SIGNALS AND SYSTEMS

Part 1

Unit-10

II B.TECH I SEM

II ECE-2

- Name of the Faculty: Mr.A.BALA RAJU, Assistant.Professor
- Name of the Course: Signals and Systems.
- Class: II B.Tech-ECE 2-I sem.
- Subject Code: EC 304PC.
- Number of Lectures hours/Week: 4
- Number of Tutorial hours/Week: 1
- Number of Credits: 4

COURSE OBJECTIVES

- This gives the basics of Signals and Systems required for all Electrical Engineering related course
- To understand the behaviour of signal in time and frequency domain
- To understand the characteristics of LTI systems
- This gives concepts of Signals and Systems and its analysis using different transform techniques.

COURSE OUTCOMES

Upon completing this course, the student will be able to

- Differentiate various signal functions.
- Represent any arbitrary signal in time and frequency domain.
- Understand the characteristics of linear time invariant systems.
- Analyse the signals with different transform technique
- Understand different sampling techniques and comparison of signals

SYLLABUS

UNIT - I

- **Signal Analysis:** Analogy between Vectors and Signals, Orthogonal Signal Space, Signal approximation using Orthogonal functions, Mean Square Error, Closed or complete set of Orthogonal functions, Orthogonality in Complex functions, Classification of Signals and systems, Exponential and Sinusoidal signals, Concepts of Impulse function, Unit Step function, Signum function

SYLLABUS

UNIT – II

- **Fourier series:** Representation of Fourier series, Continuous time periodic signals, Properties of Fourier Series, Dirichlet's conditions, Trigonometric Fourier Series and Exponential Fourier Series, Complex Fourier spectrum.
- **Fourier Transforms**: Deriving Fourier Transform from Fourier series, Fourier Transform of arbitrary signal, Fourier Transform of standard signals, Fourier Transform of Periodic Signals, Properties of Fourier Transform, Fourier Transforms involving Impulse function and Signum function, Introduction to Hilbert Transform.

SYLLABUS

UNIT – III

- **Signal Transmission through Linear Systems:** Linear System, Impulse response, Response of a Linear System, Linear Time Invariant(LTI) System, Linear Time Variant (LTV) System, Transfer function of a LTI System, Filter characteristic of Linear System, Distortion less transmission through a system, Signal bandwidth, System Bandwidth, Ideal LPF, HPF, and BPF characteristics, Causality and Paley-Wiener criterion for physical realization, Relationship between Bandwidth and rise time, Convolution and Correlation of Signals, Concept of convolution in Time domain and Frequency domain, Graphical representation of Convolution.

SYLLABUS

UNIT – IV

- **Laplace Transforms:** Laplace Transforms (L.T), Inverse Laplace Transform, Concept of Region of Convergence (ROC) for Laplace Transforms, Properties of L.T, Relation between L.T and F.T of a signal, Laplace Transform of certain signals using waveform synthesis.
- **Z–Transforms:** Concept of Z- Transform of a Discrete Sequence, Distinction between Laplace, Fourier and Z Transforms, Region of Convergence in Z-Transform, Constraints on ROC for various classes of signals, Inverse Z-transform, Properties of Z-transforms.

SYLLABUS

UNIT – V

- **Sampling theorem:** Graphical and analytical proof for Band Limited Signals, Impulse Sampling, Natural and Flat top Sampling, Reconstruction of signal from its samples, Effect of under sampling – Aliasing, Introduction to Band Pass Sampling.
- **Correlation:** Cross Correlation and Auto Correlation of Functions, Properties of Correlation Functions, Energy Density Spectrum, Parsevals Theorem, Power Density Spectrum, Relation between Autocorrelation Function and Energy/Power Spectral Density Function, Relation between Convolution and Correlation, Detection of Periodic Signals in the presence of Noise by Correlation, Extraction of Signal from Noise by Filtering.

TEXT BOOKS:

- 1. Signals, Systems & Communications - B.P. Lathi, 2013, BSP.
- 2. Signals and Systems - A.V. Oppenheim, A.S. Willsky and S.H. Nawabi, 2 Ed.

REFERENCE BOOKS:

- 1. Signals and Systems – Simon Haykin and Van Veen, Wiley 2 Ed.,
- 2. Signals and Systems – A. Rama Krishna Rao, 2008, TMH
- 3. Fundamentals of Signals and Systems - Michel J. Robert, 2008, MGH International Edition.
- 4. Signals, Systems and Transforms - C. L. Philips, J.M.Parr and Eve A.Riskin, 3 Ed., 2004, PE.
- 5. Signals and Systems – K. Deergha Rao, Birkhauser, 2018.

Day-2: **Signal –Introduction**

Can you define Communication?

Why Communication is required?

What are the different ways of Communications in olden days and now?

Communication -It is the process of exchanging the information ,message ,data or any other from one location to another location-Called as Transmitter to the Receiver(Source to the Destination)

Why Communication? - To exchange the data/ Information

-It is the Information that drives the entire world

How many Ways-There are numerous ways

- In olden days people used to communicate with each other through

speech, gestures ,graphical symbols

Drum beats, smoke signals carrier pigeons and light beams

Day-2: Signal –Introduction

What type of signals we are using Now a days?

Electrical Signals

Why Electrical Signals ?

-To transmit the information over longer distances with very high Speed($3*10^8$ m/sec)

- The interference of noise also less.

Signal:

- Signal –Signal is a time varying physical phenomenon/ *physical quantity*

  - Carries or *contains some s*et of information or DATA that can be conveyed , displayed or manipulated

  -Mathematically , Signal is represented as a function of an independent variable t

  -usually t represents "time"- denoted by f(t) or x(t)

Day 3: Examples of signals

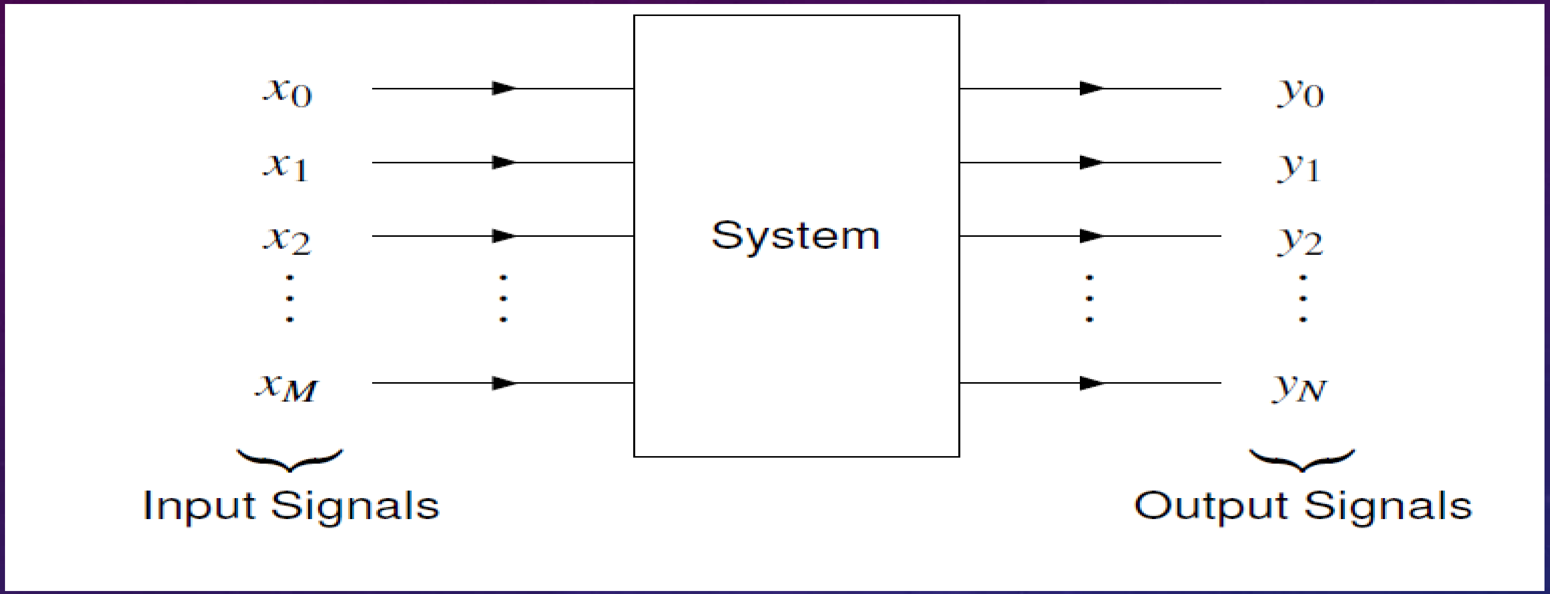
- Speech-Which we encounter for example in telephony, radio and everyday life,
- Bio-medical Signal-such as electro cardiogram(ECG-heart),electro encephalogram(EEG-brain),
- Sound and Music-Such as reproduced by the compact disc player,
- Video and Image-Which most people watch on the television, and
- Radar Signals-Which are used to determine the range of the distant targets

Some more examples of signals include:

- a voltage or current in an electronic circuit
- the position, velocity, or acceleration of an object
- a force or torque in a mechanical system
- a flow rate of a liquid or gas in a chemical process
- a digital image, digital video, or digital audio
- a stock market index

- Signal= function (Independent variable)
- $S=f(x_i)=f(x_1,x_2,x_3..........x_i)$
- x_i = number of independent variables
- If i=1 means signal is a function of one independent variable and is called as One –Dimensional Signal(1-D)- Ex-Speech signal-whose amplitude varies with time depending on the spoken word
- If i=2 means signal is a function of two independent variables and is called as Two –Dimensional Signal(2-D)- Ex-Image/Picture signal-with the horizontal and vertical co-ordinates of the image representing the two dimensions called spatial coordinates.

SYSTEMS



- It's a mathematical model of a physical process that relates the input and output
- A system is a physical device /Hard ware /soft ware that performs some operations / responds to applied input signals, to produce one or more output signals
- Input is also called as excitation and output as the response

SIGNAL PROCESSING:

- We need to process the Signals by Systems because
- To modify ,analyse the signals
- To extract additional information
- To remove the noise and interference from the signal
- To obtain the spectrum of the signal
- To transform the signal into a more suitable form
- When the signal is passed through the system then it is said to be Processed.

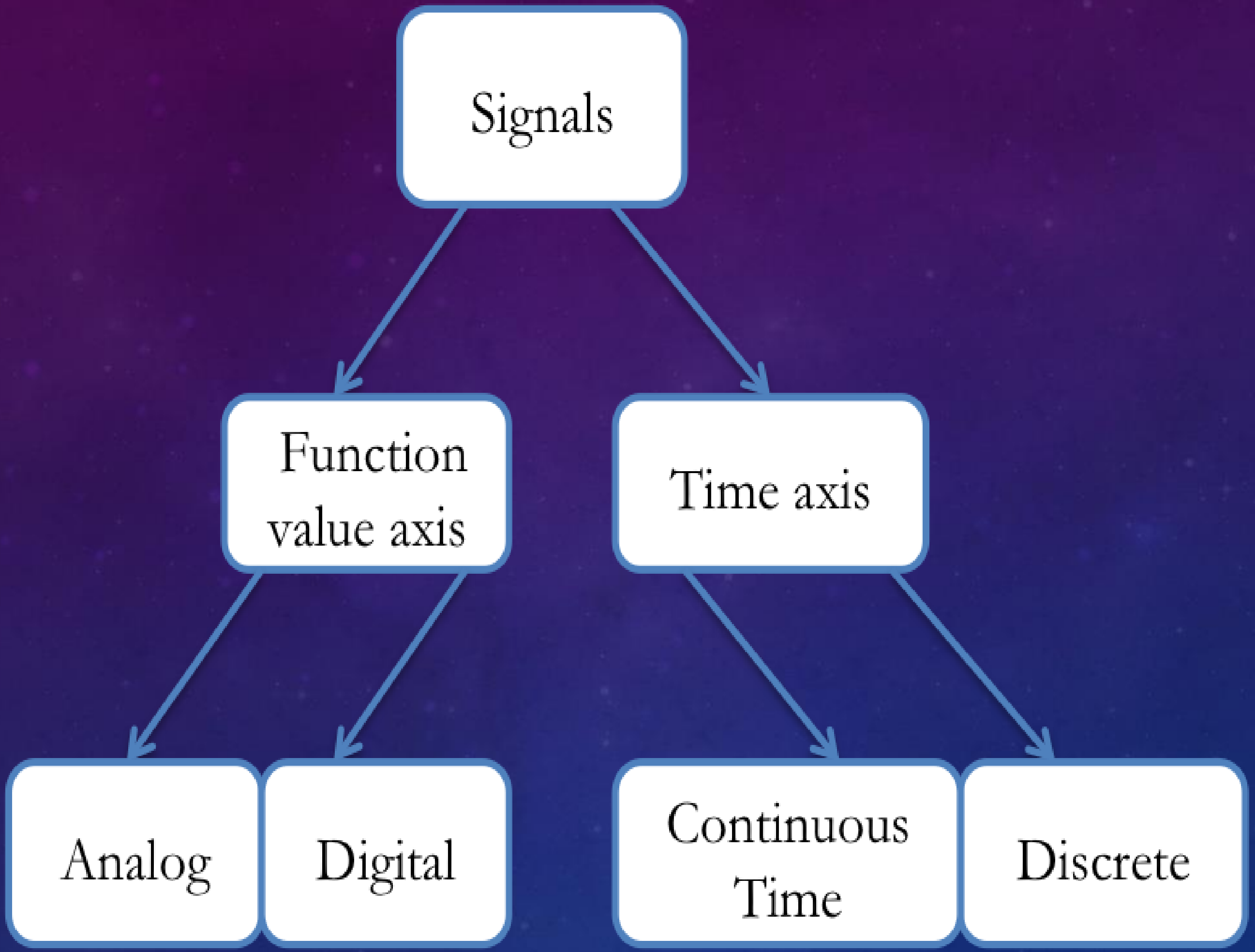
EXAMPLES OF SYSTEMS:

- Filter in a communication system
- Radar System- knowing the past location and velocity of the target the future location of the moving target is being estimated and tracked.
- Computer or the mobile phone
- Any GPP like microprocessor or Specific Digital Signal Processing Applications
- Any hydraulic, mechanical or electrical system etc.. are few examples.

DAY4-17.08.2020

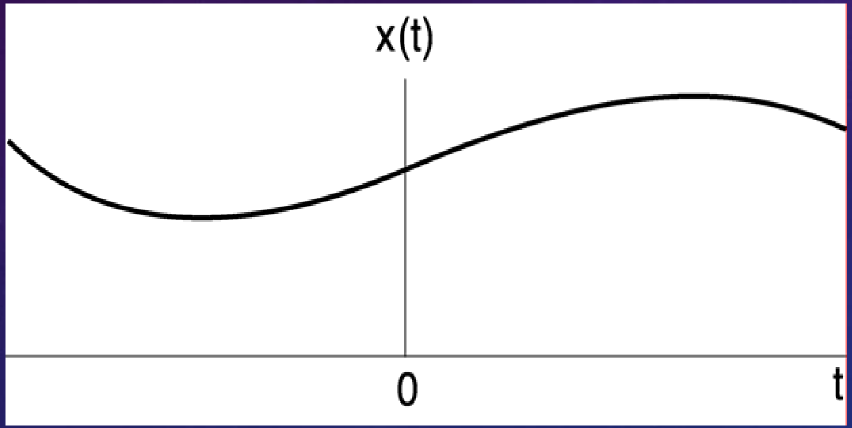
CLASSIFICATION OF SIGNALS-
SIGNALS CAN BE CLASSIFIED AS-

- CONTINUOUS TIME SIGNALS (CTS) and DISCRETE TIME SIGNALS (DTS)
- ANALOG and DIGITAL SIGNALS
- EVEN and ODD SIGNALS
- PERIODIC and NON-PERIODIC SIGNALS
- ENERGY and POWER SIGNALS
- DETERMINISTIC and RANDOM SIGNALS


Signals
Function value axis
Time axis
Analog
Digital
Continuous Time
Discrete

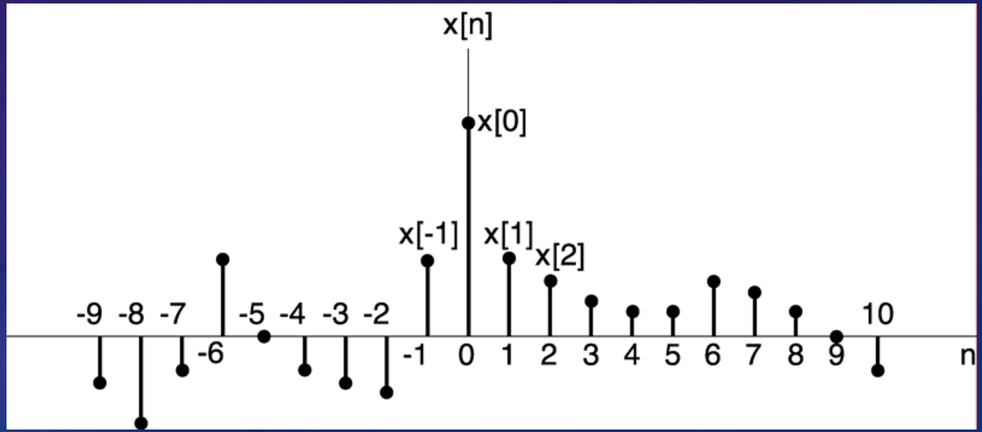
Continuous Time Signals(CTS)

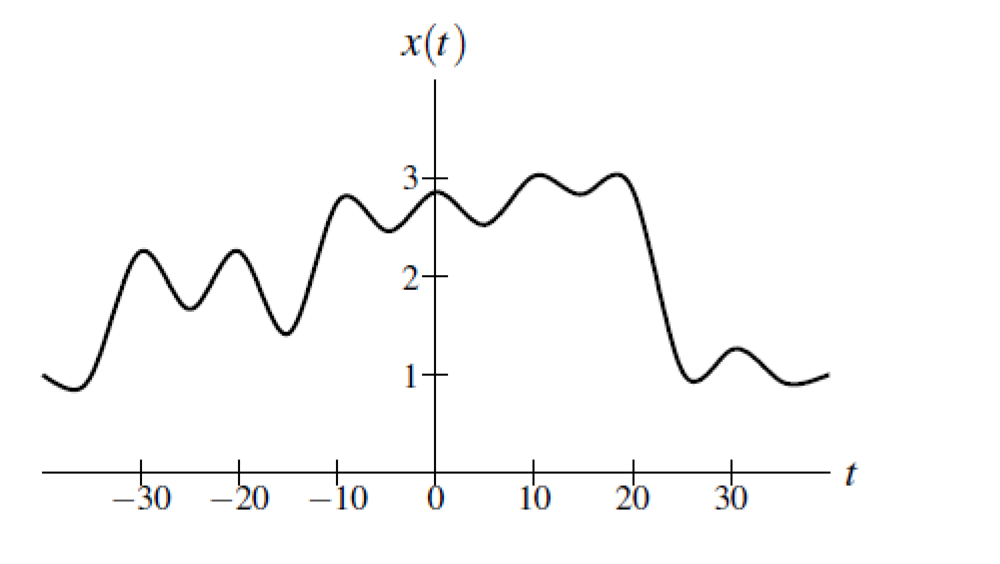
- x(t) varies continuously with respect to independent variable t
- Most of the signals in the physical world are CT signals
- E.g. voltage & current, pressure, temperature, velocity, etc.

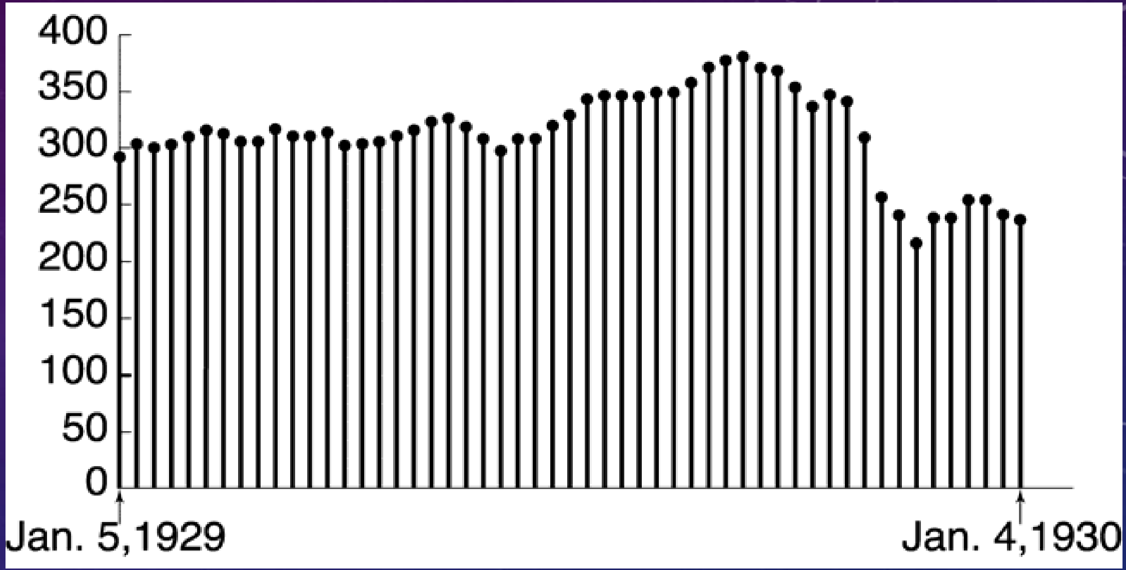


Discrete Time Signals(DTS)

- *x*[*n*], *n*—integer, time varies discretely
- Population of the *n*th generation of certain species
- Stock Market listings, Company Production of goods

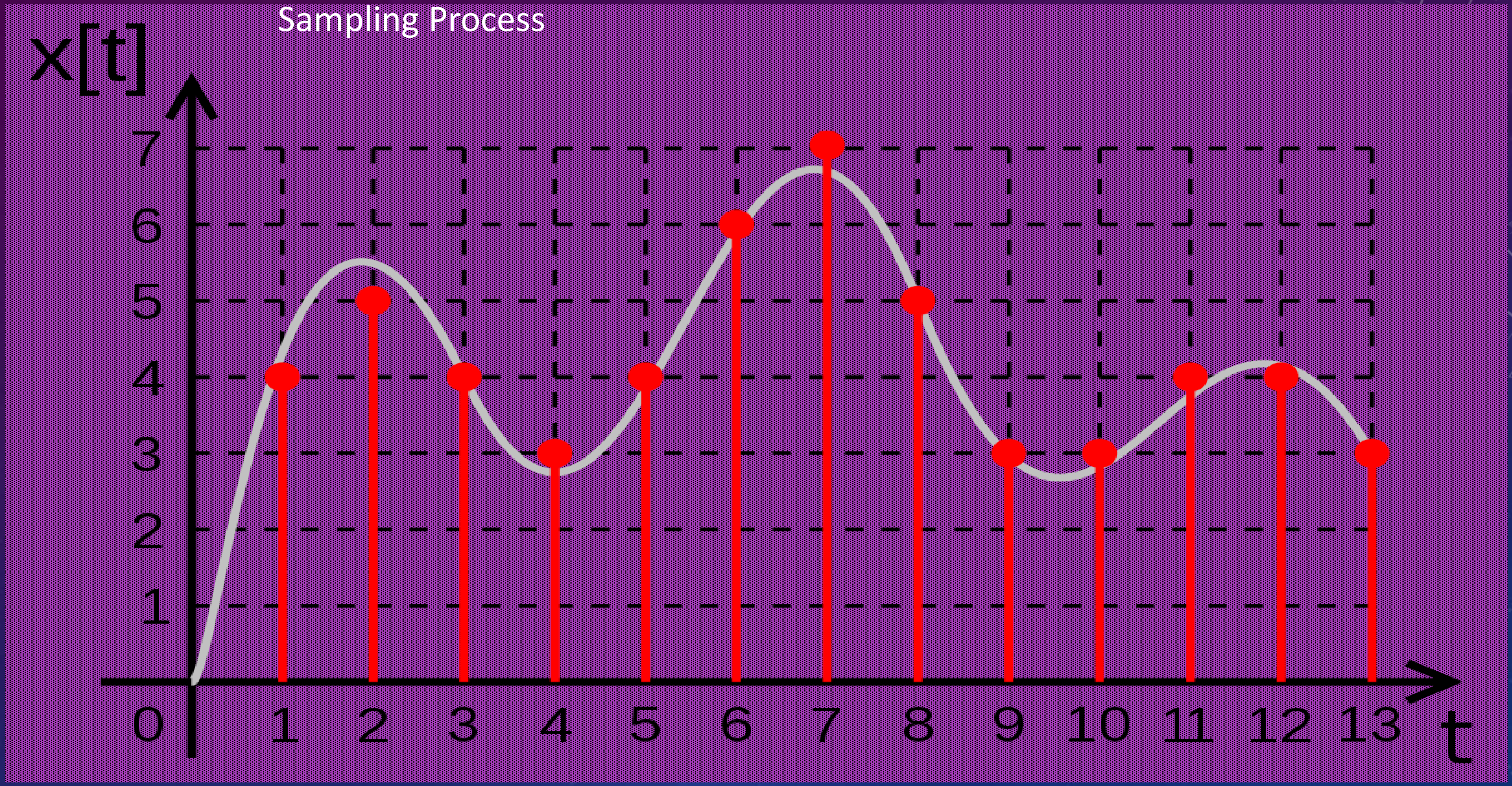






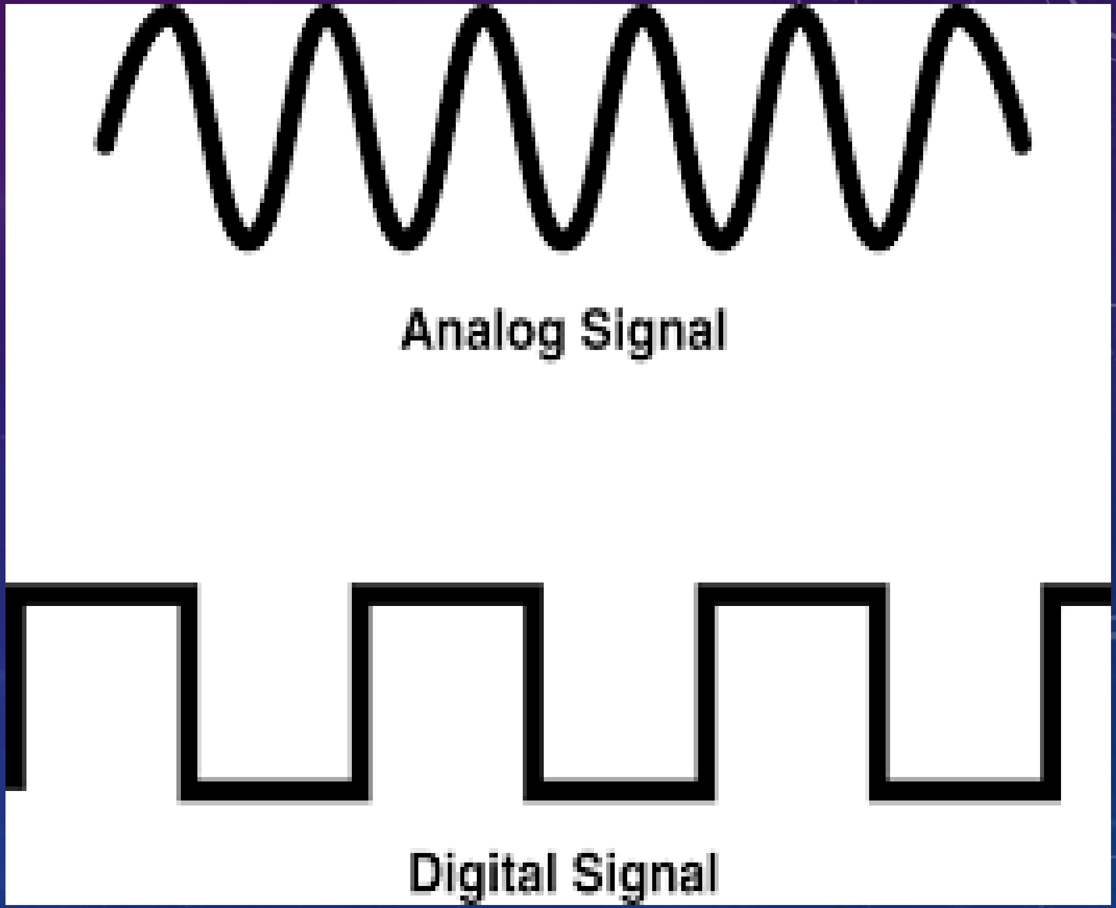
Why Discrete Time Signals?

- Can be processed by modern digital computers and digital signal processors (DSPs).


Sampling Process
x[t]
7
6
5
4
3
2
1
0
1
2
3
4
5
6
7
8
9
10
11
12
13
t

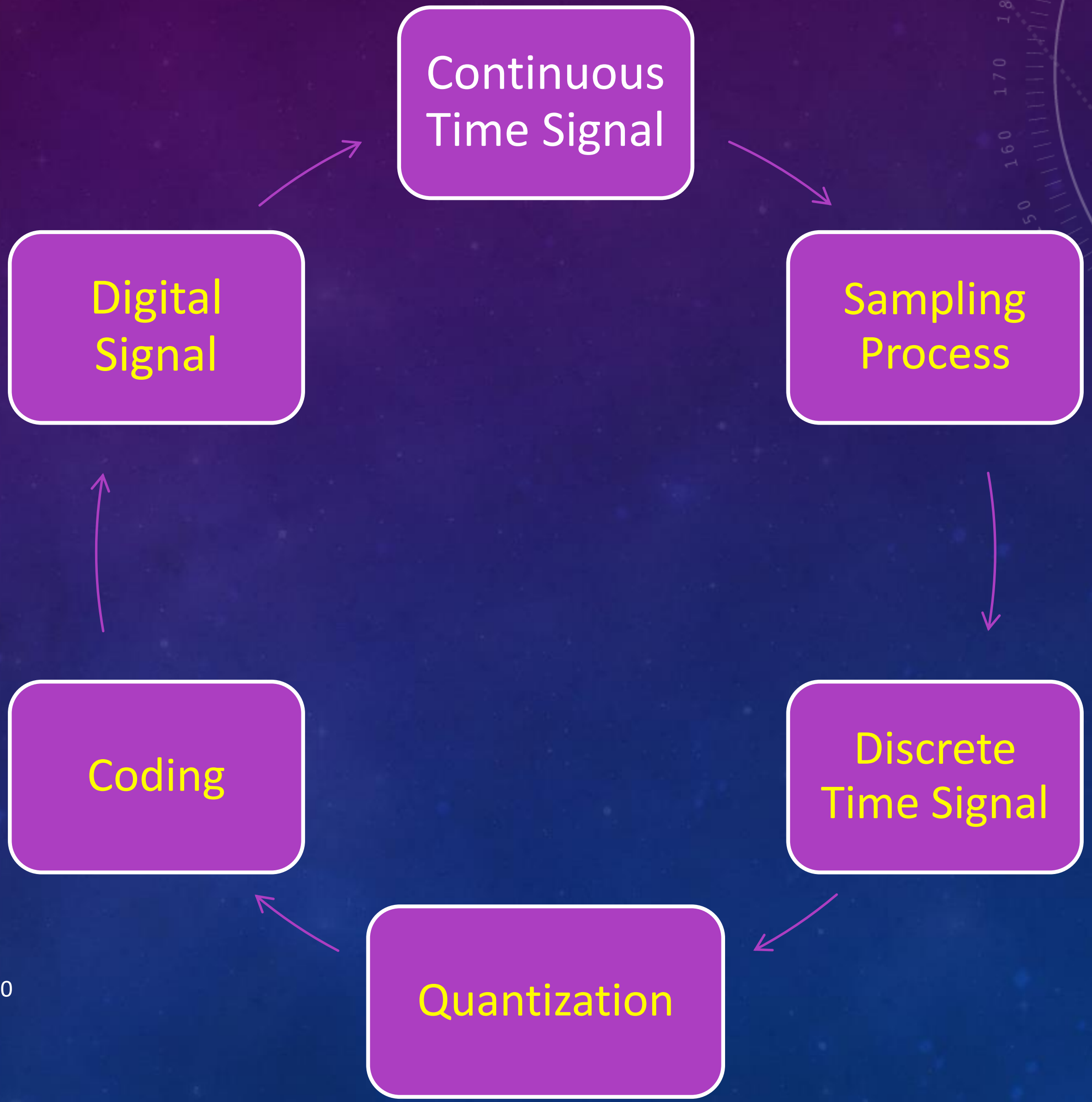
ANALOG AND DIGITAL SIGNAL

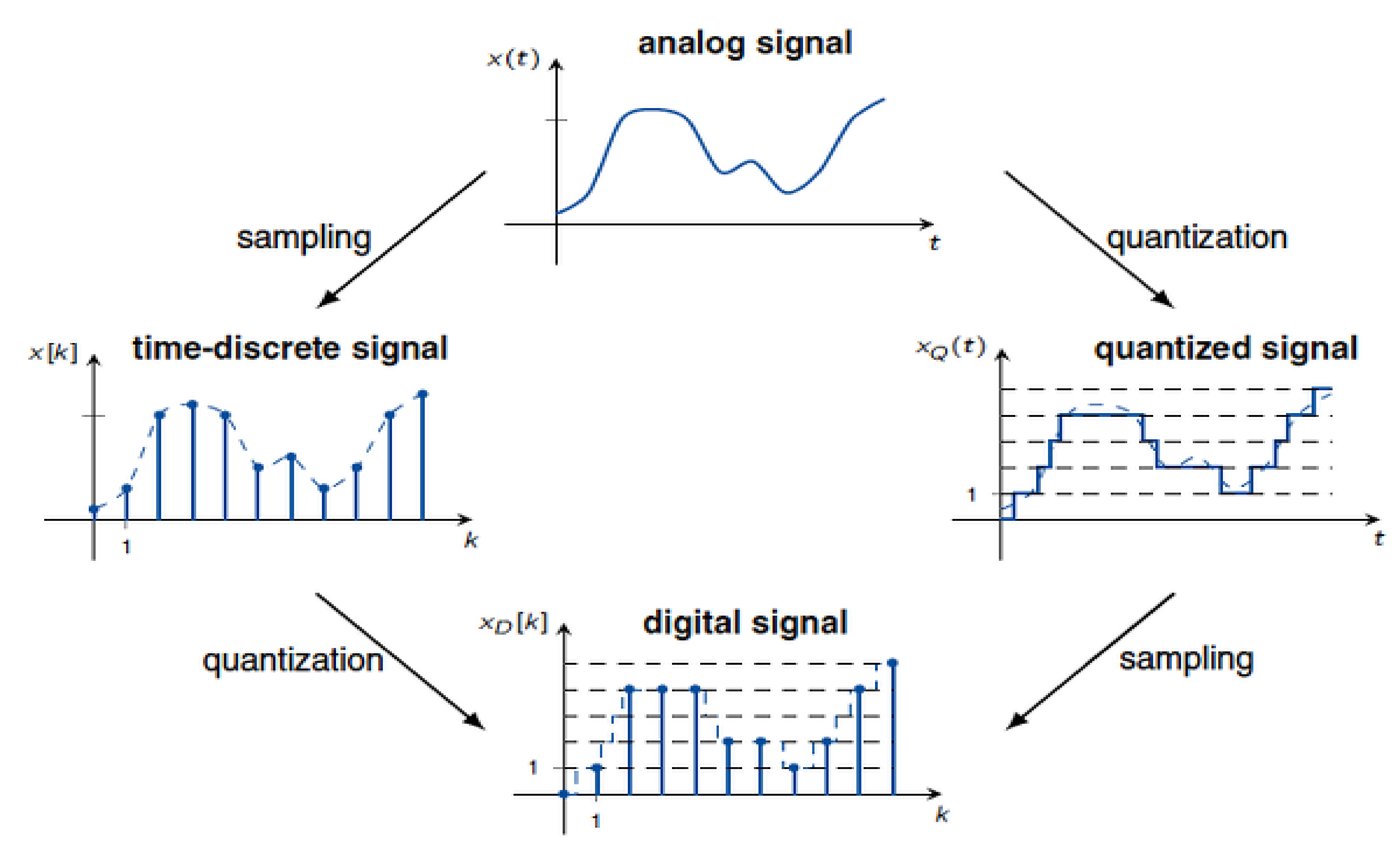
- Analog Signal: If a continuous time signal x(t) can take on any value in the continuous interval (-∞ to ∞)
- Digital Signal : If a Discrete time signal x[n] can take only a finite number of distinct values



STEPS TO CONVERT CONTINUOUS TIME SIGNAL TO DIGITAL SIGNAL

Day 5 .18.08.2020




analog signal
x(t)
t
sampling
quantization
time-discrete signal
x[k]
1
k
quantized signal
$x_Q(t)$
1
t
quantization
sampling
digital signal
$x_D[k]$
1
1
k

TRANSFORMATIONS OF THE INDEPENDENT VARIABLE

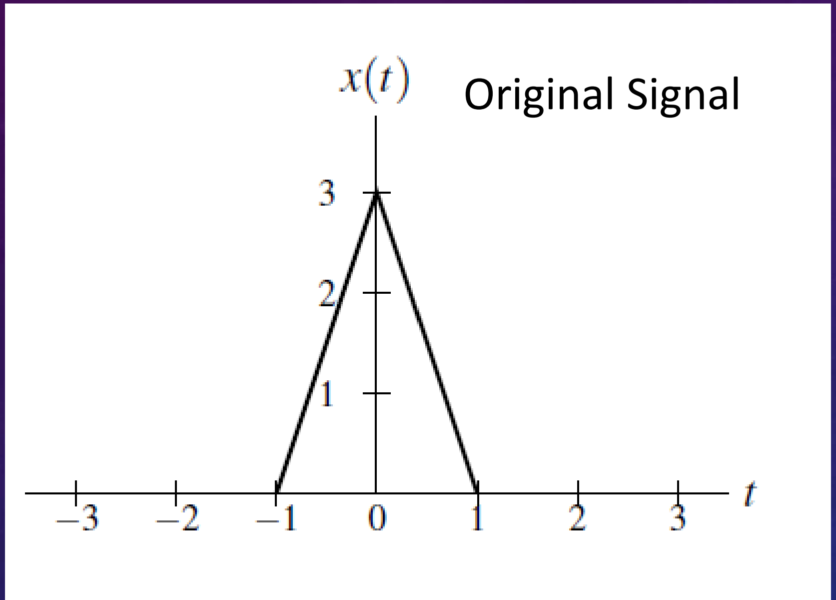
- Three Important **transformations** on Independent Variable are
  - **Time Shifting**
  - **Time Reversal**
  - **Time Scaling**

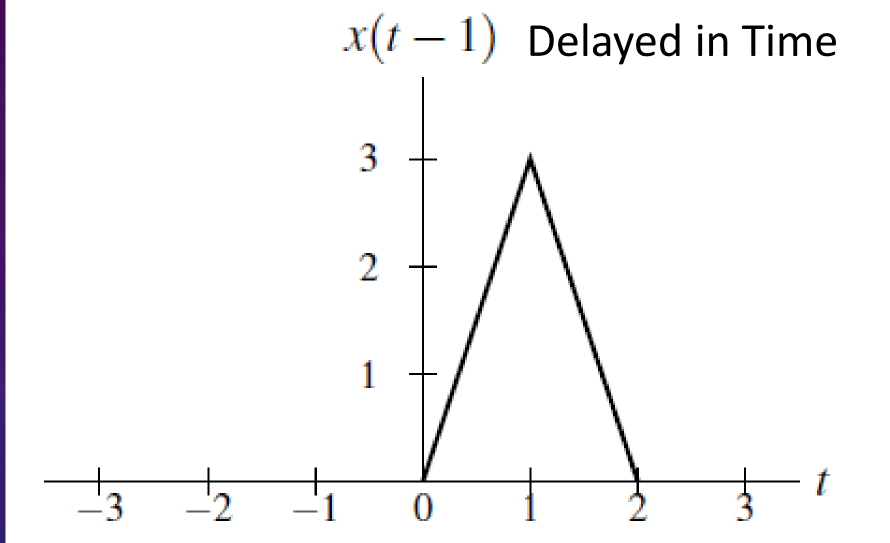
TIME SHIFTING

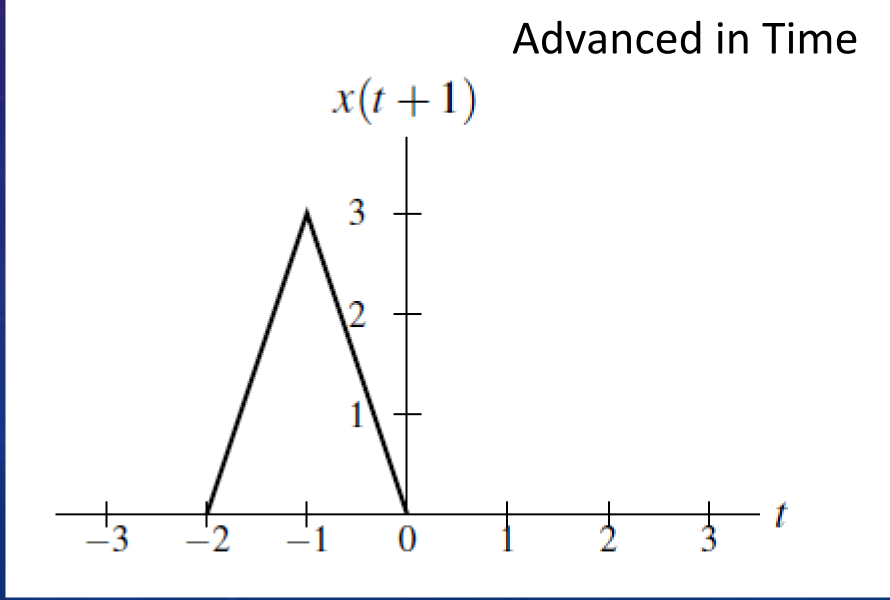
- **Time Shifting**: Time shifting (also called translation) maps the input signal x to the output signal y as given by $y(t) = x(t - t_0)$, where t_0 is a real number
- Such a transformation shifts the signal (to the left or right) along the time axis of a CTS
- If $t_0 > 0$, y is shifted to the right by $| t_0 |$, relative to x (i.e., delayed in time).
- If $t_0 < 0$, y is shifted to the left by $| t_0 |$, relative to x (i.e., advanced in time).

- **Time Shifting**: Time shifting (also called translation) maps the input signal x to the output signal y as given by $y(n) = x(n - n_0)$, where n_0 is a real number
- Such a transformation shifts the signal (to the left or right) along the time axis of DTS
- If $n_0 > 0$, y is shifted to the right by $| n_0 |$, relative to x (i.e., delayed in time).
- If $n_0 < 0$, y is shifted to the left by $| n_0 |$, relative to x (i.e., advanced in time).

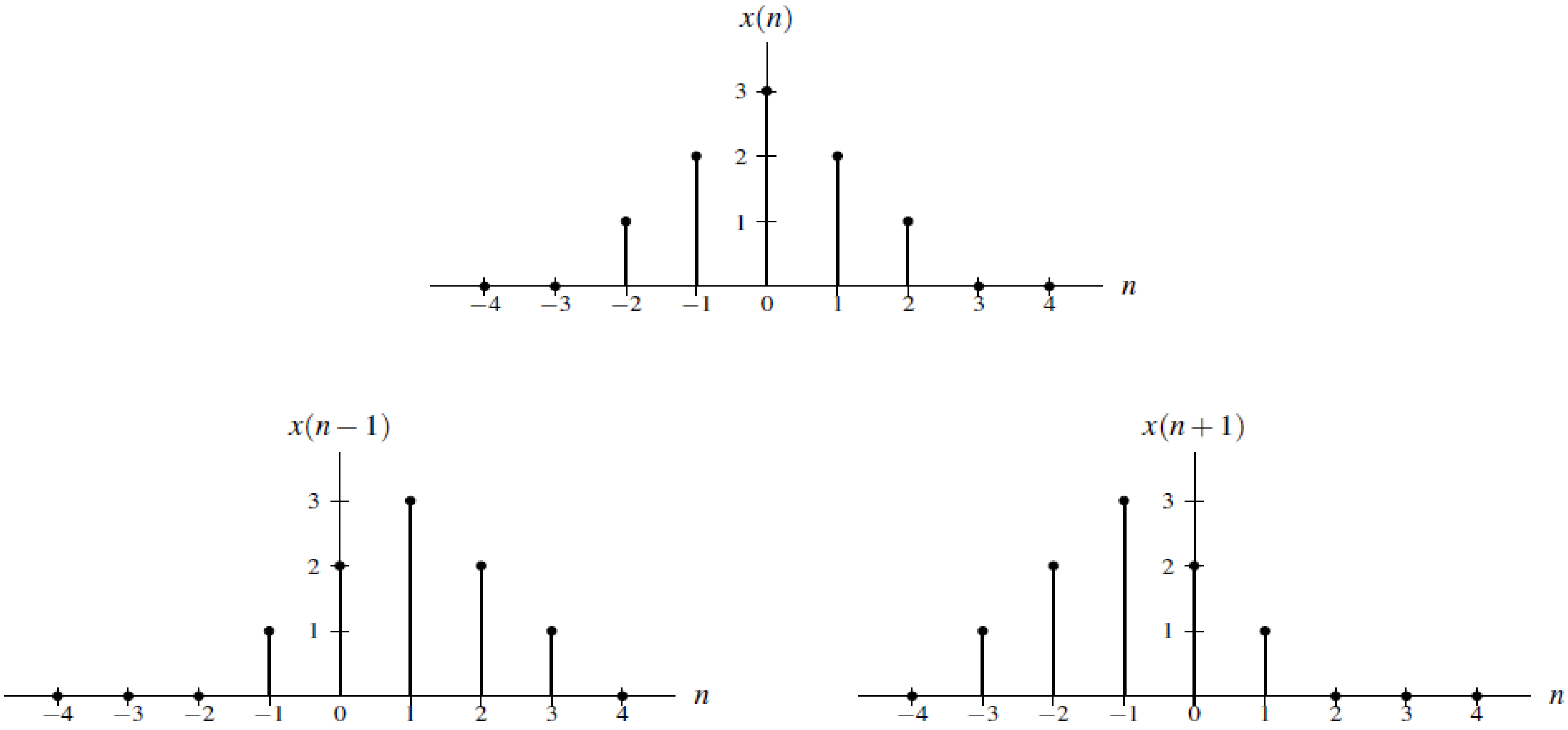
TIME SHIFTING







Time Shifting (Translation): Example

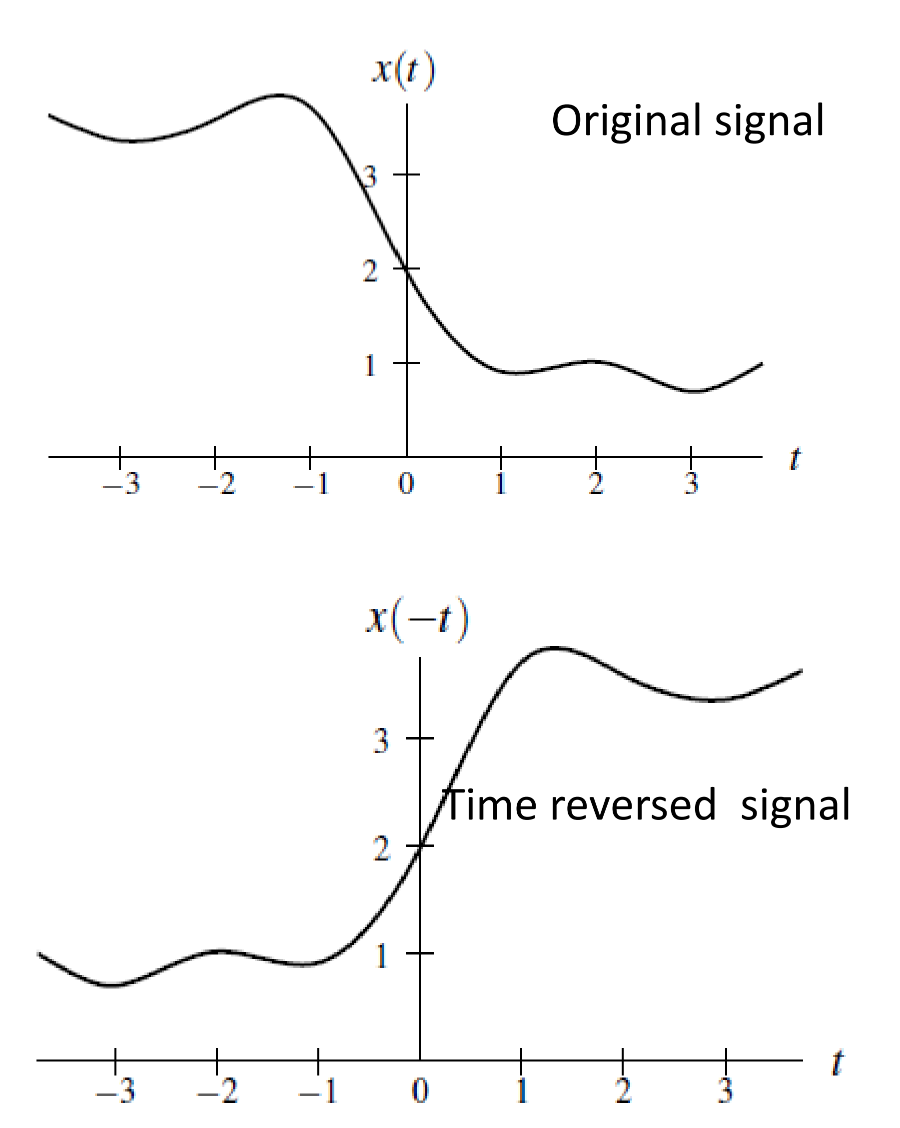


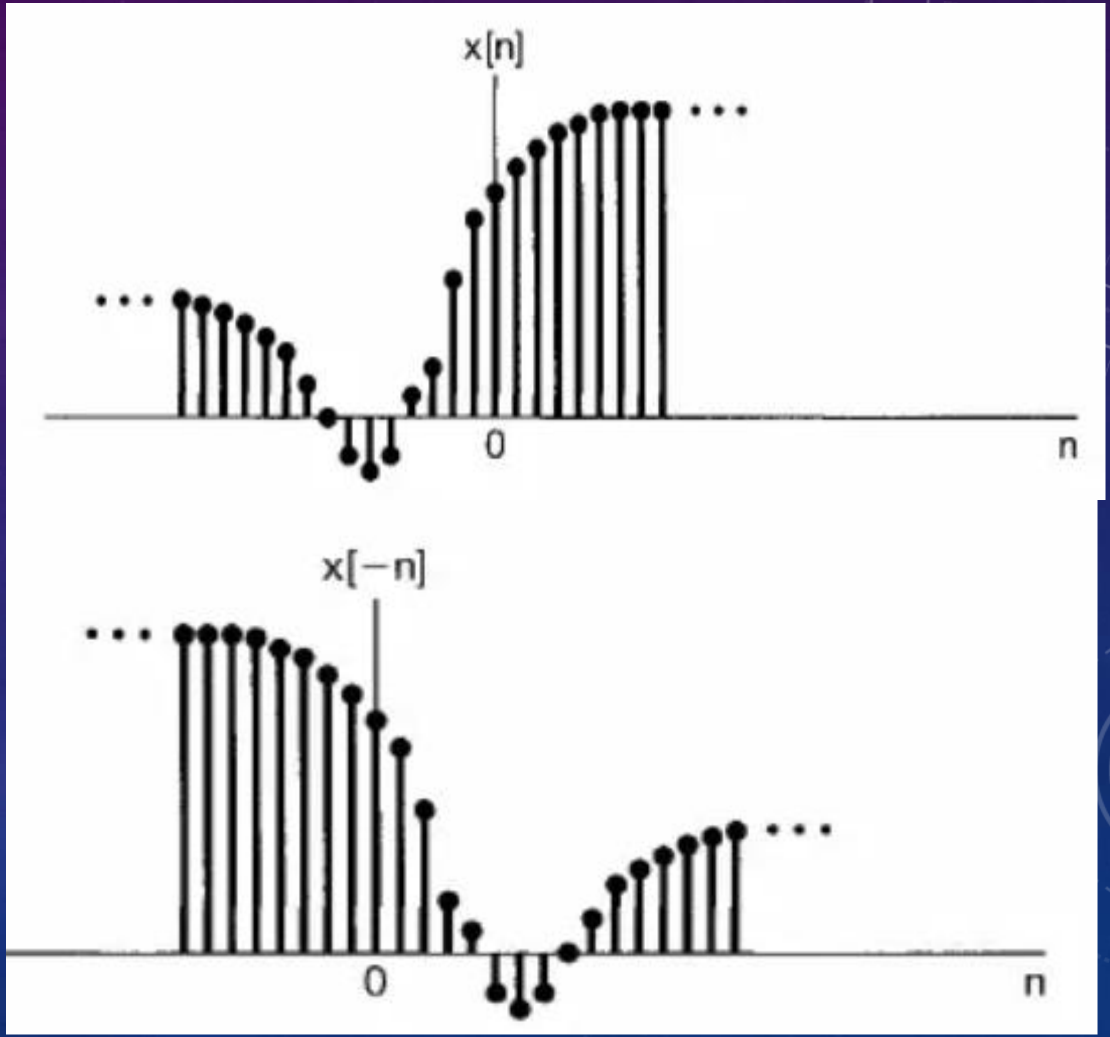
TIME REVERSAL

- **Time Reversal**: denoted by $y(t) = x(-t)$
- Obtained from the signal $x(t)$ by a reflection of the input signal x about the (vertical line)$t = 0$.
- i.e. by reversing the signal $x(t)$ – Called as a TIME REVERSAL OF CTS
- If $x(t)$ represents an audio tape recording then $x(-t)$ is the same tape recording played backwards.

- **Time Reversal**: denoted by $y(n) = x[-n]$
- Obtained from the signal $x[n]$ by a reflection of the input signal x about the (vertical line)$n = 0$.
- i.e. by reversing the signal $x[n]$ – Called as a TIME REVERSAL OF DTS

TIME REVERSAL



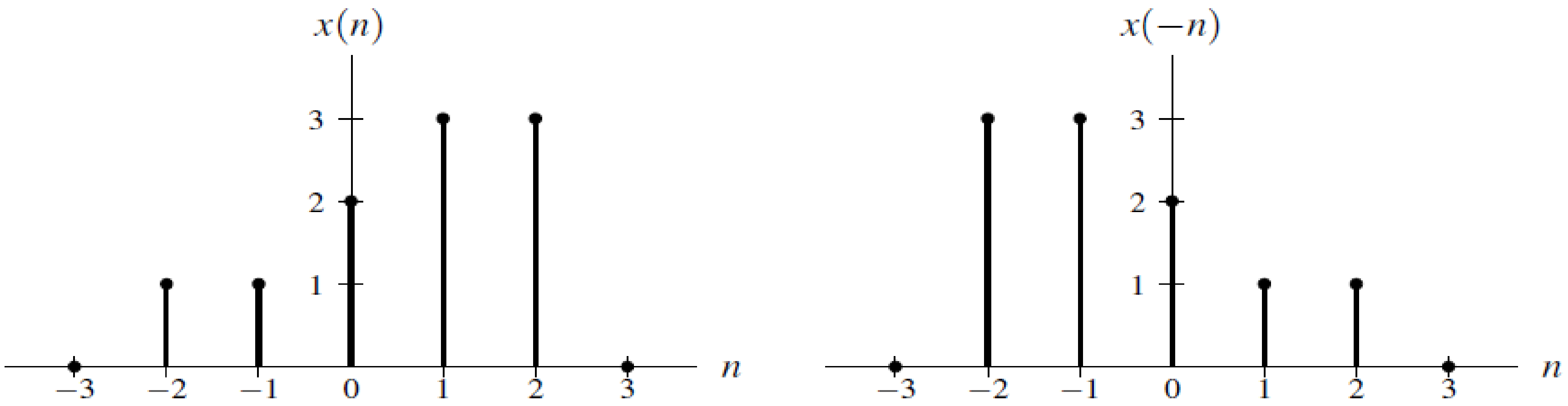


Time Reversal (Reflection)

- **Time reversal** (also known as **reflection**) maps the input signal x to the output signal y as given by

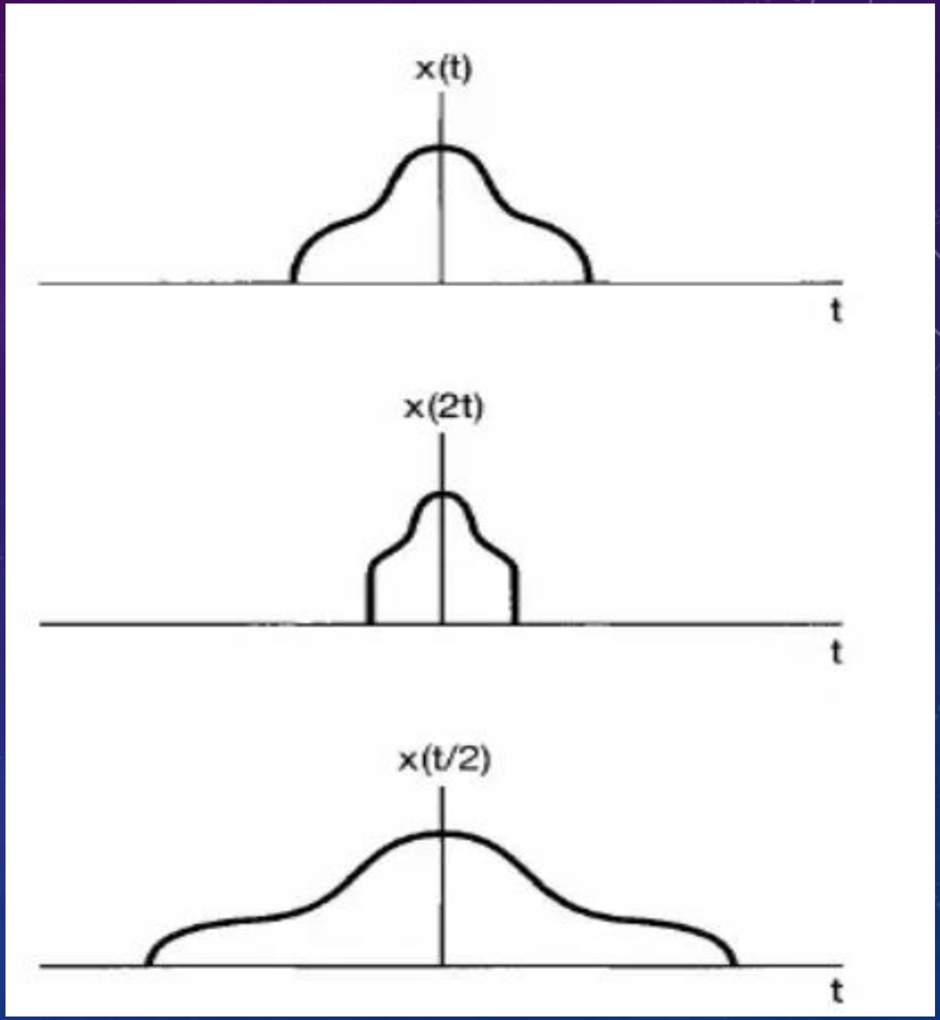
$$y(n) = x(-n).$$

- Geometrically, the output signal y is a reflection of the input signal x about the (vertical) line $n = 0$.



TIME SCALING

- **Let the three signals x(t),x(2t) and** x(t/2) which are linearly related to Input signal x to the output signal y as given by **y(t) = x(at),** where a is a strictly positive real number.
- Such a transformation is associated with a compression/expansion along the time axis
- If a > 1, y is compressed along the horizontal axis by a factor of a, relative to x.
- If a < 1, y is expanded (i.e., stretched) along the horizontal axis by a factor of 1 /a , relative to x.



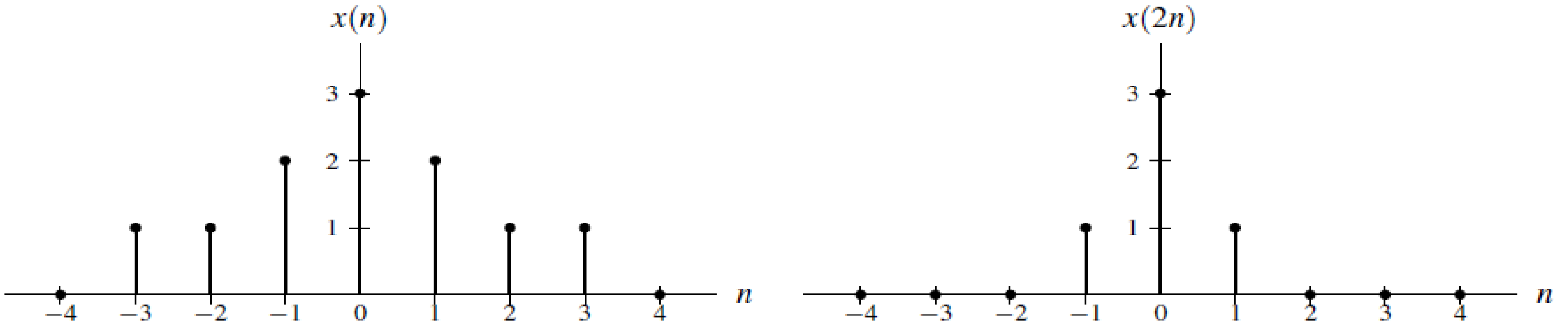
Downsampling

- **Downsampling** maps the input signal x to the output signal y as given by

$$y(n) = x(an),$$

  where a is a ***strictly positive*** integer.

- The output signal y is produced from the input signal x by keeping only every ath sample of x.



TIME SCALING

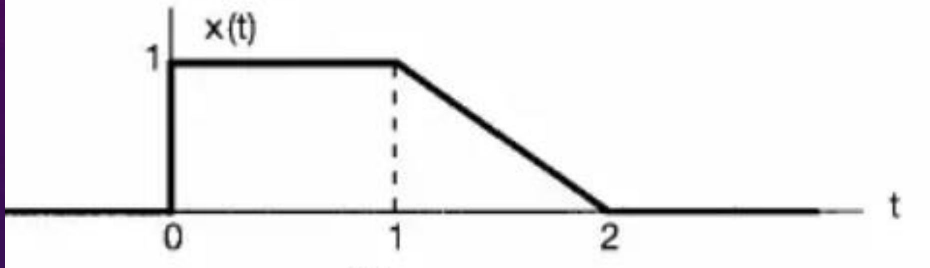
- Consider a transformation $y(t) = x(at - b)$, where a and b are real numbers and a not equal to 0.
- The above transformation can be shown to be the combination of a time-scaling operation and time-shifting operation.
- Since time scaling and time shifting do not commute, we must be particularly careful about the order in which these transformations are applied.

The above transformation has two distinct but equivalent interpretations:

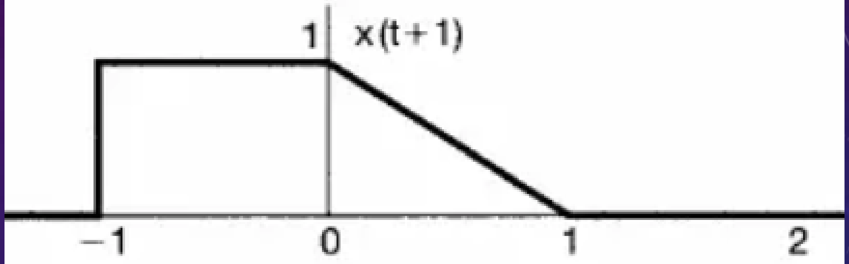
1. first, time shifting x by b, and then time scaling the result by a;
2. first, time scaling x by a, and then time shifting the result by b/a.

- Note that the time shift is not by the same amount in both cases.

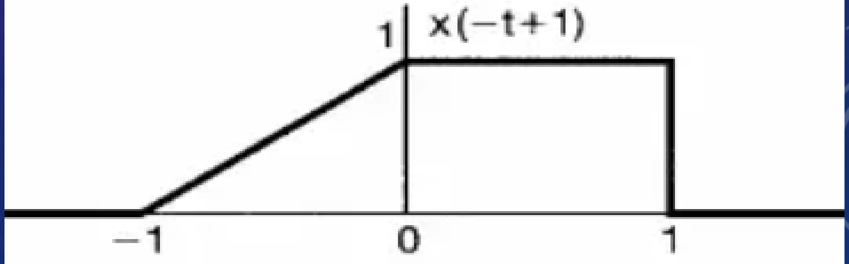
- For the signal shown ,draw x(t+1),x(-t+1),x(3/2 t) and x(3/2t+1)



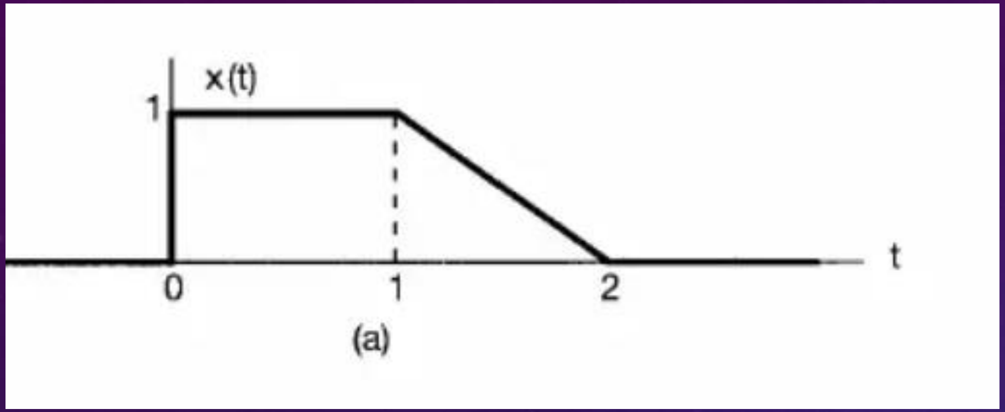
(a)

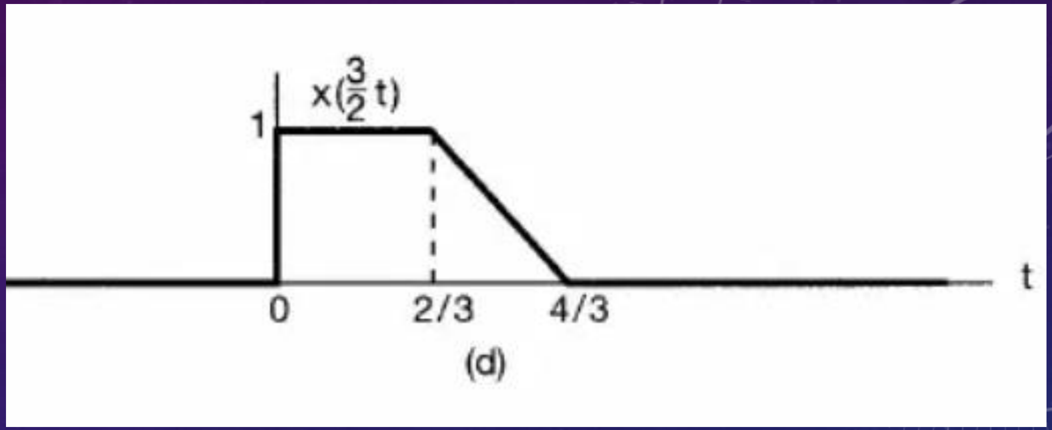


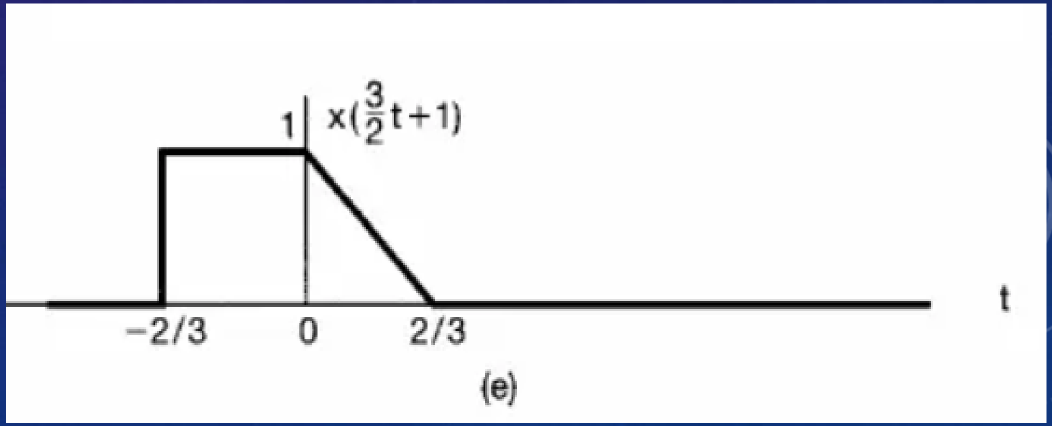
(b)



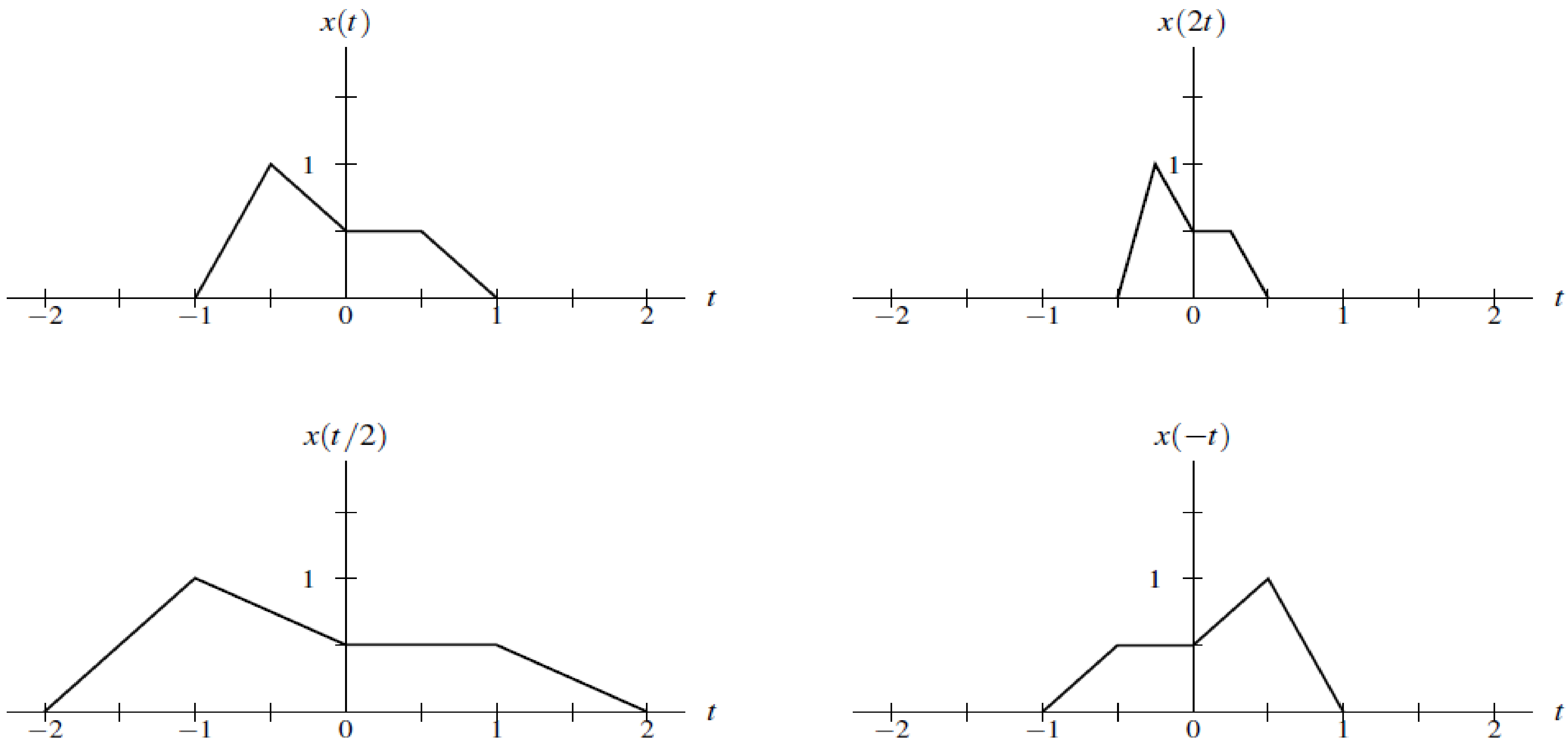
(c)


x(t)
1
0
1
2
t
(a)


x(3/2 t)
1
0
2/3
4/3
t
(d)


x(3/2 t+1)
1
−2/3
0
2/3
t
(e)

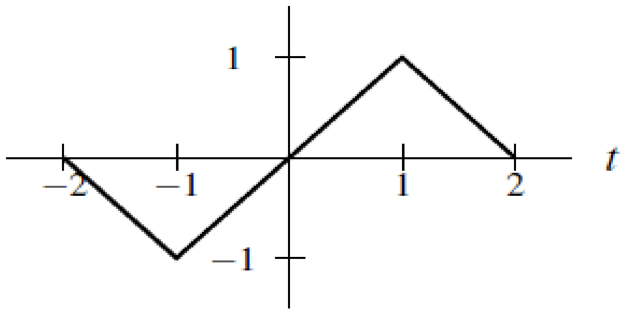
Time Scaling (Dilation/Reflection): Example



Combined Time Scaling and Time Shifting: Example

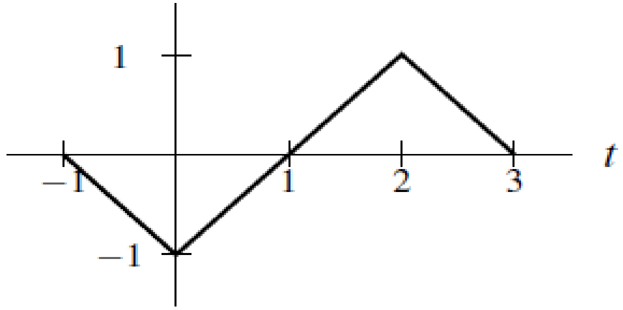
Given $x(t)$ as shown below, find $x(2t-1)$.

$x(t)$

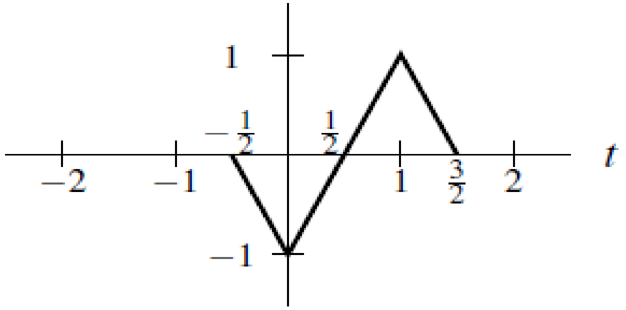


time shift by 1 and then time scale by 2

$p(t) = x(t-1)$

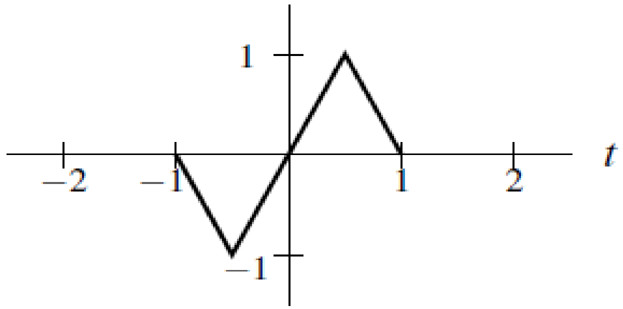


$p(2t) = x(2t-1)$

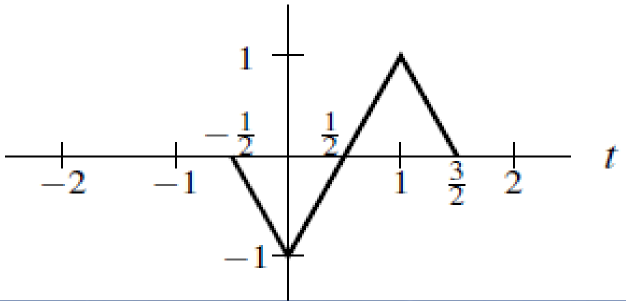


time scale by 2 and then time shift by $\frac{1}{2}$

$q(t) = x(2t)$

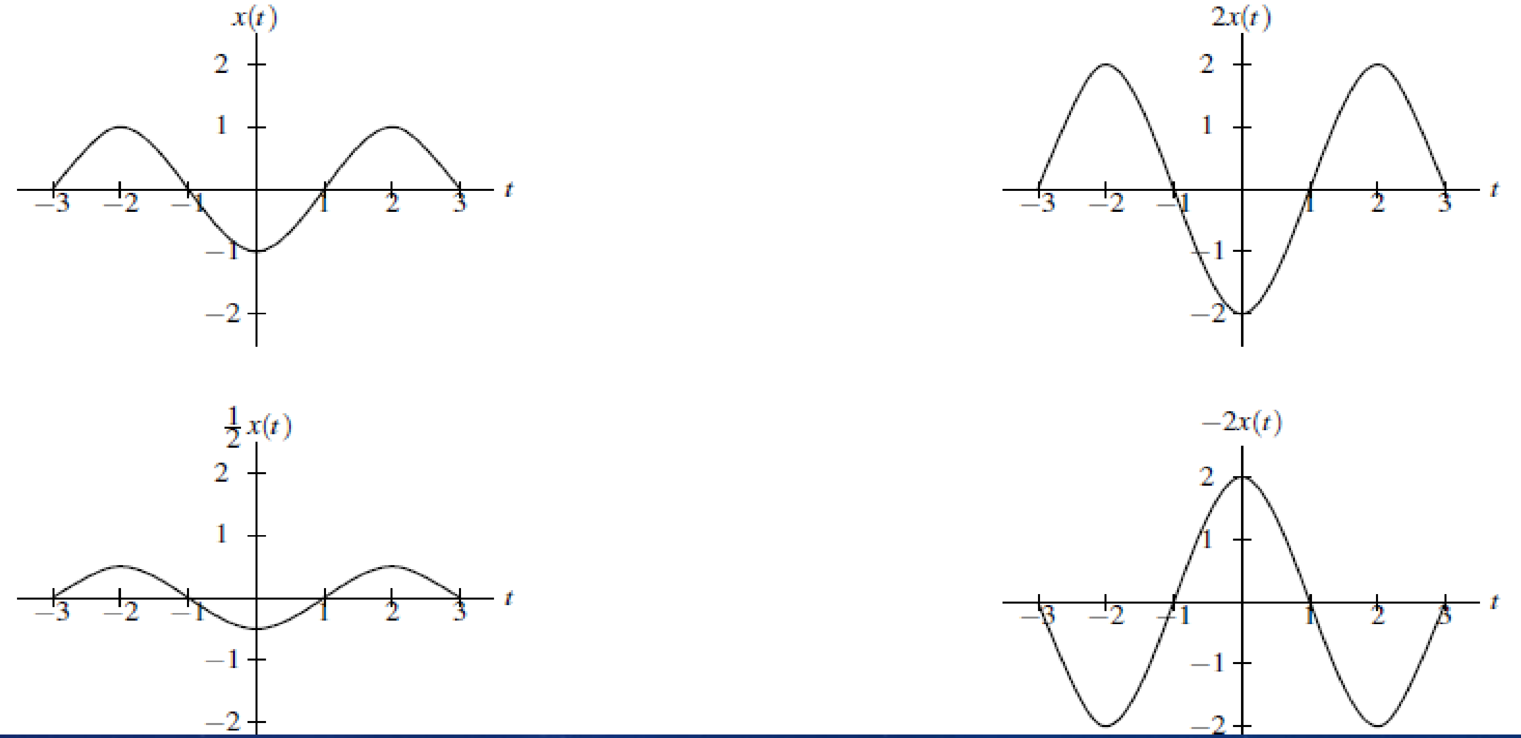


$q(t-1/2) = x(2(t-1/2)) = x(2t-1)$



Amplitude Scaling

- **Amplitude scaling** maps the input signal x to the output signal y as given by
$$y(t) = ax(t),$$
where a is a real number.
- Geometrically, the output signal y is ***expanded/compressed*** in amplitude and/or ***reflected*** about the horizontal axis.



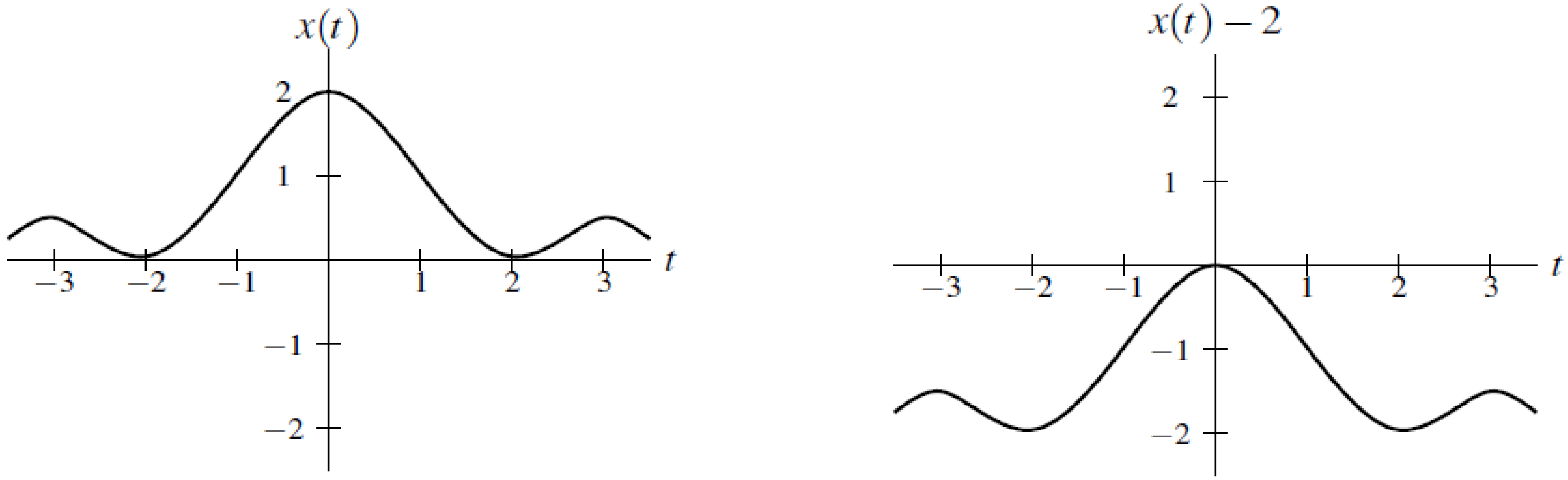
Amplitude Shifting

- **Amplitude shifting** maps the input signal x to the output signal y as given by

$$y(t) = x(t) + b,$$

where b is a real number.

- Geometrically, amplitude shifting adds a ***vertical displacement*** to x.



Even Signals

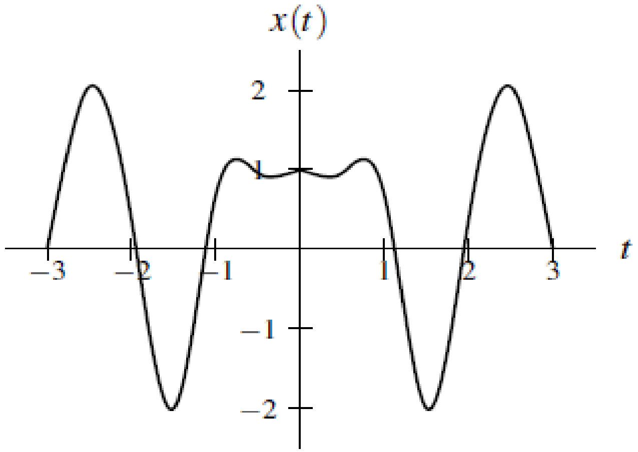
- A function x is said to be **even** if it satisfies

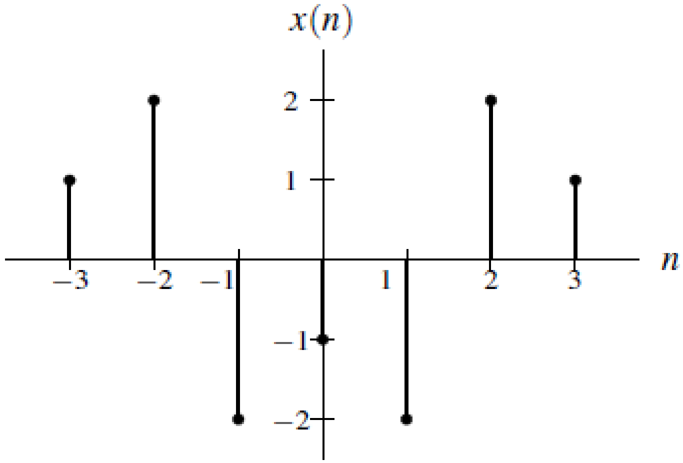
$$x(t) = x(-t) \quad \text{for all } t.$$

- A sequence x is said to be **even** if it satisfies

$$x(n) = x(-n) \quad \text{for all } n.$$

- Geometrically, the graph of an even signal is ***symmetric*** about the origin.
- Some examples of even signals are shown below.





Odd Signals

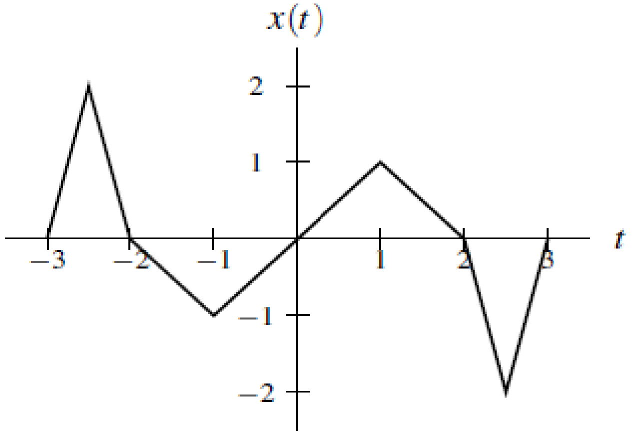
- A function x is said to be **odd** if it satisfies

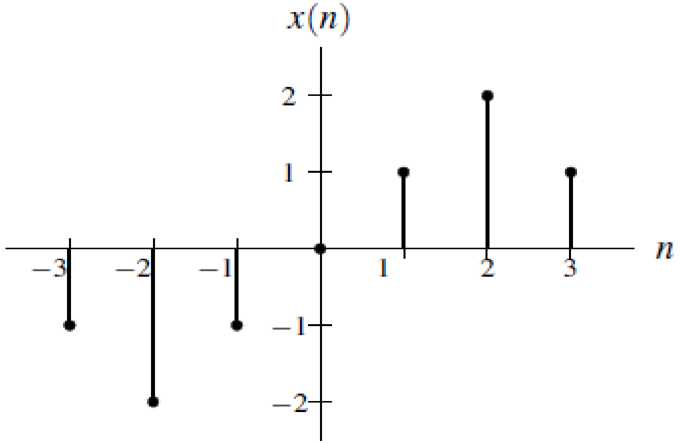
$$x(t) = -x(-t) \quad \text{for all } t.$$

- A sequence x is said to be **odd** if it satisfies

$$x(n) = -x(-n) \quad \text{for all } n.$$

- Geometrically, the graph of an odd signal is ***antisymmetric*** about the origin.
- An odd signal x must be such that $x(0) = 0$.
- Some examples of odd signals are shown below.





EVEN & ODD COMPONENT OF A SIGNAL

- Any signal x(t) can be divided into its even component and odd component
- i.e $x(t)=x_e(t)+x_o(t)$……(1)

  replacing t by –t in eqn (1)
- $x(-t)=x_e(-t)+x_o(-t)$
- $x(-t)=x_e(t)-x_o(t)$………(2)
- Adding (1) & (2)
- $x(t)+x(-t)=2x_e(t)$
- **$x_e(t)=0.5[x(t)+x(-t)]$=Even Component**
- Similarly after (1)-(2)
- **$x_o(t)=0.5[x(t)-x(-t)]$= Odd Component**

- Any sequence x[n] can be divided into its even component and odd component
- i.e $x(n)=x_e(n)+x_o(n)$……(1)

  replacing n by –n in eqn (1)
- $x(-n)=x_e(-n)+x_o(-n)$
- $x(-n)=x_e(n)-x_o(n)$………(2)
- Adding (1) & (2)
- $x(n)+x(-n)=2x_e(n)$
- **$x_e(n)=0.5[x(n)+x(-n)]$= Even Component**
- Similarly after (1)-(2)
- **$x_o(n)=0.5[x(n)-x(-n)]$= Odd Component**

Periodic Signals

- A function x is said to be **periodic** with **period** T (or T**-periodic**) if, for some strictly-positive real constant T, the following condition holds:

$$x(t) = x(t+T) \quad \text{for all } t.$$

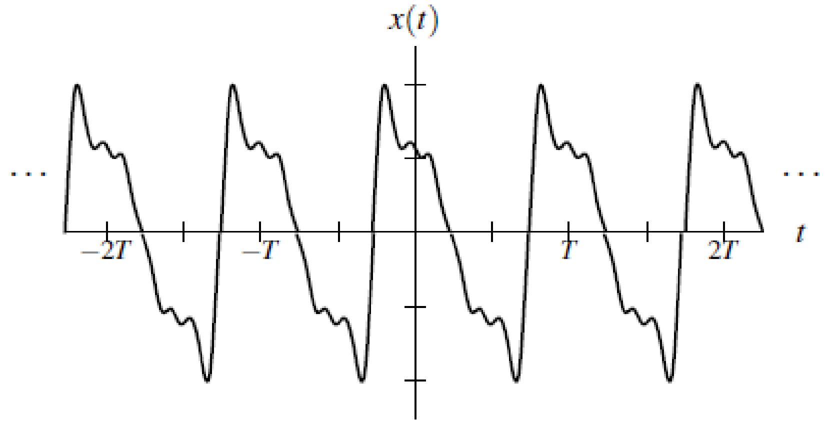
- A T-periodic function x is said to have **frequency** $\frac{1}{T}$ and **angular frequency** $\frac{2\pi}{T}$.
- A sequence x is said to be **periodic** with **period** N (or N**-periodic**) if, for some strictly-positive integer constant N, the following condition holds:

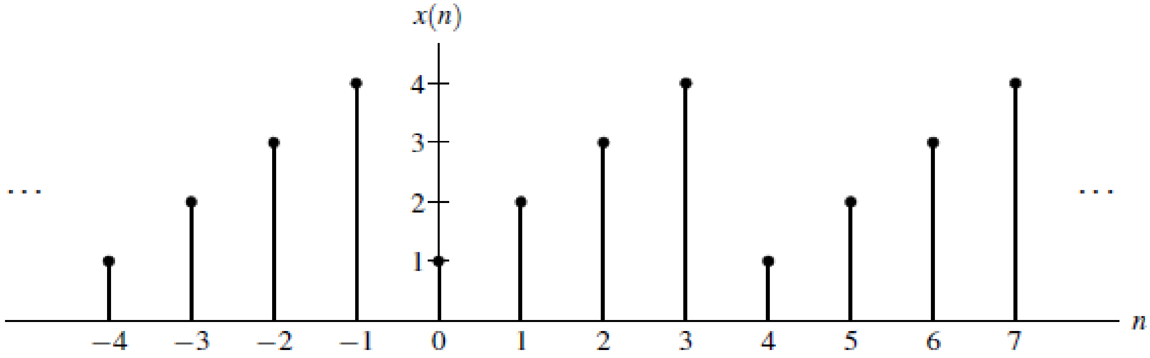
$$x(n) = x(n+N) \quad \text{for all } n.$$

- An N-periodic sequence x is said to have **frequency** $\frac{1}{N}$ and **angular frequency** $\frac{2\pi}{N}$.
- A function/sequence that is not periodic is said to be **aperiodic**.

Periodic Signals (Continued 1)

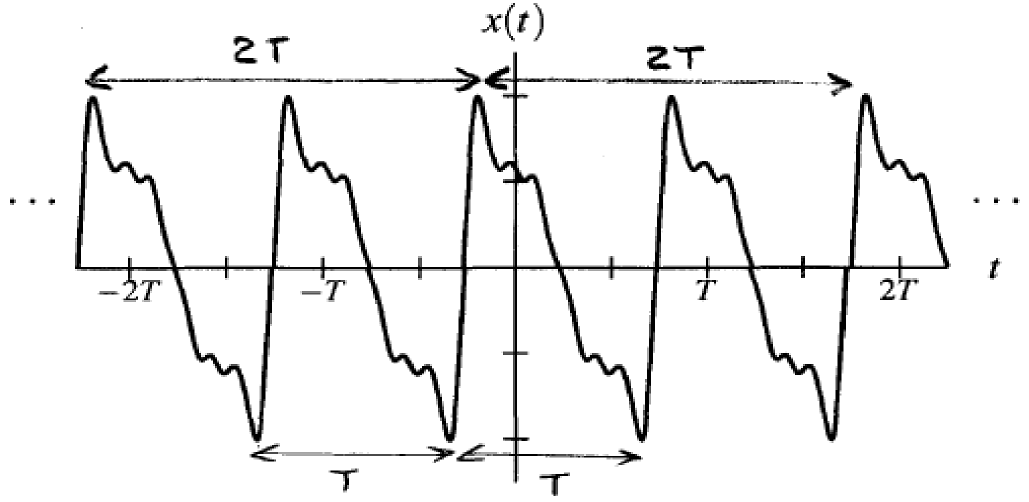
- Some examples of periodic signals are shown below.





Periodic Signals (Continued 2)

- The period of a periodic signal is ***not unique***. That is, a signal that is periodic with period T is also periodic with period kT, for every (strictly) positive integer k.



- The smallest period with which a signal is periodic is called the **fundamental period** and its corresponding frequency is called the **fundamental frequency**.

- **Sum of periodic functions.** Let x1 and x2 be periodic functions with fundamental periods T1 and T2, respectively.
- Then, the sum y = x1 +x2 is a periodic function if and only if the ratio T1/T2 is a rational number (i.e., the quotient of two integers). Suppose that T1/T2 = q/r where q and r are integers and coprime (i.e., have no common factors), then the
- fundamental period of y is rT1 (or equivalently, qT2, since rT1 = qT2).

Note that rT1 is simply the **least common multiple(LCM)** of T1 and T2.

Signal Energy and Power

- The **energy** E contained in the signal x is given by

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt.$$

- A signal with finite energy is said to be an **energy signal**.
- The **average power** P contained in the signal x is given by

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt.$$

- A signal with (nonzero) finite average power is said to be a **power signal**.

Signal Energy

- The **energy** E contained in the signal x is given by

$$E = \sum_{k=-\infty}^{\infty} |x(k)|^2 .$$

- A signal with finite energy is said to be an **energy signal**.

Sr No	Deterministic signals	Random signals
1	Deterministic signals can be represented or described by a mathematical equation or lookup table.	Random signals that cannot be represented or described by a mathematical equation or lookup table.
2	Deterministic signals are preferable because for analysis and processing of signals we can use mathematical model of the signal.	Not Preferable. The random signals can be described with the help of their statistical properties.
3	The value of the deterministic signal can be evaluated at time (past, present or future) without certainty.	The value of the random signal can not be evaluated at any instant of time.
4	Example Sine or exponential waveforms.	Example Noise signal or Speech signal

ELEMENTARY SIGNALS

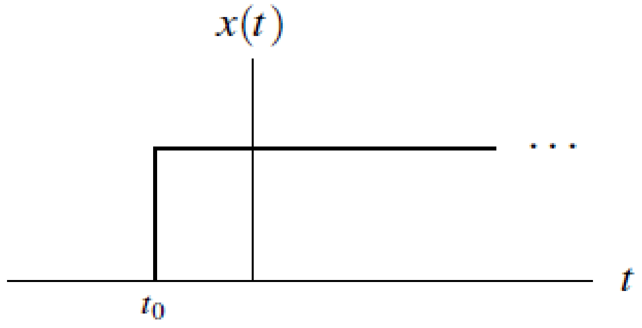
Right-Sided Signals

- A signal x is said to be **right sided** if, for some (finite) real constant t_0, the following condition holds:

$$x(t) = 0 \quad \text{for all } t < t_0$$

(i.e., x is ***only potentially nonzero to the right of*** t_0).

- An example of a right-sided signal is shown below.



- A signal x is said to be **causal** if

$$x(t) = 0 \quad \text{for all } t < 0.$$

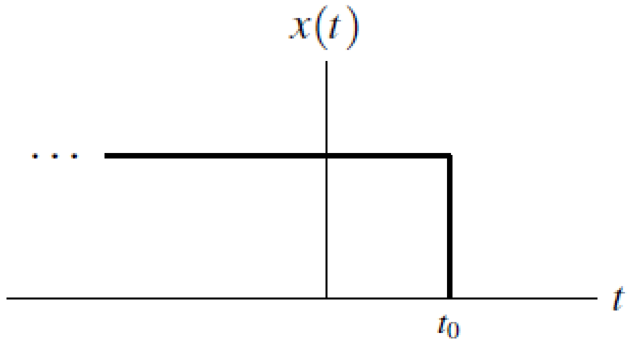
- A causal signal is a ***special case*** of a right-sided signal.
- A causal signal is not to be confused with a causal system. In these two contexts, the word "causal" has very different meanings.

Left-Sided Signals

- A signal x is said to be **left sided** if, for some (finite) real constant t_0, the following condition holds:

$$x(t) = 0 \quad \text{for all } t > t_0$$

(i.e., x is ***only potentially nonzero to the left of*** t_0).
- An example of a left-sided signal is shown below.



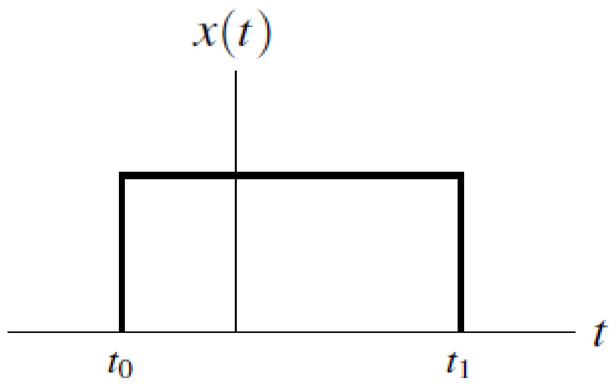
- Similarly, a signal x is said to be **anticausal** if

$$x(t) = 0 \quad \text{for all } t > 0.$$

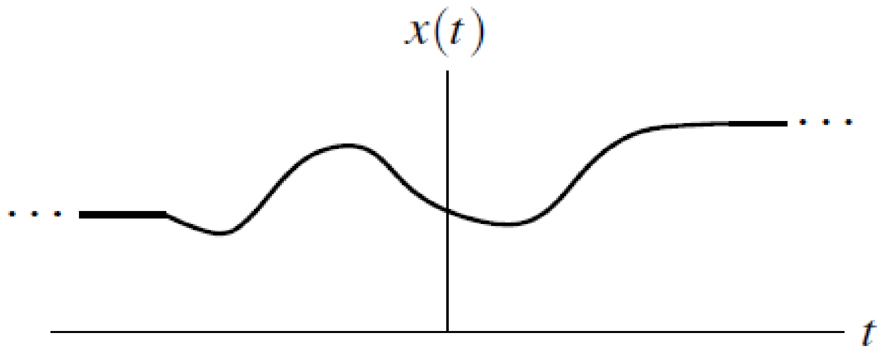
- An anticausal signal is a ***special case*** of a left-sided signal.
- An anticausal signal is not to be confused with an anticausal system. In these two contexts, the word "anticausal" has very different meanings.

Finite-Duration and Two-Sided Signals

- A signal that is both left sided and right sided is said to be **finite duration** (or **time limited**).
- An example of a finite duration signal is shown below.



- A signal that is neither left sided nor right sided is said to be **two sided**.
- An example of a two-sided signal is shown below.



Bounded Signals

- A signal x is said to be **bounded** if there exists some (*finite*) positive real constant A such that

$$|x(t)| \leq A \quad \text{for all } t$$

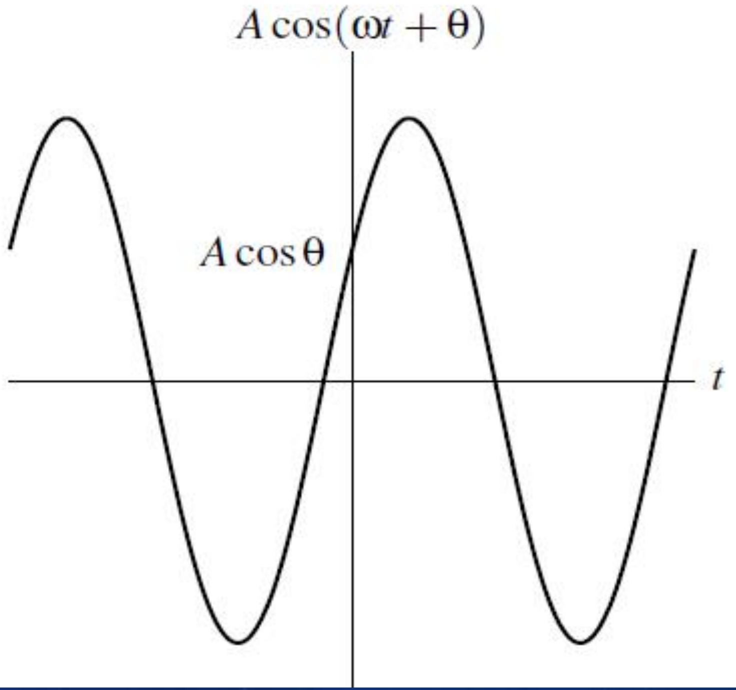
  (i.e., $x(t)$ is *finite* for all t).
- Examples of bounded signals include the sine and cosine functions.
- Examples of unbounded signals include the tan function and any nonconstant polynomial function.

Real Sinusoids

- A (CT) **real sinusoid** is a function of the form

$$x(t) = A\cos(\omega t + \theta),$$

  where A, ω, and θ are ***real*** constants.
- Such a function is periodic with ***fundamental period*** $T = \frac{2\pi}{|\omega|}$ and ***fundamental frequency*** $|\omega|$.
- A real sinusoid has a plot resembling that shown below.



Complex Exponentials

- A (CT) **complex exponential** is a function of the form

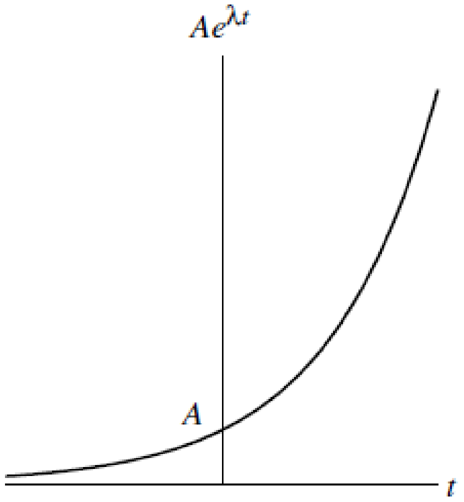
$$x(t) = Ae^{\lambda t},$$

  where A and λ are ***complex*** constants.

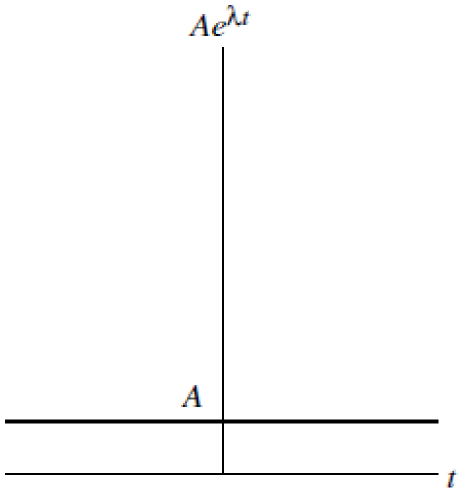
- A complex exponential can exhibit one of a number of ***distinct modes of behavior***, depending on the values of its parameters A and λ.
- For example, as special cases, complex exponentials include real exponentials and complex sinusoids.

Real Exponentials

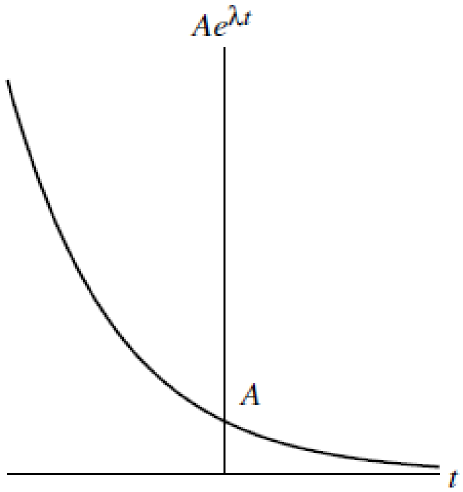
- A **real exponential** is a special case of a complex exponential $x(t) = Ae^{\lambda t}$, where A and λ are restricted to be ***real*** numbers.
- A real exponential can exhibit one of ***three distinct modes*** of behavior, depending on the value of λ, as illustrated below.
- If $\lambda > 0$, $x(t)$ ***increases*** exponentially as t increases (i.e., a growing exponential).
- If $\lambda < 0$, $x(t)$ ***decreases*** exponentially as t increases (i.e., a decaying exponential).
- If $\lambda = 0$, $x(t)$ simply equals the ***constant*** A.



$\lambda > 0$



$\lambda = 0$



$\lambda < 0$

Complex Sinusoids

- A complex sinusoid is a special case of a complex exponential $x(t) = Ae^{\lambda t}$, where A is ***complex*** and λ is ***purely imaginary*** (i.e., $\mathrm{Re}\{\lambda\} = 0$).
- That is, a (CT) **complex sinusoid** is a function of the form

$$x(t) = Ae^{j\omega t},$$

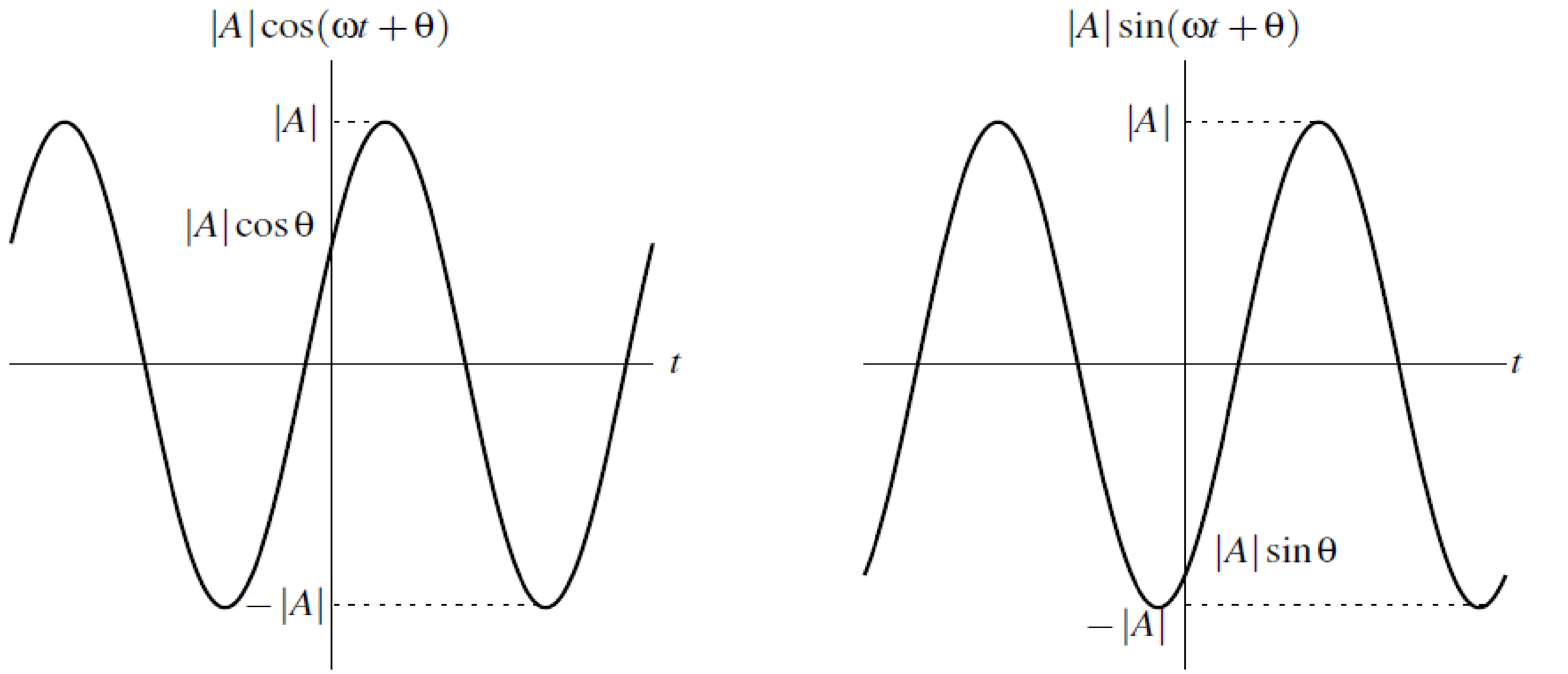
  where A is ***complex*** and ω is ***real***.
- By expressing A in polar form as $A = |A|\,e^{j\theta}$ (where θ is real) and using Euler's relation, we can rewrite $x(t)$ as

$$x(t) = \underbrace{|A|\cos(\omega t + \theta)}_{\mathrm{Re}\{x(t)\}} + j\underbrace{|A|\sin(\omega t + \theta)}_{\mathrm{Im}\{x(t)\}}.$$

- Thus, $\mathrm{Re}\{x\}$ and $\mathrm{Im}\{x\}$ are the same except for a time shift.
- Also, x is periodic with ***fundamental period*** $T = \frac{2\pi}{|\omega|}$ and ***fundamental frequency*** $|\omega|$.

Complex Sinusoids (Continued)

- The graphs of $\mathrm{Re}\{x\}$ and $\mathrm{Im}\{x\}$ have the forms shown below.



General Complex Exponentials

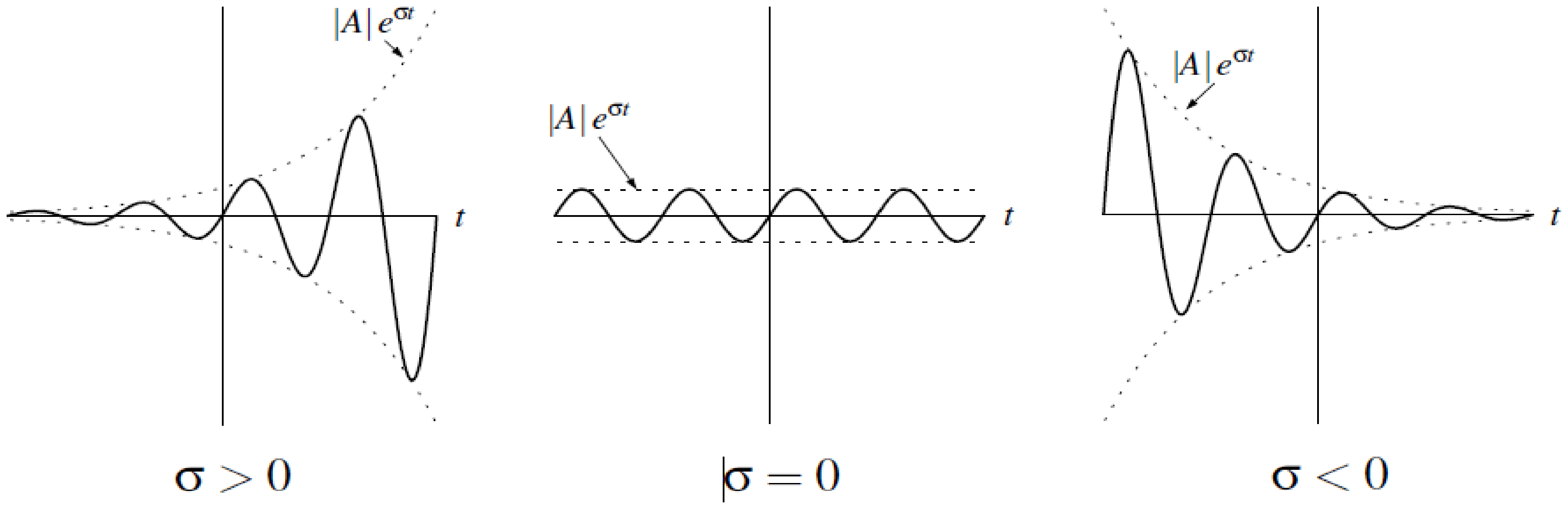
- In the most general case of a complex exponential $x(t) = Ae^{\lambda t}$, A and λ are both *complex*.
- Letting $A = |A|e^{j\theta}$ and $\lambda = \sigma + j\omega$ (where θ, σ, and ω are real), and using Euler's relation, we can rewrite $x(t)$ as

$$x(t) = \underbrace{|A|e^{\sigma t}\cos(\omega t + \theta)}_{\mathrm{Re}\{x(t)\}} + j\underbrace{|A|e^{\sigma t}\sin(\omega t + \theta)}_{\mathrm{Im}\{x(t)\}}.$$

- Thus, $\mathrm{Re}\{x\}$ and $\mathrm{Im}\{x\}$ are each the product of a real exponential and real sinusoid.
- One of *three distinct modes* of behavior is exhibited by $x(t)$, depending on the value of σ.
- If $\sigma = 0$, $\mathrm{Re}\{x\}$ and $\mathrm{Im}\{x\}$ are *real sinusoids*.
- If $\sigma > 0$, $\mathrm{Re}\{x\}$ and $\mathrm{Im}\{x\}$ are each the *product of a real sinusoid and a growing real exponential*.
- If $\sigma < 0$, $\mathrm{Re}\{x\}$ and $\mathrm{Im}\{x\}$ are each the *product of a real sinusoid and a decaying real exponential*.

General Complex Exponentials (Continued)

- The ***three modes of behavior*** for $\mathrm{Re}\{x\}$ and $\mathrm{Im}\{x\}$ are illustrated below.



Relationship Between Complex Exponentials and Real Sinusoids

- From Euler's relation, a complex sinusoid can be expressed as the sum of two real sinusoids as

$$Ae^{j\omega t} = A\cos\omega t + jA\sin\omega t.$$

- Moreover, a real sinusoid can be expressed as the sum of two complex sinusoids using the identities

$$A\cos(\omega t+\theta) = \frac{A}{2}\left[e^{j(\omega t+\theta)} + e^{-j(\omega t+\theta)}\right] \quad \text{and}$$

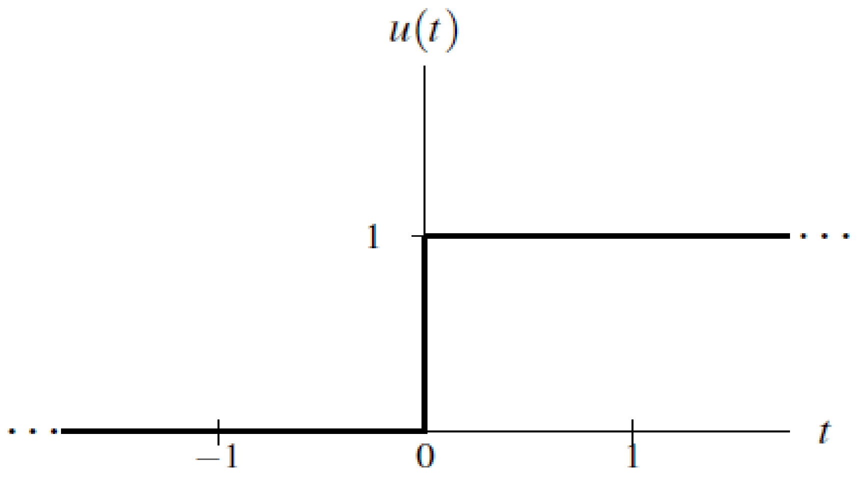
$$A\sin(\omega t+\theta) = \frac{A}{2j}\left[e^{j(\omega t+\theta)} - e^{-j(\omega t+\theta)}\right].$$

Unit-Step Function

- The **unit-step function** (also known as the **Heaviside function**), denoted u, is defined as

$$u(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

- Due to the manner in which u is used in practice, the actual ***value of $u(0)$*** is unimportant. Sometimes values of 0 and $\frac{1}{2}$ are also used for $u(0)$.
- A plot of this function is shown below.

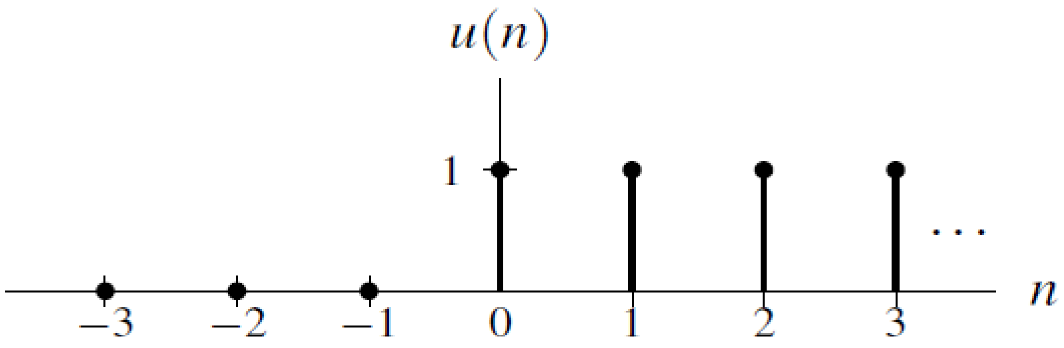


Unit-Step Sequence

- The **unit-step sequence**, denoted u, is defined as

$$u(n) = \begin{cases} 1 & \text{if } n \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

- A plot of this sequence is shown below.

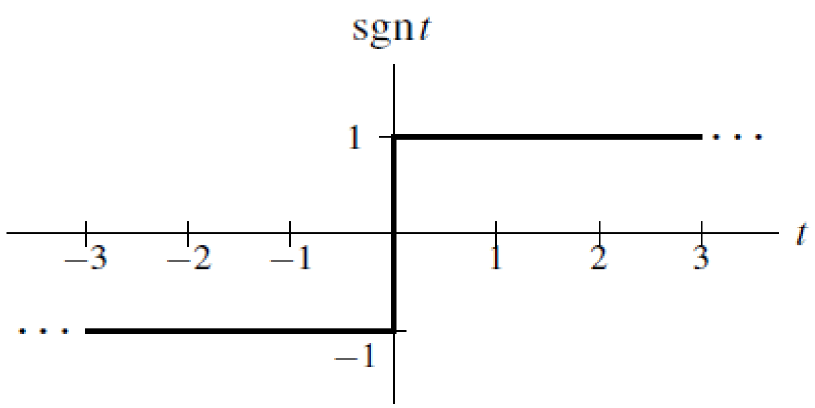


Signum Function

- The **signum function**, denoted sgn, is defined as

$$\operatorname{sgn} t = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t = 0 \\ -1 & \text{if } t < 0. \end{cases}$$

- From its definition, one can see that the signum function simply computes the *sign* of a number.
- A plot of this function is shown below.

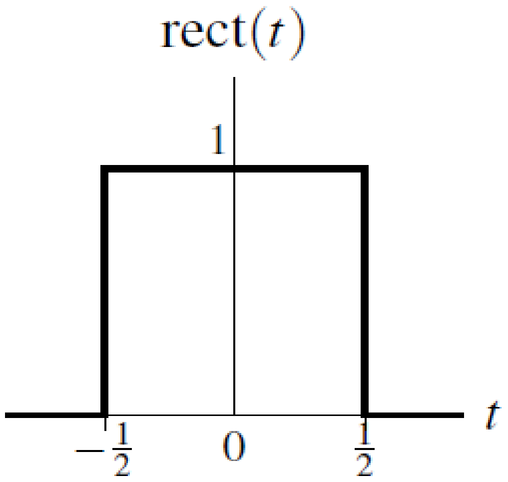


Rectangular Function

- The **rectangular function** (also called the unit-rectangular pulse function), denoted rect, is given by

$$\mathrm{rect}(t) = \begin{cases} 1 & \text{if } -\frac{1}{2} \le t < \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases}$$

- Due to the manner in which the rect function is used in practice, the actual ***value of*** $\mathrm{rect}(t)$ ***at*** $t = \pm\frac{1}{2}$ is unimportant. Sometimes different values are used from those specified above.
- A plot of this function is shown below.



Unit Rectangular Pulses

- A **unit rectangular pulse** is a sequence of the form

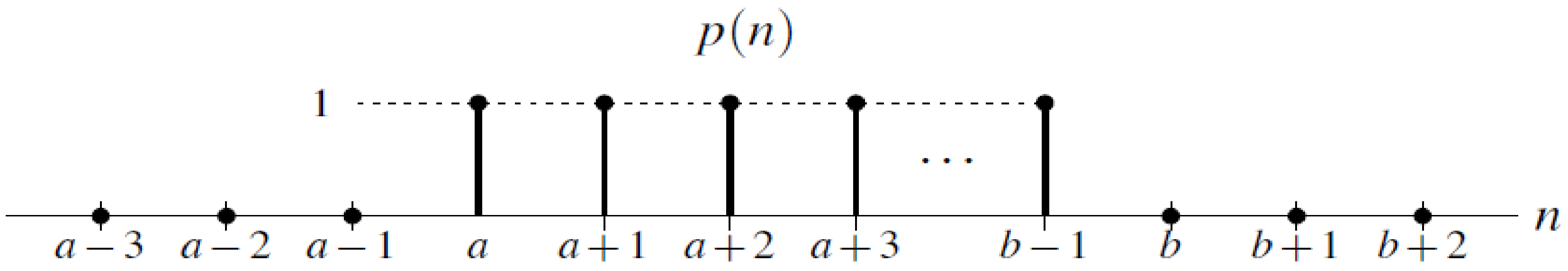
$$p(n) = \begin{cases} 1 & \text{if } a \le n < b \\ 0 & \text{otherwise} \end{cases}$$

  where a and b are integer constants satisfying $a < b$.

- Such a sequence can be expressed in terms of the unit-step sequence as

$$p(n) = u(n-a) - u(n-b).$$

- The graph of a unit rectangular pulse has the general form shown below.

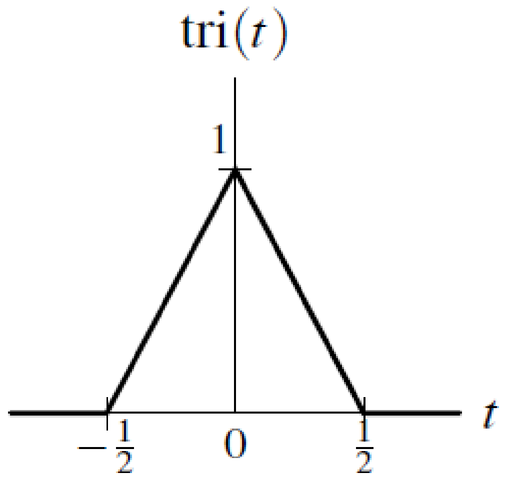


Triangular Function

- The **triangular function** (also called the unit-triangular pulse function), denoted tri, is defined as

$$\mathrm{tri}(t) = \begin{cases} 1 - 2|t| & |t| \le \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases}$$

- A plot of this function is shown below.

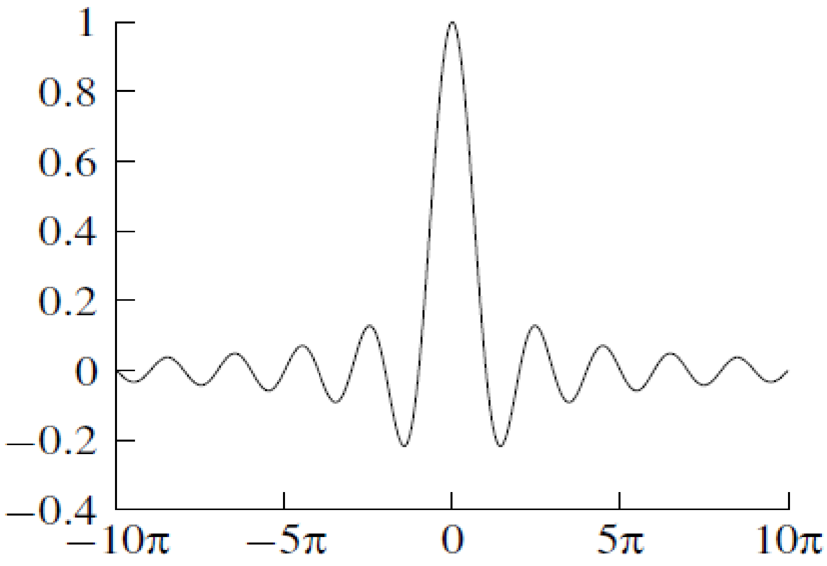


Cardinal Sine Function

- The **cardinal sine** function, denoted sinc, is given by

$$\operatorname{sinc}(t) = \frac{\sin t}{t}.$$

- By l'Hopital's rule, $\operatorname{sinc} 0 = 1$.
- A plot of this function for part of the real line is shown below. [Note that the oscillations in $\operatorname{sinc}(t)$ do not die out for finite t.]



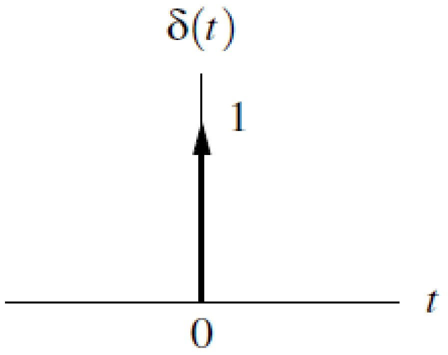
Unit-Impulse Function

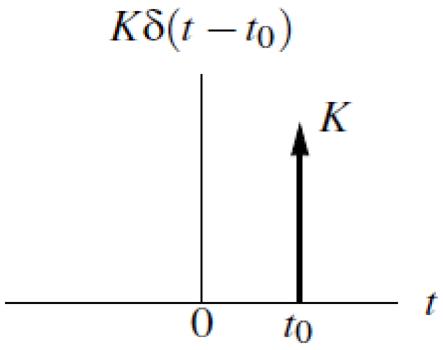
- The **unit-impulse function** (also known as the **Dirac delta function** or **delta function**), denoted δ, is defined by the following two properties:

$$\delta(t) = 0 \quad \text{for } t \neq 0 \quad \text{and}$$

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1.$$

- Technically, δ is not a function in the ordinary sense. Rather, it is what is known as a ***generalized function***. Consequently, the δ function sometimes behaves in unusual ways.
- Graphically, the delta function is represented as shown below.





Properties of the Unit-Impulse Function

- **Equivalence property.** For any continuous function x and any real constant t_0,

$$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0).$$

- **Sifting property.** For any continuous function x and any real constant t_0,

$$\int_{-\infty}^{\infty} x(t)\delta(t-t_0)dt = x(t_0).$$

- The δ function also has the following properties:

$$\delta(t) = \delta(-t) \quad \text{and}$$
$$\delta(at) = \frac{1}{|a|}\delta(t),$$

where a is a nonzero real constant.

Unit-Impulse Sequence

- The **unit-impulse sequence** (also known as the **delta sequence**), denoted δ, is defined as

$$\delta(n) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

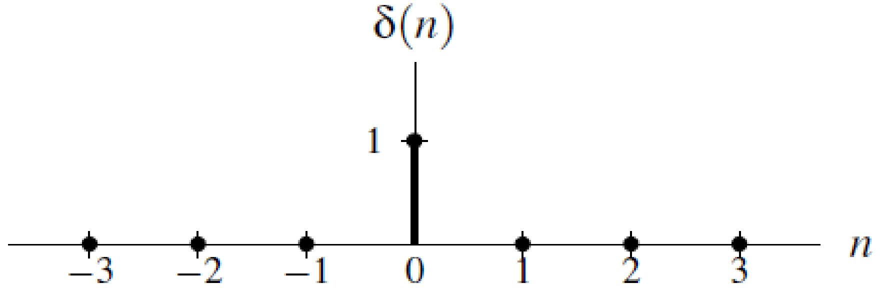
- The first-order difference of u is δ. That is,

$$\delta(n) = u(n) - u(n-1).$$

- The running sum of δ is u. That is,

$$u(n) = \sum_{k=-\infty}^{n} \delta(k).$$

- A plot of δ is shown below.



Properties of the Unit-Impulse Sequence

- For any sequence x and any integer constant n_0, the following identity holds:

$$x(n)\delta(n-n_0) = x(n_0)\delta(n-n_0).$$

- For any sequence x and any integer constant n_0, the following identity holds:

$$\sum_{n=-\infty}^{\infty} x(n)\delta(n-n_0) = x(n_0).$$

- Trivially, the sequence δ is also even.

Representing a Rectangular Pulse Using Unit-Step Functions

- For real constants a and b where $a \leq b$, consider a function x of the form

$$x(t) = \begin{cases} 1 & \text{if } a \leq t < b \\ 0 & \text{otherwise} \end{cases}$$

  (i.e., $x(t)$ is a ***rectangular pulse*** of height one, with a ***rising edge at*** a and ***falling edge at*** b).
- The function x can be equivalently written as

$$x(t) = u(t-a) - u(t-b)$$

  (i.e., the difference of two time-shifted unit-step functions).
- Unlike the original expression for x, this latter expression for x ***does not involve multiple cases***.
- In effect, by using unit-step functions, we have collapsed a formula involving multiple cases into a single expression.

Part 2
Unit-1

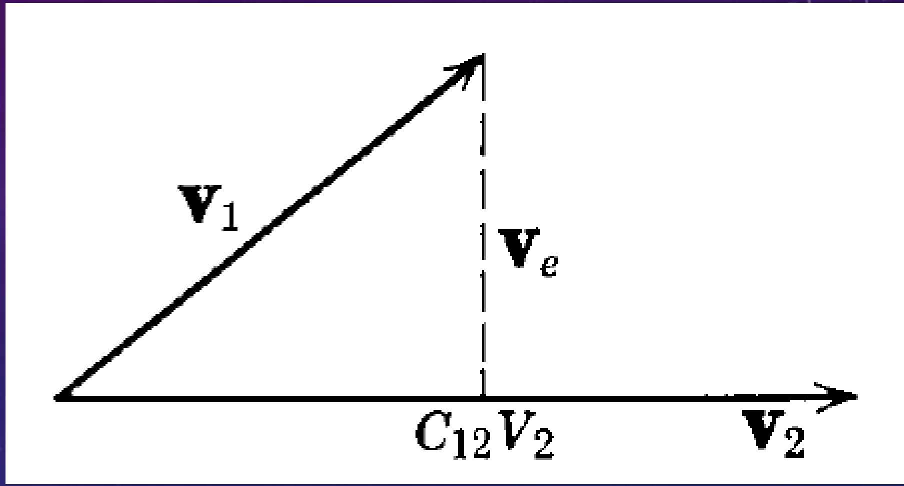
ANALOGY BETWEEN VECTORS AND SIGNALS

ANALOGY BETWEEN VECTORS AND SIGNALS

- A problem is better understood or better remembered if it can be associated with some familiar phenomenon
- Therefore we always search for analogies when studying a new problem.
- In the study of abstract problems, similarities are very helpful, particularly if the problem can be shown to be analogous to some concrete phenomenon.
- It is then easy to gain some insight into the new problem from the knowledge of the corresponding phenomenon.
- Fortunately, there is a perfect analogy between vectors and signals which leads to a better understanding of signal analysis

ANALOGY BETWEEN VECTORS AND SIGNALS

- Vectors -A vector is specified by its magnitude and direction
- Consider two vectors **V1** and **V2**
- Let the component of **V1** along V2 be given by C12V2
- How do we interpret physically the component of one vector along the other vector?

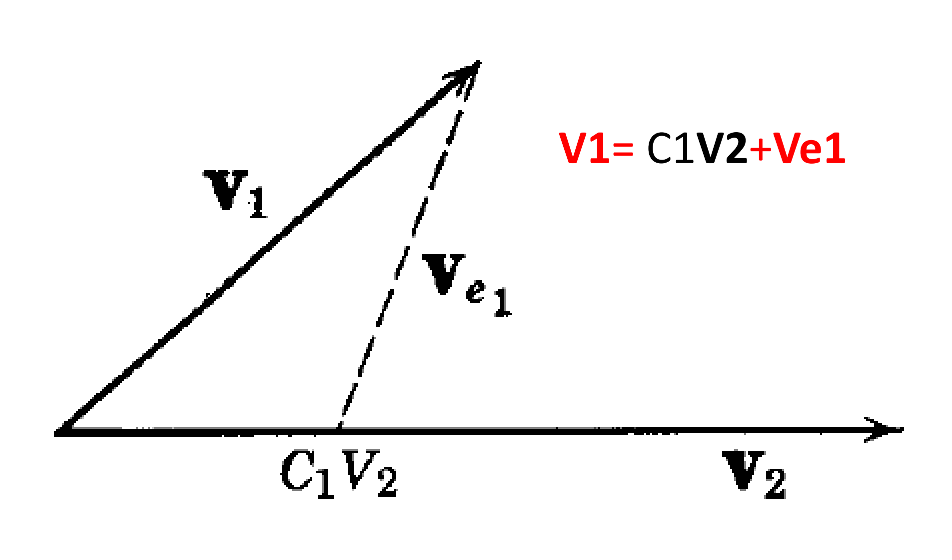


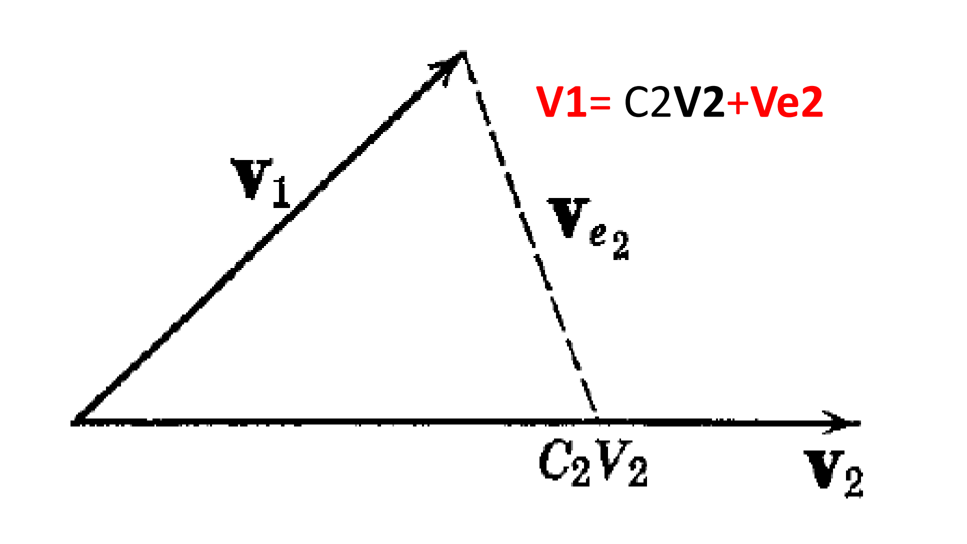
- Geometrically the component of a vector **V1** along the vector **V2** is obtained by drawing a perpendicular from the end of **V1** on to the vector **V2** as shown in the fig

ANALOGY BETWEEN VECTORS AND SIGNALS

- The vector V1 can now be expressed in terms of vector V2 as **V1**= C12V2+**Ve**
- However, this is not the only way of expressing vector **V1** in terms of vector **V2.**
- two of the infinite alternate possibilities.
- Thus **V1**= C1V2+**Ve1**

- **V1**= C2V2+**Ve2**

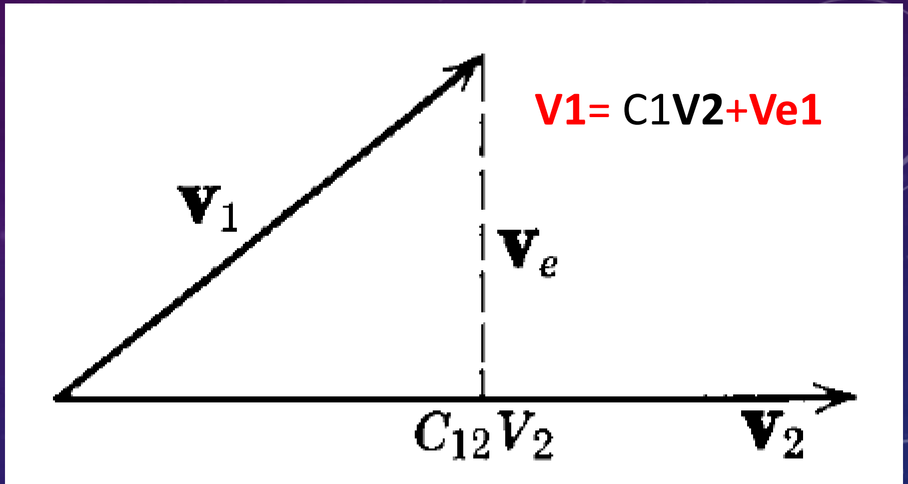




ANALOGY BETWEEN VECTORS AND SIGNALS

- In each representation, **V1** is represented in terms of **V2** plus another vector, which will be called the error vector **Ve**
- If we are asked to approximate the vector **V1** by a vector in the direction of **V2**, then **Ve** represents the error in this approximation.
- if we approximate **V1** by C12V2 then the error in the approximation is **Ve**
- If **V1 is** by C1V2 then the error is given by **Ve1 and so on**

ANALOGY BETWEEN VECTORS AND SIGNALS



- What is so unique about the representation?
- The component of a vector **V1**, along the vector **V2**, is given by **C12V2**, where C12 is chosen such that the error vector is minimum.
- Let us now interpret physically the component of one vector along another
- It is clear that the larger the component of a vector along the other vector, the more closely do the two vectors resemble each other in their directions, and the smaller is the error vector
- If the component of a vector V1 along V2, is C12V2, then the magnitude of C12 is an indication of the similarity of the two vectors

ANALOGY BETWEEN VECTORS AND SIGNALS

- If **C12 is zero**, then the vector has no component along the other vector and hence the two vectors are mutually perpendicular.
- Such vectors are known as orthogonal Vectors.
- Orthogonal vectors are thus independent vectors. I
- f the vectors are orthogonal, then the parameter **C12 is zero.**

ANALOGY BETWEEN VECTORS AND SIGNALS

- We define the dot product of two vectors A and B as A.B= AB cosθ where θ is the angle between vectors A and B.
- The Component of A along B=A cos θ =**A.B**/B and
- The Component of B along A=B cosθ =**A.B**/A
- Similarly, the component of V1 along V2
- **= V1.V2**/V2 = C12V2
- therefore

$$C_{12} = \frac{\mathbf{V}_1 \cdot \mathbf{V}_2}{V_2^{\,2}} = \frac{\mathbf{V}_1 \cdot \mathbf{V}_2}{\mathbf{V}_2 \cdot \mathbf{V}_2}$$

Note that if V1and V2 are orthogonal,
then V1.V2=0
and C12=0

SIGNALS

- The concept of vector comparison and orthogonality can be extended to signals.
- Let us consider two signals $f_1(t)$ and $f_2(t)$. Suppose we want to approximate $f_1(t)$ in terms of $f_2(t)$ over a certain interval $(t_1<t<t_2)$
- $f_1(t) = C_{12}f_2(t)$ for $(t_1<t<t_2)$
- How shall we choose C_{12} in order to achieve the best approximation?
- we must find C_{12} such that the error between the actual function and the approximated function is minimum over the interval $(t_1<t<t_2)$
- Let us define an error function $f_e(t)$ as $f_e(t)= f_1(t) – C_{12}f(t)$
- One possible criterion for minimizing the error $f_e(t)$ over the interval t_1 to t_2 is to minimize the average value of $f_e(t)$ over this interval.
- that is, to minimize

$$\frac{1}{(t_2 - t_1)} \int_{t_2}^{t_2} [f_1(t) - C_{12} f_2(t)]\, dt$$

SIGNALS

- However, this criterion is inadequate because there can be large positive and negative errors present that may cancel one another in the process of averaging and give the false indication that the error is zero
- This situation can be corrected if we choose to minimize the average (or the mean) of the square of the error instead of the error itself
- Let us designate the average of $f_e^2(t)$ by ε

$$\varepsilon = \frac{1}{(t_2 - t_1)} \int_{t_1}^{t_2} f_e^2(t)\, dt = \frac{1}{(t_2 - t_1)} \int_{t_1}^{t_2} [f_1(t) - C_{12} f_2(t)]^2\, dt$$

SIGNALS

- To find the value of C12 which will minimize error ε, we must have $\frac{d\varepsilon}{dC_{12}} = 0$
- That is

$$\frac{d}{dC_{12}}\left\{\frac{1}{(t_2 - t_1)}\int_{t_1}^{t_2}[f_1(t) - C_{12}f_2(t)]^2\,dt\right\} = 0$$

SIGNALS

- Changing the order of integration and differentiation, we get

$$\frac{1}{(t_2 - t_1)}\left[\int_{t_1}^{t_2} \frac{d}{dC_{12}} f_1^2(t)\,dt - 2\int_{t_1}^{t_2} f_1(t)\,f_2(t)\,dt + 2C_{12}\int_{t_1}^{t_2} f_2^2(t)\,dt\right] = 0$$

SIGNALS

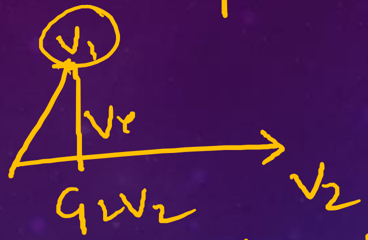
- By analogy with vectors, we say that f1(t) has a component of waveform f2(t), and this component has a magnitude C12. If C12 vanishes, then the signal f1(t) contains no component of signal f2(t),
- and we say that the two functions are orthogonal over the interval (t1, t2).
- It therefore follows that the two functions f1(t), and f2(t) are orthogonal over an interval (t1, t2) if

$$C_{12} = \frac{\int_{t_1}^{t_2} f_1(t) f_2(t)\, dt}{\int_{t_1}^{t_2} f_2^2(t)\, dt}$$

$$\int_{t_1}^{t_2} f_1(t) f_2(t)\, dt = 0$$

Analogy b/w Vectors & Signals → approximate $\bar{V}_1$, in $\bar{V}_2$

$\bar{V}_1$ & $\bar{V}_2$ → component → $c_2 V_2$



$\bar{V}_1 = c_2 V_2 + \bar{V}_e$

c_2 = similarity $\bar{V}_1$ & $\bar{V}_2$

c_2 = max = large → more similar

$c_2 = 0$ = No component = $\bar{V}_1 \cdot \bar{V}_2 = 0$

$\bar{V}_1 \cdot \bar{V}_2 = 0 = c_2 = \bar{V}_1$ & $\bar{V}_2$ — mutually orthogonal

approximate Signals/functions

$f_1(t)$, $f_2(t)$, over $(t_1 < t < t_2)$

$f_1(t) \cong c_2 f_2(t) \quad (t_1 < t < t_2)$

$f_e(t) = f_1(t) - c_2 f_2(t)$

$$\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \left[f_1(t) - c_2 f_2(t) \right]^2 dt = \text{MSE} = \varepsilon$$

$c_2 = ? \ \exists \ \dfrac{d\varepsilon}{dc_2} = 0$

$$C_2 = \frac{\int_{t_1}^{t_2} f_1(t) f_2(t)\, dt}{\int_{t_1}^{t_2} f_2^2(t)\, dt}$$

→ find the magnitude component

$f_1(t), f_2(t)$ are orthog

$C_2 = 0 = \bar{V}_1 \cdot \bar{V}_2 = 0$ = Condition for orthogonality of vectors

$$\boxed{\int_{t_1}^{t_2} f_1(t)\, f_2(t)\, dt = 0}$$

Condition for orthogonality of two signals/function (t_1, t_2)

(i) ~~Sin nwot~~ , Sin mwot — $(t_0, t_0 + \frac{2\pi}{\omega})$ → B.P. Lathi

(t_0, t_0+T)

(ii) Cos nwot, Cos mwot (t_1, t_2)

$m \neq n$

(iii) Sin nwot , Cos mwot = I = 0

Check for orthogonality

n, m are integers

$f_1(t) = \cos n\omega_0 t$,

$f_2(t) = \cos m\omega_0 t$

$I = \int_{t_1}^{t_2} f_1(t) f_2(t)\, dt = 0$

$I = \int_{t_0}^{t_0+\frac{2\pi}{\omega_0}} \cos n\omega_0 t \cdot \cos m\omega_0 t \, dt$

2 cos A cos B

$\frac{1}{2}\left[\int_{t_0}^{t_0+\frac{2\pi}{\omega_0}} \cos(n+m)\omega_0 t\, dt + \int_{t_1}^{t_2} \cos(n-m)\omega_0 t\, dt\right]$

$\frac{1}{2}\left[\frac{\sin(n+m)\omega_0 t}{(n+m)\omega_0} \Big|_{t_0}^{t_0+\frac{2\pi}{\omega_0}} + \frac{\sin(n-m)\omega_0 t}{(n-m)\omega_0} \Big|_{t_1}^{t_2} \right]$

$$\frac{1}{2}\left[\frac{\sin(n+m)\omega_0\left(t_0+\frac{2\pi}{\omega_0}\right)-\sin(n+m)\omega_0 t_0}{(n+m)\omega_0}\right]+\frac{1}{2}\left[\frac{\sin(n-m)\omega_0\left(t_0+\frac{2\pi}{\omega_0}\right)}{(n-m)\omega_0}-\frac{\sin(n-m)\omega_0 t_0}{(n-m)\omega_0}\right]$$

$\sin(n+m)(\omega_0 t_0+2\pi)$

$\frac{1}{2}\left(\sin(n+m)\omega_0 t_0 - \sin(n+m)\omega_0 t_0\right)+\frac{1}{2}\left[\frac{\sin(n-m)(\omega_0 t_0+2\pi)-\sin(n-m)\omega_0 t_0}{(n-m)\omega_0}\right]$

$$I = 0 = \int_{t_1}^{t_2} f_1(t) f_2(t)\,dt$$

$= \cos n\omega_0 t,\ \cos m\omega_0 t$ are orthogonal

$\cos\omega_0 t,\ \cos 2\omega_0 t,\ \cos n\omega_0 t$ ------ Cos n, Set

$f_1(t) = \sin n\omega_0 t, \quad f_2(t) = \cos m\omega_0 t$ $\quad \left(t_0, t_0 + \frac{2\pi}{\omega_0}\right)$

$$I = \int_{t_0}^{t_0+\frac{2\pi}{\omega_0}} \sin n\omega_0 t \cos m\omega_0 t \, dt$$

$$\boxed{I = 0}$$

both sin

cos

sin, cos

orthogonal set

set of orthogonal fns

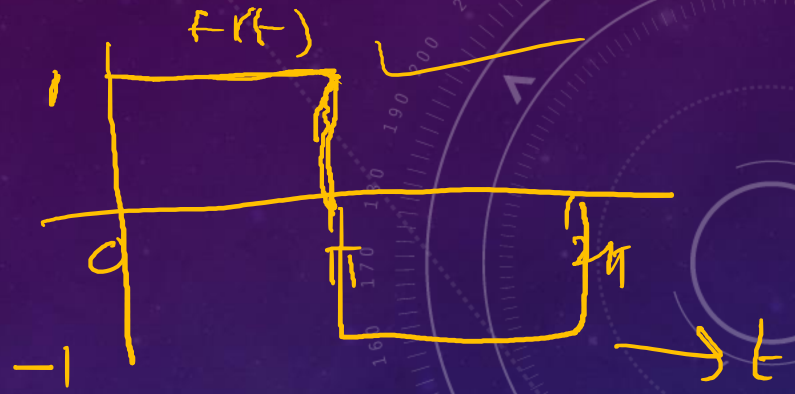
$\sin\omega_0 t, \sin 2\omega_0 t \ldots \sin n\omega_0 t$

$+ \cos\omega_0 t + \cos 2\omega_0 t + \ldots + \cos m\omega_0 t$ (3)

(*) f(t) → rectangular fn. defined as

$f(t) = 1$ for $(0 \le t \le \pi)$

$= -1$ for $(\pi \le t \le 2\pi)$



$(0, 2\pi)$

Sin t → over interval $(0, 2\pi)$

Such that MSE = $\boxed{\text{minimum}}$

Q

approximate f(t) t_1 t_2

→ Sinusoidal $(0, 2\pi)$

error minimum

$f_1(t) \approx f_2(t)$

$\exists \; C_{12} = \dfrac{\int_{t_1}^{t_2} f_1(t) f_2(t)\,dt}{\int_{t_1}^{t_2} f_2^2(t)\,dt}$

Magnitude / Error in approxi

$f_1(t) =$ rectangular fn $(0, 2\pi)$

$f_2(t) = \sin t$

$C_{12} = \dfrac{\int_{t_1}^{t_2} f_1(t) \sin t\,dt}{\int_{t_1}^{t_2} \sin^2 t\,dt}$

$\dfrac{\int_0^{\pi} 1 \cdot \sin t\,dt + \int_{\pi}^{2\pi} (-1) \cdot \sin t\,dt}{\int_0^{2\pi} \dfrac{(1-\cos 2t)}{2}\,dt}$

$\dfrac{\int_0^{\pi} \sin t\,dt - \int_{\pi}^{2\pi} \sin t\,dt}{\frac{1}{2}\int_0^{2\pi} 1 \cdot dt - \frac{1}{2}\int_0^{2\pi} \cos 2t\,dt}$

$$= \frac{\left(-\cos t\right)_0^{\pi} - \left(-\cos t\right)_{\pi}^{2\pi}}{\frac{1}{2}t\Big|_0^{2\pi} - \frac{1}{4}\cancel{\sin 2t}\Big|_0^{2\pi}}$$

$$f(t) \cong c_2 f_2(t)$$

$$\cong \frac{4}{\pi}\sin t$$

$$= \frac{-(-1-1)+(1+1)}{\frac{1}{2}\times 2\pi - \frac{1}{4}(0)} = \frac{4}{\pi} = c_2$$

$I = \int_{t_1}^{t_2} f_1(t)\, f_2(t)\, dt = 0$

no component

larger than original fn

$f_1(t)$, $f_2(t)$

$= f_2(t) = 0$

$\boxed{C_{12} = 0}$

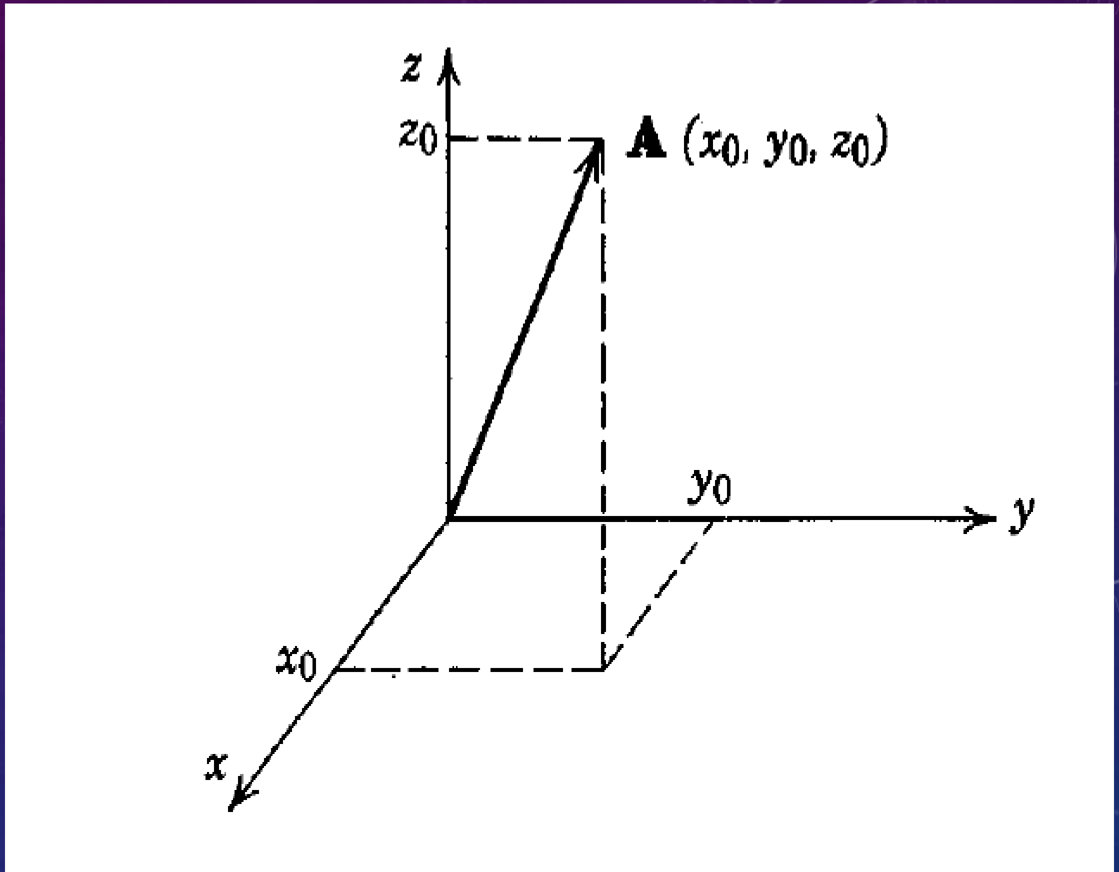
ORTHOGONAL VECTOR SPACE

- A vector A drawn from the origin to a general point (xo, yo, zo) in space has components xo, yo, and zo along the x, y, and z axes, respectively.
- We can express this vector A in terms of its components along the three mutually perpendicular axes
- $\mathbf{A}=x_o a_x+y_o a_y+z_o a_z$

The component of **A** along the x axis : $\mathbf{A} \cdot a_x$

The component of **A** along the y axis : $\mathbf{A} \cdot a_y$

The component of **A** along the z axis : $\mathbf{A} \cdot a_z$



ORTHOGONAL VECTOR SPACE

- Now we make an important observation.
- If the coordinate system has only two axes, x and y, then the system is inadequate to express a general vector A in terms of the components along these axes.
- This system can only express two components of vector A. Therefore it is necessary that to express any general vector A in terms of its coordinate components, the system of coordinates must be complete. In this case there must be three coordinate axes.

ORTHOGONAL VECTOR SPACE

- If unit vectors along these n mutually perpendicular coordinates are designated as xl, x2, . . . , xn and a general vector A in this n-dimensional space has components C1, C2,..., Cn, respectively, along these n coordinates, then
- A =C1X1+ C2X2+ C3X3 +…………………………. CnXn
- All the vectors X1, X2, . . . , Xn ,are mutually orthogonal, and the set must be complete in order for any general vector A to be represented by the above eqn.
- The constants Cr, Cr, Cu, . . . , Cn in Eq represents the magnitudes of the components of A along the vectors X1, X2, X3, . . . Xn respectively

ORTHOGONAL VECTOR SPACE

- Cr=A.Xr
- A.Xr =(C1X1+ C2X2+ C3X3 +…………………………. CnXn).Xr
- A.Xr =C1X1.Xr+ C2X2.Xr+ C3X3.Xr +……………+CrXr……………. CnXn.Xr
- A.Xr=CrXr.Xr=Cr
- We call the set of vectors (x1, x2. . . , xn) an orthogonal vector space.

  For an orthogonal vector space {xr}...(r= 1,2,...)

- If this vector space is complete, then any vector F' can be expressed as

  F =C1X1+ C2X2+ C3X3 +……………CrXr+……………. CnXn

- Where $Cr=\frac{F.Xr}{Kr}=\frac{F.Xr}{Xr.Xr}$

ORTHOGONAL SIGNAL SPACE

- Any vector can be expressed as a sum of its components along n mutually orthogonal vectors, provided these vectors formed a complete set of coordinate system.
- Similarly it is possible to express any function f(t) as a sum of its components along a set of mutually orthogonal functions if these functions form a complete set
- Called as Orthogonal Signal Space

APPROXIMATION OF A FUNCTION BY A SET OF MUTUALLY ORTHOGONAL FUNCTIONS

APPROXIMATION OF A FUNCTION BY A SET OF MUTUALLY ORTHOGONAL FUNCTIONS

- Let us consider a set of n functions g1(t), g2(t), . . . , gn(t) which are orthogonal to one another over an interval t1 to t2; that is,

$$\int_{t_1}^{t_2} g_j(t) g_k(t)\, dt = 0 \qquad j \neq k$$

$$\int_{t_1}^{t_2} g_j^2(t)\, dt = K_j$$

APPROXIMATION OF A FUNCTION BY A SET OF MUTUALLY ORTHOGONAL FUNCTIONS

- Let an arbitrary function f(t) be approximated over an interval (t1, t2) by a linear combination of these n mutually orthogonal functions.

$$f(t) \simeq C_1 g_1(t) + C_2 g_2(t) + \cdots + C_k g_k(t) + \cdots + C_n g_n(t)$$

$$= \sum_{r=1}^{n} C_r g_r(t)$$

APPROXIMATION OF A FUNCTION BY A SET OF MUTUALLY ORTHOGONAL FUNCTIONS

$$f_e(t) = f(t) - \sum_{r=1}^{n} C_r g_r(t)$$

- for the best approximation , we must find the proper values of constants C1,C2….Cn such that ε, the mean square of fe(t), is minimized. By definition ε is given by

$$\varepsilon = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \left[f(t) - \sum_{r=1}^{n} C_r g_r(t) \right]^2 dt$$

- We know that ε is a function of C1, C2, . . . , Cn
- and to minimize ε, we must have

$$\frac{\partial \varepsilon}{\partial C_1} = \frac{\partial \varepsilon}{\partial C_2} = \cdots = \frac{\partial \varepsilon}{\partial C_j} = \cdots = \frac{\partial \varepsilon}{\partial C_n} = 0$$

APPROXIMATION OF A FUNCTION BY A SET OF MUTUALLY ORTHOGONAL FUNCTIONS

Let us consider the equation:

$$\frac{\partial \varepsilon}{\partial C_j} = 0$$

Since $(t_2 - t_1)$ is constant, Eq. 1.25 may be expressed as

$$\frac{\partial}{\partial C_j}\left\{\int_{t_1}^{t_2}\left[f(t) - \sum_{r=1}^{n} C_r g_r(t)\right]^2 dt\right\} = 0$$

$$\frac{\partial}{\partial C_j}\int_{t_1}^{t_2} f^2(t)\,dt = \frac{\partial}{\partial C_j}\int_{t_1}^{t_2} C_r^{\,2} g_r^{\,2}(t)\,dt = \frac{\partial}{\partial C_j}\int_{t_1}^{t_2} C_r f(t) g_r(t)\,dt = 0$$

APPROXIMATION OF A FUNCTION BY A SET OF MUTUALLY ORTHOGONAL FUNCTIONS

$$\frac{\partial}{\partial C_j}\int_{t_1}^{t_2}[-2C_j f(t)g_j(t) + C_j^2 g_j^2(t)]\, dt = 0$$

- Changing the order of differentiation and integration we get

$$2\int_{t_1}^{t_2} f(t)g_j(t)\, dt = 2C_j\int_{t_1}^{t_2} g_j^2(t)\, dt$$

- Therefore

$$C_j = \frac{\int_{t_1}^{t_2} f(t)g_j(t)\, dt}{\int_{t_1}^{t_2} g_j^2(t)\, dt}$$

$$= \frac{1}{K_j}\int_{t_1}^{t_2} f(t)g_j(t)\, dt$$

APPROXIMATION OF A FUNCTION BY A SET OF MUTUALLY ORTHOGONAL FUNCTIONS

- Given a set of rt functions g1(t), g2(t), . . . , gn(t) mutually orthogonal over the interval (t1, t2), it is possible to approximate an arbitrary function f(t) over this interval by a linear combination of these n functions.
- f (t) ≅ C1g1(t)+c2g2(t)+……….Cn gn(t)

$$= \sum_{r=1}^{n} C_r g_r(t)$$

-
- for the best approximation, that is, the one that will minimize the
- mean of the square error over the interval, we must choose the coefficients
- C1, C2, . . . , Cn, etc., as given by Eq

$$C_j = \frac{\int_{t_1}^{t_2} f(t) g_j(t)\, dt}{\int_{t_1}^{t_2} g_j^2(t)\, dt}$$

EVALUATION OF MEAN SQUARE ERROR

EVALUATION OF MEAN SQUARE ERROR

- Let us now find the value of Mean Square Error (MSE) when optimum values of coefficients C1, C2, . . . , Cn are chosen according to Eq.

$$C_j = \frac{\int_{t_1}^{t_2} f(t) g_j(t)\, dt}{\int_{t_1}^{t_2} g_j^2(t)\, dt}$$

By Definition Mean Square Error (MSE) ε

$$\varepsilon = \frac{1}{(t_2 - t_1)} \int_{t_1}^{t_2} \left[f(t) - \sum_{r=1}^{n} C_r g_r(t) \right]^2 dt$$

EVALUATION OF MEAN SQUARE ERROR

$$\varepsilon = \frac{1}{(t_2 - t_1)}\left[\int_{t_1}^{t_2} f^2(t)\,dt + \sum_{r=1}^{n} C_r^2 \int_{t_1}^{t_2} g_r^2(t)\,dt - 2\sum_{r=1}^{n} C_r \int_{t_1}^{t_2} f(t)g_r(t)\,dt\right]$$

$$C_r = \frac{\int_{t_1}^{t_2} f(t)g_r(t)\,dt}{K_r} = \frac{\int_{t_1}^{t_2} f(t)g_r(t)\,dt}{\int_{t_1}^{t_2} g_r^2(t)\,dt}$$

$$\int_{t_1}^{t_2} f(t)g_r(t)\,dt = C_r \int_{t_1}^{t_2} g_r^2(t)\,dt = C_r K_r$$

$$\varepsilon = \frac{1}{(t_2 - t_1)}\left[\int_{t_1}^{t_2} f^2(t)\,dt + \sum_{r=1}^{n} C_r^2 K_r - 2\sum_{r=1}^{n} C_r^2 K_r\right]$$

EVALUATION OF MEAN SQUARE ERROR

$$\varepsilon = \frac{1}{(t_2 - t_1)}\left[\int_{t_1}^{t_2} f^2(t)\,dt - \sum_{r=1}^{n} C_r^2 K_r\right]$$

$$\varepsilon = \frac{1}{(t_2 - t_1)}\left[\int_{t_1}^{t_2} f^2(t)\,dt - (C_1^2 K_1 + C_2^2 K_2 + \cdots + C_n^2 K_n)\right]$$

$\boxed{f(t)} \Rightarrow$ by set of mutually orthogonal fns

W.K.T $\Rightarrow$ $\sin n\omega_0 t$, $\sin m\omega_0 t$ $\Rightarrow$ orthogonal fn $(0, 2\pi)$

$\left(t_0, t_0 + \frac{2\pi}{T}\right)$

n, m

$f(t) = c_1 g_1 + c_2 g_2 - \cdots c_n g_n(t)$

Let $\rightarrow$ $\sin t$, $\sin 2t$, $\sin 3t$ ..... $\sin rt \rightarrow$

$$I = \int_{t_0}^{t_0+\frac{2\pi}{\omega_0}} \sin t \cdot \sin 5t \, dt = \underline{\underline{0}}$$

$g_1(t) = \sin t$,

$g_2(t) = \sin 2t$ ....

$\boxed{g_r(t) = \sin rt}$

$(r = 1, 2 \ldots n)$

$g_r(t) = (r = 1, 2, \ldots n)$ — set of n orth fn

$f(t) \cong \frac{4}{\pi} \sin t$

$f(t) = 1 \quad 0 \le t < \pi$ approximate using $\sin rt \to r = 1, 2, \dots n$

$ = -1 \quad \pi \le t \le 2\pi,$ set of mutually orthogonal fn

over the interval $(0, 2\pi)$, which results in minimum MSE

$$f(t) \simeq \underline{c_1}\, g_1(t) + \underline{c_2}\, g_2(t) + \dots \underline{c_n}\, g_n(t)$$

$$c_r = \frac{\int_{t_1}^{t_2} f(t)\, g_r(t)\, dt}{\int_{t_1}^{t_2} g_r^2(t)\, dt} = \frac{\int_0^{\pi} 1 \cdot \sin rt\, dt + \int_{\pi}^{2\pi} (-1) \sin rt\, dt}{\int_0^{2\pi} \sin^2 rt\, dt}$$

$$C_r = \frac{\left[-\frac{\cos rt}{r} \right]_0^{\pi} + \frac{\cos rt}{r} \Big|_{\pi}^{2\pi}}{\int_0^{2\pi} \frac{(1-\cos 2rt)}{2}\, dt}$$

$$\frac{-\left(\frac{\cos r\pi}{r} - \frac{1}{r} \right] + \left[\frac{\cos 2\pi r}{r} - \frac{\cos \pi r}{r} \right]}{\pi}$$

$$\frac{-(-1-1)}{r} + \frac{(1+1)}{r} = \frac{4}{\pi r} \quad r = \text{odd} = C_r$$

$$\frac{1}{2} t \Big|_0^{2\pi} - \frac{\cancel{\sin 2\pi}}{\cancel{4r}}$$

$$\frac{\cancel{2\pi} - 0}{2} -$$

for $r = \text{odd}$ $\quad C_r = \frac{4}{\pi r}$

$r = \text{even}$ $\quad C_r = \underline{\underline{0}}$

$c_1 = \frac{4}{\pi}, \quad C_2 = 0, \quad c_3 = \frac{4}{3\pi},$

$c_4 = 0, \quad c_5 = \frac{4}{5\pi} - - - - -$

$f(t) \simeq \frac{4}{\pi}\sin t + \frac{4}{3\pi}\sin 3t + \frac{4}{5\pi}\sin 5t + - - - - - \frac{4}{n\pi}\sin nt$

$c_1 \quad s_1 \qquad c_3 \quad s_3 \qquad c_5 \quad s_5$

$f(t) \simeq \frac{4}{\pi}\sin t$

$\varepsilon = \text{error} =$

$$f(t) = \frac{4}{\pi}\left(\sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t + \cdots \frac{1}{r}\sin rt\right)$$

$$\varepsilon = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2}\left(\underline{\underline{f^2(t)\,dt}} - \left(\underline{\underline{C_1^2 K_1}} + C_2^2 K_2 + \cdots C_r^2 K_r\right) + \cdots \infty\right)$$

$1 \to \infty$

$0 \le t < \pi$

$f(t) = 1$

$= -1$

$\pi \le t < 2\pi$

$$C_r = \frac{4}{\pi r} \quad \text{for } r = \text{odd}$$

$$= 0 \quad \text{for } r = \text{even}$$

$$K_r = \int_0^{2\pi} \sin^2 rt\,dt = \pi$$

$$\int_{t_1}^{t_2} f^2(t)\,dt = \int_0^{\pi} 1^2\,dt + \int_{\pi}^{2\pi} (-1)^2\,dt$$

$$t\Big|_0^{\pi} + t\Big|_{\pi}^{2\pi}$$

$$\pi + \pi = \underline{\underline{2\pi}}$$

$$\varepsilon_1 = \frac{1}{2\pi - 0}\left(2\pi - \left(\frac{4}{\pi}\right)^2 \pi\right] \rightarrow \text{one term} - \underline{g_1(t)}$$

$$= 0.19$$

two trm $\quad g_1(t) \; \& \; g_3(t)$

$$\varepsilon_2 = \frac{1}{2\pi}\left((2\pi) - \left(\frac{4}{\pi}\right)^2 \pi - \left(\frac{4}{3\pi}\right)^2 \pi\right]$$

$$= 0.1 \checkmark$$

$$\varepsilon_3 = \frac{1}{2\pi}\left((2\pi) - \left(\frac{4}{\pi}\right)^2 \pi - \left(\frac{4}{3\pi}\right)^2 \pi - \left(\frac{4}{5\pi}\right)^2 \pi\right]$$

$$= 0.0675$$

$$\varepsilon_4 = 0.051$$

function $\uparrow$ $= \varepsilon$ - error $\downarrow$

$= \varepsilon = 0$

$r = 1, \cancel{3}\, n \rightarrow \alpha$

Evaluation of Mean Square error

$$C_r = \frac{1}{K_r}\int_{t_1}^{t_2} f(t)\, g_r(t)\, dt$$

$$K_r = \int_{t_1}^{t_2} g_r^2(t)\, dt$$ — which gives minimum MSE

Best approx

$$\varepsilon = \frac{1}{t_2-t_1}\left[\int_{t_1}^{t_2} f^2(t)\,dt - \left(c_1^2K_1 + c_2^2K_2 + \cdots c_n^2K_n\right)\right]$$

ε = minimum

as no. of fns in the approx $\uparrow = \varepsilon \downarrow$ $(r = 1, 2 \ldots n)$

$r = 1, 2, 3 \ldots n \ldots \ldots \infty$

$$\boxed{\varepsilon = 0}$$

$$\varepsilon = \frac{1}{t_2 - t_1}\left[\int_{t_1}^{t_2} f^2(t)\,dt - \sum_{r=1}^{\infty} C_r^2 K_r\right] = 0$$

$$\boxed{\int_{t_1}^{t_2} f^2(t)\,dt = \sum_{r=1}^{\infty} C_r^2 K_r}$$

Converging to the summation

as the no. of functions in the approxi $\uparrow$ to ∞ then Integral is converging

Summation $= \varepsilon = 0$

$$f(t) = c_1 g_1(t) + c_2 g_2(t) + \cdots + c_n g_n(t) + \cdots\cdots \infty$$

set of ortho- fns

rectangul

$\Rightarrow$ Representation of a fn by a closed (or) complete set of mutually orthogonal fns

$f(t) =$ set of sinusoids $=$

Sum of sinusoids of various freq

Complete set of mutually orthogonal fns:

Let $g_r(t) \to r = 1 \to n \to$ set of mutual orthogonal fns

said to be Complete (or) Closed set of mutually ortho. fns

If $I = \int_{t_1}^{t_2} x(t)\, g_r(t)\, dt = 0$

$x(t)$ & $g_r(t)$ are orthogonal

set of all

$g_r(t) = \sin t, \sin 2t \; \cdots \; \sin nt$

$x(t)$

$g_r(t) = \{ \sin t, \sin 2t, \ldots \sin rt \}$ — set of orthogonal

Complete ✗

Complete = $\{ g_r(t), \boxed{x(t)} \}$

$\int_{t_1}^{t_2} x(t)\, g_r(t)\, dt = 0$

$f(t) \stackrel{\checkmark}{=}$ wrp (sum of components along complete set of mutually orthogonal fns)

these

$= C_1 g_1(t) + C_2 g_2(t) + \ldots C_n g_n(t) + \ldots\ldots C\, x(t)$

$I = \int_{t_1}^{t_2} x(t)\, \sin rt\, dt$

$r=1, n \longrightarrow$

$\sin n\omega_0 t$

$\sin n\omega_0 t$ — set of ortho

$\longrightarrow \sin\omega_0, \sin 2\omega_0 t, \ldots$

$\cos n\omega_0 t \longrightarrow$ set of ortho

$\cos\omega_0 t, \cos 2\omega_0 t, \ldots$

$- \cos m\omega_0 t \rightarrow$

$\rightarrow \sin n\omega_0 t, \cos m\omega_0 t$

$$I = \int_0^{2\pi} \underbrace{\sin\omega_0 t}_{g_1(t)} \underbrace{\cos 2\omega_0 t}_{g_2(t)}\, dt = 0 = I = \text{orthog}$$

$\sin\omega_0 t, \sin 2\omega_0 t, \ldots \sin r\omega_0 t \ldots$

$$I = \int_{t_1}^{t_2} \sin n\omega t \cos m\omega t \, dt = 0$$

$g_r(t)$ — Set of orthogonal fun

$$I = \int \boxed{\frac{x(t)}{r}} g_r(t)\, dt = C_r$$

$C_r x t$

$\sin t, \sin 2t \ldots$

$\sin t$; $x(t)$ $\cos\omega t$, $\cos 2\omega t$, $\cos n\omega t$

$\cos m\omega t$, $\cos m\omega t$

Set of orthogonal fn

$= [\sin\omega t, \sin 2\omega t \ldots \sin n\omega t, \cos\omega t, \cos 2\omega t \ldots \cos m\omega t]$

Complete Set $\cong$

$f(t) = C_1 g_1(t) + \ldots C_r g_r(t)$

REPRESENTATION OF A FUNCTION BY A CLOSED OR COMPLETE SET OF MUTUALLY ORTHOGONAL FUNCTIONS

$$\varepsilon = \frac{1}{(t_2 - t_1)}\left[\int_{t_1}^{t_2} f^2(t)\,dt - (C_1{}^2K_1 + C_2{}^2K_2 + \cdots + C_n{}^2K_n)\right]$$

- It is evident from Eq that if we increase n, that is, if we approximate f(t) by a larger number of orthogonal functions, the error will become smaller.
- hence in the limit as the number of terms is made infinity, the sum $\sum_{r=1}^{\infty} C_r{}^2K_r$ may converge to the integral $\int_{t_1}^{t_2} f^2(t)\,dt$

and then ε vanishes. Thus

$$\int_{t_1}^{t_2} f^2(t)\,dt = \sum_{r=1}^{\infty} C_r{}^2K_r$$

Under these conditions $f(t)$ is represented by the infinite series:

$$f(t) = C_1 g_1(t) + C_2 g_2(t) + \cdots + C_r g_r(t) + \cdots$$

The infinite series on the right-hand side of Eq. thus converges to $f(t)$ such that the mean square of the error is zero. The series is said to *converge in the mean*. Note that the representation of $f(t)$ is now exact.

A set of functions $g_1(t), g_2(t), \ldots, g_r(t)$ mutually orthogonal over the interval (t_1, t_2) is said to be a complete or a closed set if there exists function $x(t)$ for which it is true that

$$\int_{t_1}^{t_2} x(t) g_k(t)\, dt = 0 \qquad \text{for} \quad k = 1, 2, \ldots$$

If a function $x(t)$ could be found such that the above integral is zero, then obviously $x(t)$ is orthogonal to each member of the set $\{g_r(t)\}$ and, consequently, is itself a member of the set. Evidently the set cannot be complete without $x(t)$ being its member.

Let us now summarize the results of this discussion. For a set $\{g_r(t)\}$, ($r = 1, 2, \ldots$) mutually orthogonal over the interval (t_1, t_2),

$$\int_{t_1}^{t_2} g_m(t) g_n(t)\, dt = \begin{cases} 0 & \text{if} \quad m \neq n \\ K_m & \text{if} \quad m = n \end{cases}$$

If this function set is complete, then any function $f(t)$ can be expressed as

$$f(t) = C_1 g_1(t) + C_2 g_2(t) + \cdots + C_r g_r(t) + \cdots$$

where

$$C_r = \frac{\displaystyle\int_{t_1}^{t_2} f(t) g_r(t)\, dt}{K_r} = \frac{\displaystyle\int_{t_1}^{t_2} f(t) g_r(t)\, dt}{\displaystyle\int_{t_1}^{t_2} g_r^{2}(t)\, dt}$$

- Any vector can be expressed as a sum of its components along n mutually orthogonal vectors, provided these vectors form a complete set. Similarly,
- **any function f(t) can be expressed as a sum of its components along mutually orthogonal functions, provided these functions form a closed or a complete set.**

PROBLEM

A rectangular function $f(t)$ is defined by

$$f(t) = \begin{cases} 1 & (0 < t < \pi) \\ -1 & (\pi < t < 2\pi) \end{cases}$$

- we shall now see how the approximation improves when a large number of mutually orthogonal functions are used
- It was shown previously that functions sin nωot and sin mωot are mutually orthogonal over the interval (to, t o + T) for all integral values of n and m.
- hence it follows that a set of functions Sin t, sin 2t,... sin rt, etc., are mutually orthogonal over the interval (0,2π).

$f(t) \cong C1 \sin t + C2 \sin 2t + C3 \sin 3t + \ldots\ldots\ldots\ldots\ Cn \sin nt$

The constants C_r can be evaluated by using

$$C_r = \frac{\int_0^{2\pi} f(t) \sin rt\, dt}{\int_0^{2\pi} \sin^2 rt\, dt}$$

$$= \frac{1}{\pi}\left(\int_0^{\pi} \sin rt\, dt - \int_{\pi}^{2\pi} \sin rt\, dt\right)$$

$$= \frac{4}{\pi r} \qquad \text{if } r \text{ is odd}$$

$$= 0 \qquad \text{if } r \text{ is even}$$

Thus $f(t)$ is approximated by

$$f(t) = \frac{4}{\pi}\left(\sin t + \tfrac{1}{3}\sin 3t + \tfrac{1}{5}\sin 5t + \tfrac{1}{7}\sin 7t + \cdots\right)$$

Let us evaluate the error ε in these approximations.

$$\varepsilon = \frac{1}{t_2 - t_1}\left[\int_{t_2}^{t_2} f^2(t)\,dt - C_1{}^2K_1 - C_2{}^2K_2 - \cdots\right]$$

In this case

$$(t_2 - t_1) = 2\pi$$

$$f(t) = \begin{cases} 1 & (0 < t < \pi) \\ -1, & (\pi < t < 2\pi) \end{cases}$$

Therefore

$$\int_0^{2\pi} f^2(t)\,dt = 2\pi$$

Also

$$C_r = \begin{cases} \dfrac{4}{\pi r} & \text{if } r \text{ is odd} \\ 0 & \text{if } r \text{ is even} \end{cases}$$

$$K_r = \int_0^{2\pi} \sin^2 rt \, dt = \pi$$

Therefore, for one-term approximation,

$$\varepsilon_1 = \frac{1}{2\pi}\left[2\pi - \left(\frac{4}{\pi}\right)^2 \pi\right] = 0.19$$

For two-term approximation,

$$\varepsilon_2 = \frac{1}{2\pi}\left[2\pi - \left(\frac{4}{\pi}\right)^2 \pi - \left(\frac{4}{3\pi}\right)^2 \pi\right] = 0.1$$

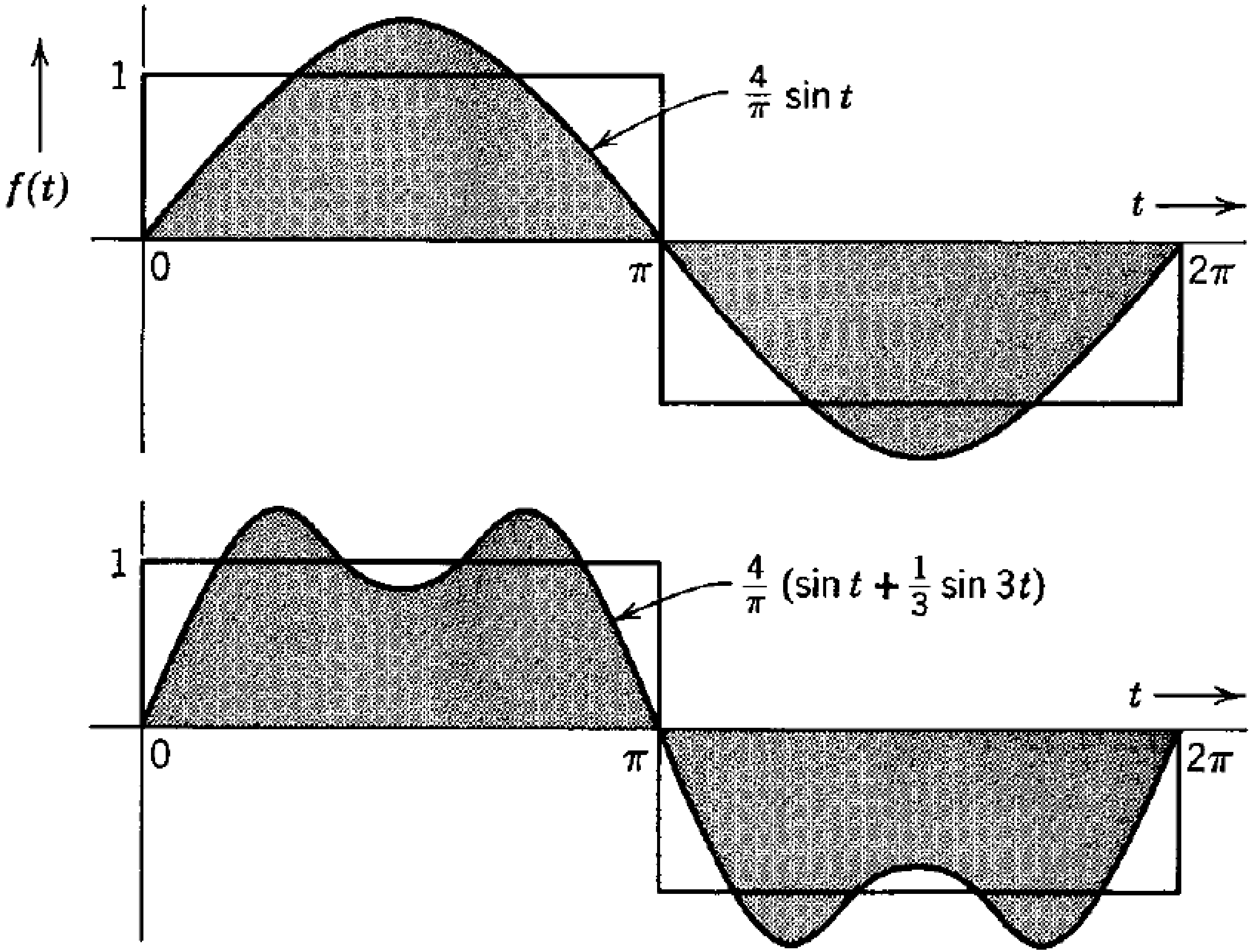
For three-term approximation,

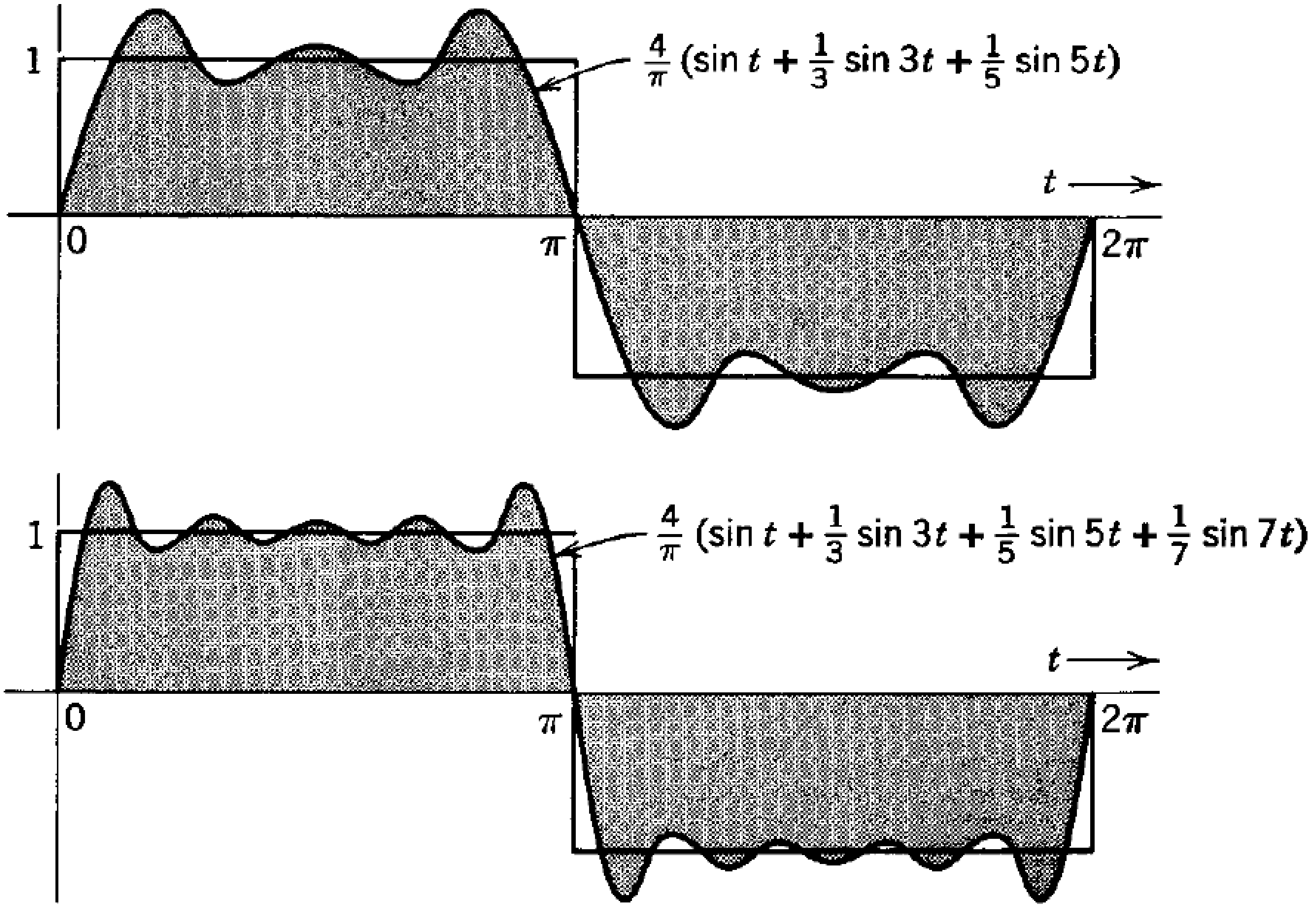
$$\varepsilon_3 = \frac{1}{2\pi}\left[2\pi - \left(\frac{4}{\pi}\right)^2 \pi - \left(\frac{4}{3\pi}\right)^2 \pi - \left(\frac{4}{5\pi}\right)^2 \pi\right] = 0.0675$$

and

$$\varepsilon_4 = \frac{1}{2\pi}\left[2\pi - \left(\frac{4}{\pi}\right)^2 \pi - \left(\frac{4}{3\pi}\right)^2 \pi - \left(\frac{4}{5\pi}\right)^2 \pi - \left(\frac{4}{7\pi}\right)^2 \pi\right] = 0.051$$

and so on.


f(t)
1
4/π sin t
t
0
π
2π
1
4/π (sin t + 1/3 sin 3t)
t
0
π
2π



Approximation of a rectangular function by orthogonal functions.

- Figure shows the actual function and. the approximated function when the function is approximated with one, two, three, and four terms, respectively
- For the given number of terms of the form sin rt, these are the optimum approximations which minimize the mean-square error.
- As we increase the number of terms, the approximation improves and the mean-square error diminishes. for infinite terms the mean-square error is zero.

ORTHOGONALITY IN COMPLEX FUNCTIONS

Orthogonality in Complex Functions

In the previous discussion, we considered only real functions of real variables. If $f_1(t)$ and $f_2(t)$ are complex functions of real variable t, then it can be shown that $f_1(t)$ can be approximated by $C_{12}f_2(t)$ over an interval (t_1, t_2).

$$f_1(t) \simeq C_{12}f_2(t)$$

The optimum value of C_{12} to minimize the mean-square error magnitude is given by†

$$C_{12} = \frac{\int_{t_1}^{t_2} f_1(t)f_2^*(t)\,dt}{\int_{t_1}^{t_2} f_2(t)f_2^*(t)\,dt}$$

where $f_2^*(t)$ is a complex conjugate of $f_2(t)$.

It is evident from Eq. 1.39 that two complex functions $f_1(t)$ and $f_2(t)$ are orthogonal over the interval (t_1, t_2) if

$$\int_{t_1}^{t_2} f_1(t) f_2^*(t)\, dt = \int_{t_1}^{t_2} f_1^*(t) f_2(t)\, dt = 0$$

For a set of complex functions $\{g_r(t)\}$, $(r = 1, 2, \ldots)$ mutually orthogonal over the interval (t_1, t_2):

$$\int_{t_1}^{t_2} g_m(t) g_n^*(t)\, dt = \begin{cases} 0 & \text{if} \quad m \neq n \\ K_m & \text{if} \quad m = n \end{cases}$$

If this set of functions is complete, then any function $f(t)$ can be expressed as

$$f(t) = C_1 g_1(t) + C_2 g_2(t) + \cdots + C_r g_r(t) + \cdots$$

where

$$C_r = \frac{1}{K_r}\int_{t_1}^{t_2} f(t)g_r^*(t)\,dt$$

If the set of functions is real, then $g_r^*(t) = g_r(t)$ and all results for complex functions reduce to those obtained for real functions

PROBLEMS on classification of Signals

$x(t) = \cos t + \sin\sqrt{2}\,t$ Periodic ?

$T = ?$

$x(t) = x_1(t) + x_2(t)$

$x_1(t) \to T_1$, $x_2(t) \to T_2$

$sum(x_1, x_2) = $ Periodic only

when $\frac{T_1}{T_2} = $ rational $=$ ratio of Integers

$LCM(T_1, T_2) = $ Period $= x(t)$

$x(t) = \cos \omega_0 t + \sin \omega_0 t$

$\omega_0 = 1$, $\omega_2 = \sqrt{2}$

$x_1(t) = \cos t = $ Periodic

$\omega_0 = 1$, $T_1 = \frac{2\pi}{\omega_0} = 2\pi$

$T_2 = \frac{2\pi}{\omega_0} = $ $\quad x_2(t) = \sin\sqrt{2}\,t$

$\sin \omega_0 t$

$T_2 = \frac{2\pi}{\sqrt{2}} = $ decimal $=$ Peri..

$T_2 = \sqrt{2}\pi = $ Periodic

$\frac{T_1}{T_2} = \frac{2\cancel{\pi}}{\sqrt{2}\,\cancel{\pi}} = \sqrt{2} = $ irrational

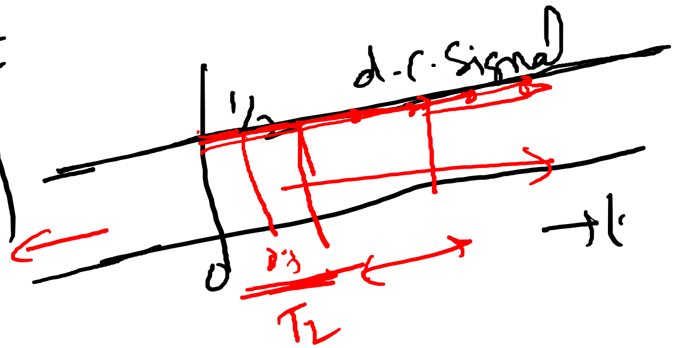
$x(t) = $ Non-Periodic

$x(t) = \sin^2 t = \frac{1-\cos 2t}{2} = \frac{1}{2} - \frac{1}{2}\cos 2t$

$= x_1(t) + x_2(t)$

$\quad\ \downarrow \qquad \downarrow$

$\quad\ T_1 \qquad T_2$



$\cos^2\theta - \sin^2\theta = \cos 2\theta$

$1 - \sin^2\theta - \sin^2\theta = 1$

$1 - 2\sin^2\theta = \cos 2\theta$

$\boxed{\frac{1-\cos 2\theta}{2}} = \sin^2\theta$

$\frac{T_1}{T_2} =$

$x_1(t) = \frac{1}{2} = \text{const} = \text{Periodic} =$ fundamental Period Can not be defined

$T = 0.5$, 0.4

$T_2 = 1$

$T_3 = 2$

π

2π

$x_2(t) = -\frac{1}{2}\cos 2t$

$\cos\omega_0 t$

$LCM(T_1, T_2)$

Period $= \pi$

$\omega_0 = 2 \Rightarrow T_2 = \frac{\cancel{2}\pi}{\cancel{2}} = \pi$

$\frac{T_1}{T_2} = \frac{T_1}{\pi} = LCM(T_1, T_2)$

π

$x_1(n) = \cos\frac{2\pi}{3}n + \sin\frac{3\pi}{4}n$

$\uparrow N_1$ $\uparrow N_2$

$\omega_1 = \frac{2\pi}{3}$, $\omega_2 = \frac{3\pi}{4}$

$N_1 = \frac{2\pi}{2\pi/3} = 3 \Rightarrow$

$\boxed{N = \frac{2\pi}{\Omega} \cdot m}$

$N_1 = 3$

$N_2 = \frac{2\pi}{3\pi/4} = \frac{8}{3} \times m$

$= \frac{8}{3} \times 3$

$\boxed{N_2 = 8}$

m = Smallest Integer which N = Integer

$\boxed{T = \frac{2\pi}{\omega}}$ = derived = Valid

x_1 — Period — $N_1 = 3$

x_2 — " — $N_2 = 8$

$x(n) = \frac{N_1}{N_2} = \frac{3}{8}$ = rational

= Periodic

Sum two DTS = always Periodic

$\frac{2\pi}{\sqrt{2}\pi} = \sqrt{2}$

Energy (or) Power Signal

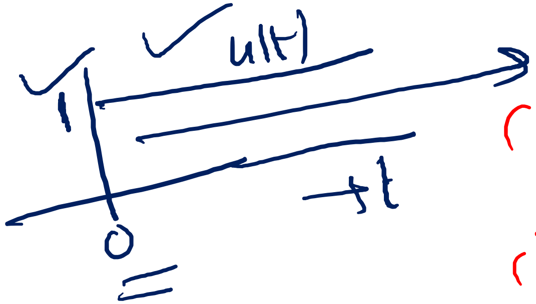
$x(t) = e^{-at}\,u(t)$

$u(t) = 1$ for $t \geq 0$

$\quad\;\; = 0$ for $t < 0$

= Limits

$|x(t)|^2 = e^{-2at}$



$E = \int_{-\infty}^{\infty} |x(t)|^2\,dt$

$= \int_{-\infty}^{\infty} e^{-2at}\cdot(1)\,dt$

$= \int_{0}^{\infty} e^{-2at}\,dt = \left.\frac{e^{-2at}}{-2a}\right|_0^{\infty} = \left(\frac{e^{-\infty} - e^{0}}{-2a}\right)$

$= \boxed{e^{-\infty} = 0,\quad e^{\infty} = \infty,\quad e^{0} = 1}$

$\frac{0-1}{-2a} = \frac{1}{2a} <$ Infinite

$\lim_{T\to\infty} \frac{1}{T\to 2a}$ E_x = finite = Energy Signal

$E_x = \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2\,dt$

avg Power $= P_x = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2\,dt$

(i) $0 < E < \infty$ = finite = Energy, Power = 0

(ii) $0 < P < \infty$ = finite = Power

Energy = ∞ = Infinite

$P = \frac{E}{T}$

$$\lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2 dt = \frac{A^2}{-2a}\left[e^{-2at}\right]_{-T/2}^{T/2}$$

Value

$$\lim_{T\to\infty} \frac{1}{T} = 0 \times \text{Value}$$

$P = 0 =$ Zero.

$E =$ finite $= P = 0 \Rightarrow x(t) =$ Energy Signal

$E = \infty$

$$x(t) = A\cos(\omega_0 t + \theta)$$

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$|x(t)|^2 = A^2 \cos^2(\omega_0 t + \theta)$$

$$= \frac{A^2\left(1 + \cos(2(\omega_0 t + \theta))\right)}{2}$$

$$= \frac{A^2}{2} + \frac{A^2}{2}\cos(2\omega_0 t + 2\theta)$$

$$\int_{-\infty}^{\infty}\left(\frac{A^2}{2} + \frac{A^2}{2}\cos(2\omega_0 t + 2\theta)\right)dt$$

$$\frac{A^2}{2} t + \frac{A^2}{2}\frac{\sin(2\omega_0 t + 2\theta)}{2\omega_0}\Bigg|_{-\infty}^{\infty}$$

$$\infty$$

$$P = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2}\left(\frac{A^2}{2} + \frac{A^2}{2}\cos(2\omega_0 t + 2\theta)\right)dt$$

e^{-at} E

$A\cos(\omega_0 t+\theta)$

Periodic

$$= \lim_{T\to\infty}\frac{A^2}{2T}\, t\Big|_{-T/2}^{T/2} + \left[\lim_{T\to\infty}\frac{1}{T}\frac{A^2}{2}\left(\frac{\sin(2\omega_0 t + 2\theta)}{2\omega_0}\right)\right]_{-T/2}^{T/2} = 0$$

$\sin(2\pi + 2\theta)$

2π

$$\lim_{T\to\infty}\frac{A^2}{2T}\{T\} +$$

$\sin(2\omega_0 \cdot \frac{T}{2} + 2\theta)$

$\sin(-(2\pi - 2\theta))$

$+\sin(2\pi - 2\theta)$

4th

$$\frac{\sin 2\theta - \sin 2\theta}{2\omega_0}$$

$\cancel{2}\omega_0 \frac{T}{\cancel{2}}$

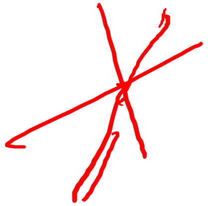
$$T = \frac{2\pi}{\omega}$$

$$\omega_0 T = 2\pi$$

$$\boxed{E = \infty}$$

$$\lim_{T\to\infty}\frac{A^2}{2\cancel{T}}\cancel{T} = \frac{A^2}{2} = \text{Power}$$

$x(t) = t \cdot u(t)$ = ramp signal $= r(t)$



$E = \int_0^{\infty} t^2 \, dt = \frac{t^3}{3}\Big|_0^{\infty} = \infty$ ✓ ~~Batch~~

$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} t^2 \, dt = \lim_{T \to \infty} \frac{1}{T} \left[\frac{t^3}{3} \right]_0^{T/2}$

$= \lim_{T \to \infty} \frac{1}{T} \left(\frac{(T/2)^3}{3} \right)$

$\lim_{T \to \infty} \frac{1}{T} \frac{T^3}{24} = \frac{T^2}{24} \to \infty$

$E_x = \infty$
$P_x = \infty$ → Infinite

Neither Energy (or) nor Power signal

$$x(n) = (-0.5)^n u(n)$$

$$|x(n)|^2 = (-0.5)^{2n} \cancel{u(n)}$$

$$|x(n)|^2 = (0.25)^n$$

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$= \sum_{n=0}^{\infty} (0.25)^n = \text{Sum of infinite terms}$$

G.P

$$\sum_{n=0}^{\infty} r^n$$

$$= \frac{1}{1-r} \quad |r|<1$$

$$= \frac{1}{1-\frac{1}{4}} = \frac{4}{3}$$

(i) $\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$ $\quad |a|<1$

(ii) $\sum_{n=0}^{N-1} a^n = \frac{1-a^N}{1-a}$ $\quad a \neq 1$

$= N$ $\quad a=1$

(iii) $\sum_{n=k}^{\infty} a^n = \frac{a^k}{1-a}$

Signal → fn of time → 1-D, 2-D, M-D → Video

1-D → Speech

2-D → Image

CTS → DTS → Quantization — Digital — finite = 0 to 1

CTS → Sampling → DTS

⇒ Analog.

infini

- Even/odd

$x(-t) = x(t)$ — even = $\frac{x(t)+x(-t)}{2}$

Eg

$x(-t) = -x(t)$ — odd = $\frac{x(t)-x(-t)}{2}$

Periodic → $\boxed{x(t) = x(t+T)}$

T = T0, 2T0 ---- NT0

fundam

CTS → $\frac{T_1}{T_2}$ =

energy/power

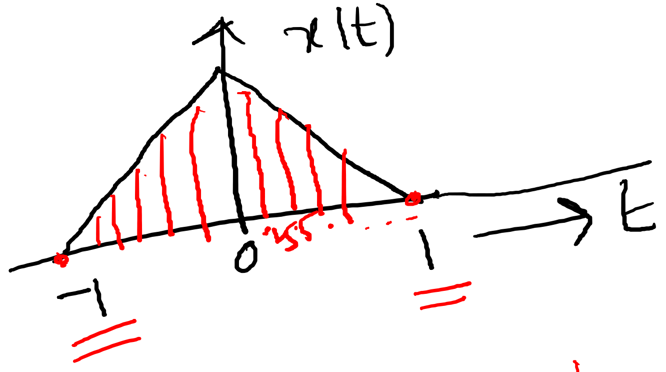
And

Non Peri - E

Period - P

Ramp - No E/P

$$x(t) = 1 - |t| \quad \text{for } -1 \le t \le 1$$
$$= 0 \quad \text{otherwise}$$



uniform sampling of $x(t)$

Sampling Interval = (i) 0.255 = T_s

(ii) $\boxed{0.5}$ sec

(iii) $\boxed{1 \text{ sec}}$

$x(t)$ — continuous

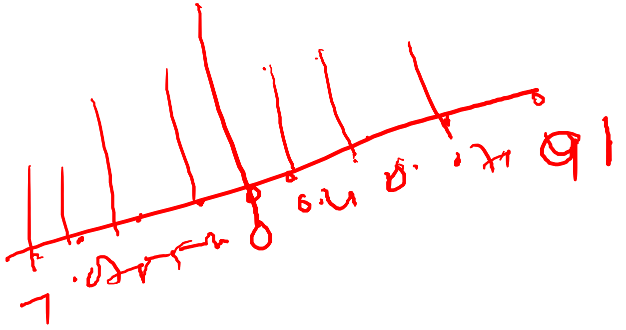
Sequence — $x(n)$

Sampling

CTS → $\boxed{\text{Sampling}}$ → DT Seq

$x(t)$ — CTS

$t =$



$$x(t) = 1 - |t|$$

= replacing $t \rightarrow nT_s$ (Sampling Interval)

n ↓ Index

$$x(n) = 1 - |nT_s|$$

$x(n) = 1 - nT_s$

(i) $T_s = 0.25\,sec$

$x(0) = 1 - 0 = 1$

$= 1 - \frac{1 \times 1}{4} = 1 - \frac{1}{4} = \frac{3}{4}$

$x(1) = 1 - 2 \times \frac{1}{4} = 1 - \frac{1}{2} = \frac{1}{2}$

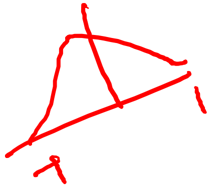
$x(2) = 1 - \frac{3}{4} = \frac{1}{4}$

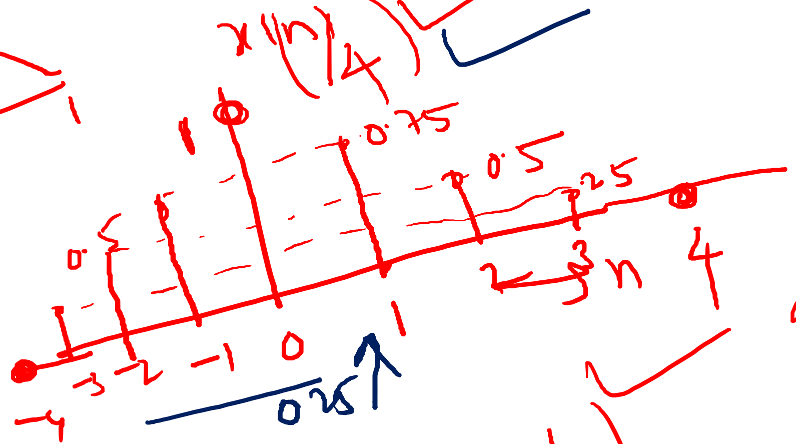
$x(3) = $

$x(4) = 1 - 4 \cdot \frac{1}{4} = 1 - 1 = 0$

$n = -4$ to 4

$x(t) \xrightarrow{Samp} x(n)$





$x(nT_s) = x(n/4)$

$x(t)$ — Sample $T_s = 0.25\,sec$

(ii) $T_s = 0.5\,sec$. $x(nT_s) = x(n/2)$

$x(0) = 1 - 0 \times \frac{1}{2} = 1$

$x(1) = 1 - \frac{1}{2} = \frac{1}{2}$

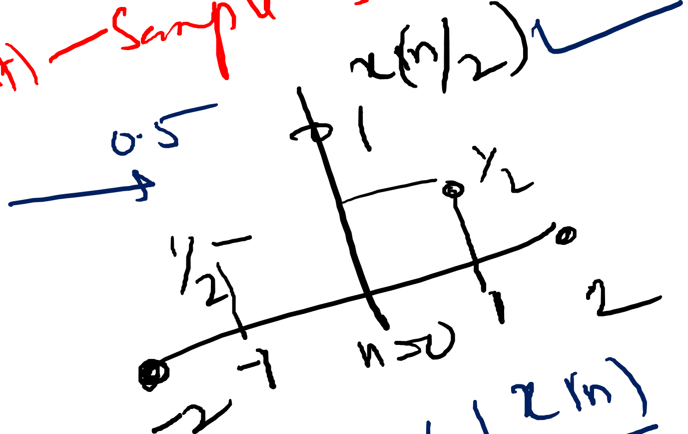
$x(2) = 1 - 2 \cdot \frac{1}{2} = 0$

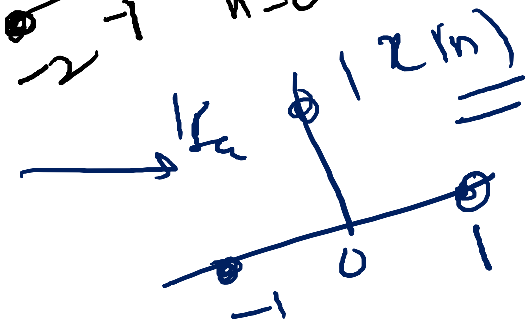
(iii) $T_s = 1\,sec$

$x(nT_s) = x(n)$

$1 - 0 \times 1 = 1 = x(0)$

$1 - 1 = 0 = x(1)$

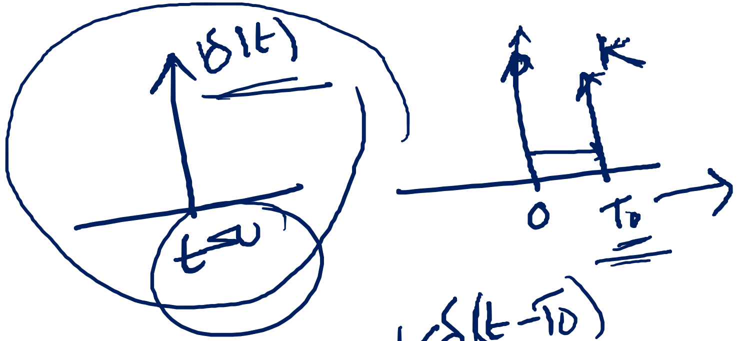




Impulse function: $\delta(t) = 0$ for $t \neq 0$

$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$ at $t=0$

Dirac Delta fn



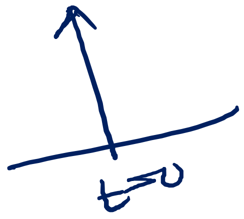
Shifted Impulse ← $K\delta(t-T_0)$

$K\delta(t-T_0)$

Properties

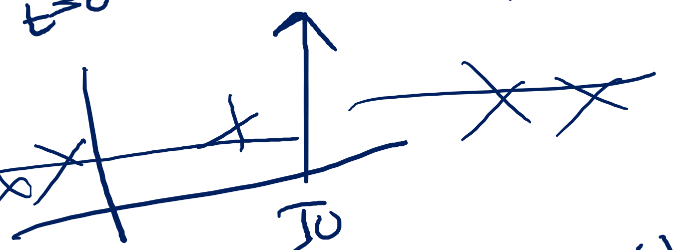
$\int_{-\infty}^{\infty} \delta(t)\,dt = 1 =$ Area

$\int_{-\infty}^{\infty} f(t)\,\delta(t)\,dt = f(0)\big|_{t=0}$ → an Impulse is occurring



$\int_{-\infty}^{\infty} f(t)\,\delta(t-T_0)\,dt = f(T_0)$

$f(t)\big|_{t=T_0}$



↓ Sifting Prop.

$\int_a^b f(t)\,\delta(t-t_0)\,dt$, $a < t_0 < b$

$= f(t)\big|_{t=t_0}$

$\int_a^b f(t)\,\delta(t-T_0)\,dt$, $a < T_0 < b$

$f(t)\,\delta(t)\,dt$

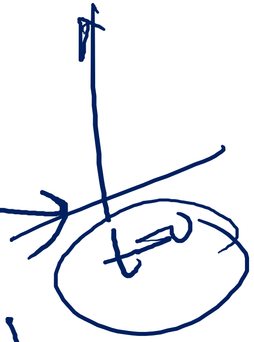
$$\int_{-1}^{1}(3t^2+1)\,\delta(t)\,dt = ?$$

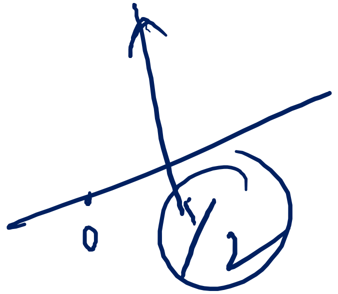
$f(t)\delta(t)dt$

$f(t)\big|_{t=0}$

$f(t)\big|$ at $t=0$

$\delta(t) \to$ at $t=0$





$$\int_{-1}^{1}(3t^2+1)\,\delta(t)\,dt = 1$$

$$(3t^2+1)\big|_{t=0} = 1$$

$\delta(t-1/2)\,dt$

$f(t)\big|_{t=1/2}$

$3(1/4)+1/2$

$$\int_{1}^{2}(3t^2+1)\,\delta(t)\,dt = 0$$

$t=0$

$\delta(t) = 0$

$\delta(t-1) =$ at $t=1$

$$\int_{2}^{\infty}(t^2+\cos\pi t)\,\delta(t-1)\,dt$$

$1+\cos\pi =$

$1-1=0$

$$\int_{-\infty}^{\infty} e^{-t}\,\delta(2t-2)\,dt$$

$$\int_{-\infty}^{\infty} e^{-t}\,\delta(2(t-1))\,dt$$

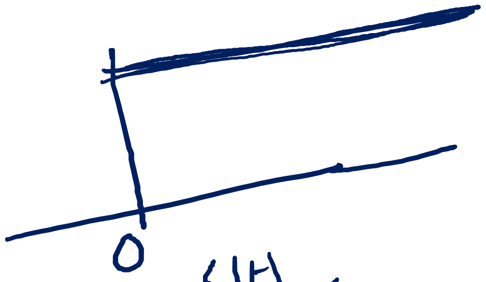
$$\int_{-\infty}^{\infty} e^{-t}\,\frac{1}{2}\,\delta(t-1)\,dt$$

$$\frac{1}{2}e^{-1} = \text{Ans}$$

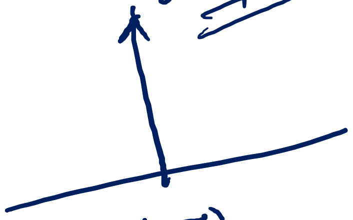
$\delta(at) =$ Scaling Prop $= \dfrac{1}{|a|}\delta(t)$

Relationship btn elementary fns

1. unit step fn
2. Impulse fn
3. Ramp fn

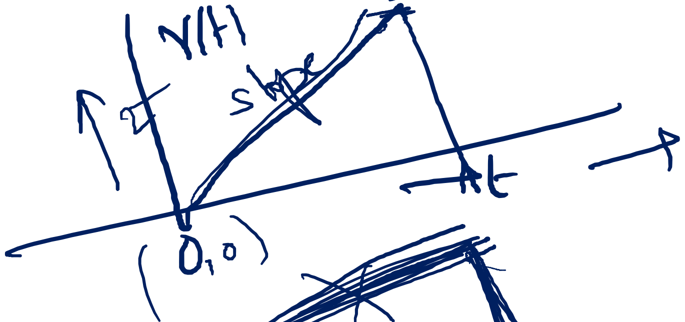


$u(t) = 1$ for $t \geq 0$
$= 0$ $\quad t < 0$



Testing of the system

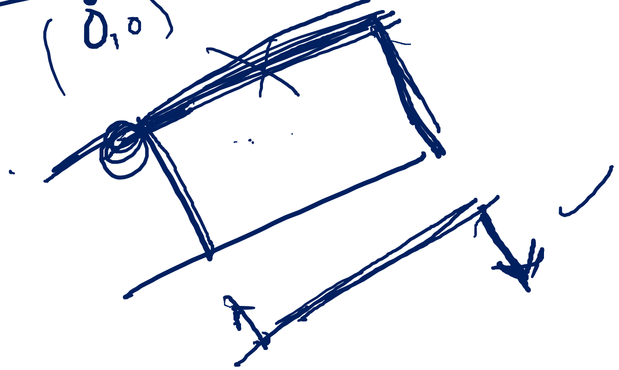
$r(t) = t$



$$\frac{dr(t)}{dt} = u(t)$$

$$\frac{du(t)}{dt} = \delta(t)$$

$$\int_{-\infty}^{t} \delta(\tau)\, d\tau = u(t)$$



Thank you