Cross Power density spectrum

Objective

To determine the relationship between two time series as a function of Frequency using Cross spectral analysis

Module Description

Using **Cross-Spectral Density** i.e. **cross**-correlation, the **power** shared by a given frequency for the two signals using its squared module, and the phase shift between the two signals at that frequency using its argument can be found

Defining the Power Spectral Density of a random Process

Let

$$X_T(t) = X(t) \qquad -T < t < T$$
$$= 0 \qquad \text{otherwise}$$
$$= X(t)rect\left(\frac{t}{2T}\right)$$

and $Y_T(t) = \begin{matrix} Y(t) for -T < t < T \\ 0 \ otherwise \end{matrix}$

where $rect\left(\frac{t}{2T}\right)$ is the unity-amplitude rectangular pulse of width $2T$ cantered at origin. As $t \to \infty, X_T(t)$ will represent the random process $X(t)$. .Similarly $Y_T(t)$

- Define $F[X_T(t)] = X_T(\omega) = \int_{-T}^{T} X_T(t).e^{-j\omega t}\ .dt = \int_{-T}^{T} X(t).e^{-j\omega t}\ .dt$
- $F[Y_T(t)] = Y_T(\omega) = \int_{-T}^{T} Y_T(t).e^{-j\omega t}\ .dt = \int_{-T}^{T} Y(t).e^{-j\omega t}\ .dt$
- Consider the generalized Parseval's relation

$$.\int_{-\infty}^{\infty} X(t).Y(t)\,.dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega).Y^*(\omega)\,.d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(\omega).X^*(\omega)\,.d\omega$$

- Therefore, the average power P_{XY} is
- $\frac{1}{2T}\int_{-T}^{T} X_T(t)Y_T(t).dt = \frac{1}{2T}\int_{-T}^{T} X(t).Y(t).dt$

- The average power is given by

$$\frac{1}{2T}E\left[\int_{-T}^{T}X(t).Y(t).dt\right]=\frac{1}{2T}E\left[\int_{-\infty}^{\infty}X^{*}{}_{T}(\omega)Y_{T}(\omega).d\omega\right]=\left[\int_{-\infty}^{\infty}\frac{E[X^{*}{}_{T}(\omega)Y_{T}(\omega)]}{2T}d\omega\right]$$

- where $E\left[\int_{-\infty}^{\infty}\frac{X^{*}{}_{T}(\omega)Y_{T}(\omega)}{2T}d\omega\right]$ is the contribution to the average powerP_{XY} at frequency ω and represents the cross power spectral density of $X_T(t)$ and $Y_T(t)$. As $T\to\infty$, the left-hand side

  in the above expression represents the average power PXY . Therefore, the cross PSD $S(\omega)$ of the process X(t) and Y(t) is defined in the limiting sense by

$$S_{xy}(\omega)=\lim_{T\to\infty}\frac{E[X^{*}{}_{T}(\omega)Y_{T}(\omega)]}{2T}$$

Relation Between cross Power-spectral Density and Cross Correlation function of the Random Processes

We have PSD

$$S_{xy}(\omega)=\lim_{T\to\infty}\frac{E[X^{*}{}_{T}(\omega)Y_{T}(\omega)]}{2T}$$

- $X^{*}{}_{T}(\omega)=\int_{-T}^{T}X(t).e^{j\omega t}.dt$ and $Y_T(\omega)=\int_{-T}^{T}Y(t).e^{-j\omega t}.dt$
- $S_{xy}(\omega)=\lim\limits_{T\to\infty}\frac{1}{2T}E\left[\int_{-T}^{T}X(t_1).e^{j\omega t_1}.dt_1.\int_{-T}^{T}Y(t_2).e^{-j\omega t_2}.dt_2\right]$
- $=\lim\limits_{T\to\infty}\frac{1}{2T}E\left[\int_{-T}^{T}\int_{-T}^{T}X(t_1)\,Y(t_2).e^{-j\omega(t_2-t_1)}.dt_1dt_2.\right]$
- $=\lim\limits_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T}E[X(t_1)\,Y(t_2)].e^{-j\omega(t_2-t_1)}.dt_1dt_2$
- $=\lim\limits_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T}R_{xy}(t_1,t_2)\,e^{-j\omega(t_2-t_1)}.dt_1dt_2$

Consider the inverse Fourier Transform of cross PSD i.e.

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}S_{xy}(\omega)\,e^{j\omega\tau}.d\omega$$

- $F^{-1}[S_{xy}(\omega)]$ $\frac{1}{2\pi}\int_{-\infty}^{\infty}[\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T}R_{xy}(t_1,t_2)\,e^{-j\omega(t_2-t_1)}.dt_1dt_2]\,e^{j\omega\tau}.d\omega$ =
- $=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T}R_{xy}(t_1,t_2)\frac{1}{2\pi}.\int_{-\infty}^{\infty}e^{-j\omega(t_2-t_1)}.d\omega.dt_1dt_2$
- Since, $F[\delta(t)]=1,\rightarrow\frac{1}{2\pi}\int_{-\infty}^{\infty}1.e^{j\omega t}\,d\omega=\delta(t)$
- On similar lines, $\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{j\omega(\tau-t_2+t_1)}\,d\omega=\delta(\tau-t_2+t_1)$
- $F^{-1}[S_{xy}(\omega)]=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T}R_{xy}(t_1,t_2)\delta(\tau-t_2+t_1).dt_1dt_2$
- *since* $\delta(\tau-t_2+t_1)=1\ at\ \tau-t_2+t_1=0\ i.e.t_2=\tau+t_1$
- $F^{-1}[S_{xy}(\omega)]=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}R_{xy}(t_1,\tau+t_1)\,dt_1$
- Let $t_1=\tau\rightarrow dt_1=d\tau$
- Hence

$$F^{-1}[S_{xy}(\omega)]=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}R_{xy}(t,t+\tau)\,dt$$

- The RHS of the above eq. is the time average of cross correlation function.
- Thus, Time average of cross-correlation function and the cross spectral density form a Fourier Transform Pair.
- If the processes X(t) and Y(t) are jointly WSS processes, the time average of $Rxy\,(t,\,t+\tau)$will be $Rxy\,(\tau)$, since it is independent of time.
- Thus, for a two jointly WSS processes, cross-correlation and cross Spectral Density form a Fourier Transform Pair.

$$\boxed{\begin{aligned}S_{xy}(\omega)&=\int_{-\infty}^{\infty}R_{xy}(\tau)e^{-j\omega\tau}\,d\tau\\R_{xy}(\tau)&=\frac{1}{2\pi}\int_{-\infty}^{\infty}S_{xy}(\omega)\,e^{j\omega\tau}.d\omega\end{aligned}}$$

Properties of cross Power Spectral Density

1. $S_{yx}(\omega) = S_{xy}(-\omega)$
2. Real part of cross spectral density is an even function of ω and imaginary part is an odd function of ω
3. $S_{xy}(\omega) = 0$ if X(t) and Y(t) are orthogonal
4. $If\ X\ t\ and\ Y(t)$ are uncorrelated and of constant mean E(X) and E(Y) respectively, then , $S_{xy}(\omega) = 2\pi E[X]\, E[Y]\, \delta(\omega)$
5. $Power\ Spectral\ density\ of\ sum\ of\ random\ processes$

   Consider the random process $Z(t) = X(t) + Y(t)$ which is the sum of two jointly WSS random processes *X(t)* and *Y(t)*. We have,

$$R_{zz}(\tau) = E[Z(t).z(t+\tau)] = E[\{x(t)+y(t)\}\{x(t+\tau)+y(t+\tau)\}]$$

$$R_{zz}(\tau) = R_{xx}(\tau) + R_{yy}(\tau) + R_{xy}(\tau) + R_{yx}(\tau)$$

Taking the Fourier transform of both sides,

$$S_{zz}(\omega) = S_{xx}(\omega) + S_{yy}(\omega) + S_{xy}(\omega) + S_{yx}(\omega)$$

Since x(t) and y(t) are orthogonal, their cross spectral density is zero.

Hence,

$$S_{zz}(\omega) = S_{xx}(\omega) + S_{yy}(\omega)$$

(1) $S_{XY}(\omega) = S^*_{YX}(\omega)$

Note that $R_{XY}(\tau) = R_{YX}(-\tau)$

$$\therefore S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau)e^{-j\omega\tau}d\tau$$

$$= \int_{-\infty}^{\infty} R_{YX}(-\tau)e^{-j\omega\tau}d\tau$$

$$= \int_{-\infty}^{\infty} R_{YX}(\tau)e^{j\omega\tau}d\tau$$

$$= S^*_{YX}(\omega)$$

(2) $\operatorname{Re}(S_{XY}(\omega))$ is an even function of ω and $\operatorname{Im}(S_{XY}(\omega))$ is an odd function of ω

We have

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau)(\cos\omega\tau + j\sin\omega\tau)d\tau$$

$$= \int_{-\infty}^{\infty} R_{XY}(\tau)\cos\omega\tau d\tau + j\int_{-\infty}^{\infty} R_{XY}(\tau)\sin\omega\tau)d\tau$$

$$= \operatorname{Re}(S_{XY}(\omega)) + j\operatorname{Im}(S_{XY}(\omega))$$

where

$\operatorname{Re}(S_{XY}(\omega)) = \int_{-\infty}^{\infty} R_{XY}(\tau)\cos\omega\tau d\tau$ is an even function of ω and

$\operatorname{Im}(S_{XY}(\omega)) = \int_{-\infty}^{\infty} R_{XY}(\tau)\sin\omega\tau d\tau$ is an odd function of ω and

(3) If $X(t)$ and $Y(t)$ are orthogonal, then

$$S_{XY}(\omega) = S_{YX}(\omega) = 0$$

If $X(t)$ and $Y(t)$ are orthogonal,

$$R_{XY}(\tau) = EX(t+\tau)Y(t)$$

$$= 0$$

$$= R_{XY}(\tau)$$

$$\therefore S_{XY}(\omega) = S_{YX}(\omega) = 0$$

(4) $X(t)$ and $Y(t)$ are uncorrelated and have constant means, then

$$S_{XY}(\omega) = S_{YX}(\omega) = \mu_X\mu_Y\delta(\omega)$$

Observe that

$$
\begin{aligned}
R_{XY}(\tau) &= EX(t+\tau)Y(t) \\
&= EX(t+\tau)EY(t) \\
&= \mu_X \mu_Y \\
&= \mu_Y \mu_X \\
&= R_{XY}(\tau)
\end{aligned}
$$

$$\therefore S_{XY}(\omega) = S_{YX}(\omega) = \mu_X \mu_Y \delta(\omega)$$

(5) The *cross power* P_{XY} between $X(t)$ and $Y(t)$ is defined by

$$P_{XY} = \lim_{T \to \infty} \frac{1}{2T} E \int_{-T}^{T} X(t)Y(t)dt$$

Applying Parseval's theorem, we get

$$P_{XY} = \lim_{T \to \infty} \frac{1}{2T} E \int_{-T}^{T} X(t)Y(t)dt$$

$$
\begin{aligned}
&= \lim_{T \to \infty} \frac{1}{2T} E \int_{-\infty}^{\infty} X_T(t)Y_T(t)dt \\
&= \lim_{T \to \infty} \frac{1}{2T} E \frac{1}{2\pi} \int_{-\infty}^{\infty} FTX_T^*(\omega)FTY_T(\omega)d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \lim_{T \to \infty} \frac{EFTX_T^*(\omega)FTY_T(\omega)}{2T} d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XY}(\omega)d\omega
\end{aligned}
$$

$$\therefore P_{XY} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XY}(\omega)d\omega$$

Similarly,

$$
\begin{aligned}
P_{YX} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{YX}(\omega)d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XY}^*(\omega)d\omega \\
&= P_{XY}^*
\end{aligned}
$$

Illustrative Problems

1.A random process is defined as $Y(t) = X(t).Cos(\omega_0 t + \theta)$, where $X(t)$ is a WSS process, ω_0 is a real constant and θ is a uniform random variable over$(0,2\pi)$ an is independent of X(t). Find the PSD of Y(t).

Soln.:

$$R_{yy}(\tau) = E[Y(t).Y(t+\tau)] = E[X(t).Cos(\omega_0 t + \theta).X(t+\tau).Cos(\omega_0(t+\tau)+\theta)]$$

Since, θ and X(t) are independent of each other,

$$R_{yy}(\tau) = E[X(t).X(t+\tau)]E[Cos(\omega_0 t + \theta).Cos(\omega_0(t+\tau)+\theta)]$$

$$=\frac{1}{2}R_{xx}(\tau)Cos(\omega_0\tau)$$

PSD of Y(t) is Fourier Transform of $R_{yy}(\tau)$. i.e

$$F\left\{\frac{1}{2}R_{xx}(\tau)Cos(\omega_0\tau)\right\} = \frac{\pi}{2}[S_{xx}(\omega+\omega_0) + S_{xx}(\omega+\omega_0)]$$

2.A random process is given by $Z(t) = A.X(t) + B.Y(t)$, where A and B are real constants and X(t) and Y(t) are jointly WSS processes.

(i) Find the Power spectrum of Z(t) (ii) Find the cross power spectrum $S_{XZ}(\omega)$

Soln.:

$$(i)R_{zz}(\tau) = E[Z(t).Z(t+\tau)] = E[\{AX(t)+BY(t)\}.[\{AX(t+\tau)+BY(t+\tau)\}.]]$$

$$=A^2R_{xx}(\tau) + ABR_{xy}(\tau) + ABR_{yx}(\tau) + B^2R_{yy}(\tau)$$

Power spectrum of Z(t) is $S_{zz}(\omega) = F[R_{zz}(\tau)]$

$$=A^2S_{xx}(\omega) + B^2S_{yy}(\omega) + ABS_{xy}(\omega) + ABS_{yx}(\omega)$$

(ii)$S_{XZ}(\omega) = F[R_{xz}(\tau)]$

$$R_{xz}(\tau) = E[X(t).Z(t+\tau)] = E[X(t)\{A.X(t+\tau) + B.Y(t+\tau)\}]$$

$$=A.E[X(t)X(t+\tau)] + BE[X(t).Y(t+\tau)]=AR_{xx}(\tau) + B.R_{xy}(\tau)$$

$$S_{XZ}(\omega) = AS_{XX}(\omega) + BS_{XY}(\omega)$$

3.A stationary random process X(t) has a spectral density $S_{xx}(\omega) = \frac{16}{\omega^2+16}$ and an independent stationary Y(t) has a spectral density $S_{yy}(\omega) = \frac{\omega^2}{\omega^2+16}$. Assuming X(t) and Y(t) are of zero mean, find the (i) PSD of U(t)=X(t)+Y(t) (ii) $S_{XY}(\omega) and\ S_{XU}(\omega)$

Soln.:

(i) PSD of U(t)= PSD of X(t) +PSD of Y(t) =1

(ii)$S_{XY}(\omega) = F\left[R_{xy}(\tau)\right] = F[E\{X(t).Y(t+\tau)\}] = F[E\{X(t)\}.E\{Y(t+\tau)\}] = 0$

$$S_{XU}(\omega) = F[R_{xU}(\tau)] = F[E\{X(t).U(t+\tau)\}] = F[E\{X(t).\{X(t+\tau)+Y(t+\tau)\}\}] =$$

$$F[E\{X(t)\}.X(t+\tau)\} + E\{X(t)Y(t+\tau)\}] = F[R_{xx}(\tau)] + F\left[R_{xy}(\tau)\right] = S_{xx}(\omega) = \frac{16}{\omega^2+16}$$

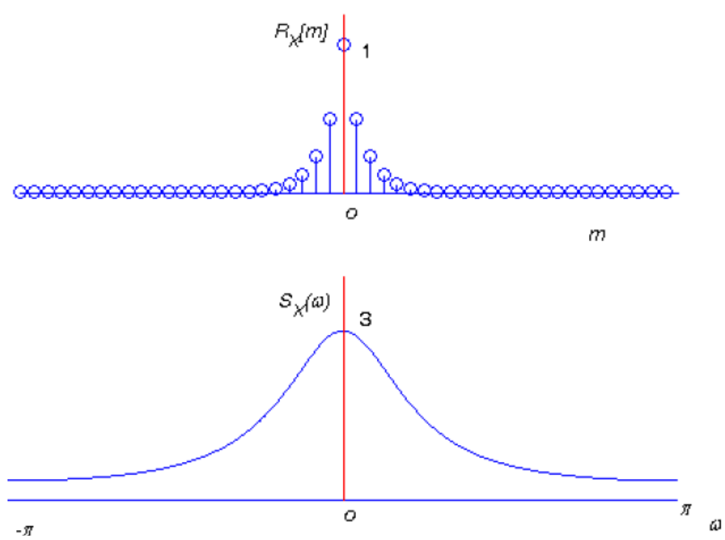
4. $R_X[m] = 2^{-|m|} \qquad m = 0, \pm1, \pm2, \pm3....$ Then

$$S_X(\omega) = \sum_{m=-\infty}^{\infty} R_X[m] e^{-j\omega m}$$

$$= 1 + \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \left(\frac{1}{2}\right)^{|m|} e^{-j\omega m}$$

$$= \frac{3}{5-4\cos\omega}$$

The plot of the autocorrelation sequence and the power spectral density is shown in Fig. below.



Exercise Problems

1.X(t) is WSS process with a PSD of $S_X(f)$. Find the PSD of Y(t)=X(2t-1).

2.The PSD of a real stationary random process X(t) is given by $S_X(f) = \begin{matrix} \frac{1}{W}\ for\ |f| \leq W \\ 0\ for\ |f| > W \end{matrix}$.

Then, find $E\left[\pi X(t)X(t-\frac{1}{4W})\right]$

3.Two random processes are given as $X(t) = Z_1(t) + 3Z_2(t-\tau)\ and\ Y(t) = 3Z_1(t-\tau) + Z_2(t+\tau)$

Where $Z_1(t)\ and\ Z_2(t)$ are independent white noise processes of zero mean and variance of 0.5. Find the autocorrelation of X(t), Y(t) and their cross correlation.

4. A real band limited random process X(t) has two sided PSD given by

$$S_x(f) = \begin{Bmatrix} \frac{10^{-6}(3000 - |f|\}watts}{Hz} for\ |f| \leq 3KHz \\ 0\ other\ wise \end{Bmatrix}$$

Where f is measured in Hz. The signal X(t) modulates a carrier cos16000πt and the resultant signal is passed through an ideal BPF of unity gain with centre frequency of 8KHz and bandwidth of 2KHz. Find the output power.

5.$X(t) = A.Cos(\omega_o t + \theta) and\ Y(t) = Z(t).Cos(\omega_o t + \theta)$ are two random processes, where A and ω_o are real positive constants. θ is a random variable and independent of Z(t), which is a random process with a constant mean $\bar{Z}$. Find the Cross spectral density of X(t) and Y(t).