UNIT 1

Part 1

UNIT- I : probability

(2) copies

Syllabus :-

1. probability introduced through sets and relative frequency
2. Experiments, sample spaces - Discrete, continuous sample spaces
3. Events
4. probability Definitions and Axioms
5. Mathematical Model of Experiments
6. probability as a relative frequency
7. Joint probability
8. conditional probability
9. Total probability
10. Baye's theorem
11. Independent Events
12. * Bernoulli trials.

Ch. RAJA
M.G.I.T

26/06/08 (UNIT-I ECC)

1. Set definition < Tabular / Rule

2. Null set (ϕ)

3. finite set / Empty set / void set (ϕ)

4. Infinite set

5. Subset ($A \subseteq B$), proper subset ($A \subset B$), superset ($B \supset A$)

6. Mutually exclusive sets (No common elements) or (Disjoint sets)

7. Universal set

8. Singleton set or Elementary set

9. Countable set

10. Uncountable set

27/06/08

1. Set operations
   - (a) $\in$, $\notin$
   - (b) $\cup$ & $\cap$
   - (c) complement

2. Algebra of sets.
   - C.L
   - A.L
   - D.L
   - Demorgan's Law
   - Duality principle

§ Terminology Associated with probability theory :-

(1) Experiment

(2) Outcome, Event

(3) Random Experiment

∮ Occurance of

∮ Equally Likely Events

∮ Mutually Exclusive Events

∮ Exhaustive Events

Independent Events : Two Events are said to be independent, if happening a failure of one does not affect the happening or failure of the other.

otherwise, the Events are said to be Dependent.

a) Sample Space :
b) Discrete Sample Space
c) Continuous Sample space

---

(1) Relative frequency definition
(2) Classical or axiomatic definition
(3) Axioms of probability

PROBABILITY AND RANDOM VARIABLES

EC 2103
4 credits

(COMMON FOR ECE & BME) II/IV B.Tech(ECE) I-Sem

UNIT-I

§ Concept of probability

§ Random variables : Discrete and continuous

§ probability distribution and density functions

§ functions of random variables

§ Joint and conditional probability density functions

§ Examples of probability density functions

§ Gaussian and Rayleigh density functions.

Books to be followed

1. probability, Random variables & Random signal principles
— peyton Z. peebles, TMH, 4th Edi...

2. probability, Random variables and stochastic processes
— papoulis & unnikrishnan
PHI, 4th Edition

3. communication systems Analog & Digital
— R.p. Singh & S.D. Sapre, TMH

4. statistical Theory of communication
— S.p. Eugene xavier, New Age publications, 20..

5. probability, statistics and Random processes.
— K. murugesan Anuradha
— p. Gurusamy publications

6. probability, Random variables & Random processes
HWEI — Hwei HSU
— Tata Mc GrawHill —

7. signals, systems & communications
— B.p. lathi, B.S. publications, 20..

8. probability methods of signal and system Analysis
— R. cooper, D.c Gillem,
oxford, 3rd Edition, 1999

9. probability and Random processes with Applications to signal processing — Henry stark and John W. Wo..
pearson Education, 3rd Editio..

§ Introduction, Applications.

§ Deterministic phenomena or predictable phenomena

§ probabilistic phenomena or unpredictable phenomena.

§ Experiment or (Trail)

§ Event or outcome

§ Random Experiment

§ Random Signal. probability def < Set Theory / Relative freq

§ Random Variable

SET DEFINITIONS

1. Set → content < Tabular method [Roster method) or / Rule method or [property method] [(Set or collection or aggregate or class) Element or Object or member]

2. Countable and uncountable sets

3. Empty or Null set (ϕ)

4. finite and Infinite sets

5. proper set ($A \subset B$) and Subset ($A \subseteq B$)

6. Mutually Exclusive or disjoint ~~Events~~ Sets.

7. universal set or sample set or Sample Space

SET Operations :-

1) Union (Sum) and Intersection (product)

2) Equality and difference

3) Complement

Algebra of Sets

1) Commutative law

4) De Morgan's law.

§ Occurrance

§ Equally likely events

§ mutually exclusive events

§ Exhaustive events

---

§ probability definition
- Relative frequency or Aposteriori probability
- classical definition or Apriori probability

§ Axioms of probability

§ Some useful laws of probability

§ Joint probability

§ Conditional probability

§ Total probability

§ Bayes theorem

§ ~~Idae~~ Independent events.

§ problems on probability

§ problems on Bayes theorem

§ previous question paper problems

— 1 — Quiz — 1 —

§ Mathematical Model of Experiments

§ Joint probability (Addition Theorem)

→ problems on dice

§ Conditional probability (Multiplication theorem or chain rule)

- § Random Variable < Definition / Example
- § conditions for a function to be a random variable
- § Types of Random Variables
  - § discrete
  - § continuous
  - § Mixed
- § probability distribution function (CDF) of discrete R.V $\rightarrow$ (cumulative D.F)
- § conditions of PDF (properties)
- § probability density function (pdf) of discrete R.V or probability mass function
- § properties of pdf
- § problems
- § Sum of two random variables
- § Joint probability distribution function $F_{X,Y}(x,y)$
- § Joint probability density function $f_{X,Y}(x,y)$
- § Conditional probability distribution function
- § conditional probability density function
- § statistical Independence
- § ... density function

Experiment : Any process of observation is referred to be experiment.

Outcome : The results of an observation are called outcome.

or

End result of the experiment is called outcome.

Random Experiment : An experiment whose outcome is not known in advance is called random experiment.

or

whose outcome cannot be predicted

"message"

Eg: roll of a die, toss of a coin, drawing a card from a deck, selecting a message signal for transmission from several messages

Random Event : is an outcome or set of outcomes of a random experiment.

Event : set of possible outcomes. / set of outcomes

REFERENCE BOOKS :-

1. principles of communication systems → by Taub and Schilling (TMH)

2. communication systems by A.B. Carlson (McGraw-Hill)

3. Electronic communication systems by Kennedy (TMH)

4. Statistical Theory of communication by S.P. Eugene Xavier New Age International publishers

5. probability, Random variables and Random signal principles → by P.Z. peebles, (Mc-Graw Hill)

CONCEPT OF PROBABILITY :-

Introduction :-

If an experiment is repeated under essentially homogeneous and similar conditions, we generally come across two types of situations. They are

1. The result or what is usually known as the outcome is unique or certain.
2. The result is not unique, but may be one of the several possible outcomes.

The phenomina covered by (i) are known as "deterministic" or "predictable" phenomina. By a deterministic phenomenon, we mean one in which the result can be predicted with certainty.

for Example : 1) for a perfect gas

$V \propto \frac{1}{P}$ i.e $PV = \underline{const}$

provided temperature remains the same.

2) The velocity 'v' of a particle after time 't' is given by $v = u + at$

where 'u' is the initial velocity and 'a' is the acceleration. This equation uniquely determines v, if the right-hand quantities are known.

3) Ohm's law $I = V/R$ or $c = E/R$

where ... ... flow of current, E is the

the conductor and R is the resistance, uniquely determines the value of C as soon as E and R are given.

A deterministic (approach) model is defined as a model which stipulates that the conditions under which an experiment is performed, and determine the outcome of the experiment. For number of situations the deterministic model suffices.

However, the phenomena covered by (2), which do not lend themselves to deterministic approach are known as "unpredictable" or "probabilistic" phenomena. For example

i) In tossing ~~of a coin~~ a coin, one is not ~~sure~~ sure if a head or tail will be obtained.

ii) If a light tube has lasted for 't' hours nothing can be said about its further life. It may fail to function any moment.

We often hear such statements

"It is likely to rain today"

"I have a fair chance of getting admission"

In each case we are not certain of the outcome, but we wish to assess the chances of our predictions coming true.

The "phrase" "probable" is often used in our daily conversation, which means "Likely". The probability of happening of an Event means that the amount of likelihood (chance) of the happening of that Event.

Short History :- Galileo (1564–1642), an Italian mathematician, was the first to attempt at a quantitative measure of the probability, while dealing with some problems related to the theory of dice in gambling.

Probability :- def < Set Theory / Relative frequency

Random Signal :- Time waveform, which can be characterized only in some probabilistic manner. In otherwords, random signal is a desired or undesired waveform.

~~Random variables~~

Random variable :- are the changing values of observation without any fixed expected time dependency.

Random Experiment :- An Experiment, whose outcome is not known in advance.

Random Event :- Random event is an outcome or a set of outcomes of random experiment.

Outcome :- The End result of an experiment.

Introduction :-

The application of probability calculations made human beings to do things with expectation. Cyclons and other natural calamities are random events, which may be occuring many times without any previous indications. These events possibility can be guessed if these are observed in certain time period.

After the observation of wind movements, temperature, pressure and other meteorological data during various years that department can be able to asses the possibility of occurance of atmospheric disturbance. To observe these similarities of all these data and to know the variations of the data, there should be certain mathematical principles. probability theory, correlation of signals, random signal analysis - all these help to access the problem.

This subject is needed for a Communication Engineer as he involved in the observation of random signal. Today as computer communication and other fields in communication, increased use, we are facing a problem of

Information measurement can be analysed using probability theory developed by mathematicians

SET DEFINITIONS :-

SET :- A set is defined as collection of objects. The objects are called elements of set, and may be anything whatsoever. we may have a set of voltages, a set of airplanes, a set of chairs, or even set of sets, Some times called as class of sets. A Set is usually denoted by a capital Letters, while an Element is represented by a lower case Letter. Thus, if "a" is an element of set "A", then we write

$\boxed{a \in A}$,

if "a" is not an element of "A", we write

$\boxed{a \notin A}$

A Set is specified by the content of two braces : { · }. Two methods exist for specifying content., 1) Tabular method or Roster method and the 2) Rule method or set Builder form

In the Tabular Method, the elements are enumerated explicitly. for example, the set of all the integers between 5 and 10 would be $\{6, 7, 8, 9\}$

In the Rule method a set's content is ... by Some rule. such as, for Example, {integers from 1 to 1000 inclusive} would be cumbersome to write explicitly using the tabular method. The rule method is usually more convenient to use when the set is very large.

§ COUNTABLE SET :- A set is said to be countable, if its elements can be put in one-to-one correspondence with the natural numbers, which are the integers 1, 2, 3, etc.

§ UnCountable Set :- If a set is not countable, it is called "Uncountable".

§ Empty Set or Null Set or void set :- If the set has no elements, then it is called as Empty set or Null set. The Empty set is represented by Symbol "ϕ".

§ finite Set :- A finite set is one that is either Empty or has elements that can be counted, with the counting process terminating. In other words, it has a finite number of elements. Eg {1, 2, 3, 4}

§ Infinite Set :- If a set is not finite, it is called "infinite" set. An infinite set having countable elements is called "Countably infinite"

Eg = {integers}

§ Singleton set: A set containing only one Element is called singleton set

$$A = \{ a \}$$

Subset :- If every element of set "A" is also an element in another set "B", 'A' is said to be contained in 'B'. "A" is known as subset of "B". and we write

$$\boxed{A \subseteq B}$$

A {2, 3}
B {2, 3, 4}

If at least one element exists in 'B' which is not in A, then 'A' is a proper subset of B, denoted by

$$\boxed{A \subset B}$$

-: The null set is clearly a subset of all other sets. $\phi \subseteq A$

Mutually Exclusive :- Two set A and B, are called disjoint or mutually exclusive, if they have no common elements.

Universal set :- The largest or all-Encompassing set of objects under discussion in a given situation is called the "Universal set". All sets are subsets of the universal set

Note : If A is subset B, then B is referred to as superset of A $B \supseteq A$.

SET OPERATIONS :-

1) Equality and Differences:

Two sets A and B are equal, if all elements in A are present in B and all elements in B are present in A ; that is if $A \subseteq B$ and $B \subseteq A$. For equal sets, we write

$$\boxed{A = B}$$

The difference of two sets A and B, denoted $A-B$, is the set containing all elements of A that are not present in B.

$$\boxed{A-B \neq B-A}$$

2) UNION And Intersection :-

The Union (call it C) of two sets A and B is written $C = A \cup B$

It is the set of all the elements of A or B or Both. The Union is sometimes called the "Sum of two sets."

The Union of N sets A_n, $n = 1, 2, 3 --- N$, become

$$C = A_1 \cup A_2 \cup A_3 --- \cup A_N$$

The Intersection (call it D) of two sets A and B is written as

$$D = A \cap B$$

It is the set of all elements common to both A and B. The intersection is also called as "product of two sets".

The intersection of N sets A_n, $n = 1, 2, 3 \dots N$, become

$$D = A_1 \cap A_2 \cap A_3 \dots A_N$$

$$= \bigcap_{n=1}^{N} A_n$$

<u>Complement :-</u>

The complement of set A, denoted by $\bar{A}$, is the set of all the elements not in A

$$\boxed{\text{Thus} \quad \bar{A} = S - A}$$

$$\bar{\phi} = S, \quad \bar{S} = \phi$$

$$A \cup A' = S \quad \text{and} \quad A \cap A' = \phi$$

§ Illustrate, intersection, Union, and complement by taking an Example with the four sets

$$S = \{ 1 \leq \text{integers} \leq 12 \}$$

$$A = \{1, 3, 5, 12\}$$

$$B = \{2, 6, 7, 8, 9, 10, 11\}$$

$$C = \{1, 3, 4, 6, 7, 8\}$$

$A \cup B$, $A \cup C$, $B \cup C$, $A \cap B$, $A \cap C$, $B \cap C$, $\bar{A}$, $\bar{B}$, $\bar{C}$

<u>Sol</u>

$$A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$$

$$A \cup C = \{1, 3, 4, 5, 6, 7, 8, 12\}$$

$$B \cup C = \{1, 2, 3, 4, 6, 7, 8, 9, 10, 11\}$$

$$A \cap B = \emptyset$$

$$A \cap C = \{1, 3\}$$

$$B \cap C = \{6, 7, 8\}$$

$$\bar{A} = \{2, 4, 6, 7, 8, 9, 10, 11\}$$

$$\bar{B} = \{1, 3, 4, 5, 12\}$$

$$\bar{C} = \{2, 5, 9, 10, 11, 12\}$$

§ Algebra of sets :-

COMMUTATIVE LAW :-

$$A \cup B = B \cup A$$

$$A \cap B = B \cap A$$

ASSOCIATIVE Law :-

$$A \cup (B \cup C) = (A \cup B) \cup C = A \cup B \cup C$$

$$A \cap (B \cap C) = (A \cap B) \cap C = A \cap B \cap C$$

Distributive law :-

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Demorgan's law :- (English mathematician) (1806 – 1871)

States that complement of a union (intersection) of two sets A and B equals the intersection (union) of the complements $\bar{A}$ and $\bar{B}$. Thus

$$\overline{(A \cup B)} = \bar{A} \cap \bar{B}$$

$$\overline{(A \cap B)} = \bar{A} \cup \bar{B}$$

Note —: If we replace unions by intersections, intersections by unions and sets by their complements, then the identity is preserved.

Q. Verify Demorgan's laws by using Example

Sets $A = \{2 < a \le 16\}$, $B = \{5 < b \le 22\}$ 24

(3 to 16) (6 to 24)

when $S = \{2 < s \le 24\}$

(3 to 24)

Sol: To verify $\overline{A \cup B} = \bar{A} \cap \bar{B}$ — ①

$\overline{A \cap B} = \bar{A} \cup \bar{B}$ — ②

$\overline{A \cap B} = \{2 < c \le 5,\ 16 < c \le 24\}$

$\bar{A} = S - A = \{16 < a \le 24\}$

$\bar{B} = S - B = \{2 < b \le 5,\ 22 < b \le 24\}$

Then $\bar{A} \cup \bar{B} = \{2 < c \le 5,\ 16 < c \le 24\}$

Demorgan law is verified

$\bar{A} = S - A = (17 \text{ to } 24)$

$\bar{B} = S - B =$ ~~(23, 24)~~ (3, 4, 5 ~~23, 24~~)

$A \cap B =$ ~~3 to 16~~ (6 to 16)

$\overline{A \cap B} =$ (3, 4, 5, 17 to ~~24~~ 24)

Duality principle :- This principle states : if in an identity we replace Unions by intersections, intersections by Unions, S by ϕ and ϕ by S, then the identity is preserved.

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Q verify Duality principle

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

for three sets $A = \{1, 2, 4, 6\}$

$B = \{2, 6, 8, 10\}$

$C = \{3 \leq c \leq 4\}$

Sol :

$$A \cap (B \cup C) = \{2, 4, 6\}$$

$$A \cap B = \{2, 6\}$$

$$A \cap C = \{4\}$$

$$(A \cap B) \cup (A \cap C) = \{2, 4, 6\}$$

$$B \cup C = \{2, 3 \leq c \leq 4, 6, 8, 10\}$$

Important Terms :-

§ out-come :- End result of the Experiment

—: Trial: Single performance of the Expt is called a trial for which there is an outcome.

§ Random Experiment :- An experiment whose outcome is not known in advance

§ Random Event :- Random Event is an outcome or a set of outcomes of a random Experiment

§ occurance :- An Event of a random Experiment is said to have occured, if the experiment terminates in an outcome that belongs to that event.

§ Equally Likely :- A Set of Events is said to be Equally Likely, if one of them cannot be expected to happen in preference to any other.

For Example: In tossing a coin the turning up of the head or tail is Equally Likely

§ Mutually Exclusive :- A Set of Events is said to be mutually Exclusive, if the occurance of one of them does not prevent the occurance of the others. For Example : In tossing a coin, Either the head or tail will turn up.

They cannot occur together or No common

(21)

Exhaustive Events :- A set of events is said to be exhaustive, if its includes all the possible outcomes. In tossing a coin, either the head or tail turn up. There is no other possibility and therefore these are the exhaustive events.

§ Sample space :- (S or U)

A Sample Space is a Set 'S', that consists of all possible outcomes of a random Experiment.

Eg : Consider the Experiment of rolling a single (unbiased) fair die. The result of the experiment is observing the number that shows up. This experiment may show any number from one through six. The set of all possible outcomes { 1, 2, 3, 4, 5, 6 } is called the Sample space, S or probability space.

The Set of likelihood of particular outcome, for Eg { 3 } is called an Event.

Sample Space → S : { 1, 2, 3, 4, 5, 6 }

Event : { 3 }

: { 6 }

: { 2, 6 }

Note : An Event is the Subset of the Sample space.

Discrete Sample space :-

In die-tossing experiment, S is a finite set of six elements, such sample spaces are said to be <u>discrete</u> and <u>finite</u>.

Continuous Sample space :- (uncountably Infinite)

Eg: obtain a number by spinning a pointer on a wheel of chance numbered from 0 to 12. Here any number s from 0 to 12 can result and $S = \{0 \leq s \leq 12\}$

Such a Sample space is called continuous sample space.

Definitions of probability :-

There are several definitions available to understand probability. Among them, two of the mostly used definitions are

1. Relative frequency definition
2. classical definition or axiomatic definition

Relative frequency definition :-

Suppose an experiment is repeated 'n' times, and one of the possible outcome 'A', occurs 'n_A' times. Now, the relative frequency of occurance of A is $\frac{n_A}{n}$. This ratio $\left(\frac{n_A}{n}\right)$ is not predictable, if n is very large [unless 'n' is very large]

We know from experience, that when an experiment is repeated too many times, the relative frequency of a particular outcome approaches a fixed limit. The limiting value of relative frequency of occurance is called probability of outcome A. i.e.

$$P(A) = \lim_{n \to \infty} \left(\frac{n_A}{n}\right)$$

This probability, determined as the limit of the relative frequency of occurance is known as

Example :- In rolling the die, let the outcome 'A' corresponds to the appearance of '3' on the die. In **four** tosses, the number 3 may not appear at all, or it may appear all the four times, or any number of times in between. Thus

$\boxed{\frac{0,1,2,3,4}{4}}$

with $n = 4$

$$\frac{n_A}{n} = 0, 1/4, 2/4, 3/4, (4/4 = 1)$$

Classical definition or Axiomatic definition :-

The probability of an Event 'A' consisting of n_A outcomes equals the ratio $\left(\frac{N_A}{N}\right)$, provided the outcomes are equally likely to occur. i.e

$$p(A) = \frac{\text{Number of possible favourable outcomes, } (N_A)}{\text{Total number of possible equally likely outcomes, } (N)}$$

for Example :- Assume that bag contains 9 balls out of which 5 are red and 4 are white. then, in a single draw, the probability to select a white ball is 4/9. This is arrived because out of 9 possible outcomes of the experiment. i.e any one of 9 balls may be drawn, four are favourable to ~~our~~ our interest. Thus, the probability of the event is determined without conducting the event. i.e prior to the Event, and is also known as ~~Apro~~ Apriori probability.

§ AXIOMS OF PROBABILITY :-

The probability of event A, denoted by P(A), is so chosen as to satisfy the following "Three" axioms :

i) $P(A) \geq 0$, non-negative, zero or

$0 \leq P(A) \leq 1$

if $P(A) = 0$, Impossible event

ii) $P(S) = 1$

Event is certain

iii) $$P\left(\bigcup_{n=1}^{N} A_n\right) = \sum_{n=1}^{N} P(A_n)$$

Some Usefull Laws of probability :-

1) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

if A and B are mutually Exclusive
[No common Elements]

$P(A \cap B) = \phi = \{ \}$

Then $\boxed{P(A \cup B) = P(A) + P(B)}$

2) if $A \cup A' = S$ and $A \cap A' = \phi$, then $\bar{A}$ is called the complement of A. i.e

$\boxed{P(\bar{A}) = 1 - P(A)}$

$\bar{A} = S - A$

$P(\bar{A}) = P(S) - P(A)$

$\bar{A} = S - A$

$P(\bar{A}) = P(S) - P(A)$

$P(\bar{A}) = 1 - P(A)$

§ **Joint Probability**: (Compound probability or law of addition of probability or Addition Theorem)

Consider an Experiment 'A' whose outcomes are $\{A_1, A_2 --- A_n\}$ and Experiment 'B' whose outcomes are $\{B_1, B_2 --- B_m\}$. If the two Experiments A and B have some common elements, they are not mutually exclusive, i.e. those elements correspond to the simultaneous or joint occurrence of the Experiments A and B.

The probability $P(A \cap B)$ or sometimes simply $P(AB)$ denotes the probability of the simultaneous occurrence of the Events A and B. This is called Joint probability or compound probability of the two Events.

$$P(A \cap B) = P(A) + P(B) - P(A \cup B)$$

It is also called as Law of addition of probability.

It may also be written that

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$\leq P(A) + P(B)$$

If the Events "A" and 'B' are mutually Exclusive

Q: One card is drawn from the deck of 52 cards. What is the probability of the card being either red or a king?
(Nov 2007 Reg JNTU-Set-1)

Solution:- Let A represent an event that the card is red, and B represents that the card is king. For a card to be either red or a king, it is required to find A+B

$P(A) = 26/52 = 1/2$

$P(B) = 4/52 = 1/13$

$P(A \cap B) = 2/52 = 1/26$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= 1/2 + 1/13 - 1/26$

$= \dfrac{13+2-1}{26} = 14/26$

$= 7/13$

$\frac{28}{52} = \frac{7}{13}$

Red - 26
Black - 26

King - 4
Red 2
Black 2

7/13
Right

§ Two dice are rolled. find the probability that the sum of the numbers thrown is 7.

36

$$S = \left\{ \begin{array}{cccccc} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{array} \right\}$$

P { getting a sum of 7 }

$$= \frac{\text{No of times getting a sum of 7}}{\text{Total number of times}}$$

$$= 6/36 = 1/6$$

Hence the required probability $= 1/6$

In a team of communication Engineers, 80% know probability theory, 75% know the information theory, and 70% know the both probability theory and information theory. Calculate the percentage of Engineers who know neither probability theory nor information theory?

Sol :

$$P(A) = 0.8$$

$$P(B) = 0.75$$

$$P(A \cap B) = 0.7$$

probability of not knowing both $= P(A' \cup B')$
$= P(\overline{A \cap B})$

$$P(A' \cup B') = P(\overline{A \cap B})$$
$$= 1 - P(A \cap B)$$
$$= 1 - 0.7$$
$$= \underline{0.3}$$

Thus, the Engineers who know neither probability theory nor information theory is 30%

§ A bag contains 12 balls numbered from 1 to 12. If a ball is taken at random, what is the probability of having a ball with a number which is multiple of either 2 or 3?

$P(A) = \{2, 4, 6, 8, 10, 12\}$: $6/12$

$P(B) = \{3, 6, 9, 12\}$: $4/12$

and $P(A \cap B) = \{6, 12\}$, $2/12$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{1}{2} + \frac{1}{3} - \frac{2}{12}$$

$$= \frac{6+4-2}{12} = 8/12 = 2/3$$

Hence, the required probability $= 2/3$

§ Conditional probability (Multiplication Theorem)

Consider the two Experiments A and B with outcomes $\{A_1, A_2, A_3 \text{---} A_n\}$ (A_i) and $\{B_1, B_2 \text{---} B_m\}$ (B_j) respectively, Some times the probability of occurrance of an Event B_j may depend on the occurrance of a related Event A_i

For Example, Assume that box contains five green colour pens and one red colour pen. Let us take a pen from the box. Then, ~~the probability of~~ without replacing the first pen, ~~getting a red pen~~ if we take another pen, the probability of getting a red pen in the second time depends on the outcome of the first draw. i.e. if we had taken a red pen on the first time itself, there is no more red pen in the box. Thus, we have a situation in which the outcome of the second Experiment is conditional on the outcome of the first Experiment.

The probability of the outcome B_j given that A_i is known is called conditional probability or Transition probability, and it is written as $P(B_j | A_i)$

If N_A represents the number of times an event A happens, N_B represents the number of times event B happens and N_{AB} represents the number of times of joint happening, then

$$P(B|A) = \frac{N_{AB}}{N_A} = \frac{N_{AB}/N}{N_A/N} = \frac{P(AB)}{P(A)}, \quad \text{for } P(A) > 0$$

Similarly $P(A|B) = \frac{N_{AB}}{N_B} = \frac{N_{AB}/N}{N_B/N}$

$$= \frac{P(AB)}{P(B)}, \quad \text{for } P(B) > 0$$

$$\boxed{\begin{aligned} \therefore\ P(B|A) &= \frac{P(AB)}{P(A)} \\ P(A|B) &= \frac{P(AB)}{P(B)} \end{aligned}}$$

this may be simply written as

$$\boxed{\begin{aligned} P(AB) &= P(A) \cdot P(B|A) \\ P(AB) &= P(B) \cdot P(A|B) \end{aligned}}$$

These expressions represent chain rule or theorem of multiplication. This is also called as law of multiplication of probability.

Total probability :-

Given that N mutually exclusive events B_n, $n = 1, 2, 3 \ldots N$, whose union equals to the sample space S. On the same sample space, the probability of any event A, P(A) can be written in terms of conditional probabilities, i.e.

$$P(A) = \sum_{n=1}^{N} P(A|B_n)\, P(B_n)$$

This result is known as the total probability of event A.

or formula for the probability of causes or theorem on probability of causes.

§ BAYES' Theorem :- [Rule of Inverse probability]

There are two forms available to Baye's theorem.

1) Using the conditional probability, we have

$$P(A/B) = \frac{P(AB)}{P(B)}$$

and $$P(B/A) = \frac{P(AB)}{P(A)}$$

Substituting $P(AB) = P(A) \cdot P(B/A)$ and

$P(AB) = P(B) \cdot P(A/B)$ respectively

in the above two equations, we arrive at,

$$\boxed{\begin{aligned} P(A/B) &= \frac{P(A) \cdot P(B/A)}{P(B)} \\ \text{and } P(B/A) &= \frac{P(B) \cdot P(A/B)}{P(A)} \end{aligned}}$$

These relations constitute Baye's Theorem, provided $P(A) \neq 0$ and $P(B) \neq 0$

2) Using the total probability, if $B_1, B_2 \text{---} B_n$ are set of mutually exclusive and Exhaustive sets of Events, then,

$$P(A) = \sum_{n=1}^{N} P(A/B_n) \cdot P(B_n)$$

Brief

$P(B) \cdot P(A/B)$

Independent Events :-

Two Events A and B are said to be statistically independent or mutually independents if the probability of occurrance of one Event is not affected by the occurrance of the other Event. i.e $P(A/B) = P(A)$ and $P(B/A) = P(B)$.

Therefore, mathematically, we can write

$$P(AB) = P(A) \cdot P(B|A)$$

$$= P(A) \cdot P(B)$$

$$\boxed{P(AB) = P(A) \cdot P(B)}$$

NOV - 2005 - Reg - JNTU - Set - 2

(1) In a single throw of two dice, what is the probability of obtaining a sum of at least 10.?

(8M)

∴ At least 10 means 10, 11, and 12

ie (5, 5) (4, 6), (6, 4) (5, 6), (6, 5), (6, 6)

∴ Required probability $= 6/36$

$\Rightarrow 1/6$

① A die is tossed, find the probabilities of the events $A = \{$ odd number shows up $\}$.

$B = \{$ number larger than 3 shows up $\}$, $A \cup B$ and $A \cap B$.

$\{1, 2, 3, 4, 5, 6\}$

Sol:

$P\{A\} = \{1, 3, 5\} = 3/6$

$P(B) = \{4, 5, 6\} = 3/6$

$A \cup B = \{1, 3, 4, 5, 6\}$, $P(A \cup B) = 5/6$

$A \cap B = \{5\}$, $P(A \cap B) = 1/6$

② In a box there are 500 colored balls: 75 black, 150 green, 175 red, 70 white and 30 blue. what are the probabilities of selecting a ball of each color.? (with replacing)

Sol:

$P(B) = 75/500$

$P(G) = 150/500$

$P(R) = 175/500$

$P(W) = 70/500$

$P(B) = 30/500$

Nov - 2007 - Reg. Jan - Set-3.

Q. When two dice are thrown, find the probability of getting the sums of 10 or 11 ?

$10 — P(A) = \{(5,5)\ (4,6)\ (6,4)\} = 3/36$

$11 — P(B) = \{(5,6)\ (6,5)\} = 2/36$

$P(A \cap B) = 0$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= 3/36 + 2/36 = 5/36$

---

at least 10 means 10, or 11 or 12

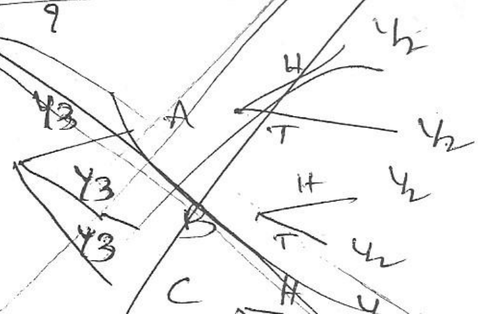
$(5,5), (4,6)\ (6,4)\ (5,6)\ (6,5)\ (6,6)$

$= 6/36 = 1/6$

Ans = 1/8

Q. ~~find the probability of three half - Rupee coins falling all heads up when tossed simultaneously ?~~

$3(\frac{1}{6} \cdot \frac{1}{2}) = \frac{1}{2}$



$\frac{1}{6} + \frac{1}{6} + \frac{1}{6}$

$= 3/6 = 1/2$

$(1/3 \times 1/2) + (1/3 + 1/2)$

8 Nov 2007 Reg-JNTU-Set-2 | Feb 2008, Suppy Set-1 (JNTU)

Nov 2006 Reg-Set-2 (JNTU)

In a box there are 100 resistors having resistance and tolerance as shown table. Let a resistor be selected from the box and assume each resistor has the same likelihood of being chosen. Three events: A as "draw a 47-Ω resistor", B as "draw a resistor with 5% tolerance, and C as draw a 100-Ω resistor. Find the (10 Marks)

1) p(A∩B) 2) P(A∩C) 3) P(B∩C)

4) P(A/B) 5) P(A|C) 6) P(B|C)

or determine joint and conditional probabilities

Resistance (Ω)	Tolerance 5%	10%	Total
22	10	14	24
47	28	16	44
100	24	8	32
Total	62	38	100

Solution 1) ~~p(A∩B)~~

$\frac{28}{100} \times \frac{100}{62}$

$P(A) = \frac{44}{100}$

$P(B) = 62/100$

$P(C) = 32/100$

$P\left(\frac{A}{B}\right) = \frac{P(A\cap B)}{P(B)}$

$\frac{28}{62}$

$P(A|C) = 0$

$P(B|C) = \frac{P(B\cap C)}{P(C)}$

$= 24$

1) $P(A\cap B) = P(47\Omega \cap 5\%) = 28/100$

2) $P(A\cap C) = P(47\Omega \cap 100\Omega) = 0$

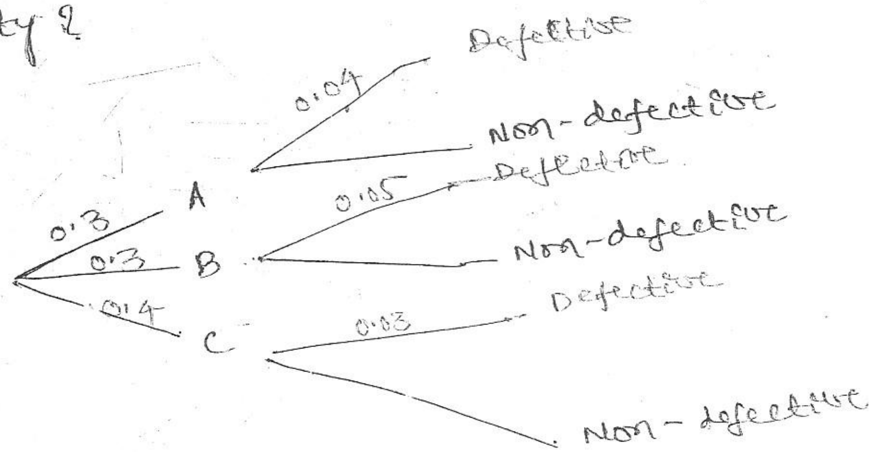
24

(1998 Regular)

§ (a) In a bolt factory machines A, B and C manufacture respectively 80%, 30% and 40% of the total of their output. 4, 5, 3 percents defective bolts. A bolt is drawn at random from the product and is found defective. What are the probabilities that it has manufactured by machines A, B and C.

(b) Explain briefly the axiomatic approach to probability?

Solution :-



The probability that a bolt drawn to be defective is

$$P(X) = 0.3\times 0.04 + 0.3\times 0.05 + 0.4\times 0.03$$

$$= 0.012 + 0.015 + 0.012$$

$$= \underline{0.039}$$

The probability that it was made by machine A is

$$P\left(\frac{A}{X}\right) = \frac{P(A).\,P\left(\frac{X}{A}\right)}{P(X)} = \frac{0.3\times 0.04}{0.039}$$

$$= 0.307$$

The probability that it was made by machine "B" is

$$P\left(\frac{B}{X}\right) = \frac{P(B) \cdot P(X/B)}{P(X)}$$

$$= \frac{0.3 \times 0.05}{0.039} = 0.384$$

The probability that it was made by machine 'C' is

$$P\left(\frac{C}{X}\right) = \frac{P(C) \cdot P(X/C)}{P(X)}$$

$$= \frac{0.4 \times 0.03}{0.039}$$

$$= 0.307$$

NOV - 2007 Reg - JNTU - Set-4
NOV - 2006 - Reg - JNTU - Set-1 & Set-3

1. find the probability of three half - Rupees coins falling all heads up when tossed simultaneously. (6M)

$2^3 = 8$ Sample points

Sol:

Sample space : { HHH, HHT, HTH, THH, HTT, THT, TTH, TTT }

∴ 1/8 Ans

(NOV - 2007 - Reg - JNTU - Set-3)

2. when two dice are thrown, find the probability of getting the sums of 10 or 11 (8M) ~~(A∪B)~~

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

P(A): getting sum = 10, i.e (5,5), (4,6) (6,4)

P(B): getting sum = 11, i.e (5,6) (6,5)

$P(A \cap B)$: 10 & 11

$$P(A) = \frac{3}{36}$$

$$P(B) = \frac{2}{36}$$

$$P(A \cap B) = 0$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{3}{36} + \frac{2}{36} - 0 = \frac{5}{36}$$

§ Mathematical Model of Experiments :-

The real experiment is defined mathematically by three things :

1. Assignment of Sample space
2. definition of Events of interest.
3. Making probability Assignments to Events, such that the axioms are satisfied.

Eg on Mathematical Model of Experiment-

Eg: Feb 2008 supply JNTU - Set-3

Two fair dice are tossed simultaneously. Let x and y denote the numbers on the first and second die respectively. find

P(A) ① $P(x+y=7)$: ~~6/36~~ $6 \times \frac{1}{36} = 1/6$

P(B) ② $P(8 < \text{sum} \leq 11)$: $9/36 = 1/4$

P(C) ③ $P(10 < \text{sum})$: $3/36 = 1/12$

$\downarrow \{(6,5), (5,6), (6,6)\}$

Exercise : $P(B \cap C) = 2/36$

$P(B \cup C) = 10/36 = 5/18$

Prob: When the dice are thrown; determine the probabilities from axiom 3 for the following Three Events.

① $A = \{\text{sum} = 7\}$

② $B = \{8 < \text{sum} \leq 11\}$

③ $C = \{10 < \text{sum}\}$ and determine

④ $P(B \cap C)$

⑤ $P(B \cup C)$

Demonstrate Duality principle

A { 1,2,4,6 }

B { 2,6,8,10 }

C { $3 \leq c \leq 4$)

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

{2,4,6}

$A \cap B = \{2,6\}$

$(A \cap C) = \{4\}$

both are same identity

or $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

① In a box there are 80 resistors each having the same size and shape. Of the 80 resistors 18 are $10\,\Omega$, 12 are $22\,\Omega$, 33 are $27\,\Omega$, and 17 are $47\,\Omega$. If the experiment is to randomly draw out one resistor from the box with each one being "Equally Likely" to be drawn, then find the probability of relative frequency of each.

(replaced)

$$P(10\,\Omega) = 18/80$$

$$P(27\,\Omega) = 33/80$$

$$P(22\,\Omega) = 12/80$$

$$P(47\,\Omega) = 17/80$$

21/04/09

62, 63, 64, ~~75~~, ~~77~~, ~~92~~, 80, ~~81~~, LE12 : 2ECE-2

(1) A set A has three elements a_1, a_2 and a_3. Determine all possible subsets of A.

$2^3 = 8$

Sol

$\{a_1, a_2, a_3\}$ — (1)

(ϕ) — (2)

$\{a_1\}$, $\{a_2\}$ $\{a_3\}$ — (5)

$\{a_1, a_2\}$ $\{a_1, a_3\}$, $\{a_2, a_3\}$ — (8)

---

(2) Two sets are given by $A = \{-6, -4, -0.5, 0, 1.6, 8\}$ and $B = \{-0.5, 0, 1, 2, 4\}$ find

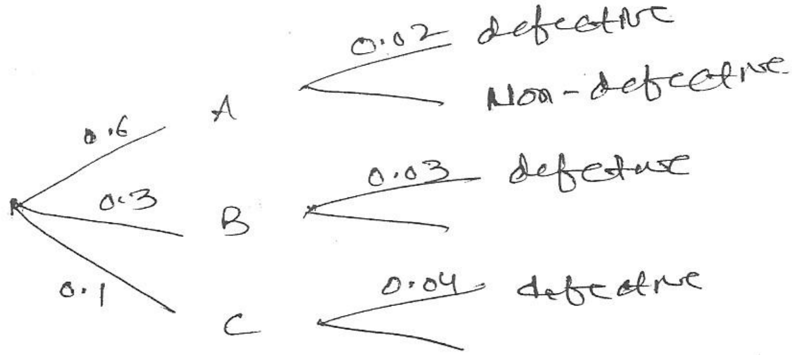
(a) $A - B$ (b) $B - A$ (c) $A \cup B$ (d) $A \cap B$.

① Three Machines A, B and C produce respectively 60%, 30% and ~~40%~~ 10% of the total number of items of a factory. The % age of defective output of these machines are respectively 2%, 3% and 4%.

ⓐ find the probability that the item is defective, when randomly selected (0.025)

ⓑ find the probability that the defective item is from machine C. (0.16)

Sol



ⓐ $P(\text{Defective item}) = (0.6\times0.02)+(0.3\times0.03)+(0.1\times0.04)$

$P(X) = \underline{0.025}$

ⓑ $P(\text{Defective } \cancel{\text{item}} \text{ is from machine } C) = \dfrac{0.1\times0.04}{0.025}$

$P(C|X) = \dfrac{P(C)\cdot P(X|C)}{P(X)}$

$= 4/25$

$= \underline{0.16}$

99 Regular

1a) A box of unmarked IC's contains 200 hex inverters 100 dual 4-input positive AND gates, 50 dual J-K flip-flops, 25 decade counters and 25 four bit shift registers.

i) If an IC is selected at random, what is the probability that it is a dual JK-flip-flop

$50/400 = 1/8$

(ii) What is the probability that an IC selected random is not a hex inverter

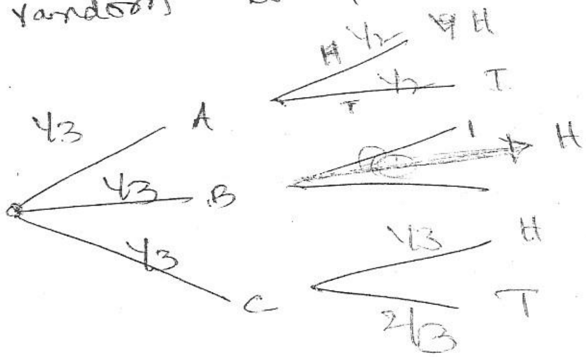
$1 - 1/2 = 1/2$

(iii) If the first IC selected is found to be a 4-bit shift register, what is the probability that the second IC selected will also be a 4-bit shift register

$$\frac{25C_1}{400C_1} \times \frac{24C_1}{399C_1} = \frac{1}{266}.$$

Q A box contains 3 coins. One coin is fair. One coin is two headed and one coin is weighted so that the probability of heads appearing is $1/3$. A coin is selected at random and tossed. Find the probability that head appears.

Solution: One coin can be drawn from 3 coins at random with a probability of $1/3$.



The probability that head appears is

$$\left(\tfrac{1}{3} \times \tfrac{1}{2}\right) + \left(\tfrac{1}{3} \times 1\right) + \left(\tfrac{1}{3} \times \tfrac{1}{3}\right)$$

$$= \tfrac{1}{6} + \tfrac{1}{3} + \tfrac{1}{9}$$

$$= \frac{3 + 6 + 2}{18} = \frac{11}{18}$$

$\frac{1}{6} + \frac{2}{9}$

$\frac{3+4}{18}$

$= \frac{7}{18}$

$1 - \frac{11}{18} = 7/18$ for tail

A1

§ A box contains 6 green balls, 4 black balls, and 10 yellow ~~balls~~ balls. ~~each~~ ~~ball~~ ~~ball~~ All balls are equally likely to be drawn. What is the probability of drawing 2 green balls from the box, if the ball on the first draw is not replaced.

$$\frac{\cancel{6}^{3}}{\cancel{20}_{4}} \times \frac{8}{19} = 3/38$$

$1.5/19$

$= \cancel{0.189}\ 0.789.$

§. prove that $p(A) = 1 - p(A')$

P:- Exercise ① peebles P. 1-5

TUTORIAL – UNIT-1

State every possible subset of the set of letters $\{a, b, c, d\}$

Sol: There are $2^4 = 16$ possible subsets (Events)

They are ϕ,

$\{a\}$

$\{b\}$

(c)

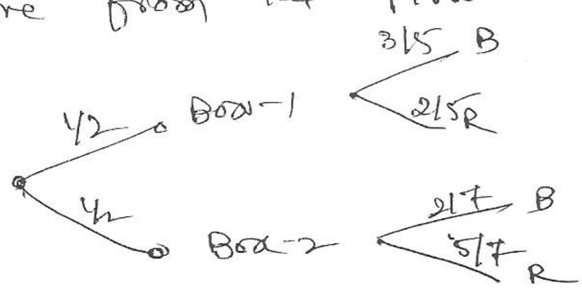
(d)

$(a,b), (a,c), (a,d), (b,c)\ (b,d)(c,d)$

$(a,b,c)\quad (a,b,d)\quad (a,c,d)\quad (b,c,d)$

$= \{a, b, c, d\}$

H-Tech (10) ② A box contains 3 blue and 2 red marbles, while another box contains 2 blue and 5 red marbles. A marble drawn at random from one of the boxes turns out to be ~~blue~~ blue. What is the probability that it came from the first box?



∴ P (selected to be blue) $= \frac{1}{2} \times \frac{3}{5} + 1/2 \times 2/7$

$= 31/70$

∴ P(selected to be from box – 1) $= \dfrac{1/2 \times 3/5}{(31/70)}$

$= \dfrac{21}{31}$ Ans

3. A Bag contains 2 green and 3 black balls. A Sample of Size 4 is made. what is the probability that the Sample is in the order $(B_1 G_2 B_3 G_4)$?

Sol :- Sample of Size 4 means that taking out four balls from the bag one by one (without replacing)

$P(B_1) = 3/5$

$P(G_2) = 2/4$

$P(B_3) = 2/3$

$P(G_4) = 1/2$

$P(B_1 G_2 B_3 G_4) = P(B_1) \cdot P(G_2/B_1) \cdot P(B_3/B_1 G_2) \cdot P(G_4/B_1 G_2 B_3)$

$= \frac{3}{5} \times \frac{2}{4} \times \frac{2}{3} \times 1/2 = 1/10$ Ans

(4) Show that, if two independent events A and B are,

(i) A' and B' are independent

(ii) A' and B are "

(iii) A and B' are "

Sol:

$$P(A \cap B) = P(A) \cdot P(B|A)$$

$$= P(A) \cdot P(B)$$

Now,

$$P(A' \cap B') = P(A \cup B)'$$

$$= 1 - P(A \cup B)$$

$$= 1 - [P(A) + P(B) - \underbrace{P(A \cap B)}_{P(A) \cdot P(B)}]$$

$$= [1 - P(B)] - P(A)[1 - P(B)]$$

$$= [1 - P(B)][1 - P(A)]$$

$$= P(B') \cdot P(A')$$

$$= P(A') \cdot P(B')$$

Hence, A' and B' are two independent event

(ii)

$$P(A' \cap B) = P(B) - P(A \cap B)$$

$$= P(B) - P(A) \cdot P(B)$$

$$= P(B)[1 - P(A)]$$

$$= P(B) \cdot P(A')$$

B and A' are independent

(iii)

$$P(B' \cap A)$$

$$= P(A) - P(A \cap B)$$

$$= P(A) - P(A) \cdot P(B)$$

$$= P(A)[1 - P(B)]$$

$$= P(A) \cdot P(B')$$

A & B' independent

(5) If A, B, and C are mutually independent Events, show that $(A \cup B)$ and C are also independent.

Sol :

$$P[(A \cup B) \cap C] = P(A \cup B) \cdot P(C)$$

$$L.H.S = P[(A \cup B) \cap C]$$

$$= P[(A \cap C) \cup (B \cap C)]$$

we know that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)$$

$$= P(A) \cdot P(C) + P(B) \cdot P(C) - P(A) \cdot P(B) \cdot P(C).$$

$$= P(C)[P(A) + P(B) - \underbrace{P(A) \cdot (B)}_{P(A \cap B)}]$$

$$= P(C)[P(A \cup B)]$$

$$= P(C) \cdot P(A \cup B) = R.H.S$$

$A \cup B$ and C are independent

feb 2008, suppy JNTU - Set - 4

Prob : From a Regular deck of 52 cards, one card is selected. Defining event A as "select a king", B as " select a Jack or queen" and C as " select hearts". verify A, B and C events are independent (pair wise) (8 Marks)

Sol : $P(A)$: probability of selecting a king $= 4/52$

$P(B)$; probability of selecting a Jack or Queen

$$\frac{4}{52} + \frac{4}{52} = 2/13$$

$2 \cdot (4/52)$ 13

$2/13$

$P(C)$: probability of selecting a heart $= 13/52$

NOTE : Two Events A and B are said to be mutually independent if they satisfy the condition

$$P(A \cap B) = P(AB) = P(A) \cdot P(B)$$

$P(A \cap B) = P(AB) =$ probability of selecting a king and Jack or Queen in one card $= 0$

$P(B \cap C) = P(BC) =$ probability of selecting a card, a Jack or queen with heart

$= 1/52 + 1/52 = 2/52$

$P(A \cap C) =$ ~~P(A)~~ $P(AC) =$ probability of selecting a King with heart $= 1/52$

$\therefore \quad P(A \cap B) = 0$

$\therefore \quad P(A) \cdot P(B) = \frac{1}{13} \cdot \frac{2}{13} = \frac{2}{169}$

$\therefore \quad P(A \cap B) \neq P(A) \cdot P(B)$

Hence the events A and B are not independent.

$P(B \cap C) = 2/52$

$P(B) \cdot P(C) = \frac{2}{\cancel{13}} \cdot \frac{\cancel{13}}{52} = 2/52$

$P(B \cap C) = P(B) \cdot P(C)$, satisfied.

Hence the events B and C are independent.

$P(A \cap C) = 1/52$

$P(A)\ P(C) = \frac{1}{\cancel{13}} \cdot \frac{\cancel{13}}{52} = 1/52$

$\therefore \quad P(A \cap C) = P(A) \cdot P(C)$, satisfied.

Hence the events A and C are independent

— x —

prob: A lot of 100 semiconductor chips contains 20 that are defective. Two chips are selected at random, without replacement, from the lot.

(a) what is the probability that the first one selected is defective?

(b) what is the probability that the second one selected is defective given that first one was defective?

(c) what is the probability that both are defective?

Sol: Let A denote the Event that the first one

(a) selected is defective : $P(A) = 20/100$
$= 0.2$

(b) $P(B|A)$; Let B denote the Event that the second one is defective.

$P(B|A) = \cancel{19/100}\ 19/99 = 0.192$

(c) The probability that both are defective is

$$P(A \cap B) = P(AB) = P(A) \cdot P(B|A)$$
$$= \frac{20}{100} \times \frac{19}{99}$$
$$= 0.0384$$

prob: Two cards are drawn at random from a deck. find the probability that both are aces

Sol,

$$p(A) = 4/52$$ (without replacing)

Spades + hearts + clubs + Diamond
13 13 13 (13)

$$p(B|A) = 3/51$$

A - 9, 10, J, Q, K

$$p(A \cap B) = p(AB) = p(A) \cdot p(B|A)$$

$$= \frac{4}{52} \times \frac{3}{51} = 1/221 \text{ Ans}$$

feb 2008 - Supply - set - 4 (JNTU)

prob: Show that two Elements cannot be both mutually Exclusive and statistically independent. what is the condition for two Events to be independent

sol: Let A and B be events in a Sample Spaces.

$$\therefore \; p(A) \neq 0 \quad \text{and} \quad p(B) \neq 0$$

$$p(A) \cdot p(B) \neq 0$$

Let A and B be mutually Exclusive Events,

$$p(A \cap B) = p(\phi) = 0$$

$$p(A \cap B) \neq p(A) \cdot p(B)$$

A and B cannot be independent.

Note: (The condition for independent is : $P(A \cap B) = P(A) \cdot P(B)$)

UNIT - I

1. Determine probabilities of system error and correct system transmission of symbols for an elementary binary communication system shown in Fig consisting of a transmitter that sends one of the two possible symbols (a 1 or a 0) over a channel to a receiver. The channel occasionally causes errors to occur

Let A, B, C are three sets which consist of elements $A = \{1, 2, 5, 7\}$, $B = \{6, 7, 8\}$, $C = \{3, 6, 8, 9\}$ check whether it satisfies distributive law.

A fair die is tossed twice. Find the probability of getting 4, 5 or 6 on the 1st toss and a 1, 2, 3 or 4 on 2nd toss.

4. If two coins are tossed simultaneously what is the probability of getting head and tail simultaneously?

5. A and B throws a pair of dice. A wins if he throws 6 before B throws 7 and B wins if he throws 7 before A throws 6. If A begins then show that his chance of winning is 30/61.

6. A letter is known to have come either from LONDON or CLIFTON, on the postmark, only the consecutive letters 'ON' are legible. What is the chance that it came from LONDON.

7. Find the probability of 4 turning atleast once in two tosses of a fair die.

A pack contains 4 white and 2 green pencils, another pack contains 3 white and 5 green pencils. If one pencil is drawn from each pack, find the probability that (i) both are white (ii) one is white and another is green

In a team of communication engineers, 80% know probability theory, 75% know information theory and 70% know both probability theory and information theory. Calculate the % of engineers who know neither probability theory nor information theory.

10. An experiment consist of rolling a single die. Two events are defined

as $A = \{$

UNIT 1

Part 2

§ RANDOM VARIABLES :— or STOCHASTIC Variables or simply a variate.

INTRODUCTION :— The probability theory gives an idea on the characteristics of outcomes of an Experiment. The outcome of the Experiment may be a descriptive one (drawing a King out of a pack of 52 cards) or a numerical one (getting 4 in the rolling of a die). It will be more convenient to define characteristics of all the outcomes of the Experiment are always numerical values instead of descriptive. Thus Random Variable concept is such an important powerful tool in solving practical probabilistic problems.

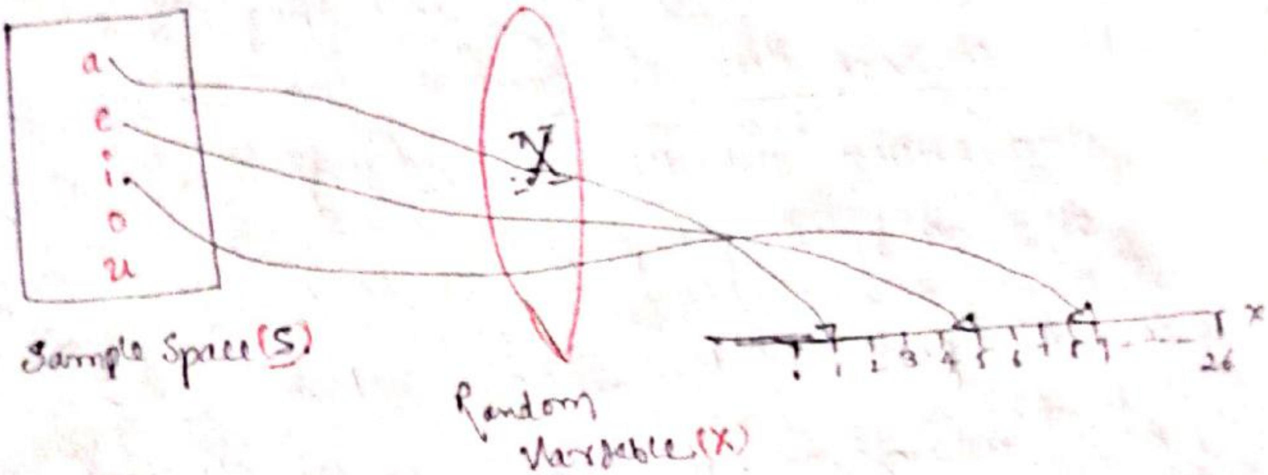
§ Definition :—

"A Random variable may be defined as a rule or functional relationship that assigns real numbers to each possible outcome of a random Experiment".

The random variable is represented by a boldface letter, X or Y etc., and the real numbers X (di) are called the values of random variable. The values of random variable are represented by a lower case letter, x or y --- etc.,

Examples:- Consider an experiment of car race.
In this particular experiment, the desired
outcome is the "name of the person who is
winning the race", All the possible outcomes
constitute the sample
of the experiment
space, S, and each and every outcome
may be written by d_i. In the car
race experiment, d_i represent the names of the
persons driving the cars. If a real number
$X(d_i)$ is given to each possible outcome,
then this representation is called a random variable.

Example :- Consider a sample space, S ~~...~~ consisting
of the alphabets "a, e, i, o, and u". we take a
random variable X in such a way that
each outcome (alphabet) corresponds to a
real number. The real number may be assigned
so as to represent the corresponding number of the
particular alphabet. For example, the letter
"a" is the first letter in English letters and
hence the real number to be assigned is "1".
The letter "e" is assigned to number "5",
"i" to "9", "o" to "15" and "u" to 21.



Thus, the random variable X can be considered to a function that maps all the elements of the space into points on the real line or some parts thereof.

a = 1
e = 5
i = 9
o = 15
u = 21

§ Conditions for a function to Be a Random Variables

The following are the necessary conditions to be satisfied by a random variable.

1) It is not necessary that the sample space points map uniquely. i.e. more than one outcome may map to a single real value.

i.e $X(d_1) = X(d_2) = -2$

But the random variable must not be multi-valued. i.e. every point in the sample space must correspond to only one real value

$X(d_1) = 1$
$X(d_1) = 2$ ×

2) The set $\{X \leq x\}$ shall be an Event for any real number x. i.e. the probability of this event, denoted by $P\{X \leq x\}$ is equal to the sum of the probabilities of the all the elementary events corresponding to $\{X \leq x\}$.

3) $P\{X = -\infty\} = 0$ and $P\{X = \infty\} = 1$

§ TYPES OF Random Variables :-

The Random variables may be a continuous, a discrete or mixed.

1) **Discrete Random Variable :-** If a Random variable assumes only a finite numbers of distinct values, then the random variable is called discrete Random variable. The Sample Space for the discrete random variable can be discrete, continuous or a mixture of discrete and continuous.

Eg:- Take an Example of tossing a coin, we know the outcomes of head or tail.

2) **Continuous Random Variable** or Non-discrete **:-** If a random variable assumes non countably (infinite) number of values, it is called continuous random variable.

Eg: Sound intensity levels of human beings, they can occupy a range of values but not definite number of values hence outcome is a continuous random variable.

§ If predictability of the Event outcome is not possible, then the output may take any value and it is called as continuous random variable

3) **Mixed Random Variable :-** is one for which some of its values are discrete and some are continuous. This type of random variable is not very common.

§ Discrete Random Variable –

— probability Distribution function
or (Cummulative Distribution function)

The probability distribution function associated with a random variable X is defined as the probability that the outcome of an Experiment will be one of the outcomes for which $X(\lambda) \leq x$, where x is any real number.

The probability $P\{X \leq x\}$ is the probability of the Event $\{X \leq x\}$. It is a number that depends on x. that is, it is a function of x, we call this function, denoted $F_X(x)$, the cummulative distribution function of the random variable X, is

$$\underset{\text{argument } (-\infty \text{ to } +\infty)}{F_X(x)} = P\{X \leq x\} \quad -\infty \leq x \leq \infty$$

$$\text{where } P(X \leq x) = \sum_i p_i = \sum_{i=1}^{N} P(X = x_i)$$

Example :- The feature of probability distribution function can become clear, if we consider the Example of throwing of two dice. There are 36 possible outcomes. Since Each die may show any number from 1 through 6. The sample space is written as

CDF or Distribution function

experiment, any real number from $-\infty$ to ∞

$$F_X(x) = P\{X \le x\}, \quad -\infty < x < \infty$$

Eg: Consider the random experiment of a tossing a dice (let X be the random variable)

Thus, the probability distribution of X is

$X = x_i$	1	2	3	4	5	6
$P(X = x_i)$	1/6	1/6	1/6	1/6	1/6	1/6

The CDF of this random variable is

$F_X(1) = P(X \le 1) = 1/6 = P(X=1)$

$F_X(2) = P(X \le 2) = P(X=1) + P(X=2)$
$= 1/6 + 1/6 = 2/6$

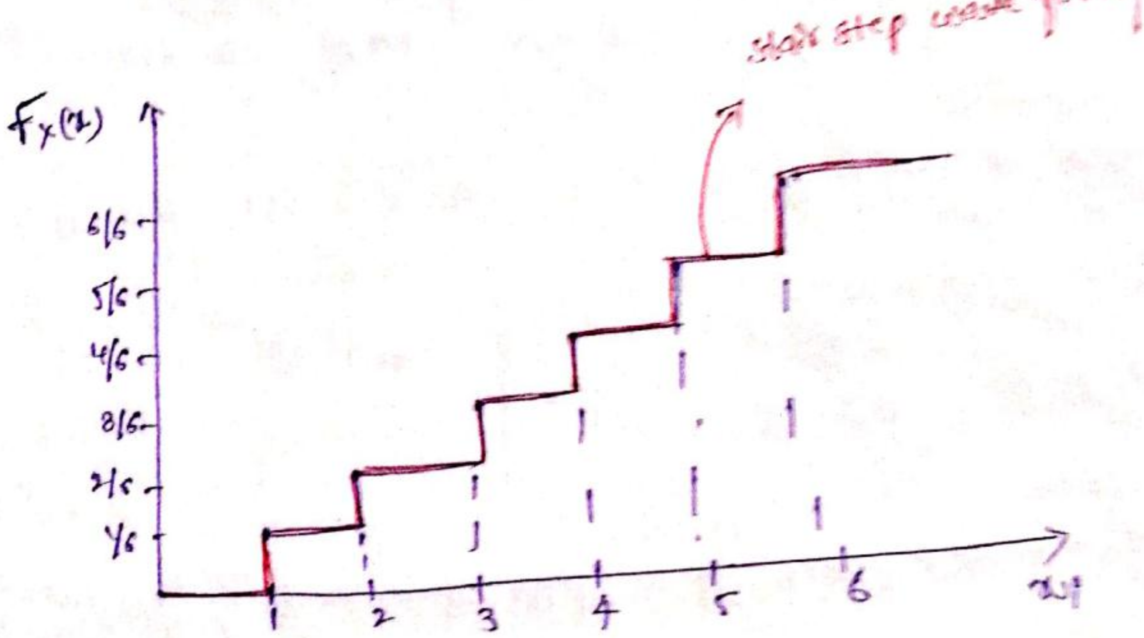
$F_X(3) = P(X \le 3) = P(X=1) + P(X=2) + P(X=3) = 3/6$

$F_X(4) = P(X \le 4) = \text{- - - - - -} = 4/6$

$F_X(5) = P(X \le 5) = \text{- - - - - -} = 5/6$

$F_X(6) = P(X \le 6) = \text{- - - - - -} = 6/6 = 1$

The above CDF is plotted for different values of x_i taken by X as.



Thus, above plot is a Stairstep form. The above random variable is discrete random variable, and for any discrete random variable, the CDF plot is a stair step (staircase) function.

Thus, CDF of a discrete random variable can be expressed as

$$F_X(x) = \sum_{i=1}^{N} P(X = x_i) \cdot u(x - x_i)$$

or

$$\sum_{i=1}^{N} P(x_i) \cdot u \cdot (x - x_i)$$

where $P(x_i)$ is the probability of $X = x_i$

and $u(x-x_i)$ is the unit step function

defined from $x = x_i$.

where $u(\cdot)$ is the unit step function defined by

$$u(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

finally, $F_X(x) = \sum_{i=1}^{N} P(x_i) . u(x - x_i)$

NOTE: 1. The CDF plot of a continuous random variable will be a continuous function.

2. The CDF plot of a mixed random variable is discontinuous, but not a staircase function.

Properties of Distribution Function:-

The following are the some of the important properties of the probability distribution functions.

1) $F_X(-\infty) = 0$ $\quad (-\infty \le x \le \infty)$ 0 to 1

2) $F_X(+\infty) = 1$ $\quad F_X(-\infty) = P(X \le -\infty)$

$P(X \le -\infty) = 0$

3) $0 \le F_X(x) \le 1$ $\quad F_X(+\infty) = P(X \le \infty)$

$= P(S)$

$= 1$

4) $F_X(x_1) \le F_X(x_2)$, if $x_1 < x_2$

Thus, this property states that $F_X(x)$ is a non-decreasing function of x

5) $P\{x_1 < X \le x_2\} = F_X(x_2) - F_X(x_1)$

6) $F_X(x^+) = F_X(x)$

This property states that $F_X(x)$ is a function continuous from right

Note: properties 1, 2, 4 and 6 may be used as tests to determine, if some function, say $G_X(x)$, could be a valid distribution function

§ probability density function (PDF) or frequency function
or probability mass function

The probability density function, $f_X(x)$ is defined as the derivative of the probability distribution function, $F_X(x)$ of the random variable X. It is called as "frequency function".
The probability density function of a discrete random variable is often known as "probability mass function."

$$\boxed{f_X(x) = \frac{d}{dx}[F_X(x)]}$$

§ probability density function : (pdf)
(or frequency function or prob mass function)

$$f_X(x) = \frac{d}{dx}\left[F_X(x)\right]$$

i.e change in CDF is named as pdf

Illustration:

consider the CDF plot

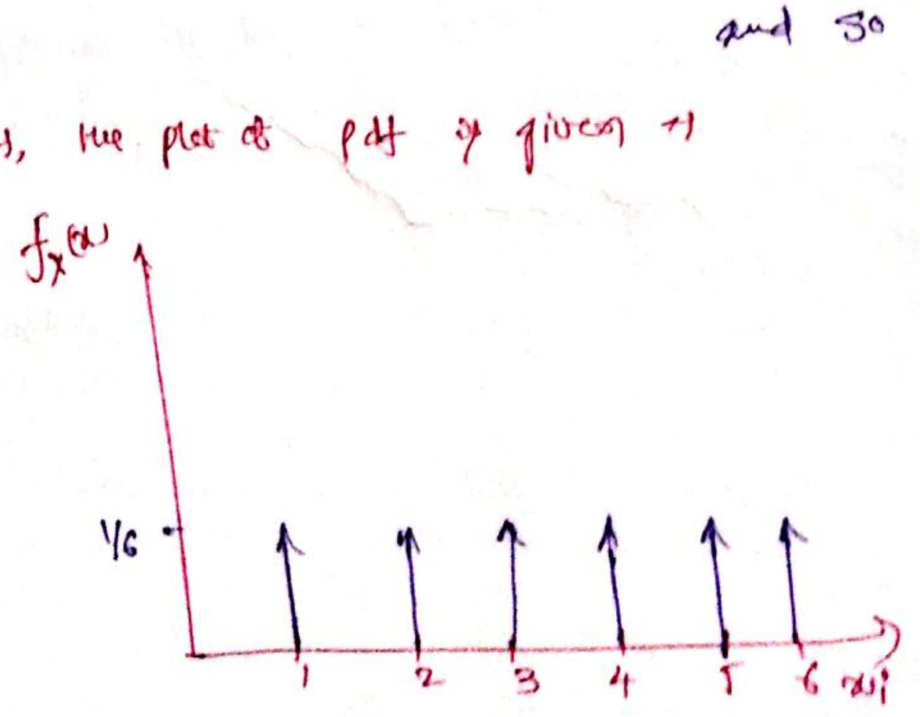
from $x_i = -\infty$ to $x_i = 0$, there is no change in CDF, hence pdf = 0

At $x_i = 1$, the CDF is changed by 1/6, the pdf at that r.v at $x_i = 1$ is 1/6

from $x_i = 1$ to $x_i = 2$, No change in CDF, and hence pdf = 0

At $x_i = 2$ CDF changing from 1/6 to 2/6 i.e by 1/6, and so pdf = 1/6

Thus, the plot of pdf is given as



Thus, the pdf of d.r.v is an impulse train.

Thus, pdf of a d.r.v can be expressed as

$$f_X(x) = \frac{d}{dx}(F_X(x)) = \frac{d}{dx}\left[\sum_{i=1}^{N} P(x_i) \cdot u(x-x_i)\right]$$

$$= \sum_{i=1}^{N} P(x_i) \cdot \delta(x-x_i)$$

$$\left[\because \frac{d}{dx}[u(x)] = \delta(x)\right.$$

Note: ① For a continuous random variable the pdf is also a continuous function, provided the derivative exists. (step-type discontinuity)

Imp. Note : for a discrete random variable, the probability and probability density function are found to be the same.

Scanned with CamSca

§ properties of density function :-

The following are the some of the important properties of the probability density functions.

1) $f_X(x) \geq 0$, for all x

2) $\int_{-\infty}^{\infty} f_X(x)dx = 1$

∴ if $f_X(x)$ is integrated, we get distribution function $F_X(x)$.

$$\int_{-\infty}^{\infty} f_X(x) = [F_X(x)]_{-\infty}^{\infty}$$

$$= F_X(\infty) - F_X(-\infty)$$

$$= 1 - 0$$

$$= 1.$$

The above properties can be used as tests to check whether the given function is a valid probability density function. Both tests must be satisfied for validity.

3) $F_X(x) = \int_{-\infty}^{\infty} f_X(x)\,dx$ $\quad \int_{-\infty}^{x} f_X(x)\,dx$

4) $P(x_1 < X \leq x_2) = \int_{x_1}^{x_2} f_X(x)\,dx$

$= [F_X(x)]_{x_1}^{x_2}$

$= F_X(x_2) - F_X(x_1)$

5) If X is a continuous random variable, $P(X=x)=0$ for all x

55

A Random variable X has the following probability function:

Values of X: x :	0	1	2	3	4	5	6
$p(x)$:	k	3k	5k	7k	9k	11k	13k

(i) find k (ii) Evaluate $P(X<4)$, $P(X\geq5)$ and $P(3<X\leq6)$

(iii) what is the smallest value of x for which $P(X\leq x)>1/2$?

(iv) " " " " k so that $P(X\leq2)>0.3$

Sol: from the axioms of probability $p(S)=1$

$$\sum_{x=0}^{6} p(x) = 1$$

$$p(0)+p(1)+p(2)+p(3)+p(4)+p(5)+p(6)=1$$

$$k+3k+5k+7k+9k+11k+13k=1$$

$$49k=1$$

$$k=1/49$$

ii) $P(X<4) = p(0)+p(1)+p(2)+p(3)$

$$= \frac{1}{49}+\frac{3}{49}+\frac{5}{49}+\frac{7}{49}$$

$$= 16/49$$

$$P(X\geq5) = P(X=5)+P(X=6)$$

$$= \frac{11}{49}+\frac{13}{49} = 24/49$$

Scanned with CamSca

$P(3 < X \leq 6) = P(X=4) + P(X=5) + P(X=6)$

$= \frac{9}{49} + \frac{11}{49} + \frac{13}{49} = \frac{33}{49}$

iii) The minimum value of x may be determined by trial method, i.e. substituting various values of x and finding the corresponding distributions ?

$P(X \leq 0) = 1/49$

$P(X \leq 1) = P(X=0) + P(X=1) = 1/49 + 3/49 = 4/49$

$P(X \leq 2) = P(X=0) + P(X=1) + P(X=2)$

$= 1/49 + 3/49 + 5/49 = 9/49$

$P(X \leq 3) = \frac{9}{49} + \frac{7}{49} = 16/49$

(0.51 > 1/2)

$P(X \leq 4) = \frac{16}{49} + \frac{9}{49} = \frac{25 \times 10}{49 \times 10} > 1/2$

~~(0.51 > 1/2)~~

$P(X \leq 5) = \frac{25}{49} + \frac{11}{49} = \frac{36}{49}$ (0.73)

$\therefore$ The smallest value of x for which

~~$P(3 < X \leq 6)$~~

$P(X \leq x) > 1/2$ is $\underline{4}$ Ans

(iv) $P(X \leq 2) > 0.3$

$\Rightarrow 9K > 0.3$

$K > \frac{0.3}{9}$

or $K > 1/30$

Scanned with CamSca

Nov-2004-Reg-Set-3 UNIT-2

① A random variable X has the discrete random variable in the set $\{-1, -0.5, 0.7, 1.5, 3\}$, the corresponding probabilities are assumed to be $\{0.1, 0.2, 0.1, 0.4, 0.2\}$. plot its distribution function and state is it a discrete or continuous distribution function. — (4M)

Sol

$F_X(x) = P\{X \leq x\}$

$F_X(-1) = P\{X < -1\} = 0$, because there are no sample space points in the set $(X < -1)$

only $X = -1$, we obtain the outcome. $\{ P(X \leq -1\} = P\{X = -1\} = 0.1$

$F_X(-1) = 0.1$

$F_X(-0.5) = P\{X \leq -0.5\}$

$= P(X = -1) + (P X = -0.5)$

$= 0.1 + 0.2$

$= 0.3$

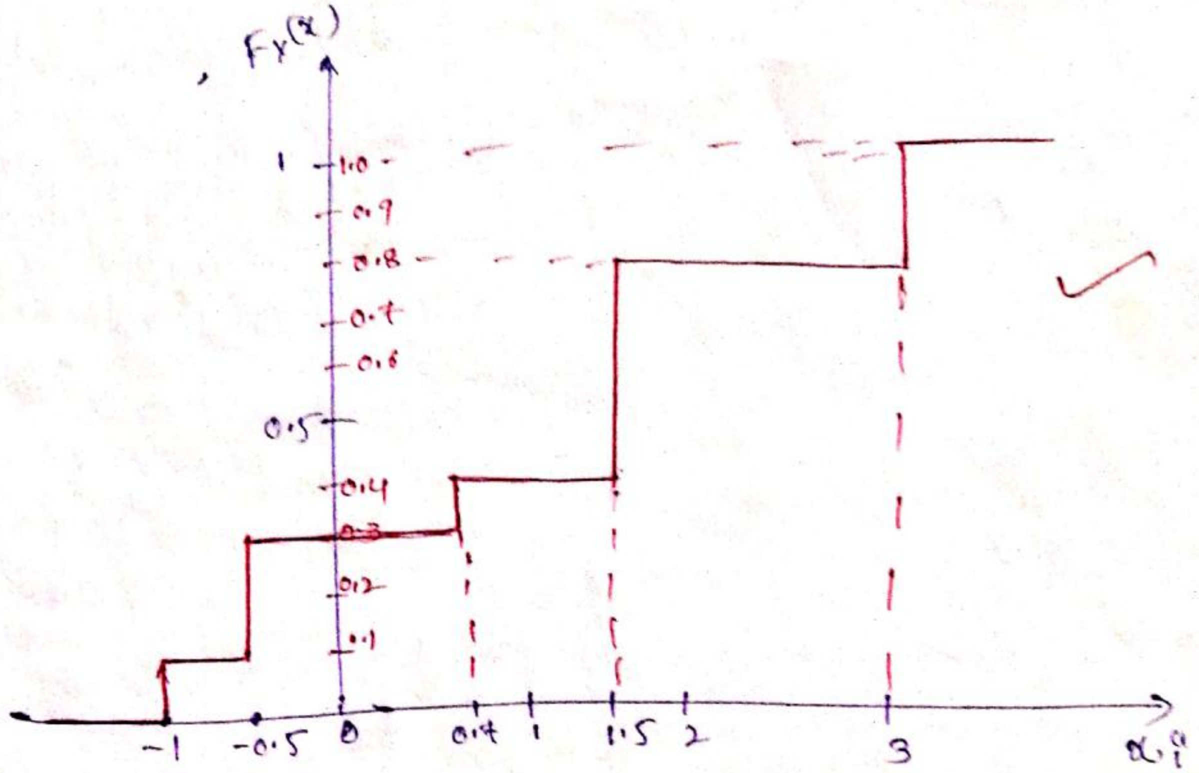
$F_X(0.7) = 0.4$

$F_X(1.5) = 0.8$

$F_X(3) = 1.0$

Scanned with CamSca

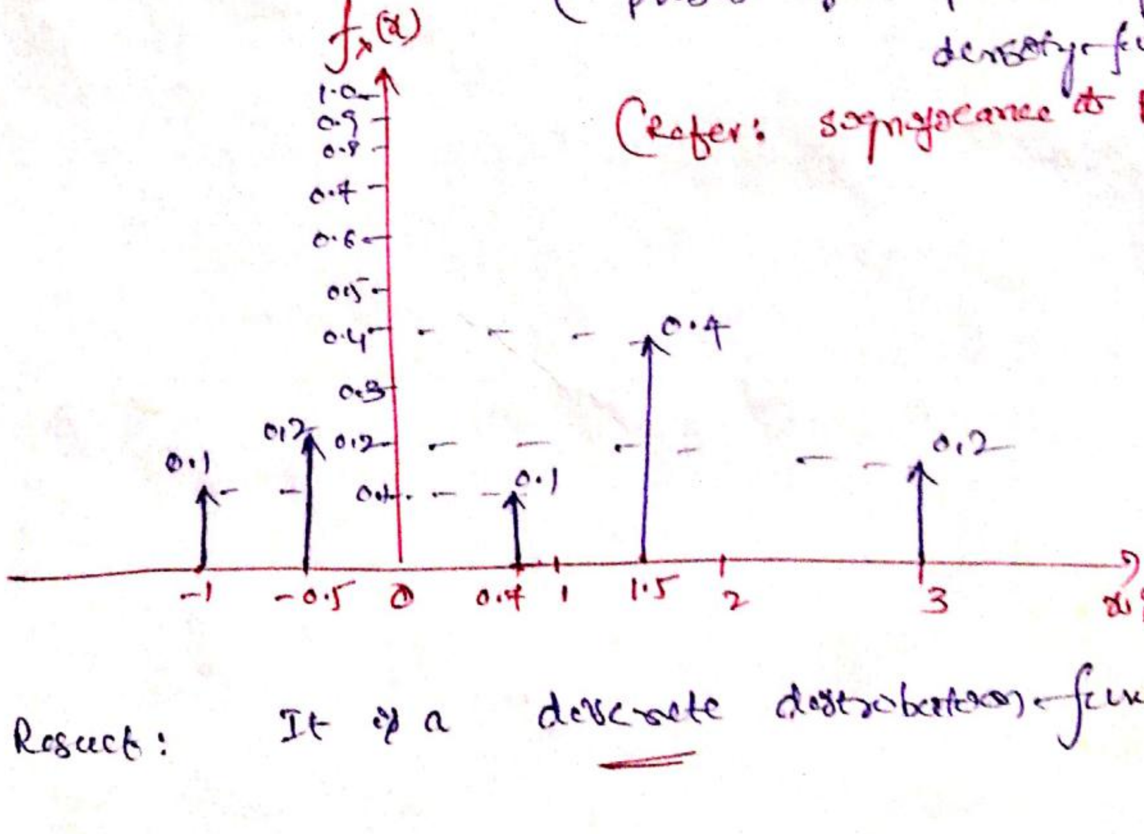
Distribution function : of discrete Random Variable



density function : of a discrete random Variable

(probability = probability density function)

(Refer: significance of pdf)



Result : It is a discrete distribution function

UNIT 1

Part 3

Swiss mathematician (1654–1705)

BERNOULLI TRIALS

$^nP_r = \frac{n!}{(n-r)!}$

$\binom{n}{r} = {^nC_r} = \frac{n!}{r!\,(n-r)!}$

Consider an Experiment for which there are only two possible outcomes on any trial.

Example : flipping a coin, hitting or missing the target in artillery, passing or failing an exam, receiving a 0 or 1 in a computer bit stream or winning or losing in a game of chance

for this type of Experiment,

Let A be the Elementary Event having one of the two possible outcomes as its element.

$\bar{A}$ is the only other possible Elementary Event.

specifically, we shall repeat the basic Experiment N times, and determine the probability that A is observed exactly K times out of the N trials.

Such repeated Experiments are called Bernoulli trials.

Assume that Events are statistically independent.

Scanned with CamScanner

Let Event A occur on any given trial with

probability $P(A) = p$ — ①

and $P(\bar{A}) = 1-p$ — ②

After N trials of the Experiment, one particular sequence of outcomes has A occurring K times, followed by $\bar{A}$ occurring $(N-K)$ times. Because of statistical independent of trials, the probability of this one sequence is

$$\underbrace{P(A) \cdot P(A) \cdots P(A) \cdots}_{\substack{K \text{ times} \\ K \text{ terms}}} \overset{P(A)}{} \underbrace{P(\bar{A}) \cdot P(\bar{A}) \cdot P(\bar{A}) \cdots P(\bar{A})}_{(N-K) \text{ terms}} = p^K \cdot (1-p)^{N-K} \quad — ③$$

Now, we only need to find the number of such sequences. Some thought will reveal that this is the number of ways of taking K objects at a time from N objects. The number is known to be the Binomial co-efficient.

(combinations) $\binom{N}{K} = \dfrac{N!}{K!\,(N-K)!}$

$$P\{A \text{ occurs Exactly } K \text{ times}\} = \binom{N}{K} p^K \cdot (1-p)^{N-K}$$

§ Some other probability distribution and Density functions:-

1. Binomial
2. poisson
3. Uniform
4. Gaussian → Exponential
5. Rayleigh

Binomial and poisson } for discrete random variables

Uniform, Gaussian, Rayleigh } continuous random variables

-

(1) **BINOMIAL :-** (after discussion of Bernoulli trials).

According to Bernoulli trials

$$P\{A \text{ occurs exactly } k \text{ times}\} = \binom{N}{k} p^{k}(1-p)^{N-k}.$$

Let $0<p<1$ and $N = 1, 2, \ldots$ then the

function $$f_x(x) = \sum_{k=0}^{N} \binom{N}{k} p^{k}(1-p)^{N-k} \cdot \delta(x-k) \quad \text{——(1)}$$

is called the binomial density function. The quantity $\binom{N}{k}$ is the binomial coefficient.

$$\binom{N}{k} = \frac{N!}{k!(N-k)!}$$

$^{n}C_{k}$

Scanned with CamScanner

The binomial density can be applied to Bernoulli trial of Experiment. also

① It applies to many games of chance

② Detection problems in radar and sonar

③ and many experiments having only two possible outcomes on any given trial.

④ also used in the study of communication systems.

By Integration of Eq ①, the binomial distribution function is found:

$$F_X(x) = \sum_{k=0}^{N} \binom{N}{k} p^k (1-p)^{N-k} \cdot u(x-k) \quad -②$$

(2) POISSON:

The poisson random variable X has a density and distribution given by.

$$f_X(x) = e^{-b} \sum_{k=0}^{\infty} \frac{b^k}{k!} \delta(x-k)$$

and $$F_X(x) = e^{-b} \sum_{k=0}^{\infty} \frac{b^k}{k!} u(x-k)$$

where $b>0$, is a real constant.

$N \to \infty$, and $p \to 0$, for the binomial case in such a way that $Np = b$, a constant, the poisson case results.

Applications:

(1) poisson R.V. applies to a wide variety of counting-type applications.

(2) used for, No of telephone calls made during a period of time.

If the time interval of interest has duration T, and the events being counted (occur) at an average rate of λ, then $\boxed{b = \lambda T}$ in the above Eq

Scanned with CamScanner

(8) Uniform :- or (Rectangular Distribution)

The Uniform probability density and distribution functions are defined by

Density → $f_X(x) = \begin{cases} 1/(b-a), & a \le x \le b \\ 0, & \text{elsewhere} \end{cases}$

Distribution → $F_X(x) = \begin{cases} 0 & x < a \\ (x-a)/(b-a), & a \le x < b \\ 1 & b \le x \end{cases}$

Applications : (1) Quantization of signal samples prior to Encoding in digital communication systems.

EXPONENTIAL :-

The exponential density and distribution functions are :

$$\text{density} \rightarrow f_X(x) = \begin{cases} \frac{1}{b} e^{-(x-a)/b}, & x > a \\ 0, & x < a \end{cases}$$

$$\text{Distribution} : F_X(x) = \begin{cases} 1 - e^{-(x-a)/b}, & x > a \\ 0 & x < a \end{cases}$$

Applications: ① The exponential density is useful in describing raindrop sizes when a ~~large~~ large number of rainstorm measurements are made.

RAYLEIGH :-

The Rayleigh density and distribution functions are:

density

$$f_X(x) = \begin{cases} \frac{2}{b}(x-a) \cdot e^{-(x-a)^2/b}, & x \geq a \\ 0, & x < a \end{cases}$$

distribution

$$F_X(x) = \begin{cases} 1 - e^{-(x-a)^2/b}, & x \geq a \\ 0, & x < a \end{cases}$$

Applications: ① The Rayleigh density describes the envelope of one type of noise when passed through a bandpass filter.

② In analysis of error in various measurement systems.

(German mathematician) Gaussian density or Normal density 17

GAUSSIAN :-

A random variable X is called gaussian, if its density function has the form

$$\boxed{f_x(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} \cdot e^{-(x-a_x)^2/2\sigma_x^2}}$$

or

where

$$f_x(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-(x-m)^2/2\sigma^2}$$

where m = mean of the random variable
σ^2 = Variance of the random variable

The Gaussian / Normal ~~distribution~~ density is denoted as $N(m, \sigma^2)$

$$f_x(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2} \quad (m=0)$$

is also called density of gaussian random variable with zero mean and variance σ^2, denoted as $N(0, \sigma^2)$

$$f_x(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$, is the density of Gaussian random variable with zero mean and unity variance, denoted by $N(0,1)$. This is referred as "Standardized normal density".

Consider $$f_x = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-m)^2/2\sigma^2}$$

Solve the eqn $\frac{d}{dx}\left(f_x(x)\right) = 0$

$$\frac{d}{dx}\left[\frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-(x-m)^2/2\sigma^2}\right] = 0$$

$$\Rightarrow \frac{1}{\sqrt{2\pi\sigma^2}}\left[-e^{-(x-m)^2/2\sigma^2} \cdot \frac{2(x-m)}{2\sigma^2}\right] = 0$$

$$\Rightarrow x - m = 0$$

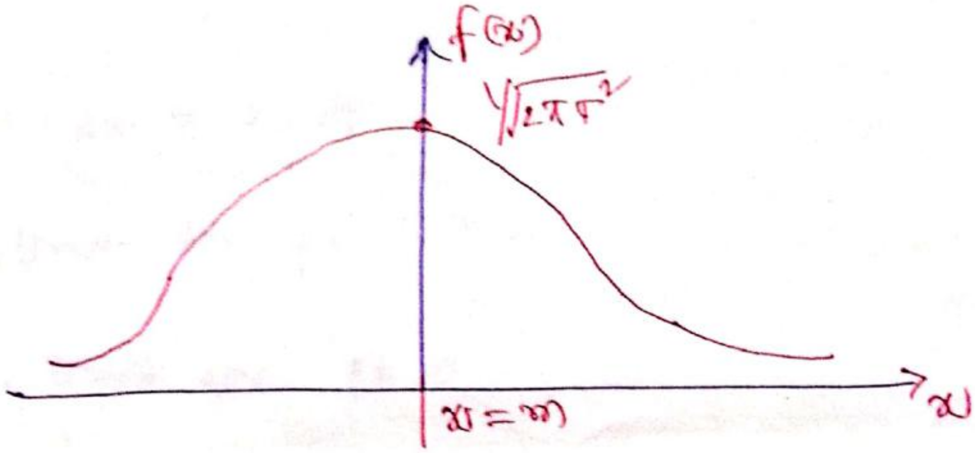
$$\therefore x = m$$

Thus, Gaussian density function has its maximum at $x = m$

The maximum value is $f(x)\Big|_{x=m} = \frac{1}{\sqrt{2\pi\sigma^2}}$

The Gaussian density function is a Bell shaped curve, having its peak at $x = m$ and is symmetrical about its mean.

It is plotted as.



The corresponding CDF of a Gaussian random variable is

$$F_X(x) = \int_{-\infty}^{x} f_X(x) \cdot dx$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \int_{-\infty}^{x} e^{-(x-m)^2/2\sigma^2} \, dx.$$

The solution of this integration can be in two different forms:

(i).

$$F_X(x) = P\{X \leq x\} = 1 - P\{X > x\}$$

$$= 1 - \int_x^\infty \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-(x-m)^2/2\sigma^2} \cdot dx$$

Let $\frac{x-m}{\sigma} = p$ $\Rightarrow$ $x - m = \sigma \cdot p$

$dx = \sigma \cdot dp$

lower limit for $p = x-m/\sigma$

upper limit for $p = \infty$

$$\therefore F_X(x) = 1 - \int_{\frac{x-m}{\sigma}}^\infty \frac{1}{\sqrt{2\pi\cancel{\sigma^2}}} \cdot e^{-p^2/2} \cdot \cancel{\sigma} \cdot dp$$

$$= 1 - \frac{1}{\sqrt{2\pi}} \cdot \int_{\frac{x-m}{\sigma}}^\infty e^{-p^2/2} \cdot dp$$

we have $Q(k) = \frac{1}{\sqrt{2\pi}} \cdot \int_k^\infty e^{-x^2/2} \cdot dx$

named as Q-function

Thus, $\frac{1}{\sqrt{2\pi}} \cdot \int_{\frac{x-m}{\sigma}}^\infty e^{-p^2/2} \cdot dp = Q\left(\frac{x-m}{\sigma}\right)$

$$\boxed{\therefore F_X(x) = 1 - Q\left(\frac{x-m}{\sigma}\right)}$$

(ii).

$$F_X(x) = P(X \le x) = 1 - P(X > x)$$

$$= 1 - \int_{x}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(x-m)^2}{2\sigma^2}} \cdot dx$$

let $\frac{x-m}{\sqrt{2}\cdot\sigma} = p$

$\Rightarrow dx = \sqrt{2}\,\sigma \cdot dp$

lower limit for $p = \frac{x-m}{\sqrt{2}\cdot\sigma}$

upper limit for $p = \infty$

$$\therefore F_X(x) = 1 - \int_{\frac{x-m}{\sqrt{2}\cdot\sigma}}^{\infty} \frac{1}{\sqrt{2\pi\cdot\sigma^2}} \cdot e^{-p^2} \sqrt{2}\cdot\sigma \cdot dp$$

$$= 1 - \frac{1}{\sqrt{\pi}} \cdot \int_{\frac{x-m}{\sqrt{2}\cdot\sigma}}^{\infty} e^{-p^2} \cdot dp$$

$$= 1 - \frac{1}{2} \cdot \frac{2}{\sqrt{\pi}} \cdot \int_{\frac{x-m}{\sqrt{2}\cdot\sigma}}^{\infty} e^{-p^2} \cdot dp$$

we have complementary Error function

$$erfc(k) = \frac{2}{\sqrt{\pi}} \cdot \int_{k}^{\infty} \cdot e^{-x^2} \cdot dx$$

Thus, $$\frac{2}{\sqrt{\pi}} \int_{\frac{x-m}{\sqrt{2}\cdot\sigma}}^{\infty} e^{-p^2} \cdot dp = erfc\left(\frac{x-m}{\sqrt{2}\cdot\sigma}\right)$$

$$\therefore \quad F_X(x) = 1 - \frac{1}{2} \cdot erfc\left(\frac{x-m}{\sqrt{2}\cdot\sigma}\right)$$

$\therefore$ Thus, the CDF of a Gaussian random variable can be expressed interms of Q-function or complementary Error function i.e. erfc(x) ie Q(x) or complementary Error function i.e. erfc(x)

→ Most of the naturally occurring phenomena are characterized by random variable distributed according to Gaussian density function.

18 10.07.08.

§ Conditional Distribution :-

Introduction :

for two Events A and B, The conditional probability of the Event A, given that the Event B, with $p(B) \neq 0$, is defined as

$$P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)} = \frac{P(AB)}{P(B)} = \frac{P(A,B)}{P(B)} \quad -①$$

Thus $p(A/B)$ is the probability of the Event A, Under the Condition that the Event B has already occured.

$$F_X(x) = P\{X \leq x\}$$

Conditional Distribution :-

Let A in Eq ① be identified as the Event $\{X \leq x\}$ for the random variable X; The resulting probability $P\{X \leq x/B\}$ is defined as the conditional Distribution function of X, which we denote

$$F_X(x/B) = P\{X \leq x/B\}$$

$$= \frac{P\{X \leq x \cap B\}}{P(B)}$$

where we use the notation $\{X \le x \cap B\}$

to imply the Joint event $\{X \le x\} \cap B$.

ie $\{X \le x, B\}$. ie $P\{X \le x, B\}$.

Properties of conditional Distribution :-

① $F_X(-\infty|B) = 0$

② $F_X(\infty|B) = \phi$ 1

③ $0 \le F_X(x|B) \le 1$

④ $F_X(x_1|B) \le F_X(x_2|B)$, if $x_1 < x_2$

⑤ $P\{x_1 < X \le x_2 | B\} = F_X(x_2|B) - F_X(x_1|B)$

$\frac{1}{B}$

⑥ $F_X(x^+|B) = F_X(x|B)$

Imp
NOTE: Conditional distribution applies to discrete, continuous and mixed random variables.

§ Conditional Density :—

—: Derivative of the conditional distribution function

we denote this density by $f_X(x/B)$ then

$$f_X(x/B) = \frac{d}{dx}\left[F_X(x/B)\right]$$

properties :-

① $f_X(x/B) \geq 0, \ \forall x$

② $\int_{-\infty}^{\infty} f_X(x/B)\, dx = 1$

③ $F_X(x/B) = \int_{-\infty}^{x} f_X(x).dx$

④ $P\left\{\frac{x_1}{B} < X \leq x_2/B\right\} = \int_{x_1}^{x_2} f_X(x/B)\, dx.$

* Methods of Defining Conditioning Events :

One method
(1) Event B, is defined in terms of R.V. X

Another method
(2) Event B, may depend on some other random variable other than X

(1) One way to define Event B in terms of X is, to be ?

$$B = \{X \leq b\}$$

$f_X(x/B)$ where b is a real number $-\infty < b < \infty$

$$F_X(x/X \leq b) = P\{X \leq x / X \leq b\}$$

$$= \frac{P\{X \leq x \cap X \leq b\}}{P(X \leq b)}$$

$$F_X(x/X \leq b) = \frac{P\{X \leq x \cap B \leq b\}}{P(X \leq b)}$$

$$= \frac{P(X \leq b)}{P(X \leq b)} \quad \leftarrow (1)$$

$$= 1 \quad , \text{ if } b \leq x$$

$$F_X(x / x \le b) = \frac{P\{X \le x \cap B \le b\}}{P(X \le b)}$$

$$= \frac{P(X \le x)}{P(X \le b)} = \frac{F_X(x)}{F_X(b)}, \quad x < b \quad — ②$$

$$= 1, \quad x \ge b \quad — ②$$

By combining Eq ① and ②, we obtain

$$F_X(x / x \le b) = \begin{cases} \dfrac{F_X(x)}{F_X(b)} & x < b \\ 1 & b \le x \end{cases} \quad — ③$$

The conditional density function derives from the derivative of Eq ③

$$f_X(x / x \le b) = \begin{cases} \dfrac{f_X(x)}{F_X(b)} = \dfrac{f_X(x)}{\int_{-\infty}^{b} f_X(x)\,dx}, & x < b \\ 0 & x \ge b \end{cases}$$

NOTE: Conditional distribution function is never smaller than the ordinary distribution function.

UNIT 1

Part 4

Theory and solved problems

5) $F_X(x)$ always lies between [c]

(a) 1 and ∞ (b) $-\infty$ to 0 (c) 0 to 1 (d) 0 to ∞

6) $f_X(x)$ is always [a]

(a) ≥ 0 (b) ≥ 1 (c) ≤ 0 (d) ≤ 1

7) $\frac{d}{dx}\left[F_X(x/B)\right] =$ [a]

(a) $f_X(x/B)$ (b) $F_X(B/x)$ (c) $f_X(B/x)$ (d) 0

8) for a D.R.V, the probability and probability density function are ——— [a]

(a) Equal (b) Not Equal (c) Equal to =0

9) The CDF plot of a C.R.V will be [b]

(a) discrete function (b) continuous function

(c) both

10) $F_X(+\infty) =$ ——— [b]

(a) 0 (b) 1 (c) >1 (d) -1

Assignment – UNIT-II

17.07.08

feb-2008-set-1 supply, (New 2008 set-4)

1. (a) Define and Explain the following density function

   (i) Binomial

   (ii) Exponential

   (iii) Uniform

   (iv) Rayleigh.

   (b) What is the density function of a random variable X, if X is gaussian. (12+4) m

set-2

2. (a) What are the conditions for the function to be a random variable? Discuss (4M)

   (b) What do you mean by continuous and discrete random variable (6M)

feb-2008-Supply set-3 and set-4

3. Discuss what do you mean by density function. What is the distribution function of mixed random variable.? (8M)

4. In an Experiment of fair wheel of chance is spun with the numbers 0 to 12 on the wheel. What is the distribution and density function explain with plot. (8M)

NOV-2007 - Reg - Set - 1

(5) (a) Define Rayleigh density and distribution function (5M)

(b) Define and Explain gaussian random variable in brief (5M)

(c) Determine whether the following is a valid distribution function. $F(x) = 1 - e^{-x/2}, \quad x = 0$

$= 0, \quad$ elsewhere (6 M)

NOV-2007 - Reg - Set - 2

(6) (a) What is binomial density function. Write the Equation for binomial distribution function (6 M)

(b) What do you mean by continuous and discrete random variable? Discuss the conditions for a function to be a random variable (10M)

NOV-2007 - Set - 3 (Reg)

(7) Define cumulative probability distribution function. And discuss distribution function specific properties. (8M)

NOV - 2007 - Set - 4 - Reg

(8) What is poisson random variable? Explain in brief - (5M)

⑨ Nov-2006 Reg-Set-1, Set 3

(a) Define conditional distribution and density functions and explain their properties. (10 M)

(b) A C.R.V. X has a PDF $f_X(x) = 3x^2$, $0 \le x \le 1$

find 'a' and 'b' such that $0 \le x \le 1$

$P(X \le a)$ (i) $P(X \le a) = P\{X > a\}$ and (6 M)

(ii) $P(X > b) = 0.05$

Nov-2006-Reg-Set-2, Set 3

⑩ Define Random Variable and give the concept of random variable

Nov-2005-Set-3 (Reg)

⑪ Define conditional PMF? (4 M)

Nov-2005-Set-2 (Reg)

⑫ A fair coin is tossed 3 times and faces showing up are observed.

(i) write the sample description space

(ii) If X is no. of heads in each of the outcomes of the expt., find the probability function

(iii) Sketch the CDF and pdf (10 M)

peebles page no: 28

Swiss mathematician (1654 – 1705)

Jacob BERNOULLI TRIALS

$^nP_r = \frac{n!}{(n-r)!}$

$\binom{n}{r} = {}^nC_r = \frac{n!}{r!\,(n-r)!}$

Consider an Experiment for which there are only two possible outcomes on any trial.

Example: flipping a coin, hitting or missing the target in artillery, passing or failing an exam,
receiving a 0 or 1 in a computer bit stream
or
winning or losing in a game of chance.

For this type of Experiment,
let $\underline{A}$ be the Elementary Event having one of the two possible outcomes as its element.
$\bar{A}$ is the only other possible Elementary Event.
Specifically, we shall repeat the basic Experiment $\underline{N}$ times, and determine the probability that A is observed, exactly $\underline{K}$ times out of the $\underline{N}$ trials.
Such repeated experiments are called Bernoulli trials.

Assume that Events are statistically independent.

Let Event A occur on any given trial with probability

$$P(A) = p \qquad \text{—} \; ①$$

and

$$P(\bar{A}) = 1 - p \qquad \text{—} \; ②$$

After N trials of the Experiment, one particular sequence of outcomes has A occurring K times, followed by $\bar{A}$ occurring $(N-K)$ times. Because of statistical independent of trials, the probability of this one sequence of

$$\underbrace{P(A) \cdot P(A) \cdots P(A)}_{\substack{K \text{ times} \\ K \text{ terms}}} \cdots \overset{P(A)}{} \underbrace{P(\bar{A}) \cdot P(\bar{A}) \cdot P(\bar{A}) \cdots P(\bar{A})}_{(N-K) \text{ terms}} = p^K \cdot (1-p)^{N-K} \qquad \text{—} \; ③$$

Now, we only need to find the number of such sequences. Some thought will reveal that this is the number of ways of taking K objects at a time from N objects. The number of known to be the Binomial co-efficient:

(combinations)

$$\binom{N}{K} = \frac{N!}{K!\,(N-K)!}$$

[illegible] $\binom{N}{\;} p^k (1-p)^{N-K}$

Note ① When N, K, and (N−K) are large, the factorials are difficult to evaluate. So approximations become useful. One approximation, called Stirling's formula, is

$$m! \simeq (2\pi m)^{1/2}\, m^{m} \cdot e^{-m} \quad , \quad m \text{ large}$$

$$\therefore\ P\{A \text{ occurs exactly } K \text{ times}\} = \binom{N}{K} p^{K} \cdot (1-p)^{(N-K)}$$

$$\simeq \frac{1}{\sqrt{2\pi N p(1-p)}} \cdot \exp\left[-\frac{(K-Np)^2}{2Np(1-p)}\right]$$

This equation is called "De-moivre - Laplace" approximation.

Note ② If N is very large and p is very small, the poisson approximation is useful.

$$\binom{N}{K} p^{K} (1-p)^{N-K} = \frac{(Np)^{K} \cdot e^{-Np}}{K!}$$

§ Some other probability distribution and Density functions:-

1. Binomial
2. poisson } for discrete random variables
3. Uniform → Exponential
4. Gaussian } continuous random variables
5. Rayleigh

① BINOMIAL :- (after discussion of Bernoulli trials).

According to Bernoulli trials

$$P\{A \text{ occurs exactly } k \text{ times}\} = \binom{N}{k} p^k (1-p)^{N-k}.$$

Let $0<p<1$ and $N = 1, 2, \ldots$ then the function

$$f_x(x) = \sum_{k=0}^{N} \binom{N}{k} p^k (1-p)^{N-k} \cdot \delta(x-k) \quad \text{— (1)}$$

is called the binomial density function. The quantity $\binom{N}{k}$ is the binomial coefficient.

$$\binom{N}{k} = \frac{N!}{k!(N-k)!}$$

nCk

The binomial density can be applied to Bernoulli trial of Experiment. also

① It applies to many games of chance

② Detection problems in radar and sonar

③ and many experiments having only two possible outcomes on any given trial.

④ also used in the study of communication systems.

By Integration of Eq ①, the binomial distribution function is found:

$$F_X(x) = \sum_{k=0}^{N} \binom{N}{k} p^k (1-p)^{N-k} \cdot u(x-k) \quad -②$$

(2) <u>POISSON :</u>

The poisson random variable X has a density and distribution given by.

$$f_X(x) = e^{-b} \sum_{k=0}^{\infty} \frac{b^k}{k!} \delta(x-k)$$

and

$$F_X(x) = e^{-b} \sum_{k=0}^{\infty} \frac{b^k}{k!} u(x-k)$$

where $b > 0$, is a real constant.

$N \to \infty$, and $p \to 0$, for the binomial case in such a way that $Np = b$, a constant, the poisson case results.

Applications :

1. poisson R.V. applies to a wide variety of <u>counting-type</u> applications.
2. used for, No of telephone calls made during a period of time.

If the time interval of interest has duration T, and the events being counted (occur) at an

(3) Uniform :- or (Rectangular Distribution)

The Uniform probability density and distribution functions are defined by

$$\text{Density} \rightarrow f_X(x) = \begin{cases} 1/(b-a), & a \leq x \leq b \\ 0, & \text{elsewhere} \end{cases}$$

$$\text{Distribution} \rightarrow F_X(x) = \begin{cases} 0 & x < a \\ (x-a)/(b-a), & a \leq x < b \\ 1 & b \leq x \end{cases}$$

Applications :- (1) Quantization of signal samples prior to Encoding in digital communication systems.

EXPONENTIAL :-

The exponential density and distribution functions are :

$$\text{density} \rightarrow f_X(x) = \begin{cases} \frac{1}{b} e^{-(x-a)/b}, & x > a \\ 0, & x < a \end{cases}$$

$$\text{Distribution} : \quad F_X(x) = \begin{cases} 1 - e^{-(x-a)/b}, & x > a \\ 0 & x < a \end{cases}$$

Applications: ① The exponential density is useful in describing raindrop sizes when a ~~large~~ large number of rainstorm measurements are made.

RAYLEIGH :-

The Rayleigh density and distribution functions are:

density

$$f_X(x) = \begin{cases} \frac{2}{b}(x-a) \cdot e^{-(x-a)^2/b}, & x \geq a \\ 0, & x < a \end{cases}$$

distribution

$$F_X(x) = \begin{cases} 1 - e^{-(x-a)^2/b}, & x \geq a \\ 0, & x < a \end{cases}$$

Applications : ① The Rayleigh density describes the envelope of one type of noise when passed through a bandpass filter.

② In analysis of errors in various measurement systems.

PROBLEMS — UNIT-II

1. The density function of a continuous Random variable X is

$$f(x) = ax \quad , \quad 0 \le x \le 1$$
$$= a \quad , \quad 1 \le x \le 2$$
$$= 3a - ax \quad , \quad 2 \le x \le 3$$
$$= 0 \quad , \quad \text{elsewhere}$$

1. find (a)
2. If X_1, X_2, X_3 are three independent observations of X, what is the probability that exactly one of these three is greater than 1.5.

Sol:

(1) $$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

$$\Rightarrow \int_0^1 ax\,dx + \int_1^2 a\,dx + \int_2^3 (3a - ax)\,dx = 1$$

$$= a\left[\frac{x^2}{2}\right]_0^1 + a\left[x\right]_1^2 + \left[3ax - \frac{ax^2}{2}\right]_2^3 = 1$$

$$= \frac{a}{2} + a + \left[\left(9a - \frac{9a}{2}\right) - (6a - 2a)\right] = 1$$

$$= \frac{a}{2} + a + \frac{a}{2} = 1$$

$$\Rightarrow a = 1/2$$

2

(2) This can be considered as a case of Binomial distribution

Let the observation greater than 1.5 be the success, and let its probability be p.

$$\therefore\ p = \int_{1.5}^{2} a \cdot dx + \int_{2}^{3} (3a - ax)\, dx$$

$$= 1/2$$

Among 3 observations, exactly one should be greater than 1.5.

This can be approximated by Binomial distribution.

Here $N = 3$, $K = 1$

$p = 1/2$ and $(1-p) = 1/2$

$\therefore$ The required probability is

$$P\{>1.5 \text{ occurs exactly one time}\} = \binom{3}{1}(1/2)^1(1/2)^2$$

$$= 3C_1 \cdot 1/2 \cdot 1/4$$

$$= 3/8$$

(1) A fair coin is tossed 4 times. Find the probability that there will appear.

(a) 2 heads (b) 1 tail and 3 heads

(c) at least one head (d) Exactly one head

(e) not more than one head, use binomial distribution

Ans: (a) 3/8 (b) 1/4 (c) 15/16 (d) 1/4 (e) 5/16

(a) $4C_2 \cdot (1/2)^2 (1/2)^{4-2}$

(b) $4C_3 (1/2)^3 (1/2)^{4-3}$

(c) $1 - P(X=0) \longleftrightarrow$ or $P(X=1) + P(X=2) + P(X=3) + P(X=4)$

? $= 1 - 4C_0 (1/2)^0 (1/2)^{4-0} = 7/8$

(d) $4C_1 . (1/2) (1/2)^{4-1}$

(e) $4C_0 (1/2)^0 (1/2)^{4-0} + 4C_1 . (1/2)^1 (1/2)^{4-1}$

$= \frac{1}{16} + \frac{4}{16} = 5/16$

$\binom{N}{K} p^K . (1-p)^{N-K}$

$4C_1 (1/2)^1 (1/2)^3$

Note: $nC_0 = 1$

$4! \quad \frac{1}{4} \cdot \frac{1}{4} = 1/4$

1. A die is thrown 6 times. find the probability that a face 3 will occur (a) exactly 3 times (b) at least two times (c) at most once.

(Use Binomial distribution)

Sol :

$N = 6$

probability of getting a 3 in one throw of the die $= 1/6$.

let it be the success and of probability p

ie $p = 1/6$

$(1-p) = 1 - 1/6 = 5/6 \longrightarrow$ failure

(a) $p(x=3) = 6C_3 (1/6)^3 (5/6)^{6-3}$

$= 0.0535$

(b) $p(x \geq 2) = 1 - p(x=0) - p(x=1)$ or $p(x=2) + p(x=3) + p(x=4) + p(x=5) + p(x=6)$

$= 1 - 6C_0 (1/6)^0 (5/6)^{6-0} - 6C_1 (1/6)^1 (5/6)^{6-1}$

$= 1 - (5/6)^6 - (5/6)^5$

$= 0.263$

max of one face(3)

(c) $p(x=0) + p(x=1) \Rightarrow 6C_0 \cdot (1/6)^0 (5/6)^{6-0} + 6C_1 (1/6)^1 (5/6)^{6-1}$

$= 0.736$

5

(1) A random variable has poisson distribution such that $p(x=3) = p(x=4)$, find $p(6)$.

Sol

$$p(x=x) = \frac{e^{-b} \cdot b^k}{k!}, \quad k = 1, 2, 3$$

$p(x=k)$

$$e^{-b} \sum_{k=0}^{\infty} \frac{b^k}{k!} \delta(x-k)$$

$$F_X(x) = P(X \le x)$$

$$P(X=3) = \frac{e^{-b} \cdot b^3}{3!}$$

$$P(X=4) = \frac{e^{-b} \cdot b^4}{4!}$$

Since $p(x=3) = p(x=4)$

$$\frac{\cancel{e^{-b}} \cdot b^3}{3!} = \frac{\cancel{e^{-b}} \cdot b^4}{4!}$$

$$\Rightarrow b = 4$$

$$\therefore \; p(6) = p(x=6) = \frac{e^{-b} \cdot b^6}{6!}$$

$$= \frac{e^{-4} \cdot (4)^6}{6!}$$

$$= \underline{0.104}$$

15.07.08

① A random variable x has the following distribution.

x_i	0	1	2	3	4	5	6	7	8
$P(x)$	a	$3a$	$5a$	$7a$	$9a$	$11a$	$13a$	$15a$	$17a$

① find 'a' (Ans: $a = 1/81$)

② find $P(x \le 3)$, $P(x > 3)$ and $P(0 < x < 5)$
$16/81$ $(1/9)$ $8/9$ ~~26/81~~ $65/81$ ~~24/81~~ $24/81$

③ find the smallest value of x, for which $P(x \le x) > 1/2$
$x = 6$

④ find the CDF of x.

$F_x(x) = P\{x \le x\}$

② Consider the experiment of tossing two dice simultaneously. Let x be the r.v. defined such that the sum of the two faces. It takes values of the plot of CDF

x_i = Sum

Sol

x_i	2	3	4	5	6	7	8	9	10	11	12
$P(x)$	$1/36$	$2/36$	$3/36$	$4/36$	$5/36$	$6/36$	$5/36$	$4/36$	$3/36$	$2/36$	$1/36$

CDF of $F_x(x) = P\{x \le x\}$

$F_x(2) = 1/36$ $F_x(3) = 3/36$, $F_x(4) = 6/36$

$F_x(5) = 10/36$, $F_x(6) = 15/36$, $F_x(7) = 21/36$

$F_X(8) = 26/36$ $F_X(9) = 30/36$, $F_X(10) = 33/36$

$F_X(11) = 35/36$, $F_X(12) = 1$

① A gaussian r.v X is having a mean 2 and variance 4. find the probability of $X \leq 1$

$m = 2$

$\sigma^2 = 4 \Rightarrow \sigma = 2$

$x = 1$

$$P(X \leq x) = 1 - Q\left(\frac{x-m}{\sigma}\right)$$

$$= 1 - Q\left(\frac{1-2}{2}\right)$$

$$= 1 - Q(-0.5)$$

since $Q(0.5) + Q(-0.5) = 1$

$$\Rightarrow Q(0.5) = 1 - Q(-0.5)$$

$$\boxed{\therefore P(X \leq x) = Q(0.5)}$$

(1) A random variable X is known to have a distribution function

$$F_X(x) = u(x)\left[1 - e^{-x^2/b}\right], \quad b > 0, \text{ a constant,}$$

find its density function

Sol:

$$f_X(x) = \frac{d}{dx}\left[F_X(x)\right]$$

$$= \frac{d}{dx}\left[u(x) \cdot \left[1 - e^{-x^2/b}\right]\right]$$

$$= u(x)\frac{d}{dx}\left[1 - e^{-x^2/b}\right] + \left[1 - e^{-x^2/b}\right]\frac{d}{dx}\left[u(x)\right]$$

$$= \underbrace{\delta(x)\left(1 - e^{-x^2/b}\right)}_{=0} + u(x) \cdot \frac{2x}{b} \cdot e^{-x^2/b}$$

$$= \therefore \; u(x) \cdot \frac{2x}{b} \cdot e^{-x^2/b}$$

The impulse term disappears because its coefficient is zero at $x = 0$, where the impulse exists.

Imp note:

Note: for a function $g(x)$ $g(x)\,\delta(x) = g(0) \cdot \delta(x)$ page no: 416 peebles. Appendix A, A-29

UNIT 1

Part 5

Problems

57

§ Is the function defined as follows a density function ?

$$f(x) = 0 \quad \text{for } x<2$$

$$= \frac{1}{18}(3+2x) \quad \text{for } 2 \le x \le 4$$

$$= 0 \quad \text{for } x > 4$$

Sol : The probability density function satisfies the relation $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

$$\therefore \int_2^4 \frac{1}{18}(3+2x)\,dx$$

$$= \frac{1}{18}\left[3x + x^2\right]_2^4$$

$$= \frac{1}{18}\left[(12+16) - (6+4)\right]$$

$$= 1/18\left[28 - 10\right] = \frac{18}{18} = 1$$

Since the relation is satisfied, it is a density function.

§ A Random Variable X has the following probability distribution

X	P(X)
0	0
1	K
2	2K
3	2K
4	3K
5	K^2
6	$2K^2$
7	$7K^2+K$

find (i) The value of K

(ii) $P(X \leq 3)$

(iii) The minimum value of 'm' such that $P(X \leq m) > 0.5$

Sol

$$P(S) = 1$$

$$0 + K + 2K + 2K + 3K + K^2 + 2K^2 + 7K^2 + K = 1$$

$$10K^2 + 9K - 1 = 0$$

$ax^2+bx+c=0$

$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$b = 9$
$a = 10$
$c = -1$

$$K = \frac{-9 \pm \sqrt{49 - 40}}{20}$$

$81+40$

$$= \frac{-9 \pm \sqrt{121}}{20}$$

$$\therefore K = \frac{-9+11}{20}, \frac{-9-11}{20}$$

$$= 2/20, -1$$

Since $\quad = 1/10, -1$

(ii)

$$p(X \leq 3) = p(X=0) + p(X=1) + p(X=2) + p(X=3)$$

$$= 0 + K + 2K + 2K$$

$$= 5K$$

$$= 5 \times \frac{1}{10} = \frac{1}{2} = 0.5$$

(iii)

$$p(X \leq m) > 0.5$$

$$m = 4$$

§ The diameter of an electric cable is assumed to be a continuous random variable X with probability density function $f(x)$

$$f(x) = 6x(1-x), \quad 0 \le x \le 1$$

(i) Check that $f(x)$ describes actually a probability density function

(ii) find $P(0 \le X \le 0.5)$

Sol (i)

$$f(x) = 6x(1-x), \quad 0 \le x \le 1$$

$$\int_0^1 f_X(x)\,dx = 1$$

$$\int_0^1 6x(1-x)\,dx$$

$$= \int_0^1 (6x - 6x^2)\,dx$$

$$= \left[3x^2 - 2x^3\right]_0^1$$

$$= (3-2) = 1$$

Since $\int_0^1 f(x)\,dx = 1$, the given function $f(x)$ is a probability density function.

(ii) $P(0 \le X \le 0.5) = \int^{0.5} 6x(1-x)\,dx = \left[3x^2 - 2x^3\right]_0^{0.5}$

§ Consider the probability function $f(x) = ae^{-b|x|}$, where X is a random variable whose allowable values ranges from $x = -\infty$ to $x = +\infty$ find (i) cdf $F(x)$ (ii) The relationship between a and b and (iii) The probability that the outcome X lies between 1 and 2.

ii)

if $x > 0$
$x < 0$

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{-\infty}^{0} ae^{bx}\,dx + \int_{0}^{\infty} ae^{-bx}\,dx = 1$$

$$\left[a \cdot \frac{e^{bx}}{b}\right]_{-\infty}^{0} + \left[-\frac{a}{b}e^{-bx}\right]_{0}^{\infty} = 1$$

$$\frac{a}{b} + \frac{a}{b} = 1$$

$$\frac{2a}{b} = 1$$

$$\boxed{2a = b}$$ ✓

iii) $$p(1 \leq X \leq 2) = \int_{1}^{2} ae^{-b|x|}\,dx$$

$$= \int_{1}^{2} ae^{-bx}\,dx$$

$$= a\left[-\frac{e^{-bx}}{b}\right]_{1}^{2} = \frac{a}{b}\left[e^{-b} - e^{-2b}\right]$$

$\frac{a}{b} = \frac{1}{2}$

$$= \frac{1}{2}\left[e^{-b} - e^{-2b}\right]$$

X is a continuous random variable with ~~distribution~~ or density

$$f(x) = kx \qquad 0 \le x \le 5$$
$$= 0 \qquad \text{elsewhere}$$

find (i) $P(1 \le x \le 3)$ (ii) $P(2 \le x \le 4)$

(iii) $P(x \le 3)$ or $P(0 \le x \le 3)$

Sol

$$\int_0^5 kx\,dx = 1$$

$$k \cdot \left[\frac{x^2}{2}\right]_0^5 = 1$$

$$k\left(\frac{25}{2}\right) = 1$$

$$\Rightarrow k = 2/25$$

i) $P(1 \le x \le 3) = \int_1^3 kx\,dx = \frac{2}{25} \cdot \left[\frac{x^2}{2}\right]_1^3$

$$= 2/25 \left[\frac{9}{2} - 1/2\right]$$

$$= \frac{\cancel{2}}{25} \times \frac{8}{\cancel{2}} = 8/25$$

ii) $\int_2^4 kx\,dx = 2/25 \cdot \left(\frac{16}{2} - \frac{4}{2}\right)$

$$= 2/25 \times \frac{12}{2} = 12/25$$

(iii) $\int_0^3 kx\,dx = 2/25\left[9/2\right] = 9/25$

xample 2.1

In a given experiment, if a random variable X has elements $X = \{1,2,3,4,5,6\}$ with probabilities $P_x(x) = \left[\frac{2}{36}, \frac{8}{36}, \frac{12}{36}, \frac{7}{36}, \frac{5}{36}, \frac{2}{36}\right]$ Draw the distribution function of X.

Solution

Given $\quad X = \{1, 2, 3, 4, 5, 6\}$

$$P_x(x) = \left\{\frac{2}{36}, \frac{8}{36}, \frac{12}{36}, \frac{7}{36}, \frac{5}{36}, \frac{2}{36}\right\}$$

Therefore, the distribution function of X is given as

$$F_x(x_1) = P(X \le x_1) = P(x_1) = \frac{2}{36}$$

$$F_x(x_2) = P(X \le x_2) = P(x_1) + P(x_2) = \frac{2}{36} + \frac{8}{36} = \frac{10}{36}$$

$$F_x(x_3) = P(X \le x_3) = P(x_1) + P(x_2) + P(x_3) = \frac{10}{36} + \frac{12}{36} = \frac{22}{36}$$

$$F_x(x_4) = P_x(X \le x_4) = P(x_1) + \cdots + P(x_4) = \frac{22}{36} + \frac{7}{36} = \frac{29}{36}$$

$$F_x(x_5) = P(X \le x_5) = P(x_1) + \ldots\ldots + P(X_5) = \frac{29}{36} + \frac{5}{36} = \frac{34}{36}$$

$$F_x(x_6) = P(X \le x_6) = P(x_1) + \ldots\ldots + P(X_6) = \frac{34}{36} + \frac{2}{36} = 1$$

Table 2.2 shows the values of distribution function $F_x(x)$ and the Fig 2.2 shows the variation of $F_x(x)$ with x.

X	1	2	3	7	5	6
$P(x)$	$\frac{2}{36}$	$\frac{8}{36}$	$\frac{12}{36}$	$\frac{7}{36}$	$\frac{5}{36}$	$\frac{2}{36}$
$F_x(x)$	$\frac{2}{36}$	$\frac{10}{36}$	$\frac{22}{36}$	$\frac{29}{36}$	$\frac{34}{36}$	1

Table 2.2: Probability and PDF of X

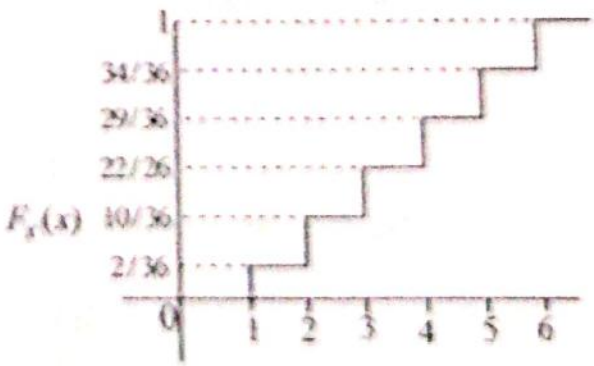


Fig 2.2: Distribution function of X

2.2.1 Expression for Distribution Function

Discrete random variable:

If X is a discrete random variable, the distribution function $F_x(x)$ is a cumulative sum of all probabilities of X up to the value x. As x is discrete, the $F_x(x)$ must have a stair case form with step functions. The amplitude of the step is equal to the probability of X at that x value.

If the value of X are $\{x_i\}$, the distribution function can be written mathematically as

$$F_X(x) = \sum_{i=1}^{N} P(x_i)u(x-x_i) \qquad \text{.....(2.3)}$$

where $u(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$

is the unit step function, and N = No. of elements in x. N may be infinite.

For example in the experiment "Tossing 3 fair coins" explained above the distribution function (from the Table 2.1) is

$$F_X(x) = P(x_1)u(x-x_1) + P(x_2)u(x-x_2) + P(x_3)u(x-x_3) + P(x_4)u(x-x_4)$$

$$F_X(x) = \frac{1}{8}\left[u(x) + 3u(x-1) + 3u(x-2) + u(x-3)\right]$$

F

X	0	1	2	3
$P(x)$	1/8	3/8	3/8	1
$F(x)$	1/8	4/8	7/8	1

Table 2.1: Probability and PDF of X

Example 2.2

A random variable X has probabilities shown in Table 2.3

a) Find the value of K.

b) Find $F_X(x), f_X(x)$ and draw the plots

X	-3	-2	-1	0	1	2
$P(x)$	0.2	$0.5K$	K	0.1	$0.3K$	K

Table 2.3: Probabilities of X

Solution

If a random variable X has probabilities, we know that, the sum of all probabilities is equal to 1. i.e.,

from eq. (2.20), $\sum_{i=1}^{6} P(x_i) = 1$

$$0.2 + 0.5K + K + 0.1 + 0.3K + K = 1$$

$$0.3 + 0.8K = 1$$

$$2.8K = 0.7$$

$$\text{Then } K = \frac{0.7}{2.8} = \frac{7}{28} = \frac{1}{4} = 0.25$$

$$\therefore \qquad P(x) = \left\{0.2, \frac{0.5}{4}, \frac{1}{4}, 0.1, \frac{0.3}{4}, \frac{1}{4}\right\}$$

$$P(x) = \{0.2, 0.125, 0.25, 0.1, 0.075, 0.25\}$$

The cumulative distribution function, $F_X(x)$ is given by

$$F_x(-3) = P(x_1) = 0.2$$

$$F_X(-2) = P(x_1) + P(x_2) = 0.2 + 0.125 = 0.325$$

$$F_X(-1) = P(x_1) + P(x_2) + P(x_3) = 0.325 + 0.25 = 0.575$$

$$F_X(0) = P(x_1) + ...P(x_4) = 0.575 + 0.1 = 0.675$$

$$F_X(1) = P(x_1) + ...P(x_5) = 0.675 + 0.075 = 0.75$$

$$F_X(2) = P(x_1) + ...P(x_6) = 0.75 + 0.25 = 1$$

The cumulative distribution function is

from eq. (2.3), $\qquad F_X(x) = \sum P(x_i)u(x - x_i)$

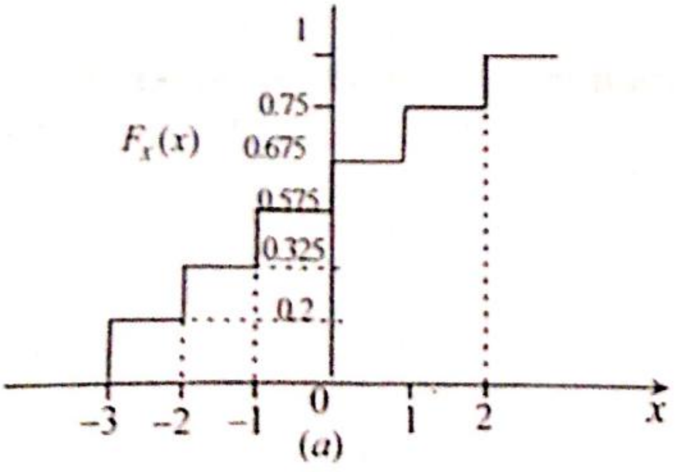
$$F_X(x) = 0.2u(x+3) + 0.125u(x+2) + 0.25u(x+1)$$
$$+0.1u(x) + 0.075u(x-1) + 0.25u(x-2)$$

and the density function of X is

from eq. (2.6), $\qquad f_X(x) = \sum P(x_i)\delta(x - x_i)$

$$f_X(x) = 0.2\delta(x+3) + 0.125\delta(x+2) + 0.25\delta(x+1)$$
$$+0.1\delta(x) + 0.075\delta(x-1) + 0.25\delta(x-2)$$

The plots of $F_X(x)$ and $f_X(x)$ are shown in Fig. 2.9.



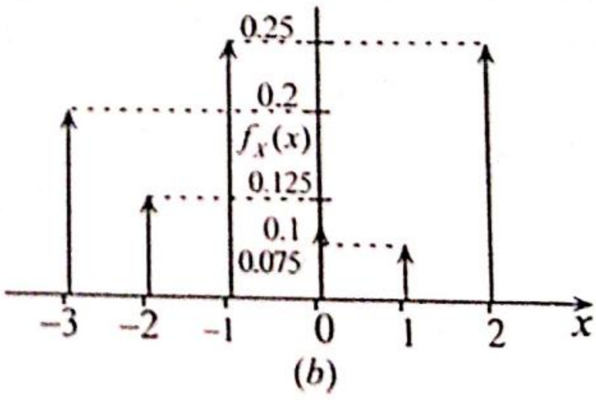


Fig 2.9 : a) Plot for $F_X(x)$ b) Plot for $f_X(x)$

ADDITIONAL PROBLEMS

Example 2.5

If the probability density of a random variable is given by

$$f_X(x) = K(1-x^2) \quad 0 < x < 1$$

Find the value K and $F_X(x)$

Solution

Given the probability density function

$$f_X(x) = K(1-x^2) \qquad 0 < x < 1$$

(a) Value of K

If the pdf is valid then

$$\int_{-\infty}^{\infty} f_X(x) = 1$$

$$\int_0^1 K(1-x^2)dx = 1$$

$$K\left(x-\frac{x^3}{3}\right)_0^1 = 1$$

$$K\left(1-\frac{1}{3}\right) = 1$$

$$\frac{2K}{3} = 1$$

$$K = \frac{3}{2} = 1.5$$

Then

$$f_X(x) = \frac{3}{2}(1-x^2) \quad 0 < x < 1$$

(ii) The probability distribution function is given by

$$F_X(x) = \int_{-\infty}^{x} f_X(x)dx$$

$$F_X(x) = \int_{0}^{x} \frac{3}{2}(1-x^2)\,dx$$

or $$F_X(x) = \frac{3}{2}\left(x - \frac{x^3}{3}\right)$$

$\therefore$ $$F_X(x) = \begin{cases} 0 & x \le 0 \\ \frac{3}{2}\left(x - \frac{x^3}{3}\right) & 0 < x < 1 \\ 1 & x \ge 1 \end{cases}$$

Example 2.15

Consider the experiment of tossing four fair coins. The random variable X is associated with the number of tails showing. Compute and sketch the cumulative distribution function of X *Aug/Sep -2007, June -2003*

Solution

Consider the experiment of tossing four fair coins

The Possible events in the sample space are

$\{HHHH, HHHT, HHTH, HTHH, THHH, THTH, TTHH, THHT,$
$HTTH, HTHT, HHTT, TTTH, TTHT, THTT, HTTT, TTTT\}$

Let X be a discrete random variable shows number of tails.

Therefore from the events,

$$X = \{0,1,1,1,1,2,2,2,2,2,2,3,3,3,3,4\}$$

The probability of each random variable is shown in Table 2.6

X	0	1	2	3	4
$P(X)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

Table 2.6: Random variable X with probabilities

The cumulative distribution function of X is

$$F_X(x) = P(X \le x)$$

$$F_X(0) = P(X \le 0) = P(0) = \frac{1}{16}$$

$$F_X(1) = P\{X \le 1\} = P(0) + P(1)$$

$$= \frac{1}{16} + \frac{4}{16} = \frac{5}{16}$$

$$F_X(2) = P[X \le 2] = F_X(1) + P(2)$$

$$= \frac{5}{16} + \frac{6}{16} = \frac{11}{16}$$

$$F_X(3) = P[X \leq 3] = F_X(2) + P(3)$$

$$= \frac{11}{16} + \frac{4}{16} = \frac{15}{16}$$

$$F_X(4) = P[X \leq 4] = F_X(3) + P(4)$$

$$= \frac{15}{16} + \frac{1}{16} = \frac{16}{16} = 1$$

$$F_X(x) = \frac{1}{16}u(x) + \frac{4}{16}u(x-1) + \frac{6}{16}u(x-2) + \frac{4}{16}u(x-3) + \frac{1}{16}u(x-1)$$

and $$f_X(x) = \frac{1}{16}\delta(x) + \frac{4}{16}\delta(x-1) + \frac{6}{16}\delta(x-2) + \frac{4}{16}\delta(x-3) + \frac{1}{16}\delta(x-4)$$

The cumulative distribution function and probability density function are plotted in Fig.2.18

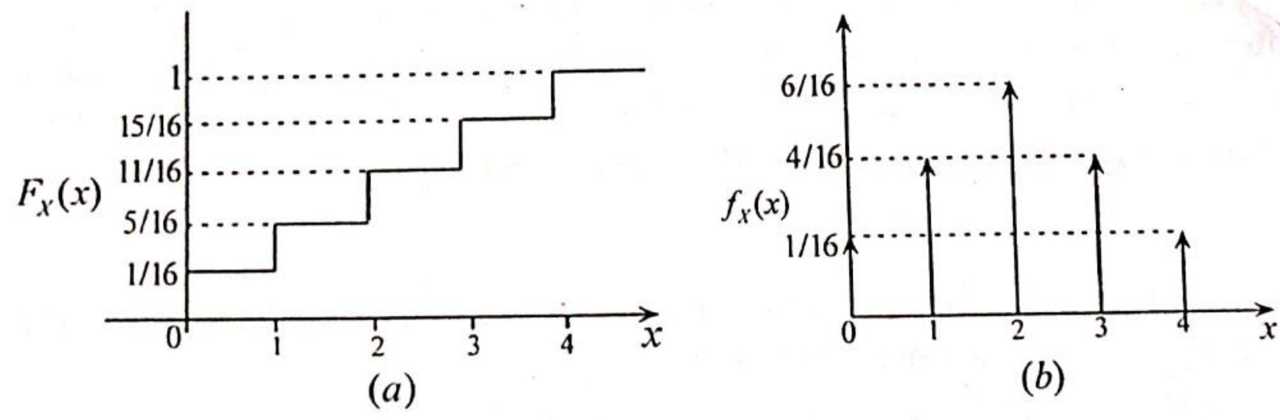


Fig. 2.18: (a) Cumulative distribution function $F_X(x)$ and b) Probability density function $f_X(x)$

Example 2.17

In experiment where the pointer on a wheel of chance is spun. The possible outcomes are the numbers from 0 to 12 marked on the wheel. The sample space consists of the numbers in the set $\{0 < S \leq 12\}$ and if the random variable X is defined as $X = X(S) = S^2$, map the elements of random variable on the real line and explain. ***Feb 2008***

Solution

Here the experiment is the pointer on a wheel of chance is spun the possible outcomes of the experiment are the numbers from 0 to 12.

i.e. the sample space is $S = \{0 \text{ to } 12\}$, $\{0 \leq x \leq 12\}$

A random variable is given as

$$X = X(s) = S^2$$

Therefore $$X = \{0 < X \leq 144\}.$$

The points in S are mapped on to the real line as a set $\{0 < x \leq 144\}$, shown in Fig.2.20

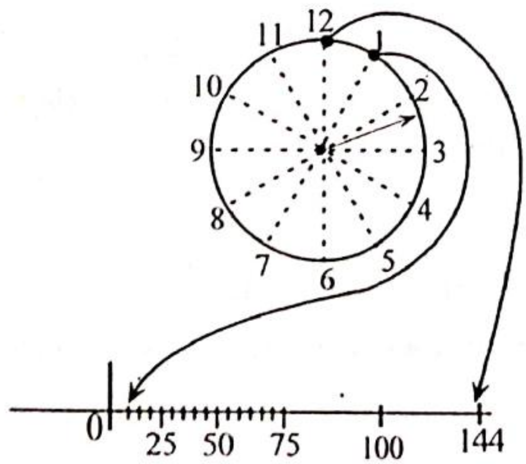


Fig. 2.20: Mapping of S on real line

Example 2.20

A continuous random variable X has a PDF $f(x)=3x^2, 0<x<1$. Find 'a' and 'b' such that (i) $P\{X=a\}=P\{X>a\}$ and (ii) $P\{X>b\}=0.05$.

Nov -2006

$P(X \leq a)$

2015 Repeated

Solution

Given the probability density function

$$f_X(x) = 3x^2 \quad 0<x<1$$

(i) $P\{X>a\} = 1-P\{X\leq a\} = 1-F_X(a)$

$$= 1-\int_0^a 3x^2\,dx$$

$$= 1-x^3 = 1-a^3$$

$P\{X=a\} = f_X(a) = 3a^2$?

$P(X=a) = F_X(x)$
$= \int_0^a 3x^2\,dx$

probability = prob. density function

Given that $P\{X=a\} = P\{X>a\}$

$$3a^2 = 1-a^3$$

$a = -2$

$$a^2(a+3) = 1$$

(ii) $P\{X>b\} = 1-P\{X\leq b\} = 0.05$

$$1-F_X(b) = 0.05$$

$$F_X(b) = \int_0^b 3x^2 dx = b^3$$

$$\therefore \quad 1-b^3 = 0.05$$

$$b^3 = 0.05 \qquad b = \sqrt[3]{0.05} = 0.3684$$

Example 2.29

Find a value for constant A such that

$$f_x(x) = \begin{cases} 0 & x < -1 \\ A(1-x^2)\cos\left(\frac{\pi x}{2}\right) & -1 \le x \le 1 \\ 0 & 1 < x \end{cases}$$

is a valid probability density function. *May-2011*

Solution

Given

$$f_X(x) = \begin{cases} 0 & x < -1 \\ A(1-x^2)\cos\left(\frac{\pi x}{2}\right) & -1 \le x \le 1 \\ 0 & 1 < x \end{cases}$$

If the function is a valid probability density function, then

$$\int_{-\infty}^{\infty} f_X(x)dx = 1$$

Now

$$\int_{-1}^{1} A(1-x^2)\cos\left(\frac{\pi x}{2}\right)dx = 1$$

$$A\left[(1-x^2)\frac{\sin\frac{\pi x}{2}}{\frac{\pi}{2}}\Bigg|_{-1}^{+1} - \int_{-1}^{1}\frac{\sin\frac{\pi x}{2}}{\frac{\pi}{2}}(-2x)dx\right]=1$$

$$A\left[0+\frac{4}{\pi}\int_{-1}^{1}x\sin\frac{\pi x}{2}dx\right]=1$$

$$\frac{4A}{\pi}\left[\frac{x\left(-\cos\frac{\pi x}{2}\right)}{\frac{\pi}{2}}\Bigg|_{-1}^{+1} - \int_{-1}^{1}\frac{\left(-\cos\frac{\pi x}{2}\right)}{\frac{\pi}{2}}dx\right]=1$$

$$\frac{8A}{\pi^2}\left[-x\cos\frac{\pi x}{2}+\frac{2}{\pi}\sin\frac{\pi x}{2}\right]_{-1}^{+1}=1$$

$$\frac{8A}{\pi^2}\left[-\cos\frac{\pi}{2}+\frac{2}{\pi}\sin\frac{\pi}{2}-\left(-(-1)\cos\left(\frac{-\pi}{2}\right)+\frac{2}{\pi}\sin\left(\frac{-\pi}{2}\right)\right)\right]=1$$

$$\frac{8A}{\pi^2}\left[0+\frac{2}{\pi}-\frac{2}{\pi}(-1)\right]=1$$

$$\frac{8A}{\pi^2}\left[\frac{4}{\pi}\right]=1$$

$$A=\frac{\pi^3}{32}$$

Example 2.32

Let x be a continuous random variable with density function.

$$f(x) = \begin{cases} \dfrac{x}{9} + K & 0 \le x \le 6 \\ 0 & \text{otherwise} \end{cases}$$

i) Find the value of K ii) Find $P[2 \le x \le 5]$ ***May-2011***

Solution

Given

$$f(x) = \begin{cases} \dfrac{x}{9} + K & 0 \le x \le 6 \\ 0 & \text{otherwise} \end{cases}$$

(i) If the function is a valid pdf then

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\int_0^6 \left(K + \frac{x}{9}\right)dx = 1$$

$$\left(Kx + \frac{x^2}{18}\right)\Big|_0^6 = 1$$

$$6K + \frac{36}{18} = 1$$

$$6K + 2 = 1$$

$$K = \frac{-1}{6}$$

$$\therefore \quad f_X(x) = \frac{x}{9} - \frac{1}{6}$$

(ii) $$P(2 \le x \le 5) = \int_2^5 \left(\frac{x}{9} - \frac{1}{6}\right) dx$$

$$= \left(\frac{x^2}{18} - \frac{x}{6}\right)\Big|_2^5$$

$$= \left(\frac{5^2}{18} - \frac{5}{6}\right) - \left(\frac{2^2}{18} - \frac{2}{6}\right) = \frac{25}{18} - \frac{5}{6} - \frac{4}{18} + \frac{2}{6}$$

$$= \frac{2}{3} = 0.667$$