Unit - IV :

Laplace tranforms : Extension of fouries tranforms.

$$FT[f(t)] = F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

existence condition of FT → dirichlet's condition

→ $\int_{-\infty}^{\infty} |f(t)|\, dt < \infty$ = absolutely integrable.

$u(t)$, $\delta(t)$, sgn (t), $\sin\omega_0 t$ → Not absolutely integrable.

$u(t) = 1$ $\quad \int_{-\infty}^{\infty} 1 \cdot dt = \infty$ = not absolutely integrable

∴ They cannot be analysed by using F.T.

∴ To overcome this we use Laplace transformation

Let $x(t)$ - signal, its Laplace tranform = $X(s)$

$$x(t) \xleftrightarrow{L.T} X(s)$$

$$X(s) = L.T\,(x(t)) = \int_{-\infty}^{\infty} x(t)\, e^{-st} dt \rightarrow ①$$

where s is the complex variable

$$s = \sigma + j\omega$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt \rightarrow ②$$

compare ① and ②

when $\sigma = 0$ i.e R.P $\{s\} = 0$ that is F.T.

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st} dt \rightarrow \text{Bilateral L.T}$$

Two sided L.T

$s = \sigma + j\omega$ = complex variable

$$X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-(\sigma + j\omega)t} dt$$

$$= \int_{-\infty}^{\infty} x(t)\, e^{-\sigma t} e^{-j\omega t} dt$$

$x(\sigma + j\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt = F \cdot T$

$x(t)$ is a func. which is absolutely not integrable

$e^{-\sigma t}$ = real exponential

if $\sigma > 0 \rightarrow$ decaying exponential

$$\int_{-\infty}^{\infty} |x(t)|\, dt = \infty \times \overset{e^{-\sigma t}}{0} = 0$$

$$X(\omega) = F \cdot T[x(t)] = X(s)\Big|_{\sigma = 0}$$

If $x(t)$ is multiplied by a real exponential with decreasing amplitude $\Rightarrow \int_{-\infty}^{\infty} |x(t)\, e^{-\sigma t}|\, dt < \infty$

Most of the signals can also be analysed by $L \cdot T$

1. Let $x(t) = e^{-at} u(t)$ find the $L \cdot T$.

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt = \int_{-\infty}^{\infty} e^{-at} u(t)\, e^{-st}\, dt$$

$$= \int_{0}^{\infty} e^{-(s+a)t}\, dt$$

$$= -\left(\frac{e^{-(s+a)t}}{s+a}\right)_0^{\infty} = -\left(0 - \left(\frac{e^{-0}}{s+a}\right)\right) = \frac{1}{s+a}$$

$$e^{-at} u(t) \xleftrightarrow{L \cdot T} \frac{1}{s+a}$$ only when $s + a > 0$.

$s > -a \quad \Rightarrow \quad e^{-(s+a)t} = e^{-\infty} = 0 \qquad Re\{s\} > -a$

2. $x(t) = -e^{-at} u(-t)$

$$X(s) = \int_{-\infty}^{\infty} -e^{-at} u(-t)\, e^{-st}\, dt \;\; \cancel{= -\int_{0}^{\infty} e^{-(a+s)t}\, dt}$$

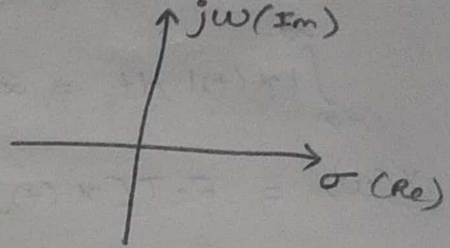
$$= -\int_{-\infty}^{0} e^{-(s+a)t}\, dt = \left(\frac{e^{-(s+a)t}}{s+a}\right)_{-\infty}^{0} = \left(\frac{1}{s+a}\right)$$

only when $s + a < 0$

(i) $-e^{-at}u(-t) \longleftrightarrow \dfrac{1}{s+a}$ $\quad Re\{s\} < -a$

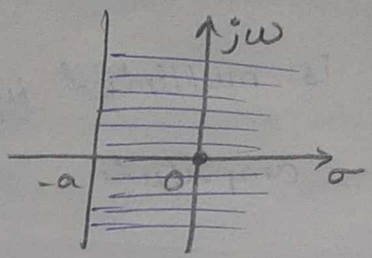
(ii) $e^{-at}\, u(t) \longleftrightarrow \dfrac{1}{s+a}$ $\quad Re\{s\} > -a$

Region of convergence (ROC): The set of values of 's' for which the LT converges.

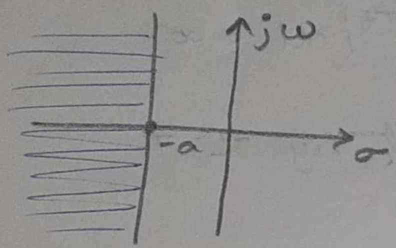


s-plane $= s = \sigma + j\omega$

$e^{-at}u(t) \longleftrightarrow \dfrac{1}{s+a}$ $\quad Re\{s\} > -a$



$-e^{-at}u(-t) \longleftrightarrow \dfrac{1}{s+a}$ $\quad Re\{s\} < -a$



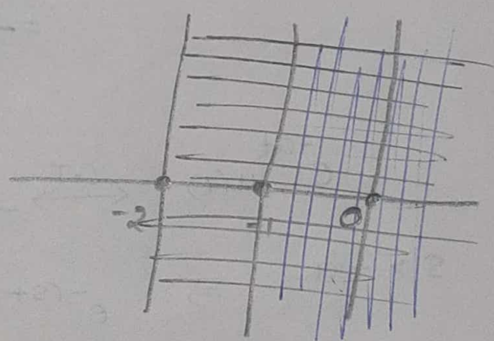
3. $3e^{-2t}u(t) - 2e^{-t}u(t) = x(t)$

$$X(s) = \int_{-\infty}^{\infty} (3e^{-2t}u(t) - 2e^{-t}u(t))\, e^{-st}\, dt$$

$$= \frac{3}{s+2} - \frac{2}{s+1}$$

$\dfrac{3}{s+2} \longrightarrow Re\{s\} > -2$

$\dfrac{2}{s+1} \longrightarrow Re\{s\} > -1$



$Re\{s\} > -2 \cap Re\{s\} > -1 \Rightarrow Re\{s\} > -1$

* $e^{-at}u(t) \overset{L.T}{\longleftrightarrow} \dfrac{1}{s+a}$

$a = 0 \Rightarrow u(t) \overset{L.T}{\longleftrightarrow} \dfrac{1}{s}$

(or) $u(t) = 1$

$X(s) = \int_0^{\infty} 1\, e^{-st}\, dt$

$= \left(\dfrac{e^{-st}}{-s}\right)_0^{\infty} = \dfrac{1}{s}$

4. Find the L.T of an impulse func. $\delta(t)$

$$\delta(t) = 1 \quad t = 0$$
$$= 0 \quad t \neq 0$$

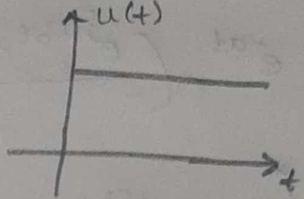
$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

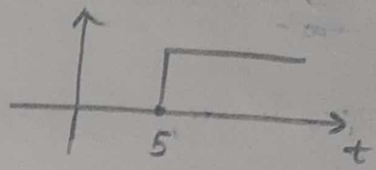
$$= \int_{-\infty}^{\infty} \delta(t)\, e^{-st}\, dt$$

$$= \left[e^{-st}\right]_{t=0} = 1$$

5. $x(t) = -u(t-5)$

$$u(t-5) = 1 \quad , \quad t \geq 5$$
$$= 0 \quad , \quad t < 5$$





$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$$= \int_{5}^{\infty} e^{-st}\, dt = \left(\frac{e^{-st}}{-s}\right)_5^{\infty}$$

$$= \frac{0 - e^{-5s}}{-s} = \frac{1}{s} e^{-5s}$$

$$u(t-5) \xleftrightarrow{L.T} \frac{1}{s} e^{-5s}$$

6. $x(t) = \sin \omega_0 t\, u(t)$

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt = \int_{-\infty}^{\infty} \sin \omega_0 t \cdot u(t) e^{-st}\, dt$$

$$= \int_{0}^{\infty} \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} \cdot e^{-st}\, dt$$

$$X(s) = \frac{1}{2j}\left(\int_0^{\infty} e^{(j\omega_0 - s)t}\, dt - \int_0^{\infty} e^{-(j\omega_0 + s)t}\, dt\right)$$

$$= \frac{1}{2j}\left(\frac{\left(e^{(j\omega_0 - s)t}\right)_0^{\infty}}{j\omega_0 - s} - \frac{\left(e^{-(j\omega_0 + s)t}\right)_0^{\infty}}{-(j\omega_0 + s)}\right)$$

$$= \frac{1}{2j}\left(-\frac{1}{s - j\omega_0}\left(e^{-(s - j\omega_0)t}\right)_0^{\infty} + \frac{1}{s + j\omega_0}\left(e^{-(j\omega_0 + s)t}\right)_0^{\infty}\right)$$

$$X(s) = \frac{1}{2j}\left(\frac{1}{s-j\omega_0} - \frac{1}{s+j\omega_0}\right)$$

$$= \frac{1}{2j}\left(\frac{2j\omega_0}{(s-j\omega_0)(s+j\omega_0)}\right) = \frac{\omega_0}{s^2+\omega_0^2}$$

$$\sin\omega_0 t \cdot u(t) \xleftrightarrow{L\cdot T} \frac{\omega_0}{s^2+\omega_0^2}, \quad Re\{s\} > 0$$

$$\cos\omega_0 t \cdot u(t) \xleftrightarrow{L\cdot T} \frac{s}{s^2+\omega_0^2} \quad Re\{s\} > 0$$

7. $x(t) = e^{-at}\cos\omega_0 t \; u(t)$

$$= e^{-at}\left(\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\right)u(t)$$

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt = \int_{-\infty}^{\infty}\left(\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\right) e^{-(a+s)t}\, u(t)\, dt$$

$$= \frac{1}{2}\int_0^{\infty}\left(e^{(j\omega_0-(a+s))t} + e^{-(j\omega_0+a+s)t}\right)dt$$

$$= \frac{1}{2}\int_0^{\infty} e^{-(s+a-j\omega_0)t}\, dt + \frac{1}{2}\int_0^{\infty} e^{-(s+a+j\omega_0)t}\, dt$$

$$X(s) = \frac{1}{2}\left(\frac{\left(e^{-(s+a-j\omega_0)t}\right)_0^{\infty}}{-(s+a-j\omega_0)} + \frac{\left(e^{-(s+a+j\omega_0)t}\right)_0^{\infty}}{-(s+a+j\omega_0)}\right)$$

$$= \frac{1}{2}\left(\frac{1}{s+a-j\omega_0} + \frac{1}{s+a+j\omega_0}\right)$$

$$= \frac{1}{2}\left(\frac{s+a+j\omega_0+s+a-j\omega_0}{(s+a)^2+\omega_0^2}\right) = \frac{s+a}{(s+a)^2+\omega_0^2}$$

$$e^{-at}\cos\omega_0 t \; u(t) \xleftrightarrow{L\cdot T} \frac{s+a}{(s+a)^2+\omega_0^2}, \quad s+a>0$$

$$Re\{s\} > -a$$

||ly $e^{at}\cos\omega_0 t \; u(t) \xleftrightarrow{L\cdot T} \dfrac{s-a}{(s-a)^2+\omega_0^2}$

$$e^{-at}\sin \omega_0 u(t) \xleftrightarrow{L.T} \frac{\omega_0}{(s+a)^2+\omega_0^2}$$

$$e^{at}\sin \omega_0 t \; u(t) \xleftrightarrow{L.T} \frac{\omega_0}{(s-a)^2+\omega_0^2}$$

8. $x(t) = \delta(t) - \frac{4}{3}e^{-t}u(t) + \frac{1}{3}e^{2t}u(t)$

$$X(s) = 1 - \frac{4}{3}\cdot\frac{1}{s+1} + \frac{1}{3}\,\frac{1}{s-2}$$

$Re\{s\}>0$ $\quad$ $Re\{s\}>-1$ $\quad$ $Re\{s\}>2$

$\therefore Re\{s\}>2$

$$X(s) = 1 - \frac{4}{3(s+1)} + \frac{1}{3(s-2)} \; ; \; Re\{s\}>2$$

9. $x(t) = t\cdot u(t)$

$$X(s) = \int_{-\infty}^{\infty} x(t)\,e^{-st}dt = \int_{-\infty}^{\infty} t\cdot u(t)\,e^{-st}dt$$

$$= \int_0^{\infty} t\,e^{-st}dt = \left(\frac{-te^{-st}}{s}\right)_0^{\infty} - \int_0^{\infty}\frac{e^{-st}}{-s}dt$$

$$= 0 - \frac{(e^{-st})_0^{\infty}}{s^2}$$

$$= 0 - \frac{(0-1)}{s^2} = \frac{1}{s^2}$$

$t\cdot u(t) \xleftrightarrow{L.T} \frac{1}{s^2}.\quad Re\{s\}>0$

10. $x(t) = t^2u(t)$

$$X(s) = \int_0^{\infty} t^2e^{-st}dt = \left(\frac{-t^2e^{-st}}{s}\right)_0^{\infty} - \int_0^{\infty}\frac{e^{-st}}{-s}(2t)dt$$

$$= 0 - 2\left(\left(\frac{te^{-st}}{s^2}\right)_0^{\infty} - \int_0^{\infty}\frac{e^{-st}}{s^2}dt\right)$$

$$= -2\left(\cdot\frac{e^{-st}}{s^3}\right)_0^{\infty} = \frac{2}{s^3}$$

$t^2u(t) \xleftrightarrow{L.T} \frac{2}{s^3}\cdot Re\{s\}>0$

* $t^n \cdot u(t) \xleftrightarrow{L \cdot T} \dfrac{n!}{s^{n+1}}$.

11. $x(t) = t\, e^{at} u(t)$

$$X(s) = \int_{-\infty}^{\infty} t\, e^{at} u(t)\, e^{-st}\, dt = \int_{0}^{\infty} t\, e^{(a-s)t}\, dt$$

$$= \frac{\left(t\, e^{(a-s)t}\right]_0^{\infty}}{a-s} - \int_0^{\infty} \frac{e^{(a-s)t}}{a-s}\, dt$$

$$= 0 - \frac{\left(e^{(a-s)t}\right]_0^{\infty}}{(a-s)^2} \quad \cancel{= \frac{1}{s^2}}$$

$$= \frac{1}{(s-a)^2} \qquad Re\{s\} > a.$$

$$t\, e^{at} u(t) \xleftrightarrow{L \cdot T} \frac{1}{(s-a)^2}$$

12.

$$x(t) = e^{-2t} u(t) + e^{-t} \cos 3t\, u(t)$$

$$X(s) = \frac{1}{s+2} + \frac{s+1}{(s+1)^2 + 9}$$

$Re\{s\} > -2$ $\qquad$ $Re\{s\} > -1$

$$X(s) = \frac{1}{s+2} + \frac{s+1}{(s+1)^2+9} \; ; \; Re\{s\} > -1$$

13.

$x(t) = \sinh \omega_0 t\; u(t)$

$$\sinh \omega_0 t = \frac{e^{\omega_0 t} - e^{-\omega_0 t}}{2}$$

$$\sinh \omega_0 t\; u(t) \xleftrightarrow{L \cdot T} \frac{\omega_0}{s^2 - \omega_0^2}$$

14.

$x(t) = \sin at \cos bt\; u(t)$

$$X(s) = \int_{-\infty}^{\infty} \frac{1}{2} \left(\sin (a+b)t + \sin(a-b)t\right) e^{-st} u(t)\, dt$$

$\cancel{x(s) = \frac{1}{2}\int_0^\infty}$ $$X(s) = \frac{1}{2}\left(\frac{(a+b)}{s^2+(a+b)^2} + \frac{(a-b)}{s^2+(a-b)^2}\right)$$

15. $x(t) = \cos^3 t = \frac{1}{4}(\cos 3t + 3\cos t)$

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$$X(s) = \frac{1}{4}\frac{s}{s^2+3^2} + \frac{3}{4}\frac{s}{s^2+1^2}$$

$$X(s) = \frac{s}{4(s^2+9)} + \frac{3s}{4(s^2+1)}$$

16. $x(t) = (4 - 6t + 3t^2 + t^3)\, u(t)$

$$X(s) = \frac{4}{s} - \frac{6}{s^2} + \frac{(3)2}{s^3} + \frac{6}{s^4}$$

$$X(s) = \frac{4}{s} - \frac{6}{s^2} + \frac{6}{s^3} + \frac{6}{s^4}$$

17. $x(t) = e^{-6|t|}$

$$X(s) = \int_{-\infty}^{\infty} e^{-6|t|} e^{-st}\, dt = \int_{-\infty}^{0} e^{(6-s)t}\, dt + \int_{0}^{\infty} e^{-(6+s)t}\, dt$$

$$= \frac{\left(e^{(6-s)t}\right)_{-\infty}^{0}}{6-s} + \frac{\left(e^{-(6+s)t}\right)_{0}^{\infty}}{-(6+s)}$$

$$= \frac{1-0}{6-s} + \frac{1}{s+6} = \frac{1}{s+6} - \frac{1}{s-6}$$

$Re\{s\} > -6$ $\quad$ $Re\{s\} < 6$

$-6 < Re\{s\} < 6$

Properties of Laplace transforms:

$$x(t) \xleftrightarrow{L.T} X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt \longrightarrow \text{Bilateral L.T}$$

Two sided L.T

$$X(s) = \int_{0}^{\infty} x(t)\, e^{-st}\, dt \longrightarrow \text{Single sided L.T}$$

one / unilateral L.T

Properties of F.T

1) Linearity property

$$a x_1(t) + b x_2(t) \xleftrightarrow{F.T} a X_1(\omega) + b X_2(\omega)$$

2) Time shifting

$$x(t) \xleftrightarrow{F.T} X(\omega)$$

$$x(t-t_0) \xleftrightarrow{F.T} e^{-j\omega t_0} X(\omega)$$

3) Frequency shifting

$$e^{j\omega_0} x(t) \longleftrightarrow X(\omega-\omega_0)$$

4) Scaling

$$x(at) \longleftrightarrow \frac{1}{a} F\left(\frac{\omega}{a}\right)$$

5) Integration in time domain

$$\int_{-\infty}^{t} x(\tau)\,d\tau = \frac{1}{j\omega} X(\omega)$$

6) Convolution in time domain

$$x_1(t) * x_2(t) \xleftrightarrow{F.T} X_1(\omega) X_2(\omega)$$

7) Convolution in freq. domain

$$x_1(t) \cdot x_2(t) \xleftrightarrow{F.T} \frac{1}{2\pi}[X_1(\omega) * X_2(\omega)]$$

8) differentiation in time domain

$$\frac{d}{dt} x(t) \xleftrightarrow{F.T} j\omega X(\omega)$$

9) differention in freq. domain

$$-jt\, x(t) \xleftrightarrow{F.T} \frac{d}{d\omega} X(\omega)$$

$$(-jt)^n x(t) \xleftrightarrow{F.T} \frac{d^n}{d\omega^n} X(\omega)$$

Properties of L.T

1) Linearity

$$a x_1(t) + b x_2(t) \xleftrightarrow{L.T} a X_1(s) + b X_2(s)$$

2) Time shifting

$$x(t) \xleftrightarrow{L.T} X(s)$$

$$x(t-t_0) \xleftrightarrow{L.T} e^{-st_0} X(s)$$

3) Frequency shifting

$$e^{s_0 t} x(t) \xleftrightarrow{L.T} X(s-s_0)$$

4) Scaling

$$x(at) \xleftrightarrow{L.T} \frac{1}{a} F\left(\frac{s}{a}\right)$$

5) Integration in time domain

$$\int_{-\infty}^{t} x(\tau)\,d\tau = \frac{1}{s} X(s)$$

6) Convolution in time domain

$$x_1(t) * x_2(t) \xleftrightarrow{L.T} X_1(s) X_2(s)$$

7) Convolution in freq. domain

$$x_1(t) \cdot x_2(t) \xleftrightarrow{L.T} \frac{1}{2\pi}[X_1(s) * X_2(s)]$$

8) differentiation in time domain

$$\frac{d}{dt} x(t) \xleftrightarrow{L.T} s X(s)$$

9) differentiation in freq. domain

$$-t\, x(t) \xleftrightarrow{L.T} \frac{d}{ds} X(s)$$

$$(-1)^n t^n x(t) \xleftrightarrow{L.T} \frac{d^n}{ds^n} X(s)$$

Initial value theorem:

The value of the function at $t=0$ is initial value

i.e for $x(t) \longrightarrow x(0)$ is initial value

Statement: If $x(t)$ & it's derivative $x'(t)$ are Laplace tranformable then $\lim_{t\to 0} x(t) = x(0) = \lim_{s\to\infty} s\cdot X(s)$

Proof: $x(t) \xleftrightarrow{L\cdot T} X(s)$ $\quad L\cdot T(x(t)) = X(s)$

$L\cdot T(x'(t)) = sX(s)$ when there are no initial conditions

$= sX(s) - x(0)$ when there is initial conditions

$$\int_0^\infty x'(t)e^{-st}dt = sX(s) - x(0)$$

$$\lim_{s\to\infty}\int_0^\infty x'(t)e^{-st}dt = \lim_{s\to\infty}(sX(s) - x(0))$$

$$\int_0^\infty x'(t)dt \lim_{s\to\infty} e^{-st} = \lim_{s\to\infty} sX(s) - \lim_{s\to\infty} x(0)$$

$$0 = \lim_{s\to\infty} sX(s) - \cancel{\lim_{s\to}}\, x(0)$$

$\therefore x(0) = \lim_{s\to\infty} sX(s).$ Initial value theorem of L·T

Final value theorem:

The value of the function at $t=\infty$ is final value

i.e $\lim_{t\to\infty} X(t) = X(\infty).$

$$\lim_{t\to\infty} x(t) = x(\infty) = \lim_{s\to 0} sX(s)$$

Proof: $L\cdot T[x'(t)] \longleftrightarrow sX(s) - x(0)$

$$\int_0^\infty x'(t)e^{-st}dt = sX(s) - x(0)$$

$$\lim_{s\to 0}\int_0^\infty x'(t)e^{-st}dt = \lim_{s\to 0} sX(s) - x(0)$$

$$\int_0^\infty x'(t)\,dt \quad \lim_{s\to 0} e^{-st}\,dt = \lim_{s\to 0} s\,X(s) - x(0)$$

$$\int_0^\infty x'(t)\,dt = \lim_{s\to 0} s\,X(s) - x(0)$$

$$(x(t))_0^\infty = \lim_{s\to 0} s\,X(s) - x(0)$$

$$x(\infty) - x(0) = \lim_{s\to 0} s\,X(s) - x(0)$$

$$\therefore x(\infty) = \lim_{s\to 0} s\,X(s)$$ Final value theorem of L.T

1. Find L.T of $te^{-at}u(t)$ using properties

let $x(t) = e^{-at}u(t)$

$$X(s) = \frac{1}{s+a}$$

$te^{-at} \rightarrow$ using differentiation in s-domain

$$t\,x(t) \longleftrightarrow \frac{-d}{ds}X(s)$$

$$\frac{-d}{ds}\left(\frac{1}{s+a}\right) \Rightarrow \frac{-(-1)}{(s+a)^2} = \frac{1}{(s+a)^2}$$

$$te^{-at}u(t) \longleftrightarrow \frac{1}{(s+a)^2}$$

2. Find the L.T of $f(t) = e^{-t}u(t) * \sin 3\pi t\,u(t)$ using convolution in time domain property

$$f_1(t) * f_2(t) \overset{L.T}{\longleftrightarrow} F_1(s)\,F_2(s)$$

$$f_1(t) = e^{-t}u(t) \longleftrightarrow \frac{1}{s+1}$$

$$f_2(t) = \sin 3\pi t\,u(t) \longleftrightarrow \frac{3\pi}{s^2+(3\pi)^2}$$

$$L.T\,[f(t)] = \frac{1}{s+1}\cdot\frac{3\pi}{s^2+(3\pi)^2}$$

3. Find LT of $t \frac{d}{dt}(e^{-t} \sin t \, u(t))$

$$\sin t \xleftrightarrow{L.T} \frac{1}{s^2+1^2}$$

$$e^{-t} \sin t \xleftrightarrow{L.T} \frac{1}{(s+1)^2+1^2}$$

we have $\frac{d}{dt} x(t) \xleftrightarrow{L.T} s\,X(s)$

$$\frac{d}{dt}(e^{-t} \sin t) \xleftrightarrow{L.T} \frac{s}{(s+1)^2+1}$$

$$t f(t) \xleftrightarrow{L.T} \frac{-d}{ds} X(s)$$

$$= \frac{-d}{ds}\left(\frac{s}{(s+1)^2+1}\right)$$

$$= \frac{-\left(((s+1)^2+1) - s\,2(s+1)\right)}{((s+1)^2+1)^2} = \frac{-(s^2+2s+2-2s^2-s)}{((s+1)^2+1)^2}$$

$$t \frac{d}{dt}(e^{-t} \sin t \, u(t)) \xleftrightarrow{L.T} \frac{s^2-s-2}{((s+1)^2+1)^2}$$

4. Find L.T of $t \sin at \, u(t)$

$$\sin at \, u(t) \xleftrightarrow{L.T} \frac{a}{s^2+a^2}$$

$$t \sin at \, u(t) \xleftrightarrow{L.T} \frac{-d}{ds}\left(\frac{a}{s^2+a^2}\right)$$

$$= \frac{2as}{(s^2+a^2)^2}.$$

5. Find L.T of $te^{-t} u(t-T)$

$$x(t) = u(t) \longleftrightarrow \frac{1}{s}$$

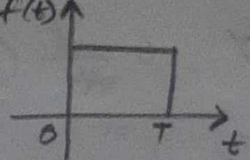
$$x(t-T) \longleftrightarrow e^{-sT} X(s) = e^{-sT} \frac{1}{s}$$

$$e^{s_0 t} x(t) \longleftrightarrow X(s-s_0) \cancel{X(s+1)}$$

$$e^{-t} u(t-T) = \frac{e^{-(s+1)T}}{s+1}$$

$$te^{-t} u(t-T) \longleftrightarrow \frac{-d}{ds} X(s) = \frac{(sT+T+1)e^{-(s+1)T}}{(s+1)^2}$$

6. Find the L·T of



$f(t) = 1 \quad 0 \le t \le T$

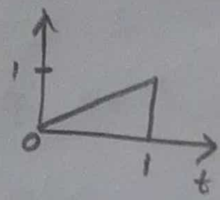
$$F(s) = \int_{-\infty}^{\infty} f(t)\, e^{-st}\, dt$$

$$= \int_0^T 1 \cdot e^{-st}\, dt = \frac{\left(e^{-st}\right)_0^T}{-s} = \frac{-e^{-sT} + 1}{s}$$

$$F(s) = \frac{1 - e^{-sT}}{s}$$

7.

$f(t) =$



$f(t) = t \quad , \quad 0 \le t \le 1$

$$F(s) = \int_{-\infty}^{\infty} f(t)\, e^{-st}\, dt = \int_0^1 t\, e^{-st}\, dt$$

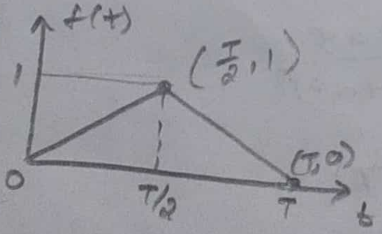
$$= \left(t \frac{e^{-st}}{-s}\right)_0^1 - \int_0^1 1 \cdot \frac{e^{-st}}{-s}\, dt$$

$$= \frac{e^{-s}}{-s} + \frac{1}{s}\left(\frac{e^{-st}}{-s}\right)_0^1$$

$$= \frac{-e^{-s}}{s} - \frac{(e^{-s} - 1)}{s^2} = \frac{1}{s^2} - \frac{e^{-s}}{s} - \frac{e^{-s}}{s^2}$$

8.

$f(t) =$



$f(t) = \frac{2}{T} t \quad 0 \le t \le T/2$

$\quad = 2 - \frac{2}{T} t \quad \frac{T}{2} \le t \le T$

$f(t) - 0 = \frac{1 - 0}{\frac{T}{2} - T}(t - T)$

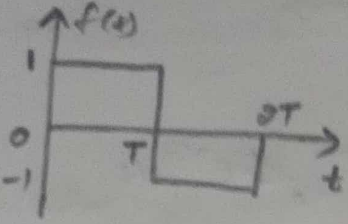
$f(t) - 0 = \frac{-2}{T}(t - T)$

$f(t) = 2 - \frac{2}{T} t$

$$F(s) = \int_0^{T/2} \frac{2}{T} t\, e^{-st} dt + \int_{T/2} \left(2 - \frac{2}{T} t\right) e^{-st} dt$$

$$= \frac{2}{T} \int_0^{T/2} t\, e^{-st} dt + 2 \int_{T/2}^{T} e^{-st} dt - \frac{2}{T} \int_{T/2}^{T} t\, e^{-st} dt$$

8. $f(t) =$



$f(t) = 1 , \quad 0 \le t \le T$

$f(t) = -1 , \quad T \le t \le 2T$

$$F(s) = \int_{-\infty}^{\infty} f(t) e^{-st} dt$$

$$= \int_0^T e^{-st} dt + \int_T^{2T} (-1) e^{-st} dt$$

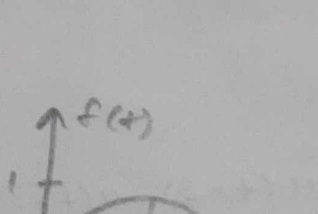
$$= \left(\frac{e^{-st}}{-s}\right)_0^T - \left(\frac{e^{-st}}{-s}\right)_T^{2T}$$

$$= \frac{e^{-sT} - 1}{-s} - \frac{(e^{-2sT} - e^{-sT})}{-s}$$

$$= \frac{1 - e^{-sT} + e^{-2sT} - e^{-sT}}{s} = \frac{e^{-2sT} - 2e^{-sT} + 1}{s}$$

$$= \frac{(e^{-sT} - 1)^2}{s}$$

9. $f(t) =$



$\Rightarrow f(t) = \sin t \quad 0 \le t \le \pi$

$= 0 \quad$ otherwise

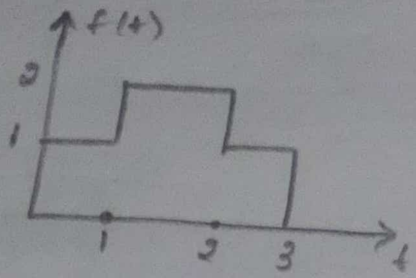
$$F(s) = \int_{-\infty}^{\infty} f(t) e^{-st} dt = \int_0^{\pi} \sin t\, e^{-st} dt$$

we have $\int_0^{\pi} \sin at\, e^{-bt} dt = \frac{1}{a^2 + b^2}\left(-b \sin at\, e^{-bt} - e^{-bt} a \cos at\right)$

$$F(s) = \frac{1}{1^2+s^2}\left(-s \sin t\, e^{-st} - e^{-st}(1)\cos t\right)_0^{\pi}$$

$$\cancel{= \frac{1}{2}(e^{-\pi} -} \quad = \frac{1+e^{-s\pi}}{1+s^2}$$

10. $f(t) =$



$f(t) = 1 \quad 0 \le t \le 1$

$= 2 \quad 1 \le t \le 2$

$= 1 \quad 2 \le t \le 3$

$$\int_0^1 1\cdot e^{-st}\,dt + \int_1^2 2\cdot e^{-st}\,dt + \int_2^3 1\, e^{-st}\,dt$$

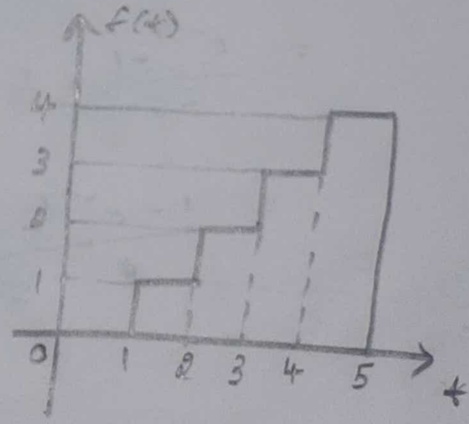
$$F(s) = \frac{1+e^{-s}-e^{-2s}-e^{-3s}}{s}$$

(or)

$$f(t) = u(t) + u(t-1) - u(t-2) - u(t-3)$$

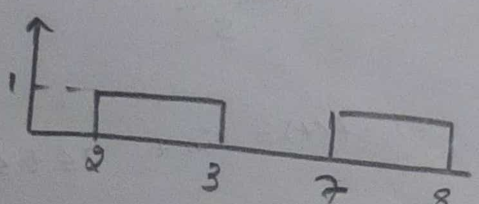
$$F(s) = \frac{1}{s} + \frac{e^{-s}}{s} - \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s}$$

11. $f(t) = u(t-1) + u(t-2) + u(t-3) + u(t-4) - 4u(t-5)$



$$F(s) = \frac{1}{s}e^{-s} + \frac{1}{s}e^{-2s} + \frac{1}{s}e^{-3s} + \frac{1}{s}e^{-4s} - \frac{4}{s}e^{-5s}$$

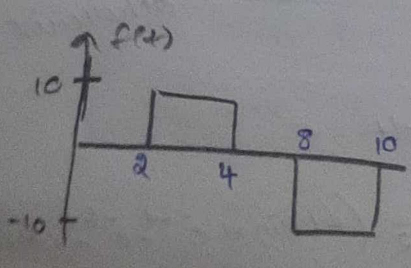
12.



$f(t) = u(t-2) - u(t-3) + u(t-7) - u(t-8)$

$$F(s) = \frac{1}{s}\left(e^{-2s} - e^{-3s} + e^{-7s} - e^{-8s}\right)$$

13.



$$f(t) = 10\,u(t-2) - 10\,u(t-4) - 10\,u(t-8) + 10(t-10)$$

14. $f(t) = e^{-at}\left(A\cos bt + \frac{B-Aa}{b}\sin bt\right)u(t)$

$= Ae^{-at}\cos bt\, u(t) + \frac{B-Aa}{b}\sin bt\, u(t)\, e^{-at}$

$= \frac{A(s+a)}{(s+a)^2+b^2} + \frac{B-Aa}{b}\cdot\frac{b}{(s+a)^2+b^2}$

$= \frac{A(s+a)}{(s+a)^2+b^2} + \frac{B-Aa}{(s+a)^2+b^2} = \frac{As+Aa+B-Aa}{(s+a)^2+b^2}$

$= \frac{As+B}{(s+a)^2+b^2}$

Inverse Laplace Transform:

$L.T[f(t)] = F(s) = \int_{-\infty}^{\infty} f(t)e^{-st}dt$

Inverse $L.T = L^{-1}[F(s)] = f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)e^{st}ds$

If $F.S = \frac{A(s)}{B(s)}$ = Rational func.

express this $F(s)$ in terms of Partial functions

$F(s) = \frac{A_1}{s-a_1} + \frac{A_2}{s-a_2} + \frac{A_3}{s-a_3} + \dots + \frac{A_K}{s-a_K}$

$L^{-1}[F(s)] = L^{-1}\left(\frac{A_1}{s-a_1}\right) + L^{-1}\left(\frac{A_2}{s-a_2}\right) + \dots + L^{-1}\left(\frac{A_K}{s-a_K}\right)$

$a_1, a_2, \dots a_K$ = poles of the transfer func's

↳ values of the func at which $F(s)$ becomes infinite

$s-a_1=0 \Rightarrow s=a_1 \Rightarrow$

$s-b_1=0 \Rightarrow s=b_1$

w.r.t $e^{-at}u(t) \longleftrightarrow \frac{1}{s+a}$ ROC $= Re\{s\} > -a$

$-e^{at}u(-t) \longleftrightarrow \frac{1}{s+a}$ ROC $= Re\{s\} < a$

$\cancel{A_K e^{a_K t}u(t) \longleftrightarrow \frac{A_K}{s-a_K} \quad ROC = Re\{s\} > a, ROC}$

(i) RH deq.

$$A_K e^{a_K t} u(t) \longleftrightarrow \frac{A_K}{s-a_K} \quad ROC = Re\{s\} > a$$

(ii) LH

$$-A_K e^{a_K t} u(-t) \longleftrightarrow \frac{A_K}{s-a_K} \quad ROC = Re\{s\} < a$$

1.

$$F(s) = \frac{-5s-7}{(s+1)(s-1)(s+2)}$$

Find the ILT

Sol. If ROC is not mentiond consider RH deq.

$$F(s) = \frac{A_1}{s+1} + \frac{A_2}{s-1} + \frac{A_3}{s+2}$$

$$F(s) = \frac{1}{s+1} - \frac{2}{s-1} + \frac{1}{s+2}$$

$$L^{-1}(F(s)) = L^{-1}\left(\frac{1}{s+1}\right) - L^{-1}\left(\frac{2}{s-1}\right) + L^{-1}\left(\frac{1}{s+2}\right)$$

$$\cancel{f(t) = e^{-2t} u(t) - 2e^{-t} u(t) + e^{2t} u(t)}$$

$$f(t) = e^{-t} u(t) - 2e^{t} u(t) + e^{-2t} u(t)$$

2. Find ILT of $\frac{2s+9}{s^2+4s+29}$

Sol.

$$F(s) = \frac{2s+9}{s^2+4s+29} = \frac{2s+4+5}{(s+2)^2+5^2}$$

$$= \frac{2(s+2)+5}{(s+2)^2+5^2}$$

$$F(s) = \frac{2(s+2)}{(s+2)^2+5^2} + \frac{5}{(s+2)^2+5^2}$$

$$f(t) = 2e^{-2t} \cos 5t + 5e^{-2t} \sin 5t.$$

3. $\frac{2(s+2)}{s^2+4s+3}$ find ILT $\quad s^2+4s+3 = (s+1)(s+3)$

$$\frac{A}{s+1} + \frac{B}{s+3}$$

$$\left.\frac{2(s+2)}{(s+1)\cancel{(s+3)}}\cancel{(s+3)}\right|_{s=-3} = 1 = B$$

$$\left.\frac{2(s+2)}{\cancel{(s+1)}(s+3)}\cancel{(s+1)}\right|_{s=-1} = 1 = A$$

$$F(s) = \underset{\substack{\downarrow \\ Re\{s\}>-1}}{\frac{1}{s+1}} + \underset{\substack{\downarrow \\ Re\{s\}>-3}}{\frac{1}{s+3}}$$

(i) Assuming RHS $Re\{s\} > -1 \cap Re\{s\} > -3 \Rightarrow Re\{s\} > -1$

$$f(t) = e^{-t}u(t) + e^{-3t}u(t)$$

(ii) Assuming LHS $Re\{s\} < -1 \cap Re\{s\} < -3$

$$f(t) = -e^{-t}u(-t) - e^{-3t}u(-t)$$

(iii)

$$\underset{\substack{\downarrow \\ LHS \\ \downarrow \\ Re\{s\}<-1}}{\frac{1}{s+1}} + \underset{\substack{\downarrow \\ RHS \\ \downarrow \\ Re\{s\}>-3}}{\frac{1}{s+3}} \qquad -3 < Re\{s\} < -1$$

$$f(t) = -e^{-t}u(-t) + e^{-3t}u(t)$$

3. Find the ILT of $F(s) = \log\left(\frac{s+a}{s+b}\right)$

Sol.

$$f(t) \xleftrightarrow{LT} F(s)$$

$$\frac{d}{ds}F(s) \longleftrightarrow -t\,f(t)$$

$$F(s) = \log\left(\frac{s+a}{s+b}\right)$$

$$-\frac{d}{ds}F(s) = \frac{-1}{\frac{s+a}{s+b}}\left(\frac{(s+b)-(s+a)}{(s+b)^2}\right)$$

$$= \frac{-(b-a)}{(s+a)(s+b)} = \frac{a-b}{(s+a)(s+b)}$$

$$= \frac{1}{s+b} - \frac{1}{s+a}$$

$$t\,f(t) = -e^{-at}u(t) + e^{-bt}u(t)$$

$$f(t) = \frac{1}{t}\left(e^{-bt}u(t) - e^{-at}u(t)\right)$$

4.

$$F(s) = s\log\left(\frac{s+a}{s+b}\right) = s\log(s+a) - s\log(s+b)$$

Sol. $f(t) = \frac{1}{t}\left(a e^{-at} - b e^{-bt} + e^{-at} - e^{-bt}\right)u(t)$

5.

$$F(s) = \frac{s^2+2s+5}{(s+3)(s+5)^2} = \frac{A}{s+3} + \frac{B}{s+5} + \frac{C}{(s+5)^2} \quad Re\{s\} > -3$$

Sol. $A = 2, \quad B = -1, \quad C = -10$

$$f(t) = 2e^{-3t}u(t) - e^{-5t}u(t) - 10\,te^{-5t}u(t)$$

6. $F(s) = \frac{s^2+6s+7}{s^2+3s+2} \quad Re\{s\} > -1$

Sol.

$$\frac{s^2+6s+7}{s^2+3s+2} = 1 + \frac{3s+5}{s^2+3s+2} = 1 + \frac{3s+5}{(s+1)(s+2)}$$

$$F(s) = 1 + \frac{2}{s+1} + \frac{1}{s+2}$$

$$f(t) = \delta(t) + 2e^{-t}u(t) + e^{-2t}u(t)$$

7. $$F(S) = \frac{2 + 2se^{-2s} + 4e^{-4s}}{s^2+4s+3}$$

$$F(S) = \frac{2}{(s+1)(s+3)} + \frac{2s\,e^{-2s}}{(s+1)(s+3)} + \frac{4e^{-4s}}{(s+1)(s+3)}$$

$$F_1(S) = \frac{2}{(s+1)(s+3)} = \frac{1}{s+1} - \frac{1}{s+3}$$

$$f_1(t) = (e^{-t} - e^{-3t})\,u(t)$$

$$F_2(S) = \frac{2s\,\cancel{e^{-2s}}}{(s+1)(s+3)} = \frac{-1}{s+1} + \frac{3}{s+3}$$

$$f_2(t) = -e^{-t}u(t) + 3e^{-3t}u(t)$$

$$F_3(s) = \frac{4}{(s+1)(s+3)} = \frac{2}{s+1} - \frac{2}{s+3}$$

$$f_3(t) = 2e^{-t}u(t) - 2e^{-3t}u(t)$$

$$u(t) \longleftrightarrow \frac{1}{s}$$

$$\cancel{F_2(s)\, e^{-2s} \Longleftrightarrow}$$

$$u(t-5) \longleftrightarrow \frac{e^{-5s}}{s}$$

$$f(t) = (e^{-t} - e^{-3t})\,u(t) + \left(3e^{-3(t-2)} - e^{-(t-2)}\right)u(t-2)$$
$$+ \left(2e^{-(t-4)} - 2e^{-3(t-4)}\right)u(t-4)$$

8. * $$F(S) = \frac{1 - e^{-2s}}{3s^2+2s}, \quad F(s) = \frac{5s + 13}{s(s^2+4s+13)}$$

~~8. $F(s) = \frac{8s+13}{s^2+4s+5}$ find the initial and final value.~~

8. $F(s) = \frac{s+3}{s^2+4s+8}$ find the initial & final value

Sol.

$$f(0) = \underset{t\to 0}{\mathrm{Lt}}\, f(t) = \underset{s\to\infty}{\mathrm{Lt}}\, sF(s)$$

$$f(\infty) = \underset{t\to\infty}{\mathrm{Lt}}\, f(t) = \underset{s\to 0}{\mathrm{Lt}}\, s\cdot F(s)$$

$$f(0) = \underset{s\to\infty}{\mathrm{Lt}}\, s\left(\frac{s+3}{s^2+4s+8}\right)$$

$$= \underset{s\to\infty}{\mathrm{Lt}}\left(\frac{\cancel{s^2}\left(1+\frac{3}{s}\right)}{\cancel{s^2}\left(1+\frac{4}{s}+\frac{8}{s^2}\right)}\right) = 1$$

$$f(\infty) = \underset{s\to 0}{\mathrm{Lt}}\left(\frac{s^2+3s}{s^2+4s+8}\right) = 0$$

9. $f(t) = 2\cdot e^{-5t}u(t)$ find initial value

Sol.

$$\cancel{f(0) = \underset{t\to 0}{\mathrm{Lt}}\, f(t)}$$

$$\cancel{= \underset{t\to 0}{\mathrm{Lt}}\, 2e^{-5t}u(t)}$$

$$F(s) = \frac{2}{s} - \frac{1}{s+5}$$

$$= \frac{2s+10-s}{s(s+5)}$$

$$= \frac{s+10}{s(s+5)}$$

~~F(s~~

$$f(0) = \underset{s\to\infty}{\mathrm{Lt}}\, \cancel{s}\left(\frac{s+10}{\cancel{s}(s+5)}\right)$$

$$= \underset{s\to\infty}{\mathrm{Lt}}\left(\frac{1+10/s}{1+5/s}\right) = 1$$

10. $f(t) = 5e^{-4t}$ find initial value

$$F(s) = 5\,\frac{1}{s+4}$$

$$f(\infty) = \underset{s\to 0}{\mathrm{Lt}}\, s\left(\frac{5}{s+4}\right) = 0$$

Laplace transform of a periodic function:

Periodic func. ⇒ $f(t) = f(t \pm T)$, $T = T_0, 2T_0, 3T_0, \ldots$

$$F(s) = \int_0^\infty f(t) e^{-st} dt = \int_0^T f(t) e^{-st} dt + \int_T^{2T} f(t) e^{-st} dt + \ldots \ldots + \infty$$

replace $t = t + T$ in 2nd

$t = t + 2T$ in 3rd

$$F(s) = \int_0^T f(t) e^{-st} dt + \int_0^T f(t+T) e^{-s(t+T)} dt + \int_0^T f(t+2T) e^{-s(t+2T)} dt + \ldots \ldots$$

$$= \int_0^T f(t) e^{-st} dt + e^{-sT} \int_0^T f(t+T) e^{-st} dt + e^{-2sT} \int_0^T f(t+2T) e^{-st} dt + \ldots$$

w.k.t $f(t) = f(t+T) = f(t+2T) + \ldots$

$$F(s) = \int_0^T f(t) e^{-st} dt + e^{-sT} \int_0^T f(t) e^{-st} dt + e^{-2sT} \int_0^T f(t) e^{-st} dt + \ldots$$

$$= \int_0^T f(t) e^{-st} dt \left(1 + e^{-sT} + e^{-2sT} + e^{-3sT} + \ldots + \infty\right) \ldots$$

$$= \frac{1}{1 - e^{-sT}} \int_0^T f(t) e^{-st} dt$$ = Laplace transform of periodic func.

Properties of ROC of Laplace transform:

1. ROC of $X(s)$ consists of $\parallel$le strips to the $j\omega$ axis in the s-plane
2. For rational L.T, the ROC does not contain any poles
3. If $x(t)$ is of finite duration and is absolutely integrable, then the ROC is the entire s-plane
4. If $x(t)$ is right sided & if the line ($Re\{s\} = \sigma_0$) is in the ROC, then all values of s for which $Re\{s\} > \sigma_0$

will also lie in the ROC

5. If $x(t)$ is left sided " " $Re\{s\} <$ sigma o "

6. If $x(t)$ is two sided & if the line $Re\{s\} =$ sigma o is in the ROC, then the ROC will consist of a strip in the s-plane that include $Re\{s\} =$ sigma o

7. If $X(s)$ of $x(t)$ is rational then its ROC is bounded by poles or extends to infinity. In addition, no poles of $X(s)$ are contained in the ROC.

8. If the L.T of $X(s)$ of $x(t)$ is rational, then if $x(t)$ is right sided, the ROC is the region in the s-plane to the right of the rightmost pole

the left sided, the ROC is the region in the s-plane to the left of the leftmost pole.

Z transforms:

Classification of signals $\longrightarrow$ Continuous time (CT) Discrete time (DT)

FS → Fourier series

- Continuous time (CT)
  - Periodic → CT FS
  - Non Periodic → CTFT
- Discrete time (DT)
  - DT periodic → DTFS
  - DT Non periodic → DTFT

CTFT → existence condition → $\int_{-\infty}^{\infty} |f(t)| dt < \infty$ ⇒ absolutely integrable

$\int_{-\infty}^{\infty} |f(t)| dt = \infty$ ⇒ Not absolutely integrable

∴ Laplace transforms is used

DTFT $\longrightarrow X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$ $\rightarrow$ existence condition

$$= \sum_{n=-\infty}^{\infty} |x(n)| < \infty$$

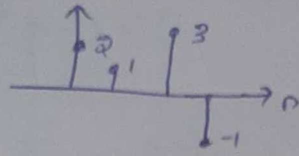
absolutely summable

If not absolutely summable then z-transform is used

Discrete time counterpart of L-transform.

z-transform: Discrete time signal $\rightarrow$ sequence $\rightarrow x[n]$

$x(n)$



sequence $x\,[2, 1, 3, -1]$

If $x(n)$ DT ~~freq~~ seq $\rightarrow$ $\boxed{ZT[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}}$

z = complex variable = polar form $z = re^{j\omega}$

s = complex variable = rectangular form = $\sigma + j\omega$

$z = re^{j\omega}$ $\Rightarrow$ r = magnitude of the complex variable z

ω = angle " " " " "

$$\text{DTFT}(x(n)) = X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} \longrightarrow ①$$

$$ZT[x(n)] = X(z) = \sum x(n)\, z^{-n}$$

replace $z = re^{j\omega}$ $\Rightarrow X(z) = \sum_{n=-\infty}^{\infty} x(n)\, (re^{j\omega})^{-n}$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, r^{-n} e^{-j\omega n} \longrightarrow ②$$

$$= \sum_{n=-\infty}^{\infty} f(n)\, e^{-j\omega n} \longrightarrow \text{DTFT}(f(n))$$

$$X(re^{j\omega}) = \sum x(n)\, r^{-n}\, e^{-j\omega n}$$

$x(n) \longrightarrow$ absolutely not summable

$r^{-n} \longrightarrow$ real exponential (decreasing & increasing)

$x(n)\, r^{-n} \longrightarrow$ absolutely summable

If $\sum_{n=-\infty}^{\infty} |x(n) r^{-n}| < \infty$ = absolutely summable

$\downarrow$

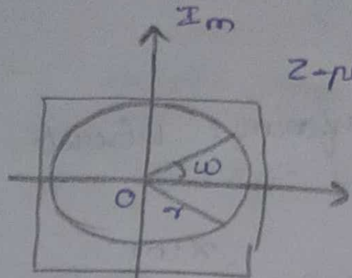
F.T is also converging.

$|z| = r =$ radius $= 1$ $=$ unit radius.

$$X(e^{j\omega}) = \Sigma x(n) e^{-j\omega n}$$

$$X(z)\Big|_{at\ r=1=|z|} = X(e^{j\omega}) = X(\omega) = \text{DTFT.}$$

$\Rightarrow$ unit circle and $j\omega$ axis are both same



If $r=1$ = radius of circle
unit circle

1. Find the Z.T of $x(n) = a^n u(n)$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$ = Bilateral / two sided z transform

$$= \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

$$= \frac{1}{1-az^{-1}} = \frac{z}{z-a}.$$

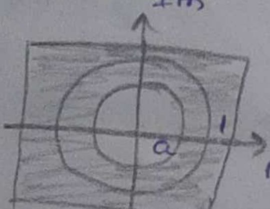
absolutely summable $= \sum_n |az^{-1}| < 1$

$\left|\frac{a}{z}\right| < 1 \Rightarrow |z| > |a| \rightarrow$ circle with radius 'a'

$X(z) = \frac{z}{z-a}$ for $|z|>|a|$, Range of values for which $X(z)$ is valid

ROC.

Right sided sequence



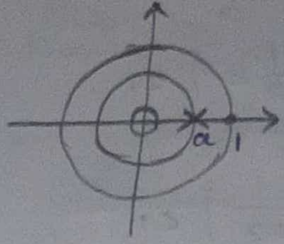
$$x(n) \xleftrightarrow{ZT} X(z)$$

$$a^n u(n) \longleftrightarrow \frac{z}{z-a}$$

$\Rightarrow$ roots of the numerator are called zeroes : $z=0 \rightarrow$ zero

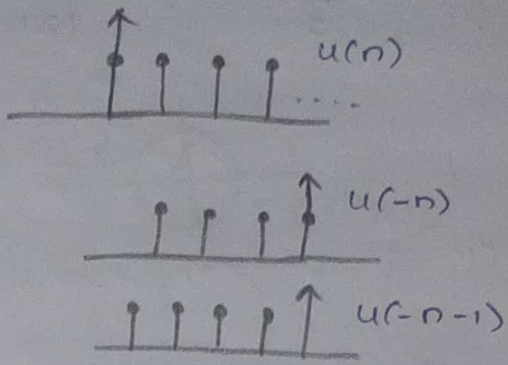
$\Rightarrow$ roots of the denominator are called poles : $z=a$

pole - zero plot : $0 < a < 1$.



2. $x(n) = -a^n u(-n-1)$

= left sided sequence
$(-\infty \text{ to } -1)$



$$X(z) = \sum_{n=-\infty}^{\infty} -a^n u(-n-1) z^{-n}$$

$$= \sum_{n=-\infty}^{-1} -a^n \cancel{u(-n-1)} z^{-n}$$

$$= -\sum_{n=1}^{\infty} a^{-n} z^n$$

$$= -\left(\sum_{n=0}^{\infty} (a^{-1}z)^n - 1\right)$$

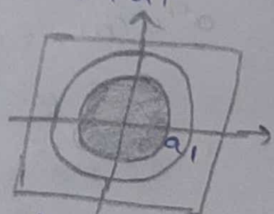
$$= 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n = 1 - \frac{1}{1-a^{-1}z}$$

$$= \frac{1-a^{-1}z-1}{1-a^{-1}z} = \frac{z}{z-a}$$

for $x(n) = -a^n u(-n-1)$, $X(z) = \frac{z}{z-a}$.

$\Sigma\, |a^{-1}z| < 1 \Rightarrow |z| < |a|$

$X(z) = \frac{z}{z-a}$ for $|z| < |a|$ for ROC



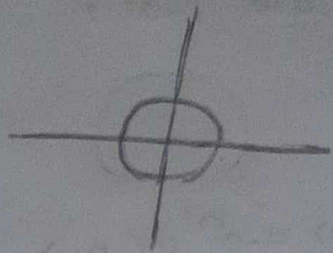
left sided sequence.

3. $x(n) = 7\left(\frac{1}{3}\right)^n u(n) - 6\left(\frac{1}{2}\right)^n u(n)$, $X(z) = ?$, ROC = ?

$$X(z) = \frac{7}{1-\frac{1}{3}z^{-1}} - \frac{6}{1-\frac{1}{2}z^{-1}}$$

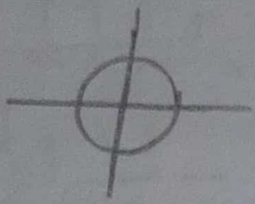
$$= \frac{7z}{z-\frac{1}{3}} - \frac{6z}{z-\frac{1}{2}} = \frac{z\left(z-\frac{3}{2}\right)}{\left(z-\frac{1}{3}\right)\left(z-\frac{1}{2}\right)}$$

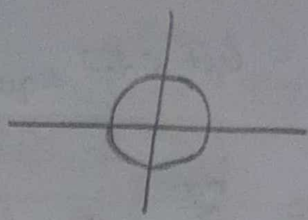
$\left|\frac{1}{3}z^{-1}\right| < 1$ $\qquad$ $\left|\frac{1}{2}z^{-1}\right| < 1$



$\left|\frac{1}{3z}\right| < 1$ $\qquad$ $\left|\frac{1}{2z}\right| < 1$

$|z| > \frac{1}{3}$ $\qquad$ $|z| > \frac{1}{2}$.





4.

$x(n) = \{1, 2, 5, 7, 0, 1\}$

$x(n) = $ 1 at $n=0$; 0 at $n=4$

2 at $n=1$; 1 at $n=5$

5 at $n=2$

7 at $n=3$

$x(n) = \delta(n) + 2(\delta(n-1)) + 5\delta(n-2) + 7\delta(n-3) + \delta(n-5)$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$X(z) = x(0) + x(1)z^{-1} + x(2)z^{-2} + \dots + \dots$$

$$= 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5}$$

5.

$x(n) = \delta(n) \Rightarrow X(z) = 1$

$a^n u(n) = x(n) \Rightarrow a = 1 \Rightarrow x(n) = u(n)$

$X(z) = \frac{z}{z-a}$, $X(z) = \frac{z}{z-1}$

* $x(n) = \cos \omega_0 n \, u(n) \longrightarrow \cos \omega_0 n = e^{j\omega_0 n} + e^{-j\omega_0 n}$

6. $x(n) = \sin \omega_0 n \, u(n)$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}.$$

$$\sin \omega_0 n = \frac{e^{j\omega_0 n} - e^{-j\omega_0 n}}{2j}$$

$$X(z) = \sum_{n=0}^{\infty} \frac{(e^{j\omega_0 n} - e^{-j\omega_0 n})}{2j} z^{-n}$$

$$= \frac{1}{2j}\sum_{n=0}^{\infty} e^{j\omega_0 n} z^{-n} - \frac{1}{2j}\sum_{n=0}^{\infty} e^{-j\omega_0 n} z^{-n}$$

$$= \frac{1}{2j}\sum_{n=0}^{\infty} (e^{j\omega} z^{-1})^n - \frac{1}{2j}\sum_{n=0}^{\infty} (e^{-j\omega} z^{-1})^n$$

$$= \frac{1}{2j}\left(\frac{1}{1-e^{j\omega}z^{-1}}\right) - \frac{1}{2j}\left(\frac{1}{1-e^{-j\omega}z^{-1}}\right)$$

$$= \frac{1}{2j}\left(\frac{z}{z-e^{j\omega}} - \frac{ze^{j\omega}}{ze^{j\omega}-1}\right)$$

$$= \frac{1}{2j}\left(\frac{\cancel{z^2e^{j\omega}} - z - \cancel{z^2e^{j\omega}} + ze^{2j\omega}}{z^2e^{j\omega} - z - ze^{2j\omega} + e^{j\omega}}\right)$$

$$= \frac{1}{2j}\left(\frac{z(-e^{-j\omega} + e^{j\omega})}{z^2(1) - ze^{-j\omega} - ze^{j\omega} + 1}\right)$$

$$X(z) = \frac{\sin\omega\, z^{-1}}{1 - z^{-1}e^{-j\omega} - z^{-1}e^{j\omega} + z^{-2}} = \frac{z\sin\omega}{z^2 - ze^{-j\omega} - ze^{j\omega} + 1}$$

* $a^n \cos\omega n\, u(n) = x(n)$

$$X(z) = \frac{1}{2}\left(\frac{1}{1-ae^{j\omega}z^{-1}} + \frac{1}{1-ae^{-j\omega}z^{-1}}\right)$$

$$= \frac{1}{2}\left(\frac{1 - ae^{-j\omega}z^{-1} + 1 - ae^{j\omega}z^{-1}}{1 - ae^{-j\omega}z^{-1} - ae^{j\omega}z^{-1} + a^2 z^{-2}}\right)$$

$$= \left(\frac{z^2 - az\cos\omega}{z^2 - 2az\cos\omega + a^2}\right)$$

* $\sin\omega n\, u(n) = x(n) \Rightarrow X(z) = \dfrac{z\sin\omega}{z^2 - 2z\cos\omega + 1}$

* $a^n \sin\omega n\, u(n) \Rightarrow X(z) = \dfrac{az\sin\omega}{z^2 - 2az\cos\omega + a^2}$

* $x(n) = \left(\frac{1}{3}\right)^n \sin\left(\frac{\pi}{4}n\right) u(n)$

$$X(z) = \frac{\frac{1}{3} z \sin \pi/4}{z^2 - 2\frac{1}{3} z \cos\frac{\pi}{4} + \left(\frac{1}{3}\right)^2}$$

$$= \frac{3z \; 1/\sqrt{2}}{9z^2 - 2z\frac{1}{\sqrt{2}} + 1} = \frac{3z}{9\sqrt{2}z^2 - 2z + \sqrt{2}}.$$

Properties of z transform:

$x(n) \longleftrightarrow X(z)$

1) Linearity property.

If $x_1(n) \longleftrightarrow X_1(z)$ & $x_2(n) \longleftrightarrow X_2(z)$

then $a x_1(n) + b x_2(n) \longleftrightarrow a X_1(z) + b X_2(z)$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_n (a x_1(n) + b x_2(n)) z^{-n}$$

$$= \sum_n (a x_1(n) z^{-n} + b x_2(n) z^{-n})$$

$$= a X_1(z) + b X_2(z)$$

2) Time shifting property:

If $x(n) \longleftrightarrow X(z)$

$x(n-n_0) \longleftrightarrow z^{-n_0} X(z)$

$X(z) = \sum_n x(n) z^{-n} \Rightarrow X(z) = \sum x(n-n_0) z^{-n}$

let $n - n_0 = l \Rightarrow n = l + n_0$

$$X(z) = \sum_l x(l) z^{-(l+n_0)} = z^{-n_0} X(z)$$

$x(n-n_0) \longleftrightarrow z^{-n_0} X(z)$

$\delta(n) \longleftrightarrow 1$, $\delta(n-k) \longleftrightarrow z^{-k}$

3) Scaling property :

If $x(n) \longleftrightarrow X(z)$ $\quad a^n x(n) \longleftrightarrow X\left(\frac{z}{a}\right)$

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$= \sum_{n} a^n x(n) z^{-n} = \sum_{n} x(n) (z^{-1} a^{1})^{n}$$

$$= \sum x(n) \left(\frac{z}{a}\right)^{-n}$$

$$= X\left(\frac{z}{a}\right)$$

4) Time reversal property:

If $x(n) \longleftrightarrow X(z)$

$x(-n) \longleftrightarrow X(z^{-1}) = X\left(\frac{1}{z}\right)$

$$X(z) = \sum x(n) z^{-n}$$

let $-n = m$

$$\Rightarrow \sum x(m) z^{m} = \sum x(m) (z^{-1})^{-m}$$

$$= X(z^{-1}) = X\left(\frac{1}{z}\right)$$

5) Convolution property :

If $x_1(n) \longleftrightarrow X(z)$ & $x_2(n) \longleftrightarrow X_2(z)$ then

$$x(n) = x_1(n) * x_2(n)$$

$$x_1(n) * x_2(n) = \sum_{K} x_1(K) x_2(n-K)$$

$$X(z) = \sum x(n) z^{-n}$$

$$= \sum (x_1(n) * x_2(n)) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left(\sum_{K=0}^{\infty} x_1(K) x_2(n-K) \right) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x_2(n-K) z^{-n} \sum_{\cancel{K=0}}^{\infty} x_1(K)$$

$$= X_2(z) \sum_{K=0} x_1(K) z^{-K} = X_1(z) X_2(z)$$

$$x_1(n) * x_2(n) \longleftrightarrow X_1(z)\, X_2(z)$$

convolution in time domain is Z.T of Multiplication in freq. domain.

6) Differentiation property:

If $x(n) \longleftrightarrow X(z)$ then

$$n\,x(n) \longleftrightarrow -z \frac{d}{dz} X(z).$$

$$X(z) = \sum_n x(n)\, z^{-n}$$

$$\frac{d}{dz} X(z) = \sum_n x(n) \frac{d}{dz}(z^{-n})$$

$$= \sum_n x(n)\,(-n)\, z^{-n-1}$$

$$= -\sum n\, x(n)\, z^{-n}\, z^{-1} \qquad = \frac{-1}{z} \sum n\, x(n)\, z^{-n}.$$

$\cancel{\frac{d}{dz} X(z) =}$

$$n\,x(n) \longleftrightarrow -z \frac{d}{dz} X(z)$$

Initial and Final Value theorem:

Initial value theorem: For a causal seq $x(n)$

$$\lim_{n\to 0} x(n) = x(0) = \lim_{z\to\infty} X(z)$$

Proof: $X(z) = \sum_{n=0}^{\infty} x(n)\, z^{-n}$

$$= x(0) + x(1)\, z^{-1} + x(2)\, z^{-2} + \dots + x(n)\, z^{-n}$$

$$\lim_{z\to\infty} X(z) = \lim_{z\to\infty} \left(x(0) + \frac{x(1)}{z} + \frac{x(2)}{z^2} + \dots + \frac{x(n)}{z^n} + \dots \right)$$

$$\therefore\ x(0) = \lim_{z\to\infty} X(z)$$

Final value theorem: For a causal seq $x(n)$

$$\lim_{n\to\infty} x(n) = x(\infty) = \lim_{z\to 1} (z-1)\, X(z)$$

proof : $X(z) = \sum_{n=0}^{\infty} x(n) z^{-n} \Rightarrow$ consider $Z T[x(n) - x(n-1)]$

$$\Rightarrow X(z) - X(z) z^{-1} = \sum_{n=0}^{\infty} (x(n) - x(n-1)) z^{-n}$$

$$= (\cancel{x(0)} - \cancel{x(-1)} + \cancel{x(1)} - \cancel{x(0)} + \cancel{x(2)} - \cancel{x(1)} + x(3) - \cancel{x(2)} \ldots) z^{-n}$$

$$X(z)(1 - z^{-1}) = X(z)\left(\frac{z-1}{z}\right)$$

$$\Rightarrow \lim_{z \to 1} X(z)(z-1) = \lim_{z \to 1} z(x(\infty))$$

$$\Rightarrow x(\infty) = \lim_{z \to 1} (z-1) X(z)$$

1. Find the Z.T of $\delta(n-k)$

w.k.t $\delta(n) \xleftrightarrow{Z.T} 1$, $\delta(n-k) \longleftrightarrow z^{-n_0} X(z)$

$\delta(n-k) \xleftrightarrow{Z.T} z^{-k}$.

2. Find the Z.T of $n\left(\frac{1}{4}\right)^n u(n)$

$$a^n u(n) \longleftrightarrow \frac{z}{z-a}$$

$$n a^n u(n) \longleftrightarrow -z \frac{d}{dz}(X(z))$$

Let $\left(\frac{1}{4}\right)^n u(n) = x(n) \Rightarrow X(z) = \frac{z}{z-\frac{1}{4}}$.

$$n\left(\frac{1}{4}\right)^n u(n) \longleftrightarrow -z \frac{d}{dz}\left(\frac{z}{z-\frac{1}{4}}\right) = -z \frac{d}{dz}\left(\frac{4z}{4z-1}\right)$$

$$-z\left(\frac{(4z-1)(4) - (4z)(4)}{(4z-1)^2}\right)$$

$$= -z\left(\frac{16z - 4 - 16z}{(4z-1)^2}\right) = \frac{4z}{(4z-1)^2}.$$

Inverse Z-transforms :

When $X(z)$ is a rational function $= \frac{A(z)}{B(z)}$

we can find Inverse Z.T using

(i) Partial fraction expansion method

(ii) Long division Method (Power series method)

(iii) Residue method

(i) Partial fraction expansion method

Let $X(z) = \frac{A(z)}{B(z)}$.

$$X(z) = \frac{A_1}{z - z_1} + \frac{A_2}{z - z_2} + \dots + \frac{A_K}{z - z_K}.$$

where $z_1, z_2, \dots z_K$ = simple poles

$$IZT[X(z)] = z^{-1}[X(z)] = z^{-1}\left(\frac{A_1}{z - z_1}\right) + z^{-1}\left(\frac{A_2}{z - z_2}\right)$$

$$+ \dots + z^{-1}\left(\frac{A_K}{z - z_K}\right)$$

w.k.t $a^n u(n) \xleftrightarrow{Z.T} \frac{z}{z-a}$ for $|z| > |a|$

for $|z| > |a|$ i.e Right hand sequence

$$z^{-1}\left(\frac{z}{z-a}\right) = a^n u(n)$$

for $|z| < |a|$ i.e left hand sequence

$$z^{-1}\left(\frac{z}{z-a}\right) = -a^n u(-n-1)$$

1. Find the IZT of $X(z) = \frac{z}{3z^2 - 4z + 1}$

$$\frac{X(z)}{z} = \frac{1}{3\left(z^2 - \frac{4}{3}z + \frac{1}{3}\right)} = \frac{1}{3(z-1)\left(z - \frac{1}{3}\right)}$$

$A_1 = 1/2$, $A_2 = -1/2$

$$\frac{X(z)}{z} = \frac{1/2}{z-1} - \frac{1/2}{z-\frac{1}{3}}.$$

$$X(z) = \frac{z}{2(z-1)} - \frac{z}{2(z-\frac{1}{3})}.$$

Note: If $X(z) = \dfrac{1}{3z^2-4z+1}$ then take $\dfrac{X(z)}{z} = \dfrac{1}{z(3z^2-4z+1)}$

Case (i) $X(z) = \frac{1}{2}\frac{z}{z-1} - \frac{1}{2}\frac{z}{z-\frac{1}{3}}$

$$\underbrace{\tfrac{1}{2}\tfrac{z}{z-1}}_{|z|>1} \ \cap\ \underbrace{\tfrac{1}{2}\tfrac{z}{z-\frac{1}{3}}}_{|z|>\frac{1}{3}} \Rightarrow |z|>1$$

$x(n) = \frac{1}{2}(1)^n u(n) - \frac{1}{2}\left(\frac{1}{3}\right)^n u(n)$

Case (ii) $X(z) = \frac{1}{2}\frac{z}{z-1} - \frac{1}{2}\frac{z}{z-\frac{1}{3}}$

$$\underbrace{\tfrac{1}{2}\tfrac{z}{z-1}}_{|z|<1} \ \cap\ \underbrace{\tfrac{1}{2}\tfrac{z}{z-\frac{1}{3}}}_{|z|<\frac{1}{3}} \Rightarrow |z|<\frac{1}{3}$$

~~Case (ii):~~ $x(n) = \frac{1}{2}(-(1)^n)u(-n-1) + \frac{1}{2}\left(\frac{1}{3}\right)^n u(-n-1)$

$= -\frac{1}{2}(1)^n u(-n-1) + \frac{1}{2}\left(\frac{1}{3}\right)^n u(-n-1)$

Case (iii): $\frac{1}{3}<|z|<1$

$x(n) = -\frac{1}{2}(1)^n u(-n-1) - \frac{1}{2}\left(\frac{1}{3}\right)^n u(n)$

2. $X(z) = \dfrac{z(z^2-4z+5)}{(z-1)(z-2)(z-3)}$

$$\frac{X(z)}{z} = \frac{z^2-4z+5}{(z-1)(z-2)(z-3)} = \frac{A}{z-1} + \frac{B}{z-2} + \frac{C}{z-3}$$

$A = \left.\dfrac{z^2-4z+5}{(z-1)(z-2)(z-3)} \times (z-1)\right|_{z=1} = \dfrac{2}{2} = 1$

$B = \left.\dfrac{z^2-4z+5}{(z-1)(z-2)(z-3)} (z-2)\right|_{z=2} = -1$

$$C = \frac{z^2-4z+5}{(z-1)(z-2)(z-3)}(z-3)\Big|_{z=3} = \frac{9-12+5}{2\times 1} = \frac{2}{2} = 1$$

$$\frac{X(z)}{z} = \frac{1}{z-1} - \frac{1}{z-2} + \frac{1}{z-3}$$

$$X(z) = \frac{z}{z-1} - \frac{z}{z-2} + \frac{z}{z-3}$$

$$z^{-1}(X(z)) = z^{-1}\left(\frac{z}{z-1}\right) - z^{-1}\left(\frac{z}{z-2}\right) + z^{-1}\left(\frac{z}{z-3}\right)$$

case (i) : for $|z| > 3$

$$x(n) = 1^n u(n) - 2^n u(n) + 3^n u(n)$$

case (ii) : for $|z| < 1$

$$x(n) = -1^n u(-n-1) + 2^n u(-n-1) - 3^n u(-n-1)$$

case (iii) : $1 < |z| < 3$

$$x(n) = 1^n u(n) + 2^n u(-n-1) - 3^n u(-n-1)$$

3.

$$X(z) = \frac{z+1}{3z^2-4z+1} = \frac{z(z+1)}{z(3z^2-4z+1)}$$

$$\frac{X(z)}{z} = \frac{z+1}{z(3z^2-4z+1)} = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{z-\frac{1}{3}}$$

$$A = \frac{z+1}{\cancel{z}(3z^2-4z+1)}\cancel{z}\Big|_{z=0} = 1$$

$B = 1$, $C = -2$

$$X(z) = \frac{z}{z} + \frac{z}{z-1} - \frac{2z}{z-\frac{1}{3}}$$

$x(n) = \delta(n) +$

(ii) **Long division method** (Power series expansion method):

$$x(z) = \frac{A(z)}{B(z)}$$

$$X(z) = \frac{1}{1-az^{-1}} \quad , \text{ROC} \Rightarrow |z| > a$$

$$\begin{array}{l} z-a \,)\, z \,(\, 1 + az^{-1} + a^2z^{-1} + \dots \\ \quad z - a \\ \quad (-)\ (+) \\ \hline \quad a \\ \quad (-)\ (\mp)\ a^2z^{-1} \\ \hline \quad a^2z^{-1} \end{array}$$

$$\begin{array}{l} 1 - az^{-1} \,)\, 1 \,(\, 1 + az^{-1} + a^2z^{-2} + \dots \\ \quad 1 - az^{-1} \\ \quad (-)\ (+) \\ \hline \quad az^{-1} \\ \quad az^{-1} \mp a^2z^{-2} \\ \quad (-)\ (+) \\ \hline \quad a^2z^{-2} \\ \quad (-)a^2z^{-2} \mp a^3z^{-3} \\ \hline \end{array}$$

$$X(z) = 1 + az^{-1} + a^2z^{-2} + \dots$$

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n} = x(0) + x(1)z^{-1} + x(2)z^{-2} + \dots\dots$$

$$\therefore x(n) = a^n u(n).$$

$u(n)$ because ROC $\Rightarrow |z| > a$

For right sided sequence ($|z| > |a|$)

Arrange $X(z)$ in decreasing powers of z (or) increasing powers of z^{-1}

For left sided sequence ($|z| < |a|$)

Arrange $X(z)$ in decreasing powers of z^{-1} (or) increasing powers of z.

$$X(z) = \frac{1}{1-az^{-1}} \quad , |z| < |a|$$

$$\begin{array}{l} -az^{-1} + 1 \,)\, 1 \,(\, -a^{-1}z - a^{-2}z^2 - a^{-3}z^3 - a^{-4}z^4 \dots\dots \\ \quad (-)\ 1 \mp a^{-1}z \\ \hline \quad a^{-1}z \\ \quad a^{-1}z - a^{-2}z^2 \\ \quad (-)\ (+) \\ \hline \quad a^{-2}z^2 \end{array}$$

$-\infty$ to -1

$\therefore u(-n-1)$

$$x(n) = -a^n u(-n-1)$$

1. $X(z) = \frac{2}{2-3z^{-1}+z^{-2}} = \frac{2}{2-\frac{3}{z}+\frac{1}{z^2}}$

$= \frac{2z^2}{2z^2-3z+1}.$

(iii) Residue Theorem (or) Inverse ZT using contour integral:

$IZT[X(z)] = x(n) = \frac{1}{2\pi j}\oint X(z)\,z^{n-1}\,dz$

$= \Sigma$ residue of $X(z)z^{n-1}$ at pole $z=z_i$.

$x(n) = R_1+R_2+R_3$

1. Find IZT of $X(z) = \frac{5z}{(z-1)(z-2)}$ using residue theorem

Sol. $x(n) = z^{-1}[X(z)] =$ Residue of $X(z)z^{n-1}$ at $z=z_i$

Residue at $z-z_i = (z-z_i)X(z)z^{n-1}\Big|_{z=z_i}$

$X(z) = \frac{5z}{(z-1)(z-2)}$

$X(z)z^{n-1} = \frac{5z}{(z-1)(z-2)}z^{n-1} = \frac{5z^n}{(z-1)(z-2)}$

poles of $X(z)z^{n-1} = \frac{5z^n}{(z-1)(z-2)}$

$\hookrightarrow$ Roots of denominator polynomial $=(z-1)(z-2)=0$

$z=1$ & $z=2$

$X(z)z^{n-1}$ has two distinct poles

$x(n) =$ Residue of $X(z)z^{n-1}$ at pole $z=z_1$ +

Residue of $X(z)z^{n-1}$ at pole $z=z_2$

$x(n) = R_1 + R_2$

$$R_1 = (z - z_1)\, X(z)\, z^{n-1} \Big|_{z=z_1}$$

$$= (z-1) \frac{5z^n}{(z-1)(z-2)} \Big|_{z=1} = -5(1)^n.$$

$$R_2 = (z-2) \frac{5z^n}{(z-1)(z-2)} \Big|_{z=2} = 5(2)^n.$$

$$x(n) = R_1 + R_2 = -5(1)^n + 5(2)^n. = 5(2^n - 1^n)$$

2. Find IZT of $X(z) = \dfrac{10z}{(z-2)(z-3)}$

Sol $X(z)\, z^{n-1} = \dfrac{10z^n}{(z-2)(z-3)}$

$$R_1 = (z-2) \frac{10z^n}{(z-2)(z-3)} \Big|_{z=2} = 10(2)^n$$

$$R_2 = (z-3) \frac{10z^n}{(z-2)(z-3)} \Big|_{z=3} = -10(3)^n$$

$$x(n) = 10(2^n - 3^n)$$

3. $X(z) = \dfrac{6z^3 - 2z^2 - z}{(z+1)^2 (z-1)^2}$ find IZT

Sol. $X(z)\, z^{n-1} = \dfrac{(6z^2 - 2z - 1)z}{(z+1)(z-1)^2} z^{n-1} = \dfrac{(6z^2 - 2z - 1)}{(z+1)(z-1)^2} z^n.$

$z_1 = -1 \rightarrow$ single pole

$z_2 = 1 \rightarrow$ 2nd order pole.

$$R_1 = \frac{(6z^2 - 2z - 1)z^n}{(z+1)(z-1)^2}(z+1) \Big|_{z=-1} = \frac{(6+2-1)(1)^n}{2^2} = \frac{7}{4}(1)^n$$

If there is nth order pole then the residue of $X(z)z^{n-1}$ at nth order pole is given by

$$\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left((z-z_0)^m \cdot X(z)z^{n-1}\right)\Big|_{at\, z=z_0}$$

$m=2 \Rightarrow$

$$\frac{1}{(2-1)!}\frac{d}{dz}\left((z-1)^2\frac{(6z^2-2z-1)z^n}{(z-1)^2(z+1)}\right)\Big|_{z=1}$$

$$=\frac{d}{dz}\left(\frac{6z^{n+2}-2z^{n+1}-z^n}{z+1}\right)\Big|_{z=1}$$

$$=\frac{(z+1)\left(6(n+2)z^{n+1}-2(n+1)z^n-nz^{n-1}\right)-(6z^{n+2}-2z^{n+1}-z^n)}{(z+1)^2}\Big|_{z=1}$$

$$=\frac{6(n+2)z^{n+2}-2(n+1)z^{n+1}-nz^n+6(n+2)z^{n+1}-2(n+1)z^n-nz^{n-1}-6z^{n+2}+2z^{n+1}+z^n}{(z+1)^2}\Big|_{z=1}$$

$$=\frac{6(n+1)z^{n+2}+4(n+3)z^{n+1}-(3n-1)z^n-nz^{n-1}}{(z+1)^2}\Big|_{z=1}$$

$$R_2=\frac{6(n+1)(1)^{n+2}+4(n+3)1^{n+1}-(3n-1)1^n-n(1)^{n-1}}{4}$$

$$x(n)=\frac{7}{4}(1)^n+R_2$$

~~Properties of ROC of Z.T:~~

Properties of ROC of Z.T

1) The ROC of $X(z)$ of a ring in the z-plane centered about the origin.

2) The ROC does not contain any poles

3) If $x[n]$ is of finite ~~[illegible]~~ duration, then the ROC is the entire z-plane, except possibly $z=0$ and/or $z=\infty$

4) If $x[n]$ is a Right sided sequence, and if the circle $|z| = r_0$ is in the ROC then all the finite values of z for which $|z| > r_0$ will ~~be~~ also be in the ROC

5) If $x[n]$ is a left sided sequence, and if the circle $|z| = r_0$ is in the ROC, then all the values of z for which $0 < |z| < r_0$ will also be in the ROC

6) If $x[n]$ is two sided and if the circle $|z| = r_0$ is in the ROC, then the ROC will consist of a ring in the z-plane that includes the circle $|z| = r_0$

7) If the Z.T $X(z)$ of $x[n]$ is rational, then its ROC is bounded by poles or extends to infinity

8) If the Z.T $X(z)$ of $x[n]$ is rational and $x[n]$ is Right sided, then the ROC is the region in the z plane outside the outermost pole - i.e outside of the circle of radius equal to the largest magnitude of the poles of $X(z)$. Furthermore if $x[n]$ is causal (i.e if it is right sided & equal to 0 for $n<0$) then the ROC also includes, $z=\infty$

9) $x[n]$ = left sided, inside the innermost nonzero pole. $x[n]$ is anti causal (i.e left sided & equal to 0 for $n>0$) then the ROC also includes $z=0$