

UNIT-5LAPLACE TRANSFORMS AND Z-TRANSFORMSLAPLACE TRANSFORMS:28/101, 2, 6, 10, 11, 12, 14, 20, 21, 23, 25, 26, 28, 33, 35,  
38, 41, 42, 48, 50, 55,

→ Laplace transform represents continuous time signals in terms of complex exponentials i.e.  $e^{-st}$ . It is used to analyze the signals or functions which are not absolutely integrable.

→ More effectively continuous time signals can be analyzed using Laplace transform.

→ Laplace transform provides broader characterization compared to Fourier Transform.

DEFINITION:

→ To transform a time domain signal  $x(t)$  to S-domain, multiply the signal by  $e^{-st}$  and then integrate from  $-\infty$  to  $\infty$ .

→ The transformed signal is represented as  $x(s)$  and transformation is denoted by letter  $\mathcal{L}$ .

Laplace transform is given as for continuous time signal  $x(t)$

i.e

$$\boxed{x(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt} \rightarrow ①$$

where 's' is complex in nature and given as  $s = \sigma + j\omega$ .

$\sigma \nearrow$  real part / attenuation constant  
 $\omega \searrow$  imaginary / complex frequency

→ If  $x(t)$  is defined for  $t \geq 0$  (i.e  $x(t)$  is causal) then

$$\boxed{\mathcal{L}\{x(t)\} = x(s) = \int_0^{\infty} x(t) e^{-st} dt} \rightarrow ②$$

TYPES OF LAPLACE TRANSFORM:

(i) Bilateral or two sided Laplace transform : If the integration is taken from  $-\infty$  to  $\infty$  as shown in eq(1) then it is called Bilateral LT

(ii) Unilateral or one sided Laplace transform : If the integration is taken from 0 to  $\infty$  as shown in eq(2) then it is called Unilateral LT

→ Useful in analysis of networks and solving differential equations.

## INVERSE LAPLACE TRANSFORM:

→ The s-domain signal  $x(s)$  can be transformed to time domain signal  $x(t)$  by using inverse laplace transform and is defined as

$$\boxed{L^{-1}[x(s)] = x(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} x(s) e^{st} ds}$$

→ The signal  $x(t)$  and  $x(s)$  are called Laplace transform pair.

$$x(t) \xrightleftharpoons[L^{-1}]{L} x(s)$$

## RELATION BETWEEN FOURIER TRANSFORM AND LAPLACE TRANSFORM:

Fourier transform is given as

$$x(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \rightarrow ①$$

→ FT can be calculated only if  $x(t)$  is absolutely integrable i.e

$$\int_{-\infty}^{\infty} |x(t)| dt < \infty \rightarrow ②$$

→ Laplace transform is written as

$$x(s) = \int_{-\infty}^{\infty} x(t) e^{-(\sigma+j\omega)t} dt \text{ putting } s = \sigma + j\omega$$

$$= \int_{-\infty}^{\infty} x(t) e^{-\sigma t} \cdot e^{-j\omega t} dt$$

$$= \left\{ \int_{-\infty}^{\infty} x(t) e^{-\sigma t} dt \right\} e^{-j\omega t} dt \rightarrow ③$$

Comparing eq(3) with eq(1), Laplace transform of  $x(t)$  is basically Fourier transform of  $x(t) e^{-\sigma t}$ .

→ If  $\sigma=0$ , then above equation i.e  $s=j\omega$  the above eqn

$$x(s) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = x(j\omega) \text{ when } s=j\omega$$

→ It is basically Fourier transform on imaginary ( $j\omega$ ) axis in s-plane.

## CONVERGENCE / REGION OF CONVERGENCE (ROC):

- From eqn  $\left\{ \int_{-\infty}^{\infty} (x(t)e^{-at}) e^{-j\omega t} dt \right\}$  we know that Laplace transform is basically the Fourier transform of  $x(t)e^{-at}$ . If Fourier transform of  $x(t)e^{-at}$  exists then Laplace transform of  $x(t)$  exists.
- $\int_{-\infty}^{\infty} |x(t)e^{-at}| dt < \infty$  must be absolutely integrable for Fourier transform to exist.
- Laplace transform of  $x(t)$  will exist, if above condition is satisfied.
- The range of values of ' $a$ ' for which Laplace transform converges is called ROC or region of convergence.

(Q4)

- The Laplace transform of a signal given by  $\int_{-\infty}^{\infty} x(t)e^{-st} dt$ . The values of ' $s$ ' for which the integral  $\int_{-\infty}^{\infty} x(t)e^{-st} dt$  converges is called ROC.

## PROBLEMS:

- (1) Calculate the Laplace transform and ROC for  $x(t) = e^{-at} u(t)$  (Right sided causal sig)

Sol  $x(t) = e^{-at} u(t)$  where  $a > 0$   
 $= e^{-at}$  for  $t \geq 0$

$$\begin{aligned} L[x(t)] = X(s) &= \int_{-\infty}^{\infty} x(t) e^{-st} dt = \int_{-\infty}^{\infty} e^{-at} \cdot u(t) e^{-st} dt \\ &= \int_0^{\infty} e^{-at} e^{-st} dt = \int_0^{\infty} e^{-(s+a)t} dt \\ &= \frac{1}{-(s+a)} \left[ e^{-(s+a)t} \right]_0^{\infty} = \frac{-1}{s+a} [0 - 1] \\ &= \frac{1}{s+a} \end{aligned}$$

ROC:  $s > -a$

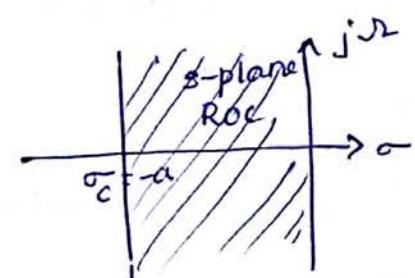
$$= - \left[ \frac{e^{-(s+a)t}}{s+a} \right]_0^\infty$$

$$= \left\{ \lim_{t \rightarrow \infty} \left[ \frac{e^{-(s+a)t}}{s+a} \right] - \lim_{t \rightarrow 0} \left[ \frac{e^{-(s+a)t}}{s+a} \right] \right\}$$

$$X(s) = + \left[ \frac{e^{-(s+a)\infty}}{-s-a} - \frac{e^{-(s+a)0}}{-s-a} \right]$$

$$s = j\omega + \sigma$$

$$\mathcal{L}\{x(t)\} = \frac{e^{-kx\infty} e^{-j\omega x\infty}}{s+a} + \frac{1}{s+a}$$



$$\text{where } k = \sigma + a = \sigma - (-a).$$

when  $\sigma > a$ ,  $k = \sigma - (-a)$  = positive  $\therefore e^{-k\infty} = e^{-\infty} = 0$

when  $\sigma < -a$ ,  $k = \sigma - (-a)$  = Negative  $\therefore e^{-k\infty} = e^{+\infty} = \infty$

$\therefore X(s)$  converges when  $\sigma > -a$  and does not converge for  $\sigma < -a$ .

$\therefore$  When  $\sigma > -a$ , the  $X(s)$  is given by

$$\mathcal{L}\{x(t)\} = X(s) = -\frac{e^{-kx\infty} e^{-j\omega x\infty}}{s+a} + \frac{1}{s+a} = \frac{-0 \times e^{-j\omega x\infty}}{s+a} + \frac{1}{s+a} = \frac{1}{s+a}$$

## Properties of L.T:

### (1) Amplitude scaling

If  $L[x(t)] = X(s)$  then  $L[Ax(t)] = Ax(s)$

Proof:

$$X(s) = L\{x(t)\} = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$\begin{aligned} L\{Ax(t)\} &= \int_{-\infty}^{\infty} A \cdot x(t) e^{-st} dt = A \int_{-\infty}^{\infty} x(t) e^{-st} dt \\ &= Ax(s). \end{aligned}$$

### (2) Linearity:

If  $L\{x_1(t)\} = X_1(s)$  and  $L\{x_2(t)\} = X_2(s)$  then  $L\{a_1x_1(t) + a_2x_2(t)\} = a_1X_1(s) + a_2X_2(s)$

Proof:

$$X_1(s) = L\{x_1(t)\} = \int_{-\infty}^{\infty} x_1(t) e^{-st} dt$$

$$X_2(s) = L\{x_2(t)\} = \int_{-\infty}^{\infty} x_2(t) e^{-st} dt$$

$$\begin{aligned} L\{a_1x_1(t) + a_2x_2(t)\} &= \int_{-\infty}^{\infty} [a_1x_1(t) + a_2x_2(t)] e^{-st} dt \\ &= a_1 \int_{-\infty}^{\infty} x_1(t) e^{-st} dt + a_2 \int_{-\infty}^{\infty} x_2(t) e^{-st} dt \\ &= a_1X_1(s) + a_2X_2(s) \end{aligned}$$

### 3) Time differentiation:

If  $L\{x(t)\} = X(s)$  then  $L\left\{\frac{dx(t)}{dt}\right\} = sX(s) - x(0)$ ; where  $x(0)$  is value of  $x(t)$  at  $t=0$

Proof:

$$X(s) = L\{x(t)\} = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$\therefore L\left\{\frac{dx(t)}{dt}\right\} = \int_0^{\infty} \frac{dx(t)}{dt} e^{-st} dt = \int_0^{\infty} e^{-st} \frac{dx(t)}{dt} dt$$

$$\left\{ \because \int uv = v \int u ds - \int \left( \frac{du}{ds} v \right) ds \right\}$$

$$= \left[ e^{-st} \cdot x(t) \right]_0^{\infty} - \int_0^{\infty} -se^{-st} x(t) dt$$

$$= e^{-\infty} x(\infty) - e^0 x(0) + s \int_0^{\infty} x(t) e^{-st} dt$$

$$= s \int_0^{\infty} x(t) e^{-st} dt - x(0) = sX(s) - x(0)$$

(4) Time Integration:

$$\text{If } L[x(t)] = X(s) \text{ then } L\left\{\int x(t) dt\right\} = \frac{x(s)}{s} + \left[\int x(t) dt\right]_{t=0}$$

Proof

$$X(s) = L\{x(t)\} = \int_0^\infty x(t) e^{-st} dt$$

$$L\left\{\int x(t) dt\right\} = \int_0^\infty \left[ \int x(t) dt \right] e^{-st} dt$$

$$= \left[ \left[ \int x(t) dt \right] \frac{e^{-st}}{-s} \right]_0^\infty - \int_0^\infty x(t) \cdot \frac{e^{-st}}{-s} dt$$

$$= \left[ \int x(t) dt \right] \Big|_{t=\infty}^0 - \left[ \int x(t) dt \right] \Big|_{t=0}^0 + \frac{1}{s} \int_0^\infty x(t) e^{-st} dt$$

$$= \frac{1}{s} \left[ \int x(t) dt \right] \Big|_{t=0}^0 + \frac{1}{s} \int_0^\infty x(t) e^{-st} dt$$

$$= \frac{x(s)}{s} + \left[ \frac{\int x(t) dt}{s} \right] \Big|_{t=0}$$

(5) Frequency shifting:

$$\text{If } L[x(t)] = X(s) \text{ then } L\{e^{\pm at} x(t)\} = X(s \mp a)$$

Proof

$$X(s) = L\{x(t)\} = \int_{-\infty}^\infty x(t) e^{-st} dt$$

$$\therefore L\{e^{\pm at} x(t)\} = \int_{-\infty}^\infty e^{\pm at} x(t) \cdot e^{-st} dt$$

$$= \int_{-\infty}^\infty x(t) \cdot e^{-(s \mp a)t} dt$$

$$= X(s \mp a).$$

(6) Time shifting

$$\text{If } L\{x(t)\} = X(s) \text{ then } L\{x(t \pm a)\} = e^{\pm as} X(s)$$

Proof:

$$x(s) = L\{x(t)\} = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$L\{x(t+a)\} = \int_{-\infty}^{\infty} x(t+a) e^{-st} dt = \int_{-\infty}^{\infty} x(\tau) e^{-s(\tau+a)} d\tau$$

put  $t+\alpha = \tau$   
 $t = \tau - a$   
 $dt = d\tau$

$$= \int_{-\infty}^{\infty} x(\tau) e^{-s\tau} \times e^{\pm as} d\tau = e^{\pm as} \int_{-\infty}^{\infty} x(\tau) e^{-s\tau} d\tau$$

$$= e^{\pm as} L\{x(t)\} = e^{\pm as} (x(s))$$

(7) Frequency differentiation:

If  $L\{x(t)\} = x(s)$  then  $L\{t x(t)\} = -\frac{d}{ds} x(s)$

Proof

$$x(s) = L\{x(t)\} = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$\frac{d}{ds} x(s) = \frac{d}{ds} \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$= \int_{-\infty}^{\infty} x(t) \left( \frac{d}{ds} e^{-st} \right) dt = \int_{-\infty}^{\infty} x(t) (-te^{-st}) dt$$

$$= \int_{-\infty}^{\infty} (-tx(t)) e^{-st} dt = L\{-tx(t)\} = -L\{tx(t)\}$$

$$\therefore L\{tx(t)\} = -\frac{d}{ds} x(s).$$

(8) Frequency Integration:

If  $L\{x(t)\} = x(s)$  then  $L\left\{\frac{1}{s} x(t)\right\} = \int_s^{\infty} x(s) ds.$

Proof

$$x(s) = L\{x(t)\} = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

On integrating above eqn w.r.t s b/w limits 0 to  $\infty$

$$\int_s^{\infty} x(s) ds = \int_s^{\infty} \left[ \int_{-\infty}^0 x(t) e^{-st} dt \right] ds = \int_{-\infty}^{\infty} x(t) \left[ \int_s^{\infty} e^{-st} ds \right] dt$$

$$= \int_{-\infty}^{\infty} x(t) \left[ \frac{e^{-st}}{-t} \right] ds = \int_{-\infty}^{\infty} x(t) \left[ \frac{e^0}{-t} - \frac{e^{-st}}{-t} \right] ds$$

$$= \int_{-\infty}^{\infty} x(t) \left[ 0 + \frac{e^{-st}}{t} \right] ds = \int_{-\infty}^{\infty} \left[ \frac{1}{t} x(t) \right] e^{-st} dt = L\left\{\frac{1}{t} x(t)\right\}$$

## 9) Time Scaling:

If  $L\{x(t)\} = X(s)$  then  $L\{x(at)\} = \frac{1}{|a|} \times \left(\frac{s}{a}\right)$

Proof

$$X(s) = L\{x(t)\} = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$\therefore L\{x(at)\} = \int_{-\infty}^{\infty} x(at) e^{-st} dt = \int_{-\infty}^{\infty} x(\tau) e^{-s\left(\frac{\tau}{a}\right)} \frac{d\tau}{a} \quad \left. \begin{array}{l} \text{put } at=\tau \\ \therefore d\tau = \frac{d\tau}{a} \end{array} \right\}$$

$$= \frac{1}{a} \int_{-\infty}^{\infty} x(\tau) e^{-\left(\frac{s}{a}\right)\tau} d\tau$$

$$= \frac{1}{a} \times \left(\frac{s}{a}\right)$$

The above transform is applicable for positive values of 'a'.

If 'a' happens to be negative

$$L\{x(at)\} = -\frac{1}{a} \times \left(\frac{s}{a}\right)$$

$$\text{In general, } L\{x(at)\} = \frac{1}{|a|} \times \left(\frac{s}{a}\right).$$

## (10) Periodicity:

If  $x(t) = x(t+nT)$  and  $x_1(t)$  be one period of  $x(t)$  and  $L\{x_1(t)\} = \int_0^T x_1(t) e^{-st} dt$

$$\text{then } L\{x(t+nT)\} = \frac{1}{1-e^{-sT}} \int_0^T x_1(t) e^{-st} dt$$

Proof

$$L\{x(t+nT)\} = \int_0^{\infty} x(t+nT) e^{-st} dt$$

$$= \int_0^T x_1(t) e^{-st} dt + \int_T^{2T} x_1(t-T) e^{-s(t-T)} dt + \int_{2T}^{3T} x_1(t-2T) e^{-s(t-2T)} dt + \dots + \int_{(P+1)T}^{PT} x_1(t-PT) e^{-s(t-PT)} dt + \dots$$

$$= \sum_{P=0}^{\infty} \int_{PT}^{(P+1)T} x_1(t-PT) e^{-s(t-PT)} dt$$

$$= \sum_{P=0}^{\infty} \int_0^T x_1(t) e^{-st} \cdot e^{-Pst} dt = \int_0^T x_1(t) e^{-st} \left( \sum_{P=0}^{\infty} e^{-Pst} \right) dt$$

$$= \int_0^T x_1(t) e^{-st} \left( \sum_{P=0}^{\infty} e^{-Pst} \right)^P dt$$

$$= \int_0^T x_1(t) e^{-st} \left( \frac{1}{1-e^{-st}} \right) dt = \frac{1}{1-e^{-st}} \int_0^T x_1(t) e^{-st} dt$$

## INVERSE LAPLACE TRANSFORM: (PARTIAL FRACTION EXPANSION)

The inverse laplace transform by partial fraction method of all three cases.

Case i) when s-domain signal  $x(s)$  has distinct poles

$$\text{Let } x(s) = \frac{k}{s(s+p_1)(s+p_2)}$$

By partial fraction

$$x(s) = \frac{k_1}{s} + \frac{k_2}{s+p_1} + \frac{k_3}{s+p_2}$$

The residues  $k_1, k_2, k_3$  are given by

$$k_1 = x(s) \times s \Big|_{s=0}, \quad k_2 = x(s) \times (s+p_1) \Big|_{s=-p_1}$$

$$k_3 = x(s) \times (s+p_2) \Big|_{s=-p_2}$$

$$L^{-1}\{x(s)\} = L^{-1}\left\{\frac{k_1}{s} + \frac{k_2}{s+p_1} + \frac{k_3}{s+p_2}\right\}$$

$$\begin{aligned} x(t) &= k_1 L^{-1}\left\{\frac{1}{s}\right\} + k_2 L^{-1}\left\{\frac{1}{s+p_1}\right\} + k_3 L^{-1}\left\{\frac{1}{s+p_2}\right\} \\ &= k_1 u(t) + k_2 e^{-p_1 t} u(t) + k_3 e^{-p_2 t} u(t). \end{aligned}$$

Case ii) when s-domain signal  $x(s)$  has multiple poles

$$x(s) = \frac{k}{s(s+p_1)^r(s+p_2)^s}$$

$$x(s) = \frac{k_1}{s} + \frac{k_2}{s+p_1} + \frac{k_3}{(s+p_1)^2} + \frac{k_4}{s+p_2}$$

The residues  $k_1, k_2, k_3, k_4$  are given by

$$k_1 = x(s) \times s \Big|_{s=0} \quad k_2 = x(s) \times (s+p_1) \Big|_{s=-p_1}$$

$$k_3 = x(s) \times (s+p_1)^2 \Big|_{s=-p_1} \quad k_4 = \frac{d}{ds} \left[ x(s) \times (s+p_2) \right]_{s=-p_2}$$

$$L^{-1}\{x(s)\} = L^{-1}\left\{\frac{k_1}{s} + \frac{k_2}{s+p_1} + \frac{k_3}{(s+p_1)^2} + \frac{k_4}{s+p_2}\right\}$$

$$= k_1 u(t) + k_2 e^{-p_1 t} u(t) + k_3 t e^{-p_2 t} u(t) + k_4 e^{-p_2 t} u(t)$$

In general

$$x(s) = \frac{k}{s(s+p_1)(s+p_2)} v \text{ then}$$

$$x(s) = \frac{k_1}{s} + \frac{k_2}{s+p_1} + \frac{k_3}{(s+p_2)^v} + \frac{k_4}{(s+p_2)^{v-1}} + \dots + \frac{k_{(v-1)2}}{s+p_2}$$

$$k_{ry} = \frac{1}{r!} \frac{d^r}{dq^r} \left[ x(s) \times (s+2)^q \right] ; r=1, 2, \dots, v-1.$$

Case iii) When s-domain signal  $x(s)$  has complex conjugate poles

$$\text{Let } x(s) = \frac{k}{(s+p_1)(s^v+bs+c)}$$

$$x(s) = \frac{k_1}{s+p_1} + \frac{k_2 s + k_3}{s^v + bs + c}$$

$$k_1 = x(s) \times (s+p_1) \Big|_{s=-p_1}$$

$$x(s) = \frac{k_1}{s+p_1} + \frac{k_2 s + k_3}{s^v + 2 \times \frac{b}{2} \cdot s + \left(\frac{b}{2}\right)^v + c - \left(\frac{b}{2}\right)^v}$$

Arranging  $s^v + bs$   
in  $(s+a)^v$ .

$$= \frac{k_1}{s+p_1} + \frac{k_2 s + k_3}{\left(s+\frac{b}{2}\right)^v + \left(c-\frac{b^v}{4}\right)^v} \quad \text{put } \frac{b}{2} = a, c - \frac{b^v}{4} = n_0^v$$

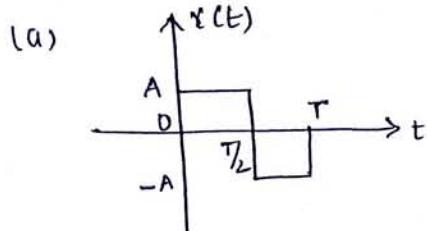
$$x(s) = \frac{k_1}{s+p_1} + k_2 \cdot \underbrace{\frac{s+a + \frac{k_3}{k_2} - a}{(s+a)^v + n_0^v}}$$

$$= \frac{k_1}{s+p_1} + k_2 \cdot \frac{s+a + k_4}{(s+a)^v + n_0^v} \quad \left( \because \text{put } \frac{k_3}{k_2} - a = k_4 \right)$$

$$\therefore x(s) = \frac{k_1}{s+p_1} + k_2 \cdot \frac{s+a}{(s+a)^v + n_0^v} + k_5 \frac{n_0^v}{(s+a)^v + n_0^v} \quad \left( \frac{k_2 k_4}{n_0^v} = k_5 \right).$$

$$x(t) = k_1 e^{-p_1 t} u(t) + k_2 e^{-at} \cos n_0 t u(t) + k_5 e^{-at} \sin n_0 t u(t),$$

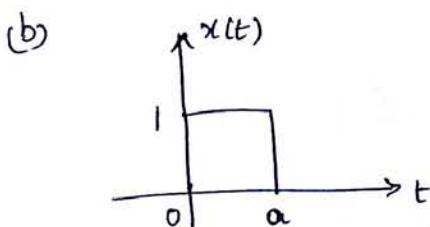
LAPLACE TRANSFORM OF CERTAIN SIGNALS USING WAVEFORM SYNTHESIS:



Proof  $x(t) = A \text{ for } 0 < t < T/2$

$$= -A \text{ for } T/2 < t < T$$

$$\begin{aligned} L\{x(t)\} &= X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt = \int_0^T x(t) e^{-st} dt \\ &= \int_0^{T/2} A e^{-st} dt + \int_{T/2}^T (-A) e^{-st} dt = \left[ \frac{A e^{-st}}{-s} \right]_0^{T/2} + \left[ \frac{-A e^{-st}}{-s} \right]_{T/2}^T \\ &= \left[ \frac{A e^{-sT/2}}{-s} - \frac{A e^0}{-s} \right] + \left[ \frac{A e^{-st}}{s} - \frac{A e^{-sT/2}}{s} \right] \\ &= -\frac{A e^{-sT/2}}{s} + \frac{A}{s} + \frac{A e^{-st}}{s} - \frac{A e^{-sT/2}}{s} \\ &= \frac{A}{s} \left[ 1 - e^{-sT/2} \right] \end{aligned}$$

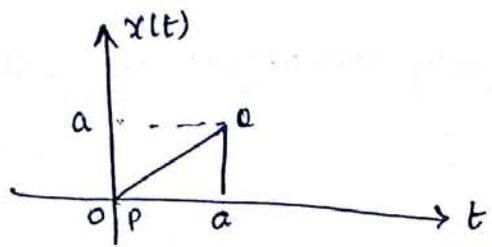


Proof  $x(t) = 1 \text{ for } 0 < t < a$

$$= 0 \text{ for } t > a$$

$$\begin{aligned} L\{x(t)\} &= X(s) = \int_0^a x(t) e^{-st} dt = \int_0^a 1 \times e^{-st} dt = \int_0^a e^{-st} dt \\ &= \left[ \frac{e^{-st}}{-s} \right]_0^a = \frac{e^{-as}}{-s} - \frac{e^0}{-s} = \frac{-e^{as}}{s} + \frac{1}{s} = \frac{1}{s} (1 - e^{-as}) \end{aligned}$$

(C)



Consider the eqn of straight line  $\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$ .

Here  $y = x(t)$ ,  $x = t$

Consider points P and Q as shown in figure

$$P = [0, 0], Q = [a, a]$$

$$t_1, x_1(t) \quad t_2, x_2(t)$$

$$\therefore \frac{x(t)-0}{0-a} = \frac{t-0}{0-a} \Rightarrow x(t) = t$$

$\therefore x(t) = t$  for  $t = 0$  to  $a$

$= 0$  for  $t > a$

$$\begin{aligned} L\{x(t)\} &= X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt = \int_0^a e^{-st} \cdot t dt \\ &= \left[ t \times \frac{e^{-st}}{-s} - \int 1 \cdot \frac{e^{-st}}{-s} dt \right]_0^a \\ &= \left[ -\frac{te^{-st}}{s} - \frac{e^{-st}}{s^2} \right]_0^a \\ &= \left[ -\frac{ae^{-sa}}{s} - \frac{e^{-sa}}{s^2} + 0 + \frac{e^0}{s^2} \right] \\ &\approx \frac{1}{s^2} - \frac{e^{-as}}{s^2} - \frac{ae^{-as}}{s} \\ &= \frac{1}{s^2} \left[ 1 - e^{-as}(1+as) \right] \end{aligned}$$

## Problems on Laplace transforms:

(i) Determine the Laplace transform of continuous time signals and their ROC

$$(a) x(t) = A u(t)$$

$$x(t) = A \text{ for } t \geq 0 \text{ bcoz } u(t) = 1 \text{ for } t \geq 0 \\ = 0 \text{ for } t < 0$$

Laplace transform

$$\begin{aligned} L\{x(t)\} = X(s) &= \int_{-\infty}^{\infty} x(t) e^{-st} dt = \int_0^{\infty} A \cdot e^{-st} dt = A \left[ \frac{e^{-st}}{-s} \right]_0^{\infty} \\ &= A \left[ \frac{e^{-(\sigma+j\omega)t}}{-s} \right]_0^{\infty} \\ &= A \left[ \frac{e^{-(\sigma+j\omega)\times\infty}}{-s} + \frac{e^0}{s} \right] = A \left[ \frac{e^{-\sigma \times \infty} \cdot e^{-j\omega \times \infty}}{-s} + \frac{e^0}{s} \right] \end{aligned}$$

when  $\sigma > 0$ ;  $e^{-\sigma \times \infty} = e^{-\infty} = 0$

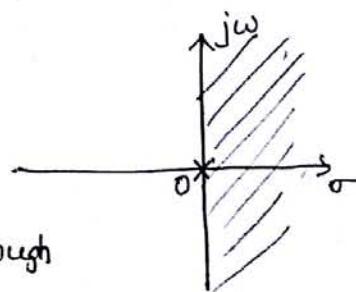
when  $\sigma < 0$ ;  $e^{-\sigma \times \infty} = e^{\infty} = \infty$

∴ we can say that  $X(s)$  converges when  $\sigma > 0$

when  $\sigma > 0$ , the  $X(s)$  is given by

$$X(s) = A \left[ \frac{x e^{-j\omega \times \infty}}{-s} + \frac{1}{s} \right] = \frac{A}{s}$$

∴  $L\{A u(t)\} = \frac{A}{s}$ ; with ROC as all point is s-plane to the right of line passing through  $\sigma = 0$



(or ROC is right half of s-plane).

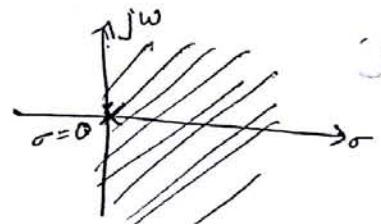
$$(2) \quad x(t) = t u(t)$$

$$\text{Sol} \quad x(t) = t \begin{cases} u(t) = 1 \text{ for } t > 0 \\ = 0 \text{ otherwise} \end{cases}$$

$$\begin{aligned} L\{x(t)\} &= X(s) = \int_0^{\infty} x(t) e^{-st} dt \\ &= \int_0^{\infty} t \cdot e^{-st} dt = t \left[ \frac{e^{-st}}{-s} \right]_0^{\infty} - \int \left[ \frac{e^{-st}}{-s} \right]_0^{\infty} \\ &= t \cdot \left[ \frac{e^{-st}}{-s} \right]_0^{\infty} - \left[ \frac{e^{-st}}{s^2} \right]_0^{\infty} \\ &= t \left[ \frac{e^{-(\sigma+j\omega)t}}{-s} \right]_0^{\infty} - \left[ \frac{e^{-(\sigma+j\omega)t}}{s^2} \right]_0^{\infty} \\ &= \left[ \infty \times \frac{e^{-(\sigma+j\omega)\infty}}{-s} - 0 - \frac{e^{-(\sigma+j\omega)\infty}}{s^2} + \frac{e^0}{s^2} \right] \\ &= \left[ \infty \times \frac{e^{-\sigma \times \infty} \times e^{-j\omega \times \infty}}{-s} - \frac{e^{-\sigma \times \infty} \cdot e^{-j\omega \times \infty}}{s^2} + \frac{1}{s^2} \right] \end{aligned}$$

when  $\sigma > 0$ , positive i.e  $e^{-\sigma \times \infty} = e^{-\infty} = 0$   
 when  $\sigma < 0$ , negative i.e  $e^{-\sigma \times \infty} = e^{\sigma \infty} = \infty$

It converges when  $\sigma > 0$



$$= \left[ \infty \times \frac{0 \times e^{-j\omega \times \infty}}{-s} - \frac{0 \times e^{-j\omega \times \infty}}{s^2} + \frac{1}{s^2} \right] = \frac{1}{s^2}$$

∴ ROC is right half of s-plane.

$$3) \quad \text{Given that } x(t) = e^{-3t} u(t) :$$

$$\text{Sol} \quad x(t) = e^{-3t} \text{ for } t \geq 0$$

$$L\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t) \cdot e^{-st} dt$$

$$= \int_0^\infty x(t) \cdot e^{-st} dt = \int_0^\infty e^{-3t} \cdot e^{-st} dt = \int_0^\infty e^{-(s+3)t} dt.$$

$$= \left[ \frac{e^{-(s+3)t}}{-(s+3)} \right]_0^\infty$$

$$= \left[ \frac{e^{-(s+3)\infty}}{-(s+3)} + \frac{e^{-(s+3) \cdot 0}}{s+3} \right] \Rightarrow \frac{e^{-(\sigma+j\omega+3)\infty}}{-(s+3)} + \frac{1}{s+3}$$

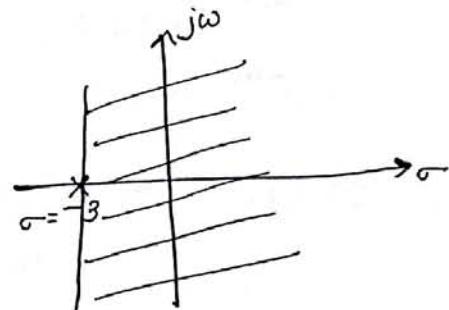
$$= \left[ \frac{e^{-(\sigma+j\omega)\infty} \cdot e^{-j\omega\infty}}{-(s+3)} + \frac{1}{s+3} \right]$$

$$\sigma+3 = \sigma - (-3)$$

if  $\sigma > -3$  = positive  $\therefore e^{-k\infty} = e^{\infty} = 0$

if  $\sigma < -3$  = negative  $\therefore e^{-k\infty} = e^{\infty} = \infty$

$\therefore$  It converges at  $\sigma > -3$ .



$$x(s) = \left[ \frac{0 \cdot e^{-j\omega\infty}}{-(s+3)} + \frac{1}{s+3} \right] = \frac{1}{s+3}$$

$$\mathcal{L}\{e^{-3t} u(t)\} = \frac{1}{s+3}; \text{ with ROC as all pts in s-plane to right of line passing through } \sigma = -3$$

$$(4) \quad x(t) = e^{-3t} u(-t)$$

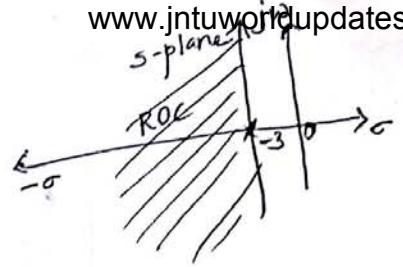
$$x(t) = e^{-3t} \text{ for } t \leq 0$$

$$\mathcal{L}\{x(t)\} = x(s) = \int_{-\infty}^0 e^{st} dt = \int_{-\infty}^0 e^{-3t} e^{st} dt$$

$$= \int_{-\infty}^0 e^{-(s+3)t} dt$$

$$= \left[ \frac{e^{-(s+3)t}}{-(s+3)} \right]_{-\infty}^0 = \frac{e^0}{-(s+3)} + \frac{e^{+(s+3)\infty}}{-(s+3)}$$

$$= \frac{-1}{s+3} + \frac{e^{+(\sigma+j\omega+3)\infty}}{s+3}$$



$$= \frac{-1}{s+3} + e^{+(\sigma+3)\times\infty} \cdot e^{-j\omega\times\infty}$$

$$k = \sigma + 3 = \sigma - (-3)$$

when  $\sigma > -3$ ; ~~positive~~ positive  $e^{(\sigma+3)\times\infty} = \infty$

when  $\sigma < -3$ ; negative  $e^{-k\infty} = e^{k\infty} = 0$

$\therefore$  ROC it converges when  $\sigma > -3 \Rightarrow \frac{-1}{s+3} + 0 \times e^{-j\omega\times\infty}$

$$\underline{L\{x(t)\}} = L\{e^{-3t}u(-t)\} = \frac{-1}{s+3} \quad \left\{ \begin{array}{l} \text{ROC as all pts in s-plane} \\ \text{to the left passing through } \sigma = -3 \end{array} \right.$$

$$(5) \quad x(t) = e^{-4|t|}$$

$$= e^{-4t} \text{ for } t \geq 0$$

$$= e^{+4t} \text{ for } t \leq 0$$

$$L\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t) \cdot e^{-st} dt = \int_{-\infty}^0 e^{4t} \cdot e^{-st} dt + \int_0^{\infty} e^{-4t} \cdot e^{-st} dt$$

$$= \int_{-\infty}^0 e^{-(s-4)t} dt + \int_0^{\infty} e^{-(s+4)t} dt$$

$$= \left[ \frac{e^{-(s-4)t}}{-(s-4)} \right]_0^{\infty} + \left[ \frac{e^{-(s+4)t}}{-(s+4)} \right]_0^{\infty}$$

$$= \left[ \frac{1}{-(s-4)} + \frac{e^{+(s-4)\times\infty}}{s-4} + \frac{e^{-(s+4)\times\infty}}{s+4} + \frac{1}{s+4} \right]$$

$$= \left[ \frac{-1}{s-4} + \frac{e^{(\sigma+j\omega-4)\times\infty}}{s-4} - \frac{e^{(s+j\omega+4)\times\infty}}{s+4} + \frac{1}{s+4} \right]$$

$$= \left[ \frac{-1}{s-4} + \frac{e^{(s-4)\times\infty} \cdot e^{j\omega\times\infty}}{s-4} - \frac{e^{-(s+4)\times\infty} \cdot e^{-j\omega\times\infty}}{s+4} + \frac{1}{s+4} \right]$$

$$s-4 \Rightarrow \sigma - (+4)$$

when  $\sigma > 4$  positive  $e^{k\infty} = \infty$

when  $\sigma < 4$  negative  $e^{-k\infty} = 0$

It converges when  $\sigma < 4$

$$\sigma + 4 \rightarrow \sigma - (-4)$$

when  $\sigma > 4$  positive  $e^{-k\infty} = 0$

when  $\sigma < -4$  negative  $e^{+k\infty} = \infty$

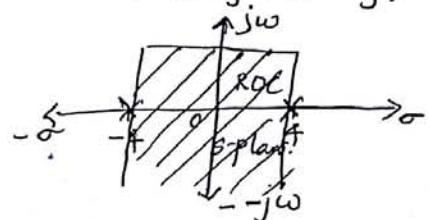
It converges when  $\sigma > -4$

when  $\sigma$  lies between -4 and +4, the  $X(s)$  is given by

$$X(s) = \left[ \frac{-1}{s+4} + \frac{e^{(\sigma-4)x\infty} e^{j\omega x\infty}}{s-4} - \frac{e^{-(\sigma+4)x\infty} e^{-j\omega x\infty}}{s+4} + \frac{1}{s+4} \right]$$

$$= \frac{-1}{s-4} + \frac{1}{s+4} \Rightarrow \frac{-s-4+s-4}{s^2-16} = \frac{-8}{s^2-16}.$$

with ROC as all points in s-plane in between the lines passing through  $\sigma = -4$  and  $\sigma = 4$ .



(6)

Determine the Laplace transform of following signals.

$$x(t) = \sin \omega_0 t u(t)$$

$$\text{Sol} \quad x(t) = \sin \omega_0 t \text{ for } t \geq 0$$

$$L\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt \quad \therefore \sin \theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$= \int_{-\infty}^{\infty} \sin \omega_0 t \cdot e^{-st} dt$$

$$= \int_0^{\infty} \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} \cdot e^{-st} dt$$

$$= \frac{1}{2j} \left[ \int_0^{\infty} e^{-(s-j\omega_0)t} dt - \int_0^{\infty} e^{-(s+j\omega_0)t} dt \right]$$

$$= \frac{1}{2j} \left\{ \left[ \frac{e^{-(s-j\omega_0)t}}{-(s-j\omega_0)} \right]_0^\infty - \left[ \frac{e^{-(s+j\omega_0)t}}{-(s+j\omega_0)} \right]_0^\infty \right\}$$

$$= \frac{1}{2j} \left[ \frac{e^{-\infty}}{-(s-j\omega_0)} + \frac{e^0}{s-j\omega_0} + \frac{e^{-\infty}}{s+j\omega_0} - \frac{e^0}{s+j\omega_0} \right]$$

$$= \frac{1}{2j} \left[ \frac{1}{s-j\omega_0} - \frac{1}{s+j\omega_0} \right] = \frac{1}{2j} \left[ \frac{s+j\omega_0 - s-j\omega_0}{s^2 - j^2\omega_0^2} \right] = \frac{2j\omega_0}{s^2 + \omega_0^2}$$

$$= \frac{\omega_0}{s^2 + \omega_0^2}$$

(7)  $x(t) = \cos \omega_0 t u(t)$

Sol  $x(t) = \cos \omega_0 t ; t \geq 0$

$$\begin{aligned}
 L\{x(t)\} &= X(s) = \int_0^\infty x(t) \cdot e^{-st} dt = \int_0^\infty \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} \cdot e^{-st} dt \\
 &= \frac{1}{2} \int_0^\infty [e^{-(s-j\omega_0)t} + e^{-(s+j\omega_0)t}] dt \\
 &= \frac{1}{2} \left[ \frac{e^{-(s-j\omega_0)t}}{-(s-j\omega_0)} + \frac{e^{-(s+j\omega_0)t}}{-(s+j\omega_0)} \right]_0^\infty \\
 &= \frac{1}{2} \left[ \frac{e^{-\infty}}{-(s-j\omega_0)} + \frac{e^0}{s-j\omega_0} - \frac{e^{-\infty}}{s+j\omega_0} + \frac{e^0}{s+j\omega_0} \right] \\
 &= \frac{1}{2} \left[ \frac{1}{s-j\omega_0} + \frac{1}{s+j\omega_0} \right] = \frac{1}{2} \left[ \frac{s+j\omega_0 + s-j\omega_0}{s^2 - j^2\omega_0^2} \right] = \frac{1}{2} \cdot \frac{2s}{s^2 + \omega_0^2}
 \end{aligned}$$

$$L\{\cos \omega_0 t u(t)\} = \frac{s}{s^2 + \omega_0^2}$$

8)  $x(t) = \cosh \omega_0 t u(t)$

Sol  $x(t) = \cosh \omega_0 t \text{ for } t \geq 0$

$$\therefore \cosh \theta = \frac{e^\theta + e^{-\theta}}{2}$$

$$\begin{aligned}
 L\{x(t)\} &= X(s) = \int_{-\infty}^\infty x(t) \cdot e^{-st} dt \\
 &= \int_0^\infty \left( \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} \right) e^{-st} dt \\
 &= \frac{1}{2} \int_0^\infty e^{j\omega_0 t} \cdot e^{-st} dt + \frac{1}{2} \int_0^\infty e^{-j\omega_0 t} \cdot e^{-st} dt \\
 &= \frac{1}{2} \left[ \int_0^\infty e^{-(s-j\omega_0)t} dt + \int_0^\infty e^{-(s+j\omega_0)t} dt \right] \\
 &\quad \frac{1}{2} \left\{ \left[ \frac{e^{-(s-j\omega_0)t}}{-(s-j\omega_0)} \right]_0^\infty + \left[ \frac{e^{-(s+j\omega_0)t}}{-(s+j\omega_0)} \right]_0^\infty \right\} \\
 &= \frac{1}{2} \left[ \frac{e^{-\infty}}{-(s-j\omega_0)} + \frac{1}{s-j\omega_0} + \frac{e^{-\infty}}{-(s+j\omega_0)} + \frac{1}{s+j\omega_0} \right]
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{2} \left[ \frac{1}{s-\omega_0} + \frac{1}{s+\omega_0} \right] \\
 &= \frac{1}{2} \left[ \frac{s+\omega_0 + s-\omega_0}{s^2 - \omega_0^2} \right] = \frac{s}{s^2 - \omega_0^2}.
 \end{aligned}$$

9)  $x(t) = e^{-at} \sin \omega_0 t u(t)$

Sol  $x(t) = e^{-at} \sin \omega_0 t \text{ for } t \geq 0$

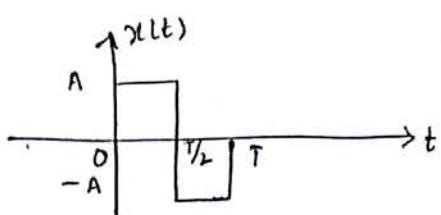
$$\begin{aligned}
 L\{x(t)\} &= X(s) = \int_0^\infty x(t) \cdot e^{-st} dt = \int_0^\infty e^{-at} \sin \omega_0 t e^{-st} dt = \int_0^\infty e^{-at} \left( \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} \right) e^{-st} dt \\
 &= \frac{1}{2j} \left[ \int_0^\infty e^{-(s+a-j\omega_0)t} - e^{-(s+a+j\omega_0)t} \right] dt \\
 &= \frac{1}{2j} \left[ \frac{e^{-(s+a-j\omega_0)t}}{-(s+a-j\omega_0)} \right]_0^\infty - \left[ \frac{e^{-(s+a+j\omega_0)t}}{-(s+a+j\omega_0)} \right]_0^\infty \\
 &= \frac{1}{2j} \left[ \frac{e^{-\infty}}{-(s+a-j\omega_0)} + \frac{e^0}{s+a-j\omega_0} + \frac{e^{-\infty}}{s+a+j\omega_0} - \frac{e^0}{s+a+j\omega_0} \right] \\
 &= \frac{1}{2j} \left[ \frac{1}{s+a-j\omega_0} - \frac{1}{s+a+j\omega_0} \right] = \frac{1}{2j} \left[ \frac{j\omega_0}{(s+a)^2 + \omega_0^2} \right] \\
 &= \frac{j\omega_0}{2j \cdot (s+a)^2 + \omega_0^2} \\
 &= \frac{\omega_0}{(s+a)^2 + \omega_0^2}.
 \end{aligned}$$

(10)  $x(t) = e^{-at} \cos \omega_0 t u(t)$

Sol  $x(t) = e^{-at} \cos \omega_0 t \text{ for } t \geq 0$

$$\begin{aligned}
 &\text{|| by} \\
 &= \frac{s+a}{(s+a)^2 + \omega_0^2}
 \end{aligned}$$

(11) Determine the Laplace transform of following signals.

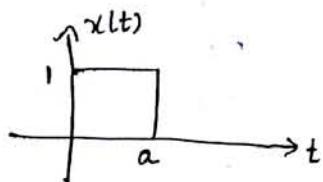
Sol

$$x(t) = A \text{ for } 0 < t < T/2$$

$$= -A \text{ for } T/2 < t < T$$

$$\begin{aligned} L\{x(t)\} = X(s) &= \int_0^{T/2} A \cdot e^{-st} dt + \int_{T/2}^T -A \cdot e^{-st} dt \\ &= A \left[ \frac{e^{-st}}{-s} \right]_0^{T/2} - A \left[ \frac{e^{-st}}{-s} \right]_{T/2}^T \\ &= A \left[ \frac{e^{-sT/2}}{-s} + \frac{1}{s} \right] - A \left[ \frac{e^{-sT}}{-s} + \frac{e^{-sT/2}}{s} \right] \\ &= -\frac{Ae^{-sT/2}}{s} + \frac{A}{s} + \frac{Ae^{-sT}}{s} - \frac{Ae^{-sT/2}}{s} \\ &= \frac{A}{s} \left[ 1 + e^{-sT} - 2e^{-sT/2} \right] = \frac{A}{s} \left[ 1 - e^{-sT/2} \right]. \end{aligned}$$

(12)

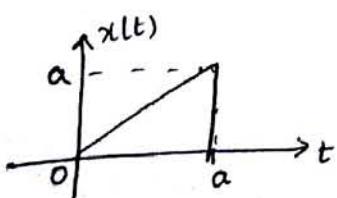


$$x(t) = 1 \text{ for } 0 < t < a$$

$$= 0 \text{ otherwise}$$

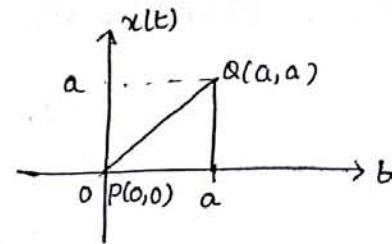
$$\begin{aligned} L\{x(t)\} = X(s) &= \int_0^a e^{-st} dt = \left[ \frac{e^{-st}}{-s} \right]_0^a \\ &= \frac{e^{-as}}{-s} + \frac{1}{s} \Rightarrow \frac{1}{s} \left[ 1 - e^{-as} \right] \end{aligned}$$

(13)



$$\frac{y-y_1}{y_1-y_2} = \frac{x-x_1}{x_1-x_2}$$

here  $(x_1, y_1) = (0, 0)$ ,  $(x_2, y_2) = (a, a)$

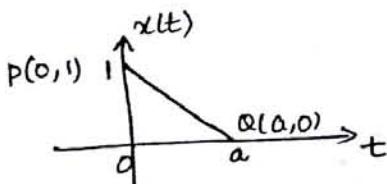


$x=t$ ,  $y=x(t)$ .

$$\Rightarrow \frac{x(t)-0}{a-0} = \frac{t-0}{a-0} \Rightarrow \frac{x(t)}{a} = \frac{t}{a} \Rightarrow x(t) = t \text{ for } 0 < t < a \\ = 0 \text{ otherwise}$$

$$\begin{aligned} L\{x(t)\} &= X(s) = \int_{-\infty}^{\infty} x(t) \cdot e^{-st} dt \\ &= \int_0^a t e^{-st} dt = \left\{ t \left[ \frac{e^{-st}}{-s} \right]_0^a - \int \frac{e^{-st}}{-s} \right\}_0^a \\ &= t \cdot \frac{e^{-st}}{-s} \Big|_0^a - \frac{e^{-st}}{s^2} \Big|_0^a \\ &= a \cdot \frac{e^{-sa}}{-s} - \frac{e^{-as}}{s^2} + \frac{1}{s^2} \\ &= \frac{1}{s^2} \left[ 1 - e^{-as} - s \cdot e^{-as} \cdot a \right] \\ &= \frac{1}{s^2} \left[ 1 - e^{-as} - as \cdot e^{-as} \right] \\ &= \frac{1}{s^2} \left[ 1 - e^{-as} (1 + as) \right] \end{aligned}$$

Q4)



$(x_1, y_1) = (0, 1)$ ,  $(x_2, y_2) = (a, 0)$ ,  $x=t$ ,  $y=x(t)$ .

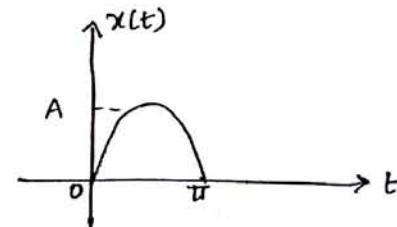
$$\frac{y-y_1}{y_1-y_2} = \frac{x-x_1}{x_1-x_2} \Rightarrow \frac{x(t)-1}{1-0} = \frac{t-0}{a-0} \Rightarrow x(t)-1 = \frac{t}{a}$$

$$x(t) = 1 - \frac{t}{a} \text{ for } 0 \leq t < a \\ = 0 \text{ for } t \geq a$$

$$\begin{aligned}
 L\{x(t)\} &= X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt \\
 &= \int_0^a \left(1 - \frac{t}{a}\right) e^{-st} dt = \int_0^a e^{-st} dt - \int_0^a \frac{t}{a} e^{-st} dt \\
 &= \left[ \frac{e^{-st}}{-s} \right]_0^a - \frac{1}{a} \left[ \int_0^a t e^{-st} dt \right] \\
 &= \frac{e^{-as}}{-s} + \frac{1}{s} - \frac{1}{a} \left[ t \cdot \frac{e^{-st}}{-s} - \frac{e^{-st}}{s^2} \right]_0^a \\
 &= \frac{e^{-as}}{-s} + \frac{1}{s} - \frac{1}{a} \cdot \frac{a \cdot e^{-as}}{-s} + \frac{1}{a s^2} - \frac{1}{a s^2} \\
 &= \cancel{\frac{e^{-as}}{-s}} + \frac{1}{s} + \cancel{\frac{e^{-as}}{s}} + \frac{e^{-as}}{a s^2} - \frac{1}{a s^2} \\
 &= \frac{1}{s} + \frac{e^{-as}}{a s^2} - \frac{1}{a s^2} = \frac{1}{a s^2} \left[ e^{-as} + as - 1 \right]
 \end{aligned}$$

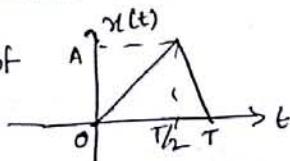
(15) Determine the Laplace transform of sinc pulse

Sol  $x(t) = A \sin t$  for  $0 < t < \pi$   
 $= 0$  for  $t > \pi$



$$\begin{aligned}
 L\{x(t)\} &= X(s) = \int_0^\pi A \sin t e^{-st} dt \\
 &= A \left[ \frac{e^{jt} - e^{-jt}}{2j} e^{-st} \right] dt = A \left[ \frac{1}{2j} \left( \int_0^\pi e^{-(s-j)t} dt - \int_0^\pi e^{-(s+j)t} dt \right) \right] \\
 &= \frac{A}{2j} \left[ \frac{e^{-(s-j)t}}{-s+j} \right]_0^\pi - \frac{A}{2j} \left[ \frac{e^{-(s+j)t}}{-s-j} \right]_0^\pi \\
 &= \frac{A}{2j} \left[ \frac{e^{-(s+j)\pi}}{s+j} - \frac{e^{-(s-j)\pi}}{s-j} \right] = \frac{A}{2j} \left[ \frac{(s-j)e^{-st} e^{-jt} - (s+j)e^{-st} e^{+jt}}{s^2 - j^2} \right]^\pi \\
 &= \frac{A}{2j(s^2 + 1)} \left[ (s-j)e^{-st} e^{-jt} - (s+j)e^{-st} e^{+jt} \right]_0^\pi \\
 &= \frac{A}{2j(s^2 + 1)} \left[ (s-j)e^{-s\pi} e^{-j\pi} - (s+j)e^{-s\pi} e^{+j\pi} - (s-j)e^{-s0} + (s+j)e^{+s0} \right] \\
 &= \frac{A}{2j(s^2 + 1)} \left[ -(s-j)e^{-s\pi} + (s+j)e^{-s\pi} - (s-j) + (s+j) \right] = \frac{A}{2j(s^2 + 1)} (2je^{\pi s} + 2j) \\
 &\Rightarrow A/s^2 (e^{\pi s} + 1)
 \end{aligned}$$

(16) Determine Laplace transform of



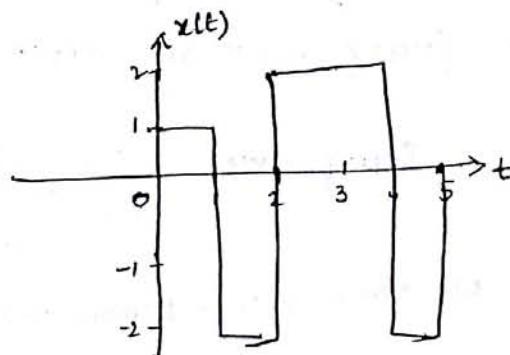
$$\text{Sol} \quad x(t) = \frac{2At}{T}; \quad 0 < t < T/2$$

$$= 2A - \frac{2At}{T} \quad T/2 < t < T.$$

$$L\{x(t)\} = X(s) = \frac{2A}{Ts} \left(1 - e^{-st/2}\right)^2.$$

(17) Determine Laplace transform of

$$\text{Sol} \quad x(t) = \begin{cases} 1 & \text{for } 0 < t < 1 \\ -2 & \text{for } 1 < t < 2 \\ 2 & \text{for } 2 < t < 4 \\ -2 & \text{for } 4 < t < 5 \\ 0 & \text{for } t > 5 \end{cases}$$



$$\begin{aligned} L\{x(t)\} = X(s) &= \int_0^1 e^{-st} dt + \int_1^2 -2e^{-st} dt + \int_2^4 2e^{-st} dt + \int_4^5 -2e^{-st} dt \\ &= \left[ \frac{e^{-st}}{-s} \right]_0^1 - 2 \left[ \frac{e^{-st}}{-s} \right]_1^2 + 2 \left[ \frac{e^{-st}}{-s} \right]_2^4 - 2 \left[ \frac{e^{-st}}{-s} \right]_4^5 \\ &= \frac{e^{-s}}{-s} + \frac{1}{s} + \frac{2e^{-2s}}{s} - \frac{e^{-s}}{s} - \frac{2e^{-4s}}{s} + \frac{2e^{-2s}}{s} + \frac{2e^{-5s}}{s} - 2 \frac{e^{-4s}}{s} \\ &= \frac{1}{s} \left[ 1 - 2e^{-s} + 4e^{-2s} - 4e^{-4s} + 2e^{-5s} \right] \end{aligned}$$

(18) Determine the Laplace transform of g(t).

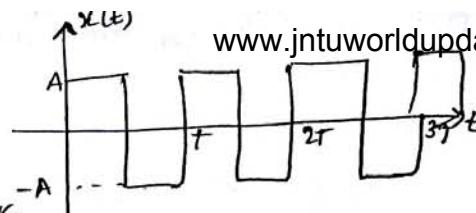
$$\text{Sol} \quad g(t) = 1 \quad \text{for } t=0 \\ = 0 \quad \text{otherwise.}$$

$$L\{g(t)\} = G(s) = \left. \frac{e^{-st}}{-s} \right|_{t=0} =$$

Determine the Laplace transform of periodic square wave

The given waveform satisfy the condition

$$x(t+nT) = x(t) \text{ and so it is periodic}$$



$$x_1(t) = A \text{ for } t = 0 \text{ to } T/2$$

$$= -A \text{ for } t = T/2 \text{ to } T$$

From periodicity property of Laplace transform

$$\text{If } X(s) = L\{x(t)\} \text{ and if } x(t) = x(t+nT) \text{ then } X(s) = \frac{1}{1-e^{-sT}} \int_0^T x_1(t) e^{-st} dt$$

$$\therefore L\{x(t)\} = X(s) = \frac{1}{1-e^{-sT}} \int_0^T x_1(t) e^{-st} dt$$

$$\text{we know } x_1(t) = \text{Laplace transform} = \frac{A}{s} \left( 1 - e^{-sT/2} \right)^r$$

$$X(s) = \frac{1}{1-e^{-sT}} \left[ \frac{A}{s} \left( 1 - e^{-sT/2} \right)^r \right]$$

$$= \frac{1}{(1-e^{-sT/2})(1+e^{-sT/2})} \left[ \frac{A}{s} \left( 1 - e^{-sT/2} \right)^r \right]$$

$$= \frac{A}{s} \left[ \frac{1 - e^{-sT/2}}{1 + e^{-sT/2}} \right]^r$$

Initial Value Theorem:

The initial value theorem states that, if  $x(t)$  and its derivative are Laplace transformable then  $\lim_{t \rightarrow 0} x(t) = \lim_{s \rightarrow \infty} sX(s)$ .

$$\text{i.e } x(0) = \lim_{t \rightarrow 0} x(t) = \lim_{s \rightarrow \infty} sX(s).$$

Proof

We know that

$$\mathcal{L} \left\{ \frac{dx(t)}{dt} \right\} = sX(s) - x(0)$$

On taking limits  $s \rightarrow \infty$  on both sides of equation.

$$\lim_{s \rightarrow \infty} \mathcal{L} \left\{ \frac{dx(t)}{dt} \right\} = \lim_{s \rightarrow \infty} [sX(s) - x(0)]$$

$$\Rightarrow \lim_{s \rightarrow \infty} \int_0^{\infty} \frac{dx(t)}{dt} e^{-st} dt = \lim_{s \rightarrow \infty} [sX(s) - x(0)]$$

$$\Rightarrow \int_0^{\infty} \frac{dx(t)}{dt} \left( \lim_{s \rightarrow \infty} e^{-st} \right) dt = \lim_{s \rightarrow \infty} sX(s) - x(0)$$

$$\Rightarrow 0 = \lim_{s \rightarrow \infty} sX(s) - x(0)$$

$$\therefore x(0) = \lim_{s \rightarrow \infty} sX(s).$$

$$\boxed{\therefore \lim_{t \rightarrow 0} x(t) = \lim_{s \rightarrow \infty} sX(s)}$$

Final Value theorem:

The final value theorem states that if  $x(t)$  and its derivative are Laplace transformable then  $\lim_{t \rightarrow \infty} x(t) = \lim_{s \rightarrow 0} sX(s)$ .

$$\text{i.e Final value of signal } x(\infty) = \lim_{t \rightarrow \infty} x(t) = \lim_{s \rightarrow 0} sX(s).$$

Proof

$$\mathcal{L} \left\{ \frac{dx(t)}{dt} \right\} = sX(s) - x(0)$$

By taking limits  $s \rightarrow 0$  on both sides of above eqn we get

$$\lim_{s \rightarrow 0} \mathcal{L} \left\{ \frac{dx(t)}{dt} \right\} = \lim_{s \rightarrow 0} \{ sX(s) - x(0) \}$$

$$\underset{s \rightarrow 0}{\text{Lt}} \int_0^{\infty} \frac{dx(t)}{dt} e^{-st} dt = \underset{s \rightarrow 0}{\text{Lt}} [sx(s) - x(0)]$$

$$\int_0^{\infty} \frac{dx(t)}{dt} \left( \underset{s \rightarrow 0}{\text{Lt}} e^{-st} \right) dt = \underset{s \rightarrow 0}{\text{Lt}} [sx(s) - x(0)]$$

$\downarrow$   
not fn of 's'

## Z - TRANSFORMS :

FUNDAMENTAL DIFFERENCES BETWEEN CONTINUOUS & DISCRETE TIME SIGNAL  
(Ref Unit-I page No: 9)

→ DISCRETE TIME SIGNAL REPRESENTATION USING COMPLEX EXPONENTIAL AND SINUSOIDAL COMPONENTS.  
(Ref Unit-I - page No: 10, 11, 12)

→ PERIODICITY PROPERTIES OF DT COMPLEX FUNCTIONS:

→ A DT complex exponential is periodic if it repeats after certain N' number of samples.

$$\text{consider } x(n) = e^{j\omega_0 n}$$

$$\therefore x(n+N) = e^{j\omega_0(n+N)}$$

$$= e^{j\omega_0 n} \cdot e^{j\omega_0 N}$$

For periodicity  $x(n) = x(n+N)$

$$\text{i.e. } e^{j\omega_0 n} = e^{j\omega_0 n} \cdot e^{j\omega_0 N} \Rightarrow e^{j\omega_0 N} = 1$$

Expressing  $e^{j\omega_0 N}$  in terms of sinusoidal functions

$$e^{j\omega_0 N} = \cos \omega_0 N + j \sin \omega_0 N \text{ by Euler's identity}$$

$$e^{j\theta} = \cos \theta + j \sin \theta$$

→ For periodicity  $e^{j\omega_0 N} = 1$  gives  $\cos \omega_0 N + j \sin \omega_0 N = 1$ . This eqn is satisfied for

$$\omega_0 N = 2\pi, 4\pi, 6\pi, 8\pi \dots$$

$$= 2\pi k \text{ where } k = \text{integer.}$$

$$(OR) \quad \frac{\omega_0}{2\pi} = \frac{k}{N} \rightarrow (1)$$

$$\text{but } \omega_0 = 2\pi f \Rightarrow \frac{\omega_0}{2\pi} = f \rightarrow (2)$$

Substitute (2) in (1)

$$\therefore f = \frac{k}{N}$$

For complex exponential to be periodic,  $\frac{\omega_0}{2\pi} = \frac{k}{N}$  (i.e. rational).

## Z-TRANSFORM (CONCEPT):

The z-transform of  $x(n)$  is denoted by  $X(z)$  and defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \quad \text{where } z \text{ is complex variable.}$$

→  $x(n)$  &  $X(z)$  is called z-transform pair represented as

$$x(n) \xleftrightarrow{z} X(z).$$

- Purpose of learning z-transform is to analyse the DT signals and systems, digital filter design and for synthesis of digital filter / systems.
- For any input sequence, the z-transform is complex. It has real & imaginary parts.



## DISTINCTION BETWEEN LAPLACE, FOURIER AND Z TRANSFORM!

z-transform given as

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \quad \text{ROC: } r_2 < |z| < r_1$$

where 'z' is defined as  $z = re^{j\omega}$  in which 'r' is magnitude of  $z$  i.e.  $|z|$  and  $\omega$  = angle of  $z$  i.e.  $\angle z$

→ putting for  $z = re^{j\omega}$  in  $X(z)$  we get

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x(n) (re^{j\omega})^{-n} \\ &= \sum_{n=-\infty}^{\infty} [x(n) r^{-n}] e^{-j\omega n}. \end{aligned} \quad \rightarrow (1)$$

→ Fourier transform is given by  $X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \rightarrow (2)$  comparing  $X(\omega)$  with  $X(z)$  we find that  $X(z)$  indicates Fourier transform of  $x(n) r^{-n}$ .

→ Let  $x(z)$  of eqn ① is evaluated on unit circle. Then  $|z| = r = 1$  i.e.  $r^{-n} = 1$

$$\therefore x(z) \Big|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} x(n) e^{-jn\omega} \rightarrow \text{Fourier transform by definition}$$

$$\boxed{x(z) \Big|_{z=e^{j\omega}} = X(j\omega)} \rightarrow R/B<1 \quad FT \in Z-T$$

→ Laplace Transform of i/p sgl  $x(t)$  defined as

$$L\{x(t)\} = X(s) = \int_0^{\infty} x(t) e^{-st} dt \rightarrow ①$$

On substituting  $s = \sigma + j\omega$  in above eqn ①

$$L\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t) e^{-(\sigma+j\omega)t} dt \rightarrow ②$$

but definition of Fourier transform of  $x(t)$  is

$$F\{x(t)\} = X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \rightarrow ③$$

By comparing equations ② & ③, we can say that-

By putting  $\sigma = 0$  in Laplace transform we get Fourier transform of continuous time signal.

$$\therefore X(j\omega) = X(s) \Big|_{s=j\omega}$$

### REGION OF CONVERGENCE IN Z-TRANSFORM :

→ Since z-transform is an infinite power series, it exists only for those values of  $z$  for which the series converges.

→ The ROC of  $x(z)$  is the set of all values of  $z$ , for which  $x(z)$  attains a finite value.

# CONSTRAINTS ON ROC FOR VARIOUS CLASSES OF SIGNALS! (UR)

## PROPERTIES OF ROC with proofs:

- (1) The ROC is a ring or disk in the z-plane centered at the origin.
- (2) The ROC cannot contain any poles.
- (3) If  $x(n)$  is a finite duration, causal sequence then the ROC is the entire z-plane except at  $z=0$ .
- (4) If  $x(n)$  is a finite duration, anti-causal sequence then the ROC is the entire z-plane except at  $z=\infty$ .
- (5) If  $x(n)$  is a finite duration, two sided sequence then the ROC is entire z-plane except at  $z=0 \& z=\infty$ .
- (6) If  $x(n)$  is an infinite duration, two sided sequence the ROC will consist of a ring in z-plane, bounded on exterior and exterior by a pole not containing any poles.
- (7) The ROC of an LTI stable system contains unit circle.
- (8) The ROC must be connected region.

Case i) Finite duration, right sided (causal) signal:

Let  $x(n)$  be finite duration signal with N-samples, defined in range  $0 \leq n \leq N-1$

$$\therefore x(n) = \{x(0), x(1), x(2), \dots, x(N-1)\}$$

$$\begin{aligned} x(z) &= \sum_{n=0}^{N-1} x(n) z^{-n} \\ &= x(0) + x(1) z^{-1} + \dots + x(N-1) z^{-(N-1)} \\ &= x(0) + \frac{x(1)}{z} + \frac{x(2)}{z^2} + \dots + \frac{x(N-1)}{z^{N-1}} \end{aligned}$$

When  $z=0$ , all the terms except the first term become infinite.  
Hence the  $x(z)$  exists for all values of  $z$  except  $z=0$ .

$\therefore$  ROC for finite duration right sided is entire z-plane except  $z=0$

Case ii) Finite duration, left sided (anticausal) signal:

Let  $x(n)$  be finite duration signal with  $N$ -samples, defined in the range

$$-(N-1) \leq n < 0$$

$$\therefore x(n) = \{x(-N+1), \dots, x(-2), x(-1), x(0)\}$$

$z$ -transform of  $x(n)$  is

$$X(z) = \sum_{n=-N+1}^0 x(n) z^{-n}$$

$$X(z) = x(-N+1) z^{N-1} + \dots + (x(-2)) z^2 + x(-1) z + x(0)$$

when  $z = \infty$ , all terms except the last term become infinite. Hence the  $X(z)$  exists for all values of  $z$ , except  $z = \infty$ .

$\therefore$  ROC of  $X(z)$  is entire  $z$ -plane except  $z = \infty$ .

Case iii) Finite duration, two sided (non-causal) signal:

Let  $x(n)$  be a finite duration signal with  $N$ -samples, defined in the range  $-M \leq n \leq M$  where  $M = \frac{N-1}{2}$ .

$$\therefore x(n) = \{x(-M), \dots, x(-2), x(-1), x(0), x(1), x(2), \dots, x(M)\}$$

$z$ -transform of  $x(n)$  is

$$X(z) = \sum_{n=-M}^M x(n) z^{-n}$$

$$= x(-M) z^M + \dots + x(-2) z^2 + x(-1) z + x(0) + x(1) z^1 +$$

$$x(2) z^2 + \dots + x(M) z^{-M}$$

$$= x(-M) z^M + \dots + x(-2) z^2 + x(-1) z + x(0) + \frac{x(1)}{z} + \frac{x(2)}{z^2} + \dots + \frac{x(M)}{z^M}.$$

when  $z = 0$ , the terms with negative power of  $z$  attain infinity

and when  $z = \infty$ , the terms with positive power of  $z$  attain infinity.

Hence  $X(z)$  converges for all values of  $z$ , except at  $z = 0$  &  $z = \infty$ .

ROC is entire  $z$ -plane except  $z = 0$  &  $z = \infty$ .

Case iv) Infinite duration, right sided (causal) signal:

$$\text{Let } x(n) = r_1^n ; n \geq 0$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} = \sum_{n=0}^{\infty} r_1^n z^{-n} = \sum_{n=0}^{\infty} (r_1 z^{-1})^n$$

$$\text{If } 0 < |r_1 z^{-1}| < 1, \text{ then } \sum_{n=0}^{\infty} (r_1 z^{-1})^n = \frac{1}{1 - r_1 z^{-1}}$$

$$\therefore X(z) = \frac{1}{1 - \frac{r_1}{z}} = \frac{z}{z - r_1}$$

{ Using infinite geometric series sum formula

$$\sum_{n=0}^{\infty} c_n = \frac{1}{1 - c}$$

$$\text{if } 0 < |c| < 1$$

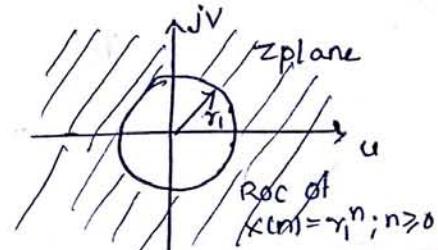
Condition to be satisfied for convergence of  $X(z)$  is

$$0 < |r_1 z^{-1}| < 1$$

$$\therefore |r_1 z^{-1}| < 1$$

$$\frac{|r_1|}{|z|} < 1 \Rightarrow |z| > |r_1|$$

represent a circle  
of radius  $r_1$  in  $z$ -plane.



$\rightarrow x(z)$  converges for all points external to the circle of radius  $r_1$  in  $z$ -plane.

$\therefore$  ROC of  $x(z)$  is exterior of the circle of radius  $r_1$  in  $z$ -plane.

Case v) Infinite duration, left sided (anticausal) signal:

$$\text{Let } x(n) = r_2^n ; n \leq 0$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} = \sum_{n=0}^{0} r_2^n z^{-n} = \sum_{n=0}^{\infty} (r_2^{-1} z)^n$$

$$\text{If } 0 < |r_2^{-1} z| < 1 \text{ then } \sum_{n=0}^{\infty} (r_2^{-1} z)^n = \frac{1}{1 - r_2^{-1} z}$$

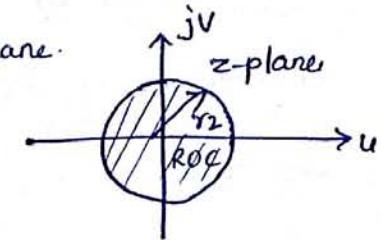
$$\begin{aligned} \therefore X(z) &= \frac{1}{1 - r_2^{-1} z} = \frac{1}{1 - \frac{z}{r_2}} = \frac{r_2}{r_2 - z} \\ &= \frac{-r_2}{z - r_2} \end{aligned}$$

Condition to be satisfied for convergence of  $X(z)$  is

$$0 < |r_2^{-1} z| < 1 \Rightarrow |r_2^{-1} z| < 1 \Rightarrow \frac{|z|}{|r_2|} < 1 \Rightarrow |z| < |r_2|$$

The term  $|r_2|$  represent a ob of radius of  $r_2$  in  $z$ -plane.

$x(z)$  converges for all pts internal to the ob of radius  $r_2$  in  $z$ -plane.  $\therefore$  ROC of  $x(z)$  is exterior of the circle of radius  $r_2$ .



Case vi) Infinite duration, two sided (anticausal) signal:

$$\text{Let } x(n) = r_1^n u(n) + r_2^n u(-n)$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^0 r_2^n z^{-n} + \sum_{n=0}^{\infty} r_1^n z^{-n}$$

$$= \sum_{n=0}^{\infty} r_2^{-n} z^n + \sum_{n=0}^{\infty} r_1^{-n} z^n$$

$$= \sum_{n=0}^{\infty} (r_2^{-1} z)^n + \sum_{n=0}^{\infty} (r_1^{-1} z)^n \quad \left. \begin{array}{l} \text{Using infinite geometric} \\ \text{series sum formula} \\ \text{if } 0 < |r_2^{-1} z| < 1 \text{ & } 0 < |r_1^{-1} z| < 1 \end{array} \right\}$$

$$= \frac{1}{1 - r_2^{-1} z} + \frac{1}{1 - r_1^{-1} z}$$

The term  $\sum_{n=0}^{\infty} (r_2^{-1} z)^n$  converges if

$$0 < |r_2^{-1} z| < 1$$

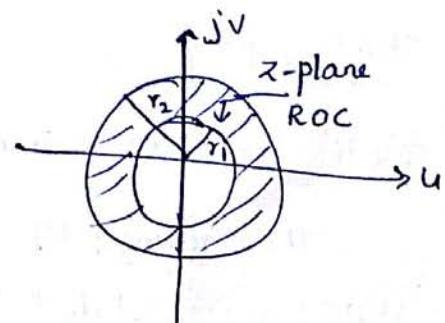
$$\therefore |r_2^{-1} z| < 1 \Rightarrow \frac{|r_2^{-1}|}{|z|} < 1 \Rightarrow |z| < |r_2|.$$

The term  $\sum_{n=0}^{\infty} (r_1^{-1} z)^n$  converges if

$$0 < |r_1^{-1} z| < 1$$

$$\therefore |r_1^{-1} z| < 1$$

$$\frac{|r_1|}{|z|} < 1 \Rightarrow |z| > |r_1|.$$



$\therefore$  ROC is the region between two circles of radius  $r_1$  &  $r_2$ .

## PROPERTIES OF Z-TRANSFORM:

### Linearity Property:

→ States that z-transform of linear weighted combination of discrete time signals is equal to similar linear weighted combination of z transform of individual discrete time signals.

Let  $z\{x_1(n)\} = X_1(z)$  and  $z\{x_2(n)\} = X_2(z)$  then by linearity property

$$z\left[a_1 x_1(n) + a_2 x_2(n)\right] = a_1 X_1(z) + a_2 X_2(z) \quad (a_1, a_2 \text{ are constants}).$$

Proof:

z-transform definition

$$X_1(z) = z\{x_1(n)\} = \sum_{n=-\infty}^{\infty} x_1(n) z^{-n}$$

$$X_2(z) = z\{x_2(n)\} = \sum_{n=-\infty}^{\infty} x_2(n) z^{-n}$$

$$\begin{aligned} z\left[a_1 x_1(n) + a_2 x_2(n)\right] &= \sum_{n=-\infty}^{\infty} [a_1 x_1(n) + a_2 x_2(n)] z^{-n} \\ &= \sum_{n=-\infty}^{\infty} [a_1 x_1(n) z^{-n} + a_2 x_2(n) z^{-n}] \\ &= \sum_{n=-\infty}^{\infty} a_1 x_1(n) z^{-n} + \sum_{n=-\infty}^{\infty} a_2 x_2(n) z^{-n} \\ &= a_1 \sum_{n=-\infty}^{\infty} x_1(n) z^{-n} + a_2 \sum_{n=-\infty}^{\infty} x_2(n) z^{-n} \\ &= a_1 X_1(z) + a_2 X_2(z). \end{aligned}$$

### Shifting property:

#### Case i) Two sided z-transform

The shifting property of z-transform states that z-transform of a shifted signal shifted by  $m$  units of time is obtained by multiplying  $z^m$  to z-transform of unshifted signal.

Let  $z\{x(n)\} = X(z)$  then  $z\{x(n-m)\} = z^m X(z)$  &  $z\{x(n+m)\} = z^m X(z)$

Proof:

$$x(z) = \sum \{ x(n) \} = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$z \{ x(n-m) \} = \sum_{n=-\infty}^{\infty} x(n-m) z^{-n}$$

$$= \sum_{p=-\infty}^{\infty} x(p) z^{-(m+p)}$$

$$= \sum_{p=-\infty}^{\infty} x(p) z^{-p} \cdot z^{-m}$$

$$= z^{-m} \sum_{p=-\infty}^{\infty} x(p) z^{-p} = z^{-m} x(z)$$

$\left\{ \begin{array}{l} \text{let } n-m=p \therefore n=p+m \\ \text{when } n \rightarrow -\infty, p \rightarrow -\infty \\ n \rightarrow \infty, p \rightarrow \infty \end{array} \right.$

$$z \{ x(n+m) \} = \sum_{n=-\infty}^{\infty} x(n+m) z^{-n}$$

$$= \sum_{p=-\infty}^{\infty} x(p) z^{-(p-m)}$$

$$= \sum_{p=-\infty}^{\infty} x(p) z^{-p} \cdot z^m$$

$$= z^m \sum_{p=-\infty}^{\infty} x(p) z^{-p} \quad (\because \text{if } p \rightarrow n)$$

$$= z^m \cdot x(z).$$

Case ii) One Sided Z-transform:

Let  $x(n)$  be discrete time signal defined in range  $0 < n < \infty$

$$z[x(n)] = x(z)$$

By shifting property

$$z[x(n-m)] = z^{-m} x(z) + \sum_{i=1}^m x(-i) z^{-(m-i)}$$

$$z[x(n+m)] = z^m x(z) - \sum_{i=0}^{m-1} x(i) z^{(m-i)}$$

Proof

$$x(z) = z \{x(n)\} = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$z \{x(n-m)\} = \sum_{n=0}^{\infty} x(n-m) z^{-m}$$

Multiply by  $z^m$  &  $z^{-m}$

$$= \sum_{n=0}^{\infty} x(n-m) z^{-n} z^m, z^{-m}$$

$$= z^{-m} \sum_{n=0}^{\infty} x(n-m) z^{-(n-m)}$$

$$= z^{-m} \sum_{p=-m}^{\infty} x(p) z^{-p} \quad \left\{ \begin{array}{l} \text{Let } n-m=p \\ p=-m \end{array} \right.$$

$$= z^{-m} \sum_{p=0}^{\infty} x(p) z^{-p} + \sum_{p=m}^{-1} x(p) z^{-p} \cdot z^{-m}$$

$$= z^{-m} \sum_{p=0}^{\infty} x(p) z^{-p} + z^{-m} \sum_{p=1}^m x(-p) z^p$$

Let  $p=n$ , in 1<sup>st</sup> summation

$$= z^{-m} \sum_{n=0}^{\infty} x(n) z^{-n} + z^{-m} \sum_{i=1}^m x(-i) z^i \quad p=i \text{ in 2<sup>nd</sup> "}$$

$$= z^m x(z) + \sum_{i=1}^m x(-i) z^{-(m-i)}$$

$$z \{x(n+m)\} = \sum_{n=0}^{\infty} x(n+m) z^{-n} = \sum_{n=0}^{\infty} x(n+m) z^{-n} z^m z^{-m} \quad (\text{multiply by } z^m \& z^{-m})$$

$$= z^m \sum_{n=0}^{\infty} x(n+m) z^{-(n+m)}$$

$$= z^m \sum_{p=m}^{\infty} x(p) z^{-p}$$

Let  $n+m=p$   
when  $n \rightarrow 0, p \rightarrow m$   
 $n \rightarrow \infty, p \rightarrow \infty$

$$= z^m \sum_{p=0}^{\infty} x(p) z^{-p} - z^m \sum_{p=0}^{m-1} x(p) z^{-p}$$

$$= z^m \sum_{n=0}^{\infty} x(n) z^{-n} - z^m \sum_{i=0}^{m-1} x(i) z^{-i}$$

$$= z^m x(z) - \sum_{i=0}^{m-1} x(i) z^{m-i} \quad \left\{ \begin{array}{l} \therefore \text{Let } p=n \text{ in 1<sup>st</sup>} \\ p=i \text{ in 2<sup>nd</sup>.} \end{array} \right.$$

(3) Multiplication by n (or Differentiation in z-domain)

Proof If  $\mathcal{Z}\{x(n)\} = X(z)$  then  $\mathcal{Z}\{nx(n)\} = -z \frac{d}{dz} X(z)$

$$\text{In general } \mathcal{Z}\{n^m x(n)\} = \left(-z \frac{d}{dz}\right)^m X(z)$$

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$\mathcal{Z}\{nx(n)\} = \sum_{n=-\infty}^{\infty} n x(n) z^{-n} = \sum_{n=-\infty}^{\infty} n x(n) z^{-n} z^{-1}$$

$$= -z \sum_{n=-\infty}^{\infty} x(n) [-nz^{-n-1}]$$

$$= -z \sum_{n=-\infty}^{\infty} x(n) \left[ \frac{d}{dz} z^{-n} \right]$$

$$= -z \frac{d}{dz} \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$\mathcal{Z}\{nx(n)\} = -z \frac{d}{dz} X(z).$$

$\left\{ \begin{array}{l} \text{Interchanging} \\ \text{summation \& differentiation} \end{array} \right.$

(4) Multiplication by an exponential sequence,  $a^n$  (scaling in z-domain)

If  $\mathcal{Z}\{x(n)\} = X(z)$  then  $\mathcal{Z}\{a^n x(n)\} = X(a^{-1}z)$

Proof  $\mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$$\mathcal{Z}\{a^n x(n)\} = \sum_{n=-\infty}^{\infty} a^n x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) (a^{-1}z)^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) (a^{-1}z)^{-n}$$

$$= X(a^{-1}z).$$

(5) Time Reversal:

If  $\mathcal{Z}\{x(n)\} = X(z)$  then  $\mathcal{Z}\{x(-n)\} = X(z^{-1})$

proof

$$\mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

$$\mathcal{Z}\{x(-n)\} = \sum_{n=-\infty}^{\infty} x(-n)z^{-n}$$

$$= \sum_{p=-\infty}^{\infty} x(p)z^p$$

$$= \sum_{p=-\infty}^{\infty} x(p)(z^{-1})^{-p}$$

$$= X(z^{-1})$$

Let  $p = -n$   
 when  $n \rightarrow -\infty, p \rightarrow \infty$   
 $n \rightarrow \infty, p \rightarrow -\infty$

(6) Conjugation:

If  $\mathcal{Z}\{x(n)\} = X(z)$  then  $\mathcal{Z}\{x^*(n)\} = X^*(z^*)$

proof

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

$$\mathcal{Z}\{x^*(n)\} = \sum_{n=-\infty}^{\infty} x^*(n)z^{-n}$$

$$= \left[ \sum_{n=-\infty}^{\infty} x(n)(z^*)^{-n} \right]^*$$

$$= [X(z^*)]^*$$

$$= X^*(z^*)$$

## INVERSE Z-TRANSFORM:

Let  $x(z)$  be z-transform of discrete time signal  $x(n)$ . The inverse z-transform is the process of recovering the discrete time sgl  $x(n)$  from its z-transform  $x(z)$ .

→ The signal  $x(n)$  can be uniquely determined from  $x(z)$  & its ROC.

It can be determined by three methods

- Direct evaluation by contour integration (or residue method)
- partial fraction expansion method
- power series expansion method.

Q1 Determine the inverse z-transform of fn  $x(z) = \frac{3+2z^{-1}+z^{-2}}{1-3z^{-1}+2z^{-2}}$  by following three methods and prove that it is unique.

Sol (i) Residue Method :

$$x(z) = \frac{3+2z^{-1}+z^{-2}}{1-3z^{-1}+2z^{-2}} = \frac{z^2(3z^2+2z+1)}{z^2(z^2-3z+2)}$$

$$\therefore x(z) = \frac{3z^2+2z+1}{z^2-3z+2}$$

$$= 3 + \frac{11z-5}{z^2-3z+2}$$

$$= 3 + \frac{11z-5}{(z-1)(z-2)}$$

$$\begin{array}{r} 3 \\ z^2-3z+2 \end{array} \overline{ \left[ \begin{array}{r} 3z^2+2z+1 \\ +3z^2-9z+6 \\ \hline 11z-5 \end{array} \right] }$$

$$\text{Let } x_1(z) = 3, \quad x_2(z) = \frac{11z-5}{(z-1)(z-2)} ; \text{ i.e. } x(z) = x_1(z) + x_2(z)$$

$$x(n) = z^{-n} \{x_2(z)\} = z^{-1} \{x_1(z)\} + z^{-1} \{x_2(z)\}$$

$$= z^{-1} \{3\} + z^{-1} \left\{ \frac{11z-5}{(z-1)(z-2)} \right\}$$

$$= 3\delta(n) + \sum_{i=1}^N \left[ (z-p_i) x_2(z) z^{p_i-1} \right]_{z=p_i} \quad \text{Using residue theorem.}$$

$$= 3S(n) + (z-1) \left. \frac{11z-5 \cdot z^{n-1}}{(z-1)(z-2)} \right|_{z=1} + (z-2) \cdot \left. \frac{11z-5}{(z-1)(z-2)} \cdot z^{n-1} \right|_{z=2}$$

$$= 3S(n) + \frac{11-5}{1-2} (1)^{n-1} + \frac{11 \times 2 - 5}{2-1} 2^{n-1}$$

$$\therefore x(n) = 3S(n) - 6u(n-1) + 17(2)^{n-1} u(n-1)$$

$$= 3S(n) + [-6 + 17(2)^{n-1}] u(n-1).$$

when  $n=0$ ,  $x(0) = 3-0+0 = 3$

when  $n=2$ ,  $x(2) = 0-6+17 \times 2^1 = 28$

when  $n=1$ ,  $x(1) = 0-6+17 \times 2^0 = 11$

when  $n=3$ ,  $x(3) = 0-6+17 \times 2^2 = 62$

$$\therefore x(n) = \{3, 11, 28, 62, 130, \dots\}$$

Method 2:

$$x(z) = \frac{3+2z^{-1}+z^{-2}}{1-3z^{-1}+2z^{-2}} = \frac{3z^{\gamma}+2z+1}{z^{\gamma}-3z+2}$$

$$\therefore \frac{x(z)}{z} = \frac{3z^{\gamma}+2z+1}{z(z-1)(z-2)}$$

$$\text{Let } \frac{x(z)}{z} = \frac{3z^{\gamma}+2z+1}{z(z-1)(z-2)} = \frac{A_1}{z} + \frac{A_2}{z-1} + \frac{A_3}{z-2}$$

$$A_1 = z \cdot \left. \frac{x(z)}{z} \right|_{z=0} = z \cdot \left. \frac{3z^{\gamma}+2z+1}{z(z-1)(z-2)} \right|_{z=0} = 0.5$$

$$A_2 = (z-1) \left. \frac{x(z)}{z} \right|_{z=1} = (z-1) \left. \frac{3z^{\gamma}+2z+1}{z(z-1)(z-2)} \right|_{z=1} = -6$$

$$A_3 = (z-2) \left. \frac{x(z)}{z} \right|_{z=2} = (z-2) \left. \frac{3z^{\gamma}+2z+1}{z(z-1)(z-2)} \right|_{z=2} = 8.5$$

$$\frac{x(z)}{z} = \frac{0.5}{z} - \frac{6}{z-1} + \frac{8.5}{z-2}$$

$$\therefore X(z) = 0.5 - 6 \cdot \frac{z}{z-1} + 8.5 \cdot \frac{z}{z-2}$$

On taking inverse z-transform

$$\begin{aligned} x(n) &= 0.5g(n) - 6.u(n) + 8.5(2)^n u(n) \\ &= 0.5s(n) + [-6 + 8.5(2)^n] u(n) \end{aligned}$$

$$\text{When } n=0, x(0) = 0.5 - 6 + 8.5 \times 2^0 = 3$$

$$n=1, x(1) = 10 - 6 + 8.5 \times 2^1 = 11$$

$$n=2, x(2) = 0 - 6 + 8.5 \times 2^2 = 28$$

$$n=3, x(3) = 0 - 6 + 8.5 \times 2^3 = 62$$

$$\therefore x(n) = \left\{ \begin{array}{l} 3, 11, 28, 62 \dots \\ \uparrow \end{array} \right\}$$

Method 3: Power Series Expansion method.

$$\begin{aligned} &\frac{3+11z^{-1}+28z^{-2}+62z^{-3}+130z^{-4}+\dots}{1-3z^{-1}+2z^{-2}} \\ &\quad \boxed{\begin{array}{r} z+2z^{-1}+z^{-2} \\ + 3-9z^{-1}+6z^{-2} \\ \hline \end{array}} \\ &\quad \boxed{\begin{array}{r} 11z^{-1}-5z^{-2} \\ + 1z^{-1}+33z^{-2}+22z^{-3} \\ \hline \end{array}} \\ &\quad \boxed{\begin{array}{r} 28z^{-2}-22z^{-3} \\ + 28z^{-2}-84z^{-3}+56z^{-4} \\ \hline \end{array}} \\ &\quad \boxed{\begin{array}{r} 62z^{-3}-56z^{-4} \\ + 62z^{-3}-166z^{-4}+124z^{-5} \\ \hline \end{array}} \\ &\quad \boxed{130z^{-4}+124z^{-5}} \\ &\quad \vdots \end{aligned}$$

$$\therefore X(z) = 3+11z^{-1}+28z^{-2}+62z^{-3}+130z^{-4}+\dots \rightarrow ①$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} = \dots + x(0) + x(1)z^{-1} + x(2)z^{-2} + \dots \rightarrow ②$$

On comparing ① & ②

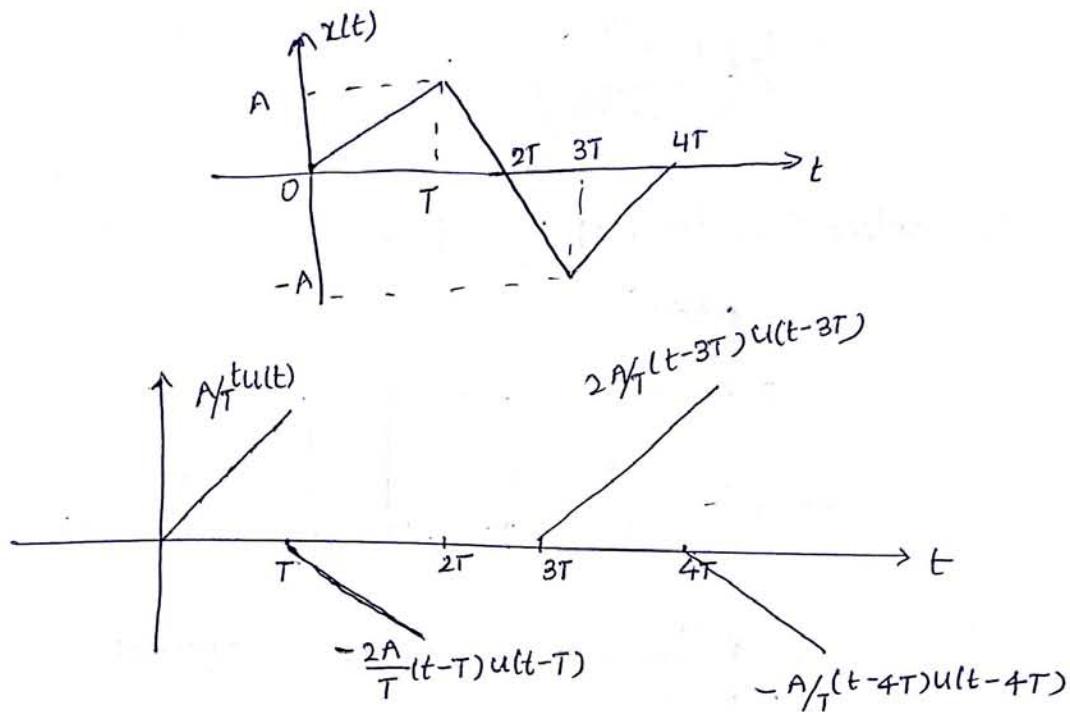
$$x(0) = 3, x(1) = 11, x(2) = 28, x(3) = 62$$

$$\therefore x(n) = \{3, 11, 28, 62 \dots\}$$

# Wave synthesis Using Laplace transform:

To express the function into singular functions and express it in synthesis

- Find the Laplace transform of the waveform.

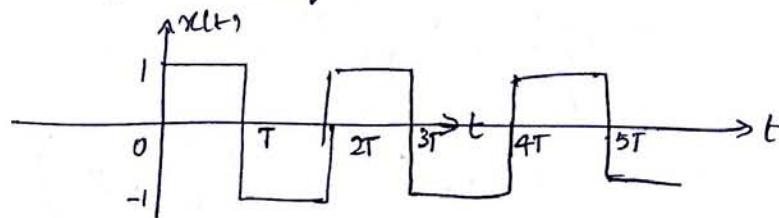


$$x(t) = \frac{A}{T} t u(t) - \frac{2A}{T} (t-T) u(t-T) + \frac{2A}{T} (t-3T) u(t-3T) - \frac{A}{T} (t-4T) u(t-4T)$$

Taking Laplace transform on both sides

$$\begin{aligned} X(s) &= \frac{A}{T} \frac{1}{s^2} - \frac{2A}{T} e^{-Ts} \frac{1}{s^2} + \frac{2A}{T} e^{-3Ts} \frac{1}{s^2} - \frac{A}{T} e^{-4Ts} \frac{1}{s^2} \\ &= \frac{A}{Ts^2} \left[ 1 - 2e^{-Ts} + 2e^{-3Ts} - e^{-4Ts} \right] \end{aligned}$$

- Find the Laplace transform

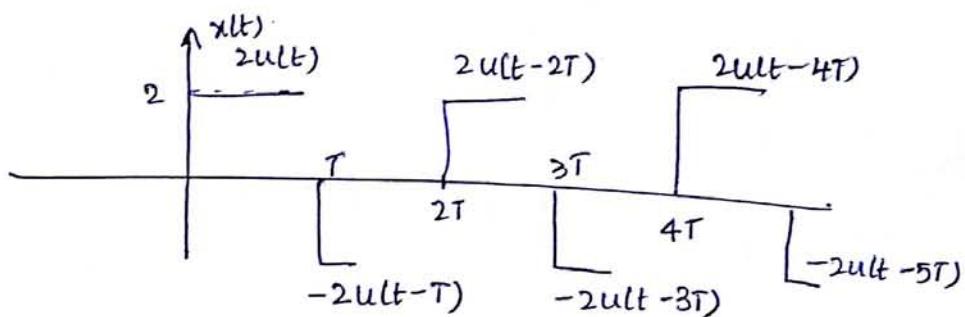
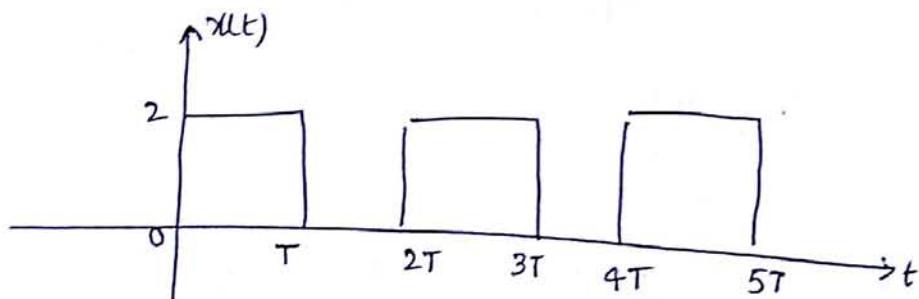


$$x(t) = u(t) - 2u(t-T) + 2u(t-2T) - 2u(t-3T) + 2u(t-4T) - 2u(t-5T).$$

$$X(s) = \frac{1}{s} - 2e^{-Ts} \frac{1}{s} + 2e^{-2Ts} \frac{1}{s} - 2e^{-3Ts} \frac{1}{s} + 2e^{-4Ts} \frac{1}{s} - 2e^{-5Ts} \frac{1}{s} + \dots$$

$$\begin{aligned}
 &= \frac{1}{s} \left[ 1 - 2e^{-Ts} (1 - e^{-Ts} + e^{-2Ts} - e^{-3Ts} + e^{-4Ts} + \dots) \right] \\
 &= \frac{1}{s} \left[ 1 - 2e^{-Ts} (1 + e^{-Ts})^{-1} \right] = \frac{1}{s} \left( 1 - \frac{2e^{-Ts}}{1 + e^{-Ts}} \right) = \frac{1}{s} \left( \frac{1 + e^{-Ts} - 2e^{-Ts}}{1 + e^{-Ts}} \right) \\
 &= \frac{1}{s} \left( \frac{1 - e^{-Ts}}{1 + e^{-Ts}} \right)
 \end{aligned}$$

③ Find the Laplace transform of waveform



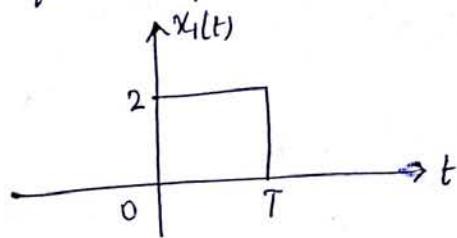
$$\begin{aligned}
 x(t) &= 2u(t) - 2u(t-T) + 2u(t-2T) - 2u(t-3T) + 2u(t-4T) - \\
 &\quad 2u(t-5T) \dots
 \end{aligned}$$

Taking Laplace transform

$$\begin{aligned}
 X(s) &= \frac{2}{s} - \frac{2e^{-Ts}}{s} + \frac{2e^{-2Ts}}{s} - \frac{2e^{-3Ts}}{s} + \frac{2e^{-4Ts}}{s} - \frac{2e^{-5Ts}}{s} \dots \\
 &= \frac{2}{s} \left[ 1 - e^{-Ts} + e^{-2Ts} - e^{-3Ts} + e^{-4Ts} - e^{-5Ts} \right] \\
 &= \frac{2}{s} \left[ 1 + e^{-Ts} \right]^{-1} = \frac{2}{s} \left( \frac{1}{1 + e^{-Ts}} \right).
 \end{aligned}$$

Another method

The given waveform is periodic with period  $2T$ .



periodicity property

$$X(s) = \frac{1}{1 - e^{-2Ts}} X_1(s)$$

$$x_1(t) = 2 [u(t) - u(t-T)]$$

$$X_1(s) = 2 \left( \frac{1}{s} - \frac{e^{-Ts}}{s} \right) = \frac{2}{s} (1 - e^{-Ts})$$

$$X(s) = \frac{1}{1 - e^{-2Ts}} \left( \frac{2}{s} \right) (1 - e^{-Ts})$$

$$= \frac{2}{s} \cdot \frac{1 - e^{-Ts}}{(1 + e^{-Ts})(1 - e^{-Ts})} = \frac{2}{s} \left( \frac{1}{1 + e^{-Ts}} \right)$$

periodicity property:

used in determining the transform of periodic time functions.

$$x(t) = x(t+nT), \text{ where } T \text{ is period}, n=0, 1, 2, \dots$$

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt.$$

$$X(s) = \int_0^T x(t) e^{-st} dt + \int_T^{2T} x(t) e^{-st} dt + \int_{2T}^{3T} x(t) e^{-st} dt + \dots + \int_{nT}^{(n+1)T} x(t) e^{-st} dt + \dots$$

$$= \int_0^T x(t) e^{-st} dt + e^{-sT} \int_0^T x(t+T) e^{-st} dt + \dots + e^{-nsT} \int_0^T x(t+nT) e^{-st} dt$$

$$x(t) = x(t+T) = x(t+2T) \dots$$

$$\begin{aligned}
 X(s) &= \int_0^T x(t) e^{-st} dt + e^{-sT} \int_0^T x(t) e^{-st} dt + \dots + e^{-nsT} \int_0^T x(t) e^{-st} dt + \dots \\
 &= [1 + e^{-sT} + e^{-2sT} + \dots + e^{-nsT}] \int_0^T x(t) e^{-st} dt \\
 &= [1 - e^{-sT}]^{-1} \int_0^T x(t) e^{-st} dt = \frac{1}{1 - e^{-sT}} X_1(s).
 \end{aligned}$$

where  $X_1(s) = \int_0^T x(t) e^{-st} dt$