→ Faraday's Law and Transformer EMF :

Faraday's law states that in a magnetic field if a closed ckt is in the motion (or) if the magnetic field is time varying then the induced electro magnetic force (emf) (or) voltage across the terminal of the closed ckt is equal to the time rate of change of magnetic flux leakage by the ckt is given by

$$\boxed{V = -N \frac{d\psi}{dt}} \quad \text{——— (1)}$$

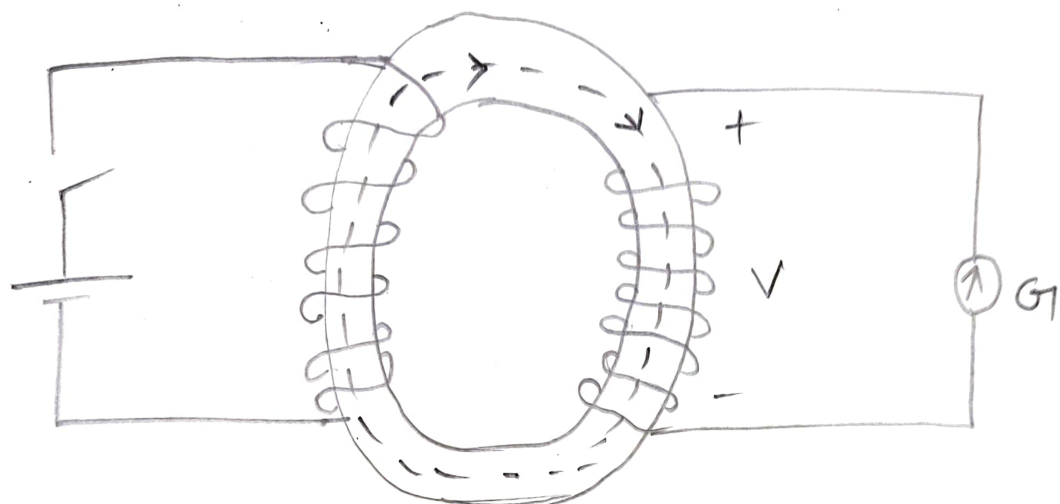
where N is the no of turns of a ckt and flux through each turn.

where &-ve sign indicates according to the Lenz's law. The induced emf will produce a magnetic field which opposes the original field. It can be observed that Faraday's law links dynamic and magnetic field. Induced emf may be of following 3 kinds

(i) Transformer EMF

(ii) Generator EMF

(iii) Both transformer & generator EMF is.



ψ = flux through each turn.

NOTE 1:-

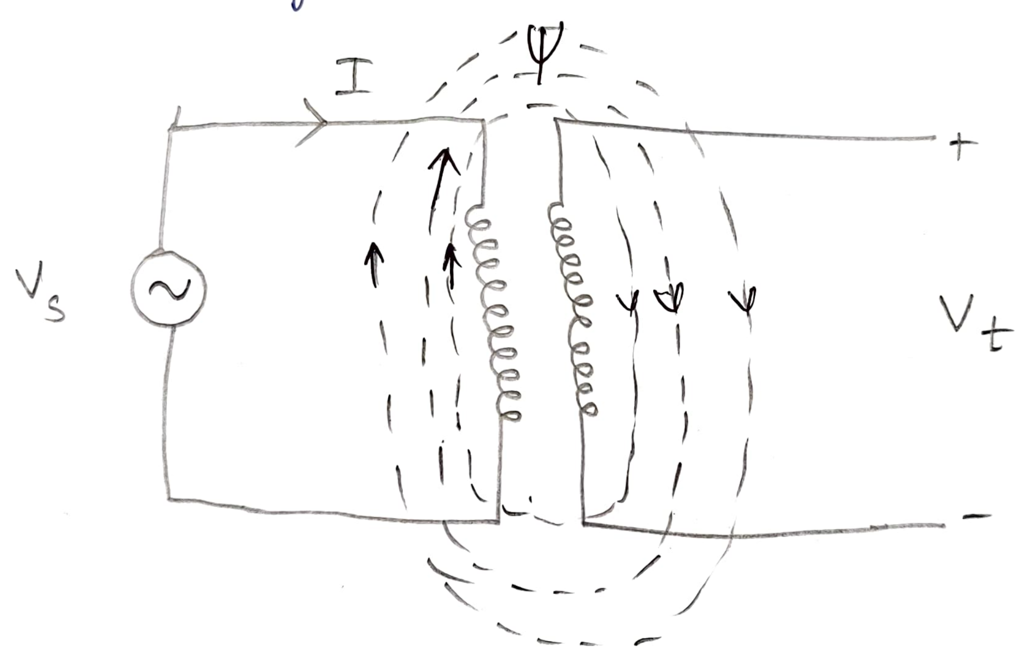
→ The -ve sign is explained by Henrich Ka Lenz's law.

simple rule to find direction of induced emf. The rule states that the direction of induced emf is always such that it tends to setup are induced current separately which produces flux opposing the flux leakage.

Note 2:- In other words the direction of the current flows in the closed ckt is such that induced magnetic field produced by a current will oppose the original magnetic field. Therefore we keep -ve symbol.

TRANFORMER EMF (or) STATIONARY LOOP IN TIME VARYING FIELD (or) STATIONARY CIRCUIT :-

Consider a transformer (stationary ckt) as shown in a figure. when a time varying current is applied it produce time varying magnetic flux in a primary coil that induced and emf in the secondary coil. This EMF is called Transformer EMF (or) ~~Stationary~~ statically induced EMF



Statically Induced EMF or Transformer emf

We know that from the faraday's law

$$V_t = -N \frac{d\psi}{dt}$$

If $N = 1$ $\Rightarrow$ $\boxed{V_t = \frac{-d\psi}{dt}}$ —— (2)

Shot on OnePlus
By APKK

In the electric field intesity potential

$$\boxed{V_t = \oint_L \vec{E}\cdot d\vec{l}} \quad (3)$$

~~$\psi = \oint$~~ $$\boxed{\psi = \int_S \bar{B}\cdot d\vec{s}} \quad (4)$$

substituting eq (4) in eq (2) we get V_t as

$$V_t = -N\frac{d\psi}{dt} \overset{N=1}{=} -\frac{d\psi}{dt} = -\frac{d}{dt}\left[\int_S \vec{B}\cdot d\vec{s}\right]$$

$$\oint_L \bar{E}\cdot d\bar{l} = -\frac{d}{dt}\left[\int_S \bar{B}\cdot d\bar{s}\right]$$

$$\boxed{\oint_L \bar{E}\cdot d\bar{l} = -\int_S \frac{d\bar{B}}{dt}\cdot d\bar{s}} \quad (5)$$

By seeing eq (5) we can apply stokes theorem and eq (5) is the transformed EMF (eq)

$$\oint \bar{E}\cdot dl = \int_S (\nabla\times\vec{E})\cdot ds$$

$$\boxed{\int_S \nabla\times\vec{E}\, d\bar{s} = \oint_{L} \vec{E}\cdot d\bar{l} = -\int_S \frac{d\bar{B}}{dt}\, d\bar{s}}$$

By comparing we get

$$\boxed{\nabla\times\vec{E} = -\frac{d\bar{B}}{dt}} \quad (6)$$

The curl of time varying electric field is equal to time rate of decreasing the flux density $\boxed{\nabla \times \vec{E} = -\frac{\partial B}{\partial t}}$

This is the 4^{th} law of Maxwell's equation and shows that the time varying field is ~~not~~ non-conservative. We know that static electric field is ~~non~~- conservative then the curl of the vector is zero $\boxed{\nabla \times E = 0}$

and $\boxed{\oint_L \vec{E} \cdot d\vec{l} = 0}$

GENERATOR EMF (OR) MOTIONAL EMF :

(Moving loop in a static Magnetic field)

Consider a stationary magnetic field is applied on a conducting ckt which is moving (or) revolving on its axis with respected to time. When the ckt cuts across the magnetic field, electromagnetic field is induced across the Circuit terminals. This EMF is Called Generator EMF and the action is called Generator action. The Circuit which is moving in a linear velocity (u) in the magnetic field uniform flux density (B) from Lorentz force equation. the force on free charge ckt is

$$\boxed{\vec{F} = Q\vec{u} \times \vec{B}}$$

when the force acting on a charge

$$\frac{\vec{F}}{Q} = \vec{u} \times \vec{B}$$

$$\frac{\vec{F}}{Q} = \vec{E}$$

$$\boxed{\vec{E} = \vec{u} \times \vec{B}} \quad \text{—— (7)}$$

The induced EMF in the ckt

$$\boxed{V_m = \oint_l \vec{E} \cdot d\vec{l} = \oint_l (\vec{u} \times \vec{B}) \cdot d\vec{l}} \quad \text{—— (8)}$$

By applying the stoke's theorem $\oint_l \vec{E} \cdot d\vec{l} = \int_S (\nabla \times \vec{E}) \cdot d\vec{s}$

$$\int_S (\nabla \times \vec{E}) \cdot d\vec{s} = \int_S \nabla \times (\vec{u} \times \vec{B}) \, d\vec{s}$$

By eliminating the surface integrals,

$$\boxed{\nabla \times \vec{E} = \nabla \times \vec{u} \times \vec{B}} \quad \text{—— (9)}$$

Both

TRANSFORMER EMF & (Moving ckts in Time varying field)
GENERATOR EMF

If a conducting ckt is moving in time varying magnetic field both Transformer EMF & Motional EMF will be present in the conductor. therefore

$$\boxed{V_m = \oint_l \vec{E} \cdot d\vec{l} = -\int_S \frac{d\vec{B}}{dt} \, d\vec{s} + \oint_l (\vec{u} \times \vec{B}) \cdot d\vec{l}}$$

moving ckts in Time varying field — (10)

Transformer EMF

Generator EMF

The first term in eqn (10) gives gives voltage induced by time varying magnetic flux. The second terms gives the voltage induced by ckt in motion.

By applying Stoke's theorem we get

$$\int_S (\nabla\times\vec{E})\,d\vec{s} = -\int_S \frac{\partial \vec{B}}{\partial t}\,d\vec{s} + \int_S \nabla\times(\vec{u}\times\vec{B})\cdot d\vec{s} \quad (11)$$

$$\boxed{\nabla\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} + \nabla\times\vec{u}\times\vec{B}} \quad (12)$$

INCONSISTENCY OF AMPERE'S LAW AND DISPLACEMENT CURRENT DENSITY (J):-

According to the Ampere's circuit law for static electromagnetic field, the current density is

$$\nabla\times H = \vec{J}.$$

Applying divergence on both sides

$$\nabla\cdot(\nabla\times\vec{H}) = \nabla\cdot\vec{J} = 0$$

since the divergence of the curl of any vector field is zero, but we know that $\nabla\cdot\vec{J} = -\frac{\partial \rho_v}{\partial t}$ of the equation $\nabla\cdot\vec{J}$ does not vanish in t varying field. Therefore amp ckt law is not consistant with time varying field.

8.7 Modified Ampere's Circuital Law for Time-Varying Fields

According to Ampere's circuital law, for static electromagnetic fields, the current density is

$$\nabla \times \vec{H} = \vec{J}.$$

Taking divergence of both sides,

$$\nabla \cdot (\nabla \times \vec{H}) = \nabla \cdot \vec{J} = 0, \tag{8.25}$$

Since the divergence of the curl of any vector field is zero. But the continuity equation for time-varying fields is given by

$$\nabla \cdot \vec{J} = \frac{-\partial \rho_v}{\partial t}. \tag{8.26}$$

Thus the quantity $\nabla . \vec{J}$ does not vanish in a time-varying fields, Eq.(8.25) and (8.26) are incompatible. Therefore, Ampere's circuital law is not consistent with time-varying fields.

James Clark Maxwell developed a modification to Ampere's law by substituting the Gauss' law in the equation of continuity,

$$\nabla \cdot \vec{J} = \frac{-\partial \rho_v}{\partial t}.$$

From Gauss' law, $$\nabla \cdot \vec{D} = \rho_v$$

$\therefore$ $$\nabla \cdot \vec{J} = \frac{-\partial}{\partial t}(\nabla \cdot D).$$

$$\nabla \cdot \vec{J} = -\nabla \cdot \frac{\partial \vec{D}}{\partial t} \tag{8.27}$$

or $$\nabla \cdot \vec{J} + \nabla \cdot \frac{\partial \vec{D}}{\partial t} = 0.$$

$$\nabla \cdot \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) = 0.$$

In integral form, using the divergence theorem,

$$\int_S \left(\frac{\partial \vec{D}}{\partial t} + \vec{J} \right) \cdot d\vec{S} = 0. \tag{8.28}$$

Thus the total current flowing through a closed surface is zero.

Therefore the total current density for time-varying fields is

$$\vec{J}_t = \vec{J} + \frac{\partial \vec{D}}{\partial t}. \tag{8.29}$$

Since $\vec{D}$ is the displacement charge density, $(\partial\vec{D}/\partial t)$ is called the displacement current density. The additional $(\partial\vec{D}/\partial t)$ term makes the Ampere's circuital law consistent with the principle of conservation of charge. Therefore, the modified Ampere's circuital law is

$\boxed{\nabla \times H = J}$ maxwell 3rd eq

$$\nabla \times \vec{H} = \vec{J}_t$$

or
$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}.$$

It is observed that a time-varying electric field will produce a magnetic field even when there is no current flow in the medium. This indicates that the displacement current density flowing through a dielectric material can develop a magnetic field.

Using Stokes' theorem, the integral form is

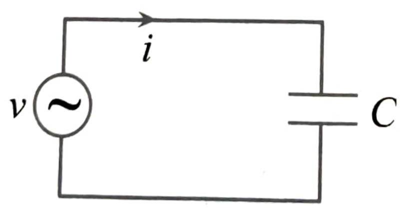
$$\oint \vec{H} \cdot d\vec{l} = \int_s \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) d\vec{S} = I_{\text{encl}}. \tag{8.30}$$

This states that the magnetomotive force around a closed path is equal to the total current enclosed by the path.

8.8 Displacement Current

Consider a current flowing through a capacitor as shown in Fig. 8.6. If i_c is the current through the circuit, then the conduction current density is $\vec{J}_c = i_c / S = \sigma \vec{E}$, where S is the area of cross section and E the electric field across the capacitor.

$J_c = \frac{i_c}{S}$



$J_d = \frac{i_d}{S}$

Fig. 8.6 *Capacitor circuit*

The current flowing through the capacitor is

$$i_d = C\frac{dV}{dt} = \frac{\varepsilon S}{d}\frac{dV}{dt},$$

where ε is the permittivity of the dielectric material in the capacitor and d the separation between plates.

We know that
$$V = \vec{E}\, d = \frac{\vec{D}\, d}{\varepsilon}.$$

So
$$i_d = \frac{\varepsilon S}{d} \cdot \frac{\partial}{\partial t}\left(\frac{\vec{D} \cdot d}{\varepsilon} \right) = S \cdot \frac{\partial \vec{D}}{\partial t}. \tag{8.31}$$

This current is called the displacement current flowing through the capacitor.

The displacement current density is $J_d = \frac{i_d}{S} = \frac{\partial \vec{D}}{\partial t}$. (8.32)

Therefore the total current density

$$J = J_c + J_d$$

or
$$J = \sigma \vec{E} + \frac{\partial \vec{D}}{\partial t} \tag{8.33}$$

This shows that a time-varying current will flow through the capacitor circuit even though there is no flow of current between the capacitor plates.

Note: The conduction current i_c flows through the conductor while the displacement current i_d flows through the capacitor.

Example 8.6 Moist soil has a conductivity of 10^{-3} siemens/m and $\varepsilon_r = 2.5$. Find J_c and J_d where, $E = 6\times10^{-6}\sin(9\times10^{9}t)$ V/ m.

Solution Given conductivity, $\sigma = 10^{-3}$ S/m; relative permittivity $\varepsilon_r = 2.5$; and electric field intensity $E = 6\times10^{-6}\sin(9\times10^{9}t)$ V/ m.

We know that conduction current density is

$$J_c = \sigma E = 10^{-3}\times 6\times10^{-6}\sin(9\times10^{9}\ t)$$

$$= 6\times10^{-9}\sin(9\times10^{9}t)\ \text{A/ m}^2$$

or
$$= 6\sin(9\times10^{9}t)\ \text{nA/ m}^2.$$

Displacement current density,

$$J_d = \frac{\partial D}{\partial t}$$

$$= \varepsilon\frac{\partial E}{\partial t} = \varepsilon_o\varepsilon_r\frac{\partial E}{\partial t}$$

$$= 8.854\times10^{-12}\times 2.5\times\frac{\partial}{\partial t}[6\times10^{-6}\sin(9\times10^{9}\ t)]$$

$$= 8.854\times10^{-12}\times2.5\times6\times10^{-6}\cos(9\times10^{9}t)\times9\times10^{9}$$

$$= 1.1952\times10^{-6}\cos(9\times10^{9}t)\ \text{A/ m}^2.$$

8.9 Ratio between Conduction Current Density and Displacement Current Density

Let $\vec{E}$ be a time-varying field given by

$$\vec{E} = E\,e^{j\omega t}. \tag{8.34}$$

$\vec{E} = Ee^{j\omega t}$

Then $$\boxed{\frac{\partial \vec{E}}{\partial t} = j\omega \vec{E}.}$$

The conduction current density is $J_c = \sigma \vec{E}$

and the displacement current density is

$\vec{D} = \varepsilon \vec{E}$

$$J_d = \frac{\partial \vec{D}}{\partial t} = \varepsilon \cdot \frac{\partial \vec{E}}{\partial t} = j\omega\varepsilon \vec{E}. \tag{8.35}$$

The ratio of conduction current density to displacement current density is

$$\frac{J_c}{J_d} = \frac{\sigma \vec{E}}{j\omega\varepsilon \vec{E}} = \frac{\sigma}{j\omega\varepsilon},$$

whose magnitude is $$\left|\frac{J_c}{J_d}\right| = \frac{\sigma}{\omega\varepsilon}. \tag{8.36}$$

At low frequencies the displacement current density J_d is usually negligible compared to conduction current density J_c. But at radio frequencies (high frequencies), in wave propagation, the value of J_d becomes comparable with J_c.

Note: The current densities depend on the medium constants σ, ε and frequency ω. The medium is classified depending on the current densities in the wave, i.e.,

1. If $\dfrac{\sigma}{\omega\varepsilon} >> 1$, the medium is a perfect conductor.
2. If $\dfrac{\sigma}{\omega\varepsilon} << 1$, the medium is a perfect dielectric.
3. If $\dfrac{\sigma}{\omega\varepsilon} = 0$, the medium is an insulator or free space.

It is also noted that a conducting material at very low frequency may become dielectric at very high frequencies.

8.10 Differences between Conduction, Convection and Displacement Currents

Table 8.1 *Differences between conduction, convection and displacement currents*

Conduction current	Convection current	Displacement current
1. Conduction current is the flow of electrons through any conducting medium.	1. Convection current is the flow of electrons through a non-conducting (insulating) medium.	1. Displacement current is the flow of charge which results due to time-varying electric field.
2. Conduction current is the current passing through the resistors and wires.	2. It is the leakage current passing through the dielectric (insulating) medium of the capacitor.	2. It is the rate of flow of charge between the capacitor plates in a capacitor circuit.
3. It is given by $I_c = \sigma \vec{E}.\vec{S}$, where σ = conductivity, $\overline{E}$ = electric field intensity and $\overline{S}$ = surface area	3. It is given by $I = \iint \vec{J} \cdot d\vec{S} = \iint \rho_v \vec{u} \cdot d\vec{S}$ where $\vec{J} = \rho_v \vec{u}$, the convection current density, and ρ_v = volume density and $\vec{u}$ = velocity.	3. It is given by $I_d = \varepsilon S \dfrac{\partial E}{\partial t}$ where, ε = permittivity of the dielectric material.
4. It is independent of frequency.	4. Its value increases with frequency.	4. It is directly proportional to frequency.
5. It obeys Ohm's law and hence has linear charge characteristics.	5. It does not obey Ohm's law and so has nonlinear characteristics.	5. It also does not obey Ohm's law and so has nonlinear characteristics.
6. Exists in both time variant and invariant cases.	6. It also exists in both time variant and invariant cases.	6. It exists only in time variant case.
7. Typical examples are current through conductors, resistors, etc.	7. Typical examples are electron beam moving through vaccum, CRT, liquids, etc.	7. Typical examples are the currents flowing through capacitor and all imperfect conductors carrying a time-varying conduction current.
8. It flows only through the closed paths in the circuits.	8. It flows only through the open paths in circuits.	8. It provides a closed path in the circuits having capacitor elements or in open circuits where the conduction current cannot flow further.

8.11 Differences between Displacement Current Density and Conduction Current Density

Table 8.2 *Differences between displacement current density and conduction current density*

	Displacement current density		Conduction current density
1.	It is defined as the displacement current at a given point, passing through a unit surface area when the surface is normal to the direction of the displacement current. It is denoted by $\vec{J}_d$.	1.	It is defined as the conduction current at a given point, passing through a unit surface area normal to the direction of the current. It is denoted by J_c.
2.	Displacement current results when a potential is applied across the dielectric medium.	2.	Conduction current results in conductors according to Ohm's law.
3.	Displacement current density is given by $\vec{J_d} = \frac{\partial \vec{D}}{\partial t}$.	3.	Conduction current density is given by $\vec{J_C} = \frac{i_C}{A} = \sigma \vec{E}$.
4.	Only displacement current density exists and conduction current density is negligible in dielectric medium.	4.	Only conduction current exists and displacement current density is negligible in conductors.
5.	Displacement current density exists only in time-varying fields.	5.	Conduction current density exists both in steady and time varying fields.

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4.	Only displacement current density exists and conduction current density is negligible in dielectric medium.	4.	Only conduction current exists and displacement current density is negligible in conductors.
5.	Displacement current density exists only in time-varying fields.	5.	Conduction current density exists both in steady and time varying fields.

8.12 Maxwell's Equations for Static Fields

As stated earlier, in the previous chapters we had considered static electric field due to charges at rest and static magnetic field due to charges in motion due to steady currents. The equations so derived are summarised below in Table 8.3..

Table 8.3 *Maxwell's equations for static electric and magnetic fields*

Point form	Integral form	
$\nabla \times \vec{E} = 0$	$\oint \vec{E} \cdot d\vec{l} = 0$ (Potential around a closed path is zero)	(8.37)
$\nabla \times \vec{H} = \vec{J}$	$\oint \vec{H} \cdot d\vec{l} = \int_s \vec{J} \cdot d\vec{S} = I_{encl}$ (Ampere's circuital law)	(8.38)
$\nabla . \vec{D} = \rho_v$	$\oint \vec{D} \cdot d\vec{S} = \int_v \rho_v \, dv = Q$ (Gauss' law)	(8.39)
$\nabla \cdot \vec{B} = 0$	$\oint \vec{B} \cdot d\vec{S} = 0$ (Magnetic field is continuous, conservation of flux)	(8.40)

The continuity equation for steady current is

$$\oint \vec{J} \cdot d\vec{S} = 0 \tag{8.41}$$

For time-varying fields, according to Faraday's law, an emf is induced either with time varying magnetic field or with generator action.

The Maxwell's equation for time-varying fields is a modification of Eq. (8.37).

$$\nabla\times\vec{E}=\frac{-\partial\vec{B}}{\partial t} \quad \text{and} \quad \oint\vec{E}\cdot d\vec{l}=-\int_S\frac{\partial\vec{B}}{\partial t}\cdot d\vec{S}. \tag{8.42}$$

James Clerk Maxwell, a professor in Cambride University, England, while performing experiments on electromagnetic fields in the year 1873, modified the laws of Faraday, Ampere and Gauss into a set of equations for time-varying fields. These equations are well known as Maxwell's equations. Maxwell developed mathematical analysis for electromagnetic waves (field theory) by defining the relationship between electric and magnetic fields using these equations.

Since the Maxwell's equations are valid at every point in free space, to analyse electromagnetic fields for specific objects and boundaries, it is convenient to convert the point forms into integral forms.

The point or differential forms are obtained by using the del operator while the integral forms are obtained by using divergence or Stokes' theorem.

The general expressions for Maxwell's equations valid for both static and time varying fields are given Table 8.4.

Table 8.4 *Maxwell's equations*

Point or Differential form	Integral form		
$\nabla\times\vec{H}=\vec{J}+\frac{\partial\vec{D}}{\partial t}$	$\int_L\vec{H}\cdot d\vec{l}=\int_S\left(\frac{d\vec{D}}{dt}+\vec{J}\right)d\vec{S}$	(modified Ampere's circuital law)	(8.43)
$\nabla\times\vec{E}=\frac{-\partial\vec{B}}{\partial t}$	$\int_L\vec{E}\cdot d\vec{l}=-\int_S\frac{\partial\vec{B}}{\partial t}\cdot d\vec{S}$	(Faraday's law of induction)	(8.44)
$\nabla\cdot\vec{D}=\rho_v$	$\oint_S\vec{D}\cdot d\vec{S}=\int_v\rho_v dv$	(Gauss' law)	(8.45)
$\nabla\cdot\vec{B}=0$	$\oint_S\vec{B}\cdot d\vec{S}=0$	(Conservation of flux)	(8.46)
The continuity equation is $\nabla\cdot\vec{J}=-\frac{\partial\rho_v}{\partial t}$	$\oint_S J\cdot dS=-\int_v\frac{\partial\rho_v}{\partial t}dv$		(8.47)

Equation (8.43) is a modified form of Ampere's circuital law applicable to time-varying electric and magnetic fields. It is observed that current density $\vec{J}$ may comprise both conduction current density (σE) and convection current density $(\rho\,\mu)$ due to the motion of free charge. Equation (8.44) is Faraday's law of electromagntic induction due to time-varying field.

Equation (8.45), Gauss' law, is valid for both static and time-varying fields and states that the divergence of charge density is equal to Q. Equation (8.46) indicate that flux is conservative and that the total outword magnetic flux through any closed surface is zero. Also, that isolated magnetic charges do not exist.

Maxwell's equations stated in words

1. The first equation states that *the magnetomotive force (mmf) around a closed path is equal to the sum of the conduction current and the displacement current through any surface bounded by the path.*

   Or

   The magnetic voltage around a closed path is equal to the electric current through the path.

2. The second equation states that *the electromotive force (emf) around a closed path is equal to the negative of the time derivative of the magnetic flux density (magnetic displacement) through any surface bounded by the path.*

   Or

   The electric voltage around a closed path is equal to the magnetic current through the path.

3. The third equation states that *the total electric flux density (electric displacement) through any surface enclosing a volume is equal to the total charge within that volume.*

4. The fourth equation states that *the net magnetic flux emerging through any closed surface is zero.*

Proof of Maxwell's equations

First Equation

Consider that time-varying electric and magnetic fields exist in a medium.

Method 1 : The continuity equation for time-varying fields is

$$\nabla \cdot \vec{J} = \frac{-\partial \rho_v}{\partial t}$$

or
$$\nabla \cdot \vec{J} + \frac{\partial \rho_v}{\partial t} = 0. \tag{8.48}$$

We know that $\nabla \times \vec{H} = \vec{J}$ (for static fields).

Then
$$\nabla \cdot \nabla \times \vec{H} = \nabla \cdot \vec{J} = 0.$$

For time varying fields, from continuity equation,

$$\nabla \cdot \nabla \times \vec{H} = \nabla \cdot J + \frac{\partial \rho_v}{\partial t}. \qquad (8.49)$$

From Gauss' law,

$$\nabla \cdot D = \rho_v.$$

Substituting in Eq.(8.49),

$$\nabla \cdot (\nabla \times \vec{H}) = \nabla \cdot \vec{J} + \frac{\partial}{\partial t}(\nabla \cdot \vec{D})$$

$$\nabla \cdot \nabla \times \vec{H} = \nabla \cdot \vec{J} + \nabla \cdot \frac{\partial \vec{D}}{\partial t}$$

$$\nabla \cdot \nabla \times \vec{H} = \nabla \cdot \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \qquad (8.50)$$

Eliminating the del operator,

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}. \qquad \text{(Proved)}$$

Method 2 : From Ampere's circuital law, we know that

$$\oint \vec{H} \cdot d\vec{l} = I_{encl} = \int_S \vec{J} \cdot d\vec{S}.$$

If $\vec{J} = \vec{J}_c + \vec{J}_d$ for time-varying fields.

$$\oint \vec{H} \cdot d\vec{l} = \int_S (\vec{J}_c + \vec{J}_d) \cdot d\vec{S}$$

$$\oint \vec{H} \cdot d\vec{l} = \int_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \cdot d\vec{S}. \qquad (8.51)$$

Applying Stokes' theorem,

$$\oint \vec{H} \cdot d\vec{l} = \int_S (\nabla \times \vec{H}) \cdot d\vec{S}. \qquad (8.52)$$

$$\therefore \quad \int_S (\nabla \times \vec{H}) \cdot d\vec{S} = \int_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \cdot d\vec{S}.$$

Eliminating surface integrals on both sides,

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}. \qquad \text{(Proved)}$$

Second Equation

Consider Faraday's law of transformer action,

$$\oint_L \vec{E}\cdot d\vec{l} = -\int_s \frac{\partial \vec{B}}{\partial t}\cdot d\vec{S}. \tag{8.53}$$

Using Stokes' theorem,

$$\int_s (\nabla\times\vec{E})\cdot d\vec{S} = -\int_s \frac{\partial \vec{B}}{\partial t}\cdot d\vec{S}. \tag{8.54}$$

Eliminating surface integrals on both sides,

or $$\nabla\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}. \quad \text{(Proved)}$$

Third Equation

According to Gauss' law, the total flux flowing out of a closed surface is equal to the charge enclosed by that surface.

That is, $$\int_s \vec{D}\cdot d\vec{S} = Q_{\text{encl}}.$$

If ρ_v is the volume charge density and V is the volume enclosed by the surface, then

$$\int_s \vec{D}\cdot d\vec{S} = \int_v \rho_v dv. \tag{8.55}$$

$\rho_v = \nabla\cdot\vec{D}$

Using divergence theorem,

$$\int_s \vec{D}\cdot d\vec{S} = \int_v (\nabla\cdot\vec{D})dv$$

$$\therefore \quad \int_v (\nabla\cdot\vec{D})dv = \int_v \rho_v\, dv. \tag{8.56}$$

Eliminating volume integrals on both sides

$$\nabla\cdot\vec{D} = \rho_v. \quad \text{(Proved)}$$

Fourth Equation

In any magnetic field, due to conservation of flux the total flux enclosed by a surface is always zero, (non-existence of monopole in the magnetic fields).

$$\int_s \vec{B}\cdot d\vec{S} = 0$$

Using divergence theorem,

$$\int_s \vec{B}\cdot d\vec{S} = \int_v (\nabla\cdot\vec{B})dv = 0. \tag{8.57}$$

Since the volume is finite, i.e., $v \neq 0$,

$$\therefore \quad \nabla\cdot\vec{B} = 0.$$

Maxwell's equations for different cases are summerised in Table 8.5, Table 8.6 and Table 8.7.

Table 8.5 *Maxwell's equations for free space and time-varying fields* ($\sigma = 0$, $J = 0$ and $\rho_v = 0$)

Point (differential) form	Integral form	
1. $\nabla\times\vec{H} = \dfrac{\partial\vec{D}}{\partial t}$	$\int_L \vec{H}\cdot d\vec{l} = \int_s \dfrac{\partial\vec{D}}{\partial t}\cdot d\vec{S}$	(8.58)
2. $\nabla\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t}$	$\int_L \vec{E}\cdot d\vec{l} = -\int_s \dfrac{\partial\vec{B}}{\partial t}\cdot d\vec{S}$	
3. $\nabla\cdot\vec{D} = 0$	$\oint_S \vec{D}\cdot d\vec{S} = 0$	
4. $\nabla\cdot\vec{B} = 0$	$\oint_S \vec{B}\cdot d\vec{S} = 0$	
Continuity equation $\nabla\cdot J = 0$		

Table 8.6 *Maxwell's equations for free space and static fields*

Point form	Integral form	
1. $\nabla\times\vec{H} = 0$	$\int_L \vec{H}\cdot d\vec{l} = 0$	(8.59)
2. $\nabla\times\vec{E} = 0$	$\int_L \vec{E}\cdot d\vec{l} = 0$	
3. $\nabla\cdot\vec{D} = 0$	$\oint \vec{B}\cdot d\vec{S} = 0$	
4. $\nabla\cdot\vec{B} = 0$	$\oint_S \vec{B}\cdot d\vec{S} = 0$	
Continuity equation $\nabla\cdot J = 0$		

Table 8.7 *Maxwell's equations for good conductor* ($\sigma >> \omega\varepsilon$, $\rho_v = 0$)

Point form	Integral form	
1. $\nabla\times\vec{H} = \vec{J}$	$\oint \vec{H}\cdot d\vec{l} = \int_s J\cdot d\vec{S} = I$	(8.60)
2. $\nabla\times\vec{E} = \dfrac{-\partial\vec{B}}{\partial t}$	$\oint \vec{E}\cdot d\vec{l} = -\int_s \dfrac{\partial\vec{B}}{dt}\cdot d\vec{S}$	
3. $\nabla\times\vec{D} = 0$	$\oint \vec{D}\cdot d\vec{S} = 0$	
4. $\nabla\cdot\vec{B} = 0$	$\oint_S \vec{B}\cdot d\vec{S} = 0$	
Continuity equation $\nabla\cdot J = 0$		

Maxwell's equations for sinusoidal fields are summarised in Table 8.8.

Table 8.8 *Maxwell's equations for sinusoidal fields*

Point form	**Integral form**
$\nabla \times \vec{H} = (\sigma + j\omega\varepsilon)\vec{E}$	$\int_L \vec{H} \cdot d\vec{l} = (\sigma + j\omega\varepsilon)\int_s \vec{E} \cdot d\vec{S}$ (8.67)
$\nabla \times \vec{E} = -j\omega\mu\vec{H}$	$\int_L \vec{E} \cdot d\vec{l} = -j\omega\mu \int_s \vec{H} \cdot d\vec{S}$
$\nabla \cdot \vec{D} = \rho_v$	$\oint_S \vec{D} \cdot d\vec{S} = \int_v \rho_v dv$
$\nabla \cdot \vec{B} = 0$	$\oint_S \vec{B} \cdot d\vec{S} = 0$
The continuity equation $\nabla \cdot \vec{J} = -j\omega\rho_v$	$\oint_S \vec{J} \cdot d\vec{S} = -j\omega\int_v \rho_v dv$

BOUNDARY CONDITIONS

So far, we considered the Electric field $\vec{E}$ in a homogeneous medium.

If field $\vec{E}$ exists in a region consisting of two different media, the condition that the field must satisfy at the interface separating the media are called boundary condition.

These conditions are helpful in determining the field on one side of the boundary if the field on the other side is known.

These conditions are helpful in determining the field on one side of the boundary if the field on the other side is known.

1) Dielectric (ε_{r_1}) and Dielectric (ε_{r_2})
2) Conductor and Dielectric
3) Conductor and free space.

→ To determine the boundary condition, we use Maxwell's eq^n

$$\text{and.} \left\{ \begin{array}{l} \oint_L \vec{E} \cdot \vec{dl} = 0 \\ \oint_S \vec{D} \cdot \vec{dS} = Q_{enc} \end{array} \right\}$$

→ Also, decompose the electric field intensity $\vec{E}$ into two orthogonal components.

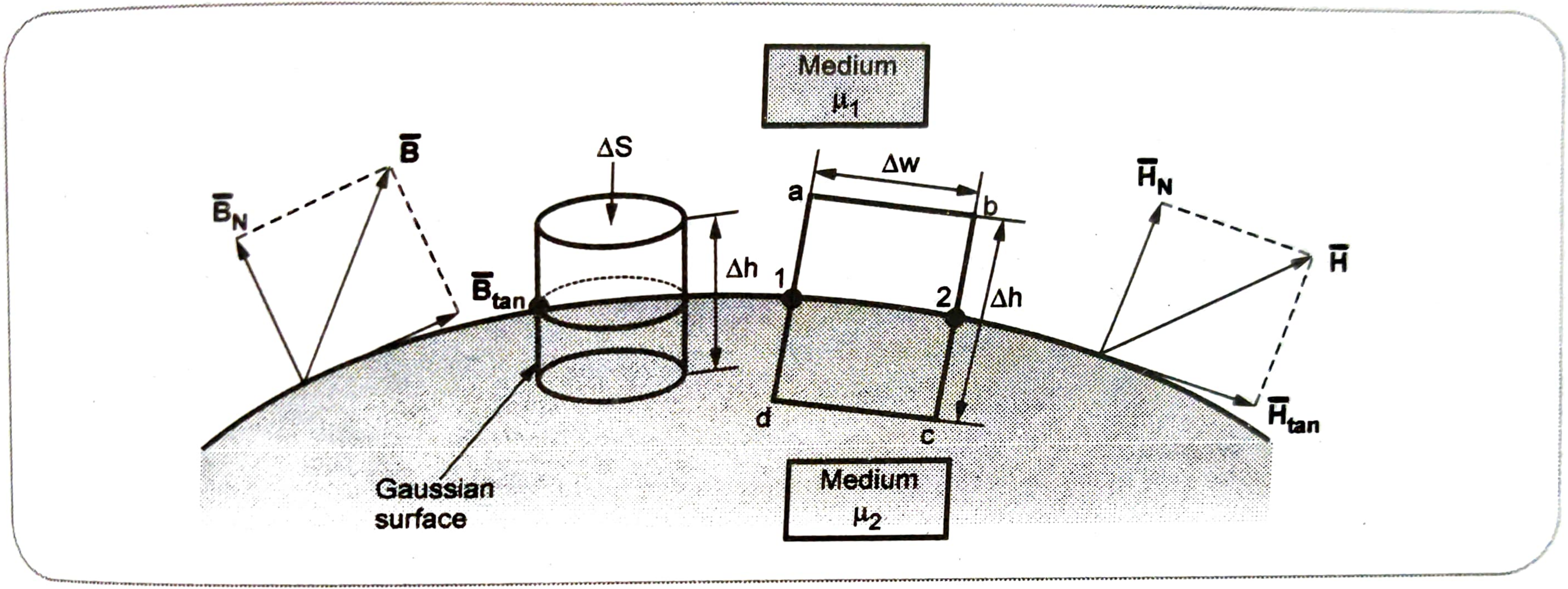
$$\left\{ \vec{E} = \vec{E_t} + \vec{E_n} \right\}$$

E_t = tangential component of $\vec{E}$

E_n = Normal " of $\vec{E}$

Similarly, Electric flux Density $\vec{D}$

$$\left\{ \vec{D} = \vec{D_t} + \vec{D_n} \right\}$$



Boundary between two magnetic materials of different permeabilities

6.12 Boundary Conditions

Maxwell's equations give the relationship between the four field vectors, $\vec{H}, \vec{E}, \vec{D}$ and $\vec{B}$, at any point within a continuous medium. The conditions of the field vectors at the boundary surface between different media can be obtained by applying Maxwell's equation at the interface. The boundary conditions can be stated as

(1) The tangential component of $\vec{E}$ is continuous at the surface.

$$E_{\tan 1} = E_{\tan 2} \tag{6.65}$$

(2) The tangential component of $\vec{H}$ is continuous across the surface except at the surface of a perfect conductor.

$$H_{\tan 1} = H_{\tan 2}. \tag{6.66}$$

At the surface of a perfect conductor, the tangential component of $\vec{H}$ is discontinuous by an amount equal to the surface current density $K\ A/m$. Since the current will flow only on the surface of a perfect conductor,

$$H_{\tan 2} - H_{\tan 1} = K. \tag{6.67}$$

(3) The normal component of $\vec{D}$ is discontinuous across the surface of the boundary by the amount of the surface charge density, i.e.,

$$D_{n1} - D_{n2} = \rho_s. \tag{6.68}$$

(4) The normal components of $\vec{B}$ is always continuous across the boundary surface. That is,

$$B_{n1} = B_{n2}. \tag{6.69}$$

Proof of boundary condition for electric field intensity ($\vec{E}$) Consider the surface of discontinuity at $x = 0$ plane between the two dielectrics media as shown in Fig. 6.6.

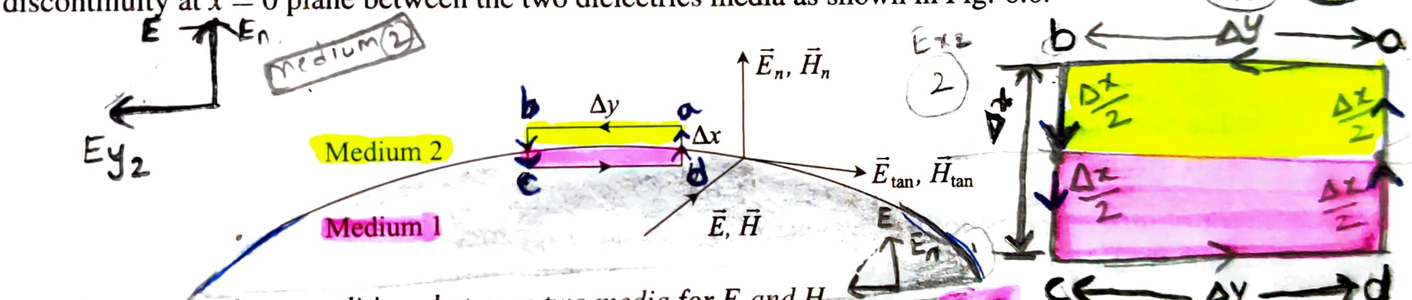


Fig. 6.6 *Boundary conditions between two media for E and H*

Consider a small rectangular surface of width Δx and length Δy on the boundary between two media. Maxwell's equation is given by

$$\oint \vec{E} \cdot d\vec{l} = -\int_s \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S}.$$

$d\vec{S}$ $\Delta x\ \Delta y$

For the rectangular path in the anti-clockwise direction,

$$E_{y2}\Delta y - E_{x2}\frac{\Delta x}{2} - E_{x1}\frac{\Delta x}{2} - E_{y1}\Delta y + E_{x1}\frac{\Delta x}{2} + E_{x2}\frac{\Delta x}{2} = -\frac{\partial B_z}{\partial t}\Delta x\Delta y, \tag{6.70}$$

where B_z is the magnetic flux density through the rectangular area $\Delta x\Delta y$. E_{y1} and E_{y2} are the tangential components of $\vec{E}$.

Now, at the boundary $\boxed{\Delta x \to 0}$,

$$E_{y2}\Delta y - E_{y1}\Delta y = 0.$$

$$E_{y2} = E_{y1} \text{ (tangential components of } \vec{E}).$$

(Proved) (6.71)

or $E_{\tan 1} = E_{\tan 2}$.

Therefore, the tangential component electric field intensity is continuous at the boundary.

Proof of boundary condition for magnetic field intensity ($\vec{H}$)

Consider Maxwell's equation,

$$\oint \vec{H} \cdot d\vec{l} = \int_s \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \cdot d\vec{S}.$$

For the rectangular path in the anti-clockwise direction,

$$H_{y2}\Delta y - H_{x2}\frac{\Delta x}{2} - H_{x1}\frac{\Delta x}{2} - H_{y1}\Delta y + H_{x3}\frac{\Delta x}{2} + H_{x4}\frac{\Delta x}{2} = (J + \frac{\partial D}{\partial t})\Delta x \Delta y \tag{6.72}$$

At the boundary, $\Delta x \to 0$, $D \neq 0$ and $J \neq 0$. Then,

$$H_{y2}\Delta y - H_{y1}\Delta y = 0.$$

$$H_{y2} = H_{y1} \text{ (tangential components of } \vec{H}).$$

i.e., $H_{\tan 1} = H_{\tan 2}$ (Proved). (6.73)

The tangential component of the magnetic field intensity is continuous at the boundary except for perfect conductor as explained below.

For perfect conductors If one of the media is a perfect conductor whose conductivity is infinity at the boundary, the current will flow only on the surface of the conductor. This current is called sheet current. In good conductors, at high frequency, the current will flow through the surface and the depth of current penetration becomes zero as the conductivity tends to infinity.

If K is the sheet current per unit width along the z direction on the conductor surface, the current density through a small depth Δx as $\Delta x \to 0$ is

$$\vec{J} = \lim_{\Delta x \to 0} \frac{K}{\Delta x} \text{ A/m}^2.$$

Therefore, Eq. (6.72) becomes $\longrightarrow$ $\left(J + \frac{\partial D}{\partial t}\right) \Delta x \, \Delta y \Rightarrow \left(\lim_{\Delta x \to 0} \frac{K}{\Delta x}\right) \Delta x \Delta y$

$D=0$

$$\cancel{H_{y2}\Delta y} - \cancel{H_{y1}\Delta y} = \lim_{\Delta x \to 0} \frac{K}{\Delta x}\Delta x \Delta y. \tag{6.74}$$

$$H_{y2} - H_{y1} = K.$$

$$H_{y1} = H_{y2} - K, \tag{6.75}$$

i.e., $\boxed{H_{\tan 2} - H_{\tan 1} = K.}$

$H_{y1} = H_{\tan 1}$
$H_{y2} = H_{\tan 2}$

Within a perfect conductor (say medium 2), the electric and magnetic fields are zero, i.e.,

$$H_{y2} = H_{\tan 2} = 0.$$

$$\therefore \quad H_{y1} = K. \tag{6.76}$$

That is, the current per unit width along surface of a perfect conductor is equal to the magnetic field strength $\vec{H}$ just outside the surface.

In Vector notation $\vec{K} = \hat{a}_n \times \vec{H}$

The magnetic field and surface current density are perpendicular to each other.

In vector notation,

$$\vec{K} = \vec{a}_n \times \vec{H}. \tag{6.77}$$

The magnetic field and surface current density are perpendicular to each other.

Proof of boundary conditions for electric flux density $(\vec{D})$ Consider a small cylindrical surface area dS with height Δx between the media at boundary $x = 0$ as shown in Fig. 6.7.

Maxwell's equation for charge density is

$$\oint \vec{D}\cdot d\vec{S} = Q_{enc} = \oint_s \vec{D}\cdot d\vec{S} = \int_v \rho_v dv, \quad \Rightarrow \underbrace{\oint \vec{D}\cdot d\vec{s}}_{\text{bottom}} + \underbrace{\oint \vec{D}\cdot d\vec{s}}_{\text{top}} + \underbrace{\oint D_{tan}\,\Delta x\,(2\pi r)}_{\text{lateral (side)}}$$

(cancel due to symmetry of cylinder) and also $\Delta x \to 0$

where dS = closed surface area of the cylinder and ρ_v = charge density enclosed by the volume.

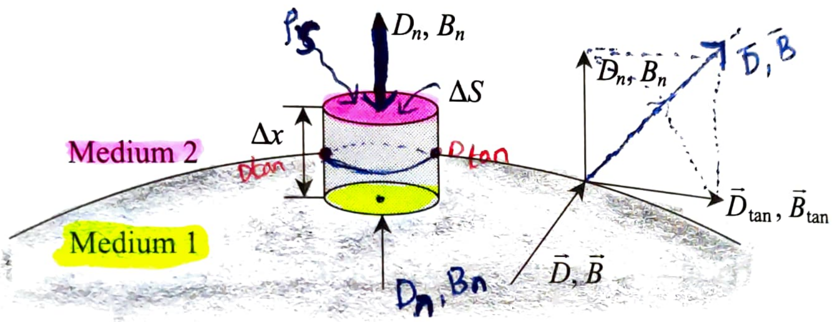


Fig. 6.7 *Boundary conditions between two media for D and B*

$\int_v dv = \Delta x\, ds$

From the Fig. 6.7,

$$D_{n1}dS - D_{n2}dS + \overbrace{\boxed{D_{tan}\Delta x(2\pi r)}}^{\text{Symmetry}} = \rho_v \Delta x dS. \tag{6.78}$$

For zero surface charge density At the boundary, as $\Delta x \to 0$, if surface charge density ρ_v is zero, then $(\rho_v = 0)$

$$D_{n1}dS - D_{n2}dS = 0 \tag{6.79}$$

or $\quad D_{n1} = D_{n2}.$

Therefore, if there is no surface charge, the normal component of flux density is continuous across the surface.

For metallic surface If there is a charge on the boundary of say, a metallic surface(medium 2), and ρ_s is a surface charge density, then $\boxed{\rho_s = \lim_{\Delta x \to 0} \rho_v \Delta x.}$

At the boundary as $\Delta x \to 0$, Eq. (6.78) becomes

ρ_v or ρ_s

$$D_{n1}dS - D_{n2}dS = \boxed{\lim_{\Delta x \to 0} \rho_v \Delta x dS}$$

or $\quad D_{n1}dS - D_{n2}dS = \boxed{\rho_s dS.}$

$$\therefore \quad D_{n1} - D_{n2} = \rho_s \qquad \text{(Proved).} \tag{6.80}$$

That is, if there is a surface charge density ρ_s, then the normal component of flux density is discontinuous across the surface by an amount equal to the surface charge density.

Since for metallic conductor $\vec{E} = 0$, therefore $D_{n2} = 0$.

$$\text{Then} \quad D_{n1} = \rho_s. \tag{6.81}$$

That is, at the boundary between conductor and a dielectric, the normal component of flux density in the dielectric is equal to the surface charge density on the conductor.

Proof of boundary conditions for magnetic flux density $(\vec{B})$

Similarly, Maxwell's equation for flux density is

$$\oint \vec{B} \cdot d\vec{S} = 0.$$

For a cylindrical closed surface shown in Fig. (6.7), since there are no isolated magnetic charges at the boundary,

$$B_{n1} dS - B_{n2} dS = 0$$

$$\text{or} \quad B_{n1} = B_{n2} \qquad \text{(Proved).} \tag{6.82}$$

That is, the normal component of the magnetic flux density is always continuous across a boundary surface.