UNIT - II

FOURIER TRANSFORMS AND SAMPLING

FOURIER TRANSFORMS:

→ In Fourier series any continuous time periodic signal f(t) can be represented as linear combination of complex exponentials and fourier coefficients are discrete.

→ The F.S can be applied to periodic signals only but F.T can also be applied to non periodic functions like rectangular pulse, step fn, ramp fn etc.

→ The fourier transform can be developed by finding fourier series of periodic function and then tending T to infinity.

→ If T tends to infinity then f(t) becomes aperiodic signal.

DERIVING FOURIER TRANSFORM FROM FOURIER SERIES:

Exponential form of fourier series is

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$$

$$F_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, e^{-jn\omega_0 t}\, dt$$

for aperiodic

$$\omega = n\omega_0 \quad (\text{continuous} \rightarrow \text{discrete})$$

$$TF_n = \int_{-\infty}^{\infty} f(t)\, e^{-jn\omega_0 t}\, dt \rightarrow (1)$$

$$\boxed{F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt} \rightarrow F.T \rightarrow (2)$$

The coefficient F_n becomes by comparing (1) & (2)

$$TF_n = F(\omega)$$

$$F_n = \frac{1}{T} F(\omega)$$

$$f(t) = \sum_{n=-\infty}^{\infty} \frac{1}{T}\cdot F(\omega)\, e^{jn\omega_0 t}$$

Other wise

$$x(t) = \sum_{k=-\infty}^{\infty} x(n)\, e^{jn\omega t} \rightarrow (1)$$

$$x(n) = \frac{1}{T}\int_{\langle T\rangle} x(t)\, e^{-jn\omega t}\, dt$$

$$Tx(n) = \int_{\langle T\rangle} x(t)\, e^{-jn\omega t}\, dt \rightarrow (2)$$

Reagranging eq (1)

$$x(t) = \frac{1}{T}\sum_{k=-\infty}^{\infty} Tx(n)\, e^{jn\omega t}$$

we know $\omega = \frac{2\pi}{T} \Rightarrow T = \frac{2\pi}{\omega}$

$$x(t) = \frac{\omega}{2\pi}\sum_{k=-\infty}^{\infty} Tx(n)\, e^{jn\omega t}$$

$$= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} Tx(n)\, e^{jn\omega t}\cdot\omega$$

As $T\rightarrow\infty$, ω becomes small. Hence $\omega \rightarrow d\omega$

→ $Tx(k)$ becomes continuous as $T\rightarrow\infty$

$$x(t) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

Eq ② can be written as

$T \to \infty$, $T X(k) \to X(\omega)$

eq ②

$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$

$$f(t) = \sum_{n=-\infty}^{\infty} \left[\frac{1}{T} F(n\omega_0) \right] e^{jn\omega_0 t}$$

$$= \frac{1}{T} \sum_{n=-\infty}^{\infty} F(n\omega_0) e^{jn\omega_0 t}$$

$$\omega_0 = \frac{2\pi}{T} \; ; \; T = \frac{2\pi}{\omega_0}$$

$$f(t) = \frac{\omega_0}{2\pi} \sum_{n=-\infty}^{\infty} F(n\omega_0) e^{jn\omega_0 t} \quad \left(\because \omega_0 = \frac{\omega}{n} \quad n\,d\omega_0 = d\omega . \right)$$

$n \to \infty$

$n\omega_0 \to$ approaches a continuous freq variable 'ω'.

$$\boxed{f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega}$$

IFT.

Note:

→ For periodic signals discrete waveform is formed and for aperiodic signals continuous waveform is formed.

→ Fourier transform of a signal $f(t)$ is defined as

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

Inverse fourier transform

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega .$$

$$F(\omega) = \underbrace{|F(\omega)|}_{\text{magnitude}} \, e^{j\overbrace{\theta(\omega)}^{\text{phase}}}$$

→ By application of F.T a signal in time domain is converted into frequency domain.

FOURIER TRANSFORM OF ANY ARBITARY SIGNALS AND STANDARD SIGNALS:

(1) Fourier transform of Single Sided Exponential signal.

The single sided exponential is defined as

$$x(t) = Ae^{-at} \quad ; \text{ for } t \geq 0$$

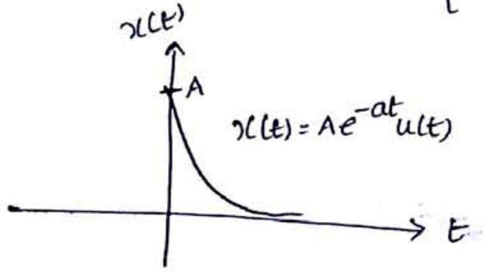
By definition of fourier transform

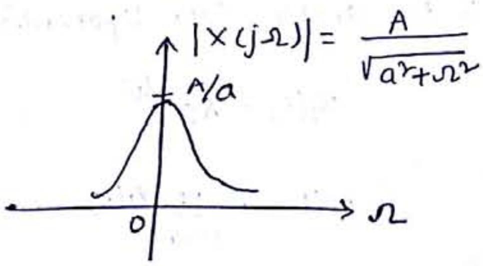
$$X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt = \int_{0}^{\infty} Ae^{-at}e^{-j\Omega t}dt$$

$$= \int_{0}^{\infty} Ae^{-(a+j\Omega)t}dt = \left[\frac{Ae^{-(a+j\Omega)t}}{-(a+j\Omega)}\right]_0^{\infty}$$

$$= \left[\frac{Ae^{-\infty}}{-(a+j\Omega)} - \frac{Ae^{0}}{-(a+j\Omega)}\right] = \frac{A}{a+j\Omega}$$

$$\therefore F\{Ae^{-at}u(t)\} = \frac{A}{a+j\Omega}$$





(2) Fourier transform of Doubled Sided Exponential Signal

$$x(t) = Ae^{-a|t|} \quad ; \text{ for all 't'}$$

$$\therefore x(t) = Ae^{at} \quad ; \text{ for } t = -\infty \text{ to } 0$$

$$= Ae^{-at} \quad ; \text{ for } t = 0 \text{ to } \infty$$

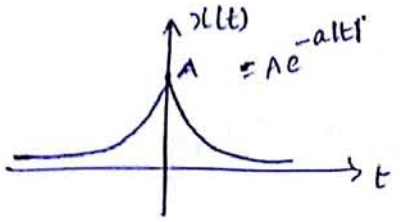
$$X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt = \int_{-\infty}^{0} Ae^{at}\cdot e^{-j\Omega t}dt + \int_{0}^{\infty} Ae^{-at}e^{-j\Omega t}dt$$

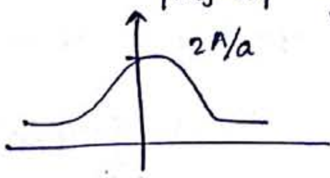
$$= \int_{-\infty}^{0} A\cdot e^{(a-j\Omega)t}dt + \int_{0}^{\infty} Ae^{-(a+j\Omega)t}dt$$

$$= \frac{Ae^{(a-j\Omega)t}}{a-j\Omega}\Bigg|_{-\infty}^{0} + \frac{Ae^{-(a+j\Omega)t}}{-(a+j\Omega)}\Bigg|_{0}^{\infty}$$

$$= \frac{Ae^{0}}{a-j\Omega} - \frac{Ae^{-\infty}}{a-j\Omega} + \frac{Ae^{-\infty}}{-(a+j\Omega)} - \frac{Ae^{0}}{-(a+j\Omega)}$$

$$= \frac{A}{a-j\Omega} + \frac{A}{a+j\Omega} = \frac{2aA}{a^2+\Omega^2}$$





(c) Fourier Transform of a Constant :

Let $x(t) = A$ (constant)

If definition of F.T is directly applied, the constant will not satisfy the condition

$$\int_{-\infty}^{\infty} |x(t)|\,dt < \infty.$$

Let $x_1(t)$ = double sided exponential sgl

$$x_1(t) = Ae^{-a|t|}$$

$$\therefore x(t) = A \lim_{a\to 0} x_1(t).$$

On taking F.T

$$F\{x(t)\} = F\{\lim_{a\to 0} x_1(t)\}$$

$$F\{x(t)\} = \lim_{a\to 0} F\{x_1(t)\}$$

$$X(j\Omega) = \lim_{a\to 0}\left[X_1(j\Omega)\right]$$

$$= \lim_{a\to 0} \frac{2aA}{\Omega^2+a^2} = 0 \text{ for all values of } \Omega \text{ except at } \Omega = 0.$$

At $\Omega=0$, the above eqn represents an impulse of magnitude k'

$$X(j\Omega) = k\delta(\Omega) \quad ; \Omega = 0$$
$$= 0 \quad ; \Omega \neq 0.$$

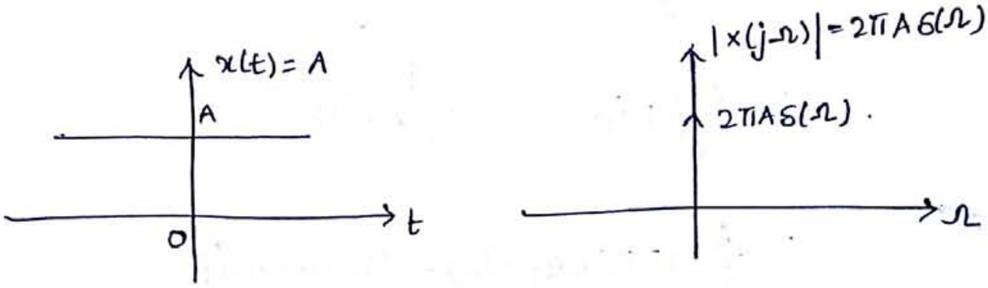
The magnitude 'k' can be evaluated as

$$k = \int_{-\infty}^{\infty} \frac{2aA}{\Omega^2+a^2}\,d\Omega = 2aA\int_{-\infty}^{\infty}\frac{1}{\Omega^2+a^2}\,d\Omega \quad \left(\because \int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}\right)$$

$$= 2aA\left[\frac{1}{a}\tan^{-1}\left(\frac{\Omega}{a}\right)\right]_{-\infty}^{\infty} = 2aA\left[\frac{1}{a}\tan^{-1}(+\infty) - \frac{1}{a}\tan^{-1}(-\infty)\right]$$

$$= 2aA\left[\frac{1}{a}\cdot\frac{\pi}{2} - \frac{1}{a}\left(-\frac{\pi}{2}\right)\right] = 2aA\left(\frac{\pi}{a}\right) = 2\pi A$$

$$\therefore F\{A\} = 2\pi A\,\delta(\Omega)$$



(d) Fourier transform of Unit step function:

$$u(t) = 1 \quad ; t \geq 0$$
$$= 0 \quad ; t < 0$$

$$\text{sgn}(t) = 2u(t) - 1 \Rightarrow u(t) = \frac{1}{2}\left[1 + \text{sgn}(t)\right]$$

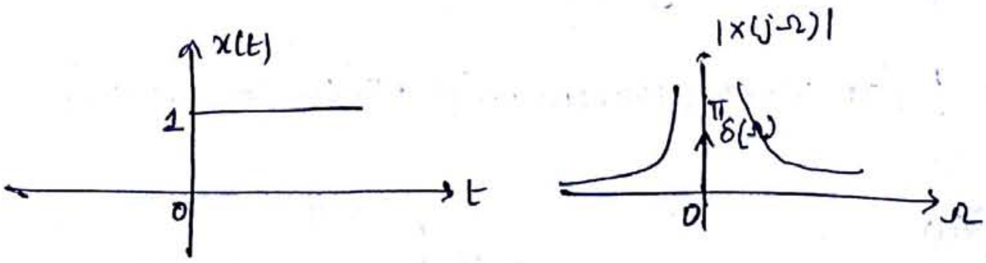
$$\therefore x(t) = u(t) = \frac{1}{2}\left[1 + \text{sgn}(t)\right]$$

$$X(j\Omega) = F\left\{\frac{1}{2}\right\} + F\left\{\frac{1}{2}\text{sgn}(t)\right\}$$

$$= \frac{1}{2}F\{1\} + \frac{1}{2}F\{\text{sgn}(t)\}$$

$$= \frac{1}{2}\left[2\pi\delta(\Omega)\right] + \frac{1}{2}\left[\frac{2}{j\Omega}\right] \qquad \left(\because F\{\text{sgn}(t)\} = \frac{2}{j\Omega}\right)$$

$$F\{u(t)\} = \pi\delta(\Omega) + \frac{1}{j\Omega}$$



(e) Fourier transform of Sinusoidal Signals:

$$x(t) = A\sin\Omega_0 t = \frac{A}{2j}\left[e^{j\Omega_0 t} - e^{-j\Omega_0 t}\right] \quad \left(\because \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}\right]$$

On taking F.T we get

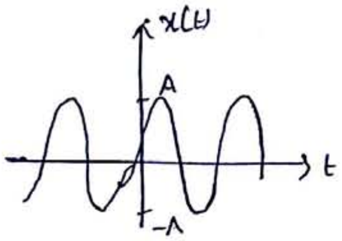
$$F\{x(t)\} = F\left\{\frac{A}{2j}\left(e^{j\omega_0 t} - e^{-j\Omega_0 t}\right)\right\}$$

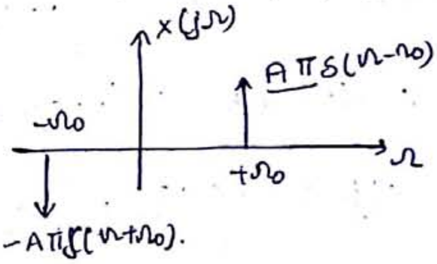
$$= \frac{A}{2j}\left[F\{e^{j\Omega_0 t}\} - F\{e^{-j\Omega_0 t}\}\right] \qquad \left\{ \frac{1}{2}e^{j\omega_0 t} \xleftrightarrow{FT} \pi\,\delta(\omega-\omega_0) \right.$$

$$= \frac{A}{2j}\left[2\pi\delta(\Omega-\Omega_0) - 2\pi\delta(\Omega+\Omega_0)\right]$$

$$= \frac{A\pi}{j}\left[\delta(\Omega-\Omega_0) - \delta(\Omega+\Omega_0)\right]$$

$$\therefore F\{A\sin\Omega_0 t\} = \frac{A\pi}{j}\left[\delta(\Omega-\Omega_0) - \delta(\Omega+\Omega_0)\right]$$





(f) Fourier transform of cosinusoidal sgl:

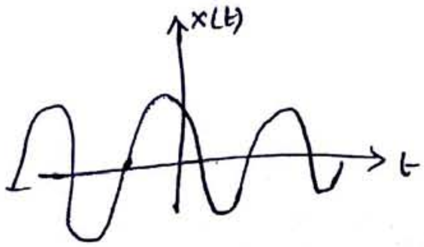
$$x(t) = A\cos\Omega_0 t = \frac{A}{2}\left[e^{j\Omega_0 t} + e^{-j\Omega_0 t}\right] \qquad \left\{ \because \text{ we know that } \; 1 \xleftrightarrow{FT} 2\pi\delta(\omega) \right.$$

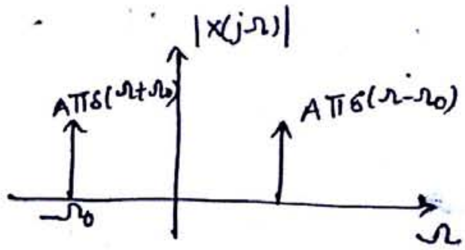
$$\frac{1}{2} \xleftrightarrow{FT} \pi\delta(\omega)$$

On taking F.T

$$F\{x(t)\} = F\left\{\frac{A}{2}\left(e^{j\Omega_0 t} + e^{-j\Omega_0 t}\right)\right\} = \frac{A}{2}\left\{F\left(e^{j\Omega_0 t}\right\} + F\left(e^{-j\Omega_0 t}\right)\right]$$

$$= \frac{A}{2}\left[2\pi\,\delta(\Omega-\Omega_0) + 2\pi\delta(\Omega+\Omega_0)\right] = A\pi\left[\delta(\Omega-\Omega_0) + \delta(\Omega+\Omega_0)\right]$$





Properties of F.T :

(1) Linearity : If $x(t) \xleftrightarrow{FT} X(\omega)$ & $y(t) \xleftrightarrow{FT} Y(\omega)$ then $z(t) = a x(t) + b y(t)$
$\xleftrightarrow{FT} Z(\omega) = aX(\omega) + bY(\omega)$

proof

$$Z(\omega) = \int_{-\infty}^{\infty} z(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} [a x(t) + b y(t)] e^{-j\omega t} dt$$

$$= a \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt + b \int_{-\infty}^{\infty} y(t) e^{-j\omega t} dt$$

$$= a X(\omega) + b Y(\omega).$$

(2) Time shift : If $x(t) \xleftrightarrow{FT} X(\omega)$ then $y(t) = x(t - t_0) \xleftrightarrow{FT} Y(\omega) = e^{-j\omega t_0} \cdot X(\omega)$

proof

$$Y(\omega) = \int_{-\infty}^{\infty} y(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} x(t - t_0) e^{-j\omega t} dt$$

put $t - t_0 = \tau \Rightarrow t = \tau + t_0$

$\therefore dt = d\tau$ and integration limits remains the same.

$$Y(\omega) = \int_{-\infty}^{\infty} x(\tau) e^{-j\omega(\tau + t_0)} d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau) e^{-j\omega\tau} d\tau \cdot e^{-j\omega t_0}$$

$$Y(\omega) = e^{-j\omega t_0} \cdot X(\omega).$$

(3) Frequency shift : If $x(t) \xleftrightarrow{FT} X(\omega)$ then $y(t) = e^{j\omega_0 t} x(t) \xleftrightarrow{FT} Y(\omega) = X(\omega - \omega_0)$

proof:

$$Y(\omega) = \int_{-\infty}^{\infty} y(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} e^{j\omega_0 t} x(t) \cdot e^{-j\omega t} dt = \int_{-\infty}^{\infty} x(t) \cdot e^{-jt(\omega - \omega_0)} dt$$

$$= X(\omega - \omega_0)$$

(4) Time Scaling:

If $x(t) \xleftrightarrow{FT} X(\omega)$ then $y(t) = x(at) \xleftrightarrow{FT} Y(\omega) = \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$.

Meaning: Compression of a signal in time domain is equivalent to expansion in frequency domain & vice versa.

proof

$$Y(\omega) = \int_{-\infty}^{\infty} y(t) \cdot e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} x(at) e^{-j\omega t} dt$$

put $at = \tau \Rightarrow t = \tau/a$

$dt = \frac{d\tau}{a}$ and limits remain the same

$$\therefore \quad Y(\omega) = \int_{-\infty}^{\infty} x(\tau) e^{-j\omega(\tau/a)} \cdot \frac{d\tau}{a}$$

$$= \frac{1}{a} \int_{-\infty}^{\infty} x(\tau) e^{-j\left(\frac{\omega}{a}\right)\tau} d\tau$$

$$= \frac{1}{|a|} X\left(\frac{\omega}{a}\right).$$

(5) Frequency differentiation:

If $x(t) \xleftrightarrow{FT} X(\omega)$ then $-jt \; x(t) \xleftrightarrow{FT} \frac{d}{d\omega} X(\omega)$

Meaning: Differentiating the frequency spectrum is equivalent to multiplying the time domain signal by complex number $-jt$.

proof

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$\frac{dX(\omega)}{d\omega} = \int_{-\infty}^{\infty} x(t) \cdot \frac{d}{d\omega}\left[e^{-j\omega t}\right] dt = \int_{-\infty}^{\infty} x(t) \cdot (-jt) e^{-j\omega t} dt$$

$$= -jt \cdot \frac{j}{j} \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} dt =$$

$$= \frac{1}{j} \int_{-\infty}^{\infty} (t x(t)) e^{-j\omega t} dt$$

$$= -j(t x(t))]$$

(6) Time Differentiation:

If $x(t) \xleftrightarrow{FT} X(\omega)$ then $\frac{dx(t)}{dt} \xleftrightarrow{FT} j\omega X(\omega)$.

Meaning : Differentiation in time domain corresponds to multiplying by $j\omega$ in frequency domain.

proof:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

$$\frac{dx(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) \cdot \frac{d}{dt}[e^{j\omega t}]\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) \cdot e^{j\omega t}\, d\omega\, (j\omega)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} [j\omega X(\omega)]\, e^{j\omega t}\, d\omega$$

Thus fourier transform is multiplied by $j\omega$

(or)

$$F\{x(t)\} = X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$F\left\{\frac{dx(t)}{dt}\right\} = \int_{-\infty}^{\infty}\left(\frac{dx(t)}{dt}\right) e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} e^{-j\omega t}\left(\frac{dx(t)}{dt}\right) dt$$

$$= \left[e^{-j\omega t} \cdot x(t)\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} (-j\omega)\, e^{-j\omega t} \cdot x(t)\, dt$$

$$= e^{-\infty} x(\infty) - e^{+\infty} x(-\infty) + j\omega \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t}\, dt$$

$$\frac{dx(t)}{dt} = j\omega \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t}\, dt$$

$$\frac{dx(t)}{dt} \xleftrightarrow{FT} j\omega X(\omega).$$

(7) Time Integration:

If $F\{x(t)\} = X(j\omega)$ then $\int_{-\infty}^{t} x(\tau)\,d\tau \xleftrightarrow{FT} \frac{1}{j\omega} X(\omega)$

proof

Let

$x(t)$ is expressed as

$$x(t) = \frac{d}{dt}\left[\int_{-\infty}^{t} x(\tau)\,d\tau\right]$$

$$\therefore \quad F[x(t)] = F\left[\frac{d}{dt}\int_{-\infty}^{t} x(\tau)\,d\tau\right]$$

By differentiation property for right hand side

$$F[x(t)] = j\omega\, F\left[\int_{-\infty}^{t} x(\tau)\,d\tau\right]$$

$$X(\omega) = j\omega\, F\left[\int_{-\infty}^{t} x(\tau)\,d\tau\right]$$

$$\frac{1}{j\omega} X(\omega) = F\left[\int_{-\infty}^{t} x(\tau)\,d\tau\right]$$

(or) $$F\left[\int_{-\infty}^{t} x(\tau)\,d\tau\right] = \frac{1}{j\omega} X(\omega).$$

(8) Modulation or Multiplication:

If $x(t) \xleftrightarrow{FT} X(\omega)$ and $y(t) \xleftrightarrow{FT} Y(\omega)$ then $z(t) = x(t)\,y(t) \xleftrightarrow{FT}$

$$Z(\omega) = \frac{1}{2\pi}\left[X(\omega) * Y(\omega)\right]$$

Proof

$$Z(\omega) = \int_{-\infty}^{\infty} z(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} x(t)\, y(t)\, e^{-j\omega t}\, dt \quad \longrightarrow (1)$$

Inverse F.T states that

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)\, e^{j\lambda t}\, d\lambda$$

putting for $x(t)$ in eq ①

$$Z(\omega) = \int_{-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)e^{j\lambda t}d\lambda\right] y(t)\,e^{-j\omega t}dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)\int_{-\infty}^{\infty} y(t)\,e^{-j(\omega-\lambda)t}\,dt\,d\lambda$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)\,Y(\omega-\lambda)\,d\lambda$$

$$= \frac{1}{2\pi}\left[X(\omega) * Y(\omega)\right]$$

Parseval's Theorem:

If $x(t) \xleftrightarrow{FT} X(\omega)$ then $E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}|X(\omega)|^2 d\omega$

$$= \int_{-\infty}^{\infty} |X(f)|^2 df.$$

proof

$$E = \int_{-\infty}^{\infty} |x(t)|^2\,dt$$

$$= \int_{-\infty}^{\infty} x(t)\cdot x^*(t)\,dt \longrightarrow ①$$

Inverse F.T states that

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\,e^{j\omega t}d\omega$$

Taking conjugate of both the sides

$$x^*(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega)\,e^{-j\omega t}\,d\omega$$

putting above eqn $x^*(t)$ in eq ①

$$E = \int_{-\infty}^{\infty} x(t)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega)\,e^{-j\omega t}\right] d\omega\cdot dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega)\int_{-\infty}^{\infty} x(t)\,e^{-j\omega t}\,dt\cdot d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega)\cdot X(\omega)\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega.$$

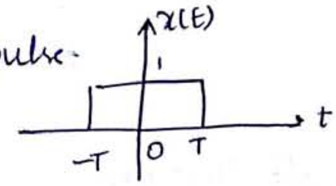
Since $\omega = 2\pi f$, $d\omega = 2\pi df$

Equation becomes

$$E = \int_{-\infty}^{\infty} |X(f)|^2 df \cdot \frac{1}{2\pi} \cdot 2\pi$$

$$\boxed{E = \int_{-\infty}^{\infty} |X(f)|^2 df}$$

2) Obtain the fourier transform of rectangular pulse.



Sol

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$= \int_{-T}^{T} 1 \cdot e^{-j\omega t} dt = \left[\frac{e^{-j\omega t}}{-j\omega}\right]_{-T}^{T} = \left[\frac{e^{-j\omega T}}{-j\omega} + \frac{e^{+j\omega T}}{-j\omega}\right]$$

$$= -\frac{1}{j\omega}\left[e^{-j\omega T} - e^{j\omega T}\right]$$

$$= \frac{1}{\omega} \cdot \frac{2}{2j}\left[e^{j\omega T} - e^{-j\omega T}\right]$$

$$= \frac{2}{\omega}\left[\frac{e^{j\omega T} - e^{-j\omega T}}{2j}\right]$$

$$= \frac{2}{\omega} \sin(\omega T)$$

we know $\text{sinc}(\theta) = \frac{\sin \pi\theta}{\pi\theta}$.

$\left\{ \because \frac{\sin\theta}{\theta} = \text{sinc}\,\theta \right.$

$$X(\omega) = \frac{2}{\omega} \sin\omega T \cdot \frac{T}{T}$$

$$= 2T \cdot \frac{\sin\omega T}{\omega T} = 2T\,\text{sinc}(\omega T)$$

→ Obtain the fourier transform of $x(t) = t e^{-at} u(t)$

Sol

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} t \cdot e^{-at} u(t) e^{-j\omega t} dt$$

$$= \int_{0}^{\infty} t \cdot e^{-at} e^{-j\omega t} dt \quad \left\{ \because u(t) = 1 \text{ for } t \geq 0 \right.$$

$$X(\omega) = \int_{0}^{\infty} t \cdot e^{-(a+j\omega)t} dt$$

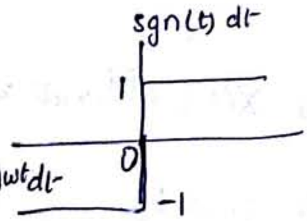
$$X(\omega) = \left[t \int e^{-(a+j\omega)t} dt - \int 1 \cdot \frac{e^{-(a+j\omega)t}}{-(a+j\omega)} dt \right]_{0}^{\infty}$$

$$= \left[\frac{t \cdot e^{-(a+j\omega)t}}{-(a+j\omega)} + \frac{1}{a+j\omega} \cdot \frac{e^{-(a+j\omega)t}}{[-(a+j\omega)]} \right]_{0}^{\infty}$$

$$= 1/(a+j\omega)^2$$

$$\therefore \; t e^{-at} u(t) \xleftrightarrow{FT} \frac{1}{(a+j\omega)^2}$$

Signum function:



$\Rightarrow u(t) - u(-t)$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} dt = \int_{-\infty}^{\infty} sgn(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{0} -1 \cdot e^{-j\omega t} dt + \int_{0}^{\infty} 1 \cdot e^{-j\omega t} dt$$

$$= -\left[\frac{e^{-j\omega t}}{-j\omega} \right]_{-\infty}^{0} + \left[\frac{e^{-j\omega t}}{-j\omega} \right]_{0}^{\infty} = \frac{1}{j\omega}\left[e^{-0} - e^{-\infty} \right] + \frac{1}{j\omega}[0-1]$$

$$= \frac{1}{j\omega}[1-0] - \frac{1}{j\omega}[0-1] = \frac{2}{j\omega}$$

(B) $x(t) = 1$

$$\int_{-\infty}^{\infty} |x(t)|\, dt = \int_{-\infty}^{\infty} dt \to \infty$$ i.e Dirichlet condition is not satisfied.

But its fourier transform can be calculated with the help of duality property.

$$\delta(t) \xleftrightarrow{FT} 1$$

Here $X(\omega) = 1$

$$\therefore \quad \delta(t) \xleftrightarrow{FT} X(\omega)$$

The duality property states that

$$X(t) \xleftrightarrow{FT} 2\pi\, x(-\omega)$$

Here $X(t) = 1$ then $x(-\omega)$ will be $\delta(-\omega)$.

$$1 \longleftrightarrow 2\pi\, \delta(-\omega)$$

Since it is impulse function $\delta(\omega)$ will be even functions of ω.

$$\delta(-\omega) = \delta(\omega)$$

$$1 \xleftrightarrow{FT} 2\pi\delta(\omega)$$

$$\therefore \text{ if } x(t) = 1,\ X(\omega) = 2\pi\delta(\omega).$$

Duality property:

If $x(t) \xleftrightarrow{FT} X(\omega)$ then $X(t) \xleftrightarrow{FT} 2\pi\, x(-\omega)$

proof

Inverse F.T is given by

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

Interchanging 't' by 'ω'

$$x(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(t) e^{j\omega t} dt$$

Interchanging ω by −ω

$$x(-\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(t)\, e^{-j\omega t} dt \Rightarrow \boxed{X(t) = 2\pi\, x(-\omega)}$$

FOURIER TRANSFORM OF UNIT IMPULSE SIGNAL AND SIGNUM FUNCTION:

→ The impulse signal is defined as

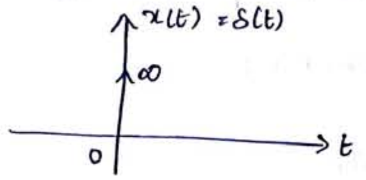
$$x(t) = \delta(t) = \infty \;;\; t=0 \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(t)\,dt = 1$$
$$= 0 \;;\; t \neq 0$$

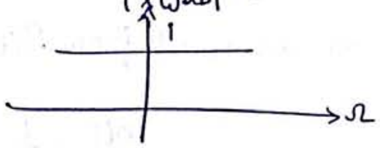
By definition of F.T

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\Omega t}\, dt = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\Omega t}\, dt$$

$$= 1 \times e^{-j\Omega t}\Big|_{t=0} = 1 \times e^{0} = 1$$

$$\therefore F\{x(t)\} = 1.$$





Signum function:

→ $x(t) = \text{sgn}(t) = 1 \;;\; t>0$
$\quad\quad = -1 \;;\; t<0$

$$\therefore \text{sgn}(t) = \underset{a\to 0}{Lt} \left[e^{-at}u(t) - e^{at}u(-t) \right]$$

$$x(t) = \text{sgn}(t) = \underset{a\to 0}{Lt} \left[e^{-at}u(t) - e^{at}u(-t) \right]$$

By definition of F.T

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\Omega t}\, dt = \int_{-\infty}^{\infty} \underset{a\to 0}{Lt} \left[e^{-at}u(t) - e^{at}u(-t) \right] e^{-j\Omega t}\, dt$$

$$= \underset{a\to 0}{Lt} \left[\int_{0}^{\infty} e^{-at} \cdot e^{-j\Omega t}\, dt - \int_{-\infty}^{0} e^{at} e^{-j\Omega t}\, dt \right]$$

$$= \underset{a\to 0}{Lt} \left[\int_{0}^{\infty} e^{-(a+j\Omega)t}\, dt - \int_{-\infty}^{0} e^{(a-j\Omega)t}\, dt \right]$$

$$= \underset{a\to 0}{Lt} \left[\left[\frac{e^{-(a+j\Omega)t}}{-(a+j\Omega)} \right]_{0}^{\infty} - \left[\frac{e^{(a-j\Omega)t}}{a-j\Omega} \right]_{-\infty}^{0} \right]$$

$$= \underset{a\to 0}{Lt}\left[\frac{e^{-\infty}}{-(a+j\Omega)} - \frac{e^{0}}{-(a+j\Omega)} - \frac{e^{0}}{a-j\Omega} + \frac{e^{-\infty}}{a-j\Omega}\right]$$

$$= \underset{a\to 0}{Lt}\left[\frac{1}{a+j\Omega} - \frac{1}{a-j\Omega}\right] = \frac{1}{j\Omega} + \frac{1}{j\Omega} = \frac{2}{j\Omega}$$

INTRODUCTION TO HILBERT TRANSFORM:

The hilbert transform of $h(t)$ is defined by

$$\hat{h}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{h(\tau)}{t-\tau}\,d\tau$$

The inverse hilbert transform by means of which the original signal $h(t)$ is recovered from $\hat{h}(t)$ is defined by

$$h(t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\hat{h}(\tau)}{t-\tau}\,d\tau$$

→ The functions $h(t)$ and $\hat{h}(t)$ are said to constitute a hilbert transform pair.

APPLICATIONS:

→ Areas of signal processing, analysis and synthesis of signals, design of filters etc.

→ To represent band pass signals

→ To realize phase selectivity in the generation of SSB systems.

→ To relate the gain & phase characteristics of linear communication channels & filters of minimum phase type.

SAMPLING:

→ It is a process of converting a continuous time signal into discrete-time signal.

→ The time interval between two subsequent sampling instants is called sampling interval.

→ Only we need to consider that the signal sampling rate must be kept high in order to reconstruct the original signal from its samples.

SAMPLING THEOREM - GRAPHICAL AND ANALYTICAL PROOF FOR BAND LIMITED SIGNALS:

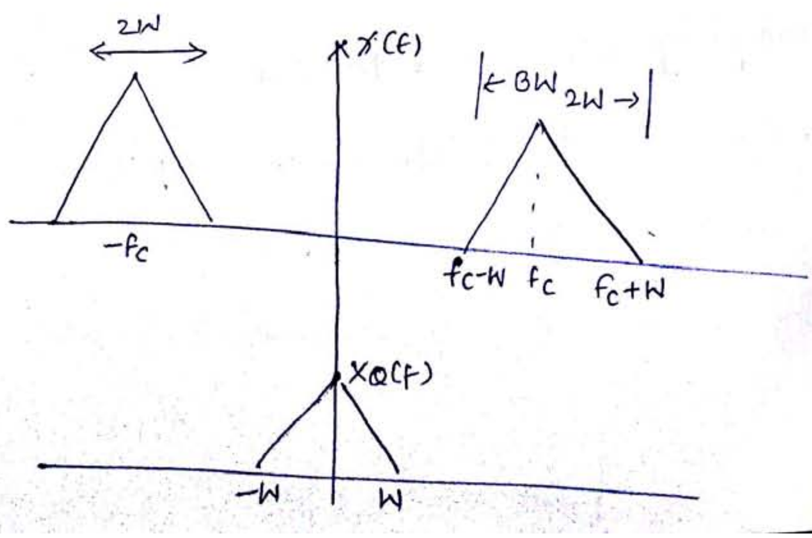
→ A band limited signal $x(t)$ with $X(j\Omega) = 0$ for $|\Omega| > \Omega_m$ is uniquely determined from its samples $x(nT)$, if the sampling frequency $f_s \geq 2f_m$ i.e sampling frequency must be at least twice the highest frequency present in the signal.

→ The time interval between successive samples is called sampling period and its reciprocal $1/T = f_s$ is called sampling rate.

proof:

Let the signal $x(t)$ be a bandpass nature. The bandpass has the bandwidth of '2W' centered around f_c. The bandpass signal is represented interms of its inphase and quadrature components as

$$x(t) = \underbrace{x_I(t)\cos(2\pi f_c t)}_{\text{phase}} - \underbrace{x_Q(t)\sin(2\pi f_c t)}_{\text{quadrature}}$$



→ The inphase and quadrature components are sampled at $f_S = 4W$ and obtained by multiplying $x(t)$ by $\cos w_c t$ and $\sin w_c t$.

→ Suppressing the sum frequencies using LPF.

→ Thus $x_I(t)$ and $x_Q(t)$ contain only low frequency components.

→ The spectrum of these components is limited b/n $-f_m$ to f_m as shown in the figure above.

$$x(t) = \sum_{n=-\infty}^{\infty} x\left(\frac{n}{4f_m}\right) \operatorname{sinc}\left[2f_m t - \frac{n}{2}\right] \cos\left(w_c\left(t - \frac{n}{4f_m}\right)\right)$$

compare reconstruction formula with that of LP signals given in the interpolation formula, we observe $x(t)$ is replaced by $x\left(\frac{n}{4f_m}\right)$.

$$x\left(\frac{n}{4f_m}\right) = x(nT_S) \longrightarrow \text{sampled version of BP signal}$$

$$T_S = 1/4f_m$$

$4f_m \longrightarrow$ samples/sec are taken.

→ Band pass signal $BW = 2f_m$ can be completely recovered by its samples.

→ Minimum sampling rate = twice the BW

$$f_S = 2 \times 2f_m = 4f_m \text{ samples/sec.}$$

SAMPLING THEOREM FOR LOW PASS SIGNAL:

→ A LP signal contains frequencies from 1Hz to some higher value.

statements:

1. A band limited signal of finite energy, which has no frequency components higher than W Hertz, is completely described by specifying the values of the signals at instants of time separated by $\frac{1}{2W}$ seconds.

2. A band limited signal of finite energy, which has no freq components higher than W hertz may be completely recovered from the knowledge of its samples taken at the rate 2W samples per second → reconstruction of sgl.

3. A CT signal can be completely represented in its samples and recovered back if the sampling frequency is twice of the highest frequency content of the signal i.e $f_s > 2W$ → higher frequency.
↳ sampling frequency

Proof: There are 2 parts
1. Representation of $x(t)$ in terms of samples
2. Reconstruction of $x(t)$ from its samples.

1. Representation of $x(t)$ in its samples $x(nT_s)$:

steps: 1. Define $x_\delta(t)$
2. F.T of $x_\delta(t)$ i.e $\delta(f)$
3. Relation b/n $X(f)$ and $X_\delta(f)$
4. Relation b/n $x(t)$ & $x(nT_s)$

1. Define $x_\delta(t)$: $x_\delta(t) = \sum_{n=-\infty}^{\infty} x(t) \cdot \delta(t - nT_s)$ → ① it is product of x_δ and impulse train $\delta(t)$ as shown in fig ①

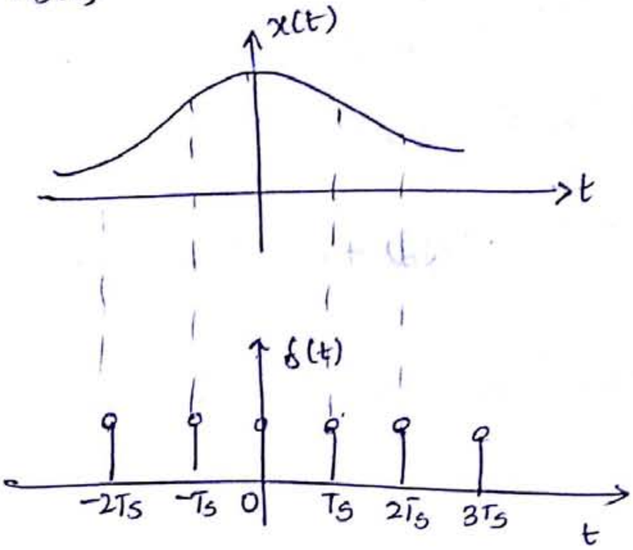
in ① ⇒ $\delta(t - nT_s)$ indicates the samples placed at $\pm T_s, \pm 2T_s \cdots$

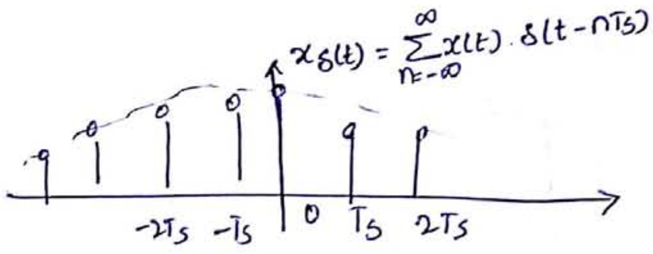
2. F.T of $x_\delta(t)$:

$$X_\delta(t) = FT\left[\sum_{n=-\infty}^{\infty} x(t)\,\delta(t - nT_s)\right]$$

Proof:

step 1: Define $x_\delta(t)$





$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x(t) . \delta(t-nT_s)$$

↙

product of $x(t)$ and impulse train . $\delta(t-nT_s)$ indicates samples placed at $\pm T_s, \pm 2T_s + \cdots$

step 2: Fourier transform of $x_\delta(t)$ i.e $X_\delta(f)$.

$$X_\delta(f) = FT\left\{ \sum_{n=-\infty}^{\infty} x(t)\, \delta(t-nT_s) \right] \longrightarrow ⓐ$$

We know that fourier transform of product or multiplication in time domain is the convolution in frequency domain.

$$X_\delta(f) = FT\{x(t)\} * FT\{\delta(t-nT_s)\} \longrightarrow ①$$

By definition $x(t) \longleftrightarrow X(f)$ and $\delta(t-nT_s) \xleftrightarrow{FT} f_s \sum_{n=-\infty}^{\infty} \delta(f-nf_s)$

$\therefore$ Eq (1) becomes

$$X_\delta(f) = X(f) * f_s \sum_{n=-\infty}^{\infty} \delta(f-nf_s)$$

$$X_\delta(f) = f_s \sum_{n=-\infty}^{\infty} X(f) * \delta(f - nf_s)$$

By shifting property of impulse function

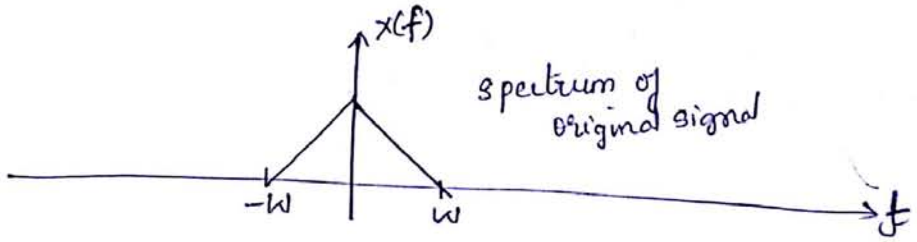
$$X_\delta(f) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s)$$

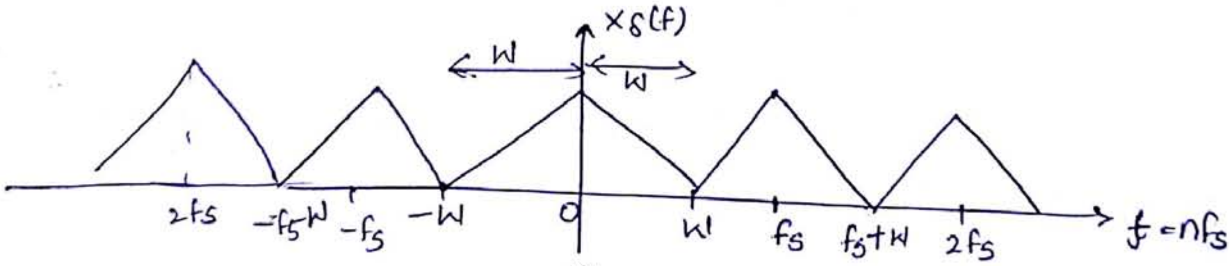
$$= + \cdots f_s X(f + 2f_s) + f_s X(f + f_s) + f_s X(f) + f_s X(f - f_s) + \cdots$$

∴ (i) The RHS of the above eqn is placed at i.e $X(f)$ is placed at $\pm f_s, \pm 2f_s, \pm 3f_s \cdots$

(ii) $X(f)$ is periodic in f_s.

(iii) If $f_s = 2W$, then spectrums $X(f)$ just touch each other





step 3: Relation between $X(f)$ and $X_\delta(f)$.

Let us assume that $f_s = 2W$ as per above diagram.

$$X_\delta(f) = f_s \cdot X(f) \qquad -W \le f \le W \;\&\; f_s = 2W$$

$$X(f) = \frac{1}{f_s} \cdot X_\delta(f) \longrightarrow ①$$

step 4: Relation between $x(t)$ and $x(nT_s)$

F.T $\qquad X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$

$$X(f) = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi f n}$$ → frequency of DT.

If we replace $X(f)$ by $X_\delta(f)$ then 'f' becomes frequency of CT signal

$$X_\delta(f) = \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-2\pi f/f_s \cdot n}$$

→ 'f' is frequency of CT signal & f/f_s = frequency of DT signal

Since $x(n) = x(nT_s)$

$$X_\delta(f) = \sum_{n=-\infty}^{\infty} x(nT_s)\, e^{-j2\pi f n T_s} \quad \rightarrow (2)$$

putting above eq (2) in eq (1)

$$X(f) = \frac{1}{f_s} \cdot \sum_{n=-\infty}^{\infty} x(nT_s)\, e^{-j2\pi f n T_s}$$

Inverse F.T of above equation gives $x(t)$

$$x(t) = IFT \left\{ \frac{1}{f_s} \sum_{n=-\infty}^{\infty} x(nT_s)\, e^{-j2\pi f n T_s} \right\}$$

Note:

(1) $x(t)$ is represented completely in terms of $x(nT_s)$

(3) Above eqn holds for $f_s = 2W$ i.e samples are taken at the rate of 2W or higher, $x(t)$ is completely represented by samples.

Part II: Reconstruction of $x(t)$ from its samples.

step 1: The IFT of $x(t) = \int_{-\infty}^{\infty} \frac{1}{f_s} \sum_{n=-\infty}^{\infty} x(nT_s)\, e^{-j2\pi f n T_s} \}\, e^{j2\pi f t}\, df$

here integration can be taken from $-W$ to $f \le W$. Since $X(f) = \frac{1}{f_s} X_\delta(f)$ for $-W \le f \le W$

$$x(t) = \int_{-W}^{W} \frac{1}{f_s} \sum_{n=-\infty}^{\infty} x(nT_s)\, e^{-j2\pi f n T_s} \cdot e^{j2\pi f t}\, dt$$

Interchanging the order of summation & integration

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \cdot \frac{1}{f_s} \int_{-W}^{W} e^{j2\pi f(t - nT_s)}\, df$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s) \cdot \frac{1}{f_s} \left[\frac{e^{j2\pi f(t-nT_s)}}{j2\pi(t-nT_s)} \right]_{-W}^{W}$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s) \cdot \frac{1}{f_s} \left\{ \frac{e^{j2\pi W(t-nT_s)} - e^{-j2\pi W(t-nT_s)}}{j2\pi(t-nT_s)} \right\}$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s) \cdot \frac{1}{f_s} \frac{\sin 2\pi W(t-nT_s)}{\pi(t-nT_s)}$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s) \cdot \frac{\sin \pi(2Wt - 2WnT_s)}{\pi(f_s t - nT_s \cdot f_s)}$$

Here $f_s = 2W$ hence $T_s = \frac{1}{f_s} = \frac{1}{2W}$

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \cdot \frac{\sin \pi(2Wt - n)}{\pi(2Wt - n)}$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s) \operatorname{sinc}(2Wt - n) \quad \left\{ \because \frac{\sin \pi\theta}{\pi\theta} = \operatorname{sinc}\theta \right\}$$

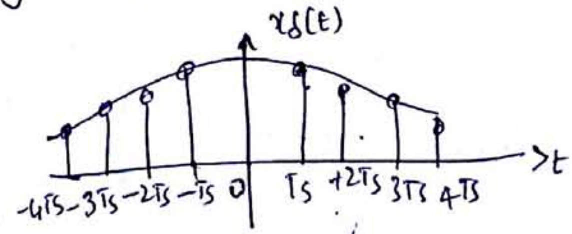
$$x(t) = \cdots + x(-2T_s)\operatorname{sinc}(2Wt+2) + x(-T_s)\operatorname{sinc}(2WT+1) + x(0)\operatorname{sinc}(2Wt) + x(T_s)\operatorname{sinc}(2Wt-1) \cdots$$

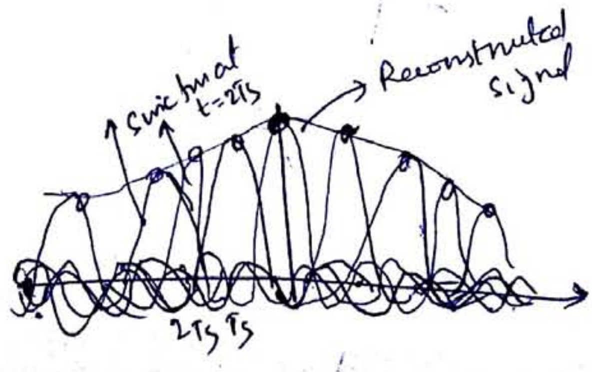
(i) The samples $x(nT_s)$ are weighted by sinc functions

(ii) The sinc function is the interpolating function.

Step 3: Reconstruction of $x(t)$ by LPF

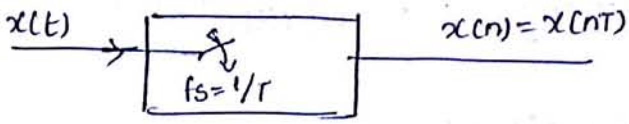
When the interpolated signal is passed through low pass filter of bandwidth $-W \le f \le W$ then the reconstructed waveform is obtained as shown in figure.



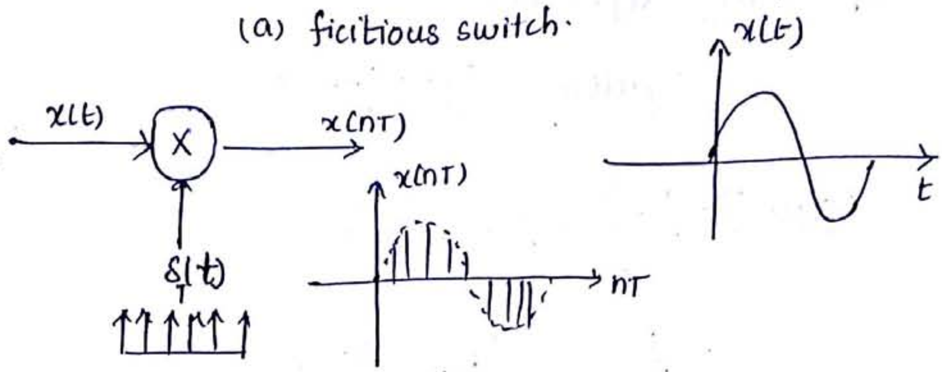


TYPES OF SAMPLING:

Impulse Sampling:



(a) ficitious switch.



→ The switch is closed for short interval of time T, during which the signal is available at the output.

→ If input is $x(t)$ & output is $x(nT), n=0,\pm1,\pm2\ldots$ i.e $x(nT)$ is called sample sequence of $x(t)$ where T is time interval b/w successive samples and sampling frequency $f_s = 1/T$ HZ.

→ $$x_s(t) = x(t)\,\delta_T(t)$$

where $\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT)$.

$$x_{(s)}(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\delta(t-nT)$$

Applying Fourier transform

$$F[x_s(t)] = F\left[\sum_{n=-\infty}^{\infty} x(nT)\,\delta(t-nT)\right]$$

$$= \sum_{n=-\infty}^{\infty} x(nT)\, F[\delta(t-nT)]$$

we have $F[\delta(t-nT)] = \int_{-\infty}^{\infty} \delta(t-nT)\, e^{-j\omega t}\,dt = e^{-j\omega nT}$

but $\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT) = \frac{1}{T}\sum_{m=-\infty}^{\infty} e^{jm\Omega_0 t}$.

$$x_s(t) = x(t)\left[\frac{1}{T}\sum_{m=-\infty}^{\infty} e^{jm\Omega_0 t}\right] = \frac{1}{T}\sum_{m=-\infty}^{\infty} x(t)e^{jm\Omega_0 t}$$

$$F[x_s(t)] = \frac{1}{T}\sum_{m=-\infty}^{\infty} F\left[x(t)e^{jm\Omega_0 t}\right]$$

If $F[x(t)] = X(j\Omega)$ then

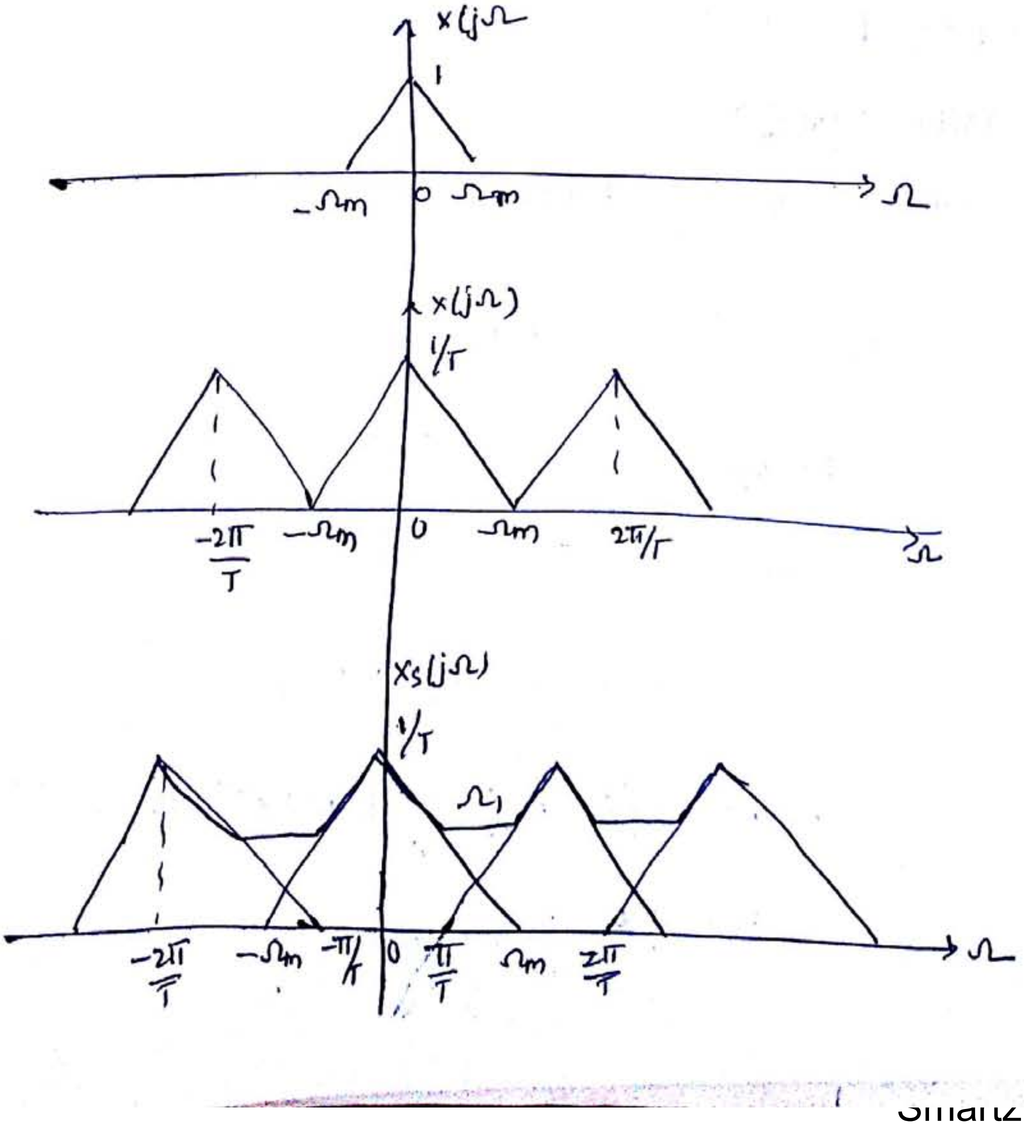
$$F\left[x(t)e^{jm\Omega_0 t}\right] = X(j(\Omega - m\Omega_0))$$

$$F[x_s(t)] = \frac{1}{T}\sum_{m=-\infty}^{\infty} X(j(\Omega - m\Omega_0))$$

$$= \frac{1}{T}\sum_{m=-\infty}^{\infty} X\left(j\left(\Omega - \frac{2\pi m}{T}\right)\right).$$

→ Now consider a signal $x(t)$ band limited to f_m. That is the highest frequency component of $x(t)$ is f_m.

$$X(j\Omega) = 0 \text{ for } |\Omega| > \Omega_m.$$



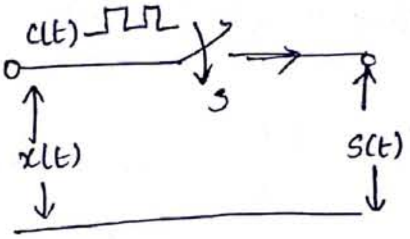
Natural Sampling: (Chopper sampling)

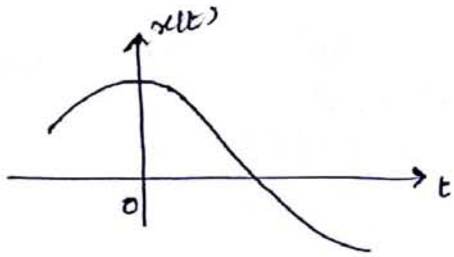
In natural sampling the pulse has a finite width, τ. It is also called as chopper sampling because the waveform of the sampled signal appears to be chopped off from the original signal waveform.

→ Let $x(t)$ be analog continuous time signal that has to be sampled at the rate of f_s Hz.

→ A sampled signal $s(t)$ is obtained by multiplication of a sampling function and signal $x(t)$.

→ Sampling function is denoted by $c(t)$ with pulses of width τ and frequency equal to f_s.



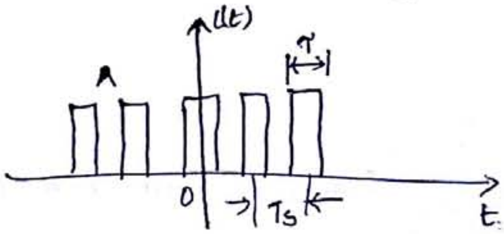


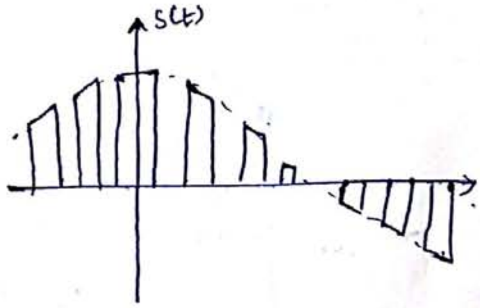
→ When $c(t)$ goes high, a switch 's' is closed.

$$s(t) = x(t) \quad \text{when } c(t) = A$$

$$s(t) = 0 \quad \text{when } c(t) = 0$$

$$s(t) = c(t) \cdot x(t) \rightarrow ①$$





Exponential Fourier series for periodic waveform is given as

$$x(t) = \sum_{n=-\infty}^{\infty} x(n) e^{jn\omega_0 t}$$

$$x(t) = \sum_{n=-\infty}^{\infty} x(n) e^{jn2\pi t/T_0} \rightarrow ②$$

For periodic pulse train of $c(t)$

$$T_0 = T_s = \frac{1}{f_s}$$

(or) $f_0 = f_s = \frac{1}{T_0} = \frac{1}{T_s}$ = frequency of $c(t)$.

Equation (2) will be [with $x(t) = c(t)$]

$$c(t) = \sum_{n=-\infty}^{\infty} c(n)\, e^{j2\pi n t \cdot f_s} \longrightarrow (3)$$

rectangular pulse train.

$$c_n = \frac{TA}{T_0} \operatorname{sinc}(f_n T') \quad \text{and } T = \text{pulse width} = \tau \longrightarrow (4)$$

$f_n = n f_s$ — Harmonic frequency

putting eq (4) in (3)

$$c(t) = \sum_{n=-\infty}^{\infty} \frac{TA}{T_s} \operatorname{sinc}(f_n \tau)\, e^{j2\pi f_s n t} \longrightarrow (5)$$

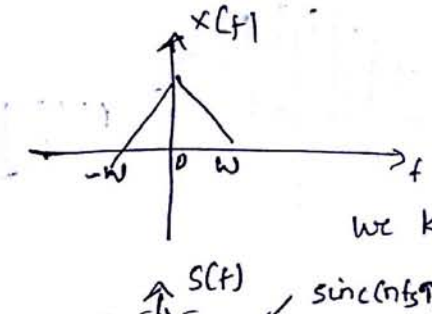
Substituting (5) in eq (1), we have

$$s(t) = \frac{\tau A}{T_s} \sum_{n=-\infty}^{\infty} \operatorname{sinc}(f_n \tau)\, e^{j2\pi f_s n t} \cdot x(t) \longrightarrow (6)$$

Represents naturally sampled signals.

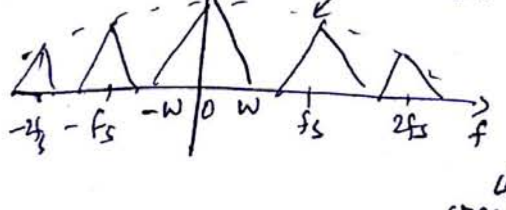
Fourier transform of $s(t)$ is obtained

$$S(f) = FT\{s(t)\}$$



$$= \frac{\tau A}{T_s} \sum_{n=-\infty}^{\infty} \operatorname{sinc}(f_n \tau)\, FT\left\{ e^{j2\pi f_s n t} \cdot x(t) \right\}$$

we know frequency shifting property



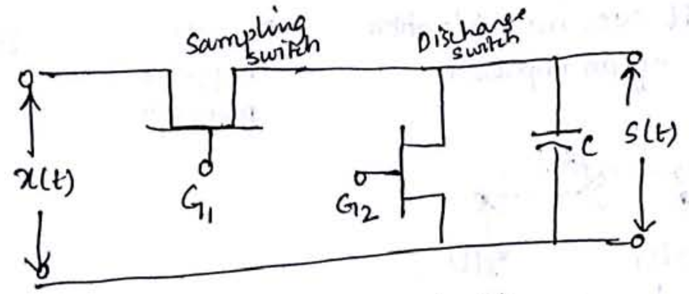
$$e^{j2\pi f_s n t} \cdot x(t) \longleftrightarrow X(f - f_s n)$$

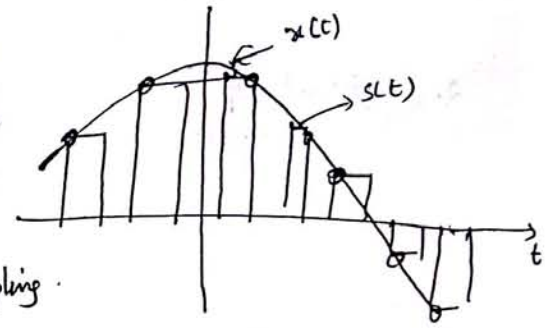
$$S(f) = \frac{\tau A}{T_s} \sum_{n=-\infty}^{\infty} \operatorname{sinc}(f_n \tau)\, X(f - f_s n).$$

spectrum of Naturally sampled sgl.

Flat top Sampling or Rectangular Pulse Sampling:

→ This is also practically possible method but easy method to get flat top samples.

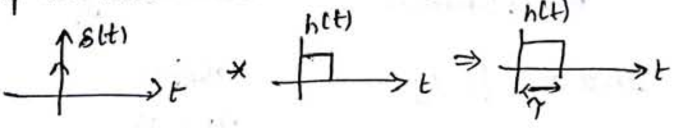
→ 



sample & hold circuit generating flat top sampling.

→ The width of the pulse in flat top sampling and natural sampling is increased to reduce the transmission bandwidth.

→ Here the top of the samples remain const which is equal to instanteaneous value of the base band.



$\therefore x(t) * \delta(t) = x(t)$

$h(t) * \delta(t) = h(t)$.

→ $$y(t) = s(t) * h(t) = h(t) * \left[x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \right] \rightarrow (1)$$

Applying F.T

$$Y(f) = H(f) \left[X(f) \cdot \frac{1}{T_s} \sum_{n=-\infty}^{\infty} \delta(f - nf_s) \right. \rightarrow (2)$$

$H(f)$ ↙ sinc fn.

→ We can say that the primary effect of flat top sampling is an attenuation of high frequency components. This effect is called aperature effect.

→ This can be compensated by an equalizing filter with transfer fn

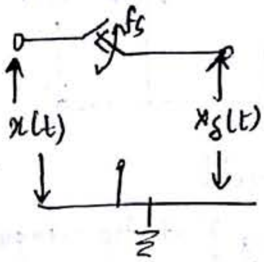
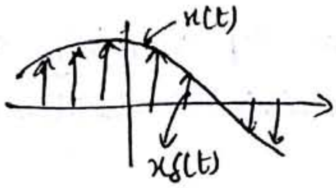
$$H_{eq}(f) = \frac{1}{|H(f)|} \rightarrow (3)$$

if $\tau \ll T_s$ then $H(f)$ will essentially be constant over the based band.

$$\therefore Y(f) = H(f) f_s \sum_{k=-\infty}^{\infty} X(f - kf_s)$$

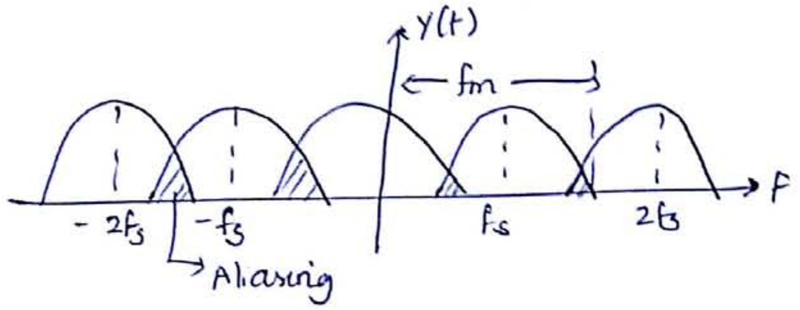
where $H(f) = \tau \operatorname{sinc}(f\tau) e^{-j2\pi f \tau} \rightarrow (5)$

Comparsion of Various Sampling Techniques

S.No	Parameter of comparsion	Ideal or instantaneous sampling	Natural sampling	Flat top sampling
1	Principle of sampling	It uses multiplication by an impulse fn	It uses chopping principle	It uses sample and hold circuit
2.	Circuit of Sampler			
3.	Waveforms			
4.	Realizability	This is not pratically possible method	Used Practically	Used practically
5.	Sampling rate	tends to infinity	satisfies nyquist rate	satisfies nyquist criteria
6.	Noise interference	Noise interference is maximum	is minimum	is maximum
7.	Time domain representation	$x_\delta(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t-nT_s)$	$s(t) = \frac{\tau A}{T_s}\sum_{n=-\infty}^{\infty} x(t)\sin(nf_s\tau)\, e^{j2\pi n f_s t}$	$s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\, h(t-nT_s)$
8.	Frequency domain representation	$X_\delta(t) = f_s\sum_{n=-\infty}^{\infty} X(f-nf_s)$	$S(f) = \frac{\tau A}{T_s}\sum_{n=-\infty}^{\infty} \text{sinc}(nf_s\tau)\, X(f-nf_s)$	$S(f) = f_s\sum_{n=-\infty}^{\infty} X(f-nf_s)H(f)$

EFFECTS OF UNDERSAMPLING - ALIASING :

→ When a continuous time band-limited signal is sampled at a rate lower than nyquist rate, $f_s < 2f_m$ then successive cycles of the spectrum $Y(f)$ of the sampled signal $y(t)$ overlap with each other as shown in below



→ Hence the signal is undersampled when $f_s < 2f_m$ and some amount of aliasing is produced.

→ Aliasing is the phenomenon in which a high freq component in the frequency spectrum of signal takes identity of a LF component in the spectrum of sampled signal.

→ Due to aliasing, it is not possible to recover the original signal $x(t)$ from sampled signal $y(t)$ by LPF.

→ The signal is distorted and the information signal contains a large no. of frequencies deciding the sampling frequency is always a problem.

→ The signal is 1st passed through an LPF which blocks all the freq/s above f_m Hz → band limiting of original signal.

→ This LPF is called pre-alias filter, it is used to prevent aliasing effect.

To avoid aliasing

(1) pre-alias filter must be used to limit the band of frequencies of the signal to f_m Hz

(2) Sampling frequency f_s must be selected such that $f_s \geq 2f_m$.