CHAPTER 2

Equations of First Order But Not of the First Degree

2.1 INTRODUCTION

A differential equation in which we have only first order derivative *i.e.*, $\frac{dy}{dx}$ will be of first order as already defined, but since it is of higher degree it may involve $\frac{dy}{dx}$ raised to any power. In this chapter, we study differential equations of the first order and degree higher than the first.

2.2 DEFINITION

The general form of the first order differential equation of degree $n > 1$ is

$$P_0\left(\frac{dy}{dx}\right)^n + P_1\left(\frac{dy}{dx}\right)^{n-1} + P_2\left(\frac{dy}{dx}\right)^{n-2} + \ldots\ldots + P_{n-1}\left(\frac{dy}{dx}\right) + P_n = 0 \quad \ldots(1)$$

where $P_0, P_1, P_2, \ldots\ldots, P_{n-1}, P_n$ are functions of x and y.

It will be convenient for us to denote $\frac{dy}{dx}$ by p and so (1) becomes

$$P_0 p^n + P_1 p^{n-1} + P_2 p^{n-2} + \ldots\ldots + P_{n-1} p + P_n = 0$$

Such equations can be solved by various methods given in this chapter. In each of these methods, the given problem is reduced to that of solving one or more equations of the first order and the first degree (already discussed in Chapter 1)

2.3 METHOD I : EQUATIONS SOLVABLE FOR p

Let $P_0 p^n + P_1 p^{n-1} + P_2 p^{n-2} + \ldots\ldots + P_{n-1} p + P_n = 0 \quad \ldots(1)$

be the given differential equation of the first order and degree $n > 1$.

Resolving the left hand side of (1) into n linear factors, we have

$$[p - f_1(x,y)][p - f_2(x,y)]\ldots\ldots[p - f_n(x,y)] = 0 \quad \ldots(2)$$

Equating each factor equal to zero, we obtain n equations of the first order and the first degree, namely,

$$p = \frac{dy}{dx} = f_1(x,y), p = \frac{dy}{dx} = f_2(x,y), \ldots\ldots, p = \frac{dy}{dx} = f_n(x,y) \quad \ldots(3)$$

Let the solutions of these equations will be of the type

$$F_1(x,y,c_1) = 0, F_2(x,y,c_2) = 0, \ldots\ldots, F_n(x,y,c_n) = 0$$

where $c_1, c_2, \ldots\ldots, c_n$ are the arbitrary constants of integration.

We have already stated that constants occurring in the differential equation can be made to take any value whatsoever we like. Hence there is no loss of generality if we replace all the constants $c_1, c_2, \ldots\ldots, c_n$ by another constant c.

Hence the 'n' solutions become

$$F_1(x,y,c)=0, F_2(x,y,c)=0, \ldots\ldots, F_n(x,y,c)=0$$

Combining the above equations the solution of differential equation (1) can be written as

$$F_1(x,y,c)\cdot F_2(x,y,c)\cdot \ldots\ldots\cdot F_n(x,y,c)=0$$

Note : Since the given equation (1) is of the first order, its general solution cannot have more than one arbitary constant.

SOLVED EXAMPLES

Example 1. Solve the following differential equations

(*i*) $p^2 = ax^3$, where $p=\dfrac{dy}{dx}$ (*ii*) $\left(\dfrac{dy}{dx}\right)^3 - ax^4 = 0$

(*iii*) $p^2-5p+6=0$, where $p=\dfrac{dy}{dx}$ (*iv*) $p^2-7p+12=0$

(*v*) $p^2-2p\cosh x+1=0$ where $p=\dfrac{dy}{dx}$

Solution : (*i*) The given differential equation can be written as

$p=\pm(ax^3)^{1/2}=\pm a^{1/2}x^{3/2}$

Here $p=\dfrac{dy}{dx}=\pm a^{1/2}x^{3/2}$

Separating the variables, we get

$dy=\pm a^{1/2}x^{3/2}dx$

$\therefore$ The solution is $\int dy=\pm a^{1/2}\int x^{3/2}dx$ $\left[\text{Apply}\int x^n dx=\dfrac{x^{n+1}}{n+1}\right]$

$$\Rightarrow y=\pm a^{1/2}\left(\frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1}\right)+c=\pm\frac{2}{5}a^{1/2}x^{5/2}+c=\pm\frac{1}{5}(2a^{1/2}x^{5/2}+5c)$$

$\Rightarrow 5(y+c)=\pm 2a^{1/2}x^{5/2}$

Squaring on both sides, we get

$25(y+c)^2=(\pm 2a^{1/2}x^{5/2})^2=4ax^5$, which is the required solution.

(*ii*) Given $\left(\dfrac{dy}{dx}\right)^3=ax^4$ so that $\dfrac{dy}{dx}=(ax^4)^{1/3}=a^{1/3}x^{4/3}$

$\Rightarrow dy=a^{1/3}x^{4/3}dx$ (Variables Separable)

$\therefore$ The solution is

$\int dy=\int a^{1/3}x^{4/3}dx+c$

i.e., $y=a^{1/3}\dfrac{x^{\frac{4}{3}+1}}{\frac{4}{3}+1}+c=\dfrac{3}{7}a^{1/3}x^{7/3}+c$ $\left[\because \int x^n dx=\dfrac{x^{n+1}}{n+1}\right]$

or $y+c=\frac{3}{7}a^{1/3}x^{7/3}$ or $7(y+c)=3\,a^{1/3}x^{7/3}$

Cubing on both sides, we get

$343(y+c)^3=(3a^{1/3}x^{7/3})^3=27ax^7$

which is the required solution.

(*iii*) Resolving into linear factors, $p^2-5p+6=0$ becomes $(p-2)(p-3)=0$

$\Rightarrow p-2=0$ and $p-3=0 \Rightarrow p=2$ and $p=3$

So $p=\frac{dy}{dx}=2$ and $p=\frac{dy}{dx}=3$

$\Rightarrow dy=2\,dx$ and $dy=3\,dx$

Integrating, $\int dy=\int 2\,dx+c$ and $\int dy=\int 3\,dx+c$

$\Rightarrow y=2x+c$ and $y=3x+c \Rightarrow y-2x-c=0, y-3x-c=0$

$\therefore$ The required solution is $(y-2x-c)(y-3x-c)=0$

(*iv*) Given differential equation is

$p^2-7p+12=0 \Rightarrow (p-3)(p-4)=0$

$\Rightarrow p-3=0$ and $p-4=0 \Rightarrow p=3$ and $p=4$

So $\frac{dy}{dx}=3$ and $\frac{dy}{dx}=4$

$\Rightarrow dy=3\,dx$ and $dy=4\,dx$

Integrating, $y=3x+c$ and $y=4x+c$

$\Rightarrow y-3x-c=0$ and $y-4x-c=0$

Hence the required combined solution is

$(y-3x-c)(y-4x-c)=0$, c being an arbitrary constant.

(*v*) We know that $\cosh x=\frac{e^x+e^{-x}}{2}$

$\therefore$ The given differential equation can be written as

$$p^2-2p\left(\frac{e^x+e^{-x}}{2}\right)+1=0 \Rightarrow p^2-p(e^x+e^{-x})+1=0$$

$$\Rightarrow p^2-pe^x-pe^{-x}+e^x.e^{-x}=1 \qquad [\because e^xe^{-x}=e^0=1]$$

$$\Rightarrow p(p-e^x)-e^{-x}(p-e^x)=0$$

$$\Rightarrow (p-e^x)(p-e^{-x})=0 \Rightarrow p-e^x=0 \text{ and } p-e^{-x}=0$$

$$\Rightarrow p=\frac{dy}{dx}=e^x \text{ and } p=\frac{dy}{dx}=e^{-x}$$

$$\Rightarrow dy=e^x dx \text{ and } dy=e^{-x}dx$$

Integrating, $\int dy = \int e^x dx + c$ and $\int dy = \int e^{-x} dx + c$

$\Rightarrow y = e^x + c$ and $y = (-e^{-x}) + c$

$\Rightarrow y - e^x - c = 0$ and $y + e^{-x} - c = 0$

Hence the required solution is

$(y - e^x - c)(y + e^{-x} - c) = 0$, c being an arbitrary constant.

Example 2. Solve (*i*) $y\left(\frac{dy}{dx}\right)^2 + (x - y)\frac{dy}{dx} - x = 0$

(*ii*) $x^2p^2 + xyp - 6y^2 = 0$

(*iii*) $p(p + y) = x(x + y)$

Solution : (*i*) Given equation can be written as

$yp^2 + (x - y)p - x = 0$ ... (1)

which is quadratic in p.

So (1) can be resolved into linear factors

$yp^2 + xp - yp - x = 0$

i.e., $yp(p - 1) + x(p - 1) = 0$

i.e., $(p - 1)(yp + x) = 0$

Its components are $p - 1 = 0$ and $yp + x = 0$

or $\frac{dy}{dx} = 1$ and $y\frac{dy}{dx} = -x$

or $dy = dx$ and $y\,dy = -x\,dx$

Integrating, we get

$\int dy = \int dx + c$ and $\int y\,dy = -\int x\,dx + c$

i.e., $y = x + c$ and $\frac{y^2}{2} = \frac{-x^2}{2} + c$

or $y - x - c = 0$ and $x^2 + y^2 - c = 0$

Hence the general solution is

$(y - x - c)(x^2 + y^2 - c) = 0$

(*ii*) Given equation is

$x^2p^2 + xyp - 6y^2 = 0$ ... (1)

which is quadratic in p.

So (1) can be resolved into linear factors.

$$x^2p^2+3xyp-2xyp-6y^2=0$$
$$\Rightarrow xp\,(xp+3y)-2y(xp+3y)=0$$
$$\Rightarrow (xp+3y)(xp-2y)=0$$

Its components are $xp+3y=0$ and $xp-2y=0$

$$\Rightarrow x\frac{dy}{dx}=-3y \text{ and } x\frac{dy}{dx}=2y$$

$$\Rightarrow \frac{dy}{y}=-3\frac{dx}{x} \text{ and } \frac{dy}{y}=\frac{2dx}{x} \text{ (Variables Separable)}$$

Integrating, we get

$$\int\frac{dy}{y}=(-3)\int\frac{dx}{x}+c \text{ and } \int\frac{dy}{y}=2\int\frac{dx}{x}+c$$
$$\Rightarrow \log y=-3\log x+\log c \text{ and } \log y=2\log x+\log c$$
$$\Rightarrow \log y+3\log x=\log c \text{ and } \log y-2\log x=\log c$$
$$\Rightarrow \log yx^3=\log c \text{ and } \log\left(\frac{y}{x^2}\right)=\log c$$
$$\Rightarrow yx^3=c \text{ and } \frac{y}{x^2}=c$$

$\therefore$ The required combined solution is

$$(yx^3-c)\left(\frac{y}{x^2}-c\right)=0$$

(*iii*) Given equation is

$$p(p+y)=x(x+y)$$

i.e. $p^2+py-x^2-xy=0$

i.e., $(p^2-x^2)+y(p-x)=0$

i.e. $(p-x)(p+x)+y(p-x)=0$

or $(p-x)(p+x+y)=0$

Its components are $p-x=0$ and $p+x+y=0$

$\Rightarrow \frac{dy}{dx}=x$ and $\frac{dy}{dx}+y=-x$, which is linear.

Here I.F. $= e^{\int 1.\,dx}=e^x$

$\therefore$ The solution is

$$y=\int x\,dx+c \text{ and } ye^x=-\int xe^x\,dx+c$$

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$$\Rightarrow y=\frac{x^2}{2}+c \text{ and } ye^x=-(x-1)e^x+c \text{ or } y=-(x-1)+ce^{-x}$$

$$\Rightarrow y-\frac{x^2}{2}-c=0 \text{ and } y+(x-1)-ce^{-x}=0$$

Hence the general solution is

$$(2y-x^2-c)(y+x-1-ce^{-x})=0$$

Example 3. Solve $\frac{dy}{dx}-\frac{dx}{dy}=\frac{x}{y}-\frac{y}{x}$

Solution : Given equation is

$$p-\frac{1}{p}=\frac{x}{y}-\frac{y}{x} \text{ where } p=\frac{dy}{dx}$$

$$\text{or } \frac{p^2-1}{p}=\frac{x}{y}-\frac{y}{x} \text{ or } p^2+p\left(\frac{y}{x}-\frac{x}{y}\right)-1=0$$

which is quadratic in p. Here $a=1, b=\frac{y}{x}-\frac{x}{y}$ and $c=-1$

$$\therefore p=\frac{b\pm\sqrt{b^2-4ac}}{2a}=\frac{-\left(\frac{y}{x}-\frac{x}{y}\right)\pm\sqrt{\left(\frac{y}{x}-\frac{x}{y}\right)^2+4}}{2}$$

$$=\frac{-\left(\frac{y}{x}-\frac{x}{y}\right)\pm\sqrt{\left(\frac{y}{x}-\frac{x}{y}\right)^2+4.\frac{y}{x}.\frac{x}{y}}}{2}=\frac{-\left(\frac{y}{x}-\frac{x}{y}\right)\pm\sqrt{\left(\frac{y}{x}+\frac{x}{y}\right)^2}}{2}$$

$$=\frac{\left(\frac{x}{y}-\frac{y}{x}\right)\pm\left(\frac{x}{y}+\frac{y}{x}\right)}{2} \quad \left[\because (a-b)^2+4ab=(a+b)^2\right]$$

$$=\frac{x}{y} \text{ or } -\frac{y}{x}$$

Now $p=\frac{x}{y}$ and $p=\frac{-y}{x}$

i.e., $\frac{dy}{dx}=\frac{x}{y}$ and $\frac{dy}{dx}=-\frac{y}{x}$

i.e., $x\,dx=y\,dy$ and $\frac{dy}{y}=\frac{-dx}{x}$

Integrating, $\frac{x^2}{2}=\frac{y^2}{2}+c$ and

$$\log y=-\log x+\log c \Rightarrow \log x+\log y=\log c \Rightarrow \log xy=\log c$$

or $x^2-y^2-c=0$ and $xy-c=0$

$\therefore$ The required solution is $(x^2-y^2-c)(xy-c)=0$

Example 4. (*i*) Solve $p^2 + 2py\cot x = y^2$

(*ii*) If the curve whose differential equation is $p^2 + 2py\cot x = y^2$ passes through $\left(\frac{\pi}{2}, 1\right)$, show that the equation of the curve is given by $\left(2y - \sec^2\frac{x}{2}\right)\left(2y - \operatorname{cosec}^2\frac{x}{2}\right) = 0$.

Solution : (*i*) Given $p^2 + 2py\cot x = y^2$ or $p^2 + (2y\cot x)p - y^2 = 0$

which is quadratic in p.

Here $a = 1, b = 2y\cot x, c = -y^2$

Solving it for p, we get

$$p = \frac{-2y\cot x \pm \sqrt{4y^2\cot^2 x + 4y^2}}{2} = \frac{-2y\cot x \pm \sqrt{4y^2(\cot^2 x + 1)}}{2}$$

$$= \frac{-2y\cot x \pm 2y\operatorname{cosec} x}{2} = -y\cot x \pm \operatorname{cosec} x \left[\because 1 + \cot^2 A = \operatorname{cosec}^2 A\right]$$

Its components are

$$p = -y\cot x + y\operatorname{cosec} x \text{ or } \frac{dy}{dx} = -y(\cot x - \operatorname{cosec} x) \quad \text{... (1)}$$

$$\text{and } p = -y\cot x - y\operatorname{cosec} x \text{ or } \frac{dy}{dx} = -y(\cot x + \operatorname{cosec} x) \quad \text{... (2)}$$

From (1), we have

$$\frac{dy}{y} = -\left(\frac{\cos x}{\sin x} - \frac{1}{\sin x}\right)dx = \left(\frac{1-\cos x}{\sin x}\right)dx = \frac{2\sin^2(x/2)}{2\sin(x/2)\cos(x/2)} = \tan\left(\frac{x}{2}\right)dx$$

$$\left[\because 1 - \cos\theta = 2\sin^2(\theta/2) \text{ and } \sin A = 2\sin(A/2)\cos(A/2)\right]$$

$$\text{Integrating, } \log y = \frac{\log\sec\left(\frac{x}{2}\right)}{1/2} + \log c = 2\log c\sec\left(\frac{x}{2}\right) = \log c\sec^2\left(\frac{x}{2}\right)$$

$$\Rightarrow y = c\sec^2\left(\frac{x}{2}\right) \quad \text{... (3)}$$

From (2), we have

$$\frac{dy}{y} = -\left(\frac{\cos x}{\sin x} + \frac{1}{\sin x}\right)dx = -\left(\frac{1+\cos x}{\sin x}\right)dx = \frac{-\left(2\cos^2\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}dx = -\cot\left(\frac{x}{2}\right)dx$$

$$\left[\because 1 + \cos\theta = 2\cos^2(\theta/2)\right]$$

$$\text{Integrating, } \log y = \frac{-\log\sin(x/2)}{1/2} + c = -2\log\sin\left(\frac{x}{2}\right) + \log c$$

$$= \log c\left[\sin\left(\frac{x}{2}\right)\right]^{-2} = \log c\operatorname{cosec}^2\left(\frac{x}{2}\right)$$

$$\Rightarrow y = c\,\text{cosec}^2\left(\frac{x}{2}\right) \qquad \dots (4)$$

Equations (3) and (4) constitute the required solution.

Otherwise, combining these into one, the required general solution can be written as

$$\left(y - c\sec^2\left(\frac{x}{2}\right)\right)\left(y - c\,\text{cosec}^2\left(\frac{x}{2}\right)\right) = 0$$

(*ii*) From part (*i*), the general equation of the curve is

$$\left(y - c\sec^2\left(\frac{x}{2}\right)\right)\left(y - c\,\text{cosec}^2\left(\frac{x}{2}\right)\right) = 0 \qquad \dots (5)$$

Since it is passing through $\left(\frac{\pi}{2}, 1\right)$, we have

$$(1-2c)(1-2c) = 0 \Rightarrow (1-2c)^2 = 0 \Rightarrow c = \frac{1}{2}$$

Putting $c = \frac{1}{2}$ in (5), the equation of the required curve is

$$\left(y - \frac{1}{2}\sec^2\frac{x}{2}\right)\left(y - \frac{1}{2}\text{cosec}^2\frac{x}{2}\right) = 0$$

$$\Rightarrow \left(2y - \sec^2\frac{x}{2}\right)\left(2y - \text{cosec}^2\frac{x}{2}\right) = 0$$

EXERCISE 2.1

Solve the following differential equations :

1. $p^2 = x^5$
2. $x^2 = 1 + p^2$
3. $(p - 2x)(p + y) = 0$
4. $p^2 + p - 6 = 0$
5. $p^2 - 6p + 8 = 0$
6. $xp^2 + (y - x)p - y = 0$
7. $x^2\left(\frac{dy}{dx}\right)^2 + 3xy\frac{dy}{dx} + 2y^2 = 0$
8. $4xp^2 = (3x - a)^2$
9. $p(p + x) = y(x + y)$
10. $p(p - y) = x(x + y)$
11. $p^3 + 2xp^2 - y^2p^2 - 2xy^2p = 0$
12. $xyp^2 - (x^2 + y^2)p + xy = 0$

ANSWERS

1. $49(y + c)^2 = 4x^7$
2. $y = \pm\frac{x}{2}\sqrt{x^2 - 1} \mp \frac{1}{2}\log\left(x + \sqrt{x^2 - 1}\right) + c$
3. $(x^2 - y - c)(x + \log y + c) = 0$
4. $(y + 3x - c)(y - 2x - c) = 0$
5. $(y - 2x - c)(y - 4x - c) = 0$
6. $(y - x + c)(xy + c) = 0$

7. $(xy-c)(x^2y-c)=0$

8. $(y+c)^2 = x(x-a)^2$

9. $(y-ce^x)(x+y+1-ce^{-x})=0$

10. $(x^2+2y-c)(x+y+1-ce^x)=0$

11. $(y-c)(y+x^2-c)\left(x+\frac{1}{y}-c\right)=0$

12. $(x^2-y^2-c)(y-cx)=0$

2.4. METHOD II : EQUATIONS SOLVABLE FOR y

If the given equation is solvable for y, then we can express y explicitly as a function of x and p. Thus, an equation solvable for y can be put in the form

$$y = f(x, p) \quad \text{... (1)}$$

Differentiating (1) w.r.t. x and writing p for $\frac{dy}{dx}$, we get

$$\frac{dy}{dx} = p = \mathrm{F}\left(x, p, \frac{dp}{dx}\right) \quad \text{...(2)}$$

which is a differential equation of the first order in two variables p and x.

Let its solution be

$$\phi(x, p, c) = 0, \quad \text{... (3)}$$

c being an arbitrary constant.

Eliminating p between (1) and (3), we will get a relation between x, y and c which will be the required solution.

In some cases, the elimination of p between (1) and (3) is not possible. In that case (1) and (3) together constitute the solution, giving the values of x and y in terms of the parameter p and c *i.e.*, in the form

$$x = \mathrm{F}_1(p, c), y = \mathrm{F}_2(p, c) \quad \text{...(4)}$$

These two equations together form the general solution of (1) in the parametric form, where p is the parameter.

Note : In some problems (2) can be expressed as $f_1(x, p).\, f_2\left(x, p, \frac{dp}{dx}\right) = 0$

In such cases we ignore the first factor $f_1(x, p)$ which does not involve $\frac{dp}{dx}$ and proceed with $f_2\left(x, p, \frac{dp}{dx}\right) = 0$.

SOLVED EXAMPLES

Example 1 : Solve the following differential equations

(*i*) $y=(x-a)p-p^2$ (*ii*) $y=(1+p)x+p^2$

(*iii*) $y=x+a\tan^{-1}p$ (*iv*) $y=2px-p^2$

Solution : (*i*) Given equation is

$$y=(x-a)p-p^2 \quad \text{... (1)}$$

Differentiating (1) w.r.t. x and denoting $\frac{dy}{dx}$ by p, we get

$$\frac{dy}{dx}=p=p+(x-a)\frac{dp}{dx}-2p\frac{dp}{dx}$$

$$\Rightarrow 0=(x-a)\frac{dp}{dx}-2p\frac{dp}{dx}$$

$$\Rightarrow \frac{dp}{dx}(x-a-2p)=0 \Rightarrow \frac{dp}{dx}=0$$

Integrating, we get

$$p=c \qquad \text{... (2)}$$

Eliminating p between (1) and (2), we get

$$y=(x-a)c-c^2$$

which is the required solution.

(*ii*) Given equation is

$$y=(1+p)x+p^2 \qquad \text{... (1)}$$

Differentiating (1) w.r.t. x and denoting $\frac{dy}{dx}$ by p, we get

$$\frac{dy}{dx}=p=(1+p)+x\frac{dp}{dx}+2p\frac{dp}{dx}$$

$$\Rightarrow 0=1+(x+2p)\frac{dp}{dx} \Rightarrow (x+2p)\frac{dp}{dx}=-1$$

$$\Rightarrow \frac{dp}{dx}=\frac{-1}{x+2p} \text{ or } \frac{dx}{dp}=-x-2p$$

$$\text{or } \frac{dx}{dp}+x=-2p \qquad \text{... (2)}$$

which is linear equation in 'x'.

Here $P=1$ and $Q=-2p$

Now I. F. $=e^{\int P dp}=e^{\int 1.dp}=e^p$

$\therefore$ The solution of (2) is $x\,(\text{I.F.})=\int Q\,(\text{I.F.})\,dp+c$

i.e., $xe^p=\int(-2p)\,e^p dp+c=(-2)\int p\,e^p dp+c$ (Apply Integration by parts)

$$=(-2)(p-1)\,e^p+c \quad \left[\because \int te^t dt=(t-1)e^t\right]$$

$$\text{or } x=2(1-p)+ce^{-p} \qquad \text{... (3)}$$

Putting for x in (1), we get

$$y=(1-p)\left[2(1-p)+ce^{-p}\right]+p^2 \qquad \text{... (4)}$$

Equations (3) and (4) together give the required solution.

(*iii*) Given equation is

$$y=x+a\tan^{-1}p \qquad \text{... (1)}$$

Differentiating (1) w.r.t. x and denoting $\frac{dy}{dx}$ by p, we get

$$\frac{dy}{dx}=p=1+\frac{a}{1+p^2}\cdot\frac{dp}{dx}$$

$$\Rightarrow \frac{a}{1+p^2}\frac{dp}{dx}=p-1 \Rightarrow \frac{dp}{dx}=\frac{(p-1)(1+p^2)}{a}$$

Separating the variables, we get

$$\frac{dx}{a}=\frac{dp}{(p-1)(1+p^2)}=\left(\frac{A}{p-1}+\frac{Bp+C}{p^2+1}\right)dp \text{, using partial fractions}$$

Here $A=\frac{1}{2}, B=-\frac{1}{2}, C=-\frac{1}{2}$

$$\therefore \frac{dx}{a}=\frac{dp}{2(p-1)}-\frac{p+1}{2(p^2+1)}dp$$

or $dx=\frac{a}{2}\left[\frac{1}{p-1}-\frac{p}{p^2+1}-\frac{1}{p^2+1}\right]dp$

Integrating, $x=\frac{a}{2}\left[\log(p-1)-\frac{1}{2}\log(p^2+1)-\tan^{-1}p+\log c\right]$

$$=\frac{a}{2}\left[\log\left(\frac{c(p-1)}{\sqrt{p^2+1}}\right)-\tan^{-1}p\right] \quad \text{... (2)}$$

Putting for x in (1), we get

$$y=\frac{a}{2}\left[\log\left\{\frac{c(p-1)}{\sqrt{p^2+1}}\right\}-\tan^{-1}p\right]+a\tan^{-1}p$$

$$=\frac{a}{2}\left[\log\left\{\frac{c(p-1)}{\sqrt{p^2+1}}\right\}+\tan^{-1}p\right] \quad \text{... (3)}$$

Equations (2) and (3) together give the required solution in parametric form.

(iv) Given $y=2px-p^2$ ... (1)

Diff. (1) w.r.t. x, we get

$$\frac{dy}{dx}=p=2p+2x\frac{dp}{dx}-2p\frac{dp}{dx}$$

$$\Rightarrow p+2(x-p)\frac{dp}{dx}=0 \Rightarrow p\frac{dx}{dp}+2(x-p)=0$$

$$\Rightarrow \frac{dx}{dp}+\frac{2}{p}x-2=0 \Rightarrow \frac{dx}{dp}+\frac{2}{p}x=2 \quad \text{... (2)}$$

which is linear equation in 'x'.

Here $P=\frac{2}{p}$ and $Q=2$

Now I. F. $= e^{\int P dp} = e^{2\int \frac{1}{p} dp} = e^{2\log p} = e^{\log p^2} = p^2 \quad \left[\because n\log m = \log m^n\right]$

Thus the solution of (2) is

$x(\text{I.F.}) = \int Q\,(\text{I.F.})\, dp + c$

i.e., $xp^2 = \int 2p^2 dp + c = \dfrac{2p^3}{3} + c$

or $x = \dfrac{2}{3}p + cp^{-2}$ ... (3)

Putting this value of x in (1), we get

$$y = 2p\left[\frac{2}{3}p + cp^{-2}\right] - p^2 = \frac{4}{3}p^2 + 2cp^{-1} - p^2$$

$$= \left(\frac{4}{3} - 1\right)p^2 + 2cp^{-1} = \frac{1}{3}p^2 + 2cp^{-1} \quad \text{... (4)}$$

Equations (3) and (4) together give the required solution in parametric form.

Note : In general, the equations of the form $y = xf(p) + \phi(p)$, known as Lagrange's equation, are solvable for y and lead to linear equation in $\dfrac{dx}{dp}$ (see the above problem).

Example 2 . Solve the following differential equations

(*i*) $y + px = x^4p^2$

(*ii*) $x^2\left(\dfrac{dy}{dx}\right)^4 + 2x\dfrac{dy}{dx} - y = 0$ (or) $y = 2px + p^4x^2$

Solution : (*i*) The given equation can be written as

$y = -px + x^4p^2$ ... (1)

Diff. (1) w.r.t. x and denoting $\dfrac{dy}{dx}$ by p, we get

$$\frac{dy}{dx} = p = -p - x\frac{dp}{dx} + x^4.2p\frac{dp}{dx} + p^2.4x^3$$

$$\Rightarrow 2p + x\frac{dp}{dx} = 2px^3\left(x\frac{dp}{dx} + 2p\right)$$

$$\Rightarrow \left(2p + x\frac{dp}{dx}\right) - 2px^3\left(2p + x\frac{dp}{dx}\right) = 0$$

$$\Rightarrow \left(2p + x\frac{dp}{dx}\right)(1 - 2px^3) = 0$$

Neglecting the factor which does not involve $\dfrac{dp}{dx}$, we have

$$2p + x\frac{dp}{dx} = 0$$

$$\Rightarrow \frac{2}{x}dx + \frac{dp}{p} = 0$$

Integrating, we get

$2\log x + \log p = \log c$ or $\log px^2 = \log c$ or $px^2 = c$

$$\therefore\ p = \frac{c}{x^2} \qquad \text{... (2)}$$

Eliminating p between (1) and (2), we get

$$y = -x\left(\frac{c}{x^2}\right) + x^4 . \frac{c^2}{x^4} = \frac{-c}{x} + c^2 = \frac{-c + c^2 x}{x}$$

or $xy = -c + c^2 x$

which is the required solution.

(*ii*) Given equation can be written as

$$y = 2px + p^4 x^2 \text{ where } p = \frac{dy}{dx} \qquad \text{... (1)}$$

Diff. (1) w.r.t. x, we get

$$\frac{dy}{dx} = p = 2p + 2x\frac{dp}{dx} + p^4 . 2x + x^2 . 4p^3 \frac{dp}{dx}$$

$$\Rightarrow -p - 2xp^4 = (2x + 4x^2 p^3)\frac{dp}{dx}$$

$$\Rightarrow -p(1 + 2xp^3) = 2x(1 + 2xp^3)\frac{dp}{dx}$$

$$\Rightarrow p(1 + 2xp^3) + 2x(1 + 2xp^3)\frac{dp}{dx} = 0$$

$$\Rightarrow (1 + 2xp^3)\left(p + 2x\frac{dp}{dx}\right) = 0 \qquad \text{... (2)}$$

Neglecting the first factor which does not involve $\frac{dp}{dx}$, (2) reduces to

$$p + 2x\frac{dp}{dx} = 0 \Rightarrow 2x\frac{dp}{dx} = -p$$

$$\Rightarrow \frac{dp}{p} = -\frac{1}{2x}dx$$

Integrating, we get

$$\log p = -\frac{1}{2}\log x + \log c$$

$$\Rightarrow 2\log p = -\log x + \log c = \log\left(\frac{c}{x}\right)$$

$$\Rightarrow p^2 = \frac{c}{x} \qquad \text{... (3)}$$

Putting this value of p in (1), we get

$$y = 2x\left(\frac{c}{\sqrt{x}}\right) + \left(\frac{c^2}{x^2}\right)x^2 = 2\sqrt{x}c + c^2$$

$$\Rightarrow y - c^2 = 2\sqrt{x}c$$

Squaring, $(y - c^2)^2 = 4cx$

which is the required solution.

Example 3. Solve the following differential equations

(*i*) $y = 2px + f(xp^2)$, where $p = \frac{dy}{dx}$

(*ii*) $y = p\tan p + \log\cos p$, where $p = \frac{dy}{dx}$

(*iii*) $y = p\sin p + \cos p$, where $p = \frac{dy}{dx}$

Solution. (*i*) Given $y = 2px + f(xp^2)$ ... (1)

Diff. (1) w.r.t. x and writing p for $\frac{dy}{dx}$, we get

$$\frac{dy}{dx} = p = 2p + 2x\frac{dp}{dx} + f'(xp^2).\left[p^2 + 2xp\frac{dp}{dx}\right]$$

$$\Rightarrow p + 2x\frac{dp}{dx} + p^2 f'(xp^2) + 2xp.f'(xp^2)\frac{dp}{dx} = 0$$

$$\Rightarrow p\left[1 + p.f'(xp^2)\right] + 2x\frac{dp}{dx}\left[1 + p.f'(xp^2)\right] = 0$$

$$\Rightarrow \left[1 + p.f'(xp^2)\right]\left[p + 2x\frac{dp}{dx}\right] = 0 \quad \text{... (2)}$$

Neglecting the first factor which does not involve $\frac{dp}{dx}$, (2) reduces to

$$p + 2x\frac{dp}{dx} = 0 \Rightarrow 2x\frac{dp}{dx} = -p \Rightarrow 2\frac{dp}{p} = \frac{-dx}{x}$$

Integrating, $2\log p = -\log x + \log c \Rightarrow \log p^2 + \log x = \log c$

$$\Rightarrow p^2 x = c \text{ or } p = \frac{c}{\sqrt{x}}$$

Putting this value of p in (1), we get

$$y = 2x\left(\frac{c}{\sqrt{x}}\right) + f(c^2) \text{ or } y = 2c\sqrt{x} + f(c^2)$$

which is the required solution.

(ii) Given $y = p\tan p + \log\cos p$ ... (1)

Diff. (1) w.r.t. x, we get

$$\frac{dy}{dx} = p = \frac{dp}{dx}\tan p + p\sec^2 p\frac{dp}{dx} + \frac{1}{\cos p}(-\sin p)\frac{dp}{dx}$$

or $p = \frac{dp}{dx}\tan p + p\sec^2 p\frac{dp}{dx} - \tan p\frac{dp}{dx}$

$$= p\sec^2 p\frac{dp}{dx}$$

or $\sec^2 p\frac{dp}{dx} = 1$ or $dx = \sec^2 p\, dp$

Integrating, we get

$$\int dx = \int \sec^2 p\, dp + c \Rightarrow x = \tan p + c \qquad ...(2)$$

Equations (1) and (2) together form the required solution in parametric form.

(iii) Given $y = p\sin p + \cos p$ ... (1)

Differentiating (1) w.r.t. x and writing p for $\frac{dy}{dx}$, we get

$$\frac{dy}{dx} = p = p\cos p\frac{dp}{dx} + \frac{dp}{dx}\sin p - \sin p\frac{dp}{dx}$$

$$\Rightarrow p = p\cos p\frac{dp}{dx} \Rightarrow p\left(1 - \cos p\frac{dp}{dx}\right) = 0 \qquad ...(2)$$

Rejecting the first factor since it does not involve $\frac{dp}{dx}$, (2) gives

$$1 - \cos p\frac{dp}{dx} = 0 \Rightarrow \cos p\frac{dp}{dx} = 1 \Rightarrow \cos p\, dp = dx$$

Integrating, $\sin p = x + c$ or $x = c + \sin p$ ... (3)

(1) and (3) together form the required solution in parametric form.

EXERCISE 2.2

Solve the following differential equations :

1. $y = xp^2 + p$
2. $y = 2px + p^n$
3. $y = 2px + \tan^{-1}(xp^2)$
4. $y = \sin p - p\cos p$
5. $4y = x^2 + p^2$
6. $x = yp + ap^2$ (or) $y = \frac{x}{p} - ap$
7. $p^3 + p = e^y$ **[Hint: Take logarithm]**
8. $yp^2 - 2xp + y = 0$
9. $x^3p^2 + x^2py + a^3 = 0$
10. $y = 3x + \log p$
11. $y = x\left[p + \sqrt{(1+p^2)}\right]$
12. $y = 3px + 4p^2$

ANSWERS

1. $x = (\log p - p + c)(p-1)^{-2}, y = xp^2 + p$

2. $x = \frac{c}{p^2} - \frac{np^{n-1}}{n+1}, y = \frac{2c}{p} + \frac{1-n}{1+n}p^n$

3. $y = 2\sqrt{cx} + \tan^{-1} c$

4. $x = c - \cos p, y = \sin p - p\cos p$

5. $\log(p-x) = \frac{x}{p-x}, 4y = x^2 + p^2$

6. $x = \frac{p}{\sqrt{1+p^2}}(c + a\sin^{-1} p), y = \frac{1}{\sqrt{1+p^2}}(c + a\sin^{-1} p) - ap$

7. $x = -\frac{1}{p} + 2\tan^{-1} p + c, y = \log p + \log(1+p^2)$

8. $x = \frac{c}{p^2}(1+p^2), y = \frac{2c}{p}$

9. $c^2 + cxy + a^3 x = 0$ $\left(\text{Here } p = \frac{c}{x^2}\right)$

10. $y = 3x + \log\left(\frac{3}{1-ce^{3x}}\right)$

11. $x^2 + y^2 = 2cx$ (or) $x^2 + y^2 = cx$

12. $x = \frac{-8}{5}p + cp^{-3/2}, y = 3cp^{-1/2} - \frac{4}{5}p^2$

2.5 METHOD III : EQUATIONS SOLVABLE FOR x

If the given equation is solvable for x, then we can express x explicitly as a function of y and p. Thus the equation of this type can be written as

$$x = f(y, p) \qquad \text{... (1)}$$

Differentiating (1) w.r.t. y and writing $\frac{1}{p}$ for $\frac{dx}{dy}$, we get

$$\frac{dx}{dy} = \frac{1}{p} = F\left(y, p, \frac{dp}{dy}\right) \qquad \text{... (2)}$$

This equation when solved will give us a relation of the form

$$\phi(y, p, c) = 0, \ c \text{ being an arbitrary constant} \qquad \text{... (3)}$$

Eliminating p between (1) and (3), we get the required solution of (1) in the form

$$\psi(x, y, c) = 0$$

If the elimination of p between (1) and (3) is not possible, then we solve (1) and (3) to express x and y in terms of p and c in the form

$$x = f_1(p, c), y = f_2(p, c) \qquad \text{... (4)}$$

These two equations together will be the solution of (1) in terms of parameter p.

Sometimes even the form (4) of the desired solution is not possible. In that case (1) and (3) together constitute the solution giving x and y in terms of the parameter p.

Note 1 : In some problems equation (2) can be expressed as

$$F_1(y, p)\, F_2\left(y, p, \frac{dp}{dy}\right) = 0$$

In such cases, we ignore the first factor $F_1(y, p)$ which does not involve $\frac{dp}{dy}$ and proceed with $F_2\left(y, p, \frac{dp}{dy}\right) = 0$

Note 2 : If instead of ignoring the factor $F_1(y, p)$, we eliminate p between (1) and $F_1(y, p) = 0$ we obtain an equation involving no constant c.

This is known as singular solution of (1).

SOLVED EXAMPLES

Example 1. Solve the following differential equations

(*i*) $x = 3y - \log p$ (*ii*) $p^3 - 4xyp + 8y^2 = 0$

(*iii*) $x = 4(p + p^3)$

Solution. (*i*) Given $x = 3y - \log p$, where $p = \frac{dy}{dx}$ ... (1)

Differentiating (1) w.r.t. 'x' and writing $\frac{1}{x}$ for $\frac{dp}{dy}$, we get

$$\frac{dx}{dy} = \frac{1}{p} = 3 - \frac{1}{p}\frac{dp}{dy}$$

$$\Rightarrow \frac{1}{p} + \frac{1}{p}\frac{dp}{dy} = 3 \Rightarrow \frac{1}{p}\left(1 + \frac{dp}{dy}\right) = 3 \Rightarrow 1 + \frac{dp}{dy} = 3p$$

$$\Rightarrow \frac{dp}{dy} = 3p - 1 \Rightarrow dy = \frac{dp}{3p-1} \text{ (Variables Separable)}$$

Integrating, we get

$$\int dy = \int \frac{dp}{3p-1} + c$$

$$\Rightarrow y = \frac{1}{3}\log(3p-1) + \log c = \log(3p-1)^{1/3} c$$

or $e^y = c(3p-1)^{1/3}$

cubing, $ce^{3y} = 3p - 1$ or $p = \frac{1 + ce^{3y}}{3}$ ... (2)

Eliminating p between (1) and (2), we get

$$x = 3y - \log\left(\frac{1 + ce^{3y}}{3}\right)$$

which is the required solution.

(*ii*) Given equation is

$$p^3 - 4xyp + 8y^2 = 0 \qquad \text{... (1)}$$

Solving (1) for x, we get

$$4xyp = p^3 + 8y^2$$

$$\Rightarrow x = \frac{p^3 + 8y^2}{4py} = \frac{p^2}{4y} + \frac{2y}{p} \qquad \text{... (2)}$$

Differentiating (2) w.r.t. y and writing $\frac{1}{p}$ for $\frac{dx}{dy}$, we get

$$\frac{dx}{dy} = \frac{1}{p} = \frac{-p^2}{4y^2} + \frac{2p}{4y}\cdot\frac{dp}{dy} + \frac{2}{p} - \frac{2y}{p^2}\frac{dp}{dy}, \text{ using Product Rule}$$

$$\Rightarrow \frac{p^2}{4y^2} - \frac{1}{p} - \frac{dp}{dy}\left(\frac{p}{2y} - \frac{2y}{p^2}\right) = 0$$

$$\Rightarrow \left(\frac{p^2}{4y^2} - \frac{1}{p}\right) - \frac{2y}{p}\frac{dp}{dy}\left(\frac{p}{2y} - \frac{2y}{p^2}\right)\cdot\frac{p}{2y} = 0$$

$$\Rightarrow \left(\frac{p^2}{4y^2} - \frac{1}{p}\right) - \frac{2y}{p}\cdot\frac{dp}{dy}\left(\frac{p^2}{4y^2} - \frac{1}{p}\right) = 0$$

$$\Rightarrow \left(\frac{p^2}{4y^2} - \frac{1}{p}\right)\left(1 - \frac{2y}{p}\frac{dp}{dy}\right) = 0 \qquad \text{... (3)}$$

Neglecting the first factor which does not involve $\frac{dp}{dy}$,(3) reduces to

$$1 - \frac{2y}{p}\cdot\frac{dp}{dy} = 0 \Rightarrow \frac{2y}{p}\cdot\frac{dp}{dy} = 1 \Rightarrow \frac{2}{p}dp = \frac{dy}{y}$$

Integrating, $2\log p = \log y + \log c$

$$\Rightarrow \log p^2 = \log cy \Rightarrow p^2 = cy \qquad \text{... (4)}$$

We now eliminate p between (1) and (4).

Equation (1) can be written as

$$p(p^2 - 4xy) = -8y^2$$

Squaring, $p^2(p^2 - 4xy)^2 = 64y^4$

or $cy(cy - 4xy)^2 = 64y^4$, using (4) ... (5)

Let $c = 4c_1$, where c_1 is new arbitrary constant. Then (5) gives

$$4c_1y\,(4c_1y - 4xy)^2 = 64y^4$$

$$\Rightarrow 64c_1y\,(c_1y - xy)^2 = 64y^4 \Rightarrow c_1y\,(c_1y - xy)^2 = y^4$$

$$\Rightarrow c_1y^2(c_1 - x)^2 = y^4 \Rightarrow c_1(c_1 - x)^2 = y^2$$

or $y^2 = c(c - x)^2$

(*iii*) Given equation is $x = 4(p + p^3)$ ... (1)

Diff. (1) w.r.t. 'y' and writing $\frac{1}{p}$ for $\frac{dx}{dy}$, we get

$$\frac{dx}{dy} = \frac{1}{p} = 4\left(\frac{dp}{dy} + 3p^2\frac{dp}{dy}\right) = 4(1+3p^2)\frac{dp}{dy}$$

Separating the variables, we get

$$dy = 4p(1+3p^2)\,dp = 4(p+3p^3)\,dp$$

Integrating, we get

$$\int dy = 4\int (p+3p^3)\,dp \Rightarrow y = 4\left(\frac{p^2}{2} + \frac{3p^4}{4}\right) + c = 2p^2 + 3p^4 + c \quad \text{... (2)}$$

Both (1) and (2) constitute the solution of the given differential equation.

Example 2. Solve the following differential equations

(*i*) $y = 2px + p^3y^2$ (*ii*) $y^2 \log y = xyp + p^2$ (*iii*) $p = \tan\left(x - \frac{p}{1+p^2}\right)$

Solution. (*i*) Given equation can be written as

$$2x = \frac{y}{p} - y^2p^2$$

Diff. w.r.t. 'y' and writing $\frac{1}{p}$ for $\frac{dx}{dy}$, we get

$$2\frac{dx}{dy} = \frac{2}{p} = \frac{1}{p} - \frac{y}{p^2}\frac{dy}{dp} - 2yp^2 - 2py^2\frac{dp}{dy}$$

$$\Rightarrow \frac{2}{p} - \frac{1}{p} + 2yp^2 = \frac{-y}{p}\frac{dp}{dy}\left(\frac{1}{p} + 2yp^2\right)$$

$$\Rightarrow \left(\frac{1}{p} + 2yp^2\right) + \frac{y}{p}\frac{dp}{dy}\left(\frac{1}{p} + 2yp^2\right) = 0$$

$$\Rightarrow \left(\frac{1}{p} + 2yp^2\right)\left(1 + \frac{y}{p}\frac{dp}{dy}\right) = 0$$

Neglecting the first factor which does not involve $\frac{dp}{dy}$, the above equation reduces to

$$1 + \frac{y}{p}\frac{dp}{dy} = 0 \Rightarrow \frac{y}{p}\frac{dp}{dy} = -1 \Rightarrow \frac{dp}{dy} = \frac{-p}{y}$$

$$\Rightarrow \frac{dp}{p} = -\frac{dy}{y}$$

Integrating, $\log p = -\log y + \log c$ or $\log p + \log y = \log c$ or $py = c$

$$\therefore p = \frac{c}{y}$$

Putting for p in the given equation, we get

$$y = 2.\frac{c}{y}.x + y^2\left(\frac{c^3}{y^3}\right) = \frac{2cx + c^3}{y}$$

or $y^2 = 2cx + c^3$

which is the required solution.

(*ii*) Given equation can be written as

$$x = \frac{y\log y}{p} - \frac{p}{y} \quad \text{... (1)}$$

Diff. (1) w.r.t. 'y' and writing $\frac{1}{p}$ for $\frac{dx}{dy}$, we get

$$\frac{1}{p} = \frac{p\left[\log y + y.\frac{1}{y}\right] - y\log y.\frac{dp}{dy}}{p^2} - \frac{y.\frac{dp}{dy} - p}{y^2}, \text{ using Quotient Rule}$$

$$= \frac{(1+\log y)}{p} - \frac{y\log y}{p^2}\frac{dp}{dy} - \frac{1}{y}\frac{dp}{dy} + \frac{p}{y^2}$$

$$= \frac{1}{p} + \frac{1}{p}\log y - \frac{y}{p^2}\log y\frac{dp}{dy} - \frac{1}{y}.\frac{dp}{dy} + \frac{p}{y^2}$$

$$\Rightarrow \frac{1}{p}\log y + \frac{p}{y^2} = \left(\frac{y}{p^2}\log y + \frac{1}{y}\right)\frac{dp}{dy}$$

$$\Rightarrow \frac{p}{y^2}\left(1 + \frac{y^2}{p^2}\log y\right) = \frac{1}{y}\frac{dp}{dy}\left(1 + \frac{y^2}{p^2}\log y\right)$$

$$\Rightarrow \left(1 + \frac{y^2}{p^2}\log y\right)\left(\frac{p}{y^2} - \frac{1}{y}\frac{dp}{dy}\right) = 0 \quad \text{... (2)}$$

Neglecting the first factor which does not involve $\frac{dp}{dy}$, (2) reduces to

$$\frac{p}{y^2} - \frac{1}{y}\frac{dp}{dy} = 0$$

$$\Rightarrow \frac{p}{y} = \frac{dp}{dy} \Rightarrow \frac{dp}{p} = \frac{dy}{y}$$

Integrating, $\log p = \log y + \log c \Rightarrow p = cy$ ... (3)

Putting the value of p in the given equation, we get

$$y^2\log y = xy(cy) + c^2y^2 = y^2(xc + c^2)$$

$\Rightarrow \log y = cx + c^2$, which is the required solution.

(*iii*) Given equation is

$$p = \tan\left(x - \frac{p}{1+p^2}\right) \quad \text{... (1)}$$

(1) can be written as

$$\tan^{-1} p = x - \frac{p}{1+p^2}$$

$$\Rightarrow x = \tan^{-1} p + \frac{p}{1+p^2} \quad \text{... (2)}$$

Diff. (2) w.r.t. 'y' and writing $\frac{1}{p}$ for $\frac{dx}{dy}$, we get

$$\frac{dx}{dy} = \frac{1}{p} = \frac{1}{1+p^2}\frac{dp}{dy} + \frac{(1+p^2).1 - p(2p)}{(1+p^2)^2}\frac{dp}{dy}$$

$$\Rightarrow \frac{1}{p} = \frac{1}{1+p^2}\frac{dp}{dy} + \frac{1-p^2}{(1+p^2)^2}\frac{dp}{dy} = \frac{(1+p^2)+(1-p^2)}{(1+p^2)^2}\frac{dp}{dy}$$

$$\Rightarrow \frac{1}{p} = \frac{2}{(1+p^2)^2}\frac{dp}{dy} \Rightarrow dy = \frac{2p}{(1+p^2)^2}dp$$

Integrating, $\int dy = \int (1+p^2)^{-2}(2p)\,dp$

$$\Rightarrow y = \frac{(1+p^2)^{-2+1}}{-2+1} + c \qquad \left[\because \int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1}\right]$$

$$\Rightarrow y = \frac{-1}{1+p^2} + c \text{ or } y + (1+p^2)^{-1} = c \quad \text{... (3)}$$

Equations (2) and (3) which represent the values of x and y in terms of parameter p, together give the required solution.

EXERCISE 2.3

Solve the following differential equations

1. $x = y + a\log p$
2. $x = y + p^2$
3. $x = py + p^2$
4. $x(1+p^2) = 1$
5. $x + \frac{p}{\sqrt{1+p^2}} = a$
6. $p^3 - 2xyp + 4y^2 = 0$
7. $y = 2px + p^2 y$
8. $y = 2px + p^3 y$
9. $yp^2 - 2xp + y = 0$
10. $p^3 - p(y+3) + x = 0$

ANSWERS

1. $y = c - a\log(1-p), x = c - a\log(1-p) + a\log p$
2. $y = c - \left[\frac{p^2}{2} + p + \log(p-1)\right], x = c - 2[p + \log(p-1)]$
3. $y = \frac{c+\sin^{-1} p}{\sqrt{1-p^2}} - p, x = \frac{p(c+\sin^{-1} p)}{\sqrt{1-p^2}}$

4. $y = c - \tan^{-1} p + \frac{p}{1+p^2}, x = (1+p^2)^{-1}$

5. $(x-a)^2 + (y-c)^2 = 1$

6. $16yc^3 = (1-2xc)^2$

7. $y^2 = c(2x+c)$

8. $y^2 = 2cx + c^3$

9. $y^2 = 2cx - c^2$

10. $y = \frac{c}{\sqrt{1-p^2}} - (1-p^2)$

2.6. METHOD IV : CLAIRAUT'S EQUATION

Definition : An equation of the form $y = px + f(p)$ is known as Clairaut's equation.

General solution of Clairaut's Equation :

To show that the solution of Clairaut's equation $y = px + f(p)$ is $y = cx + f(c)$ which is obtained by replacing p by c, where c is an arbitrary constant.

Proof. We have $y = px + f(p)$ ... (1)

Differentiating (1) w.r.t. x, we get

$$\frac{dy}{dx} = p = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$$

$$\Rightarrow p - p = 0 = [x + f'(p)]\frac{dp}{dx} \quad \text{... (2)}$$

Ignoring the first factor which does not involve $\frac{dp}{dx}$, (2) reduces to

$$\frac{dp}{dx} = 0 \text{ or } p = c \quad \text{... (3)}$$

Eliminating p between (1) and (3), we get $y = cx + f(c)$ which is the required solution.

Note : If we eliminate p between the given equation and the relation $x + f'(p) = 0$, we get another solution which is free from an arbitrary constant and it is called singular solution of the given equation.

Working Rule : The solution of Clairaut's equation is obtained on replacing p by c.

Singular Solution of Clairaut's Equation :

To obtain the singular solution, perform the following :

1. Find the general solution of Clairaut's equation by replacing p by c
   i.e. $y = cx + f(c)$ ...(1)

2. Differentiating (1) w.r.t. 'c', we get $x + f'(c) = 0$ ... (2)

3. Eliminate 'c' from (1) and (2) which will be the singular solution.

SOLVED EXAMPLES

Example 1. Solve the following differential equations.

(*i*) $y = px + p^n$ (*ii*) $xp^2 - yp + 2 = 0$ (*iii*) $y = px + \log p$

(*iv*) $p = \log(px - y)$ (*v*) $p = \tan(px - y)$ (*vi*) $(y - px)(p - 1) = p$

(*vii*) $\cos y \cos px + \sin y \sin px = p$ (*viii*) $\sin px \cos y = \cos px \sin y + p$

Solution. (*i*) The given equation is in Clairaut's form $y = px + f(p)$.

So replacing p by c. Its general solution is $y = cx + c^n$, where c is an arbitrary constant.

(*ii*) Given equation can be written as

$yp = xp^2 + 2$ or $y = px + \dfrac{2}{p}$

which is Clairaut's form and hence the solution is obtained by putting $p = c$ and is

$y = cx + \dfrac{2}{c}$

(*iii*) Given equation is $y = px + \log p$ which is the Clairaut's equation.

$\therefore$ Its solution is $y = cx + \log c$, c being an arbitrary constant.

(*iv*) Given equation can be written as

$e^p = px - y$ or $y = px - e^p$ which is the Clairaut's equation.

So replacing p by c, the required general solution is

$y = cx - e^c$, where c is an arbitrary constant.

(*v*) Given equation can be written as $\tan^{-1} p = px - y$ or $y = px - \tan^{-1} p$

which is the Clairaut's equation.

So replacing p by c, the required general solution is $y = cx - \tan^{-1} c$.

(*vi*) Given equation can be written as $y - px = \dfrac{p}{p-1}$ or $y = px + \dfrac{p}{p-1}$

which is the Clairaut's equation.

$\therefore$ Its solution is $y = cx + \dfrac{c}{c-1}$, c being an arbitrary constant.

(*vii*) Given equation can be written as

$\cos(y - px) = p$ $[\because \cos(A - B) = \cos A \cos B + \sin A \sin B]$

i.e., $y - px = \cos^{-1} p$ or $y = px + \cos^{-1} p$

which is the Clairaut's equation and hence the solution is obtained by putting $p = c$ and is $y = cx + \cos^{-1} c$

(*viii*) Given equation can be written as

$\sin px \cos y - \cos px \sin y = p$

i.e., $\sin(px - y) = p$ $[\because \sin(A - B) = \sin A \cos B - \cos A \sin B]$

or $px - y = \sin^{-1} p$ or $y = px - \sin^{-1} p$

which is Clairaut's equation and hence the solution is obtained by putting $p = c$ and is $y = cx - \sin^{-1} c$, where c is an arbitrary constant.

Example 2. Solve $y = px + p^2$ and obtain the singular solution.

Solution. Given $y = px + p^2$ ... (1)

Differentiating (1) w.r.t. x, we get

$$\frac{dy}{dx} = p = p + x\frac{dp}{dx} + 2p\frac{dp}{dx}$$

$$\Rightarrow 0 = (x + 2p)\frac{dp}{dx}$$

Taking $\frac{dp}{dx} = 0$, we get $p = c$ and putting in the given equation, we get $y = cx + c^2$ as the required solution.

Taking $x + 2p = 0$ or $p = -\frac{x}{2}$

Eliminating p between this and the given equation we shall get the singular solution as

$$y = \frac{-x^2}{2} + \frac{x^2}{4} = \frac{-2x^2 + x^2}{4} = \frac{-x^2}{4}$$

$$\Rightarrow x^2 + 4y = 0$$

Example 3. Find the general and singular solution of the equations :

(*i*) $y = px + \frac{a}{p}$ (*ii*) $y = px - \sqrt{1 + p^2}$ (*iii*) $\sin(px - y) = p$

Solution. (*i*) Given equation is

$$y = px + \frac{a}{p} \quad ...(1)$$

Differentiating (1) w.r.t. x, we get

$$\frac{dy}{dx} = p = p + x\frac{dp}{dx} - \frac{a}{p^2}\frac{dp}{dx}$$

$$\Rightarrow \frac{dp}{dx}\left(x - \frac{a}{p^2}\right) = 0$$

Taking $\frac{dp}{dx} = 0$ we get $p = c$ and putting in the given equation, we get $y = cx + \frac{a}{c}$ as the general solution.

To find singular solution :

Taking $x - \frac{a}{p^2} = 0$ or $p^2 = \frac{a}{x}$ or $p = \sqrt{\frac{a}{x}}$ ... (2)

Eliminating p between (1) and (2), we get

$$y=\left(\sqrt{\frac{a}{x}}\right)x+a\left(\sqrt{\frac{x}{a}}\right)=\sqrt{ax}+\sqrt{ax}=2\sqrt{ax}$$

or $y^2=4ax$ which is a singular solution.

Alternate Method to find Singular solution :

The given equation is of Clairaut's form and hence the solution is obtained by putting $p=c$

and is $y=cx+\dfrac{a}{c}$ ...(1)

To find the singular solution,

Differentiating (1) w.r.t. 'c' giving

$$0=x-\frac{a}{c^2}\Rightarrow c^2=\frac{a}{x} \text{ or } c=\sqrt{\frac{a}{x}} \qquad ...(2)$$

Eliminating 'c' between (1) and (2), we get

$$y=\left(\sqrt{\frac{a}{x}}\right)x+a\left(\sqrt{\frac{x}{a}}\right)=\sqrt{ax}+\sqrt{ax}=\sqrt{2ax}$$

Squaring, $y^2=4ax$

which is the desired singular solution.

(ii) Given $y=px-\sqrt{1+p^2}$ ...(1)

which is in Clairut's form. So replacing p by c,

the general solution is $y=cx-\sqrt{1+c^2}$ ...(2)

To find the singular solution, differentiate (2) w.r.t. c, giving

$$0=x-\frac{1}{2\sqrt{1+c^2}}(2c)\Rightarrow x=\frac{c}{\sqrt{1+c^2}}\Rightarrow\sqrt{1+c^2}=\frac{c}{x} \qquad ...(3)$$

Squaring, $1+c^2=\dfrac{c^2}{x^2}$ or $c^2\left(\dfrac{1}{x^2}-1\right)=1$

or $c^2\left(\dfrac{1-x^2}{x^2}\right)=1$ or $c^2=\dfrac{x^2}{1-x^2}$

$$\therefore c=\frac{x}{\sqrt{1-x^2}} \qquad ...(4)$$

Now substituting the value of c in (2), we get

$y=\dfrac{x^2}{\sqrt{1-x^2}}-\dfrac{c}{x}$, using (3)

$=\dfrac{x^2}{\sqrt{1-x^2}}-\dfrac{1}{\sqrt{1-x^2}}$, using (4) or $y=\frac{x^2-1}{\sqrt{1-x^2}}=\frac{-(1-x^2)}{\sqrt{1-x^2}}=-(\sqrt{1-x^2})$

or $y+\sqrt{1-x^2}=0$

which is the desired singular solution.

(iii) Given equation can be written as

$\sin^{-1}p=px-y$ or $y=px-\sin^{-1}p$ ... (1)

which is in Clairuat's form $y=px+f(p)$

So replacing p by c, the required general solution is

$y=cx-\sin^{-1}c$, c being an arbitrary constant. ... (2)

To find the singular solution, differentiate (2) w.r.t. c giving

$0=x-\frac{1}{\sqrt{1-c^2}} \Rightarrow x=\frac{1}{\sqrt{1-c^2}}$

or $1-c^2=\frac{1}{x^2}$ or $c^2=1-\frac{1}{x^2}$

i.e., $c^2=\frac{x^2-1}{x^2}$ or $c=\frac{1}{x}\sqrt{x^2-1}$

Now substituting this value of c in (2), we get

$y=\sqrt{x^2-1}-\sin^{-1}\left(\frac{\sqrt{x^2-1}}{x}\right)$, which is the desired singular solution.

EXERCISE 2.4

1. Solve the following differential equations

(i) $y=px+p^3$ (ii) $y=px+\log p$ (iii) $y=px+\sin^{-1}p$

(iv) $y=px+a\tan^{-1}p$ (v) $y=px+p-p^2$ (vi) $(y-px)^2=1+p^2$

(vii) $\frac{(y-px)^2}{1+p^2}=a^2$ (viii) $xp^2-yp+a=0$ (ix) $(x-a)p^2+(x-y)p$...

(x) $p^2(x^2-a^2)-2pxy+y^2-b^2=0$

2. Solve $y=px+p-p^2$ and obtain the singular solution.

3. Solve $y=(x-a)p-p^2$ and obtain the singular solution.

ANSWERS

1. (i) $y=cx+c^3$ (ii) $y=cx+\log c$ (iii) $y=cx+\sin^{-1}c$

(iv) $y=cx+a\tan^{-1}c$ (v) $y=cx+c-c^2$ (vi) $(y-cx)^2(1+c^2)=$...

UNIT VI

DIFFERENTIAL EQUATIONS OF SECOND AND HIGHER ORDER

Non - homogeneous linear differential equations of second and higher order with constant coefficients with R.H.S. term of the type e^{ax}, sin ax, cos ax, polynomial in x, e^{ax} V (x), x V (x). Method of variation of parameters

6.1 DIFFERENTIAL EQUATIONS OF SECOND AND HIGHER ORDER WITH CONSTANT COEFFICIENTS :

Definition : An equation of the form

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots\ldots + P_{n-1} \frac{dy}{dx} + P_n y = X$$

where $P_1, P_2, \ldots\ldots P_n$ are functions of x and X is a function of x only is called a linear differential equation of n^{th} order.

A linear differential equation with constant coefficients is of the form

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots\ldots\ldots + a_n y = X \quad \ldots\ldots\ldots\ldots (1)$$

where $a_1, a_2, \ldots\ldots a_n$ are all constants and X is a function of x only.

The operator D :

Let D be the symbol which denotes differentiation with respect to x say, of the function which immediately follows it i.e., $D = \frac{d}{dx}$

$$D(y) = \frac{dy}{dx}$$

$$D(Dy) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2 y}{dx^2}$$

i.e., $$D^2 y = \frac{d^2 y}{dx^2}$$

$$D^n y = \frac{d^n y}{dx^n}$$

In symbolic form, the equation (1) can be written as

$$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots\ldots + a_{n-1} D + a_n) y = X$$

$$f (D) y = X \qquad \ldots\ldots\ldots\ldots (2)$$

So, f (D) can be considered as a polynomial in D.

i.e., $f (D) = D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots\ldots\ldots + a_{n-1} D + a_n$

Theorem - 1 : If $y = y_1, y = y_2 \ldots\ldots\ldots y = y_n$ are n linearly independent solutions of the differential equation

$$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots\ldots\ldots + a_n) y = 0 \qquad \ldots\ldots\ldots (1)$$

then $y = c_1 y_1 + c_2 y_2 + c_3 y_3 + \ldots\ldots\ldots + c_n y_n$ is also a solution where $c_1, c_2, \ldots\ldots, c_n$ are constants.

Proof : Given $y = y_1, y = y_2, \ldots\ldots\ldots, y_n$ are solutions of equation (1)

So, $y_1, y_2, \ldots\ldots\ldots, y_n$ satisfy the equation

$$\left.\begin{array}{l} D^n y_1 + a_1 D^{n-1} y_1 + a_2 D^{n-2} y_1 + \ldots\ldots\ldots + a_n y_1 = 0 \\ D^n y_2 + a_1 D^{n-1} y_2 + a_2 D^{n-2} y_2 + \ldots\ldots\ldots + a_n y_2 = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ D^n y_n + a_1 D^{n-1} y_n + a_2 D^{n-2} y_n + \ldots\ldots + a_n y_n = 0 \end{array}\right\} \ldots\ldots\ldots (2)$$

$$\begin{aligned} \text{Now } & D^n y + a_1 D^{n-1} y + a_2 D^{n-2} y + \ldots\ldots\ldots + a_n y \\ &= D^n (c_1 y_1 + c_2 y_2 + \ldots\ldots\ldots + c_n y_n) \\ &\quad + a_1 D^{n-1} (c_1 y_1 + c_2 y_2 + \ldots\ldots + c_n y_n) \\ &\quad + a_2 D^{n-2} (c_1 y_1 + c_2 y_2 + \ldots\ldots + c_n y_n) \\ &\quad + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\quad + a_n (c_1 y_1 + c_2 y_2 + \ldots\ldots\ldots + c_n y_n) \\ &= c_1 (D^n y_1 + a_1 D^{n-1} y_1 + a_2 D^{n-2} y_1 + \ldots\ldots\ldots + a_n y_1) \\ &\quad + c_2 (D^n y_2 + a_1 D^{n-1} y_2 + a_2 D^{n-2} y_2 + \ldots\ldots\ldots + a_n y_2) \\ &\quad + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\quad + c_n (D^n y_n + a_1 D^{n-1} y_n + a_2 D^{n-2} y_n + \ldots\ldots\ldots + a_n y_n) \\ &= c_1 (0) + c_2 (0) + \ldots\ldots\ldots + c_n (0) \\ &= 0 \qquad \text{by (2)} \end{aligned}$$

This implies that

$y = c_1 y_1 + c_2 y_2 + \ldots\ldots\ldots\ldots + c_n y_n$ is also the solution of equation (1) which is the general solution or complete solution as it contains n arbitrary constants.

6.2 COMPLETE SOLUTION :

To find the complete solution of the differential equation of n^{th} order with constant coefficients $\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + a_2 \frac{d^{n-2}y}{dx^{n-2}} + \dots\dots\dots + a_n y = X$

i.e., $$f(D)\, y = X \qquad \dots\dots\dots (1)$$

The complete solution is given by the following theorem.

Theorem - 2 : If $y = f(x)$ is general solution of $f(D)\, y = 0$ .... (2)
and $y = \phi(x)$ is any solution of equation of (1),
then $y = f(x) + \phi(x)$
is the complete solution of equation (1).

Proof : Since $y = f(x)$ is the general solution of the equation $f(D)\, y = 0$

$$\therefore \quad f(D)\, f(x) = 0 \qquad \dots\dots\dots (i)$$

Also $y = \phi(x)$ is a particular solution of the equation $f(D)\, y = X$

$$f(D)\, \phi(x) = X \qquad \dots\dots\dots (ii)$$

Adding (i) and (ii), we get

$$f(D)\,[\,f(x) + \phi(x)\,] = X$$

Thus $y = f(x) + \phi(x)$ satisfy the equation (1). The part $y = f(x)$ is called the complementary function (C.F.) which contains n arbitrary constants.

The part $y = \phi(x)$ is called the particular integral (P.I.) which does not contain any arbitrary constant.

The complete solution of equation $f(D)\, y = X$ is

$$y = \text{C. F.} + \text{P. I.}$$

Thus, to solve the equation (1), first we have to find C.F. and then P.I.

6.3 AUXILIARY EQUATION :

Consider the differential equation

$$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots\dots + a_n)\, y = 0 \qquad \dots\dots\dots (1)$$

Assume $y = e^{mx}$ is a solution of (1)

then $\quad Dy = m e^{mx}$

$$D^2 y = m^2 e^{mx}$$

$$D^n y = m^n e^{mx}$$

Substituting these values in (1), we get

$$(m^n + a_1 m^{n-1} + a_2 m^{n-2} + \dots\dots + a_n)\, e^{mx} = 0$$

$$\text{i.e., } m^n + a_1 m^{n-1} + a_2 m^{n-2} + \dots\dots + a_n = 0, \; e^{mx} \neq 0 \qquad \dots\dots (2)$$

Equation (2) is called auxiliary equation of the differential equation (1)

6.4 COMPLEMENTARY FUNCTION :

Let $m_1, m_2, \ldots\ldots m_n$ be the roots of the auxiliary equation (2)

Case - (i) : If the roots are real and distinct :

These roots satisfy the equation

$$D^n + a_1D^{n-1} + a_2D^{n-2} + \ldots\ldots + a_n = 0 \qquad \ldots\ldots (3)$$

Let $D^n + a_1D^{n-1} + a_2D^{n-2} + \ldots\ldots + a_n = (D - m_1)(D - m_2) \ldots\ldots (D - m_n)$

$D^n + a_1D^{n-1} + a_2D^{n-2} + \ldots\ldots + a_n = 0$

$\Rightarrow (D - m_1)(D - m_2) \ldots\ldots (D - m_n) = 0$

Consider $(D - m_1)\, y = 0$ i.e., $\dfrac{dy}{dx} - m_1 y = 0$

$$\frac{dy}{y} = m_1 dx$$

Integrating, $\log y = m_1 x + c$

$$y = e^{m_1x+c} = e^{m_1x} \cdot e^c = c_1 e^{m_1x} \qquad \text{where } c_1 = e^c$$

Similarly $(D - m_2)\, y = c_2 e^{m_2x}$

$(D - m_3)\, y = c_3 e^{m_3x}$

$(D - m_n)\, y = c_n e^{m_nx}$

Hence the complete solution is

$$y = c_1 e^{m_1x} + c_2 e^{m_2x} + \ldots\ldots + c_n e^{m_nx} \qquad \ldots\ldots (4)$$

Case - (ii) : If two roots are equal, say $m_1 = m_2$

The solution (4) becomes

$$y = (c_1 + c_2)\, e^{m_1x} + c_3 e^{m_3x} + \ldots\ldots + c_n e^{m_nx}$$

$$= c e^{m_1x} + c_3 e^{m_3x} + \ldots\ldots + c_n e^{m_nx}$$

This is not complete solution since it contains (n – 1) arbitrary constants.

Now consider $(D - m_1)(D - m_1)\, y = 0$

Let $(D - m_1)\, y = u$

$(D - m_1)\, u = 0$ i.e., $\dfrac{du}{dx} - m_1 u = 0$

$\Rightarrow u = c_1 e^{m_1x}$ by case (i)

But $u = (D - m_1)\, y$

$\therefore (D - m_1)\, y = c_1 e^{m_1x}$ or $\dfrac{dy}{dx} - m_1 y = c_1 e^{m_1x}$

which is a linear equation in y, $P = -m$, $Q = c_1 e^{m_1 x}$

I. F. $= e^{\int -m_1\, dx} = e^{-m_1 x}$

Its solution is $y e^{-m_1 x} = \int c_1 e^{m_1 x} \cdot e^{-m_1 x} \cdot dx + c_2$

$$= c_1 x + c_2$$

$\therefore\ y = (c_1 x + c_2)\, e^{-m_1 x}$

Thus, the complete solution for $m = m_1 = m_2$ is

$$y = (c_1 x + c_2)\, e^{-m_1 x} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

If three roots are equal $m = m_1 = m_2 = m_3$

$$y = (c_1 x^2 + c_2 x + c_3)\, e^{-m_1 x} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

Case - (iii) : If two roots of Auxiliary equation are imaginary $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$

The solution (4) becomes

$$y = c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x} + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x}$$

$$= c_1 e^{\alpha x} \cdot e^{i\beta x} + c_2 e^{\alpha x} \cdot e^{-i\beta x} + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x}$$

$$= e^{\alpha x} (c_1 e^{i\beta x} + c_2 e^{-i\beta x}) + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x}$$

$$= e^{\alpha x} \{ [c_1 (\cos \beta x + i \sin \beta x)] + [c_2 (\cos \beta x - i \sin \beta x)] \} + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x}$$

$$= e^{\alpha x} [(c_1 + c_2) \cos \beta x + i (c_1 - c_2) \sin \beta x] + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x}$$

$$= e^{\alpha x} (A \cos \beta x + B \sin \beta x) + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x}$$

where $A = c_1 + c_2$

$B = c_1 - c_2$

Note - 1 : If $\alpha + i\beta$ and $\alpha - i\beta$ are the two roots, the solution is $e^{\alpha x} (A \cos \beta x + B \sin \beta x)$

Note – 2 : If $\alpha + i\beta$ and $\alpha - i\beta$ are repeated, then the solution is

$$e^{\alpha x} [(A_1 + B_1 x) \cos \beta x + (A_2 + B_2 x) \sin \beta x]$$

1. If $m_1, m_2 \ldots\ldots m_n$ are real and distinct

   C.F. $= c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots\ldots\ldots + c_n e^{m_n x}$

2. If $m_1 = m_2$

   C.F. $= (c_1 + c_2 x) e^{m_1 x}$

3. If $m_1 = m_2 = m_3$

   C.F. $= (c_1 + c_2 x + c_3 x^2)\, e^{m_1 x}$

4. If $m_1 = \alpha + i\beta,\ m_2 = \alpha - i\beta$

$$\text{C.F.} = e^{m_1 x}\,(A\cos\beta x + B\sin\beta x)$$

SOLVED EXAMPLES – 6.1

EXAMPLE - 1 :

$$\text{Solve } \frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0$$

Solution : Auxiliary equation, $m^3 - 6m^2 + 11m - 6 = 0$

$$(m-1)(m^2 - 5m - 6) = 0$$

$$(m-1)(m-2)(m-3) = 0$$

$$m = 1, 2, 3$$

Solution : $y = c_1e^x + c_2e^{2x} + c_3e^{3x}$

EXAMPLE - 2 :

$$\frac{d^3y}{dx^3} - 5\frac{d^2y}{dx^2} + 8\frac{dy}{dx} - 4y = 0$$

Solution : Auxiliary equation, $m^3 - 5m^2 + 8m - 4 = 0$

$$(m-2)(m^3 - 3m + 2) = 0$$

$$(m-2)(m-2)(m-1) = 0$$

$m = 1, 2, 2$

Solution : $y = (c_1x + c_2)\,e^{2x} + c_3e^x$

EXAMPLE - 3 :

$$\text{Solve } \frac{d^3y}{dx^3} - 8y = 0$$

Solution : Auxiliary equation, $m^3 - 8 = 0$

$$m = 2,\ m = -1 \pm \sqrt{3}\,i$$

Solution : $y = e^{-x}\,(c_1\cos\sqrt{3}\,x + c_2\sin\sqrt{3}\,x)$

EXAMPLE - 4 :

$$\text{Solve } \frac{d^4y}{dx^4} - \frac{d^3y}{dx^3} - 9\frac{d^2y}{dx^2} - 11\frac{dy}{dx} - 4y = 0$$

Solution : Auxiliary equation, $m^4 - m^3 - 9m^2 - 11m - 4 = 0$

$$(m+1)^3(m-4) = 0$$

$$m = -1, -1, -1, 4$$

Solution : $y = (c_1 + c_2x + c_3x^2)\,e^{-x} + c_4e^{4x}$

EXAMPLE - 5 :

Solve $\dfrac{d^4y}{dx^4} + 4y = 0$

Solution : Auxiliary equation, $m^4 + 4 = 0$

$$m^4 + 4m^2 + 4 - 4m^2 = 0$$

$$(m^2 + 2)^2 - 4m^2 = 0$$

$$(m^2 + 2 - 2m)(m^2 + 2 + 2m) = 0$$

$$m^2 - 2m + 2 = 0 \qquad m = \frac{+2 \pm \sqrt{4-8}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$$

$$m^2 + 2m + 2 = 0 \qquad m = \frac{-2 \pm \sqrt{4-8}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

The solution $y = e^x (c_1 \cos x + c_2 \sin x) + e^{-x}(c_3 \sin x + c_4 \sin x)$

EXERCISE 6.1

Solve the following equations :

1. $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 4y = 0$ **Ans : $y = (c_1 + c_2x)e^{-2x}$**
2. $(D^3 - 9D^2 + 23D - 15)y = 0$ **Ans : $y = c_1e^x + c_2e^{3x} + c_3e^{5x}$**
3. $\dfrac{d^3y}{dx^3} + 2\dfrac{d^2y}{dx^2} - 8\dfrac{dy}{dx} = 0$ **Ans : $y = c_1 + c_2e^{2x} + c_3e^{-4x}$**
4. $\dfrac{d^3y}{dx^3} - \dfrac{d^2y}{dx^2} + 8\dfrac{dy}{dx} + 12y = 0$ **Ans : $y = (c_1 + c_2x)e^{2x} + c_3e^{-3x}$**
5. $(D^3 + 6D^2 + 12 + 8)y = 0$ given that for $x = 0$, $y = 1$ and $y' = -2$, $y'' = 2$
   Ans : $y = (1 - x^2)e^{-3x}$
6. $(D^2 - 6D + 13)y = 0$ **Ans : $(c_1 \cos 2x + c_2 \sin 2x)e^{3x}$**
7. $(D^4 + 18D^2 + 81)y = 0$ **Ans : $y = (c_1 + c_2x)\cos 3x + (c_3 + c_4x)\sin 3x$**
8. $(D^3 - 14D + 8)y = 0$ **Ans : $y = c_1e^{-4x} + e^{2x}[c_2 \cosh x\sqrt{2} + c_3 \sinh x\sqrt{2}]$**
9. $\dfrac{d^2x}{dt^2} + \mu x = 0,\ \mu > 0$ given that $x = a$ and $\dfrac{dx}{dt} = 0$
   when $t = \dfrac{\pi}{\sqrt{\mu}}$ **Ans : $x = -a \cos t\sqrt{\mu}$**
10. $(D^4 - 2D^3 + 2D^2 - 2D + 1)y = 0$ **Ans : $y = (c_1 + c_2x)e^x + (c_3 \cos x + c_4 \sin x)$**

6.5 PARTICULAR INTEGRAL :

Definition : Inverse operator $\frac{1}{f(D)}$

$\frac{1}{f(D)}$ X is that function of x, which when operated upon by f (D) gives X

$$\text{Then } f(D)\left\{\frac{1}{f(D)}X\right\} = X$$

f (D) and $\frac{1}{f(D)}$ are inverse operators.,

Since f (D) y = X, the particular integral (P. I.) can be written as $y = \frac{1}{f(D)}X$

Meaning of $\frac{1}{D}X$:

$\frac{1}{D}X$ is that function of x which when operated on by D gives X

$$D\left(\frac{1}{D}\right)X = X$$

Since, D stands for differential $\frac{d}{dx}$

$\frac{1}{D}$ stands for integration

$$\frac{1}{D}X = \int X\,dx$$

Formula for $\frac{1}{D-a}X$:

$$\text{Let } \frac{1}{D-a}X = y$$

Operating both sides by (D – a)

$$(D-a)\cdot\frac{1}{(D-a)}X = (D-a)\,y$$

$$X = (D-a)\,y \Rightarrow \frac{dy}{dx} - ay = X$$

I. F. $e^{-\int a\,dx} = e^{-ax}$

Solution $y \cdot e^{-ax} = \int X e^{-ax}\,dx$

$$y = e^{ax}\int X e^{-ax}\,dx$$

$$\boxed{\frac{1}{D-a}X = e^{ax}\int X e^{-ax}\,dx}$$

Note : $\frac{1}{D+a}X = e^{-ax}\int X e^{ax}\,dx$

Example - 1 : $\frac{1}{D-3}x = e^{3x}\int e^{-3x}x\,dx$

$$= e^{3x}\left[-x\frac{e^{-3x}}{3} + \int \frac{e^{-3x}}{3}dx\right]$$

$$= e^{3x}\left[-\frac{xe^{-3x}}{3} - \frac{e^{-3x}}{9}\right] = -\frac{x}{3} - \frac{1}{9}$$

Example - 2 : Find the particular integral of $(D^2 - 4D + 3)\,y = e^{2x}$.

$$\text{P. I.} = \frac{1}{D^2-4D+3}e^{2x} = \frac{1}{2}\left\{\frac{1}{D-3} - \frac{1}{D-1}\right\}e^{2x}$$

$$= \frac{1}{2}\left[\frac{1}{D-3}e^{2x} - \frac{1}{D-1}e^{2x}\right]$$

$$= \frac{1}{2}\left\{e^{3x}\int e^{-3x}e^{2x}\,dx - e^{x}\int e^{-x}e^{2x}\,dx\right\}$$

$$= \frac{1}{2}\left\{e^{3x}\int e^{-x}\,dx - e^{x}\int e^{x}\,dx\right\}$$

$$= \frac{1}{2}\left\{e^{3x}\left(-e^{-x}\right) - e^{x}e^{x}\right\}$$

$$= \frac{1}{2}(-2e^{2x}) = -e^{2x}$$

Another Method :

$$\text{P. I.} = \frac{1}{D^2-4D+3}e^{2x} = \frac{1}{(D-1)(D-3)}e^{2x}$$

$$= \frac{1}{D-1}\frac{1}{D-3}e^{2x}$$

$$= \frac{1}{D-1}e^{3x}\int e^{-3x}\cdot e^{2x}\,dx$$

$$= \frac{1}{D-1}e^{3x}\int e^{-x}\,dx$$

$$= \frac{1}{D-1}e^{3x}\left(\frac{e^{-x}}{-1}\right)$$

$$= \frac{1}{D-1}(-e^{2x}) = e^{x}\int e^{-x}(-e^{2x})\,dx$$

$$= -e^{x}\int e^{x}\,dx$$

$$= -e^{x}\cdot e^{x} = -e^{2x}$$

6.6 PARTICULAR CASES :

Let the given differential equation be $f(D)\,y = X$

(I) When $X = e^{aX}$, to find P. I.

$$D(e^{ax}) = ae^{ax},\; D^2(e^{ax}) = a^2e^{ax} \;\ldots\ldots\ldots\; D^n(e^{ax}) = a^n e^{ax}$$

Now $f(D)\,e^{ax} = (D^n + a_1D^{n-1} + a_2D^{n-2} + \ldots\ldots\ldots + a_n)\,e^{ax}$

$$= (a^n + a_1a^{n-1} + a_2a^{n-2} + \ldots\ldots\ldots + a_n)\,e^{ax}$$

$$= f(a)\,e^{ax}$$

$$f(D)\,e^{ax} = f(a)\,e^{ax}$$

Operating on both sides by $\frac{1}{f(D)}$

$$\frac{1}{f(D)}\,f(D)\cdot e^{ax} = \frac{1}{f(D)}\,f(a)\cdot e^{ax}$$

$$\Rightarrow e^{ax} = \frac{1}{f(D)}\,f(a)\,e^{ax}$$

$$\Rightarrow \frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax},\quad f(a)\neq 0$$

$$\boxed{\frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax}}$$

Case of failure : If $f(a) = 0$, then $D - a$ is a factor of $f(D)$

Let $f(D) = (D - a)\,\phi(D)$ and $\phi(a) \neq 0$

$$\text{P.I.} = \frac{1}{f(D)} e^{ax} = \frac{1}{D-a} \cdot \frac{1}{\phi(D)} \cdot e^{ax}$$

$$= \frac{1}{D-a} \cdot \frac{1}{\phi(a)} \cdot e^{ax}$$

$$= \frac{1}{\phi(a)} \cdot \frac{1}{D-a} \cdot e^{ax}$$

$$= \frac{1}{\phi(a)} e^{ax} \int e^{-ax} e^{ax}\, dx$$

$$= \frac{1}{\phi(a)} e^{ax} \int dx = \frac{1}{\phi(a)} e^{ax} \cdot x$$

If $f(a) = 0$, $\dfrac{1}{f(D)} e^{ax} = \dfrac{xe^{ax}}{\phi(a)}$ .... (i)

But $f(D) = (D - a)\,\phi(D)$ differentiating with respect to D

$f'(D) = (D - a)\,\phi'(D) + \phi(D)$

If $D = a$, $f'(a) = \phi(a)$

The formula (i) becomes $\dfrac{1}{f(D)} e^{ax} = \dfrac{xe^{ax}}{f'(a)}$, $f'(a) \neq 0$

If $f'(a) = 0$, then $\dfrac{1}{f(D)} e^{ax} = x^2 \cdot \dfrac{1}{f''(a)} e^{ax}$

P.I. $\boxed{\dfrac{1}{f(D)} e^{ax} = \dfrac{1}{f(a)} e^{ax}}$, $f(a) \neq 0$

$\boxed{\dfrac{1}{f(D)} e^{ax} = \dfrac{xe^{ax}}{f'(a)}}$, if $f(a) = 0$, $f'(a) \neq 0$

$\boxed{\dfrac{1}{f(D)} e^{ax} = \dfrac{x^2 e^{ax}}{f''(a)}}$, if $f'(a) = 0$, $f''(a) \neq 0$

SOLVED EXAMPLES – 6.2

EXAMPLE - 1 :

Solve $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} + 4x = e^{3x}$

Solution : A.E. $m^3 - 3m + 4 = 0$

$$(m+1)(m-2)(m-2) = 0$$

$$m = -1, 2, 2$$

C.F. $c_1e^{-x} + (c_2 + c_3x)e^{2x}$

$$\text{P.I..} = \frac{1}{D^3 - 3D^2 + 4}e^{3x}$$

$$= \frac{1}{3^3 - 3 \cdot 3^2 + 4}e^{3x} = \frac{e^{3x}}{4}$$

Complete solution y = C. F. + P. I.

$$y = c_1e^{-x} + (c_2 + c_3x)e^{2x} + \frac{e^{3x}}{4}$$

EXAMPLE - 2 :

Solve $(D^3 + 3D^2 + 3D + 1)y = e^{-x}$

Solution : A.E. $m^3 + 3m^2 + 3m + 1 = 0$

$(m+1)^3 = 0$ $\quad m = -1, -1, -1$

C.F. = $(c_1 + c_2x + c_3x^2)e^{-x}$

$$\text{P.I.} = \frac{1}{(D+1)^3}e^{-x} \qquad f(-1) = 0$$

$$= \frac{x \cdot 1}{3(D-1)^2}e^{-x} \quad \text{(differentiating with respect to D and D} = -1\text{)}$$

$$= \frac{x^2 \cdot 1}{3 \cdot 2\,(D-1)}e^{-x} \quad \text{(Again differentiating with respect to D and D} = -1\text{)}$$

$$= \frac{x^3 \cdot 1}{3 \cdot 2 \cdot 1}e^{-x} \quad \text{(Again differentiating with respect to D and D} = -1\text{)}$$

$$= \frac{e^{-x}x^3}{6}$$

Complete solution $y = (c_1 + c_2x + c_3x^2)e^{-x} + \dfrac{e^{-x}x^3}{6}$

EXAMPLE - 3 :

Solve $\dfrac{d^3y}{dx^3} + y = (e^x + 1)^2$

Solution : Given equation in symbolic form is

$$(D^3 + 1)y = (e^x + 1)^2$$

A.E. $m^3 + 1 = 0$

$$(m + 1)(m^2 - m + 1) = 0$$

$$m = -1, \frac{1 \pm i\sqrt{3}}{2}$$

$$C.F. = c_1e^{-x} + e^{1/2\,x}\left(c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right)$$

$$P.I. = \frac{1}{D^3 + 1}(e^{2x} + 2e^x + e^{0 \cdot x})$$

$$= \frac{1}{2^3 + 1}e^{2x} + 2\frac{e^x}{1^2 + 1} + \frac{1}{0 + 1}e^{0 \cdot x}$$

$$= \frac{1}{9}e^{2x} + e^x + 1$$

Complete solution $y = C.F. + P.I.$

$$y = c_1e^{-x} + e^{\frac{1}{2}x}\left(c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right) + \frac{1}{9}e^{2x} + e^x + 1$$

EXAMPLE - 4 :

Solve $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 3y = 4e^{3x}$

Given $y(0) = -1,\ y'(0) = 3$ ***(JNTU 2007, 2005, 2004)***

Solution : Auxiliary equation is $m^2 - 4m + 3 = 0$

$$(m - 1)(m - 3) = 0$$

$$m = 1, 3$$

$$C.F = c_1e^x + c_2e^{3x}$$

$$\text{P.I.} = \frac{1}{D^2 - 40 + 3}\, 4e^{3x}$$

$$= 4\, \frac{x}{2D - 4}\, e^{3x} \quad \text{(Differentiating with respect to D)}$$

$$= \frac{4\,xe^{3x}}{2\cdot 3 - 4}$$

$$= 2xe^{3x}$$

The complete solution = C.F. + P.I.

$$y = c_1e^{x} + c_2e^{3x} + 2xe^{3x}$$

EXAMPLE - 5 :

Solve $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 5y = -2\cos hx$ ***(JNTU 2007, 2005)***

Solution : A.E. is $m^2 + 4m + 5 = 0$

$$m = -\frac{4 \pm \sqrt{16 - 20}}{2} = -2 \pm i$$

$$\text{C.F.} = e^{-2x}(\, c_1 \cos x + c_2 \sin x\,)$$

$$\text{P.I.} = \frac{1}{D^2 + 4D + 5}\,(\,-2\cos hx\,)$$

$$= -2\, \frac{1}{2\left(D^2 + 4D + 5\right)}\,(\, e^{x} + e^{-x}\,)$$

$$= \frac{1}{D^2 + 4D + 5}\,(\,-e^{x} - e^{-x}\,)$$

$$= \frac{1}{D^2 + 4D + 5}\,(\,-e^{x}\,) - \frac{1}{D^2 + 4D + 5}\, e^{-x}$$

$$= -\frac{e^{x}}{10} - \frac{e^{-x}}{2}$$

Complete Solution

$$y = e^{-2x}\,(\, c_1 \cos x + c_2 \sin x\,) - \frac{e^{x}}{10} - \frac{e^{-x}}{2}$$

EXAMPLE - 6 :

Solve $(D^3 - 6D^2 + 11D - 6)y = e^{-2x} + e^{-3x}$ ***(JNTU 2008)***

Solution : A.E. $m^3 - 6m^2 + 11m - 6 = 0$

$(m-1)(m^2 - 5m + 6) = 0$

$(m-1)(m-2)(m-3) = 0$

$m = 1, 2, 3$

$C.F = c_1e^x + c_2e^{2x} + c_3e^{3x}$

$$P.I = \frac{1}{(D-1)(D-2)(D-3)}(e^{-2x} + e^{-3x})$$

$$= \frac{e^{-2x}}{(-3)(-4)(-5)} + \frac{e^{-3x}}{(-4)(-5)(-6)}$$

$$= \frac{e^{-2x}}{-60} + \frac{e^{-3x}}{-120}$$

Complete Solution $y = CF + PI$

$$y = c_1e^x + c_2e^{2x} + c_3e^{3x} - \frac{e^{-2x}}{60} - \frac{e^{-3x}}{120}$$

EXERCISE 6.2

1. $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{-x}$ **Ans :** $y = c_1e^{-x} + c_2e^{-2x} + xe^{-x}$

2. $(D^2 - 2D + 1)y = e^x + 1$ **Ans :** $y = (c_1 + c_2x)e^x + \frac{x^2}{2}e^x + 1$

3. $(D^3 - 1)y = (e^x + 1)^2$ **Ans :** $y = c_1e^x + e^{x/2}\left(c_2\cos\frac{\sqrt{3}}{2}x + c_3\sin\frac{\sqrt{3}}{2}x\right) + \frac{1}{7}e^{2x} + \frac{2}{3}e^x + 1$

4. $(D^3 - 4D^2 + 5D - 2)y = e^x$ **Ans :** $y = (c_1 + c_2x)e^x + c_3e^{2x} - \frac{x^2}{2}e^x$

5. $\frac{d^2y}{dx^2} + 2p\frac{dy}{dx} + (p^2 + q^2)y = e^{kx}$ **Ans :** $y = e^{-px}(c_1\cos qx + c_2\sin qx) + \frac{e^{kx}}{(p+k)^2 + q^2}$

6. $\frac{d^2y}{dx^2} - 2k\frac{dy}{dx} + k^2y = e^x$ Ans : $y = (c_1 + c_2x)e^{kx} + \frac{e^x}{(k-1)^2}$

7. $(D^2 - D - 6)y = e^x \cosh 2x$ Ans : $y = c_1 e^{3x} + c_2 e^{-2x} + \frac{1}{10}e^{3x} - \frac{1}{8}e^{-x}$

8. $(D^2 - 9)y = e^{-3x} + 1 + e^{3x}$ Ans : $y = c_1 e^{3x} + c_2 e^{-3x} + \frac{x}{6}e^{3x} - \frac{1}{9} - \frac{x}{6}e^{-3x}$

II. When X = sin ax

$$\text{P. I.} = \frac{1}{f(D)}\sin ax$$

Consider $D(\sin ax) = a\cos ax$

$$D^2(\sin ax) = -a^2\sin ax$$

$$D^3(\sin ax) = -a^3\cos ax$$

$$D^4(\sin ax) = (-a^2)^2\sin ax \ldots\ldots\ldots\ldots$$

$$f(D) = D^n + a_1D^{n-1} + a_2D^{n-2} + \ldots\ldots\ldots + a_n$$

$$f(D^2) = (D^2)^n + a_1(D^2)^{n-1} + a_2(D^2)^{n-2} + \ldots\ldots\ldots + a_n$$

$$f(D^2)\sin ax = \{(-a^2)^n + a_1(-a^2)^{n-2} + \ldots\ldots\ldots + a_n\}\sin ax$$

$$= f(-a^2)\sin ax$$

operating on both sides by $\frac{1}{f(D^2)}$

$$\frac{1}{f(D^2)}f(D^2)\sin ax = \frac{1}{f(D^2)}f(-a^2)\sin ax$$

$$\Rightarrow \sin ax = \frac{1}{f(D^2)}f(-a^2)\sin ax$$

$$\Rightarrow \frac{1}{f(D^2)}\sin ax = \frac{1}{f(-a^2)}\sin ax, \quad f(-a^2) \neq 0$$

Formula : $\boxed{\frac{1}{f(D^2)}\sin ax = \frac{1}{f(-a^2)}\sin ax}$ $f(-a^2) \neq 0$

Formula : $$\boxed{\frac{1}{f\left(D^2\right)} \cos ax = \frac{1}{f\left(-a^2\right)} \cos ax} \qquad f(-a^2) \neq 0$$

Case of failure : If $f(-a^2) = 0$, the above method fails

$\cos ax + i \sin ax = e^{iax}$

$$\frac{1}{f\left(D^2\right)} (\cos ax + i \sin ax) = \frac{1}{f\left(D^2\right)} e^{iax}$$

Replacing D by ia, $f(i^2a^2) = f(-a^2) = 0$

$$= \frac{x e^{iax}}{f'\left(-a^2\right)} \text{ since } \frac{1}{f(D)} e^{ax} = \frac{xe^{ax}}{f'(a)} \qquad f(a) = 0$$

$$= \frac{1}{f'\left(-a^2\right)} [x \cos ax + ix \sin ax]$$

Equating the real and imagining parts

$$\frac{1}{f\left(D^2\right)} \cos ax = \frac{x \cos ax}{f'\left(-a^2\right)}, \quad f'(-a^2) \neq 0$$

$$\frac{1}{f\left(D^2\right)} \sin ax = \frac{x \sin ax}{f'\left(-a^2\right)}, \quad f'(-a^2) \neq 0$$

Again if $f'(-a^2) = 0$, $$\frac{1}{f\left(D^2\right)} \cos ax = \frac{x^2 \cos ax}{f''\left(-a^2\right)}, \quad f''(-a^2) \neq 0$$

$$\frac{1}{f\left(D^2\right)} \sin ax = \frac{x^2 \sin ax}{f''\left(-a^2\right)}, \quad f''(-a^2) \neq 0$$

Particular cases :

$$\frac{1}{D^2 + a^2} \cos ax = \frac{x}{2a} \sin ax$$

$$\frac{1}{D^2 + a^2} \sin ax = -\frac{x}{2a} \cos ax$$

SOLVED EXAMPLES – 6.3

EXAMPLE - 1 :

Solve $(D^2 - 5D + 6)\,y = \sin 3x$

Solution : A.E. $m^2 - 5m + 6 = 0$

$$(m-3)(m-2) = 0; \quad m = 2, 3$$

$$\text{C.F.} = c_1 e^{2x} + c_2 e^{3x}$$

$$\text{P.I.} = \frac{1}{D^2 - 5D + 6}\sin 3x = \frac{1}{-9 - 5D + 6}\sin 3x$$

$$= -\frac{1}{5D+3}\sin 3x$$

$$= -\frac{1}{5D+3}\cdot\frac{5D-3}{5D-3}\sin 3x$$

$$= -\frac{5D-3}{25D^2-9}\sin 3x$$

$$= -\frac{5D-3}{25(-9)-9}\sin 3x$$

$$= \frac{1}{234}(5D-3)\sin 3x$$

$$= \frac{1}{234}[5D(\sin 3x) - 3(\sin 3x)]$$

$$= \frac{1}{234}[5 \cdot 3\cos 3x - 3\sin 3x]$$

$$= \frac{1}{78}(5\cos 3x - \sin 3x)$$

Complete solution : $y = c_1 e^{2x} + c_2 e^{3x} + \frac{1}{78}(5\cos 3x - \sin 3x)$

EXAMPLE - 2 :

Solve $(D^3 + D)\,y = \cos x$

Solution : A.E. $m^3 + m = 0 \qquad m(m^2+1) = 0$

$$m = 0, i, -i$$

$$\text{C.F.} = c_1 + (c_2\cos x + c_3\sin x)$$

$$\text{P.I.} = \frac{1}{D^3+D} = \frac{1}{D^2+1}\cdot\frac{1}{D}\cos x = \frac{1}{D^2+1}\int\cos x\,dx = \frac{1}{D^2+1}\sin x$$

$$\frac{1}{D^2+1}\sin x = -\frac{x}{2}\cos x$$

Complete solution $c_1 + (c_2\cos x + c_3\sin x) - \frac{x}{2}\cos x$

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EXAMPLE - 3 :

Solve $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + y = a\cos 2x$

Solution : A.E. $m^2 - 4m + 1 = 0$ $\qquad m = 2 \pm \sqrt{3}$

$$\text{C.F.} = e^{2x}\left[c_1 e^{\sqrt{3}x} + c_2 e^{-\sqrt{3}x}\right]$$

$$\text{P.I.} = \frac{1}{D^2 - 4D + 1}\, a\cos 2x$$

$$a\,\frac{1}{-2^2 - 4D + 1}\cos 2x$$

$$= -a\,\frac{1}{4D + 3}\cos 2x$$

$$= -a\,\frac{4D - 3}{(4D+3)(4D-3)}\cos 2x$$

$$= -a\,\frac{4D - 3}{16D^2 - 9}\cos 2x$$

$$= -a\,\frac{4D - 3}{16\left(-2^2\right) - 9}\cos 2x$$

$$= \frac{a}{73}\,(4D - 3)\cos 2x$$

$$= \frac{a}{73}\,[\,4D\cos 2x - 3\cdot\cos 2x\,]$$

$$= \frac{a}{73}\,[\,-8\sin 2x - 3\cos 2x\,]$$

Complete solution : $y = e^{2x}\left[c_1 e^{\sqrt{3}x} + c_2 e^{-\sqrt{3}x}\right] + \dfrac{a}{73}\,[\,+8\sin 2x + 3\cos 2x\,]$

EXAMPLE - 4 :

Solve $(D^2 - 4)\,y = 2\cos^2 x$

Solution : A. E. $m^2 - 4 = 0,\ m = \pm 2$

$$\text{C.F.} = c_1 e^{2x} + c_2 e^{-2x}$$

$$\text{P.I.} = \frac{1}{D^2 - 4}\,2\cos^2 x = \frac{1}{D^2 - 4}\cdot\frac{2(1 + \cos 2x)}{2}$$

$$= \frac{1}{D^2 - 4}[1 + \cos 2x]$$

$$= \frac{1}{D^2 - 4} e^{0x} + \frac{1}{D^2 - 4} \cos 2x$$

$$= -\frac{1}{4} + \frac{1}{-2^2 - 4} \cos 2x = -\frac{1}{4} - \frac{1}{8} \cos 2x$$

Complete solution : $y = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{4} - \frac{1}{8} \cos 2x$

EXAMPLE - 5 :

Solve $(D^2 - 4D + 3)y = \sin 3x \cos 2x$

Solution : A.E. $m^2 - 4m + 3 = 0$

$m = 1, 3$

C.F. $= c_1 e^x + c_2 e^{3x}$

P.I. $= \frac{1}{D^2 - 4D + 3} \sin 3x \cos 2x$

$$= \frac{1}{2} \frac{1}{D^2 - 4D + 3} (\sin 5x + \sin x)$$

$$= \frac{1}{2}\left[\frac{1}{(-5^2) - 4D + 3} \sin 5x + \frac{1}{(-1^2) - 4D + 3} \sin x\right]$$

$$= \frac{1}{2}\left[-\frac{1}{4D + 22} \sin 5x + \frac{1}{2 - 4D} \sin x\right]$$

$$= \frac{1}{4}\left[\frac{1}{1 - 2D} \sin x - \frac{1}{2D + 11} \sin 5x\right]$$

$$= \frac{1}{4}\left[\frac{1 + 2D}{1 - 4D^2} \sin x - \frac{2D - 11}{4D^2 - 121} \sin 5x\right]$$

$$= \frac{1}{4}\left[\frac{1 + 2D}{1 - 4(-1^2)} \sin x - \frac{2D - 11}{4(-5^2) - 121} \sin 5x\right]$$

$$= \frac{1}{4}\left[\frac{1}{5}(\sin x + 2\cos x) + \frac{1}{221}(10\cos 5x - 11\sin 5x)\right]$$

$$= \frac{1}{20}(\sin x + 2\cos x) + \frac{1}{884}(10\cos 5x - 11\sin 5x)$$

$$y = c_1 e^x + c_2 e^{3x} + \frac{1}{20}(\sin x + 2\cos x) + \frac{1}{884}(10\cos 5x - 11\sin 5x)$$

EXAMPLE - 6 :

Solve $(D^2 + 2D + 10)\, y = -37 \sin 3x$

Solution : A.E. $m^2 + 2m + 10 = 0, \quad m = -1 \pm 3i$

C.F. $= e^{-x}\,(c_1 \cos 3x + c_2 \sin 3x)$

$$\text{P.I.} = \frac{1}{D^2 + 2D + 10}(-37 \sin 3x)$$

$$= -37 \frac{1}{-3^2 + 2D + 10} \sin 3x$$

$$= -37 \frac{1}{1 + 2D} \sin 3x$$

$$= -37 \frac{1 - 2D}{1 - 4D^2} \sin 3x$$

$$= -37 \frac{1 - 2D}{1 - 4\left(-3^2\right)} \sin 3x$$

$$= -(\sin 3x - 6 \cos 3x)$$

Complete solution : $y = e^{-x}\,(c_1 \cos 3x + c_2 \sin 3x) + 6 \cos 3x - \sin 3x$

EXAMPLE - 7 :

Show that the solution of $\dfrac{d^2x}{dt^2} + 4x = a \sin pt$, given $x = 0, \dfrac{dx}{dt} = 0$ when $t = 0$ is

$$x = \frac{a\left(\sin pt - \frac{1}{2} p \sin 2t\right)}{4 - p^2} \text{ if } p \neq 2$$

Solution : $(D^2 + 4)\, x = a \sin pt$

A.E. $m^2 + 4 = 0, \quad m = +2i, -2i$

C.F. $= c_1 \cos 2t + c_2 \sin 2t$

$$\text{P.I.} = \frac{1}{D^2 + 4} a \sin pt = a \frac{1}{\left(-p^2\right) + 4} \sin pt, \text{ if } p \neq 2$$

$$= \frac{a}{4 - p^2} \sin pt$$

Complete solution : $x = c_1 \cos 2t + c_2 \sin 2t + \dfrac{a}{4 - p^2} \sin pt$ .... (1)

Given that $x = 0$, when $t = 0$

from (1) $\quad 0 = c_1 \Rightarrow c_1 = 0$ .... (2)

$$\frac{dx}{dt} = -2\,c_1 \sin 2t + 2\,c_2 \cos 2t + \frac{ap}{4-p^2} \cos pt \qquad \text{.........(3)}$$

Given that $\frac{dx}{dt} = 0$ when $t = 0$

From (3), $0 = 2c_2 + \frac{ap}{4-p^2} \quad \Rightarrow c_2 = \frac{-ap}{2\left(4-p^2\right)}$ ....(4)

From (1), (3) and (4), $x = \frac{-ap}{2\left(4-p^2\right)} \sin pt + \frac{a}{4-p^2} \sin pt$

i.e., $x = \frac{a\left(\sin pt - \frac{1}{2} p \sin pt\right)}{4-p^2}$, if $p \neq 2$

EXAMPLE - 8 :

Solve $(D^2 + 9)\, y = \cos 3x$

Solution : A.E. $m^2 + 9 = 0, \quad m = +3i, -3i$

C.F. $= c_1 \cos 3x + c_2 \sin 3x$

P.I. $= \frac{1}{D^2+9} \cos 3x$ case of failure $f(a) = 0$

$= \frac{x}{2D} \cos 3x$

$= \frac{x}{2} \frac{1}{D} \cos 3x$

$= \frac{x}{2} \frac{\sin 3x}{3} \qquad \left(\frac{1}{D} \text{ stands for integration}\right)$

$= \frac{1}{6} x \sin 3x$

Another Method : $e^{i\,3x} = \cos 3x + i \sin 3x$

$\cos 3x =$ real part of $e^{i\,3x}$

P.I. $\frac{1}{D^2+9} \cos 3x =$ Real part of $\frac{1}{D^2+9} e^{i3x}$

Replacing D by i 3, $D^2 = -9$ which in case of failure

$=$ Real part of $\frac{x}{2D} e^{i3x}$

$$= \text{Real part of } \frac{x}{2}\frac{1}{D}e^{i3x}$$

$$= \text{Real part of } \frac{x}{2}\frac{e^{i3x}}{i3}$$

$$= \text{Real part of } \frac{x}{2}\frac{1}{3}\frac{1}{i}(\cos 3x + i\sin 3x)$$

$$= \text{Real part of } \frac{x}{6}\left(\frac{\cos 3x}{i} + \sin 3x\right)$$

$$= \frac{x}{6}\sin 3x$$

Complete solution : $c_1 \cos 3x + c_2 \sin 3x + \frac{x}{6}\sin 3x$

EXAMPLE - 9 :

Solve $\frac{d^3y}{dx^3} + 4\frac{dy}{dx} = \sin 2x$ **(JNTU 2007, 2005)**

Solution : A.E $m^3 = 4m = 0$

$m = 0, \pm 2i$

C.F $C_1 + C_2 \cos 2x + C_3 \sin 2x$

P.I. $\frac{1}{D^3 + 4D}\sin 2x = \frac{\sin 2x}{-4D + 4D}$ (o case)

$$= \frac{x}{3D^2 + 4}\sin 2x$$

$$= \frac{x}{3\left(-2^2\right) + 4}\sin 2x = -\frac{x\sin 2x}{8}$$

Complete Solution y = EF + PI

$$y = C_1 + C_2 \cos 2x + C_3 \sin 2x - \frac{x\sin 2x}{8}$$

EXAMPLE - 10 :

Solve $(D^2+4)y = \sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t,\ y(0)=1,\ y'(0)=\frac{3}{35}$

(JNTU 2005, 2004)

Solution : A.E $m^2+4=0,\ m=\pm 2i$

C.F $= c_1\cos 2t + c_2\sin 2t$

$$\text{P.I} = \frac{1}{D^2+4}\sin t + \frac{1}{3}\frac{1}{D^2+4}\sin 3t + \frac{1}{5}\frac{1}{D^2+4}\sin 5t$$

$$= \frac{\sin t}{-1+4} + \frac{1}{3}\frac{\sin 3t}{(-9+4)} + \frac{\sin 5t}{5(-25)+4}$$

$$= \frac{\sin t}{3} - \frac{\sin 3t}{15} - \frac{\sin 5t}{105}$$

Complete solution y = C.F + P.I.

$$y = c_1\cos 2t + c_2\sin 2t + \frac{\sin t}{3} - \frac{\sin 3t}{15} - \frac{\sin 5t}{105}$$

Given $y(0)=1,\ c_1=1$

$$y'(t) = -2c_1\sin 2t + 2c_2\cos 2t + \frac{\cos t}{3} - \frac{3\cos 3t}{15} - \frac{5\cos 5t}{105}$$

$$y'(0) = \frac{3}{35} \Rightarrow 2c_2 + \frac{1}{3} - \frac{1}{5} - \frac{1}{21} = \frac{3}{35}$$

$\Rightarrow c_2 = 0$

$$y = \cos 2t + \frac{\sin t}{3} - \frac{\sin 3t}{15} - \frac{\sin 5t}{105}$$

EXAMPLE - 11 :

Solve $y'' + 4y' + 4y = 4\cos x + 3\sin x$

$y(0)=1,\ y'(0)=0$

(JNTU 2007, 2004)

Solution : A.E. $m^2+4m+4=0$

$(m+2)^2=0,\ m=-2,-2$

C.F. $= (C_1 + C_2 x)e^{-2x}$

$$\text{P.I.} = \frac{1}{D^2+4D+4}(4\cos x + 3\sin x)$$

$$= \frac{4}{-1+4D+4}\cos x + \frac{3}{-1+4D+4}\sin x$$

$$= \frac{4\cos x}{4D+3} + \frac{3\sin x}{4D+3}$$

$$= \frac{4(3-4D)}{9-16D^2}\cos x + \frac{3(3-4D)}{9-16D^2}\sin x$$

$$= \frac{4}{25}(3\cos x + 4\sin x) + \frac{3}{25}(3\sin x - 4\cos x) = \sin x$$

Complete solution is y = C.F + P.I.

$$y = (c_1 + c_2 x)e^{-2x} + \sin x$$

Given $y(0) = 0 \Rightarrow c_1 = 1$

$y'(0) = 1 \Rightarrow -2c_1 + c_2 + 1 = 0$

$\Rightarrow c_2 = 1$

The solution is $y = (1 + x)e^{-2x} + \sin x$

EXAMPLE 12 :

Solve $y'' + 4y' + 20y = 23\sin t - 15\cos t$

$y(0) = 0,\ y'(0) = -1$ **(JNTU 2007, 2006, 2004)**

Solution : A.E. $m^2 + 4m + 20 = 0$

$$m = \frac{4 \pm \sqrt{16-80}}{2} = -2 \pm 4i$$

$$\text{C.F.} = e^{-2t}(c_1\cos 4t + c_2\sin 4t)$$

$$\text{P.I.} = \frac{1}{D^2+4D+20}23\sin t - \frac{1}{D^2+4D+20}15\cos t$$

$$= \frac{23}{-1+4D+20}\sin t - \frac{15}{-1+4D+20}\cos t$$

$$= \frac{23(4D-19)}{(4D+19)(4D-19)}\sin t - \frac{15(4D-19)}{(4D+19)(4D-19)}\cos t$$

$$= \frac{23(4\cos t - 19\sin t)}{16D^2 - 19^2} - \frac{15(-4\sin t - 19\cos t)}{16D^2 - 19^2}$$

$$= \cos t - \sin t$$

Complete solution

$y = e^{-2t}(c_1 \cos 4t + c_2 \sin 4t) + \cos t - \sin t$

Given $y(0) = 0 \Rightarrow c_1 + 1 = 0;\ c_1 = -1$

$y'(t) = -e^{-2t} 4c_1 \sin 4t - 2e^{-2t} c_1 \cos 4t + 4c_2 e^{-2t} \cos 4t - 2e^{-2t} \sin 4t - \sin t - \cos t$

$y'(0) = -1 \ \Rightarrow -2c_1 + 4c_2 - 1 = -1$

$$\Rightarrow 2 + 4c_2 = 0 \qquad c_2 = -\frac{1}{2}$$

$$y = e^{-2t}\left(-\cos 4t - \frac{1}{2}\sin 4t\right) + \cos t - \sin t$$

III. Particular Integral when $X = x^k$ where k is a positive integer :

$$\text{P. I.} = \frac{1}{f(D)} x^k$$

Reduce $\frac{1}{f(D)}$ to the form $\frac{1}{1+\phi(D)}$ by takingout the lowest degree term from $f(D)$.

Now write $\frac{1}{f(D)}$ as $[1 + \phi(D)]^{-1}$ and expand it in ascending powers of D, regarding D as a number by Binomial theorem upto the term containing D^k. Then operate on x^k with each term of the expansion $[1 + \phi(D)]^{-1}$

Use the formulae :

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots\ldots\ldots$$

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots\ldots\ldots\ldots$$

for the expansion of $[1 + \phi(D)]^{-1}$

SOLVED EXAMPLES – 6.4

EXAMPLE - 1 :

Solve $(D^3 - 3D + 2)\,y = x$

Solution : A.E. $m^3 - 3m + 2 = 0$

$$m = 1, 1, -2$$

$$\text{C.F.} = (c_1 + c_2 x)\,e^x + c_3 e^{-2x}$$

$$\text{P.I.} = \frac{1}{D^3 - 3D + 2}x = \frac{1}{2\left[1 - \left(\frac{3D - D^3}{2}\right)\right]}x$$

$$= \frac{1}{2}\left[1 - \frac{3D - D^3}{2}\right]^{-1}x$$

$$= \frac{1}{2}\left[1 + \frac{3D - D^3}{2} + \ldots\ldots\right]x$$

$$= \frac{1}{2}\left[x + \frac{3}{2}\right]$$

Complete solution : $y = (c_1 + c_2x)e^x + c_3e^{-2x} + \frac{1}{2}\left[x + \frac{3}{2}\right]$

EXAMPLE - 2 :

Solve $(D^2 + D + 1)y = x^3$

Solution : A.E. $m^2 + m + 1 = 0$

$$m = -\frac{1}{2} + \frac{\sqrt{3}}{2}i,\ -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

$$\text{C.F.} = e^{-x/2}\left[c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right]$$

$$\text{P.I.} = \frac{1}{D^2 + D + 1}x^3 = [1 + (D^2 + D)]^{-1}x^3$$

$$= [1 - (D^2 + D) + (D^2 + D)^2 - (D^2 + D)^3 + \ldots\ldots]x^3$$

$(1 - D + D^3)x^3 = x^3 - 3x^2 + 6$ expanding up to D^3

Complete solution : $y = e^{-x/2}\left[c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right] + x^3 - 3x^2 + 6$

EXAMPLE - 3 :

Solve $(D^2 + D - 2)y = 2(1 + x - x^2)$

Solution : A.E. $m^2 + m - 2 = 0,\ m = -2, 1$

$$\text{C.F.} = c_1e^x + c_2e^{-2x}$$

$$\text{P.I.} = \frac{1}{D^2 + D - 2}2(1 + x - x^2)$$

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$$= 2\,\frac{1}{-2\left[1-\frac{D^2+D}{2}\right]}(1+x-x^2)$$

$$= -\left[1-\frac{D^2+D}{2}\right]^{-1}(1+x-x^2)$$

expand as far as terms containing D^2

$$= -\left[1+\frac{D^2+D}{2}+\frac{D^4+D^2+2D^3}{4}+\ldots\ldots\right](1+x-x^2)$$

$$= -\left[1+x-x^2+\frac{1}{2}(-2+1-2x)+\frac{1}{4}(-2)\right] = x^2$$

Complete solution : $y = c_1e^x + c_2e^{-2x} + x^2$

EXAMPLE - 4 :

Solve $(D^3 - D^2 - D + 1)\,y = 1 + x^2$

Solution : A.E. $m^3 - m^2 - m + 1 = 0$

$$(m-1)^2(m+1) = 0$$

$$m = 1, 1, -1$$

C.F. $= c_1e^{-x} + (c_2 + c_3x)\,e^x$

$$\text{P.I.} = \frac{1}{(D-1)^2(D+1)}(1+x^2)$$

$$= (1-D)^{-2}(1+D)^{-1}(1+x^2)$$

$$= (1+2D+3D^2+\ldots\ldots)(1-D+D^2-D^3+\ldots\ldots)(1+x^2)$$

$$= (1+D+2D^2)(1+x^2) \quad \text{expanding as far as terms containing } D^2$$

$$= x^2 + 2x + 5$$

Complete solution : $y = c_1e^{-x} + (c_2 + c_3x)\,e^x + x^2 + 2x + 5$

EXAMPLE - 5 :

Solve $D^2(D^2+4)\,y = 320\,(x^3 + 2x^2)$

Solution : A.E. $m^2(m^2+4) = 0,\ m = 0, 0, 2i, -2i$

C.F. $= c_1 + c_2x + c_3\cos 2x + c_4\sin 2x$

$$\text{P.I.} = \frac{1}{D^2\left(D^2+4\right)}\,320\,(x^3+2x^2)$$

$$= \frac{1}{4D^2}\left(1+\frac{D^2}{4}\right)^{-1} 320\,(x^3+2x^2)$$

comment

$$= \frac{1}{4D^2}\left(1 - \frac{D^2}{4} + \frac{D^4}{16} - \frac{D^6}{64} + \ldots\ldots\right) 320\,(x^3 + 2x^2)$$

$$= \frac{1}{4}\left(\frac{1}{D^2} - \frac{1}{4} + \frac{D^2}{16} - \ldots\ldots\right) 320\,(x^3 + 2x^2)$$

expanding up to the term containing x^3

$$= 80\left[\frac{x^5}{20} - \frac{1}{4}\left(x^3 + 2x^2\right) + \frac{1}{16}(6x + 4)\right]$$

$$= 4x^5 - 20x^3 - 40x^2 + 30x + 20 \qquad \left(\frac{1}{D} \text{ stands for integration}\right)$$

Complete solution :

$$y = c_1 + c_2 x + c_3 \cos 2x + c_4 \sin 2x + 4x^5 - 20x^3 - 40x^2 + 30x + 20$$

IV. Particular Integral when $X = e^{ax}\,V$ where V is a function of x.

Let V_1 be a function of x, then

$$D\,(e^{ax}V_1) = e^{ax}\,DV_1 + ae^{ax}V_1 = e^{ax}\,(D + a)\,V_1$$

$$D^2\,(e^{ax}V_1) = e^{ax}D\,(D + a)\,V_1 + ae^{ax}\,(D + a)\,V_1$$

$$= e^{ax}\,(D + a)^2\,V_1$$

Integral $D^n\,(e^{ax}V_1) = e^{ax}\,(D + a)^n\,V_1$

Hence $f(D)\,e^{ax}\,V_1 = e^{ax}\,f(D + a)\,V_1$

Let $\quad f(D + a)\,V_1 = V,\ V_1 = \dfrac{1}{f(D + a)}\,V$

$$f(D)\,e^{ax}\cdot\frac{1}{f(D + a)}\,V = e^{ax}V$$

Operating by $\dfrac{1}{f(D)}$ on both sides

$$\frac{1}{f(D)}\cdot f(D)\,e^{ax}\,\frac{1}{f(D + a)}\,V = \frac{1}{f(D)}\,e^{ax}\,V$$

$$\frac{1}{f(D)}\,e^{ax}\,V = e^{ax}\,\frac{1}{f(D + a)}\,V$$

$$\boxed{\frac{1}{f(D)}\,e^{ax} = e^{ax}\,\frac{1}{f(D + a)}\,V}$$

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Note :

Replace D by D + a, write e^{ax} out of the operator $\frac{1}{f(D)}$, and find $\frac{1}{f(D+a)}$ V

SOLVED EXAMPLES – 6.5

EXAMPLE - 1 :

Solve $(D^2 + 2)\, y = e^x \cos x$

Solution : A.E. $m^2 + 2 = 0,\ m = +\sqrt{2}\, i,\ -\sqrt{2}\, i$

C.F. $= c_1 \cos \sqrt{2}x + c_2 \sin \sqrt{2}\, x$

$$\text{P.I.} = \frac{1}{D^2+2} e^x \cos x$$

$$= e^x \frac{1}{(D+1)^2 + 2} \cos x$$

$$= e^x \frac{1}{D^2 + 2D + 3} \cos x = e^x \frac{1}{2D+2} \cos x$$

$$= e^x \frac{1}{2} \frac{D-1}{D^2 - 1} \cos x = e^x \cdot \frac{1}{2} \cdot \frac{1}{(-2)} (-\sin x - \cos x)$$

$$= e^x \frac{1}{4} (\sin x + \cos x)$$

Complete solution : $y = (c_1 \cos \sqrt{2}x + c_2 \sin \sqrt{2}\, x) + e^x \frac{1}{4} (\sin x + \cos x)$

EXAMPLE - 2 :

Solve $\frac{d^2y}{dx^3} - 6\frac{dy}{dx} + 13y = 8\, e^{3x} \sin 2x$

Solution : Given equation $(D^2 - 6D + 13)\, y = 8\, e^{3x} \sin 2x$

A.E. $m^2 - 6m + 13 = 0$

$m = 3 + 2i,\ 3 - 2i$

$$\text{P.I.} = 8 \frac{1}{D^2 - 6D + 13} e^{3x} \sin 2x$$

$$= 8\, e^{3x} \frac{1}{(D+3)^2 - 6(D+3) + 13} \sin 2x$$

$$= 8\, e^{3x} \frac{1}{D^2 + 4} \sin 2x$$

$$= 8\, e^{3x} x \frac{1}{2D} \sin 2x \qquad \boxed{[\, f(a) = 0 \text{ case of failure} \,]}$$

ment

$$= 8e^{3x}\ \frac{x}{2} \int \sin 2x\, dx$$

$$= 8\, e^{3x}\ \frac{x}{2}\left(\frac{-\cos 2x}{2}\right) = -\,2x\, e^{3x} \cos 2x$$

Complete solution : $y = e^{3x}\,(\,c_1 \cos 2x + c_2 \sin 2x\,) - 2x\, e^{3x} \cos 2x$

EXAMPLE - 3 :

Solve $(D^3 - 7D - 6\,)\, y = \cosh x \cos x$

Solution : A.E. $m^3 - 7m - 6 = 0,$

$$(\,m + 1\,)\,(\,m^2 - m - 6\,) = 0 \ \Rightarrow\ m = -\,1, -\,2, 3$$

$$m = -\,1, -\,2, +\,3$$

$$\text{C.F.} = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{+3x}$$

$$\text{P.I.} = \frac{1}{D^3 - 7D - 6} \cosh x \cos x$$

$$= \frac{1}{D^3 - 7D - 6}\left(\frac{e^x + e^{-x}}{2}\right) \cos x$$

$$= \frac{1}{2} \cdot \frac{1}{D^3 - 7D - 6}\, e^x \cos x + \frac{1}{2} \cdot \frac{1}{D^3 - 7D - 6}\, e^{-x} \cos x$$

Now $\dfrac{1}{D^3 - 7D - 6}\, e^x \cos x = e^x\, \dfrac{1}{(D+1)^3 - 7(D+1) - 6} \cos x$

$$= e^x\, \frac{1}{D^3 + 3D^2 - 4D - 12} \cos x$$

$$= e^x\, \frac{1}{(\,-D - 3 - 4D - 12\,)} \cos x \quad \text{Replace } D^2 \text{ by } -\,1^2$$

$$= -\frac{1}{5}\, e^x\, \frac{1}{D + 3} \cos x = -\frac{1}{5}\, e^x\, \frac{D - 3}{D^2 - 9} \cos x$$

$$= -\frac{1}{5}\, e^x \cdot \frac{1}{(-1 - 9)}\, (\,D - 3\,) \cos x = \frac{e^x}{50}\,(\,-\sin x - 3 \cos x\,)$$

In the same way, $\dfrac{1}{D^3 - 7D - 6}\, e^{-x} \cos x = \dfrac{e^{-x}}{34}\,(\,3 \cos x - 5 \sin x\,)$

Complete solution :

$$y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{+3x} - \frac{e^x}{100}\,(\,\sin x + 3 \cos x\,) + \frac{1}{68}\, e^{-x}\,(\,3 \cos x - 5 \sin x\,)$$

EXAMPLE - 4 :

Solve $(D^3 - 3D^2 + 3D - 1)\,y = x^2 e^x$

Solution : A.E. $m^3 - 3m^2 + 3m - 1 = 0$

$$(m-1)^3 = 0,\ m = +1, +1, +1$$

C.F. = $(c_1 + c_2x + c_3x^2)\,e^x$

$$\text{P.I.} = \frac{1}{D^3 - 3D^2 + 3D - 1}\, x^2 e^x$$

$$= e^x \frac{1}{(D+1)^3 - 3(D+1)^2 + 3(D+1) - 1}\, x^2$$

$$= e^x \left(\frac{1}{D^3}\right) x^2$$

$$= e^x \frac{1}{D^2}\frac{1}{D}\, x^2 = e^x \frac{1}{D^2} \int x^2\, dx = e^x \frac{1}{D^2}\left(\frac{x^3}{3}\right)$$

$$= \frac{e^x}{3}\frac{1}{D}\left(\frac{1}{D} x^3\right) = \frac{e^x}{3}\frac{1}{D}\left(\frac{x^4}{4}\right)$$

$$= \frac{e^x}{12}\int x^4 = \frac{e^x}{12}\frac{x^5}{5} = \frac{x^5 e^x}{60}$$

Complete Solution : $y = e^{-x}(c_1 + c_2x + c_3x^2) + \dfrac{x^5 e^x}{60}$

EXAMPLE - 5 :

Solve $\dfrac{d^2y}{dx^2} - 7\dfrac{dy}{dx} + 6y = e^{2x}(1 + x)$

Solution : Given equation $(D^2 - 7D + 6)\,y = e^{2x}(1 + x)$

A.E. $m^2 - 7m + 6 = 0$

$m = 1, 6$

C.F. = $c_1e^x + c_2e^{6x}$

$$\text{P.I.} = \frac{1}{D^2 - 7D + 6}\, e^{2x}(1 + x)$$

$$= e^{2x} \frac{1}{(D+2)^2 - 7(D+2) + 6}(1 + x)$$

$$= e^{2x} \frac{1}{D^2 - 3D - 4}(1 + x)$$

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$$= e^{2x} \frac{1}{-4\left[1 - \frac{D^2 - 3D}{4}\right]}(1 + x)$$

$$= \frac{e^{2x}}{-4}\left[1 - \frac{D^2 - 3D}{4}\right]^{-1}(1 + x)$$

$$= -\frac{e^{2x}}{4}\left[1 - \frac{D^2 - 3D}{4} + \left(\frac{D^2 - 3D}{4}\right)^2 + \ldots\ldots\right](1 + x)$$

expanding upto the term containing D

$$= -\frac{e^{2x}}{4}\left(1 - \frac{3D}{4}\right)(1 + x)$$

$$= -\frac{e^{2x}}{4}\left[1 + x - \frac{3D}{4}(1 + x)\right] = -\frac{e^{2x}}{4}\left[1 + x - \frac{3}{4}\right]$$

$$= -\frac{e^{2x}}{16}(4x + 1)$$

Complete solution : $y = 4e^x + c_2e^{6x} - \frac{e^{2x}}{16}(1 + 4x)$

EXAMPLE 6 :

Solve $(D^3 - 7D^2 + 14D - 8)y = e^x \cos 2x$ *(JNTU 2007, 2004)*

Solution : A.E. $m^3 - 7m^2 + 14m - 8 = 0$

$(m - 1)(m - 2)(m - 4) = 0$

$m = 1, 2, 4$

C.F. $= c_1e^x + c_2e^{2x} + c_3e^{4x}$

$$\text{P.I.} = \frac{1}{D^3 - 7D^2 + 14D - 8} e^x \cos 2x$$

$$= e^x \frac{1}{(D + 1)^3 - 7(D + 1)^2 + 14(D + 1) - 8} \cos 2x$$

$$= e^x \frac{1}{D^3 - 4D^2 + 3D} \cos 2x$$

$$= e^x \frac{1}{D(-4) - 4(-4) + 3D} \cos 2x$$

$$= e^x \frac{1}{16 - D} \cos 2x$$

$$= e^x \frac{(16 + D)}{256 - D^2} \cos 2x$$

$$= e^x \frac{(16 + D)}{256 + 4} \cos 2x$$

$$= e^x \frac{(16 \cos 2x - 2 \sin 2x)}{260}$$

$$= e^x \frac{(8 \cos x - \sin 2x)}{130}$$

Complete Solution

$$y = c_1 e^x + c_2 e^{2x} + c_3 e^{3x} + \frac{e^x}{130} (8 \cos x - \sin 2x)$$

EXAMPLE - 7 :

Solve $(D^2 + 2D - 3)y = x^2 e^{-3x}$ **(JNTU 2006)**

Solution : A.E. $m^2 + 2m - 3 = 0$

$m = 1, -3$

C.F. $= C_1 e^x + C_2 e^{-3x}$

$$\text{P.I.} = \frac{1}{D^2 + 2D - 3} x^2 e^{-3x}$$

$$= e^{-3x} \frac{1}{(D-3)^2 + 2(D-3) - 3} x^2$$

$$= e^{-3x} \frac{1}{D^2 - 4D} x^2$$

$$= \frac{e^{-3x}}{-4D} \left(1 - \frac{D}{4}\right)^{-1} x^2$$

$$= \frac{e^{-3x}}{-4D} \left(1 + \frac{D}{4} + \frac{D^2}{16}\right) x^2$$

$$= \frac{e^{-3x}}{-4D} \left(x^2 + \frac{x}{2} + \frac{1}{8}\right)$$

$$= \frac{e^{-3x}}{-4} \left(\frac{x^3}{3} + \frac{x^2}{4} + \frac{x}{8}\right)$$

Complete Solution

$$y = c_1 e^x + c_2 e^{-3x} - \frac{e^{-3x}}{4} \left(\frac{x^3}{3} + \frac{x^2}{4} + \frac{x}{8}\right)$$

EXAMPLE - 8 :

Solve $(D^3 - 4D^2 - D + 4)\, y = e^{3x} \cos 2x$ **(JNTU 2007, 2006, 2004S, 2003S)**

Solution : A.E. $= m^3 - 4m^2 - m + 4 = 0$

$$(m-1)(m^2 - 3m - 4) = 0$$

$$(m-1)(m-4)(m+1) = 0$$

$m = 1, -1, 4$

C.F. $= c_1e^x + c_2e^{-x} + c_3e^{4x}$

$$\text{P.I.} = \frac{1}{D^3 - 4D^2 - D + 4} e^{3x} \cos 2x$$

$$= e^{3x} \frac{1}{(D+3)^3 - 4(D+3)^2 - (D+3) + 4} \cos 2x$$

$$= e^{3x} \frac{1}{D^3 + 5D^2 + 2D - 8} \cos 2x$$

$$= e^{3x} \frac{1}{-4D + 20 + 2D - 8} \cos 2x$$

$$= e^{3x} \frac{1}{-2(D+14)} \cos 2x$$

$$= \frac{e^{3x}}{-2} \frac{(D-14)}{(D+14)(D-14)} \cos 2x$$

$$= \frac{e^{3x}}{-2} \frac{(D-14)}{D^2 - 196} \cos 2x = \frac{e^{3x}}{400} (-2 \sin 2x - 14 \cos 2x)$$

Complete Solution

$$y = c_1e^x + c_2e^{-x} + c_3\, e^{4x} - \frac{e^{3x}}{200} (\sin 2x + 7 \cos 2x)$$

V. Particular Integral when $X = xV$ when V is a function of x :

Let V_1 be a function of x, then $D(xV_1) = x\, DV_1 + V_1$

$D^2(xV_1) = D(x\, DV_1 + V_1) = x^2D^2V_1 + 2\, DV_1$

Similarly $D^3(x V_1) = xD^3V_1 + 3D^2V_1$

In general $D^n(xV_1) = xD^nV_1 + nD^{n-1} V_1$

$$= xD^nV_1 + \left(\frac{d}{dD} D^n\right) V_1$$

$\therefore\ f(D)\, x\, V_1 = x\, f(D)\, V_1 + f'(D)\, V_1$ .... (1)

Now let $f(D)\ V_1 = V$

$$V_1 = \frac{1}{f(D)} V \qquad \text{..................... (2)}$$

Operating by $\frac{1}{f(D)}$ on both sides, from (1)

$$\frac{1}{f(D)} f(D)\, xV_1 = \frac{1}{f(D)} x\, f(D)\, V_1 + \frac{1}{f(D)} f'(D)\, V_1$$

$$xV_1 = \frac{1}{f(D)} x\, f(D)\, V_1 + \frac{1}{f(D)} f'(D)\, V_1$$

By (2), $x \frac{1}{f(D)} V = \frac{1}{f(D)} xV + \frac{1}{f(D)} f'(D) \frac{1}{f(D)} V$

$$\frac{1}{f(D)} xV = x \frac{1}{f(D)} V - \frac{1}{f(D)} \left[f'(D) \cdot \frac{1}{f(D)} V \right]$$

$$\boxed{\frac{1}{f(D)} xV = x \frac{1}{f(D)} V - \frac{f'(D)}{\{f(D)\}^2} V} \quad \text{or} \quad \boxed{\frac{1}{f(D)} xV = \left[x - \frac{1}{f(D)} f'(D) \right] \frac{1}{f(D)} V}$$

Note : This method can be applied only when the degree of x is 1.

SOLVED EXAMPLES – 6.6

EXAMPLE - 1 :

Solve $(D^2 + 2D + 1)\, y = x \cos x$

Solution : A.E. $m^2 + 2m + 1 = 0$, $m = -1, -1$

C.F. $= (c_1 + c_2 x)\, e^{-x}$

$$\text{P.I.} = \frac{1}{D^2 + 2D + 1} x \cos x$$

$$= \left[x - \frac{1}{f(D)} f'(D) \right] \frac{1}{f(D)} x \cos x, \quad f(D) = D^2 + 2D + 1$$

$$= \left[x - \frac{1}{(D+1)^2} 2(D+1) \right] \frac{1}{-1^2 + 2D + 1} x \cos x$$

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$$= \left[x - \frac{2}{D+1}\right] \frac{1}{2D} \cos x$$

$$= \left[x - \frac{2}{D+1}\right] \frac{1}{2} \int \cos x \, dx = \left(x - \frac{2}{D+1}\right) \frac{1}{2} \sin x$$

$$= \frac{x}{2} \sin x - \frac{1}{D+1} \sin x$$

$$= \frac{x}{2} \sin x - \frac{D-1}{D^2 - 1} \sin x$$

$$= \frac{x}{2} \sin x - \frac{(D-1) \sin x}{-1^2 - 1}$$

$$= \frac{x}{2} \sin x + \frac{1}{2} (D - 1) \sin x$$

$$= \frac{x}{2} \sin x + \frac{1}{2} (D \sin x) - \frac{\sin x}{2}$$

$$= \frac{x}{2} \sin x + \frac{1}{2} \cos x - \frac{\sin x}{2}$$

$$= \frac{x}{2} \sin x + \frac{1}{2} (\cos x - \sin x)$$

EXAMPLE - 2 :

(JNTU 1998)

Solve $(D^2 + 4) y = x \sin x$

Solution : A.E. $m^2 + 4 = 0$, $m = 2i, -2i$

C.F. $= c_1 \cos 2x + c_2 \sin 2x$

$$\text{P.I.} = \frac{1}{D^2 + 4} x \sin x$$

$$= \left\{x - \frac{1}{D^2 + 4} \cdot 2D\right\} \frac{1}{D^2 + 4} \sin x$$

$$= \left\{x - \frac{1}{D^2 + 4} \cdot 2D\right\} \frac{1}{-1^2 + 4} \sin x$$

$$= \left\{x - \frac{1}{D^2 + 4} \cdot 2D\right\} \frac{1}{3} \sin x$$

$$= \frac{x}{3} \sin x - \frac{1}{D^2 + 4} \cdot \frac{2}{3} D \sin x$$

$$= \frac{x}{3}\sin x - \frac{1}{D^2+4}\cdot\frac{2}{3}\cos x$$

$$= \frac{x}{3}\sin x - \frac{2}{3}\frac{1}{D^2+4}\cos x$$

$$= \frac{x}{3}\sin x - \frac{2}{3}\frac{1}{-1^2+4}\cos x$$

$$= \frac{x}{3}\sin x - \frac{2}{3}\cdot\frac{1}{3}\cos x$$

$$= \frac{x}{3}\sin x - \frac{2}{9}\cos x$$

Complete Solution : $y = c_1\cos 2x + c_2\sin 2x + \frac{x}{3}\sin x - \frac{2}{9}\cos x$

EXAMPLE - 3 :

Solve $(D^2+1)\,y = x^2\sin 2x$

Solution : A.E. $m^2+1=0$, $m=+i,-i$

C.F. $= c_1\cos x + c_2\sin x$

$$\text{P.I.} = \frac{1}{D^2+1}x^2\sin 2x$$

$$= \text{Imaginary part of } \frac{1}{D^2+1}x^2e^{i2x}$$

$$= \text{Imaginary part of } e^{i2x}\frac{1}{(D+2i)^2+1}x^2$$

$$= \text{I.P. of } e^{i2x}\frac{1}{D^2+4iD-3}x^2$$

$$= \text{I.P. of } e^{i2x}\frac{1}{-3\left[1-\frac{D^2+4iD}{3}\right]}x^2$$

$$= -\frac{1}{3}\text{ I.P. of } e^{i2x}\left[1-\frac{D^2+4iD}{3}\right]^{-1}x^2$$

$$= -\frac{1}{3}\text{ I.P. of } e^{i2x}\left[1+\frac{D^2+4iD}{3}+\left(\frac{D^2+4iD}{3}\right)^2+\ldots\right]x^2$$

ment

$$= -\frac{1}{3}\text{ I.P. of } e^{i2x}\left[1+\frac{D^2+4iD}{3}-\frac{16D^2}{9}\right]x^2 \quad \text{expanding upto } D^2 \text{ terms}$$

$$= -\frac{1}{3}\text{ I.P. of } e^{i2x}\left[x^2+\frac{2}{3}+\frac{4i}{3}(2x)-\frac{16}{9}\cdot 2\right]$$

$$= -\frac{1}{3}\text{ I.P. of } (\cos 2x + i\sin 2x)\left[\left(x^2-\frac{26}{9}\right)+\frac{8i}{3}x\right]$$

$$= -\frac{1}{3}\left[\frac{8x}{3}\cos 2x+\left(x^2-\frac{26}{9}\right)\sin 2x\right]$$

Complete Solution : $y = c_1\cos x + c_2\sin x - \frac{1}{3}\left[\frac{8x}{3}\cos 2x+\left(x^2-\frac{26}{9}\right)\sin 2x\right]$

MISCELLANEOUS EXAMPLES – 6.7

EXAMPLE - 1 :

Solve $(D^2 + a^2)\, y = \sec ax$

Solution : A.E. $m^2 + a^2 = 0,\ m = +ai, -ai$

C.F. = $c_1\cos ax + c_2\sin ax$

$$\text{P.I.} = \frac{1}{D^2+a^2}\sec ax$$

$$= \frac{1}{(D+ai)(D-ai)}\sec ax$$

$$= \frac{1}{2ai}\left[\frac{1}{D-ai}-\frac{1}{D+ai}\right]\sec ax$$

$$= \frac{1}{2ai}\left[\frac{1}{D-ai}\sec ax-\frac{1}{D+ai}\sec ax\right]$$

$$= \frac{1}{2ai}\left[e^{iax}\int e^{-iax}\sec ax\,dx - e^{-aix}\int e^{iax}\sec ax\,dx\right]$$

$$= \frac{1}{2ai}\left[e^{iax}\int(\cos ax - i\sin ax)\sec x\,dx - e^{-iax}\int(\cos ax + i\sin ax)\sec ax\,dx\right]$$

$$= \frac{1}{2ai}\left[e^{iax}\int(1 - i\tan ax)\,dx - e^{-iax}\int(1 + i\tan x)\,dx\right]$$

$$= \frac{1}{2ai}\left[e^{iax}\left\{x - \frac{i}{a}\log \sec ax\right\} - e^{-iax}\left\{x + \frac{i}{a}\log \sec ax\right\}\right]$$

$$= \frac{1}{2ai}\left[(\cos ax + i \sin ax)\left\{ x - \frac{i}{a}\log \sec ax \right\} - (\cos ax - i \sin ax)\left\{ x + \frac{i}{a}\log \sec ax \right\}\right]$$

$$= \frac{1}{2ai}\left[2\,ix \sin ax - \frac{2i}{a}\cos ax \log \sec ax \right]$$

$$= \frac{x}{a}\sin ax - \frac{1}{a^2}\cos ax \log \sec ax$$

$$= \frac{x}{a}\sin ax + \frac{1}{a^2}\cos ax \log \cos ax$$

Complete solution :

$$y = c_1 \cos ax + c_2 \sin ax + \frac{x}{a}\sin ax + \frac{1}{a^2}\cos ax \log \cos ax$$

EXAMPLE - 2 :

Solve $(D^2 - 2D + 1)\, y = xe^x \sin x$

Solution : A.E. $m^2 - 2m + 1 = 0$, $m = 1, 1$

C.F. $= (c_1 + c_2 x)\, e^x$

$$\text{P.I.} = \frac{1}{(D-1)^2}\, e^x x \sin x$$

$$= e^x \frac{1}{(D+1-1)^2}\, x \sin x$$

$$= e^x \frac{1}{D^2}\, x \sin x$$

$$= e^x \left[\text{I. P. of } \frac{1}{D^2}\, xe^{ix} \right]$$

$$= e^x \left[\text{I. P. of } e^{ix} \frac{1}{(D+i)^2}\, x \right]$$

$$= e^x \left[\text{I. P. of } e^{ix} \frac{1}{D^2 + 2iD - 1}\, x \right]$$

$$= e^x \left[\text{I. P. of } e^{ix} \frac{1}{-1\left[1-\left(D^2+2iD\right)\right]} \right] x$$

$$= e^x \,[\, \text{I.P. of } e^{ix} (-1)\{1-(D^2+2iD)\}^{-1} x \,]$$

$$= e^x \,[\, \text{I.P. of } e^{ix} (-1)\{1+D^2+2iD\} x \,]$$

expanding upto terms containing D only

$$= e^x \,[\, \text{I.P. of } e^{ix} (-1)\{x+2i\,Dx\}\,]$$

$$= e^x \,[\, \text{I.P. of } e^{ix} (-1)\{x+2i\}\,]$$

$$= e^x \,[\, \text{I.P. of } (\cos x + i\sin x)(-x-2i)\,]$$

$$= -e^x (2\cos x + x\sin x)$$

EXAMPLE - 3 :

Solve $(D^2+5D+6)y = e^{-2x}\sec^2 x\,(1+2\tan x)$

Solution : A.E. $m^2+5m+6=0$

$(m+3)(m+2)=0,\ m=-2,-3$

C.F. $= y = c_1 e^{-2x} + c_2 e^{-3x}$

$$\text{P. I.} = \frac{1}{D+3}\cdot\frac{1}{D+2} e^{-2x}\sec^2 x\,(1+2\tan x)$$

$$= \frac{1}{D+3}\cdot e^{-2x}\int e^{2x} e^{-2x}\cdot \sec^2 x\,(1+2\tan x)\,dx$$

$$= \frac{1}{D+3} e^{-2x}\int \sec^2 x\,(1+2\tan x)\,dx$$

$$= \frac{1}{D+3}\int e^{3x} e^{-2x}(\tan x + \tan^2 x)\,dx$$

$$= e^{-3x}\int e^x\{(\tan x+\sec^2 x)-1\}\,dx$$

$$= e^{-3x}\left[\int e^x(\tan x+\sec^2 x)\,dx - \int e^x\,dx\right]$$

$$= e^{-3x}[e^x\tan x - e^x] = e^{-2x}[\tan x - 1]$$

$$\left[\text{formula} \int e^x[f(x)+f'(x)]\,dx = e^x f(x)\right]$$

$$y = c_1 e^{-2x} + c_2 e^{-3x} + e^{-2x}(\tan x - 1)$$

EXAMPLE - 4 :

Solve $(D^2+3D+2)\sin e^x$

Solution : A.E. $m^2+3m+2=0$

$m=-1,-2$

C.F. = $y = c_1 e^{-x} + c_2 e^{-2x}$

P.I. $= \dfrac{1}{(D+1)(D+2)} \sin e^x$

$= \left(\dfrac{1}{D+1} - \dfrac{1}{D+2} \right) \sin e^x$

$= \dfrac{1}{D+1} \sin e^x - \dfrac{1}{D+2} \sin e^x$

$= e^{-x} \int e^x \sin e^x \, dx - e^{-2x} \int e^{2x} \sin e^x \, dx$

Put $e^x = t$

$e^x \, dx = dt$

P.I. $= e^{-t} \int \sin t \, dt - e^{-2t} \int t \sin t \, dt$

$= e^{-t} (-\cos t) - e^{-2t} [t(-\cos t) - (1)(-\sin t)]$

$= -e^{-t} \cos t - e^{-2t} [-e^x \cos e^x + \sin e^x]$

$= -e^{2x} \sin e^x$

The complete solution : $y = c_1 e^{-x} + c_2 e^{-2x} - e^{2x} \sin e^x$

EXAMPLE - 5 :

Solve $(D^2 - 4D + 4) y = 8x^2 e^{2x} \sin 2x$

Solution : A.E. $m^2 - 4m + 4 = 0$

$m = 2, 2$

C.F. $= (c_1 + c_2 x) e^{2x}$

P.I. $= \dfrac{1}{(D-2)^2} 8x^2 e^{2x} \sin 2x$

$= 8 e^{2x} \dfrac{1}{(D+2-2)^2} x^2 \sin 2x$

$= 8 e^{2x} \dfrac{1}{D^2} x^2 \sin 2x$

$= 8 e^{2x}$ Integrating part of $\dfrac{1}{D^2} x^2 e^{i2x}$

$= 8 e^{2x}$ Integrating part of $e^{i2x} \dfrac{1}{(D+i2)^2} x^2$

$= 8e^{2x}$ Integrating part of $e^{i2x} \dfrac{1}{D^2 + 4iD - 4} x^2$

$= 8e^{2x}$ I.P. of $e^{2ix} \dfrac{1}{-4\left[1 - \dfrac{D^2 + 4iD}{4}\right]} x^2$

$$= 8e^{2x} \text{ I.P. of } e^{i2x} \left(-\frac{1}{4}\right)\left[1 - \frac{D^2 + 4iD}{4}\right]^{-1} x^2$$

$$= 8e^{2x} \text{ I.P. of } e^{i2x} \left(-\frac{1}{4}\right)\left[1 + \frac{D^2 + 4iD}{4} + \left(\frac{D^2 + 4iD}{4}\right)^2 + \ldots\ldots\right] x^2$$

$$= 8e^{2x} \text{ I.P. of } e^{i2x} \left(-\frac{1}{4}\right)\left[1 + \frac{D^2}{4} + \frac{4i}{4} D + \frac{16i^2D^2}{16}\right] x^2$$

expanding upto the term containing D^2

$$= 8e^{2x} \text{ I.P. of } e^{i2x} \left(-\frac{1}{4}\right)\left[x^2 + \frac{1}{2} + i2x - 2\right]$$

$$= -2e^{2x} \text{ I.P. of } (\cos 2x + i \sin 2x) \left[x^2 - \frac{3}{2} + i2x\right]$$

$$= -2\, e^{2x} \left[2x \cos 2x + x^2 \sin 2x - \frac{3}{2} \sin 2x \right]$$

$$= -2\, e^{2x} \left[2x \cos 2x + \left(x^2 - \frac{3}{2}\right) \sin 2x\right]$$

EXERCISE 6.3

Solve the following equations.

1. $(D^2 - D + 1)\, y = \cos 2x$ **Ans :** $y = c_1 + c_2 \cos x + c_3 \sin x - \frac{x}{2} \cos x$

2. $\frac{d^4y}{dx^4} - a^4 y = \sin ax$

   Ans : $y = c_1 e^{ax} + c_2 e^{-ax} + c_3 \cos ax + c_4 \sin ax + \frac{1}{4a^2} x \cos ax$

3. $(D^4 + 10\, D^2 + 9)\, y = 96 \sin 2x \cos x$

   Ans : $y = c_1 \cos x + c_2 \sin x + c_3 \cos 3x + c_4 \sin 3x + x(\cos 3x - 3 \cos x)$

4. $(D^2 - 4)y = 2\cos^2 x$ Ans : $y = c_1e^{2x} + c_2e^{-2x} - \frac{1}{4} - \frac{1}{8}\cos 2x$

5. $D^2(D^2 + 1)y = \sin x + e^{-x}$

Ans : $y = (c_1 + c_2x) + (c_3\cos 4x + c_4\sin 4x) - \frac{x}{2} + \frac{1}{2}e^{-x}$

6. $(D^2 - 6D + 9)y = 54x + 18$ Ans : $y = (c_1 + c_2x)e^{3x} + 6x + 6$

7. $(D^2 + 3D + 2)y = e^{-2x} + x^2 + \sin x$

Ans : $y = c_1e^{-x} + c_2e^{-2x} - xe^{-2x} + \frac{1}{2}\left(x^2 - 3x + \frac{7}{2}\right)$

8. $(D^3 + D)y = \cos x$ Ans : $y = c_1 + c_2\cos x + c_3\sin x - \frac{x}{2}\cos x$

9. $(D^2 + D + 1)y = x^3$ Ans : $e^{3x}(c_1\cos 2x + c_2\sin 2x) - 2xe^{3x}\cos 2x$

10. $(D^3 - 7D - 6)y = (1 + x^2)e^{2x}$

Ans : $y = c_1e^{-x} + c_2e^{-2x} + c_3e^{3x} - e^{2x}\frac{1}{12}\left(x^2 + \frac{5}{6}x + \frac{169}{72}\right)$

11. $(D^2 - 2D + 1)y = x^2e^{3x}$ Ans : $y = (c_1 + c_2x)e^x + e^{3x}\frac{1}{4}\left(x^2 - 2x + \frac{3}{2}\right)$

12. $(D^2 - 2D + 1)y = \frac{3e^x}{x^2}$ Ans : $y = (c_1 + c_2x)e^x - 3e^x\log x$

13. $(D^4 - 2D^3 - 3D^2 + 4D + 4)y = x^2e^x$

Ans : $y = e^{-x}(c_1 - c_2x) + e^{2x}(c_3 + c_4x) + e^x \cdot \frac{1}{4}\left(x^2 + x + \frac{7}{2}\right)$

14. $(D^2 + 2D + 1)y = 2x + x^2$

Ans : $y = (c_1 + c_2x)e^{-x} + x^2 - 2x + 2$

15. $(D^4 - 1)y = e^x\cos x$ Ans : $y = c_1e^x + c_2e^{-x} + c_3\cos x + c_4\sin x - \frac{1}{5}e^x\cos x$

16. $(D - 1)^3y = e^x(\cos 2x + x^2)$

Ans : $y = (c_1 + c_2x + c_3x^2)e^x + e^x\frac{(2x^5 - 15\sin 2x)}{120}$

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Assignment

17. $(D^2 + 4)\,y = x \sin x$

Ans : $y = c_1 \cos 2x + c_2 \sin 2x + \dfrac{x \sin x}{3} - \dfrac{2 \cos x}{9}$

18. $(D^2 - 1)\,y = x^2 \cos x$

Ans : $y = c_1 e^x + c_2 e^{-x} + x \sin x + (1 + x^2)\,\dfrac{(\cos x)}{2}$

19. $(D^2 - 2D + 1)\,y = x e^x \sin x$

20. $(D^2 + 1)\,y = x^2 e^{2x} + x \cos x$

Ans : $(c_1 + c_2 x)\; e^x - e^x\,(x \sin x + 2 \cos x)$

Ans : $y = c_1 \cos x + c_2 \sin x + \dfrac{\left(25x^2 - 40x + 22\right)}{125}\, e^{2x} + \dfrac{x^2 \sin x}{4} + \dfrac{x \cos x}{4}$

21. $(D^2 - 4D + 4)\,y = 4\,(e^{2x} - \cos 2x)$ **Ans :** $y = (c_1 + c_2 x)\,e^{2x} + 2x^2 e^{2x} + \dfrac{1}{2} \sin 2x$

22. $(D^3 - 3D^2 + 4D - 2)\,y = e^x + \cos x$

Ans : $y = c_1 e^x + e^x\,(c_2 \cos x + c_3 \sin x) + x\,e^x + \dfrac{3 \sin x + \cos x}{10}$

23. $(D^2 + 4)\,y = 4 \tan 2x$ **Ans :** $y = c_1 \cos 2x + c_2 \sin 2x - \log\left\{\tan\left(\dfrac{\pi}{4} + x\right)\right\} \cos 2x$

24. $(D^2 + 1)\,y = \operatorname{cosec} x$ **Ans :** $y = c_1 \cos x + c_2 \sin x - x \cos x + \sin x \log(\sin x)$

25. $(D^2 + 1)\,y = \sin 3x - \cos^2\left(\dfrac{x}{2}\right)$ **Ans :** $y = c_1 e^{-x} + e^{x/2}\left(c_2 \cos \dfrac{\sqrt{3}}{2}x + c_3 \sin \dfrac{\sqrt{3}}{2}x\right)$

$$+ \frac{\sin 3x + 27 \cos 3x}{730} - \frac{1}{2} + \frac{\sin x - \cos x}{4}$$

6.7 EQUATIONS REDUCIBLE TO LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS :

Cauchy's Homogeneous Linear Equations :

An equation of the form

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \ldots\ldots\ldots + a_{n-1} x \frac{dy}{dx} + a_n y = X \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

where $a_1, a_2, \ldots\ldots, a_n$ are constants and X is a function of x is called Cauchy's homogeneous linear equation of order n.

Equation (1) can be reduced to a linear equation with constant coefficients by changing the independent variable.

Let $x = e^z$ or $z = \log x,\ x > 0$

$$\frac{dz}{dx} = \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz} \qquad \Rightarrow x\frac{dy}{dx} = \frac{dy}{dz}$$

Denoting $\quad \frac{d}{dz} = D$

$$x\frac{dy}{dx} = \frac{dy}{dx} = Dy \qquad \ldots\ldots\ldots\ldots (2)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right)$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right)$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx}$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dx^2}$$

$$x^2\frac{d^2y}{dx^2} = -\frac{dy}{dz} + \frac{d^2y}{dx^2} = D^2y - Dy = D(D-1)y \qquad \ldots\ldots\ldots\ldots (3)$$

Similarly $x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)y \qquad \ldots\ldots\ldots\ldots (4)$

Substituting these values in the equation (1), a linear differential equation with constant coefficients can be obtained and can be solved by the methods discussed already.

6.8 LEGENDRE'S EQUATION :

An equation of the form

$$(a+bx)^n\frac{d^ny}{dx^n} + a_1(a+bx)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \ldots\ldots\ldots + a_{n-1}(a+bx)\frac{dy}{dx} + a_ny = X \qquad \ldots\ldots\ldots\ldots (1)$$

where $a_1, a_2, \ldots\ldots\ldots a_n$ are constants and X is a function of x, is called Legendre's linear equation.

Equation (1) can be reduced to linear differential equation with constant coefficients.

Substitute $a + bx = e^z$ or $z = \log(a + bx),\ a + bx > 0$

Denote $\dfrac{d}{dz} = D \qquad \dfrac{dz}{dx} = \dfrac{b}{a + bx}$

$$\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} = \frac{b}{a+bx}\cdot\frac{dy}{dz} \Rightarrow (a+bx)\frac{dy}{dx} = b\,Dy \qquad \ldots\ldots\ldots\ldots (2)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{b}{a+bx}\frac{dy}{dz}\right) = -\frac{b^2}{(a+bx)^2}\frac{dy}{dz} + \frac{b}{a+bx}\cdot\frac{d^2y}{dz^2}\cdot\frac{dy}{dx}$$

$$= -\frac{b^2}{(a+bx)^2}\frac{dy}{dz} + \frac{b}{(a+bx)}\frac{d^2y}{dz^2}\cdot\frac{b}{a+bx}$$

$$= \frac{b^2}{(a+bx)^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)$$

$$(a+bx)^2\frac{d^2y}{dx^2} = b^2(D^2y - Dy) = b^2D(D-1)y \qquad \ldots\ldots\ldots\ldots\ldots (3)$$

$$\text{Similarly } (a+bx)^3\frac{d^3y}{dx^3} = b^3D(D-1)(D-2)y \qquad \ldots\ldots\ldots\ldots\ldots (4)$$

Substituting these values in equation (1), a linear differentiating equation with constant coefficients can be obtained and can be solved by the methods already discussed.

SOLVED EXAMPLES – 6.8

EXAMPLE - 1 :

$$\text{Solve } x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 2\log x \qquad \ldots\ldots\ldots\ldots\ldots (1) \quad \textit{(JNTU 2006)}$$

Solution : Put $x = e^z$ so that $z = \log x,\ x > 0$

and denoting $D = \dfrac{d}{dz}$

$$x\frac{dy}{dx} = Dy,\quad x^2\frac{d^2y}{dx^2} = D(D-1)y$$

Substituting these values in the given equation

$[D(D-1) - D + 1]\,y = 2z$

$$(D^2 - 2D + 1)\, y = 2z \qquad \text{.......................} (2)$$

A.E. $m^2 - 2m + 1 = 0, \quad m = 1, 1$

C.F. $= (c_1 + c_2 z)\, e^z$

$$\text{P.I.} = \frac{1}{(D-1)^2}\, 2z = 2\,(D-1)^{-2} z$$

$= 2\,(1 + 2D + \text{.........})\, z$ expanding terms containing D,

$= 2\,(z + 2Dz)$

$= 2z + 4$

The genral solution of (2) is

$$y = (c_1 + c_2 z)\, e^z + 2z + 4$$

Substituting $z = \log x$, the complete solution of (1) is

$$y = (c_1 + c_2 \log x)\, x + 2 \log x + 4$$

EXAMPLE - 2 :

Solve $x^2 \dfrac{d^2y}{dx^2} + x \dfrac{dy}{dx} + y = \log x \sin(\log x)$

Solution : Given equation is $x^2 \dfrac{d^2y}{dx^2} + x \dfrac{dy}{dx} + y = \log x \sin(\log x)$ .... (1)

Put $x = e^z$ or $z = \log x$, $x > 0$ and $\dfrac{d}{dz} = D$

$$x \frac{dy}{dx} = Dy, \quad x^2 \frac{d^2y}{dx^2} = D(D-1)\, y$$

$$[\,D(D-1) + D + 1\,]\, y = z \sin z$$

$$(D^2 + 1)\, y = z \sin z \qquad \text{...............} (2)$$

A.E. $m^2 + 1 = 0 \qquad m = -i, i$

C.F. $= c_1 \cos z + c_2 \sin z$

$$\text{P.I.} = \frac{1}{D^2 + 1}\, z \sin z$$

$$= \text{I.P. of } \frac{1}{D^2 + 1}\, z e^{iz}$$

$$= \text{I.P. of } e^{iz} \frac{1}{(D+i)^2 + 1}\, z$$

$$= \text{I.P. of } e^{iz} \frac{1}{D^2 + 2iD - 1 + 1} z$$

$$= \text{I.P. of } e^{iz} \frac{1}{2iD\left(1 + \frac{D}{2i}\right)} z$$

$$= \text{I.P. of } e^{iz} \frac{1}{2iD} \left(1 + \frac{D}{2i}\right)^{-1} z$$

$$= \text{I.P. of } e^{iz} \frac{1}{2iD} \left[1 - \frac{D}{2i}\right] z \quad \text{expanding upto terms containing D,}$$

$$= \text{I.P. of } e^{iz} \frac{1}{2i} \frac{1}{D} \left[z - \frac{1}{2i}\right]$$

$$= \text{I.P. of } e^{iz} \frac{1}{2i} \left[\frac{1}{D} z - \frac{1}{D}\left(\frac{1}{2i}\right)\right]$$

$$= \text{I.P. of } e^{iz} \cdot \frac{1}{2i} \left[\frac{z^2}{2} - \frac{z}{2i}\right]$$

$$= \text{I.P. of } (\cos z + i \sin z) \; \frac{1}{2} (-i) \left[\frac{z^2}{2} - \frac{z}{2i}\right], \quad \left(\frac{1}{i} = -i\right)$$

$$= \frac{-1}{2} \text{ I.P. of } \left[i \frac{z^2}{2} \cos z - \frac{z}{2} \cos z - \sin z \cdot \frac{z^2}{2} - i \frac{z}{2} \sin z \right]$$

$$= \frac{-1}{2} \text{ I.P. of } \left[+ i \cos z \cdot \frac{z^2}{2} - \frac{z}{2} \cos z - \sin z \cdot \frac{z^2}{2} - \frac{iz}{2} \sin z \right]$$

$$= -\frac{1}{2} \left[\frac{z^2}{2} \cos z - \frac{z}{2} \sin z \right]$$

$$= -\frac{z^2}{4} \cos z + \frac{z}{4} \sin z$$

The general solution of (2) is

$$y = c_1 \cos z + c_2 \sin z - \left(\frac{z^2}{4}\right) \cos z + \left(\frac{z}{4}\right) \sin z$$

The general solution of (1) is

$$y = c_1 \cos(\log x) + c_2 \sin(\log x) - \frac{(\log x)^2}{4}\cos(\log x) + \left(\frac{\log x}{4}\right)\sin(\log x)$$

EXAMPLE - 3 :

Solve $x^2 \dfrac{d^2y}{dx^2} + 4x\dfrac{dy}{dx} + 2y = e^x$

Solution : Given equation is $x^2 \dfrac{d^2y}{dx^2} + 4x\dfrac{dy}{dx} + 2y = e^x$ .... (1)

Put $x = e^z$ so that $z = yx$ and $\dfrac{d}{dz} = D$

$x\dfrac{dy}{dx} = Dy,\ x^2\dfrac{d^2y}{dx^2} = D(D-1)y$

$[D(D-1) + 4D + 2]\,y = e^{e^z}$

$(D^2 + 3D + 2)\,y = e^{e^z}$ .... (2)

A.E. $m^2 + 3m + 2 = 0,\ m = -1, -2$

C.F. $= c_1e^{-z} + c_2e^{-2z}$

$$\text{P.I.} = \frac{1}{D^2 + 3D + 2}\,e^{e^z}$$

$$= \frac{1}{(D+1)(D+2)}\,e^{e^z}$$

$$= \left[\frac{1}{D+1} - \frac{1}{D+2}\right]e^{e^z} \qquad \text{............... (3)}$$

Let $\dfrac{1}{D+1}\,e^{e^z} = U \Rightarrow (D+1)\,U = e^{e^z}$

$$\Rightarrow \frac{dU}{dz} + U = e^{e^z}$$

This equation is linear in u.

I.F. $e^{\int dz} = e^z$

$U \cdot e^z = \int e^{e^z} \cdot e^z\,dz = e^{e^z}$

$\therefore \quad U = e^{-z}\,e^{e^z}$ .... (4)

Let $\frac{1}{D+2} e^{e^z} = V$

$(D+2)V = e^{e^z}$ i.e., $\frac{dV}{dz} + 2V = e^{e^z}$

This is linear equation in V

I.F. $e^{\int 2\,dz} = e^{2z}$

$V \cdot e^{2z} = \int e^{e^z} \cdot e^{2z}\,dz = e^{e^z}(e^z - 1)$

$V = e^{-2z}(e^z - 1)\, e^{e^z}$ .... (5)

From (3), (4) and (5)

P. I. $= e^{-z} e^{e^z} - e^{2z}(e^z - 1)\, e^{e^z}$

The general solution of (2) is

$$y = c_1 e^{-z} + c_2 e^{-2z} + e^{e^z}\left[\frac{1}{e^z} - e^{-2z}\left(e^z - 1\right)\right]$$

The complete solution of (1) is

$$y = \frac{c_1}{x} + \frac{c_2}{x_2} + \frac{1}{x} e^x - \frac{1}{x^2}(x-1)e^x$$

i.e., $y = \frac{c_1}{x} + \frac{c_2}{x^2} + \frac{1}{x^2} e^x$

EXAMPLE - 4 :

Solve $(1+x)^2 \frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 4 \cos \log(1+x)$

Solution : Given equation is $(1+x)^2 \frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 4 \cos \log(1+x)$ .... (1)

Put $1 + x = e^z$ or $z = \log(1+x)$ and $\frac{d}{dz} = D$

$$(1+x)\frac{dy}{dx} = Dy,$$

$$(1+x)^2 \frac{d^2y}{dx^2} = D(D-1)y$$

Substituting in (1),

$$[D(D-1) + D + 1]\, y = 4 \cos z$$

$$(D^2 + 1)\, y = 4 \cos z \quad \text{............ (2)}$$

A.E. $m^2 + 1 = 0, \ m = -i, i$

C.F. $= c_1 \cos z + c_2 \sin z$

P.I. $= \dfrac{1}{D^2 + 1} 4 \cos z \qquad [f(-a^2) = 0]$

$= \dfrac{4 \cdot z}{2D} \cos z = 2 z \sin z$

The general solution of (2) is $y = c_1 \cos z + c_2 \sin z + 2z \sin z$

The complete solution of (1) is

$$y = c_1 \cos \log(1 + x) + c_2 \sin \log(1 + x) + 2 \log(1 + x) \sin(1 + x)$$

EXAMPLE - 5 :

Solve $(1 + 2x)^2 \dfrac{d^2y}{dx^2} - 6(1 + 2x) \dfrac{dy}{dx} + 16 y = 8(1 + 2x)^2$

Solution : Given equation is

$$(1 + 2x)^2 \frac{d^2y}{dx^2} - 6(1 + 2x) \frac{dy}{dx} + 16 y = 8(1 + 2x)^2 \qquad \dots\dots\dots\dots (1)$$

Put $1 + 2x = e^z$ or $z = \log(1 + 2x), \ \dfrac{d}{dz} = D$

$(1 + 2x) \dfrac{dy}{dx} = 2 \cdot Dy, \ (1 + 2x)^2 \dfrac{d^2y}{dx^2} = 2^2 D(D - 1) y = 4D(D - 1) y$

Substituting in (1), the equation reduces to

$[4D(D - 1) - 6 \cdot 2D + 16] y = 8 e^{2z}$

$[4D^2 - 4D - 12D + 16] y = 8 e^{2z}$

$$(D^2 - 4D + 4) y = 2 e^{2z} \qquad \dots\dots\dots\dots (2)$$

A.E. $m^2 - 4m + 4 = 0, \ m = 2, 2$

C.F. $= (c_1 + c_2 z) e^{2z}$

P.I. $= \dfrac{1}{(D - 2)^2} 2 e^{2z} \qquad [f(a) = 0, \ f'(a) = 0]$

$2 \cdot \dfrac{z}{2(D - 2)} e^{2z} = 2 \cdot \dfrac{z^2}{2} \cdot e^{2z} = z^2 e^{2z}$

The general solution of (2) is

$y = (c_1 + c_2 z) e^{2z} + z^2 e^{2z}$

The complete solution of (1) is

$y = [c_1 + c_2 \log(1 + 2x)](1 + 2x)^2 + (1 + 2x)^2 [\log(1 + 2x)]^2$

EXERCISE 6.4

Solve the following equations.

1. $\left(x^2 \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} + 5\right) y = x^2 \sin(\log x)$

   Ans : $y = [c_1 \cos(\log x) + c_2 \sin(\log x)]\, x^2 - \frac{\left[x^2 \log x \cos(\log x)\right]}{2}$

2. $x^4 \frac{d^3y}{dx^3} + 2x^3 \frac{d^2y}{dx^2} - x^2 \frac{dy}{dx} + xy = \frac{1}{x}$ *[JNTU 2000]*

   Ans : $y = (c_1 + c_2 \log x)\, x + \frac{c_3}{x} + \frac{1}{4x} \log x$

3. $x^2 \frac{d^2y}{dx^2} - x \frac{dy}{dx} + 2y = x \log x$ *[JNTU 1993]*

   Ans : $y = [c_1 \cos(\log x) + c_2 \sin(\log x)]\, x + x \log x$

4. $x^2 \frac{d^2y}{dx^2} - x \frac{dy}{dx} + y = \log x$ *[JNTU 1995]*

   Ans : $y = (c_1 + c_2 \log x)\, x + \log x + 4$

5. $\frac{d^2y}{dx^2} + \frac{1}{x} \frac{dy}{dx} = \frac{12 \log x}{x^2}$

   Ans : $y = c_1 + c_2 \log x + 2(\log x)^2$

6. $x^2 \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} - 4y = x^2 + 2 \log x$ *[JNTU 1995/S]*

   Ans : $y = c_1 x^4 + \frac{c_2}{x} - \frac{x^2}{6} - \frac{1}{2}\left(\log x + \frac{3}{4}\right)$

7. $(3x + 2)^2 \frac{d^2y}{dx^2} + 3(3x + 2) \frac{dy}{dx} - 36y = 3x^2 + 4x + 1$

   Ans : $c_1 (3x + 2)^{-2} + c_2 (3x + 2)^2 + \frac{\left[(3x + 2)^2 \log(3x + 2) + 1\right]}{108}$

8. $(1+x)^2 \frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = \sin 2[\log(1+x)]$ **[JNTU 1996]**

Ans : $y = c_1 \cos(\log x) + c_2 \sin(\log x) + \frac{x}{2}$

9. $(2x+1)^2 \frac{d^2y}{dx^2} - 2(2x+1)\frac{dy}{dx} - 12y = 6x$

Ans : $y = c_1(2x+1)^3 + \frac{c_2}{(2x+1)} - \frac{3}{16}(2x+1) + \frac{1}{4}$

10. $(x+1)^2 \frac{d^2y}{dx^2} - 3(x+1)\frac{dy}{dx} + 4y = x^2 + x + 1$

Ans : $y = [c_1 + c_2 \log(x+1)](x+1)^2 + [\log(x+1)]^2 (x+1)^2 - (x+1) + \frac{1}{4}$

6.9 METHOD OF VARIATION OF PARAMETERS :

The method of variation of parameter can be used to find the complete solution of a linear equation whose complementary function is known.

Consider the linear equation of second order

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = X \quad \ldots\ldots\ldots (1)$$

Let the complementary function be

$$y = A\phi(x) + B\Psi(x) \quad \ldots\ldots\ldots (2)$$

where A and B are arbitrary constants.

The solution (2) satisfies the equation $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$ .... (3)

So, substituting (2) in (3), we get

$$[A\phi''(x) + B\psi''(x)] + [(A\phi'(x) + B\psi'(x)] + Q[A\phi(x) + B\psi(x)] = 0$$

or $A[\phi''(x) + P\phi'(x) + Q\phi(x)] + B[\psi''(x) + P\psi'(x) + Q\psi(x)] = 0$

$$\phi''(x) + P\phi'(x) + Q\phi(x) = 0 \quad \ldots\ldots\ldots (4)$$

$$\psi''(x) + P\psi'(x) + Q\psi(x) = 0 \quad \ldots\ldots\ldots (5)$$

Considering A and B are functions of x, let us suppose that

$$y = A\phi(x) + B\psi(x) \quad \ldots\ldots\ldots (6)$$

is the complete solution of (1)

$$\frac{dy}{dx} = A\,\phi'(x) + B\,\psi'(x) + \phi(x)\frac{dA}{dx} + \psi(x)\frac{dB}{dx}$$

Let A and B satisfy the equation

$$\phi(x)\frac{dA}{dx} + \psi(x)\frac{dB}{dx} = 0 \qquad \text{.............. (7)}$$

$$\frac{dy}{dx} = A\,\phi'(x) + B\,\psi'(x)$$

$$\frac{d^2y}{dx^2} = A\,\phi''(x) + B\,\psi''(x) + \phi'(x)\frac{dA}{dx} + \psi'(x)\frac{dB}{dx}$$

Substituting in (1), we get

$$\left[A\,\phi''(x) + B\,\psi''(x) + \frac{dA}{dx}\phi'(x) + \frac{dB}{dx}\psi'(x)\right]$$

$$+ P\left[A\,\phi'(x) + B\,\psi'(x)\right] + Q\left[A\,\phi(x) + B\,\psi(x)\right] = X$$

i.e., $A\left[\phi''(x) + P\,\phi'(x) + Q\,\phi(x)\right] + B\left[\psi''(x) + P\,\psi'(x) + Q\,\psi(x)\right]$

$$+ \phi'(x)\frac{dA}{dx} + \psi'(x)\frac{dB}{dx} = X$$

By (4) and (5), the coefficients of A and B are zero.

$$\phi'(x)\frac{dA}{dx} + \psi'(x)\frac{dB}{dx} = X \qquad \text{.............. (8)}$$

By (7) and (8), we get

$$\frac{dA}{dx}\left[\phi(x)\,\psi'(x) - \phi'(x)\,\psi(x)\right] = -R\,\psi(x)$$

$$\therefore \quad \frac{dA}{dx} = \frac{-X\psi(x)}{\phi(x)\,\psi'(x) - \phi'(x)\,\psi(x)}$$

Integrating $\boxed{A = \int \frac{-X\,\psi(x)}{\phi(x)\,\psi'(x) - \phi'(x)\,\psi(x)}\,dx + c_1}$

Similarly; by (7) and (8)

$$\frac{dB}{dx} = \frac{X\,\phi(x)}{\phi(x)\,\psi'(x) - \phi'(x)\,\psi(x)}$$

Integrating $\boxed{B = \int \frac{X\,\phi(x)}{\phi(x)\,\psi'(x) - \phi'(x)\,\psi(x)}\,dx + c_2}$

Substituting the values of A and B in (6), the complete solution can be obtained.

SOLVED EXAMPLES - 6.9

EXAMPLE - 1 :

Solve $(D^2 + a^2)\, y = \tan ax$

Solution : A.E. $m^2 + a^2 = 0 \quad \Rightarrow m = \pm ai$

C.F. $= c_1 \cos ax + c_2 \sin ax$ .... (1)

Let P.I. $A\,\phi(x) + B\,\psi(x)$ .... (2)

where $\phi(x) = \cos ax,\ \psi(x) = \sin ax$

$$\phi(x)\,\psi'(x) - \psi'(x)\,\phi(x) = \cos ax\,(a \cos ax) - \sin ax\,(-a \sin ax)$$

$$= a\,(\cos^2 ax + \sin^2 ax) = a$$

$$A = \int \frac{-X\,\psi(x)}{\phi(x)\,\psi'(x) - \psi'(x)\,\phi(x)}\, dx \qquad \text{where } X = \tan ax$$

$$= -\int \frac{\tan ax \cdot \sin ax}{a} = -\frac{1}{a}\int \frac{\sin^2 ax}{\cos ax}\, dx$$

$$= -\frac{1}{a}\int \frac{1 - \cos^2 ax}{\cos ax}\, dx$$

$$= -\frac{1}{a}\left[\int \sec ax\, dx - \int \cos ax\, dx\right]$$

$$= -\frac{1}{a^2}\log(\sec ax + \tan ax) + \frac{1}{a^2}\sin ax$$

$$B = \frac{X\,\phi(x)}{\phi(x)\,\psi'(x) - \psi'(x)\,\phi(x)}\, dx$$

$$= \int \frac{\tan ax \cos ax}{a}\, dx$$

$$= \frac{1}{a}\int \sin ax\, dx = -\frac{1}{a^2}\cos ax$$

Substituting the values A and B in (2), we get the complete solution is

$$y = c_1 \cos ax + c_2 \sin ax - \left[\frac{1}{a^2}\log(\sec ax + \tan x) + \frac{1}{a^2}\sin ax\right]\cos ax$$

$$-\frac{1}{a^2}\cos ax \cdot \sin ax$$

$$y = c_1 \cos ax + c_2 \sin ax - \frac{1}{a^2}\log(\sec ax + \tan ax)\cos ax$$

Assignment

EXAMPLE - 2 :

Solve $\frac{d^2y}{dx^2} + y = \text{cosec } x$ by the method of variation of parameters.

(JNTU 2007, 2004)

Solution : Given equation $(D^2 + 1)y = \text{cosec } x$

A.E. $m^2 + 1 = 0 \quad \Rightarrow m = \pm i$

C.F. $= c_1 \cos x + c_2 \sin x$

Let P.I. $= A\,\phi(x) + B\,\psi(x)$ .... (1)

where $\phi(x) = \cos x$ and $\psi(x) = \sin x$ .... (2)

$$\phi(x)\,\psi'(x) - \psi(x)\,\phi'(x) = \cos x \cos x + \sin x \sin x$$
$$= \cos^2 x + \sin^2 x = 1$$

$$A = -\int \frac{\psi(x)X}{\phi(x)\psi'(x) - \psi(x)\phi'(x)}\,dx \qquad \text{where } X = \text{cosec } ax$$

$$= -\int \frac{\sin x \cdot \text{cosec } ax}{1}\,dx = -\int dx = -x$$

$$B = \int \frac{\phi(x)X}{\phi(x)\psi'(x) - \psi(x)\phi'(x)}\,dx$$

$$= \int \frac{\cos x \text{ cosec } x}{1}\,dx$$

$$= \int \cot x\,\phi\,dx = \log(\sin x)$$

Substituting the values A and B in (2)

P. I. $= -x\cos x + \log(\sin x)\sin x$

Complete solution : $y = c_1 \cos x + c_2 \sin x - x\cos x + \sin x \log \sin x$

EXAMPLE - 3 :

Solve $(D^2 - 3D + 2)y = e^{4x}$ by method of variation of parameters.

Solution : A.E. $m^3 - 3m + 2 = 0$

$m = 1, 2$

C.F. $= c_1e^x + c_2e^{2x}$

Let P.I. $= Ae^x + Be^{2x}$ where $e^x = \phi(x)$, $e^{2x} = \psi(x)$

$\phi(x)\,\psi'(x) - \psi(x)\,\phi'(x)$

$$= e^x \cdot 2e^{2x} - e^{2x} \cdot e^x = e^{2x} \cdot e^x = e^{3x}$$

$$A = -\int \frac{\psi(x)X}{\phi(x)\psi'(x) - \psi(x)\phi'(x)}\,dx = -\int \frac{e^{2x} \cdot e^{4x}}{e^{3x}}\,dx = -\int e^{3x}\,dx = -\frac{e^{3x}}{3}$$

$$B=\int \frac{\phi(x)X}{\phi(x)\psi'(x)-\psi(x)\phi'(x)}\,dx=\int \frac{e^{x}\cdot e^{4x}}{e^{3x}}\,dx=\int e^{2x}\,dx=\frac{e^{3x}}{2}$$

$$\text{P.I.}=-\frac{e^{-3x}}{3}\cdot e^{x}+\frac{e^{2x}}{3}e^{2x}=-\frac{e^{-4x}}{3}+\frac{e^{4x}}{2}=\frac{e^{4x}}{6}$$

Complete solution = C.F. + P.I.

Complete solution $y=c_1e^{x}+c_2e^{2x}+\dfrac{e^{4x}}{6}$

EXAMPLE - 4 :

Solve $\dfrac{d^2y}{dx^2}-y=\dfrac{2}{1+e^{x}}$ by the method of variation of parameter.

Solution : Given equation $(D^2-1)\,y=\dfrac{2}{1+e^{x}}$

A.E. $m^2-1=0$ $\quad m=+1,-1$

C.F. $=c_1e^{x}+c_2e^{-x}$

Let P.I. $=Ae^{x}+Be^{-x}$ where $\phi(x)=e^{x};\ \psi(x)=e^{-x}$

$\phi(x)\,\psi'(x)-\psi(x)\,\phi'(x)=e^{x}(-e^{-x})-(e^{-x})\,e^{x}=-2$

$$A=-\int \frac{\psi(x)X}{\phi(x)\psi'(x)-\psi(x)\phi'(x)}\,dx=-\int \frac{e^{-x}\cdot\frac{2}{1+e^{x}}}{-2}=\frac{2}{2}\int \frac{1}{e^{x}\left(1+e^{x}\right)}dx$$

$$=\int\left(\frac{1}{e^{x}}-\frac{1}{1+e^{x}}\right)dx=\int e^{-x}\,dx-\int \frac{e^{-x}}{e^{-x}+1}\,dx=-e^{-x}+\log(e^{-x}$$

$$B=\int \frac{\phi(x)X}{\phi(x)\psi'(x)-\psi(x)\phi'(x)}\,dx=\int \frac{e^{x}\cdot\frac{2}{1+e^{x}}}{2}\,dx=-\int \frac{e^{x}\,dx}{1+e^{x}}=-\log(1$$

P.I. $=-1+e^{x}\log(e^{-x}+1)-e^{-x}(\log e^{x}+1)$

Complete solution :

$$y=c_1e^{x}+c_2e^{-x}-1+e^{x}\log(e^{-x}+1)-e^{-x}(\log e^{x}+1)$$

EXAMPLE - 5

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Roots of A.E. $f(m)=0$	C.F. (complimentary function)	Examples
1) Roots are real and distinct $m_1 \neq m_2 \neq m_3$	$y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 \cdot e^{m_3 x}$	eg: roots are $-1, 2, 3$ $y_c = c_1 e^{-x} + c_2 e^{2x} + c_3 e^{3x}$
2) Roots are real and equal $m_1 = m_2 = m_3$	$y_c = (c_1 + c_2 x + c_3 x^2) e^{mx}$	eg: The roots are $-1, -1, -1$, $y_c = (c_1 + c_2 x + c_3 x^2) e^{-x}$
3) Roots are complex numbers. $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$	$y_c = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$	eg: $1+2i$, $1-2i$, then $y_c = e^{x}(c_1 \cos 2x + c_2 \sin 2x)$
4) complex roots are repeated. $m_1 = m_2 = \alpha + i\beta$ $m_3 = m_4 = \alpha - i\beta$	$y_c = e^{\alpha x}\left[(c_1 + c_2 x)\cos \beta x + (c_3 + c_4 x) \sin \beta x\right]$	eg: $m_1 = m_2 = -2+3i$ $m_3 = m_4 = -2-3i$ Then $y = e^{-2x}\left[(c_1 + c_2 x) \sin 3x + (c_3 + c_4 x) \cos 3x\right]$
5) Roots are real and complex. m_1 real root $m_2 = \alpha + i\beta$, $m_3 = \alpha - i\beta$	$y_c = c_1 \cdot e^{m_1 x} + e^{\alpha x}\left[c_2 \cos \beta x + c_3 \sin \beta x\right]$	
6) Roots are surds. $m_1 = \alpha + \sqrt{\beta}$ $m_2 = \alpha - \sqrt{\beta}$	$y_c = e^{\alpha x}\left[c_1 \cosh \sqrt{\beta}\, x + c_2 \sinh \sqrt{\beta}\, x\right]$	

Solve:

1) $y''' - 9y'' + 23y' - 15y = 0$

Given DE is $y''' - 9y'' + 23y - 15y = 0$ — ①

eq-① can be written as

$(D^3 - 9D^2 + 23D - 15)y = 0$ — ②

Eq② it is in the form of $f(D)y = 0$

where $f(D) = D^3 - 9D^2 + 23D - 15$

The auxillary equation of eq ② is

$f(m) = 0$

$m^3 - 9m^2 + 23 - 15 = 0$

The roots are $1, 3, 5$

Hence the complimentary function is

$y_c = c_1 e^x + c_2 e^{3x} + c_3^{5x}$.

∴ The General solution is

$y = y_c$

$y = c_1 e^x + c_2 e^{3x} + c_3^{5x}$

Solve $\dfrac{d^3x}{dt^3} - 2\dfrac{d^2x}{dt^2} - 3\dfrac{dx}{dt} = 0$

Given DE is $\dfrac{d^3x}{dt^3} - 2\dfrac{d^2x}{dt^2} - 3\dfrac{dx}{dt} = 0$

Eq ① can be written as $(D^3 - 2D^2 - 3D) = 0 = f(D)$

The auxillary equation of eq ② is

$f(m) = 0$

$m^3 - 2m^2 - 3m = 0$

The roots are $-1, 0, 3$

Hence the complimentary function y_c is

$$y_c = c_1 e^{0x} + c_2 e^{-1x} + c_3 e^{3x}$$

The general solution is

$$y = y_c$$

$$y = y_c = c_1 + c_2 e^{-x} + c_3 e^{3x}$$

The given DE is

3) $$\frac{d^3y}{dx^3} - \frac{3dy}{dx} + 2y = 0 \quad - (1)$$

The equation (1) can be written as $y''' - 3y' + 2y = 0$

$$= f(D\, y) \sim (2)$$

The eq(1) c_h ~~auxillary~~ equation is

$$f(y) = f(D)$$

$$D^3 - 3D + 2 = 0$$

Auxillary equation is

$$f(m) = 0 = m^3 - 3m + 2 = 0$$

$$m_1 = 1, \quad m_2 = 1, \quad m_3 = -2$$

The general equation is

$$(c_1 + c_2 x)e^{x} + c_3 e^{-2x}$$

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Type-1

when $Q(x)$ is an exponential function.

$$Q(x) = e^{ax} \text{ then}$$

PI is

$$y_p = \frac{1}{f(D)} Q(x)$$

$$= \frac{1}{f(D)} e^{ax}$$

Replace D by a

$$y_p = \frac{1}{f(a)} \cdot e^{ax} \text{ pro } f(a) \neq 0$$

If $f(a)=0$ then $f(D) = (D-a)^r \phi(D)$ where $\phi(a) \neq 0$

$$y_p = \frac{1}{f(D)} Q(x)$$

$$= \frac{1}{(D-a)\phi(D)} \cdot e^{ax}$$

$$y_p = \frac{x^r}{r!\,\phi(a)} e^{ax}$$

Solve $y'' + 4y' + 4y = e^{-2x}$

$$(D^2+4D+4)y = e^{-2x}$$

~~$f(D)=0$~~ $f(D)y = Q(x)$

where $f(D) = D^2+4D+4$,

$Q(x) = e^{-2x}$

$f(m) = 0$

$m^2+4m+4=0$

$m = -2, -2$

$y_c = (c_1 + c_2 x)e^{-2x}$

$$y_p = \frac{1}{f(D)} \cdot Q(x)$$

$$= \frac{1}{D^2+4D+4} \cdot e^{-2x}$$

Replace D by -2, we get.

$$\frac{x}{2D+4} \cdot e^{-2x}$$

$$y_p = \frac{x^2}{2} \cdot e^{-2x}.$$

Q) solve $(D^3 - 5D^2 + 8D - 4)y = e^{2x} + 2\cosh x + 10$

~~where $f(D)=$~~ Given DE is

$$(D^3 - 5D^2 + 8D - 4)y = e^{2x} + 2\cosh x + 10 \quad \text{--- (1)}$$

Now comparing eq (1) with $f(D)y = Q(x)$

where $f(D) = D^3 - 5D^2 + 8D - 4$,

$Q(x) = e^{2x} + 2\cosh x + 10$

The auxillary equation of eq -(1) is

$f(m) = 0$ i.e. $m^3 - 5m^2 + 8m - 4 = 0$

Roots are $m_1 = 1$,

$m_2 = 2$

$m_3 = 2$

$\therefore$ The complimentary function for eq ① is

$$y_c = c_1 \cdot e^{x} + (c_2 + c_3 x) e^{2x}$$

<u>To find P.I :</u>

$$y_p = \frac{1}{f(D)} \phi(x)$$

$$= \frac{1}{D^3 - 5D^2 + 8D - 4} \left[e^{2x} + 2\cosh x + 10 \right]$$

$$\boxed{\cosh x = \frac{e^{x} + e^{-x}}{2}}$$

$$\Rightarrow \frac{1}{D^3 - 5D^2 + 8D - 4} e^{2x} + \frac{1}{D^3 - 5D^2 + 8D - 4} 2\left[\frac{e^{x} + e^{-x}}{2}\right] + \frac{1}{D^3 - 5D^2 + 8D - 4} \times 10 e^{0x}$$

Replace D with 2 — Replace D by 0

$$= \frac{x}{3D^2 - 10D + 8} e^{2x} + \frac{1}{D^3 - 5D^2 + 8D - 4} \cdot e^{x} + \frac{1}{D^3 - 5D^2 + 8D - 4} e^{-x} + \left(\frac{-1}{4}\right)10$$

Replace D by 1

$$= \frac{x^2}{6D - 10} \cdot e^{2x} + \frac{x}{3D^2 - 10D + 8} \cdot e^{x} + \frac{1}{(-1)^3 - 5(-1)^2 + 8(-1) - 4} e^{-x} - \frac{5}{2}$$

$$y_p = \frac{x^2}{2} e^{2x} + x e^{x} - \frac{1}{18} e^{-x} - \frac{5}{2}$$

$$y_p = \frac{1}{2}\left[x^2 e^{2x} + 2x e^{x} - \frac{1}{9} \cdot e^{-x} - 5\right]$$

$\therefore$ The GS is

$y = CF + PI$

i.e $y = y_c + y_p$

$$y = c_1 e^{x} + (c_2 + c_3 x) e^{2x} + \frac{1}{2}\left(x^2 e^{2x} + 2x e^{x} - \frac{1}{9} e^{-x} - 5\right)$$

---

2) solve $(D^2 + 4D + 13)y = 2e^{-x}$

3) solve $((D+2)(D-1)^2) y = e^{-2x} + 2\sinh x$

4) solve $(D^2 - a^2) y = \sinh ax$

5) solve $y'' - y' - 2y = 3e^{2x} + e^{-x}$

---

2) solve $(D^2 + 4D + 13) y = 2e^{-x}$

Given DE is

$(D^2 + 4D + 13) y = 2e^{-x}$ — ①

Now comparing ① with $f(D) y = Q(x)$

where $f(D) = D^2 + 4D + 13$

$Q(x) = 2 e^{-x}$

The auxillary of eq ① is

$f(m) = m^2 + 4m + 13$

Roots are $m_1 = -2+3i, -2-3i$

The complementary function is

~~$y_c = CF$~~ $y_c = e^{-2x}[c_1 \cos 3x + c_2 \sin 3x]$

To find PI

$$y_p = \frac{1}{f(D)} Q(x)$$

$$= \frac{1}{D^2+4D+13} [2 e^{-x}]$$

$$y_p = \frac{1}{1-4+13} \cdot 2 \cdot e^{-x}$$ [replace D by -1]

$$= \frac{2}{1-4+13} \cdot e^{-x} = \frac{e^{-x}}{5}$$

$\therefore$ The GS is $y = y_c + y_p = e^{-2x}[C_1 \cos 3x + C_2 \sin 3x] + \frac{e^{-x}}{5}$

---

3). $((D+2)(D-1)^2)y = e^{-2x} + 2\sinh x$

The given DE is

$$((D+2)(D-1)^2)y = e^{-2x} + 2\sinh x \quad -(1)$$

Now comparing ~~to eqn~~ (1) with $f(D)y = Q(x)$

Here $f(D) = (D+2)(D-1)^2$

$Q(x) = e^{-2x} + 2\sinh x$

The Auxillary equation of (1) is

$$f(m) = (m+2)(m-1)^2$$

here roots are $m = -2, 1, 1$

The complementary function is

$$y_c = c_1 e^{-2x} + e^{x}(c_2 + c_3 x)$$

~~The~~ To find PI

$$y_p = \frac{1}{f(D)} \cdot Q(x)$$

$$= \frac{1}{(D+2)(D-1)^2} \cdot (e^{-2x} + 2\sinh x)$$

$$= \frac{1}{(D+2)(D-1)^2} \cdot e^{-2x} + \frac{2\sinh x}{(D+2)(D-1)^2}$$

$$\Rightarrow \frac{e^{-2x}}{(D+2)(D-1)^2} + \frac{1}{(D+2)(D-1)^2} \cdot 2\left[\frac{e^{x} - e^{-x}}{2}\right]$$

$$\Rightarrow \frac{e^{-2x}}{(D+2)(D-1)^2} + \frac{e^{x}}{(D+2)(D-1)^2} - \frac{e^{-x}}{(D+2)(D-1)^2}$$

$$\Rightarrow \frac{e^{-2x}}{(D+2)(D^2-2D+1)} + \frac{e^{x}}{(D+2)(D^2-2D+1)} - \frac{e^{-x}}{(D+2)(D^2-2D+1)}$$

$$\Rightarrow \frac{e^{-2x}}{D^3-2D^2+D+2D^2-4D+2} + \frac{e^{x}}{D^3-2D^2+D+2D^2-4D+2} - \frac{e^{-x}}{D^3-2D^2+D+2D^2-4D+2}$$

$$\Rightarrow \frac{e^{-2x}}{D^3-3D+2} + \frac{e^{x}}{D^3-3D+2} - \frac{e^{-x}}{D^3-3D+2}$$ → Replace D with -1.

Replace D with -2 ~~Replace D with -1~~ D with 1

$$\Rightarrow \frac{x\, e^{-2x}}{3D^2-3} + \frac{x \cdot e^{x/2}}{3D^2-3} - \frac{e^{-x/2}}{-1+3+2}$$

Replace d with -2

$$\Rightarrow \frac{x \cdot e^{-2x}}{9} + \frac{x \cdot x \cdot e^{x/2}}{6D} - \frac{e^{-x/2}}{4}$$

Replace D with (1)

$$y_p = \frac{x}{8} \cdot e^{-2x} + \frac{x^2}{6} \cdot e^{x}/2 - \frac{1}{4} \cdot e^{-x/2}$$

The general solution :

$$y = y_c + y_p$$

$$y = c_1 e^{-2x} + e^{x} (c_2 + c_3 x)$$
$$+ \frac{x}{8} \cdot e^{-2x} + \frac{x^6}{6} \cdot e^{x/2} - \frac{1}{4} e^{-x/2}$$

---

4) $(D^2 - a^2)\, y = \sinh ax$

The given DE is

$$(D^2 - a^2)\, y = \sinh ax \quad -(1)$$

Now comparing (1) with $f(D) \cdot y = Q(x)$

here $f(D) = D^2 - a^2$

$Q(x) = \sinh ax$

The auxillary equation of (1) is

$f(m) = m^2 - a^2$

The roots are $m = \pm a$

The complementary function is

$$y_c = c_1 e^{ax} + c_2 e^{-ax}$$

To find P I

$$y_p = \frac{1}{f(D)} \cdot Q(x)$$

$$\frac{1}{D^2 - a^2} \cdot \sinh ax$$

$$\frac{1}{D^2 - a^2} \left[\frac{e^{ax} - e^{-ax}}{2} \right]$$

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$\frac{1}{D^2-a^2} \cdot \frac{e^{ax}}{2} - \frac{e^{-ax}}{2} \times \frac{1}{D^2-a^2}$

Replace D by a.

$\frac{x}{2D} \cdot \frac{e^{ax}}{2} - \frac{e^{-ax}}{2} \cdot \frac{x}{2D}$

$=$ Replace D with 'a'

$\frac{x}{2a} \cdot \frac{e^{ax}}{2} - \frac{e^{-ax}}{2} \cdot \frac{x}{2a}$

$\frac{x}{4a}\left[e^{ax} - e^{-ax}\right]$

The general solution is.

$y_1 = c_1 e^{ax} + c_2 e^{-ax} + \frac{x}{4a}\left[e^{ax} - e^{-ax}\right]$

---

5) Solve $y'' - y' - 2y = 3e^{2x} + e^{-x}$

$(D^2 - D - 2)y = 3e^{2x} + e^{-x}$ — ①

Now comparing ① with $f(D)y = Q(x)$

here $f(D) = D^2 - D - 2$

$Q(x) = 3e^{2x} + e^{-x}$

The auxillary equation of ① is

$f(m) = m^2 - m - 2$

The roots are

$m = 2, -1$

The complementary function is

$y_c = c_1 e^{2x} + c_2 e^{-x}$

To find the P.I

$$y_p = \frac{1}{f(D)} Q(x)$$

$$= \frac{1}{D^2-D-2} \cdot (3 \cdot e^{2x} + e^{-x})$$

$$\frac{1}{D^2-D-2} \cdot 3 \cdot e^{2x} + \frac{1}{D^2-D-2} \cdot e^{-x}$$

put $D=2$ put $D=-1$

$$\frac{x}{2D-1} \cdot 3 \cdot e^{2x} + \frac{x}{2D-1} \cdot e^{-x}$$

put $D=2$ put $D=-1$

$$\frac{x}{3} \cdot 3 \cdot e^{2x} + \frac{x}{-3} \cdot e^{-x}$$

$$x \cdot e^{2x} - \frac{x}{3} \cdot e^{-x}$$

$$y = y_c + y_p = c_1 \cdot e^{2x} + c_2 e^{-x} + x \cdot e^{2x} - \frac{x}{3} e^{-x}$$

Type - II : when $Q(x) = \cos bx / \sin bx$

$$y_p = \frac{1}{f(D)} Q(x)$$

$$= \frac{1}{f(D)} \sin bx / \cos bx$$

Replace D^2 by $-b^2$

$$y_p = \frac{1}{f(-b^2)} \sin bx / \cos bx \quad \text{where } f(-b^2) \neq 0$$

when $f(-b^2) = 0$, Then $-b^2$ is root of $f(D)$

Then $y_p = \frac{x}{f'(D)} \sin bx / \cos bx$ where $f'(-b^2) \neq 0$

Q) Solve $y'' + 2y' + y = \cos 2x$

Sol. Given D E is $y'' + 2y' + y = \cos 2x$ — ①

$(D^2 + 2D + 1)y = \cos 2x$ — ②

comparing eq ② with $f(D) \cdot y = Q(x)$, where

$f(D) = D^2 + 2D + 1$,

$Q(x) = \cos 2x$

The auxillary equation of eq - ② is

$f(m) = 0$

$m^2 + 2m + 1 = 0$

$(m+1)^2 = 0$

∴ The roots are $-1, -1$

The complementary function $y_c = (c_1 + c_2 x) e^{-x}$

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To find particular Integral

$$y_p = \frac{1}{f(D)} \cdot Q(x)$$

$$= \frac{1}{D^2+2D+1} \cos 2x$$

= Replace D^2 by -4

$$= \frac{1}{-4+2D+1} \cos 2x$$

$$\Rightarrow \frac{1}{2D-3} \cos 2x$$

$$= \frac{2D+3}{(2D-3)(2D+3)} \cos 2x$$

$$\Rightarrow \frac{(2D+3)}{4D^2-9} \cdot \cos 2x$$

Replace D^2 by -4.

$$\frac{(2D+3)}{4(-4)-9} \cos 2x \Rightarrow y_p = \frac{-1}{25}(-4\sin 2x + 3\cos 2x)$$

$$\Rightarrow y_p = \frac{1}{25}(4\sin 2x - 3\cos 2x)$$

$$\therefore y = y_c + y_p$$

The general solution is

$$y = y_c + y_p$$

$$y = (c_1 + c_2 x) \cdot e^{-x} + \frac{1}{25}(4\sin 2x - 3\cos 2x)$$

---

2). $(D^2+9)y = \cos 4x + \cos 3x$.

Sol Given DE is $(D^2+9)y = \cos 4x + \cos 3x$ —①

The given equation is in the form.

$$f(D)y = Q(x)$$

Here $f(D)$ is D^2+9

and $Q(x) = \cos 4x + \cos 3x$

The auxillary equation of eq ① is

$$D^2+9=0$$

$$D^2 = -9$$

$$D = 3i, -3i$$

To find particular Integral

$$y_p = \frac{1}{f(D)} \cdot Q(x)$$

$$= \frac{1}{D^2+9} \cdot Q(x)$$

$$\Rightarrow \frac{1}{D^2+9} \cdot \cos 4x + \cos 3x$$

$$\Rightarrow \frac{1}{D^2+9} \cdot \cos 4x + \frac{1}{D^2+9} \cos 3x$$

Replace D^2 by -16 Replace D^2 by -9.

$$\Rightarrow \frac{1}{-16+9} \cdot \cos 4x + \frac{x}{2D} \cdot \cos 3x$$

$$\Rightarrow \frac{-1}{7} \cos 4x + \frac{x}{2} \cdot \frac{\sin 3x}{3}$$

$$y_p \Rightarrow -\frac{1}{7} \cdot \cos 4x + \frac{x}{6} \cdot \sin 3x$$

The general solution is:

$$y = y_c + y_p$$

Type - III.

when $Q(x) =$ polynomial $g(x)$, then.

$$y_p = \frac{1}{f(D)} \cdot Q(x)$$

$$\Rightarrow \frac{1}{f(D)} \cdot g(x)$$

$$= \frac{1}{1 \pm \phi(D)} \cdot g(x)$$

$$y_p = (1 \pm \phi(D))^{-1} \cdot g(x)$$

Note: $(1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + \cdots$

$(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$

$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$

$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3$

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Q $\Rightarrow y'' + y' = x^2 + 2x + 4$

The auxilliary equation $f(m) = m^2 + m$

$f(m) = 0$

$m^2 + m = 0$

$m(m+1) = 0$

$m = 0, -1$

The $y_c = c_1 \cdot e^{0x} + c_2 e^{-x}$.

The PI is $\frac{1}{f(D)} \cdot Q(x)$

$\Rightarrow \frac{1}{D^2 + D}(x^2 + 2x + 4)$

$\frac{1}{D}[D+1]^{-1}(x^2 + 2x + 4)$

$\frac{1}{D}[1 - D + D^2 - D^3 + \cdots](x^2 + 2x + 4]$

$\Rightarrow \frac{1}{D}[x^2 + 2x + 4 - 2x - 2 + 2]$

$\Rightarrow \frac{1}{D}[x^2 + 4] =$

$\Rightarrow \frac{x^3}{3} + 4x$

The general solution is: $y = y_c + y_p$

$\Rightarrow c_1 + c_2 e^{-x} + \frac{x^3}{3} + 4x$

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6) $y''' + 2y'' - y' - 2y = 1 - 4x^3 + e^{2x}$

$f(D)y = Q(x)$

$D \cdot E = (D^3 + 2D^2 - D - 2)\, y = 1 - 4x^3 + e^{2x}$

$AE \quad f(m) = 0$

$m^3 + 2m - m - 2 = 0$

$m(m+2) - 1(m+2) = 0$

$m = 1, -2$

$y_c = c_1 \cdot e^{x} + c_2 \cdot e^{-2x}$

$y_p = \frac{1}{f(D)} \cdot Q(x)$

$= \frac{e^{2x}}{D^3 + 2D^2 - D - 2} + \frac{1 - 4x^3}{D^3 + 2D^2 - D - 2}$

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$$y_p = \frac{e^{2x}}{8+8-4} + \frac{1-4x^3}{D^3+2D^2-\cancel{D}-2}$$

$$y_p = \frac{e^{2x}}{12} + \frac{1-4x^3}{-2\left[\frac{D^3+2D^2-D}{2}+1\right]}$$

$$y_p = -\frac{1}{2}\left(1+\frac{D^3+2D^2-D}{2}\right)^{-1}(1-4x^3)$$

$$y_p = -\frac{1}{2}\left(1-4x^3+\frac{1}{2}(-24)-24x+\frac{1}{2}\cdot 2x^2\right)$$

$$y_p = -\frac{1}{2}\left(1-4x^3-12-24x+6x^2\right)$$

$$= -\frac{1}{2}+2x^3+6+12x-3x^2$$

$$y = y_c + y_p$$

$$= 1+\frac{11}{2}+12x+2x^3-3x^2+\frac{e^{2x}}{12}+c_1e^{x}+c_2e^{-2x}$$

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Type-4 when $Q(x) = e^{ax} V(x)$, then the P.I is

$$y_p = \frac{1}{f(D)} \cdot Q(x)$$

$$= \frac{1}{f(D)} \cdot e^{ax} \cdot V(x)$$

Replace D by $D+a$ in $f(D)$ we get.

$$y_p = e^{ax} \cdot \frac{1}{f(D+a)} \cdot V(x)$$

Solve:

$$(D^2 - 4D + 4) y = e^{-2x} \cdot \cos x.$$

$$y_c = (c_1 + c_2 x) \cdot e^{2x}$$

PI: $$y_p = \frac{1}{f(D)} \cdot Q(x)$$

$$= \frac{1}{D^2 - 4D + 4} e^{-2x} \cos x.$$

Replace D by $(D + (-2)) = D - 2$ in $f(D)$ we get.

$$= e^{-2x} \frac{1}{(D-2)^2 - 4(D-2) + 4} \cos x$$

$$= e^{-2x} \cdot \frac{1}{D^2 - 4D + 4 - 4D + 8 + 4} \cdot \cos x$$

$$= e^{-2x} \cdot \frac{1}{D^2 - 8D + 16} \cdot \cos x$$

Replace D^2 by -1

$$= e^{-2x} \cdot \frac{1}{-1 - 8D + 16} \cdot \cos x$$

$$= e^{-2x} \cdot \frac{1}{15-8D} \cdot \cos x$$

$$= e^{-2x} \cdot \frac{(15+8D)}{225-64D^2} \cdot \cos x$$

$$= \frac{e^{-2x}}{225-64D^2} [15+8D] \cdot \cos x$$

$$y_p = \frac{e^{-2x}}{289} [15\cos x - 8\sin x]$$

$\therefore$ The G.S is.

$$y = y_c + y_p$$

$$y = (c_1 + c_2 x) \cdot \frac{e^{-2x}}{289} [15\cos x - 8\sin x]$$

Type - 5

when $Q(x) = x \cdot V(x)$

Then PI is $y_p = \frac{1}{f(D)} \cdot x \cdot V(x)$

$$= \left[x - \frac{f'(D)}{f(D)}\right] \frac{1}{f(D)} \cdot V(x)$$

when $Q(x) = x^m \cdot V(x)$, m is a positive integer and V is any function of 'x'

working rule for finding particular integral of $f(D) \cdot y = Q(x)$

where $Q(x) = x^m \cdot \sin ax$

(or)

$Q(x) = x^m \cdot \cos ax$.

we know that $e^{i\theta} = \cos\theta + i\sin\theta$.

$\therefore \sin\theta = IP(e^{i\theta})$

$\cos\theta = RP(e^{i\theta})$

$\therefore I_p = \frac{1}{f(D)} \cdot x^m \cdot \sin ax$

$= \frac{1}{f(D)} \cdot x^m \cdot (IP \cdot e^{iax})$

$IP \cdot e^{iax} \cdot \frac{1}{f(D+ia)} \cdot x^m$.

Solve $(D^2+3D+2)y = x \cdot e^x \sin x$.

Sol) The given DE is

$$(D^2+3D+2)y = x \cdot e^x \sin x \quad -① $$

① is in the form $f(D) \cdot y = Q(x)$

$$f(D) = D^2+3D+2$$
$$Q(x) = x \cdot e^x \sin x$$

The auxillary equation is

$$f(m) = 0$$
$$m^2+3m+2=0$$
$$m=-1, \ m=-2.$$

$$y_c = c_1 \cdot e^{-x} + c_2 \cdot e^{-2x}.$$

To find PI:

$$y_p = \frac{1}{f(D)} \cdot Q(x)$$

$$y_p = \frac{1}{D^2+3D+2} \cdot e^x \cdot x \cdot \sin x.$$

Replace D by D+1.

$$= e^x \cdot \frac{1}{(D+1)^2+3(D+1)+2} \cdot x \cdot \sin x.$$

$$= e^x \cdot \frac{1}{D^2+2D+1+3D+3+2} \cdot x \cdot \sin x.$$

$$= e^x \cdot \frac{1}{D^2+5D+6} \cdot x \cdot \sin x.$$

we know that $\sin x = I.P. \, e^{ix}$.

$$y_p = e^x \cdot \frac{1}{D^2+5D+6} \, x \cdot (I.P. \, e^{ix})$$

$$= e^{x} \cdot IP \cdot \frac{1}{D^2+5D+6} \cdot x \cdot e^{ix}$$

Replace D by $D+i$

$$= e^{x} \cdot IP \cdot e^{ix} \cdot \frac{1}{(D+i)^2+5(D+i)+6} \cdot x$$

$$= e^{x} \cdot IP \cdot e^{ix} \cdot \frac{1}{D^2-1+2iD+5D+5i+6} \cdot x$$

$$\Rightarrow e^{x} IP \cdot e^{ix} \cdot \frac{1}{D^2+(2i+5)D \cdot 5i+5} \cdot x$$

$$\Rightarrow e^{x} \cdot IP \cdot e^{ix} \cdot \frac{1}{(5i+5)+(2i+5)D+D^2} \cdot x$$

$$= e^{x} \cdot IP \cdot \frac{e^{ix}}{5i+5}\left[\frac{1}{1+\frac{D^2+(2i+5)D}{5i+5}}\right] \cdot x$$

$$= \frac{e^{x}}{5} \cdot IP \cdot \frac{e^{ix}}{(i+1)} \cdot \frac{(i-1)}{(i-1)}\left[1-\frac{D^2+(2i+5)D}{5(i+1)}+\cdots\right] x$$

$$= \frac{e^{x}}{5} \cdot IP \cdot \frac{(i-1) \cdot e^{ix}}{-2}\left[1-\frac{(2i+5) \cdot D}{5(i+1)}\right] x$$

$$= \frac{e^{x}}{10} \cdot IP \cdot (1-i) e^{ix}\left[x-\frac{(2i+5)(i-1)}{5(i+1)(i-1)}\right]$$

$$\Rightarrow \frac{e^{x}}{10} \cdot IP(1-i) \cdot e^{ix}\left[x+\frac{-2-2i+5i-5}{10}\right]$$

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$$\frac{e^x}{10} IP(1-i)\, e^{ix}\left[\frac{10x+3i-7}{10}\right]$$

$$= \frac{e^x}{10} \cdot IP\,[1-i]\,[\cos x + i\sin x)\,(10x-7+i3)$$

$$\frac{e^x}{100} \cdot IP\,(1-i)\,(\cos x + i\sin x)\,(10x-7+i3)$$

$$= \frac{e^x}{100} \cdot IP\,[\cos x + \sin x] + i\,[\sin x - \cos x]\,(10x-7+i3)$$

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20/02/2020

given DE is.

$(D^2+3D+2)y = x \cdot e^x \cdot \sin x \quad -①$

The ① is in the form $f(D)y = Q(x)$

here $f(D) = D^2+3D+2$

$Q(x) = x \cdot e^x \cdot \sin x$.

The auxillary equation is

$f(m)=0$

$m^2+3m+2=0$

$m^2+2m+m+2=0$

$m(m+2)+1(m+2)=0$

$(m+1)(m+2)=0$

$m=-1,-2$

$y_c = c_1 \cdot e^{-x} + c_2 \cdot e^{-2x}$.

The PI is.

Let $y_p = \frac{1}{f(D)} \cdot Q(x)$

$= \frac{1}{D^2+3D+2} \cdot x \cdot e^x \cdot \sin x$.

$\frac{1}{f(D)} \cdot V(x) = \left[x - \frac{1}{f(D)} \cdot f'(D)\right] \cdot \frac{1}{f(D)} \cdot V(x)$

$= \left[x - \frac{2D+3}{D^2+3D+2}\right] \cdot \frac{1}{D^2+3D+2} \cdot e^x \cdot \sin x$.

Replace D by D+1.

$= e^x \left[x - \frac{2(D+1)+3}{(D+1)^2+3(D+1)+2}\right] \cdot \frac{1}{(D+1)^2+3(D+1)+2} \cdot \sin x$.

$= e^x \cdot \left[x - \frac{2D+5}{D^2+5D+6}\right] \cdot \frac{1}{D^2+5D+6} \cdot \sin x$.

Replace D^2 by -1

$$= e^{x} \cdot \left[x - \frac{2D+5}{D^2+5D+6}\right] \cdot \frac{1}{5D+5} \cdot \sin x$$

$$= e^{x}\left[x - \frac{2D+5}{D^2+5D+6}\right] \frac{(D-1)}{5(D+1)(D-1)} \cdot \sin x$$

$$= \frac{e^{x}}{5} \cdot \left[x - \frac{2D+5}{D^2+5D+6}\right] \cdot \left(\frac{D-1}{D^2-1} \cdot \sin x\right)$$

$$= \frac{e^{x}}{5}\left[x - \frac{2D+5}{D^2+5D+6}\right]\left(\frac{\cos x - \sin x}{-2}\right)$$

$$= \frac{e^{x}}{10}\left[x - \frac{2D+5}{D^2+5D+6}\right](\sin x - \cos x)$$

$$= \frac{e^{x}}{10}\left[x(\sin x - \cos x) - \frac{2D+5}{D^2+5D+6}(\sin x - \cos x)\right]$$

$$= \frac{x \cdot e^{x}}{10}(\sin x - \cos x) - \frac{e^{x}}{10}\left[\frac{1}{D^2+5D+6} \cdot (2\cos x + 2\sin x + 5\sin x - 5\cos x\right]$$

$$= \frac{x \cdot e^{x}}{10}(\sin x - \cos x) - \frac{e^{x}}{10}\left[\frac{1}{D^2+5D+6} \cdot (7\sin x - 3\cos x)\right]$$

$$= \frac{x \cdot e^{x}}{10}[\sin x - \cos x] - \frac{e^{x}}{10}\left[\frac{7}{D^2+5D+6}\sin x - \frac{3}{D^2+5D+6} \cdot \cos x\right]$$

Replace D^2 by -1 Replace D^2 by -1

$$= \frac{x \cdot e^{x}}{10}[\sin x - \cos x] - \frac{e^{x}}{10}\left[\frac{7}{5(D+1)}\sin x - \frac{3}{5(D+1)}\cos x\right]$$

$$= \frac{x \cdot e^x}{10}\left[\sin x - \cos x\right] - \frac{e^x}{10}\left(\frac{7(D-1)}{5(D^2-1)}\sin x - \frac{3(D-1)}{5(D^2-1)}\cos x\right)$$

$$= \frac{x \cdot e^x}{10}(\sin x - \cos x) - \frac{e^x}{50}\left(\frac{7(D-1)}{D^2-1}\sin x - \frac{3(D-1)\cos x}{D^2-1}\right)$$

$$= \frac{x \cdot e^x}{10}(\sin x - \cos x) + \frac{e^x}{50 \times 2}\left(7\cos x - 7\sin x + 3\sin x + 3\cos x\right)$$

$$= \frac{x \cdot e^x}{10}(\sin x - \cos x) + \frac{e^x}{100}\left(10\cos x - 4\sin x\right)$$

$$= \frac{x \cdot e^x}{10}(\sin x - \cos x) + \frac{e^x}{50}\left(5\cos x - 2\sin x\right)$$