Unit – II –Operations on Multiple Random Variables

In many situations and applications, multiple random variables are required for analysis than a single random variable. Analysis of two random variables is very much needed and it can be extended to multiple random variables.

Joint Probability Distribution Function: Consider two random variables X and Y. And let two events be A$\{X \le x\}$ and B$\{Y \le y\}$ Then the joint probability distribution function for the joint event $\{X \le x, Y \le y\}$ is defined as $F_{X,Y}(x, y) = P\{X \le x, Y \le y\} = P(A \cap B)$

For discrete random variables, if $X = \{x_1, x_2, x_3, \ldots, x_n\}$ and $Y = \{y_1, y_2, y_3, \ldots, y_m\}$ with joint probabilities $P(x_n, y_m) = P\{X = x_n, Y = y_m\}$ then the joint probability distribution function is

$$F_{X,Y}(x,y) = \sum_{n=1}^{N} \sum_{m=1}^{M} P(x_n, y_m) u(x - x_n)\, u(y - y_m)$$

Similarly for N random variables X_n, where n=1, 2, 3 … N the joint distribution function is given as $F_{x_1,x_2,x_3,\ldots x_n}(x_1,x_2,x_3,\ldots x_n) = P\{X_1 \le x_1, X_2 \le x_2, X_3 \le x_3, \ldots\ldots\ldots\ldots X_n \le x_n\}$

Properties of Joint Distribution Functions: The properties of a joint distribution function of two random variables X and Y are given as follows.

(1) $F_{X,Y}(-\infty,-\infty) = 0$

$F_{X,Y}(x,-\infty) = 0$

$F_{X,Y}(-\infty, y) = 0$

(2) $F_{X,Y}(\infty,\infty) = 1$

(3) $0 \le F_{X,Y}(x, y) \le 1$

(4) $F_{X,Y}(x, y)$ is a monotonic non-decreasing function of both x and y.

(5) The probability of the joint event $\{x_1 \le X \le x_2, y_1 \le Y \le y_2\}$ is given by

$P\{x_1 \le X \le x_2, y_1 \le Y \le y_2\} = F_{X,Y}(x_2, y_2) + F_{X,Y}(x_1, y_1) - F_{X,Y}(x_1, y_2) - F_{X,Y}(x_2, y_1)$

(6) The marginal distribution functions are given by $F_{X,Y}(x, \infty) = F_X(x)$ and $F_{X,Y}(\infty, y) = F_Y(y)$.

Joint Probability Density Function: The joint probability density function of two random variables X and Y is defined as the second derivative of the joint distribution function. It can be expressed as

$$f_{X,Y}(x, y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y}$$

It is also simply called as joint density function. For discrete random variables $X = \{x_1, x_2, x_3, \ldots, x_n\}$ and $Y = \{y_1, y_2, y_3, \ldots, y_m\}$ the joint density function is

By direct integration, the joint distribution function can be obtained in terms of density as

$$f_{X,Y}(x,y) = \sum_{n=1}^{N} \sum_{m=1}^{M} P(x_n, y_m)\delta(x - x_n)\, \delta(y - y_m)$$

$$F_{X,Y(x,y)} = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{X,Y(x,y)}\, dx\, dy$$

For N random variables Xn, n=1,2,…N, The joint density function becomes the N-fold partial derivative of the N-dimensional distribution function. That is,

$$f_{X1,\,X2,\,X3\ldots\ldots XN}(X_1,X_2,X_3\ldots Xn) = \frac{\partial^2 F_{X1,X2,X3\ldots XN(x1,X2,X3\ldots\ldots XN)}}{\partial x_1 \partial x_2 \partial x_3 \ldots\ldots \partial x_n}$$

By direct integration the N-Dimensional distribution function is

$$F_{X1,\ X2,\ X3\ldots\ldots XN}(X_1,X_2,X_3\ldots Xn) = \int_{-\infty}^{x1}\int_{-\infty}^{x2}\int_{-\infty}^{x3}\ldots\int_{-\infty}^{xN} f_{X1.X2.X3\ldots XN(x1,x2,x3\ldots\ldots xN)}\, dx_1\ dx_2\ dx_3\ldots\ldots dx_N$$

Properties of Joint Density Function: The properties of a joint density function for two random variables X and Y are given as follows:

(1) $f_{X,Y}(x, y) \geq 0$ A Joint probability density function is always non-negative.

(2) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y(x,y)}\, dx\, dy = 1$ i.e. the area under the density function curve is always equals to one.

(3) *The joint distribution function is always equals to*

$$F_{X,Y(x,y)} = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{X,Y(x,y)}\, dx\, dy$$

(4) The probability of the joint event $\{x1 \leq X \leq x2,\ y1 \leq Y \leq y2\}$ is given as

$$P\{x_1 \leq X \leq x_2,\ y_1 \leq Y \leq y_2\} = \int_{x1}^{x2}\int_{y1}^{y2} f_{X,Y(x,y)}\, dx\, dy$$

(5) The marginal distribution function of X and Y are

$$F_{X(x)} = \int_{-\infty}^{x}\int_{-\infty}^{\infty} f_{X,Y(x,y)}\, dx\, dy$$

$$F_{Y(y)} = \int_{-\infty}^{\infty}\int_{-\infty}^{y} f_{X,Y(x,y)}\, dx\, dy$$

(6) The marginal density functions of X and Y are

$$f_{X(x)} = \int_{-\infty}^{\infty} f_{X,Y(x,y)}\ dy$$

$$f_{Y(y)} = \int_{-\infty}^{\infty} f_{X,Y(x,y)}\ dx$$

Conditional Density and Distribution functions:

Point Conditioning: Consider two random variables X and Y. The distribution of random variable X when the distribution function of a random variable Y is known at some value of y is defined as the conditional distribution function of X. It can be expressed as

$$F_X(x/\,Y=y) = \frac{\int_{-\infty}^{x} f_{X,Y}(x,y)\ dx}{f_Y(y)}$$

and the conditional density function of X is

$$f_X(x/\,Y=y) = \frac{d}{dx}\left[F_X(x/\,Y=y)\right]$$

$$= \frac{\int_{-\infty}^{x} \frac{d}{dx} f_{X,Y}(x,y)}{f_Y(y)}$$

$$f_X(x/\,Y=y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} \text{ or we can simply write } f_X(x/y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

Similarly, the conditional density function of Y is

$$f_Y(y/x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

For discrete random variables, Consider both X and Y are discrete random variables. Then we know that the conditional distribution function of X at a specified value of y_k is given by

$$F_X(x/(y-\Delta y<Y<y+\Delta y)) = \frac{\sum_{j=y-\Delta y}^{y+\Delta y} \sum_{i=1}^{N} P(x_i,y_j)\, u(x-x_i)\, u(y-y_j)}{\sum_{j=y-\Delta y}^{y+\Delta y} P(y_j)\, u(y-y_j)}$$

At $y = y_k$, $\Delta y \rightarrow 0$

$$F_X(x/Y=y_k) = \sum_{i=1}^{N} \frac{p(x_i,y_j)}{p(y_k)}\, u(x-x_i)$$

Then the conditional density function of X is

$$f_X(x/Y=y_k) = \sum_{i=1}^{N} \frac{p(x_i,y_j)}{p(y_k)}\,\delta(x-x_i)$$

Similarly, for random variable Y the conditional distribution function at x = xk is

$$F_Y(y/x_k) = \sum_{i=1}^{N} \frac{p(x_k,y_j)}{p(x_k)}\,u(y-y_j)$$

And conditional density function is

$$f_Y(y/x_k) = \sum_{i=1}^{N} \frac{p(x_k,y_j)}{p(x_k)}\,\delta(y-y_j)$$

Interval Conditioning: Consider the event B is defined in the interval y1 ≤ Y ≤ y2 for the random variable Y i.e. B = { y1 ≤ Y ≤ y2}. Assume that P(B) =P(y1 ≤ Y ≤ y2) 0, then the conditional distribution function of x is given by

$$F_X(x/\ y_1 \le Y \le y_2) = \frac{\int_{y1}^{y2}\int_{-\infty}^{x} f_{X,Y(x,y)}\ dx\,dy}{\int_{y1}^{y2} f_{Y(y)}\,dy}$$

We know that the conditional density function

$$\int_{y1}^{y2} f_{Y(y)}\,dy = \int_{y1}^{y2}\int_{-\infty}^{\infty} f_{X,Y(x,y)}\,dx\,dy$$

$$\text{Or}\quad F_X(x/\ y_1 \le Y \le y_2) = \frac{\int_{y1}^{y2}\int_{-\infty}^{x} f_{X,Y(x,y)}\ dx\,dy}{\int_{y1}^{y2}\int_{-\infty}^{\infty} f_{X,Y(x,y)}\,dx\,dy}$$

By differentiating we can get the conditional density function of X as

$$f_X(x/\ y_1 \le Y \le y_2) = \frac{\int_{y1}^{y2} f_{Y(y)}\,dy}{\int_{y1}^{y2}\int_{-\infty}^{\infty} f_{X,Y(x,y)}\,dx\,dy}$$

Similarly, the conditional density function of Y for the given interval x1 ≤ X ≤ x2 is

$$f_Y(y/(\ x_1 \le X \le x_2)) = \frac{\int_{x1}^{x2} f_{Y(y)}\,dx}{\int_{x1}^{x2}\int_{-\infty}^{\infty} f_{X,Y(x,y)}\,dx\,dy}$$

Statistical Independence of Random Variables: Consider two random variables X and Y with events A= {X≤ x } and B = {Y ≤ y} for two real numbers x and y. The two random variables are said to be statistically independent if and only if the joint probability is equal to the product of the individual probabilities.

P {X≤x ,Y ≤ y} P {X≤x } P {Y ≤ y} Also the joint distribution function is

$$F_{X,Y(x,y)} = F_{X(x)} F_{Y(y)}$$

And the joint density function is

$$f_{X,Y(x,y)} = f_{X(x)} f_{Y(y)}$$

These functions give the condition for two random variables X and Y to be statistically independent. The conditional distribution functions for independent random variables are given by

$$F_X\,(x/\,Y=y) = F_X\,(x/\,y) = \frac{F_{X,Y(x,y)}}{F_{Y(y)}} = \frac{F_{X(x)} F_{Y(y)}}{F_{Y(y)}}$$

Therefore F_X (x/

y) = F_X (x) Also

F_Y (y/ x) = F_Y (y)

Similarly, the conditional density functions for independent random variables are

$$f_X\,(x/\,y) = f_{X(x)}$$

$$f_Y\,(y/\,x) = f_{Y(y)}$$

Hence the conditions on density functions do not affect independent random variables

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Sum of two Random Variables: The summation of multiple random variables has much practical importance when information signals are transmitted through channels in a communication system. The resultant signal available at the receiver is the algebraic sum of the information and the noise signals generated by multiple noise sources. The sum of two independent random variables X and Y available at the receiver is W =X+Y

If F_X (x) and F_Y (y) are the distribution functions of X and Y respectively, then the probability distribution function of W is given as F_W (w) =P {W≤w }= P {X+Y≤w }.Then the distribution function

$$F_{W(w)} = \int_{-\infty}^{\infty} \int_{-\infty}^{x} f_{X,Y^{(x,y)}}\, dx\, dy$$

is

Since X and Y are independent random variables,

$$f_{X,Y}(x,y) = f_X(x) f_Y(y)$$

Therefore

$$F_W(w) = \int_{-\infty}^{\infty} f_Y(y) \int_{-\infty}^{w-y} f_X(x) \, dx \, dy$$

Differentiating using Leibniz rule, the density function is

$$f_W(w) = \frac{dF_W(w)}{dw} = \int_{-\infty}^{\infty} f_Y(y) \frac{d}{dw} \int_{-\infty}^{w-y} f_X(x) \, dx \, dy$$

$$f_W(w) = \int_{-\infty}^{\infty} f_Y(y) \; f_X(w-y) \; dy$$

Similarly it can be written as

$$f_W(w) = \int_{-\infty}^{\infty} f_X(x) \; f_Y(w-x) \; dx$$

This expression is known as the convolution integral. It can be expressed as

$$f_W(w) = f_X(x) * f_Y(y)$$

Hence the density function of the sum of two statistically independent random variables is equal to the convolution of their individual density functions.

Sum of several Random Variables: Consider that there are N statistically independent random variables then the sum of N random variables is given by W=X1+X2+X3+…+XN.

Then the probability density function of W is equal to the convolution of all the individual density functions. This is given as

$$f_W(w) = f_{X_1}(x_1) * f_{X_2}(x_2) * f_{X_3}(x_3) * \ldots * f_{X_N}(x_N)$$

Central Limit Theorem: It states that the probability function of a sum of N independent random variables approaches the Gaussian density function as N tends to infinity. In practice, whenever an observed random variable is known to be a sum of large number of random variables, according to the central limiting theorem, we can assume that this sum is Gaussian random variable.

Equal Functions: Let N random variables have the same distribution and density functions. And Let Y=X1+X2+X3+…+XN. Also let W be normalized random variable

$$W = \frac{Y-\bar{Y}}{\sigma_Y} \text{ Where } Y=\sum_{n=1}^{N} X_n \text{ , } \bar{Y} = \sum_{n=1}^{N} \overline{X_n} \text{ and } {\sigma_Y}^2 = \sum_{n=1}^{N} {\sigma_{X_n}}^2$$

So

$$W = \frac{\sum_{n=1}^{N} X_n - \sum_{n=1}^{N} \overline{X_n}}{\left[\sum_{n=1}^{N} {\sigma_{X_n}}^2\right]^{1/2}}$$

Since all random variables have same distribution

$${\sigma_{X_n}}^2 = {\sigma_X}^2, \quad \left[\sum_{n=1}^{N} {\sigma_{X_n}}^2\right]^{\frac{1}{2}} = \sqrt{{\sigma_X}^2} = \sqrt{N}\ \sigma_X \text{ and } \overline{X_n} = \bar{X}$$

Therefore

$$W = \frac{1}{\sqrt{N}\,\sigma_X} \sum_{n=1}^{N} (X_n - \bar{X})$$

Then W is Gaussian random variable.

Unequal Functions: Let N random variables have probability density functions, with mean and variance. The central limit theorem states that the sum of the random variables W=X1+X2+X3+…+XN have a probability distribution function which approaches a Gaussian distribution as N tends to infinity.

Function of joint random variables: If g(x,y) is a function of two random variables X and Y with joint density function fx,y(x,y) then the expected value of the function g(x,y) is given as

$$\overline{g} = E[g(x,y)]$$

$$\overline{g} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{X,Y(x,y)dx\,dy}$$

Similarly, for N Random variables X1, X2, . . . XN With joint density function fx1,x2, . . . Xn(x1,x2, . . . xn), the expected value of the function g(x1,x2, . . . xn) is given as

$$\overline{g} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(x_1, x_2, \ldots x_N) f_{x_1, x_2, \ldots x_N}(x_1, x_2, \ldots x_N)\, dx_1 dx_2 \ldots dx_N$$

Joint Moments about Origin: The joint moments about the origin for two random variables, X, Y is the expected value of the function g(X,Y) =E(X^n,Y^k) and is denoted as mnk.. Mathematically,

$$m_{nk} = E[X^n\, Y^k] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^n y^k f_{X,Y(x,y)dx\,dy}$$

Where n and k are positive integers. The sum n+k is called the order of the moments. If k=0, then

$$m_{10} = E[X] = \bar{X} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f_{X,Y(x,y)dx\,dy}$$

$$m_{01} = E[Y] = \bar{Y} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f_{X,Y(x,y)dx\,dy}$$

The second order moments are m20= E[X2] ,m02= E[Y2] and m11 = E[XY]

For N random variables X1, X2, . . . XN, the joint moments about the origin is defined as

$$m_{n1,n2,\ldots nN} = E[X_1^{\,n_1}, X_2^{\,n_2}, \ldots X_N^{\,n_N}]$$

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} X_1^{\,n_1}, X_2^{\,n_2}, \ldots X_N^{\,n_N} f_{x_1, x_2, \ldots x_N}(x_1, x_2, \ldots x_N)\, dx_1 dx_2 \ldots dx_N$$

Where n1,n2, . . . nN are all positive integers.

Correlation: Consider the two random variables X and Y, the second order joint moment m11 is called the Correlation of X and Y. I t is denoted as RXY. RXY = m11 = E [XY] =

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_{X,Y^{(x,y)dx\,dy}}$$

For discrete random variables

$$\mathrm{R_{XY}} = \sum_{n=1}^{N}\sum_{m=1}^{M} x_n\ y_m\ P_{XY}{}^{(x_n,y_m)}$$

Properties of Correlation:

1. If two random variables X and Y are statistically independent then X and Y are said to be uncorrelated. That is RXY = E[XY]= E[X] E[Y].

Proof: Consider two random variables, X and Y with joint density function fx,y(x,y)and marginal density functions fx(x) and fy(y). If X and Y are statistically independent, then we know that fx,y(x,y) = fx(x) fy(y).

The correlation is

$$\mathrm{R_{XY}} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_{x,y^{(X,Y)}}\ \mathrm{dx\ dy}.$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_{x^{(X)}}f_{y^{(Y)}}\ \mathrm{dx\ dy}.$$

$$=\int_{-\infty}^{\infty} x\ f_{X^{(X)}}\ \mathrm{dx}\ \ \int_{-\infty}^{\infty} y\ f_{y^{(Y)}}\ \mathrm{dy}.$$

$$\mathrm{R_{XY}} = \mathrm{E[XY]= E[X]\ E[Y]}.$$

2. If the Random variables X and Y are orthogonal then their correlation is zero. i.e. RXY = 0.

Proof: Consider two Random variables X and Y with density functions fx(x) and fy(y). If X and Y are said to be orthogonal, their joint occurrence is zero. That is fx,y(x,y)=0. Therefore the correlation is

$$\mathrm{R_{XY}} = \mathrm{E[XY]} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_{x,y^{(X,Y)}}\ \mathrm{dx\ dy} = 0.$$

Joint central moments: Consider two random variables X and Y. Then the expected values of the function g(x,y)=$(x-\overline{X})^n(y-\overline{Y})^k$ are called joint central moments. Mathematically μ_{nk} =E[$(x-\overline{X})^n(y-\overline{Y})^k$]

= $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x-\overline{X})^n(y-\overline{Y})^k f_{x,y(x,y)}$ dx dy =0. Where n, k are positive integers0,1,2,… The order of the central moment is n+k. The 0th Order central moment is μ_{00} = E[1]=1. The first order central moments are μ_{10} =E[x-$\bar{X}$] =E[$\bar{X}$] -E[$\bar{X}$]=0and μ_{01} =E[y-$\bar{Y}$] =E[$\bar{Y}$] -E[$\bar{Y}$]=0. The second order central moments are

μ_{20} =E[$(x-\overline{X})^2$]=σ_{X^2}, μ_{02} =E[$(y-\overline{Y})^2$]=σ_{Y^2} and μ_{11} =E[$(x-\overline{X})^1(y-\overline{Y})^1$]= σ_{XY}

For N random Variables X_1, X_2, . . . X_N , the joint central moments are defined as $\mu_{n1,n2,...nN} = E[(x_1-\overline{X_1})^{n_1}(x_2-\overline{X_2})^{n_2}\ldots(x_N-\overline{X_N})^{n_N}]$

$=$

$\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}(x_1-\overline{X_1})^{n_1}(x_2-\overline{X_2})^{n_2}\ldots(x_N-$
$XN)\ nNfx1,x2,\ldots xN(x1,x2,\ldots xN)\ dx1dx2\ldots dxN$

The order of the joint central moment n_1+n_2+ . . .+n_N.

Covariance: Consider the random variables X and Y. The second order joint central moment μ_{11} is called the covariance of X and Y. It is expressed as C_{XY}= σ_{XY} = μ_{11}=E[x-$\bar{X}$] E[y-$\bar{Y}$]

C_{XY} = $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x-\overline{X})^1(y-\overline{Y})^1 f_{x,y(x,y)}$ dx dy

For discrete random variables X and Y, C_{XY} = $\sum_{n=1}^{N}\sum_{k=1}^{K}(x_n-\overline{X_n})^1(y_k-\overline{Y_k})^1 P(x_n, y_k)$

Correlation coefficient: For the random variables X and Y, the normalized second order Central moment is called the correlation coefficient It is denoted as ρ and is given by

$$\rho = \frac{\mu_{11}}{\sqrt{\mu_{20}\mu_{02}}} = \frac{C_{XY}}{\sqrt{\sigma_X{}^2\sigma_Y{}^2}} = \frac{C_{XY}}{\sigma_X\sigma_Y} = \frac{\sigma_{XY}}{\sigma_X\sigma_Y} = \frac{E[x-\bar{X}]\,E[y-\bar{Y}]}{\sigma_X\sigma_Y}.$$

Properties of ρ: 1. The range of correlation coefficient is $-1 \le \rho \le 1$.

2. If X and Y are independent then $\rho=0$.

3. If the correlation between X and Y is perfect then $\rho \pm 1$.

4. If X=Y, then $\rho=1$.

Properties of Covariance:

1. If X and Y are two random variables, then the covariance is

2. If two random variables X and Y are independent, then the covariance is zero. i.e. CXY = 0.

$$C_{XY}=R_{XY}-\bar{X}\,\bar{Y}$$

Proof: If X and Y are two random variables, We know that

$$C_{XY} = E[x-\bar{X}]\,E[y-\bar{Y}]$$

$$= E[XY-\bar{X}Y-\bar{Y}X-\bar{X}\bar{Y}]$$

$$= E[XY]-E[\bar{X}Y]-E[\bar{Y}X]-E[\bar{X}\bar{Y}]$$

$$= E[XY]-\bar{X}E[Y]-\bar{Y}E[X]-\bar{X}\bar{Y}E[1]$$

$$= E[XY]-\bar{X}\,\bar{Y}-\bar{Y}\,\bar{X}+\bar{X}\,\bar{Y}$$

$$= E[XY]-\bar{X}\,\bar{Y}$$

But the converse is not true.

Proof: Consider two random variables X and Y. If X and Y are independent, We know that E[XY]=E[X]E[Y] and the covariance of X and Y is

$C_{XY}=R_{XY}-\bar{X}\,\bar{Y}$

$= E[XY]-\bar{X}\,\bar{Y}$

$= E[X]\,E[Y] - \bar{X}\,\bar{Y}$

$= C_{XY}=\bar{X}\,\bar{Y}-\bar{X}\,\bar{Y} = 0.$

3. If X and Y are two random variables, Var(X+Y) = Var(X) + Var(Y) + 2 CXY.

Proof: If X and Y are two random variables, We know that $Var(X)=\sigma_X{}^2 = E[X^2]-E[X]^2$

Then $Var(X+Y) = E[(X+Y)^2]-(E[X+Y])^2$

$= E[X^2+Y^2+2XY]-(E[X] +E[Y])^2$

$= E[X^2]+E[Y^2]+2E[XY]-E[X]^2-E[Y]^2-2E[X]E[Y]$

$= E[X^2]- E[X]^2+E[Y^2] -E[Y]^2+2(E[XY]-E[X]E[Y])$

$= \sigma_X{}^2 +\sigma_Y{}^2+ 2\, C_{XY}.$

Therefore $Var(X+Y) = Var(X) + Var(Y) + 2\, C_{XY}.$ hence proved.

4. If X and Y are two random variables, then the covariance of X+a,Y+b, Where ‘a’and ‘b’ are constants is Cov (X+a,Y+b) = Cov (X,Y) = CXY.

Proof: If X and Y are two random variables, Then

$Cov(X+a,Y+b)=E[((X+a)-(\overline{X+a})\ (Y+b)-\overline{Y+b})]$

$= E[(X+a-\bar{X}-a)(Y+b-\bar{Y}-b)]$

$= E[(X -\bar{X})(Y-\bar{Y})]$

Therefore $Cov (X+a,Y+b) = Cov (X,Y) = C_{XY}.$ hence proved.

5. If X and Y are two random variables, then the covariance of aX,bY, Where 'a'and 'b' are constants is Cov (aX,bY) = abCov (X,Y) = abCXY.

<u>Proof</u>: Proof: If X and Y are two random variables, Then

Cov(aX,bY)=E[((aX)-($\overline{aX}$) (bY-$\overline{bY}$)]

= E[a(X -$\bar{X}$)b(Y-$\bar{Y}$)]

= E[ab(X -$\bar{X}$)(Y-$\bar{Y}$)]

Therefore Cov (aX,bY) = abCov (X,Y) = abC$_{XY}$. hence proved.

6. If X, Y and Z are three random variables, then Cov (X+Y,Z) = Cov (X,Z) + Cov (Y,Z).

<u>Proof</u>: We know that Cov(X+Y,Z)=E[((X+Y)-($\overline{X+Y}$) (Z-$\bar{Z}$)]

= E[X+Y- $\bar{X} - \bar{Y}$ (Z-$\bar{Z}$)]

= E[((X $-\overline{X}$) + (Y $-\overline{Y}$)) (Z-$\bar{Z}$)]
= E[(X $-\overline{X}$)(Z $-\bar{Z}$)] + E[(Y $-\overline{Y}$) (Z-$\bar{Z}$)]

Therefore Cov (X+Y,Z) = Cov (X,Z) + Cov (Y,Z). hence proved.

<u>Joint characteristic Function:</u> The joint characteristic function of two random variables X and Y is defined as the expected value of the joint function g(x,y)=$e^{j\omega 1X}e^{j\omega 2Y}$. It can be expressed as $\square_{X,Y(\omega1,\omega2)}$ = E[$e^{j\omega 1X}e^{j\omega 2Y}$.]= $e^{j\omega 1X+j\omega 2Y}$. Where $\omega1$ *and* $\omega2$ *are* real variables.

Therefore $\square_{X,Y(\omega1,\omega2)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{j\omega 1X+j\omega 2Y} f_{x,y(X,Y)}$ dx dy.

This is known as the two dimensional Fourier transform with signs of $\omega1$ *and* $\omega2$ are reversed for the joint density function. So the inverse Fourier transform of the joint characteristic function gives the joint density function again the signs of $\omega1$ *and* $\omega2$ are reversed. i.e. The joint density function is $f_{x,y(X,Y)}= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \square_{X,Y(\omega1,\omega2)} e^{-(j\omega 1X+j\omega 2Y)}$ dω1 dω2.

Joint Moment Generating Function: the joint moment generating function of two random variables X and Y is defined as the expected value of the joint function g(x,y)=$e^{\vartheta 1X}e^{\vartheta 2Y}$.It can be expressed as

$M_{X,Y(\vartheta 1,\vartheta 2)}$ = E[$e^{\vartheta 1X}e^{\vartheta 2Y}$]= $e^{\vartheta 1X+\vartheta 2Y}$. Where $\vartheta 1$ *and* $\vartheta 2$ *are* real variables.

Therefore $M_{X,Y(\vartheta 1,\vartheta 2)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\vartheta 1X+\vartheta 2Y} f_{x,y(X,Y)}$ dx dy.

And the joint density function is

$f_{x,y(X,Y)}= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} M_{X,Y(\vartheta 1,\vartheta 2)}\, e^{-(\vartheta 1X+\vartheta 2Y)}$ d$\vartheta 1$ d$\vartheta 2$.

Gaussian Random Variables:

(2 Random variables): If two random variables X and Y are said to be jointly Gaussian, then the joint density function is given as

$$f_{x,y(X,Y)}=\frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}\exp\{\frac{-1}{2(1-\rho^2)}[\frac{(x-\overline{X})^2}{\sigma_X{}^2}-\frac{2\rho((X-\bar{X})(Y-\bar{Y})}{\sigma_X\sigma_Y}+\frac{(y-\overline{Y})^2}{\sigma_Y{}^2}]\}$$

This is also called as bivariate Gaussian density function.

N Random variables: Consider N random variables X_n, n=1,2, . . . N. They are said to be jointly Gaussian if their joint density function(N variate density function) is given by

$$f_{x_1,x_2,..x_N}(x_1,x_2,..x_N) = \frac{1}{(2\pi)^{N/2}|C_X|^{1/2}}\ \exp\{\frac{-[X-\bar{X}]^t[C_X]^{-1}[X-\bar{X}]}{2}\}$$

Where the covariance matrix of N random variables is

$$[C_X] = \begin{bmatrix} C_{11} & C_{12}\cdots & C_{1N} \\ C_{21} & C_{22}\ldots & C_{2N} \\ \vdots & & \vdots \\ C_{N1} & C_{N2}\ldots & C_{NN} \end{bmatrix},\ [X-\bar{X}] = \begin{bmatrix} X_1-\bar{X}_1 \\ X_2-\bar{X}_2 \\ \vdots \\ X_N-\bar{X}_{1N} \end{bmatrix}$$

$[X-\bar{X}]^t$ = transpose of $[X-\bar{X}]$

$|C_X|$= determinant of $[C_X]$

And $[[C_X]^{-1}]$ = inverse of $[C_X]$.

The joint density function for two Gaussian random variables X_1 and X_2 can be derived by substituting N=2 in the formula of N Random variables case.

Properties of Gaussian Random Variables:

1. The Gaussian random variables are completely defined by their means, variances and covariances.
2. If the Gaussian random variables are uncorrelated, then they are statistically independent.
3. All marginal density functions derived from N-variate Gaussian density functions are Gaussian.
4. All conditional density functions are also Gaussian.
5. All linear transformations of Gaussian random variables are also Gaussian.

Linear Transformations of Gaussian Random variables: Consider N Gaussian random variables Yn, n=1,2, . . .N. having a linear transformation with set of N Gaussian random variables Xn, n=1,2, . . .N. The linear transformations can be written as

$$\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_N \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12}\cdot\cdot\cdot & a_{1N} \\ a_{21} & a_{22}\cdot\cdot\cdot & a_{2N} \\ \vdots & & \vdots \\ a_{N1} & a_{N2}\cdot\cdot\cdot & a_{NN} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \end{bmatrix}$$

The transformation T is

$$[T] = \begin{bmatrix} a_{11} & a_{12}\cdot\cdot\cdot & a_{1N} \\ a_{21} & a_{22}\cdot\cdot\cdot & a_{2N} \\ \vdots & & \vdots \\ a_{N1} & a_{N2}\cdot\cdot\cdot & a_{NN} \end{bmatrix}$$

Therefore [Y]=[T] [X]. Also with mean values of X and Y. $[Y-\bar{Y}] = [T]\,[X-\bar{X}]$.

And $[X-\bar{X}]=[T]^{-1}\,[Y-\bar{Y}]$.