4(b). NETWORK FUNCTIONS

Introduction:-

A Network function gives the relation between currents (or) voltages at different parts of the network. It is broadly classified as driving point and transfer functions. It is associated with terminals & ports.

Driving point Functions:-

If excitation and response are measured at the same ports, the N/w function is known as the driving-point function. For a one-port network, only one voltage and current are specified and hence only one network function and its reciprocal (Driving point impedance & admittance) can be defined.

1. **Driving point Impedance Function:-** It is defined as the ratio of voltage transform at one port to the current transform at the same port. It is denoted by Z(s), $Z(s) = \frac{V(s)}{I(s)}$

2. **Driving point Admittance function:-** It is defined as the ratio of current transform at one port to the voltage transform at the same port. It is denoted by Y(s), $Y(s) = \frac{I(s)}{V(s)}$

For a 2-port N/w:-

The driving point impedance function at port-1

$$Z_{11}(s) = \frac{V_1(s)}{I_1(s)}$$

The driving point admittance function at port 1

$$Y_{11}(s) = \frac{I_1(s)}{V_1(s)}$$

The d.P.I.F at port-2

$$Z_{22}(s) = \frac{V_2(s)}{I_2(s)}$$

The d.P.A.f at port-2

$$Y_{22}(s) = \frac{I_2(s)}{V_2(s)}$$

Transfer Functions:-

The transfer function is used to describe networks which have at least two ports. It relates a voltage (or) current at one port to the voltage (or) current at another port. These functions are also defined as the ratio of response transform to an excitation transform. Thus, there are four possible form of transfer functions.

1. Voltage Transfer Function:- It is defined as the ratio of the voltage transform at one port to the voltage transform at another port. It is denoted by $G(s)$

$$G_{21}(s) = \frac{V_2(s)}{V_1(s)} \rightarrow \text{Forward voltage transfer function,}$$

$$G_{12}(s) = \frac{V_1(s)}{V_2(s)} \rightarrow \text{Reverse voltage transfer function}$$

2. Current Transfer Function:- It is defined as the ratio of the current transform at one port to the current transform at another port. It is denoted by $\alpha(s)$

$$\alpha_{21}(s) = \frac{I_2(s)}{I_1(s)} \rightarrow \text{forward current Transfer function.}$$

$$\alpha_{12}(s) = \frac{I_1(s)}{I_2(s)} \rightarrow \text{Reverse current Transfer function.}$$

3. Transfer Impedance Function:- It is defined as the ratio of the voltage transform at one port to the current transform at another port. It is denoted $z(s)$

$$z_{21}(s) = \frac{V_2(s)}{I_1(s)} \rightarrow \text{forward Transfer impedance function.}$$

$$z_{12}(s) = \frac{V_1(s)}{I_2(s)} \rightarrow \text{Reverse Transfer impedance function}$$

4. Transfer Admittance Function:- It is defined as the ratio of the current transform at one port to the voltage transform at another port. It is denoted by $Y(s)$

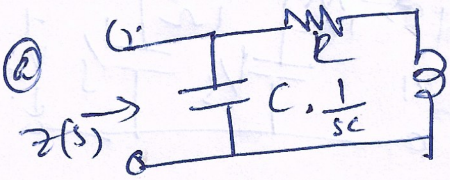
$$Y_{21}(s) = \frac{I_2(s)}{V_1(s)} \rightarrow \text{Forward Transfer Admittance function.}$$

$$Y_{12}(s) = \frac{I_1(s)}{V_2(s)} \rightarrow \text{Reverse Transfer Admittance function.}$$

Problems on Network Functions:-

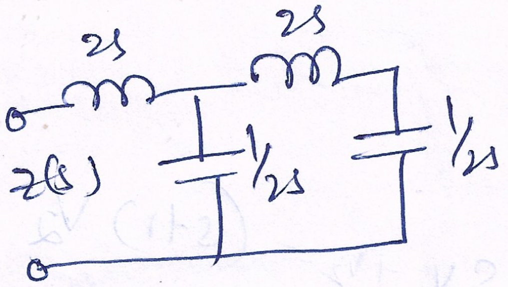
1. Determine the driving-point impedance function for one-port network

(a)



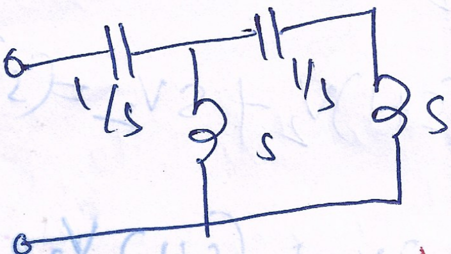
$$Z(s) = \frac{(R+sL)\frac{1}{sC}}{R+sL+\frac{1}{sC}} = \frac{\frac{1}{C}\left(s+\frac{R}{L}\right)}{s^2+\frac{R}{L}s+\frac{1}{LC}}$$

(b)



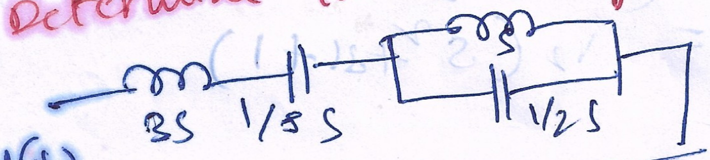
$$Z(s) = \frac{16s^4+12s^2+1}{8s^3+4s}$$

(c)



$$Z(s) = \frac{s^4+3s^2+1}{2s^3+s}$$

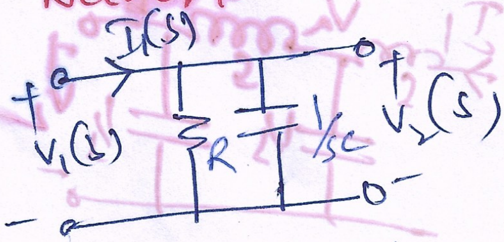
2. Determine the driving point admittance



$$Z(s) = \frac{30s^4+22s^2+1}{5s(2s^2+1)}$$

$$Y(s) = \frac{5s(2s^2+1)}{30s^4+22s^2+1}$$

3. Find the transfer impedance function $Z_{12}(s)$ for the Network

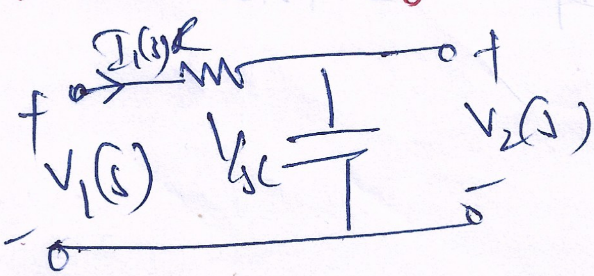


$$V_2(s) = I_1(s) \times \frac{R\,\frac{1}{sC}}{R+\frac{1}{sC}}$$

$$\frac{V_2(s)}{I_1(s)} = \frac{R}{RCs+1}$$

$$Z_{12}(s) = \frac{1}{C\left(s+\frac{1}{RC}\right)}$$

4. Find voltage transfer function of the two port network



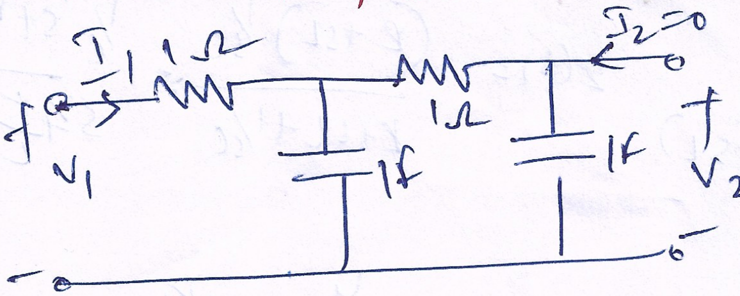
$$V_2(s) = V_1(s) \times \frac{\frac{1}{sC}}{R+\frac{1}{sC}} = V_1(s)\frac{1}{sCR+1} = V_1(s)\frac{\frac{1}{RC}}{s+\frac{1}{RC}}$$

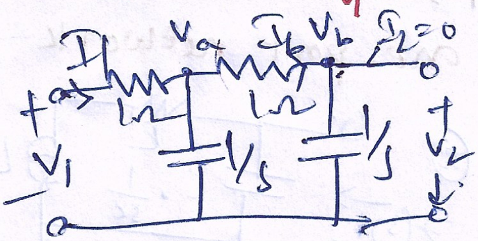
$$\boxed{\frac{V_2(s)}{V_1(s)} = \frac{\frac{1}{RC}}{s+\frac{1}{RC}}}$$

N.F of ladder networks

1. For the N/W, Determine transfer function $\frac{V_2}{V_1}$

(a)





$V_b = V_2$

$I_b = \frac{V_2}{1/s} = sV_2$

$V_a = 1 I_b + V_2 = sV_2 + V_2 = (s+1)V_2$

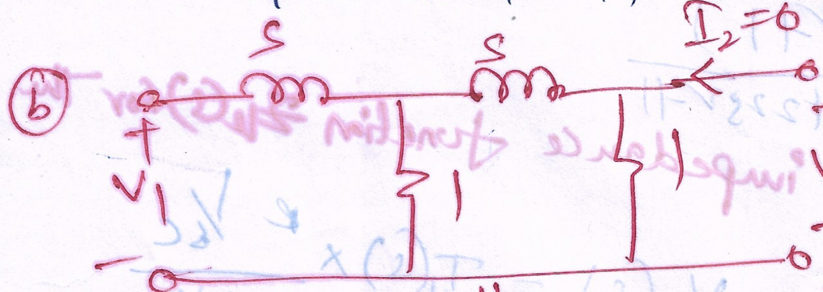
$I_1 = \frac{V_a}{1/s} + I_b = s(s+1)V_2 + sV_2 = (s^2+2s)V_2$

$V_1 = 1 I_1 + V_a = (s^2+2s)V_2 + (s+1)V_2$

$= V_2(s^2+3s+1)$

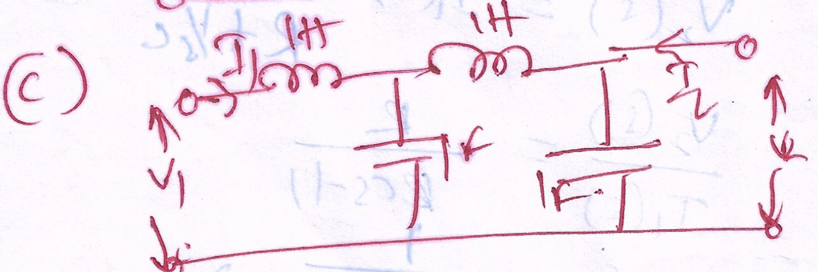
$\frac{V_2}{V_1} = \frac{1}{s^2+3s+1}$

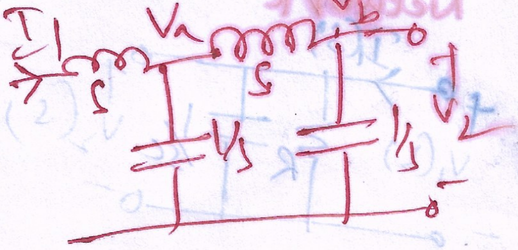
(b)



$\frac{V_2}{V_1} = \frac{1}{s^2+3s+1}$

(c)

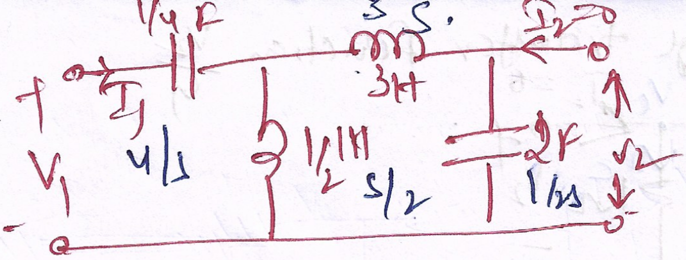




$\frac{V_2}{V_1} = \frac{1}{s^4+3s^2+1}$ $\quad \frac{V_1}{I_1} = \frac{s^4+3s^2+1}{s^3+2s}$

$\frac{V_2}{I_1} = \frac{1}{s^3+2s}$

(a)

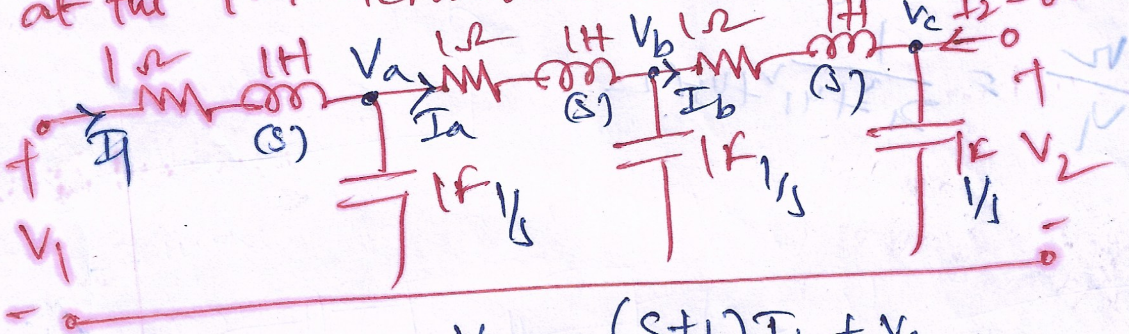


$$\frac{V_1}{I_1} = \frac{6s^4+57s^2+8}{14s^3+2s}$$

$$\frac{V_2}{V_1} = \frac{s^2}{6s^4+57s^2+8}$$

$$\frac{V_2}{I_1} = \frac{s}{14s^2+2}$$

2. For the N/w below, find the driving point impedance at the 1-1' terminal with 2-2' open.



$V_c = V_2$

$I_b = \frac{V_2}{1/s} = sV_2$

$$V_b = (s+1)I_b + V_2 = (s+1)sV_2 + V_2 = (s^2+s+1)V_2$$

$$I_a = \frac{V_b}{1/s} + I_b = sV_b + sV_2 = s(s^2+s+1)V_2 + sV_2 = (s^3+s^2+2s)V_2$$

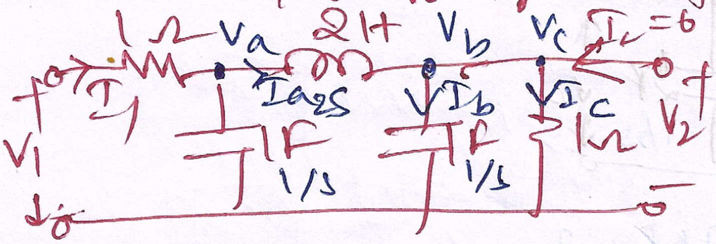
$$V_a = (s+1)I_a + V_b = (s+1)(s^3+s^2+2s)V_2 + (s^2+s+1)V_2 = (s^4+2s^3+4s^2+3s+1)V_2$$

$$I_1 = \frac{V_a}{1/s} + I_a = sV_a + (s^3+s^2+2s)V_2 = (s^5+2s^4+5s^3+4s^2+3s)V_2$$

$$V_1 = (s+1)I_1 + V_a = (s^6+3s^5+8s^4+11s^3+11s^2+6s+1)V_2$$

$$Z_{11} = \frac{V_1}{I_1} = \frac{s^6+3s^5+8s^4+11s^3+11s^2+6s+1}{s^5+2s^4+5s^3+4s^2+3s}$$

3. Determine the voltage transfer function $\frac{V_2}{V_1}$



$$V_c = V_b = 2$$

$$I_a = I_b + I_c = \frac{V_2}{1/s} + \frac{V_2}{1} = (s+1)V_2$$

$$V_a = 2sI_a + V_2 = 2s(s+1)V_2 + V_2$$
$$= (2s^2+2s+1)V_2$$

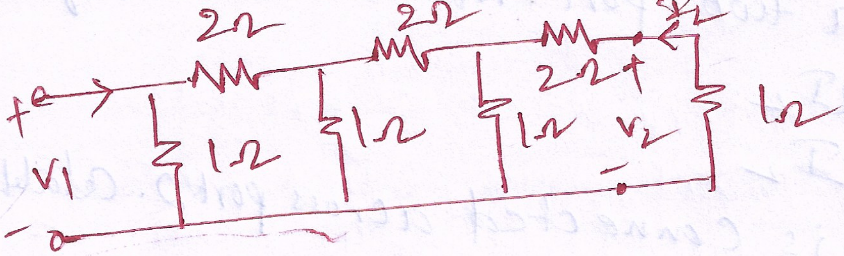
$$I_1 = \frac{V_a}{1/s} + I_a = (2s^3+2s^2+2s+1)V_2$$

$$V_1 = 1I_1 + V_a = (2s^3+4s^2+4s+2)V_2$$

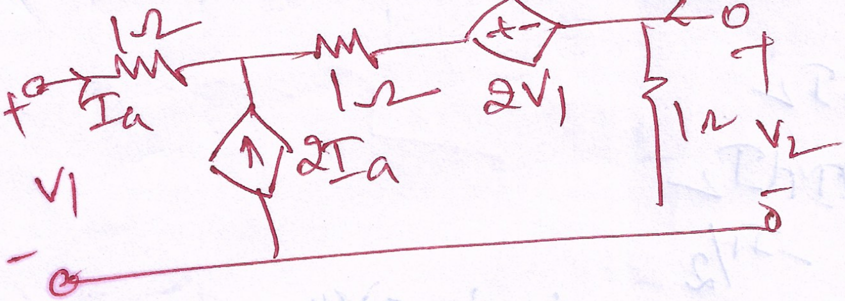
$$\frac{V_2}{V_1} = \frac{1}{2s^3+4s^2+4s+2}$$

Ladder N|ws

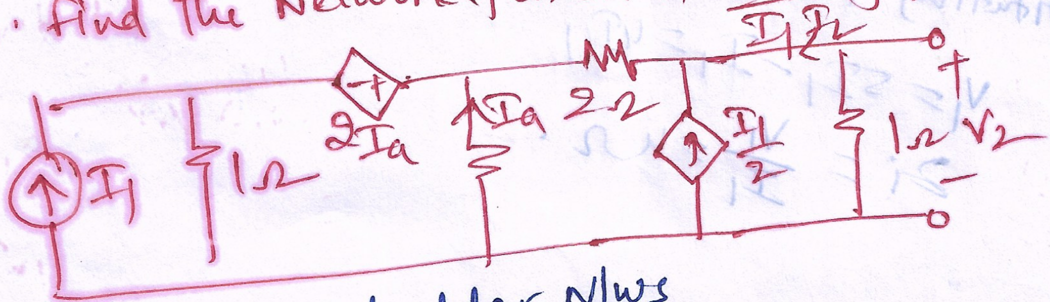
1. For the resistive two port N|w, find $\frac{V_2}{V_1}$, $\frac{V_2}{I_1}$, $\frac{I_2}{V_1}$ & $\frac{I_2}{I_1}$



2. Find the N|w function $\frac{V_2}{V_1}$ $I_2=0$ for the network shown below

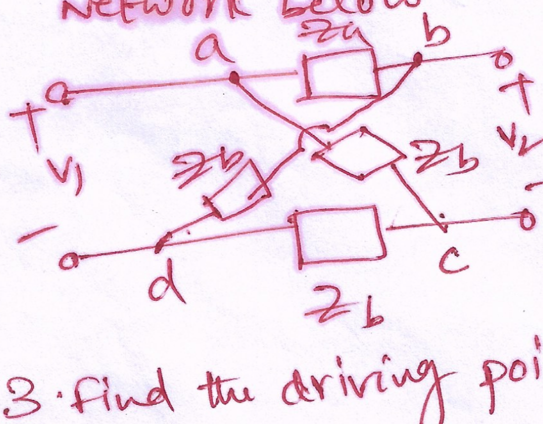


3. Find the Network function $\frac{I_2}{I_1}$ for the network below

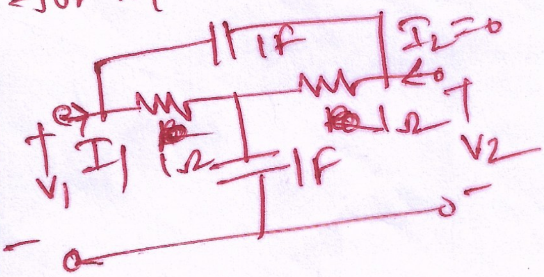


Non-Ladder N|ws

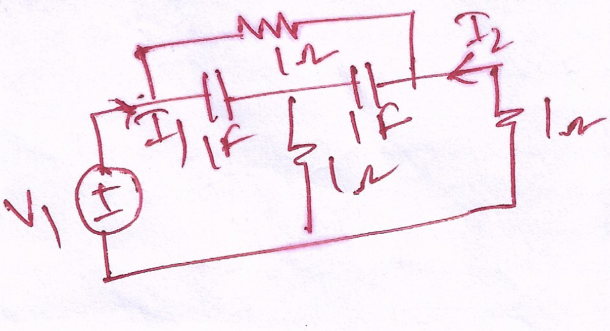
1. Find Z_{11}, Z_{12} & G_{12} for the network below.



2. Find $Z_{11}(s)$, $G_{12}(s)$ & $Z_{21}(s)$ for N|w below.



3. Find the driving point admittance Y_{11} & transfer admittance Y_{12}



Poles & zeros of Network Functions:-

The Network function N(s) may be written as

$$N(s) = \frac{P(s)}{Q(s)} = \frac{a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots a_{n-1} s + a_n}{b_0 s^m + b_1 s^{m-1} + b_2 s^{m-2} + \cdots b_{m-1} s + b_m}$$

where $a_0, a_1, a_2 \ldots a_n$ & $b_0, b_1, b_2, \ldots b_m$ are the coefficients of the polynomials P(s) & Q(s). They are real and positive ~~and~~ for a passive network which consists of passive elements. If the numerator & denominator of polynomial N(s) are factorised, the Network function can be written as

$$N(s) = \frac{P(s)}{Q(s)} = \frac{a_0 (s-z_1)(s-z_2)\cdots(s-z_n)}{b_0 (s-P_1)(s-P_2)\cdots(s-P_n)}$$

where $z_1, z_2, \ldots z_n$ are the ~~N~~ roots for $P(s)=0$ & $P_1, P_2, \ldots P_n$ are the m roots for $Q(s)=0$ and $\frac{a_0}{b_0} = H$ is constant called the scale factor.

→ $z_1, z_2 \ldots z_n$ & $P_1, P_2 \ldots P_n$ are the complex frequencies. $z_1, z_2 \ldots z_n$ in the Network function are called zeros and are denoted by O. Similarly $P_1, P_2 \ldots P_n$ in the N/w function are called poles and are denoted by X.

→ The N/F is ~~completel~~ becomes zero when 's' is equal to any one of the zeros. N(s) becomes infinite when s is equal to any one of the poles. The N/F is completely defined by its poles and zeros and scale factor.

→ If the poles (or) zeros are not repeated, then the function is said to have simple poles & zeros

→ If the poles ~~(or)~~ zeros are repeated, then the function is said to have multiple poles, multiple zeros.

→ When $n > m$, then $(n-m)$ zeros are at $s = \infty$ & for $m > n$, $(m-n)$ poles are at $s = \infty$

→ The total number of zeros is equal to the total number of poles. For any network function, poles and zeros at zero and infinity are taken into account in addition to finite poles and zeros.

→ poles & zeros are critical frequencies. The N/w function becomes infinite at poles, while the N/w function becomes zero at zeros. The N/w F has a finite, non-zero value at other frequencies.

Consider the Network function

$$N(s) = \frac{(s+1)^2 (s+5)}{(s+2)(s+3+j2)(s+3-j2)}$$

That has double ~~poles~~ zeros at $s = -1$ & zero at $s = -5$ & three finite poles at $s = -2$, $s = -3+j2$ & $s = -3-j2$ and it is shown as pole zero plot in fig ①.

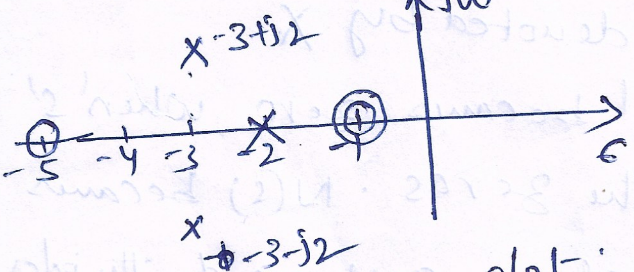


Fig ①. pole zero plot:

→ The N/w function is said to be stable when the real parts of the poles and zeros are negative. otherwise the poles & zeros must lies within the left half of the s-plane.

Necessary conditions for a driving point function:-

The restrictions on pole & zero locations in the driving-point with common factors in $P(s)$ & $Q(s)$ cancelled are listed below:

1. The coefficients in the polynomials $P(s)$ & $Q(s)$ of the network function $N(s) = P(s)/Q(s)$ must be real & positive.
2. The poles & zeros, if complex (or) imaginary, must occur in conjugate pairs.
3. The real part of all poles and zeros must be negative (or) zero, i.e, the poles & zeros must lie in left half of 's' plane.
4. If the real part is zero, then the pole & zero must be simple.
5. The polynomials $P(s)$ & $Q(s)$ may not have any missing terms between the highest & the lowest degrees, unless all even (or) all odd terms are missing.
6. The degree of $P(s)$ & $Q(s)$ may differ by either zero (or) one only.
7. The terms of lowest degree in $P(s)$ & $Q(s)$ may differ in degree by one at most. This condition prevent multiple poles & zeros at $s=0$

Necessary conditions for transfer function:

The restriction on poles & zero location in transfer functions with common factors in $P(s)$ & $Q(s)$ cancelled are listed below.

1. ⓐ The coefficients in the polynomials $P(s)$ & $Q(s)$ must be real.

(b) The coefficients in $Q(s)$ must be positive, but some of the coefficients in $P(s)$ may be negative.

2. The poles and zeros, if complex (or) imaginary, must occur in conjugate pairs.

3. The real part of poles must be negative (or) zero. If the real part is zero, then the pole must be simple.

4. The polynomial $Q(s)$ may not have any missing terms between the highest & the lowest degree unless all even (or) all odd terms are missing.

5. The polynomial $P(s)$ may have missing terms between the lowest & the highest degree, and some of the coefficients may be negative.

6. The degree of $P(s)$ may be as small as zero, independent of the degree of $Q(s)$.

7. For the voltage & current T/Fs, the maximum degree of $P(s)$ must equal the degree of $Q(s)$.

8. For the transfer impedance & admittance functions, the maximum degree of $P(s)$ must equal the degree of $Q(s)$ plus one.

SIGNIFICANCE OF POLES & ZEROS :-

Poles and zeros are critical frequencies. At poles, the network function become infinite, while at zeros, the network function becomes zero. At other frequencies, the network function has a finite non-zero value.

Poles and zeros provide useful information in network functions.

Driving point Impedance :-

$Z(s) = \frac{V(s)}{I(s)}$, A pole of $Z(s)$ implies zero current for a finite voltage which means an open circuit. A zero of $Z(s)$ implies no voltage for a finite current (or) short circuit.

Consider $Z(s) = \frac{1}{sC}$

The above function has a pole at $s=0$ and zero at $s=\infty$.

Therefore, the above function represented by capacitor acts an open circuit at "pole frequency" and acts as short circuit at "zero" frequency.

Driving point admittance :-

$Y(s) = \frac{I(s)}{V(s)}$, A pole of $Y(s)$ implies zero voltage for a finite value of current which gives a short circuit. A zero of $Y(s)$ implies zero current for a finite value of voltage which gives an open circuit

Voltage transform ratio:-

$$G_{21}(s) = \frac{V_2(s)}{V_1(s)}$$

$$V_2(s) = G_{21}(s)\, V_1(s)$$

To obtain output voltage, we require the product of input and transfer function. The expression for ~~$G_{21}(s)$~~ $G_{21}(s) \cdot V_1(s)$ is obtained ~~from~~ in the form of a ratio of polynomial in s

By making use of partial fractions, we can obtain a pole of either $G_2(s)$ (or) $V_1(s)$ and no repeated roots.

$$G_{21}(s)\, V_1(s) = \sum_{i=1}^{n} \frac{A}{s-a_i} + \sum_{j=1}^{m} \frac{A}{s-a_j}$$

where n & m are the number of poles of $G_{21}(s)$ & $V_1(s)$ respectively.

→ The frequencies a_i from the natural complex frequencies corresponding to free oscillations and depend on the network function $G_{21}(s)$. While the frequencies a_j constitute the complex frequencies corresponding to the forced oscillations and depend on the driving force $V_1(s)$.

→ The poles determine the time variation of the response (time-domain behaviour) and the zeros determine the magnitude response.

Time Domain Behaviour from the pole-zero plot

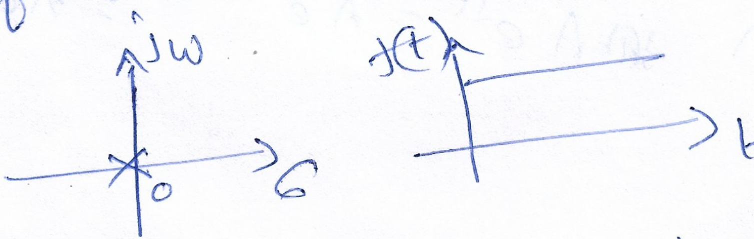
The time-domain behaviour of a system can be determined from the pole-zero plot.

Consider a N/w function

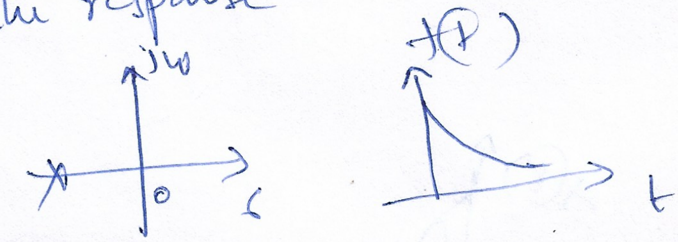
$$F(S) = \frac{H\,(S-z_1)(S-z_2)\cdots(S-z_n)}{(S-P_1)(S-P_2)\cdots(S-P_m)}$$

The poles of this function determine the time-domain behaviour of f(t). The function f(t) can be determined from the knowledge of poles, zeroes & scale factor H.

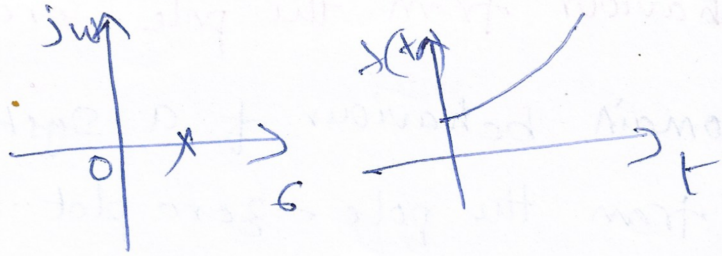
(i) when pole is at origin i.e, at $S=0$ the function f(t) represents steady-state response of the circuit i.e, dc value



(ii) when pole lies in the left half of the s-plane, the response decreases exponentially

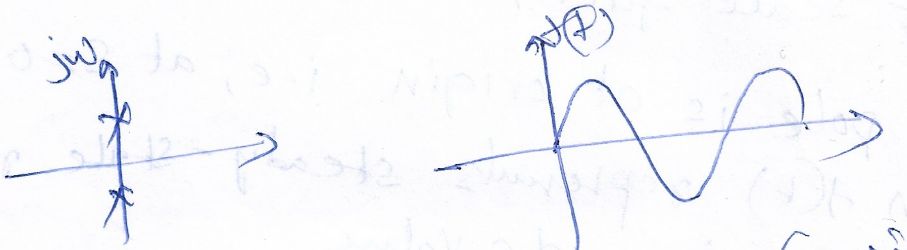


(iii) when pole in the right half of the s-plane the response increases exponentially. A pole in the right-half plane gives rise to unbounded response and unstable system.

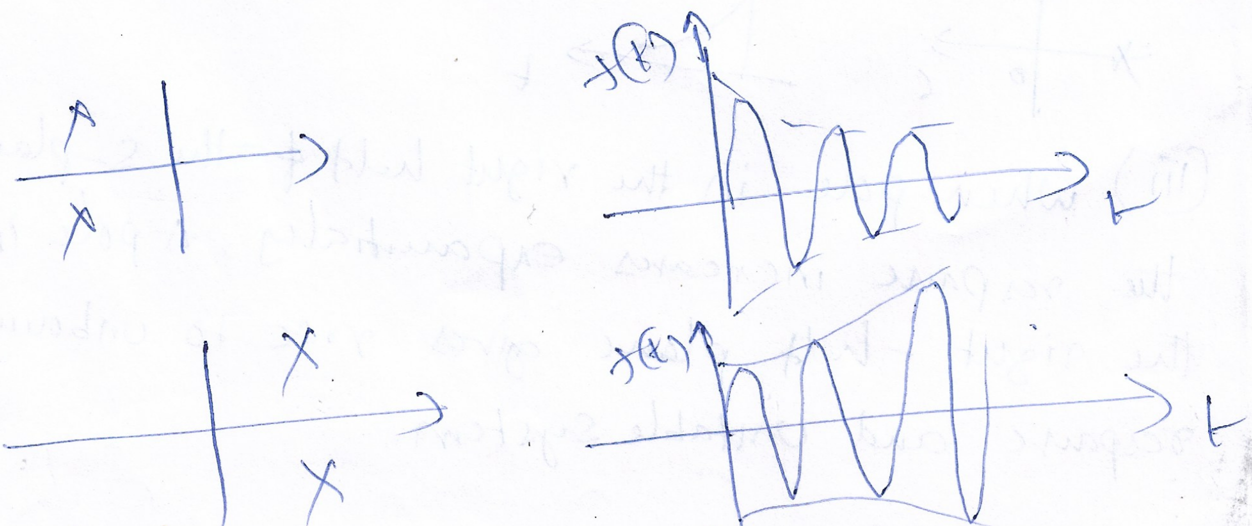


(iv) For $s = 0 \pm j\omega_n \rightarrow x(t) = A e^{\pm j\omega_n t}$

$= A(\cos\omega_n t \pm j\sin\omega_n t)$



(v) $s = \sigma \pm j\omega_n$ $\quad x(t) = A e^{st} = A e^{(\sigma_n \pm j\omega_n t)} = A e^{\sigma_n t} e^{j\omega_n t}$



1. obtain the pole-zero plot of the following equation.

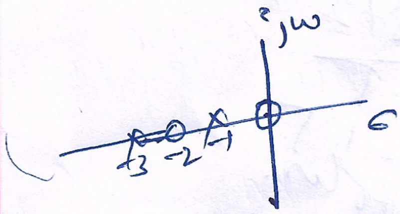
(a) $F(s) = \dfrac{s(s+2)}{(s+1)(s+3)}$ (b) $F(s) = \dfrac{s(s+1)}{(s+2)^2(s+3)}$

(c) $F(s) = \dfrac{s(s+2)}{(s+1+j1)(s+1-j1)}$ (d) $F(s) = \dfrac{(s+1)(s+5)}{(s+2)(s+3+j2)(s+3-j2)}$

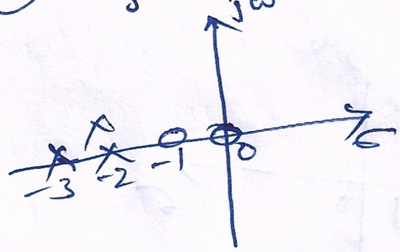
(e) $F(s) = \dfrac{s^2+4}{(s+2)(s^2+9)}$

2. ~~Determi~~

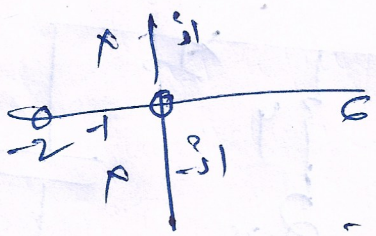
(a) $s=0,\ s=-2 \rightarrow$ zeros.
$s=-1,\ s=-3 \rightarrow$ poles.



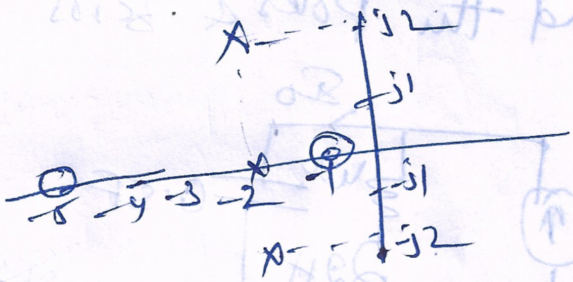
(b) $s=0,\ s=-1 \rightarrow$ zeros
$s=-2, -2, s=-3$ poles.



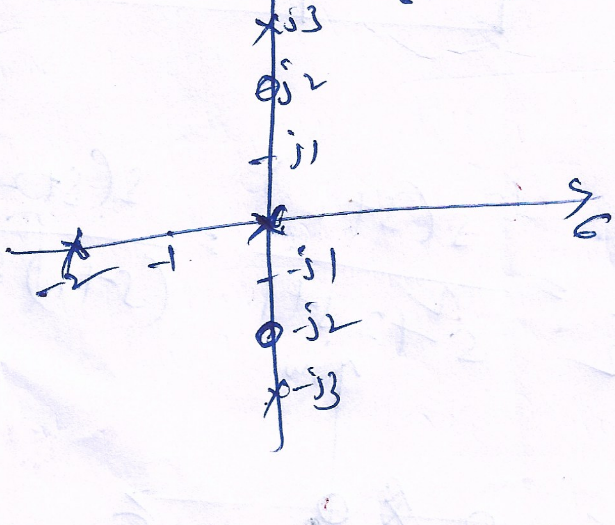
(c) zeros $s=0,\ s=-2$
poles $s=-1-j,\ s=-1+j$



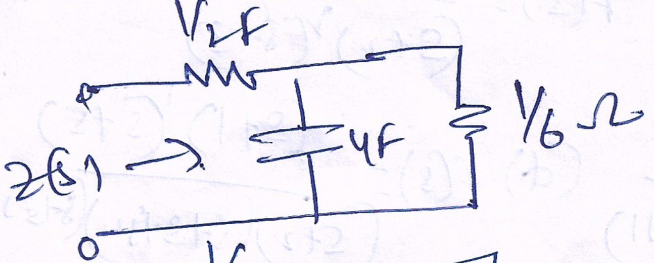
(d) $s=-1, -1$ $s=-5$ zeros
$s=-2$ $s=-3+j2$
$s=-3-j2$

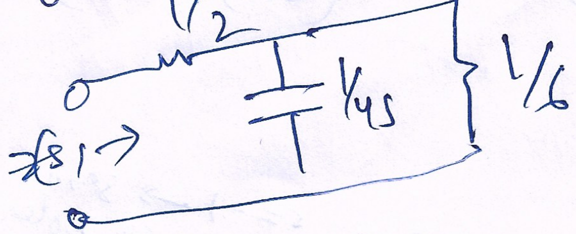


(e) $s=j2,\ s=-j2 \rightarrow$ zeros
$s=-2,\ s=j3\ s=-j3 \rightarrow$ poles



2. Determine the poles and zeros of the impedance function $Z(s)$ in the network



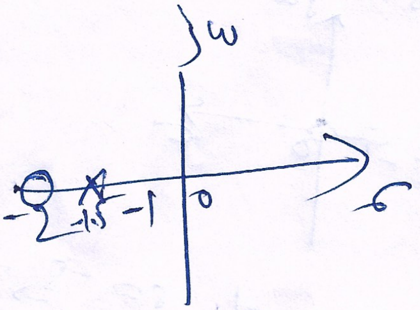


$$Z(s) = \frac{\frac{1}{4s} \times \frac{1}{6}}{\frac{1}{4s} + \frac{1}{6}} + \frac{1}{2}$$

$$= \frac{1}{2} + \frac{1}{4s+6}$$

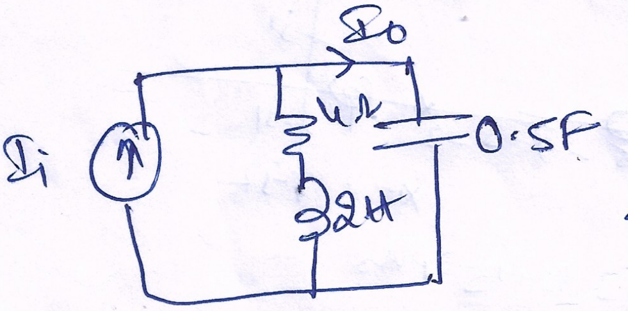
$$= \frac{4s+8}{2(4s+6)} = \frac{s+2}{2s+3}$$

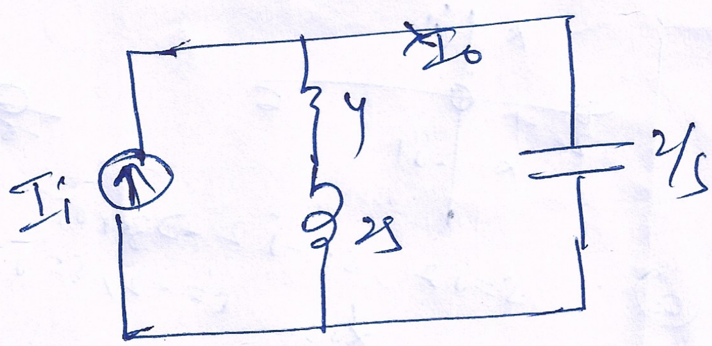
$$= \frac{0.5(s+2)}{s+1.5}$$



$s = -2$ (zero), $s = -1.5$ (poles)

3. Find the poles & zeros of function $\frac{I_o}{I_i}$





$$I_o = I_i \frac{4+2s}{4+2s+2/s}$$

$$I_o/I_i = \frac{s(4+2s)}{4s+2s^2+2} = \frac{s(s+2)}{s^2+2s+1} = \frac{s(s+2)}{(s+1)(s+1)}$$

zeros → $s = 0, -2$,

poles → $s = -1, -1$