Unit 1 : Signal Analysis

⇒ We use electrical signals now-a-days (3×10^8 m/sec)

⇒ Signal is a time varying physical phenomena/quantity

⇒ Represented as $f(t)$

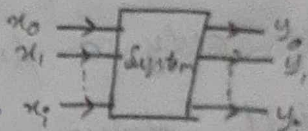
⇒ Ex: Speech, Music, Video, Image etc.

⇒ $S = f(x_i) = f(x_1, x_2, \cdots x_i)$ If $i=1$, 1-D signal (Speech)

If $i=2$, 2-D signal (Picture)

System :

⇒ Is a physical device / Hard ware / soft ware that performs some operations / responds to applied input signals to produce one or more output signals.



Input = excitation , Output = response.

Signal Processing :

⇒ Process the signals using systems to

(1) modify, analyse the signal

(2) To extract additional info

(3) To remove noise, interferance

(4) To obtain spectrum of signal

(5) To tranform it into more suitable form

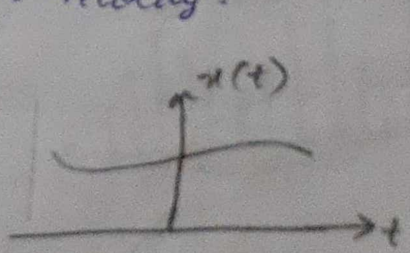
(6) We get processed signals after that.

Ex: Filter in communication system

Radar system – knowing past velocity, estimating the future velocity.

Types of Signals :

Continuous Time Signals (CTS)



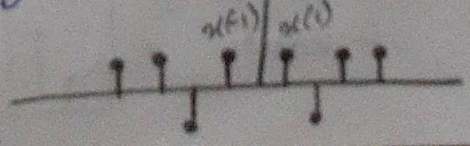
$x(t)$ varies continuously w.r.t, t

Ex: Voltage, Current, P, T, velocity

Discrete Time Signals (DTS)

$x(n)$ Varies discretely

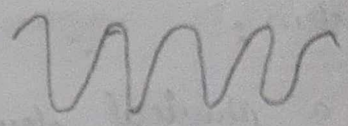
Ex: Stock Market



- Signals
  - Time axis
    - ~~Analog to CST~~ CTS
    - ~~Digital~~ ~~DS~~ DTS
  - Function value axis
    - Analog
    - Digital

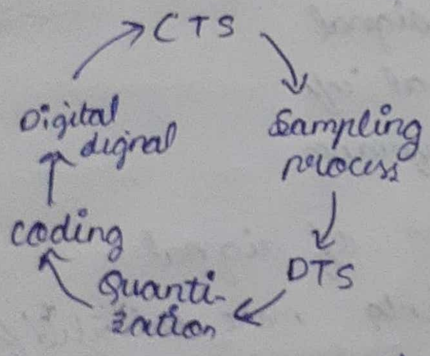
=> DTS can be processed by modern digital computers and digital signal processors.

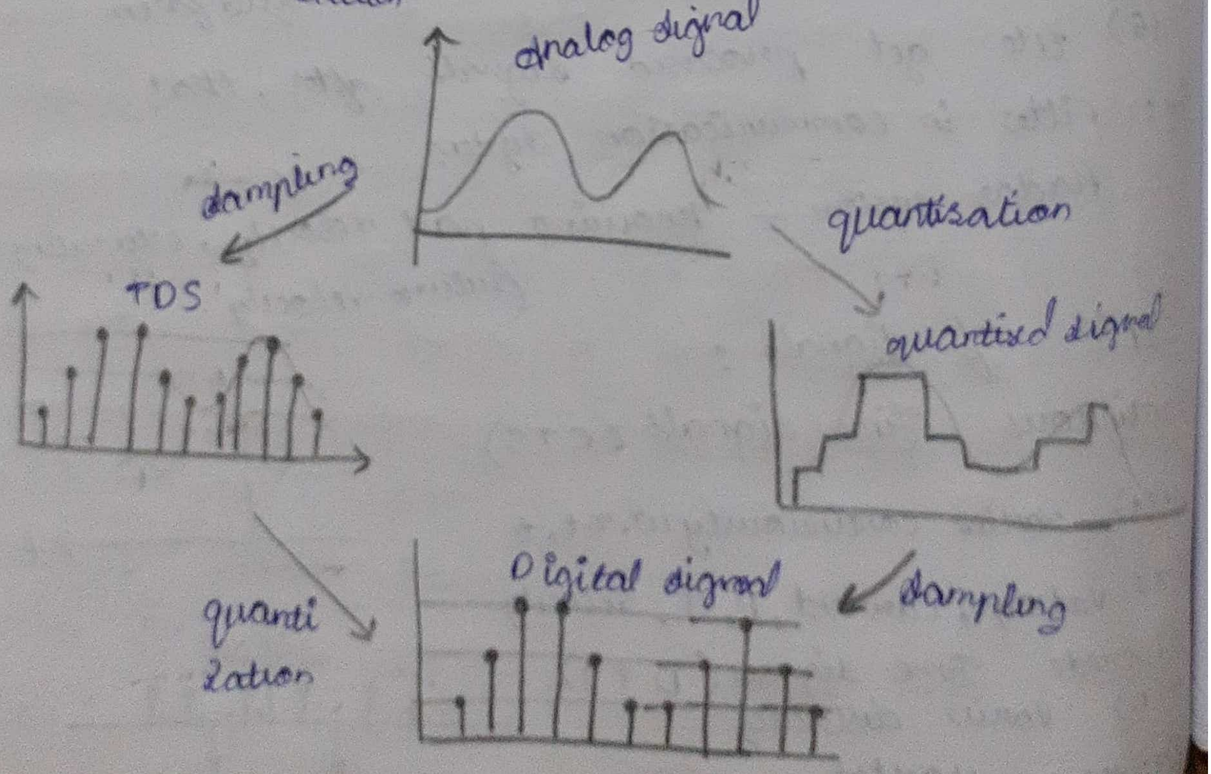
Analog signal: If a continuous-time signal $x(t)$ can take on any value in the interval $(-\infty$ to $\infty)$



Digital signal: If a DTS $x(n)$ can take only a finite numbers of distinct values.

Convert CTS to ~~DTS~~ Digital signal:





Transformations of the independent Variable:

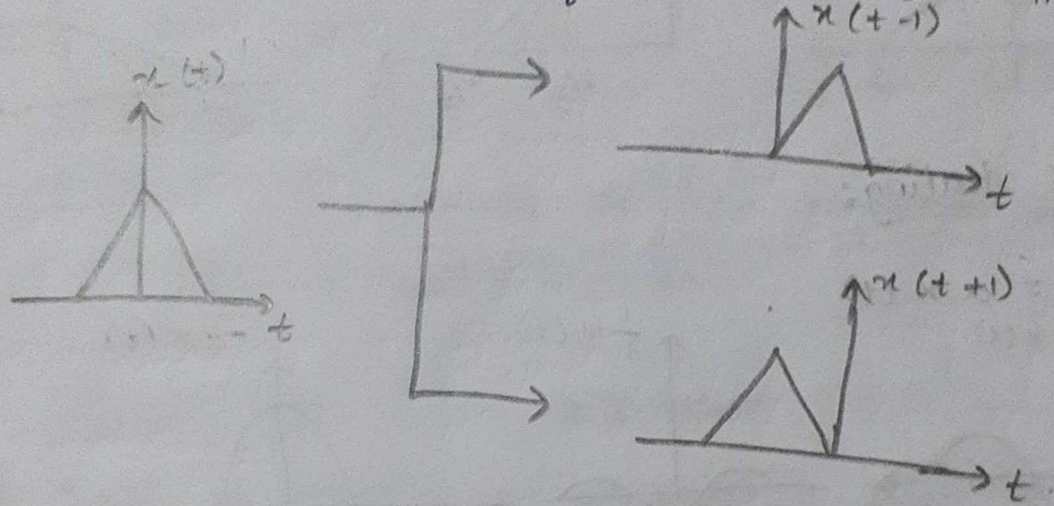
1. **Time shifting :**

=> It maps input (x) to output (y) as $y(t) = x(t - t_0)$

=> shifts signal to left or right.

=> $t_0 > 0 \rightarrow$ shifted to right (delay in time)

=> $t_0 < 0 \rightarrow$ " " left (advanced " ")

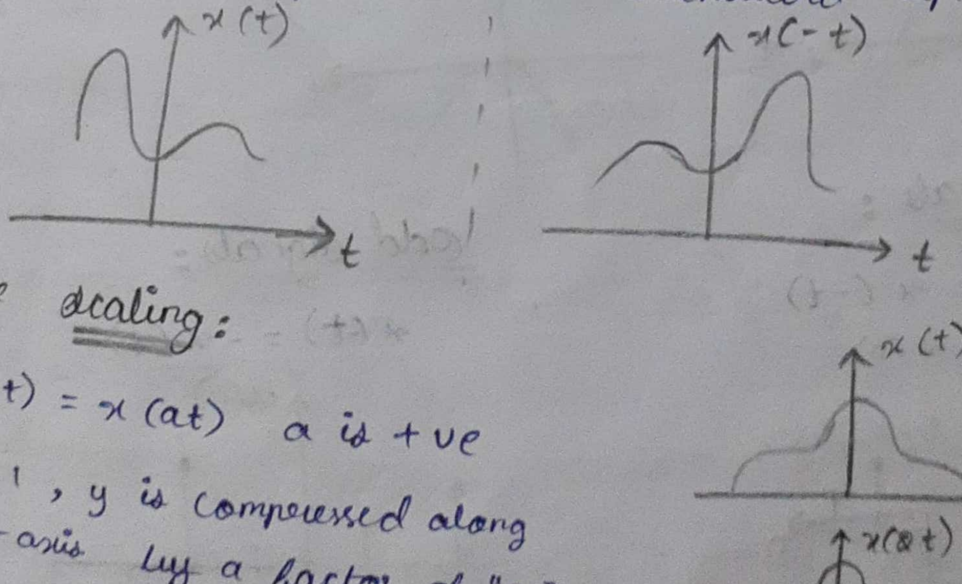


2. **Time Reversal :**

=> $y(t) = x(-t)$

=> By reflecting original signal to vertical line ($t = 0$)

=> $x(t)$ = audio tape $x(-t)$ = backward tape

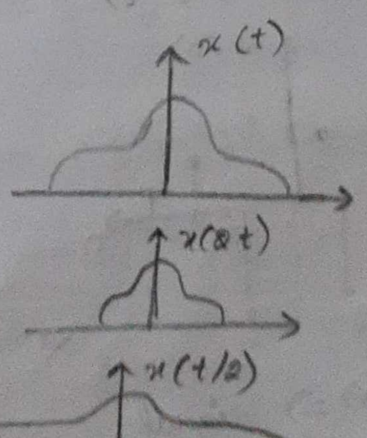


3. **Time scaling :**

=> $y(t) = x(at)$ a is +ve

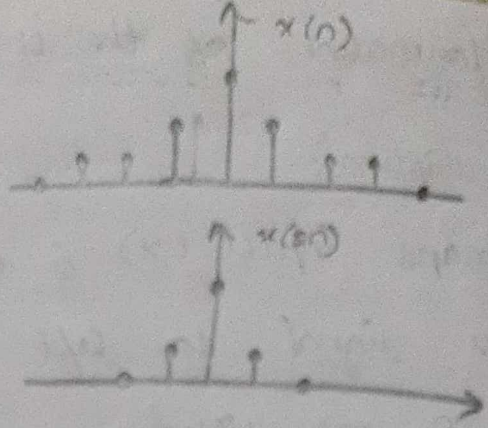
=> $a > 1$, y is compressed along x-axis by a factor of "a"

=> $a < 1$, y is expanded



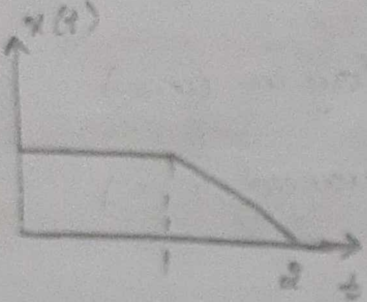
Downsampling:

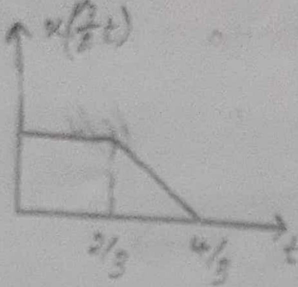
$\Rightarrow y(n) = x(an)$

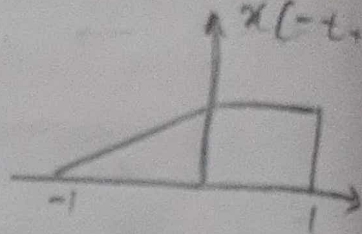


y is produced from x by keeping only a^{th} sample of x

1.

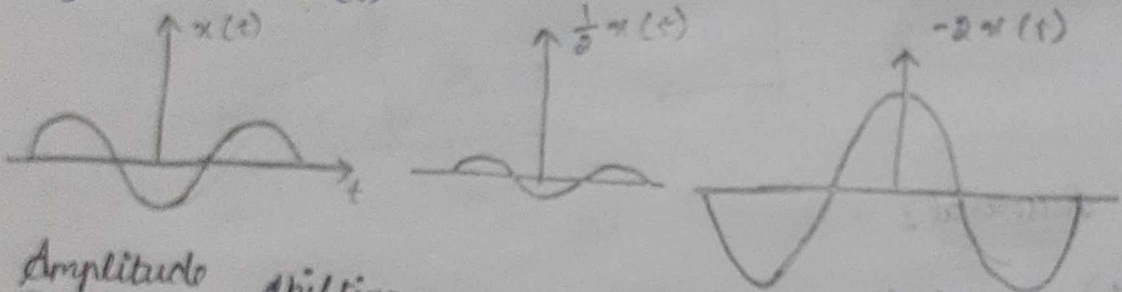






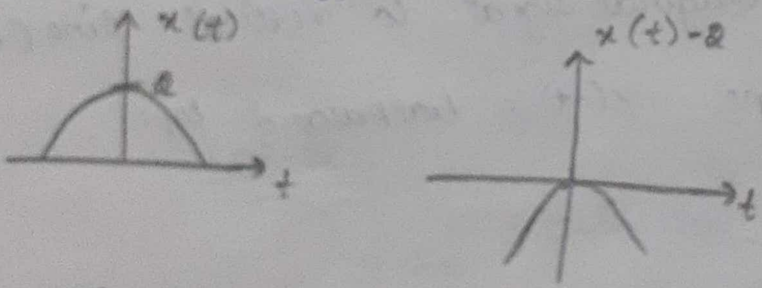
Amplitude scaling:

$\Rightarrow y(t) = a\,x(t)$



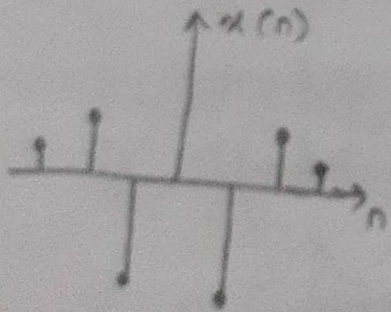
Amplitude shifting:

$\Rightarrow y(t) = x(t) + b.$



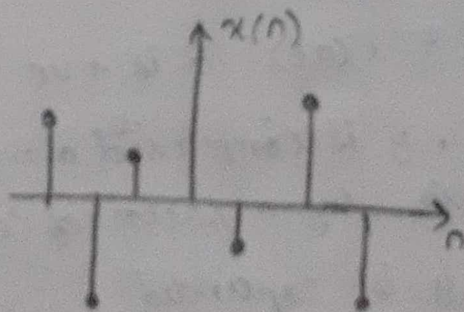
Even signals:

$x(t) = x(-t)$



Odd signals:

$x(-t) = -x(t)$



$x(t) \Rightarrow$ signal , $x(n) \Rightarrow$ sequence.

Even and odd components of a signal:

$x(t) = x_o(t) + x_e(t) \longrightarrow ①$

$x(-t) = x_o(-t) + x_e(-t)$

$x(-t) = -x_o(t) + x_e(-t) \longrightarrow ②$

Same applies for sequence.

$x(t) + x(-t) = 2x_e(t)$

$x_e(t) = \frac{1}{2}(x(t) + x(-t))$, $x_o(t) = \frac{1}{2}(x(t) - x(-t))$

even component odd component.

Periodic signals:

=) A function x is said to be periodic with period T if $x(t) = x(t+T)$

=) If not periodic then aperiodic.

=) If a x is periodic with period T then it is also periodic with period KT. $(K>0)$

=) smallest period = fundamental period.

=) x_1, x_2 are two periodic functions

then $y = x_1 + x_2$ is periodic if and only if

T_1/T_2 is a rational number

i.e $\frac{T_1}{T_2} = \frac{q}{r}$

y is periodic with period rT_1 / qT_2

rT_1 = LCM of T_1 & T_2

Signal Energy and Power:

=) Energy contained in signal x

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

A signal with finite energy = energy signal

$$E = \sum^{\infty} |x(k)|^2$$

$\Rightarrow$ Avg Power contained in signal x

$$P = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2 \, dt$$

A signal with finite avg power = power signal

~~Elementary signals~~:

1. $x(t) = \sin \frac{2\pi}{3} t \qquad x(t) = \sin \omega_0 t$

$\omega_0 = \frac{2\pi}{T}, \quad \omega_0 = \frac{2\pi}{3}$

$\therefore T = 3$

$\because x(t+T) = \sin \frac{2\pi}{3}(t+3)$

$= \sin\left(\frac{2\pi}{3}t + 2\pi\right) = \sin \frac{2\pi}{3} t$

2. $x(t) = \cos \frac{\pi}{3} t + \sin \frac{\pi}{4} t.$

$x(t) = x_1(t) + x_2(t)$

$\omega_1 = \frac{\pi}{3}, \quad \omega_2 = \frac{\pi}{4}$

$T_1 / T_2 = 6/8 = 3/4$

$4T_1 = 3T_2$

3. $x(t) = \sin^2(t)$

$= \frac{1 - \cos 2(t)}{2} \qquad = \underset{T_1}{\frac{1}{2}} - \underset{T_2}{\frac{\cos 2(t)}{2}}$

$T_2 = \frac{2\pi}{2} = \pi$

$\frac{T_1}{T_2} = \frac{\pi \text{ or } 1 \text{ or } 1.5}{\pi}$

4. $x(n) = \cos \frac{2\pi}{3} n + \sin \frac{3\pi}{4} n$

$\omega_1 = \frac{2\pi}{3}, \quad \omega_2 = \frac{3\pi}{4}$

$N_1 = 3, \quad N_2 = 8/3$

$N = \frac{2\pi}{\Omega} m$

$N_2 = \frac{8}{3} m$

$= \frac{8}{3} \times 3$

$N_2 = 8.$

$x_1 \rightarrow$ periodic $\rightarrow N_1 = 3$, $x_2 \longrightarrow$ periodic $\longrightarrow N_2 = 2$

$$x(n) = \frac{N_1}{N_2} = \frac{3}{2} = \text{rational} \quad \therefore \text{periodic}$$

1. Find if $x(t) = e^{-at} u(t)$ energy signal or periodic signal

$$\boxed{\begin{array}{l} \text{If } 0 < E < \infty = \text{finite} \rightarrow \text{Energy}, \text{ power} = 0 \\ \text{If } 0 < P < \infty = \text{finite} \longrightarrow \text{Power}, \text{ Energy} = \infty \end{array}}$$

Sol. $u(t) = 1$ for $t > 0$

$= 0$ for $t < 0$

$$|x(t)|^2 = e^{-2at}$$

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} e^{-2at} dt = \left(\frac{-e^{-2at}}{2a}\right)_0^{\infty}$$

$$= \frac{-e^{-\infty}}{2a} + \frac{e^{0}}{2a} = \frac{1}{2a}$$

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt = \lim_{T \to \infty} \frac{1}{T} \int_0^{T/2} e^{-2at} dt$$

$$= \lim_{T \to \infty} \frac{1}{T} \left(\frac{-e^{-2at}}{2a}\right)_0^{T/2} \quad \cancel{= \lim_{T \to \infty} \frac{1}{T} (}$$

$$= 0$$

E = finite , $P = 0$

2. $x(t) = A \cos(\omega_0 t + \theta)$

$$E = \int_{-\infty}^{\infty} A^2 \cos^2(\omega_0 t + \theta) dt$$

$$|x(t)|^2 = A^2 \left(\frac{1 + \cos 2(\omega t + \theta)}{2}\right)$$

$$= \frac{A^2}{2} + \frac{A^2 \cos 2(\omega t + \theta)}{2}$$

$$E = \int_{-\infty}^{\infty} \left(\frac{A^2}{2} + \frac{A^2 \cos 2(\omega t + \theta)}{2}\right) dt$$

$$E = \frac{A^2}{2}(t)_0^{\infty} + \frac{A^2}{2}\frac{(\sin 2(\omega t+\theta))_0^{\infty}}{2\omega}$$

$$E = \infty$$

$$P = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\left(\frac{A^2}{2}+\frac{A^2\cos(2\omega t+2\theta)}{2}\right)dt$$

$$= \lim_{T\to\infty}\frac{1}{T}\frac{A^2}{2}(T) + \frac{A^2}{2}\frac{(\sin 2(\omega t+\theta))_{-T/2}^{T/2}}{2\omega}$$

$$= \lim_{T\to\infty}\frac{A^2}{2} + \frac{A^2}{4\omega}(\sin(2\pi+2\theta) - \sin(-2\pi+2\theta))$$

$$= \lim_{T\to\infty}\frac{A^2}{2} + \frac{A^2}{4\omega}(\sin 2\theta - \sin 2\theta)$$

$$= \frac{A^2}{2}$$

$\therefore x(t)$ is a Power signal.

3. $x(t) = t\,u(t)$

$$E = \int_{-\infty}^{\infty} t^2\,dt = \left(\frac{t^3}{3}\right)_0^{\infty} = \infty$$

$$P = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} t^2\,dt = \lim_{T\to\infty}\frac{1}{T}\left(\frac{t^3}{3}\right)_0^{T/2} = \lim_{T\to\infty}\frac{1}{T}\times\frac{T^3}{24}$$

$$= \infty$$

Neither one $\therefore$ Ramp signal

4. $x(n) = -(0.5)^n u(n)$

$$|x(n)|^2 = (0.25)^n.$$

$$E = \sum_{-\infty}^{\infty}(0.25)^n = \frac{1}{1-\frac{1}{4}} = \frac{4}{3}$$

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$$

$$\sum_{n=0}^{N-1} a^n = \frac{1-a^N}{1-a}$$

$$\sum_{n=k}^{\infty} a^n = \frac{a^k}{1-a}$$

1) CTS + sampling = DTS

2) DTS + quantisation + code = Digital signal

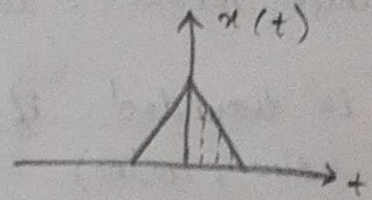
If the signal differs in y-axis it is analog → infinite

sum of 2CTS need not be always periodic

" " 2DTS are always periodic

1. $x(t) = 1 - |t|$ for $-1 \le t \le 1$

$= 0$ otherwise



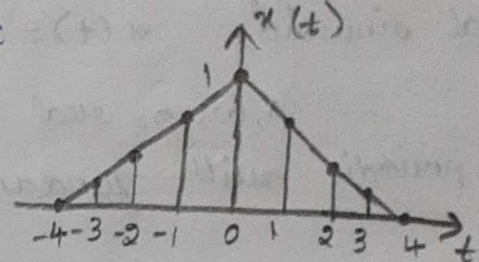
uniform sampling of $x(t)$

sampling interval = (i) 0.25 sec

(ii) 0.5 sec

(iii) 1 sec

(i) $T_s = 0.25$ sec



$= \frac{1}{4}$, $x(n) = 1 - n\,T_s$

$x(0) = 1$

$x(1) = 1 - \frac{1}{4} = \frac{3}{4}$, $x(2) = 1 - \frac{1}{2} = \frac{1}{2}$

$x(3) = 1 - \frac{3}{4} = \frac{1}{4}$, $x(4) = 1 - 1 = 0$

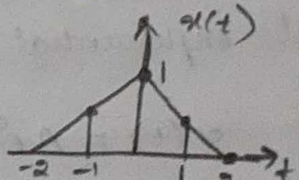
(ii) $T_s = 0.5$ sec

$= \frac{1}{2}$

$x(n) = 1 - n\,T_s$

$x(0) = 1$, $x(1) = 1 - \frac{1}{2} = \frac{1}{2}$

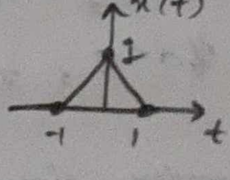
$x(2) = 1 - 1 = 0$



iii) $T_s = 1$ sec

$x(n) = 1 - n\,T_s$

$x(0) = 1 - 0 = 1$, $x(1) = 1 - 1 = 0$.

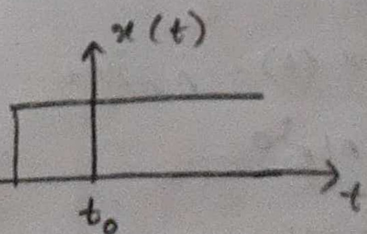


Elementary signals.

i) Right sided signals:

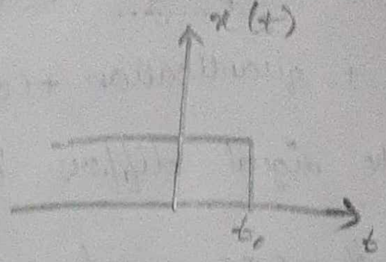
x is potentially non zero to the right of t_0

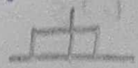
$t < t_0$



Left sided system:

x is only potentially non-zero to the left of t_0, $t > t_0$



=) A signal which both left sided & right sided is said to be finite duration. → 

=) which neither = two sided. → 

=) x is bounded if $|x(t)| \leq A$ $\quad$ A = +ve real const.
(sine, cosine)

=) unbounded (tan, non constant polynomial func)

=) Real sinusoid $\quad x(t) = A\cos(\omega t + \theta)$ / $A\sin(\omega t + \theta)$

A, ω, θ = real consts.

periodic with fundamental period $T = \frac{2\pi}{|\omega|}$

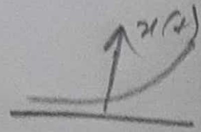
=) Complex exponential:, $x(t) = Ae^{\lambda t}$.

A, λ = complex consts (real exp, complex sinusoids)

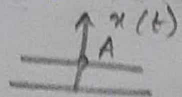
=) Real exponential:

$x(t) = Ae^{\lambda t}$ $\quad$ A, λ = real numbers.

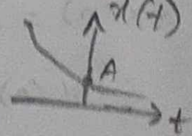
(i) If $\lambda > 0$, $x(t) \uparrow$ exponentially as $t \uparrow$



(ii) If $\lambda = 0$, $x(t) = A$ (const)



(iii) If $\lambda < 0$, $x(t) \downarrow$ exponentially as $t \uparrow$.



=) Complex sinusoids:

$x(t) = Ae^{\lambda t}$ $\quad$ A = complex, λ = purely imaginary

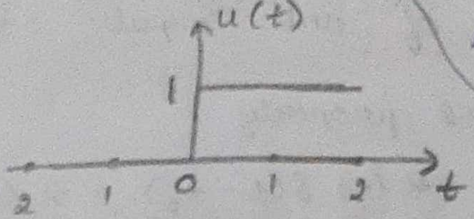
$x(t) = Ae^{j\omega t}$ $\quad$ A = complex, ω = real

$A = |A|e^{j\theta}$

$x(t) = |A|\cos(\omega t + \theta) + j\,A\sin(\omega t + \theta)$

Unit Step Function:

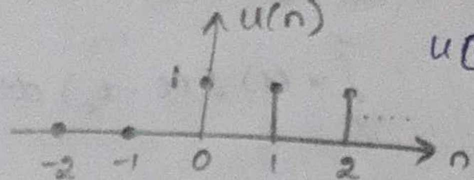
$$u(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$



Step function
$u(n) = A, n \geq 0$
$= 0$, others

Unit Step Sequence:

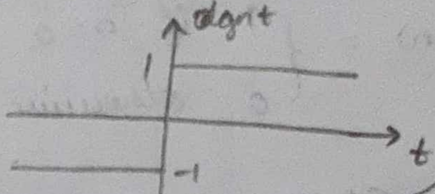
$$u(n) = \begin{cases} 1 & \text{if } n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

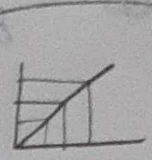


$$u[n-k] = \begin{cases} 1, & n \geq k \\ 0, & n < k \end{cases}$$

Signum Function:

$$\text{sgn } t = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t = 0 \\ -1 & \text{if } t < 0 \end{cases}$$

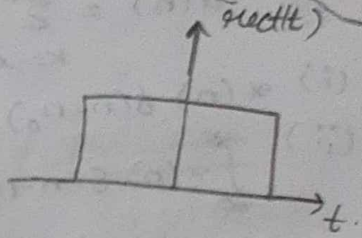




Ramp signal
$r(n) = An, n \geq 0$
$= 0$ others

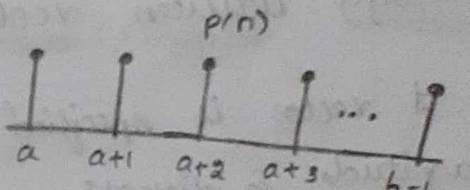
Rectangular Function:

$$rect(t) = \begin{cases} 1 & \text{if } \frac{-1}{2} \leq t < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$



Unit Rectangular ~~Fun~~ Pulses:

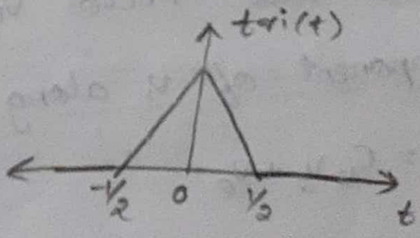
$$P(n) = \begin{cases} 1 & \text{if } a < n < b \\ 0 & \text{otherwise} \end{cases}$$



$P(n) = u(n-a) - u(n-b)$

Triangular Function.

$$tri(t) = \begin{cases} 1 - 2|t|, & |t| \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$



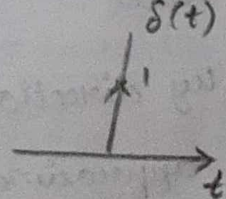
Cardinal Sine Function:

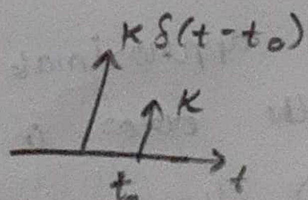
$$sinc(t) = \frac{\sin t}{t}.$$

Unit Impulse Function:

$\delta(t) = 0$ for $t \neq 0$ and

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$





Sequence ⇒ $\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$, $\delta[n-K] = \begin{cases} 1, & n = K \\ 0 & n \neq K \end{cases}$

Properties of unit Impulse Function :

(i) Equivalence property

$$x(t)\,\delta(t-t_0) = x(t_0)\,\delta(t-t_0)$$

(ii) Sifting property

$$\int_{-\infty}^{\infty} x(t)\,\delta(t-t_0)\,dt = x(t_0)$$

(iii) $\delta(t) = \delta(-t)$, (iv) $\delta(at) = \frac{1}{|a|}\delta(t)$

$$\delta(n) = \begin{cases} 1 & \text{if } n=0 \\ 0 & \text{otherwise} \end{cases}$$ Impulse sequence

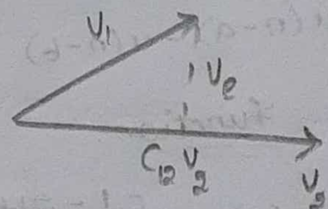
$$\delta(n) = u(n) - u(n-1)$$

$$u(n) = \sum_{k=-\infty}^{n} \delta(k)$$

(i) $x(n)\,\delta(n-n_0) = x(n_0)\,\delta(n-n_0)$

(ii) $\cancel{\int_{-\infty}^{\infty} x(n)\,\delta(n-n_0)\,d.}$ $\sum_{n=-\infty}^{\infty} x(n)\,\delta(n-n_0) = x(n_0)$

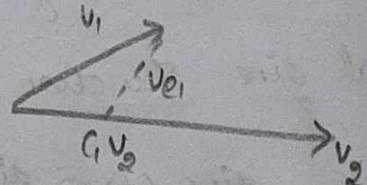
Analogy between vectors and signals:



=) A vector is specified by its magnitude & direction

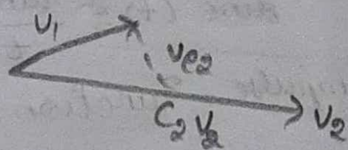
=) Consider two vectors V_1 & V_2

=) Component of V_1 along V_2 = $C_{12}V_2$



=) $V_1 = C_{12}V_2 + V_e$

=) $V_1 = C_1V_2 + V_{e1}$, $V_1 = C_2V_2 + V_{e2}$



=) where V_e = error vector.

=) approximate V_1 by direction of V_2, then V_e represents the error in this approximation

=) If V_1 by $C_{12}V_2$ then V_e

=) If V_1 by C_1V_2 then V_{e1} and so on

=) C_{12} is chosen such that the error vector is min

=) $C_{12} \uparrow$, $V_e \downarrow$

=) $|C_{12}|$ = indication of the similarity of two vectors.

=) $C_{12} = 0$, V_1 & V_2 are $\perp$er

=) They are orthogonal vectors (Independent vectors)

=) $A \cdot B = AB \cos\theta$, θ = angle b/w A and B

=) component of A along B = $A \cos\theta = \frac{A \cdot B}{B}$

=) " " B " A $= B\cos\theta = \frac{A \cdot B}{A}$

=) component of V_1 along V_2 = $C_{12}V_2 = \frac{V_1 \cdot V_2}{V_2}$

$$\therefore C_{12} = \frac{V_1 \cdot V_2}{V_2^2}$$

Signals:

let $f_1(t)$ & $f_2(t)$ are two signals

$$f_1(t) = C_{12} f_2(t) \quad \text{for } t_1 < t < t_2$$

=) ~~f_{12} is~~ let $f_e(t) = f_1(t) - C_{12} f_2(t)$

=) To minimise the error $f_e(t)$ over interval t_1 to t_2 is to minimise value of $f_e(t)$ over t_1 to t_2

$\therefore$ that is to minimise $\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} (f_1(t) - C_{12} f_2(t))\, dt$

=) But we may get error = 0 since cancelling out so. we can use avg of sqr of errors instead.

$$\mathcal{E} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} f_e^2(t)\, dt = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} (f_1(t) - C_{12} f_2(t))^2 dt.$$

$\Rightarrow$ C_{12} value can be found which will minimise ε

$$\frac{d\varepsilon}{dC_{12}} = 0$$

$$\frac{d}{dC_{12}}\left(\frac{1}{t_2-t_1}\int_{t_1}^{t_2}\left(f_1(t) - C_{12}f_2(t)\right)^2 dt\right) = 0$$

$$\frac{1}{t_2-t_1}\left[\int_{t_1}^{t_2}\frac{d}{dC_{12}}f_1^2(t) - 2\int_{t_1}^{t_2}f_1(t)f_2(t)\,dt + 2C_{12}\int_{t_1}^{t_2}f_2^2(t)\,dt\right] = 0$$

$$C_{12} = \frac{\int_{t_1}^{t_2}f_1(t)f_2(t)\,dt}{\int_{t_1}^{t_2}f_2^2(t)\,dt}$$

1. ~~Find~~ orthogonal if $\int_{t_1}^{t_2} f_1(t)f_2(t)\,dt = 0$

2. Find orthogonal or not

$\sin n\omega_0 t$, $\sin m\omega_0 t$. $\left(t_0, t_0+\frac{2\pi}{\omega_0}\right)$

$$I = \int_{t_1}^{t_2} f_1(t)f_2(t)\,dt = \int_{t_0}^{t_0+\frac{2\pi}{\omega_0}} \sin n\omega_0 t \sin m\omega_0 t\,dt$$

$$= \frac{1}{2}\int_{t_0}^{t_0+\frac{2\pi}{\omega_0}} \cos(n-m)\omega_0 t\,dt - \frac{1}{2}\int_{t_0}^{t_0+\frac{2\pi}{\omega_0}} \cos(n+m)\omega_0 t\,dt$$

$$= \frac{1}{2}\frac{\left(\sin(n-m)\omega_0 t\right)_{t_0}^{t_0+\frac{2\pi}{\omega_0}}}{(n-m)\omega_0} - \frac{1}{2}\frac{\left(\sin(n+m)\omega_0 t\right)_{t_0}^{t_0+\frac{2\pi}{\omega_0}}}{(n+m)\omega_0}$$

$$= \frac{1}{2\omega_0}\left(\frac{\sin(n-m)(\cancel{2\pi}+\omega_0 t_0)}{(n-m)\omega_0} - \frac{\sin(n-m)(\omega_0 t_0)}{(n-m)\omega_0}\right.$$

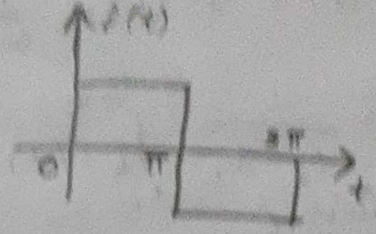
$$\left. - \frac{\sin(n+m)(\cancel{2\pi}+\omega_0 t_0)}{(n+m)\omega_0} + \frac{\sin(n+m)(\omega_0 t_0)}{(n+m)\omega_0}\right)$$

$$= \frac{1}{2\omega_0}\left(\frac{\sin(n-m)\omega_0 t_0}{(n-m)\omega_0} - \frac{\sin(n-m)\omega_0 t_0}{n-m}\right)$$

$= 0$

$\therefore$ orthogonal

Q. $f_1(t)$ = rectangular function defined as

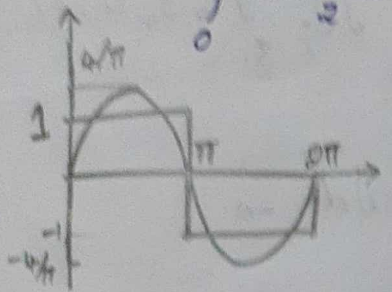


$$f_1(t) = 1 \quad \text{for } (0 < t < \pi)$$
$$= -1 \quad \text{for } (\pi < t < 2\pi)$$

$f_2(t) = \sin t$ over $(0, 2\pi)$

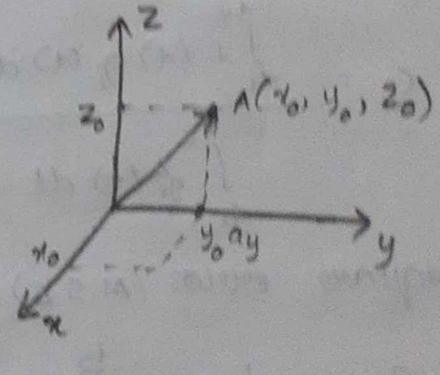
$$C_{12} = \frac{\int_{t_1}^{t_2} f_1(t) \sin t \, dt}{\int_{t_1}^{t_2} \sin^2 t \, dt} = \frac{\int_0^{\pi} \sin t \, dt + \int_{\pi}^{2\pi} -\sin t \, dt}{\int_0^{2\pi} \sin^2 t \, dt}$$

$$C_{12} = \frac{(-\cos t)_0^{\pi} - (-\cos t)_{\pi}^{2\pi}}{\int_0^{2\pi} \frac{1-\cos 2t}{2} dt} = \frac{-(-1-1) + (1+1)}{\frac{1}{2}\left((t)_0^{2\pi} - \left(\frac{\sin 2t}{2}\right)_0^{2\pi}\right)}$$



$$= \frac{4 \times 2}{(2\pi) - 0} = \frac{4}{\pi} = C_{12}$$

Orthogonal Vector space:



$\Rightarrow \bar{V}_1 \cdot \bar{V}_2 = 0$

$|a_x| = |a_y| = |a_z| = 1$

$\bar{A} = x_0 a_x + y_0 a_y + z_0 a_z$

$\bar{A} = C_1 x_1 + C_2 x_2 + \cdots + C_n x_n$

component of $\bar{A}$ along x-axis $= \dfrac{V_1 \cdot V_2}{|V_2|} = \dfrac{\bar{A} \cdot a_x}{|a_x|}$

$$= \frac{x_0}{|a_x|}$$

~~$\bar{A}\bar{a}_x = x_0$~~

$f_1(t) \cong C_{12} f_2(t)$

$f_e(t) = f_1(t) - C_{12} f_2(t)$

$$\varepsilon = MSE = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} (f_1(t) - C_{12} f_2(t))^2 dt$$

$$f(t) \cong \sum_{r=1}^{n} C_r g_r(t) \qquad g_r(t) = \text{orthogonal functions}$$

$$f_e(t) = f(t) - \sum_{r=1}^{n} C_r\, g_r(t)$$

$$\varepsilon = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \left[f(t) - \sum_{r=1}^{n} C_r\, g_r(t) \right]^2 dt.$$

To minimize ε,

$$\frac{\partial \varepsilon}{\partial C_1} = \frac{\partial \varepsilon}{\partial C_2} = \frac{\partial \varepsilon}{\partial C_3} = \cdots = \frac{\partial \varepsilon}{\partial C_n} = 0$$

$$\frac{\partial}{\partial C_j} \left(\int_{t_1}^{t_2} \left[f(t) - \sum_{r=1}^{n} C_r\, g_r(t) \right]^2 dt \right) = 0$$

$$\Rightarrow \cancel{\frac{\partial}{\partial C_j} \int_{t_1}^{t_2} f^2(t)\, dt = \frac{\partial}{\partial C_j} \int_{t_1}^{t_2} C_r^2\, g_r^2(t)\, dt = \frac{\partial}{\partial C_j} \int_{t_1}^{t_2} C_r\, g_r(t)\, f(t) = 0}$$

$$\frac{\partial}{\partial C_j} \int_{t_1}^{t_2} \left[-2 C_j\, f(t)\, g_j(t) + C_j^2\, g_j^2(t) \right] dt = 0$$

$$2 \int_{t_1}^{t_2} f(t)\, g_j(t)\, dt = 2 C_j \int_{t_1}^{t_2} g_j^2(t)\, dt.$$

$$C_j = \frac{\int_{t_1}^{t_2} f(t)\, g_j(t)\, dt}{\int_{t_1}^{t_2} g_j^2(t)\, dt}.$$

Mean square error (MSE): $C_r = \dfrac{\int_{t_1}^{t_2} f(t)\, g_r(t)\, dt}{\int_{t_1}^{t_2} g_r^2(t)\, dt}$, $K_r = \int_{t_1}^{t_2} g_r^2(t)\, dt$

$$\varepsilon = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \left(f(t) - \sum_{r=1}^{n} C_r\, g_r(t) \right)^2 dt$$

$$= \frac{1}{t_2 - t_1} \left(\int_{t_1}^{t_2} f_1^2(t)\, dt - 2 \sum_{r=1}^{n} C_r \int_{t_1}^{t_2} f(t)\, g_r(t)\, dt + \sum_{r=1}^{n} C_r^2 \int_{t_1}^{t_2} g_r^2(t)\, dt \right)$$

$$= \frac{1}{t_2 - t_1} \left(\int_{t_1}^{t_2} f_0^2(t)\, dt - 2 \sum_{r=1}^{n} C_r^2 K_r + \sum_{r=1}^{n} C_r^2 K_r \right)$$

$$\varepsilon = \frac{1}{t_2 - t_1} \left[\int_{t_1}^{t_2} f^2(t)\, dt - \left(C_1^2 K_1 + C_2^2 K_2 + \cdots + C_n^2 K_n \right) \right]$$

$$C_r = \begin{cases} \frac{4}{\pi r}, & r = \text{odd} \\ 0 & \text{for } r = \text{even} \end{cases}$$

for $f(t) = \begin{cases} 1 & 0<t<\pi \\ -1 & \pi<t<2\pi \end{cases}$

$f_1(t) \cong C_{12} f_2(t)$

$g(t) = \sin rt$

$$f(t) \cong \frac{4}{\pi} \sin t + \frac{4}{3\pi} \sin 3t + \frac{4}{5\pi} \sin 5t + \cdots + \frac{4}{n\pi} \sin nt$$

$$f(t) = \frac{4}{\pi} \left(\sin t + \frac{1}{3} \sin 3t + \frac{1}{5} \sin 5t + \cdots + \frac{1}{n} \sin nt \right)$$

$$\mathcal{E} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} (f^2(t)\, dt - (C_1^2 K_1 + C_2^2 K_2 + \cdots + C_r^2 K_r)) \to \infty$$

$$\mathcal{E}_1 = \frac{1}{2\pi} \left(2\pi - \left(\frac{4}{\pi}\right)^2 \pi \right) = 0.19$$

$$\mathcal{E}_2 = \frac{1}{2\pi} \left(2\pi - \left(\frac{4}{\pi}\right)^2 \pi - \left(\frac{4}{3\pi}\right)^2 \pi \right) = 0.1$$

$f(t) \uparrow \quad \mathcal{E} \downarrow$

$\Rightarrow$ If $n = \infty$, $\sum_{r=1}^{\infty} C_r^2 K_r = \int_{t_1}^{t_2} f^2(t)\, dt$

thus $\mathcal{E}$ vanishes

series is said to converge in the mean

$$\Rightarrow \int_{t_1}^{t_2} g_m(t)\, g_n(t)\, dt = \begin{cases} 0 & \text{if } m \neq n \\ K_m & \text{if } m = n \end{cases}$$

$g_r(t)$ are mutually orthogonal vectors.

$\Rightarrow f(t) = C_1 g_1(t) + C_2 g_2(t) + \cdots + C_r g_r(t) + \cdots$

$$C_r = \frac{\int_{t_1}^{t_2} f(t)\, g_r(t)\, dt}{\int_{t_1}^{t_2} g_r^2(t)\, dt} = \frac{\int_{t_1}^{t_2} f(t)\, g_r(t)\, dt}{K}$$

Any func $f(t)$ can be expressed as a sum of its components along mutually orthogonal functions, provided these functions form a closed set.

Orthogonality in complex functions:

$f_1(t)$ & $f_2(t)$ are complex functions then

$$f_1(t) \simeq C_{12} f_2(t)$$

$$C_{12} = \frac{\int_{t_1}^{t_2} f_1(t)\, f_2^*(t)\, dt}{\int_{t_1}^{t_2} f_2(t)\, f_2^*(t)\, dt}$$

orthogonal if $\int_{t_1}^{t_2} f_1(t)\, f_2^*(t)\, dt = \int_{t_1}^{t_2} f_1^*(t)\, f_2(t)\, dt = 0$

Mutually orthogonal over (t_1, t_2)

$$\int_{t_1}^{t_2} g_m(t)\, g_n^*(t)\, dt = \begin{cases} 0 & \text{if } m \neq n \\ K_m & \text{if } m = n \end{cases}$$

If real functions then $g_r^*(t) = g_r(t)$