Propagation of a plane wave in an Ionized Medium

If a plane wave, traveling in the *z*-direction, and having its electric field along the x-axis and given by $E_x = E \sin \omega t$, moves in a region containing free electrons, the electric field will act on each electron with a force F_x given by:

$$F_x = -eE_x \tag{1}$$

The consequent acceleration is in the x-direction, and, from Newton's Law, is given by:

$$m\frac{d^2x}{dt^2} = -eE_x = -eE\sin \omega t \tag{2}$$

Integration gives the velocity of the electron as:

$$\frac{dx}{dt} = \frac{e}{m\omega}E\cos \omega t \tag{3}$$

If there are N electrons per cubic meter in the space, each carrying a charge –e, the current density I represented by this motion of electrons is:

$$I = -Ne\frac{dx}{dt} = -\frac{Ne^2}{m\omega}E\cos \omega t \tag{4}$$

This current must be included in the Maxwell field equations, and is a particular case of the term represented generally by σE. Thus the first Maxwell equation for plane wave has now an additional term, and becomes:

$$-\frac{Ne^2}{m\omega}E\cos\omega t+k_0\frac{\partial}{\partial t}(E\sin\omega t)=-\frac{\partial H_y}{\partial z} \qquad (5)$$

Which simplifies to:

$$\left(k_0-\frac{Ne^2}{m\omega^2}\right)\omega E\cos\omega t=-\frac{\partial H_y}{\partial z} \qquad (6)$$

This is now the same equation as would be given by first curl equation of Maxwell with k_0 replaced by $k_0 - Ne^2/m\omega^2$, and the medium can be said to have an effective permittivity which is less than that of free space. The ionized region thus has a relative permittivity of $1-Ne^2/k_0m\omega^2$.

What is known in optics as the refractive index is the square root of the relative permittivity and so we can obtain the refractive index 'n' of an ionized medium as:

$$n=\left(1-\frac{Ne^2}{k_0m\omega^2}\right)^{1/2} \qquad (7)$$

For an electron $e=1.59\times10^{-10}$ coulombs, m = 9 × 10–31 kg, and with f in Hertz, above equation becomes:

$$n=\left(1-\frac{81N}{f^2}\right)^{1/2} \qquad (8)$$

Equation (7) holds equality well for ions or electrons, but from its form it is clear that, for equal numbers and charges, electrons, being very much lighter than ions, will be much more effective in determining the refractive index.

In equation (7) ‘n’ is a function of frequency, and so the velocity of the wave in the medium is also frequency dependent. This is the same phenomenon as occurs in a wave-guide, giving rise to two velocities, a phase velocity v_p and a group velocity v_g. The phase velocity is $v_p = c/(k_r \mu_r)^{1/2}$, so that, μ, being unity:

$$v_p = \frac{c}{k_r^{1/2}} = \frac{c}{\left(1 - \frac{81N}{f^2}\right)^{1/2}} \tag{9}$$

Since $v_p v_g = c^2$, we can write down the expression for the group velocity as:

$$v_g = c\left[1 - \left(\frac{81N}{f^2}\right)^{1/2}\right] \tag{10}$$

In the collisions which take place between the electrons and the gas molecules, however, energy is dissipated in the form of heat, and this energy must be supplied by the wave. The effect may be considered as a ‘frictional’ force on the electron proportional to its velocity, and proportional to the collision frequency ν.

It can be shown that in this case the refractive index of the medium becomes:

$$n = \left\{ 1 - \frac{Ne^2}{k_0 m \left(\nu^2 + \omega^2 \right)} \right\}^{1/2} \tag{11}$$

and the medium has in addition an effective conductivity σ given by:

$$\sigma = \frac{Ne^2}{m \left(\nu^2 + \omega^2 \right)} \tag{12}$$

The collision frequency is dependent on the gas pressure, and is therefore a function of height. Values of νs are estimated to vary from about 6×10^8 per second at a height of 50 km to about 100 per second at a height of 400 km, these heights being the limits of the ionosphere. Below the lower edge of the ionosphere the electron density drops off rapidly; it is therefore around this lower edge that the conductivity is greatest.

In the higher regions of the ionosphere ω is normally much greater than ν, so that the expression (11) approximates to equation (7), and the value of σ becomes negligible.