where $X_T(\omega)$ is a Fourier Transform of $X(t)$ in the interval $[-T, T]$

7.2.1 Average Power of the Random Process

The average power P_{XX} of a WSS random process $X(t)$ is defined as the time average of its second moment or autocorrelation function at $\tau = 0$.

Mathematically

$$P_{XX} = A\{E[X^2(t)]\} \qquad \text{.....(7.4)}$$

$$P_{XX} = \underset{T\to\infty}{lt}\ \frac{1}{2T}\int_{-T}^{T} E[X^2(t)]\,dt$$

or

$$P_{XX} = R_{XX}(\tau)\big|_{\tau=0} \qquad \text{.....(7.5)}$$

we know that from power density spectrum,

$$R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)e^{j\omega\tau}\,d\omega$$

at $\tau = 0$

$$P_{XX} = R_{XX}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)d\omega$$

$\therefore$ The average power of $X(t)$ is

$$P_{XX} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)d\omega \qquad \text{.....(7.6)}$$

Example 7.1

Determine which of the following functions are valid power density spectrums and why?

(a) $\frac{\cos 8(\omega)}{2+\omega^4}$ (b) $e^{-(\omega-1)^2}$ (c) $\frac{\omega^2}{\omega^6+3\omega^2+3}$

Solution

(a) Given $S_{XX}(\omega)=\frac{\cos 8(\omega)}{2+\omega^4}$

From the properties of psd,

(i) The given function $S_{XX}(\omega)$ is real, and positive,

(ii) $S_{XX}(-\omega)=\frac{\cos 8(-\omega)}{2+(-\omega)^4}=\frac{\cos 8\omega}{2+\omega^4}=S_{XX}(\omega)$.

The function is even.

Hence the given function is valid psd.

(b) Given $S_{XX}(\omega)=e^{-(\omega-1)^2}$

From the properties of psd,

(i) $S_{XX}(-\omega)=e^{-(\omega-1)^2}=e^{-(-\omega-1)^2}=e^{-(\omega+1)^2}$

$\therefore\ S_{XX}(-\omega)\neq S_{XX}(\omega)$

$\therefore$ The given function is not valid psd,

(c) Given $S_{XX}(\omega)=\frac{\omega^2}{\omega^6+3\omega^2+3}$

From the properties of psd,

(i) The given function is real and positive

(ii) $$S_{XX}(-\omega)=\frac{(-\omega)^2}{(-\omega)^6+3(-\omega)^2+3}=\frac{\omega^2}{\omega^6+3\omega^2+3}$$

$\therefore$ $S_{XX}(-\omega)=S_{XX}(\omega)$, even function

$\therefore$ The given function is valid psd.

Example 7.2

The power density spectrum of a baseband random process $X(t)$ is

$$S_{XX}(\omega) = \frac{2}{\left[1+\left(\frac{\omega}{2}\right)^2\right]^2}$$

Find the *rms* bandwidth

Solution

Given power density spectrum

$$S_{XX}(\omega) = \frac{2}{\left[1+\left(\frac{\omega}{2}\right)^2\right]^2}$$

Now, the rms bandwidth is

$$W_{rms}^2 = \frac{\int_{-\infty}^{\infty} \omega^2 S_{XX}(\omega) d\omega}{\int_{-\infty}^{\infty} S_{XX}(\omega) d\omega}$$

$$\therefore \quad \int_{-\infty}^{\infty} \omega^2 S_{XX}(\omega) d\omega = \int_{-\infty}^{\infty} \frac{2\omega^2}{\left(1+\frac{\omega^2}{4}\right)^2} d\omega$$

$$= \int_{-\infty}^{\infty} \frac{32\omega^2}{(4+\omega^2)^2} d\omega$$

$$= 32\int_{-\infty}^{\infty} \frac{\omega^2}{(4+\omega^2)^2} d\omega$$

$$= 32\left[\frac{-\omega}{2(4+\omega^2)} + \frac{1}{2\times 2}\tan^{-1}\left(\frac{\omega}{2}\right)\right]_{-\infty}^{\infty} \rightarrow \text{Doubt}$$

$$= 32\left[0+\frac{\pi}{4}\right] = 8\pi$$

and $$\int_{-\infty}^{\infty} S_{XX}(\omega) d\omega = \int_{-\infty}^{\infty} \frac{2}{\left(1+\frac{\omega^2}{4}\right)^2} d\omega$$

$$= \int_{-\infty}^{\infty} \frac{32}{(4+\omega^2)^2} d\omega$$

$$= 32\left[\frac{\omega}{8(4+\omega^2)} + \frac{1}{2\times 8}\tan^{-1}\left(\frac{\omega}{2}\right)\right]_{-\infty}^{\infty}$$

$$= 32\left[0 + \frac{\pi}{16}\right] = 2\pi$$

$$\therefore \quad W_{rms}^2 = \frac{8\pi}{2\pi} = 4\pi$$

$$W_{rms}^2 = \sqrt{4\pi} = 2\sqrt{\pi} = 3.55 \, rad/\sec$$

ADDITIONAL PROBLEMS

Example 7.4 ✓

Consider the random process

$$X(t) = A\cos(\omega t + \theta)$$

Where A and ω are real constants and θ is a random variable uniformly distributed over $[0, 2\pi]$

Find the average power P_{XX}

Solution

We know that

$$P_{XX} = A\{E[X^2(t)]\}$$

Now $$X(t) = A\cos(\omega t + \theta)$$

and $$f_\theta(\theta) = \begin{cases} \dfrac{1}{2\pi} & 0 < \theta \leq 2\pi \\ 0 & \text{Otherwise} \end{cases}$$

$$\therefore \quad E[X^2(t)] = \int_{-\infty}^{\infty} X^2(t) f_\theta(\theta) d\theta$$

$$= \int_0^{2\pi} A^2 \cos^2(\omega t + \theta) \frac{1}{2\pi} d\theta$$

$$= \frac{A^2}{2\pi} \left[\int_0^{2\pi} \frac{1 + \cos(2\omega t + 2\theta)}{2} d\theta \right]$$

$$= \frac{A^2}{4\pi} \left[\int_0^{2\pi} d\theta + \int_0^{2\pi} \cos(2\omega t + 2\theta) d\theta \right]$$

$$= \frac{A^2}{4\pi} \left[2\pi + \frac{(-)\sin(2\omega t + 2\theta)}{2} \Big|_0^{2\pi} \right]$$

$$= \frac{A^2}{4\pi} [2\pi - 4\sin\ (2\omega t)]$$

$$E[X^2(t)] = \frac{A^2}{2} - \frac{A^2}{\pi} \sin 2\omega t$$

The time average power is

$$P_{XX} = A\left[E[X^2(t)] \right] = \underset{T\to\infty}{lt} \frac{1}{2T} \int_{-T}^{T} \left[\frac{A^2}{2} - \frac{A^2}{\pi} \sin(2\omega t) \right] dt$$

$$= \frac{1}{2T} \frac{A^2}{2} (2T) - 0$$

$$P_{XX} = \frac{A^2}{2}$$

Example 7.5

The psd of $X(t)$ is given by

$$S_{XX}(\omega) = \begin{cases} 1 + \omega^2 & \text{for } |\omega| < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find out the autocorrelation function. *May-2*

Solution

The autocorrelation function is

$$R_{XX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) e^{j\omega\tau} d\omega$$

$$R_{XX}(\tau)=\frac{1}{2\pi}\int_{-1}^{1}(1+\omega^2)e^{j\omega\tau}\,d\omega$$

$$R_{XX}(\tau)=\frac{1}{2\pi}\left[\int_{-1}^{1}e^{j\omega\tau}\,d\omega+\int_{-1}^{1}\omega^2e^{j\omega\tau}\,d\omega\right]$$

Now take, $$\int_{-1}^{1}\omega^2e^{j\omega\tau}\,d\omega=\frac{e^{j\omega\tau}}{j\tau}\,\omega^2\Bigg|_{-1}^{1}-\int_{-1}^{1}\frac{e^{j\omega\tau}}{j\tau}(2\omega)\,d\omega$$

$$=\frac{e^{j\tau}-e^{-j\tau}}{j\tau}-\left[\frac{2e^{j\omega\tau}}{(j\tau)^2}\,\omega\Bigg|_{-1}^{1}-\int_{-1}^{1}\frac{2e^{j\omega\tau}}{(j\tau)^2}\,d\omega\right]$$

$$=\frac{e^{j\tau}-e^{-j\tau}}{j\tau}-\frac{2(e^{j\tau}+e^{-j\tau})}{-\tau^2}+\frac{2}{-\tau^2}\cdot\frac{e^{j\omega\tau}}{j\tau}\Bigg|_{-1}^{1}$$

$$\therefore\quad\int_{-1}^{1}\omega^2e^{j\omega\tau}\,d\omega=\frac{e^{j\tau}-e^{-j\tau}}{j\tau}+\frac{2(e^{j\tau}+e^{-j\tau})}{\tau^2}-\frac{2(e^{j\tau}-e^{-j\tau})}{j\tau^3}$$

Also, $$\int_{-1}^{1}e^{j\omega\tau}\,d\omega=\frac{e^{j\omega\tau}}{j\tau}\Bigg|_{-1}^{1}=\frac{e^{j\tau}-e^{-j\tau}}{j\tau}$$

$$\therefore\quad R_{XX}(\tau)=\frac{1}{2\pi}\left[\frac{e^{j\tau}-e^{-j\tau}}{j\tau}+\frac{e^{j\tau}-e^{-j\tau}}{j\tau}+\frac{2}{\tau^2}(e^{j\tau}+e^{-j\tau})-\frac{2}{j\tau^3}(e^{j\tau}-e^{-j\tau})\right]$$

$$=\frac{1}{2\pi}\left[\frac{2\sin\tau}{\tau}+\frac{2\sin\tau}{\tau}+\frac{4\cos\tau}{\tau^2}-\frac{4}{\tau^3}\sin\tau\right]$$

$$R_{XX}(\tau)=\frac{1}{\pi}\left[\frac{2\sin\tau}{\tau}+\frac{2\cos\tau}{\tau^2}-\frac{2}{\tau^3}\sin\tau\right]$$

$$R_{XX}(\tau)=\frac{2}{\pi\tau^3}\left[\tau^2\sin\tau+\tau\cos\tau-\sin\tau\right]$$

Example 7.7

Find out the psd of a *WSS* random process $X(t)$ whose autocorrelation function is $R_{XX}(\tau) = ae^{-b|\tau|}$.

Solution

we know that the power spectral density,

$$S_{XX}(\omega) = \text{Fourier Transform of } R_{XX}(\tau)$$

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} a e^{-b|\tau|} e^{-j\omega\tau} d\tau$$

$$= a \int_{-\infty}^{0} e^{b\tau} e^{-j\omega\tau} + a \int_{0}^{\infty} e^{-b\tau} e^{-j\omega\tau} d\tau$$

$$= a \int_{-\infty}^{0} e^{\tau(b-j\omega)} d\tau + a \int_{0}^{\infty} e^{-\tau(b+j\omega)} d\tau$$

$$= a \left. \frac{e^{(b-j\omega)\tau}}{b-j\omega} \right|_{-\infty}^{0} + a \left. \frac{e^{-(b+j\omega)\tau}}{-(b+j\omega)} \right|_{0}^{\infty}$$

$$= \frac{a}{b-j\omega}[1-0] - \frac{a}{b+j\omega}[0-1]$$

$$= \frac{a}{b-j\omega} + \frac{a}{b+j\omega} = \frac{a(b+j\omega+b-j\omega)}{b^2+\omega^2}$$

$$S_{XX}(\omega) = \frac{2ab}{b^2+\omega^2}$$

UNIVERSITY PROBLEMS

Example 7.10

Find out the autocorrelation function and power spectral density of the random process $X(t) = A\cos(\omega_0 t + \theta)$, where θ is a random variable over the ensemble and it is uniformly distributed over the range $(0, 2\pi)$.

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Solution

Given the random process

$$X(t) = A\cos(\omega_0 t + \theta)$$

and $$f_\theta(\theta) = \frac{1}{2\pi} \quad 0 \le \theta \le 2\pi$$

(i) The autocorrelation function is

$$\begin{aligned}
R_{XX}(\tau) &= E[X(t)X(t+\tau)] \\
&= E[A\cos(\omega_0 + \theta)A\cos(\omega_0(t+\tau)+\theta)] \\
&= A^2 E[\cos(\omega_0 t + \theta)\cos(\omega_0 t + \omega_0\tau + \theta)] \\
&= \frac{A^2}{2} E[\cos(2\omega_0 t + \omega_0\tau + 2\theta) + \cos\omega_0\tau]
\end{aligned}$$

$$= \frac{A^2}{2}\{E[\cos\omega_0\tau] + E[\cos(2\omega_0 t + \omega_0\tau + 2\theta)]\}$$

$$= \frac{A^2}{2}\int_0^{2\pi}\frac{\cos\omega_0\tau}{2\pi}\,d\theta + \frac{A^2}{2}\int_0^{2\pi}\frac{1}{2\pi}[\cos(2\omega_0 t + \omega_0\tau + 2\theta)]d\theta$$

$$= \frac{A^2}{2}\cos\omega_0\tau + \frac{A^2}{4\pi}\left[\frac{\sin(2\omega_0 t + \omega_0\tau + 2\theta)}{2}\right]_0^{2\pi}$$

$$= \frac{A^2}{2}\cos\omega_0\tau + 0$$

$$\therefore \quad R_{XX}(\tau) = \frac{A^2}{2}\cos\omega_0\tau$$

(ii) The power spectral density is

$$S_{XX}(\omega) = \text{Fourier Transform of } R_{XX}(\tau)$$

$= \int_{-\infty}^{\infty} R_{XX}(\tau)\, e^{-j\omega\tau}\, d\tau$

$$= \frac{A^2}{2}\int_{-\infty}^{\infty}\cos\omega_0\tau\, e^{-j\omega\tau}\,d\tau = \frac{A^2}{2}\int_{-\infty}^{\infty}\left(\frac{e^{j\omega_0\tau} + e^{-j\omega_0\tau}}{2}\right)e^{-j\omega\tau}\,d\tau$$

$$= \frac{A^2}{4}\left[\int_{-\infty}^{\infty} e^{-j(\omega-\omega_0)\tau}\,d\tau + \int_{-\infty}^{\infty} e^{-j(\omega+\omega_0)\tau}\,d\tau\right]$$

$$= \frac{A^2}{4}\,2\pi[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)]$$

$$S_{XX}(\omega) = \frac{A^2\pi}{2}[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)]$$

Example 7.14

Find out the cross power spectral density. If (a) $R_{XY}(\tau)=\frac{A^2}{2}\sin(\omega_0\tau)$ and (b) $R_{XY}(\tau)$ $=\frac{A^2}{2}\cos(\omega_0\tau)$.

Solution

Given

(a) $$R_{XY}(\tau)=\frac{A^2}{2}\sin(\omega_0\tau)$$

The power spectral density

$$S_{XY}(\omega) = F\left[\frac{A^2}{2}\sin\omega_0\tau\right]$$

$$S_{XY}(\omega) = \frac{jA^2\pi}{2}[(\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]$$

(b)

$$R_{XX}(\tau) = \frac{A^2}{2}\cos(\omega_0\tau)$$

$$S_{XY}(\omega) = F\left[\frac{A^2}{2}\cos(\omega_0\tau)\right]$$

$$S_{XY}(\omega) = \frac{A^2\pi}{2}[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)]$$

Both power density spectrums are shown in Fig. 7.5

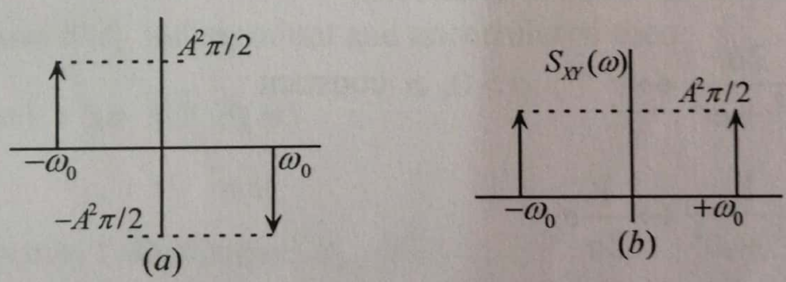

Fig.7.5: (a) psd for $\frac{A^2}{2}$ *sin* $\omega_0\tau$ *and (b) psd for* $\frac{A^2}{2}\cos\omega_0\tau$