Compensation design using Root Locus

13.8 Designing Lead Compensator using Root Locus

The procedure to design lead compensator is,

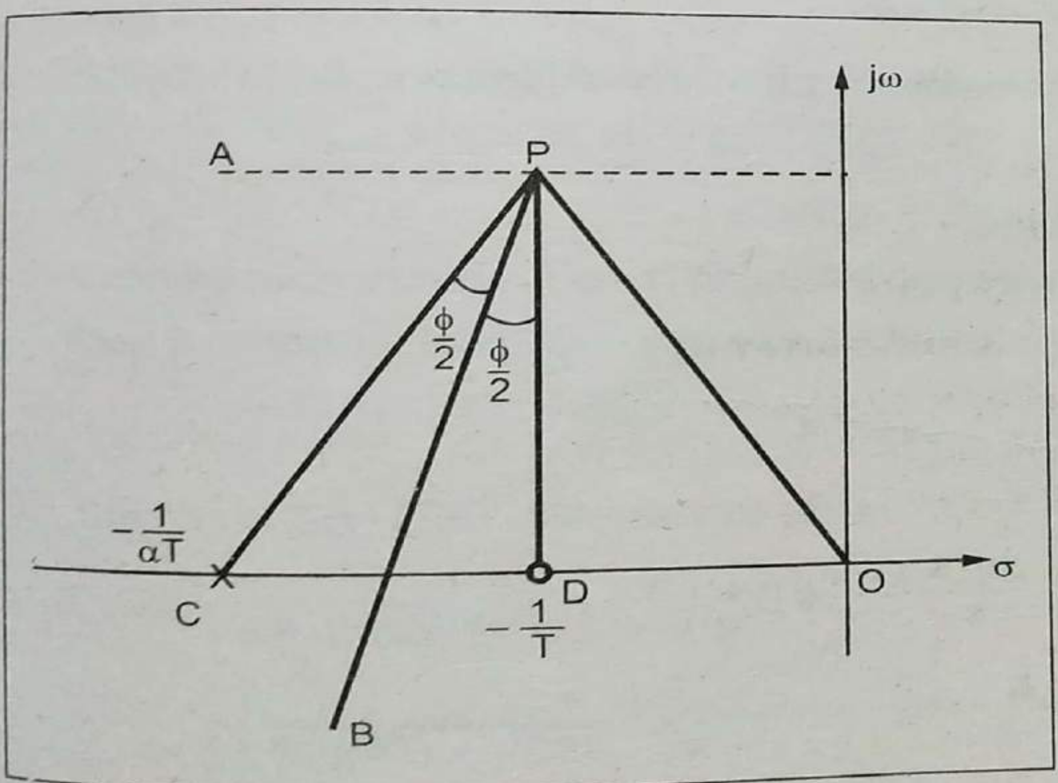
Step 1 : From the given specifications, find the desired locations of the dominant closed loop poles.

Step 2 : Assume the lead compensator as,

$$G_c(s) = K_c \alpha \frac{(1+Ts)}{(1+\alpha Ts)}, \alpha < 1$$

K_c is determined from the requirement of open loop gain.

Step 3 : Find the sum of the angles at the desired location of one of the dominant closed loop poles with the open loop poles and zeros of the original system. This angle must be an odd multiple of 180°. If it is not, calculate the necessary angle ϕ to be added to get the sum as an odd multiple of 180°. This ϕ must be contributed by lead compensator. If ϕ is more than 60° then two or more lead networks may be needed. This ϕ, helps to determine values of α and T.



Step 4 : To determine α and T for known ϕ, draw the horizontal line from one of the dominant closed loop pole say P. Join origin to P, as shown in the Fig. 13.24. Bisect the angle between the lines PA and PO. Draw the two line PC and PD that makes angle $\pm\frac{\phi}{2}$ with the bisector PB. The intersection of PC and PD with the negative real axis gives the necessary pole and zero of compensator.

Step 5 : The open loop gain can be determined by applying the magnitude condition at point P.

Step 6 : Check that the compensated system satisfies all the specifications. If not, adjust the compensator pole and zero till all the specifications are satisfied.

Ex. 13.4 *Design a suitable lead compensator for a system with unity feedback and having open loop transfer function :*

$$G(s) = \frac{K}{s(s+1)(s+4)}$$

to meet the specifications :

1. *Damping ratio $\xi = 0.5$*
2. *Undamped natural frequency $\omega_n = 2$ rad/sec*

Sol. : Let us sketch the root locus of the uncompensated system.

$$P = 3,\ Z = 0, N = P = 3, P - Z = 3$$

Starting points : 0, –1, –4

Terminating points : ∞, ∞, ∞

The angles of asymptotes : 60°, 180°, 300°

0 – 1 – 4

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Starting points : 0, –1, –4

Terminating points : ∞, ∞, ∞

The angles of asymptotes : 60°, 180°, 300°

Centroid : $$\frac{0-1-4}{3} = -1.67$$

Breakaway point : $1 + G(s)\,H(s) = 0$

$$\therefore \quad 1 + \frac{K}{s\,(s+1)\,(s+4)} = 0$$

$$\therefore \quad s^3 + 5\,s^2 + 4s + K = 0$$

$$\therefore \quad K = -s^3 - 5s^2 - 4s$$

$$\therefore \quad \frac{dK}{ds} = -3s^2 - 10s - 4 = 0$$

$$\therefore \quad s^2 + 3.33\,s + 1.33 = 0$$

$$\therefore \quad s = -0.464,\ -2.86$$

The point $s = -0.464$ is valid breakaway point.

Intersection with imaginary axis : The Routh's array is,

s	1	4
s	5	K
s	$\frac{20-K}{5}$	0

$$20 - K = 0$$

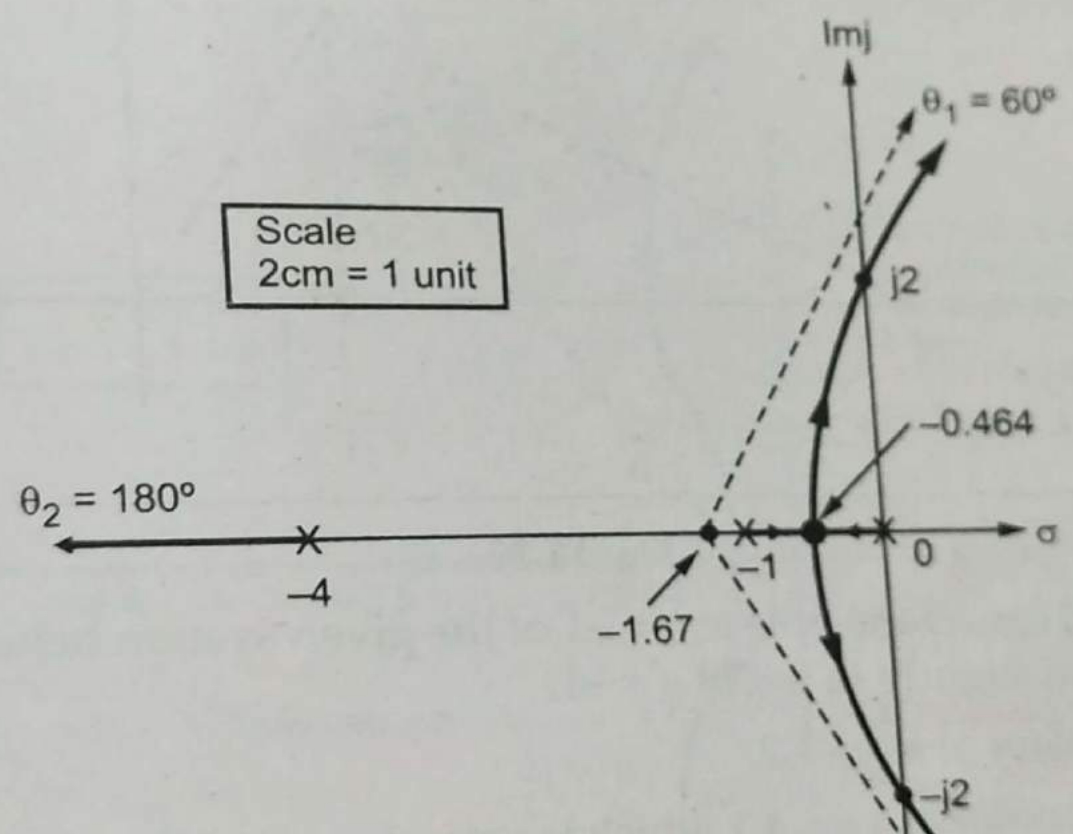
$$K_{mar} = 20$$

$\therefore$ $$5s^2 + K_{mar} = 0 \text{ is auxiliary equation}$$

$\therefore$ $$5\,s^2 = -20$$

$\therefore$ $$s = \pm j2 \quad \text{are intersection points}$$

$\therefore$ The root locus is as shown in the Fig. 13.25.



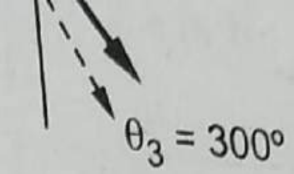


Fig. 13.25

Step 1 : $\xi = 0.5$ and $\omega_n = 2$

The desired dominant closed loop poles are,

$$= -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$$

$$= -1 \pm j\,1.73$$

It can be seen that dominant poles are not on the root locus shown in the Fig. 14.25.

Step 2 : Assume lead compensator with

$$G_c(s) = K_c\,\alpha\,\frac{(1+Ts)}{(1+\alpha\,Ts)}$$

Step 3 : $\angle G(s)\,H(s)$ at dominant pole

$$\therefore\ \angle\frac{K}{s\,(s+1)\,(s+4)} \text{ at } s = -1 + j\,1.73 \text{ is,}$$

$$\frac{\angle K}{\angle -1+j1.73\ \angle j1.73\ \angle 3+j1.73}$$

$$\frac{\angle K}{\quad} = \frac{\angle K}{\quad}$$

$$= \frac{0^\circ}{120.029^\circ \; 90^\circ \; 29.97^\circ} = -240^\circ$$

$\therefore$ Angle to be contributed by lead compensator is,

$$-180^\circ - (-240^\circ) = 60^\circ$$

Step 4 : Find locations of pole and zero.

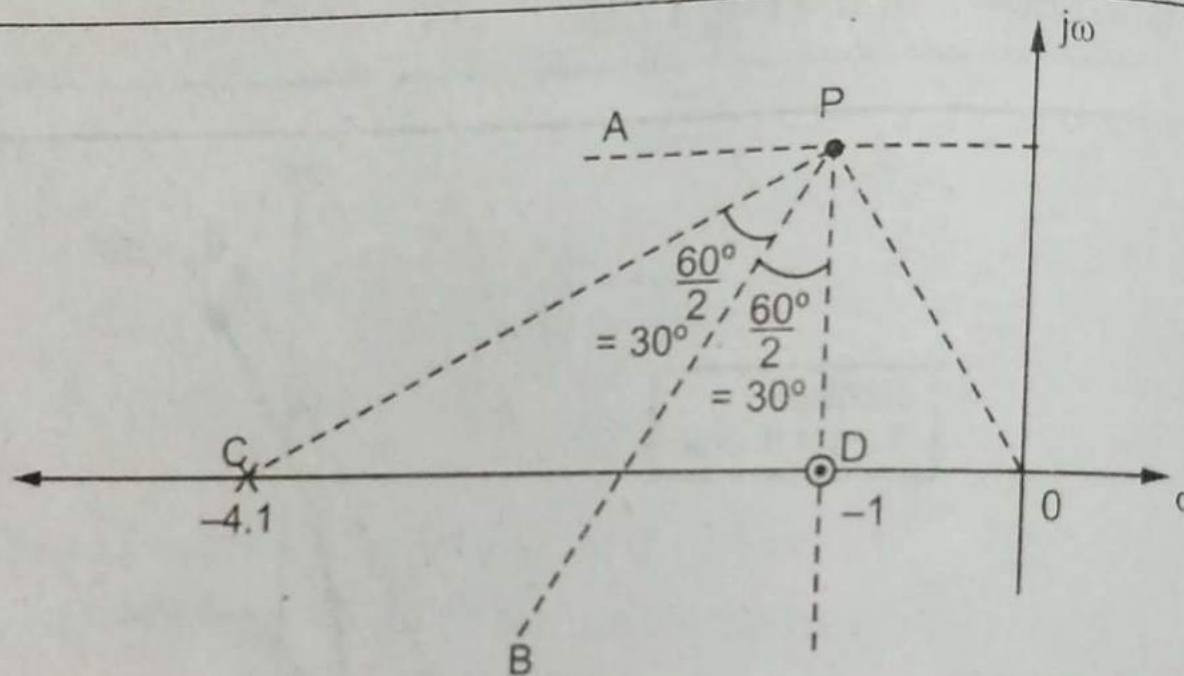


Fig. 13.26

$\therefore$ Zero at s= –1 will cancel the pole at s = –1 of the given system hence we will select the zero closer to s = –1 but slightly to left of s = –1.

$\therefore$ Zero of compensator at s = – 1.2

and pole of compensator ...
hence we will select it at s = – 4.6.

$$\therefore \quad \frac{1}{T} = 1.2 \quad \therefore T \doteq 0.833$$

and

$$\frac{1}{\alpha T} = 4.6 \quad \therefore \alpha T = 0.2173$$

$$\therefore \quad \alpha = 0.26 < 1$$

The transfer function of the compensated system is,

$$G_c(s)\, G(s) = \frac{K\,(1+0.833\,s)}{s\,(1+s)\,(s+4)\,(1+0.2173\,s)}$$

Step 5 : Use the magnitude condition to obtain value of K at s=–1 + j 1.73.

$$\left|G_c(s)\, G(s)\right|_{s=-1+j\,1.73} = 1$$

$$\therefore \quad \frac{K\,[1+0.833\,(-1+j\,1.73)\,]}{(-1+j\,1.73)\,(\,j\,1.73)\,(3+j\,1.73)\,[1+0.213\,(-1+j\,1.73)]} = 1$$

$$\therefore \quad \frac{K\left|0.167+j\,1.44\right|}{\left|-1+j\,1.73\right|\left|0+j\,1.73\right|\left|3+j\,1.73\right|\left|0.7827+j\,0.3759\right|} = 1$$

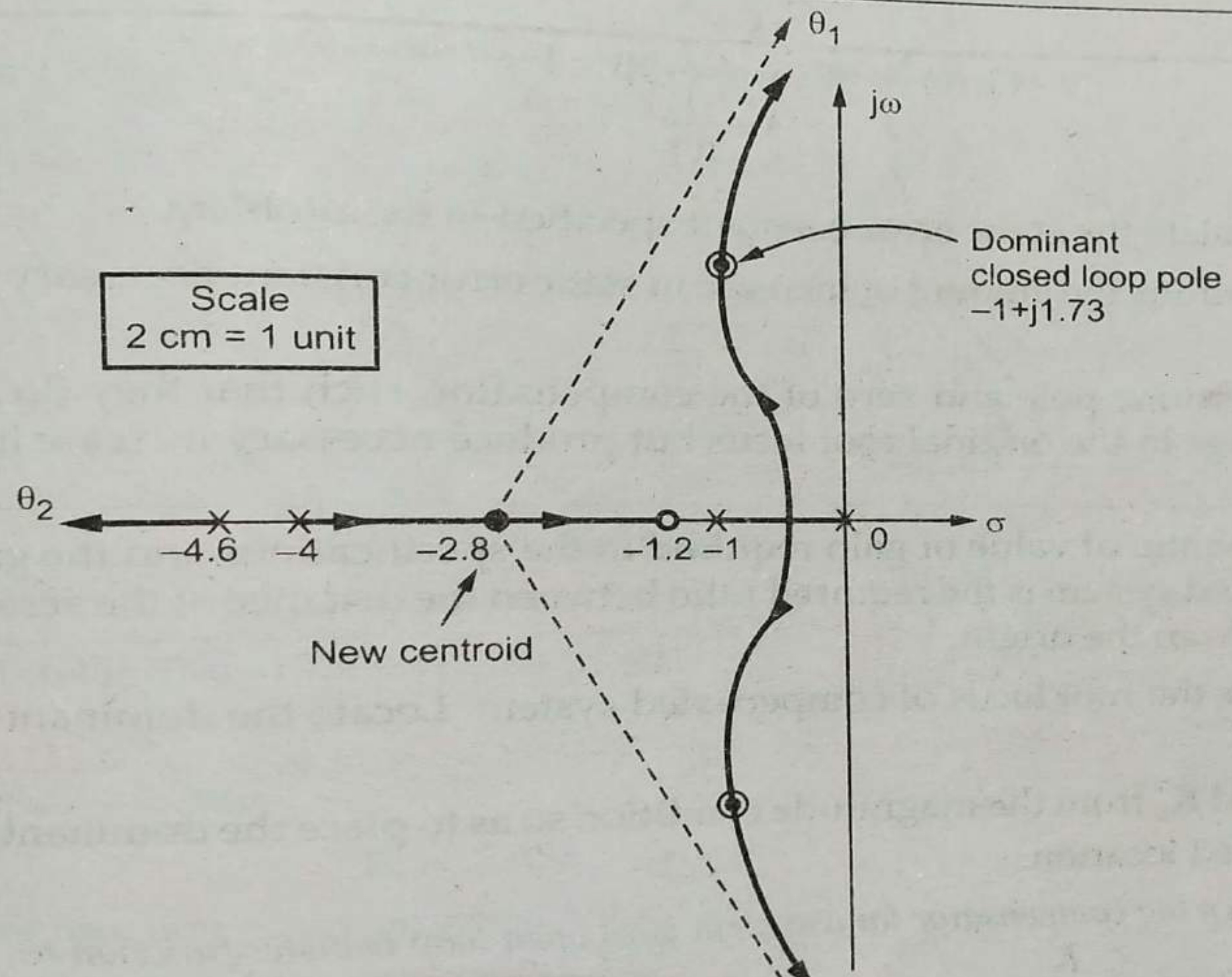
$$\therefore \quad \frac{K\times 1.449}{1.998\times 1.73\times 3.46\times 0.8682} = 1$$

$$\therefore \quad K = 7.166$$

$$\therefore \quad G_c(s)\,G(s) = \frac{7.166\,(1+0.833\,s)}{s\,(1+s)\,(s+4)\,(1+0.2173\,s)}$$

$$\therefore \quad G_c(s)\,G(s) = \frac{27.47\,(s+1.2)}{s\,(s+1)\,(s+4)\,(s+4.6)}$$

The root locus of the compensated system is shown in the Fig. 13.27. Students are expected to calculate breakaway points and intersection with the imaginary axis.



Steps to design the lag compensator are,

Step 1 : Draw the root locus of the uncompensated system and locate the domin
sed loop poles on the root locus.

Step 2 : Assume the lag compensator having transfer function,

$$G_C(s) = \hat{K}_c \frac{s + \frac{1}{T}}{s + \frac{1}{\beta T}}, \beta > 1$$

Step 3 : Calculate the static error constant specified in the problem.

Step 4 : Determine the amount of increase in static error constant necessary to satisfy
cification.

Step 5 : Determine pole and zero of the compensator, such that they do not prod
reciable change in the original root locus but produce necessary increase in static er
stant.

Note that the ratio of value of gain required in the specifications and the gain found
uncompensated system is the required ratio between the distance of the zero from ori
that of pole from the origin.

Step 6 : Draw the root locus of compensated system. Locate the dominant closed lo
es.

...13.5 *Design a lag compensator for a system with open loop transfer function as,*

$$G(s) = \frac{K}{s(s+1)(s+4)}$$

to meet the following specifications :

1. Damping ratio=0.5 *2. Velocity error constant ≥ 5sec*

3. Settling time=10 sec

Sol.: Let us draw the root locus of the uncompensated system. This is obtained in Ex.4 which is shown again in the Fig. 13.28 (See Fig. on next page)

From the given specifications, $\xi = 0.5$

and $$T_s = \frac{4}{\xi\omega_n} = 10$$

$\therefore$ $$\omega_n = 0.8 \text{ rad/sec}$$

Hence the dominant closed loop poles are,

$$= -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$$

$$= -0.4 \pm j\,0.693$$

The gain K at dominant pole $-0.4 + j\,0.693$ can be obtained by magnitude condit

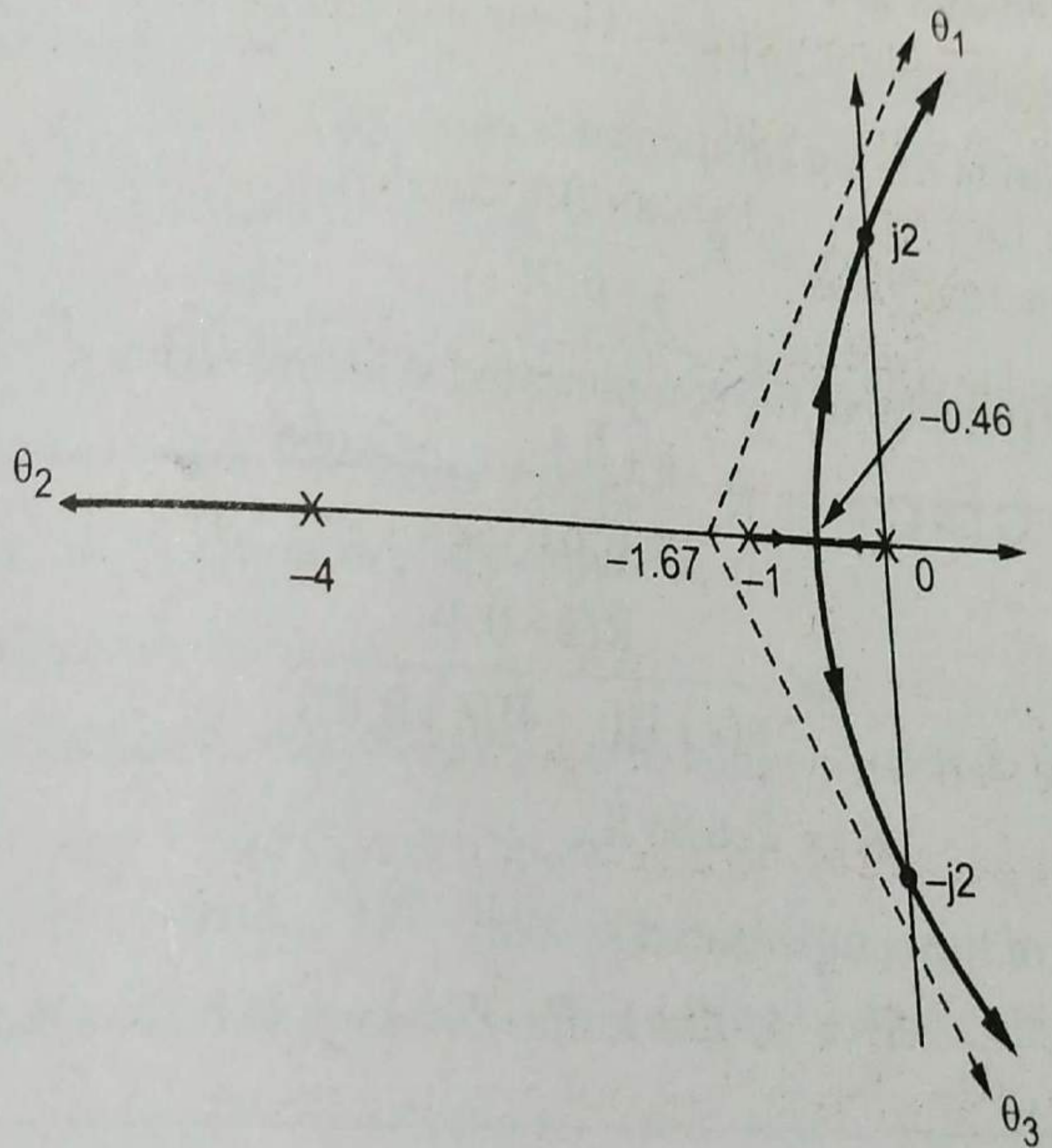


Fig. 13.28

$$\therefore \frac{K}{|-0.4+j0.693||0.6+j0.693||3.6+j0.693|} = 1$$

$$\therefore \frac{K}{0.8 \times 0.9167 \times 3.667} = 1$$

$$\therefore \quad K = 2.688$$

The transfer function of uncompensated system is,

$$G(s) = \frac{2.688}{s(s+1)(s+4)}$$

$$\therefore \quad K_v = \lim_{s \to 0} s\,G(s) = \lim_{s \to 0} \frac{s.2.688}{s(s+1)(s+4)}$$

$$= 0.672$$

It is desired to have K_v of 5 sec^{-1}. The factor by which static error constant is to be increased is,

$$\text{Factor} = \frac{K_v \text{ desired}}{K_v \text{ of uncompensated system}}$$

$$= \frac{5}{0.672}$$

$$= 7.44$$

Let us choose this factor be 10.

$$\therefore \quad \beta = 10$$

Place the zero and pole of the lag compensator very close to the origin.

Let zero of compensator at $s = -0.1$

$\therefore$ Pole of compensator at $s = -0.01$

So transfer function of the lag compensator is,

$$G_c(s) = \hat{K}_c \frac{s+0.1}{s+0.01}$$

Hence the transfer function of the compensated system becomes,

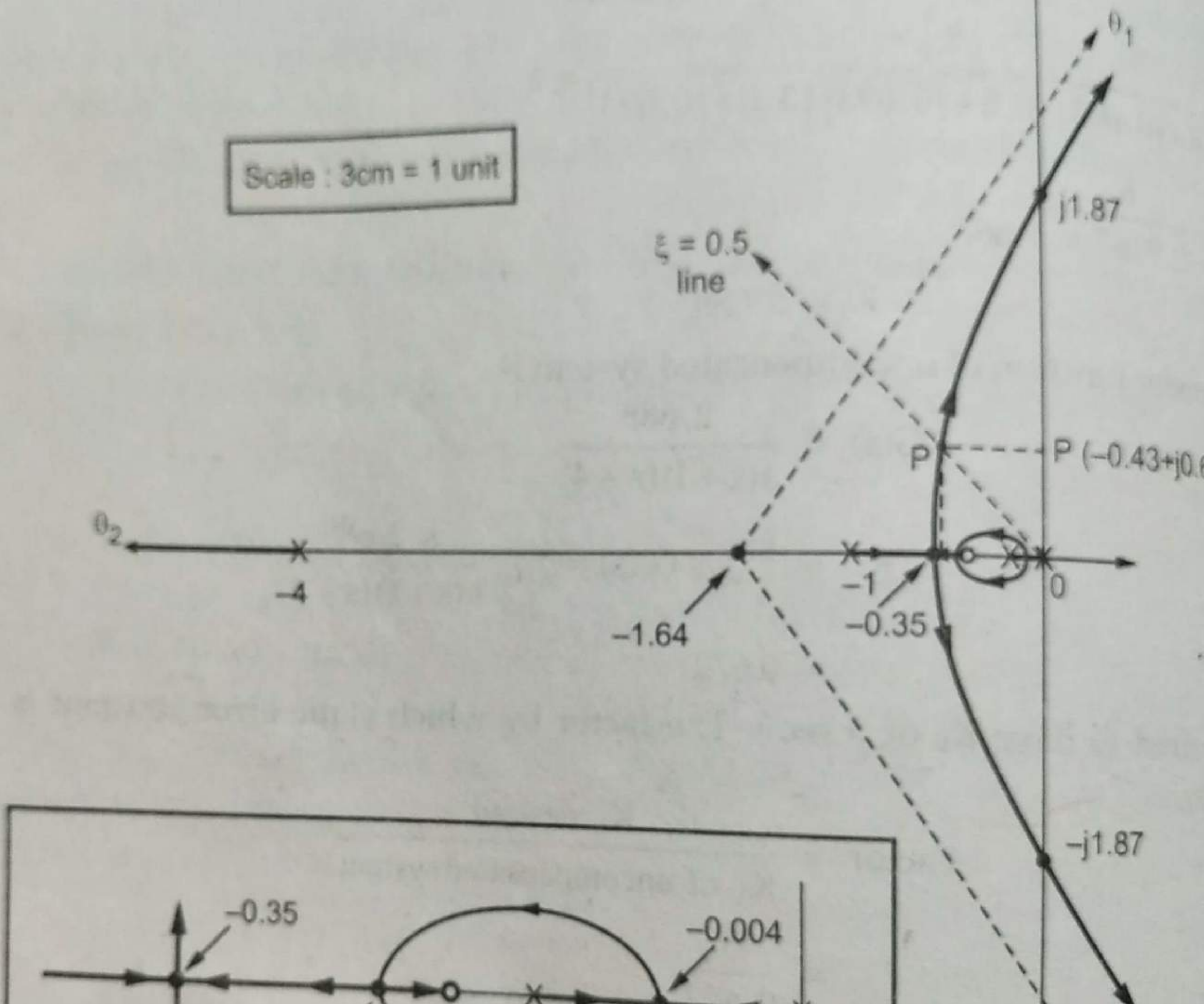
$$G_c(s)\,G(s) = \hat{K}_c \frac{s+0.1}{s+0.01} \cdot \frac{2.688}{s(s+1)(s+4)}$$

$$= \frac{K(s+0.1)}{s(s+1)(s+4)(s+0.01)}$$

where $$K = 2..688\,\hat{K}_c$$

Draw the root locus of the compensated system.

$$P = 4,\ Z = 1,\ P - Z = 4 - 1 = 3,\ N = 4$$



Scale : 3cm = 1 unit
θ1
j1.87
ξ = 0.5
line
P
P (−0.43+j0.
θ2
−4
−1.64
−1
−0.35
0
−j1.87
−0.35
−0.004

Starting points $\rightarrow 0, -1, -4, -0.01$

Terminating points $\rightarrow -0.1, \infty, \infty, \infty$

Angles of asymptotes = 60°, 180°, 300°

$$\text{Centroid} = -1.64$$

For breakaway point get $\frac{dK}{ds} = 0$. This gives the equation,

$$s^4 + 3.47\, s^3 + 1.85\, s^2 + 0.27\, s + 0.0013 = 0$$

This gives three approximate valid breakaway points as,

$s = -0.004, -0.26$ and -0.3511

Hence the root locus of compensated system is as shown in the Fig. 13.29 (See Fig. on previous page).

The new dominant pole is $-0.43 + j\,0.67$.

Apply magnitude condition at this new location of dominant closed loop pole.

$$|G_c(s)\,H(s)|_{s=-0.43+j0.67} = 1$$

$$\therefore \frac{|K|\,|-0.33+j\,0.67|}{|-0.43+j\,0.67|\,|0.57+j\,0.67|\,|3.57+j\,0.67|\,|-0.42+j\,0.67|} = 1$$

$$\therefore \frac{K \times 0.7468}{0.7961 \times 0.8786 \times 3.6323 \times 0.7907} = 1$$

$$\therefore \quad K = 2.69$$

But $\quad K = 2.688$

$$\therefore \quad \hat{K}_c = 1.0019$$

Hence the compensated system has the transfer function,

$$\frac{2.69(s+0.1)}{}$$