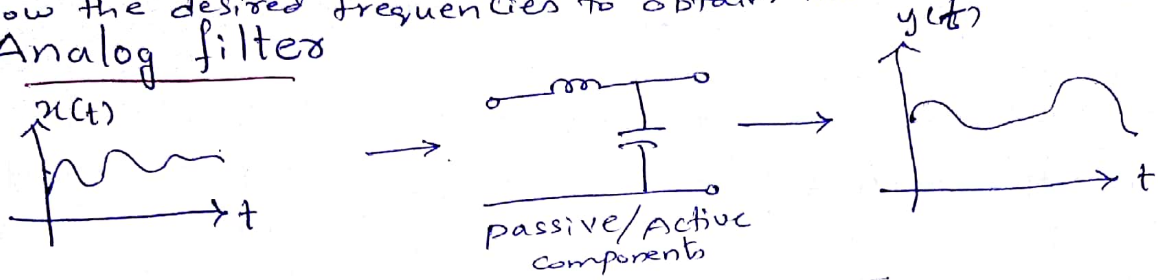
Unit III

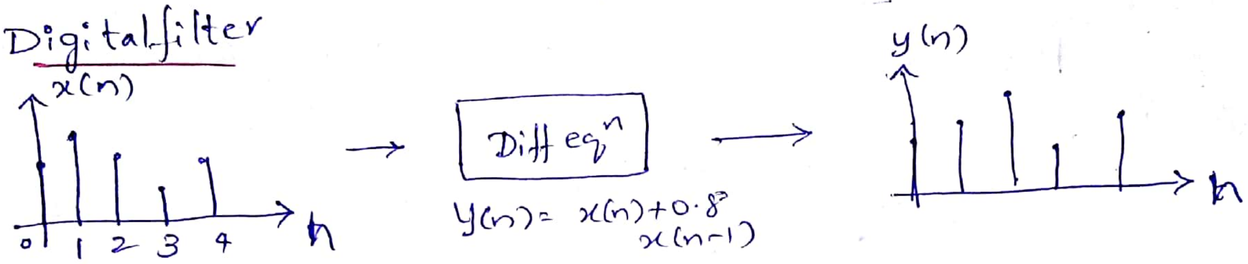
IIR Digital Filters.

A filter rejects unwanted frequencies from the i/p signal & allow the desired frequencies to obtain the required shape of o/p.

Analog filter



Digital filter



Analog filters	Digital filters.
1. It processes analog inputs and generates analog output	1. It processes discrete data and generates discrete data.
2. Combination of active or passive electronic components	2. Combination of digital elements like adder, multiplier, delay unit.
3. It is described by a differential eqn. $\frac{1}{SRC+1}$	3. It is described by a diff eqn.
4. Not flexible	4. Flexible
5. Not easy to port	5. Portable
6. Changes with change in environmental (temp, ageing of components).	6. Negligible effect.
7. Less immune to noise	7. Highly immune to noise
8. No quantization errors	8. Quantization error arises
9. Problem with impedance matching	9. No problem with impedance matching.

The impulse response h(n) for a realizable digital filter is $\boxed{h(n) = 0 \text{ for } n < 0}$ → Causal and for stability it must satisfy the condition

$$\boxed{\sum_{n=0}^{\infty} |h(n)| < \infty}$$

Design of digital filters from analog filters.

1. For the given specifications design an analog filter fun^n $H_a(s)$. Techniques used are i) Butterworth
   ii) Chebyshev — Type-1, Type-2
2. Transform the analog transfer fun^n into digital $H(z)$. Techniques used are i) Impulse Invariant
   ii) Bilinear Transformation

Transformation techniques (Analog to digital filter).

i) Impulse Invariant :-

$$H_a(s) = \sum_{K=1}^{N} \frac{C_K}{s-P_K}$$

P_K are poles of the analog filter
C_K are coefficients in the partial fraction expansion

Steps:
1) Analog Transfer fun^n $H_a(s)$
2) ILT $h(t)$
3) $t=nT$ (discrete)
4) Z-Transform $H(z)$

The inverse laplace transform is

$$h_a(t) = \sum_{K=1}^{N} C_K e^{P_K t} \qquad t \geq 0$$

Sampling $h_a(t)$ periodically at $t=nT$ we get

$$h(n) = h_a(nT)$$

$$= \sum_{K=1}^{N} C_K e^{P_K nT}$$

1) $H_a(s)$
2) $h(t)$
3) $h(n)$
4) $H(z)$

Taking z-transform we get

$$H(z) = \sum_{n=0}^{\infty} \sum_{K=1}^{N} C_K e^{P_K nT} z^{-n}$$

$$= \sum_{K=1}^{N} C_K \sum_{n=0}^{\infty} \left(e^{P_K T} z^{-1}\right)^n$$

$$\boxed{H(z) = \sum_{K=1}^{N} \frac{C_K}{1-e^{P_K T} z^{-1}}}$$

$$\boxed{\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}}$$

Poles of analog filter $s = P_K$

Poles of digital filter $z = e^{P_K T}$

we know $z = r e^{j\omega} = e^{sT} = e^{(\sigma + j\Omega)T}$

$\Rightarrow r = e^{\sigma T}$ & $\omega = \Omega T$

Mapping of points:- s-plane & z-plane.

a) $\sigma = 0$ then $r = 1$ (Imaginary axis of s-plane is mapped onto unit circle of z-plane)

b) $\sigma < 0$, then $0 < r < 1$ (LHS of s-plane mapped onto inside the unit circle)

c) $\sigma > 0$ then $r > 1$ (RHS of s-plane mapped onto outside the unit circle).

$s_1 = \sigma + j\Omega$

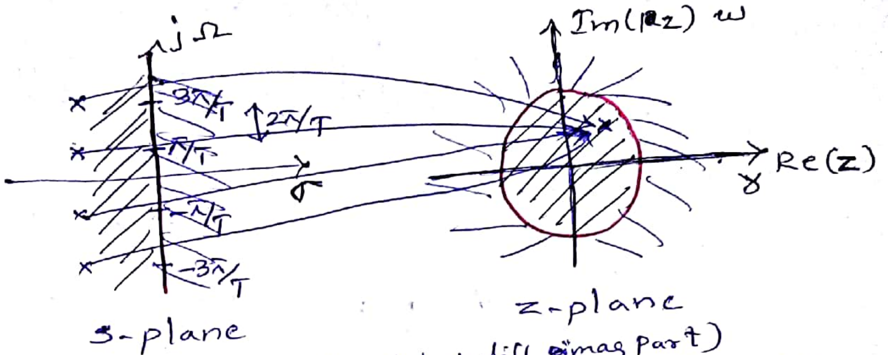
$z_1 = e^{(\sigma + j\Omega)T}$

$s_2 = \sigma + j\left(\Omega + \frac{2\pi}{T}\right)$

$z_2 = e^{\sigma + j\left(\Omega + \frac{2\pi}{T}\right)T}$

$= e^{(\sigma + j\Omega)T} \cdot e^{j2\pi}$

$= e^{(\sigma + j\Omega)T}$



∴ Two poles s_1 & s_2 (having same real part but diff imag part) are mapped to same location in z-plane. Infinite no. of poles having same real part & different imag part differed by multiples of 2π map to same location in z-plane. This leads to aliasing in freq domain. But impulse invariant mapping preserves the stability of the filter.

1. Obtain the IIR digital filter transfer funⁿ using impulse invariant method for the following analog filter transfer funⁿ.

a) $H_a(s) = \dfrac{s + 0.1}{(s+0.1)^2 + 9}$. b) $H_a(s) = \dfrac{1}{(s+2)(s+4)}$

Sol:- a) $H_a(s) = \dfrac{A}{(s + 0.1 + j3)} + \dfrac{B}{s + 0.1 - j3} = \dfrac{1}{2(s+0.1+j3)} + \dfrac{1}{2(s+0.1-j3)}$

Applying IIM

$$H(z) = \frac{1}{2\left(1 - e^{-(0.1+j3)T}z^{-1}\right)} + \frac{1}{2\left(1 - e^{-(0.1-j3)T}z^{-1}\right)}$$

b) $H_a(s) = \dfrac{A}{(s+2)} + \dfrac{B}{(s+4)} = \dfrac{1}{2(s+2)} - \dfrac{1}{2(s+4)}$

$$H(z) = \frac{1}{2(1 - e^{-2T}z^{-1})} - \frac{1}{2(1 - e^{-4T}z^{-1})}.$$

ii) Bilinear Transformation method (BLT):

Consider an analog linear-filter with system funⁿ

$$H(s) = \frac{b}{s+a} \quad ; \quad \frac{Y(s)}{X(s)} = \frac{b}{s+a}$$

$\Rightarrow sY(s) + aY(s) = bX(s)$

Taking inverse laplace transform

$y'(t) + a\,y(t) = bx(t)$. $\longrightarrow$ (1)

Trapezoidal formula $\rightarrow$ $y(t) = \int_0^t y'(\tau)\,d\tau + y(t_0)$.

Replace $t = nT$, $t_0 = nT - T$

$y(nT) = \frac{T}{2}\left[y'(nT) + y'(nT-T)\right] + y(nT-T) \rightarrow$ (2)

$y'(nT) = b(x(nT)) - a\,y(nT)$. $\rightarrow$ (3)

Substitute eqⁿ (3) in eqⁿ (2)

$$y(nT) = \frac{T}{2}\left[b\,x(nT) - a\,y(nT) + b\,x(nT-T) - a\,y(nT-T)\right] + y(nT-T)$$

which implies

$$y(nT) + \frac{aT}{2}y(nT) - \left(1 - \frac{aT}{2}\right)y(nT-T) = \frac{bT}{2}\left[x(nT) + x(nT-T)\right]$$

$$y(n)\left(1 + \frac{aT}{2}\right) - y(n-1)\left(1 - \frac{aT}{2}\right) = \frac{bT}{2}\left[x(n) + x(n-1)\right]$$

z-transform of this difference eqⁿ is

$$\left(1 + \frac{aT}{2}\right)Y(z) - \left(1 - \frac{aT}{2}\right)z^{-1}Y(z) = \frac{bT}{2}\left[1 + z^{-1}\right]X(z).$$

The system funⁿ is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\frac{bT}{2}(1+z^{-1})}{1 + \frac{aT}{2} - \left(1 - \frac{aT}{2}\right)z^{-1}} = \frac{\frac{bT}{2}(1+z^{-1})}{(1-z^{-1}) + \frac{aT}{2}(1+z^{-1})}$$

Dividing & multiplying by $\frac{T}{2}(1+z^{-1})$ we get

$$\Rightarrow H(z) = \frac{b}{a + \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)} \quad \text{where } H(s) = \frac{b}{s+a}$$

$\Rightarrow s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$ This is called bilinear transformation.

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Let $z = re^{j\omega}$ & $s = \sigma + j\Omega$

$$s = \frac{2}{T}\left(\frac{z-1}{z+1}\right)$$

$$= \frac{2}{T}\left(\frac{re^{j\omega}-1}{re^{j\omega}+1}\right) = \frac{2}{T}\left[\frac{r\cos\omega - 1 + jr\sin\omega}{r\cos\omega + 1 + jr\sin\omega}\right]$$

$$= \frac{2}{T}\left[\frac{r\cos\omega - 1 + jr\sin\omega}{r\cos\omega + 1 + jr\sin\omega}\right]\left[\frac{r\cos\omega + 1 - jr\sin\omega}{r\cos\omega + 1 - jr\sin\omega}\right]$$

$$= \frac{2}{T}\left[\frac{r^2\cos^2\omega - 1 + r^2\sin^2\omega + j2r\sin\omega}{(r\cos\omega+1)^2 + r^2\sin^2\omega}\right]$$

$$s = \frac{2}{T}\left[\frac{r^2-1}{1+r^2+2r\cos\omega} + j\frac{2r\sin\omega}{1+r^2+2r\cos\omega}\right]$$

$$\Rightarrow \sigma = \frac{2}{T}\left[\frac{r^2-1}{1+r^2+2r\cos\omega}\right]; \quad \Omega = \frac{2}{T}\left[\frac{2r\sin\omega}{1+r^2+2r\cos\omega}\right]$$

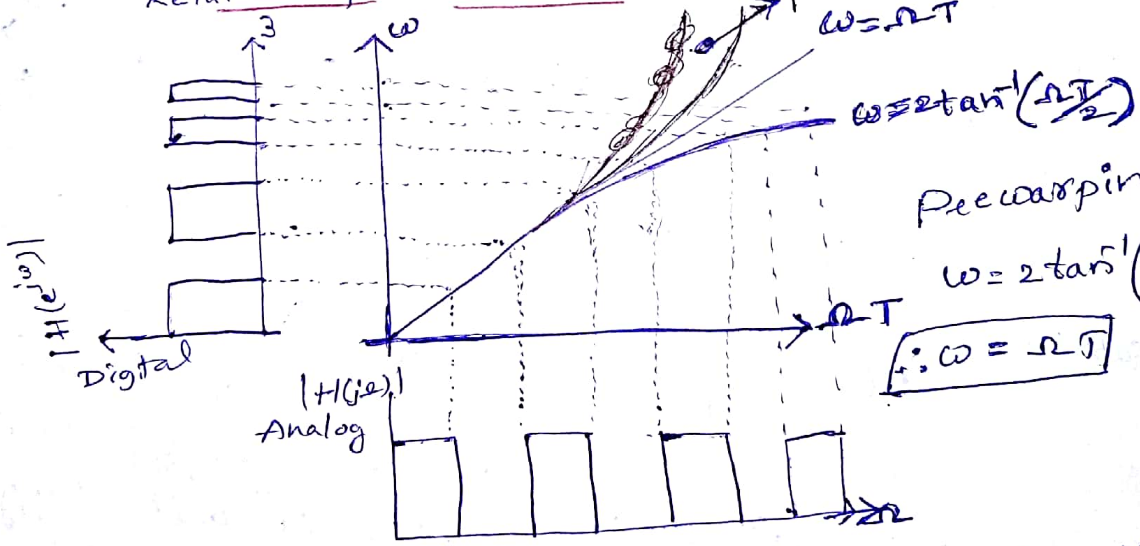
for $r=1$; $\sigma = 0$

1) $r < 1 \Rightarrow \sigma < 0$
2) $r > 1 \Rightarrow \sigma > 0$
3) $r = 1 \Rightarrow \sigma = 0$.

$$\Omega = \frac{2}{T}\left[\frac{\sin\omega}{1+\cos\omega}\right] = \frac{2}{T}\left[\frac{2\sin\frac{\omega}{2}\cos\frac{\omega}{2}}{2\cos^2\frac{\omega}{2}}\right]$$

$$\Omega = \frac{2}{T}\tan\frac{\omega}{2} \Rightarrow \omega = 2\tan^{-1}\left(\frac{\Omega T}{2}\right)$$

Relationship b/w Ω & ω.



Prewarping:
$\omega = 2\tan^{-1}\left(\tan\left(\frac{\Omega T}{2}\right)\right)$
$\boxed{\therefore \omega = \Omega T}$

Effect on mag. response due to warping effect.

→ For low freqs the relationship b/w Ω & ω are linear.
→ For high freqs the " " Ω & ω are non-linear.
Distortion introduced in high freqs is known as warping effect.
→ The no. of passbands are same for both analog & digital filters.
→ But the center freqs and B.W. of high frequencies will tend to reduce disproportionately.

Prewarping: The warping effect can be eliminated by prewarping the analog ~~filter~~ frequencies. ($\Omega = \frac{2}{T}\tan\frac{\omega}{2}$.)

Probm:

2. Compute the digital filter transfer funn for $H_a(s) = \dfrac{3s}{s^2+0.5s+2}$

Assume $T = 1$ sec.

sol:

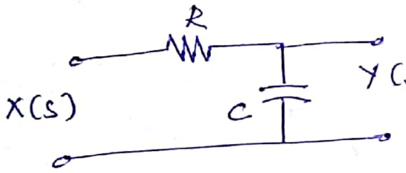
$$s = \frac{2}{T}\left[\frac{z-1}{z+1}\right]$$

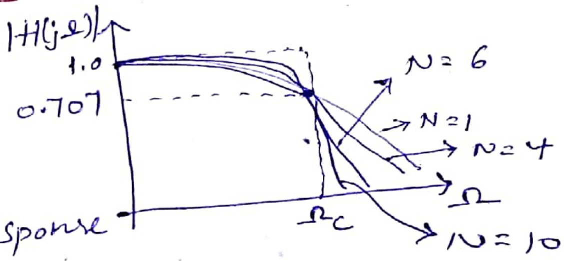
$$H(z) = \frac{3\left(\frac{2}{1}\right)\left(\frac{z-1}{z+1}\right)}{\left(\frac{2(z-1)}{(z+1)}\right)^2 + 0.5\left(\frac{2(z-1)}{(z+1)}\right) + 2} = \frac{6\left(\frac{z-1}{z+1}\right)}{4\left(\frac{z-1}{z+1}\right)^2 + \left(\frac{z-1}{z+1}\right) + 2}$$

$$= \frac{6(z-1)(z+1)}{4(z-1)^2 + (z^2-1) + 2(z^2+1+2z)} = \frac{6(z^2-1)}{7z^2-4z+5}$$

Analog filter design techniques:

1) **Butterworth Low pass filter:** Transfer funn derivation





If the order is high the response resembles ideal values.

$$H(s) = \frac{\frac{1}{sC}}{R+\frac{1}{sC}} = \frac{X_C}{R+X_C} = \frac{1}{sRC+1} = \frac{1}{s+1} \quad (RC=1).$$

$$\Omega_c = \frac{1}{RC}$$

Substitute $s = j\Omega$ ∴

$$H(j\Omega) = \frac{1}{1+j\frac{\Omega}{\Omega_c}} \Rightarrow |H(j\Omega)| = \frac{1}{\sqrt{1+\left(\frac{\Omega}{\Omega_c}\right)^2}} \rightarrow 1^{st} \text{ order}$$

Let $\Omega_c = 1$ rad/sec (Normalization).

$$|H(j\Omega)|^2 = \frac{1}{1+\Omega^2} \quad ; \text{ For } N^{th} \text{ order } |H(j\Omega)|^{2} = \frac{1}{1+(\Omega^2)^N}$$

Substitute $\Omega = \frac{s}{j}$

$$|H(s)|^2 = H(s)\,H(-s) = \frac{1}{1+\left(\frac{s}{j}\right)^{2N}}$$

N - odd : $1 - s^{2N} = 0$

$\Rightarrow s = (1)^{1/2N}$

$\Rightarrow s^{2N} = 1 = e^{j2\pi k}$

$\Rightarrow S_K = e^{j2\pi k/2N}$; $k = 1, 2 \ldots 2N$

N - odd : $1 + (-s^2)^N = 0$ → poles

N - even : $1 + s^{2N} = 0$

$s^{2N} = -1$

$\Rightarrow S_K = (-1)^{\frac{1}{2N}} e^{j\frac{(2k-1)\pi}{2N}}$

$k = 1, 2, \ldots 2N$

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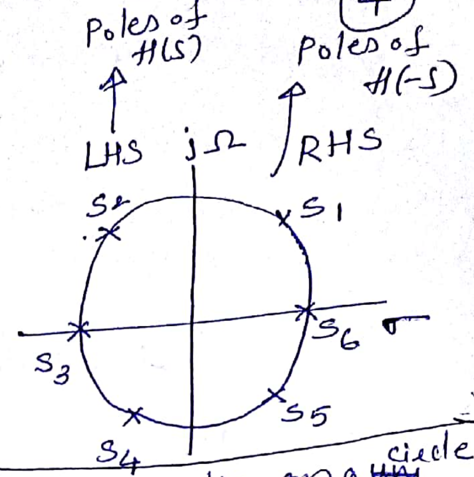
Let us find the poles for N=3.



$S_1 = e^{j\pi/3} = 0.5 + j(0.866)$

$S_2 = e^{j2\pi/3} = -0.5 + j0.866$

$S_3 = e^{j\pi} = -1$

$S_4 = e^{+j\frac{4\pi}{3}} = -0.5 - j0.866$

$S_5 = e^{j5\pi/3} = 0.5 - j0.866$

$S_6 = e^{j2\pi} = 1$

$\boxed{\text{Hence poles lie on a circle for Butterworth filter}}$

To ensure stability, consider only poles that lie on the left half of S-plane.

$\Rightarrow H(s) = \dfrac{1}{(s+1)(s^2+s+1)}$ Transfer funⁿ of 3rd order Butterworth filter whose cutoff freq is $\Omega_c = 1$ rad/sec

To consider only left hand side poles $s_k = e^{j\phi_k}$ where $\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N}$ → N odd, $k = 1, 2 \ldots N$

for $N=2$

$s_1 = e^{j\pi/4}$ } RHS
$s_2 = e^{j3\pi/4}$ } LHS
$s_3 = e^{j5\pi/4}$ } LHS
$s_4 = e^{j7\pi/4}$ } RHS

$\phi_k = \frac{\pi}{2} + \frac{(2k+1)\pi}{2N}$ → N even

Butterworth polynomials

N	Denominator of H(s)
1	$s+1$
2	$s^2+\sqrt{2}s+1$
3	$(s+1)(s^2+s+1)$
4	$(s^2+0.76535s+1)(s^2+1.8477s+1)$

Design steps for digital lowpass Butterworth filter

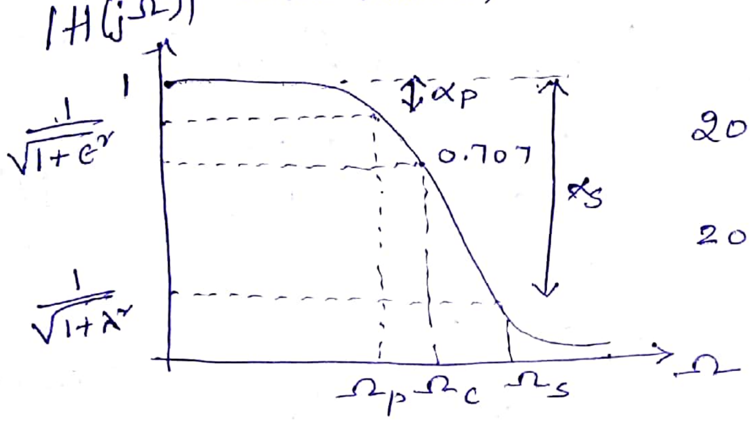
1) Compute the order (N) & cutoff freq Ω_c from analog specifications.
2) Cal. the poles by using $s_K = e^{j\phi_K}$ where $\phi_K = \frac{\pi}{2} + \frac{(2K-1)\pi}{2N}$ → odd, $K = 1, 2, \ldots N$; $\phi_K = \frac{\pi}{2} + \frac{(2K+1)\pi}{2N}$ → N even
3) Compute analog transfer funⁿ H(s) from the poles (Normalized one) i.e. $H(s) = \dfrac{1}{(s-s_1)(s-s_2)(s-s_3)\ldots(s-s_N)}$.
4) Replace 's' with $\frac{s}{\Omega_c}$ to get actual transfer funⁿ $H_a(s)$.

5) Use IIM or BLT methods (transformation) to get $H(z)$.

6) Realize the $H(z)$ using canonical form/direct form (Use realization technique).

Specifications needed to design analog filter:

1. Passband freq. Ω_p
2. Stopband freq. Ω_s
3. Passband ~~ripple factor~~ (attentuation) α_p
4) ~~Pass~~ Stop band ~~ripple factor~~ (attentuation) α_s.



$$20\log\frac{1}{\sqrt{1+\epsilon^2}} = -\alpha_p.$$

$$20\log\frac{1}{\sqrt{1+\lambda^2}} = -\alpha_s.$$

$$\Rightarrow \epsilon = \sqrt{10^{0.1\alpha_p}-1}$$

$$\lambda = \sqrt{10^{0.1\alpha_s}-1}$$

$$|H(j\Omega)| = \frac{1}{\sqrt{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}}} = \frac{1}{\sqrt{1+\epsilon^2\left(\frac{\Omega}{\Omega_p}\right)^{2N}}} = \frac{1}{\sqrt{1+\lambda^2\left(\frac{\Omega}{\Omega_s}\right)^{2N}}}$$

1) To find order N:

$$\sqrt{1+\epsilon^2\left(\frac{\Omega}{\Omega_p}\right)^{2N}} = \sqrt{1+\lambda^2\left(\frac{\Omega}{\Omega_s}\right)^{2N}}$$

$$\Rightarrow \epsilon\left(\frac{\Omega}{\Omega_p}\right)^{N} = \lambda\left(\frac{\Omega}{\Omega_s}\right)^{N}$$

$$\Rightarrow \left(\frac{\Omega_s}{\Omega_p}\right)^{N} = \frac{\lambda}{\epsilon}$$

Apply log on both sides

$$N\log\frac{\Omega_s}{\Omega_p} = \log\frac{\lambda}{\epsilon}$$

$$\Rightarrow N = \frac{\log\left(\sqrt{\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}\right)}{\log\frac{\Omega_s}{\Omega_p}}$$

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2) Calculate cut-off freq Ω_c:

$$\frac{1}{\sqrt{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}}} = \frac{1}{\sqrt{1+\epsilon^2\left(\frac{\Omega}{\Omega_p}\right)^{2N}}}$$

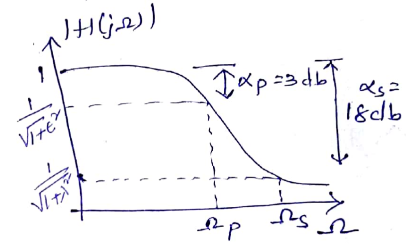
$$\Rightarrow 1+\left(\frac{\Omega}{\Omega_c}\right)^{2N} = 1+\epsilon^2\left(\frac{\Omega}{\Omega_p}\right)^{2N}$$

$$\Rightarrow (\Omega_p)^{N} = \epsilon(\Omega_c)^{N}$$

$$\Rightarrow \Omega_c = \frac{\Omega_p}{\epsilon^{1/N}} = \frac{\Omega_p}{\left(\sqrt{10^{0.1\alpha_p}-1}\right)^{1/N}}.$$

Problem:

3. For the given specifications find the order of analog lowpass butterworth filter $\alpha_p = 3db$, $\alpha_s = 18db$, $f_p = 1KHz$, $f_s = 2KHz$.



sol:- $f_p = 1KHz$

$\Omega_p = 2\pi(1000) = 2000\pi$

$f_s = 2KHz$; $\Omega_s = 4000\pi$

1) Order (N):

$$N = \frac{\log\sqrt{\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\log\left(\frac{\Omega_s}{\Omega_p}\right)} = 2.981 \simeq 3.$$

$\therefore$ Order is 3.

2) Cut-off freq (Ω_c):

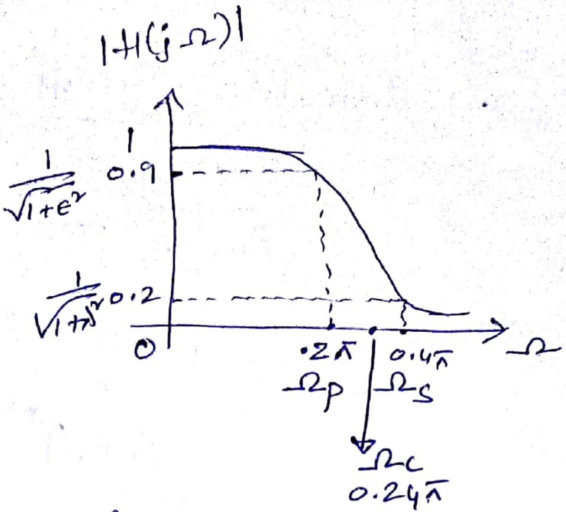
$$\Omega_c = \frac{\Omega_p}{\left(\sqrt{10^{0.1\alpha_p}-1}\right)^{1/N}} = \frac{2000\pi}{(0.997)^{1/3}} = .2000\pi.$$

$$= \frac{\Omega_p}{\epsilon^{1/N}} \quad (\epsilon=1) \simeq \Omega_p$$

4. For the given specifications, design an analog Butterworth filter $0.9 \le |H(j\Omega)| \le 1$ for $0 \le \Omega \le 0.2\pi$

$|H(j\Omega)| \le 0.2$ for $0.4\pi \le \Omega \le \pi$

Sol:- $\Omega_p = 0.2\pi \; ; \; \Omega_s = 0.4\pi$



$$\frac{1}{\sqrt{1+\epsilon^2}} = 0.9 \; ; \; \frac{1}{\sqrt{1+\lambda^2}} = 0.2$$

$$\Rightarrow \epsilon = 0.484 \; ; \; \lambda = 4.898$$

Step 1) Order (N):

$$N = \frac{\log \frac{\lambda}{\epsilon}}{\log \frac{\Omega_s}{\Omega_p}} = \frac{\log\left(\frac{4.898}{0.484}\right)}{\log\left(\frac{0.4\pi}{0.2\pi}\right)} = 3.339 \simeq 4.$$

Step 2) Cal. cut-off freq (Ω_c):

$$\Omega_c = \frac{\Omega_p}{(\epsilon)^{1/N}} = \frac{0.2\pi}{(0.484)^{1/4}} = 0.24\pi$$

Step 3) Find poles :- $\quad s_k = e^{j\phi_k}$

$$\phi_k = \pi/2 + \frac{(2k-1)\pi}{2N} \quad \text{for } k = 1, 2 \ldots N$$

$$s_1 = e^{j(\pi/2 + \pi/8)} = -0.382 + j\,0.923$$

$$s_2 = e^{j(\pi/2 + 3\pi/8)} = -0.9238 + j\,0.382$$

$$s_3 = e^{j(\pi/2 + 5\pi/8)} = -0.9238 - j\,0.382$$

$$s_4 = e^{j(\pi/2 + 7\pi/8)} = -0.382 - j\,0.923.$$

$$\Rightarrow H(s) = \frac{1}{(s-s_1)(s-s_2)(s-s_3)(s-s_4)} = \frac{1}{(s^2 + 1.847s + 1)(s^2 + 0.765s + 1)}$$

Step 4) Replace 's' by $\frac{s}{\Omega_c}$:

$$H_a(s) = \frac{1}{\left(\left(\frac{s}{0.24\pi}\right)^2 + \left(\frac{1.847s}{0.24\pi}\right) + 1\right)\left(\left(\frac{s}{0.24\pi}\right)^2 + 0.76\left(\frac{s}{0.24\pi}\right) + 1\right)}$$

5. Design digital low pass butterworth filter satisfying the constraints $0.707 \le |H(e^{j\omega})| \le 1$ for $0 \le \omega \le \pi/2$

$$|H(e^{j\omega})| \le 0.2 \quad \text{for } \frac{3\pi}{4} \le \omega \le \pi$$

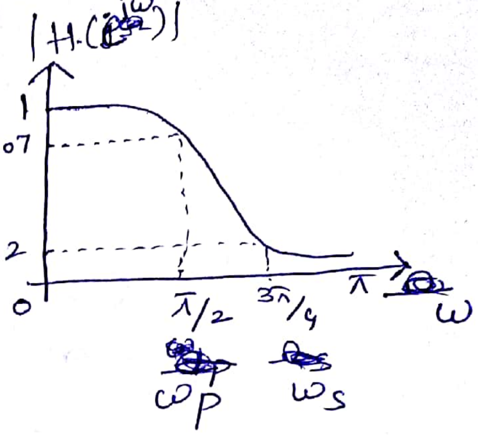
using i) Impulse invariant ii) Bilinear transformation method for T=1sec.

Sol:-

i) $\Omega_p = \pi/2 = \omega_p$

$\Omega_s = 3\pi/4 = \omega_s$

$\boxed{\text{Relationship b/w digital \& analog frequencies } \omega = \Omega T}$



$$\frac{1}{\sqrt{1+\epsilon^2}} = 0.707 \Rightarrow \epsilon = 1 \qquad \frac{1}{\sqrt{1+\lambda^2}} = 0.2 \Rightarrow \lambda = 4.89$$

step 1:- Order (N):

$$N = \frac{\log\left(\frac{\lambda}{\epsilon}\right)}{\log\left(\frac{\Omega_s}{\Omega_p}\right)} = 3.914 \simeq 4$$

step 2: Cut-off freq Ω_c:

$$\Omega_c = \frac{\Omega_p}{\epsilon^{1/N}} = \frac{\pi/2}{(1)^{1/4}} = \pi/2$$

step 3: Poles of the filter:

$$s_k = e^{j\phi_k}$$

$$H(s) = \frac{1}{(s^2+0.76537s+1)(s^2+1.847s+1)}$$

step 4: Replace s by $\frac{s}{\Omega_c}$

$$H_a(s) = \frac{1}{\left(\left(\frac{2s}{\pi}\right)^2 + 0.765\times\left(\frac{2s}{\pi}\right)+1\right)\left(\left(\frac{2s}{\pi}\right)^2+1.847\left(\frac{2s}{\pi}\right)+1\right)}$$

$$= \frac{(\pi/2)^4}{(s^2+1.202s+2.465)(s^2+2.902s+2.465)}$$

$$= \frac{0.7253+j(1.754)}{s-(-1.45-j0.6)} + \frac{0.7253-j(1.754)}{s-(-1.45+j0.6)} + \frac{-0.7253-0.3j}{s-(-0.6-j1.45)} + \frac{-0.7253+0.3j}{s-(0.6+j1.45)}$$

$$H(z) = \frac{0.7253+j(1.754)}{1-0.55e^{-j0.6}z^{-1}}$$

Step 5:- Find $H(z)$:

$$\frac{1}{s-p} \rightarrow \frac{1}{1-e^{PT}z^{-1}}$$

$$\Rightarrow H(z) = \frac{0.7253+j(1.754)}{1-e^{-1.45-j0.6}z^{-1}} + \frac{0.7253-j1.754}{1-e^{-1.45+j0.6}z^{-1}}$$

$$+ \frac{-0.7253-0.3j}{1-e^{-0.6-j1.45}z^{-1}} + \frac{-0.7253+0.3j}{1-e^{0.6+j1.45}z^{-1}}.$$

ii) Using BLT method:

Pre-warp the analog frequencies.

$$\Omega = \frac{2}{T}\tan\frac{\omega}{2}$$

$$\Rightarrow \Omega_p = \frac{2}{T}\tan\frac{\omega_p}{2} = 2\tan\left(\frac{\pi}{2\times 2}\right) = 2 \text{ rad/sec}$$

$$\Omega_s = 2\tan\frac{\omega_s}{2} = 2\tan\left(\frac{3\pi}{4\times 2}\right) = 4.828 \text{ rad/sec.}$$

Step 1: Find order (N) :

$$N = \frac{\log(4.89)}{\log\left(\frac{4.828}{2}\right)} = 1.8 \approx 2$$

step 2: Find cut-off freq (Ω_c) :

$$\Omega_c = \frac{\Omega_p}{\epsilon^{1/N}} = 2.$$

step 3: Find poles.

$$s_k = e^{j\phi_k}$$

$$\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N} \quad ; k = 1, 2 \ldots N$$

$$\Rightarrow s_1 = -0.707 + j0.707 = e^{j(\frac{\pi}{2}+\pi/4)}$$

$$s_2 = -0.707 - j0.707 = e^{j(\pi/2+3\pi/4)}$$

$$\Rightarrow H(s) = \frac{1}{s^2+\sqrt{2}s+1}$$

step 4: Replace 's' with 's/Ω_c'

$$H_a(s) = \frac{1}{\left(\frac{s}{2}\right)^2+\sqrt{2}\frac{s}{2}+1} = \frac{1}{\frac{s^2}{4}+\frac{s}{\sqrt{2}}+1} = \frac{4}{s^2+2.82s+4}$$

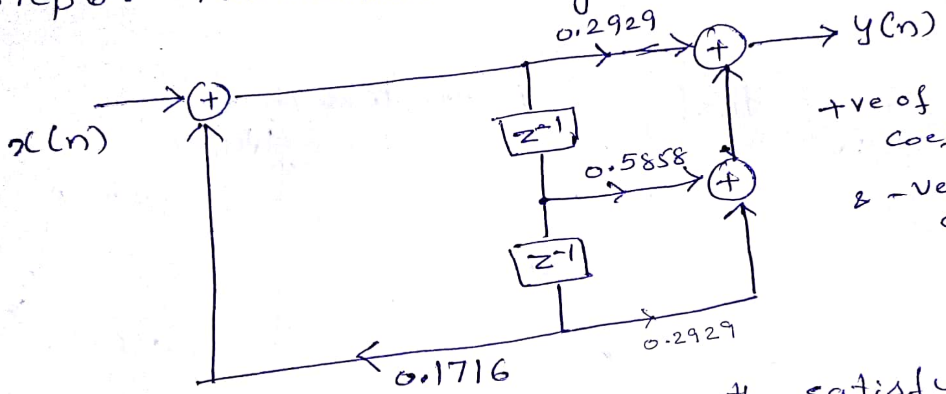
Step 5: Obtain $H(z)$.

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$$\Rightarrow H(z) = \frac{4}{\left(2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right)^2 + 2.82\left(2\times\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right) + 4}$$

$$= \frac{0.2929\,(1+z^{-1})^2}{1+0.1716 z^{-2}} = \frac{0.2929 + 0.5858 z^{-1} + 0.2929 z^{-2}}{1+0.1716 z^{-2}}$$

Step 6: Realization using Direct form - II :



+ve of num coeff & -ve of den coeff.

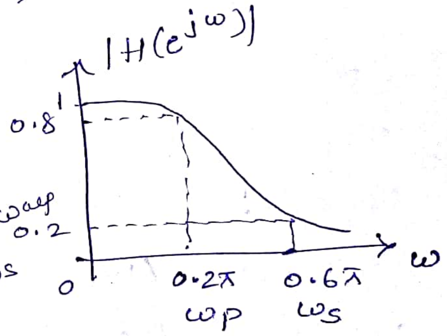
6. Design a digital LPF butterworth satisfying the following specifications: $0.8 \le |H(e^{j\omega})| \le 1$; $0 \le \omega \le 0.2\pi$

$|H(e^{j\omega})| \le 0.2$; $0.6\pi \le \omega \le \pi$

using BLT for $T = 1\,sec$.

Sol:-

Convert digital to analog specification using BLT :



$$\left.\begin{aligned}\Omega_p &= \frac{2}{T}\tan\left(\frac{\omega_p}{2}\right) = 0.649 \\ \Omega_s &= \frac{2}{T}\tan\left(\frac{\omega_s}{2}\right) = 2.752\end{aligned}\right\} \text{Pre-warp freqs}$$

1) Order $N = 2$

2) $\Omega_c = \dfrac{\Omega_p}{\epsilon^{1/N}} = 0.739$

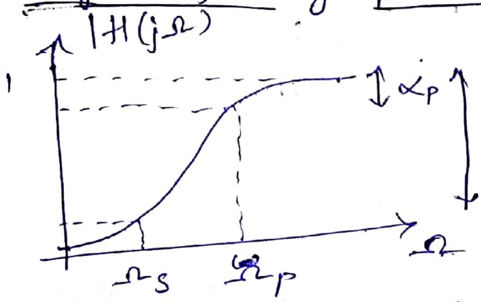
3) $H_a(s) = \dfrac{1}{s^2 + \sqrt{2}s + 1}$

4) $H_a(s)\Big|_{s=\frac{s}{\Omega_c}} = \dfrac{1}{\left(\frac{s}{0.739}\right)^2 + \sqrt{2}\times\frac{s}{0.739} + 1} = \dfrac{0.546}{s^2 + 1.045s + 0.546}$

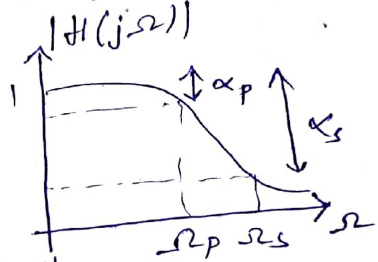
5) $$H(z)\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)} = \frac{0.546}{4\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 1.045\left(2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right) + 0.546}$$

$$= \frac{0.546(1+z^{-1})^2}{4(1-z^{-1})^2 + 2.09(1-z^{-2}) + 0.546(1+z^{-1})^2}$$

Design of High pass Butterworth filter:



$|H(j\Omega)|_{HPF} = 1 - |H(j\Omega)|_{LPF}$



step 1: Exchange edge frequencies.

Step 2: Convert digital freq to analog freqs using transformation technique.

Step 3: Compute Order (N) & Cut-off freq (Ω_c).

Step 4: Find poles to obtain transfer funⁿ $H(s)$

Step 5: Replace 's' by '$\frac{\Omega_c}{s}$' for High pass filter

Step 6: Use transformation technique to obtain $H(z)$.

Step 7: Realize the filter using one of the realization techniques.

7. Design a butterworth digital high pass filter with the following specifications.

$|H(e^{j\omega})| \leq 0.2$; $0 \leq \omega \leq 0.2\pi$

$0.8 \leq |H(e^{j\omega})| \leq 1$; $0.6\pi \leq \omega \leq \pi$ using

Impulse invariant method & $T = 1$ sec.

(8)

Sol:- 1) Exchange edge frequencies:

$\Omega_p = 0.2\pi \qquad \Omega_s = 0.6\pi$

2) Convert digital to analog frequencies using IIM.

$\Rightarrow \omega_p = \Omega_p T = 0.2\pi$ rad/sec

$\omega_s = \Omega_s = 0.6\pi$ rad/sec.

3) Order :

$$N = \frac{\log \frac{\lambda}{\epsilon}}{\log\left(\frac{\Omega_s}{\Omega_p}\right)} = 1.706 \approx 2 \qquad \epsilon = 0.75 \quad \lambda = 4.89$$

Cutt off freq Ω_c: $\Omega_c = \frac{\Omega_p}{\epsilon^{1/N}} = 0.725$ rad/sec.

4) Find poles to obtain $H(s)$:

$$H(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$$

5) Replace 's' with '$\frac{\Omega_c}{s}$':

$$H_a(s) = \frac{1}{\left(\frac{0.725}{s}\right)^2 + \sqrt{2}\left(\frac{0.725}{s}\right) + 1} = \frac{s^2}{(s^2 + 1.025s + 0.526)}$$

Divide num by den.

$$H_a(s) = 1 - \frac{1.025s + 0.526}{s^2 + 1.025s + 0.526} = 1 - \frac{0.5129}{s + (0.513 + j0.513)} + \frac{0.5129}{s + (0.513 - j0.513)}$$

6) Transformation to obtain $H(z)$:

$$H(z) = H_a(s)\Big|_{\frac{1}{s-p} \to \frac{1}{1 - e^{pT}z^{-1}}}$$

$$= 1 - \frac{0.5129}{1 - e^{-0.513 - j0.513}z^{-1}} + \frac{0.5129}{1 - e^{-0.513 + j0.513}z^{-1}}$$

$$= 1 - \frac{1.0258 - 0.535z^{-1}}{1 - 1.643z^{-1} + 0.3538z^{-2}}$$

8. For the given LPF gain $|H_a(j\Omega)|^2 = \frac{1}{1+64\Omega^6}$, Determine the analog system transfer fun?.

Sol:- $|H_a(j\Omega)|^2 = \frac{1}{1+(2\Omega)^6} \cong \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}}$

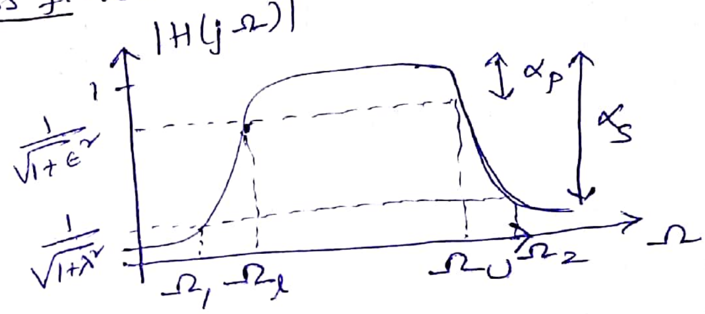
$= \frac{1}{1+\left(\frac{\Omega}{1/2}\right)^{2\times 3}} \Rightarrow N=3$ & $\Omega_c = 0.5$ rad/sec

For $N=3$

$$H(s) = \frac{1}{(s+1)(s^2+s+1)}$$

$$H_a(s)\Big|_{s=\frac{s}{\Omega_c}} = \frac{1}{\left(\frac{s}{0.5}+1\right)\left(\left(\frac{s}{0.5}\right)^2+\left(\frac{s}{0.5}\right)+1\right)} = \frac{0.125}{(s+0.5)(s^2+0.5s+0.25)}$$

Design of Bandpass filter :-



order:

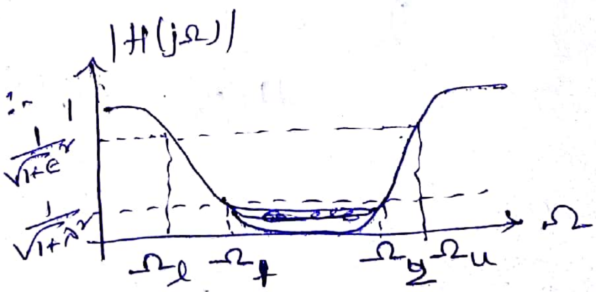
$$N = \frac{\log\left(\frac{\lambda}{\epsilon}\right)}{\log \Omega_r}$$

$\Omega_r = \min\{|A|, |B|\}$

$$A = \frac{-\Omega_1^2 + \Omega_l\Omega_u}{\Omega_1(\Omega_u - \Omega_l)} \quad ; \quad B = \frac{\Omega_2^2 - \Omega_l\Omega_u}{\Omega_2(\Omega_u - \Omega_l)}$$

$$s \rightarrow \frac{s^2 + \Omega_l\Omega_u}{s(\Omega_u - \Omega_l)}$$

Design of Bandstop filter :-



order:- $N = \frac{\log\left(\frac{\lambda}{\epsilon}\right)}{\log \Omega_r}$

$\Omega_r = \min\{|A|, |B|\}$

$$A = \frac{\Omega_1(\Omega_u - \Omega_l)}{-\Omega_1^2 + \Omega_l\Omega_u} \qquad s \rightarrow \frac{s(\Omega_u - \Omega_l)}{s^2 + \Omega_l\Omega_u}.$$

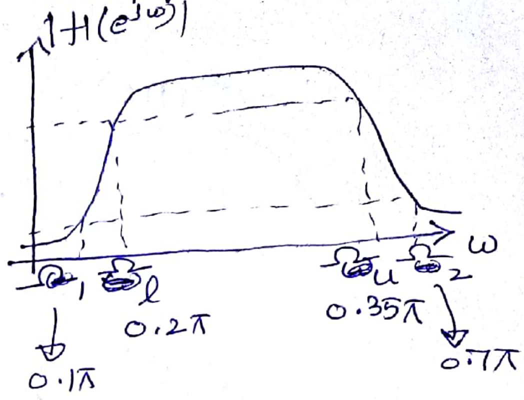
$$B = \frac{\Omega_2(\Omega_u - \Omega_l)}{-\Omega_2^2 + \Omega_l\Omega_u}$$

9. Design a Band pass Butterworth filter for the following specifications: $\alpha_p = -3dB$ $\quad 0.2\pi \leq \omega \leq 0.35\pi$

$\alpha_s = -20dB \quad 0 \leq \omega \leq 0.1\pi$

$0.7\pi \leq \omega \leq \infty$

using Bilinear transformation technique $T = 1 sec$.



Sol:- Step1:- Convert digital to analog frequencies.

$$\Omega_1 = \frac{2}{T}\tan\frac{\omega_1}{2} = 0.3167 \, rad/sec$$

$$\Omega_l = \frac{2}{T}\tan\frac{\omega_l}{2} = 0.6498 \, rad/sec$$

$$\Omega_u = \frac{2}{T}\tan\frac{\omega_u}{2} = 1.225 \, rad/sec$$

$$\Omega_2 = \frac{2}{T}\tan\frac{\omega_2}{2} = 3.925 \, rad/sec$$

step2: Find order (N)

$$N = \frac{\log\sqrt{\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\log \Omega_r}$$

$\Omega_r = \min(|A|, |B|)$.

$$A = \frac{-\Omega_1^2 + \Omega_l\Omega_u}{\Omega_1(\Omega_u - \Omega_l)} = 3.8$$

$$B = \frac{\Omega_2^2 - \Omega_l\Omega_u}{\Omega_2(\Omega_u - \Omega_l)} = 6.462 \; ; \; \Omega_r = \min(|3.8|, |6.4|) = 3.8$$

$$\Rightarrow N = \frac{\log\sqrt{\frac{10^{0.1\times 20}-1}{10^{0.1\times 3}-1}}}{\log(3.8)} = 1.722 \approx 2$$

step3: Find poles to obtain $H(s)$

$$H(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$$

step4: Replace s by $\frac{s^2 + \Omega_l\Omega_u}{s(\Omega_u - \Omega_l)}$

$$\Rightarrow H_a(s) = \frac{1}{\left(\frac{s^2 + 0.64\times 1.225}{s(1.225 - 0.649)}\right)^2 + \sqrt{2}\left(\frac{s^2 + 0.649\times 1.225}{s(1.225 - 0.649)}\right) + 1}$$

$$= \frac{s^2 \times (0.331)}{(s^2 + 0.795)^2 + \sqrt{2}(0.576)(s^2 + 0.795) + s^2(0.331)}$$

$$= \frac{0.331(s^2)}{s^4 + 0.814s^3 + 1.926s^2 + 0.647s + 0.632}$$

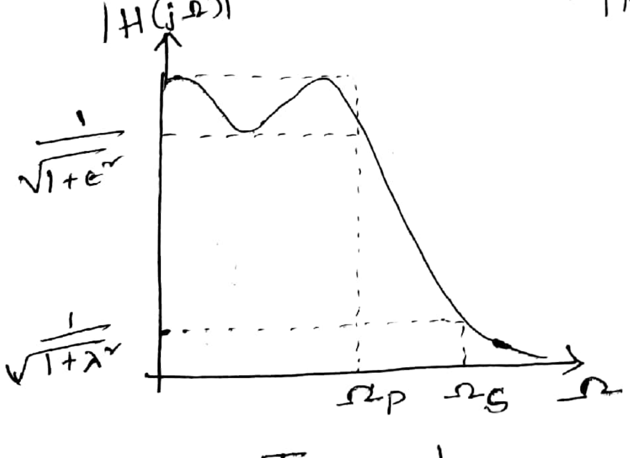
Step5: Replace s by $\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$ to obtain $H(z)$

$$\Rightarrow H(z) = \frac{0.331\left(4\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2\right)}{16\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^4 + 0.814\left(8\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^3\right) + 1.926\left(4\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2\right) + 0.647\left(2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right) + 0.632}$$

10. Design a Butterworth Band stop filter with the following specifications. $\alpha_p = -2dB$, $\alpha_s = -10dB$; $T_s = 1sec$. passband edge frequencies 0.07π & 0.8π & stopband edge frequences 0.2π & 0.3π using IIM.

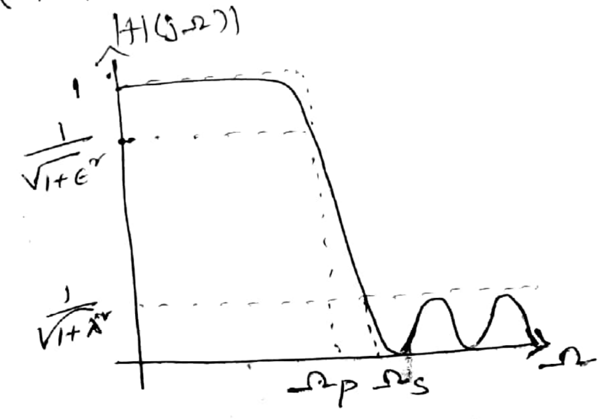
Analog low-pass Chebyshev filter design.

Two types: Type-1 (All pole filter)
Type-2 (Poles & zeros)



Type-1

Equiripple in passband
Monotonic in stopband
Charac's



Type-2

Equiripple in stopband
Montonic in passband.

Differences b/w Butterworth & Chebyshev Type I

Butterworth

1. Order: $N = \dfrac{\log \frac{\lambda}{\epsilon}}{\log\left(\frac{\Omega_s}{\Omega_p}\right)}$

2. Poles: $s_K = e^{j\left(\frac{\pi}{2} + \frac{(2K-1)\pi}{2N}\right)}$

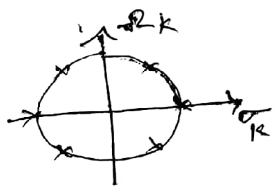
   $K = 1, 2, \ldots N$

3. $H(s) = \dfrac{1}{(s-s_1)(s-s_2)\cdots(s-s_N)}$

4. $|H(j\Omega)| = \dfrac{1}{\sqrt{1+\epsilon^2\left(\frac{\Omega}{\Omega_p}\right)^{2N}}}$

5. Poles form a Circle.

6. N is high



Chebyshev Type-I

1. $N = \dfrac{\cosh^{-1}\left(\frac{\lambda}{\epsilon}\right)}{\cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)}$

2. $s_K = a\cos\phi_K + jb\sin\phi_K$

   $\phi_K = \frac{\pi}{2} + \frac{(2K-1)\pi}{2N}$

   $a = \Omega_p\left(\dfrac{\mu^{1/N} - \mu^{-1/N}}{2}\right)$ (Minor axis)

   $b = \Omega_p\left(\dfrac{\mu^{1/N} + \mu^{-1/N}}{2}\right)$ (Major axis)

   where $\mu = \epsilon^{-1} + \sqrt{1+\epsilon^{-2}}$

3. Denominator of $H(s)$ is obtained using the above poles. Numerator of $H(s)$ depends on N.

   $N \to$ odd substitute $s=0$ in den and find the value.

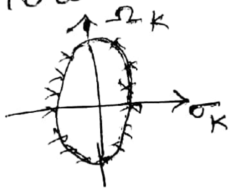
   $N \to$ even substitute $s=0$ in den and divide by $\sqrt{1+\epsilon^2}$.

4. $|H(j\Omega)| = \dfrac{1}{\sqrt{1+\epsilon^2 C_N^2\left(\frac{\Omega}{\Omega_p}\right)}}$.

   $C_N(x) = \cos(N\cos^{-1}x)$; $|x| \le 1$ PB
   $|x| > 1$ SB

5. Poles form an ellipse.

6. N is low



Chebyshev Type-II filter

1. $N = \dfrac{\cosh^{-1}\left(\frac{\lambda}{\epsilon}\right)}{\cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)}$

2. Zeros are located on imaginary axis

   $s_K = j\dfrac{\Omega_s}{\sin\phi_K}$ $K = 1, 2, \ldots N$

3) Poles are located at (x_K, y_K)

   $x_K = \dfrac{\Omega_s \sigma_K}{\sigma_K^2 + \Omega_K^2}$ $K = 1, 2, \ldots N$

   $y_K = \dfrac{\Omega_s \Omega_K}{\sigma_K^2 + \Omega_K^2}$ $K = 1, 2, \ldots N$

   $\sigma_K = a\cos\phi_K$ $\Omega_K = b\sin\phi_K$

   $\mu = \lambda + \sqrt{1+\lambda^2}$

4. $|H(j\Omega)|^2 = \dfrac{1}{1+\epsilon^2\left[\dfrac{C_N^2\left(\frac{\Omega_s}{\Omega_p}\right)}{C_N^2\left(\frac{\Omega_s}{\Omega}\right)}\right]}$

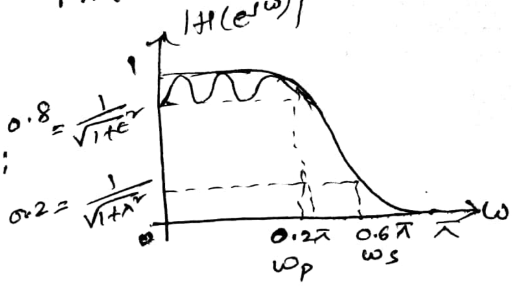
Problem

11. Design a digital low pass chebyshev Type-I filter for the given specifications using BLT method. $0.8 \le |H(e^{j\omega})| \le 1;\ 0 \le \omega \le 0.2\pi$

$|H(e^{j\omega})| \le 0.2;\ 0.6\pi \le \omega \le \pi$

Sol:- $\dfrac{1}{\sqrt{1+\epsilon^2}} = 0.8 \Rightarrow \epsilon = 0.75$

$\dfrac{1}{\sqrt{1+\lambda^2}} = 0.2 \Rightarrow \lambda = 4.89$



1. Digital to analog freq Conversion;

$\Omega_p = 2\tan\frac{\omega_p}{2} = 0.649\ rad/sec$

$\Omega_s = 2\tan\frac{\omega_s}{2} = 2.752\ rad/sec$

2. Order (N):

$N = \dfrac{\cosh^{-1}\left(\frac{\lambda}{\epsilon}\right)}{\cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)} = 1.2 \simeq 2$

3. Poles (S_K): $S_K = a\cos\phi_K + j\,b\sin\phi_K$

$\phi_K = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N}$; $\phi_1 = \frac{3\pi}{4}$ & $\phi_2 = \frac{5\pi}{4}$

$\mu = \epsilon^{-1} + \sqrt{1+\epsilon^{-2}} = 3.$

$a = \Omega_p\dfrac{(\mu^{1/N} - \mu^{-1/N})}{2}$ & $b = \Omega_p\dfrac{(\mu^{1/N} + \mu^{-1/N})}{2} = 0.749$

$\Rightarrow a = 0.394$

$\boxed{S_1 = -0.264 + j0.53\ \&\ S_2 = -0.264 - j0.53}$.

4. To obtain numerator:

$N \to$ even ; sub $s=0$ in den $\frac{}{\sqrt{1+\epsilon^2}}$

$H(s) = \dfrac{1}{(s-(-0.264+j0.53))(s-(-0.264-j0.53))}$

Sub $s=0$

$H(s) = \dfrac{\cancel{1}}{(0.264-j0.53)(0.264+j0.53)} = 0.35$

$\therefore$ Num $= \dfrac{0.35}{\sqrt{1+\epsilon^2}} = \dfrac{0.35}{\sqrt{1+(0.75)^2}} = 0.28.$

$$\Rightarrow H(s) = \frac{0.28}{(s+0.264)^2 + (0.53)^2}$$

5. Replace s with $\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$ to get $H(z)$.

$$\therefore H(z) = \frac{0.28}{\left(2\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 0.264\right)^2 + (0.53)^2}$$

12. Design a chebyshev filter with a max. passband att^n of 2.5dB at $\Omega_p = 20$ rad/sec and the stopband att^n of 30dB at $\Omega_s = 50$ rad/sec.

Sol:- 1) Order: $N = \dfrac{\cosh^{-1}(\lambda/\epsilon)}{\cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)} = 2.72 \approx 3.$

2) Poles: $s_K = a\cos\phi_K + b\sin\phi_K$ $\quad K = 1,2,3.$

$\phi_K = \frac{\pi}{2} + \frac{(2K-1)\pi}{2N}$ $\quad K = 1,2,3.$

$\Rightarrow \phi_1 = 120^\circ \quad \phi_2 = 180^\circ, \ \phi_3 = 240^\circ$

$a = \Omega_p \left[\frac{\mu^{1/N} - \mu^{-1/N}}{2}\right] = 6.6 \qquad \mu = 2.65.$

$b = \Omega_p \left[\frac{\mu^{1/N} + \mu^{-1/N}}{2}\right] = 21.06$

$s_1 = -3.3 + j18.23;\ s_2 = -6.6;\ s_3 = -3.3 - j18.23.$

3) Den of $H(s) = (s+6.6)(s^2 + 6.6s + 343.2)$

3) Num of $H(s)$: sub $s = 0$ in Dem

$\Rightarrow H(0) = 6.6 \times 343.2 = 2265.27$

$$\Rightarrow H(s) = \frac{2265.27}{(s+6.6)(s^2 + 6.6s + 343.2)}$$

Prewarping: The effect of the non-linear compression at high freqs can be compensated by prewarping the critical freqs.

Bilinear transformation:

Advs: 1. It provides one-one mapping
2. No aliasing
3. Stable

Disadv: 1. Compression at high freqs.
2. Neither impulse response nor phase response is preserved.

Butterworth	Chebyshev (Type-I)
1. Mag. response ↓s monotonically as freq increases.	1. It exhibits ripple in the passband and monotonically ↓ing in the stopband.
2. The transition B.W. is more	2. B.W. is less.
3. Poles lies on a circle	3. Poles lie on an ellipse.

4. For given design specifications the no. of poles in Butterworth filter is more when compared to Chebyshev.
i.e. Order is high for Butterworth.

Mind map of Unit III

Design of digital filter
- Analog filter design → Butterworth, Chebyshev (Type-1, Type-2)
- Transformation tech → Impulse invariant (aliasing), Bilinear Transformation (warping)

Specifications to design a filter
- α_p, α_s, ω_p, ω_s.
- Order N, ω_c.
- Poles and find $H(s)$ & $H_a(s)$
- Transformation to digital $H(z)$
- Realize the filter using structures.

Impulse Invariant	Bilinear transformation
1. It is many-to-one mapping	1. It is one-one mapping
2. The relation b/w analog & digital freqs is linear.	2. The relation b/w analog & digital freqs is non-linear.
3. Aliasing effect. To prevent it analog filters are band-limited	3. No aliasing effect. Prewarping effect. The mag. response can be preserved by prewarping analog freqs.

Assignment-3 IIR filters.

1. Discuss the steps in design of IIR filter using bilinear transformation for any one type of filter.

2. Design a digital second order Low-pass Butterworth filter with cut-off freq. 2.2 KHz using Bilinear transformation Sampling rate 8KHz.

3. Design a Butterworth high pass filter satisfying:

   $f_p = 0.32\,Hz$; $\alpha_p = .5\,dB$

   $f_s = 0.16\,Hz$; $\alpha_s = 30\,dB$; $F = 1\,Hz$ (sampling freq).

4. For the analog transfer funⁿ $H(s) = \frac{2}{(s+2)(s+3)}$. Det. $H(z)$ using impulse invariance method. Assume $T = 1\,sec$.

5. Write the design steps to design an analog Chebyshev LPF.

6. Distinguish b/w Butterworth and Chebyshev (Type-1) filters.

7. Why impulse invariant method is not preferred in the design of IIR filter other than lowpass filters?

8. What is warping effect? what is its effect on mag. and phase response? write short note on pre-warping.

9. Design a chebyshev analog LPF that have $-3\,dB$ cut-off freq of 100 rad/sec and a stopband attⁿ of 25 dB or greater for all radian freq's past 250 rad/sec. ~~Plot 20 log |H(jΩ)| for your~~

10. Design a digital Butterworth filter satisfying the constraints.

    $$0.75 \le |H(e^{j\omega})| \le 1 \quad \text{for } 0 \le \omega \le \pi/2$$

    $$|H(e^{j\omega})| \le 0.2 \quad \text{for } \frac{3\pi}{4} \le \omega \le \pi$$

    with $T = 1\,sec$ using impulse invariance method.