UNIT - V

Waveguides: Electromagnetic Spectrum and Bands.

Introduction

- Microwaves are electromagnetic waves, whose Frequency's Range from 1 GHz to 100 GHz. Hence the history of microwaves is a part of the Evolution of the Electromagnetic waves.

- James Clerk Maxwell's (1831 – 1879) unified all previous know Results, experimental and theoretical on electromagnetic waves in Four Equations and predicted the Existence of electromagnetic wave.

→ Heinrich Rudolf Hertz (1857-1937) Experimentally Confirmed Maxwell's prediction.

→ Guglielme Marconi (1874-1937) transmitted information on an Experimental basis at microwave freq

→ George C. South worth (1930) Really Carried at Max Conis experiment on a Commercial basis.

→ During World war-II (1945) based on the previous developments, radar was invented and was explaited for military application.

→ Micro waves (MW's) are so called since they are defined in terms of there wave length in the sense that micro refers, to tinyness.

tinyness refering to the wave length and the period of a cycle of a cm wave.

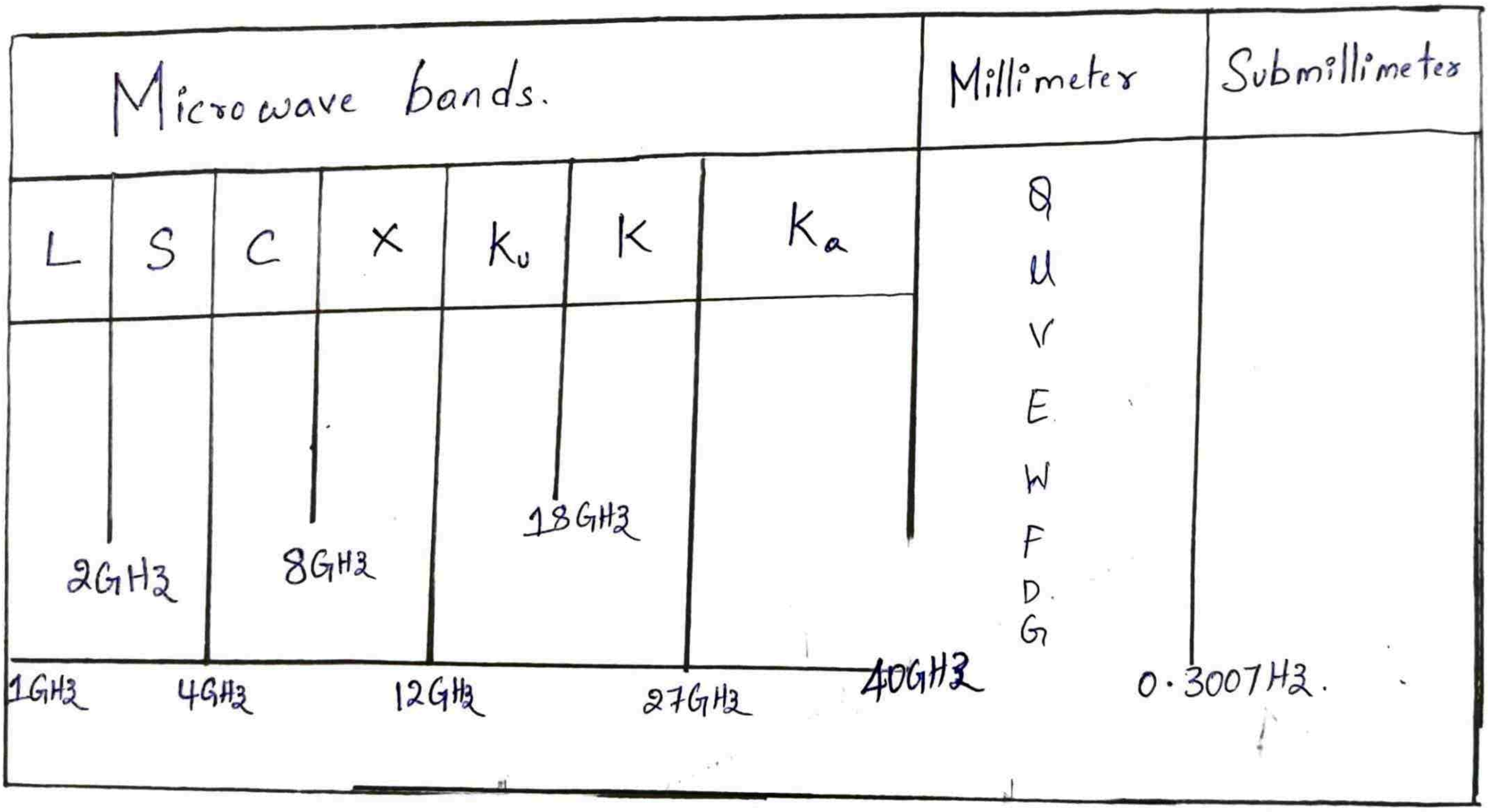
→ the wave length (λ) of a cm waves at micro wave frequencies are very short. typically from a few tens of CM to a fraction of a mm.

→ In short, a microwave is a signal that has a wavelength of 1 foot or less i,e $\lambda \leq 30.5$ cm $\cong$ 1 foot.

It this converts to a frequency is of 984 MHz, (i.e) approximately = 1 GHz. So all frequency above approximately 1000 GHz are Microwave (1000 MHz - 1 GHz to about 1000 GHz frequency.

Electromagnetic Spectrum and Bands.

(i) Microwave bands



Prefix	Power of Ten	Symbol
Exa	10^{18}	E
Peta	10^{15}	P
Tera	10^{12}	T
Giga	10^{9}	G
Mega	10^{6}	M
Kilo	10^{3}	K
hecto	10^{2}	h
Deca	10	da

Department of Electronics and Communication Engineering,MRCE

Electro Magnetic Frequency Spectrum, Band and Application

ELF	SLF	VLF	LF	MF	HF	VHF	UHF	SHF	EHF	Infra Red light	Visible light spectrum (Light)	X-ray	Gamma rays	Cosmic rays
Extremely low Frequency	Super Low Frequency	Very low Frequency	Low freque	Medium Frequ	High Freq	Very High frequ	Ultra High Freq	Super High Freq	Extremely High freq					

30 Hz, 300 Hz, 3 KHz, 30 KHz, 300 KHz, 3 MHz, 30 MHz, 300 MHz, 3 GHz, 30 GHz, 300 GHz, 3 THz

Abbreviation	Frequencies	Wave length	Application
ELF	30 Hz - 300 Hz	10mm - 1mm	ELF - SUB MARINES communication
VLF	3 KHz — 30 KHz	100 km — 10 km	VLF - Maritime Radio, Navigation, Point to Point communication
LF	30 KHz — 300 KHz	10 km - 1 km	LF - Maritime Radio, Navigation
MF	300 KHz — 3 MHz	1 km - 100 m	MF - AM Raido, Aviation Radio, Navigation
HF	3 MHz — 30 MHz	100 m - 10 m	HF - Shortwave Radio
VHF	30 MHz — 300 MHz	10 m — 1 m	VHF - VHF Television, FM Radio.
UHF	300 MHz - 3 GHz	1 m - 10 cm	UHF - UHF Television, mobile phones, GPS, Wi-Fi, 4G
SHF	3 GHz - 30 GHz	10 cm - 1 cm	SHF - Satellite Communication, Wi-Fi
EHF	30 GHz - 300 GHz	1 cm - 1 mm	EHF - Radio Astronomy, Satellite communication.

Microwaves are electromagnetic waves whose frequencies range from 1 GHz to 1000 GHz (1 GHz = 10^9 Hz). For comparison, the signal from an AM Radio station is 1 MHz (Mega Hertz = 10^6 Hz) and the signal from FM radio station is 100 MHz.

Microwaves (μw's) are so called since they are defined in terms of their wavelength in the sense that micro refers to tinyness—tinyness referring to the wavelength and the period of a cycle of a cm wave. In other words, the wavelength (λ) of cm waves at microwave frequencies are very short; typically from a few tens of cm to a fraction of a mm. In short, a microwave is a signal that has a wavelength of 1 foot or less i.e. $\lambda \leq 30.5$ cms $\cong$ 1 foot. This converts to a frequency of 984 MHz, approximately = 1 GHz. So, all frequencies above approximately 1000 MHz (1 GHz) to about 1000 GHz are microwave frequencies.

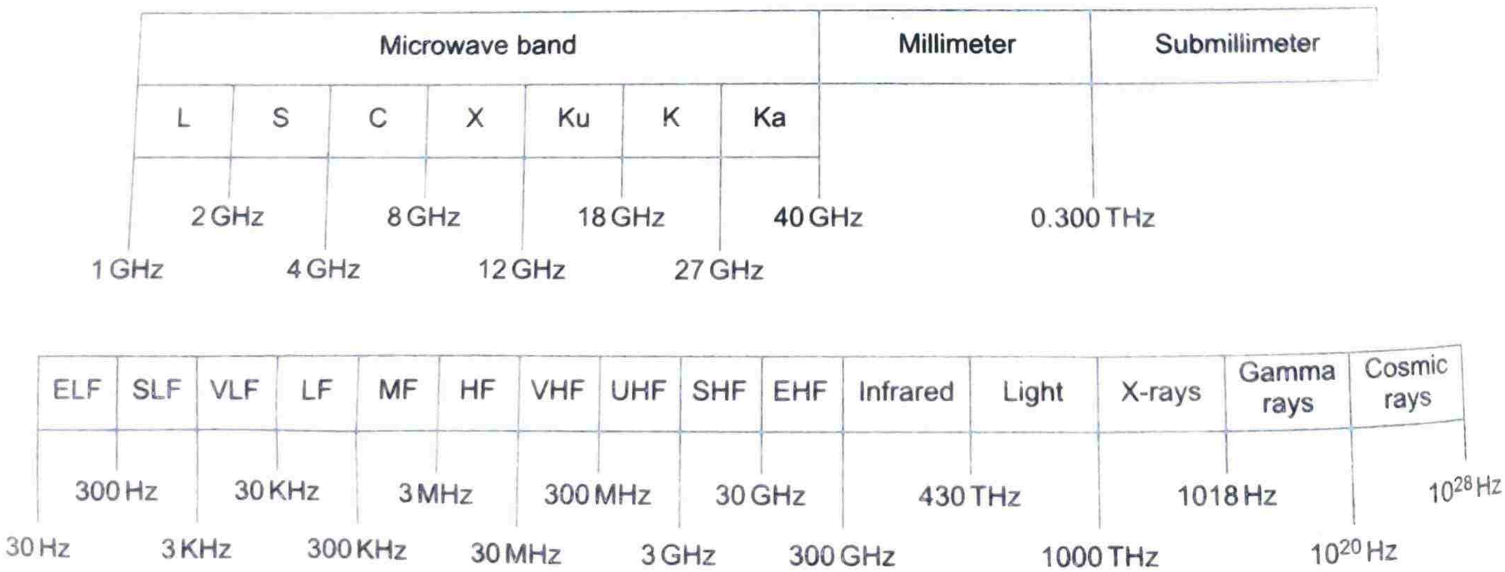


Fig. 1.1 Electromagnetic frequency spectrum

The higher frequency edge of microwaves borders on the infrared and visible-light regions or spectrum. This explains why microwaves behave more like rays of light than ordinary radio waves. Due to this unique behaviour, microwave frequencies are classified separately from radio waves. Fig 1.1 shows the various available electromagnetic frequency spectrum.

The visible light is above infrared region and falls between 430 THz and 1 PHz (Peta Hz, 1 peta $= 10^{15}$ Hz). This region includes the fibre optics and laser region. This region above 1 PHz includes X-rays, Gamma Rays and Cosmic rays. The various metric prefixes such as Mega, Micro are given in Table 1.1.

Table 1.1 Metric Prefixes

Prefix	Power of Ten	Symbol
Exa	10^{18}	E
Peta	10^{15}	P
Tera	10^{12}	T
Prefix	*Power of Ten*	*Symbol*
Giga	10^{9}	G
Mega	10^{6}	M
Kilo	10^{3}	K
milli	10^{-3}	m
micro	10^{-6}	μ
nano	10^{-9}	n
pico	10^{-12}	p
femto	10^{-15}	f
atto	10^{-18}	a

IEEE - Industry Standards

S.No	Band Designation	Frequency Designation
1.	HF (High frequency)	3 to 30 MHz.
2.	VHF (Very High frequency)	30 to 3 MHz.
3.	UHF (Ultra High frequency)	0.3 to 1 GHz.
4.	L	1 to 2 GHz.
5.	S	2 to 4 GHz.
6.	C	4 to 8 GHz.
7.	X	8 to 12 GHz.
8.	Ku	12 to 18 GHz.
9.	K	18 to 27 GHz.
10.	Ka	27 to 40 GHz.
11.	V	40 to 75 GHz.
12.	W	75 to 110 GHz.
13.	mm	110 to 300 GHz.

Table (2) CCIR Band Designations

Electrom

Band	Frequency Range*	Band Designation (or) Designation
1.	3 Hz – 30 Hz 30 – 300 Hz	VHF (Ultra Low Frequency) ELF (Extremely Low Frequency)
2.	300 Hz – 3000 Hz (0.3 – 3 kHz)	VF (voice Frequency).
3.	3 kHz – 30 kHz	VLF (Very low frequency).
4.	30 kHz – 300 kHz	LF (Low frequency).
5.	300 kHz – 3000 kHz (0.3 – 3 MHz)	MF (Medium Frequency).
6.	3 MHz – 30 MHz	HF (High frequency).
7.	30 MHz – 300 MHz	VHF (Very high frequency).
8.	300 MHz – 3000 MHz (0.3 – 3 GHz)	UHF (ultra high frequency).
9.	3 GHz – 30 GHz	SHF (super high frequency).
10	30 GHz – 300 GHz	EHF (Extreme high frequency).
11.	300 GHz – 3000 GHz (0.3 – 3 THz)	Infrared light
12	3 THz – 30 THz	Unassigned.
13.	30 THz – 300 THz	Visible-light spectrum.
14.	30.0 THz – 3000 THz (0.3 – 3 PHz)	Ultraviolet light
15.	3 PHz – 30 PHz	X-ray.
16.	30 PHz – 300 PHz	Unassigned.
17.	300 PHz – 3000 PHz (0.3 – 3 EHz)	Gamma rays.
18.	3 EHz – 30 EHz	Cosmic rays
19.		

* Voice frequency → Telephone / base band frequency.

* electromagnetic waves (radio waves) in the frequency range Between 30 hertz and 300 herts (SLF: Super Low frequency) 30 Hz to 300 Hz

Table 1.2 CCIR Band Designations

Band	Frequency Range	Band Designation
1.	3 Hz–30 Hz	Ultra Low frequency (ULF)
2.	30 Hz–300 Hz	Extra low frequency (ELF)
3.	300 Hz–3000 Hz (3 kHz)	Voice frequency (VF) Telephone/baseband frequencies
4.	3 KHz–30 KHz	Very Low frequency (VLF)
5.	30 KHz–300 KHz	Low frequency (LF)
6.	300 KHz–3000 KHz (3 MHz)	Medium frequency (MF)
7.	3 MHz–30 MHz	High frequency (HF)
8.	30 MHz–300 MHz	Very High frequency (VHF)
9.	300 MHz–300 MHz (3 GHz)	Ultra High frequency (UHF)
10.	3 GHz–30 GHz	Super High frequency (SHF)
11.	30 GHz–300 GHz	Extreme High frequency (EHF)
12.	300 GHz–3000 GHz (3 THz)	Intrared light
13.	3 THz–30 THz	Infrared light
14.	30 THz–300 THz	Infrared light
15.	300 THz–3 PHz	Visible light
16.	3 PHz–30 PHz	Ultra voilet light
17.	30 PHz–300 PHz	X-rays
18.	300 PHz–3 EHz	Gamma rays
19.	3 EHz–30 EHz	Cosmic rays.

Table 1.3 IEEE/Industry Standards

Band Designation	Frequency Range (GHz)
UHF	0.3–3.0
L	1.1–1.7
LS	1.7–2.6
S	2.6–3.9
C	3.9–8.0
X	8.0–12.5
Ku	12.5–18.0
K	18–26
Ka	26–40
Q (Millimeter)	33–50
U (Millimeter)	40–60
M (Millimeter)	50–75
E (Millimeter)	60–90
F (Millimeter)	90–140
G (Millimeter)	140–220
R (Millimeter)	220–300
Sub-millimeter	> 330

No	Millimeter Wave band	Frequency range.
1.	Q band	30 to 50 GHz.
2.	U band	40 to 60 GHz
3.	V band	50 to 75 GHz.
4.	E band	60 to 90 GHz.
5.	W band	75 to 110 GHz.
6.	F. band	90 to 140 GHz.
7.	D. band	110 to 170 GHz.
8.	G. band	170 to 300 GHz.

Submillimeter $\geq$ 300 GHz To 3000 GHz

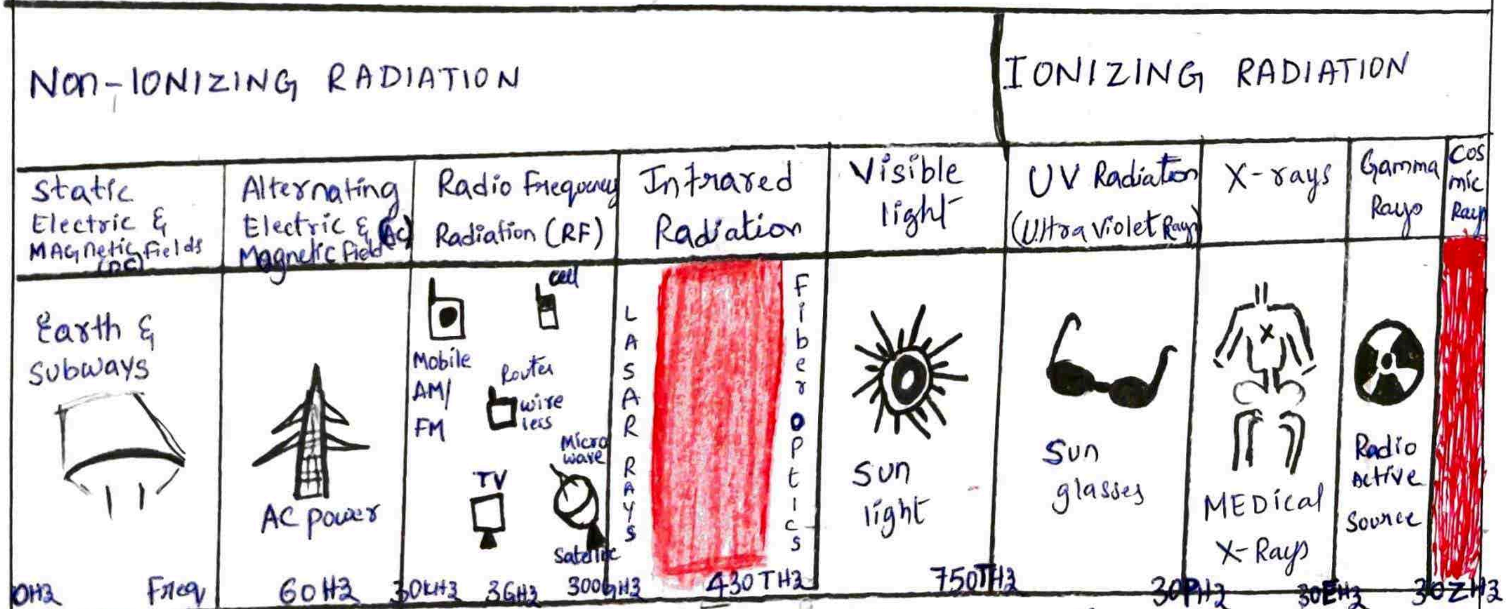
S.No	Prefix	Power of Ten (or) Factor	Symbol.
1.	Milli	10^{-3}	m
2.	centi	10^{-2}	c
2.	Micro	10^{-6}	μ
3.	nano	10^{-9}	n
4.	Pico	10^{-12}	P
5.	femto	10^{-15}	f
6.	alto	10^{-18}	a

Electromagnetic Spectrum:

The whole Range of Frequencies / wavelength of the Electromagnetic wave arranged in the an Order is know as Electromagnetic Spectrum.

- The Electromagnetic Spectrum extends from low Frequencies (long wavelength) used for electrical power distribution, to radio Frequencies for radio, TV, cell phones, microwave ovens. and wireless network through to short wavelength high gamma radiation at the opposite end of the Frequency spectrum.

ElectroMagnetic OR FREQUENCY SPECTRUM.



K-Kilo → 10^3 ; M-Mega → 10^6 ; G-Giga → 10^9 ; T-tera → 10^{12} ; P-Peta → 10^{15} ; E-Exa-10^{18} ; Z-zetta-10^{21} ; Y-Yotta-10^{24}

ado Astronomy,
communication.

Electromagnetic Spectrum and Bands.

The electromagnetic (EM) spectrum is the range of all types of EM radiation

"Rabbits Mate In Very Unusual eXpensive Gardens" (memonic rembering)

Gamma-ray	X-ray	Ultraviolet Waves (U.V rays)	Visible	Infrared Wave (IR rays)	Micro Wave.	Radio Wave
PET Scan	Medical (Body X-ray)	UV light from the Sun		TV Remote Control	Aircraft communication	AM radio
Terrestrial gamma-ray flashes	Airport Security Scanner			Night Vision goggles	Microwave oven	Amateur Radio

Table 1.4 US/Military Microwave Frequency Bands

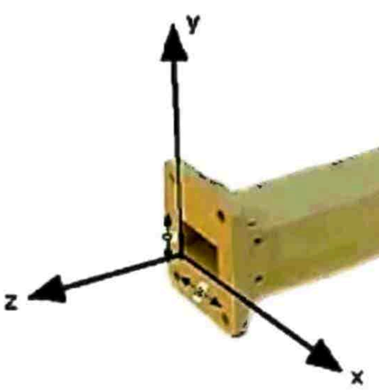
Band Designation	Frequency Range (GHz)
A	0.1–0.25
B	0.25–0.5
C	0.5–1.0
D	1.0–2.0
E	2.0–3.0
F	3.0–4.0
G	4.0–6.0
H	6.0–8.0
I	8.0–10.0
J	10.0–20.0
K	20.0–40.0
L	40.0–60.0
M	60.0–100.0
N	100.0–140.0

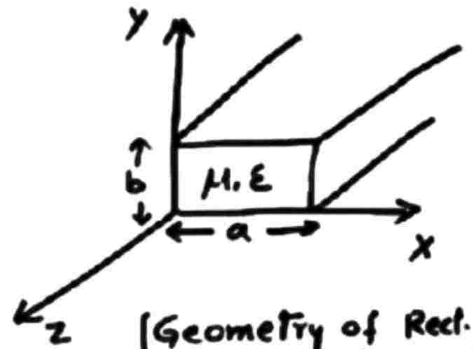
Table 1.5 Propagation characteristics and applications of various bands

Band	Frequecy	Wavelength	Propagation characteristics	Applications
ELF	30–300 Hz	10–1 Mm	Penetration into earth and sea. Very low	Communication with submarines.
VLF	3–30 kHz	100–10 km	Surface wave upto 1000 km. Sky wave in the night extends range. Low attenuation both during day and night. Very reliable.	Long distance point to-point communication.
LF	30–300 kHz	10–1 km	Surface wave and sky wave at night. Surface wave attenuation greater than VHF	Point-to-point marine communication, time standard frequency broadcast.

(Contd...)

Band	Frequecy	Wavelength	Propagation characteristics	Applications
MF	300–3000 kHz	1000–100 m	Ground wave during day and in addition sky wave at night. Attenuation high in daytime and low at night time.	Broadcasting and marine communication.
HF	3–30 MHz	100–10 m	Reflection from ionosphere and varies as per time of day. Season frequency. Quality generally poor	Moderate and long distance communication of all types.
VHF	30–300 MHz	10–1 m	Space wave line of sight.	Television FM service, aviation and police
UHF	300–3000 MHz	100–10 cm	Same as VHF. Affected by tall objects like hills skyscrapers.	Short distance communication including Radar.
SHF	3–30 GHz	10–1 cm	Same as VHF, UHF suffers atmosphere attenuation above 10 GHz	Radar, microwave and space communication
EHF	30–300Hz	10–1 cm	Same as above (SHF)	Same as above





[Geometry of Rect. Waveguide]

4.3.3 Propagation of Waves in Rectangular Waveguides

Consider a rectangular waveguide situated in the rectangular coordinate system with its breadth along x-axis, width along y-axis and the wave is assumed to propagate along the z-direction. Waveguide is filled with air as dielectric as shown in Fig. 4.27.

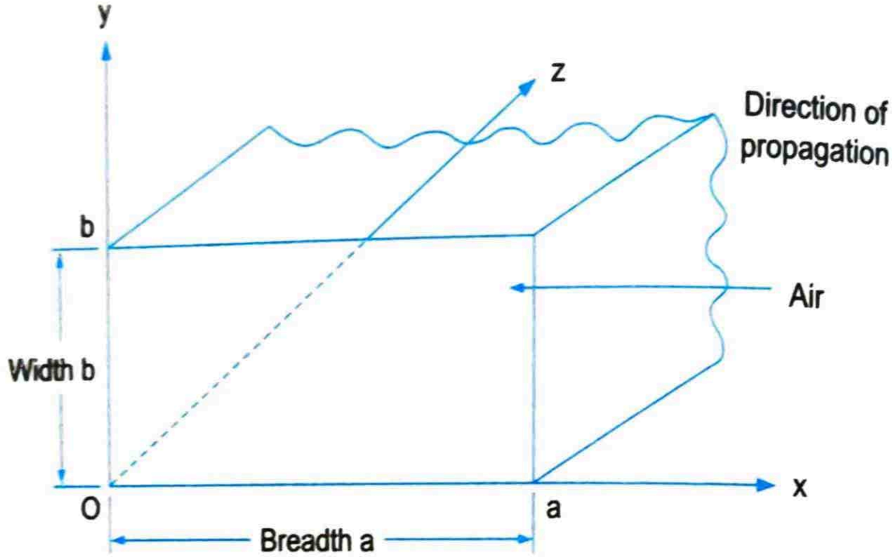


Fig. 4.27 Propagation through a rectangular waveguide.

The wave equation for TE and TM waves are given by

$$\Delta^2 \mathbf{H}_z = -\omega^2\mu\varepsilon \mathbf{H}_z \quad \text{for TE wave } (\mathbf{E}_z = 0)$$

$$\Delta^2 \mathbf{E}_z = -\omega^2\mu\varepsilon \mathbf{E}_z \quad \text{for TM wave } (\mathrm{H}_z = 0)$$

Expanding $\Delta^2\mathbf{E}_z$ in rectangular coordinate system

$$\frac{\partial^2 \mathbf{E}_z}{\partial x^2} + \frac{\partial^2 \mathbf{E}_z}{\partial y^2} + \frac{\partial^2 \mathbf{E}_z}{\partial z^2} = -\omega^2\mu\varepsilon \mathbf{E}_z \qquad ...(4.23)$$

Since the wave is propagating in the 'z' direction we have the operator.

$$\frac{\partial^2}{\partial z^2} = \gamma^2$$

Substituting this operator in Eq. 4.23, we get

$$\frac{\partial^2 \mathbf{E}_z}{\partial x^2} + \frac{\partial^2 \mathbf{E}_z}{\partial y^2} + \gamma^2 \mathbf{E}_z = -\omega^2\mu\varepsilon \mathbf{E}_z \qquad ...(4.24)$$

or

$$\frac{\partial^2 \mathbf{E}_z}{\partial x^2} + \frac{\partial^2 \mathbf{E}_z}{\partial y^2} + (\gamma^2 + \omega^2\mu\varepsilon)\, \mathbf{E}_z = 0 \qquad ...(4.25)$$

Let $\gamma^2 + \omega^2\mu\varepsilon = h^2$, be a constant, then Eq. 4.25 can be rewritten as

$$\frac{\partial^2 \mathbf{E}_z}{\partial x^2} + \frac{\partial^2 \mathbf{E}_z}{\partial y^2} + h^2 \mathbf{E}_z = 0 \quad \text{for TM wave} \qquad ...(4.26)$$

Similarly, $\dfrac{\partial^2 \mathbf{H}_z}{\partial x^2} + \dfrac{\partial^2 \mathbf{H}_z}{\partial y^2} + h^2 \mathbf{H}_z = 0$ for TM wave ...(4.27)

By solving the above partial differential equations, we get solutions for E_z and H_z. Using Maxwell's equation, it is possible to find the various components along x and y directions $[\mathbf{E}_x, \mathbf{H}_x, \mathbf{E}_y, \mathbf{H}_y]$

From Maxwell's 1st equation, we have

$$\nabla \times \mathbf{H} = j\omega\varepsilon \mathbf{E}$$

Expanding $\nabla \times \mathbf{H}$,

i.e.,
$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \mathbf{H}_x & \mathbf{H}_y & \mathbf{H}_z \end{vmatrix} = j\omega\varepsilon\ [\hat{i}\mathbf{E}_x + \hat{j}\mathbf{E}_y + \hat{k}\mathbf{E}_z]$$

Replacing $\dfrac{\partial}{\partial z} = -\gamma$ (an operator), we get

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & -\gamma \\ \mathbf{H}_x & \mathbf{H}_y & \mathbf{H}_z \end{vmatrix} = j\omega\varepsilon\ [\hat{i}\mathbf{E}_x + \hat{j}\mathbf{E}_y + \hat{k}\mathbf{E}_z]$$

Equating coefficients of $\hat{i}$, $\hat{j}$ and $\hat{k}$ (after expanding), we get

$$\frac{\partial \mathbf{H}_z}{\partial y} + \gamma \mathbf{H}_y = j\omega\varepsilon \mathbf{E}_x \qquad ...(4.28)$$

$$\frac{\partial \mathbf{H}_z}{\partial x} + \gamma \mathbf{H}_x = -j\omega\varepsilon \mathbf{E}_y \qquad ...(4.29)$$

$$\frac{\partial \mathbf{H}_y}{\partial x} - \frac{\partial \mathbf{H}_x}{\partial y} = j\omega\varepsilon \mathbf{E}_z \qquad ...(4.30)$$

Similarly from Maxwell's 2nd equation, we have

$$\nabla \times \mathbf{E} = -j\omega\mu \mathbf{H}$$

Expanding $\nabla \times \mathbf{E}$, we get

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \mathbf{E}_x & \mathbf{E}_y & \mathbf{E}_z \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & -\gamma \\ \mathbf{E}_x & \mathbf{E}_y & \mathbf{E}_z \end{vmatrix} = -j\omega\mu\ [\hat{i}\mathbf{H}_x + \hat{j}\mathbf{H}_y + \hat{k}\mathbf{H}_z]$$

Expanding and equating coefficients of $\hat{i}, \hat{j}$ and $\hat{k}$, we get

$$\frac{\partial \mathbf{E}_z}{\partial y} + \gamma \mathbf{E}_y = -j\omega\mu \mathbf{H}_x \quad ...(4.31)$$

$$\frac{\partial \mathbf{E}_z}{\partial x} + \gamma \mathbf{E}_x = +j\omega\mu \mathbf{H}_y \quad ...(4.32)$$

$$\frac{\partial \mathbf{E}_y}{\partial x} - \frac{\partial \mathbf{E}_x}{\partial y} = -j\omega\mu \mathbf{H}_z \quad ...(4.33)$$

Combining Eq. 4.28 and Eq. 4.32 to eliminate H_y, we get an expression for E_x. From Eq. 4.32,

$$\mathbf{H}_y = \frac{1}{j\omega\mu}\frac{\partial \mathbf{E}_z}{\partial x} + \frac{\gamma}{j\omega\mu}\mathbf{E}_x$$

Substituting for H_y in Eq. 4.28, we get

$$\frac{\partial \mathbf{H}_z}{\partial y} + \frac{\gamma}{j\omega\mu}\frac{\partial \mathbf{E}_z}{\partial x} + \frac{\gamma^2}{j\omega\mu}\mathbf{E}_x = j\omega\varepsilon \mathbf{E}_x$$

or
$$\mathbf{E}_x\left[j\omega\varepsilon - \frac{\gamma^2}{j\omega\mu}\right] = \frac{\gamma}{j\omega\mu}\frac{\partial \mathbf{E}_z}{\partial x} + \frac{\partial \mathbf{H}_z}{\partial y}$$

Multiplying by $j\omega\mu$, we get

$$\mathbf{E}_x\,[-\omega^2\mu\varepsilon - \gamma^2) = \gamma\frac{\partial \mathbf{E}_z}{\partial x} + j\omega\mu\frac{\partial \mathbf{H}_z}{\partial y}$$

or
$$\mathbf{E}_x\,[-(\gamma^2 + \omega^2\mu\varepsilon)] = \gamma\frac{\partial \mathbf{E}_z}{\partial x} + j\omega\mu\frac{\partial \mathbf{H}_z}{\partial y}$$

where $\gamma^2 + \omega^2\mu\varepsilon = h^2$

Dividing by $-h^2$ throughout, we get

$$\mathbf{E}_x = \frac{-\gamma}{h^2}\frac{\partial \mathbf{E}_z}{\partial x} - \frac{j\omega\mu}{h^2}\frac{\partial \mathbf{H}_z}{\partial y} \quad ...(4.34)$$

Similarly
$$\mathbf{E}_y = \frac{-\gamma}{h^2}\frac{\partial \mathbf{E}_z}{\partial y} + \frac{j\omega\mu}{h^2}\frac{\partial \mathbf{H}_z}{\partial x} \quad ...(4.35)$$

and
$$\mathbf{H}_x = \frac{-\gamma}{h^2}\frac{\partial \mathbf{H}_z}{\partial x} + \frac{j\omega\varepsilon}{h^2}\frac{\partial \mathbf{E}_z}{\partial y} \quad ...(4.36)$$

and
$$\mathbf{H}_y = \frac{-\gamma}{h^2}\frac{\partial \mathbf{H}_z}{\partial y} - \frac{j\omega\varepsilon}{h^2}\frac{\partial \mathbf{E}_z}{\partial x} \quad ...(4.37)$$

These, equations give a general relationship for field components within a waveguide.

Rectangular Wave guides - Solution of wave Equation in Rectangular co-ordinate

Points to remember:

Note: If the medium for propagation of Electromagnetic wave is air or free space. the Maxwell's equation are

1. Maxwell's 1st equation

$$\boxed{\nabla \times \bar{H} = j\omega\varepsilon E}$$

2. Maxwell's 2nd equation

$$\boxed{\nabla \times \bar{E} = -j\omega\mu H}$$

3. Maxwell's 3rd equation

$$\boxed{\nabla^2 E = -\omega^2 \mu \varepsilon \bar{E}}$$

Resolving $\bar{E}$ into 3 mutually perpendicular directions we get

$$\boxed{\bar{E} = E_x \hat{i} + E_y \hat{j} + E_z \hat{k}}$$

LHS $\quad \nabla^2 E = \nabla^2 (E_x \hat{i} + E_y \hat{j} + E_z \hat{k})$

RHS $\quad -\omega^2 \mu \varepsilon \bar{E} = -\omega^2 \mu \varepsilon (E_x \hat{i} + E_y \hat{j} + E_z \hat{k})$

where $\hat{i}, \hat{j}, \hat{k}$ are unit vectors along x, y, z directions respectively.

Equating co-efficient of $\hat{i}$, $\hat{j}$ & $\hat{k}$ on The both si... ProPa...

$$\nabla^2(E_x\hat{i}+\hat{j}E_y+\hat{k}E_z) = -\omega^2\mu\varepsilon[E_x\hat{i}+\hat{j}E_y+\hat{k}E$$

Con...

the...

$$\nabla^2 E_x = -\omega^2\mu\varepsilon E_x$$

$$\nabla^2 E_y = -\omega^2\mu\varepsilon E_y$$

$$\nabla^2 E_z = -\omega^2\mu\varepsilon E_z$$

lly Similarly $\nabla^2\bar{H} = -\omega^2\mu\varepsilon\bar{H}$

$$\nabla^2\bar{H}_x = -\omega^2\mu\varepsilon\bar{H}_x$$

$$\nabla^2\bar{H}_y = -\omega^2\mu\varepsilon\bar{H}_y.$$

$$\nabla^2\bar{H}_z = -\omega^2\mu\varepsilon\bar{H}_z$$

In general wave equation can be written as

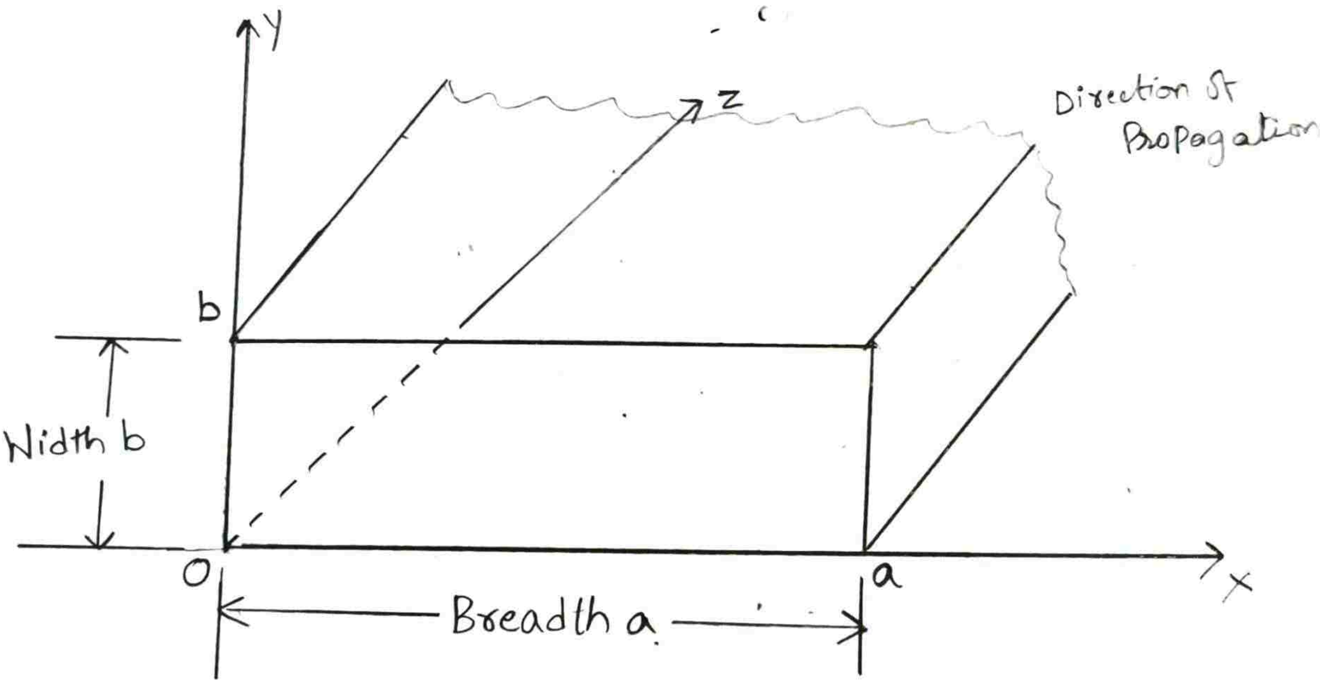
$$\nabla^2 E = -\omega^2\mu\varepsilon E.$$

$$\nabla^2 H = -\omega^2\mu\varepsilon H$$

Propagation of waves in Rectangular wave guides

Consider a Rectangular waveguide situated in the rectangular co-ordinate system with its Breadth (a) along x-axis, width (b) along y-axis & the wave is assumed to propagate along the z-direction.

* let us consider waveguide is filled with air as dielectric as show in fig.



Propagation through a rectangular waveguide

* the wave equation for TE and TM waves are given by, as the wave propagates in the z-direction.

$$\nabla^2 H_z = -\omega^2 \mu \epsilon H_z \text{ for TE wave } (E_z = 0$$

$$\nabla^2 E_z = -\omega^2 \mu \varepsilon E_z \text{ for TM wave } (H_z = 0,$$

For TM Waves :-

Expending $\nabla^2 E_z = -\omega^2 \mu \varepsilon E_z$ in Rectangular Coordinate System

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} = -\omega^2 \mu \varepsilon E_z \quad \text{--- (1)}$$

let us Consider

$$\frac{\partial^2}{\partial z^2} = \gamma^2$$

where "γ" is an Operator.

Since the wave is propagating in the z-direction we have the Operator.

Substituting the operator in the equation (1)

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} = -\omega^2 \mu \varepsilon E_z \quad \text{--- (1)}$$

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \gamma^2 E_z = -\omega^2 \mu \varepsilon E_z \quad \text{--- (2)}$$

$E_z = 0$

$H_z = 0$

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \gamma^2 E_z + \omega^2 \mu \varepsilon E_z = 0$$

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + E_z(\gamma^2 + \omega^2 \mu \varepsilon) = 0 \quad \text{--- (3)}$$

let $\boxed{\gamma^2 + \omega^2 \mu \varepsilon = h^2}$, be a Constant, then equation can be re-write as

TM wave
$$\boxed{\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + E_z h^2 = 0} \quad \text{--- (4)}$$

For TE wave:

Expanding $\nabla^2 H_z = -\omega^2 \mu \varepsilon H_z$ in Rectangular co-ordinate System.

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + \frac{\partial^2 H_z}{\partial z^2} = -\omega^2 \mu \varepsilon H_z \quad \text{--- (5)}$$

$$\text{let } \frac{\partial^2}{\partial z^2} = \gamma^2$$

Substituting the operator in equation (5)

$$\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + \gamma^2 H_z = -\omega^2 \mu \varepsilon H_z \quad \text{--- (6)}$$

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + \gamma^2 H_z = -\omega^2 \mu \varepsilon H_z$$

e

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + \gamma^2 H_z + \omega^2 \mu \varepsilon H_z = 0$$

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + H_z (\gamma^2 + \omega^2 \mu \varepsilon) = 0$$

$$\boxed{\gamma^2 + \omega^2 \mu \varepsilon = h^2}$$

TE wave.

$$\boxed{\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + h^2 H_z = 0} \quad \text{--- (7)}$$

By solving the above partial differential equation (4) & (5), we get solutions for E_z and H_z

Using the Maxwell's equation, it is possible to find the various components along 'x' and 'y' direction, they are E_x, H_x, E_y, H_y.

(*) From maxwell's 1st equation, we have

$$\boxed{\nabla \times \bar{H} = j\omega\varepsilon\bar{E}} \Rightarrow (8), (9), (10)$$

Expanding $\nabla \times \bar{H}$,

i,e
$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} = j\omega\varepsilon\left[E_x \hat{i} + \hat{j} E_y + \hat{k} E_z\right]$$

Replacing $\frac{\partial}{\partial z} = -\gamma$ (an operator), we get

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & -\gamma \\ H_x & H_y & H_z \end{vmatrix} = j\omega\varepsilon\left[\hat{i}\,E_x + \hat{j}\,E_y + \hat{k}\,E_z\right]$$

Equating coefficients of $\hat{i}, \hat{j}, \hat{k}$ (after expanding), we get

$$\hat{i}\left[\frac{\partial}{\partial y}H_z - (-\gamma)H_y\right] - \hat{j}\left[\frac{\partial}{\partial x}H_z + \gamma H_x\right] + \hat{k}\left[\frac{\partial}{\partial x}H_y - \frac{\partial}{\partial y}H_x\right] =$$

$$j\omega\varepsilon\left[\hat{i}\,E_x + \hat{j}\,E_y + \hat{k}\,E_z\right].$$

$$\hat{i}\left[\frac{\partial}{\partial y}H_z + \gamma H_y\right] - \hat{j}\left[\frac{\partial H_z}{\partial x} + \gamma H_x\right] + \hat{k}\left[\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right] =$$

$$\hat{i}\left[j\omega\varepsilon E_x\right] + \hat{j}\left[j\omega\varepsilon E_y\right] + \hat{k}\left[j\omega\varepsilon E_z\right]$$

$$\boxed{\frac{\partial H_z}{\partial y} + \gamma H_y = j\omega\varepsilon E_x} \longrightarrow (8)$$

$$\boxed{\frac{\partial H_z}{\partial x} + \gamma H_x = j\omega\varepsilon E_y} \longrightarrow (9)$$

$$\boxed{\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j\omega\varepsilon E_z} \longrightarrow (10)$$

(*) Similary from Maxwell's (2)nd Equation we have

$$\boxed{\nabla \times E = -j\omega\mu H} \Rightarrow (11), (12),$$

Expanding $\nabla \times E$, we get

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \partial/\partial y & \partial/\partial z \\ E_x & E_y & E_z \end{vmatrix} = -j\omega\mu\left[H_x \hat{i} + H_y \hat{j} + H_z \hat{k}\right]$$

$$\boxed{\frac{\partial}{\partial z} = -\gamma}$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & -\gamma \\ E_x & E_y & E_z \end{vmatrix} = -j\omega\mu\left[\hat{i} H_x + \hat{j} H_y + \hat{k} H_z\right]$$

Expanding and equating co-efficient of $\hat{i}$, $\hat{j}$ and $\hat{k}$, we get

$$\hat{i}\left[\frac{\partial}{\partial y}E_z - (-\gamma)E_y\right] - \hat{j}\left[\frac{\partial}{\partial x}E_z - (-\gamma)E_x\right] + \hat{k}\left[\frac{\partial}{\partial x}E_z + \gamma E_x\right] =$$

$$\hat{i}\left[-j\omega\mu H_x\right] + \hat{j}\left[-j\omega\mu H_y\right] +$$

$$\hat{k}\left[-j\omega\mu H_z\right]$$

$$\boxed{\frac{\partial E_z}{\partial y} + \gamma E_y = -j\omega\mu H_x} \rightarrow (11)$$

$$\boxed{\frac{\partial E_z}{\partial x} + \gamma E_x = +j\omega\mu H_y} \rightarrow (12)$$

$$\boxed{\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\mu H_z} \rightarrow (13)$$

ave

(13) Combining Equation (8) and Equation (12) to Eliminate Hy, we get an Expression for Ex

From equation (12)

$$\frac{\partial E_z}{\partial x} + \gamma E_x = +j\omega\mu H_y \quad \text{—— (12)}$$

$$\boxed{H_y = \frac{1}{j\omega\mu}\frac{\partial E_z}{\partial x} + \frac{\gamma}{j\omega\mu} E_x}$$

Subsituting for Hy in the equation (8)

$$\boxed{\frac{\partial H_z}{\partial y} + \gamma H_y = j\omega\varepsilon E_x} \quad \text{—— (8)}$$

$$\frac{\partial H_z}{\partial y} + \gamma\left[\frac{1}{j\omega\mu}\frac{\partial E_z}{\partial x} + \frac{\gamma}{j\omega\mu}E_x\right] = j\omega\varepsilon E_x$$

$$\frac{\partial H_z}{\partial y} + \frac{\gamma}{j\omega\mu}\frac{\partial E_z}{\partial x} + \underline{\frac{\gamma^2}{j\omega\mu}E_x} = \underline{j\omega\varepsilon E_x}$$

$$E_x\left[j\omega\varepsilon - \frac{\gamma^2}{j\omega\mu}\right] = \frac{\partial H_z}{\partial y} + \frac{\gamma}{j\omega\mu}\frac{\partial E_z}{\partial x}$$

multiplying jwμ on Both sides,

we get.   substi

$$E_x\left[j^2\omega^2\mu\varepsilon-\gamma^2\right]=\gamma\frac{\partial E_z}{\partial x}+j\omega\mu\frac{\partial H_z}{\partial y}$$

$$E_x\left[-\omega^2\mu\varepsilon-\gamma^2\right]=\gamma\frac{\partial E_z}{\partial x}+j\omega\mu\frac{\partial H_z}{\partial y}$$

$$E_x\left[-(\gamma^2+\omega^2\mu\varepsilon)\right]=\gamma\frac{\partial E_z}{\partial x}+j\omega\mu\frac{\partial H_z}{\partial y}$$

$$\boxed{h^2=\gamma^2+\omega^2\mu\varepsilon}$$

$$E_x\left[-h^2\right]=\gamma\frac{\partial E_z}{\partial x}+j\omega\mu\frac{\partial H_z}{\partial y}$$

Dividing by $-h^2$ on Both the side.

$$\boxed{E_x=\frac{-\gamma}{h^2}\frac{\partial E_z}{\partial x}-\frac{j\omega\mu}{h^2}\frac{\partial H_z}{\partial y}} \quad \text{—— (14)}$$

Ily Combing the equation (9) and equation (11) to eliminate H_x, we get an expression for E_y.

From eq (11)

$$\frac{\partial E_z}{\partial y}+\gamma E_y=-j\omega\mu H_x \quad \text{—— (11)}$$

$$H_x=\frac{-1}{j\omega\mu}\frac{\partial E_z}{\partial y}-\frac{\gamma}{j\omega\mu}E_y$$

Substituting for H_x in Equation (9) we get

$$\boxed{\frac{\partial H_z}{\partial x} + \gamma H_x = -j\omega\varepsilon E_y} \quad (9)$$

$$\boxed{H_x = \frac{-1}{j\omega\mu}\frac{\partial E_z}{\partial y} - \frac{\gamma}{j\omega\mu}E_y} \rightarrow (11) \text{ modified}$$

(modified) $\boxed{\text{Eq. (11) modified in Eq (9)}}$

$$\frac{\partial H_z}{\partial x} + \gamma\left[\frac{-1}{j\omega\mu}\frac{\partial E_z}{\partial y} - \frac{\gamma}{j\omega\mu}E_y\right] = -j\omega\varepsilon E_y$$

$$\frac{\partial H_z}{\partial x} - \frac{\gamma}{j\omega\mu}\frac{\partial E_z}{\partial y} - \frac{\gamma^2}{j\omega\mu}E_y = -j\omega\varepsilon E_y$$

$$\frac{\partial H_z}{\partial x} - \frac{\gamma}{j\omega\mu}\frac{\partial E_z}{\partial y} = -j\omega\varepsilon E_y + \frac{\gamma^2}{j\omega\mu}E_y$$

$$\frac{\partial H_z}{\partial x} - \frac{\gamma}{j\omega\mu}\frac{\partial E_z}{\partial y} = E_y\left[\frac{\gamma^2}{j\omega\mu} - j\omega\varepsilon\right]$$

multiplying by $j\omega\mu$ on both side we get

$j^2 = -1$

$$j\omega\mu\frac{\partial H_z}{\partial x} - \gamma\frac{\partial E_z}{\partial y} = E_y\left[\gamma^2 + \omega^2\mu\varepsilon\right]$$

$$\boxed{\gamma^2 + \omega^2\mu\varepsilon = h^2}$$

$$j\omega\mu\frac{\partial H_z}{\partial x} - \gamma\frac{\partial E_z}{\partial y} = E_y\left[h^2\right]$$

$$\boxed{E_y = \frac{j\omega\mu}{h^2}\frac{\partial H_z}{\partial x} - \frac{\gamma}{h^2}\frac{\partial E_z}{\partial y}} \quad \text{— (15)}$$

H_x

* Combining equation (9) and (11) to Eliminate E_y, we get an expression for H_x from Equation (11)

Eq (9) $\boxed{\frac{\partial H_z}{\partial x} + \gamma H_x = -j\omega\varepsilon E_y}$

For Equation (11)

Modified Eq (9) $\boxed{\frac{-1}{j\omega\varepsilon}\frac{\partial H_z}{\partial x} - \frac{\gamma}{j\omega\varepsilon} H_x = E_y}$

Eq (9) in Eq (11) Modified

$$\boxed{\frac{\partial E_z}{\partial y} + \gamma E_y = -j\omega\mu H_x} \quad \text{— eq (11)}$$

$$\frac{\partial E_z}{\partial y} + \gamma\left[\frac{-1}{j\omega\varepsilon}\frac{\partial H_z}{\partial x} - \frac{\gamma}{j\omega\varepsilon} H_x\right] = -j\omega\mu H_x$$

$$\frac{\partial E_z}{\partial y} - \frac{\gamma}{j\omega\varepsilon}\frac{\partial H_z}{\partial x} - \frac{\gamma^2}{j\omega\varepsilon} H_x = -j\omega\mu H_x$$

$$\frac{\partial E_z}{\partial y} - \frac{\gamma}{j\omega\varepsilon}\frac{\partial H_z}{\partial x} = -j\omega\mu H_x + \frac{\gamma^2}{j\omega\varepsilon} H_x$$

$$\frac{\partial E_z}{\partial y} - \frac{\gamma}{j\omega\varepsilon}\frac{\partial H_z}{\partial x} = H_x\left[-j\omega\mu + \frac{\gamma^2}{j\omega\varepsilon}\right]$$

multiplying Both the side by $j\omega\varepsilon$

$$j\omega\varepsilon\frac{\partial E_z}{\partial y} - \gamma\frac{\partial H_z}{\partial x} = H_x\left[-j^2\omega^2\varepsilon\mu + \gamma^2\right]$$

$$= H_x\left[-(-1)\omega^2\varepsilon\mu + \gamma^2\right] \qquad \boxed{j^2 = -1}$$

(15) $$H_x[\omega^2\mu\varepsilon + \gamma'^2] = j\omega\varepsilon \frac{\partial E_z}{\partial y} - \gamma \frac{\partial H_z}{\partial x}$$

$$\boxed{\omega^2\mu\varepsilon + \gamma'^2 = h^2}$$

$$H_x[h^2] = j\omega\varepsilon \frac{\partial E_z}{\partial y} - \gamma \frac{\partial H_z}{\partial x}$$

$$\boxed{H_x = \frac{j\omega\varepsilon}{h^2} \frac{\partial E_z}{\partial y} - \frac{\gamma}{h^2} \frac{\partial H_z}{\partial x}}$$

Combining equation (8) and (12) to Eliminate E_x

$$\boxed{\frac{\partial H_z}{\partial y} + \gamma H_y = j\omega\varepsilon E_x} \quad \text{— Eq (8).}$$

$$\boxed{\frac{\partial E_z}{\partial x} + \gamma E_x = j\omega\mu H_y} \quad \text{— Eq (12)}$$

from Eq (8) Modified as $$\boxed{E_x = \frac{1}{j\omega\varepsilon} \frac{\partial H_z}{\partial y} + \frac{\gamma}{j\omega\varepsilon} H_y}$$

Put Eq (8) in eq (12)

$$\frac{\partial E_z}{\partial x} + \gamma\left[\frac{1}{j\omega\varepsilon} \frac{\partial H_z}{\partial y} + \frac{\gamma}{j\omega\varepsilon} H_y\right] = j\omega\mu H_y$$

$$\frac{\partial E_z}{\partial x} + \frac{\gamma}{j\omega\varepsilon} \frac{\partial H_z}{\partial y} + \frac{\gamma'^2}{j\omega\varepsilon} H_y = j\omega\mu H_y$$

$$\frac{\partial E_z}{\partial x} + \frac{\gamma}{j\omega\varepsilon}\frac{\partial H_z}{\partial y} + \frac{\gamma^2}{j\omega\varepsilon} H_y = j\omega\mu H_y$$

these
gl

$$\frac{\partial E_z}{\partial x} + \frac{\gamma}{j\omega\varepsilon}\frac{\partial H_z}{\partial y} = j\omega\mu H_y - \frac{\gamma^2}{j\omega\varepsilon} H_y$$

$$\frac{\partial E_z}{\partial x} + \frac{\gamma}{j\omega\varepsilon}\frac{\partial H_z}{\partial y} = H_y\left(j\omega\mu - \frac{\gamma^2}{j\omega\varepsilon}\right)$$

multiplying Both side by $j\omega\varepsilon$

$$j\omega\varepsilon\frac{\partial E_z}{\partial x} + \gamma\frac{\partial H_z}{\partial y} = H_y\left[j^2\omega^2\varepsilon\mu - \gamma^2\right]$$

$(j^2 = -1)$

$$j\omega\varepsilon\frac{\partial E_z}{\partial x} + \gamma\frac{\partial H_z}{\partial y} = H_y\left[-\omega^2\varepsilon\mu - \gamma^2\right]$$

$$j\omega\varepsilon\frac{\partial E_z}{\partial x} + \gamma\frac{\partial H_z}{\partial y} = -H_y\left[\gamma^2 + \omega^2\varepsilon\mu\right]$$

$\boxed{\omega^2\varepsilon\mu + \gamma^2 = h^2}$

$$j\omega\varepsilon\frac{\partial E_z}{\partial x} + \gamma\frac{\partial H_z}{\partial y} = -H_y\left[h^2\right]$$

$$\boxed{H_y = \frac{-j\omega\varepsilon}{h^2}\frac{\partial E_z}{\partial x} - \frac{\gamma}{h^2}\frac{\partial H_z}{\partial y}} \quad \text{---} (17)$$

These equations from 14 to 17 give general relationship for field components with in a wave guide.

$$\boxed{E_x = -\frac{\gamma}{h^2}\frac{\partial E_z}{\partial x} - \frac{j\omega\mu}{h^2}\frac{\partial H_z}{\partial y}} \quad \text{—— (14)}$$

$$\boxed{E_y = -\frac{\gamma}{h^2}\frac{\partial E_z}{\partial y} + \frac{j\omega\mu}{h^2}\frac{\partial H_z}{\partial x}} \quad \text{—— (15)}$$

$$\boxed{H_x = -\frac{\gamma}{h^2}\frac{\partial H_z}{\partial x} + \frac{j\omega\varepsilon}{h^2}\frac{\partial E_z}{\partial y}} \quad \text{—— (16)}$$

$$\boxed{H_y = -\frac{\gamma}{h^2}\frac{\partial H_z}{\partial y} - \frac{j\omega\varepsilon}{h^2}\frac{\partial E_z}{\partial x}} \quad \text{—— (17)}$$

The above equations 14, 15, 16 & 17 are expressions for fields in Rectangular waveguides.

4.3.5 TE and TM Modes

The electromagnetic wave inside a waveguide can have an infinite number of patterns which are called *modes*. We know that an electromagnetic wave consists of magnetic and electric fields which are always perpendicular to each other. The fields in the waveguide which make up these mode patterns must obey certain physical laws. At the surface of a conductor, the electric field cannot have a component parallel to the surface. This indicates that the electric field must always be perpendicular to the surface at a conductor. The magnetic field on the other hand is always parallel to the surface of the conductor and cannot have a component perpendicular to it at the surface.

In general, there are two kinds of modes in a waveguide. In the first type, the electric field is always transverse to the direction of propagation and is called the ***Transverse Electric*** or ***TE*** wave. In the second type, the magnetic field is always transverse to the direction of propagation and is called the ***Transverse Magnetic*** or ***TM*** wave. Thus in a TE mode, no electric line is in direction of propagation *i.e.*, $\mathbf{E}_z = 0$, if z is the direction of the propagation. But $\mathbf{H}_z \neq 0$. In a TM mode, no magnetic line is in direction of propagation *i.e.*, $\mathbf{H}_z = 0$ but $\mathbf{E}_z \neq 0$.

Field Patterns

Figure 4.28 shows the field pattern for a TE wave. Solid lines depict electric field lines or voltage lines and dotted lines depict magnetic field lines.

We use subscript for designating a particular mode. TE_{mn} or TM_{mn} where 'm' indicates the number of half wave variations of the electric field (or magnetic field in a TM) across the wider dimension a, of the waveguide and 'n' indicates the number across the narrow dimension b. Referring to TE pattern shown in Fig. 4.28 it can be seen that the voltage varies from '0' to maximum and maximum to '0' across the wide dimension a. This is one half variation. Hence $m = 1$. Across the narrow dimension, there is no variation in voltage v. Hence $n = 0$. Therefore this mode is TE_{10} mode. The mode having the *highest cutoff wavelength* is known as dominant mode of the waveguide and all other modes are called higher modes. For example TE_{10} is the dominant mode for TE waves. It is the mode which is used for practically all electromagnetic transmission in a rectangular waveguide. Dominant mode is almost always a low-loss, distortion less transmission and higher modes result in a significant loss of power and also undesirable harmonic distortion. The radiation pattern for TE mode is shown in Fig. 4.29. Sketches of some higher order TE modes are shown in Fig. 4.30.

4.3.6 Propagation of TM Waves in Rectangular Waveguide

For TM wave, $\mathbf{H}_z = 0; \mathbf{E}_z \neq 0$

The wave equation of a TM wave is

$$\frac{\partial^2 \mathbf{E}_z}{\partial x^2} + \frac{\partial^2 \mathbf{E}_z}{\partial y^2} + h^2 \mathbf{E}_z = 0 \qquad \text{...(4.38)}$$

This is a *partial differential equation* (p.d.e) which can be solved to get the different field components $\mathbf{E}_x$, $\mathbf{E}_y$, $\mathbf{H}_x$ and $\mathbf{H}_y$ by 'separation of variables method'.

Let us assume a solution

$$\mathbf{E}_z = XY \qquad \text{...(4.39)}$$

where, X is a pure function of 'x' only.

Y is a pure function of 'y' only.

Since X and Y are independent variables,

$$\frac{\partial^2 \mathbf{E}_z}{\partial x^2} = \frac{\partial^2 (XY)}{\partial x^2} = Y\frac{\partial^2 x}{\partial x^2}$$

$$\frac{\partial^2 \mathbf{E}_z}{\partial y^2} = \frac{\partial^2 (XY)}{\partial y^2} = Y\frac{\partial^2 y}{\partial y^2}$$

Using these two in Eq. 4.38, we get

$$Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} + h^2XY = 0 \qquad \text{...(4.40)}$$

Dividing throughout by XY, we get

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + h^2 = 0 \qquad \text{...(4.41)}$$

$\frac{1}{X}\frac{d^2X}{dx^2}$ is a pure function of x only.

$\frac{1}{Y}\frac{d^2Y}{dy^2}$ is a pure function of y only.

The sum of these is a constant. Hence each term must be equal to a constant separately since X and Y are independent variables.

We use separation of variables method to solve the differential Eq. 4.41.

Let $$\frac{1}{X}\frac{d^2X}{dx^2} = -B^2 \qquad \text{...(4.42)}$$

and $$\frac{1}{Y}\frac{d^2Y}{dy^2} = -A^2 \qquad \text{...(4.43)}$$

where $-A^2$ and $-B^2$ are constants.

Substituting Eqs. 4.42 and 4.43 in Eq. 4.41, we get

$$-B^2 - A^2 + h^2 = 0$$

or $$h^2 = A^2 + B^2 \qquad \text{...(4.44)}$$

Equations 4.42 and 4.43 are ordinary 2nd order differential equations, the solutions of which are given by,

$$X = C_1 \cos Bx + C_2 \sin Bx \qquad \text{...(4.45)}$$

$$Y = C_3 \cos Ay + C_4 \sin Ay \qquad \text{...(4.46)}$$

where C_1, C_2, C_3, and C_4 are constants which can be evaluated by applying the boundary conditions.

The complete solution is given by Eq. 4.39.

i.e., $$\mathbf{E}_z = X.Y$$

Substituting the values of X and Y from Eq. 4.45 and 4.46, we get

$$\mathbf{E}_z = [C_1, \cos Bx + C_2 \sin Bx]\ [C_3 \cos Ay + C_4 \sin Ay] \qquad ...(4.47)$$

Boundary Conditions

Since the entire surface of the rectangular waveguide acts as a short circuit or ground for electric field, $\mathbf{E}_z$, $= 0$ all along the boundary walls of the waveguide. Since there are four walls, as shown in Fig. 4.31 there are four boundary conditions.

1st boundary condition: [Bottom plane or bottom wall]

We know that $\mathbf{E}_z = 0,$ all along the bottom wall.

i.e., $$\mathbf{E}_z = 0 \text{ at } y = 0 \ \forall\ x \longrightarrow 0 \text{ to } a$$

$\forall$ stands "for all" and $x \longrightarrow 0$ to a means x varying between 0 to a.

2nd boundary condition: [Left side plane or left side wall]

$$\mathbf{E}_z = 0 \text{ at } x = 0 \ \forall\ y \longrightarrow 0 \text{ to } b$$

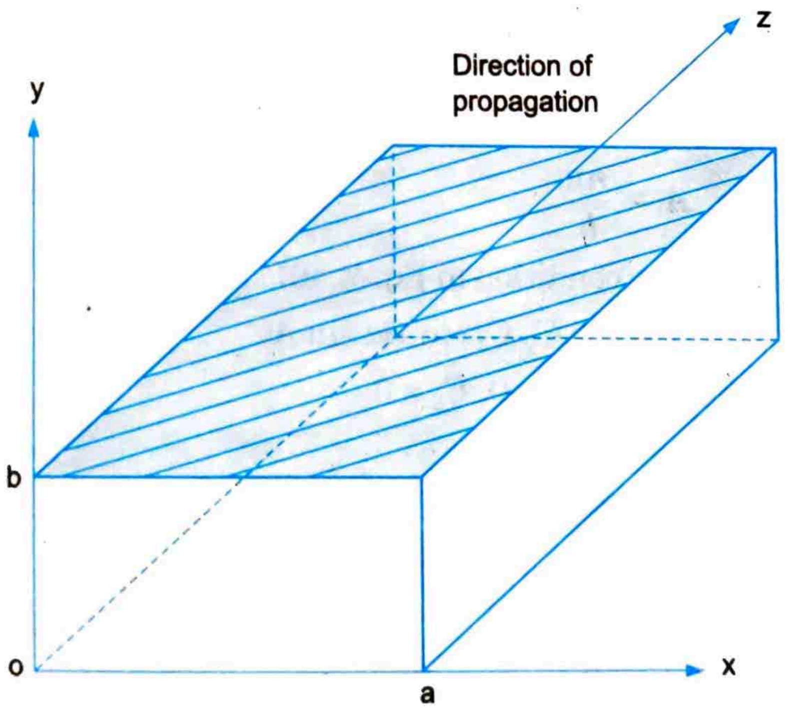


Fig. 4.31

3rd boundary condition : [Top plane or top wall]

$$\mathbf{E}_z = 0 \text{ at } y = b \ \forall\ x \longrightarrow 0 \text{ to } a$$

4th boundary condition : [Right side plane or right side wall]

$$\mathbf{E}_z = 0 \text{ at } x = a \ \forall\ y \longrightarrow 0 \text{ to } b$$

(*i*) Substituting 1st boundary condition in Eq. 4.47, given by

$$\mathbf{E}_z = [C_1 \cos Bx + C_2 \sin Bx]\ [C_3 \cos Ay + C_4 \sin Ay]$$

We have, $E_z = 0$ at $y = 0 \; \forall \; x \longrightarrow 0$ to a.

or $\quad 0 = [C_1 \cos Bx + C_2 \sin Bx] \, C_3 \qquad [\because \cos 0 = 1, \sin 0 = 0]$

This is true for all ($\forall$) $x \longrightarrow 0$ to a

$$C_1 \cos Bx + C_2 \sin Bx \neq 0; \quad \therefore \; C_3 = 0$$

Using this in Eq. 4.47, the solution reduces to,

$$E_z = [C_1 \cos Bx + C_2 \sin Bx] \, [C_4 \sin Ay] \qquad ...(4.48)$$

(ii) Substituting 2nd boundary condition in Eq. 4.48 above, we get

$$E_z = 0 = C_1 \, C_4 \sin Ay \; \forall \; y \longrightarrow 0 \text{ to } b \qquad [\because \cos 0 = 1 \text{ and } \sin 0 = 0]$$

Since $\sin Ay \neq 0$ and $C_4 \neq 0$.

$$C_1 = 0$$

Now using this in Eq. 4.48, the solution further reduces to,

$$E_z = C_2 \, C_4 \sin Bx \sin Ay \qquad ...(4.49)$$

(iii) Substituting 3rd boundary condition in Eq. 4.49 above, we get

$$E_z = 0 = C_2 \, C_4 \sin Bx \sin Ab \; [\text{at } y = b, \forall \; x \longrightarrow 0 \text{ to } a]$$

Since $\sin Bx \neq 0$ $C_4 \neq 0$, $C_2 \neq 0$, otherwise there would be no solution

$$\sin Ab = 0$$

or $\quad Ab = a$ multiple of $\pi = n\pi$

where n is a constant, $n = 0, 1, 2, ...$

$$\therefore \quad A = \frac{n\pi}{b} \qquad ...(4.50)$$

(iv) Substituting 4th boundary condition in Eq. (4.49),

$$E_z = 0 = C_2 \, C_4 \sin Ba \sin Ay \qquad [\text{at } x = a, \forall \; y \longrightarrow 0 \text{ to } b]$$

Since $\quad \sin Ay \neq 0, \; C_4 \neq 0, \; C_2 \neq 0$

$$\sin Ba = 0$$

or $\quad Ba = m\pi$

where m is another constant, $m = 0, 1, 2, ...$

$$\therefore \quad B = \frac{m\pi}{a} \qquad ...(4.51)$$

Now the complete solution is given by, Eq. 4.49

$$E_z = C_2 C_4 \sin Bx \sin Ay$$

where A and B are as in Eqs. 4.50 and 4.51.

i.e.,

$$E_z = C_2 C_4 \sin\left(\frac{m\pi}{a}\right) x \sin\left(\frac{n\pi}{b}\right) y \cdot e^{-\gamma z} \cdot e^{+j\omega t}$$

where, $e^{-\gamma z}$ = propagation along 'z' direction

$e^{j\omega t}$ = sinusoidal variation w.r.t. 't'

Let $C = C_2 C_4$, some other constant.

$$\therefore \quad \mathbf{E}_z = \mathbf{C} \sin\left(m \frac{\pi}{a}\right)x \sin\left(n \frac{\pi}{b}\right)y\, e^{j\omega t - \gamma z} \qquad ...(4.52)$$

Since $\mathbf{E}_z$ is known $\mathbf{E}_x$, $\mathbf{E}_y$, $\mathbf{H}_x$ and $\mathbf{H}_y$ are given by the following equations [from Eqs. 4.31 to 4.34]

$$\mathbf{E}_x = \frac{-\gamma}{h^2}\frac{\partial \mathbf{E}_z}{\partial x} - \frac{j\omega\mu}{h^2}\frac{\partial \mathbf{H}_z}{\partial y}$$

$$\mathbf{E}_x = \frac{-\gamma}{h^2}\frac{\partial \mathbf{E}_z}{\partial x} \quad \text{(as for a TM wave } \mathbf{H}_z = 0)$$

$$\therefore \quad \mathbf{E}_x = \frac{-\gamma}{h^2} C\left(\frac{m\pi}{a}\right)\cos\left(\frac{m\pi}{a}\right)x \sin\left(\frac{n\pi}{b}\right)y\, e^{j\omega t - \gamma z} \qquad ...(4.53)$$

and

$$\mathbf{E}_y = \frac{-\gamma}{h^2}\frac{\partial \mathbf{E}_z}{\partial y} + \frac{j\omega\mu}{h^2}\frac{\partial \mathbf{H}_z}{\partial x}$$

$$\therefore \quad \mathbf{E}_y = \frac{-\gamma}{h^2} c\left(\frac{n\pi}{b}\right)\sin\left(\frac{m\pi}{a}\right)x \cos\left(\frac{n\pi}{b}\right)y\, e^{j\omega t - \gamma z} \qquad ...(4.54)$$

$$\mathbf{H}_x = \frac{-\gamma}{h^2}\frac{\partial \mathbf{H}_z}{\partial x} + \frac{j\omega\varepsilon}{h^2}\frac{\partial \mathbf{E}_z}{\partial y}$$

$$\therefore \quad \mathbf{H}_x = \frac{j\omega\varepsilon}{h^2} C\left(\frac{n\pi}{b}\right)\sin\left(\frac{m\pi}{a}\right)x \cos\left(\frac{n\pi}{b}\right)y\, e^{j\omega t - \gamma z} \qquad ...(4.55)$$

$$\mathbf{H}_y = \frac{-\gamma}{h^2}\frac{\partial \mathbf{H}_z}{\partial y} + \frac{j\omega\varepsilon}{h^2}\frac{\partial \mathbf{E}_z}{\partial x}$$

$$\therefore \quad \mathbf{H}_y = \frac{j\omega\varepsilon}{h^2} C\left(\frac{m\pi}{a}\right)\cos\left(\frac{m\pi}{a}\right)x \sin\left(\frac{n\pi}{b}\right)y\, e^{j\omega t - \gamma z} \qquad ...(4.56)$$

4.3.7 TM Modes in Rectangular Waveguides

Depending on the values of m and n, we have various modes in TM waves. In general, we represent the modes as TM_{mn} where m and n are as defined earlier.

Various TM_{mn} Modes

TM_{00} mode : $m = 0$ and $n = 0$

If $m = 0$ and $n = 0$ are substituted in E_x, E_y, H_x *and* H_y, (Eqs. 4.53 to 4.56), we see that all of them vanish and hence TM_{00} mode cannot exist.

TM_{01} mode : $m = 0$ and $n = 1$

Again, all field components vanish and hence TM_{01} mode cannot exist.

TM_{10} mode : $m = 1$ and $n = 0$

Even now, all field components vanish and hence TM_{10} mode cannot exist.

TM_{11} mode : $m = 1$ and $n = 1$

Now we have all the four components E_x, E_y, H_x and H_y, *i.e.*, TM_{11} mode exists and for all higher values of m and n, the components exist *i.e.*, all higher modes do exist.

4.3.9 Propagation of TE Waves in a Rectangular Waveguide

The TE_{mn} modes in a rectangular waveguide are characterised by $E_z = 0$. In other words the 'z' component of the magnetic field, H_z, must exist in order to have energy transmission in the guide.

The wave equation (Helmhottz equation) for TE wave is given by

$$\Delta^2 H_z = -\omega^2 \mu\varepsilon H_z$$

i.e.,
$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + \frac{\partial^2 H_z}{\partial z^2} = -\omega^2 \mu\varepsilon H_z$$

or
$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + \gamma^2 H_z + \omega^2 \mu\varepsilon H_z = 0 \qquad \left[\because \frac{\partial^2}{\partial z^2} = \gamma^2\right]$$

or
$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + (\gamma^2 + \omega^2 \mu\varepsilon)\, H_z = 0$$

or
$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + h^2 H_z = 0 \qquad \text{...(4.74)}$$

This is a *partial differential equation whose solution can be assumed.*

Assume a solution $H_z = XY$. Where

X is a pure function of 'x' only.

Y is a pure function of 'y' only.

Substituting for H_z in Eq. 4.70, we get

$$Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} + h^2XY = 0$$

Dividing throughout by XY, we get

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + h^2 = 0 \qquad ...(4.75)$$

Here $\frac{1}{X}\frac{d^2X}{dx^2}$ is purely a function of x,

and $\frac{1}{Y}\frac{d^2Y}{dx^2}$ is purely a function of y,

Equating each of these items to a constant, we get

$$\frac{1}{X}\frac{d^2X}{dx^2} = -B^2$$

and

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = -A^2$$

where $-B^2$ and $-A^2$ are constants.

Substituting these in Eq. 4.75 above, we get

$$-B^2 - A^2 + h^2 = 0$$

$$\therefore \qquad h^2 = A^2 + B^2 \qquad ...(4.76)$$

Solving fox X and Y by separation of variable method.

$$X = C_1 \cos Bx + C_2 \sin Bx$$

$$Y = C_3 \cos Ay + C_4 \sin Ay$$

Therefore the complete solution is, $H_z = XY$

i.e.,

$$H_z = (C_1 \cos Bx + C_2 \sin Bx)(C_3 \cos Ay + C_4 \sin Ay) \qquad ...(4.77)$$

where C_1, C_2, C_3 and C_4 are constants which can be evaluated by applying boundary conditions.

Boundary Conditions

As in case of TM waves, we have four boundaries for TE waves also, as shown in Fig. 4.40.

Here since we are considering a TE wave,

$E_z = 0$ but we have components along x and y direction.

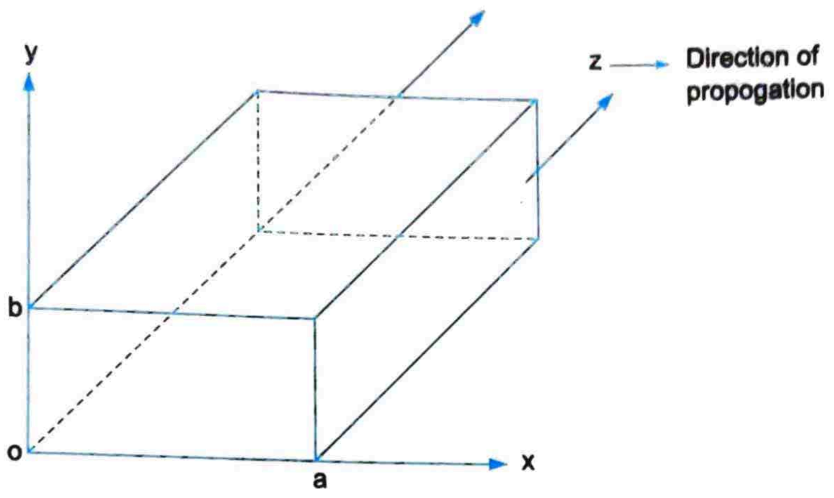


Fig. 4.40

$E_x = 0$ all along bottom and top walls of the waveguide.

$E_y = 0$ all along left and right walls of the waveguide.

1st Boundary Condition :

$$E_x = 0 \text{ at } y = 0 \;\forall\; x \longrightarrow 0 \text{ to } a \text{ (bottom wall)}$$

2nd Boundary Condition :

$$E_x = 0 \text{ at } y = b \;\forall\; x \longrightarrow 0 \text{ to } a \text{ (top wall)}$$

3rd Boundary Condition :

$$E_y = 0 \text{ at } x = 0 \;\forall\; y \longrightarrow 0 \text{ to } b \text{ (left side wall)}$$

4th Boundary Condition :

$$E_y = 0 \text{ at } x = a \;\forall\; y \longrightarrow 0 \text{ to } b \text{ (right side wall)}$$

(i) Substituting 1st Boundary Condition in Eq. 4.77.

Since, 1st Boundary Condition is

$$E_x = 0 \text{ at } y = 0 \;\forall\; x \longrightarrow 0 \text{ to } a, \text{ let us write } E_x \text{ in terms of } H_z.$$

From Eq. 4.31, we have

$$E_x = \frac{-\gamma}{h^2}\frac{\partial E_z}{\partial x} - \frac{j\omega\mu}{h^2}\frac{\partial H_z}{\partial y}.$$

Since $E_z = 0$, the 1st term = 0.

$$\therefore \qquad E_x = \frac{-j\omega\mu}{h^2}\frac{\partial}{\partial y}\left[(C_1 \cos Bx + C_2 \sin Bx)(C_3 \cos Ay + C_4 \sin Ay)\right]$$

i.e.,

$$E_x = \frac{-j\omega\mu}{h^2}(C_1 \cos Bx + C_2 \sin Bx)(-AC_3 \sin Ay + AC_4 \cos Ay)$$

Substituting 1st Boundary condition in the above equation we get

$$0 = \frac{-j\omega\mu}{h^2}(C_1 \cos Bx + C_2 \sin Bx)(0 + AC_4)$$

Since $(C_1 \cos Bx + C_2 \sin Bx) \neq 0, A \neq 0$.

$$C_4 = 0$$

Substituting the value of C_4 in Eq. 4.77, the solution reduces to,

$$H_z = (C_1 \cos Bx + C_2 \sin Bx)(C_3 \cos Ay) \quad \text{...(4.78)}$$

(ii) 3rd Boundary condition :

$$E_y = 0 \text{ at } x = 0 \;\forall\; y \longrightarrow 0 \text{ to } b.$$

From Eq. 4.32 We have,

$$E_y = \frac{-\gamma}{h^2}\frac{\partial E_z}{\partial y} + \frac{j\omega\mu}{h^2}\frac{\partial H_z}{\partial x}.$$

Since $E_z = 0$ and substituting the value of H_z from Eq. 4.78, we get

$$E_y = \frac{j\omega\mu}{h^2}\frac{\partial}{\partial x}[(C_1 \cos Bx + C_2 \sin Bx)\, C_3 \cos Ay]$$

i.e.,

$$E_y = \frac{j\omega\mu}{h^2}[(-BC_1 \sin Bx + BC_2 \cos Bx)\, C_3 \cos Ay]$$

Substituting the 3rd boundary condition,

$$x = 0, \forall\; y \longrightarrow 0 \text{ to } b \text{ in the above equation.}$$

$$0 = \frac{j\omega\mu}{h^2}(0 + BC_2)\, C_3 \cos Ay.$$

Since, $\cos Ay \neq 0, B \neq 0, C_3 \neq 0$.

$$C_2 = 0$$

Substituting the value of C_2 in Eq. 4.78, the solution now reduces to,

$$H_z = C_1 C_3 \cos Bx \cos Ay \quad \text{...(4.79)}$$

(iii) 2nd Boundary Condition :

$$E_x = 0 \text{ at } y = b \;\forall\; x \longrightarrow 0 \text{ to } a.$$

From Eq. 4.31, we have

$$E_x = \frac{-\gamma}{h^2}\frac{\partial E_z}{\partial x} - \frac{j\omega\mu}{h^2}\frac{\partial H_z}{\partial y}$$

$$= \frac{-j\omega\mu}{h^2}\frac{\partial}{\partial y}[C_1 C_3 \cos Bx \cos Ay] \quad (\because E_z = 0)$$

$$= \frac{+j\omega\mu}{h^2} C_1 C_3 A \cos Bx \sin Ay.$$

Substituting 2nd Boundary condition, in the above equation, we get

$$0 = \frac{j\omega\mu}{h^2} C_1 C_3 A \cos Bx \sin Ab$$

$$\cos Bx \neq 0,\ C_1,\ C_3 \neq 0$$

$\therefore \qquad \sin Ab = 0$ or $Ab = n\pi$

where $n = 0, 1, 2, ..., \infty$.

or $$A = \frac{n\pi}{b} \qquad ...(4.80)$$

(*iv*) 4th Boundary condition :

$$E_y = 0 \text{ at } x = a \ \forall\ y \longrightarrow 0 \text{ to } b.$$

$$E_y = \frac{-\gamma}{h^2}\frac{\partial E_z}{\partial y} + \frac{j\omega\mu}{h^2}\frac{\partial H_z}{\partial x}$$

$$E_y = \frac{j\omega\mu}{h^2}\frac{\partial}{\partial x}[C_1 C_3 \cos Bx \cos Ay]$$

($\because \quad E_z = 0$ and $H_z = C_1 C_3 \cos Bx \cos Ay$)

i.e., $$E_y = \frac{-j\omega\mu}{h^2} C_1 C_3 B \sin Bx \cos Ay.$$

Substituting the boundary condition

$$0 = \frac{-j\omega\mu}{h^2} C_1 C_3 B \sin Bx \cos Ay \Rightarrow \forall\ y \longrightarrow 0 \text{ to } b.$$

$$\cos Ay \neq 0,\ C_1,\ C_3 \neq 0$$

$\therefore \qquad \sin Ba = 0$

$\therefore \qquad Ba = m\pi$ where $m = 0, 1, 2, ..., \infty$

$\therefore$ $$B = \frac{m\pi}{a} \qquad ...(4.81)$$

The complete solution is (as per Eq. 4.79),

$$H_z = C_1 C_3 A \cos Bx \cos Ay$$

Substituting for A and B from Eqns. 4.80 and 4.81, we get.

$$H_z = C_1 C_3 \cos\left(\frac{m\pi}{a}\right)x \cos\left(\frac{n\pi}{b}\right)y.$$

Let $\qquad C_1 C_3 = C$ (another constant)

$\therefore$ $$H_z = C \cos\left(\frac{m\pi}{a}\right)x \cos\left(\frac{n\pi}{b}\right)y \cdot e^{(j\omega t - \gamma z)} \qquad ...(4.82)$$

Thus it can be seen that for a TM wave E_z has sine-sine components (as per Eq. 4.52) and for a TE wave H_z has *cosine-cosine* components (as per Eq. 4.82).

Field Components

$$E_x = \frac{-\gamma}{h^2}\frac{\partial E_z}{\partial x} - \frac{j\omega\mu}{h^2}\frac{\partial H_z}{\partial y}$$

Here 1st term = 0 since $E_z = 0$ for TM wave

i.e.,
$$E_x = \frac{j\omega\mu}{h^2} \cdot C \cdot \left(\frac{n\pi}{b}\right)\cos\left(\frac{m\pi}{a}\right)x \sin\left(\frac{n\pi}{b}\right)y \cdot e^{(j\omega t - \gamma z)} \qquad \ldots(4.83)$$

$$E_y = \frac{-\gamma}{h^2}\frac{\partial E_z}{\partial y} + \frac{j\omega\mu}{h^2}\frac{\partial H_z}{\partial x}$$

Again, 1st term = 0 since $E_z = 0$ for TM wave.

$\therefore$
$$E_y = \frac{-j\omega\mu}{h^2} C\left(\frac{m\pi}{a}\right)\sin\left(\frac{m\pi}{a}\right)x \cos\left(\frac{n\pi}{b}\right)y \cdot e^{(j\omega t - \gamma z)} \qquad \ldots(4.84)$$

Similarly,

$$H_x = \frac{-\gamma}{h^2}\frac{\partial H_z}{\partial x} + \frac{j\omega\varepsilon}{h^2}\frac{\partial E_z}{\partial y}$$

$\therefore$
$$H_x = \frac{+\gamma}{h^2} C\left(\frac{m\pi}{a}\right)\sin\left(\frac{m\pi}{a}\right)x \cos\left(\frac{n\pi}{b}\right)y \cdot e^{(j\omega t - \gamma z)} \qquad \ldots(4.85)$$

and
$$H_y = \frac{-\gamma}{h^2}\frac{\partial H_z}{\partial x} - \frac{j\omega\varepsilon}{h^2}\frac{\partial E_z}{\partial x}$$

$\therefore$
$$H_y = \frac{-\gamma}{h^2} C \cdot \left(\frac{n\pi}{b}\right)^2 \cos\left(\frac{m\pi}{a}\right)x \cdot \sin\left(\frac{n\pi}{b}\right)y \cdot e^{j\omega t - \gamma z} \qquad \ldots(4.86)$$

4.3.10 TE Modes in Rectangular Waveguides

TE_{mn} is the general mode and the specific modes are given by various values of m and n as discussed below.

(a) ***TE_{00} mode*** : $m = 0, n = 0$

All field components vanish therefore it cannot exist.

(b) ***TE_{01} mode*** : $m = 0, n = 1$

$$E_y = 0, H_x = 0, E_x \text{ and } H_y \text{ exist.}$$

(c) ***TE_{10} mode*** : $m = 1, n = 0$

$$E_x = 0, H_y = 0, E_y \text{ and } H_x \text{ exist.}$$

Therefore TE_{10} mode exists.

(d) ***TE_{11} mode*** : $m = 1, n = 1$

This also exists and even higher modes.

5.5 FIELD EXPRESSIONS FOR TM_{mnp} AND TE_{mnp} MODES IN A RECTANGULAR CAVITY RESONATOR

Case 1. TM waves : The field expressions can be obtained by considering a rectangular cavity resonator as shown in Fig. 5.5.

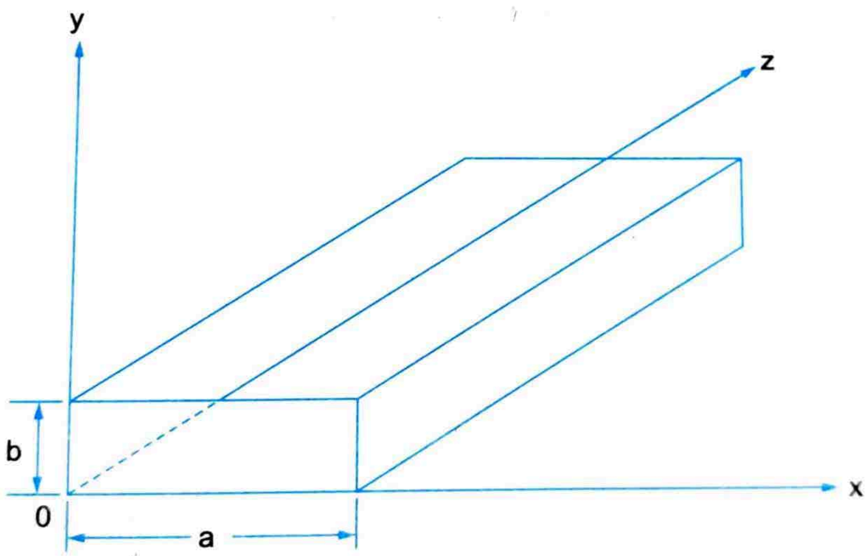


Fig. 5.5

For a TM wave

$$H_z = 0 \text{ and } E_z \neq 0$$

The wave equation is given by

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \gamma^2 E_z = -\omega^2 \mu\varepsilon E_z$$

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + E_z[\gamma^2 + \omega^2 \mu\varepsilon] = 0 \quad \text{...(5.7)}$$

Let $\gamma^2 + \omega^2\mu\varepsilon = h^2$

Equation 5.7 becomes

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + h^2 E_z = 0$$

This is a partial differential equation of 2nd order. The solution can be obtained by separation of variables method

Let $E_z = XY$ be the required solution ...(5.8)

$$\frac{\partial^2 XY}{\partial x^2} + \frac{\partial^2 XY}{\partial y^2} + h^2 XY = 0$$

where, X is a function of x only.

Y is a function of y only.

$$Y\frac{d^2 X}{dx^2} + X\frac{d^2 Y}{dy^2} + h^2 XY = 0 \quad \text{...(5.9)}$$

Dividing throughout by XY, we get

$$\frac{1}{X}\frac{d^2 X}{dx^2} + \frac{1}{Y}\frac{d^2 Y}{dy^2} + h^2 = 0 \quad \text{...(5.10)}$$

Let $$\frac{1}{X}\frac{d^2 X}{dx^2} = -B^2 \quad \text{and} \quad \frac{1}{Y}\frac{d^2 Y}{dy^2} = -A^2 \quad \text{...(5.11)}$$

$\therefore$ Equation 5.10 becomes

$$-A^2 - B^2 + h^2 = 0$$

or

$$h^2 = A^2 + B^2.$$

The solution for Eqs in 5.11 are given by

$$\left.\begin{aligned} X &= C_1 \cos Bx + C_2 \sin Bx \\ Y &= C_3 \cos Ay + C_4 \sin Ay \end{aligned}\right\} \quad \text{...(5.12)}$$

The constants C_1, C_2, C_3, C_4 can be determined by the application of boundary conditions.

From Eq. 5.8 the required solution is

$$E_z = XY$$

Substituting for X and Y from Eq. 5.12

$$E_z = (C_1 \cos Bx + C_2 \sin Bx)(C_3 \cos Ay + C_4 \sin Ay) \quad ...(5.13)$$

We shall now apply Boundary conditions to determine the constants C_1, C_2, C_3 and C_4.

Applying 1st Boundary condition (Bottom wall)

$E_z = 0$, for $y = 0$ and all values of x varying from 0 to a.

Substituting these values in Eq. 5.13, we get

$$0 = (C_1 \cos Bx + C_2 \sin Bx)(C_3 \cos 0 + C_4 \sin 0)$$

$$0 = (C_1 \cos Bx + C_2 \sin Bx)\, C_3$$

For all values of x varying from 0 to a

$$\cos Bx \neq 0, \text{ and } \sin Bx \neq 0$$

$\therefore$ Only $C_3 = 0$

Substituting $C_3 = 0$ in Eq. 5.13

$$E_z = (C_1 \cos Bx + C_2 \sin Bx)(C_4 \sin Ay) \quad ...(5.14)$$

is the intermediate solution.

Applying 2nd boundary condition (Left side wall)

$E_z = 0$ for $x = 0$ and all values of y varying from 0 to b.

Substituting these in Eq. 5.14, we get

$$0 = (C_1 \cos 0 + C_2 \sin 0)(C_4 \sin Ay)$$

$$0 = C_1 C_4 \sin Ay$$

Since $\sin Ay \neq 0$ for all values of y varying from 0 to b, only $C_1 = 0$. With $C_1 = 0$, Eq. 5.14 becomes

$$E_z = C_2 \sin Bx\, C_4 \sin Ay. \quad ...(5.15)$$

This is the intermediate solution.

Applying 3rd boundary condition

$E_z = 0$ for $y = b$ and for all values of x varying from 0 to a.

Putting these values in Eq. 5.15.

$$0 = C_2 \sin Bx\, C_4 \sin Ab.$$

$\sin Bx \neq 0$ since x varies from 0 to a.

C_2 and $C_4 \neq 0$, therefore only $\sin Ab = 0$

$$Ab = \text{multiple of } \pi \text{ radian}$$

or

$$Ab = n\pi \text{ where } n = 0, 1, 2, 3, \ldots$$

$$\therefore \quad A = \frac{n\pi}{b} \quad ...(5.16)$$

where, b = height of the waveguide.

Applying 4th boundary condition

i.e., $E_z = 0$ at $x = a$ and y varies from 0 to b.

Putting these values in Eq. 5.15

$$0 = C_2 C_4 \sin Ba \sin Ay$$

$$C_2 \neq 0 \text{ and } C_4 \neq 0 \text{ and also } \sin Ay \neq 0$$

therefore only

$$\sin Ba = 0$$

$$Ba = m\pi \quad \text{where } m = 0, 1, 2, 3, \ldots$$

$$\therefore \quad B = \frac{m\pi}{a} \qquad \ldots(5.17)$$

where, a = the width of the waveguide.

Substituting the values of A and B from Eqs. 5.16 and 5.17 in Eq. 5.15, we get

$$E_z = C_2 C_4 \sin\left(\frac{m\pi}{a}\right)x \sin\left(\frac{n\pi}{b}\right)y \cdot e^{j\omega t} \cdot e^{-\gamma z} \qquad \ldots(5.18)$$

for the wave propagating along the positive z direction.

Similarly,

$$E_z = C_2 C_4 \sin\left(\frac{m\pi}{a}\right)x \sin\left(\frac{n\pi}{b}\right)y \cdot e^{j\omega t} \cdot e^{\gamma z} \qquad \ldots(5.19)$$

for the wave propagating in the negative z direction.

We also know when the wave propagates, $\gamma = j\beta$. Adding the fields of two travelling waves *i.e.*, one in positive 'z' direction and the other in the negative 'z' direction, we obtain

$$E_z = C_2 C_4 \sin\left(\frac{m\pi}{a}\right)x \sin\left(\frac{n\pi}{b}\right)y e^{j(\omega t \pm \beta z)} \qquad \ldots(5.20)$$

Let A^+ be the amplitude constant for the wave propagating in the positive z direction and A^- be the amplitude constant for the wave propagating in the negative direction

Then,

$$E_z = [A^+ e^{-j\beta z} + A^- e^{j\beta z}] \qquad \ldots(5.21)$$

To make E_z vanish at $z = 0$ we must choose $A^- = A^+$, so that the Eq. 5.21 reduces to

$$E_z = [A^+ e^{-j\beta z} + A e^{-j\beta z}]\, C_2 C_4 \sin\left(\frac{m\pi}{a}\right)x \sin\left(\frac{n\pi}{b}\right)y e^{j\omega t} \qquad \ldots(5.22)$$

In order that E_z vanishes at $z = 0$ and $z = d$, choose

$$A^+ = A^- = A$$

Then Eq. 5.22 reduces to

$$E_z = C_2 C_4 A[e^{-j\beta d} + e^{j\beta d}] \sin\left(\frac{m\pi}{a}\right)x \sin\left(\frac{n\pi}{b}\right)y e^{j\omega t} \qquad \ldots(5.23)$$

But

$$e^{-i\theta} + e^{+i\theta} = 2\cos\theta$$

$$\therefore \quad e^{-j\beta d} + e^{+j\beta d} = 2\cos\beta d$$

With these values, the Eq. 5.23 becomes

$$E_z = C_2 C_4\, 2A \cos\beta d \sin\left(\frac{m\pi}{a}\right)x \sin\left(\frac{n\pi}{b}\right)y \cdot e^{j\omega t} \qquad \ldots(5.24)$$

$E_z = 0$ all along the surface of the resonator. Eq. 5.24 becomes,

$$0 = C_2C_4\, 2A \cos \beta d \sin\left(\frac{m\pi}{a}\right)x \sin\left(\frac{n\pi}{b}\right)y \cdot e^{j\omega t}$$

Since $C_2\, C_4$,

$$\sin\left(\frac{m\pi}{a}\right)x \text{ and } \sin\left(\frac{n\pi}{b}\right)y \neq 0 \text{ and } A \neq 0 \text{ only } \cos \beta d = 0$$

$$\therefore \qquad \beta = \frac{p\pi}{d} \qquad ...(5.25)$$

Now, Eq, 5.24 becomes (with this value of β)

$$E_z = C \sin\left(\frac{m\pi}{a}\right)x \sin\left(\frac{n\pi}{b}\right)y \cos\left(\frac{p\pi}{d}\right)z e^{-j\omega t - \gamma z}$$

where $C = 2C_2\, C_4\, A$

$$\therefore \qquad E_z = C \sin\left(\frac{m\pi}{a}\right)x \sin\left(\frac{n\pi}{b}\right)y \cos\left(\frac{p\pi}{d}\right)z e^{j\omega t - \gamma z}$$

$$E_z(TM_{mnp}) = C \sin\left(\frac{m\pi}{a}\right)x \sin\left(\frac{n\pi}{b}\right)y \cos\left(\frac{p\pi}{d}\right)z e^{j\omega t - \gamma z} \qquad ...(5.26)$$

where

$m = 0, 1, 2, 3, ...$ represents the number of half wave variations in the x direction
$n = 0, 1, 2, 3, ...$ represents the number of half wave variations in the y direction
$p = 1, 2, 3, ...$ represents the number of half wave variations in the z direction

Case 2. For TE Waves : TM_{mnp} mode of propagation in a rectangular waveguide is shown with components in Fig. 5.6.

For TE_{mn} wave to propagate $E_z = 0$ and $H_z \neq 0$. We know from Maxwell's equation.

$$\nabla^2 H_z = -\omega^2 \mu\varepsilon H_z$$

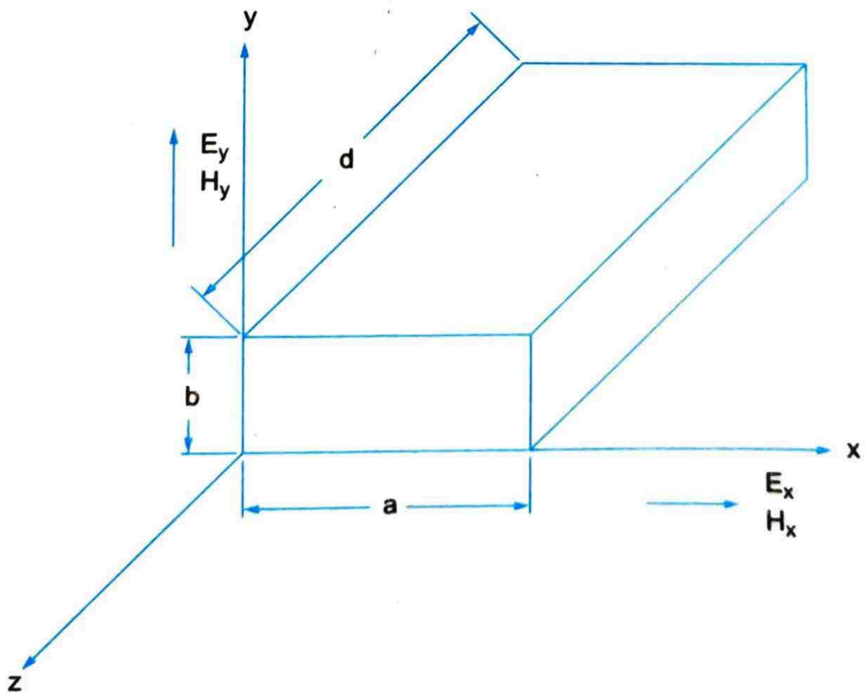


Fig. 5.6

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + \frac{\partial^2 H_z}{\partial z^2} = -\omega^2 \mu\varepsilon H_z$$

But $$\frac{\partial^2}{\partial z^2} = \gamma^2 \quad \text{an operator.}$$

$\therefore$ $$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + \gamma^2 H_z + \omega^2 \mu\varepsilon H_z = 0$$

or $$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + H_z[\gamma^2 + \omega^2 \mu\varepsilon] = 0 \quad \text{...(5}$$

Let $$\gamma^2 + \omega^2\mu\varepsilon = h^2$$

Now Eq. 5.28 becomes

$\therefore$ $$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + h^2 H_z = 0$$

This is a partial differential equation of 2nd order.

Let $$H_z = XY \quad \text{...(5.2}$$

where X is a function of x alone, Y is a function of y alone.

$$Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} + h^2 XY = 0$$

Dividing by XY throughout,

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + h^2 = 0$$

Let $$\frac{1}{X}\frac{d^2X}{dx^2} = -A^2 \quad \text{...(5.30}$$

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = -B^2 \quad \text{...(5.30}$$

where A^2 and B^2 are constants.

$\therefore$ $$-A^2 - B^2 = -h^2$$

or $$h^2 = A^2 + B^2$$

The solutions for above Eqs. 5.30a, b are

$$X = C_1 \cos Bx + C_2 \sin Bx$$
$$Y = C_3 \cos Ay + C_4 \sin Ay \quad \text{...(5.3}$$

The constants C_1, C_2, C_3 and C_4 are Determined by applying Boundary Conditions.

1st Boundary Condition (Bottom Wall)

$E_x = 0$ for $y = 0$ and all values of x varying from 0 to a. We know,

$$E_x = \frac{-\gamma}{h^2}\frac{\partial E_z}{\partial x} \frac{-j\omega\mu}{h^2}\frac{\partial H_z}{\partial y}$$

But $E_z = 0$ for a TE wave

$$\therefore \qquad E_x = \frac{-j\omega\mu}{h^2}\frac{\partial H_z}{\partial y}$$

But $H_z = XY$ from Eq. 5.29.

Using Eq. 5.31, H_z reduces to

$$H_z = (C_1 \cos Bx + C_2 \sin Bx)\,(C_3 \cos Ay + C_4 \sin Ay) \qquad \text{...(5.32)}$$

$$E_x = \frac{-j\omega\mu}{h^2}\frac{\partial}{\partial y}\,(C_1 \cos Bx + C_2 \sin Bx)\,(C_3 \cos Ay + C_4 \sin Ay) \qquad \text{...(5.33)}$$

But $E_x = 0$ for $y = 0$ and all values of x varying from 0 to a.

Substituting these values in Eq. 5.33 after differentiating with respect to y, Eq. 5.33 becomes

$$0 = \frac{-j\omega\mu}{h^2}\,[(C_1 \cos Bx + C_2 \sin Bx)\,(-AC_3 \sin Ay + AC_4 \cos Ay)] \qquad \text{...(5.34)}$$

$$C_1 = \text{and } C_2 \neq 0$$

Since x takes a values from 0 to a,

$$\cos Bx \quad \text{and} \quad \sin Bx \neq 0 \quad \text{and} \quad y = 0$$

With these values Eq. 5.34 becomes

$$0 = \frac{-j\omega\mu}{h^2}\,[(C_1 \cos Bx + C_2 \sin Bx)\,AC_4]$$

Therefore only condition is that $C_4 = 0$ since x has a value 0 to a

Therefore, the intermediate solution is

$$H_z = (C_1 \cos Bx + C_2 \sin Bx)\,(C_3 \cos Ay) \qquad \text{...(5.35)}$$

2nd Boundary Condition (Left Side Wall)

$Ey = 0$ for $x = 0$ and y varying from 0 to b.

We know that,

$$E_y = \frac{-\gamma}{h^2}\frac{\partial E_z}{\partial y} + \frac{j\omega\mu}{h^2}\frac{\partial H_z}{\partial x}$$

But $E_z = 0$ for a TE wave, and substituting for H_z from Eq. 5.35, we get

$$E_y = \frac{j\omega\mu}{h^2}\frac{\partial}{\partial x}\,[(C_1 \cos Bx + C_2 \sin Bx)\,C_3 \cos Ay]$$

i.e.,

$$E_y = \frac{j\omega\mu}{h^2}\,[(-C_1 \sin Bx + BC_2 \cos Bx)\,C_3 \cos Ay] \qquad \text{...(5.36)}$$

Substituting the Boundary condition,

$E_y = 0$ for $x = 0$ and y varying from 0 to b, Eq. 5.36 becomes,

$$0 = \frac{j\omega\mu}{h^2}\,[(0 + BC_2)\,C_3 \cos Ay]$$

Now, $\cos Ay$ and $C_3 \neq 0$

Therefore only solution is $C_2 = 0$

With this, the intermediate solution Eq. 5.35 becomes

$$H_z = (C_1 \cos Bx\, C_3 \cos Ay) \quad ...(5.37)$$

3rd Boundary Condition (Top Wall)

$E_x = 0$ for $y = b$ and x varies from 0 to a.

We know that,

$$E_x = \frac{-\gamma}{h^2}\frac{\partial E_z}{\partial x}\frac{-j\omega\mu}{h^2}\frac{\partial H_z}{\partial y}$$

But, $E_z = 0$ for TE wave

$$\therefore \quad E_x = \frac{-j\omega\mu}{h^2}\frac{\partial H_z}{\partial y}$$

From Eq. 5.37,

$$H_z = C_1 \cos Bx\, C_3 \cos Ay$$

$$\therefore \quad E_x = -\frac{j\omega\mu}{h^2}\frac{\partial}{\partial y}\,[C_1 \cos Bx\, C_3 \cos Ay] \quad ...(5.38)$$

Substituting the boundary condition,

$E_x = 0$ for $y = b$ and x varies from 0 to a in Eq. 5.8, we get

$$0 = \frac{j\omega\mu}{h^2}\,[-\,C_1\, C_3\, A \cos Bx \sin Ay]$$

Here, $\cos Bx \neq 0$ and $C_1\, C_3 \neq 0$

Therefore, only $\sin Ay = 0$

But $y = b$. Putting $y = b$, we get

$$\sin Ab = 0 \text{ or } Ab = n\pi \text{ where } n = 0, 1, 2, 3, \ldots$$

or

$$A = \frac{n\pi}{b}$$

4th Boundary Condition (Right Side Wall)

$E_y = 0$ for $x = a$ and y varying from 0 to a.

$$E_y = \frac{-\gamma}{h^2}\frac{\partial E_z}{\partial y} + \frac{j\omega\mu}{h^2}\frac{\partial H_z}{\partial x}$$

Since, $E_z = 0$ for TE wave, and $H_z = C_1 \cos Bx\, C_3 \cos Ay$, we have

$$E_y = \frac{j\omega\mu}{h^2}\frac{\partial}{\partial x}\,[C_1 \cos Bx\, C_3 \cos Ay]$$

i.e.,

$$E_y = \frac{j\omega\mu}{h^2}\,[-\,C_1\, B \sin Bx \cdot C_3 \cos Ay]$$

Substituting Boundary Condition,

$E_y = 0$ for $x = a$ and y varying from 0 to b, we get

$$0 = [-C_1 B \sin Ba\, C_3 \cos Ay]$$

Here, $C_1 \neq 0$; $C_3 \neq 0$ and also $\cos A_y \neq 0$.

Therefore only $\sin Ba = 0$

or

$$Ba = m\pi \quad \text{where } m = 0, 1, 2 \ldots$$

or

$$B = \frac{m\pi}{a}$$

Therefore, the solution for H_z is given by Eq. 5.37. The values of A and B are substituted to give,

$$H_z = C_1 C_3 \cos\left(\frac{m\pi}{a}\right)x \cos\left(\frac{n\pi}{b}\right)y \cdot e^{j(\omega t - \gamma z)} \quad \ldots(5.39)$$

Since $\gamma = j\beta$ for a wave propagating along the positive z direction

$$H_z = C \cos\left(\frac{m\pi}{a}\right)x \cos\left(\frac{n\pi}{b}\right)y \cdot e^{j(\omega t - \beta z)} \quad \ldots(5.40)$$

where, $C = C_1 C_3$

and for the wave propagating along the negative z direction,

$$H_z = C \cos\left(\frac{m\pi}{a}\right)x \cos\left(\frac{n\pi}{b}\right)y \cdot e^{j(\omega t - j\beta z)} \quad \ldots(5.41)$$

The amplitude constant along the positive 'z' direction is represented by A^+, and that along the negative 'z' direction by A^-. Adding the two travelling waves to obtain the fields of standing wave we have from Eqs. 5.40 and Eq. 5.41.

$$H_z = (A^+ e^{-j\beta z} + A^- e^{+j\beta z}) \cos\left(\frac{m\pi}{a}\right)x \cos\left(\frac{n\pi}{b}\right)y e^{j\omega t} \quad \ldots(5.42)$$

To make E_y vanish at $z = 0$ and $z = d$ we must make $A^+ = -A^-$ or $A^- = -A^+$ and also we know that

$$E_y = -\frac{\gamma}{h^2}\frac{\partial E_z}{\partial y} + \frac{j\omega\mu}{h^2}\frac{\partial H_z}{\partial x}$$

For TE waves, $E_z = 0$,

$\therefore$

$$E_y = \frac{j\omega\mu}{h^2}\frac{\partial H_z}{\partial x}$$

$$E_y = \frac{j\omega\mu}{h^2}\left[\frac{\partial}{\partial x}(A^+ e^{-j\beta z} + A^- e^{j\beta z}) \cos\left(\frac{m\pi}{a}\right)x \cos\left(\frac{n\pi}{b}\right)y\right] e^{j\omega t}$$

Since E_y vanishes at $z = 0$ and $z = d$, we have

$$0 = \left[(A^+ e^{-j\beta z} + A^- e^{j\beta z}) - \left(\frac{m\pi}{a}\right) \sin\left(\frac{m\pi}{a}\right)x \cos\left(\frac{n\pi}{b}\right)y\right] e^{j\omega t} \quad \ldots(5.43)$$

But $\quad \sin\left(\frac{m\pi}{a}\right)x$ and $\cos\left(\frac{n\pi}{b}\right)y \neq 0$

Therefore only $\quad A^+ e^{-j\beta z} + A^- e^{+j\beta z} = 0$

To make $E_y = 0$ choose $A^- = -A^+$. Putting these in Eq. 5.43, we get

$$0 = (A^+e^{-j\beta z} - A^+e^{j\beta z}) = A^+[e^{-j\beta z} - e^{+j\beta z}]$$

We know that $(e^{-i\theta} - e^{+i\theta}) = -2i \sin\theta$

$\therefore$ $$0 = -2jA^+ \sin \beta z$$

since $A^+ \neq 0$ only $\sin \beta d = 0$ with $z = d$

or $$\beta = \frac{\pi}{d}$$

or $$\beta = \frac{p\pi}{d} \text{ where } p = 1, 2, 3, \ldots$$

$\therefore$ $$H_z = -2jA^+ \cos\left(\frac{m\pi}{a}\right)x \cos\left(\frac{n\pi}{b}\right)y \sin\left(\frac{p\pi}{d}\right)z e^{(j\omega t - \gamma z)}$$

Putting $-2jA^+ = C$, another constant,

$$H_z = C \cos\left(\frac{m\pi}{a}\right)x \cos\left(\frac{n\pi}{b}\right)y \sin\left(\frac{p\pi}{d}\right)z e^{j(\omega t - \gamma z)}$$

For TE_{mnp} mode,

$$H_z = C \cos\left(\frac{m\pi}{a}\right)x \cos\left(\frac{n\pi}{b}\right)y \sin\left(\frac{p\pi}{d}\right)z e^{j(\omega t - \gamma z)} \quad \ldots(5.4$$

4.3.11 Wave Impedance (Z_z) in TM and TE Waves

Wave impedance is defined as the ratio of the strength of electric field in one transverse direction to the strength of the magnetic field along the other transverse direction as shown in Fig. 4.41.

i.e.,
$$Z_z = \frac{E_x}{H_y} = \frac{-E_y}{H_x}$$

or
$$Z_z = \frac{\sqrt{E_x^2 + E_y^2}}{\sqrt{H_x^2 + H_y^2}} \qquad ...(4.87)$$

1. Wave impedance of a TM wave in rectangular waveguide :

$$Z_z = Z_{TM} = \frac{E_x}{H_y} = \frac{\dfrac{-\gamma}{h^2}\dfrac{\partial E_z}{\partial x} - \dfrac{j\omega\mu}{h^2}\dfrac{\partial H_z}{\partial y}}{\dfrac{-\gamma}{h^2}\dfrac{\partial H_z}{\partial y} - \dfrac{j\omega\varepsilon}{h^2}\dfrac{\partial E_z}{\partial x}}$$

for a TM wave $H_z = 0$ and $\gamma = j\beta$

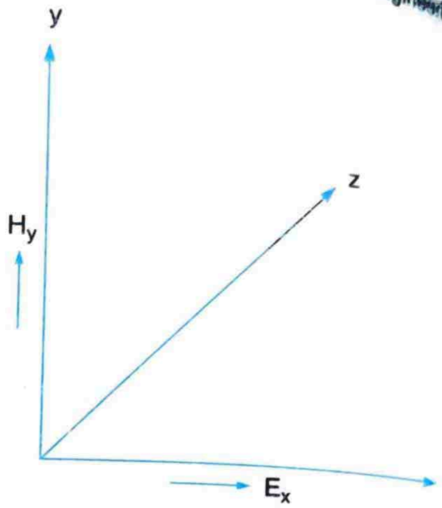


Fig. 4.41

$$\therefore \quad Z_{TM} = \frac{\dfrac{-\gamma}{h^2}\dfrac{\partial E_z}{\partial x}}{\dfrac{-j\omega\varepsilon}{h^2}\dfrac{\partial E_z}{\partial x}} = \frac{\gamma}{j\omega\varepsilon} = \frac{j\beta}{j\omega\varepsilon}$$

or $$Z_{TM} = \frac{\beta}{\omega\varepsilon}$$

We know that $\beta = \sqrt{\omega^2\mu\varepsilon - \omega_c^2\mu\varepsilon}$

$$\therefore \quad Z_{TM} = \frac{\sqrt{\omega^2\mu\varepsilon - \omega_c^2\mu\varepsilon}}{\omega\varepsilon} = \frac{\sqrt{\mu\varepsilon}\cdot\sqrt{\omega^2 - \omega_c^2}}{\varepsilon\cdot\omega}$$

$$= \frac{\sqrt{\mu}\cdot\cancel{\sqrt{\varepsilon}}\cdot\sqrt{\omega^2-\omega_c^2}}{\sqrt{\varepsilon}\cdot\cancel{\sqrt{\varepsilon}}\cdot\omega} = \frac{\sqrt{\mu}\cdot\omega\sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2}}{\sqrt{\varepsilon}\;\omega} = \sqrt{\frac{\mu}{\varepsilon}}\sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2}$$

$\boxed{\omega_c = 2\pi f_c}$

$\omega = 2\pi f$

$$= \sqrt{\frac{\mu}{\varepsilon}}\sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2} = \sqrt{\frac{\mu}{\varepsilon}}\cdot\sqrt{1-\left(\frac{f_c}{f}\right)^2}$$

or $$Z_{TM} = \sqrt{\frac{\mu}{\varepsilon}}\cdot\sqrt{1-(\lambda_o/\lambda_c)^2}$$

$\lambda_0 = \frac{1}{f}$

$\lambda_c = \frac{1}{f_c}$

For air, $$\sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_0\,\mu_r}{\varepsilon_0\,\varepsilon_r}}$$

$$= \sqrt{\frac{4\pi\times10^{-7}}{1/36\pi\times10^{-9}}} = \sqrt{4\pi\times36\pi\times10^2}$$

$$= 2\times6\pi\times10 = 120\,\pi = 377\,\Omega = \eta$$

where η is the intrinsic impedance of free space.

$$\therefore \quad Z_{TM} = \eta\sqrt{1-(\lambda_o/\lambda_c)^2} \qquad ...(4.88)$$

Since λ_o is always less than λ_c for wave propagation $Z_{TM} < \eta$.

This shows that wave impedance for a TM wave is always less than free space impedance.

2. Wave impedance of TE wave is rectangular waveguide.

$$Z_z = Z_{TE} = \frac{E_x}{H_y} = \frac{\dfrac{-\gamma}{h^2}\dfrac{\partial E_z}{\partial x} - \dfrac{j\omega\mu}{h^2}\dfrac{\partial H_z}{\partial y}}{\dfrac{-\gamma}{h^2}\dfrac{\partial H_z}{\partial y} - \dfrac{j\omega\varepsilon}{h^2}\dfrac{\partial E_z}{\partial x}}$$

For TE waves $E_z = 0$ and $\gamma = j\beta$

$$\therefore \quad Z_{TE} = \frac{\dfrac{-j\omega\mu}{h^2}\dfrac{\partial H_z}{\partial y}}{\dfrac{-\gamma}{h^2}\dfrac{\partial H_z}{\partial y}} = \frac{j\omega\mu}{\gamma}$$

$$= \frac{j\omega\mu}{j\beta} = \frac{\omega\mu}{\beta}$$

$$\therefore \quad Z_{TE} = \frac{\omega\mu}{\sqrt{\mu\varepsilon}\sqrt{\omega^2 - \omega_c^2}}$$

$$= \frac{\eta}{1-(\omega_c/\omega^2)} = \frac{\eta}{\sqrt{1-(f_c/f)^2}}$$

or

$$Z_{TE} = \frac{\eta}{1-(\lambda_o/\lambda_c)^2} \qquad \text{...(4.89)}$$

Therefore $Z_{TE} > \eta$ as $\lambda_o < \lambda_c$ for wave propagation. This shows that wave impedance for a TE wave is always greater than free space impedance.

Cutoff Frequency of a Waveguide (Waveguide as a High Pass Filter)

From Eqs. 4.50 and 4.51, we know that

$$h^2 = \gamma^2 + \omega^2\mu\varepsilon = A^2 + B^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$

i.e.,

$$\gamma^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - \omega^2\mu\varepsilon$$

or

$$\gamma = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - \omega^2\mu\varepsilon} = \alpha + j\beta$$

At lower frequencies,

$$\omega^2\mu\varepsilon < \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$

γ then becomes real and positive and equal to the attenuation constant 'α' *i.e.*, the wave is completely attenuated and there is no phase change. Hence the wave can not propagate.

However, at higher frequencies,

$$\omega^2\mu\varepsilon > \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$

γ becomes imaginary, there will be phase change β and hence the wave propagates. At the transition, γ becomes zero and the propagation just starts. The frequency at which γ just becomes zero is defined as the *cutoff frequency* (or *threshold frequency*) 'f_c'.

i.e., At $f = f_c$, $\gamma = 0$ or $\omega = 2\pi f = 2\pi f_c = \omega_c$

$\therefore$

$$0 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - \omega_c^2\mu\varepsilon$$

or

$$\omega_c = \frac{1}{\sqrt{\mu\varepsilon}}\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^{1/2}$$

or

$$f_c = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^{1/2}$$

or

$$f_c = \frac{c}{2\pi}\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^{1/2} \qquad \because c = \frac{1}{\sqrt{\mu\varepsilon}}$$

$\therefore$

$$f_c = \frac{c}{2}\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]^{1/2} \qquad ...(4.57)$$

The cutoff wavelength (λ_c) is

$$\lambda_c = \frac{c}{f_c} = \frac{c}{\frac{c}{2}\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]^{1/2}}$$

or

$$\lambda_{c_{m,n}} = \frac{2ab}{\sqrt{m^2b^2 + n^2a^2}} \quad ...(4.58)$$

All wavelengths greater than λ_c are attenuated and those less than λ_c are allowed to propagate inside the waveguide.

TM Modes in Rectangular Waveguides

Consider the various TM_{mn} modes for various values of m and n.

1. **TM_{11} mode:** Minimum possible mode $m = 1$, $n = 1$.

   From Eq. 4.58 we know that

$$\lambda_{c_{mn}} = \frac{2\,ab}{\sqrt{m^2b^2 + n^2a^2}}$$

   Now putting $m = n = 1$, we get

$$\lambda_{c_{11}} = \frac{2\,ab}{\sqrt{a^2 + b^2}}$$

2. **TM_{12} mode:** Now putting $m = 1$, $n = 2$ in Eq. 4.58, we get

$$\lambda_{c_{12}} = \frac{2\,ab}{\sqrt{b^2 + 4a^2}}$$

3. **TM_{21} mode:** Now putting $m = 2$, $n = 1$ in Eq. 4.58, we get

$$\lambda_{c_{21}} = \frac{2\,ab}{\sqrt{4b^2 + a^2}}$$

   By inspection, it is clear that, $\lambda_{c_{11}} > \lambda_{c_{12}}$ and $\lambda_{c_{11}} > \lambda_{c_{21}}$ and so on.

4.3.14 Dominant Mode and Degenerate Modes in Rectangular Waveguides

As already discussed, the walls of the waveguides can be considered as nearly perfect conductors. Therefore, the boundary conditions require that electric field be normal *i.e.*, perpendicular, to the waveguide walls. The magnetic fields must be tangential *i.e.*, parallel to the waveguide walls. Because of these boundary conditions a zero subscript can exist in the TE mode but not in the TM mode. For *e.g.*, TE_{10}, TE_{01}, TE_{20} etc. modes can exist in a rectangular waveguide but only the TM_{11}, TM_{12}, TM_{21} etc. modes can exist. Also the cutoff frequency relationship shows that the physical size of the waveguide determines the propagation of modes depending on the values of m and n. The minimum cutoff frequency for a rectangular waveguide is obtained for a dimension $a > b$ for $m = 1$ and $n = 0$, *i.e.*, TE_{10} mode is the dominant mode for a rectangular waveguide. (Since for TM_{mn} modes $m \neq 0$ or $n \neq 0$, the lowest order mode TE_{10} is the dominant mode for $a > b$).

Some of the higher order modes, having the same cutoff frequency are called degenerate modes. For a rectangular waveguide TE_{mn}/TM_{mn} modes for which both $m \neq 0$, $n \neq 0$ will always be degenerate modes. For a square guide in which $a = b$, all the TE_{pq}, TE_{qp}, TM_{pq} and TM_{qp} modes are together degenerate modes. It is necessary that higher order degenerate modes are not supported by the

Dominant Mode : Dominant mode is that mode for which the cutoff wavelength (λ_c) assumes a maximum value.

From Eq. 4.58 we know that $\lambda_{c_{mn}} = \dfrac{2ab}{\sqrt{m^2b^2 + n^2a^2}}$

For **$\mathbf{TE_{01}}$ mode** $\quad \lambda_{c_{01}} = \dfrac{2ab}{\sqrt{a^2}} = 2b$

For **$\mathbf{TE_{10}}$ mode** $\quad \lambda_{c_{10}} = \dfrac{2ab}{\sqrt{b^2}} = 2a$

For **$\mathbf{TE_{11}}$ mode** $\quad \lambda_{c_{11}} = \dfrac{2ab}{\sqrt{a^2 + b^2}}$

Of these $\lambda_{c_{10}}$ has the maximum value since 'a' is the larger dimension. Hence TE_{10} mode is the dominant mode in rectangular waveguides.

The other expressions for β, V_p, V_g, and λ_g remain the same as for TM waves. (Eqs. 4.65 to 4.69)

i.e.,

$$\beta = \frac{2\pi}{\lambda_g} = \sqrt{\omega^2\mu\varepsilon - \omega_c^2\mu\varepsilon}$$

$$V_p = \frac{c}{\sqrt{1 - (\lambda_o/\lambda_c)^2}},$$

$$V_g = c\sqrt{1 - (\lambda_o/\lambda_c)^2}$$

and

$$\lambda_g = \frac{\lambda_o}{\sqrt{1 - (\lambda_o/\lambda_c)^2}}$$

guide in the operating band of frequencies to avoid undesirable components appearing at the output alongwith losses.

Also it may be necessary to prevent the conversion of a particular waveguide mode to another. Such mode conversion usually results from waveguide irregularities or from impedance structure used in transmission line. Such mode conversion can be supported by

(i) Choosing suitable waveguide dimension (the undesired mode/modes having cutoff frequency above the desired modes can be suppressed)

(ii) Using mode filters (undesired modes can be suppressed by providing a metallic plate or vane in the waveguide where undesired modes have tangential electric field lines).

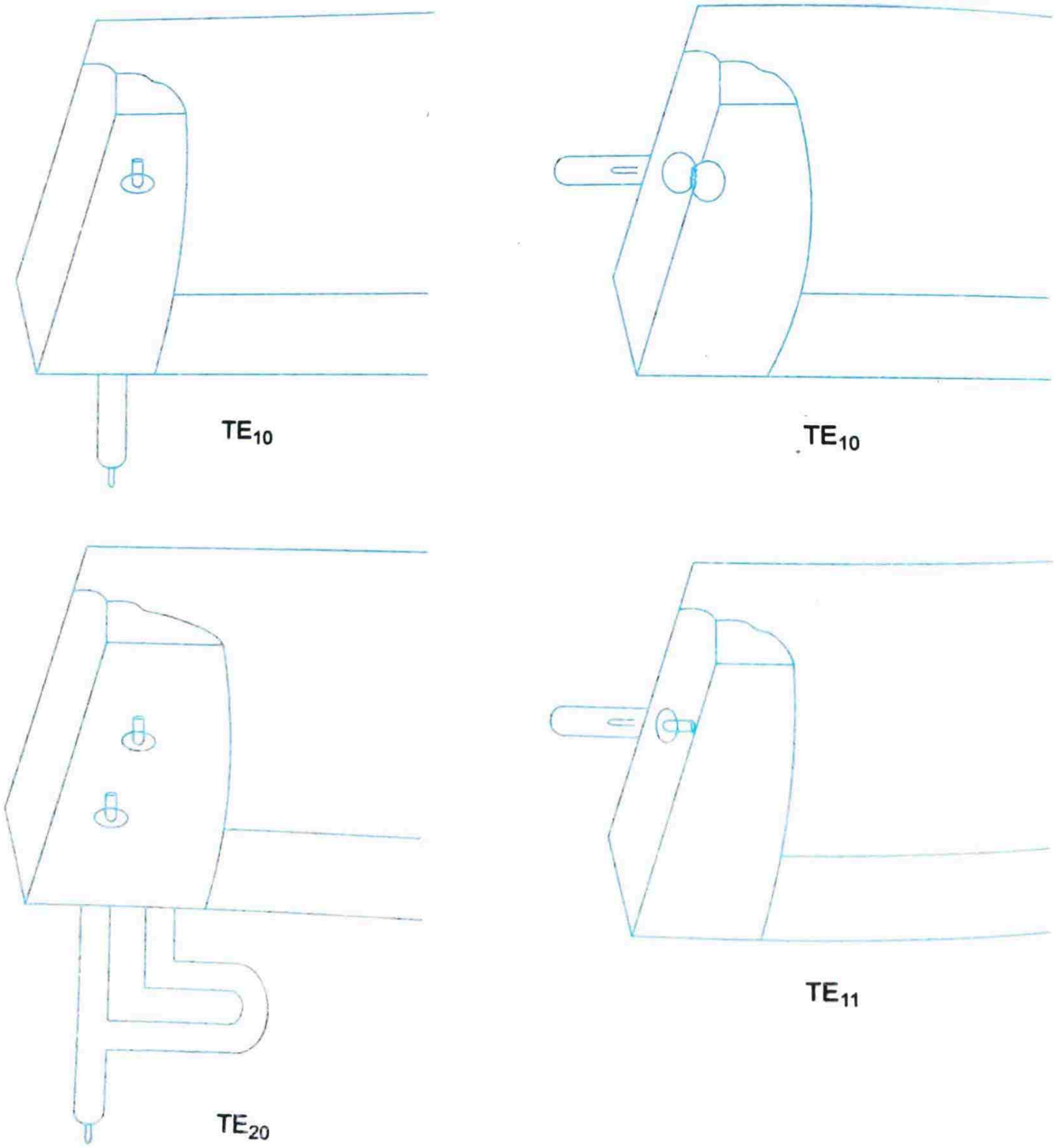


Fig. 4.42 Excitation of mode in rectangular waveguides.

The various modes in a waveguide can be excited by various launching devices. Fig. 4.42 illustrates how the TE_{10}, TE_{20} and TM_{11} modes are launched in rectangular waveguides. These launching devices are in fact, antennas. At the receiving end of the waveguide, a similar launching device (receiving antenna) is used to convert the e.m. fields within the waveguide back to voltage and currents on a transmission line. Hence, one must know which waveguide mode was used to launch the e.m. fields at the transmitting end of the waveguide. If more than one mode exists at a particular frequency in the waveguide, then discontinuities such as bends, joints etc. would cause e.m. energy to be transferred from one mode to another. This results in an additional loss in the waveguide since it will not be recovered in the receiving end.

Field Patterns

Figure 4.28 shows the field pattern for a TE wave. Solid lines depict electric field lines or voltage lines and dotted lines depict magnetic field lines.

We use subscript for designating a particular mode. TE_{mn} or TM_{mn} where 'm' indicates the number of half wave variations of the electric field (or magnetic field in a TM) across the wider dimension a, of the waveguide and 'n' indicates the number across the narrow dimension b. Referring to TE pattern shown in Fig. 4.28 it can be seen that the voltage varies from '0' to maximum and maximum to '0' across the wide dimension a. This is one half variation. Hence $m = 1$. Across the narrow dimension, there is no variation in voltage v. Hence $n = 0$. Therefore this mode is TE_{10} mode. The mode having the *highest cutoff wavelength* is known as dominant mode of the waveguide and all other modes are called higher modes. For example TE_{10} is the dominant mode for TE waves. It is the mode which is used for practically all electromagnetic transmission in a rectangular waveguide. Dominant mode is almost always a low-loss, distortion less transmission and higher modes result in a significant loss of power and also undesirable harmonic distortion. The radiation pattern for TE mode is shown in Fig. 4.29. Sketches of some higher order TE modes are shown in Fig. 4.30.

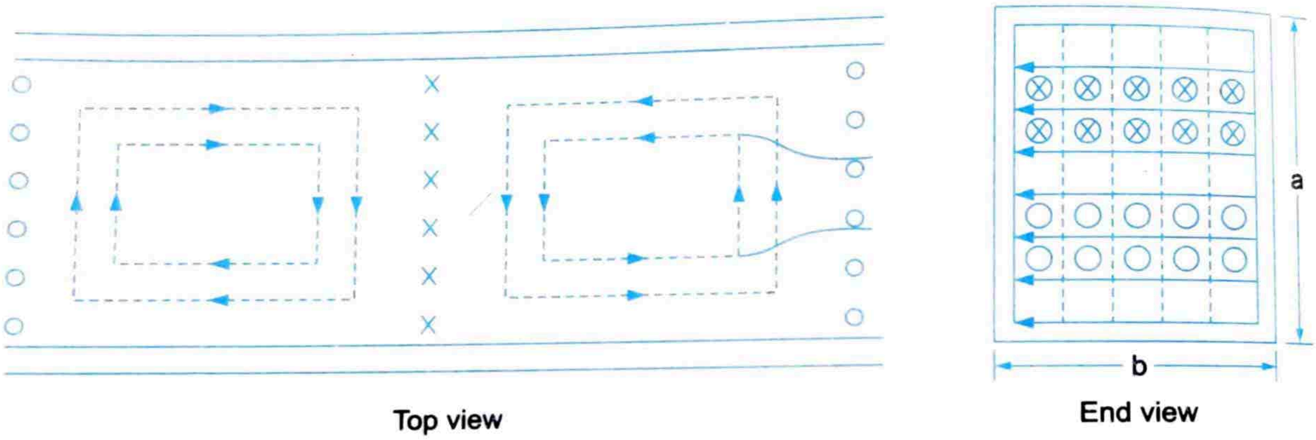


Fig. 4.28 Field pattern of a TE wave.

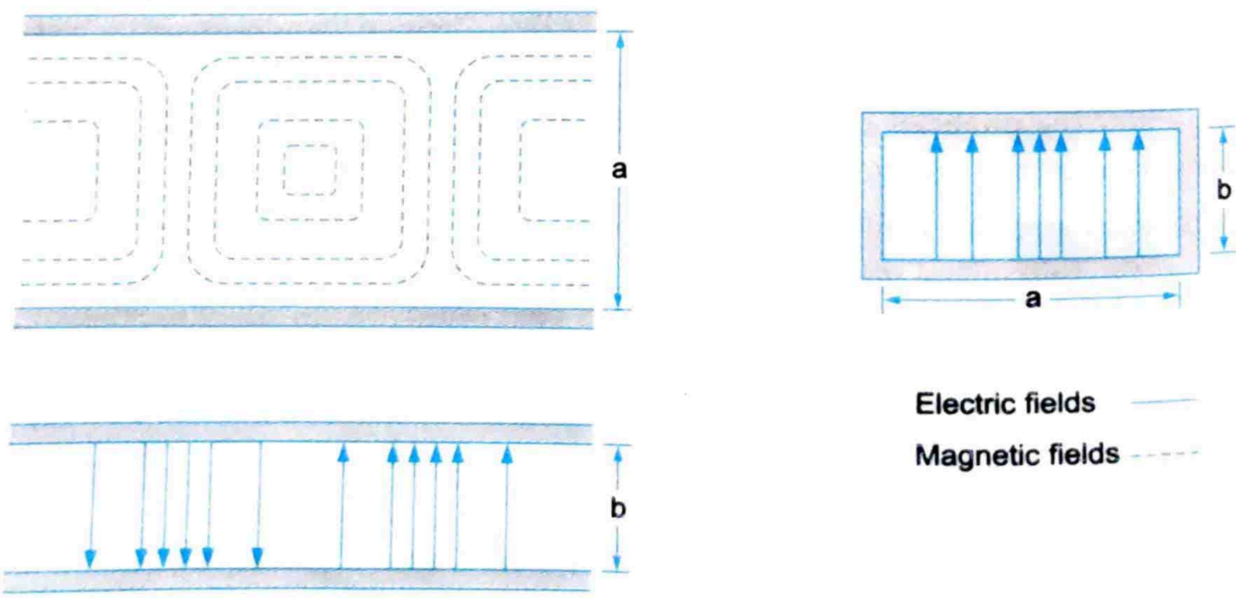


Fig. 4.29 Radiation pattern for TE_{10} mode.

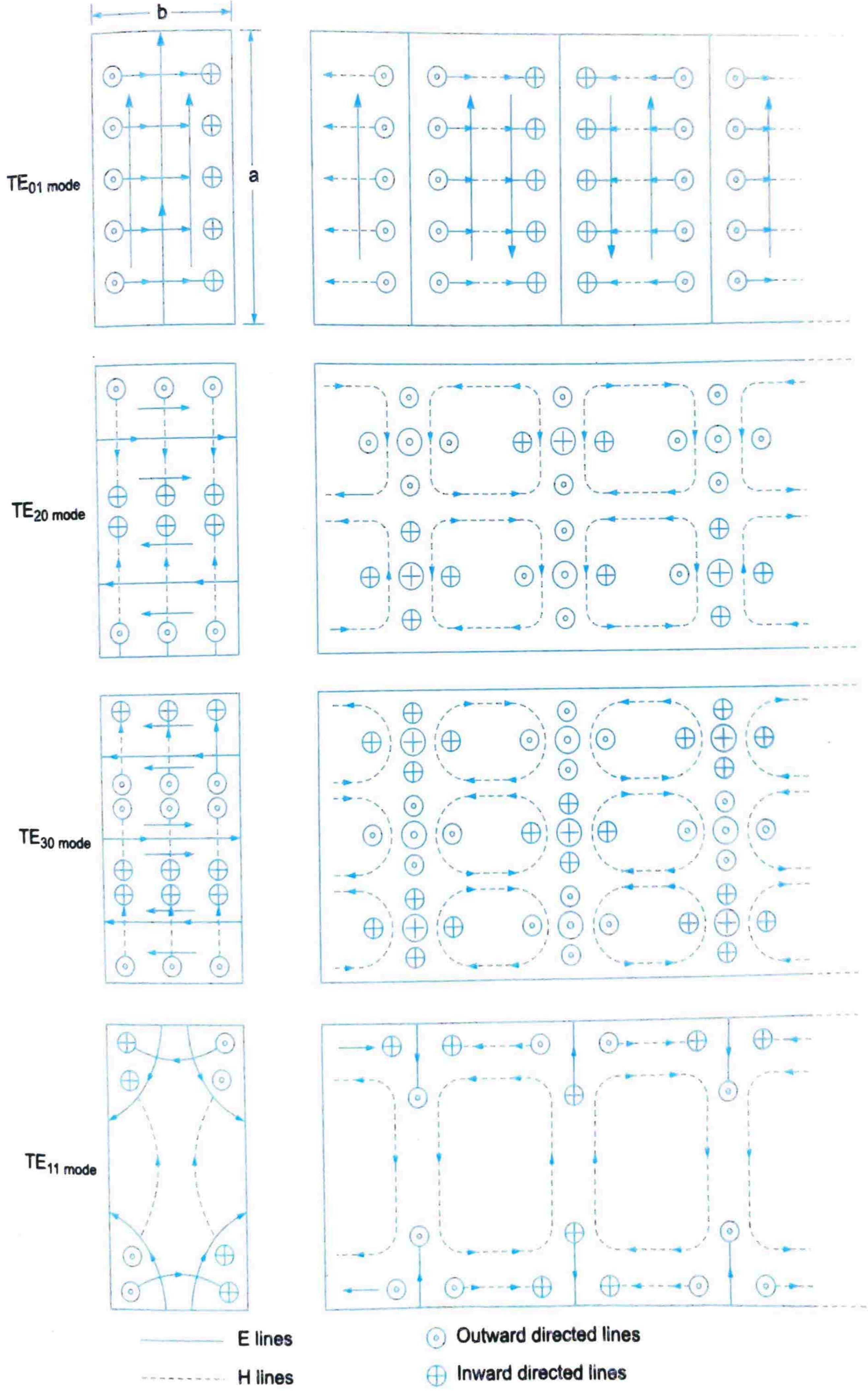


Fig. 4.30 Field pattern of higher order modes.

TM Modes in Rectangular Waveguides

Consider the various TM_{mn} modes for various values of m and n.

1. **TM_{11} mode:** Minimum possible mode $m = 1, n = 1$.
   From Eq. 4.58 we know that

$$\lambda_{c_{mn}} = \frac{2\,ab}{\sqrt{m^2b^2 + n^2a^2}}$$

   Now putting $m = n = 1$, we get

$$\lambda_{c_{11}} = \frac{2\,ab}{\sqrt{a^2 + b^2}}$$

2. **TM_{12} mode:** Now putting $m = 1, n = 2$ in Eq. 4.58, we get

$$\lambda_{c_{12}} = \frac{2\,ab}{\sqrt{b^2 + 4\,a^2}}$$

3. **TM_{21} mode:** Now putting $m = 2, n = 1$ in Eq. 4.58, we get

$$\lambda_{c_{21}} = \frac{2\,ab}{\sqrt{4\,b^2 + a^2}}$$

   By inspection, it is clear that, $\lambda_{c_{11}} > \lambda_{c_{12}}$ and $\lambda_{c_{11}} > \lambda_{c_{21}}$ and so on.

The field pattern of TM_{11} mode is shown in Fig. 4.39.

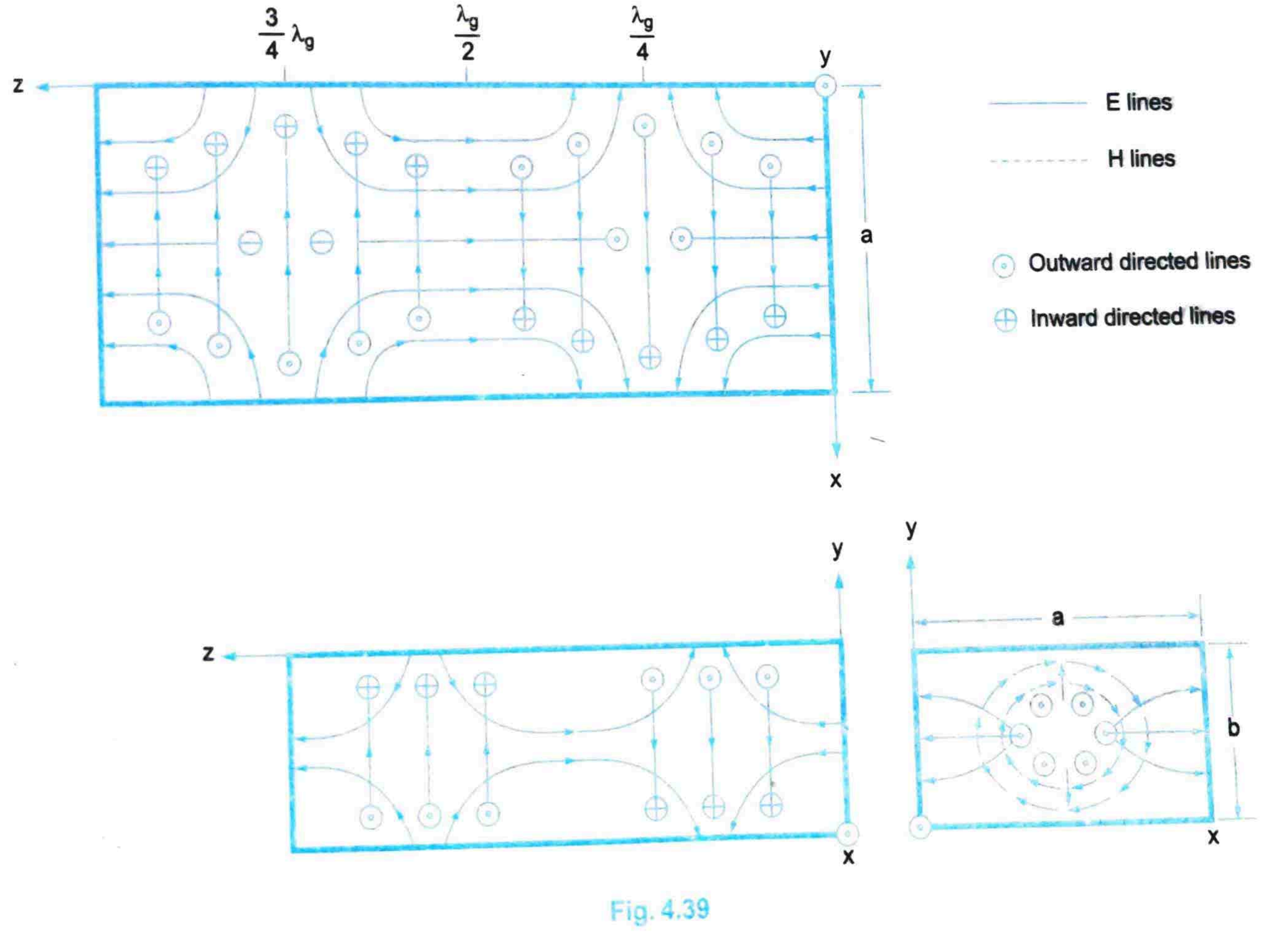


Fig. 4.39

4.3.8 Guide Wavelength, Group and Phase Velocity

Here we define the guide wavelength, group velocity and phase velocity relevant for transmission of a wave in a waveguide.

Guide Wavelength (λ_g)

It is defined as the distance travelled by the wave in order to undergo a phase shift of 2π radians. This is shown by Fig. 4.32.

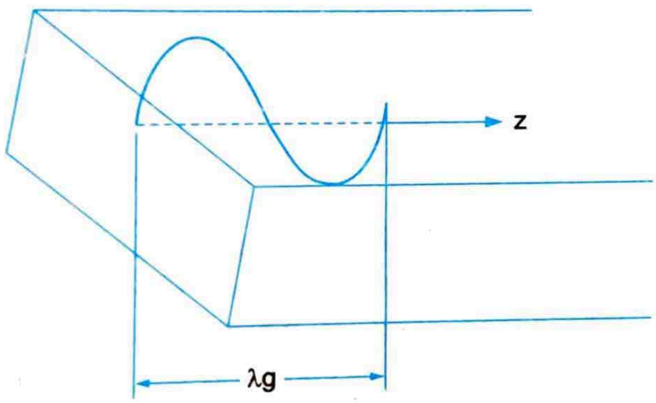


Fig. 4.32

It is related to the phase constant by the relation

$$\lambda_g = \frac{2\pi}{\beta} \qquad \text{...(4.59)}$$

The wavelength in the waveguide is different from the wavelength in free space. In fact, it is related to free space wavelength λ_o and cutoff wavelength λ_c by $\frac{1}{\lambda_g^2} = \frac{1}{\lambda_o^2} - \frac{1}{\lambda_c^2}$ (to be proved in section 4.3.8.5)

or
$$\lambda_g = \frac{\lambda_o}{\sqrt{1-(\lambda_o/\lambda_c)^2}} \qquad \text{...(4.60)}$$

This equation is true for any mode in a waveguide of any cross-section, provided λ_c corresponds to the mode and the cross-section of the waveguide.

From the above relation, it is clear that if $\lambda_o << \lambda_c$, the denominator is approximately equal to '1' and $\lambda_g = \lambda_o$. As λ_o *symbol* λ_c, λ_g increases and reaches infinity when $\lambda_o = \lambda_c$. When $\lambda_o > \lambda_c$, it is evident that λ_g is imaginary which is nothing but no propagation in the waveguide.

Phase Velocity (V_p)

We have just seen that wave propagates in the waveguide when guide wavelength λ_g is greater than the free space wavelength λ_o. Since the velocity of propagation is the product of λ and f, it follows that in a waveguide, $V_p = \lambda_g f$ where V_p is the phase velocity. But the speed of light is equal to product of λ_o and f. This V_p is greater than the speed of light since $\lambda_g > \lambda_o$. This is contradicting since no signal can travel faster than the speed of light. However, the wavelength in the guide is the length of the cycle and V_p represents the velocity of the phase. In fact it is defined as the *rate at which the wave changes its phase in terms of the guide wavelength.*

i.e.,

$$V_p = \frac{\lambda_g}{\text{unit time}} = \lambda_g \cdot f = \frac{2\pi f \cdot \lambda_g}{2\pi} = \frac{2\pi f}{2\pi/\lambda_g}$$

i.e.,

$$V_p = \frac{\omega}{\beta} \qquad ...(4.61)$$

where, $\omega = 2\pi f$, $\beta = \dfrac{2\pi}{\lambda_g}$

Since no intelligence or modulation travel at this velocity, V_p is termed as phase velocity.

Group Velocity (V_g)

If there is modulation in the carrier, the modulation envelope actually travels at velocity slower than that of carrier alone and of course slower than speed of light. The velocity of modulation envelope is called the group velocity V_g. This happens when a modulated signal travels in a waveguide, the modulation goes on slipping backward with respect to the carrier.

It is defined as the *rate at which the wave propagates through the waveguide* and is given by

$$V_g = \frac{d\omega}{d\beta} \qquad ...(4.62)$$

Expression for Phase Velocity and Group Velocity

1. **Expression for V_p:** From Eq. 4.61 we know that $V_p = \dfrac{\omega}{\beta}$

Also,

$$h^2 = \gamma^2 + \omega^2\mu\varepsilon = A^2 + B^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$

and

$$\gamma = \alpha + j\beta$$

For wave propagation, $\gamma = j\beta$ $\qquad$ ($\because$ attenuation, $\alpha = 0$)

$$\therefore \quad \gamma^2 = (j\beta)^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - \omega^2\mu\varepsilon \quad ...(4.63)$$

At $f = f_c$, $\omega = \omega_c$, $\gamma = 0$

$$\therefore \quad \omega_c^2\,\mu\varepsilon = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \quad ...(4.64)$$

Using Eq. 4.64 in Eq. 4.63, we get

$$\gamma^2 = (j\beta)^2 = \omega_c^2\mu\varepsilon - \omega^2\mu\varepsilon$$

$$\therefore \quad \gamma^2 = \beta^2 = \omega^2\mu\varepsilon - \omega_c^2\mu\varepsilon$$

$$\text{or} \quad \beta = \sqrt{\omega^2\mu\varepsilon - \omega_c^2\mu\varepsilon}$$

$$\beta = \sqrt{\mu\varepsilon(\omega^2 - \omega_c^2)} = \sqrt{\mu\varepsilon}\,\sqrt{\omega^2 - \omega_c^2} \quad ...(4.65)$$

$$\therefore \quad V_p = \frac{\omega}{\beta} = \frac{\omega}{\sqrt{\mu\varepsilon}\,\sqrt{\omega^2 - \omega_c^2}} = \frac{1}{\sqrt{\mu\varepsilon}}\,\frac{1}{\sqrt{1 - (\omega_c/\omega)^2}}$$

$$i.e., \quad V_p = \frac{c}{\sqrt{1 - (f_c/f)^2}}$$

We also know that, f (any frequency) $= c/\lambda_o$ where λ_o is free space δ wavelength and f_c [cutoff frequency] $= c/\lambda_c$ where λ_c is cutoff wavelength

$$\frac{f_c}{f} = \frac{c}{\lambda_c}\cdot\frac{\lambda_o}{c} = \frac{\lambda_o}{\lambda_c}$$

$$\therefore \quad V_p = \frac{c}{\sqrt{1 - (\lambda_o/\lambda_c)^2}} \quad ...(4.66)$$

2. Expression for V_g : From Eq. 4.62 we know that, $V_g = \dfrac{d\omega}{d\beta}$

But from Eq. 4.65, $\beta = \sqrt{\mu\varepsilon\,(\omega^2 - \omega_c^2)}$

Now differentiating β w.r.t. 'ω', we get

$$\frac{d\beta}{d\omega} = \frac{1}{2\sqrt{(\omega^2 - \omega_c^2)\,\mu\varepsilon}}\cdot 2\,\omega\mu\varepsilon$$

$$\frac{d\beta}{d\omega} = \frac{\sqrt{\mu\varepsilon}}{\sqrt{1 - (\omega_c/\omega)^2}} = \frac{\sqrt{\mu\varepsilon}}{\sqrt{1 - (f_c/f)}}$$

$$\text{or} \quad V_g = \frac{d\omega}{d\beta} = \frac{\sqrt{1 - (f_c/f)^2}}{\sqrt{\mu\varepsilon}}$$

$$\text{or} \quad V_g = c\sqrt{1 - \left(\frac{\lambda_o}{\lambda_c}\right)^2} \quad ...(4.67)$$

Consider the product of V_p and V_g

i.e.,
$$V_p \cdot V_g = \frac{c}{\sqrt{1-(\lambda_o/\lambda_c)^2}} \cdot c \cdot \sqrt{1-(\lambda_o/\lambda_c)^2}$$

∴
$$V_p V_g = c^2 \qquad \text{...(4.68)}$$

ALITER: Now we give another method for proving $V_p\, V_g = c^2$

As per earlier discussion we know that V_p is greater than the speed of light by the ratio λ_g/λ_o

∴
$$V_p = \frac{\lambda_g}{\lambda_o} c$$

The group velocity V_g is shortened by the same ratio.

i.e.,
$$V_g = \frac{\lambda_o}{\lambda_g} c$$

∴
$$V_p\, V_g = \frac{\lambda_g}{\lambda_o} \cdot c \cdot \frac{\lambda_o}{\lambda_g} \cdot c = c^2$$

Relation between λ_g, λ_o, and λ_c

We know that
$$V_p = \lambda_g \cdot f = \frac{\lambda_g}{\lambda_o} \cdot c$$

and also
$$V_p = \frac{c}{\sqrt{1-(\lambda_o/\lambda_c)^2}}$$

∴
$$\frac{c}{\sqrt{1-(\lambda_o/\lambda_c)^2}} = \frac{\lambda_g}{\lambda_o} \cdot c$$

or
$$\lambda_g = \frac{\lambda_o}{\sqrt{1-(\lambda_o/\lambda_c)^2}} \qquad \text{...(4.69)}$$

4.3.22 Phase Velocity, Group Velocity, Guide Wavelength and Wave Impedance

The relations for phase velocity, group velocity and guide wavelength remain the same as in case of a rectangular waveguide both for TE and TM modes.

i.e.,
$$V_g = \frac{\omega}{\beta} = \frac{V_g}{\sqrt{1-(\lambda_o/\lambda_c)^2}} \qquad \text{...(4.134)}$$

$$\lambda_g = \frac{\lambda_o}{\sqrt{1-(\lambda_o/\lambda_c)^2}} \quad \text{...(4.135)}$$

$$Z_{z_{TE}} = \frac{\omega\mu}{\beta} = \frac{\eta}{\sqrt{1-(\lambda_o/\lambda_c)^2}} \quad \text{...(4.136)}$$

where, $\lambda = \frac{V\rho}{f}$ and $n = \sqrt{\frac{\mu}{\varepsilon}}$

$$Z_{z_{TM}} = \frac{\beta}{\omega\mu} = \eta\sqrt{1-(\lambda_o/\lambda_c)^2} \quad \text{...(4.137)}$$

as before.

4.3.12 Power Transmission in Rectangular Waveguide

The power transmitted through a waveguide and the power loss in the guide walls can be calculated by means of complex Poynting theorem. We assume that the waveguide is terminated in such a way that there is no reflection from the receiving end or that the waveguide is infinitely long as compared with its wavelength.

The power transmitted P_{tr}, through a waveguide given by,

$$P_{tr} = \oint \mathrm{P} \cdot ds = \oint \frac{1}{2}(E \times H^*) \cdot ds \qquad \text{...(4.90)}$$

For a lossless dielectric, the time average power flow through a rectangular waveguides is

$$P_{tr} = \frac{1}{2Z_z} \int_a |E|^2 da = \frac{Z_z}{z} \int_a |H|^2 \, da$$

where $Z_z = \dfrac{E_x}{H_y} = \dfrac{-E_y}{H_x}$

$$|E|^2 = |E_x|^2 + |E_y|^2$$

$$|H|^2 = |H_x|^2 + |H_y|^2$$

For TM_{mn} mode, the average power transmitted through a rectangular waveguide of dimensions a and b is

$$P_{tr} = \frac{1}{2\eta\sqrt{1-(\lambda_o/\lambda_c)^2}} \int_0^b \int_0^a |E_x|^2 + |E_y|^2 \, dx\, dy \qquad \text{...(4.91)}$$

For TE_{mn} modes,

$$Z_z = \frac{\eta}{\sqrt{1-(\lambda_o/\lambda_c)^2}}$$

$\therefore$

$$P_{tr} = \frac{\sqrt{1-(\lambda_o/\lambda_c)^2}}{2\eta} \int_0^b \int_0^a |E_x|^2 \,|+|E_y|^2 \, dx\, dy \qquad \text{...(4.92)}$$

4.3.4 Propagation of TEM waves

We know for a TEM wave,

$$\mathbf{E}_z = 0 \text{ and } \mathbf{H}_z = 0.$$

Substituting, these values in Eqs. 4.34 to 4.37 all the field components along x and y directions $\mathbf{E}_x$, $\mathbf{E}_y$, $\mathbf{H}_x$, $\mathbf{H}_y$ vanish and hence a TEM *wave cannot exist inside a waveguide*.

Consider a rectangular waveguide situated in the rectangular coordinate system with its breadth along x-axis, width along y-axis and the wave is assumed to propagate along the z-direction. Waveguide is filled with air as dielectric as shown in Fig. 4.27.

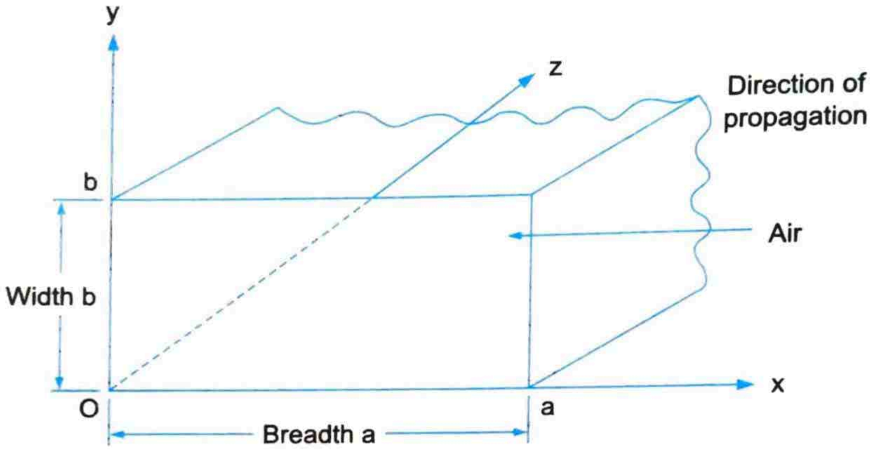


Fig. 4.27 Propagation through a rectangular waveguide.

The wave equation for TE and TM waves are given by

$$\Delta^2 \mathbf{H}_z = -\omega^2 \mu\varepsilon \mathbf{H}_z \quad \text{for TE wave } (\mathbf{E}_z = 0)$$

$$\Delta^2 \mathbf{E}_z = -\omega^2 \mu\varepsilon \mathbf{E}_z \quad \text{for TM wave } (\mathrm{H}_z = 0)$$

Expanding $\Delta^2 \mathbf{E}_z$ in rectangular coordinate system

$$\frac{\partial^2 \mathbf{E}_z}{\partial x^2} + \frac{\partial^2 \mathbf{E}_z}{\partial y^2} + \frac{\partial^2 \mathbf{E}_z}{\partial z^2} = -\omega^2 \mu\varepsilon \mathbf{E}_z \qquad \text{...(4.23)}$$

Since the wave is propagating in the 'z' direction we have the operator.

$$\frac{\partial^2}{\partial z^2} = \gamma^2$$

Substituting this operator in Eq. 4.23, we get

$$\frac{\partial^2 \mathbf{E}_z}{\partial x^2} + \frac{\partial^2 \mathbf{E}_z}{\partial y^2} + \gamma^2 \mathbf{E}_z = -\omega^2 \mu\varepsilon \mathbf{E}_z \qquad \text{...(4.24)}$$

or

$$\frac{\partial^2 \mathbf{E}_z}{\partial x^2} + \frac{\partial^2 \mathbf{E}_z}{\partial y^2} + (\gamma^2 + \omega^2 \mu\varepsilon)\, \mathbf{E}_z = 0 \qquad \text{...(4.25)}$$

Let $\gamma^2 + \omega^2\mu\varepsilon = h^2$, be a constant, then Eq. 4.25 can be rewritten as

$$\frac{\partial^2 \mathbf{E}_z}{\partial x^2} + \frac{\partial^2 \mathbf{E}_z}{\partial y^2} + h^2 \mathbf{E}_z = 0 \quad \text{for TM wave} \qquad \text{...(4.26)}$$

Similarly, $\dfrac{\partial^2 \mathbf{H}_z}{\partial x^2} + \dfrac{\partial^2 \mathbf{H}_z}{\partial y^2} + h^2 \mathbf{H}_z = 0$ for TM wave ...(4.27)

By solving the above partial differential equations, we get solutions for E_z and H_z. Using Maxwell's equation, it is possible to find the various components along x and y directions [$\mathbf{E}_x$, $\mathbf{H}_x$, $\mathbf{E}_y$, $\mathbf{H}_y$]

From Maxwell's 1st equation, we have

$$\nabla \times \mathbf{H} = j\omega\varepsilon \mathbf{E}$$

Expanding $\nabla \times \mathbf{H}$,

i.e.,
$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \mathbf{H}_x & \mathbf{H}_y & \mathbf{H}_z \end{vmatrix} = j\omega\varepsilon\ [\hat{i}\mathbf{E}_x + \hat{j}\mathbf{E}_y + \hat{k}\mathbf{E}_z]$$

Replacing $\dfrac{\partial}{\partial z} = -\gamma$ (an operator), we get

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & -\gamma \\ \mathbf{H}_x & \mathbf{H}_y & \mathbf{H}_z \end{vmatrix} = j\omega\varepsilon\ [\hat{i}\mathbf{E}_x + \hat{j}\mathbf{E}_y + \hat{k}\mathbf{E}_z]$$

Equating coefficients of $\hat{i}$, $\hat{j}$ and $\hat{k}$ (after expanding), we get

$$\frac{\partial \mathbf{H}_z}{\partial y} + \gamma \mathbf{H}_y = j\omega\varepsilon \mathbf{E}_x \quad ...(4.28)$$

$$\frac{\partial \mathbf{H}_z}{\partial x} + \gamma \mathbf{H}_x = -j\omega\varepsilon \mathbf{E}_y \quad ...(4.29)$$

$$\frac{\partial \mathbf{H}_y}{\partial x} - \frac{\partial \mathbf{H}_x}{\partial y} = j\omega\varepsilon \mathbf{E}_z \quad ...(4.30)$$

Similarly from Maxwell's 2nd equation, we have

$$\nabla \times \mathbf{E} = -j\omega\mu \mathbf{H}$$

Expanding $\nabla \times \mathbf{E}$, we get

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \mathbf{E}_x & \mathbf{E}_y & \mathbf{E}_z \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & -\gamma \\ \mathbf{E}_x & \mathbf{E}_y & \mathbf{E}_z \end{vmatrix} = -j\omega\mu\ [\hat{i}\mathbf{H}_x + \hat{j}\mathbf{H}_y + \hat{k}\mathbf{H}_z]$$

Expanding and equating coefficients of $\hat{i}$, $\hat{j}$ and $\hat{k}$, we get

$$\frac{\partial \mathbf{E}_z}{\partial y} + \gamma \mathbf{E}_y = -j\omega\mu \mathbf{H}_x \quad ...(4.31)$$

$$\frac{\partial \mathbf{E}_z}{\partial x} + \gamma \mathbf{E}_x = +j\omega\mu \mathbf{H}_y \quad ...(4.32)$$

$$\frac{\partial \mathbf{E}_y}{\partial x} - \frac{\partial \mathbf{E}_x}{\partial y} = -j\omega\mu \mathbf{H}_z \quad ...(4.33)$$

Combining Eq. 4.28 and Eq. 4.32 to eliminate H_y, we get an expression for E_x. From Eq. 4.32,

$$\mathbf{H}_y = \frac{1}{j\omega\mu}\frac{\partial \mathbf{E}_z}{\partial x} + \frac{\gamma}{j\omega\mu}\mathbf{E}_x$$

Substituting for H_y in Eq. 4.28, we get

$$\frac{\partial \mathbf{H}_z}{\partial y} + \frac{\gamma}{j\omega\mu}\frac{\partial \mathbf{E}_z}{\partial x} + \frac{\gamma^2}{j\omega\mu}\mathbf{E}_x = j\omega\varepsilon \mathbf{E}_x$$

or

$$\mathbf{E}_x\left[j\omega\varepsilon - \frac{\gamma^2}{j\omega\mu}\right] = \frac{\gamma}{j\omega\mu}\frac{\partial \mathbf{E}_z}{\partial x} + \frac{\partial \mathbf{H}_z}{\partial y}$$

Multiplying by $j\omega\mu$, we get

$$\mathbf{E}_x\,[-\omega^2\mu\varepsilon - \gamma^2) = \gamma\frac{\partial \mathbf{E}_z}{\partial x} + j\omega\mu\frac{\partial \mathbf{H}_z}{\partial y}$$

or

$$\mathbf{E}_x\,[-(\gamma^2 + \omega^2\mu\varepsilon)] = \gamma\frac{\partial \mathbf{E}_z}{\partial x} + j\omega\mu\frac{\partial \mathbf{H}_z}{\partial y}$$

where $\gamma^2 + \omega^2\mu\varepsilon = h^2$

Dividing by $-h^2$ throughout, we get

$$\mathbf{E}_x = \frac{-\gamma}{h^2}\frac{\partial \mathbf{E}_z}{\partial x} - \frac{j\omega\mu}{h^2}\frac{\partial \mathbf{H}_z}{\partial y} \quad ...(4.34)$$

Similarly

$$\mathbf{E}_y = \frac{-\gamma}{h^2}\frac{\partial \mathbf{E}_z}{\partial y} + \frac{j\omega\mu}{h^2}\frac{\partial \mathbf{H}_z}{\partial x} \quad ...(4.35)$$

and

$$\mathbf{H}_x = \frac{-\gamma}{h^2}\frac{\partial \mathbf{H}_z}{\partial x} + \frac{j\omega\varepsilon}{h^2}\frac{\partial \mathbf{E}_z}{\partial y} \quad ...(4.36)$$

and

$$\mathbf{H}_y = \frac{-\gamma}{h^2}\frac{\partial \mathbf{H}_z}{\partial y} - \frac{j\omega\varepsilon}{h^2}\frac{\partial \mathbf{E}_z}{\partial x} \quad ...(4.37)$$

These, equations give a general relationship for field components within a waveguide.

We know for a TEM wave,

$$\mathbf{E}_z = 0 \text{ and } \mathbf{H}_z = 0.$$

Substituting, these values in Eqs. 4.34 to 4.37 all the field components along x and y directions $\mathbf{E}_x$, $\mathbf{E}_y$, $\mathbf{H}_x$, $\mathbf{H}_y$ vanish and hence a TEM *wave cannot exist inside a waveguide.*

4.2.4 Microstrip Line

Microstrip line is an unsymmetrical strip line that is nothing but a parallel plate transmission line having dielectric substrate, the one face of which is metallised ground and the other (top) face has a thin conducting strip of certain width 'w' and thickness 't'. This is shown in Fig. 4.4. The top ground plane is not present in a microstrip as compared to a strip line. Sometimes a cover plate is used for shielding purposes but it is kept much farther away than the ground plane so as not to affect the microstrip field lines as shown in Fig. 4.5.

There are certain advantages of microstrip lines over strip lines, coaxial lines, and waveguides.

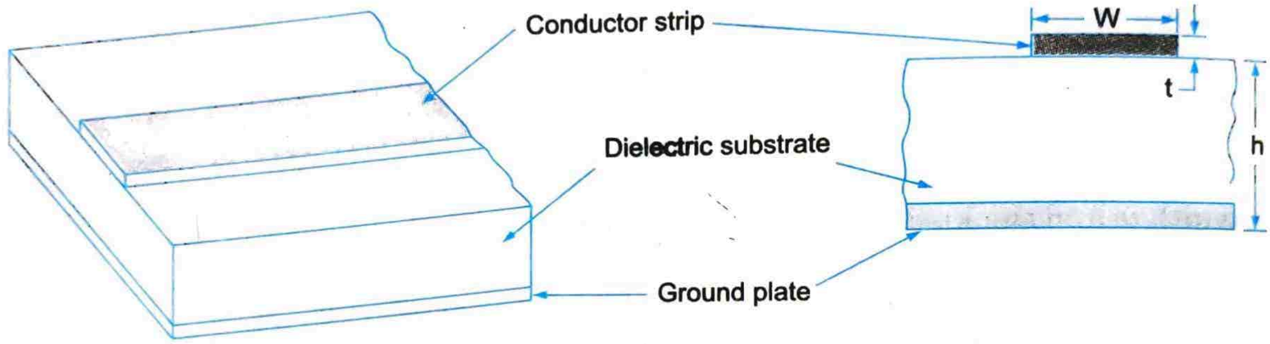


Fig. 4.4 Microstrip line.

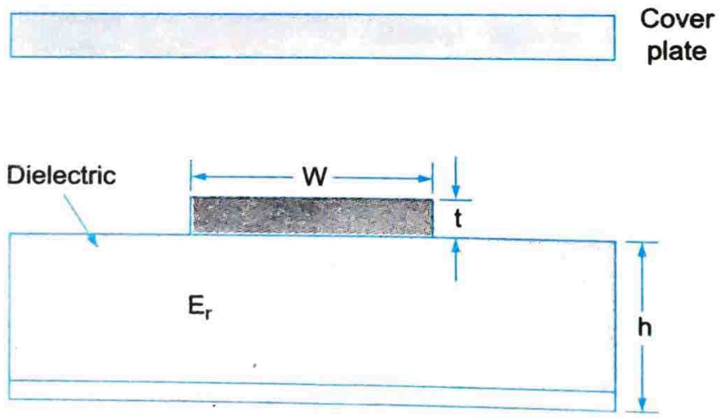


Fig. 4.5 Microstrip line with a cover plate.

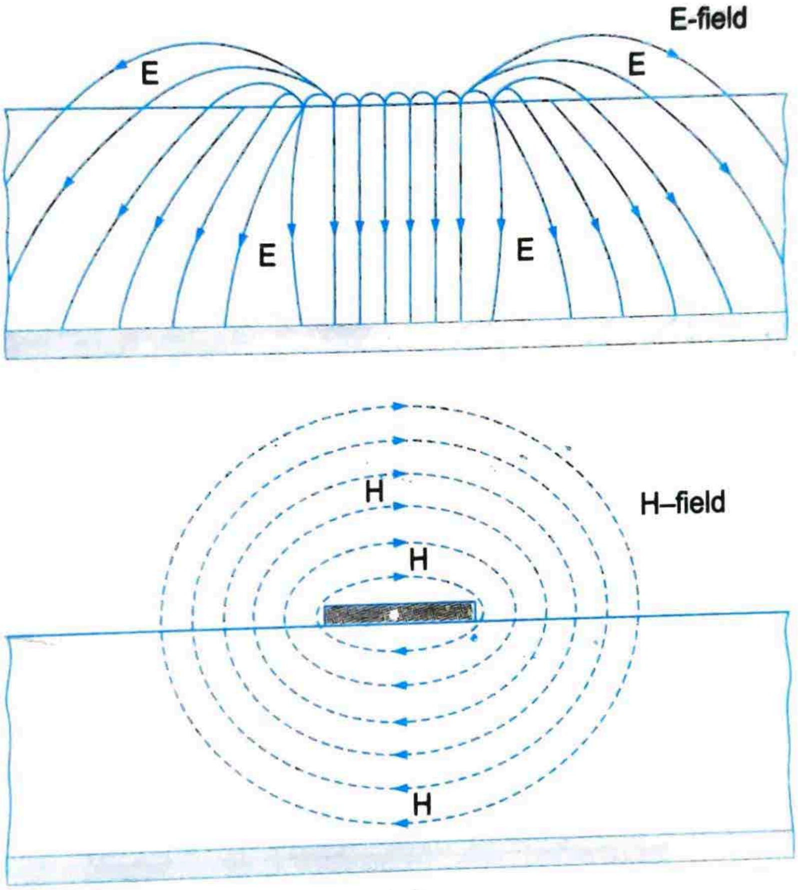
1. Complete conductor pattern may be deposited and processed on a single dielectric substrate which is supported by a single metal ground plane. Thus fabrication costs would be substantially lower than strip line, coaxial or waveguide circuits.
2. Due to the planar nature of the microstrip structure, both packaged and unpackaged semiconductor chips can be conveniently attached to the microstrip element.
3. Also there is an easy access to the top surface making it easy to mount passive or active discrete devices and also for making minor adjustments after the circuit has been fabricated. This also allows access for probing and measurement purposes.

However, microstrips have some limitations too.

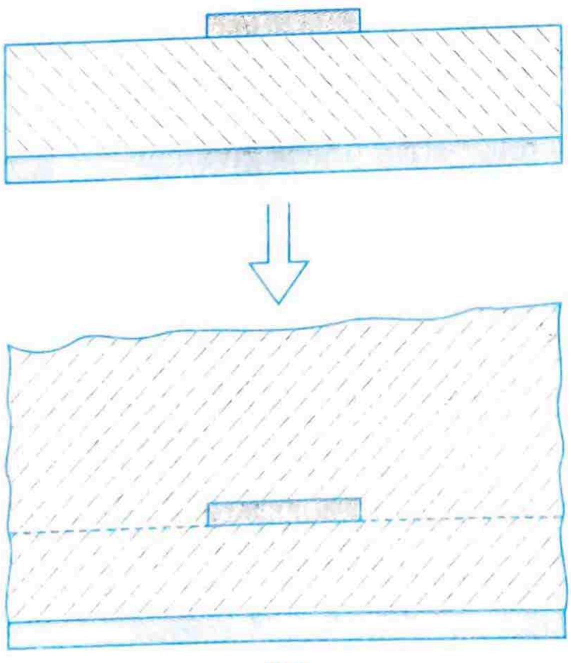
1. Due to the openness of the microstrip structure, they have higher radiation losses or interference due to nearby conductors. These can be reduced by choosing thin substrates with high dielectric constants.

2. Because of the proximity of the air-dielectric air interface with the microstrip conductor at the interface, a discontinuity in the electric and magnetic fields is generated. This results in a microstrip configuration that becomes a mixed dielectric transmission structure with unpure TEM modes propagating. This makes the analysis complicated.

The approximate distribution of electric and magnetic field is shown in Fig. 4.6.



(a)



(b)

Fig. 4.6 Approximate electric and magnetic field in a microstrip line.

We can see that there is a concentration of fields below the microstrip element. The electric flux crossing the air dielectric boundary is small and although a pure TEM mode cannot exist, a small deviation from TEM mode does exist which can be neglected.

The characteristic impedance of a microstrip is a function of the strip line width (w), thickness (t) and the distance between the line and the ground plane (h). In fact, the variation of characteristic impedance in terms of w/h ratio is shown in Fig. 4.7.

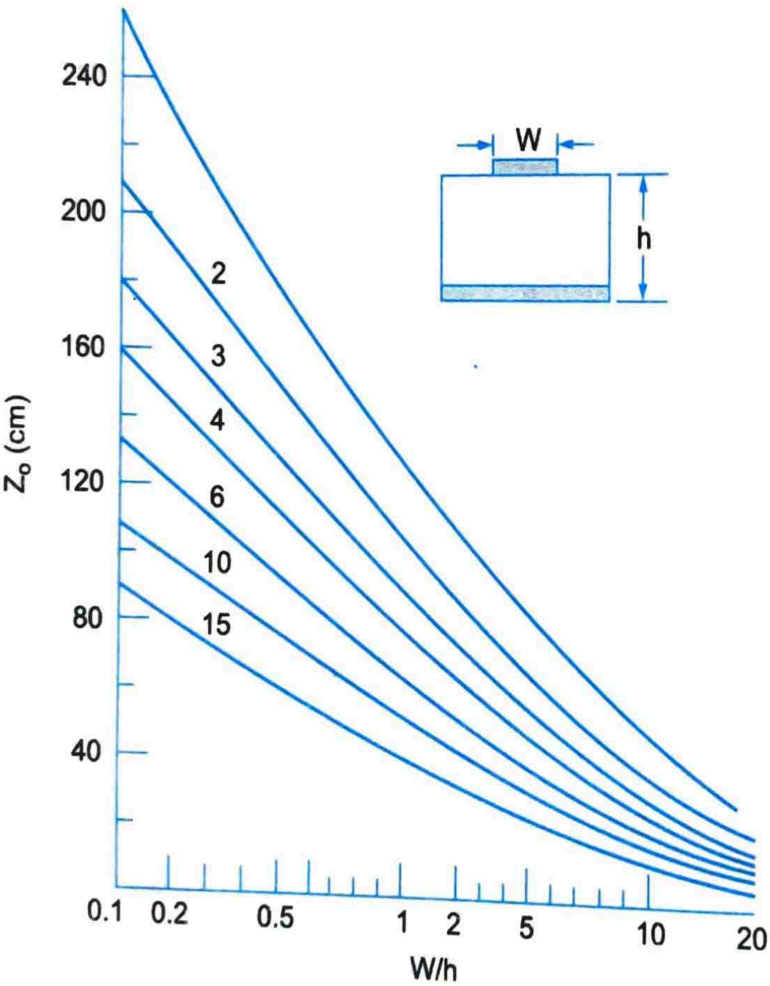


Fig. 4.7 Z_0 vs w/h ratios.

Empirical relation for Z_o for a microstrip line is given by

$$Z_o = \frac{60}{\sqrt{\varepsilon_r}} \ln\left(\frac{4h}{d}\right) \text{for } h >> d \qquad ...(4.14)$$

where, ε_r = dielectric constant of the dielectric medium

h = distance between the microstrip line and the ground plane

d = diameter of the wire (wire over ground transmission line).

Effective dielectric constant (ε_{re}) also has an empirical relation given by (due to Digiacomo)

$$\varepsilon_{re} = 0.475\,\varepsilon_r + 0.67 \qquad ...(4.15)$$

where, ε_r = relative dielectric constant of the board material.

ε_{re} = effective relative dielectric constant for a microstrip line.

Since the cross-section of microstrip line is rectangular, diameter (d) also has an empirical relation given by (due to Springfield),

$$d = 0.67\, w\left(0.8 + \frac{t}{w}\right) \quad \text{...(4.16)}$$

where symbols have their usual significance.

The phase velocity of a microstrip line is given by

$$V_p = V_c / \sqrt{\varepsilon_{re}} \quad \text{...(4.17)}$$

where, V_c = velocity of electromagnetic waves.

ε_{re} is given by an empirical relation (due to Schmeiter)

$$\varepsilon_{re} = \frac{\varepsilon_r + 1}{2} + \frac{\varepsilon_r - 1}{2}\left(1 + \frac{10h}{w}\right)^{-1/2} \quad \text{...(4.18)}$$

Hence design of microstrip is quite complex as it has to take care of so many factors discussed above like w, h, ε_r, ε_{re}, ε_{eff} etc.

Taking into account the relationships for ε_{re} and d (from Eqs. 4.15 and 4.16), Z_o can be written as,

$$Z_o = \frac{87}{\sqrt{\varepsilon_r + 1.41}} \ln\left[\frac{5.98h}{0.8w + t}\right] \text{ for } h < 0.8\, w \quad \text{...(4.19)}$$

If $w >> h$, *i.e.*, for a wide microstrip line, Z_o is given by (as per Assadourian)

$$Z_o = \frac{377}{\sqrt{\varepsilon_r}} \cdot \frac{h}{w} \quad \text{...(4.20)}$$

The microstrip lines have a power handling capacity of a few watts which is quite adequate for most microwave circuits. Microstrip lines offer advantage of miniaturization but for long transmission lengths, they suffer from excessive attenuation per unit length. The attenuation of a microstrip depends upon the electric properties of the substrate and the conductors and also on the frequency. The attenuation constant α, is given by

$$\alpha = \alpha_d + \alpha_c \quad \text{...(4.21)}$$

where, α_d = dielectric attenuation constant (due to dielectric in substrate)

α_c = ohmic attenuation constant (due to ohmic skin losses in conductor and the ground plane)

Radiation loss of a microstrip line depends on the substrate thickness and dielectric constant as well as its geometry.

The quality factor Q of a microstrip line is very high which may be the requirement for high quality resonant MICs. It is however limited by the radiation losses of the substrate and with low dielectric constant. The Q of a microstrip line is given by

$$Q_d = \frac{1}{\tan\theta} \quad \text{...(4.22)}$$

where, θ = dielectric loss tangent.