}{r}\bar{r}$ where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$.
3. Prove that $\nabla(r^n)= nr^{n-2}\bar{r}$.
4. Find the directional derivative of $f = xy+yz+zx$ in the direction of vector $\bar{i} + 2\bar{j} + 2\bar{k}$ at the point (1,2,0).
5. Find the directional derivative of the function $xy^2+yz^2+zx^2$ along the tangent to the curve $x =t$, $y = t^2$, $z = t^3$ at the point (1,1,1).
6. Find the directional derivative of the function $f = x^2-y^2+2z^2$ at the point P =(1,2,3) in the direction of the line $\overline{PQ}$ where Q = (5,0,4).
7. Find the greatest value of the directional derivative of the function $f = x^2yz^3$ at (2,1,-1).
8. Find the directional derivative of xyz^2+xz at (1, 1 ,1) in a directional of the normal to the surface $3xy^2+y= z$ at (0,1,1).
9. Find the directional derivative of $2xy+z^2$ at (1,-1,3) in the direction of $\bar{i} + 2\bar{j} + 3\bar{k}$.
10. Find the directional derivative of $\phi = x^2yz+4xz^2$ at (1,-2,-1) in the direction 2i-j-2k.

11. If the temperature at any point in space is given by t = xy+yz+zx, find the direction in which temperature changes most rapidly with distance from the point (1,1,1) and determine the maximum rate of change**.**

12. Find the directional derivative of $\phi(x,y,z) = x^2yz+4xz^2$ at the point (1,-2,-1) in the direction of the normal to the surface $f(x,y,z) = x \log z-y^2$ at (-1,2,1).

13. Find a unit normal vector to the given surface $x^2y+2xz = 4$ at the point (2,-2,3).

14. Evaluate the angle between the normals to the surface $xy= z^2$ at the points (4,1,2) and (3,3,-3).

15. Find a unit normal vector to the surface $x^2+y^2+2z^2 = 26$ at the point (2, 2 ,3).

16. Find the values of a and b so that the surfaces $ax^2-byz = (a+2)x$ and $4x^2y+z^3= 4$ may intersect orthogonally at the point (1, -1,2).

17. (or) Find the constants a and b so that surface $ax^2-byz=(a+2)x$ will orthogonal to $4x^2y+z^3=4$ at the point (1,-1,2).

18. Find a unit normal vector to the surface $z= x^2+y^2$ at (-1,-2,5)

19. Find the angle of intersection of the spheres $x^2+y^2+z^2 =29$ and $x^2+y^2+z^2 +4x-6y-8z-47 =0$ at the point (4,-3,2).

20. Find the angle between the surfaces $x^2+y^2+z^2 =9$, and $z = x^2+y^2- 3$ at point (2,-1,2).

21. **If** $\bar{f} = xy^2\bar{i} + 2x^2yz\bar{j} - 3yz^2\bar{k}$ find div $\bar{f}$ at(1, -1, 1).

22. find div $\bar{f}$ = grad$(x^3+y^3+z^3-3xyz)$

23. **:** If $\bar{f} = (x+3y)\bar{i} + (y-2z)\bar{j} + (x+pz)\bar{k}$ is solenoidal, find *P*.

24. Find div $\bar{f} = r^n\bar{r}$. Find n if it is solenoidal?

25. Evaluate$\nabla.\left(\frac{\bar{r}}{r^3}\right)$ where $\bar{r} = xi + yj + zk \; and \; r = |\bar{r}|$.

26. Find div $\bar{r}$. where $\bar{r}. = x\bar{i} + y\bar{j} + z\bar{k}$

27. **if** $\bar{f} = xy^2\bar{i} + 2x^2yz\,\bar{j} - 3yz^2\,\bar{k}$ find curl $\bar{f}$ at the point (1,-1,1).

28. Find curl $\bar{f}$ where $\bar{f}$ = grad$(x^3+y^3+z^3-3xyz)$

29. Prove that if $\bar{r}$ is the position vector of an point in space, then $r^n\bar{r}$ is Irrotational. (or) Show that $(r^n\bar{r}) = 0$

30. **If** $\bar{a}$ is a constant vector, prove that curl $\left(\frac{\bar{a}x\bar{r}}{r^3}\right) = -\frac{\bar{a}}{r^3} + \frac{3\bar{r}}{r^5}(\bar{a}.\bar{r})$.

31. Show that the vector $(x^2-yz)\bar{i} + (y^2-zx)\,\bar{j} + (z^2-xy)\bar{k}$ is irrotational and find its scalar potential

32. Find constants a,b and c if the vector $\bar{f} = (2x+3y+az)\bar{i} + (bx+2y+3z)\,\bar{j} + (2x+cy+3z)\bar{k}$ is Irrotational

33. Find constants a,b,c so that the vector $\bar{A}=(x+2y+az)\bar{i}+(bx-3y-z)\,\bar{j}+(4x+cy+2z)\bar{k}$ is Irrotational. Also find ϕ such that $\bar{A}=\nabla\phi$.
34. Prove that div.(grad r^m)= $m(m+1)r^{m-2}$
35. Show that $\nabla^2[f(r)]= \frac{d^2 f}{dr^2}+\frac{2}{r}\frac{df}{dr}=f^{11}(r)+\frac{2}{r}f^{1}(r)$
36. If $\bar{a}$ is a differentiable function and ϕ is a differentiable scalar function. Then prove that div($\phi\bar{a}$)= (grad ϕ).$\bar{a}$ +ϕ div $\bar{a}$ or $\nabla.(\phi\bar{a})=(\nabla\phi)\bar{a}+\phi(\nabla.\bar{a})$
37. prove that curl ($\phi\bar{a}$)= (grad ϕ)x$\bar{a}$ +ϕ curl $\bar{a}$
38. Prove that grad ($\bar{a}.\bar{b}$)= $(\bar{b}.\nabla)\bar{a}+(\bar{a}.\nabla)\bar{b}+\bar{b}x\,curl\,\bar{a}+\bar{a}x\,curl\,\bar{b}$
39. Prove that div ($\bar{a}$ x $\bar{b}$)= $\bar{b}.curl\,\bar{a}-\bar{a}.curl\,\bar{b}$

UNIT – V

Line integral:

3. If $\bar{F}$ (x^2-27) $\bar{i}$ -6yz $\bar{j}$ +8xz^2 $\bar{k}$, evaluate $\int \bar{F}.d\bar{r}$ from the point (0,0,0) to the point (1,1,1) along the Straight line from (0,0,0) to (1,0,0), (1,0,0) to (1,1,0) and (1,1,0) to (1,1,1).

4. If $\bar{F}$ (5xy-6x^2) $\bar{i}$ (2y-4x) $\bar{j}$, evaluate $\int_C \bar{F}.d\bar{r}$ along the curve C in xy plane y=$x^3$from (1,1) to (2,8).

5. Find the work done by the force $\bar{F}=z\bar{i}+x\bar{j}+y\bar{k}$, when it moves a particle along the arc of the curve $\bar{r}$ = cost $\bar{i}$ + sint $\bar{j}$ -t $\bar{k}$ from t = 0 to t = 2π

6. Find $\int_c \bar{F}.d\bar{r}$ where $\bar{F}$ = $x^2y^2\bar{i}$ +y$\bar{i}$ and the curve y^2=4x in the xy-plane from (0,0) to (4,4).

7. If $\bar{F}$ =3xy$\bar{i}$ -5z $\bar{j}$ +10x$\bar{k}$ evaluate $\int_C \bar{F}.d\bar{r}$ along the curve x=t^2+1,y=2 t^2, z = t^3 from t = 1 to t= 2.

8. If $\bar{F}$ =y$\bar{i}$ +z$\bar{j}$ +x$\bar{k}$, find the circulation of $\bar{F}$ round the curve c where c is the circule x^2 +y^2 =1, z=0.

9. If $\phi = x^2yz^3, evaluate \int_c \phi d\bar{r}$ along with curve x= t, y =2t, z=3t from t = 0 to t=1.

10. If $\phi = 2xy^2z + x^2y, evaluate \int_c \phi d\bar{r}$ where c is the curve x=t, y=t^2, z= t^3 from t=0 to t=1.

11. Find the work done by the force $\bar{F}=(x^2-yz)\bar{i}+(y^2-zx)\bar{j}+(z^2-xy)\bar{k}$ in taking particle from (1,1,1) to 3,-5,7).

12. Find the work done by the force $\bar{F} = (2y+3)i + (zx)\,j + (yz - x)\,k$ when it moves a particle from the point (0,0,0) to (2,1,1) along the curve $x = 2t^2$, $y = t$, $z = t^3$

Surface integral

4. Evaluate $\int \bar{F}.\mathrm{ndS}$ where $\bar{F} = zi + xj - 3y^2zk$ and S is the surface $x^2 + y^2 = 16$ included in the first octant between z = 0 and z = 5.
5. If $\bar{F} = zi + xj - 3y^2zk$, evaluate $\int_S \bar{F}.\mathrm{ndS}$ where S is the surface of the cube bounded by x = 0, x = a, y = 0, y= a, z = 0, z = a.

Gauss Divergence theorem.

1. Verify Gauss Divergence theorem for $\bar{F} = (x^3 - yz)\bar{\imath} - 2x^2y\bar{\jmath} + z\bar{k}$ taken over the surface of the cube bounded by the planes x = y = z = a and coordinate planes
2. Compute $\int (ax^2 + by^2 + cz^2)dS$ over the surface of the sphere $x^2+y^2+z^2 = 1$
6. By transforming into triple integral, evaluate $\int\int x^3\, dy\, dz + x^2y\, dz\, dx + x^2 dx\, dy$ where S is the closed surface consisting of the cylinder $x^2+y^2 = a^2$ and the circular discs z= 0 , z= b.
7. Applying Gauss divergence theorem, Prove that $\int \bar{r}\ .\bar{n}dS = 3V$ or $\int \bar{r}\ .d\bar{s} = 3V$
8. Show that $\int_S (ax\bar{\imath} + by\bar{\jmath} + cz\bar{k}).\bar{n}dS = \frac{4\pi}{3}(a+b+c)$, where S is the surface of the sphere $x^2+y^2+z^2=1$
9. Using Divergence theorem, evaluate $\int\int_S (x\, dy\, dz + y\, dz\, dx + z\, dx\, dy)$, where S: $x^2+y^2+z^2=a^2$
10. Apply divergence theorem to evaluate $\int\int_S (x+z)dy\, dz + (y + z_{dz}dx + (x+y)dxdy$ where S is the surface of the sphere $x^2+y^2+z^2=4$
11. Evaluate $\int_S \bar{F}.\bar{n}ds$, if $F = xy\bar{\imath} + z^2\bar{\jmath} + 2yz\bar{k}$ over the tetrahedron bounded by x=0, y=0, z=0 and the plane x+y+z=1.
12. Use divergence theorem to evaluate $\int_S \int \bar{F}.d\bar{S}$ where $\bar{F} = x^3i + y^3j + z^3k$ and S is the surface of the sphere $x^2+y^2+z^2 = r^2$
13. Use divergence theorem to evaluate $\int\int_S \bar{F}.dS$ where $\bar{F} = 4xi - 2y^2j + z^2k$ and S is the surface bounded by the region $x^2+y^2=4$, z=0 and z=3.
14. Verify divergence theorem for $\bar{F} = x^2i + y^2j + z^2k$ over the surface S of the solid cut off by the plane x+y+z=a in the first octant.

15. Verify divergence theorem for $2x^2yi-y^2j+4xz^2k$ ove the region of first octant of the cylinder $y^2+z^2=9$ and x=2.
16. Evaluate $\int_S \int \bar{F}.\bar{n}ds,\ where\ \bar{F} = 2x^2yi - y^2j + 4xz^2k$ and S is the closed surface of the region in the octant bounded by the cylinder $y^2+z^2 = 9$ and the planes x=0, x=2, y=0, z=0
17. Use Divergence theorem to evaluate $\int\int(x\bar{\imath} + y\bar{\jmath} + z^2\bar{k}).\bar{n}.ds\ Where$ S is the surface bounded by the cone $x^2+y^2=z^2$ in the plane z = 4
18. Use Gauss Divergence theorem to evaluate $\int\int_S (yz^2\bar{\imath} + zx^2\bar{\jmath} + 2z^2\bar{k}).ds, where$ S is the closed surface bounded by the xy plane and the upper half of te sphere $x^2+y^2+z^2=a^2$
19. Verify Gauss divergence theorem for $\bar{F} = x^3\bar{\imath} + y^3\bar{\jmath} + z^3\bar{k}$ taken over the cuve bounded by x = 0, x = a, y= 0, y = a, z = 0, z = a.
20. Evaluate $\iint_s xdy\,dz + y\,dz\,dx + zdxdy$ over $x^2 + y^2 + z^2=1$
21. Compute $\iint(a^2x^2 + b^2y^2 + c^2z^2)^{\frac{1}{2}}\,dS$ over the ellipsoid $a^2x^2 + b^2y^2 + c^2z^2=1$
22. Find $\int_s \bar{F}.\bar{n}dS$ where $\bar{F} = 2x^2\bar{\imath} - y^2\bar{\jmath} + 4xz\bar{k}$ and S is the region in the first octant bounded by $y^2 + z^2=9$ and x=0,x=2.
23. Find $\int_s (4.x\bar{\imath} - 2y^2\bar{\jmath} + z^2k).\bar{n}\,dS$ Where S Is the region bounded by $x^2 + y^2 =4$, z=0 and z=3.
24. Verity divergence theorem for $F=6z\bar{\imath} + (2x+y)\bar{\jmath} - x\bar{k}$, taken over the region bounded by the surface of the cylinder $x^2 + y^2 =9$ included in z=0, z=8, x=0 and y=0. [JNTU 2007 S(Set No.2)]

Green's theorem

1. Verify Green's theorem in plane for $\oint(3x^2 - 8y^2)dx + (4y - 6xy)dy$ where C is the region bounded by $y=\sqrt{x}$ and $y=x^2$.
2. Evaluate by Green's theorem $\int_c(y - \sin x)dx + \cos x\,dy$ where C is the triangle enclosed by the lines y=0, $x=\frac{\pi}{2}$, $\pi y = 2x$.
3. Evaluate by Green's theorem for $\oint_c (x^2 - \cosh y)dx + (y + \sin x)dy$ where C is the rectangle with vertices $(0,0), (\pi, 0), (\pi, 1), (0,1)$.
4. A Vector field is given by $\bar{F} = (\sin y)\boxed{\bar{\imath}} + x(1 + \cos y)\boxed{\bar{\jmath}}$.
5. Evaluate the line integral over the circular path $x^2+y^2 = a^2$, z=0

   Directly (ii) By using Green's theorem

6. Show that area bounded by a simple closed curve C is given by $\frac{1}{2}\oint xdy - ydx$ and hence find the area of

   (i)The ellipse x=$acos\theta$ ⟦$\theta, y = bsin\theta$ ⟦θ $(i.e)$ $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$⟧ ⟧

   (II)The Circle x=$a\cos\theta$, $y = a\sin\theta$ $(i.e) x^2 + y^2 = a^2$
7. Verify Green's theorem for $\int_c [(xy + y^2)dx + x^2 dy]$, where C is bounded by y=x and y= x^2
8. Using Green's theorem evaluate $\int_c (2xy - x^2)dx + (x^2 + y^2)dy$, Where "C" is the closed curve of the region bounded by y=x^2 and $y^2 = x$
9. Verify Green's theorem for $\int_c [(3x^2 - 8y^2)dx + (4y - 6xy)dy]$ where c is the region bounded by x=0, y=0 and x+y=1.
10. Apply Green's theorem to evaluate $\oint_c (2x^2 - y^2)dx + (x^2 + y^2)\, dy, where\ c\ is$
11. The boundary of the area enclosed by the x-axis and upper half of the circle $x^2 + y^2 = a^2$
12. Verify Green's theorem in the plane for $\int_C (x^2 - xy^3)\, dx + (y^2 - 2xy)dy$
13. Evaluate $\oint_c (3x + 4y)dx + (2x - 3y)\, dy$ where c is the circle $x^2 + y^2 = 4$
14. Verify Green's theorem in the plane for $\oint (x^2 - xy^3)dx + (y^2 - 2xy)dy$ where c is the square with vertices (0,0), (2,0), (2,2) and (0,2). [JNTU Sep 2008, 2008S, JNTU(H) 2009(Set No.1)]
15. Use Green's theorem to evaluate $\oint_c x^2(1 + y)dx + (y^3 + x^3)\, dy$ where c is the square bounded by y=$\pm 1\ and\ x = \pm 1$.
16. Find the area bounded by one arc of the cycloid x=a $(\theta - sin\theta), y = a(1 - cos\theta), a > 0\ and\ the\ x - axis.$
17. Find the area bounded by the hypocycloid $x^{2/3} + y^{2/3} = a^{2/3}, a > 0.$
18. Find $\oint (x^2 - y^2)dx + dx + 3xy^2dy$ where c is the circle $x^2 + y^2 = 4$ in xy plane.

Stokes theorem

1. Verify Stokes theorem for $\bar{F} = -y^3\bar{\imath} + x^3\bar{\jmath}$, Where S is the circular disc $x^2 + y^2 \leq 1, z = 0.$
2. If $\bar{F} = y\bar{\imath} + (x - 2xz)\bar{\jmath} - xy\bar{k}, evaluate\ \int_s (\nabla \times \bar{F}).\bar{n}ds\ Where$ Sis the surface of sphere $x^2 + y^2 + z^2 = a^2, above\ the\ xy - plane.$
3. Verify Stokes theorem for $\bar{F} = (2x - y)\bar{\imath} - yz^2\bar{\jmath} - y^2z\bar{k}$ over the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$bounded by the projection of the xy-plane

4. Verify Stokes theorem for the function $\bar{F} = x^2\,\bar{\imath} + xy\,\bar{\jmath}$ integrated round the square in the plan z=0 whose sides are along the lines x=0, y=0, x=a, y=a.
5. Apply Stokes theorem, to evaluate $\oint_c (ydx + zdy + xdz)$ where c is the curve of intersection of the sphere $x^2 + y^2 + z^2 = a^2$ and x+z=a.
6. Apply the Stoke's theorem and show that $\int_S \int curl\ \bar{F}.\bar{n}d\bar{s} = 0\ where\ \bar{F}$ is any vector and S $= x^2 + y^2 + z^2 = 1$
7. Use Stoke's theorem to evaluate $\int \int_S curl\ \bar{F}.\bar{n}\ dS$ over the surface if the paraboloid z+ $x^2 + y^2 = 1, z \geq 0$ where $\bar{F} = y\,\bar{\imath} + z\,\bar{\jmath} + x\,\bar{k}$.
8. Verify Stoke's theorem for $\bar{F} = (x^2 + y^2)\bar{\imath} - 2xy\bar{\jmath}$ taken round the rectangle bounded by the lines x=$\pm a, y = 0, y = b$.
9. Verify Stoke's theorem for $\bar{F} = (y - z + 2)\bar{\imath} + (yz + 4)\bar{\jmath} - xz\bar{k}$ where S is the surface of the cube x =0, y=0, z=0, x=2, y=2,z=2 above the xy plane.
10. Verify the Stoke's theorem for $\bar{F} = y\bar{\imath} + z\bar{\jmath} + x\bar{k}$ and surface is the part of the sphere $x^2 + y^2 + z^2 = 1\ above\ the\ xy\ plane.$
11. Verify Stoke's theorem for $\bar{F} = (x^2 + y^2)\bar{\imath} + 2xy\bar{\jmath}$ over the box bounded by the planes x=0,x=a,y=0,y=b.
12. Verify Stoke's theorem for $\bar{F}=(x^2 - y^2)\bar{\imath}$ + 2xy$\bar{\jmath}$ over the box bounded by the planes x=0,x=a,y=0,y=b,z=c
13. Using Stroke's theorem evaluate the integral $\int_C \bar{F}\,.\,d\bar{r}$ where$\bar{F}$=2$y^2\,\bar{\imath}$+3$x^2\,\bar{\jmath}$ -(2x+z)$\bar{k}$ and C is the boundary of the triangle whose vertices are (0,0,0),(2,0,0),(2,2,0).

14.Evaluate $\iint_s (\nabla \times \bar{F}).\bar{n}dS$, where $\bar{F}$=(x^2 +y-4)$\bar{\imath}$ + 3xy$\bar{\jmath}$+(2xz+z^2)$\bar{k}$ and S is the surface of,

(i). The hemisphere $x^2 + y^2 + z^2$=16 above the xy-plane , (ii) The paraboloid Z=4 - ($x^2 + y^2$) above the xy-plane

OBJECTIVE QUESTIONS

UNIT-I

1. For the differential equation $\frac{dy}{dx}+\frac{y}{x}=\frac{y^2}{x^2}$,which of the following is not applicable

   a) It is Bernoulli equation b) It is Homogeneous c) It is not exact d) Solution is $y=2x^2/(1+x^2)$

2. The particular solution of the equation $y^{'}\sin x = yIny$ satisfying the initial condition $y(\pi/2)=e$ is

   a) $e^{\tan(x/2)}$ b) $e^{\cot(x/2)}$ c) $In\tan(x/2)$ d) $In\cot(x/2)$

3. The initial value problem $x\frac{dy}{dx}=y, y(0)=0, x\geq 0$ has

   a) No solution b)A unique solution c) Exactly two solutions d) Unaccountably many solutions

4. For the differential equation $xy^{'}-y=0$ which of the following is not an integrating factor?

   i. a) $1/x^2$ b) $1/y^2$ c) $1/xy$ d) $1/(x+y)$

5. An integrating factor for $ydx-xdy=0$ is

   2. a)*x/y* b)*y/x* c)*1/x² y²* d)*1/(x² +y²).*

6. The particular integral of $\frac{d^2y}{dx^2}+y=\cos x$ is

   a. a) $\frac{1}{2}\sin x$ b) $\frac{1}{2}\cos x$ c) $\frac{1}{2}x\cos x$ d) $\frac{1}{2}x\cos x$

7. The general solution of dy/dx=e^{x+y} is
a) $e^x+e^y=c$ b)$e^x+e^{-y}=c$ c)$e^{-x}+e^y=c$ d)$e^{-x}+e^{-y}=c$

8. Find the differential equation corresponding to y=$ae^x+be^{2x}+ce^{3x}$
a) $y^{111}-6y^{11}+11y^1-6y=0$ b) $y^{111}+y^{11}-3y^1=0$ c) $y^{11}+2y^1+y=0$ d) $y^{111}-2y^{11}+3y^1+y=0$

9. Find the differential equation of the family of curves y=e^x(Acosx+Bsinx)
a) $y^{11}-2y^1+3y=0$ b) $y^{11}-3y^1+y=0$ c) $y^{11}-2y^1+3y=0$ d) none

10. Form the differential equation by eliminating the arbitrary constant y2=(x-c)2
a) $(y^1)^2=1$ b) $y^{11}+2y^1=2$ c) $(y^1)^2=0$ d) none

UNIT-II OBJECTIVEQUESTIONS

1. The P.I of $(D^2-5D+6)y=e^{2x}$ is
a)-x e^{2x} b) x e^{2x} c) e^{2x} d) 0

2. P.I of $(D+1)^2y=x$ is
a) x b) x-2 c) $(x+1)^2$ d) $(x+2)^2$

3. P.I of $1/D^2+D+1(\sin x)$=
a) sinx b) -cosx c) 1/3 sinx d)1-cosx

4. P.I of $(D-1)^4y=e^x$ is

a) $x^4/4!(e^x)$ b) x^4e^x c) e^x d) $e^x/4$

5. The value of 1/D-2(sinx) is
a) -1/5(cosx+sinx) b) 1/5(cosx) c) 1/5(sinx) d) 1/5(cosx-2sinx)

6. The value of $1/D^2+4(\sin 2x)$ is

a) 1/5 (sin2x) b) $-1/5 \sin^2 x$ c) 1/5(cos2x) d) -1/4 cos2x

7.$1/D^2-1(e^x)$=
a) $1/2(xe^x)$ b)$-1/2(xe^x)$ c) $x^2/2(e^x)$ d) none

8. $1/D+2(x+e^x)$=

a)$-x/4-1/16+e^x/3$ b) $x/4+1/16-e^x/3$ c) $x/4-1/16+e^x$ d) none

9. P.I of $(D^4-1)y=e^x\cos x$ (b)
a) $-e^x\cos y/6$ b) $-e^x\cos x/5$ c) $-e^x\cos x/3$ d) $e^x\cos x/5$

10 .C.F of $(D-1)^2y=\sin 2x$ is

a) $(c_1+c_2x)e^x$ b)$(c_1+c_2x)e^{-2x}$ c)$c_1x+c_2e^x$ d) none

11. P.I of $(D^2+1)y=x^2e^{3x}$ is

a) $e^{3x}/250\,(25x^2+30x+30)$ b) $e^{3x}/250\,(25x^2-30x-30)$ c) $e^{3x}/250\,(25x^2-30x+30)$ (d) $e^{3x}/25\,(25x^2-30x+30)$

12. P.I of $(D^2-2D+1)y=\cosh x$ is
a) $x^2/4(e^x)+e^{-x}/8$ b) $x^2/4(e^{-x})+e^x/8$ c)$x^2/4(e^x)$ d) $c_1e^x+c_2e^{-x}$

13. If 30% of the ratio active substance disappears in 10days ,how long will it take for 90% of it to disappear
a) 34.5days b) 64.5days c)100.0days d)55.5days

UNIT - III

1. $\iint xydxdy$ Over the region bounded by X – Axis , ordinate x=2a and the parabola $x^2 = 4ay$ is

a) $\frac{a^4}{3}$ b) $\frac{a^6}{16}$ c) $\frac{a^4}{32}$ d) $\frac{a^4}{16}$

2. $\iiint(x^2 + y^2 + z^2)dzdydx$ Where V is the Volume of v The cube bounded by the co-originate planes $x = y = z = a$

a) $\frac{a^4}{3}$ b) $\frac{a^5}{16}$ c) $\frac{a^5}{32}$ d) $\frac{a^5}{1}$

3. $\int_0^1\int_y^1\int_0^{1-x} xdzdxdy$ is

a) $\frac{1}{3}$ b) $\frac{1}{16}$ c) $\frac{1}{32}$ d) $\frac{1}{12}$

4. $\int_0^1\int_0^{1-x}\int_0^{1-x-y} dxdydz$ is

a) $\frac{1}{3}$ b) $\frac{1}{16}$ c)0 d) $\frac{1}{12}$

5. $\iint ydxdy$ over the region bounded by the parabolas $y^2 = 4x, x^2 = 4y$

a) $\frac{48}{5}$ b) $\frac{1}{16}$ c)0 d) $\frac{1}{12}$

6. $\int_0^2 \int_0^x y\, dy\, dx$

(a) $\frac{4}{3}$ (b) $\frac{4}{5}$ (c) $\frac{2}{3}$ (d) $\frac{2}{5}$

7. $\int_{x=0}^{a} \int_{y=0}^{b} (x^2 + y^2)$ dx dy

(a) $\frac{1}{3}(a^2+b^2)$ (b) $\frac{a}{3}(a^2+b^2)$ (c) $\frac{b}{3}(a^2+b^2)$ (d) $\frac{ab}{3}(a^2+b^2)$

8. $\int_0^1 \int_0^1 \frac{dx\, dy}{\sqrt{(1-x^2)(1-y^2)}}$

(a) $\frac{\pi}{2}$ (b) $\frac{\pi^2}{2}$ (c) $\frac{\pi^2}{4}$ (d) $\frac{\pi}{4}$

9. $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}$dxdy

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{8}$

10 $\int_0^\pi \int_0^{a\sin\theta} r\, dr\, d\theta$

(a) $\frac{\pi a^2}{4}$ (b) $\frac{\pi a}{4}$ (c) $\frac{\pi a^2}{2}$ (d) $\frac{\pi a}{2}$

=

11. $\int_0^2\int_0^{\sqrt{2x-x^2}}\frac{x}{\sqrt{x^2+y^2}}dxdy$

(A)5/2

(B)4/3

(C)3/2

(D)2/3

12. $\int_0^a \int_0^{\sqrt{x^2+y^2}} dx\, dy$ after changing to polar co-ordinates is

(a) $\int_0^{\frac{\pi}{2}} \int_0^a r^3 \sin\theta\, dr\, d\theta$ (b) $\int_0^{\pi} \int_0^a r^2 \sin\theta\, dr\, d\theta$

(c) $\int_0^{\frac{\pi}{2}} \int_0^a r \sin\theta\, dr\, d\theta$ (d) None

13 $\int_0^a \int_0^{\frac{a\sqrt{b^2-y^2}}{b}} xy\, dx\, dy =$

(a) $\int_0^a \int_0^{\frac{b\sqrt{a^2-x^2}}{b}} xy\, dx\, dy$ (b) $\int_0^{\frac{a\sqrt{b^2-y^2}}{b}} \int_0^b yx\, dy\, dx$

(c) $\int_0^a \int_0^b xy\, dx\, dy$ (d) $\frac{a^2b^2}{8}$

14. $\int_0^1 \int_1^2 \int_2^3 x\, y\, z\, dx\, dy\, dz$

(a) $\frac{15}{2}$ (b) $\frac{15}{4}$ (c) $\frac{15}{9}$ (d) $\frac{15}{8}$

15. The area enclosed by the parabolas x^2 = y and y^2=x is

(a) $\frac{2}{3}$ (b) $\frac{1}{3}$ (c) $\frac{3}{2}$ (d) $\frac{3}{4}$

16. The area of the region bounded by y^2 = 4ax and x^2 = 4ay is

(a) $\frac{4a^2}{3}$ (b) $\frac{8a^2}{3}$ (c) $\frac{16a^2}{3}$ (d) $\frac{25a^2}{3}$

UNIT – IV

1. A unit vector normal to the surface $xy^3z^2 = 4$ at the point (-1,-1,2) is ( )

a) $-\frac{1}{\sqrt{11}}(\vec{i}+3\vec{j}-\vec{k})$ b) $\frac{1}{\sqrt{11}}(\vec{i}+3\vec{j}-\vec{k})$

c) $-\frac{1}{\sqrt{11}}(\vec{i}-3\vec{j}+\vec{k})$ d) $\frac{1}{\sqrt{11}}(\vec{i}-3\vec{j}+\vec{k})$

2.The greatest rate of increase of $u = x^2 + yz^2$ at the point (1,-1,3) is ( )

a) $\sqrt{79}$ b)11 c) $\sqrt{89}$ d) $4\sqrt{7}$

3. Directional derivative is maximum along ( )

a) Tangent to the surface b) Normal to the surface
c) any unit vector d) Coordinate axis

4 If for a vector function $\vec{F}, div\vec{F} = 0$ then $\vec{F}$ is called ( )

a)Irrotational b)Conservative c)Solenoidal d)Rotational

5. For a vector function $\vec{F}$,there exists a scalar potential only when ( )

a) $div\vec{F} = 0$ b) $grad(div\vec{F}) = 0$

c) $curl\vec{F} = 0$ d) $\vec{F}curl\vec{F} = 0$.

6. If $\vec{a}$ is a constant vector and $\vec{r} = x\vec{i} + y\vec{j} + \vec{z}k$,then $\nabla X(\vec{a}X\vec{r})$ is equal to ( )

a)0 b) $\vec{a}$ c) $2\vec{a}$ d)-2 $\vec{a}$

7.Using the following integral,work done by a force $\vec{F}$ can be calculated: ( )

a) Line Integral b)Surface Integral
c)Volume Integral d) None of these

8. The gradient of a differentiable scalar field is ( )
a) Irrotational b) Solenoidal
c)Both Irrotional and Solenoidal d) None

9. Gauss Divergence theorem is a relation between ( )
a) A line integral and a surface integral
b) a surface integral and a volume integral
c) A line integral and a volume integral
d) two volume integrals

10. Green's theorem in the plane is applicable to ( )
a)*xy*-plane b)*yz*-plane c)*zx*-plane d)all of these

UNIT – V

(1) For any closed surface S, $\iint_s curl\ \bar{F}\ \bar{n} dS$ =

(a) 0 (b) 2 $\bar{F}$ (c) $\bar{n}$ (d) $\oint \bar{F}\ d\bar{r}$

(2) If S is any closed surface enclosing a volume V and $\bar{F} = x\bar{\imath} + 2y\,\bar{\jmath}$ +3z$\bar{k}$ then $\iint_s \bar{F}\ \bar{n} dS$ =

(a) V (b)3V (c)6V (d)None

(3) If $\bar{r} = x\bar{\imath} + y\,\bar{\jmath}$ +z$\bar{k}$ then $\oint \bar{r}\ d\bar{r}$ =

(a) 0 (b) $\bar{r}$ (c) x (d) None

(4) $\int \bar{r}$ x $\bar{n}\, dS$ =

(a) 0 (b) r (c) 1 (d) None

(5) $\int_s \bar{r}\ .\ \bar{n}\ dS$ =

(a) V (b) 3V (c) 4V (d) None

(6) If $\bar{n}$ is the unit outward drawn normal to any closed surface then $\int_s$ div $\bar{n}\ dV$ =

(a) S (b)2S (c) 3S (d) None

(7) $\oint f\ \nabla f\ .d\bar{r}$ =

(a) f (b)2f (c) 0 (d) None

(8) The value of the line integral $\int grad(x + y - z) d\bar{r}$ from (0, 1,-1) to (1 , 2, 0) is

(a) -1 (b) 0 (c) 2 (d) 3

(9) A necessary and sufficient condition that the line integral $\int_c A\ .\ dr$ =0 for every closed curve c is that

(a) div A=0 (b)div A 0 (c) curl A=0 (d) curl A 0

(10) If $\bar{F}$ =axi + byj+ czk where a, b, c are constants then $\iint \bar{F}\ .\bar{n} dS$ where S is the surface of the unit sphere is

(a) 0 (b) $\frac{4}{3}\pi(a + b + c)$ (c) $\frac{4}{3}\pi(a + b + c)^2$ (d) none

EXTERNAL QUESTION PAPERS

Code No : 1800BS02

MALLA REDDY ENGINEERING COLLEGE FOR WOMEN

(Autonomous Institution – UGC Govt. of India)

Permanently Affiliated to JNTUH, Approved by AICTE

Accredited by NBA and NAAC with A grade– ISO 9001-2015Certified

B.TECH I YEAR II Semester Regular Examinations, April 2019

MATHEMATICS-II

Time: 3 hours **(**Common to all Branches**)** **Max Marks: 70**

Note: This question paper Consists of 5 Sections. Answer **FIVE** Questions, Choosing ONE Question from each SECTION and each Question carries 14 marks.

SECTION-I

1. (a) Solve the exact differential equation $(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy$. (6M)

   (b) Find the general and singular solution of the equation $y = px - \sqrt{1+p^2}$. (8M)

OR

2. (a) Solve the Bernoulli's equation $xy\,(1+xy^2)\dfrac{dy}{dx} = 1$. (8M)

   (b) Solve $y = xp^2 + p$. (6M)

SECTION-II

3. (a) Using the method of variation of parameter, solve $\dfrac{d^2y}{dx^2} + 4y = \tan 2x$. (8M)

   (b) Solve $(D^2 + 5D - 6)y = e^x$. (6M)

OR

4. (a) Solve $x^2\dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + 4y = x^2 + \cos(\log x)$. (8M)

   (b) Solve $(D^2 + 3D + 2)y = x^2$. (6M)

SECTION-III

5. (a) Change the order of integration of $\int_0^a\int_y^a \dfrac{x}{x^2+y^2}dxdy$ and then evaluate. (6M)

   (b) By changing into Polar coordinates, evaluate $\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dxdy$. (8M)

OR

6. (a) Using double integral find the area bounded by $y = x$ and $y = x^2$. (6M)

   (b) Find the volume of the tetrahedron bounded by the plane $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$ and the coordinate plane $x = 0,\ y = 0,\ z = 0$. (8M)

SECTION-IV

7. (a) Find the directional derivative of $\phi = 2xy + z^2$ at the point (1, -1, 3) in the direction of $\vec{r} = \vec{i} + 2\vec{j} + 2\vec{k}$. (6M)

   (b) Find the value of $'a'$ if the vector $(ax^2 y + yz)\vec{i} + (xy^2 - xz^2 y)\vec{j} + (2xyz - 2x^2 y^2)\vec{k}$ has zero divergence. Find the curl of the above vector which zero divergence. (8M)

OR

8. (a) If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $r = |\vec{r}|$, prove that $div(r^n \vec{r}) = (n+3)r^n$ and $curl(r^n \vec{r}) = 0$. (6M)

   (b) Show that $\vec{F} = (6xy + z^3)\vec{i} + (3x^2 - z)\,j + (3xz^2 - y)\vec{k}$ is irrotational and find the scalar potential function ϕ such that $\vec{F} = \nabla\phi$. (8M)

SECTION-V

9. (a) Find $\int_C \vec{F}.d\vec{r}$ where $\vec{F} = (2y+3)\vec{i} + xz\,j + (yz - x)\vec{k}$ along the line joining the points (0, 0, 0) to (2, 1, 1). (6M)

   (b) Evaluate $\int_C [(2xy - x^2)dx + (x + y^2)dy]$ using Green's theorem, where C is the closed curve formed by $y^2 = x, y = x^2$. (8M)

OR

10. State Stoke's theorem. Verify Gauss divergence theorem for $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$ taken over the cube bounded by $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1.$ (14M)

*****

Code No : 1800BS02

MALLA REDDY ENGINEERING COLLEGE FOR WOMEN

(Autonomous Institution – UGC Govt. of India)

Permanently Affiliated to JNTUH, Approved by AICTE

Accredited by NBA and NAAC with A grade– ISO 9001-2015Certified

B.TECH I YEAR II Semester Advanced Supplementary Examinations, August 2019

MATHEMATICS-II

Time: 3 hours (Common to all Branches) **Max Marks: 70**

Note: This question paper Consists of 5 Sections. Answer **FIVE** Questions, Choosing ONE Question from each SECTION and each Question carries 14 marks.

SECTION-I

1. (a) Solve the Differential equation $y(y^2 - 2x^2)dx + x(2y^2 - x^2)dy = 0$ (7M)

   (b) If the temperature of a body is changing from $100^0 C\ to\ 70^0 C$ in 15 minutes, find when the temperature will be $40^0 C$, if the temperature of air is $30^0 C$. (7M)

OR

2. (a) Solve $x^2\left(\frac{dy}{dx}\right)^2 + xy\frac{dy}{dx} - 6y^2 = 0$ (7M)

   (b) Solve $Siny\ Cos^2 x = Cos^2 y\ p^2 + Sinx\,.\,Cosx\,.\,Cosy\,.\,p$ (7M)

SECTION-II

3. (a) Find the general solution of the differential equation $y'' - 6y' - 7y = 0$ (5M)

   (b) Solve $y'' + 6y' + 8y = 130\ Cos3t$ (9M)

OR

4. (a) Write a note on "Method of variation of parameters" and evaluate $(D^2 - 2D)y = e^x Sinx$ (7M)

   (b) Solve $x^3\frac{d^3y}{dx^3} + 2x^2\frac{d^2y}{dx^2} + 2y = 10\left(x + \frac{1}{x}\right)$ (7M)

SECTION-III

5. (a) Change the order of integration in $\int_0^1 \int_{x^2}^{2-x} xy\ dx$ and hence evaluate the double integral (7M)

   (b) Find the area of a plate in the form of a quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (7M)

OR

6. Find the volume of the greatest rectangular parallelopiped that can be inscribed in an ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ (14M)

SECTION-IV

7. (a) Find $div\,\bar{F}$ where $\bar{F} = grad(\,x^3 + y^3 + z^3 - 3xyz)$ (7M)

   (b) In what direction from $(3,1,-2)$ is the directional derivative of $f(x,y,z)=x^2y^2z^4$ is maximum and what is its magnitude (7M)

OR

8. (a) Prove that $\nabla\times(\nabla\times\bar{a})=\nabla(\nabla.\,\bar{a})-\nabla^2\bar{a}$ (7M)

   (b) Find $div\ \bar{f}$, where $\bar{f}=r^n\ \bar{r}.$ find n if it is Solenoidal (7M)

SECTION-V

9. Verify Green's theorem for $\int_C\left[(xy+y^2)dx+x^2dy\right]$, where C is bounded by the curves $y=x\ and\ y=x^2$ (14M)

OR

10. If $\overline{F}=(x^2-27)\bar{i}-6yz\bar{j}+8xz^2\overline{k}$, evaluate $\int_C\overline{F}.\,d\bar{r}$ from the point $(0,0,0)$ to $(1,1,1)$ along the straight line from $(0,0,0)$ to $(1,0,0)$, $(1,0,0)$ to $(1,1,0)$ and $(1,1,0)$ to $(1,1,1)$ (14M)

*****