Boolean Algebra

binary operators, 1 & 0
→ AND, OR, NOT (unary)

Axioms(postulates) of Boolean Algebra

Set of logical expressions that we accept without proof and upon which we can build a set of useful theorems

Addition
binary $1+1 = (2)_{10} = 10$ Decimal
Carry Sum

	AND Operation	OR Operation	NOT (unary) Operation
Axiom 1 (Postulate 1)	$0 \cdot 0 = 0$	$0+0=0$	$\bar{1}=0$
Axiom 2 (Postulate 2)	$0 \cdot 1 = 0$	$0+1=1$	$\bar{0}=1$
Axiom 3 (Postulate3)	$1 \cdot 0 = 0$	$1+0=1$	
Axiom 4 (Postulate 4)	$1 \cdot 1 = 1$	$1+1=1$	

Complement
$\bar{1} = 0$ (or) $1' = 0$

Complementation Law \ (Inversion Law)

1. $\bar{0} = 1$
2. $\bar{1} = 0$
3. If $A = 0$ Then $\bar{A} = 1$
4. If $A = 1$ then $\bar{A} = 0$
5. $\bar{\bar{A}} = A$ (Double inversion $\bar{\bar{A}}$)

AND law

AND law

1. $A \cdot 0 = 0$ (Null law)
2. $A \cdot 1 = A$ (Identity law)
3. $A \cdot A = A$
4. $A \cdot \bar{A} = 0$

OR law

$\boxed{\begin{array}{l} A+0 = A \\ A+1 = 1 \end{array}}$

A

1. $A + 0 = A$ (Null law)
2. $A + 1 = 1$

if $A = 0$ then $0+1 = 1$
$A = 1$ then $1+1 = 1$

3. $A + A = A$
4. $A + \bar{A} = 1$

Commutative Law

1. $A + B = B + A$
2. $AB = BA$

$A+B = B+A$ $\quad$ $AB = BA$

A	B	A+B	B+A	AB	BA
0	0	0	0	0	0
0	1	1	1	0	0
1	0	1	1	0	0
1	1	1	1	1	1

Associative Law

(A+B)+C = $A + (B+C)$

Associative Law

Law ① $(A+B)+C = A+(B+C)$

② $(AB)C = A(BC)$

Distributive Law

① $A(B+C) = AB+AC$

② $A+BC = (A+B)(A+C)$

Redundant literal rule

① $A+\bar{A}B = A+B$

$A+\bar{A}B$

③ literals

$\breve{A}+\breve{B} \rightarrow$ 2 literal

A = normal

$\bar{A}$ = Complement

$A+\bar{A}\bar{B}+A\bar{C}$ ④ = ④ literals

$A+\bar{B}\bar{C} + \bar{A}C + B\bar{D}+D\bar{B}$

⑧ literals

3 literals 2 literals

① $A+\bar{A}B = A+B$

Proof

$A+\bar{A}B = (A+\bar{A})(A+B)$

$1.(A+B)$

$= A+B$

$A+\bar{A} = ?$

$A+\bar{A} = 1$

$1.A = A$

$\overline{B} + BC = \,?$

$(\overline{B}+B)\,(\overline{B}+C)$

$1 \cdot (\overline{B}+C) \Rightarrow \overline{B}+C$

① ② ③ 3 literals ② literals

② $A(\overline{A}+B) = AB$

$A \cdot \overline{A} = 0$

$0 + AB = AB$

Proof

Distributive Law $A(\overline{A}+B) = A\cdot\overline{A} + AB$

$\downarrow$

$0 + AB$

$= AB$

Idempotence Law

Law ① $A \cdot A = A$

② $A + A = A$

Absorption law

① $A + AB = A$

$A(1+B)$

$A \cdot 1 \quad (\because 1+B = 1)$

$= A$

② $A \cdot (A+B) = A$

$A \cdot A + AB$

$A + AB$

$A + AB$

$A(\underbrace{1+B})$

$A \cdot 1 = A$

Consensus theorem (Included factor theorem)

Theorem 1

$$AB + \bar{A}C + BC = AB + \bar{A}C$$

(LHS) = (R.H.S)

$$\boxed{\begin{array}{l} A+\bar{A}=1 \\ 1+A=1 \end{array}}$$

$$AB + \bar{A}C + BC = AB + \bar{A}C + \underbrace{BC(A+\bar{A})}$$

$$= AB + \bar{A}C + ABC + \bar{A}BC$$

$$= AB(\underbrace{1+C}_{1}) + \bar{A}C(\underbrace{1+B}_{1})$$

$$= AB + \bar{A}C$$

This theorem can be extended to any no of literals

② $AB + \bar{A}C + BCD = AB + \bar{A}C$

LHS = RHS

$AB + \bar{A}C + BCD(A+\bar{A})$

$AB + \bar{A}C + ABCD + \bar{A}BCD$

$AB(1+CD) + \bar{A}C(1+BD)$

$$\boxed{\begin{array}{l} 1+A=1 \\ 1+CD=1 \\ 1+BD=1 \end{array}}$$

$AB.1 + \bar{A}C.1$

$AB + \bar{A}C$

$1 \cdot A = A$

Demorgen's theorem

Law ① $\overline{A+B} = \bar{A} \cdot \bar{B}$

② $\overline{A.B} = \bar{A} + \bar{B}$ sum

NOR $\overline{A+B}$

A	B	A+B	$\overline{A+B}$
0	0	0	1 ✓
0	1	1	0
1	0	1	0
1	1	1	0

A	B	$\bar{A}$	$\bar{B}$	$\bar{A} \cdot \bar{B}$
0	0	1	1	1 ✓
0	1	1	0	0
1	0	0	1	0
1	1	0	0	0

Duality theorem

0 becomes 1

1 becomes 0

AND becomes OR

OR becomes AND

<u>Note</u> :- Variables are not complemented in this process.

① $ABCD + ABD$, reduce this Boolean expression

$$f = \underbrace{\overset{①②③④}{ABCD}}_{} + \underbrace{ABD}_{} \quad \text{①②③④} \rightarrow 4 \text{ literals}$$

$$= ABD(C+1)$$

①②③ → ③ literals

$$= ABD$$

$1 + A = 1$

$1 + C = 1$

② $f = \underline{AB\bar{C}} + \underline{A\bar{B}C} + \underline{ABC} + \underline{ABCD}$

$$f = AB(\underbrace{\bar{C} + CD}) + AC(\underbrace{\bar{B}+B}_{1})$$

$$= AB(\bar{C}+D) + AC(1)$$

$$= \underbrace{AB\bar{C}} + ABD + \underbrace{AC}$$

$$= A(C + \bar{C}B) + ABD$$

$$= A(C+B) + ABD$$

$$= AC + \underbrace{AB + ABD}$$

$$= AC + AB(1+D)$$

$$= AC + AB \checkmark \rightarrow ④ \text{ literals}$$

$$\boxed{\begin{aligned} &= A + \bar{A}B = A+B \\ &(A+\bar{A})(A+B) \\ &(A+B) \checkmark \end{aligned}}$$

$$\boxed{\begin{aligned} &\bar{C} + CD = \bar{C}+D \\ &(\bar{C}+C)(\bar{C}+D) \\ &1.(\bar{C}+D) \end{aligned}}$$

③ $A + \bar{A}B + A\bar{B}$, reduce this Boolean function to minimum no. of literals function to minimum no. of literals

$$\underbrace{A}_{①} + \overline{A}B_{②③} + \underbrace{A\overline{B}}_{④} \rightarrow 4 \text{ literals}$$

$$A(\underbrace{1+\overline{B}}_{1}) + \overline{A}B$$

$$A \cdot 1 + \overline{A}B$$

$$A + \overline{A}B = A + B$$

$$A + \overline{A}B$$

$$= A + B \rightarrow 2 \text{ literals}$$

④ $f = \overline{(\overline{A}\overline{B}) + (\overline{A}) + AB}$, reduce this expression in to minimum no. of literals

$$\overline{AB} = \overline{A} + \overline{B}$$

$$\overline{A+B} = \overline{A} \cdot \overline{B}$$

$$\overline{\overline{A}} = A$$

$$A \cdot A = A$$

$$A \cdot \overline{A} = 0$$

$$f = \overline{\overline{AB}} \cdot \overline{\overline{A}} \cdot \overline{AB}$$

$$= (AB) \cdot A \cdot (\overline{A} + \overline{B})$$

$$= AB \cdot (\overline{A} + \overline{B})$$

$$= \underbrace{AB \cdot \overline{A}}_{0} + \underbrace{AB \cdot \overline{B}}_{0}$$

$$= 0$$

⑤ $f = AB + \overline{AC} + A\overline{B}C(AB + C)$, reduce this Boolean expression in to minimum no. of literals

$$f = AB + \overline{AC} + \underbrace{A\overline{B}C \cdot AB}_{0} + \underbrace{A\overline{B}C \cdot C}$$

$$= AB + \overline{AC} + A\overline{B}C$$

$$= A(B + \overline{B}C) + \overline{AC}$$

$$\boxed{\begin{aligned} \because B \cdot \overline{B} &= 0 \\ A \cdot A &= A \\ C \cdot C &= C \end{aligned}}$$

$$A + \overline{A}B = A + B$$

$$= A\underbrace{(B+\overline{B}C)} + \overline{AC}$$

$$= A(B+C) + \overline{AC}$$

$$= AB + AC + (\overline{A}+\overline{C})$$

$$= AB + AC + \overline{A} + \overline{C}$$

$$= \underbrace{\overline{A} + AB} + \overline{C} + CA$$

$$= \overline{A} + B + \overline{C} + A$$

$$= \underbrace{\overline{A} + A} + B + \overline{C}$$

$$= 1 + B + \overline{C} \Rightarrow 1$$

$$\boxed{\begin{array}{l} A+\overline{A}B = A+B \\ B+\overline{B}C = B+C \\ \overline{AC} = \overline{A}+\overline{C} \\ 1+A = 1 \\ 1+B+\overline{C} = 1 \end{array}}$$

(6) Determine dual of a given Boolean function

$$f = A\underset{\text{AND}}{\cdot}B \underset{\text{OR}}{+} \overline{A}\underset{\text{AND}}{\cdot}\overline{B}$$

$$f = (A\underset{\text{OR}}{+}B)\underset{\text{AND}}{\cdot}(\overline{A}\underset{\text{OR}}{+}\overline{B})$$

$$\underset{\text{AND}}{AB} = A\underset{\text{OR}}{+}B$$

$$\overline{A}\underset{\text{AND}}{\cdot}\overline{B} = \overline{A}\underset{\text{OR}}{+}\overline{B}$$

$$f = AB \underset{\text{OR}}{+} \overline{A}\,\overline{B}$$

↓ Duality → AND

$$f = AB + \overline{A}\cdot\overline{B}$$

↓ Dual of f

$$f = (A+B)\cdot(\overline{A}+\overline{B})$$

$$f = (A+B)\cdot(\bar{A}+B)$$

$$f = (A+B)(\bar{A}+\bar{B})$$

$$= \underbrace{A\cdot\bar{A}}_{0} + A\bar{B} + B\bar{A} + \underbrace{B\cdot\bar{B}}_{0}$$

$$= A\bar{B} + \bar{A}B$$

$f = AB + \bar{A}\bar{B}$ its Dual of f as

$$A\bar{B} + \bar{A}B$$

(6) $f = A\bar{B} + \bar{A}B$, determine the dual of this function

$$f = A\bar{B} + \bar{A}B$$

AND → OR, OR → AND, AND → OR

Dual

$$(A+\bar{B})\cdot(\bar{A}+B)$$

$$\underbrace{A\bar{A}}_{0} + AB + \bar{A}\bar{B} + \underbrace{B\bar{B}}_{0}$$

$$f = AB + \bar{A}\bar{B}$$

Complement of a function

① Take dual

② determine complement : from step 1 ... each literal

② determine Complement

complement each literal

$f = AB + \bar{A}\bar{B}$ determine Complement of f

① dual of f

$f = (A+B)(\bar{A}+\bar{B})$

② complement each literal from step 1

$\bar{f} = (\bar{A}+\bar{B})(A+B)$

$f = \overline{AB + \bar{A}\bar{B}}$

$= \overline{AB} \cdot \overline{\bar{A}\bar{B}}$

$= (\bar{A}+\bar{B}) \cdot (\bar{\bar{A}}+\bar{\bar{B}})$

$= (\bar{A}+\bar{B})(A+B)$