Random Processes

(Stochastic processes)

Introduction:

Random process is conceptually an extension of random variable.

Random variable when expressed as function of time is called Random process.

Random process (Definition):

It is defined as collection / ensemble / family of time with probability measure associated.

Generally, Random processes is represented by $X(t, \lambda)$ (or) $X(t)$.

Interpretations:

(1) $X(t, \lambda)$ is a family of functions of time, where t & λ are variables.

(2) If λ is fixed, then Random process is function of time only, then it is

Single time function.

3) If 't' is fixed, the Random process is function of λ only and hence Random process will represent Random Variable of time 't'.

4) If both 't' & 'λ' are fixed, then $x(t, \lambda)$ is a number (or) constant.

Classification of Random Processes

There are 2 types of Random processes

1) Continous Random Processes
2) Discrete Random process

1) A Continous Random process is one in which random variable 'x' is continous ~~in 't'~~ and 't' can assume any value between t_1 and t_2.

2) A discrete Random process is one in which Random variable 'x' can assume only certain Specified values while 't' is continous.

The Random process furthured classified as

1) Deterministic Random process

2) Non - Deterministic Random Process

1) If ~~x~~ the future values of any Sample function can be predicted from the knowledge of past values, then it is called Deterministic Random process.

2) If the future values of any Sample function can be not be predicted from the knowledge of past values, then it is called Non-Deterministic Random process.

11/10/12 **Equal ~~x~~ Random Processes**

Two random process $x(t)$ & $y(t)$ are equal every where, if their respective Samples $x(t, \lambda i)$ and $y(t, \lambda i)$ are identical for every λi. ~~Two~~

Two Stochastic processes $x(t)$ & $y(t)$ are equal in mean Square sense iff

$$E\left[|x(t) - y(t)|^2\right] = 0$$

Statistics of Random Processes:

First Order Distribution & Density functions:

For any given time t_1; the distribution function associated with the random variable $x_1 = X(t_1)$ is given by $F_X(x_1;t_1) = P\{X(t_1) \le x_1\}$

this is called first order distribution function of Random process $X(t_1)$.

The derivative of first order distributive function with respect to x_1 is called 1st order density function ie;

$$f_X(x_1;t_1) = \frac{d}{dx_1}[F_X(x_1;t_1)]$$

Second Order joint distribution & density functions:

For two Random processes $x_1 = X(t_1)$ and $x_2 = X(t_2)$. the second order joint distribution function is given by

$$F_X(x_1, x_2; t_1, t_2) =$$

$$P\{X(t_1) \le x_1, X(t_2) \le x_2\}$$

The corresponding second order joint density function is given by

$$f_X(x_1, x_2; t_1, t_2) = \frac{\partial^2}{\partial x_1 \cancel{\partial y_2}} \left[F_X(x_1, x_2; t_1, t_2)\right]$$

$\partial x_1 \cdot \partial x_2$

N^{th} order joint distribution & density functions &

The above definitions of joint distributions and joint density functions of random processes can be extended to 'N' number of Random processes. They are respectively known as N^{th} order joint distribution function & N^{th} order joint density ~~funct~~ function.

The N^{th} order joint distribution function of N number of Random processes $X(t_1), X(t_2), \ldots X(t_N)$ is given by $F_X(x_1, x_2, \ldots x_n; t_1, t_2, \ldots t_N)$

$= P\{X(t_1) \leq x_1; X(t_2) \leq x_2; \ldots X(t_N) \leq x_N\}$

The corresponding n^{th} order joint density function can be written as

$$f_X(x_1, x_2, \ldots x_N; t_1, t_2, \ldots t_N) = \frac{\partial^N [F_X(x_1, x_2, \ldots x_N; t_1, t_2, \ldots t_N)]}{\partial x_1 \partial x_2 \ldots \partial x_N}$$

Statistical Independances

Two random processes $X(t)$ & $Y(t)$ are statistically independant if their random variable group $X(t); X(t_1), X(t_2), \ldots X(t_N)$ is independant of another group $Y(t); Y(t_1'), Y(t_2'), \ldots Y(t_N')$ for any choice of times $t_1, t_2, \ldots$ so on and $t_1', t_2', \ldots t_N'$.

Independance means the joint density be factorable by groups ie;

$$f_{XY}(x_1, x_2, \ldots, x_N, y_1, y_2, y_3, \ldots y_N; t_1, t_2, \ldots t_N; t_1', t_2', \ldots t_N')$$

$$= f_X(x_1, x_2, \ldots x_N; t_1, t_2, \ldots t_N) \cdot f_Y(y_1, y_2, \ldots y_N; t_1', t_2' \ldots t_N').$$

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Statistical properties of Random Processes

~~the random~~ process

Mean: The mean of the random process $x(t)$ = Expected value of the random process $x(t)$. It is denoted as

$$\overline{X(t)} = m(t) = E[X(t)]$$

$$= \int_{-\infty}^{\infty} x f_X(x;t)\,dx$$

where $f_X(x;t)$ is first order density function of random process $X(t)$

The mean of the random process is also called Ensemble average of $X(t)$

2) Auto Correlation :

Consider the random process $x(t)$ and Let x_1 & x_2 are two random variables define at times t_1 & t_2 respectively with joint density function

$$f_{x_1 x_2}(x_1, x_2; t_1, t_2).$$

$x(t)$ t_2 $x(t_2)$ t_1 $t_1 = x(t_1)$

The correlation of x_1 & x_2 is

$$E[x_1 \cdot x_2] = E[x(t_1) \cdot x(t_2)]$$

is called the autocorrelation function random process $x(t)$

It is denoted as

$$R_{x_1 x_2}(t_1, t_2) = E[x(t_1) \cdot x(t_2)]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_{x_1 x_2}(x_1, x_2; t_1, t_2)\, dx_1\, dx_2$$

3) Cross Correlation:

Consider the two random process $X(t)$ & $Y(t)$ defined at two different instants of time, becomes $X(t_1)$ & $Y(t_2)$ with joint density function,

$X_1 = X(t_1)$

$Y_2 = Y(t_2)$

$$= f_{XY}(x, y; t_1, t_2)$$

then the correlation of X & Y

$$\text{is} = E[XY] = E[X_1 \cdot Y_2]$$

$$= E[X(t_1) \cdot Y(t_2)]$$ is called cross correlation function of the random process $X(t)$ & $Y(t)$.

It is denoted as

$$R_{XY} = E[X(t_1) \cdot Y(t_2)]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy \, f_{XY}(x, y; t_1, t_2)\, dx\, dy$$

Stationary Random Processes

– A random process is said to be stationary if all it's statistical properties, such as mean, variance, moments donot change with time.

There are three types of stationary Random processes

1) I^{st} order Stationary Random process
2) II^{nd} order Stationary Random Process (WSS) wide or weak
3) N^{th} order Stationary Random Process.

II^{nd} order is also called as
wide-Sense stationary R.P /
(WSS)
weak-Sense stationary R.P
(WSS)

N^{th} order is Strict Sense stationary R.P
(SSS)

I^{st} order stationary R.Ps

Random process is said to be stationary to order one, or a I^{st} order stationary if its first order density function does not change with time or shift in time value.

If $x(t)$ is a random process

$$f_x(x_1; t_1) = f_x(x_1; t_1 + \Delta t)$$

where Δt is the shift in time value

$$\boxed{\overline{x(t)} = E[x(t)] = m(t) = \text{constant}}$$

From the above equation, we can say that $f_x(x_1; t_1)$ is independant of t_1, so the mean value of the process is constant. therefore the condition for 1^{st} order stationary process is it's mean value is constant at any time instant i.e;

$$\overline{x(t)} = E[x(t)] = m(t) = \text{Constant.}$$

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II^{nd} order stationary Random Process

~~A~~ Random process is stationary to order 2 ~~as~~ or 2nd order stationary if it's 2nd order joint density function does not change with time or shift in time value i.e $f_X(x_1, x_2; t_1, t_2)$

$= f_X(x_1, x_2; t_1+\Delta t, t_2+\Delta t) \;\forall\; t_1, t_2, \Delta t$

the condition for stationary ~~for~~ & 2nd order Random process is it's auto correlation function ~~depends~~ is ~~only on~~ τ function of τ & τ only, but independant of time t.

Consider the R.P $X(t)$ defined at two different instants of time ~~t_1~~ & t_2, then corresponding Random process are $X(t_1)$ & $X(t_2)$ respectively.

Then the auto Correlation function is

$$R_{XX}(t_1, t_2) = E[X(t_1).X(t_2)]$$

Let $\tau = t_2 - t_1$, then $R_{XX}(t_1, t_1+\tau) =$

$$E[X(t_1).X(t_1+\tau)] = R_{XX}(\tau)$$

$t_2 = t_1 + \tau$

$t_1 = t$

i.e; $R_{XX}(\tau)$ should be independant of time

Wide Sense Stationary: (weak Sense Stationary).

If a Random process $X(t)$ is a 2nd order stationary Random process, then it is called wide Sense stationary.

* The conditions for wide Sense stationary process are:

1) It's mean is constant

$$E[X(t)] = \text{constant}$$

2) It's autocorrelation function is function of τ only, but independant of time t i.e;

$$E[X(t).X(t+\tau)] = R_{XX}(\tau);$$ function of τ only, but independant of time t.

Jointly Wide Sense Stationary:

Consider two Random processes $X(t)$ & $Y(t)$ are said to be jointly (w.s.s) if it satisfies the following conditions:

1) $E[X(t)] = \text{constant}$

2) It's cross correlation function is a function of τ but independant of time t.

i.e; $E[X(t).Y(t+\tau)] = R_{XY}(\tau)$; function of τ independant of t.

The Cross Correlation function can be written as Consider two Random processes $x(t)$ and $y(t)$ defined at two different instants of time t_1 & t_2, then their Cross Correlation function can be written as

$$R_{xy}(t_1, t_2) = E[x(t_1) \cdot y(t_2)]$$

Let $\tau = t_2 - t_1$ & $t_1 = t$

$$R_{xy}(t_1, t_1+\tau) = E[x(t)\, y(t+\tau)] = R_{xy}(\tau)$$

3) N^{th} order Stationary processes

Random process is Said to be stationary to order N or N^{th} order stationary, if it's N^{th} order joint density function does not change with time or shift in time value.

Consider N number of Random processes $x(t_1), x(t_2), \ldots x(t_n)$ then

$$f_x(x_1, x_2, \ldots x_N; t_1, t_2, \ldots t_N)$$

$$= f_x(x_1, x_2, \ldots x_N; t_1+\Delta t, t_2+\Delta t, \ldots t_N+\Delta t)$$

$$\forall\ t_1, t_2, t_3, \ldots t_N$$

the N^{th} order stationary process is also called strict sense stationary Random process (S.S.S)

Note: All strict sense stationary processes are also a W.S.S but the converse is not true.

Time Averages of Random Processes:

Consider the Random process $x(t)$, Let $x(t)$ $X(t)$ be a sample function exist for all time in a given sample space. the average value of $x(t)$ $X(t)$ taken over all time is called time average of $x(t)$ ~~$X(t)$~~. It is also called mean value of $E[X(t)]$. It is expressed as $\bar{x}$

$$\bar{x} = A[x(t)] = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)\,dt$$

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Time Auto correlation function:

Consider the random process $X(t)$. $X(t_1)$ & $X(t_2)$ are two random processes defined at two different instants of time t_1 & t_2 respectively. Let $x(t_1)$ & $x(t_2)$ are sample functions of random processes $X(t_1)$ & $X(t_2)$.

The time average of product of $x(t_1)$ & $x(t_2)$ is called time auto correlation $x(t)$. i.e;

$$R_{xx}(t_1,t_2) = A\left[x(t_1) \cdot x(t_2)\right]$$

$t_2 = t_1+\tau$

& $t_1 = t$

Let $\tau = t_2 - t_1$

Let $t_1 = t$

$$R_{xx}(\tau) = A\left[x(t)\, x(t+\tau)\right]$$

$$= \underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T} x(t)\, x(t+\tau)\,dt$$

Time Cross Correlation functions:

Consider the random processes $X(t)$ & $Y(t)$.

Let $X(t_1)$ & $Y(t_2)$ are random process defined at t_1 & t_2. Let $x(t_1)$ & $y(t_2)$ are two sample functions of the random processes $X(t_1)$ & $Y(t_2)$ respectively. ~~The~~

The time average of product of two sample functions $x(t_1)$ & $y(t_2)$ is called Time cross correlation function of the random processes ~~x(t) & y(t)~~ X(t) & Y(t).

$$R_{xy}(t_1,t_2) = A[x(t_1) \cdot y(t_2)]$$

let $\tau = t_2 - t_1$

$t_1 = t$

$$R_{xy}(\tau) = A[x(t)\, y(t+\tau)]$$

$$= \underset{T\to\infty}{Lt} \frac{1}{2T} \int_{-T}^{T} x(t)\, y(t+\tau)\, dt$$

NOTE: Time cross Corelation function - $\square_{xy}$

Time auto Correlation function - $\square_{xx}$

Ergodic Theorem:

Ergodic Theorem states that for any given random process 'x(t)', all the time averages of Sample functions of X(t) = ensemble (or) statistical averages of random process X(t).

Ergodic process:

~~A~~ ~~pr~~ random process which satisfies the ergodic theorem is called ergodic process.

They are 2 types of Ergodic processes -

1) Mean Ergodic process

2) ~~Correlation~~ Auto Correlation Ergodic process

③ Cross Correlation Ergodic process

Mean Ergodic process:

A random process $x(t)$ is said to mean Ergodic (or) Ergodic in mean if the time average of any sample function $x(t)$ is equal to the statistical average $\bar{X}$.

$$E[X(t)] = \bar{X} = A(x(t)) = \bar{x}$$

Auto Correlation Ergodic process:

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The Random process $x(t)$ is said to be Time Auto Correlation in Ergodic (or) Ergodic in Auto correlation iff the time auto correlation function of any sample function $x(t)$ is equal to the statistical auto correlation function of Random process $x(t)$.

i.e; $A[x(t)\,x(t+\tau)] = E[X(t)\cdot X(t+\tau)]$

$$R_{xx}(\tau) = R_{XX}(\tau)$$

$$\square_{xx}(\tau) = R_{XX}(\tau)$$

Cross Correlation Ergodic process:

Two random processes $X(t)$ & $Y(t)$ are said to be cross correlation ergodic (or) ergodic in cross correlation iff the time cross correlation function of sample functions of $x(t)$ & $y(t)$ is $\boxed{\text{equal}}$ to the statistical cross correlation function of two random processes $X(t)$ & $Y(t)$.

i.e; $A[x(t)\,y(t+\tau)] = E[X(t)\,Y(t+\tau)]$

$$\Rightarrow R_{xy}(\tau) = R_{XY}(\tau)$$

$$\Rightarrow \square_{xy}(\tau) = R_{XY}(\tau)$$

Auto Correlation and it's properties:

Definition: Consider the random process $X(t)$ defined at two different instants of t_1 & t_2 results in two random processes

$x(t_1)$ & $x(t_2)$. The auto correlation function of Random process $x(t)$ can be written as

$$R_{xx}(t_1,t_2) = E[x(t_1)\cdot x(t_2)]$$

Let $\tau = t_2 - t_1$

$$= E[x(t)\, x(t+\tau)]$$

$$= R_{xx}(\tau)$$

Properties of Auto Correlation function:

1) The mean square value of Random process $x(t)$ is $E[(x(t))^2] = R_{xx}(0)$, It is Equal to the power of the process

Proof: we know that

$$R_{xx}(\tau) = E[x(t)\, x(t+\tau)]$$

if $\tau = 0$

$$R_{xx}(\tau) = E[x(t)\cdot x(t)]$$

$$= E[(x(t))^2]$$

2) The auto correlation function is maximum at the origin i.e; $|R_{xx}(\tau)| \leq R_{xx}(0)$

3) $R_{xx}(\tau)$ is even function of τ i.e; $R_{xx}(-\tau) = R_{xx}(\tau)$

proof: We know that

$$R_{xx}(\tau) = E[x(t)\cdot x(t+\tau)]$$

Let $\tau = -\tau$

$$R_{xx}(-\tau) = E[x(t)\, x(t-\tau)]$$

Let $t-\tau = u$

$$R_{xx}(-\tau) = E[x(u)\cdot x(u+\tau)]$$

$$= R_{xx}(\tau)$$

4) If $x(t)$ has a periodic, then its auto correlation function is also periodic.

5) If the random process $z(t)$ is a sum of two random processes $x(t)$ & $y(t)$ i.e; $z(t) = x(t) + y(t)$ then

$$R_{zz}(\tau) = R_{xx}(\tau) + R_{xy}(\tau) + R_{yx}(\tau) + R_{yy}(\tau)$$

proof: Consider

$$R_{zz}(\tau) = E[z(t)\cdot z(t+\tau)]$$

$$= E[(x(t) + y(t))(x(t+\tau) + y(t+\tau))]$$

$$= E[x(t)\cdot x(t+\tau) + x(t)y(t+\tau) + y(t)x(t+\tau) + y(t)y(t+\tau)]$$

$$= R_{xx}(\tau) + R_{xy}(\tau) + R_{yx}(\tau) + R_{yy}(\tau)$$

Cross Correlation function and it's properties

Consider the two random processes $x(t)$ & $y(t)$ and it is defined at two different instants of time 't_1' & 't_2' resulting a random processes $x(t_1)$ & $y(t_2)$. The cross correlation function between Random proesses $x(t)$ & $y(t)$ can be written as

$$R_{xy}(t_1, t_2) = E[x(t_1)\cdot x(t_2)]$$

Let $\tau = t_2 - t_1$

$t_1 = t$

$$= E[x(t)\cdot y(t+\tau)]$$

$$= R_{xy}(\tau)$$

Properties

1) $R_{xy}(\tau) = R_{yx}(-\tau)$

proof: We know that

$$R_{xy}(t_1, t_2) = E[X(t_1)\, X(t_2)]$$

Let $\tau = t_2 - t_1$

$t_1 = t$

$$R_{xy}(\tau) = E[X(t) \cdot Y(t+\tau)]$$

Let $\tau = -\tau$

$$= E[X(t) \cdot Y(t-\tau)]$$

Let $t - \tau = u$

$$= E[X(u+\tau) \cdot Y(u)]$$

$$= R_{yx}(\tau)$$

2) If $R_{xx}(\tau)$ & $R_{yy}(\tau)$ are the auto correlation functions of $X(t)$ & $Y(t)$ respectively. then the cross correlation satisfies the inequality

$$|R_{xy}(\tau)| \leq \sqrt{R_{xx}(0) \cdot R_{yy}(0)}$$

3) If two Random processes $X(t)$ & $Y(t)$ are statistically independant and are atleast wide sense stationary then $R_{XY}(\tau) = \bar{X}\bar{Y}$.

~~Sol~~ proof: we know that

$$R_{XY}(\tau) = E[X(\tau)\, Y(t+\tau)]$$

Independant

$$= E[X(\tau)] . E[Y(t+\tau)]$$

$$= \bar{X} . \bar{Y}$$

4) A Random process $Y(t)$ is given as

$$Y(t) = X(t) \cos(\omega t + \theta)$$

where $X(t)$ is a Wide Sense Stationary random process, ω is a constant and θ is a random phase independent of $X(t)$, uniformly distributed on $(-\pi, \pi)$ find out

ⓐ $E[Y(t)]$ ⓑ $R_{YY}(\tau)$

ⓐ $E[Y(t)]$

Sol: Given, $Y(t) = X(t) \cos(\omega t + \theta)$

Since the phase 'θ' is uniformly distributed on $(-\pi, \pi)$, then the density fn. is

$$f_\theta(\theta) = \begin{cases} \frac{1}{2\pi}, & -\pi \le \theta \le \pi \\ 0, & \text{elsewhere} \end{cases}$$

Mean value of $E[Y(t)] = E[Y(t)]$

$= E[X(t)\cos(\omega t+\theta)]$

since θ & $X(t)$ are independant

$= E[X(t)] \cdot E[\cos(\omega t+\theta)]$

$$E[\cos(\omega t+\theta)] = \int_{-\pi}^{\pi} \cos(\omega t+\theta) f_\theta(\theta)\, d\theta$$

$$= \int_{-\pi}^{\pi} \frac{\cos(\omega t+\theta)}{2\pi}\, d\theta$$

$$= \frac{1}{2\pi}\left[\sin(\omega t+\theta)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left[\sin(\omega t+\pi) - \sin(\omega t-\pi)\right]$$

$$= \frac{1}{2\pi}\left[\sin(\omega t) + \sin(\omega t)\right]$$

$$= 0.$$

$\therefore\ E[Y(t)] = E[X(t)] \cdot 0$

$= 0$

ⓑ $R_{YY}(\tau) = E[Y(t)\,Y(t+\tau)]$

$$= E\left[X(t)\cos(\omega t+\theta) \cdot X(t+\tau)\cos(\omega(t+\tau)+\theta)\right]$$

$$= E\left[x(t)\cos(\omega t+\theta)\cdot x(t+\tau)\cos(\omega(t+\tau)+\theta)\right]$$

$$= E\left[\underbrace{x(t)x(t+\tau)}_{R_{XX}(\tau)}\right]E\left[\cos(\omega t+\theta)\cos(\omega(t+\tau)+\theta)\right]$$

$$= \int_{-\pi}^{\pi}\frac{\cos(\omega t+\theta)\cos(\omega(t+\tau)+\theta)}{2\pi}\,d\theta$$

$$= \int_{-\pi}^{\pi}\frac{\cos(2\omega t+\omega\tau+2\theta)+\cos(\omega\tau)}{4\pi}\,d\theta$$

$2\cos A\cos B$

$\cos(A+B)+\cos(A-B)$

$$= \frac{\cos(2\omega t+\omega\tau)-\cos(2\omega t+\omega\tau)}{4\pi} + \frac{\cos(\omega\tau)\pi+\cos(\omega\tau)\pi}{4\pi}$$

$$= 0+\frac{\cos(\omega\tau)\times \cancel{2}}{\cancel{4}\,2}$$

$= \frac{1}{2}R_{xx}(\tau)\cos\omega\tau$, it is a fn. of τ only but independant of time 't'.

2) A random process is given as $x(t)=At$ where ~~an~~ A is an uniformly distributed random variable on (0,2) find whether $x(t)$ is WSS (or) not. The conditions for WSS are ① $E[x(t)]=$ const. ② $R_{xx}(\tau)=$ function of τ only but independant of time

Sol: ① $E[X(t)] = \int_0^2 \frac{At}{2}\, dA$ $\int_0^2 x(t) f_A(A)\, dA$

$= \frac{1}{2} t \left[\frac{A^2}{2}\right]_0^2$

$dA = \frac{1}{2},\ 0 \le A \le 2$
$= 0,$ otherwise

$= \frac{t}{2}\left[\frac{4}{2}\right]$

$= t$

② $R_{XX}(\tau) = E[X(t) \cdot X(t+\tau)]$

$f_A(A) = 1/2$

$= E[At \cdot A(t+\tau)]$

$= \int_0^2 \frac{At \cdot (At + A\tau)}{2}\, dA$

$= \int_0^2 \frac{A^2 t^2}{2}\, dA + \int_0^2 \frac{A^2 t \tau}{2}\, dA$

$= \left(\frac{A^3}{3}\right)_0^2 \left[\frac{t^2}{2}\right] + \left(\frac{A^3}{3}\right)_0^2 t\tau$

$= \frac{8}{3}\frac{t^2}{2} + \frac{8}{3}\frac{t\tau}{2}$

$= \frac{4t^2}{3} + \frac{4t\tau}{3}$

∴ the above two conditions are not satisfied x(t) is not a wide sense stationary process.

3) A Random process is described by $x(t) = A$ where A is a random variable distributed on $(0,1)$.
(i) classify the process.

Sol

$$E[x(t)] = \int_0^1 A \, dA \qquad \int_0^1 x(t) f_A(A) \, dA$$

$f_A(A) = \frac{1}{1} = 1$

$0 \le A \le 1$

$$= \left[\frac{A^2}{2}\right]_0^1$$

$$= \frac{1}{2}.$$

$$R_{XX}(\tau) = E[x(t) \cdot x(t+\tau)]$$

$$= E[A \cdot A]$$

$$= \int_0^1 A^2 \, dA$$

$$= \left[\frac{A^3}{3}\right]_0^1$$

$$= \frac{1}{3}.$$

It is not a wide sense stationary.

4) Consider a random process $X(t) = A\sin(\omega t + \theta)$ where A is a constant and 'θ' is uniformly distributed over $-\pi$ to π. Check whether $X(t)$ is a W.S.S. or not.

Sol

$$E[X(t)] = \int_{-\pi}^{\pi} \frac{A\sin(\omega t + \theta)}{2\pi} \, d\theta$$

$$= \frac{-A\left[\cos(\omega t + \theta)\right]_{-\pi}^{\pi}}{2\pi}$$

$$= \frac{A\left[-\cos(\omega t) + \cos(\omega t)\right]}{2\pi}$$

$$= 0.$$

$$R_{xx}(\tau) = E\left[x(t)\, x(t+\tau)\right]$$

$$= \int_{-\pi}^{\pi} \frac{2\sin(2\omega t + \omega\tau + 2\theta) + \sin(\omega\tau)}{4\pi}\, d\theta$$

$$= \frac{\left[\sin(2\omega t + 2\theta + \omega\tau)\right]_{-\pi}^{\pi}}{4\pi}\, d\theta +$$

$$\frac{\left(\sin(\omega\tau)\right]_{-\pi}^{\pi}\, d\theta}{4\pi}$$

$$= \frac{+\cos\left[2\omega t + \omega\tau\right]}{4\pi} \mp \frac{\left[\cos(2\omega t + \omega\tau)\right]}{4\pi}$$

$$+ 0 - \frac{\cos(\omega\tau)\left[\pi + \pi\right]}{4\pi}$$

$$= 0 + \left[-\cos(\omega\tau)\right]$$

$$R_{xx}\tau = \frac{-1}{2} R_{xx}(\tau) \cos(\omega\tau)$$

∴ the given function is a wss process.

2) prove that the random process $x(t) = A\cos(\omega t + \theta)$ is a W.S.S. & if it is assumed that 'ω' is a const. and 'θ' is uniformly distributed variable in the interval $(0, 2\pi)$.

Sol

$$\Rightarrow f_\theta(\theta) = \frac{1}{2\pi}$$

$$\Rightarrow E[X(t)] = \int_0^{2\pi} \frac{A\cos(\omega t+\theta)}{2\pi} d\theta$$

$$= \frac{A\left[\sin(\omega t+\theta)\right]_0^{2\pi}}{2\pi}$$

$$= \frac{A\left[-\sin(\omega t)+\sin\omega t\right]}{2\pi}$$

$$= 0.$$

$$\Rightarrow R_{XX}(\tau) = E\left[X(t)\cdot X(t+\tau)\right]$$

$$= R_{XX}(\tau)$$

$$= A\int_0^{2\pi} \frac{\sin(2\omega t+\omega\tau+2\theta)+\cos(\omega\tau)}{2\pi}$$

$$= \frac{A}{2\pi}(0) + \frac{A\cos(\omega\tau)[2\pi]}{2\pi \cdot 2}$$

$$= \frac{A}{2}\cos(\omega\tau)$$

$$\therefore R_{XX}(\tau) = \frac{A}{2}\cos(\omega\tau)$$

6) A stationary ergodic process have auto correlation function with periodic components is $R_{XX}(\tau) = 25 + \frac{4}{1+6\tau^2}$. Find the mean & variance of $X(t)$.

sol $E(x(t)) = \overline{x}^2$ for

$\underset{(\tau)\to\infty}{Lt}\ R_{xx}(\tau) = \overline{x}^2$

$\Rightarrow \boxed{\overline{X} = \sqrt{25} = 5}$

$\sigma_x^2 = E[X(t)^2] - (E[x(t)])^2$

$\underset{(\tau)\to\infty}{Lt}\ R_{xx}(0) = 0$

$E[X(t)^2] = \sigma_x^2 + (E[x(t)])^2$

$25 + 4 = \sigma_x^2 + 25$

$\boxed{\sigma_x^2 = 4}$

20/10/14

Covariance functions for Random processess

Auto Covariance function: the auto covariance function is a measure of dependance between two Random variables of the Random process $x(t)$.

The Same concept of Random variables can be extended to Random processes.

Consider two Random processes $x(t)$ and $x(t+\tau)$ defined at two intervals of time t_1 & t_2.

Then the auto covariance function can be expressed as $C_{XX}(t,t+\tau)$

$$C_{XX}(t,t+\tau) = E\big[[X(t) - E(X(t))][X(t+\tau) - E(X(t+\tau))]\big]$$

$E[x(t) \cdot x(t+\tau)] - E[x(t)]\, E[x(t+\tau)] - E[x(t)]\, E[x(t+\tau)] +$

$$= R_{XX}(t,t+\tau) - X(t)\cancel{E[X(t+\tau)]}$$
$$- E[X(t)]\,X(t+\tau) + \cancel{E(X(t))X(t+\tau)}$$

$$= R_{XX}(t,t+\tau) - E[X(t)]\,E[X(t+\tau)]$$

NOTE 1: If $X(t)$ is atleast w.s.s. Random process, then it becomes

$$C_{XX}(\tau) = R_{XX}(\tau) - \bar{X}^2$$

NOTE 2: At $\tau = 0$, $C_{XX}(0) = R_{XX}(0) - \bar{X}^2$

$$= E(X^2) - (E(X))^2$$

$$= \sigma_X^2.$$

$\therefore$ At $\tau = 0$, auto covariance function becomes the variance of the Random process.

Cross Covariance function:

If the two Random processes $X(t)$ & $Y(t)$ have random variables at time t & $t+\tau$, then the Cross Covariance of Random processes is written as

$$C_{xy}(t,t+\tau) = E\left[\left[X(t) - E[X(t)]\right]\left[Y(t+\tau) - E[Y(t+\tau)]\right]\right]$$

$$C_{xy}(t,t+\tau) = R_{xy}(t,t+\tau) - E[X(t)] \cdot E[Y(t+\tau)]$$

Case 1: If the random processes are atleast wide Sense stationary, then the above equation becomes

$$C_{xy}(\tau) = R_{xy}(\tau) - \bar{X} \cdot \bar{Y}$$

Case 2: If two Random processes X & Y are uncorrelated, then

$$\cancel{C_{xy}(\tau) = R_{xy}(\tau)}\ C_{xy}(t, t+\tau) = 0$$

$$\therefore R_{xy}(t,t+\tau) = E[X(t)] \cdot E[Y(t+\tau)]$$

This is the condition for two random processes are statistically independant.

∴ The independant Random processes are uncorrelated.

1) For a given Random process $x(t)$, the mean value is $\bar{x} = 6$ and auto correlation is

$R_{xx}(\tau) = 36 + 25e^{-|\tau|}$

ⓐ Avg. power of the process.

ⓑ Variance of $x(t)$.

~~Sol~~ NOTE: Mean Square value of $x(t)$ is $E[x^2(t)] = R_{xx}(0)$, it is equal to the power of the process $x(t)$.

Sol:

Average power is $E[x^2(t)] = R_{xx}(0)$

$\Rightarrow R_{xx}(\tau) = 36 + 25e^{-|\tau|}$

$\Rightarrow \boxed{R_{xx}(0) = 61}$

Variance of $x(t) \Rightarrow \sigma_x^2 = E[x(t)^2] - (E[x])^2$

$= 61 - 36$

$= 25.$

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2) The auto correlation function of stationary random processes $X(t)$ is given by $R_{XX}(\tau) = 36 + \frac{16}{1+8\tau^2}$. Find mean, mean square & variance of the process.

Sol) NOTE: If a random process $X(t)$ has non-zero mean value i.e., $E[X(t)] \neq 0$ and ergodic with no periodic components is

$$\underset{|\tau|\to\infty}{Lt} R_{XX}(\tau) = \overline{X}^2$$

mean $\Rightarrow$ $\overline{X}^2 = \underset{\tau\to\infty}{Lt} R_{XX}(\tau) = 36 + \frac{16}{1+8\tau^2}$

$$\overline{X}^2 = 36$$

$$\boxed{\overline{X} = 6}$$

Mean Square $\Rightarrow$ $E[X^2(t)] = \underset{\tau\to\infty}{Lt} R_{XX}(0) = 52$

$$\boxed{E[X^2(t)] = 52}$$

variance $\Rightarrow$ $\sigma_X^2 = E[X^2(t)] - (E[X(t)])^2$

$$= 52 - 36$$

$$\boxed{\sigma_X^2 = 16}$$

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3) for a stationary ergodic random process with no periodic components, the auto correlation function is given as $R_{xx}(\tau) = 36 + \frac{5}{1+7\tau^2}$. Find the mean & variance of Random process.

Sol) Mean $= \underset{\tau\to\infty}{Lt} \; R_{xx}(\tau) = \bar{X}^2$

$$\bar{X}^2 = 36$$

$$\boxed{\bar{X} = 6}$$

variance $= \underset{R\to\infty}{Lt} \; R_{xx}(0) = E[x^2(t)]$

$$\sigma_x^2 = E[x^2(t)] - (E[x(t)])^2$$

$$= 51 - 36$$

$$\boxed{\sigma_x^2 = 5}$$

4) If $x(t)$ is a random process with mean '3' and auto correlation function is $9 + 4e^{-0.2\tau}$ find the Average power ~~mean~~ and variance of $x(t)$.

Sol) Average power $\rightarrow E[x^2(t)] = 13$

variance $\rightarrow \sigma_x^2 = 13 - 9 = 4$.

A stationary process has an auto correlation function is given by $R(\tau) = \frac{25\tau^2 + 36}{6.25\tau^2 + 4}$

Find the mean value, mean saqare & variance of $x(t)$?

Sol) mean value $\Rightarrow \bar{x}^2 \Rightarrow$ 2

Mean square $\Rightarrow$ + 9

variance $\Rightarrow$ 5

1) The collection of all the Sample functions is referred as <u>ensemble</u>.

If $R_{xy} = 0$, then x & y are <u>independant & orthogonal</u>

2) If $F_X(x_1, x_2; t_1, t_2)$ is referred to as <u>Second order joint distribution function</u> then the corresponding joint density function is $\frac{\partial^2 (f_X(x_1, x_2; x_1, x_2))}{\partial x_1 \partial x_2}$

3) If the future values of Sample function cannot be predicted based on its past values, the process is referred to as <u>non deterministic (or) Unpredictable process</u>.

5) Two random processes are defined as $x(t) = A\cos(\omega t+\theta)$ and $y(t) = B\sin(\omega t+\theta)$ where 'θ' is an uniform random variable on $(0, 2\pi)$; A, B & ω are constants.

(a) Find out correlation function $R_{xy}(t, t+\tau)$

(b) verify whether $x(t), y(t)$ are jointly W.S.S (cond)

Sol: $\because \theta$ is uniformly distributed over $(0, 2\pi)$

$$f_\theta(\theta) = \begin{cases} 1/2\pi & , 0 \le \theta \le 2\pi \\ 0 & , \text{elsewhere} \end{cases}$$

Conditions for Jointly W.S.S P are:

(i) $E[x(t)] = \text{Const}$ (ii) $E[y(t)] = \text{Const}$

(iii) $R_{xy}(t, t+\tau)$ is function of 'τ' only but independant of time 't'.

(a) $$R_{xy}(t, t+\tau) = E[x(t) \cdot y(t+\tau)]$$

$$= E[A\cos(\omega t+\theta) \cdot B\sin(\omega(t+\tau)+\theta)]$$

$$= \frac{AB}{2} E[2\cos(\omega t+\theta) \cdot \sin(\omega(t+\tau)+\theta)]$$

$$= \frac{AB}{2} E\left[\sin\left(\frac{2\omega t+\omega\tau+2\theta}{2}\right) + \sin\left(\frac{\omega\tau}{2}\right)\right]$$

$$= \frac{AB}{2} E\left[\sin\left(\frac{2\omega t+\omega\tau+2\theta}{2}\right)\right] + \frac{AB}{2} E\left[\sin\left(\frac{\omega\tau}{2}\right)\right]$$

$$= 0 + \frac{AB}{2}\sin(\omega\tau),$$

$$= \frac{AB}{2}\sin(\omega\tau);$$ function of 'τ' only and independant of time.

(b) $$E[X(t)] = E[A\cos(\omega t+\theta)]$$

$$= A\int_0^{2\pi} \frac{\cos(\omega t+\theta)}{2\pi}\, d\theta$$

$$= \frac{A}{2\pi}\left[-\sin(\omega t+\theta)\right]_0^{2\pi}$$

$$= 0.$$

$$E[Y(t)] = E[A\sin(\omega t+\theta)]$$

$$= A\int_0^{2\pi} \frac{\sin(\omega t+\theta)}{2\pi}\, d\theta$$

$$= \frac{A}{2\pi}\left[\cos(\omega t+\theta)\right]_0^{2\pi}$$

$$= 0.$$

$\therefore$ Above three Conditions are verified, therefore the given random processes jointly w.s.s. processes.

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6) If a Random process $x(t) = A\cos\omega t + B\sin\omega t$ where A & B are uncorrelated zero mean random variable and variance σ^2 find ⓐ Auto Correlation ⓑ Show that $x(t)$ is W.S.S.

Sol) Given that,

A & B are zero mean random variables

where $E[A]$ & $E[B] = 0$ & $E[AB] = 0$, $\sigma_A^2 = \sigma^2 = E[A^2]$

$\sigma_B^2 = \sigma^2 = E[B^2]$

ⓐ Auto Correlation:

$$R_{xx}(\tau) = E[x(t)\,x(t+\tau)]$$

$$= E[(A\cos\omega t + B\sin\omega t).(A\cos\omega(t+\tau) + B\sin\omega(t+\tau))]$$

$$R_{xx}(\tau) = E[A^2\cos\omega(t+\tau)\cos\omega t + AB\cos\omega t\sin\omega(t+\tau) + AB\sin\omega t\cos\omega(t+\tau) + B^2\sin\omega t\sin\omega(t+\tau)]$$

$$= E[A^2(\cos\omega t\cos\tau + \sin\omega t\sin\omega\tau)\cos\omega t + 0 + 0 + B^2(\sin\omega t(\sin\omega t\cos\omega\tau + \cos\omega t\sin\omega\tau)]$$

$$= E[A^2(\cos^2\omega t\cos\omega\tau) + A^2(\cos\omega t\sin\omega t\sin\omega\tau) + B^2(\sin\omega^2 t\cos\omega\tau + B^2(\sin\omega t\cos\omega t\sin\omega\tau)]$$

$= E[A^2(\cos^2\omega t \cos\omega\tau) + 0 + 0 + B^2(\sin^2\omega t \cos\omega\tau)]$

$= E[\cancel{A^2}\, A^2\cos^2\omega t \cos\omega\tau] + E[B^2\sin^2\omega t\cos\omega\tau]$

$\boxed{R_{XX}(\tau) = \sigma^2 \cos\omega\tau}$

7) If $Y_1(t) = X_1\cos\omega t + X_2\sin\omega t$ and $Y_2(t) = X_1\sin\omega t + X_2\cos\omega t$, where X_1 & X_2 are '0' mean independant Random variable with unity variance. Show that the Random process $Y_1(t)$ & $Y_2(t)$ are individually w.s.s but not jointly w.s.s.

Sol) Given, $E[X1] = E[X2] = 0$; $E[X1] \cdot E[X2] = 0$

$\sigma_{X_1}^2 = \sigma_{X_2}^2 = 1$

$E[Y1] = E[X_1\cos\omega t + X_2\sin\omega t]$

$E[Y1] = 0$

$E[Y2] = E[X_1\sin\omega t + X_2\cos\omega t]$

$E[Y2] = 0$

$E[Y_1(t)\cdot Y_2(t+\tau)] = R_{Y_1Y_2}$

$R_{Y_1Y_2} = E[(X_1\cos\omega t + X_2\sin\omega t)(X_1\sin\omega(t+\tau) + X_2\cos\omega(t+\tau))]$

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$$E[Y_1Y_2] = E[X_1^2 \cos\omega t \sin\omega(t+\tau) + X_2^2 \sin\omega t \cos\omega(t+\tau)]$$

$$E[Y_1Y_2] = E[X_1^2 \sin(\omega t + \omega t + \omega\tau)] + E(X_2^2 \cos(\omega t + \omega t + \omega\tau))$$

$$E[Y_1Y_2] = E[X_1X_2] \cos\omega\tau$$

$$E[Y_1Y_2] = \cos(\omega t + \tau).$$

$\therefore Y_1(t)$ & $Y_2(t)$ are not jointly w.s.s but individually ~~is~~. w.s.s

4) Let $X(t)$ be a Randomprocess which is w.s.s then $\underline{E[X(t)]}$ = const and $\underline{R_{xx}(\tau)}$ = Const.

5) If a process is stationary to all the orders $n = 1, 2, 3, \dots N$ then $x_i = X(t_i)$ where $i = 1, 2, \dots N$ is called <u>strict sense stationary process</u>.