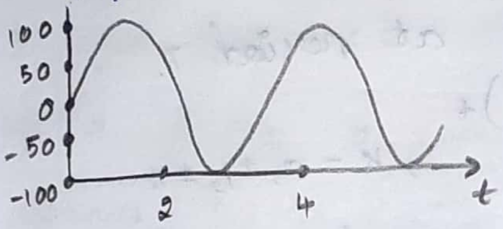
Unit - 2: Fourier Series and Transforms

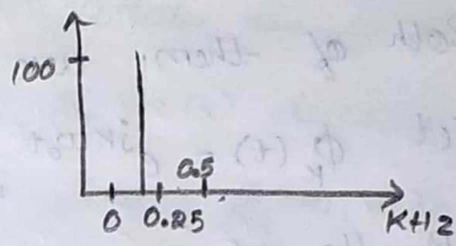
Signals can have dual personalities

1) Time domain perspective
(i) depicts its wave form
(ii) signal width (duration)
(iii) Rate at which the waveform decays

2) Frequency domain perspective
(i) In terms its sinusoidal components
(ii) Their relative amplitudes & phases

Amplitude $= 100$, $f = 200Hz$





Fourier series representation :

=) Resolving signal into frequency (sinusoidal) components is called frequency Analysis of signal

=) Decomposition in sinusoidal / complex exponential comps.

=) with such decomposition, a signal is said to be represented in the frequency domain.

=) The basic mathematical representation of periodic signals is the fourier series.

=) linear weighted sum of harmonically related sinusoids or complex exponentials.

=) Decomposition of periodic signals into sinusoids or complex exponentials

=) such a decomposition of the non-periodic / energy signals is called fourier transform representation.

Fourier Series Representation of Continuous-Time Periodic Signals:

$x(t) = x(t+T)$ periodic for period T.

Two basic periodic signals

$x(t) = \cos \omega_0 t$, $x(t) = e^{j\omega_0 t}$.

sinusoidal complex exponential

Both of them are periodic at period T.

let $\phi_K(t) = e^{jK\omega_0 t} = e^{jK\left(\frac{2\pi}{T}\right)t}$, $K = 0, \pm 1, \pm 2 \dots$

is the set of harmonically related complex exponentials

=) Each is periodic with period T

$$x(t) = \sum_{K=-\infty}^{+\infty} a_K e^{jK\omega_0 t}$$ (Fourier series representation)

~~Disc~~ =) $K = 0$ term = const

=) $K = 1, -1$ fundamental components / first harmonic components.

period $= T$

=) $K = 2, -2$ are periodic with ~~pe~~ half the period of fundamental components and are referred to as second harmonic components.

Dirichlet's Conditions:

condition 1 : $x(t)$ is absolutely integrable over one period.

$$\int_T |x(t)|\, dt < \infty$$

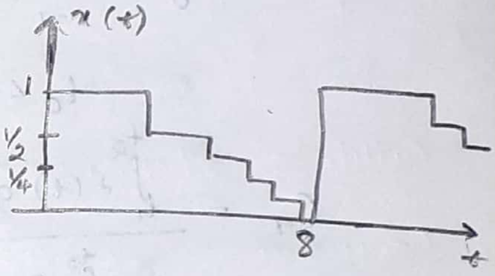
Condition 2: In a finite time interval, $x(t)$ has a finite no. of maxima & minima.

Condition 3: In a finite time interval, $x(t)$ has only a finite no. of discontinuities.

=) $x(t) = \frac{1}{t}$ violates condition 1 at $0 < t \le 1$

=) $x(t) = \sin\left(\frac{2\pi}{t}\right)$ meets condition 1 but not 2 $0 < t \le 1$

=) A signal periodic 8 that violates the condition 3 for $0 < t < 8$ the value of $x(t)$ decreases by a factor 2 whenever the distance from t to 8 decreases by a factor 8.



$x(t) = 1 \;, \; 0 \le t < 4$

$x(t) = 1/2 \;, \; 4 \le t < 6$

$x(t) = 1/4 \;, \; 6 \le t < 7$

$x(t) = 1/8 \;, \; 7 \le t < 7.5$

Types of fourier series Representation :

(i) Trigonometric Fourier series (TFS) representation

(ii) Exponential Fourier series (EFS) representation

Trigonometric Fourier series (TFS) representation :

=) $\sin \omega_0 t$, $\sin 2\omega_0 t \ldots$ form an orthogonal set over any interval $(t_0, t_0 + T)$ $T = 2\pi/\omega_0$

=) This set, however, is not complete.

=) This is evident from the fact a function $\cos \omega_0 t$ is orthogonal to $\sin \omega_0 t$ over same interval

=) ∴ To complete the set we should add sine as well as cosine functions.

→) For $n=0$, $\sin n\omega_0 t = 0$. but $\cos n\omega_0 t = 1$

→) ∴ $1, \cos\omega_0 t, \cos 2\omega_0 t, \ldots, \cos n\omega_0 t, \ldots \sin\omega_0 t,$ $\sin 2\omega_0 t, \ldots, \sin n\omega_0 t, \ldots$ etc are functions in orthogonal set.

$$\therefore f(t) = a_0 + a_1 \cos\omega_0 t + a_2 \cos 2\omega_0 t + \ldots + a_n \cos n\omega_0 t + \ldots + b_1 \sin\omega_0 t + b_2 \sin 2\omega_0 t + \ldots + b_n \sin n\omega_0 t + \ldots$$

$$t_0 < t < t_0 + \frac{2\pi}{\omega_0}.$$

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

$$t_0 < t < t_0 + T. \qquad f(t) = C_{12}\, g_r(t)$$

$$C_r = \frac{\int_{t_1}^{t_2} f(t)\, g_r(t)\, dt}{\int_{t_1}^{t_2} g_r^2(t)\, dt}$$

$$b_n = \frac{\int_{t_1}^{t_2} f(t) \sin n\omega t\, dt}{\int_{t_1}^{t_2} \sin^2 n\omega t\, dt}, \quad a_n = \frac{\int_{t_1}^{t_2} f(t) \cos n\omega t\, dt}{\int_{t_1}^{t_2} \cos^2 n\omega t\, dt}$$

$$\cancel{b_n} \quad \int_{t_0}^{t_0+T} \sin^2 n\omega t\, dt = \frac{T}{2}$$

$$b_n = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \sin n\omega t\, dt, \quad a_n = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \cos n\omega t\, dt.$$

$$a_0 = \frac{\int_{t_0}^{t_0+T} f(t)\, dt}{\int_{t_0}^{t_0+T} dt} = \frac{1}{T} \int_{t_0}^{t_0+T} f(t)\, dt = \text{avg. value of } f(t)$$

Exponential Fourier Series (EFS) Representation :

=) A set of exponential functions $(e^{jn\omega_0 t})$ $(n=0, \pm1, \pm2, \dots)$ is orthogonal over an interval $(t_0, t_0+\frac{2\pi}{\omega_0})$ for any value of t_0

$$I = \int_{t_0}^{t_0+\frac{2\pi}{\omega_0}} (e^{jn\omega_0 t})(e^{jm\omega_0 t})^* \, dt$$

$$= \int_{t_0}^{t_0+\frac{2\pi}{\omega_0}} e^{jn\omega_0 t} \, e^{-jm\omega_0 t} \, dt$$

If $n=m$ =) $I = \int_{t_0}^{t_0+\frac{2\pi}{\omega_0}} dt = \frac{2\pi}{\omega_0}$

If $n \neq m$ =) $I = \frac{1}{j(n-m)\omega_0}\left(e^{j(n-m)\omega_0 t}\right)_{t_0}^{t_0+\frac{2\pi}{\omega_0}}$

$$= \frac{1}{j(n-m)\omega_0} e^{j(n-m)\omega_0 t_0}\left(e^{j(m-n)2\pi} - 1\right)$$

$\because n, m =$ integers , $e^{j2\pi(n-m)} = 1$ $\therefore I = 0$

$$\therefore \int_{t_0}^{t_0+\frac{2\pi}{\omega_0}} (e^{jn\omega_0 t})(e^{jm\omega_0 t})^* \, dt = \begin{cases} \frac{2\pi}{\omega_0} & \text{if } m=n \\ 0 & \text{if } m \neq n \end{cases}$$

2) It is a complete set.

~~$f(t) = F_0 +$~~ $\because e^{jn\omega_0 t} = \cos n\omega_0 t + j \sin n\omega_0 t.$

=) Any func. can be expressed as sum of mutually orthogonal complex exponential func's

$$f(t) \cong F_0 + F_1 e^{j\omega_0 t} + F_2 e^{j2\omega_0 t} + \dots + F_n e^{jn\omega_0 t} + \dots$$
$$+ F_{-1} e^{-j\omega_0 t} + F_{-2} e^{-j2\omega_0 t} + \dots + F_{-n} e^{-jn\omega_0 t}$$

$$f(t) = F_0 + \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} \quad (n \neq 0)$$

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} \quad (t_0 < t < t_0 + T)$$

$$F_n = \frac{\int_{t_0}^{t_0+T} f(t)\, e^{-jn\omega_0 t}\, dt}{\int_{t_0}^{t_0+T} e^{jn\omega_0 t}\, e^{-jn\omega_0 t}\, dt} = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)\, e^{-jn\omega_0 t}\, dt$$

Relation b/w TFS and EFS :

TFS & EFS are not two different series but two ways of expressing a series.

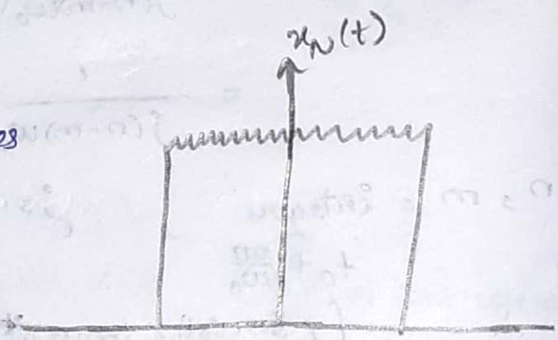
$a_0 = F_0$

$a_n = F_n + F_{-n}$ $\qquad\qquad F_n = \frac{1}{2}(a_n - jb_n)$

$b_n = j(F_n - F_{-n})$

Gibb's Phenomenon:

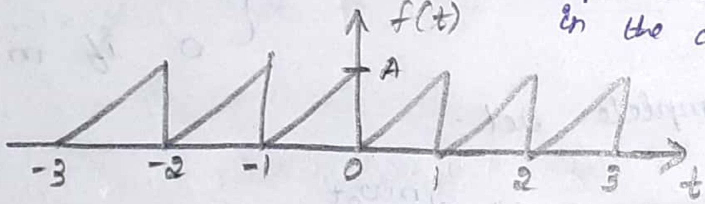
Convergence of the Fourier series representation of a square wave.



The occurance of overshot / Ripple at the points of discontinuities in the approx of F.S is called Gibb's phenomenon

$-T_1$ $\qquad T_1$

1.



$T = 1 - 0 = 2 - 1 = 0 - (-1) = 1$

$T = 1 = \frac{2\pi}{\omega_0}$

TFS $\Rightarrow f(t) \cong a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$

$a_0 = \frac{1}{T}\int_0^T f(t)\, dt.$

$y_2 - y_1 = m(x_2 - x_1) \Rightarrow f(t) - 0 = \frac{A-0}{1-0}(t-0)$

$f(t) = At \quad$ for $(0 \le t \le 1)$

$$a_0 = \frac{1}{T}\int_0^1 At\, dt = \frac{A}{1}\left(\frac{t^2}{2}\right)_0^1 = \frac{A}{2}$$

$$a_n = \frac{2}{T}\int_0^1 At \cos n\omega t\, dt$$

$$= 2A\left(\left(t \frac{\sin n\omega t}{n\omega}\right)_0^1 - \int_0^1 \frac{\sin n\omega t}{n\omega}\, dt\right)$$

$$= 2A\left(\frac{\sin n\omega}{n} + \frac{1}{n^2\omega^2}(\cos n\omega t)_0^1\right)$$

$$= 2A\left(0 + \frac{1}{n^2\omega^2}(\cos 2\pi n - 1)\right)$$

$$= 2A \frac{1}{n^2\omega^2}(1-1) = 0$$

$$b_n = \frac{2}{T}\int_0^1 At \sin n\omega t\, dt$$

$$= 2A\left(\left(-\frac{t\cos n\omega t}{n\omega}\right)_0^1 + \int_0^1 \frac{\cos n\omega t}{n\omega}\, dt\right)$$

$$= 2A\left(-\left(\frac{\cos n\omega}{n\omega} - 0\right) + \frac{1}{n^2\omega^2}(\sin n\omega t)_0^1\right)$$

$$= 2A\left(-\frac{\cos 2\pi n}{2\pi n} + \frac{1}{n^2\omega^2}(\sin 2\pi n - 0)\right)$$

$$= 2A\left(\frac{-1}{2\pi n}\right) = \frac{-A}{\pi n}$$

$$f(t) = \frac{A}{2} + 0 - \frac{A}{\pi}\sum_{n=1}^{\infty} \frac{\sin 2\pi n t}{n}$$

$$f(t) = \frac{A}{2} - \frac{A}{\pi}\sum_{n=1}^{\infty} \frac{\sin 2\pi n t}{n} \quad \text{is TFS.}$$

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$$

$$F_0 = \frac{1}{T}\int_0^T f(t)\, dt = \frac{1}{1}\int_0^1 At\, dt = \frac{A}{2}$$

$$F_n = \frac{1}{T}\int_0^T f(t) e^{-jn\omega_0 t}\, dt$$

$$F_n = \frac{1}{1}\int_0^1 A t \cdot e^{-j(n 2\pi t)}\, dt.$$

$$= A\left(\left(t\,\frac{e^{-j2\pi n t}}{-j2\pi n}\right)_0^1 - \int_0^1 1\cdot\frac{e^{-j2\pi n t}}{-j2\pi n}\, dt\right)$$

$$= A\left(\frac{e^{-j2\pi n}}{-j2\pi n} - \frac{1}{(j2\pi n)^2}\left(e^{-j2\pi n t}\right)_0^1\right)$$

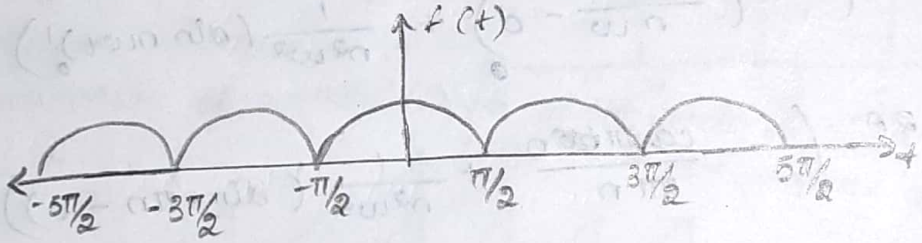
$$= A\left(\frac{e^{-j2\pi n}}{-j2\pi n} + \frac{1}{(2\pi n)^2}\left(e^{-j2\pi n} - 1\right)\right)$$

$$e^{-j2\pi n} = \cos 2\pi n - j \sin 2\pi n = 1$$

$$F_n = A\left(\frac{-1}{j2\pi n}\right) = \frac{-A}{j2\pi n}$$

$$f(t) = \frac{A}{2} - \frac{A}{j2\pi}\sum_{n=-\infty}^{\infty}\frac{e^{-2\pi n j}}{n}$$ is EFS

<u>Full wave / Half wave rectified cosine func:</u>



$$f(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

$$a_0 = \frac{1}{T}\int_0^T f(t)\,dt = \frac{1}{T}\int_{-\pi/2}^{\pi/2}\cos t\, dt = \frac{1}{\pi}(\sin t)_{-T/2}^{T/2}$$

$$= \frac{2}{\pi}$$

$$a_n = \frac{1}{T}\int_{-\pi/2}^{\pi/2} f(t)\cos n\omega_0 t\, dt$$

$$\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{\pi} = 2$$

$$= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}(\cos t)(\cos 2n t)\, dt$$

$$a_n = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}\left(\cos(2n+1)t + \cos(2n-1)t\right)dt$$

$$= \frac{1}{2\pi}\left(\frac{\sin(2n+1)t}{2n+1}\right)_{-\pi/2}^{\pi/2} + \frac{1}{2\pi}\left(\frac{\sin(2n-1)t}{2n-1}\right)_{-\pi/2}^{\pi/2}$$

$$= \frac{1}{2\pi}\frac{1}{2n+1}\left(\sin\left(n\pi + \frac{\pi}{2}\right) - \sin\left(-\frac{\pi}{2} - n\pi\right)\right)$$

$$+ \frac{1}{2\pi}\cdot\frac{1}{2n-1}\left(\sin\left(n\pi - \frac{\pi}{2}\right) - \sin\left(-n\pi + \frac{\pi}{2}\right)\right)$$

$$= \frac{1}{2\pi}\left(\frac{2\cos n\pi}{2n+1} + \frac{-2\cos n\pi}{2n-1}\right)$$

$$= \frac{\cos n\pi}{\pi}\left(\frac{2n-1-(2n+1)}{(2n-1)(2n+1)}\right) = \frac{-2\cos n\pi}{(2n-1)(2n+1)}$$

$$= \frac{-2(-1)^n}{(2n-1)(2n+1)}.$$

$$f(t) = \frac{2}{\pi} + 4\sum_{n=1}^{\infty}\left(\frac{(-1)^{n+1}}{(2n-1)(2n+1)}\right)$$

1. Odd functions have only sine terms. ($a_n = 0, b_n \neq 0$)
Even functions have only cosine terms ($a_n \neq 0, b_n = 0$)

Proof: even func. $\Rightarrow f(-t) = f(t)$ - symmetrical about vertical axis

odd func. $\Rightarrow f(-t) = -f(t)$. Anti-symmetrical

w.k.t

$\cos n\omega_0 t$ = even func.

$\sin n\omega_0 t$ = odd func.

$$I = \int_{-T}^{T} f_e(t)\,dt = \int_{-T}^{0} f_e(t)\,dt + \int_{0}^{T} f_e(t)\,dt$$

$$= 2\int_0^T f_e(t)\,dt.$$

$$I = \int_{-T}^{T} f_o(t)\,dt = \int_{-T}^{0} f_o(t)\,dt + \int_0^T f_o(t)\,dt$$

$$= -\int_0^T f_o(t)\,dt + \int_0^T f_o(t)\,dt = 0$$

(i) $f_e(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} \underbrace{\underbrace{f_e(t)}_{even}\,\underbrace{\cos n\omega_0 t}_{even}}_{even}\, dt = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\, dt$$

$$a_n = \frac{4}{T}\int_{0}^{T/2} f_e(t) \cos n\omega_0 t\, dt$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} \underbrace{\underbrace{f_e(t)}_{even}\,\underbrace{\sin n\omega_0 t}_{odd}}_{odd}\, dt = 0$$

i.e

even funcs have only cosine terms.

(ii) $f(t) = f_o(t)$

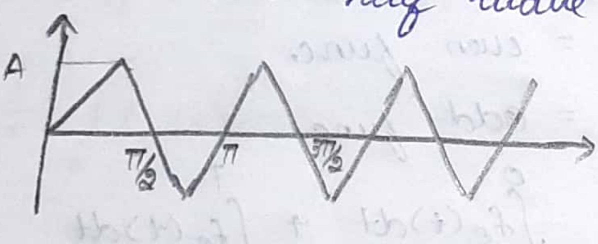
$$f_o(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} \underbrace{\underbrace{f_o(t)}_{odd}\,\underbrace{\cos n\omega_0 t}_{even}}_{odd}\, dt = 0$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} \underbrace{\underbrace{f_o(t)}_{odd}\,\underbrace{\sin n\omega_0 t}_{odd}}_{even}\, dt = \frac{4}{T}\int_{0}^{T/2} f_o(t) \sin n\omega_0 t\, dt.$$

~~$f_o(t)$~~ i.e odd funcs have only sine terms.

2. Functions with half wave symmetry have only odd harmonics.



even $= f(-t) = f(t)$, odd $= f(-t) = -f(t)$

Half wave symmetry $\Rightarrow f(t \pm T/2) = -f(t)$

$$a_n = \frac{2}{T}\int_{0}^{T} f(t) \cos n\omega_0 t\, dt.$$

$$a_n = \frac{2}{T}\int_0^{T/2} f(t)\cos n\omega_0 t\, dt + \frac{2}{T}\int_{T/2}^{T} f(t)\cos n\omega_0 t\, dt.$$

$$= \frac{2}{T}\int_0^{T/2} f(t)\cos n\omega_0 t\, dt + \frac{2}{T}\int_{T/2}^{T} f(t+T/2)\cos n\omega_0 (t+T/2)\, dt$$

$$f(\cancel{t} \pm T/2) = -f(t)$$

$$a_n = \frac{2}{T}\int_0^{T/2} f(t)\cos n\omega_0 t\, dt - \frac{2}{T}\int_0^{T/2} f(t)\cos n\omega_0 t \cos n\pi t\, dt.$$

$n =$ even $\Rightarrow I = 0$

$n =$ odd $\Rightarrow I = \frac{4}{T}\int_0^{T/2} f(t)\cos n\omega_0 t\, dt.$ $\boxed{\because \cos n\pi t = (-1)^n.}$

$=$ odd harmonics.

$b_n \Rightarrow$ $n =$ even $\Rightarrow I = 0$

$n =$ odd $\Rightarrow I = \frac{4}{T}\int_0^{T/2} f(t)\sin n\omega_0 t\, dt.$

$\therefore$ Half wave symmetry have only odd harmonics.

$\Rightarrow$

TFS $\Rightarrow a_0, a_n, b_n$ coefficients

EFS $\Rightarrow F_n$ coefficient, $F_0 \rightarrow$ dc value, avg value

$e^{jn\omega_0 t}$, $e^{-jn\omega_0 t} \longrightarrow n^{th}$ harmonics

$|F_n| = |F_{-n}|$ same magnitude but diff phase.

$$f(t) = \sin\omega_0 t = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} = \frac{1}{2j}e^{j\omega_0 t} - \frac{1}{2j}e^{-j\omega_0 t}$$

$$F_1 = \frac{1}{2j},\ F_{-1} = \frac{1}{2j},\ F_0 = 0$$

1. $f(t) = 1 + \sin\omega_0 t + 2\cos\omega_0 t + \cos(2\omega_0 t + \pi/4)$

$$= 1 + \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} + e^{j\omega_0 t} + e^{-j\omega_0 t} + \frac{e^{j(2\omega_0 t + \pi/4)} + e^{-j(2\omega_0 t + \pi/4)}}{2}$$

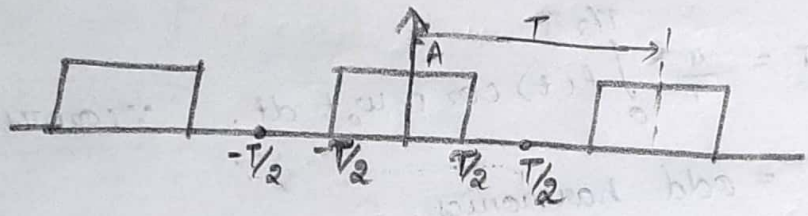
$$f(t) = 1 + \left(1 + \frac{1}{2j}\right)e^{j\omega_0 t} + \left(1 - \frac{1}{2j}\right)e^{-j\omega_0 t} + \frac{1}{2}e^{j2\omega_0 t}e^{j\pi/4}$$
$$+ \frac{1}{2}e^{-j2\omega_0 t}e^{-j\pi/4}$$

$F_0 = 1$, $F_1 = 1 + \frac{1}{2j}$, $F_{-1} = 1 - \frac{1}{2j}$.

$F_2 = \frac{1}{2}e^{j\pi/4}$, $F_{-2} = \frac{1}{2}e^{-j\pi/4}$

$F_2 = \frac{1+j}{2\sqrt{2}}$, $F_{-2} = \frac{1-j}{2\sqrt{2}}$

2.



$f(t) = A$ for $-\tau/2 < t < \tau/2$

$f(t) = 0$ for otherwise.

$$f(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

$$a_0 = \frac{1}{T}\int_{-\tau/2}^{\tau/2} A\,dt = \frac{A}{T}\left(\frac{\tau}{2} + \frac{\tau}{2}\right) = \frac{A\tau}{T}$$

$$a_n = \frac{2}{T}\int_{-\tau/2}^{\tau/2} f(t)\cos n\omega_0 t\,dt$$

$$= \frac{2}{T}\int_{-\tau/2}^{\tau/2} A\cos n\omega_0 t\,dt = \frac{2A}{T}\int_{-\tau/2}^{\tau/2}\cos n\omega_0 t\,dt$$

$$= \frac{2A}{T}\left(\frac{\sin n\omega_0 t}{n\omega_0}\right)_{-\tau/2}^{\tau/2}$$

$\frac{\sin x}{x} = Sa(x)$

$$= \frac{2A}{T}\,\frac{\sin n\omega_0 \tau/2 - \sin n\omega_0(-\tau/2)}{n\omega_0}$$

$$= \frac{4A}{T}\,\frac{\sin n\omega_0 \tau/2}{n\omega_0}$$

$$a_n = \frac{2}{T}\,\frac{\cancel{2}\sin(n\omega_0 \tau/2)}{n\omega_0 \tau/2} \times \tau/\cancel{2} = \frac{2\tau}{T}\,Sa(n\omega_0 \tau/2)$$

$$b_n = \frac{2}{T}\int_{-\tau/2}^{\tau/2} A \sin n\omega_0 t \, dt = \frac{2A}{T}\,\frac{(-\cos n\omega_0 t)_{-\tau/2}^{\tau/2}}{n\omega_0}$$

$$= \frac{-2A}{T}\,\frac{\cos n\omega_0 \tau/2 - \cos n\omega_0(-\tau/2)}{n\omega_0}$$

$$b_n = 0.$$

$$\rightarrow f(t) = \frac{A\tau}{T} + \sum_{n=1}^{\infty} \frac{2\tau}{T} Sa(n\omega_0 \tau/2)\cos n\omega_0 t. \Rightarrow TFS$$

$$EFS = f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$$

$$F_n = \frac{1}{T}\int_{-\tau/2}^{\tau/2} f(t)\, e^{-jn\omega_0 t} dt = \frac{A}{T}\left(\frac{e^{-jn\omega_0 t}}{-jn\omega_0}\right)_{-\tau/2}^{\tau/2}$$

$$\cancel{= \frac{A}{T}\,\frac{e^{-jn\omega_0 \tau} - 1}{-jn\omega_0}}$$

$$\cancel{F_n = \frac{A}{T}\,\frac{\cos n\omega_0\tau - j\sin n\omega_0 \tau - 1}{-jn\omega_0}}$$

$$F_n = \frac{A}{T}\,\frac{e^{-jn\omega_0\tau/2} - e^{jn\omega_0\tau/2}}{-jn\omega_0} = \frac{A}{T}\,\frac{(-2j\sin n\omega_0\tau/2)}{-jn\omega_0}$$

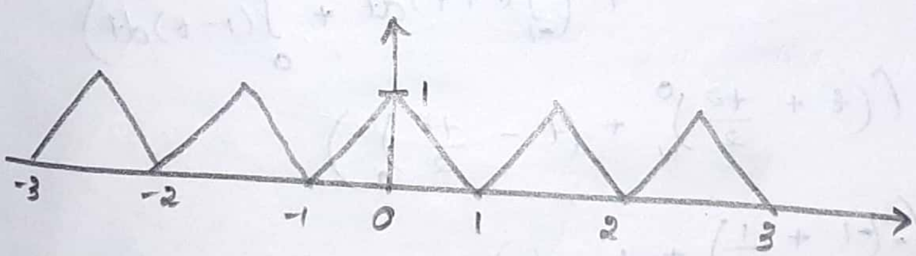
$$= \frac{A\tau/2}{T}\,\frac{2\sin n\omega_0\tau/2}{n\omega_0\tau/2}$$

$$F_n = \frac{A\tau}{T} Sa(n\omega_0\tau/2)$$

$$\rightarrow f(t) = \sum_{n=-\infty}^{\infty} \frac{A\tau}{T} Sa(n\omega_0\tau/2) e^{jn\omega_0 t}$$

$\Rightarrow EFS.$

3.



$$T = 1-(-1) = 2$$

$$f(t) - 0 = \frac{1-0}{0+1}(t+1) \Rightarrow f(t) = t+1 \quad \text{for } -1 < t < 0$$

$$f(t) - 0 = \frac{1-0}{0-1}(t-1) \Rightarrow f(t) = 1-t \quad \text{for } 0 < t < 1$$

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$\therefore \quad f(t) = 1 - |t|$

$$a_0 = \frac{1}{T}\int_{-1}^{1} f(t)\,dt = \frac{1}{T}\left(\int_{-1}^{0}(1+t)\,dt + \int_{0}^{1}(1-t)\,dt\right)$$

$$a_0 = \frac{1}{T}\left(\left(t + \frac{t^2}{2}\right)_{-1}^{0} + \left(t - \frac{t^2}{2}\right)_{0}^{1}\right)$$

$$= \frac{1}{T}\left(-\left(-1 + \frac{1}{2}\right) + \left(1 - \frac{1}{2}\right)\right) = \frac{1}{T} = \frac{1}{2}$$

$$a_n = \frac{2}{T}\int_{-1}^{1} f(t)\cos n\omega_0 t\,dt = \frac{2}{T}\left(\int_{-1}^{0}(1+t)\cos n\omega_0 t\,dt + \int_{0}^{1}(1-t)\cos n\omega_0 t\,dt\right)$$

$$a_n = \frac{2}{T}\left(\int_{-1}^{0}\cos n\omega_0 t\,dt + \left(\frac{t\sin n\omega_0 t}{n\omega_0}\right)_{-1}^{0} - \int_{-1}^{0}\frac{\sin n\omega_0 t}{n\omega_0}\,dt + \int_{0}^{1}\cos n\omega_0 t\,dt - \left(\frac{t\sin n\omega_0 t}{n\omega_0}\right)_{0}^{1} + \int_{0}^{1}\frac{\sin n\omega_0 t}{n\omega_0}\,dt\right)$$

$$a_n = \frac{2}{T}\left(\left(\frac{\sin n\omega_0 t}{n\omega_0}\right)_{-1}^{0} + \frac{\sin n\omega_0}{n\omega_0}\right.$$

$\vdots$

$$F_0 = \frac{1}{T}\int_{-1}^{1}(1 - |t|)\,dt = \frac{1}{T}\left(\int_{-1}^{0}(1+t)\,dt + \int_{0}^{1}(1-t)\,dt\right)$$

$$F_0 = \frac{1}{T}\left(\left(t + \frac{t^2}{2}\right)_{-1}^{0} + \left(t - \frac{t^2}{2}\right)_{0}^{1}\right)$$

$$= \frac{1}{T}\left(-\left(-1 + \frac{1}{2}\right) + 1 - \frac{1}{2}\right) = \frac{1}{2}(2-1) = \frac{1}{2}$$

$$F_n = \frac{1}{T}\int_{-1}^{1}(1 - |t|)e^{-jn\omega_0 t}\,dt$$

$$= \frac{1}{T}\left(\int_{-1}^{0}(1+t)e^{-jn\omega_0 t}\,dt + \int_{0}^{1}(1-t)e^{-jn\omega_0 t}\,dt\right)$$

$$F_n = \frac{1}{T}\left(\left(\frac{e^{-jn\omega_0 t}}{-jn\omega_0}\right)_{-1}^{0} + \left(\frac{t\, e^{-jn\omega_0 t}}{-jn\omega_0}\right)_{-1}^{0} - \int_{-1}^{0} \frac{e^{-jn\omega_0 t}}{-jn\omega_0}\, dt\right.$$

$$\left. + \left(\frac{e^{-jn\omega_0 t}}{-jn\omega_0}\right)_{0}^{1} - \left(\frac{t\, e^{-jn\omega_0 t}}{-jn\omega_0}\right)_{0}^{1} + \int_{0}^{1} \frac{e^{-jn\omega_0 t}}{-jn\omega_0}\, dt\right)$$

$$F_n = \frac{1}{T}\left(\frac{1 - e^{jn\omega_0}}{-jn\omega_0} + \left(0 + \frac{e^{jn\omega_0}}{-jn\omega_0}\right) - \frac{\left(e^{-jn\omega_0 t}\right)_{-1}^{0}}{(jn\omega_0)^2}\right.$$

$$+ \frac{e^{-jn\omega_0}}{-jn\omega_0} - 1 -$$

⋮

Properties of Fourier series:

1. Linearity property:

$$x(t) \xleftrightarrow{F.S} a_n \quad , \quad y(t) \xleftrightarrow{F.S} b_n$$

$$A x(t) + B y(t) \longrightarrow A a_n + B b_n$$

$$x(t) = \sum_{n=-\infty}^{\infty} a_n e^{jn\omega_0 t} \qquad a_n = \frac{1}{T}\int_T x(t) e^{-jn\omega_0 t}$$

2. Time shifting property:

If $x(t)$ is periodic with $T \xrightarrow{F.S} a_n$

If $x(t)$ is shifted by t_0 secs $\rightarrow x(t - t_0) \xrightarrow{F.S} ?$

$$a_n = \frac{1}{T}\int x(t)\, e^{-jn\omega_0 t}\, dt$$

$$b_n \rightarrow x(t - t_0) \Rightarrow b_n = \frac{1}{T}\int x(t - t_0)\, e^{-jn\omega_0 t}\, dt$$

$$t - t_0 = \tau \Rightarrow t = t_0 + \tau \qquad dt = d\tau$$

$$b_n = \frac{1}{T}\int x(\tau)\, e^{-jn\omega_0 (t_0 + \tau)}\, d\tau$$

$$b_n = e^{-jn\omega_0 t_0}\, \frac{1}{T}\int x(\tau)\, e^{-jn\omega_0 \tau}\, d\tau$$

$$b_n = e^{-jn\omega_0 t_0}\, a_n$$

$x(t), x(t - t_0) \longrightarrow$ same magnitude, diff phase

3. Time Reversal property :

$x(t) \longrightarrow T \longrightarrow a_n$ then $x(-t) \longrightarrow T \longrightarrow a_{-n}$

$$x(t) = \sum_{n=-\infty}^{\infty} a_n e^{-jn\omega_0 t}$$

$$x(-t) = \sum_{m=-\infty}^{\infty} a_{-m} e^{jm\omega_0 t} \qquad x(-t) \rightarrow a_{-n}$$

If $x(t)$ = even $\Rightarrow a_{-n} = a_n$

$x(t)$ = odd $\Rightarrow a_{-n} = -a_n$

4. Conjugation & conjugation symmetry :

$x(t) \longrightarrow T \longrightarrow a_n$ then $x^*(t) \longrightarrow T \longrightarrow ?$

$$x(t) = \sum_{n=-\infty}^{\infty} a_n e^{jn\omega_0 t}$$

$$x^*(t) = \sum_{n=-\infty}^{\infty} a_n^* e^{-jn\omega_0 t} = \sum_{m=-\infty}^{\infty} a_{-m}^* e^{jm\omega_0 t}$$

$$x^*(t) \longrightarrow a_{-n}^*$$

If $x(t)$ = real $\Rightarrow x(t) = x^*(t)$, $a_n = a_{-n}^*$

$x(t)$ = real & equal $\Rightarrow x(t) = x(-t)$, $a_n = a_{-n} = a_n^*$

5. Time Scaling Property :

$x(t) \longrightarrow T \longrightarrow a_n$ $\qquad x(t) = \sum_{n=-\infty}^{\infty} a_n e^{jn\omega_0 t}$

$x(\alpha t) \longrightarrow ?$

$$x(\alpha t) = \sum_{n=-\infty}^{\infty} a_n e^{jn\omega_0(\alpha t)}$$

$$x(\alpha t) = \sum_{n=-\infty}^{\infty} a_n e^{jn(\alpha\omega_0)t}$$

No change in a_n but period will change.

Period = T/α $\left(\because T = \frac{2\pi}{\alpha\omega_0}\right)$

Parseval's Relation:

Explains the relation in Power signal in time domain & freq. domain (F.S coefficients)

$x(t) \longrightarrow$ Periodic $\longrightarrow$ Power signals $\rightarrow$ Power Relation

$x(t) \longrightarrow a_n \Rightarrow \frac{1}{T}\int |x(t)|^2 dt = \sum_{n=-\infty}^{\infty} |a_n|^2$

avg Power = sum of the power of all freq components

Proof:

$$P_{avg} = \frac{1}{T}\int_0^T |x(t)|^2 dt = \frac{1}{T}\int_0^T x(t)\, x^*(t)\, dt$$

w.k.t $\quad x(t) = \sum_{n=-\infty}^{\infty} a_n e^{jn\omega_0 t}$

$$x^*(t) = \sum_{n=-\infty}^{\infty} a_n^* (e)^{-jn\omega_0 t}$$

$$P_{avg} = \frac{1}{T}\int_0^T x(t)\left(\sum_{n=-\infty}^{\infty} a_n^* e^{-jn\omega_0 t}\right) dt$$

$$= \sum_{n=-\infty}^{\infty} a_n^* \frac{1}{T}\int_0^T x(t)\, e^{-jn\omega_0 t} dt$$

$$= \sum_{n=-\infty}^{\infty} a_n^* \cdot a_n = \sum_{n=-\infty}^{\infty} |a_n|^2$$

$\Rightarrow$ frequency = 2 then negative freq = 2 but phase opp.

Complex Fourier spectrum:

Spectrum $\longrightarrow$ dst. of freq.

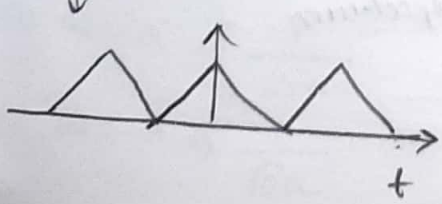
Fourier $\longrightarrow$ F.S / F.T

Complex $\longrightarrow$ The F.S coefficients of any periodic signal is complex

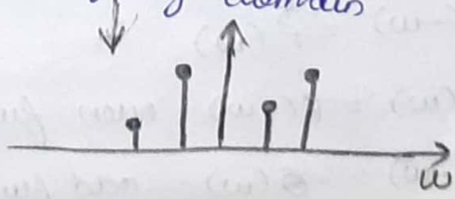
Freqs $\Rightarrow 0, \omega_0, -\omega_0, 2\omega_0, -2\omega_0 \ldots$

$$f(t) = F_0 + F_1 e^{j\omega_0 t} + F_2 e^{j2\omega_0 t} + \ldots + F_n e^{jn\omega_0 t} + \ldots + F_{-1} e^{-j\omega_0 t} + F_{-2} e^{-j2\omega_0 t} + \ldots + F_{-n} e^{-jn\omega_0 t} + \ldots$$

Time domain
↓



Frequency domain
↓



freq Vs magnitude = magnitude spectrum (symmetric about vertical axis, even fn's)

freq Vs phase = phase spectrum (Anti sym, odd fn's)

F.S → periodic , F.T → Non-periodic

F.S → Representing f(t) as discrete sum of complex exponentials. Signal is continuous & Periodic spectrum is discrete. Only few freqs.

F.T → Continuous but not periodic. Spectrum is continuous. Infinite no. of freqs. Spectral density function.

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} , \quad F_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, e^{-jn\omega_0 t}\, dt.$$

$f_T(t) \rightarrow$ as $T \rightarrow \infty \Rightarrow \lim_{T\to\infty} f_T(t) = f(t)$

with $T = \infty \Rightarrow$ only one cycle

as $T \uparrow$ to ∞ $\quad \omega \downarrow \quad \therefore$ more no. of freq are present.

$$F(\omega) = F.T[f(t)] = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

where $f(t) =$ Inverse FT $[F(\omega)] = F^{-1}[F(\omega)]$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega.$$

signal → Time domain → wave form
signal → Freq. domain → Periodic ⇄ Fourier series → f(t) exclusively
Freq. domain → Non-periodic → Fourier transform

We need two spectrum to describe $F(\omega)$

(i) Magnitude spectrum (ii) Phase spectrum.

$$F(-\omega) = F^*(\omega)$$

$|F(\omega)| = |F(-\omega)|$ even func. of ω

$\theta(-\omega) = -\theta(\omega)$ odd fun of ω

Existence conditions of Fourier transforms:

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

F.S → Dirichlet's conditions

The F.T said to be existing if $|F(\omega)|$ is finite

$$= \int_{-\infty}^{\infty} |f(t)|\, dt \leq \infty \text{ (finite)}$$

$$= \int_{-\infty}^{\infty} |f(t)|\, dt < \infty \text{ (finite)} \rightarrow \text{absolutely integrable}$$

This absolute integrability is sufficient condition but not necessary condition because there are some funcs which are not absolutely integrable but still $F(\omega)$

(i) $\int_{-\infty}^{\infty} |f(t)|\, dt < \infty$

(ii) $f(t)$ must have finite no. of Maxima's & Minimas

(iii) " " " " " discontinuties

1. $f(t) = e^{-at} u(t)$ → Find F.T

$$F(\omega) = \int_{-\infty}^{\infty} e^{-at} u(t) \cdot e^{-j\omega t}\, dt$$

$u(t) = 1$ for $t > 0$

$= 0$ for $t < 0$

$$= \int_{0}^{\infty} e^{-at}\, e^{-j\omega t}\, dt = \int_{0}^{\infty} e^{-(a+j\omega)t}\, dt$$

$$= \frac{1}{-(a+j\omega)} \left(e^{-(a+j\omega)t}\right)_0^{\infty}$$

$$= \frac{1}{-(a+j\omega)} (0 - 1) = \frac{1}{a+j\omega}$$

$|F(\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}$ = magnitude , $F(\omega)$ = complex.

$\omega = 0 \Rightarrow |F(\omega)| = \frac{1}{a}$

$\omega = a \Rightarrow \frac{1}{\sqrt{2}a}$

$\omega = -a \Rightarrow \frac{1}{\sqrt{2}a}$

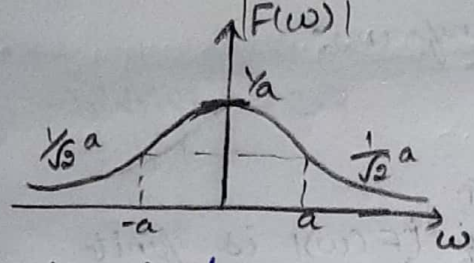
Phase $= a + jb = \tan^{-1}(b/a)$

$\theta(\omega) = -\tan^{-1}(b/a)$

$\omega = 0 \Rightarrow -\tan^{-1}(0/a) = 0$

$\omega = a \Rightarrow -\tan^{-1}(1) = -\pi/4$

$\omega = -a \Rightarrow -\tan^{-1}(-1) = \pi/4$

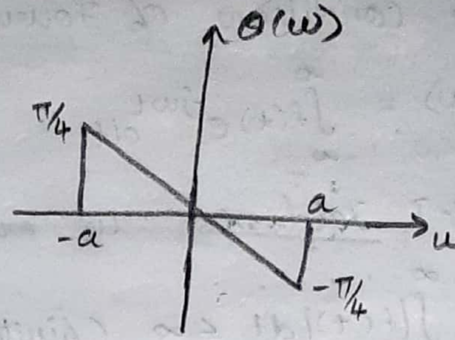


Magnitude spectrum

$f(t) = e^{-at}u(t)$

$F(\omega) = \frac{1}{a+j\omega}$

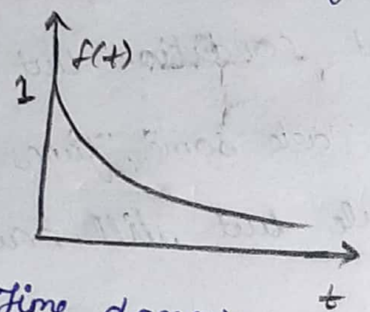
symmetric, even func



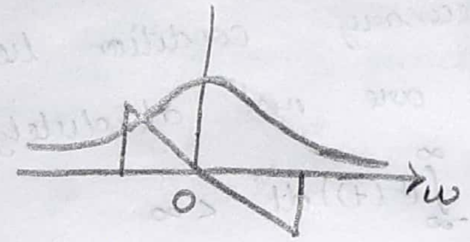
Phase spectrum

$\theta(-\omega) = -\theta(\omega)$

odd func, Anti-symmetric



Time domain



Frequency domain

2. Double sided exponential signal $f(t) = e^{-a|t|}$

$$f(t) = e^{-at} \quad \text{for} \quad t > 0$$
$$= e^{at} \quad \text{for} \quad t < 0$$

$$FT[f(t)] = F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{0} f(t)\, e^{-j\omega t}\, dt + \int_{0}^{\infty} f(t) e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{0} e^{(a-j\omega)t}\, dt + \int_{0}^{\infty} e^{-(a+j\omega)t}\, dt$$

$$F(\omega) = \frac{1}{a-j\omega}\left(e^{(a-j\omega)t}\right)_{-\infty}^{0} + \frac{1}{-(a+j\omega)}\left(e^{-(a+j\omega)}\right)_{0}^{\infty}$$

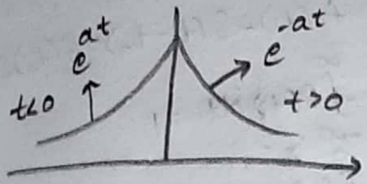
$$= \frac{1}{a-j\omega}(1-0) - \frac{1}{a+j\omega}(0-1)$$

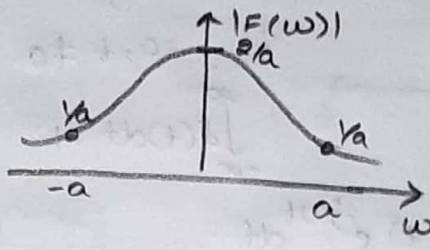
$$= \frac{1}{a-j\omega} + \frac{1}{a+j\omega} = \frac{2a}{a^2+\omega^2}$$

$$\theta(\omega) = \tan^{-1}(0) = 0$$

$$|F(\omega)| = \frac{2a}{a^2+\omega^2} \Rightarrow \omega = 0 \Rightarrow \frac{2}{a}$$

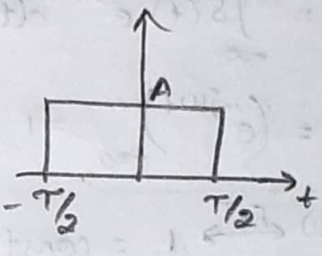
$$\omega = a \Rightarrow 1/a \quad , \omega = -a \Rightarrow 1/a$$





4. Gate function $\rightarrow G_\tau(t) \longrightarrow$ Rectangular Pulse

τ = width of pulse.



$G_\tau(t) = f(t) = A \quad$ for $-T/2 < t < T/2$

$= 0 \quad$ otherwise

$\int_{-\infty}^{\infty} |f(t)|\, dt$ = finite = F.T

$$\int_{-\infty}^{\infty} A\, dt = A\,(t)_{-\infty}^{\infty} = \infty = \text{Infinite}.$$

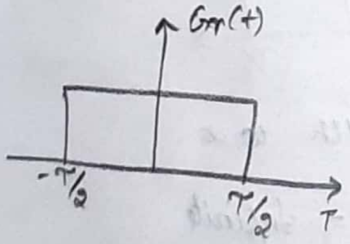
$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt = \int_{-T/2}^{T/2} A\, e^{-j\omega t}\, dt$$

$$= A\left[\frac{e^{-j\omega t}}{-j\omega}\right]_{-T/2}^{T/2} = \frac{A}{-j\omega}\left(e^{-j\omega T/2} - e^{j\omega T/2}\right)$$

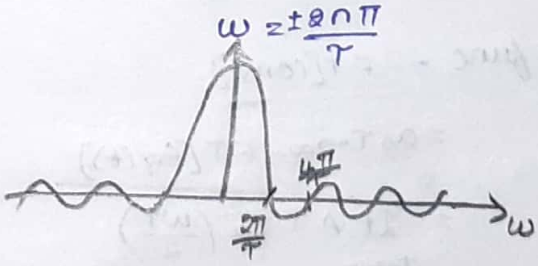
$$= \frac{2A \sin \omega T/2}{\omega}$$

$$F(\omega) = AT\, Sa(\omega T/2)$$

$$\frac{\sin(\omega T/2)}{\omega T/2} = 0 \Rightarrow \sin(\omega T/2) = 0 \Rightarrow \sin(\omega T/2) = \pm \sin n\pi$$



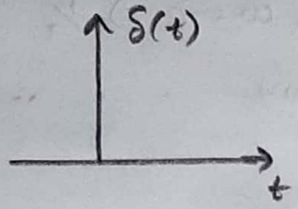
$\xleftrightarrow{FT}$



$$f(t) = G_\tau(t) \Rightarrow F(\omega) = AT\, Sa(\omega T/2)$$

Gate $\longleftrightarrow$ Sampling.

4. Impulse Function :



$\delta(t) = 1 , t = 0$

$= 0 , t \neq 0$

$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$

Properties of Impulse :

(i) $f(t) \cdot \delta(t-t_0) = f(t_0)\,\delta(t-t_0)$

(ii) $\int_{-\infty}^{\infty} \delta(t)\,dt = 1$

(iii) $\int_{-\infty}^{\infty} f(t)\,\delta(t)\,dt = [f(t)]_{t=0}$

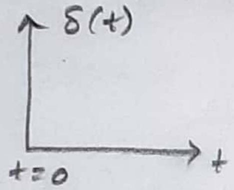
(iv) $\int_{-\infty}^{\infty} f(t)\,\delta(t-t_0)\,dt = f(t_0)$

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\,dt$$

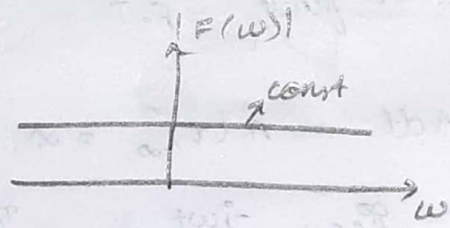
$$= \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\,dt$$

$$= \left(e^{-j\omega t}\right)_{t=0} = 1$$

$$\boxed{\delta(t) \xleftrightarrow{F.T} 1 = \text{const}}$$



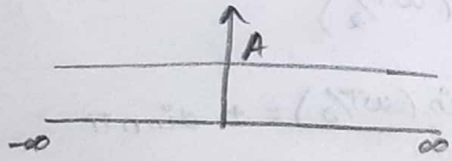
$\longleftrightarrow$

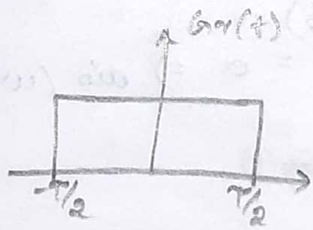


1) Impulse func. has a uniform spectral density over the entire freq. Range.

2) F.T of Impulse - contains all the freq components with the same relative amplitude - const.

Fourier transform of a constant = A.





Gate func = FT[const]

$= \text{as } T \to \infty \quad FT[G_T(t)]$

$= \lim_{T \to \infty} A\,T\,Sa\left(\frac{\omega T}{2}\right)$

T = width to ∞

$\int_{-\infty}^{\infty} A\,dt$ = Infinite

absolute Integral

↳ Indirect.

From the property Impulse func.

$$F.T[const] = \lim_{T\to\infty} AT\, Sa(\omega T/2)$$

$$\boxed{(i)\ \lim_{K\to\infty} \frac{K}{\pi} Sa(Kt) = \delta(t)}$$

$$= \lim_{T\to\infty} A \frac{T}{2} \times 2\, Sa(\omega T/2)$$

$$= \lim_{T\to\infty} A \frac{T}{2\pi} \times 2\pi\, Sa(\omega T/2)$$

$$= 2\pi A \lim_{T\to\infty} \frac{(T/2)}{\pi} Sa(\omega T/2)$$

$$= 2\pi A\, \delta(\omega)$$

$f(t) \longleftrightarrow F(\omega)$

const $= A \longleftrightarrow 2\pi A\, \delta(\omega) \longrightarrow$ Impulse at $\omega = 0$

$A = 1 \Rightarrow 2\pi . \delta(\omega)$

$1 \longleftrightarrow 2\pi\, \delta(\omega)$

$A \longleftrightarrow 2\pi A\, \delta(\omega)$

Fourier Transforms:

(i) $e^{-at} u(t) \longleftrightarrow \frac{1}{a+j\omega}$

(ii) $e^{-a|t|} \longleftrightarrow \frac{2a}{a^2+\omega^2}$

(iii) $\delta(t) \longleftrightarrow 1$

(iv) $G_T(t) \longleftrightarrow AT\, Sa\left(\frac{\omega T}{2}\right)$

(v) constant $\longleftrightarrow 2\pi A\, \delta(\omega)$

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

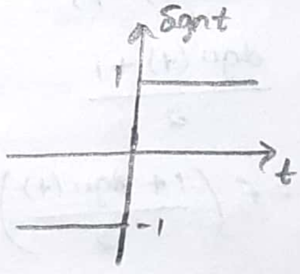
$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega$$

constant $\rightarrow$ d.c $\rightarrow$ freq $\Rightarrow \omega = 0$.

5. Signum Function:

$sgn(t) = 1$ for $t>0$

$= -1$ for $t<0$



$e^{-at} \underset{(a=0)}{=1} \rightarrow$ decaying ~~state~~ exp

$e^{at} =$ increasing exp $= (a=0) \Rightarrow 1$

$$\int_0^{\infty} 1.dt = \infty$$

$t>0 \Rightarrow e^{-at} = 0$

$t<0 \Rightarrow e^{-at} = \infty$

e^{at} $\rightarrow t<0 = 0$
$\rightarrow t>0 = \infty$

$(a=0) = 1$

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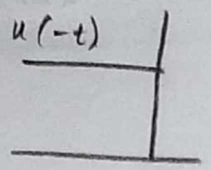
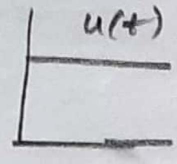
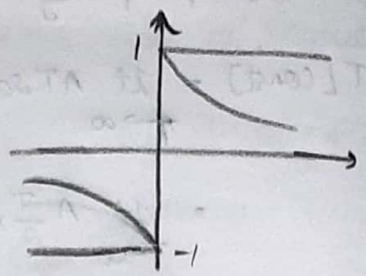
$$sgn(t) = \lim_{a\to 0}\left(e^{-at}u(t) - e^{at}u(-t)\right)$$







$$sgn(t) = \lim_{a\to 0}\left(e^{-at}u(t) - e^{at}u(-t)\right)$$

$$FT(sgn(t)) = \int_{-\infty}^{\infty} sgn(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty}\left(\lim_{a\to 0}\left(e^{-at}u(t) - e^{at}u(-t)\right)\right)$$

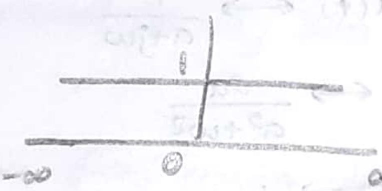
$$= \lim_{a\to 0}\int_{0}^{\infty} e^{-at} e^{-j\omega t}\, dt - \lim_{a\to 0}\int_{-\infty}^{0} e^{at} e^{-j\omega t}\, dt$$

$$= \lim_{a\to 0}\frac{1}{a+j\omega} - \lim_{a\to 0}\frac{1}{a-j\omega} = \frac{1}{j\omega} + \frac{1}{j\omega} = \frac{2}{j\omega}.$$

$$sgn(t) \longleftrightarrow \frac{2}{j\omega}.$$

6. Unit Step Function :

$$\cancel{def}\quad u(t) = 1 \quad \text{for } t > 0$$
$$= 0 \quad \text{for } t < 0$$



$$u(-t) = 1 - u(t)$$

$$sgn(t) = \left(u(t) - u(-t)\right)$$

$$= \left(u(t) - (1 - u(t))\right)$$

$$= \left(2u(t) - 1\right)$$

$$u(t) = \frac{sgn(t) + 1}{2}$$

$$F(u(t)) = f\left(\frac{1 + sgn(t)}{2}\right)$$

$$= \frac{1}{2}F(1) + \frac{1}{2}f(sgn(t))$$

$$= \frac{1}{2}\, 2\pi\, \delta(\omega) + \frac{1}{2}\left(\frac{2}{j\omega}\right)$$

$$= \pi\,\delta(\omega) + \frac{1}{j\omega}. \qquad u(t) \longleftrightarrow \pi\,\delta(\omega) + \frac{1}{j\omega}.$$

at $t=0$ $\quad sgn(t)=0 \Rightarrow u(t)=\frac{1}{2} \rightarrow$ discontinuity.

1. Fourier transform of $\cos\omega_0 t$ & $\sin\omega_0 t$

$$\cos\omega_0 t = \frac{e^{j\omega_0 t}+e^{-j\omega_0 t}}{2} \qquad \sin\omega_0 t = \frac{e^{j\omega_0 t}-e^{-j\omega_0 t}}{2}$$

Dirichlet's conditions $\rightarrow \int_0^\infty |\cos\omega_0 t|\,dt + \int_0^\infty \sin\omega_0 t\,dt$

over one period $\int_{-T/2}^{T/2} f(t)e^{-j\omega_0 t}$

$$\lim_{T\to\infty}\int_{-T/2}^{T/2} f(t)e^{-j\omega_0 t}\,dt$$

$$F[\cos\omega_0 t] = F\left[\frac{e^{j\omega_0 t}+e^{-j\omega_0 t}}{2}\right].$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega_0 t}\,dt = \lim_{T\to\infty}\int_{-T/2}^{T/2}(e^{j\omega_0 t}+e^{-j\omega_0 t})e^{-j\omega_0 t}\,dt.$$

$$= \lim_{T\to\infty}\int_{-T/2}^{T/2}\left(e^{j(\omega_0-\omega)t}+e^{-j(\omega+\omega_0)t}\right)dt$$

$$= \lim_{T\to\infty}\left(\frac{e^{j(\omega_0-\omega)t}}{j(\omega_0-\omega)}\right)_{-T/2}^{T/2} + \lim_{T\to\infty}\left(\frac{e^{-j(\omega+\omega_0)t}}{-j(\omega+\omega_0)}\right)_{-T/2}^{T/2}$$

$$= \lim_{T\to\infty}\left(\frac{e^{j(\omega_0-\omega)T/2}-e^{-j(\omega_0-\omega)T/2}}{2j(\omega_0-\omega)} + \frac{e^{j(\omega+\omega_0)T/2}-e^{-j(\omega+\omega_0)T/2}}{2j(\omega+\omega_0)}\right)$$

$$= \lim_{T\to\infty}\left(\frac{\sin(\omega_0-\omega)T/2}{(\omega_0-\omega)} + \frac{\sin(\omega_0+\omega)T/2}{\omega_0+\omega}\right)$$

$$= \lim_{T\to\infty}\left(\frac{T}{2}Sa((\omega-\omega_0)T/2) + \frac{T}{2}(Sa(\omega+\omega_0)T/2)\right)$$

$$= \pi\,\delta(\omega_0-\omega) + \pi\,\delta(\omega_0+\omega) \qquad \lim_{K\to\infty}\frac{K}{\pi}Sa(K\omega)=\delta(\omega)$$

$$\cos\omega_0 t \longleftrightarrow \pi(\delta(\omega_0-\omega)+\delta(\omega_0+\omega))$$

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$||^{ly}$ for $\sin \omega_0 t \xleftrightarrow{F.T} j\pi\,(\delta(\omega+\omega_0) - \delta(\omega-\omega_0))$

$f(t) = e^{j\omega_0 t} = \cos\omega_0 t + j\sin\omega_0 t$

$F(\omega) = F[e^{j\omega_0 t}] = F[\cos\omega_0 t] + j\,F(\sin\omega_0 t)$

$= \pi\,(\delta(\omega+\omega_0) + \delta(\omega-\omega_0)) + j(j\pi(\delta(\omega+\omega_0) - \delta(\omega-\omega_0))$

$= 2\pi\,\delta(\omega-\omega_0)$

$F(e^{j\omega_0 t}) \longleftrightarrow 2\pi\,\delta(\omega-\omega_0)$

* F.T of a periodic func

$f(t) = f(t+T)$ $\quad T = \frac{2\pi}{\omega}$

$F(f(t)) = F\left(\sum_{n=-\infty}^{\infty} F_n\, e^{jn\omega_0 t}\right)$

$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} \longrightarrow$ EFS Repth

$F(f(t)) = \sum_{n=-\infty}^{\infty} F_n\, F(e^{jn\omega_0 t})$

$= \sum_{n=-\infty}^{\infty} F_n\, 2\pi\,(\delta(\omega - n\omega_0))$

$= \ldots + 2\pi\,\delta(\omega+2\omega_0) + 2\pi\,\delta(\omega+\omega_0) + 2\pi\,\delta\omega$

$+ 2\pi\,\delta(\omega-\omega_0) + 2\pi\,\delta(\omega-2\omega_0) + \ldots$

What is F.T of periodic train of Impulse?

$\delta_T(t) = \delta(t) + \delta(t-T) + \delta(t-2T) + \ldots$

$+ \delta(t+T) + \delta(t+2T) + \ldots$

$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT)$

$\delta_T(t) = \sum_{n=-\infty}^{\infty} F_n\, e^{jn\omega_0 t}$ $\qquad F_n = \frac{1}{T}\int_{-T/2}^{T/2} \delta_T(t)\, e^{-jn\omega_0 t}\, dt.$

$$F[\delta_T(t)] = F\left[\sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}\right] \qquad F_n = \frac{1}{T}\int_{-T/2}^{T/2} \delta_T(t)\, e^{-jn\omega_0 t}\, dt$$

$$= \sum_{n=-\infty}^{\infty} F_n F(e^{jn\omega_0 t}) \qquad = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)$$

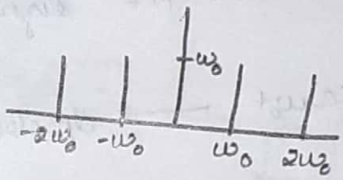
$$= \sum_{n=-\infty}^{\infty} \frac{1}{T} F(e^{jn\omega_0 t}) \qquad F_n = \frac{1}{T}$$

$$= \sum_{n=-\infty}^{\infty} \frac{1}{T}\, 2\pi\, \delta(\omega - n\omega_0) = \sum_{n=-\infty}^{\infty} \omega_0\, \delta(\omega - n\omega_0)$$

F.T of periodic train of impulses $\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$

$$= \sum_{n=-\infty}^{\infty} \omega_0\, \delta(\omega - n\omega_0)$$

height $= \omega_0$



1) $e^{-at} u(t) \longleftrightarrow \dfrac{1}{a + j\omega}$

2) $e^{-a|t|} \longleftrightarrow \dfrac{2a}{a^2 + \omega^2}$

3) $G_\tau(t) \longleftrightarrow A\tau\, Sa\left(\frac{\omega\tau}{2}\right)$

4) Constant $\longleftrightarrow 2\pi A\, \delta(\omega)$

5) $\delta(t) \longleftrightarrow 1$

6) $1 \longleftrightarrow 2\pi\, \delta(\omega)$

7) $\cos\omega_0 t \longleftrightarrow \pi(\delta(\omega + \omega_0) + \delta(\omega - \omega_0))$

8) $\sin \omega_0 t \longleftrightarrow j\pi(\delta(\omega + \omega_0) - \delta(\omega - \omega_0))$

9) $e^{jn\omega_0 t} \longleftrightarrow 2\pi\, \delta(\omega - n\omega_0)$

10) Periodic func $\Rightarrow 2\pi \sum F_n\, \delta(\omega - n\omega_0)$

11) $\delta_T(t) \longleftrightarrow \sum_{n=-\infty}^{\infty} \omega_0\, \delta(\omega - n\omega_0)$

$\downarrow$

$\sum_{n=-\infty}^{\infty} \delta(t - nt)$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, dt, \qquad F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt.$$

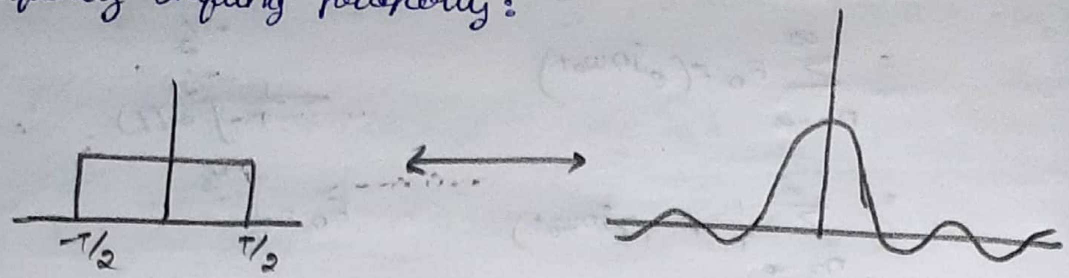
~~1. Find F.T of e^{-at} → double sided expone~~

1. Find F.T of $G_\tau(t) \cdot \cos\omega_0 t$

$$F_T\left(\frac{1}{2} G_\tau(t)\, e^{j\omega_0 t} + \frac{1}{2} G_\tau(t)\, e^{-j\omega_0 t}\right)$$

$$= \frac{1}{2} F(\omega - \omega_0) + \frac{1}{2} F(\omega + \omega_0)$$

Frequency shifting property:



$G_T(t) \cdot \cos \omega_0 t \longleftrightarrow$

Signal transmission $\rightarrow$ Modulation $\rightarrow f(t)$ - L F signal

$f(t) \cdot$ HF signal $\rightarrow$ carries signal $\rightarrow \cos\omega_0 t / \sin\omega_0 t$ $\frac{1}{T} \times$

$f(t) \cos\omega_0 t \longrightarrow$ spectrum of LF's $\rightarrow$ shifted to H.F

2) Differentiation in time domain Property.

$f(t) \rightleftarrows F(\omega)$ then

I F T = Inverse FT

$$\frac{d f(t)}{dt} \longleftrightarrow j\omega \cdot F(\omega)$$

w.k.t the $IFT[F(\omega)] = f(t)$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \cdot e^{j\omega t} d\omega$$

apply diff on both sides

$$\frac{d}{dt} f(t) = \frac{1}{2\pi} \frac{d}{dt} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$

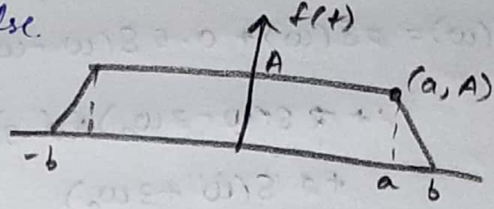
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \frac{d}{dt} e^{j\omega t} d\omega$$

$$\frac{df(t)}{dt} = \frac{1}{2\pi} \int_{-\infty}^{\infty} j\omega F(\omega) e^{j\omega t} d\omega \qquad \frac{df(t)}{dt} \longleftrightarrow j\omega F(\omega)$$

$$\frac{d^2 f(t)}{dt^2} \longleftrightarrow (j\omega)^2 F(\omega) \qquad \frac{d^n f(t)}{dt^n} \longleftrightarrow (j\omega)^n F(\omega)$$

differentiation of ramp = step (pulse)

" " pulse = impulse.



$$\frac{f(t)-0}{t+b} = \frac{A-0}{-a+b}$$

$$f(t) = \left(\frac{A}{b-a}\right)(t+b) \qquad -b \le t \le -a$$

$$u(t) = \frac{dr(t)}{dt} = \text{unit step}$$

$$f(t) = A \qquad -a < t < a \quad (t=0)$$

$$\int_{-\infty}^{t} u(t)\,dt = r(t)$$

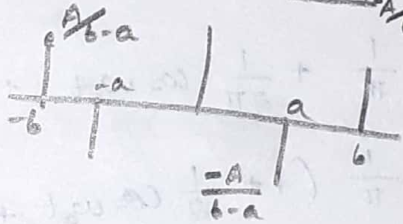
$$f(t) = \frac{A}{b-a}(b-t) \qquad a \le t \le b.$$

$$\int_{-\infty}^{t} \delta(t)\,dt = u(t)$$

$$\delta(t) = \frac{du(t)}{dt}$$

$$\frac{dr(t)}{dt} = u(t) = \text{step}$$

$$\frac{du(t)}{dt} = \delta(t) = \text{impulse}$$



$$\frac{d^2f(t)}{dt^2} = \left(\frac{-A}{b-a}\right)\delta(t-a) + \frac{A}{b-a}\delta(t-b) + \left(\frac{-A}{b-a}\right)\delta(t+a)$$

$$(j\omega)^2 F(\omega) = \frac{-A}{b-a}e^{-j\omega a} + \frac{A}{b-a}e^{-j\omega b} - \frac{A}{b-a}e^{j\omega a} + \frac{A}{b-a}e^{j\omega b}$$

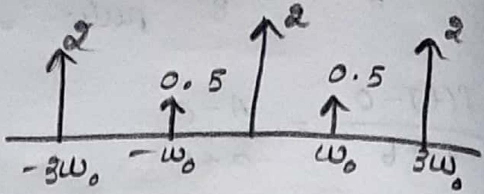
$$-\omega^2 F(\omega) = \frac{A}{b-a}\left((e^{j\omega b} + e^{-j\omega b}) - (e^{j\omega a} + e^{-j\omega a})\right)$$

$$-\omega^2 F(\omega) = \frac{A}{b-a}(2\cos\omega b - 2\cos\omega a)$$

$$F(\omega) = \frac{2A}{b-a(\omega^2)}(\cos\omega a - \cos\omega b)$$

2. Determine IFT of spectrum shown below.

$$F(\omega) = 2\delta(\omega) + 0.5\,\delta(\omega-\omega_0) + 2\,\delta(\omega-3\omega_0) + 0.5\,\delta(\omega+\omega_0) + 2\,\delta(\omega+3\omega_0)$$



$\omega+\omega_0 = 0 \Rightarrow \omega = -\omega_0$

$\omega-\omega_0 = 0 \Rightarrow \omega = \omega_0$

$f(t) \longleftrightarrow F(\omega)$

$e^{-at}u(t) \longleftrightarrow \dfrac{1}{a+j\omega}$

(i) $A \longleftrightarrow 2\pi A\,\delta(\omega)$, (ii) $\cos\omega_0 t \longleftrightarrow \pi(\delta(\omega-\omega_0)+\delta(\omega+\omega_0))$

$1 \longleftrightarrow 2\pi\,\delta(\omega)$ $\qquad \cos 3\omega_0 t \longleftrightarrow \pi(\delta(\omega-3\omega_0)+\delta(\omega+3\omega_0))$

$$F(\omega) = \frac{1}{\pi}2\pi\,\delta(\omega) + \frac{1}{2}\frac{\pi}{\pi}(\delta(\omega-\omega_0)+\delta(\omega+\omega_0)) + 2\frac{\pi}{\pi}(\delta(\omega-3\omega_0)+\delta(\omega+3\omega_0))$$

Apply I.F.T

$$f(t) = \frac{1}{\pi} + \frac{1}{2\pi}\cos\omega_0 t + \frac{2}{\pi}\cos 3\omega_0 t.$$

$$f(t) = \frac{1}{\pi}\left(1+\frac{1}{2}\cos\omega_0 t + 2\cos 3\omega_0 t\right) \longleftrightarrow \text{spectrum.}$$

Differentiation in freq domain:

$$\frac{dF(\omega)}{d\omega} \longleftrightarrow -jt\,f(t) \qquad \frac{d^nF(\omega)}{d\omega^n} \longleftrightarrow (-jt)^n f(t)$$

$$\frac{d^2F(\omega)}{d\omega^2} \longleftrightarrow (-jt)^2 f(t)$$

Integration in time domain:

If $f(t) \xleftrightarrow{F.T} F(\omega)$ then

$$\int_{-\infty}^{t} f(t)\,dt \xleftrightarrow{F.T} \frac{1}{j\omega}F(\omega)$$

Sol. $\phi(\tau) = \displaystyle\int_{-\infty}^{\infty} f(\tau)\,d\tau$

$$\frac{d\phi(\tau)}{dt} = f(\tau)$$

Apply F.T on both sides

$$(j\omega)\,\phi(\omega) = F(\omega)$$

$$\phi(\omega) = \frac{F(\omega)}{j\omega}$$

$$FT\left[\int_{-\infty}^{t} f(\tau)\,d\tau\right] \longleftrightarrow \frac{1}{j\omega}F(\omega)$$

$$\int_{-\infty}^{\infty} f(t)\,dt = \text{Value}$$
$\neq$ function

Time scaling :

If $f(t) \longleftrightarrow F(\omega)$ then

$$f(at) \longleftrightarrow \frac{1}{a}F\left(\frac{\omega}{a}\right).$$

sol. $FT[f(t)] = \int_{-\infty}^{\infty} f(t)\,e^{-j\omega t}\,dt$

$$FT[f(at)] = \int_{-\infty}^{\infty} f(at)\,e^{-j\omega t}\,dt$$

let $at = x \Rightarrow t = \frac{x}{a} \Rightarrow dt = \frac{dx}{a}$

$$FT[f(at)] = \frac{1}{a}\int_{-\infty}^{\infty} f(x)\,e^{-j\left(\frac{\omega}{a}\right)x}\,dx$$

$$\cancel{= \frac{1}{a}\int_{-\infty}^{\infty} f(x)} = \frac{1}{a}F\left(\frac{\omega}{a}\right)$$

a = const , $at \rightarrow$ scaling ; $a \rightarrow$ compression in T D $\downarrow$ expansion in F D & expansion in T D $\downarrow$ compression in F D

Convolution theorem: If $f_1(t) \xleftrightarrow{F.T} F_1(\omega)$ & $f_2(t) \xleftrightarrow{F.T} F_2(\omega)$

then $f_1(t) * f_2(t) = \int_{-\infty}^{\infty} f_1(\tau)\,f_2(t-\tau)\,d\tau$

is called convolution in time domain denoted by *

convolution in time domain prop: If $f_1(t) \longleftrightarrow F_1(\omega)$ and $f_2(t) \longleftrightarrow F_2(\omega)$ then convolution of $f_1(t)$ & $f_2(t)$ => $f_1(t) * f_2(t) = \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau) d\tau$

$\updownarrow F.T$

$F_1(\omega) \cdot F_2(\omega)$

$$\int_{-\infty}^{\infty} F_1(\lambda) F_2(\omega-\lambda) d\lambda = F_1(\omega) * F_2(\omega)$$

sol. $F[f(t)] = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$

$$FT[f_1(t) * f_2(t)] = \int_{-\infty}^{\infty} (f_1(t) * f_2(t)) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau) d\tau \right) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} f_1(\tau) d\tau \int_{-\infty}^{\infty} f_2(t-\tau) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} f_1(\tau) \cdot e^{-j\omega \tau} F_2(\omega) d\tau$$

$$= F_1(\omega) \cdot F_2(\omega)$$

convolution in time domain $\xleftrightarrow{F.T}$ Multiplication in frequency domain.

1. Convolution in freq domain.

If $f_1(t) \longleftrightarrow F_1(\omega)$, $f_2(t) \longleftrightarrow F_2(\omega)$ then

$$2\pi (f_1(t) \cdot f_2(t)) \xleftrightarrow{F.T} \int_{-\infty}^{\infty} F_1(\lambda) F_2(\omega-\lambda) d\lambda = F_1(\omega) * F_2(\omega)$$

sol. $FT[f(t)] = F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$

$$F.T[f_1(t) \cdot f_2(t)] = \int_{-\infty}^{\infty} f_1(t) f_2(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} F_1(\lambda) e^{j\lambda t} d\lambda \right) f_2(t) e^{-j\omega t} dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F_1(\lambda) d\lambda \int_{-\infty}^{\infty} f_2(t) e^{-j\omega t} e^{j\lambda t} dt.$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F_1(\lambda) d\lambda \int_{-\infty}^{\infty} f_2(t) e^{-j(\omega-\lambda)t} dt.$$

$$F.T[f_1(t).f_2(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(\alpha)\,F_2(\omega-\alpha)\,d\alpha$$ $f_1(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(\alpha)e^{j\alpha t}d\alpha$

$\int f_2(t)e^{-j\omega t}dt$

$$= \frac{1}{2\pi}[F_1(\omega) * F_2(\omega)]$$ $= F_2(\omega)$

$$2\pi[f_1(t).f_2(t)] = F_1(\omega) * F_2(\omega)$$

Multiplication in Time domain $\xleftrightarrow{F.T}$ convu. in freq domain

1. Convolution in time domain $\xleftrightarrow{F.T}$ Multiplication in freq. domain

$f_1(t) * f_2(t) \longleftrightarrow F_1(\omega).F_2(\omega)$

2. Multiplication in time domain $\xleftrightarrow{F.T}$ convolution in freq domain

$2\pi[f_1(t).f_2(t)] \longleftrightarrow F_1(\omega) * F_2(\omega)$

* Parseval's theorem / Relation / Prop of Fourier tranform

w.r.t Parseval's relation of Periodic signal ← Power signals

Power in T.D = Power in Fourier series coefficients

$$\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}|f(t)|^2dt = \sum_{n=-\infty}^{\infty}|F_n|^2$$

↑ Power spectral density.

i.e $f(t)$ ⟶ Periodic signal

F.T ⟵ Non-periodic signal → energy signals

∴ ~~Power~~ Parseval's relation N-P signal = energy relationship

⇒ The energy of Non-Periodic signal

$$E = \int_{-\infty}^{\infty}|f(t)|^2dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}|F(\omega)|^2d\omega$$

sol,

$$E = \int_{-\infty}^{\infty}|f(t)|^2dt = \int_{-\infty}^{\infty}f(t)\,f^*(t)\,dt$$

$$E = \int_{-\infty}^{\infty}f(t)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}F^*(\omega)e^{-j\omega t}d\omega\right)dt.$$

$$E = \frac{1}{2\pi}\int_{-\infty}^{\infty} F^*(\omega)\left(\int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt\right) d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F^*(\omega)\, F(\omega)\, d\omega$$

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega.$$

energy in T.D   energy in F.D → energy distribution w.r.t unit bandwidth (Freq)

→ energy spectral density

Parseval's relation of F.T

~~Time scaling~~

Duality Property:

$f(t) \longleftrightarrow F(\omega)$ then $2\pi f(-\omega) \longleftrightarrow F(t)$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega$$

$$2\pi f(t) = \int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega$$

Replace ω by x ~~and t by $-\omega$~~

$$\cancel{2\pi f(\omega)}\ 2\pi f(t) = \int_{-\infty}^{\infty} F(x)\, e^{jxt}\, dx$$

Replace t by $-\omega$

$$2\pi f(-\omega) = \int_{-\infty}^{\infty} F(x)\, e^{-jx\omega}\, dx$$

$$2\pi f(-\omega) = \int_{-\infty}^{\infty} F(t)\, e^{-j\omega t}\, dt.$$

$$2\pi f(-\omega) \longleftrightarrow F(t)$$

If $f(\omega)$ is an even func $\Rightarrow f(-\omega) = f(\omega)$

$$2\pi f(\omega) = F(t)$$

1. Find Inverse F.T of $\frac{1}{(a+j\omega)^2}$

Sol. we k.t

$$f_1(t) = e^{-at}u(t) \longleftrightarrow \frac{1}{a+j\omega} = F_1(\omega)$$

$$f_2(t) = e^{-at}u(t) \longleftrightarrow \frac{1}{a+j\omega} = F_2(\omega)$$

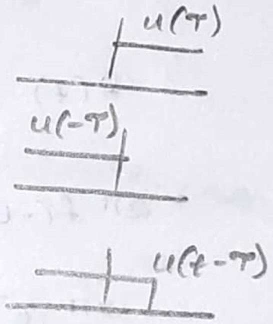
$$F(\omega) = F_1(\omega) \cdot F_2(\omega) = \frac{1}{a+j\omega} \cdot \frac{1}{a+j\omega} = \frac{1}{(a+j\omega)^2}$$

The IFT of $F(\omega) = F_1(\omega) F_2(\omega)$ which is the multiplication in freq. domain is equivalent to convolution in time domain i.e $f_1(t) * f_2(t) = f(t)$

$$\therefore F^{-1}[F(\omega)] = F^{-1}[F_1(\omega) F_2(\omega)] = e^{-at}u(t) * e^{-at}u(t)$$

convolution of $f_1(t)$ & $f_2(t)$ is defined as

$$\int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau)\, d\tau$$



$$f(t) = \int_{-\infty}^{\infty} e^{-a\tau} u(\tau)\, e^{-a(t-\tau)} u(t-\tau)\, d\tau$$

$$= \int_{0}^{t} e^{-a\tau} e^{-at} e^{a\tau}\, d\tau = e^{-at} \int_{0}^{t} d\tau = te^{-at}$$

$$\therefore t\, e^{-at} \longleftrightarrow \frac{1}{(a+j\omega)^2}$$

2. Find the F.T of $f(t) \cos\omega_0 t$

Sol. WKT $\cos\omega_0 t \longleftrightarrow \pi(\delta(\omega-\omega_0) + \delta(\omega+\omega_0))$

$f_1(t) = f(t)$ $\quad f_2(t) = \cos\omega_0 t$

$$F.T[f_1(t) f_2(t)] = \frac{1}{2\pi}[F_1(\omega) * F_2(\omega)]$$ i.e the F.T of multiplication of two T.D functions is equivalent to convulution in freq domain

$\therefore$ we need to perform convolution in freq. domain of $f(t)$ & $\cos\omega_0 t$.

$\therefore$ F.T $[f(t)\cos\omega_0 t] = \frac{1}{2\pi}\,[F(\omega) * \pi(\delta(\omega-\omega_0)+\delta(\omega+\omega_0))]$

$= \frac{1}{2\pi} F(\omega) * \pi\,\delta(\omega-\omega_0) + \frac{1}{2\pi} F(\omega) * \pi\,\delta(\omega+\omega_0)$

$= \frac{1}{2}[F(\omega-\omega_0) + F(\omega+\omega_0)]$

$\because$ convolution of any function with impulse func. yields the function itself

$f(t) * \delta(t) \longleftrightarrow F(\omega).\ (-F(\omega))$

$f(t) * \delta(t-t_0) \longleftrightarrow$

$f(t-t_1) * \delta(t-t_2) \longleftrightarrow$

3. Find the F.T. $f(t) = \frac{1}{a^2+t^2}$ (duality prop)

F.T $= e^{-a|t|} \xleftrightarrow{F.T} \frac{2a}{a^2+\omega^2}$

$f(t) = e^{-a|t|}$

$\longleftrightarrow \frac{2a}{a^2+\omega^2} = F(\omega)$

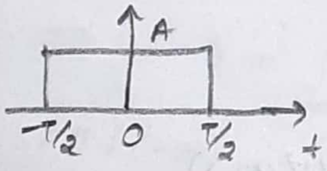
$2\pi f(-\omega) = 2\pi\, e^{-a|-\omega|}$

$\longleftrightarrow \frac{2a}{a^2+t^2}$

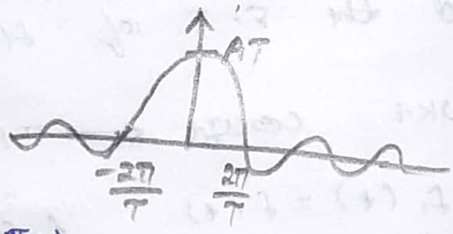
$2\pi\, e^{-a|-\omega|}$

$\longleftrightarrow \frac{2a}{a^2+t^2}$

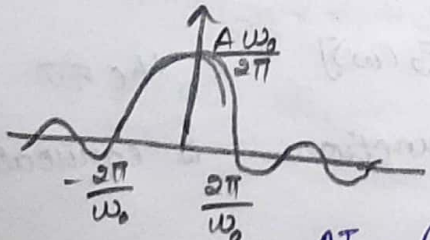
$\frac{\pi}{a} e^{-a|\omega|} \longleftrightarrow \frac{1}{a^2+t^2}$



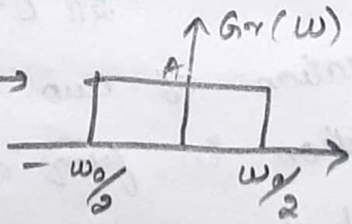
$\longleftrightarrow$



$G_T(t) \longleftrightarrow AT\,Sa(\omega T/2)$



$\longleftrightarrow$



$\frac{AT}{2\pi} Sa\left(\frac{\omega_0 T}{2}\right) \longleftrightarrow G_T(\omega)$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_0/2}^{\omega_0/2} A\, e^{j\omega t}\, d\omega$$

$$= \frac{A}{2\pi}\left(\frac{e^{j\omega t}}{jt}\right)_{-\omega_0/2}^{\omega_0/2} = \frac{A}{2\pi}\left(\frac{e^{j\omega_0 t/2} - e^{-j\omega_0 t/2}}{jt}\right)$$

$$= \frac{A}{2\pi j t}\left(2j \sin\left(\frac{\omega_0 t}{2}\right)\right) = \frac{A}{\pi}\,\frac{\sin\left(\frac{\omega_0 t}{2}\right)}{\frac{\omega_0 t}{2}}\,\frac{\omega_0}{2}$$

$$= \frac{A\omega_0}{2\pi}\, Sa\left(\frac{\omega_0 t}{2}\right)$$

If $\frac{\sin(\omega_0 t/2)}{\omega_0 t/2} = 0 \Rightarrow \frac{\omega_0 t}{2} = \pm n\pi$

$$t = \pm\frac{2n\pi}{\omega_0}$$

$$= \pm\frac{2\pi}{\omega_0}, \pm\frac{4\pi}{\omega_0} \ldots\ldots$$

Sampling & Rectangular func dual.

4. Find IFT of $sgn(\omega)$ & $u(\omega)$ $\longleftrightarrow$ duality prop

$sgn(t) \longleftrightarrow \frac{2}{j\omega}$

$f(t) \qquad F(\omega)$

$2\pi f(-\omega) \longleftrightarrow F(t)$

$2\pi\, sgn(-\omega) \longleftrightarrow \frac{2}{jt}$

w.k.t $\Rightarrow$ Signum func is an odd func

$sgn(-\omega) = -sgn(\omega)$

$-sgn(\omega) = \frac{1}{j\pi t}$

$sgn(\omega) = \frac{j}{\pi t}$

I.F.T $(u(\omega)) \Rightarrow u(t) = \frac{1 + sgn(t)}{2}$

$u(\omega) = \frac{1 + sgn(\omega)}{2}$

$I.F.T(u(\omega)) = \frac{1}{2}F^{-1}(1) + \frac{1}{2}F^{-1}(sgn(\omega))$

$$F^{-1}(u(\omega)) = \frac{1}{2}F^{-1}(1) + \frac{1}{2}F^{-1}(\mathrm{Sgn}(\omega))$$

$$u(t) = \frac{1}{2}\delta(t) + \frac{1}{2}\frac{j}{\pi t}.$$

5. Damped sinusoidal wave

$$f(t) = e^{-at}\sin\omega_0 t\, u(t)$$

$$F.T\left(e^{-at}\sin\omega_0 t\, u(t)\right) = ?$$

$$= \frac{e^{-at}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right)}{2j} u(t)$$

$$= \frac{e^{-at}e^{j\omega_0 t}u(t)}{2j} - \frac{e^{-at}e^{-j\omega_0 t}u(t)}{2j}$$

$$= \frac{e^{-at}u(t)e^{j\omega_0 t}}{2j} - \frac{e^{-at}u(t)e^{-j\omega_0 t}}{2j}$$

$$\sin\omega_0 t = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}$$

$$e^{-at}u(t) \longleftrightarrow \frac{1}{a+j\omega} \qquad (f(t) \longleftrightarrow F(\omega))$$

$$f(t)e^{j\omega_0 t} \longleftrightarrow F(\omega-\omega_0)$$

$$f(t)e^{-j\omega_0 t} \longleftrightarrow F(\omega+\omega_0)$$

$$F\left(\frac{e^{-at}u(t)e^{j\omega_0 t}}{2j}\right) - \frac{1}{2j}F\left(e^{-at}u(t)e^{-j\omega_0 t}\right)$$

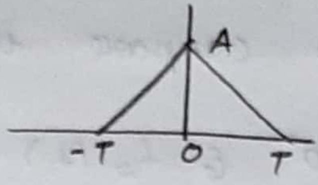
By using freq shifting prop

$$= \frac{1}{2j}\left(\frac{1}{a+j(\omega-\omega_0)} - \frac{1}{a+j(\omega+\omega_0)}\right)$$

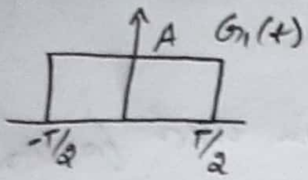
$$= \frac{\omega_0}{(a+j\omega)^2+\omega_0^2}$$

$$e^{-at}\sin\omega_0 t\, u(t) \longleftrightarrow \frac{\omega_0}{(a+j\omega)^2+\omega_0^2}$$

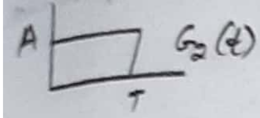
6. Find F.T of Triangular pulse



sol

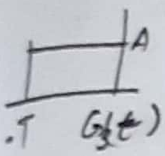


$$G_1(t) \longleftrightarrow AT\,Sa(\omega T/2)$$

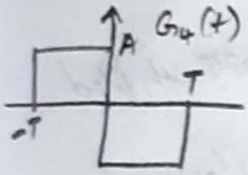


$$G_2(t) = G_1(t - T/2) \longleftrightarrow AT\,Sa(\omega T/2)\,e^{-j\omega T/2}$$

$$f(t-t_0) \longleftrightarrow e^{-j\omega t_0} F(\omega)$$



$$G_3(t) = G_1(t + T/2) \longleftrightarrow AT\,Sa\left(\frac{\omega T}{2}\right) e^{j\omega T/2}$$



$$= G_3(t) - G_2(t) = AT\,Sa\left(\frac{\omega T}{2}\right)\left(e^{j\omega T/2} - e^{-j\omega T/2}\right)$$

$$G_4(t) = AT\,Sa\left(\frac{\omega T}{2}\right) 2j \sin\left(\frac{\omega T}{2}\right)$$

$$\int_{-\infty}^{t} f(\tau)\,d\tau = \frac{1}{j\omega} F(\omega)$$

$$G_4(t) \longleftrightarrow G_4(\omega)$$

$$\int_{-\infty}^{t} G_4(\tau)\,d\tau \longleftrightarrow \frac{1}{j\omega} G_4(\omega)$$

$$\longleftrightarrow \frac{1}{j\omega} AT\,Sa(\omega T/2)\, 2j \sin\left(\frac{\omega T}{2}\right)$$

$$\longleftrightarrow 2AT\,Sa\left(\frac{\omega T}{2}\right) \frac{\sin(\omega T/2)}{\omega T/2} \cdot T/2$$

$$\longleftrightarrow AT^2 Sa^2\left(\frac{\omega T}{2}\right)$$

Hilbert transformation :

$F(\omega) \longrightarrow$ freq. spectrum

signals can be of different frequencies

(i) $f_1(t), f_2(t), \ldots f_n(t) \longrightarrow$ compare by means of freq

$\downarrow F_1(\omega)$ $\quad \downarrow F_2(\omega)$ $\quad \downarrow F_n(\omega)$

(ii) Can also compose using phase

$f_1(t) = 0°$ & $f_2(t) = 180°$

180° → phase shift which is used for comparision of two signals → Ideal phase

Ideal phase of 180°

2) we want to shift the signal ±90°

If all the freq. component of $f(t)$ are shifted by ±90°, phase shift → H.T is performed.

⇒ Hilbert transformer → ckt which introduce +90° phase for all the negative freq components & −90° phase shift to all positive freq components without modifying its amplitude values.

⇒ If $f(t)$ = is a signal then its H T is denoted $\hat{f}(t)$

$$\hat{f}(t) = HT(f(t)) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(\tau)}{t-\tau}\, d\tau$$

If $\hat{f}(t)$ is given then $f(t) = \frac{-1}{\pi}\int_{-\infty}^{\infty} \frac{\hat{f}(\tau)}{t-\tau}\, d\tau$

$$f_1(t) * f_2(t) = \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau)\, d\tau$$

$$\hat{f}(t) = f(t) * \frac{1}{\pi t} = \int_{-\infty}^{\infty} f(\tau) \frac{1}{\pi(t-\tau)}\, d\tau$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(\tau)}{t-\tau}\, d\tau$$

$f(t) \longleftrightarrow F(\omega)$

$\hat{f}(t) \longleftrightarrow \hat{F}(\omega)$

$\frac{1}{\pi t} \longleftrightarrow -j\,\mathrm{sgn}(\omega)$

$sgn(t) \longleftrightarrow \frac{2}{j\omega}$, $2\pi\, sgn(-\omega) \longleftrightarrow \frac{2}{jt}$

$j\, sgn(-\omega) = \frac{1}{\pi t}$, $sgn(\omega)$ = is an odd func

$-j\, sgn(\omega) = \frac{1}{\pi t}$.

$\hat{F}(\omega) = -F(\omega)\, j\, sgn(\omega)$

$\hat{F}(\omega) = -j\, sgn(\omega)\, F(\omega)$

$sgn(\omega) = 1$ for $\omega > 0$

$= 0$ for $\omega = 0$

$= -1$ for $\omega < 0$.

Applications:

=) Generation of Band pass signal

=) Comparision

$\omega = +ve$ = -90° phase shift is introduced

$\omega = -ve$ = $+90^\circ$ " " " "

} → HT

$f(t) \rightarrow$ its HT $\rightarrow \hat{f}(t) = f(t) * \frac{1}{\pi t}$

$\hat{f}(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(\tau)}{t-\tau}\, d\tau$

~~Conv in T.D = multiplication F.D~~

~~$f_1(t) \longleftrightarrow F_1(\omega)$, f_2 (~~

$-j\, sgn(\omega) \longleftrightarrow \frac{1}{\pi t}$.