UNIT - 1

Laplace Transforms

$f(x) = x^2$

$D = \frac{d}{dx}$ $\qquad D\{f(x)\} = \frac{d}{dx}(x^2) = 2x$

$I = \int (\ ) dx$ $\qquad$ // $\qquad = g(x)$

lly $\int f(x)dx = \int x^2 dx = \frac{x^3}{3} = h(x)$

$D \rightarrow f(x) \rightarrow g(x)$

$I \rightarrow f(x) \rightarrow h(x)$

$\boxed{I \cdot T}$ — $f(x) \rightarrow g(s)$] $\boxed{\text{Laplace Transforms}}$

$L[f(t)] = \bar{f}(s)$

$y'' + y = 0$

with $y(0) = 1$

$\boxed{IVP}$

L.T

ODE

g.sol

$x = 2, y = 1$

P.sol

directly

Def:- Let $f(t)$ be a fn. defined for all +ve values of t. Then the Laplace transform of $f(t)$ is defined by

$$L[f(t)] = \bar{f}(s) = \int_0^\infty e^{-st} f(t)\, dt \rightarrow (1)$$

provided the integral exists. Here the parameter s - real or complex no.

(1) can be written as,

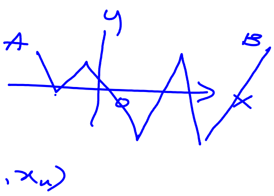
$$f(t) = L^{-1}[\bar{f}(s)] \rightarrow (2)$$

where $f(t)$ is said to be inverse L.T. of $\bar{f}(s)$.

L.T.OP $\quad L \rightarrow f(t) \rightarrow \bar{f}(s)$

I.LT operator $\quad L^{-1} \rightarrow \bar{f}(s) \rightarrow f(t)$

Piece-wise Continuous fn.:-



$f(t)$ $\quad [a, b]$

$a = x_0, \quad b = x_n$

$[x_0, x_1), (x_1, x_2) \ldots (x_{n-1}, x_n)$

i) $f(t)$ must be continuous in each sub-interval

ii) at end pt.s

$x_0, x_1, x_2 \cdots x_{n-1}, x_n \rightarrow$ finite values

Piecewise continuous in $[a,b]$.

$\downarrow$

Sectionally

Fn.s of Exponential Order:-

A fn. $f(t)$ is said to be of exponential order 'a' if

$$\lim_{t\to\infty} e^{-at} f(t) = \text{a finite quantity}$$

Sufficient cond. for the existance of the L.T. of $f(t)$:

If a fn. $f(t)$ is i) Piece-wise continuous on a closed interval $[a,b]$ and

ii) of exponential order,

then its Laplace transform exists.

Properties of Laplace Transforms:-

i) Linearity Property:-

If $L\{f(t)\} = \bar{f}(s)$ and $L\{g(t)\} = \bar{g}(s)$ then

$$L\{c_1 f(t) + c_2 g(t)\} = c_1 \bar{f}(s) + c_2 \bar{g}(s)\text{, where}$$

c_1, c_2 are constants.

P:- By def,

$$L\{f(t)\} = \bar{f}(s) = \int_0^\infty e^{-st} f(t)\,dt \rightarrow (1)$$

let $$L\{c_1 f(t) + c_2 g(t)\} = \int_0^\infty [c_1 f(t) + c_2 g(t)]\, e^{-st}\,dt$$

$$= \int_0^\infty \{e^{-st} c_1 f(t) + e^{-st} c_2 g(t)\}\,dt$$

$$= c_1 \int_0^\infty e^{-st} f(t)\,dt + c_2 \int_0^\infty e^{-st} g(t)\,dt$$

$$= c_1 \bar{f}(s) + c_2 \bar{g}(s) \quad (\because (1))$$

$$\therefore L\{c_1 f(t) + c_2 g(t)\} = c_1 \bar{f}(s) + c_2 \bar{g}(s)$$

Laplace Transforms of Elementary fn.s:-

i) $L[1] = \frac{1}{s}$

$L\{f(t)\} = \bar{f}(s) = \int_0^\infty e^{-st} f(t)\,dt \rightarrow (1)$

Sol:- By def, $L\{f(t)\} = \bar{f}(s) = \int_0^\infty e^{-st} f(t)\, dt \rightarrow (1)$

let $f(t) = 1$ in (1), then

$$L[1] = \bar{f}(s) = \int_0^\infty e^{-st} \cdot 1\, dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$= \frac{0 - 1}{-s} = \frac{1}{s} \quad (\because e^{-\infty} = 0)$$

$$\boxed{\therefore L[1] = 1/s}$$

2) $L[k] = k/s$, $k \neq 0$

Sol:- By def

$$L[k] = \int_0^\infty e^{-st} (k)\, dt = k \left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$= -\frac{k}{s}(0 - 1) = k/s$$

$\therefore L[k] = k/s$.

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3) $L[t^n] = \frac{n!}{s^{n+1}}$, $n = 0, 1, 2, 3, \ldots$

Sol:- By def, $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt \rightarrow (1)$

let $f(t) = t^0 = 1 \Rightarrow L[1] = 1/s$

$$\therefore L[t^0] = \frac{0!}{s^{0+1}} = \frac{1}{s} \rightarrow (2)$$

let $f(t) = t$ $(\because n = 1)$ in (1), then

$$L[t] = \int_0^\infty \underset{v}{e^{-st}} \underset{u}{t}\, dt =$$

[$\int uv\, dx = uv_1 - u'v_2 + u''v_3 + \cdots$

where u, v are fn. of x, $'$, $''$, ... diff, 1, 2, 3 - Int.]

$$L[t] = \left[t \cdot \cancel{\left(\frac{e^{-st}}{-s}\right)} - 1 \cdot \left(\frac{e^{-st}}{+s^2}\right) + 0\right]_0^\infty$$

$$= -\frac{1}{s^2}\{0 - 1\} = \frac{1}{s^2}$$

$$\Rightarrow L[t^1] = \frac{1}{s^2} = \frac{1!}{s^{1+1}} \longrightarrow (3)$$

$$\left.\begin{aligned} &\text{Illy } L[t^2] = \frac{2!}{s^3}, \ldots \\ &L[t^{n-1}] = \frac{(n-1)!}{s^n} \end{aligned}\right] \rightarrow (4)$$

$$L[t^n] = \int_0^\infty e^{-st} t^n \, dt$$

$$= \left\{ \cancel{t^n \left(\frac{e^{-st}}{-s}\right)} - \int_0^\infty n t^{n-1} \left(\frac{e^{-st}}{-s}\right) dt \right\}_0^\infty$$

$$L[t^n] = \frac{n}{s} \int_0^\infty e^{-st} t^{n-1} dt$$

$$= \frac{n}{s} L[t^{n-1}] \quad (\because \text{By def})$$

$$= \frac{n}{s} \left\{ \frac{(n-1)!}{s^n} \right\} \quad (\because 4)$$

$$\therefore L[t^n] = \frac{n!}{s^{n+1}}, \quad n = 0, 1, 2, \ldots$$

Gamma fn:- The Gamma fn. Γn is defined by

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx, \quad n > 0 \rightarrow (A)$$

Note:-

1) $\Gamma n+1 = n \Gamma n$

2) $\Gamma n+1 = n!$, n - +ve integer

3) $\Gamma 1 = 1$

4) $\Gamma 1/2 = \sqrt{\pi}$

4) $$L[t^n] = \frac{\Gamma n+1}{s^{n+1}} \quad \forall n.$$

Sol:- By def,

$$L[t^n] = \int_0^\infty e^{-st} t^n \, dt \rightarrow (1)$$

let $st = x \Rightarrow t = \frac{x}{s} \Rightarrow dt = \frac{dx}{s}$ subs in (1), then

$$L[t^n] = \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^n \frac{dx}{s}$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-x} x^{(n+1)-1} dx$$

$$= \frac{1}{s^{n+1}} \Gamma(n+1) \quad (\because \text{By Gamma fn. } (A))$$

$$\therefore L[t^n] = \frac{\Gamma n+1}{s^{n+1}} \quad \forall n$$

Note:-

1) when n - +ve integer, $\Gamma n+1 = n!$,

$$\therefore L[t^n] = \frac{n!}{s^{n+1}}$$

5) $$L[e^{at}] = \frac{1}{s-a}, \quad (s-a > 0)$$

5) $L[e^{at}] = \frac{1}{s-a}$, $(s-a>0)$

Sol:-

By def, $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt \rightarrow (1)$

let $L[e^{at}] = \int_0^\infty e^{-st} e^{at}\, dt$

$= \int_0^\infty e^{(a-s)t}\, dt = \int_0^\infty e^{-(s-a)t}\, dt$

$= \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty = \frac{-1}{s-a}\{0-1\}$

$\therefore L[e^{at}] = \frac{1}{s-a}$, $s-a>0$

6) $L[e^{-at}] = \frac{1}{s+a}$ if $s>a$

7) $L\{\sin at\} = \frac{a}{s^2+a^2}$, $s>0$

P:-

By def, $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt \rightarrow (1)$

let $L[\sin at] = \int_0^\infty e^{-st} \sin at\, dt \rightarrow (2)$

w.k.that

$\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2+b^2}\{a\sin bx - b\cos bx\} \rightarrow (3)$

By using eqn (3), eqn (2) reduces to

$L[\sin at] = \left\{\frac{e^{-st}}{s^2+a^2}\{-s\sin at - a\cos at\}\right\}_0^\infty$

$= \frac{-1}{s^2+a^2}\{0-1[-a]\}$

$L[\sin at] = \frac{a}{s^2+a^2}$, $s>a$

H.W 8) $L[\cos at] = \frac{s}{s^2+a^2}$, $s>a$

$\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2+b^2}\{a\cos bx + b\sin bx\}$

9) $L[\cosh at] = \frac{s}{s^2-a^2}$ if $s>|a|$

P:- w.k.that $\cosh at = \frac{e^{at}+e^{-at}}{2} \rightarrow (a)$

P:- w.k.that $\cosh at = \frac{\quad}{2}$ $\rightarrow$ (a)

Let $L[\cosh at] = L\left[\frac{e^{at}+e^{-at}}{2}\right]$ (∵ a)

$= \frac{1}{2}\{L[e^{at}] + L[e^{-at}]\}$ (∵ by Linearity Pr.)

$= \frac{1}{2}\left\{\frac{1}{s-a} + \frac{1}{s+a}\right\}$

$= \frac{1}{2}\left\{\frac{s+\cancel{a}+s-\cancel{a}}{s^2-a^2}\right\} = \frac{1}{\cancel{2}}\left\{\frac{\cancel{2}s}{s^2-a^2}\right\}$

$\therefore L[\cosh at] = \frac{s}{s^2-a^2}, \quad s > |a|$

|||ly $L[\sinh at] = \frac{a}{s^2-a^2}, \quad s > |a|$

Pbls:- Find Laplace Transform of the foll. fn-s

1) $f(t) = (t^2+1)^2$

$L[f(t)] = L[(t^2+1)^2] = L[t^4+2t^2+1]$

$= L[t^4] + 2L[t^2] + L[1]$ (∵ by L.P)

$= \frac{4!}{s^5} + 2\cdot\frac{2!}{s^3} + \frac{1}{s}$

$L[(t^2+1)^2] = \frac{24}{s^5} + \frac{4}{s^3} + \frac{1}{s}$

2) $L[\sin 2t \cos t]$

Sol:- $2\sin a\cos b = \sin(a+b) + \sin(a-b)$ $\rightarrow$ (a)

Let $L[\sin 2t \cos t] = \frac{1}{2} L[\sin 3t + \sin t]$

$= \frac{1}{2}\{L[\sin 3t] + L[\sin t]\}$

$= \frac{1}{2}\left\{\frac{3}{s^2+9} + \frac{1}{s^2+1}\right\}$

$$\begin{array}{r} 3s^2+3 \\ s^2+9 \\ \hline 4s^2+12 \end{array}$$

$L[\sin 2t \cos t] = \frac{2s^2+6}{(s^2+9)(s^2+1)}$

3) $L[(\sin t + \cos t)^2] = L[1 + \sin 2t] = \frac{1}{s} + \frac{2}{s^2+4}$

4) $L[(\sin t - \cos t)^3] = ?$

Sol:- $(\sin t - \cos t)^3 = \sin^3 t - 3\sin^2 t\cos t + 3\sin t\cos^2 t - \cos^3 t$

$= \sin^3 t - \cos^3 t - 3(1-\cos^2 t)\cos t + 3\sin t(1-\sin^2 t)$

$\ldots + 3\cos^3 t$

$= \sin^3 t - \cos^3 t - 3\cos t + 3\cos^3 t$
$+ 3\sin t - 3\sin^3 t$

$= 2\cos^3 t - 2\sin^3 t - 3\cos t + 3\sin t$

$= \frac{2}{4}[\cos 3t + 3\cos t] - \frac{2}{4}[3\sin t - \sin 3t]$
$- 3\cos t + 3\sin t$

$(\sin t - \cos t)^3 = \frac{1}{2}\cos 3t - \frac{3}{2}\cos t + \frac{3}{2}\sin t + \frac{1}{2}\sin 3t \rightarrow (1)$

$L[(\sin t - \cos t)^3] = \frac{1}{2}L[\cos 3t] - \frac{3}{2}L[\cos t]$
$+ \frac{3}{2}L[\sin t] + \frac{1}{2}L[\sin 3t]$

$\boxed{\frac{1}{2}\left[\frac{3(1-s)}{s^2+1} + \frac{s+3}{s^2+9}\right]}$

H.W

5) $f(t) = \frac{e^{-at}-1}{a}$

6) $f(t) = \sinh^3 2t$

7) $f(t) = \frac{1}{\sqrt{\pi t}}$

8) $f(t) = 2^t$

5) $L\left[\frac{e^{-at}-1}{a}\right] = \frac{1}{a}\left\{\frac{1}{s+a} - \frac{1}{s}\right\} = \frac{-1}{s(s+a)}$ $(\because L.F$

6) w.k.that $\sinh 2t = \frac{e^{2t} - e^{-2t}}{2}$

$\sinh^3 2t = \frac{1}{8}\left\{e^{2t} - e^{-2t}\right\}^3$

$L[\sinh^3 2t] = \frac{48}{(s^2-4)(s^2-36)}$

7) $L\left[\frac{1}{\sqrt{\pi t}}\right] = \frac{1}{\sqrt{\pi}} \frac{\Gamma 1/2}{s^{1/2}} = \frac{1}{\sqrt{s}}$ $(\because \Gamma 1/2 = \sqrt{\pi})$

8) $L[2^t] = \frac{1}{s - \log 2}$

9) P.T the fn. $f(t) = t^2$ is of exponential order 3.

Sol:- let $\lim_{t\to\infty} t^2 e^{-3t} = 0$ = finite ž" $(\because e^{-\infty} = 0)$

$\left[\frac{t^2}{e^{3t}} - L\text{-}H\right]$

$\therefore t^2$ is of exponential order 3.

10) P.T the L.T of e^{t^3} does not exists

Sol:- let $f(t) = e^{t^3}$

let $\lim_{t\to\infty} e^{-at} e^{t^3} = \lim_{t\to\infty} e^{t(t^2-a)} = \infty$

$\therefore t^3$ is not an exponential order.

$\therefore e^{t^3}$ is not an exponential order.

$\therefore$ we can'not find $L[e^{t^3}]$.

11) Obtain the L.T of the fn. $f(t) = \begin{cases} (t-1)^2, & t > 1 \\ 0, & 0 < t < 1 \end{cases}$

Sol:- By def,

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt \rightarrow (1)$$

$$= \cancel{\int_0^1 e^{-st}(0)\, dt} + \int_1^\infty e^{-st}(t-1)^2\, dt$$

$$L[f(t)] = \int_1^\infty e^{-st}(t-1)^2\, dt \rightarrow (2)$$

let $t-1 = u \Rightarrow t = 1+u \Rightarrow dt = du$
when $t=1 \Rightarrow u=0$
when $t=\infty \Rightarrow u=\infty$ $\Big\}$ in (2)

$$\Rightarrow L[f(t)] = \int_0^\infty e^{-s(1+u)} u^2\, du$$

$$= e^{-s} \int_0^\infty e^{-su} u^2\, du = e^{-s} L[t^2] \quad (\because (1))$$

$$L[f(t)] = e^{-s} \frac{2!}{s^3} = \frac{2e^{-s}}{s^3}$$

12) $f(t) = \begin{cases} \sin t, & 0 < t < \pi \\ 0, & t > \pi \end{cases}$

Sol:- By def, $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt \rightarrow (1)$

$(0,\infty) \longrightarrow (0,\pi), (\pi,\infty)$

$$\therefore L[f(t)] = \int_0^\pi e^{-st} \sin t\, dt + \cancel{\int_\pi^\infty e^{-st}(0)\, dt}$$

$$= \left\{ \frac{e^{-st}}{s^2+1} \left[-s \cancel{\sin t} - \cos t \right] \right\}_0^\pi$$

$$= \frac{1}{s^2+1} \left\{ e^{-s\pi}(1) + 1 \right\} \quad (\because \sin n\pi = 0) \quad n = 0, 1, 2 \ldots$$

$$\therefore L[f(t)] = \frac{e^{-s\pi} + 1}{s^2+1}$$

II. First Shifting Thm:-

If $L[f(t)] = \bar{f}(s)$, then $L[e^{at} f(t)] = \bar{f}(s-a)$, $s-a > 0$.

Sol:- let, $L[f(t)] = \bar{f}(s) = \int_0^\infty e^{-st} f(t)\, dt \rightarrow (1)$

Sol:- Let $L[f(t)] = \bar{f}(s) = \int_0^\infty e^{-st} f(t)\, dt \rightarrow (1)$

Let $L[e^{at} f(t)] = \int_0^\infty e^{-st} e^{at} f(t)\, dt \quad (\because (1))$

$$= \int_0^\infty e^{-(s-a)t} f(t)\, dt$$

$\therefore L[e^{at} f(t)] = \bar{f}(s-a) \quad (\because (1))$

Note:-

1) $L[e^{-at} f(t)] = \bar{f}(s+a), \quad s+a>0$

2) $L[e^{at}] = L[e^{at}.1] = \frac{1}{s-a} \qquad (\because L(1) = \frac{1}{s})$

3) $L[e^{at} t^n] = \frac{\overline{|n+1}}{(s-a)^{n+1}} \quad \forall n \qquad (s \rightarrow s-a)$

4) $L[e^{at} \sin bt] = \frac{b}{(s-a)^2 + b^2}$

5) $L[e^{-at} \cos bt] = \frac{s+a}{(s+a)^2 + b^2}$

6) $L[e^{-at} \sinh bt] = \frac{b}{(s+a)^2 - b^2}$

7) $L[e^{at} \cosh bt] = \frac{s-a}{(s-a)^2 - b^2}$

8) $L[\sinh at\, f(t)] = \frac{1}{2}\{\bar{f}(s-a) - \bar{f}(s+a)\}$

9) $L[\cosh at\, f(t)] = \frac{1}{2}[\bar{f}(s-a) + \bar{f}(s+a)]$

Thm: $L[t \sin at] = \frac{2as}{(s^2+a^2)^2}$ & $L[t \cos at] = \frac{s^2-a^2}{(s^2+a^2)^2}$

P:- w.know that $L[t] = \frac{1}{s^2} = \bar{f}(s) \rightarrow (1)$

By First Shifting Thm,

$L[e^{at} f(t)] = \bar{f}(s-a)$ where $L[f(t)] = \bar{f}(s) \rightarrow (2)$

Let $L[e^{iat} t] = \bar{f}(s-ia) = \frac{1}{(s-ia)^2} \quad (\because (1) \& (2))$

$$= \frac{1}{(s-ia)^2} \times \frac{(s+ia)^2}{(s+ia)^2}$$

$$= \frac{s^2 - a^2 + 2ias}{(s^2+a^2)^2}$$

$\Rightarrow L[(\cos at + i \sin at) t] = \frac{s^2-a^2}{\ldots} + i \frac{2as}{\ldots}$

$$\Rightarrow L\left[(\cos at + i \sin at)\, t\right] = \frac{s^2-a^2}{(s^2+a^2)^2} + i\,\frac{2as}{(s^2+a^2)^2}$$

Comparing real & imaginary parts of the above eqns on both sides,

$$\boxed{\begin{aligned} &\Rightarrow L[t\cos at] = \frac{s^2-a^2}{(s^2+a^2)^2} \\ &\Rightarrow L[t\sin at] = \frac{2as}{(s^2+a^2)^2} \end{aligned}}$$

Pbms Find the foll.

1) $L\left[t^3 e^{-3t}\right]$

Sol:- $L[t^3] = \frac{3!}{s^4} = \bar{f}(s) \rightarrow (1)$

By 1st shifting Thm,

$$L[e^{-3t}.t^3] = \bar{f}(s+3)$$

$$\therefore L[t^3 e^{-3t}] = \frac{3!}{(s+3)^4} = \frac{6}{(s+3)^4}$$

2) $L\left[e^{-t}\sin^2 t\right]$

Sol:- $L[\sin^2 t] = L\left[\frac{1-\cos 2t}{2}\right]$

$$= \frac{1}{2}\left\{L(1) - L[\cos 2t]\right\} \quad (\because \text{by } L.P)$$

$$= \frac{1}{2}\left\{\frac{1}{s} - \frac{s}{s^2+4}\right\} = \bar{f}(s) \rightarrow (1)$$

Then, by 1st shifting property,

$$L[e^{-t}\sin^2 t] = \bar{f}(s+a) = \frac{1}{2}\left\{\frac{1}{s+1} - \frac{s+1}{(s+1)^2+4}\right\}$$

$$= \frac{1}{2}\left\{\frac{\cancel{s^2}+\cancel{2s}+5-\cancel{s^2}-1-\cancel{2s}}{(s+1)(s^2+2s+5)}\right\}$$

$$L[e^{-t}\sin^2 t] = \frac{2}{(s+1)(s^2+2s+5)}$$

3) $L\left[t e^{2t}\sin 3t\right]$

Sol:- W.k.that $L[t\sin 3t] = \frac{6s}{(s^2+9)^2}$

$$\Rightarrow L\left[e^{2t}.t\sin 3t\right] = \frac{6(s-2)}{\{(s-2)^2+9\}^2}$$

4) $L\left[e^{4t}\sin 2t\cos t\right]$ 5) $L[\cosh at\sin bt]$

4) $L\left[e^{4t}\sin 2t\cos t\right]$

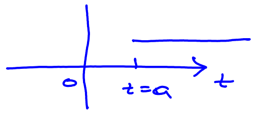
$$= \frac{1}{2}\left\{\frac{3}{s^2-8s+25} + \frac{1}{s^2-8s+17}\right\}$$

5) $L\left[\cosh at \sin bt\right]$

$$= \frac{b\,(s^2+a^2+b^2)}{\left((s-a)^2+b^2\right)\left((s+a)^2+b^2\right)}$$

Unit step Function:– The unit step fn. is defined as

$$u(t-a) = H(t-a) = \begin{cases} 1 & \text{if } t > a \\ 0 & \text{if } t < a \end{cases}$$



It is also known as Heavisides unit fn. & $H(t-a)$

let $L[u(t-a)] = \int_0^\infty e^{-st}\, u(t-a)\, dt$ (∵ by def)

$$= \int_0^a e^{-st}\,\cancel{(0)}\, dt + \int_a^\infty e^{-st}\, 1\, dt$$

$$= \left(\frac{e^{-st}}{-s}\right)_a^\infty = \frac{0 - e^{-as}}{-s}$$

$$\boxed{\therefore\ L[u(t-a)] = L[H(t-a)] = \frac{e^{-as}}{s}}$$

Second shifting Thm:– If $L[f(t)] = \bar{f}(s)$, and $a > 0$, then $L\{f(t-a)\,u(t-a)\} = e^{-as}\bar{f}(s)$, where $u(t-a)$ is a unit step fn.

(or) $L\{f(t-a)\,H(t-a)\} = e^{-as}\bar{f}(s)$, where $H(t-a)$ – Heavisides fn.

Sol:– Given that

$$L[f(t)] = \bar{f}(s) = \int_0^\infty e^{-st} f(t)\, dt \rightarrow (1)$$

let $L[f(t-a)\,u(t-a)] = \int_0^\infty e^{-st}\{f(t-a)\,\underline{\underline{u(t-a)}}\}\, dt$

$$= \int_0^a e^{-st}\,\cancel{f(t-a)}\,(0)\, dt + \int_a^\infty e^{-st} f(t-a)\, 1\, dt$$

(∵ By def of unit step fn.)

$$L[f(t-a)\,u(t-a)] = \int_a^\infty e^{-st} f(t-a)\, dt \rightarrow (2)$$

let $t-a = u \Rightarrow t = u+a \Rightarrow dt = du$ Sub these values

when $t = a \Rightarrow u = 0$ } in ②
" $t = \infty \Rightarrow u = \infty$

$$\therefore L[f(t-a)\,u(t-a)] = \int_0^\infty e^{-s(u+a)} f(u)\,du$$

$$= \int_0^\infty e^{-su} e^{-as} f(u)\,du$$

$$= e^{-as} \int_0^\infty e^{-su} f(u)\,du$$

$$\boxed{\begin{aligned} &\therefore L[f(t-a)u(t-a)] = e^{-as}\bar{f}(s) \\ &\text{(or)}\quad L[f(t-a)\,H(t-a)] = e^{-as}\bar{f}(s) \end{aligned}}$$ (∵ ①)

Another form of Second shifting Translation:-

If $L[f(t)] = \bar{f}(s)$ and $g(t) = \begin{cases} f(t-a), & t > a \\ 0, & t < a \end{cases}$

then $L[g(t)] = e^{-as}\bar{f}(s)$.

P:- By def,

$$L[g(t)] = \int_0^\infty e^{-st} g(t)\,dt \rightarrow ①$$

Given that $g(t) = \begin{cases} f(t-a), & t > a \\ 0, & t < a \end{cases} \rightarrow ②$

Subu eqn ② in eqn ①, then we have

$$L[g(t)] = \int_0^a e^{-st}(0)\,dt + \int_a^\infty e^{-st} f(t-a)\,dt$$

$$L[g(t)] = \int_a^\infty e^{-st} f(t-a)\,dt \rightarrow ③$$

let $t - a = u \Rightarrow t = u + a \Rightarrow dt = du$ } in ③
when $t = a \Rightarrow u = 0$
" $t = \infty \Rightarrow u = \infty$

$$L[g(t)] = e^{-as}\int_0^\infty e^{-su} f(u)\,du$$

$$\boxed{\therefore L[g(t)] = e^{-as}\bar{f}(s)}$$

Pblms

1) Find $L[g(t)]$, where $g(t) = \begin{cases} \cos(t - \pi/3), & t > \pi/3 \\ 0, & t < \pi/3 \end{cases}$

Sol:- Given that

Sol:- Given ...

$$g(t) = \begin{cases} f(t-\pi/3), & t > \pi/3 \\ 0, & t < \pi/3 \end{cases} \rightarrow (1)$$

where $f(t) = \cos t \rightarrow (2)$

w.k.that $L[f(t)] = \bar{f}(s) = L[\cos t]$

$$\bar{f}(s) = \frac{s}{s^2+1} \rightarrow (3)$$

By second shifting property, we have

$L[g(t)] = e^{-as}\bar{f}(s)$ where $a = \pi/3$

$$L[g(t)] = e^{-\frac{\pi s}{3}} \frac{s}{s^2+1} \quad (\because (3))$$

2) Find $L[(t-2)^3 u(t-2)]$.

Sol:- $L[f(t-a)u(t-a)] = e^{-as}\bar{f}(s) = e^{-as}L[f(t)] \rightarrow (1)$

let $f(t) = t^3 \Rightarrow L[t^3] = \bar{f}(s) = \frac{3!}{s^4} = \frac{6}{s^4}$

subn this in (1), where $a=2$

$$\therefore L[(t-2)^3 u(t-2)] = e^{-2s} \cdot \frac{6}{s^4}.$$

3) $L[e^{-3t}u(t-2)]$

Sol:- $L[e^{-3t}u(t-2)] = L[e^{-3(t-2)} \cdot \underline{\underline{e^{-6}}}\, u(t-2)]$

$$= e^{-6} L[e^{-3(t-2)} u(t-2)]$$

$$= e^{-6}\{e^{-2s} \cdot L[e^{-3t}]\}$$

$$L[e^{-3t}u(t-2)] = e^{-6-2s} \cdot \frac{1}{s+3} = \frac{e^{-2(s+3)}}{s+3}$$

Change of Scale Property:-

If $L[f(t)] = \bar{f}(s)$, then $L\{f(at)\} = \frac{1}{a}\bar{f}(s/a)$.

P:- By def,

$$L[f(t)] = \bar{f}(s) = \int_0^\infty e^{-st} f(t)\, dt \rightarrow (1)$$

let $L[f(at)] = \int_0^\infty e^{-st} f(at)\, dt \quad (\because (1)) \rightarrow (2)$

let $at = u \Rightarrow t = \frac{u}{a} \Rightarrow dt = \frac{du}{a}$? Subn let $at = u \Rightarrow t = \frac{u}{a} \Rightarrow dt = \frac{du}{a}$ } Sub$_{n}$ in (2)

when $t = 0 \Rightarrow u = 0$

" $t = \infty \Rightarrow u = \infty$

$$L[f(at)] = \int_0^\infty e^{-s(u/a)} f(u) \frac{du}{a}$$

$$= \frac{1}{a}\int_0^\infty e^{-(s/a)u} f(u)\,du$$

$$\therefore L[f(at)] = \frac{1}{a}\bar{f}(s/a) \qquad (\because (1))$$

Note:-

1) $L[f(t/a)] = a\bar{f}(as) \qquad (\because t/a = u)$

Pbl$_{ms}$

1) If $L[f(t)] = \frac{9s^2-12s+15}{(s-1)^3}$, find $L[f(3t)]$ using change of scale property.

Sol:- By change of scale property,

$$L[f(3t)] = \frac{1}{3}\bar{f}(s/3) \rightarrow (1)$$

Given that

$$L[f(t)] = \bar{f}(s) = \frac{9s^2-12s+15}{(s-1)^3} \rightarrow (2)$$

sub$_{n}$ eqn (2) in eqn (1), then

$$L[f(3t)] = \frac{1}{3}\left\{\frac{9(s/3)^2-12(s/3)+15}{\left(\frac{s}{3}-1\right)^3}\right\}$$

$$= \frac{1}{\cancel{3}}\frac{\cancel{3^3}}{\cancel{3^2}} \cdot \frac{9s^2-4s+5}{(s-3)^3}$$

$$\therefore L[f(3t)] = \frac{9s^2-4s+5}{(s-3)^3}$$

---

2) If $L[f(t)] = \frac{1}{s}e^{-1/s}$, then P.T $L[e^{-t}f(3t)] = \frac{e^{-3/(s+1)}}{s+1}$.

Sol:- Given that $\bar{f}(s) = \frac{1}{s}e^{-1/s} \rightarrow (1)$

$$L[f(3t)] = \frac{1}{3}\bar{f}(s/3) \qquad (\because \text{by 2nd sh.. P.})$$

$$= \frac{1}{3}\frac{1}{(s/3)}e^{-1/(s/3)}$$

$$\Rightarrow L[f(3t)] = \frac{1}{s}e^{-3/s}$$

$-3/(s+1)$

$$\Rightarrow L\left[e^{-t} f(3t)\right] = \frac{1}{s+1} e^{-3/(s+1)} \quad (\because \text{ by } 1^{st} \text{ Shifting Prop})$$

H.W

3) If $L[\sin t] = \frac{1}{s^2+1}$, Find $L[\sin 3t] = \frac{3}{s^2+9}$

4) If $L\left[\frac{\sin t}{t}\right] = \tan^{-1}(1/s)$, Find $L\left[\frac{\sin at}{t}\right]$.

Sol:- $L[f(at)] = L\left\{\frac{\sin at}{at}\right\}$ where $f(t) = \frac{\sin t}{t}$

$$\Rightarrow \frac{1}{a}\bar{f}(s/a) = \frac{1}{a} L\left(\frac{\sin at}{t}\right)$$

$$\Rightarrow \cancel{\frac{1}{a}} \tan^{-1}(a/s) = \cancel{\frac{1}{a}} L\left[\frac{\sin at}{t}\right]$$

$$\Rightarrow L\left(\frac{\sin at}{t}\right) = \tan^{-1}(a/s)$$

---

4. Laplace Transform of Derivatives:- (multiplication by s)

If $f(t)$ is continuous and exponential order, and $f'(t)$ is sectionally continuous then

$$L[f'(t)] = s\bar{f}(s) - f(0)$$

p:- Given that $f(t)$ is continuous & exponential order.

$\therefore L[f(t)]$ - exists.

By def, $L[f(t)] = \bar{f}(s) = \int_0^\infty e^{-st} f(t)\, dt \rightarrow (1)$

Let $L[f'(t)] = \int_0^\infty \underset{u}{\underline{e^{-st}}}\, \underset{v}{\underline{f'(t)}}\, dt$

$$= \left\{e^{-st} \cdot f(t) - \int_0^\infty -s e^{-st} \cdot f(t)\, dt\right\}_0^\infty$$

$$= \left\{(0 - f(0)) + s\int_0^\infty e^{-st} f(t)\, dt\right\}$$

$$\therefore L[f'(t)] = s\bar{f}(s) - f(0) \quad (\because (1))$$

Note:-

1) $L[f''(t)] = s^2\bar{f}(s) - sf(0) - f'(0)$

Sol:- let $g(t) = f'(t) \Rightarrow g'(t) = f''(t)$

$L[g'(t)] = sL[g(t)] - g(0)$

$$= s\,L[f'(t)] - f'(0)$$

$$= s\{s\bar{f}(s) - f(0)\} - f'(0)$$

$$\therefore L[f''(t)] = s^2\bar{f}(s) - sf(0) - f'(0)$$

$$\rightarrow L[f'''(t)] = s^3\bar{f}(s) - s^2 f(0) - s f'(0) - f''(0)$$

$$\rightarrow L[f^n(t)] = s^n\bar{f}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - sf^{n-2}(0) - f^{n-1}(0)$$

Note:-

1) $L[f^n(t)] = s^n\bar{f}(s)$ if $f(0) = f'(0) = \ldots f^{n-1}(0) = 0$

Pblms

I. Using the thm on derivatives, find L.T. of the following fn.s

a) $f(t) = t^2$ b) $f(t) = e^{at}$ c) $f(t) = \sin at$

a) let
$$\left.\begin{array}{ll} f(t) = t^2 & \Rightarrow f(0) = 0 \\ f'(t) = 2t & \Rightarrow f'(0) = 0 \\ f''(t) = 2 & \Rightarrow f''(0) = 2 \end{array}\right\} \rightarrow (1)$$

$$f'''(t) = f^{iv}(t) = \ldots = 0$$

By Laplace transform of derivatives,

$$L[f''(t)] = s^2\bar{f}(s) - sf(0) - f'(0) \rightarrow (2)$$

$$L[2] = s^2 L[t^2] - \cancel{s(0)} - (0) \quad (\because (1))$$

$$\Rightarrow s^2 L[t^2] = L[2] = \frac{2}{s} \quad \left(\because L[2] = 2L[1] = 2\cdot\frac{1}{s}\right)$$

$$\Rightarrow L[t^2] = \frac{2}{s\cdot s^2} = \frac{2!}{s^3}$$

$$\therefore L[t^2] = \frac{2!}{s^3}$$

$$\boxed{L[t^n] = \frac{n!}{s^{n+1}}}$$

b)
$$\left.\begin{array}{ll} f(t) = e^{at} & \Rightarrow f(0) = e^0 = 1 \\ f'(t) = ae^{at} & \Rightarrow f'(0) = a \\ f''(t) = a^2e^{at} & \Rightarrow f''(0) = a^2 \end{array}\right\} \rightarrow (1)$$

By Laplace transform of derivatives,

$$L[f''(t)] = s^2\bar{f}(s) - sf(0) - f'(0) \rightarrow (2)$$

From eqns (1) & (2), we have

$$L[a^2 e^{at}] = s^2 L[e^{at}] - s\cdot 1 - a$$

$$\Rightarrow a^2 L[e^{at}] = s^2 L[e^{at}] - (s+a)$$

$$\Rightarrow (s^2-a^2) L[e^{at}] = (s+a)$$

$$\Rightarrow L[e^{at}] = \frac{s+a}{s^2-a^2} = \frac{\cancel{s+a}}{(s-a)\cancel{(s+a)}}$$

$$\therefore L[e^{at}] = \frac{1}{s-a}, \quad s-a>0$$

---

c) $f(t) = \sin at$

$$L[f''(t)] = s^2 L[f(t)] - s f(0) - f'(0)$$

$$L[-a^2 \sin at] = s^2 L[\sin at] - \cancel{s(0)} - a\cdot 1$$

$$-a^2 L[\sin at] = s^2 L[\sin at] - a$$

$$\Rightarrow (s^2+a^2) L[\sin at] = a$$

$$\therefore L[\sin at] = \frac{a}{s^2+a^2}$$

---

2) If $L\left[2\sqrt{\frac{t}{\pi}}\right] = 1/s^{3/2}$ then S.T

$$L\left[\frac{1}{\sqrt{t\pi}}\right] = 1/s^{1/2}$$

Sol:- Given that $L\left[2\sqrt{\frac{t}{\pi}}\right] = \frac{1}{s^{3/2}} \rightarrow (1)$

$$\left.\begin{aligned} &\text{let } f(t) = 2\sqrt{\frac{t}{\pi}} \Rightarrow f(0) = 0 \\ &f'(t) = \frac{\cancel{2}}{\sqrt{\pi}}\cdot\frac{1}{\cancel{2}}\, t^{\frac{1}{2}-1} = \underline{\underline{\frac{1}{\sqrt{\pi t}}}} \end{aligned}\right\} \rightarrow (2)$$

By L.T of derivatives,

$$L[f'(t)] = s L[f(t)] - f(0) \qquad \left(\because \bar{f}(s) = L[f(t)]\right)$$

$$L\left[\frac{1}{\sqrt{\pi t}}\right] = s\, L\left[2\sqrt{\frac{t}{\pi}}\right] - 0 \quad (\because (2))$$

$$= s\cdot\frac{1}{s^{3/2}} \quad (\because (1))$$

$$\therefore L\left[\frac{1}{\sqrt{\pi t}}\right] = \frac{1}{\sqrt{s}} = 1/s^{1/2}$$

Laplace Transforms of Integrals:-

If $L[f(t)] = \bar{f}(s)$ then $L\left[\int_0^t f(u)\,du\right] = \frac{1}{s}\bar{f}(s)$.

$\because$ Given that $L[f(t)] = \bar{f}(s) \rightarrow (1)$

P:- Given that $L[f(t)] = \bar{f}(s) \rightarrow (1)$

let $\phi(t) = \int_0^t f(u)\,du \xrightarrow{(2)} \phi(0) = 0$

$\Rightarrow \phi'(t) = f(t)$ $\Big] \rightarrow (3)$

w.k.that $\phi'(t) = f(t)$

$\Rightarrow L[\phi'(t)] = L[f(t)]$

$\Rightarrow L[\phi'(t)] = \bar{f}(s) \rightarrow (4)$

By L'' T'' of derivatives,

$L[\phi'(t)] = s\,L[\phi(t)] - \phi(0)$

$\Rightarrow \bar{f}(s) = s\,L\left[\int_0^t f(u)\,du\right] - 0$ (∵ (2), (3) & (4))

$\Rightarrow L\left[\int_0^t f(u)\,du\right] = \frac{1}{s}\bar{f}(s)$

$\therefore L\left[\int_0^t f(u)\,du\right] = \frac{\bar{f}(s)}{s}$

Note:-

1) $L\left[\int_0^t \int_0^t f(u)\,du\,du\right] = \frac{\bar{f}(s)}{s^2}$

2) $L\left[\int_0^t \int_0^t \int_0^t f(u)\,du\,du\,du\right] = \frac{\bar{f}(s)}{s^3}$

lly $L\left[\underbrace{\int_0^t \int_0^t \cdots \int_0^t}_{(n\text{-times})} f(u)\,du \ldots du\right] = \frac{\bar{f}(s)}{s^n}$

Pb'' s

1) Find $L\left[\int_0^t e^{-t}\cos t\,dt\right]$

Sol:- Let $f(t) = e^{-t}\cos t \rightarrow (a)$

w.k.that, $L[\cos t] = \frac{s}{s^2+1} \rightarrow (b)$

Then by first shifting Translation,

$L[e^{-t}\cos t] = \frac{s+1}{(s+1)^2+1} = \frac{s+1}{s^2+2s+2}$

$$\therefore L[f(t)] = L[e^{-t}\cos t] = \bar{f}(s) = \frac{s+1}{s^2+2s+2} \longrightarrow (1)$$

$\therefore$ By L.T. of Integrals,

$$L\left[\int_0^t f(u)\,du\right] = \frac{\bar{f}(s)}{s}$$

$$\therefore L\left[\int_0^t e^{-t}\cos t\,dt\right] = \frac{(s+1)}{s(s^2+2s+2)}$$

---

2) Find $L\left[\int_0^t\int_0^t \cosh at\,dt\,dt\right]$. $\left(\frac{1}{s(s^2-a^2)}\right)$

---

3) If $L[t\sin at] = \frac{2as}{(s^2+a^2)^2}$, then find

$L[\sin at + at\cos at]$ $\left(= \frac{2as^2}{(s^2+a^2)^2}\right)$

Laplace Transform of $t^n f(t)$:- (Multiplication by t^n)

If $L\{f(t)\} = \bar{f}(s)$, then $L\{tf(t)\} = (-1)\frac{d}{ds}\{\bar{f}(s)\}$.

Sol:- Given that,

$$L\{f(t)\} = \bar{f}(s) = \int_0^\infty e^{-st} f(t)\,dt \longrightarrow (1)$$

Diff. eqn (1) w.r.t 's' on both sides,

$$\frac{d}{ds}\{\bar{f}(s)\} = \frac{d}{ds}\left\{\int_0^\infty e^{-st} f(t)\,dt\right\} \longrightarrow (2)$$

Leibnitz's Rule for differentiation,

$$\frac{d}{dt}\left\{\int_a^b f(x,t)\,dx\right\} = \int_a^b \frac{\partial}{\partial t}\{f(x,t)\}\,dx \longrightarrow (4)$$

From eqns (2) & (4), we have

$$\frac{d}{ds}\{\bar{f}(s)\} = \int_0^\infty \frac{\partial}{\partial s}\left\{e^{-st} f(t)\right\} dt$$

$$= \int_0^\infty -t\,e^{-st} f(t)\,dt$$

$$= -\int_0^\infty e^{-st}\,tf(t)\,dt$$

$$= \int_0^\infty -t\, e^{-st} f(t)\, dt$$

$$= -\int_0^\infty e^{-st}\, t f(t)\, dt$$

$$\frac{d}{ds}\{\bar{f}(s)\} = -L[t f(t)] \quad (\because ①)$$

$$\therefore L[t f(t)] = (-1)\frac{d}{ds}\{\bar{f}(s)\}$$

Note:-

1) $L[t^2 f(t)] = (-1)^2 \frac{d^2}{ds^2}\{\bar{f}(s)\}$

Sol:- Let $g(t) = t f(t)$, then

$$L[t^2 f(t)] = L[t \cdot t f(t)] = L[t\, g(t)]$$

$$= (-1)\frac{d}{ds}\{\bar{g}(s)\} = (-1)\frac{d}{ds}\{L[g(t)]\}$$

$$= (-1)\frac{d}{ds}\{L[t f(t)]\}$$

$$= (-1)\frac{d}{ds}\{(-1)\frac{d}{ds}[\bar{f}(s)]\}$$

$$L[t^2 f(t)] = (-1)^2 \frac{d^2}{ds^2}[\bar{f}(s)]$$

---

2) $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}\{\bar{f}(s)\}$

Pb's

P.T

1) $L[t \sin at] = \frac{2as}{(s^2+a^2)^2}$.

Sol:- W.k.that $L[\sin at] = \frac{a}{s^2+a^2} = \bar{f}(s) \to ①$

By $L[t f(t)]$,

$$L[t \sin at] = (-1)\frac{d}{ds}\{\bar{f}(s)\}$$

$$= (-1)\frac{d}{ds}\left\{\frac{a}{s^2+a^2}\right\}$$

$$= (-1)\frac{(-a)(2s)}{(s^2+a^2)} = \frac{2as}{(s^2+a^2)^2}$$

Find

2) $L[t^3 e^{-3t}]$.

Sol:- W.k.that $L[e^{-3t}] = \frac{1}{s+3} = \bar{f}(s) \to ①$

let $L[t^3 \cdot e^{-3t}] = (-1)^3 \frac{d^3}{ds^3} \{\bar{f}(s)\}$

$$= (-1) \frac{d^3}{ds^3} \left\{ \frac{1}{s+3} \right\}$$

$$= (-1) \frac{d^2}{ds^2} \left\{ \frac{-1}{(s+3)^2} \right\}$$

$$= \frac{d^2}{ds^2} \left\{ \frac{1}{(s+3)^2} \right\} = \frac{d}{ds} \left\{ \frac{-2}{(s+3)^3} \right\}$$

$$\therefore L[t^3 e^{-3t}] = (-2) \cdot \frac{(-3)}{(s+3)^4} = \frac{3!}{(s+3)^4} = \frac{6}{(s+3)^4}$$

---

3) Find $L[t e^{-t} \sin 3t]$.

Sol:- w.k. that $L[t \sin 3t] = \frac{6s}{(s^2+9)^2} = \bar{f}(s)$ $\rightarrow$ (1)

$$L[e^{-t} \cdot t \sin 3t] = \bar{f}(s+1)$$

$$= \frac{6(s+1)}{[(s+1)^2+9]^2}$$

$$\therefore L[t e^{-t} \sin 3t] = \frac{6(s+1)}{(s^2+2s+10)^2}$$

---

4) If $L\{t^{1/2}\} = \frac{\sqrt{\pi}}{2 s^{3/2}}$ then find $L[t^{-1/2}] = ?$

Sol:- $L[t f(t)] = -\frac{d}{ds} [\bar{f}(s)]$

$$\Rightarrow L[t f(t)] = -\frac{d}{ds} \{L(f(t))\} \rightarrow (1)$$

let $f(t) = t^{-1/2}$ subn in (1), then

$$L[t \cdot t^{-1/2}] = -\frac{d}{ds} \{L(t^{-1/2})\}$$

$$\Rightarrow L(t^{1/2}) = \quad "$$

$$\Rightarrow \frac{\sqrt{\pi}}{2 s^{3/2}} = -\frac{d}{ds} \left\{ L\left(\frac{1}{\sqrt{t}}\right) \right\}$$

$$\Rightarrow L\left(\frac{1}{\sqrt{t}}\right) = -\int \frac{\sqrt{\pi}}{2 s^{3/2}} ds$$

$$= -\frac{\sqrt{\pi}}{2} \cdot \left(\frac{s^{-3/2+1}}{-\frac{3}{2}+1} \right)$$

$$= \cancel{\sqrt{\pi}} \frac{s^{-1/2}}{(-1/2)} = \frac{\sqrt{\pi}}{\sqrt{s}} = \sqrt{\frac{\pi}{s}}$$

$$\therefore L\left[1/\sqrt{t} \right] = \sqrt{\frac{\pi}{s}}.$$

H.W

5) Find $L[t^2 \sin at] = -\frac{d}{ds}\left\{ \frac{2as}{(s^2+a^2)^2} \right\}$

$$\left(A \cdot \frac{2a(3s^2-a^2)}{(s^2+a^2)^3} \right)$$

Division by t:-

If $L[f(t)] = \bar{f}(s)$, then $L\left[\frac{f(t)}{t}\right] = \int_s^\infty \bar{f}(s)\,ds$,

provided the integral exists.

P:- Given that

$$L[f(t)] = \bar{f}(s) = \int_0^\infty e^{-st} f(t)\,dt \rightarrow (1)$$

Integrating eqn (1) w.r.t 's' from 's' to '∞', then

$$\int_s^\infty \bar{f}(s)\,ds = \int_s^\infty \int_0^\infty e^{-st} f(t)\,dt\,ds \rightarrow (2)$$

which is a double integral.

By changing the order of integration in eqn (2), then

$$\int_s^\infty \bar{f}(s)\,ds = \int_0^\infty \left\{ \int_s^\infty e^{-st} f(t)\,ds \right\} dt$$

$$= \int_0^\infty \left[\frac{e^{-st}}{-t} \right]_s^\infty f(t)\,dt = \int_0^\infty \left[\frac{0 - e^{-st}}{-t} \right] f(t)\,dt$$

$$= \int_0^\infty \frac{e^{-st}}{t} f(t)\,dt = \int_0^\infty e^{-st} \left(\frac{f(t)}{t} \right) dt$$

$$\therefore \int_s^\infty \bar{f}(s)\,ds = L\left[\frac{f(t)}{t}\right] \quad (\because (1))$$

$$\therefore L\left[\frac{f(t)}{t}\right] = \int_s^\infty \bar{f}(s)\,ds$$

Note:-

1) $L\left[\frac{f(t)}{t^2}\right] = \int_s^\infty \int_s^\infty \bar{f}(s)\,ds\,ds$

2) $L\left[\frac{f(t)}{t^n}\right] = \int_s^\infty \int_s^\infty \cdots \int_s^\infty \bar{f}(s)\,ds\,ds \ldots ds$
(n-times)

Pbms

1) Find $L\left[\frac{\sin at}{t}\right]$.

Sol:- W.k.that $L[\sin at] = \frac{a}{s^2+a^2} = \bar{f}(s) \rightarrow (1)$

By using L. T. of division by t,

$$L\left[\frac{\sin at}{t}\right] = \int_s^\infty \bar{f}(s)\,ds$$

$$= \int_s^\infty \frac{a}{s^2+a^2}\,ds \quad (\because (1))$$

$$= \cancel{a}\cdot\frac{1}{\cancel{a}}\left[\tan^{-1}(s/a)\right]_s^\infty = \tan^{-1}\infty - \tan^{-1}\frac{s}{a}$$

$$= \frac{\pi}{2} - \tan^{-1}\frac{s}{a} = \cot^{-1}(s/a)$$

Note:-

$\therefore L\left[\frac{\sin 2t}{t}\right] = \cot^{-1}(s/2) = \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{2}\right)$

2) Find $L\left[\frac{1-e^{t}}{t}\right]$

Sol:- $L\left[\frac{1-e^t}{t}\right] = \int_s^\infty L[1-e^t]\,ds = \int_s^\infty \left[\frac{1}{s} - \frac{1}{s-1}\right] ds$

$$= \left[\log s - \log(s-1)\right]_s^\infty = \left[\log\left(\frac{s}{s-1}\right)\right]_s^\infty$$

$$= \left[0 - \log\left(\frac{s}{s-1}\right)\right] = \log\left(\frac{s-1}{s}\right)$$

$$\therefore L\left[\frac{1-e^t}{t}\right] = \log(1 - 1/s).$$

3) $L\left[\frac{1-\cos t}{t}\right] = \int_s^\infty L[1-\cos t]\,ds$

$$= \int_s^\infty \left[\frac{1}{s} - \frac{s}{s^2+1}\right] ds$$

$$= \left[\log s - \frac{1}{2}\log(s^2+1)\right]_s^\infty$$

$$= \left[\log \frac{s}{\sqrt{s^2+1}}\right]_s^\infty \quad = 0 - \log \frac{s}{\sqrt{s^2+1}}$$

$$= \log \left(\frac{\sqrt{s^2+1}}{s}\right).$$

H.W

4) $L\left[\frac{\sin 3t \cos t}{t}\right]$ 5) $\left[\frac{\cos 2t - \cos 3t}{t}\right]$