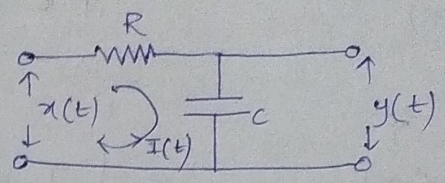
S AND S

ASSIGNMENT-04

1 (a) Find out the output voltage of the RC LPF shown in fig for an input voltage of $x(t) = te^{-at}u(t)$

From fig,



$$x(t) - i(t)R - \frac{1}{C}\int i(t)\,dt = 0$$

Applying F.T on b.s

$$X(\omega) = I(\omega)R + \frac{1}{C}\cdot\frac{1}{j\omega} I(\omega) \rightarrow (1)$$

$$X(\omega) = I(\omega)\left[R + \frac{1}{j\omega C}\right]$$

Also ~~given~~ from fig, $y(t) - \frac{1}{C}\int i(t)\,dt = 0$

Applying F.T on b.s

$$Y(\omega) = \frac{1}{C}\cdot\frac{1}{j\omega} I(\omega) \rightarrow (2)$$

$$\frac{(2)}{(1)}, \quad \frac{Y(\omega)}{X(\omega)} = \frac{\cancel{I(\omega)}\left[\frac{1}{\cancel{j\omega C}}\right]}{\cancel{I(\omega)}\left[\frac{j\omega RC + 1}{\cancel{j\omega C}}\right]} = \frac{1}{1 + j\omega RC}$$

We know that, $\frac{Y(\omega)}{X(\omega)} = H(\omega)$

$$\therefore H(\omega) = \frac{1}{1 + j\omega RC}$$

$$h(t) = F^{-1}\left[H(\omega)\right] = F^{-1}\left[\frac{1}{1 + j\omega RC}\right]$$

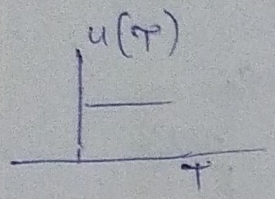
$$= F^{-1}\left[\frac{1}{RC\left(\frac{1}{RC} + j\omega\right)}\right]$$

$$h(t) = \frac{1}{RC} e^{-1/RC\, t} u(t)$$

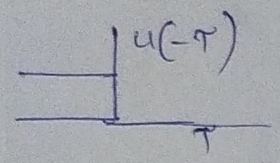
$$\therefore\ y(t) = x(t) * h(t)$$

$$= \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

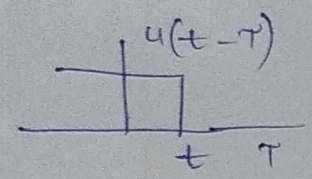
$$= \int_{-\infty}^{\infty} \tau e^{-a\tau} u(\tau) \frac{1}{RC} e^{-1/RC(t-\tau)} u(t-\tau)\, d\tau$$

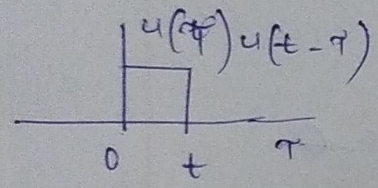


$$= \frac{1}{RC} \int_{0}^{t} \tau e^{-a\tau} e^{-t/RC} e^{\tau/RC}\, d\tau$$



$$= \frac{1}{RC} e^{-t/RC} \int_{0}^{t} \tau e^{\tau\left(\frac{1}{RC}-a\right)}\, d\tau$$





$$= \frac{1}{RC} e^{-t/RC} \left[\left[\tau \frac{e^{\tau\left(\frac{1}{RC}-a\right)}}{\frac{1}{RC}-a} \right]_0^t - \int_0^t \frac{e^{\tau\left(\frac{1}{RC}-a\right)}}{\left(\frac{1}{RC}-a\right)}\, d\tau \right]$$

$$= \frac{1}{RC} e^{-t/RC} \left[t \frac{e^{t\left(\frac{1}{RC}-a\right)}}{\left(\frac{1}{RC}-a\right)} - \frac{1}{\left(\frac{1}{RC}-a\right)} \left[\frac{e^{\tau\left(\frac{1}{RC}-a\right)}}{\left(\frac{1}{RC}-a\right)} \right]_0^t \right]$$

$$= \frac{1}{RC} e^{-t/RC} \left[tRC \frac{e^{t\left(\frac{1}{RC}-a\right)}}{1-aRC} - \frac{1}{\left(\frac{1}{R^2C^2}+a^2-\frac{2a}{RC}\right)} \left[e^{t\left(\frac{1}{RC}-a\right)} - 1 \right] \right]$$

$$= \frac{t}{1-aRC} e^{-t/RC} e^{t/RC} e^{-at} - \frac{1}{RC} e^{-t/RC} \left(\frac{R^2C^2}{1+a^2R^2C^2-2aRC} \right) \left[e^{\frac{t}{RC}-at} - 1 \right]$$

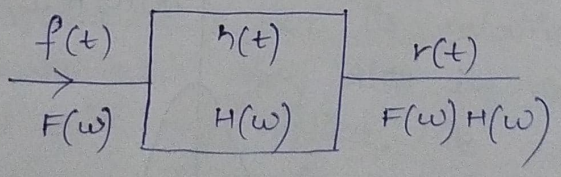
$$y(t) = \frac{t e^{-at}}{1-aRC} - \frac{RC}{1+a^2R^2C^2-2aRC} \left[e^{-at} - e^{-t/RC} \right]$$

(b) Explain the filter characteristics of linear systems.

For a given system, an input signal $f(t)$ gives rise to a response signal $r(t)$, thus processing the signal $f(t)$ in such a way that is characteristic of the system. The spectral density function of the input signal is given by $F(\omega)$ whereas the spectral density function of the response is given by $F(\omega)H(\omega)$. The system therefore modifies the spectral density function of the I/p signal. It is evident that the system acts as a kind of filter to various frequency components. Some frequency components are boosted in strength, some are attenuated, and some may remain unaffected.

Similarly, each frequency component undergoes a different amount of phase shift in the process of transmission. Thus the system modifies the spectral density function according to its filter characteristics. The modification is carried out according to the T/F $H(\omega)$, which represents the response of the system to various frequency components.

$\therefore$ $H(\omega)$ acts as a weighting function to different frequencies. The resultant response has the spectral density $F(\omega)H(\omega)$.



i.e., W.K.t, $y(t) = r(t) = f(t) * h(t)$

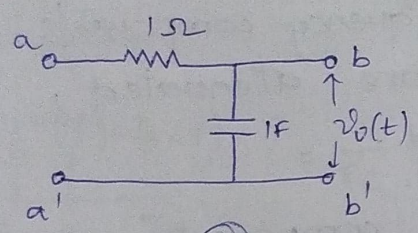
Applying F.T on b.s.

F.T [convolution in T.D] = multiplication in F.D

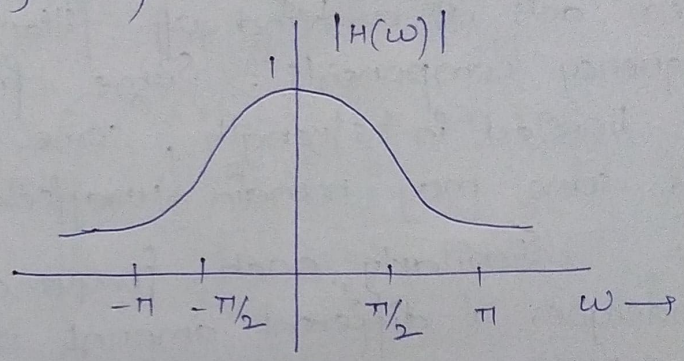
$$\Rightarrow \boxed{Y(\omega) = R(\omega) = F(\omega).H(\omega)}$$

Let us consider a simple R-C N/w shown in fig (a). A square pulse shown in fig (c) is applied at the I/p terminals aa' of this N/w. The response is the O/p voltage $v_o(t)$ observed across the O/p terminals of this N/w. The spectral density function of the I/p signal is shown in fig (d). The T/F $H(\omega)$ of the N/w, relating the O/p voltage to the I/p voltage, is obviously $1/(j\omega+1)$.

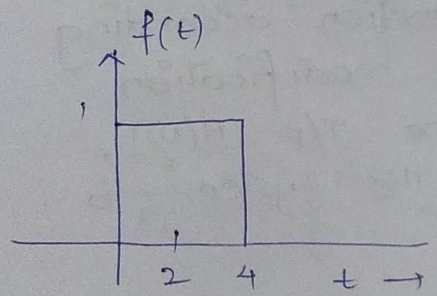
$$\therefore H(\omega) = 1/(j\omega+1)$$



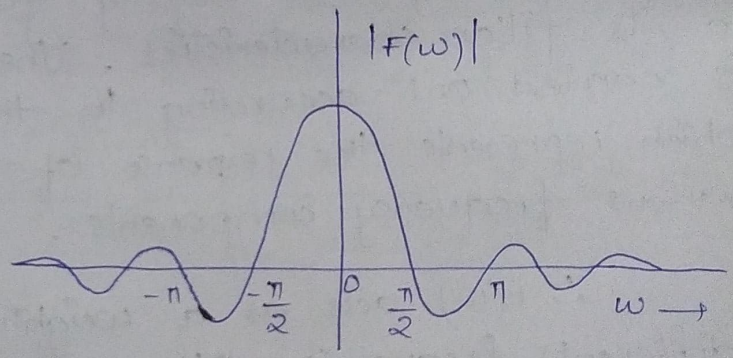
(a)



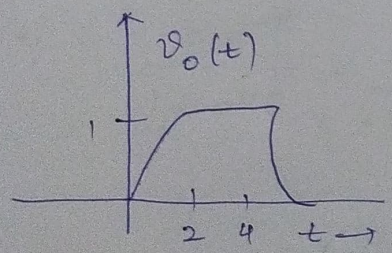
(b)



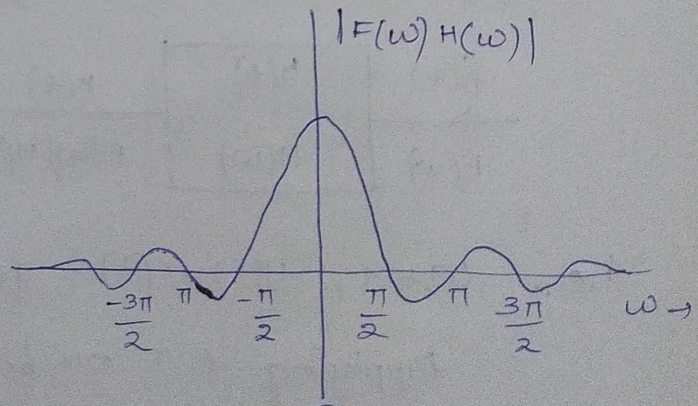
(c)



(d)



(e)



(f)

We can observe that this N/w attenuates high frequencies and allows lower frequencies to pass with a relatively small attenuation. Thus the N/w is the simplest form of a LPF. The high frequency components in the I/p spectrum experience severe attenuation compared to the low frequency components. Also comparing (d) and (f), it clearly shows that the attenuation of high frequency components is caused by the N/w. The response function $v_o(t)$ is obviously a distorted replica of the I/p signal, the distortion being caused by the fact that the N/w does not allow equal access to the transmission of all frequency components of the I/p signal. Since the N/w does not allow the high frequency components, the o/p voltage cannot change at a rapid rate, and hence it rises and falls sluggishly compared to the I/p signal.

(2) (a) Derive the condition of distortionless transmission through LTI system.

For a distortionless transmission we require that the response be an exact replica of the I/P signal. This replica may, of course, have a different magnitude. The important thing is the waveform and not its relative magnitude. In general, there may also be some time delay associated with the replica. We may therefore say that a signal $f(t)$ is transmitted without distortion if the response is $Kf(t-t_0)$. It is evident that the response is the exact replica of the input with a magnitude K times the original signal and delayed by t_0 seconds.

Thus if $f(t)$ is the I/p signal, for distortionless transmission we need the response $r(t)$ to be,

$$r(t) = k f(t - t_0)$$

From time shifting property we have,

$$R(\omega) = k F(\omega) e^{-j\omega t_0} \rightarrow (1)$$

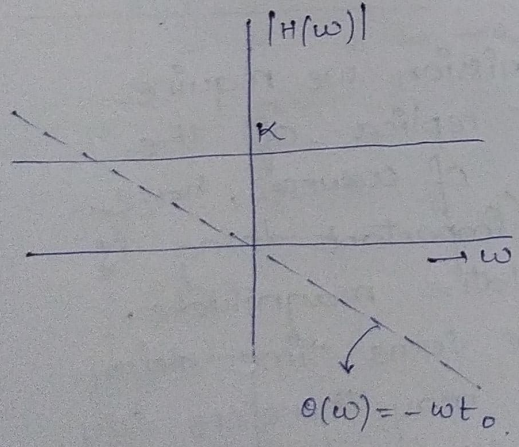
Also we have, $R(\omega) = F(\omega) H(\omega) \rightarrow (2)$

$\therefore$ For a distortionless transmission,

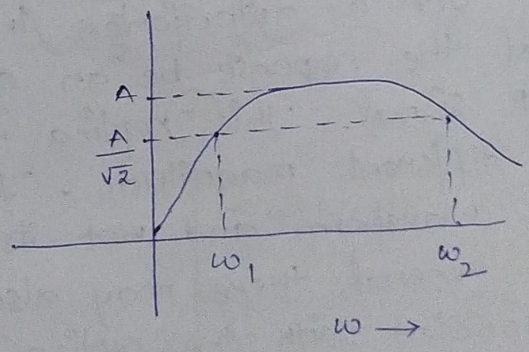
$$H(\omega) = k e^{-j\omega t_0} \rightarrow (3) \quad [\text{from } (1) \text{ \& } (2)]$$

$\therefore$ To achieve distortionless transmission through a system, the T/F must be of the form shown in eqn (3). It is evident that $|H(\omega)|$, the magnitude of the T/F is k, and is constant for all frequencies. The phase shift, on the other hand, is proportional to frequency.

i.e., $\theta(\omega) = -\omega t_0$



(a)



(b)

The magnitude and phase characteristics of a distortionless transmission.

i.e., System must have constant magnitude characteristic and phase shifting must be proportional to frequency to have a distortionless transmission.

(b) Define linear, causal and time invariant systems with expressions.

(i) Linear system :- A system is said to be linear system, if it satisfies superposition theorem.

where superposition theorem involves two properties

→ Additive property

If $x(t) \rightarrow y(t)$

$x_1(t) \rightarrow y_1(t)$ and $x_2(t) \rightarrow y_2(t)$

Then, $\underbrace{x_1(t) + x_2(t)}_{x(t)} \longrightarrow \underbrace{y_1(t) + y_2(t)}_{y(t)}$

→ Scaling (or) homogenity property

If $x(t) \rightarrow y(t)$

then, $a x(t) \rightarrow a y(t)$.

∴ The condition for linearity is,

For CTS, $a x_1(t) + b x_2(t) \longrightarrow a y_1(t) + b y_2(t)$

For DTS, $a x_1(n) + b x_2(n) \longrightarrow a y_1(n) + b y_2(n)$

(or) $\sum_{i=1}^{n} a_i x_i(n) \longrightarrow \sum_{i=1}^{n} a_i y_i(n)$

(ii) Time invariant system :- Time invariant systems are the systems where, the behaviour/characteristic of the system does not depend on time. i.e., I/p - o/p characteristics of the system does not vary w.r.t time.

They are also known as shift variant system.

(iii) Causal systems :- If the response of the system is depending on present and past values of the input, but not on future values of the input, then the system is known as causal system.

i.e., If $y(t) = f$(present and past I/P), then the system is causal system.

→ Causal systems are non-anticipative.

Condition for causality is,

I.R. $h(n) = 0$ for $n < 0$.

(3) (a) Derive the relation b/w risetime and bandwidth?

A sharp change in signal amplitude implies rapid variations of the signal with time. This gives rise to high frequency components in the signal. A signal which is relatively smooth contains predominantly low frequencies. We know that if a signal with a jump discontinuity is transmitted through a LPF, the discontinuity is rounded off in the output. The change occurs gradually. If a unit step function $u(t)$ is applied to an ideal LPF, the output will show a gradual rise. The rise time of the output will depend upon the cutoff frequency of the filter. We shall now show that the rise time is inversely proportional to the cutoff frequency of the filter. The smaller the cutoff frequency, the more gradually will the signal rise at the o/p

This can be done easily by considering the response of the LPF to a unit step function $u(t)$. If $H(\omega)$ is the T/F of the ideal LPF, then,

$$H(\omega) = G_{2W}(\omega)\, e^{-j\omega t_0}$$

Also,

$$u(t) \longleftrightarrow \pi\delta(\omega) + \frac{1}{j\omega}$$

If $r(t)$ is the response of the LPF to $u(t)$, then

$$R(\omega) = \left[\pi\delta(\omega) + \frac{1}{j\omega}\right] \mathcal{C}_{2W}(\omega)\, e^{-j\omega t_0}$$

Note that, $H(0) = 1$,

Hence $\delta(\omega) H(\omega) = H(0)\delta(\omega)$ and

$$R(\omega) = \pi\delta(\omega) + \frac{1}{j\omega} \mathcal{C}_{2W}(\omega)\, e^{-j\omega t_0}$$

$$r(t) = F^{-1}\left[\pi\delta(\omega) + \frac{1}{j\omega} \mathcal{C}_{2W}(\omega)\, e^{-j\omega t_0}\right]$$

$$= \frac{1}{2} + F^{-1}\left(\frac{1}{j\omega} \mathcal{C}_{2W}(\omega)\, e^{-j\omega t_0}\right)$$

$$= \frac{1}{2} + \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\mathcal{C}_{2W}(\omega)}{j\omega}\, e^{j\omega(t-t_0)}\, d\omega$$

$$= \frac{1}{2} + \frac{1}{2\pi}\int_{-W}^{W} \frac{e^{j\omega(t-t_0)}}{j\omega}\, d\omega$$

$$= \frac{1}{2} + \frac{1}{2\pi}\int_{-W}^{W} \frac{\cos[\omega(t-t_0)]}{j\omega}\, d\omega + \frac{1}{2\pi}\int_{-W}^{W} \frac{\sin[\omega(t-t_0)]}{\omega}\, d\omega$$

Note that the first integral vanishes because the integrand $\left\{\frac{\cos(\omega(t-t_0))}{\omega}\right\}$ is an odd function of ω. The integrand in the second integral is an even function of ω.

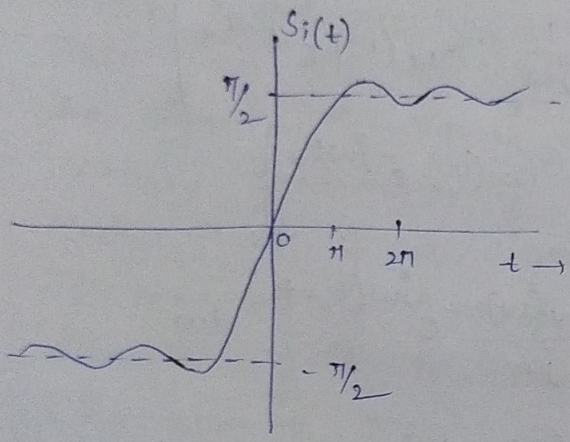
$$\therefore\ r(t) = \frac{1}{2} + \frac{1}{\pi}\int_{0}^{W} \frac{\sin[\omega(t-t_0)]}{\omega}\, d\omega$$

$$= \frac{1}{2} + \frac{1}{\pi}\int_{0}^{W(t-t_0)} \frac{\sin x}{x}\, dx$$

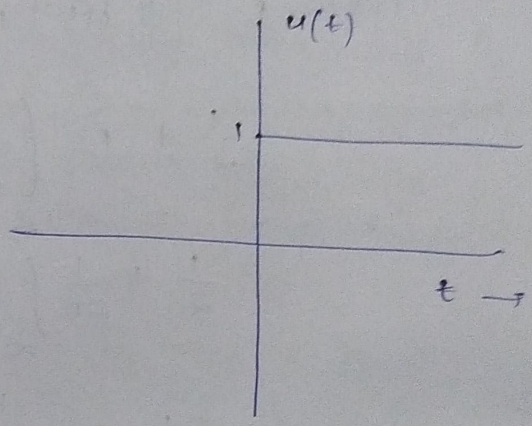
$$= \frac{1}{2} + \frac{1}{\pi} \int_0^{W(t-t_0)} Sa(x)\, dx$$

The integral that appears on the RHS cannot be evaluated in a closed form but must be obtained by expanding the sampling function in a power series. This integral is tabulated in standard tables and is given a special name sine integral denoted by $Si(x)$

$$Si(x) = \int_0^{a} Sa(\tau)\, d\tau$$



(a)



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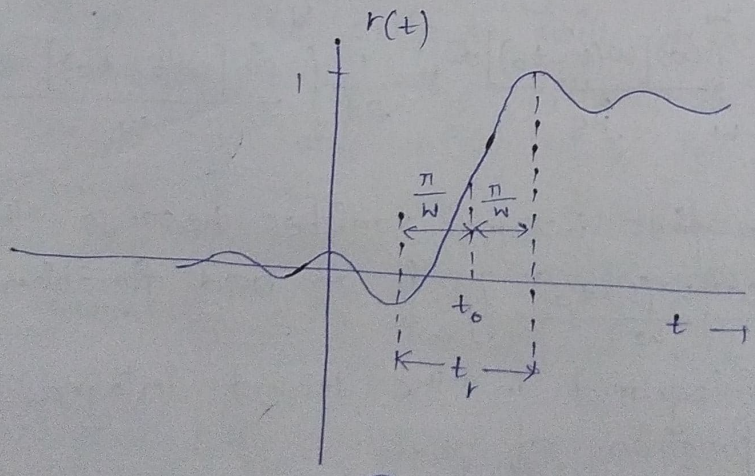


(c)

The function $Si(x)$ is shown in fig (a).
The response $r(t)$ can now be expressed as,

$$r(t) = \frac{1}{2} + \frac{1}{\pi} Si\left[W(t-t_0)\right]$$

The cutoff frequency of the LPF is W.

From fig, we can observe that as the cutoff frequency W is lowered, the o/p r(t) rises more slowly. If we define the rise time t_r as the time required for the o/p to reach ~~the~~ from the minimum to the maximum, then it is evident from fig (c) that,

$$\boxed{t_r = \frac{2\pi}{W} = \frac{1}{B}}$$

where B is the filter bandwidth in Hz.

Thus the rise time is inversely proportional to the cutoff frequency (in Hz) of the filter.

(b) Explain causality and paley-wiener criterion for physical realization.

A Physically realizable system cannot have a response before the driving function is applied. This is known as the causality condition. This condition may be expressed alternatively. A unit I.R h(t) of a physically realizable system must be causal.

A signal is said to be causal if $h(t)=0, t<0$. Thus, the I.R h(t) of a physically realizable system must be zero for $t<0$.

This is the time domain criterion of physical realizability. In the frequency domain this criterion implies that a necessary and sufficient condition for a magnitude function $|H(\omega)|$ to be physically realizable is that,

$$\int_{-\infty}^{\infty} \frac{|\ln|H(\omega)||}{1+\omega^2} d\omega < \infty \longrightarrow (1)$$

The magnitude function $|H(\omega)|$ must, however, be square-integrable before the Paley-wiener criterion is valid; that is,

$$\int_{-\infty}^{\infty} |H(\omega)|^2 d\omega < \infty$$

A system whose magnitude function violates the paley-weiner criterion has a non causal I.R, that is, the response exists forever in the past prior to the application of the driving function.

We can draw some significant conclusions from the paley-wiener criterion. It is evident that the magnitude function $|H(\omega)|$ may be zero at some discrete frequencies, but it cannot be zero over a finite band of frequencies since this will cause the integral in eqn (1) to become infinite.

∴ We can conclude that ideal filters are not physically realizable.

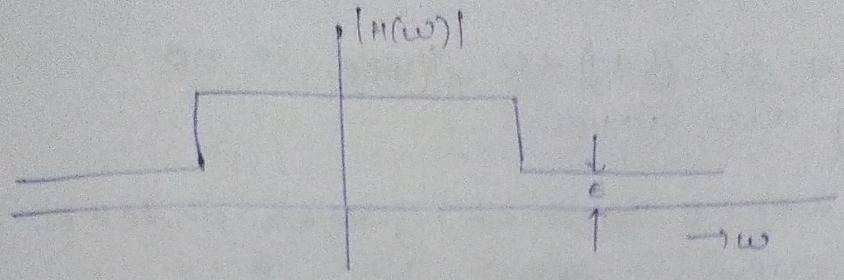
Also we can conclude from (1) that the amplitude function cannot fall off to zero faster than a function of exponential order. Thus,

$$|H(\omega)| = Ke^{-\alpha|\omega|} \text{ is permissible.}$$

But the gaussian function,

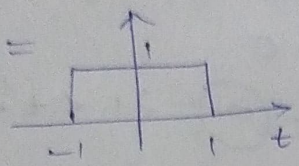
$$|H(\omega)| = Ke^{-\alpha\omega^2}$$

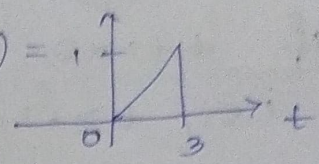
is not realizable. In short, a realizable magnitude characteristic cannot have too great a total attenuation. Thus, the LPF characteristic shown in fig is physically realizable for arbitrarily small values of 'ε'.



→ physically realizable filter characteristic.

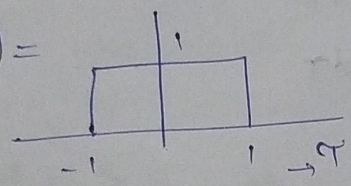
(4) (a) Find the graphical convolution of

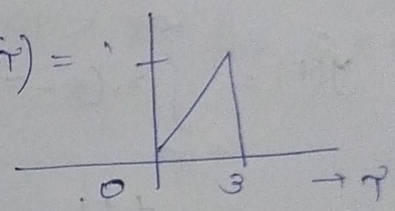
$x_1(t) =$ 

& $x_2(t) =$ 

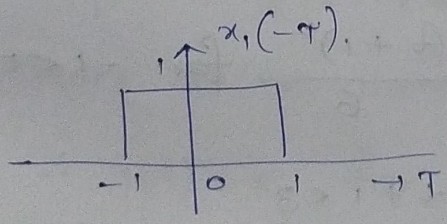
i.e., $x_1(t) = 1$ for $-1 \le t \le 1$ and $x_2(t) = \frac{t}{3}$ for $0 \le t \le 3$.

Step (1) Replace 't' by τ.

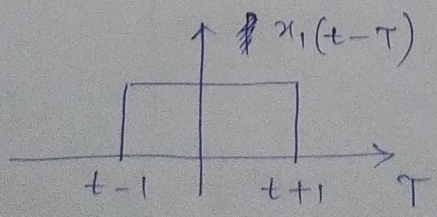
$x_1(\tau) =$ 

$x_2(\tau) =$ 

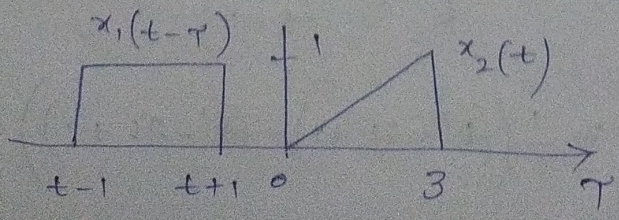
Step (2) Get $x_1(-\tau)$ by folding $x_1(\tau)$ about vertical axis.



Step (3) Shift $x_1(-\tau)$ by 't' (i.e., adding t)



Step (4) $x_1(t-\tau) \cdot x_2(t)$ [product]



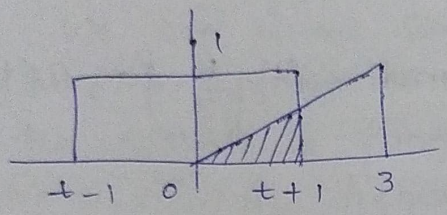
→ Now, as long as $(t+1) < 0$, there is no intersection of two curves.

$\therefore$ For $t+1 < 0 \Rightarrow t < -1$, area under the product curve is zero.

i.e., $y(t) = \int_{-\infty}^{\infty} x_1(t-\tau)\, x_2(\tau)\, d\tau = 0$, for $t < -1$

→ Now, if $t+1 > 0$ and $t-1 < 0$

i.e., $t > -1$ and $t > 1$ or $-1 < t < 1$



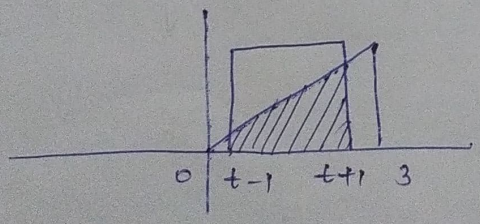
$$y(t) = \int_{-\infty}^{\infty} x_1(t-\tau)\, x_2(\tau)\, d\tau$$

$$= \int_{0}^{t+1} 1 \cdot \frac{\tau}{3}\, d\tau = \frac{1}{3}\left[\frac{\tau^2}{2}\right]_0^{t+1}$$

$$y(t) = \frac{(t+1)^2}{6} \quad \text{for } -1 < t < 1$$

→ Now if $t-1 > 0$ and $t+1 < 3$

i.e., $t > 1$ and $t < 2$ or $1 < t < 2$



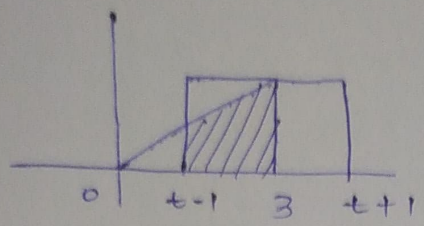
$$y(t) = \int_{t-1}^{t+1} \frac{\tau}{3}\, d\tau = \frac{1}{6}\left(\tau^2\right)_{t-1}^{t+1}$$

$$= \frac{1}{6}\left((t^2+2t+1) - (t^2-2t+1)\right)$$

$$y(t) = \frac{4t}{6} = \frac{2t}{3} \text{ for } 1 \le t < 2$$

→ If $t-1 < 3$ and $t+1 > 3$

$t < 4$ and $t > 2$ or $2 \le t < 4$

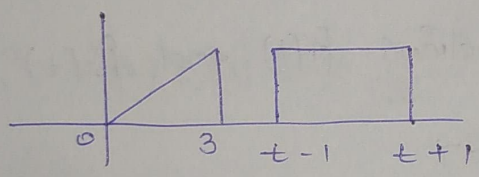


$$y(t) = \int_{t-1}^{3} \frac{\tau}{3}\, d\tau = \frac{1}{6}\left(\tau^2\right)_{t-1}^{3}$$

$$= \frac{1}{6}\left(9 - (t^2 - 2t + 1)\right)$$

$$y(t) = \frac{-t^2 + 2t + 8}{6} \text{ for } 2 \le t < 4$$

→ If $t-1 > 3 \Rightarrow t > 4$

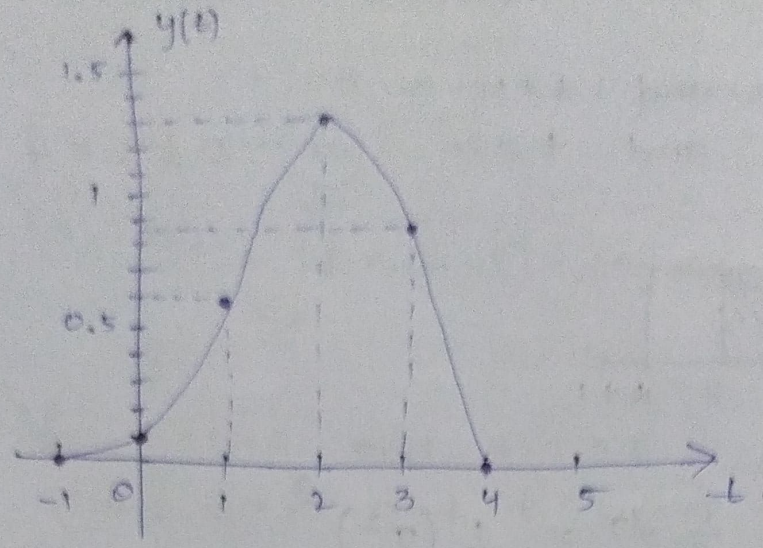


There is no common point

$\therefore y(t) = 0$ for $t > 4$

$$y(t) = \begin{cases} 0, & t < -1 \\ \frac{(t+1)^2}{6}, & -1 \le t < 1 \\ \frac{2}{3}t, & 1 \le t < 2 \\ \frac{-t^2+2t+8}{6}, & 2 \le t < 4 \\ 0, & t > 4 \end{cases}$$

Graphical representation is given by,



(b) Define the convolution theorem in both time domain and frequency domain and state the properties of convolution.

The convolution theorem is perhaps one of the most powerful tools in frequency analysis. It permits the easy derivation of many important results.

Let us consider two functions $f_1(t)$ and $f_2(t)$,

then $f(t) = \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau) d\tau$

The convolution integral is also expressed as,

$$f(t) = f_1(t) * f_2(t)$$

We have two theorems –

(i) <u>Time - Convolution theorem :-</u>

If $f_1(t) \longleftrightarrow F_1(\omega)$

and $f_2(t) \longleftrightarrow F_2(\omega)$

then $\int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau) d\tau \longleftrightarrow F_1(\omega) F_2(\omega)$

i.e., $f_1(t) * f_2(t) \longleftrightarrow F_1(\omega)\, F_2(\omega)$

PROOF :- $F[f_1(t) * f_2(t)] = \int_{-\infty}^{\infty} e^{-j\omega t} \left[\int_{-\infty}^{\infty} f_1(\tau)\, f_2(t-\tau)\, d\tau \right] dt$

$$= \int_{-\infty}^{\infty} f_1(\tau) \left[\int_{-\infty}^{\infty} e^{-j\omega t} f_2(t-\tau)\, dt \right] d\tau$$

$$= \int_{-\infty}^{\infty} f_1(\tau)\, e^{-j\omega\tau}\, F_2(\omega)\, d\tau$$

[∵ from time shifting]

$\therefore F[f_1(t) * f_2(t)] = F_1(\omega)\, F_2(\omega)$

(ii) Frequency convolution theorem :-

If $f_1(t) \longleftrightarrow F_1(\omega)$

and $f_2(t) \longleftrightarrow F_2(\omega)$

then $f_1(t)\, f_2(t) \longleftrightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} F_1(u)\, F_2(\omega - u)\, du$

i.e., $f_1(t)\, f_2(t) \longleftrightarrow \frac{1}{2\pi} \left[F_1(\omega) * F_2(\omega) \right]$

PROOF : Let, $f(t) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$ $\quad [f(t) = f_1(t)\, f_2(t)]$

$$= \int_{-\infty}^{\infty} f_1(t)\, f_2(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} f_1(\lambda)\, e^{j\lambda t}\, d\lambda \cdot f_2(t)\, e^{-j\omega t}\, dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f_1(\lambda) \int_{-\infty}^{\infty} f_2(t)\, e^{j\lambda t - j\omega t}\, dt \cdot d\lambda$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f_1(\lambda) \int_{-\infty}^{\infty} f_2(t)\, e^{-j(\omega-\lambda)t}\, dt\, d\lambda$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f_1(\lambda)\, f_2(\omega-\lambda)\, d\lambda$$

$$\therefore\ f_1(t) \cdot f_2(t) = \int_{-\infty}^{\infty} f_1(u)\, f_2(\omega-u)\, du$$

(or) $$f_1(t)\, f_2(t) = \frac{1}{2\pi}\left[f_1(\omega) * f_2(\omega) \right]$$

$\therefore$ We can conclude that the convolution of two functions in the time domain is equal to the multiplication of their spectra in the frequency domain and that multiplication of two functions in the time domain is equal to the convolution of their spectra in the frequency domain.

→ <u>PROPERTIES OF CONVOLUTION</u> :-

(i) Commutative :-

$$x(t) * h(t) = h(t) * x(t)$$
$$x(n) * h(n) = h(n) * x(n)$$

(or) $$x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k) = h(n) * x(n)$$

(ii) Associative :-

$$(x(t) * h_1(t)) * h_2(t) = x(t) * (h_1(t) * h_2(t))$$
$$(x(n) * h_1(n)) * h_2(n) = x(n) * (h_1(n) * h_2(n))$$

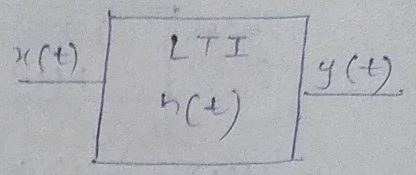
(iii) Distributive :-

$$x(t) * (h_1(t) + h_2(t)) = x(t) * h_1(t) + x(t) * h_2(t)$$
$$x(n) * (h_1(n) + h_2(n)) = x(n) * h_1(n) + x(n) * h_2(n)$$

(5) (a) The T/F of an LTI system $H(\omega)$ is $\frac{16}{4+j\omega}$. Find the response $y(t)$ whose i/p is $x(t) = u(t)$

Given, $x(t) = u(t)$

$$H(\omega) = \frac{16}{4 + j\omega}$$



$$h(t) = F^{-1}[H(\omega)] = F^{-1}\left[\frac{16}{4 + j\omega}\right]$$

$$h(t) = 16\, e^{-4t}\, u(t)$$

$$y(t) = x(t) * h(t)$$

$$= \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

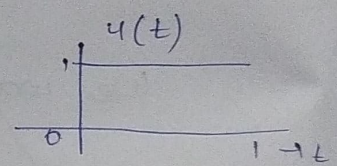
$$= \int_{-\infty}^{\infty} 16\, e^{-4\tau}\, u(\tau)\, u(t-\tau)\, d\tau$$

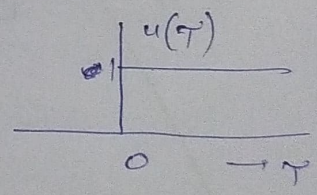
$$= 16 \int_{0}^{t} e^{-4\tau}\, 1\, d\tau$$

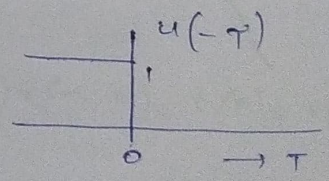
$$= \cancel{16}^{4} \left[\frac{e^{-4\tau}}{-\cancel{4}}\right]_{0}^{t}$$

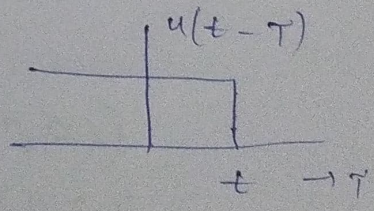
$$= -4\left[e^{-4t} - 1\right]$$

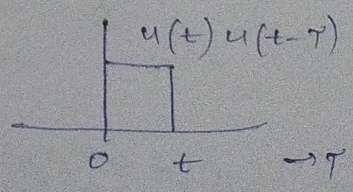
$$\boxed{y(t) = 4 - 4e^{-4t}}$$



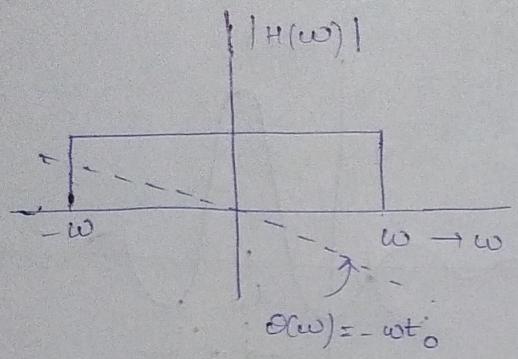




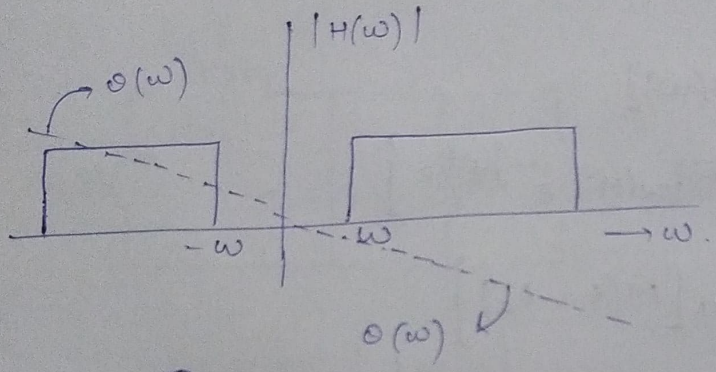




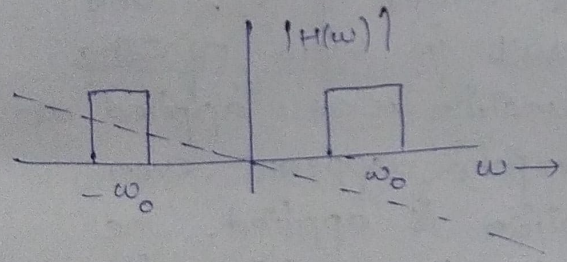
(5) (b) Sketch the frequency response of ideal LPF, HPF and BPF. Explain why ideal filters are not practically realizable.



(a) Ideal LPF.



(b) Ideal HPF



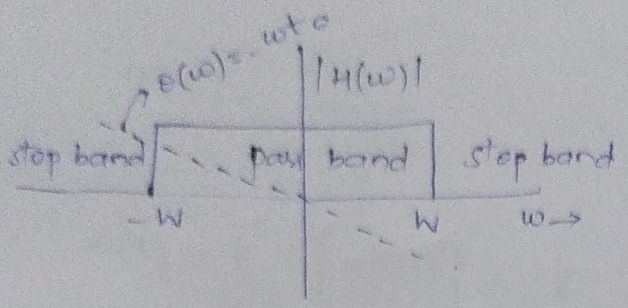
(c) Ideal BPF

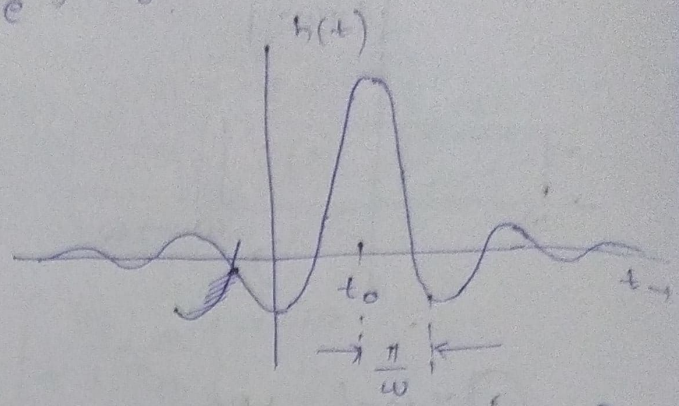
- An ideal LPF transmits, without any distortion all of the signals of frequencies below a certain frequency W radians per second. The signals of frequencies above W radians per second are completely attenuated. The frequency response of a LPF is thus a gate function $g_{2W}(\omega)$. The corresponding phase function for distortionless transmission is $-\omega t_0$

The T/F of such a filter is evidently given by,

$$H(\omega) = |H(\omega)|\, e^{j\theta(\omega)}$$

$$H(\omega) = \Theta_{2W}(\omega)\, e^{-j\omega t_0}$$





[An ideal LPF filter characteristic and impulse response.]

$$\therefore h(t) = F^{-1}[H(\omega)]$$

$$= F^{-1}\left[\Theta_{2W}(\omega)\, e^{-j\omega t_0}\right]$$

$$h(t) = \frac{W}{\pi} S_a\left[W(t - t_0)\right]$$

From fig, we can observe that the impulse response exists for negative values of t. This is certainly a strange result in view of the fact that the driving function was applied at $t = 0$. The response therefore appears even before the driving function is applied. The system seems to anticipate the driving function. Unfortunately, it is impossible in practical to build a system with such foresight.

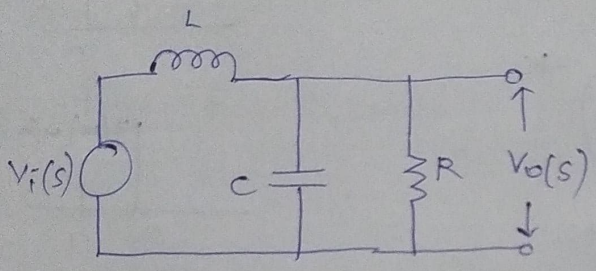
$\therefore$ We must conclude that although an ideal LPF is very desirable, it cannot be physically realizable.

IIIly we can conclude that all ideal ~~low pa~~ filters are physically unrealizable.

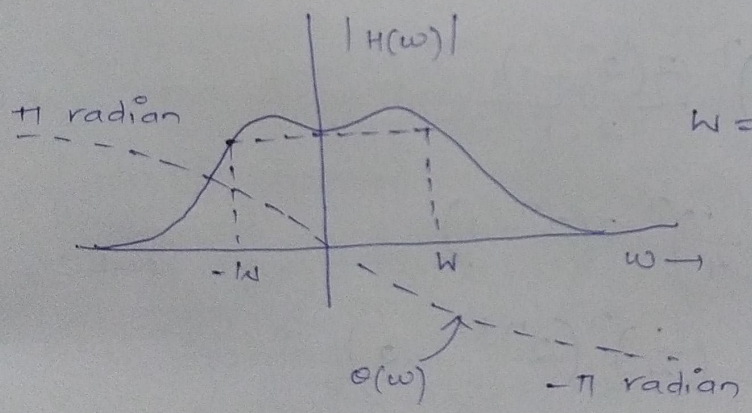
In practice we are satisfied with filters having characteristic closer to ideal filter. A simple LPF is shown in fig. The T/F of the this filter is given by,

$$H(\omega) = \frac{\dfrac{1}{\left(\frac{1}{R} + j\omega C\right)}}{j\omega L + \dfrac{1}{\left(\frac{1}{R} + j\omega C\right)}}$$

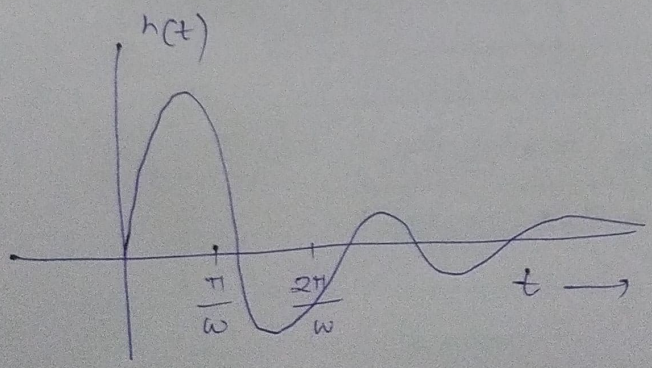
$$H(\omega) = \frac{1}{1 - \omega^2 LC + j\omega L/R}$$



$R = \sqrt{\frac{L}{C}}$



$W = \sqrt{\frac{1}{LC}}$

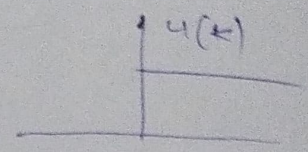


A simple realizable low pass filter and its T/F and I.R

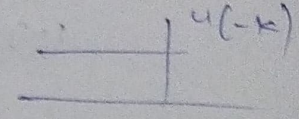
(6) (a) Determine the convolution of two sequences $x(n) = \left(\frac{1}{2}\right)^n u(n)$ and $h(n) = \left(\frac{1}{4}\right)^n u(n)$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

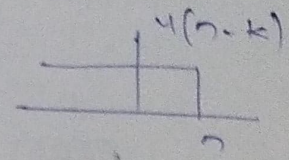
$$= \sum_{k=-\infty}^{\infty} \left(\frac{1}{2}\right)^k u(k) \left(\frac{1}{4}\right)^{n-k} u(n-k)$$



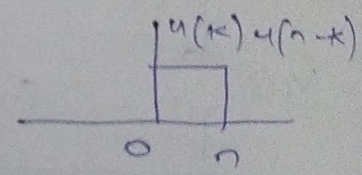
$$= \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{2n-2k}$$



$$= \sum_{k=0}^{n} \left(\frac{1}{2}\right)^{2n-k}$$



$$= \sum_{k=0}^{n} \left(\frac{1}{4}\right)^n \left(\frac{1}{2}\right)^{-k}$$



$$= \left(\frac{1}{4}\right)^n \sum_{k=0}^{n} 2^k$$

$$= \left(\frac{1}{4}\right)^n \frac{2(2^n - 1)}{2-1}$$

$$= \left(\frac{1}{4}\right)^n 2(2^n - 1)$$

$$= \left(\frac{1}{4}\right)^n (2^{n+1} - 2)$$

(b) Explain the input - output relationship of an LTI system and define I.R and T/F of an LTI system.

Convolution will give the I/p - o/p relationship of an LTI system.

Let $f_1(t)$ and $f_2(t)$ be the inputs, then the convolution of $f_1(t)$ and $f_2(t)$ is denoted by,

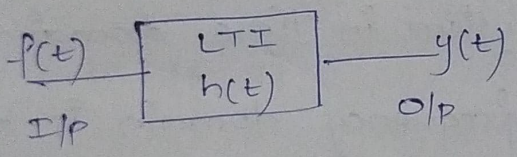
$$f(t) = f_1(t) * f_2(t) = \int_{-\infty}^{\infty} f_1(t) f_2(t-\tau) d\tau$$ which is a time domain representation.

We can also obtain the response by using frequency convolution,

$$f(t) = f_1(t) f_2(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F_1(\lambda) F_2(\omega - \lambda) d\lambda$$

$$= \frac{1}{2\pi} \left[F_1(\omega) * F_2(\omega) \right]$$

let,



Then, $y(t) = f(t) * h(t)$

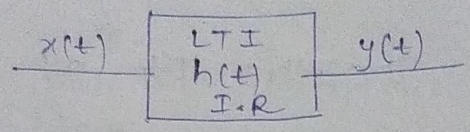
Applying F.T on b.s.

$$\boxed{Y(\omega) = F(\omega) \cdot H(\omega)}$$

$$\Rightarrow \boxed{H(\omega) = \frac{Y(\omega)}{F(\omega)} = \frac{\text{F.T of O/P}}{\text{F.T of I/P}}}$$

<u>Impulse Response</u> :- It is the response of the system when the impulse function $\delta(t)$ or delta function is applied at the input of the system.

Time response of an LTI system is nothing but the impulse response of the LTI system.



i.e., if $x(t) = \delta(t) =$ Impulse function
then, $y(t) = h(t) =$ Impulse response.

Response of an LTI system in time domain = Convolution of the I/p signal and the impulse response.

i.e., $r(t) = f(t) * h(t)$

$$= \int_{-\infty}^{\infty} f(\tau)\, h(t-\tau)\, d\tau$$

→ <u>Transfer function</u> :-

W.K.t., $h(t) =$ I.R, will describe the behaviour / characteristics of a system in time domain

∴ The response in frequency domain is known as transfer function of an LTI system.

(or) T/F is the frequency domain counterpart of I.R $h(t)$

$$H(\omega) = T/F = F.T[h(t)]$$

W.K.t., $y(t) = r(t) = f(t) * h(t)$

Applying F.T on b.s.

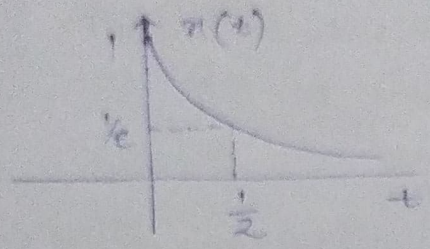
$$R(\omega) = F(\omega) \cdot H(\omega)$$

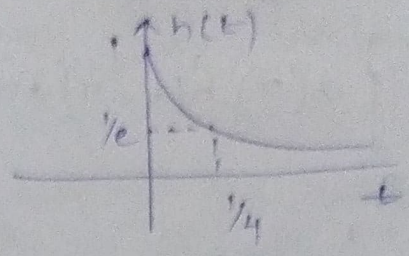
$$\Rightarrow \boxed{H(\omega) = \frac{R(\omega)}{F(\omega)} = \frac{\text{F.T of O/P}}{\text{F.T of I/P}}}$$

I.R $\boxed{h(t) = F^{-1}[H(\omega)]}$

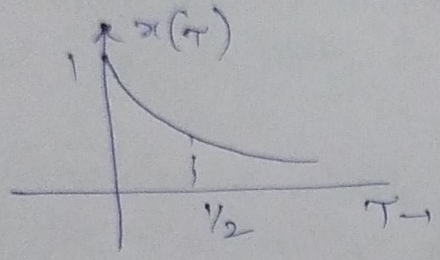
7 (a) Find the convolution of the signals $x(t) = e^{-2t} u(t)$ and $h(t) = e^{-4t} u(t)$ using graphical method.

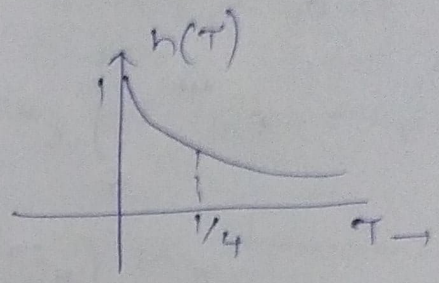
$x(t) = e^{-2t} u(t)$, $h(t) = e^{-4t} u(t)$



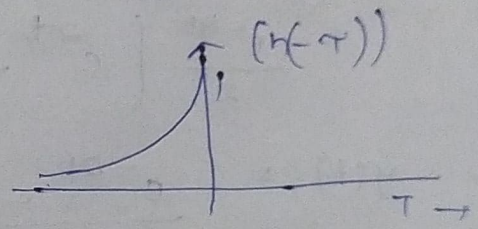


Step 1 Replace 't' by τ

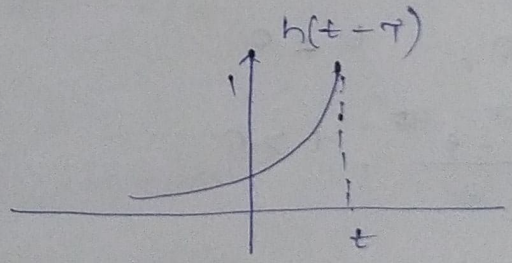




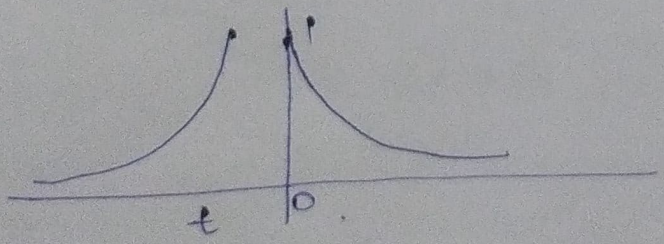
Step 2 Get $h(-\tau)$ by folding $h(\tau)$ about vertical axis



Step 3 Get $h(t-\tau)$



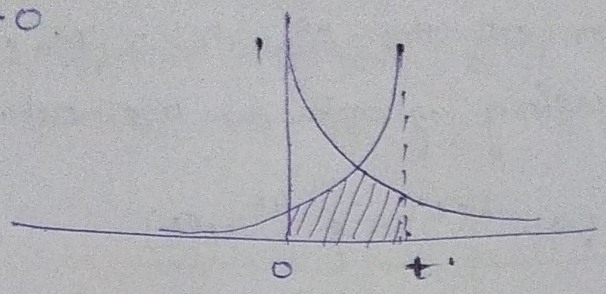
Step 4 $x(\tau)\ h(t-\tau)$



As long as $t \leq 0$,

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau) = 0.$$

for $t>0$.



$$y(t) = \int_0^t x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_0^t e^{-2\tau}\, e^{-4(t-\tau)}\, d\tau.$$

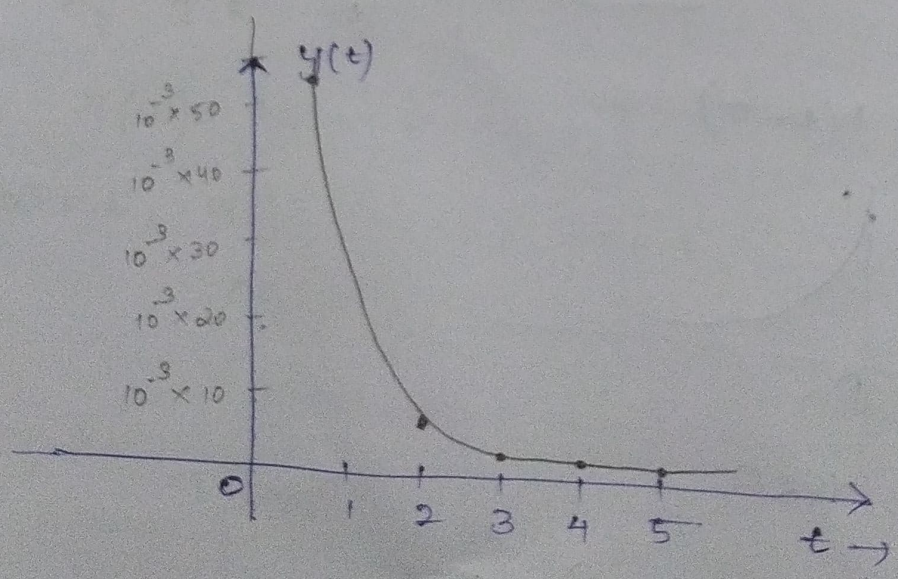
$$= e^{-4t} \int_0^t e^{2\tau}\, d\tau$$

$$= e^{-4t} \left[\frac{e^{2\tau}}{2} \right]_0^t$$

$$= \frac{e^{-4t}}{2} \left[e^{2t} - 1 \right]$$

$$y(t) = \frac{e^{-2t} - e^{-4t}}{2}, \quad t>0.$$

$$y(t) = \begin{cases} 0, & t \le 0 \\ \dfrac{e^{-2t} - e^{-4t}}{2}, & t>0. \end{cases}$$



(b) Distinguish b/w signal bandwidth and system bandwidth.

System bandwidth is the range of frequencies which a system can carry, or simply, it is the difference between the maximum and minimum frequencies.

Signal bandwidth is the range of frequencies accosiated with a signal.

As the name indicates, signal bandwidth is associated with the signal, and system bandwidth is associated with the system.

(8) (a) Consider a stable LTI system characterized by the differential equation

$$\frac{dy(t)}{dt} + 2y(t) = x(t).$$ Find its impulse response.

Applying F.T on b.s.

$$j\omega Y(\omega) + 2Y(\omega) = X(\omega)$$

$$(j\omega + 2)Y(\omega) = X(\omega)$$

$$\frac{Y(\omega)}{X(\omega)} = \frac{1}{2 + j\omega} \rightarrow (1)$$

We also have, $H(\omega) = \frac{Y(\omega)}{X(\omega)} \rightarrow (2)$

From (1) and (2),

$$H(\omega) = \frac{1}{2 + j\omega}$$

$$h(t) = F^{-1}[H(\omega)]$$

$$= F^{-1}\left[\frac{1}{2 + j\omega}\right]$$

I.R., $h(t) = e^{-2t} u(t)$

(b) Find the T/F of the system governed by the following I.R., $h(t) = u(t) + 0.5e^{-6t}u(t) + 0.2e^{-3t}\cos t\, u(t)$

$$h(t) = u(t) + 0.5e^{-6t}u(t) + 0.2e^{-3t}\cos t\, u(t)$$

$$H(s) = F^{-1}[h(t)]$$

$$= F^{-1}\left[u(t) + 0.5e^{-6t}u(t) + 0.2e^{-3t}\cos t\, u(t)\right]$$

$$H(s) = \frac{1}{s} + \frac{0.5}{s+6} + 0.2\left(\frac{s+3}{(s+3)^2 + 1^2}\right)$$

9 (a) Determine system $y(t) = x(t)\cos\omega_c t$, whether it is (a) memoryless (b) causal (c) linear (d) time invariant (e) stable.

$$y(t) = x(t)\cos\omega_c t$$

(a) At $t=0$, $y(0) = x(0)$

Here o/p at a given time is dependent only on the I/p at the same time

$\therefore$ System is memoryless.

(b) O/P depends on only present I/P.

$\therefore$ System is causal.

(c) $y(t) = x(t)\cos\omega_c t$

$x_1(t) \rightarrow y_1(t) = x_1(t)\cos\omega_c t$

$x_2(t) \rightarrow y_2(t) = x_2(t)\cos\omega_c t$

I/P $\rightarrow$ $ax_1(t) + bx_2(t) = x_3(t)$

Also, $x_3(t) \rightarrow y_3(t) = x_3(t)\cos\omega_c t$

$$y_3(t) = [ax_1(t) + bx_2(t)]\cos\omega_c t$$

$$y_3(t) = ax_1(t)\cos\omega_c t + bx_2(t)\cos\omega_c t \rightarrow \text{LHS}$$

$$ay_1(t) = ax_1(t)\cos\omega_c t$$

$$by_2(t) = bx_2(t)\cos\omega_c t$$

$$ay_1(t) + by_2(t) = ax_1(t)\cos\omega_c t + bx_2(t)\cos\omega_c t \rightarrow \text{RHS}$$

LHS = RHS.

$\therefore$ System is linear.

(d) $y(t) = x(t)\cos\omega_c t$

$y_1(t) = x_1(t - t_0)\cos\omega_c t$ $\quad [\because y_1(t) = T[x(t - t_0)]]$

$y_2(t) = x_1(t - t_0)\cos\omega_c(t - t_0)$ $\quad [\because y_2(t) = y(t - t_0)]$

$y_1(t) \neq y_2(t)$

$\therefore$ System is not time invariant

(e) Stable

if $x(t) = \delta(t)$, then $y(t) = h(t)$.

$\Rightarrow h(t) = \delta(t)\cos\omega_c t$

System is stable, if $\int_{-\infty}^{\infty} h(t)\,dt < \infty$

$$\int_{-\infty}^{\infty} \delta(t)\cos\omega_c t = (\cos\omega_c t)_{t=0} = 1$$

$\therefore$ System is stable

(5) A system has I/P - O/P relation given by $y[n] = n\,x[n]$. Determine whether the system is (a) memoryless (b) causal (c) linear (d) time invariant (e) stable

(a) $y[n] = n\,x[n]$

For $n=1$, $y[1] = x[1]$

$n=2$, $y[2] = 2\,x[2]$

Here O/P at a given time is dependent only on the I/P at the same time

$\therefore$ System is memoryless.

(b) $y[n] = n\,x[n]$

for $n=1$, $y[1] = x[1]$

Here O/P depends only on present I/P

$\therefore$ System is causal.

(c) $y[n] = n\,x[n]$

$x_1[n] \rightarrow y_1[n] = n\,x_1[n]$

$x_2[n] \rightarrow y_2[n] = n\,x_2[n]$

$a\,x_1[n] + b\,x_2[n] = x_3[n]$

Also, $x_3[n] \rightarrow y_3[n] = n\,x_3[n]$

$= n[a\,x_1[n] + b\,x_2[n]]$

$= an\,x_1[n] + bn\,x_2[n] \rightarrow$ LHS

$a\,y_1[n] = an\,x_1[n]$

$b\,y_2[n] = bn\,x_2[n]$

$a\,y_1[n] + b\,y_2[n] = an\,x_1[n] + bn\,x_2[n] \rightarrow$ RHS

LHS = RHS

$\therefore$ System is linear.

(d) $y[n] = n\,x[n]$

$y_1[n] = n\,x_1[n-n_0]$

$y_2[n] = (n-n_0)\,x_1[n-n_0]$

$y_1[n] \neq y_2[n]$

$\therefore$ System is not time invariant.

(e) for $x[n] = \delta[n]$, then $y[n] = h[n]$

$\Rightarrow h[n] = n\,\delta[n]$

$$\sum_{n=-\infty}^{\infty} h[n] = \sum_{n=-\infty}^{\infty} n\,\delta[n] = 0 < \infty$$

$\therefore$ System is stable.