(EM) ELECTROMAGNETIC WAVE PROPAGATION

"Maxwell's equation related to EM waves"

"Waves are mean of Transporting energy or Information"
Ex: radio waves, TV signals, radar beams, light rays etc.

Three Fundamental Characteristics:

1) They all travel at high velocity.
2) In travelling, they assume the properties of wave.
3) They radiate outward from a source.

Important Points to Remember

- Medium of wave propagation
  - Dieletric medium
    - **Free space**: $\sigma = 0$, $\varepsilon = \varepsilon_0$, $\mu = \mu_0$
    - **Lossless dielectric**: $\sigma = 0$, $\varepsilon = \varepsilon_0 \varepsilon_r$, $\mu = \mu_0 \mu_r$, $\sigma << \omega\varepsilon$
    - **Lossy dielectric**: $\sigma \neq 0$, $\varepsilon = \varepsilon_0 \varepsilon_r$, $\mu = \mu_0 \mu_r$
  - Conducting medium
    - **Perfect conductor**: $\sigma \cong \infty$, $\varepsilon = \varepsilon_0$, $\mu = \mu_0 \mu_r$, $\sigma >> \omega\varepsilon$

1) Free Space : $(\sigma = 0 \quad ; \varepsilon = \varepsilon_0 \quad ; \mu = \mu_0)$

2) Lossless dielectric : $(\sigma = 0 \quad ; \varepsilon = \varepsilon_0 \varepsilon_r \quad ; \mu = \mu_0 \mu_r)$

or

$(\sigma \ll \omega \varepsilon)$

* 3) Lossy dielectric : $(\sigma \neq 0 \quad ; \varepsilon = \varepsilon_0 \varepsilon_r \quad ; \mu = \mu_0 \mu_r)$

4) Good Conductor : $(\sigma \simeq \infty \quad ; \varepsilon = \varepsilon_0 \quad ; \mu = \mu_0 \mu_r)$

or

$(\sigma \gg \omega \varepsilon)$

Note: i) ω = angular velocity of wave

* ii) Case (3) is the most general case and will be

second-order differential equation in terms of a single variable in three dimensions with time and special domains which can be obtained from Maxwell's equations.

The wave equation can be either source free, called a homogeneous wave equation, or with source terms, called a non-homogeneous wave equation. It should be noted that wave homogeneity is not in any way related to medium homogeneity. In this chapter, we will assume that source generators (applied external sources) do not exist in the medium (i.e., $\rho_v = 0$). So we will consider only homogeneous wave equations throughout our discussion.

We will first develop electromagnetic wave equations in time domain using Maxwell's equations. Later, we will consider plane wave propagation in different media such as conductors and dielectrics and derive wave equations in phasor and time harmonic domains. We will also discuss wave velocity, propagation constant, wave impedance, depth of penetration and polarization. Wave radiation will be explained in the next chapter.

7.2 Wave Equations in a Homogeneous Medium

Consider a medium with constants ε, μ and σ. Let the medium be homogeneous, linear, isotropic and source free. A homogeneous medium is one for which the quantities ε, μ and σ are constant throughout the medium.

A medium is linear with respect to the field if the flux density is proportional to the field intensity. Otherwise, the medium is non-linear. A homogeneous medium is always linear.

A medium is isotropic if the permittivity ε is a scalar quantity so that $\vec{E}$ and $\vec{D}$ are always in the same direction.

A medium is source free if there is no source generator in it.

The field relations are

$$\boxed{\vec{D} = \varepsilon \vec{E}}, \boxed{\vec{B} = \mu \vec{H}}, \text{ and } \boxed{\vec{J} = \sigma \vec{E}}.$$

7.2.1 Wave Equations for a Perfect Dielectric (Free Space)

Consider a wave propagating in a free space medium.

Free space is a perfect dielectric with $\rho_v = 0$, $J = 0$, $\boxed{\sigma = 0}$, $\boxed{\varepsilon = \varepsilon_0}$ and $\boxed{\mu = \mu_0.}$ Maxwell's equations for free space are

$$\boxed{\nabla \times \vec{E} = \frac{-\partial \vec{B}}{\partial t}.} \tag{7.1}$$

$$\boxed{\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t}.} \tag{7.2}$$

$$\nabla \cdot \vec{D} = 0. \tag{7.3}$$

$$\nabla \cdot \vec{B} = 0. \tag{7.4}$$

Differentiating Eq. (7.1) and Eq. (7.2),

$$\boxed{\nabla \times \frac{\partial \vec{E}}{\partial t} = \frac{-\partial^2 \vec{B}}{\partial t^2}}$$

$B = u_0 \vec{H}$

and $$\boxed{\nabla \times \frac{\partial \vec{H}}{\partial t} = \frac{\partial^2 \vec{D}}{\partial t^2}.}$$

But $\boxed{\vec{D} = \varepsilon_0 \vec{E}}$ and $\boxed{\vec{B} = \mu_0 \vec{H}.}$

$$\therefore \quad \left(\nabla \times \frac{\partial \vec{E}}{\partial t}\right) = -\mu_0 \frac{\partial^2 \vec{H}}{\partial t^2} \tag{7.5}$$

$$\text{and} \quad \left(\nabla \times \frac{\partial \vec{H}}{\partial t}\right) = \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}. \tag{7.6}$$

Taking curl on both sides of Eq. (7.1),

$$\nabla \times \nabla \times \vec{E} = -\nabla \times \frac{\partial \vec{B}}{\partial t} = -\mu_0 \left(\nabla \times \frac{\partial \vec{H}}{\partial t}\right).$$

$\boxed{\vec{B} = \mu_0 \vec{H}}$

From Eq. (7.6),

$$\nabla \times \nabla \times \vec{E} = -\mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}.$$

~~A×B×C~~ =

From vector identity, we know that

$$\boxed{\nabla \times \nabla \times \vec{E} = \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E}.}$$

$$\nabla(\cancel{\nabla \cdot \vec{E}}_{0}) - \nabla^2 \vec{E} = -\mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}.$$

$\nabla \cdot \vec{E} \Rightarrow \dfrac{\nabla \cdot D}{\varepsilon_0}$

$\vec{E} = \dfrac{D}{\varepsilon_0}$

But from eq (7.3)

$\boxed{\nabla \cdot D = 0}$

$\nabla \cdot \vec{E} = 0$

Since $$\boxed{\nabla \cdot \vec{E} = \frac{1}{\varepsilon_0} \nabla \cdot \vec{D} = 0,}$$

$$\therefore \quad \boxed{\nabla^2 \vec{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}.}$$

Similarly, $$\nabla \times \nabla \times \vec{H} = \varepsilon_0 \left(\nabla \times \frac{\partial \vec{E}}{\partial t}\right).$$

$$-\nabla^2 \vec{H} = -\mu_0 \varepsilon_0 \frac{\partial^2 \vec{H}}{\partial t^2}$$

or $$\nabla^2 \vec{H} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{H}}{\partial t^2}.$$

$\therefore$ The partial differential equations (7.7)

$$\nabla^2 \vec{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\text{and} \quad \nabla^2 \vec{H} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{H}}{\partial t^2} \tag{7.8}$$

are called wave equations for free space.

The wave velocity in free space is $c = \dfrac{1}{\sqrt{\mu_0 \varepsilon_0}} = 3 \times 10^8$ m/s.

7.2.2 Wave Equations for a Conducting Medium

Consider a wave propagating in a conducting medium ($J \neq 0$, $\rho_v = 0$).

For a conducting medium, Maxwell's equations are

$\vec{B} = \mu \vec{H}$

$\vec{J} = \sigma \vec{E}$

$\vec{D} = \varepsilon \vec{E}$

$$\boxed{\nabla \times \vec{E} = \frac{-\partial \vec{B}}{\partial t} = -\mu \frac{\partial \vec{H}}{\partial t},}$$

$$\text{and} \quad \boxed{\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} = \sigma \vec{E} + \varepsilon \frac{\partial \vec{E}}{\partial t}.} \tag{7.9}$$

Taking partial differentiation,

$$\boxed{\nabla \times \frac{\partial \vec{E}}{\partial t} = -\mu \frac{\partial^2 \vec{H}}{\partial t^2}}$$

$$\text{and} \quad \boxed{\nabla \times \frac{\partial \vec{H}}{\partial t} = \sigma \frac{\partial \vec{E}}{\partial t} + \varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}.} \tag{7.10}$$

Taking curl on both sides of Eq. (7.9),

$$\nabla \times \nabla \times \vec{E} = -\mu \left(\nabla \times \frac{\partial \vec{H}}{\partial t} \right).$$

Laplacian of a Vector

$$\boxed{\nabla^2 \vec{E} = \nabla(\nabla \cdot \vec{E}) - \nabla \times \nabla \times \vec{E}}$$

Substituting Eq. (7.10) in the above equation,

$$\nabla \times \nabla \times \vec{E} = -\mu \left(\sigma \frac{\partial \vec{E}}{\partial t} + \varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} \right). \tag{7.11}$$

$$\boxed{\nabla \times \nabla \times \vec{E} = \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E}}$$

(0)

$\nabla \cdot E = 0$

Since the medium is homogenous,

$$\nabla \cdot \vec{D} = 0, \quad \nabla \cdot \vec{E} = 0 \quad \text{and} \quad \boxed{\nabla \times \nabla \times \vec{E} = -\nabla^2 \vec{E}.}$$

$$\therefore \quad \nabla^2 \vec{E} = \mu \sigma \frac{\partial \vec{E}}{\partial t} + \mu \varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\text{or} \quad \nabla^2 \vec{E} - \mu \varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} - \mu \sigma \frac{\partial \vec{E}}{\partial t} = 0. \tag{7.12}$$

Similarly, for magnetic field intensity, taking curl on both sides of Eq. (7.9),

$$\nabla \times \nabla \times \vec{H} = \sigma(\nabla \times \vec{E}) + \varepsilon \left(\nabla \times \frac{\partial \vec{E}}{\partial t} \right).$$

$$\text{or} \quad \nabla^2 \vec{H} - \mu\,\varepsilon \frac{\partial^2 \vec{H}}{\partial t^2} - \mu\,\sigma \frac{\partial \vec{H}}{\partial t} = 0. \qquad (7.13)$$

Equations Eq. (7.12) and Eq. (7.13) are called wave equations for a conducting medium.

Note: For free space, the wave equations can be obtained by substituting $\sigma = 0$ in Eq. (7.12) and Eq. (7.13).

7.3 Time Harmonic Wave Equations (Phasor Notation) (Lossless Medium)

To obtain wave equations in phasor or time harmonic notation, we start with electric and magnetic fields in phasor form. Assume that the medium is homogeneous, linear, isotropic and source free.

The fields can be written in phasor form as,

$$\vec{E} = |E| e^{j\omega t} \quad \text{and} \quad \vec{H} = |H| e^{j\omega t}.$$

$$\therefore \quad \frac{\partial \vec{E}}{\partial t} = j\omega \vec{E} \quad \text{and} \quad \frac{\partial^2 \vec{E}}{\partial t^2} = -\omega^2 \vec{E}. \qquad (7.14)$$

$$\text{Also,} \quad \frac{\partial \vec{H}}{\partial t} = j\omega \vec{H} \quad \text{and} \quad \frac{\partial^2 \vec{H}}{\partial t^2} = -\omega^2 \vec{E}. \qquad (7.15)$$

where ω rad/s is the wave angular frequency.

We know from Eq. (7.12) that the wave equation for an electric field in the time domain for a conducting medium is

$$\nabla^2 \vec{E} - \mu\,\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} - \mu\,\sigma \frac{\partial \vec{E}}{\partial t} = 0.$$

$$\frac{\partial}{\partial t} = j\omega$$

Substituting Eq. (7.14) in this equation,

$$\nabla^2 \vec{E} - \mu\,\varepsilon(-\omega^2)\vec{E} - \mu\,\sigma(j\omega)\vec{E} = 0.$$

$$\nabla^2 \vec{E} + \omega^2 \mu\,\varepsilon \vec{E} - \mu\,\sigma\, j\omega \vec{E} = 0.$$

$$\nabla^2 \vec{E} - (j\omega\mu\,\sigma - \omega^2 \mu\,\varepsilon)\vec{E} = 0.$$

Similarly, for a magnetic field,

$$\nabla^2 \vec{H} - (j\omega\mu\,\sigma - \omega^2 \mu\,\varepsilon)\vec{H} = 0.$$

Therefore, the time harmonic wave equations for a conducting medium are

$$\nabla^2 \vec{E} - (j\omega\mu\,\sigma - \omega^2 \mu\,\varepsilon)\vec{E} = 0 \qquad (7.16)$$

$$\text{and} \quad \nabla^2 \vec{H} - (j\omega\mu\,\sigma - \omega^2 \mu\,\varepsilon)\vec{H} = 0.$$

Note: For a lossless medium ($\sigma = 0$), the wave equations become

$$\nabla^2 \vec{E} + \omega^2 \mu \varepsilon \vec{E} = 0$$

and $$\nabla^2 \vec{H} + \omega^2 \mu \varepsilon \vec{H} = 0.$$

These equations are called Helmholtz equations.

Q.2 Explain the concept of uniform plane wave. Explain its transverse nature. [JNTU : Marks 3]

Ans. : • In uniform plane waves the electric field vector $\bar{E}$ and magnetic field vector $\bar{H}$. lie in the some plane. Also the different planes along the direction consisting of $\bar{E}$ and $\bar{H}$ vectors are parallel to each other along the direction of propagation of wave.

• A uniform plane wave with field vector $\bar{E}$ and $\bar{H}$ is illustaned in Fig. Q.2.1.

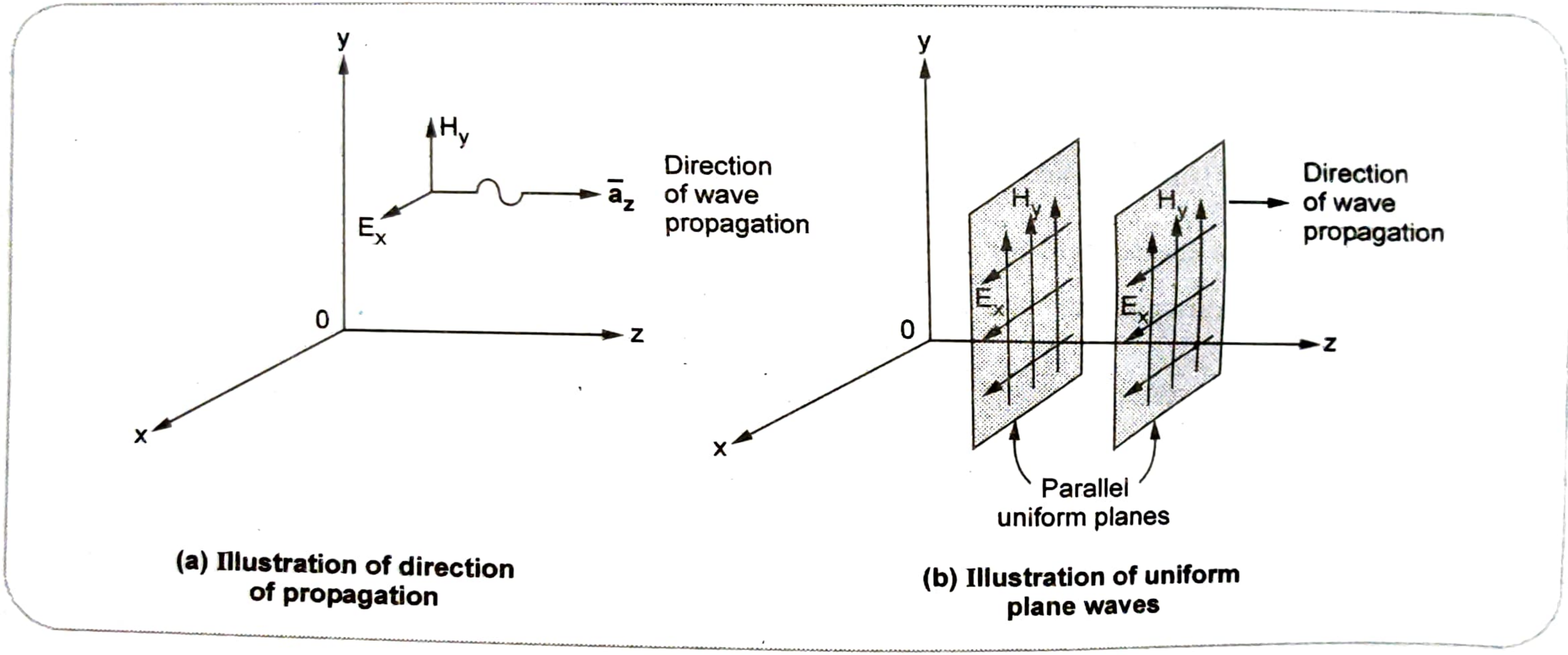

Fig. Q.2.1

• Electric field vector is in $\hat{a}_x$ direction while magnetic field is in $\hat{a}_y$ direction. That means $\bar{E}$ and $\bar{H}$ lie in the x-y plane.

• $\bar{E}$ and $\bar{H}$ are functions of z and + only, As $\bar{E}$ and $\bar{H}$ are are mutually perpendicular to each other the electromagnetic waves are transverse in nature.

7.4 Uniform Plan Wave Propagation

It is very difficult to solve the wave equations without considering the conditions of the medium and wave properties. To simplify the solution, the following set of conditions can be assumed

(1) The fields have a single dimension in space and varies perpendicular to the direction propagation.
(2) Fields are time harmonic, i.e., there are sinusoidal time variations.
(3) The medium is homogeneous, source free and lossless ($\rho_v = 0,\ \sigma = 0$).

These assumptions can be used in many practical cases. A wave which satisfies the above assumptions is called a uniform plane wave.

A uniform plane wave is defined as an electromagnetic wave in which electric and magnetic field intensities are directed in fixed directions in space and constant on infinite planes perpendicular to the direction of propagation.

That is

(1) the plane wave has no electric and magnetic field components along the direction propagation.
(2) E and H fields have constant amplitude and phase on infinite planes perpendicular to the direction propagation.

We cannot generate a uniform plane wave in practice, since it is not possible to keep constant E and H fields on infinite planes with infinite energy. However many practical waves can be approximated as uniform plane waves. For example, the radiation received by a small antenna at far distances can be approximated as a plane wave, since the wave front becomes almost spherical and a very small portion of the sphere acts like a plane at the receiving antenna.

Uniform plane waves have great importance in practical cases. They can also be considered for propagation through an isotropic medium.

7.5 Solution for the Uniform Plane Wave Equation

Consider a uniform plane wave travelling in the z direction in free space. Assume that in Cartesian coordinate system, the electric field vector is oriented along the x direction and the magnetic field vector is oriented along the y direction. The fields are independent in the x and y directions and depend only on z and t.

For the electric field, we know that

$$\nabla^2 \vec{E} = \frac{\partial^2 \vec{E}}{\partial x^2} + \frac{\partial^2 \vec{E}}{\partial y^2} + \frac{\partial^2 \vec{E}}{\partial z^2}.$$

Since for a plane wave, $\dfrac{\partial^2 \vec{E}}{\partial x^2} = \dfrac{\partial^2 \vec{E}}{\partial y^2} = 0,$

$$\nabla^2 \vec{E} = \frac{\partial^2 \vec{E}}{\partial z^2}.$$

$\therefore$ The plane wave equation for a conducting medium is

$$\nabla^2 \vec{E} - (j\omega\mu\sigma - \omega^2\mu\varepsilon)\vec{E} = 0.$$

or
$$\frac{d^2 \vec{E}}{dz^2} - (j\omega\mu\sigma - \omega^2\mu\varepsilon)\vec{E} = 0. \tag{7.18}$$

For free space or lossless medium ($\sigma = 0$ $\mu = \mu_0$ and $\varepsilon = \varepsilon_0$), the plane wave equation becomes

$$\frac{d^2 \vec{E}}{dz^2} + \omega^2\mu_0\varepsilon_0\vec{E} = 0.$$

or
$$\frac{d^2 \vec{E}}{dz^2} + \beta^2\vec{E} = 0, \tag{7.19}$$

where $\beta = \omega\sqrt{\mu\,\varepsilon}$ rad/m is called the phase shift constant.

When the uniform plane wave is propagating in the z direction, the electric field and magnetic field may have components in the x and y directions but not in the z direction.

i.e., $E_z = 0$, and $H_z = 0$.

Let the electric field be in the x direction with component E_x, then the wave equation becomes

$$\frac{d^2 \vec{E}_x}{dz^2} + \beta^2\vec{E}_x = 0. \tag{7.20}$$

This is a second-order differential equation which represents the uniform plane wave propagating in the z direction and the electric field in the x direction.

The general solution for this equation is

$$\vec{E}_x(z) = E^+ e^{-j\beta z} + E^- e^{j\beta z}, \tag{7.21}$$

where E^+ and E^- are complex arbitrary constants which can be determined from the boundary conditions of the media. E^+ represents the electric field component when the wave is propagating in the positive z direction and E^- represents the electric field component when the wave is propagating in the negative z direction. Assume that the initial phase angle of the wave is zero.

In time domain, the wave can be expressed as

$$\vec{E}_x = E_x(z,t) = \Re e[\vec{E}_x(z)e^{j\omega t}], \tag{7.22}$$

where $\Re e$ represents the real value of the function.

$$E_x(z,t) = \Re e\left[\left(E^+ e^{-j\beta z} + E^- e^{j\beta z}\right)e^{j\omega t}\right]$$

$$= \Re e\left[E^+ e^{j(\omega t - \beta z)} + E^- e^{j(\omega t + \beta z)}\right].$$

or $E_x = E_x(z,t) = E^+ \cos(\omega t - \beta z) + E^- \cos(\omega t + \beta z)$ V/m. (7.23)

If the wave propagates in free space without bounds, it travels only in the forward direction. Then the electric field component is

$$\vec{E}_x(z) = E^+ e^{-j\beta z}$$

or $E_x = E^+ \cos(\omega t - \beta z)$. (7.24)

Similarly, for the magnetic field in the y direction

$$H_y = H_y(z,t) = H^+ \cos(\omega t - \beta z) + H^- \cos(\omega t + \beta z) \text{ A/m.} \quad (7.25)$$

If the wave travels only in the forward direction,

$$\vec{H}_y(z) = H^+ e^{-j\beta z}$$

or $H_y = H^+ \cos(\omega t - \beta z)$. (7.26)

Note: Eq. (7.23) and Eq. (7.25) represent a solution for the uniform plane wave equation when the wave is travelling in free space along both the +ve and −ve directions of z. Eq. (7.24) and Eq. (7.26) represent the same but the wave is travelling only in the forward z direction.

Phase velocity The velocity at which the constant phase of the wave travels is called phase velocity.

From Eq. (7.26), the phase of the wave is

$$\omega t - \beta z = \text{constant}$$

divide by β on the Both side. Then

or $z = \dfrac{\omega t}{\beta} - \dfrac{\text{constant}}{\beta}$.

Differentiating, $\dfrac{dz}{dt} = \dfrac{\omega}{\beta}$.

The phase velocity of the wave is

$$v_p = \frac{dz}{dt} = \frac{\omega}{\beta}$$

$$\boxed{\beta = \omega\sqrt{\mu\varepsilon}}$$

or $v_p = \dfrac{\omega}{\beta} = \dfrac{1}{\sqrt{\mu\varepsilon}}$ m/s. (7.27)

Therefore, the phase velocity is material dependent. It depends on permittivity and permeability of the medium.

For free space,

$$\boxed{v_p = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 3 \times 10^8 \text{ m/s.}} \quad (7.28)$$

It is equal to the velocity of light, $\boxed{c = v_p}$

In materials, the phase velocity is, from Eq. (7.27), $\boxed{v_p = \dfrac{c}{\sqrt{\mu_r \varepsilon_r}}}$.

Thus the phase velocity in materials is lower than the velocity of light.

Wave velocity The velocity at which the energy of a wave propagates is called wave velocity. It is the real speed of the wave whereas phase velocity is not real. In unbounded lossless space, the wave velocity is equal to the phase velocity. For bounded medium, they may be different.

Wave length and frequency Wavelength λ is defined as the distance travelled by the sinusoidal wave through a full cycle of 2π radians.

i.e., $\beta\lambda = 2\pi$

or $$\boxed{\lambda = \frac{2\pi}{\beta}}. \tag{7.29}$$

If T is the time period of one full cycle, then the frequency is $\boxed{f = \frac{1}{T} \text{ Hz}}$ and angular frequency is $\boxed{\omega = 2\pi f}$.

Therefore the velocity of wave propagation is $v_p = f\lambda = \dfrac{\omega}{\beta}$ m/s.

$\boxed{V_p = \dfrac{\omega}{\beta} \text{ m/s}}$

1. Derive expression for $\eta_0 = \frac{\bar{E}}{\bar{H}} \Omega$

2. Prove that $\bar{E} . \bar{H} = 0$

3. Prove that $\bar{E} \times \bar{H} = \frac{E^2}{\eta} \bar{a}_z$

Characteristic Impedance or Intrinsic impedance (η_0)

- When an EM wave is propagating in a medium, the ratio between the electric field and magnetic field is a constant, and is called as intrinsic impedance or characteristic impedance or wave impedance, η_0

$$\boxed{\eta_0 = \frac{\bar{E}}{\bar{H}}\,\Omega}$$

- Consider a plane wave propagating in the forward z- direction in lossless medium
- For a uniform plane wave propagating along the z – direction, the conditions are,

$$E_z = 0 \qquad \frac{\partial E_x}{\partial y} = 0 \qquad \frac{\partial E_y}{\partial x} = 0 \qquad \frac{\partial H_x}{\partial y} = 0 \qquad \frac{\partial H_y}{\partial x} = 0$$

$$H_z = 0$$

7.6 Characteristic Impedance

When an EM wave is travelling in a medium, the ratio between the electric field and the magnetic field is a constant. This ratio is called the characteristic impedance or wave impedance or intrinsic impedance of the medium.

It is denoted as

$$\eta = \frac{\vec{E}}{\vec{H}}\ \Omega. \tag{7.30}$$

Relationship between $\vec{E}$ and $\vec{H}$ in a lossless medium Consider a plane wave propagating in the forward z direction in a lossless medium, i.e.,

$$\boxed{J = 0.}$$

For a uniform plane wave along the z direction, the conditions are

$$E_z = 0,\quad H_z = 0,\quad \frac{\partial E_x}{\partial y} = 0,\quad \frac{\partial E_y}{\partial x} = 0,$$

and $$\frac{\partial H_x}{\partial y} = 0,\quad \frac{\partial H_y}{\partial x} = 0. \tag{7.31}$$

We know that the Maxwell's equations for a lossless medium are

$\boxed{J=0}$ $\vec{B} = \varepsilon\vec{E}$

$$\nabla \times \vec{H} = \varepsilon \frac{\partial \vec{E}}{\partial t} \qquad \nabla\times\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} = 0 + \varepsilon\frac{\partial \vec{E}}{\partial t} \tag{7.32}$$

$\vec{B} = \mu\vec{H}$

and $$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}. \qquad \nabla\times\vec{E} = -\frac{\partial B}{\partial t} = -\mu\frac{\partial \vec{H}}{\partial t} \tag{7.33}$$

Expanding Eq. (7.31) in Cartesian coordinates,

$\boxed{E_z = 0}$ $\boxed{H_z = 0}$

$$\begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} = \varepsilon\left[\frac{\partial E_x}{\partial t}\vec{a}_x + \frac{\partial E_y}{\partial t}\vec{a}_y + \frac{\partial E_z}{\partial t}\vec{a}_z\right].$$

$$a_x\left(\overset{(0)}{\frac{\partial H_z}{\partial y}} - \frac{\partial H_y}{\partial z}\right) - \hat{a}_y\left(\overset{(0)}{\frac{\partial H_z}{\partial x}} - \frac{\partial H_x}{\partial z}\right)$$

Substituting the conditions from Eq. (7.30),

$$\vec{a}_x\left(\frac{-\partial H_y}{\partial z}\right) - \vec{a}_y\left(\frac{-\partial H_x}{\partial z}\right) = \varepsilon\frac{\partial E_x}{\partial t}\vec{a}_x + \varepsilon\frac{\partial E_y}{\partial t}\vec{a}_y.$$

$$\frac{-\partial H_y}{\partial z}\vec{a}_x + \frac{\partial H_x}{\partial z}\vec{a}_y = \varepsilon\frac{\partial E_x}{\partial t}\vec{a}_x + \varepsilon\frac{\partial E_y}{\partial t}\vec{a}_y. \tag{7.34}$$

wave along z direction

Similarly, expanding Eq. (7.32) in Cartesian coordinates,

$$\frac{-\partial E_y}{\partial z}\vec{a}_x + \frac{\partial E_x}{\partial z}\vec{a}_y = -\mu\frac{\partial H_x}{\partial t}\vec{a}_x - \mu\frac{\partial H_y}{\partial t}\vec{a}_y.$$

- We know that the Maxwell's equations for a lossless medium are

$$\nabla \times \bar{H} = \varepsilon \frac{\partial \bar{E}}{\partial t} \longrightarrow (1)$$

$$\nabla \times \bar{E} = -\mu \frac{\partial \bar{H}}{\partial t} \longrightarrow (2)$$

- Expanding equation (1), in cartesian co-ordinates, we get,

$$\begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} = \varepsilon \left[\frac{\partial E_x}{\partial t} \bar{a}_x + \frac{\partial E_y}{\partial t} \bar{a}_y + \frac{\partial E_z}{\partial t} \bar{a}_z \right]$$

$$\bar{E} = E_x \bar{a}_x + E_y \bar{a}_y + E_z \bar{a}_z$$

$$\begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} = \varepsilon \left[\frac{\partial E_x}{\partial t} \bar{a}_x + \frac{\partial E_y}{\partial t} \bar{a}_y + \frac{\partial E_z}{\partial t} \bar{a}_z \right]$$

$$\bar{a}_x \left[\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \right] - \bar{a}_y \left[\frac{\partial H_z}{\partial x} - \frac{\partial H_x}{\partial z} \right] + \bar{a}_z \left[\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right] = \varepsilon \left[\frac{\partial E_x}{\partial t} \bar{a}_x + \frac{\partial E_y}{\partial t} \bar{a}_y + \frac{\partial E_z}{\partial t} \bar{a}_z \right]$$

- Substituting the conditions, we get,

$$\bar{a}_x \left[\cancelto{0}{\frac{\partial (0)}{\partial y}} - \frac{\partial H_y}{\partial z} \right] - \bar{a}_y \left[\cancelto{0}{\frac{\partial (0)}{\partial x}} - \frac{\partial H_x}{\partial z} \right] + \bar{a}_z \left[\cancelto{0}{\frac{\partial H_y}{\partial x}} - \cancelto{0}{\frac{\partial H_x}{\partial y}} \right] = \varepsilon \left[\frac{\partial E_x}{\partial t} \bar{a}_x + \frac{\partial E_y}{\partial t} \bar{a}_y + \cancelto{0}{\frac{\partial (0)}{\partial t}} \bar{a}_z \right]$$

$$\bar{a}_x\left(-\frac{\partial H_y}{\partial z}\right)-\bar{a}_y\left(-\frac{\partial H_x}{\partial z}\right)=\varepsilon\left[\frac{\partial E_x}{\partial t}\bar{a}_x+\frac{\partial E_y}{\partial t}\bar{a}_y\right]$$

$$-\frac{\partial H_y}{\partial z}\bar{a}_x+\frac{\partial H_x}{\partial z}\bar{a}_y=\varepsilon\left[\frac{\partial E_x}{\partial t}\bar{a}_x+\frac{\partial E_y}{\partial t}\bar{a}_y\right] \longrightarrow (3)$$

- Similarly, Expanding equation (2), we get,

$$\begin{vmatrix}\bar{a}_x & \bar{a}_y & \bar{a}_z\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ E_x & E_y & E_z\end{vmatrix}=-\mu\left[\frac{\partial H_x}{\partial t}\bar{a}_x+\frac{\partial H_y}{\partial t}\bar{a}_y+\frac{\partial H_z}{\partial t}\bar{a}_z\right]$$

$$\bar{a}_x\left[\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}\right]-\bar{a}_y\left[\frac{\partial E_z}{\partial x}-\frac{\partial E_x}{\partial z}\right]+\bar{a}_z\left[\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}\right]=-\mu\left[\frac{\partial H_x}{\partial t}\bar{a}_x+\frac{\partial H_y}{\partial t}\bar{a}_y+\frac{\partial H_z}{\partial t}\bar{a}_z\right]$$

- Substituting the conditions, we get,

$$\bar{a}_x\left[\cancel{\frac{\partial(0)}{\partial y}}^{0}-\frac{\partial E_y}{\partial z}\right]-\bar{a}_y\left[\cancel{\frac{\partial(0)}{\partial x}}^{0}-\frac{\partial E_x}{\partial z}\right]+\bar{a}_z\left[\cancel{\frac{\partial E_y}{\partial x}}^{0}-\cancel{\frac{\partial E_x}{\partial y}}^{0}\right]=-\mu\left[\frac{\partial H_x}{\partial t}\bar{a}_x+\frac{\partial H_y}{\partial t}\bar{a}_y+\cancel{\frac{\partial(0)}{\partial t}}^{0}\bar{a}_z\right]$$

$$\bar{a}_x\left(-\frac{\partial E_y}{\partial z}\right)-\bar{a}_y\left(-\frac{\partial E_x}{\partial z}\right)=-\mu\left[\frac{\partial H_x}{\partial t}\bar{a}_x+\frac{\partial H_y}{\partial t}\bar{a}_y\right]$$

$$-\frac{\partial E_y}{\partial z}\bar{a}_x+\frac{\partial E_x}{\partial z}\bar{a}_y=-\mu\left[\frac{\partial H_x}{\partial t}\bar{a}_x+\frac{\partial H_y}{\partial t}\bar{a}_y\right]$$

$$-\frac{\partial E_y}{\partial z}\bar{a}_x+\frac{\partial E_x}{\partial z}\bar{a}_y=-\mu\frac{\partial H_x}{\partial t}\bar{a}_x-\mu\frac{\partial H_y}{\partial t}\bar{a}_y \longrightarrow (4)$$

From equations (3), and (4); equating with respective unit vector components , with phasor notations, we get

$$-\frac{\partial H_y}{\partial z} = \varepsilon \frac{\partial E_x}{\partial t} = j\omega\varepsilon E_x$$

$$\frac{\partial}{\partial t} = j\omega$$

$$\frac{\partial H_x}{\partial z} = \varepsilon \frac{\partial E_y}{\partial t} = j\omega\varepsilon E_y$$

$$\frac{\partial E_y}{\partial z} = \mu \frac{\partial H_x}{\partial t} = j\omega\mu H_x$$

$$\frac{\partial E_x}{\partial z} = -\mu \frac{\partial H_y}{\partial t} = -j\omega\mu H_y \longrightarrow (5)$$

- Since, the wave is propagating in the forward z-direction, the electric field component is,

$$E_x = E_x(z) = E^+ e^{-j\beta z}$$

$$\frac{\partial E_x}{\partial z} = (-j\beta)(E^+ e^{-j\beta z})$$

$$\frac{\partial E_x}{\partial z} = -j\beta E_x$$

- From equation (5), $\frac{\partial E_x}{\partial z} = -\mu \frac{\partial H_y}{\partial t} = -j\omega\mu H_y$

$$\frac{\partial E_x}{\partial z} = -j\beta E_x = -j\omega\mu H_y$$

$$-j\beta E_x = -j\omega\mu H_y$$

$$\beta E_x = \omega\mu H_y$$

$$\frac{E_x}{H_y} = \frac{\omega\mu}{\beta}$$

$$\frac{E_x}{H_y} = \frac{\omega\mu}{\omega\sqrt{\mu\varepsilon}}$$

Since, $\beta = \omega\sqrt{\mu\varepsilon}$

$$\frac{E_x}{H_y} = \sqrt{\frac{\mu}{\varepsilon}}$$

- Since, the wave is propagating in the forward z-direction, the electric field component is,

$$E_x = E_x(z) = E^+ e^{-j\beta z}$$

$$\frac{\partial E_x}{\partial z} = (-j\beta)(E^+ e^{-j\beta z})$$

$$\frac{\partial E_x}{\partial z} = -j\beta E_x$$

- From equation (5), $\frac{\partial E_x}{\partial z} = -\mu \frac{\partial H_y}{\partial t} = -j\omega\mu H_y$

$$\frac{\partial E_x}{\partial z} = -j\beta E_x = -j\omega\mu H_y$$

$$-j\beta E_x = -j\omega\mu H_y$$

$$\beta E_x = \omega\mu H_y$$

$$\frac{E_x}{H_y} = \frac{\omega\mu}{\beta}$$

$$\frac{E_x}{H_y} = \frac{\omega\mu}{\omega\sqrt{\mu\varepsilon}}$$

Since, $\beta = \omega\sqrt{\mu\varepsilon}$

$$\frac{E_x}{H_y} = \sqrt{\frac{\mu}{\varepsilon}}$$

$$and, \frac{E_y}{H_x} = -\sqrt{\frac{\mu}{\varepsilon}}$$

Equating the respective unit vector components with phasor notation, we get

$$\frac{-\partial H_y}{\partial z} = \varepsilon \frac{\partial E_x}{\partial t} = j\omega\varepsilon\, E_x,$$

$$\frac{\partial}{\partial t} = j\omega$$

$$\frac{\partial H_x}{\partial z} = \varepsilon \frac{\partial E_y}{\partial t} = j\omega\varepsilon E_y,$$

$$\frac{\partial E_y}{\partial z} = \mu \frac{\partial H_x}{\partial t} = j\omega\mu H_x,$$

and
$$\boxed{\frac{\partial E_x}{\partial z} = -\mu \frac{\partial H_y}{\partial t} = -j\omega\mu H_y.} \tag{7.35}$$

Since the wave is propagating in the forward z direction, the electric field component is

$$E_x = E_x(z) = E^+ e^{-j\beta z}$$

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BOTH

or
$$\frac{\partial E_x}{\partial z} = -j\beta E_x.$$

From Eq. (7.35),
$$\boxed{\frac{\partial E_x}{\partial z}} = -j\beta E_x = \boxed{-j\omega\mu H_y.}$$

$$\Rightarrow -j\beta E_x = -j\omega\mu H_y$$

Then
$$\frac{E_x}{H_y} = \frac{\omega\mu}{\beta} = \frac{\omega\mu}{\omega\sqrt{\varepsilon\mu}}$$

$$\Rightarrow \frac{E_x}{H_y} = \frac{\omega\mu}{\beta}$$

or
$$\frac{E_x}{H_y} = \sqrt{\frac{\mu}{\varepsilon}}.$$

$$\boxed{\beta = \omega\sqrt{\varepsilon\mu}}$$

Similarly, we can show that
$$\frac{E_y}{H_x} = -\sqrt{\frac{\mu}{\varepsilon}}. \tag{7.36}$$

The total electric and magnetic fields of the wave propagating in the z direction can be written as

$$E = \sqrt{E_x^2 + E_y^2}.$$

$$H = \sqrt{H_x^2 + H_y^2}.$$

The ratio is
$$\frac{E}{H} = \frac{\sqrt{E_x^2 + E_y^2}}{\sqrt{H_x^2 + H_y^2}} = \sqrt{\frac{\mu}{\varepsilon}}.$$

η

Therefore, the magnitude of the intrinsic impedance of the lossless medium is

$$\eta = \frac{E}{H} = \sqrt{\frac{\mu}{\varepsilon}} \text{ ohms.} \tag{7.37}$$

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_0 \mu_r}{\varepsilon_0 \varepsilon_r}}$$

For free space, μ_r=1, ε_r=1, $\mu_0 = 4\pi \times 10^{-7}$, $\varepsilon_0 = 8.854 \times 10^{-12}$, $\eta \rightarrow \eta_0$

$$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = \sqrt{\frac{4\pi \times 10^{-7}}{8.854 \times 10^{-12}}}$$

$$\boxed{\eta_0 = 120\pi\Omega = 377\Omega}$$

Thus the relationship between E and H depends only on medium properties.

Also, $\frac{E_x}{H_y} = \eta$ and $\frac{-E_y}{H_x} = \eta$. (7.38)

For free space:

The intrinsic impedance for free space is

$$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}$$

$$= \sqrt{4\pi \times 10^7 \times 36\pi 10^9} = 377\Omega = 120\pi\Omega.$$

$$\therefore \quad \eta_0 = 377\Omega. \tag{7.39}$$

Note 1: The dot product of E and H is always zero.

That is, $\vec{E} \cdot \vec{H} = (E_x \vec{a}_x + E_y \vec{a}_y)(H_x \vec{a}_x + H_y \vec{a}_y)$

$$= E_x H_x + E_y H_y.$$

From Eq. (7.37),

$$\vec{E} \cdot \vec{H} = \eta H_y H_x - \eta H_x H_y = 0. \tag{7.40}$$

Thus, E and H are always perpendicular to each other.

Note 2: The cross product of E and H fields represents signal power.

It is given by $\vec{E} \times \vec{H} = (E_x \vec{a}_x + E_y \vec{a}_y) \times (H_x \vec{a}_x + H_y \vec{a}_y)$

$$= \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ E_x & E_y & 0 \\ H_x & H_y & 0 \end{vmatrix} = (E_x H_y - H_x E_y)\vec{a}_z$$

$$= [\eta H_y H_y - H_x(-\eta H_x)]\vec{a}_z$$

$$\vec{E} \times \vec{H} = (\eta H_y^2 + \eta H_x^2)\vec{a}_z \tag{7.41}$$

$$= \eta(H_y^2 + \eta H_x^2)\vec{a}_z.$$

$$\therefore \quad \vec{E} \times \vec{H} = \eta H^2 \vec{a}_z$$

or $$\vec{E} \times \vec{H} = \frac{E^2}{\eta}\vec{a}_z$$

Thus the signal power $\vec{E} \times \vec{H}$ flows along the direction of propagation.

The $\bar{E}$ and $\bar{H}$ are perpendicular to each other. i.e, $\bar{E}.\bar{H} = 0$

- If $\bar{E}.\bar{H} = 0$, it is said to be electric and magnetic fields are perpendicular to each other.

Let, $\bar{E} = E_x\bar{a}_x + E_y\bar{a}_y + E_z\bar{a}_z$

$$\bar{H} = H_x\bar{a}_x + H_y\bar{a}_y + H_z\bar{a}_z$$

$$\bar{E}\,.\,\bar{H} = (E_x\bar{a}_x + E_y\bar{a}_y + \cancelto{0}{E_z\bar{a}_z}).(H_x\bar{a}_x + H_y\bar{a}_y + \cancelto{0}{H_z\bar{a}_z})$$

$$\bar{E}\,.\,\bar{H} = E_xH_x\cancelto{1}{(\bar{a}_x.\bar{a}_x)} + E_yH_y\cancelto{1}{(\bar{a}_y.\bar{a}_y)}$$

$$\bar{E}\,.\,\bar{H} = E_xH_x + E_yH_y$$

From, $\dfrac{E_x}{H_y} = \sqrt{\dfrac{\mu}{\varepsilon}}$ and, $\dfrac{E_y}{H_x} = -\sqrt{\dfrac{\mu}{\varepsilon}}$

$$\bar{E}\,.\,\bar{H} = (H_y\sqrt{\frac{\mu}{\varepsilon}})H_x + (-H_x\sqrt{\frac{\mu}{\varepsilon}})H_y$$

$$\bar{E}\,.\,\bar{H} = \sqrt{\frac{\mu}{\varepsilon}}\,H_xH_y - \sqrt{\frac{\mu}{\varepsilon}}\,H_xH_y$$

$$\bar{E}\,.\,\bar{H} = 0$$

The $\bar{E} \times \bar{H}$ represents the signal power

$$\text{Let, } \bar{E} = E_x\bar{a}_x + E_y\bar{a}_y + E_z\bar{a}_z$$
$$\bar{H} = H_x\bar{a}_x + H_y\bar{a}_y + H_z\bar{a}_z$$

If the wave propagates in z – direction, $E_z = 0$ and $H_z = 0$.

$$\text{Therefore, } \bar{E} = E_x\bar{a}_x + E_y\bar{a}_y$$
$$\bar{H} = H_x\bar{a}_x + H_y\bar{a}_y$$

$$\bar{E} \, X \, \bar{H} = (E_x\bar{a}_x + E_y\bar{a}_y) \, X \, (H_x\bar{a}_x + H_y\bar{a}_y)$$

$$= \begin{vmatrix} \overline{a_x} & \bar{a}_y & \bar{a}_z \\ E_x & E_y & 0 \\ H_x & Hy & 0 \end{vmatrix}$$

$$\bar{E} \times \bar{H} = \bar{a}_x \left(E_y(0) - H_y(0)\right) - \bar{a}_y\left(E_x(0) - H_x(0)\right) + \bar{a}_z(E_x H_y - E_y H_x)$$

$$\bar{E} \times \bar{H} = (E_x H_y - E_y H_x)\bar{a}_z$$

$$\text{From, } \frac{E_x}{H_y} = \sqrt{\frac{\mu}{\varepsilon}} \qquad \text{and, } \frac{E_y}{H_x} = -\sqrt{\frac{\mu}{\varepsilon}}$$

$$E_x = H_y\sqrt{\frac{\mu}{\varepsilon}} \qquad \text{and, } E_y = -H_x\sqrt{\frac{\mu}{\varepsilon}}$$

$$\bar{E} \times \bar{H} = (H_y\sqrt{\frac{\mu}{\varepsilon}})H_y - (-H_x\sqrt{\frac{\mu}{\varepsilon}})H_x)\bar{a}_z$$

$$\bar{E} \times \bar{H} = (H_y^2\sqrt{\frac{\mu}{\varepsilon}} + H_x^2\sqrt{\frac{\mu}{\varepsilon}})\,\bar{a}_z$$

$$\bar{E} \times \bar{H} = (H_y^2 + H_x^2)\sqrt{\frac{\mu}{\varepsilon}}\,\bar{a}_z$$

$$\bar{E} \times \bar{H} = H^2\sqrt{\frac{\mu}{\varepsilon}}\,\bar{a}_z$$

$$\boxed{\bar{E} \times \bar{H} = H^2\,\eta\,\bar{a}_z}$$

$$\bar{E} \times \bar{H} = \left(\frac{E}{\eta}\right)^2 \eta\,\bar{a}_z$$

$$\boxed{\bar{E} \times \bar{H} = \frac{E^2}{\eta}\,\bar{a}_z}$$

$$\eta = \sqrt{\frac{\mu}{\varepsilon}}$$

$$\eta = \frac{E}{H}$$

$$\mathrm{H} = \frac{E}{\eta}$$

* Sinusoidal variations:

the sine or sinusoidal wave is a curve that describes a smooth repetative oscillation.

* we can define the sine wave as "the wave form in which the amplitude is always proportional to sine of its dis-placement angle at every point of time.
* All waves can be made by adding up sine waves.
* the oscillation of light waves and sound waves can be described with sinusoidal functions in time (frequency) and in space (wave number).
* For a pure radio wave, a sinusoidal variation, is the field is varying in the shape of a sine wave millions or more times per second.
* A sinusoidal signal is the only periodic signal where it retains its wave shape when added to another sinusoidal signal of the same frequency with arbitrary initial phase and amplitude.

Sinusoidal Variations (or) Phasor Notation

The electric & Magnetic field Varys Sinusoidally with respect to time.

So for Sinusoidal time Variations, the maxwell's Equations will be Expressed in phasor Notation

The phasor form of Electric field is

$$\vec{E} = |E| e^{j\omega t}$$

$$\boxed{\vec{E}_x(z) = E^+ e^{-j\beta z} + E^- e^{-j\beta z}}$$

$$\boxed{\frac{\partial^2 E_x}{\partial z^2} + \beta^2 E_x = 0}$$

2nd order differential Equation. w.r.to uniform plane wave propagating in z-direction & The electric field in the x-direction

E^+ & E^- are Complex arbitrary Constants

$$E_x(z,t) = Re\{(\vec{E}_x(z) e^{j\omega t})\},$$

where Re represents the real value of the function

$$E_x(z,t) = Re\{(E^+ e^{-j\beta z} + E^- e^{j\beta z}) e^{j\omega t}\}$$

$$= Re\{E^+ e^{+j(\omega t - \beta z)} + E^- e^{j(\omega t + \beta z)}\}$$

$$= Re\{E^+[(\cos(\omega t - \beta z) + j\sin(\omega t - \beta z)] +$$

$$E^-[(\cos(\omega t + \beta z) + j\sin(\omega t + \beta z)]\}$$

If imaginary form $= 0$

$$\because j\sin(\omega t - \beta z) = 0$$

$$j\sin(\omega t + \beta z) = 0$$

$$\vec{E}_x = \vec{E}_x(z,t) = E^+ \cos(\omega t - \beta z) + E^- \cos(\omega t + \beta z) \;(V/m)\; \text{Volt/meter}$$

Amplitude · wave · amplitude · angular frequency

E^+ +ve z direction · E^- -Ve z direction

$\beta \Rightarrow$ phase variation of signal. (or)

$\beta \Rightarrow$ phase shift constant

Ily

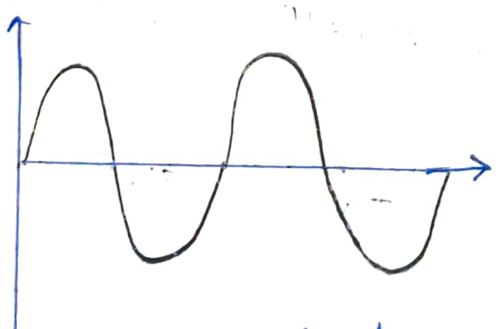
$$H_y = H_y(z,t) = H^+ \cos(\omega t - \beta z) + H^- \cos(\omega t + \beta z) \quad \text{Ampere/meter}$$

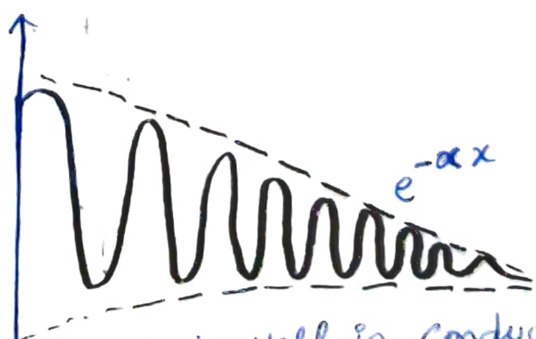
If The wave propagates in free space without bound, it travels then the electric field component is only in forward direction

$$\boxed{\vec{E}_x(z) = E^+ e^{-j\beta z}} \text{ (or) } \boxed{\vec{E}_x = E^+ \cos(\omega t - \beta z)}$$

$$\boxed{\vec{H}_y(z) = H^+ e^{-j\beta z}} \text{ (or) } \boxed{H_y = H^+ \cos(\omega t - \beta z)}$$

Lossless medium	Conducting Medium (or) (Losses)
$\sigma = 0 \quad \rho_v = 0$	$\sigma \neq 0 \quad \rho_v = 0$
$\nabla^2 E + \mu_0 \varepsilon_0 E \omega^2 = 0$	$\nabla^2 E - (j\omega\mu\sigma - \omega^2 \mu\varepsilon)\vec{E} = 0$ $\nabla^2 E - \gamma^2 E = 0 \quad \gamma^2 = j\omega\mu(\sigma + j\omega\varepsilon)$





* Wave traversal without any Attenuation
$\gamma = \alpha + j\beta$ where α = attenuation

* when wave travell in conductive Medium its progressively Amplitude decreased w.r.t. the factor of $e^{-\alpha x}$

WAVE PROPAGATION IN LOSSLess Media

7.3 Time Harmonic Wave Equations (Phasor Notation) (Lossless Medium)

To obtain wave equations in phasor or time harmonic notation, we start with electric and magnetic fields in phasor form. Assume that the medium is homogeneous, linear, isotropic and source free.

The fields can be written in phasor form as,

$$\vec{E} = |E|e^{j\omega t} \quad \text{and} \quad \vec{H} = |H|e^{j\omega t}.$$

$$\frac{\partial E}{\partial t} = (J\omega)\, E\, e^{j\omega t} \qquad \frac{\partial^2 E}{\partial t} = -\omega^2 E$$

$$\therefore \quad \frac{\partial \vec{E}}{\partial t} = j\omega\vec{E} \quad \text{and} \quad \frac{\partial^2 \vec{E}}{\partial t^2} = -\omega^2\,\vec{E}. \tag{7.14}$$

$$\frac{\partial E}{\partial t} = J\omega\vec{E}$$

$$\boxed{J^2 = -1}$$

$$\text{Also,} \quad \frac{\partial \vec{H}}{\partial t} = j\omega\vec{H} \quad \text{and} \quad \frac{\partial^2 \vec{H}}{\partial t^2} = -\omega^2\,\vec{E}. \tag{7.15}$$

z

where ω rad/s is the wave angular frequency.

We know from Eq. (7.12) that the wave equation for an electric field in the time domain for a conducting medium is

$$\nabla^2\vec{E} - \mu\,\varepsilon\frac{\partial^2\vec{E}}{\partial t^2} - \mu\,\sigma\frac{\partial\vec{E}}{\partial t} = 0.$$

Substituting Eq. (7.14) in this equation,

$$\frac{\partial}{\partial t} = j\omega$$

$$\nabla^2\vec{E} - \mu\,\varepsilon(-\omega^2)\vec{E} - \mu\,\sigma(j\,\omega)\vec{E} = 0.$$

$$\nabla^2\vec{E} + \omega^2\,\mu\,\varepsilon\,\vec{E} - \mu\,\sigma\,j\,\omega\,\vec{E} = 0.$$

$$\nabla^2\vec{E} - (j\,\omega\,\mu\,\sigma - \omega^2\,\mu\,\varepsilon)\vec{E} = 0.$$

Similarly, for a magnetic field,

$$\nabla^2\vec{H} - (j\,\omega\,\mu\,\sigma - \omega^2\,\mu\,\varepsilon)\vec{H} = 0.$$

Therefore, the time harmonic wave equations for a conducting medium are

$$\nabla^2\vec{E} - (j\,\omega\,\mu\,\sigma - \omega^2\,\mu\,\varepsilon)\vec{E} = 0$$

$$\text{and} \quad \nabla^2\vec{H} - (j\,\omega\,\mu\,\sigma - \omega^2\,\mu\,\varepsilon)\vec{H} = 0. \tag{7.16}$$

wave

Note: For a lossless medium ($\sigma = 0$), the wave equations become

$$\nabla^2 \vec{E} + \omega^2 \mu \varepsilon \vec{E} = 0$$

and $$\nabla^2 \vec{H} + \omega^2 \mu \varepsilon \vec{H} = 0. \tag{7.17}$$

These equations are called Helmholtz equations.

Wave. = $E_x(z, t)$

Wave = function [space, time]

= function [x, y, z, time]

constant

7.7 Wave Propagation in a Conducting Medium

Consider a uniform plane wave propagating in a conducting medium. The wave equations are given by (From Eq. (7.16)),

$$\nabla^2 E - (j\omega\mu\sigma - \omega^2\mu\varepsilon)E = 0,$$

$$\nabla^2 H - (j\omega\mu\sigma - \omega^2\mu\varepsilon)H = 0,$$

or $\nabla^2 E - \gamma^2 E = 0$, and $\nabla^2 H - \gamma^2 H = 0$, (7.42)

where $\gamma^2 = j\omega\mu(\sigma + j\omega\varepsilon)$ and γ is called the propagation constant of a medium at a given frequency of the wave. It is a complex quantity given by

$$\gamma = \alpha + j\beta, \tag{7.43}$$

where α is called attenuation constant and β is called phase shift constant.

When a wave is propagating in a conducting medium, its power reduces due to losses. The amount of attenuation in the wave can be represented by the attenuation constant (α). Its units are nepers (Np) or decibels (dB).

1 neper = 8.686 dB.

For a lossless medium, $\alpha = 0$.

A change in phase shift of the wave can be represented by the phase shift constant (β). Its units are radians (rad).

So $\gamma = \alpha + j\beta = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}$. (7.44)

Therefore, the plane wave equation for the electric field component can be written as

$$\frac{d^2 \vec{E}_x}{dz^2} - \gamma^2 \vec{E}_x = 0. \tag{7.45}$$

This is a second-order differential equation which represents the uniform plane wave propagating in the z direction and the electric field vector in the x direction.

The general solution for this wave equation is

$$\vec{E}_x(z) = E^+ e^{-\gamma z} + E^- e^{\gamma z}, \tag{7.46}$$

where E^+ and E^- are complex arbitrary constants which can be determined from the boundary conditions of the media. E^+ represents the electric field component when the wave is propagating along the positive z direction and E^- represents the electric field component when the wave is propagating in the negative z direction. Assume that the initial phase angle of the wave is zero.

In time domain, the wave can be expressed as

$$E_x(z,t) = \Re e\left[\vec{E}_x(z) e^{j\omega t}\right], \tag{7.47}$$

where $\Re e$ represents the real value of the function.

② W.P.

$(\alpha + j\beta)z$ $\quad$ γ $(\alpha + j\beta)z$

$$E_x(z,t) = \Re e\left[(E^+ e^{-\gamma z} + E^- e^{\gamma z})e^{j\omega t}\right].$$

$(\gamma = \alpha + j\beta)$

$$E_x(z,t) = \Re e\left[E^+ e^{-\alpha z} e^{j(\omega t - \beta z)} + E^- e^{\alpha z} e^{j(\omega t + \beta z)}\right].$$

$$E_x = E_x(z,t) = E^+ e^{-\alpha z}\cos(\omega t - \beta z) + E^- e^{\alpha z}\cos(\omega t + \beta z) \text{ V/m.} \quad (7.48)$$

If the wave propagates without bounds, it travels only in the forward direction. The electric field component is

$$\vec{E}_x(z) = \Re e\left[E^+ e^{-\alpha z} e^{j(\omega t - \beta z)}\right]$$

$e^{j(\omega t + \beta z)} = \cos(\omega t + \beta z)$

or

$$\boxed{E_x = E^+ e^{-\alpha z}\cos(\omega t - \beta z).} \quad (7.49)$$

✓ $e^{j(\omega t - \beta z)} = \cos(\omega t - \beta z)$

Similarly, for the magnetic field in the y direction,

$$H_y = H_y(z,t) = H^+ e^{-\alpha z}\cos(\omega t - \beta z) + H^- e^{\alpha z}\cos(\omega t + \beta z) \text{ A/m.} \quad (7.50)$$

If the wave travels only in the forward direction,

$$\vec{H}_y(z) = \Re e\left[H^+ e^{-\alpha z} e^{j(\omega t - \beta z)}\right]$$

or

$$\boxed{H_y = H^+ e^{-\alpha t}\cos(\omega t - \beta z).} \quad (7.51)$$

Note: Eq. (7.48) and Eq. (7.50) represent a solution for the uniform plane wave equation when the wave is travelling in a conducting medium along both +ve and −ve directions of z. Eq. (7.49) and Eq. (7.51) represent the same but the wave travels only in the forward z direction.

Conductors & Dielectrics - Characterization

⇒ characteristic Impedance in Conducting Medium (or) Imperfect Dielectric (or) Lossy Dielectric

(*) Consider a plane wave propagating in the forward z-direction in a lossy conducting medium, i.e $J \neq 0$

For a Uniform plane wave, the conditions are:

$$E_z = 0, \quad H_z = 0 \quad \Bigg| \quad \frac{\partial E_x}{\partial y} = 0 \ ; \ \frac{\partial E_y}{\partial x} = 0 \quad \Bigg| \quad \frac{\partial H_x}{\partial y} = 0 \ , \ \frac{\partial H_y}{\partial x} = 0$$

We know that Maxwell's Equation for a lossy medium are

$$\boxed{\nabla \times \vec{H} = \vec{J} + \frac{\partial D}{\partial t} = \sigma \vec{E} + \varepsilon \frac{\partial \vec{E}}{\partial t}} \quad \Bigg| \ \because \vec{D} = \varepsilon \vec{E}, \ \vec{J} = \sigma \vec{E}$$

$$\boxed{\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\mu \frac{\partial \vec{H}}{\partial t}} \quad \Bigg| \ \vec{B} = \mu \vec{H}$$

Also by Expanding $\nabla \times \vec{H}$ & $\nabla \times \vec{E}$ in Cartesian Co-ordinates system

$$\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} =$$

Equating the respective unit Vector components with phasor Notation in The Relationship B/w E & H we can write

$$\boxed{\frac{\partial E_x}{\partial E_z} = -\mu \frac{\partial H_y}{\partial t} = -j\omega\mu H_y}$$

from eq (7.35)

Since the wave is propagating in the forward z-direction in a lossy Medium, the electric field components is

$$E_x = E_x(z) = E^+ e^{-j\gamma z}$$

$$\frac{\partial E_x}{\partial z} = -\gamma E_x$$

from Eq (7.35) $\frac{\partial E_x}{\partial z} = -\gamma E_x = -j\omega\mu H_y$

Then $\frac{E_x}{H_y} = \frac{j\omega\mu}{\gamma} = \frac{j\omega\mu}{\sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}}$

$\because \gamma = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}$

$$\frac{E_x}{H_y} = \frac{\sqrt{j\omega\mu} \cdot \sqrt{j\omega\mu}}{\sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}}$$

$$\Rightarrow \frac{E_x}{H_y} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}$$

||y

$$\frac{E_y}{H_x} = \frac{-j\omega\mu}{\gamma} = \frac{-j\omega\mu}{\sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}} = -\sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}$$

ie total Electric & Magnetic field of the wave Propagating in z-direction can be written as

$$E = \sqrt{E_x^2 + E_y^2}$$

$$H = \sqrt{H_x^2 + H_y^2}$$

$$E_x = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} \cdot H_y$$

$$E_y = -\sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} \cdot H_x$$

$$E_x^2 = \left(\sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}\right)^2 (H_y)^2$$

$$E_y^2 = \left(\sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}\right)^2 (H_x)^2$$

the ratio is

$$\frac{E}{H} = \frac{\sqrt{E_x^2 + E_y^2}}{\sqrt{H_x^2 + H_y^2}}$$

$$\frac{E}{H} = \frac{\sqrt{E_x^2 + E_y^2}}{\sqrt{H_x^2 + H_y^2}} = \frac{\sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}(H_y)^2 + \frac{j\omega\mu}{\sigma + j\omega\varepsilon}(H_x)^2}}{\sqrt{H_x^2 + H_y^2}}$$

$$\frac{E}{H} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} \left[\frac{\cancel{\sqrt{(H_x)^2 + (H_y)^2}}}{\cancel{\sqrt{(H_x)^2 + (H_y)^2}}}\right]$$

$$* \quad \boxed{\eta = \frac{E}{H} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}} \quad \text{ohms}$$

$\eta \Rightarrow$ characteristic Impedance (or) Intrinsic Impedance. of lossy or Conductor (Good) Condition $[\sigma \neq 0;\ \sigma \cong \infty]$

Note: For a non-conducting Medium (or) Lossless Medium ($\sigma = 0$)

$$\boxed{\eta = \sqrt{\frac{\mu}{\varepsilon}} \text{ ohms}}$$

Complex Permittivity (ε_c) and lossy Tangent for lossy medium

A lossy medium (conductor or dielectric) is a poor Insulator. It has both polarization of charges for field establishment and free charges for conduction. Both displacement current and conduction current will flow through the lossy Medium.

The wave equation for a lossy medium is

$$\nabla^2 \vec{E} = j\omega\mu\,(\sigma + j\omega\varepsilon)\vec{E},$$

and that for a lossless material ($\sigma = 0$) is

$$\nabla^2 E = j\omega\mu\,(j\omega\varepsilon_c)\vec{E}$$

comparing the above equations for the same field

$$j\omega\varepsilon_c = \sigma + j\omega\varepsilon$$

(or)

$$\varepsilon_c = \left(\frac{\sigma}{j\omega}\right) + \varepsilon \quad F/m$$

where ε_c is called Complex Permittivity

$$\varepsilon_c = \varepsilon - j\frac{\sigma}{\omega}$$

$$\boxed{\varepsilon_c = \varepsilon - j\,\varepsilon''} \quad F/m$$

ε'' addition term $\boxed{\varepsilon'' = \frac{\sigma}{\omega}}$ result due to power losses in Material.

Loss tangent ($\tan\theta_{loss}$):

the ratio between the imaginary and real part of the Complex permittivity ε_c is called loss tangent of the material.

$\because \boxed{\varepsilon'' = \frac{\sigma}{\omega}}$

$$\tan\theta_{loss} = \frac{\varepsilon''}{\varepsilon} = \frac{\sigma}{\omega\varepsilon}$$

$$\boxed{\begin{array}{c} \varepsilon'' = \frac{\sigma}{\omega} \\ \varepsilon = \varepsilon \end{array}}$$

and the loss Angle is

$$\theta_{loss} = \tan^{-1}\left(\frac{\sigma}{\omega\varepsilon}\right)$$

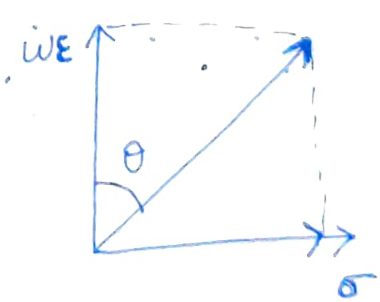
Losses tangent indicates both damping and ohmic losses in the material.

It is a measure of the power loss in the medium.

If ε_c is real, the material is lossless.

θ = Loss angle

$\tan\theta$ = Loss tangent



Characteristic Impedance (η_0): in Dielectric (or) Perfect Dielectric Medium (or) Free Space (or) Source Free Region (or) Homogeneous Medium

Consider a plane wave propagating in a lossless (or) perfect dielectric medium ie $\alpha = 0$; $\sigma = 0$

$$\boxed{\nabla^2 \vec{E} + \omega^2 \mu \varepsilon \vec{E} = 0}$$

$$\gamma = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)} = \sqrt{j\omega\mu(0 + j\omega\varepsilon)}$$

$$\boxed{\gamma = \sqrt{j^2 \omega^2 \mu \varepsilon}}$$

$$\gamma^2 = -1 . \omega^2 \mu \varepsilon$$

$$\therefore \boxed{\gamma^2 = -\omega^2 \mu \varepsilon}$$

$$\boxed{\gamma = \sqrt{-\omega^2 \mu \varepsilon}}$$

$$\gamma = j\omega\sqrt{\mu\varepsilon}$$

$$\boxed{\gamma = j\beta}$$

we know that $\boxed{\beta = \omega\sqrt{\mu\varepsilon}}$

$$\boxed{\beta = \omega\sqrt{\mu\varepsilon}}$$

$$\gamma = \alpha + j\beta$$
$$0 + j\beta$$

or $\boxed{\gamma = j\beta}$

Phase Velocity: (V_p) in Good Conductor in Term of ω, μ, σ:

we know that the Velocity of the wave is

$$V_p = \frac{\omega}{\beta}$$

But $\beta = \sqrt{\frac{\omega\mu\sigma}{2}}$ (from wave propagation in Good conductor)

$$V_p = \frac{\omega}{\sqrt{\frac{\omega\mu\sigma}{2}}}$$

phase velocity is Good Conductor is V_p

$$V_p = \sqrt{\frac{2\omega}{\mu\sigma}}$$

Note: the Speed of the wave decreases as the Conductivity (σ) of the Medium increases at a given frequency.

Intrinsic Impedance: In terms of angular wave frequency & conductivity
(η)

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}$$

for a good conductor $\boxed{\sigma >> \omega\varepsilon}$ (or) $\boxed{\frac{\sigma}{\omega\varepsilon} >> 1}$

$\boxed{\frac{\omega\varepsilon}{\sigma} << 1}$

$\cong 0$

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma} + \frac{j\omega\mu}{j\omega\varepsilon}}$$

for good conductor $\sigma >> \omega\varepsilon$

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma}}$$

(or)

$$\eta = \sqrt{\frac{\omega\mu}{\sigma}} \cdot \sqrt{j}$$

$$\boxed{\eta = \sqrt{\frac{\omega\mu}{\sigma}} \angle 45^\circ}$$

$$= \sqrt{\omega \mu \sigma}\left[\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right]$$

$$\gamma = \sqrt{\frac{\omega \mu \sigma}{2}} + j\sqrt{\frac{\omega \mu \sigma}{2}}$$

$$\gamma = \alpha + j\beta$$

$$\therefore \alpha = \sqrt{\frac{\omega \mu \sigma}{2}} \quad \text{and} \quad \beta = \sqrt{\frac{\omega \mu \sigma}{2}}$$

Thus, for good conductors, α and β are equal.

Note: $\alpha \propto \sqrt{\sigma}$ α is proportional to $\sqrt{\sigma}$

Square root of conductivity (σ)

the losses are very high and the wave attenuates rapidly.

α = attenuation constant

β = phase shift constant

γ = propagation constant

Imperfect Dielectric

7.8 Characteristic Impedance in a Lossy Dielectric or Conducting Medium

Intrinsic

Consider a plane wave propagating in the forward z direction in a lossy conducting medium, i.e., $J \neq 0$.

For a uniform plane wave, the conditions are

$$E_z = 0 \quad H_z = 0, \quad \frac{\partial E_x}{\partial y} = 0, \quad \frac{\partial E_y}{\partial x} = 0,$$

and $$\frac{\partial H_x}{\partial y} = 0, \quad \frac{\partial H_y}{\partial x} = 0. \tag{7.52}$$

We know that Maxwell's equations for a lossy medium are

$$\nabla \times \vec{H} = \sigma \vec{E} + \varepsilon \frac{\partial \vec{E}}{\partial t}, \tag{7.53}$$

and $$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}. \tag{7.54}$$

Good conductor

Lossy dielectric $\sigma \neq 0$ of conductivity

Also from Eq. (7.35),

$$\frac{\partial E_x}{\partial z} = -\mu \frac{\partial H_y}{\partial t} = -j\omega\mu H_y. \tag{7.55}$$

Since the wave is propagating in the forward z direction in a lossy medium, the electric field component is

$$E_x = E_x(z) = E^+ e^{-j\gamma z}. \tag{7.56}$$

$\gamma = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}$

$$\frac{\partial E_x}{\partial z} = -\gamma E_x.$$

from eq of E & H (7.35)

$\boxed{\frac{\partial E_x}{\partial z} = -j\omega\mu H_y}$

$\therefore$ $\boxed{-\gamma E_x = -j\omega\mu H_y}$

From Eq. (7.35), $\boxed{\frac{\partial E_x}{\partial z} = -\gamma E_x = -j\omega\mu H_y.}$

Then $\frac{E_x}{H_y} = \frac{j\omega\mu}{\gamma} = \frac{j\omega\mu}{\sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}}$

or $$\frac{E_x}{H_y} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}. \tag{7.57}$$

Similarly, we can show that $$\frac{E_y}{H_x} = \frac{-j\omega\mu}{\gamma} = \frac{-j\omega\mu}{\sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}} = -\sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}. \tag{7.58}$$

The total electric and magnetic fields of the wave propagating in the z direction can be written as

$$E = \sqrt{E_x^2 + E_y^2}.$$

$$H = \sqrt{H_x^2 + H_y^2}.$$

The ratio is $\frac{E}{H} = \frac{\sqrt{E_x^2 + E_y^2}}{\sqrt{H_x^2 + H_y^2}} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}.$

Therefore, the intrinsic impedance of the lossy medium is

$$\eta = \frac{E}{H} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} \text{ ohms.} \tag{7.59}$$

—×—

For a non-conducting medium or lossless medium, $\sigma = 0$.

$$\therefore \quad \eta = \sqrt{\frac{\mu}{\varepsilon}} \text{ ohms}$$

which is same as Eq. (7.37).

Complex permittivity and loss tangent for lossy medium

ε

A lossy medium (conductor or dielectric) is a poor insulator. It has both polarization of charges for field establishment and free charges for conduction. Both displacement current and conduction current will flow through the lossy medium.

The wave equation for a lossy medium is

$$\nabla^2\vec{E} = j\omega\mu(\sigma + j\omega\varepsilon)\vec{E},$$

and that for a lossless material ($\sigma = 0$) is

$$\nabla^2\vec{E} = (j\omega\mu)(j\omega\varepsilon_c)\vec{E}.$$

Comparing the above equations for the same field,

$$j\omega\varepsilon_c = \sigma + j\omega\varepsilon$$

$\frac{1}{j} = -j$

or $$\varepsilon_c = \varepsilon - j\frac{\sigma}{\omega} = \varepsilon - j\varepsilon'' \text{ F/m},$$

where ε_c is called complex permittivity. The additional term $\boxed{\varepsilon'' = \frac{\sigma}{\omega}}$ results due to power losses in the material.

Loss tangent: The ratio between the imaginary and real parts of the complex permittivity ε_c is called loss tangent of the material. It can be expressed as

$$\tan\theta_{\text{loss}} = \frac{\varepsilon''}{\varepsilon} = \frac{\sigma}{\omega\varepsilon}$$

and the loss angle is $\theta_{\text{loss}} = \tan^{-1}\left(\frac{\sigma}{\omega\varepsilon}\right)$.

Losses tangent indicates both damping and ohmic losses in the material. It is a measure of the power loss in the medium. If ε_c is real, the material is lossless.

Free space or Source free

7.9 Wave Propagation in Perfect Dielectric Medium

Consider a plane wave propagating in a lossless or perfect dielectric medium, i.e., $\alpha = 0$.

The wave equation is

$$\nabla^2\vec{E} + \omega^2\mu\varepsilon\vec{E} = 0,$$

where $\gamma^2 = -\omega^2\mu\varepsilon$

$\gamma = \alpha + j\beta$

or $$\gamma = \sqrt{-\omega^2\mu\varepsilon} = j\omega\sqrt{\mu\varepsilon} = j\beta. \qquad (7.60)$$

$\gamma = j\beta$

Therefore, for perfect dielectrics, the phase shift constant is

$$\beta = \omega\sqrt{\mu\varepsilon}$$

and the velocity of propagation is

$$\boxed{v_p = \frac{\omega}{\beta}.}$$

For free space,

$$\beta = \omega\sqrt{\mu_0\,\varepsilon_0}.$$

$$\therefore \quad v_0 = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu_0\,\varepsilon_0}} = 3\times 10^8 \text{ m/s.} \qquad (7$$

Wavelength is $\boxed{\lambda = \dfrac{2\pi}{\beta}}$

and intrinsic impedance is

$$\boxed{\eta = \frac{E}{H} = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 120\pi\ \Omega.} \qquad (7.$$

Example 7.2 A uniform plane wave at a frequency of 1 GHz is travelling in a large block Teflon ($\varepsilon_r = 2.1$, $\mu_r = 1$ and $\sigma = 0$). Determine γ, η, β and λ.

Solution Given:

$$\varepsilon_r = 2.1, \quad \mu_r = 1, \quad \sigma = 0 \quad \text{and} \quad f = 1\times 10^9 \text{ Hz.}$$

Since $\sigma = 0$, for a lossless dielectric medium, $\alpha = 0$.

We know that

$$\beta = \omega\sqrt{\mu\varepsilon} = \omega\sqrt{\mu_0\,\mu_r\,\varepsilon_0\,\varepsilon_r}$$

$$\beta = \frac{\omega}{c}\sqrt{\mu_r\,\varepsilon_r} = \frac{2\pi\times 10^9}{3\times 10^8}\sqrt{1\times 2.1} = 30.35 \text{ rad/m.}$$

Phase velocity, $v_p = \dfrac{\omega}{\beta} = \dfrac{2\pi f}{\beta} = \dfrac{2\pi\times 10^9}{30.35} = 2\times 10^8$ m/s.

Impedance $\eta = \sqrt{\dfrac{\mu}{\varepsilon}} = \sqrt{\dfrac{\mu_0\,\mu_r}{\varepsilon_0\varepsilon_r}} = \dfrac{120\pi}{\sqrt{2.1}}$

$$= 260\ \Omega.$$

Wavelength $\lambda = \dfrac{2\pi}{\beta} = \dfrac{2\pi}{30.35} = 0.2$ m.

7.11 Wave Propagation in Good Conductors

Consider a wave propagating in a good conductor. For good conductors, the conductivity is very high compared to the wave frequency, i.e., $\sigma >> \omega\varepsilon$.

Now, $\gamma = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}$.

$\sigma >> \omega \varepsilon$

$$\gamma = \sqrt{j\omega\mu\sigma\left(1 + \frac{j\omega\varepsilon}{\sigma}\right)}$$

$$\boxed{\frac{\sigma}{\omega\varepsilon} >> 1} \text{ (or) } \frac{\omega\varepsilon}{\sigma} << 1$$

$$= \sqrt{j\omega\mu\sigma(1+0)}$$

$$\frac{\omega\varepsilon}{\sigma} \simeq 0$$

$\gamma = \sqrt{j\omega\mu\sigma} = \sqrt{\omega\mu\sigma} \cdot \sqrt{j}$

Since $\sigma >> \omega\varepsilon$, we can neglect the ~~imaginary~~ Real part.

Hence, $\gamma = \sqrt{j\omega\mu\sigma} = \sqrt{\omega\mu\sigma}\angle 45°$.

$$= \sqrt{\omega\mu\sigma}\left[\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right] = \sqrt{\frac{\omega\mu\sigma}{2}} + j\sqrt{\frac{\omega\mu\sigma}{2}}.$$

sin(45°) + j cos(45°)

$$\boxed{\sqrt{j} = \left[\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right]}$$

$$\therefore \quad \alpha = \sqrt{\frac{\omega\mu\sigma}{2}} \tag{7.68}$$

$$\text{and} \quad \beta = \sqrt{\frac{\omega\mu\sigma}{2}}. \tag{7.69}$$

Thus, for good conductors, α and β are equal.

Note: In a highly conducting medium, since α is proportional to $\sqrt{\sigma}$, the losses are very high and the wave attenuates rapidly.

Intrinsic impedance We know that the intrinsic impedance for a conducting medium is

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}.$$

For a good conductor, $\sigma >> \omega\varepsilon$.

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma}}$$

$$\text{or} \quad \eta = \sqrt{\frac{\omega\mu}{\sigma}}\angle 45°. \tag{7.70}$$

Phase velocity We know that the velocity of the wave is

$$v_p = \frac{\omega}{\beta} = \frac{\omega}{\sqrt{\frac{\omega\mu\sigma}{2}}}.$$

Phase velocity in good conductors is

$$v_p = \sqrt{\frac{2\omega}{\mu\sigma}}. \tag{7.71}$$

Note: The speed of the wave decreases as the conductivity of the medium increases at a given frequency.

Example 7.4 Determine the phase velocity of propagation, attenuation constant, phase constant and intrinsic impedance for a forward travelling wave in a large block of copper at 1 MHz ($\sigma = 5.8 \times 10^7$, $\varepsilon_r = \mu_r = 1$).

7.12 Wave Propagation in Good Dielectrics

Consider a wave propagating in a good dielectric medium. For good dielectrics, the conductivity is very low compared to $\omega\varepsilon$, i.e., $\sigma << \omega\varepsilon$.

The expression for the propagation constant is

$$\gamma = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}$$

$$= (\sqrt{j\omega\mu})(\sqrt{j\omega\varepsilon})\sqrt{1 + \frac{\sigma}{j\omega\varepsilon}}.$$

Take common $j\omega\varepsilon$ from $(\sigma + j\omega\varepsilon)$

$$\gamma = j\omega\sqrt{\mu\varepsilon}\sqrt{1 + \frac{\sigma}{j\omega\varepsilon}}. \quad (7.72)$$

Now, using binomial expansion,

$$\sqrt{1 + \frac{\sigma}{j\omega\varepsilon}} \approx 1 + \frac{\sigma}{2j\omega\varepsilon} + \left(\frac{-1}{4}\right)\frac{\sigma^2}{2j^2\omega^2\varepsilon^2} + \cdots$$

Since $\dfrac{\sigma}{\omega\varepsilon} \ll 1$, higher order terms can be neglected

$$\sigma << \omega\varepsilon \Rightarrow \frac{\sigma}{\omega\varepsilon} << 1$$

Binomial Expansion

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!} \cdot x^2 + \cdots$$

$$\left(1 + \frac{\sigma}{j\omega\varepsilon}\right)^{1/2}$$

$$n = \frac{1}{2}$$

$$x = \frac{\sigma}{j\omega\varepsilon}$$

Therefore a good approximation is

$$\sqrt{1+\frac{\sigma}{j\omega\varepsilon}} \approx 1+\frac{\sigma}{2j\omega\varepsilon}+\frac{\sigma^2}{8\omega^2\varepsilon^2}. \tag{7.73}$$

Substituting this value in Eq. (7.72),

$$\gamma = j\omega\sqrt{\mu\varepsilon}\left[1+\frac{\sigma}{2j\omega\varepsilon}+\frac{\sigma^2}{8\omega^2\varepsilon^2}\right].$$

$$\alpha+j\beta = j\omega\sqrt{\mu\varepsilon}+\frac{\cancel{j\omega}\sqrt{\mu\varepsilon}\sigma}{2\cancel{j\omega}\varepsilon}+\frac{j\omega\sqrt{\mu\varepsilon}\sigma^2}{8\omega^2\varepsilon^2}.$$

or $$\alpha+j\beta = j\omega\sqrt{\mu\varepsilon}+\frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}+\frac{j\sigma^2\sqrt{\mu\varepsilon}\,\omega}{8\omega^2\varepsilon^2}. \tag{7.74}$$

Equating real and imaginary terms, the attenuation contant is

$$\alpha = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}} \tag{7.75}$$

and the phase shift constant is

$$\beta = \omega\sqrt{\mu\varepsilon}\left(1+\frac{\sigma^2}{8\omega^2\varepsilon^2}\right), \tag{7.76}$$

Note: For a perfect or lossless dielectric, $\sigma = 0$, and $\beta = \omega\sqrt{\mu\varepsilon}$.

Intrinsic impedance We know that the intrinsic impedance for a lossy medium is

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma+j\omega\varepsilon}}$$

$$= \sqrt{\frac{\mu}{\varepsilon}}\sqrt{\frac{1}{\left(1+\frac{\sigma}{j\omega\varepsilon}\right)}} = \sqrt{\frac{\mu}{\varepsilon}}\left(1+\frac{\sigma}{j\omega\varepsilon}\right)^{-1/2}.$$

$\boxed{n = -1/2}$ $\boxed{x = \frac{\sigma}{j\omega\varepsilon}}$

Using the binomial theorem expansion, $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots$ $\boxed{\frac{1}{j} = -j}$

$$\eta = \sqrt{\frac{\mu}{\varepsilon}}\left(1-\frac{\sigma}{2j\omega\varepsilon}+\ldots..\right).$$

Neglecting higher order terms, the intrinsic impedance for good dielectrics is

$$\eta = \sqrt{\frac{\mu}{\varepsilon}}\left(1+j\frac{\sigma}{2\omega\varepsilon}\right). \tag{7.77}$$

Thus, when a wave is travelling in a lossy dielectric medium, due to conductivity a reactive component is added in the impedance.

Phase velocity We know that the velocity of the wave is

$$v_p = \frac{\omega}{\beta} = \frac{\omega}{\omega\sqrt{\mu\varepsilon}\left(1 + \frac{\sigma^2}{8\omega^2\varepsilon^2}\right)}.$$

$$v_p = \frac{1}{\sqrt{\mu\varepsilon}}\left(1 + \frac{\sigma^2}{8\omega^2\varepsilon^2}\right)^{-1}.$$

$(1+x)^n = 1 + n.x + \frac{n(n-1)}{2!}.x^2$

Now, using binomial expansion and neglecting higher order terms, phase velocity in good dielectrics is

$$v_p = \frac{v_0}{\sqrt{\mu_r\varepsilon_r}}\left(1 - \frac{\sigma^2}{8\omega^2\varepsilon^2}\right). \quad (7.78)$$

$\mu = \mu_0 \mu_r$
$\varepsilon = \varepsilon_0 \varepsilon_r$

For practical dielectrics, since the conductivity σ is very small, the velocity of propagation reduces slighty compared to free space velocity v_0. As conductivity σ of the medium increases, the loss of the energy in the wave increases and the velocity decreases. For example, the phase velocity in sea water is less than the phase velocity in rain water.

7.14 Polarization of a Uniform Plane Wave

Polarization of a uniform plane wave is defined as *the time-varying behaviour of the electric field strength $\vec{E}$ at a given fixed point in space*.

The fields of the uniform plane wave have directions in the medium. Depending on the direction or orientation of the electric field strength, there are three types of polarization.

1. Linear polarization
2. Elliptical polarization
3. Circular polarization

7.14.1 Linear Polarization

A wave is said to be linearly polarized, if the electric field strength remains along a straight line at a given point in space.

Linear polarization is divided into two types:

(a) horizontal polarization and

(b) vertical polarization.

Consider a uniform plane wave travelling in the z direction with electric and magnetic fields laying on the $x-y$ plane.

The electric field vector in time domain for linear polarization is

$$\vec{E}(z,t) = E_1 e^{-\alpha t}\cos(\omega t - \beta z)\,\vec{a}_x + E_2 e^{-\alpha t}\cos(\omega t - \beta z)\,\vec{a}_y \tag{7.82}$$

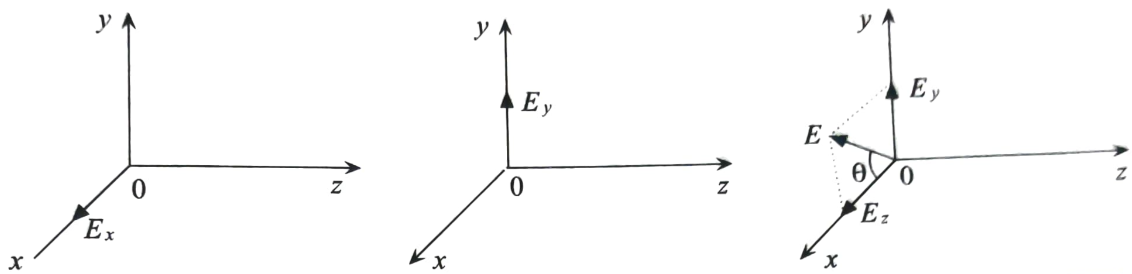
$$\vec{E}(z,t) = E_x\vec{a}_x + E_y\vec{a}_y.$$

$$E_x = E_1 e^{-\alpha t}\cos(\omega t - \beta z)\ ;\quad E_y = E_2 e^{-\alpha t}\cos(\omega t - \beta z)$$

Thus the electric field strength $\vec{E}$ has two components E_x and E_y. If E_x is present and E_y is zero, the wave is said to be *polarized* in the x direction—this phenomena is called horizontal polarization.

Similarly, if E_y is present and E_x is zero, the wave is said to be *polarized* in the y direction—this phenomena is called vertical polarization.

If both E_x and E_y present and are in-phase, the resultant vector makes an angle $\theta = \tan^{-1}(E_y/E_x)$ with the x-axis. This angle is constant with time. Therefore, in linear polarization, the direction of the resultant vector is constant with time as shown in Fig. 7.2.



(*a*) Horizontal polarization (*b*) Verical polarization (*c*) Polarized at angle θ

Fig. 7.2 *Linear polarization*

7.14.2 Circular Polarization

Consider a uniform plane wave travelling in the z-direction with electric field vector components $\vec{E}_x$ and $\vec{E}_y$ not in-phase, that is, they reach their maximum values at different instants of time. Then the direction of the resultant electric vector $\vec{E}$ will vary with time.

If $\vec{E}_x$ and $\vec{E}_y$ have equal magnitude with a 90° phase difference, then the locus of the resultant vector $\vec{E}$ is a circle. This wave is called a circularly polarized wave.

Proof Consider a uniform plane wave travelling in the z direction in a lossless medium $\alpha = 0$. The electric field vector in the time domain is

$$\vec{E}(z, t) = E_1 \cos(\omega t - \beta z)\vec{a}_x + E_2 \cos(\omega t - \beta z + \phi)\vec{a}_y, \tag{7.83}$$

where the $\vec{E}_y$ component leads the $\vec{E}_x$ component by an angle ϕ.

The electric field vector lies in the $x-y$ plane. The vector $\vec{E}$ has two components, $\vec{E}_x$ and $\vec{E}_y$, along the x- and y-axes respectively.

$$\therefore \quad \vec{E}(z, t) = E_x\vec{a}_x + E_y\vec{a}_y. \tag{7.84}$$

For circular polarization, $\vec{E}_x$ and $\vec{E}_y$ have equal magnitudes and $\phi = 90°$ phase difference

i.e., $E_1 = E_2 = E_a$ (say).

$$\therefore \quad \vec{E}(z, t) = E_a \cos(\omega t - \beta z)\vec{a}_x + E_a \cos(\omega t - \beta z + 90°)\vec{a}_y$$

at a fixed point $z = 0$ (say).

$$\vec{E}(0, t) = E_a \cos(\omega t)\vec{a}_x - E_a \sin(\omega t)\,\vec{a}_y.$$

From Eq. (7.84), $E_x = E_a \cos(\omega t)$ and $E_y = -E_a \sin(\omega t)$.

So, $\frac{E_x}{E_a} = \cos \omega t, \quad \frac{E_y}{E_a} = -\sin \omega t.$

Squaring and adding the components,

$$E_x^2 + E_y^2 = E_a^2 \cos^2(\omega t) + E_a^2 \sin^2(\omega t)$$

or $E_x^2 + E_y^2 = E_a^2.$ (7.85)

This equation represents the locus of a circle as shown in Fig. 7.3.
Therefore the wave is circularly polarized. (Proved)

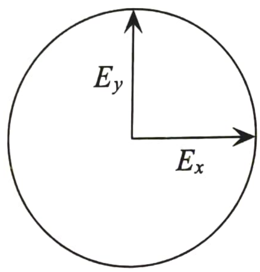


Fig. 7.3 *Locus of circular polarization*

Direction of rotation In circular polarization, if $\vec{E}_y$ leads $\vec{E}_x$ by $90°(\phi = 90°)$, the direction of rotation is a left-handed screw advancing in the z direction as shown in Fig. 7.4(a). Hence the wave is called a left-circularly polarized wave.

Similarly, if $\vec{E}_y$ lags behind $\vec{E}_x$ by $90°(\phi = -90°)$, the direction of rotation is a right-handed screw advancing in the z direction as shown in Fig. 7.4(b). Then the wave is called a right-circularly polarized wave.

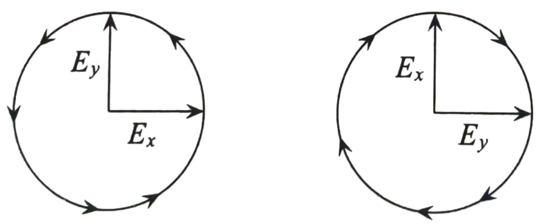


Fig. 7.4 *Left-circularly polarization, (b) right-circularly polarization*

7.14.3 Elliptical Polarization

Consider a uniform plane wave travelling in the z-direction with electric field vector components $\vec{E}_x$ and $\vec{E}_y$ not in-phase, that is, they reach their maximum values at different instants of time. Then the direction of the resultant electric vector $\vec{E}$ will vary with time.

If $\vec{E}_x$ and $\vec{E}_y$ have different magnitudes with a 90° phase difference, then the locus of the resultant vector $\vec{E}$ is an ellipse. The wave is called an elliptical polarized wave.

Proof Consider a uniform plane wave travelling in the z direction in a lossless medium $\alpha = 0$. The electric field vector in time domain is

$$\vec{E}(z, t) = E_1\cos(\omega t - \beta z)\vec{a}_x + E_2\cos(\omega t - \beta z + \phi)\vec{a}_y,$$

where the $\vec{E}_y$ component leads the $\vec{E}_x$ component by an angle ϕ.

The electric field vector lies in the $x-y$ plane. The vector $\vec{E}$ has two components, $\vec{E}_x$ and $\vec{E}_y$, along the x- and y-axes respectively.

$$\therefore \quad \vec{E}(z, t) = E_x\vec{a}_x + E_y\vec{a}_y. \tag{7.86}$$

For elliptical polarization, $\vec{E}_x$ and $\vec{E}_y$ have different magnitudes and $\phi = 90^\circ$ phase.

$$\therefore \quad \vec{E}(z, t) = E_1\cos(\omega t - \beta z)\vec{a}_\chi + E_2\cos(\omega t - \beta z + 90^\circ)\vec{a}_y.$$

At a fixed point $z = 0$ (say),

$$\vec{E}(0, t) = E_1\cos(\omega t)\vec{a}_x - E_2\sin(\omega t)\vec{a}_y.$$

From Eq. (7.86), $\quad E_x = E_1\cos(\omega t),$

$$E_y = -E_2\sin(\omega t).$$

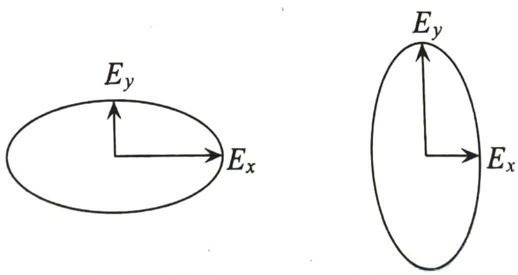
So, $\quad \dfrac{E_x}{E_1} = \cos\omega t, \quad \dfrac{E_y}{E_2} = -\sin\omega t.$

Squaring and adding the components, since $\sin^2\omega t + \cos^2\omega t = 1$,

$$\frac{E_x^2}{E_1^2} + \frac{E_y^2}{E_2^2} = 1. \tag{7.87}$$

This equation represents the locus of an ellipse as shown in Fig. 7.5.

The wave is elliptically polarized. (Proved)

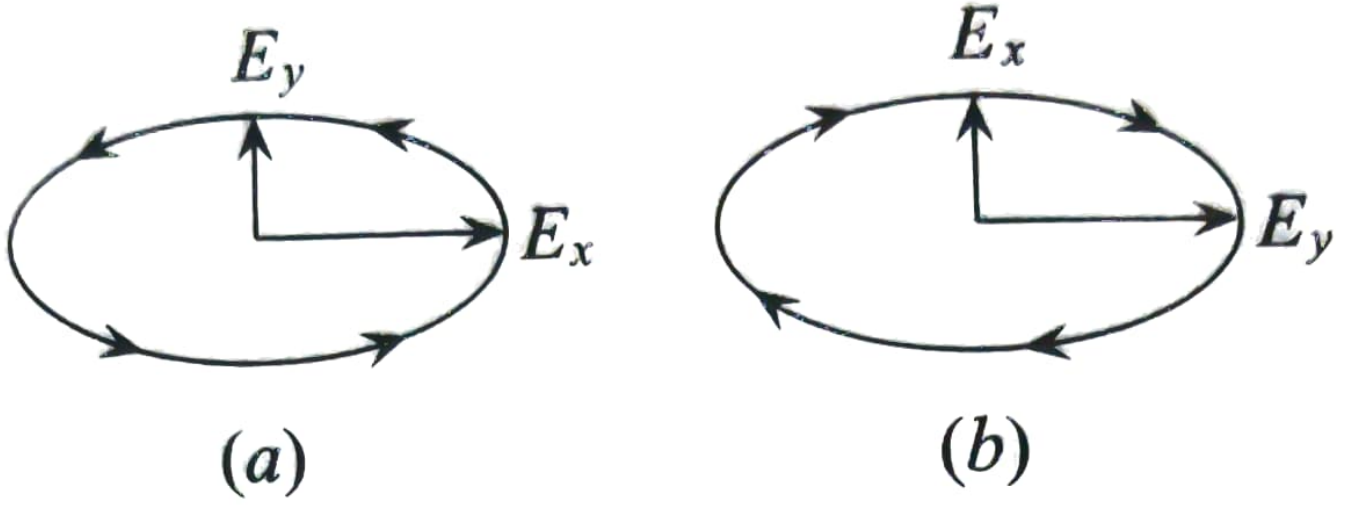


(a) $E_x > E_y$ (b) $E_x < E_y$

Fig. 7.5 *Locus of elliptical polarization*

Direction of rotation In elliptical polarization, if $\vec{E}_y$ leads $\vec{E}_x$ by 90° ($\phi = 90^\circ$), the direction of rotation is a left-handed screw advancing in the z direction as shown in Fig. 7.6(a). Hence the wave is called a left-elliptically polarized wave.

Similarly, if $\vec{E}_y$ lags behind $\vec{E}_x$ by 90° ($\phi = -90^\circ$), the direction of rotation is a right-handed screw advancing in the z direction as shown in Fig. 7.6(b). Then the wave is called a right-elliptically polarized wave.



Left-elliptical polarization, (*b*) *right-elliptical polarization*