12/2/2019 UNIT-III Tuesday

Multivariate Calculus

Multiple Integrals

→ It is an extension of a definite integral to a fn of several variables (i.e., double integral - fn of 2 variables, triple integral - fn of 3 var.)

→ These are useful in evaluating area, mass, volume of plane and solid regions.

→ Double integral is a definite integral of the fn of 2 variables in xy plane.

→ Consider a region R in the xy plane bounded by one or more curves. Let $f(x,y)$ be a fn defined at all points of region R. Let the region 'R' be divided into small subdivisions each of area $\delta R_1, \delta R_2 -- \delta R_n$ which are pair wise non overlapping.

Let $f(x_i, y_i)$ is an arbitary fnal value in the sub region R_i. Then consider the sum,

$$= f(x_1,y_1)\delta R_1 + f(x_2,y_2)\delta R_2 + --- + f(x_n,y_n)\delta R_n$$

$$= \sum_{i=1}^{n} f(x_i, y_i)\delta R_i$$

If this sum → finite limit as $n \to \infty$ such that max $\delta R_i \to 0$ irrespective of the choice x_i, y_i, the limit is called double integral of $f(x,y)$ over the region R and is denoted by $\iint_R f(x,y)\,dR$ (or) $\iint_R f(x,y)\,dx\,dy$.

* <u>Evaluation of double integral</u>:

$$\lim_{\substack{n\to\infty \\ \delta R_i \to 0}} \left\{ \sum_{i=1}^{n} f(x_i, y_i)\, \delta R_i \right\} = \iint_R f(x,y)\,dx\,dy.$$

To evaluate double integral as a limit of a sum i.e., eq Ⓐ, always it will be difficult. So these are evaluated by 2 successive integrations.

<u>Case (i)</u>:

If R be region

$R: a \le x \le b,\ c \le y \le d$. (square/Rectangle)
(constant limits)

Order of integration is immaterial. provided the limits are to be changed accordingly.

Here, $\iint_R f(x,y)\,dR = \int_a^b \int_c^d f(x,y)\,dy\,dx$

$$= \int_c^d \int_a^b f(x,y)\,dx\,dy.$$

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Case-2:

Let R be the region bounded by $a \le x \le b$, $g_1(x) \le y < g_2(x)$ (in one with variable limits, other with constant limits).

$$\iint_a f(x,y)\,de = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx.$$

Case-3:

Let R be the region bounded by $h_1(y) \le x \le h_2(y)$ and $c \le y \le d$

$$\iint_R f(x,y)\,dR = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy.$$

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Q Evaluate the integrals.

1) $\int_2^3 \int_1^2 \frac{dx\,dy}{xy}$

$= \int_2^3 \int_1^2 x^{-1}y^{-1}\,dx\,dy$

$= \int_2^3 \frac{dy}{y} \cdot \int_1^2 \frac{dx}{x} = (\log y)_2^3 \cdot (\log x)_1^2$

$= \log 3/2 \cdot \log 2 = \log 2 \cdot (\log 3/2).$

2. $\int_0^2 \int_0^x y\,dy\,dx.$

Sol. $\cancel{\int_0^x dx \cdot \int^2 y\,dy}$

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$$\int_0^2 \left[\frac{y^2}{2}\right]_0^x dx = \int_0^2 \frac{x^2}{2} dx = \frac{1}{2}\left[\frac{x^3}{3}\right]_0^2$$

$$= \frac{1}{6}(8) = \frac{4}{3}.$$

3. $$\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)\, dx\, dy$$

$$\int_0^1 \int_x^{\sqrt{x}} x^2 dx\, dy + \int_0^1 \int_x^{\sqrt{x}} y^2 dx\, dy.$$

$$\int_0^1 \left(\frac{x^3}{3}\right)_x^{\sqrt{x}} dy + \int_0^1 [y^2 x]_x^{\sqrt{x}} dy$$

$$\frac{x\sqrt{x} - x^3}{3}(1-0) + (\sqrt{x}-x)\frac{1}{3}$$

$$\frac{x\sqrt{x} - x^3 + \sqrt{x} - x}{3} \Rightarrow \text{Wrong.}$$

$$\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)\, dy\, dx$$

$$= \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_x^{\sqrt{x}} dx.$$

$$= \cancel{\int_0^1 \left(x^2 + \frac{x\sqrt{x}}{3} - \frac{x^3}{3}\right) dx} \quad \int_0^1 \left\{x^2(\sqrt{x}-x) + \frac{x^{3/2} - x^3}{3}\right\} dx$$

$$= \cancel{\frac{x^3}{3} - \frac{x^4}{12} + \frac{1}{3}} \quad \int_0^1 \left(x^{5/2} - \frac{4}{3}x^3 + \frac{1}{3}x^{3/2}\right) dx$$

$$= \left\{\frac{x^{7/2}}{7/2} - \frac{4}{3}\frac{x^4}{4} + \frac{1}{3}\frac{x^{5/2}}{5/2}\right\}_0^1$$

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4. $\int_0^2 \int_0^x e^{x+y}\,dy\,dx$.

$= \int_0^2 \int_0^x e^x . e^y\,dy\,dx$

$= \int_0^2 e^x (e^y)_0^x\,dx = \int_0^2 e^x (e^x - 1)\,dx$.

$= \int_0^2 e^{2x} dx - \int_0^2 e^x dx = \left[\frac{e^{2x}}{2}\right]_0^2 - \left[e^x\right]_0^2$

$= \frac{e^4}{2} - \frac{1}{2} - \left[e^2 - 1\right] = \frac{e^4}{2} - e^2 - \frac{1}{2} + 1$

$= \frac{e^4}{2} - e^2 + \frac{1}{2}$.

5. $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$.

$\int_0^\infty \int_0^\infty e^{-x^2-y^2}\,dx\,dy = \int_0^\infty \int_0^\infty \frac{e^{-x^2}}{e^{y^2}}\,dx\,dy$.

we know that, $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

$\int_0^\infty \frac{\sqrt{\pi}}{2} e^{-y^2} dy = \frac{\sqrt{\pi}}{2} \times \frac{\sqrt{\pi}}{2} = \frac{\pi}{4}$.

6. $\int_0^4 \int_0^{x^2} e^{y/x}\,dy\,dx$.

$\int_0^4 \left[e^{y/x}\right]_0^{x^2} dx = \int_0^4 (e^x - e^0)\,dx$

$(= \int_0^4 (e^x - 1)\,dx = (e^x)_0^4 - (x)_0^4 = e^4 - 1 - x^4)$

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$$\int_0^4 x(e^x-1)\,dx = \int_0^4 \underset{u}{x}\,\underset{v}{e^x}\,dx - \int_0^4 x\,dx$$

$$= \left[xe^x - e^x\right]_0^4 - \left[\frac{x^2}{2}\right]_0^4$$

$$= 4e^4 - e^4 - [0-1] - 8$$

$$= 3e^4 + 1 - 8 = 3e^4 - 7.$$

7. Evaluate $\iint_R y\,dx\,dy$ where R is a region bounded by the parabolas $y^2 = 4x, x^2 = 4y$

Sol: $y^2 = 4x$

$x^2 = 4y \Rightarrow x = 2\sqrt{y}$.

$y^2 = 4(2\sqrt{y})$

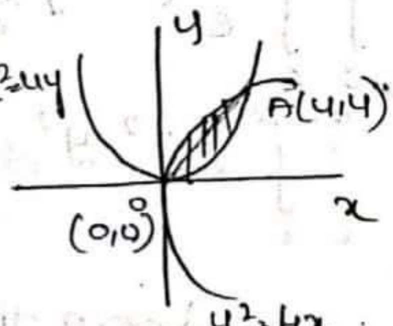
$y^2 = 8\sqrt{y} \Rightarrow y^2 - 8\sqrt{y} = 0$

$\Rightarrow y^4 = 64y$.

$\Rightarrow y^4 - 64y = 0$

$\Rightarrow y(y^3 - 64) = 0$

$\Rightarrow y = 0, y = 4, \cancel{4}$. $\Rightarrow$ Wrong



$x = 2\sqrt{0} \Rightarrow x = 0$

$x = 2\sqrt{4} \Rightarrow x = 4$

$$\int_0^4\int_0^4 y\,dx\,dy = \int_0^4 4y\,dy = \cancel{4}^{2}\left[\frac{y^2}{\cancel{2}}\right]_0^4$$

$= 2(16)$

$= 32$ sq.units.

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Sol: Let $I = \iint_R y\,dx\,dy$ —— (1)

where $R: y^2 = 4x$ & $x^2 = 4y$

Let $y^2 = 4x$

$\Rightarrow x = \frac{y^2}{4} \longrightarrow$ (a)

Let $x^2 = 4y$

$\frac{y^4}{4} = 4y \Rightarrow y = 4$

(a) $\Rightarrow x^2 = 16 \Rightarrow x = 4$

$A(4,4)$

Method-1: x: Constant limits : 0 to 4

y: variable limits : $\frac{x^2}{4}$ to $2\sqrt{x}$

$$I = \int_0^4 \int_{x^2/4}^{2\sqrt{x}} y\,dy\,dx = \int_0^4 \left(\frac{4x - \frac{x^4}{16}}{2}\right)dx$$

$$= \left\{ 2\cdot\left(\frac{x^2}{2}\right) - \frac{1}{32}\left(\frac{x^5}{5}\right)\right\}_0^4 \quad (2^2)^5$$

$$= 16 - \frac{32}{5} = \frac{48}{5}$$

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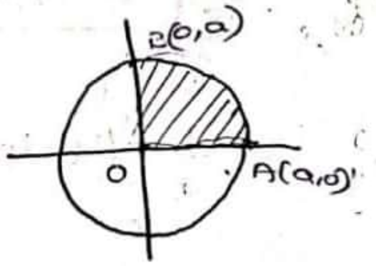
Friday

8. $\iint xy\,dx\,dy$ where over +ve quadrant of the circle $x^2+y^2 = a^2$



Sol: $$\int_0^a \int_0^{\sqrt{a^2-y^2}} xy\,dx\,dy$$

$$= \int_0^a \left[\frac{x^2}{2}\right]_0^{\sqrt{a^2-y^2}} y\,dy$$

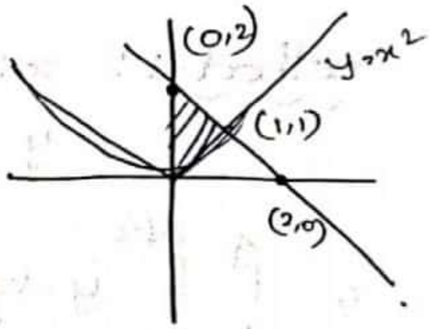
$$= \int_0^a \left(\frac{a^2-y^2}{2}\right) y\,dy = \frac{1}{2}\left[\int_0^a a^2 y\,dy - \int_0^a y^3\,dy\right]$$

$$= \frac{1}{2}\left[\frac{a^2y^2}{2} - \frac{y^4}{4}\right]_0^a$$

$$= \frac{1}{2}\left[\frac{a^4}{2} - \frac{a^4}{4}\right] = \frac{a^4}{8} \text{ sq. units.}$$

Q. Evaluate $\iint y\,dx\,dy$ where R is a region bounded by y-axis, $y=x^2$ & line $x+y=2$. in Q_1.

Sol: Let R be the region bounded by pts OAB.



$x : (c.L) \to 0$ to 1

$y : (v.L) \to x^2$ to $2-x$.

$$\int_0^1 \int_{x^2}^{2-x} y\,dx\,dy = \cancel{\int_0^1 (y x)\,dy}$$

$$\int_0^1 \int_{x^2}^{2-x} y\,dy\,dx = \int_0^1 \left[\frac{y^2}{2}\right]_{x^2}^{2-x} dx$$

$$= \frac{1}{2}\int_0^1 \frac{(2-x)^2 - x^4}{}\,dx \quad \left[= \frac{1}{2}\int_0^1 2^2 + \cancel{x^2} - 4x - \cancel{x^2}\,dx\right.$$

$$= \frac{1}{2}\int_0^1 (4-4x)\,dx = 2\left[\int dx - \int x\,dx\right]_0^1$$

$$= 2\left[x - \frac{x^2}{2}\right]_0^1$$

$$= 2\left[1 - \frac{1}{2}\right] = \cancel{\left.\right]_0^2} \cancel{y}$$

$$= \frac{1}{2}\int_0^1 (2^2 + x^2 - 4x - x^4)\,dx = \frac{16}{15}.$$

<u>Method-2:</u>

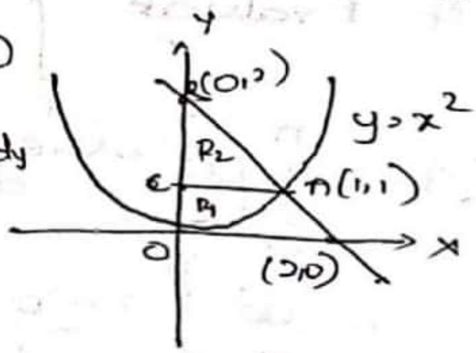
y : C.L ; x : V.L

upper limit of x is not constant.

upto to pt A up limit of x lies on curve and after pt A, up limit lies on straight line.

$\therefore$ we need to divide the region into 2 sub regions by taking || line to x-axis through the pt A.

$$\iint_R y\,dx\,dy = \iint_{R_1} y\,dx\,dy + \iint_{R_2} y\,dx\,dy \quad \text{---} (1)$$



R_1 : (OAC) y : C.L : 0 to 1

x : V.L : 0 to $\sqrt{y}$

R_2 = (CAB) y : C.L = 1 to 2

x : V.L = 0 to $2-y$.

$$\iint_R y\,dx\,dy = \int_0^1\int_0^{\sqrt{y}} y\,dx\,dy + \int_1^2\int_0^{2-y} y\,dx\,dy.$$

$$= \int_0^1 y[x]_0^{\sqrt{y}}\,dy + \int_1^2 [x]_0^{2-y}\,y\,dy$$

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$$= \int_0^1 y\sqrt{y}\, dy + \int_1^2 (2-y)y\, dy$$

$$= \int_0^1 y^{3/2} dy + \int_1^2 (2y - y^2) dy$$

$$= \left[\frac{y^{3/2}}{\left(\frac{3}{2}+1\right)}\right]_0^1 + \left[\frac{2y^2}{2} - \frac{y^3}{3}\right]_1^2$$

$$= \frac{2}{5} + \left[4 - \frac{8}{3} - \left(1 - \frac{1}{3}\right)\right]$$

$$= \frac{2}{5} + \left[4 - \frac{8}{3} - 1 + \frac{1}{3}\right]$$

$$= \frac{2}{5} + \left[3 + \frac{1}{3} - \frac{8}{3}\right] = \frac{16}{15}$$

Q. Evaluate $\int_0^{\pi}\int_0^{a\sin\theta} r\,dr\,d\theta$.

Sol: $\int_0^{\pi} \left(\frac{r^2}{2}\right)_0^{a\sin\theta} d\theta = \frac{1}{2}\int_0^{\pi} a^2\sin^2\theta\, d\theta.$

$$= \frac{a^2}{2}\int_0^{\pi} \frac{1-\cos 2\theta}{2}\, d\theta$$

$$= \frac{a^2}{4}\left[\int_0^{\pi} d\theta - \int_0^{\pi} \cos 2\theta\, d\theta\right]$$

$$= \frac{a^2}{4}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi}$$

$$= \frac{a^2}{4}\left[\pi - \frac{\sin 2\pi}{2}\right] = \frac{a^2}{4}[\pi] = \frac{\pi a^2}{4}$$

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Q. $\int_0^\infty \int_0^{\pi/2} e^{-r^2} r\,d\theta\,dr$

$$\int_0^\infty \int_0^{\pi/2} e^{-r^2} r\,d\theta\,dr = \int_0^\infty e^{-r^2}(r\,dr) \cdot \int_0^{\pi/2} d\theta.$$

$$= \frac{\pi}{2}\int_0^\infty e^{-r^2} r\,dr. = \frac{\pi}{4}\int_0^\infty e^{-r^2}(2r)dr$$

$$= \left[\frac{e^{-r^2}}{-1}\right]_0^\infty \cdot \frac{\pi}{4}.$$

$$= \frac{-\pi}{4}\left[e^{-\infty^2} - e^0\right] = \frac{\pi}{4}.$$

Q. Show that $\iint_R r^2\sin\theta\,dr\,d\theta = \frac{2a^3}{3}$ where

Sol: R is semicircle $r = 2a\cos\theta$.

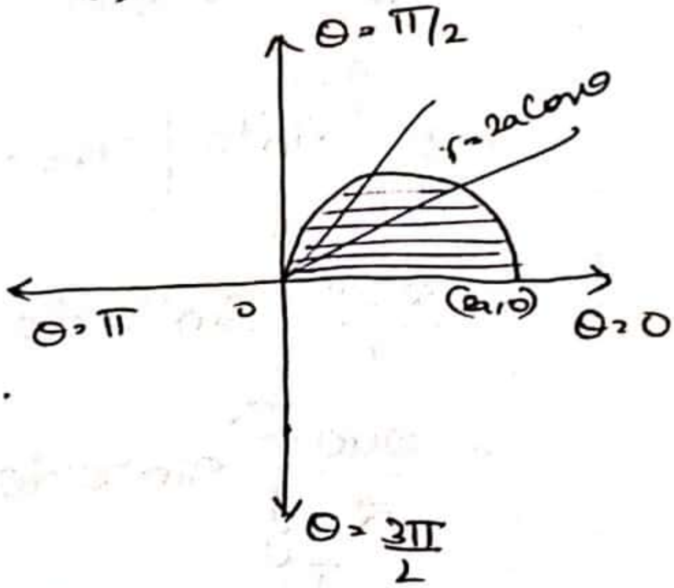
Let R: $r = 2a\cos\theta$ —— (1)

Let $\theta = 0$ in (1)

$\Rightarrow r = 2a$

$(2a, 0)$

Let $\theta = \frac{\pi}{2}$ in (1)



$r = 0$ to $2a\cos\theta$.

$$I = \int_0^{\pi/2}\int_0^{2a\cos\theta} r^2\sin\theta\,dr\,d\theta$$

$$= \int_0^{\pi/2} \frac{(8a^3\cos^3\theta - 0)}{3}\sin\theta\,d\theta$$

$$= -\frac{8a^3}{3}\int_0^{\pi/2} \cos^3\theta\,(-\sin\theta\,d\theta) = -\frac{8a^3}{3}\left[\frac{\cos^4\theta}{4}\right]_0^{\pi/2}$$

$$= \frac{2a^3}{3}.$$

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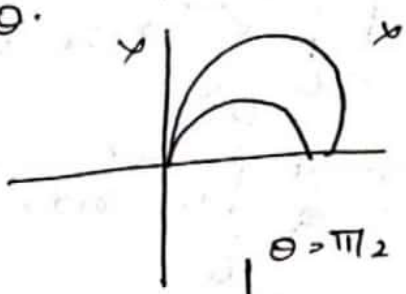
Q. Evaluate $\iint_R r^3 dr d\theta$, R is region bounded by R: $2\sin\theta$ & $r = 4\sin\theta$.

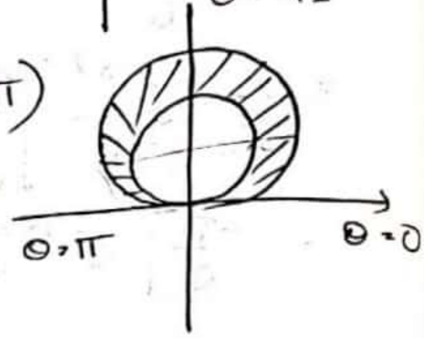
Sol. Let $r = 2\sin\theta$

$\Rightarrow (0,0), (2, \pi/2), (0, \pi)$



Let $r = 4\sin\theta$

$(r,\theta) = (0,0), (4, \pi/2), (0,\pi)$



θ: 0 to π.

r: $2\sin\theta$ to $4\sin\theta$

$$I = \int_0^{\pi}\int_{2\sin\theta}^{4\sin\theta} r^3 dr d\theta.$$

$$\int_0^{\pi}\left[\frac{r^4}{4}\right]_{2\sin\theta}^{4\sin\theta} d\theta = \frac{1}{4}\int_0^{\pi}(4^4\sin^4\theta - 2^4\sin^4\theta)\, d\theta$$

$$= \frac{1}{4}\int_0^{\pi}\sin^4\theta\,[256-16]\,d\theta.$$

$$\begin{array}{r} 256 \\ 16 \\ \hline 240 \end{array}$$

$$= \frac{240}{4}\int_0^{\pi}\sin^4\theta\, d\theta.$$

$$= \frac{240}{2}\int_0^{\pi/2}\sin^4\theta\, d\theta = 120\int_0^{\pi/2}\sin^4\theta\, d\theta$$

$$= 120\int_0^{\pi/2}\sin^4\theta\cos^0\theta\, d\theta.$$

$\cancel{m+1=4}$, $\cancel{n+1=0}$

$\cancel{m=5, n=1}$

$= \cancel{\frac{120}{2}\beta(5,1)}$

$2m-1=4 \quad ; \quad 2n-1=0$

$m=\frac{5}{2}, n=\frac{1}{2}$.

$= 60\,\beta\left(\frac{5}{2},\frac{1}{2}\right)$.

$= 60 \times \frac{\Gamma\frac{5}{2}\,\Gamma\frac{1}{2}}{\Gamma 3} = \frac{60}{2} \times \Gamma\frac{5}{2}\,\frac{\sqrt{\pi}}{2}$

$= 30\,\Gamma\frac{5}{2}\,\frac{\sqrt{\pi}}{2} = 30\,\Gamma\left(\frac{3}{2}+1\right)\frac{\sqrt{\pi}}{2} = \frac{45\pi}{2}$.

* Change of order of integration:

In some of the above problems we consider $\iint_R f(x,y)\,dx\,dy$ and in some problems we consider $\iint_R f(x,y)\,dy\,dx$. Whenever these 2 integrals exist, they are equal.

Instead of these 2 notations, consider 1 particular form then in some probs. by taking particular form. The evaluation becomes simple. But in some probs. unless we take one of the 2-forms only, we can evaluate the given integral.

Q. Evaluate $\int_0^\infty\int_x^\infty \frac{e^{-y}}{y}\,dy\,dx$.

Let $\int_0^\infty\int_x^\infty \frac{e^{-y}}{y}\,dy\,dx = I$ ——— (1)

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The given integral in eq ① can't be evaluated in given order. Therefore we need to change the order of integration.

Let R: $x : 0$ to ∞

$y : x$ to ∞

y : C.L: 0 to ∞

x : V.L : 0 to y

$$I = \int_0^\infty \int_0^y \frac{e^{-y}}{y}\, dx\, dy$$

$$= \int_0^\infty \frac{e^{-y}}{y} (x)_0^y\, dy = \int_0^\infty e^{-y} dy = 1$$

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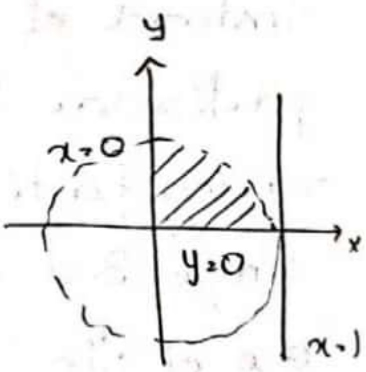
2. Evaluate by using change of order of integration.

$$I = \int_0^1 \int_0^{\sqrt{1-x^2}} y^2\, dy\, dx$$



Let R: $x : 0$ to 1

$y : 0$ to $\sqrt{1-x^2}$

Let $y = \sqrt{1-x^2}$

$\Rightarrow y^2 = 1 - x^2$

$\Rightarrow x^2 + y^2 = 1$

which is circle.

y : (C.L) : 0 to 1

x : (V.L) : 0 to $\sqrt{1-y^2}$

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$$\int_0^1\int_0^{\sqrt{1-y^2}} y^2\,dx\,dy = \int_0^1 y^2[x]_0^{\sqrt{1-y^2}}\,dy.$$

$$= \int_0^1 y^2\left[\sqrt{1-y^2}\right]dy.$$

$$= \int_0^1 y^2(1-y^2)^{1/2}\,dy.$$

$y^2 = t$

$2y\,dy = dt$

$2\sqrt{t}\,dy = dt$

$$= \int_0^1 t(1-t)^{1/2}\frac{dt}{2\sqrt{t}}.$$

$$= \frac{1}{2}\int_0^1 t^{1/2}(1-t)^{1/2}\,dt.$$

$$= \frac{1}{2}\beta\left(\frac{3}{2},\frac{3}{2}\right)$$

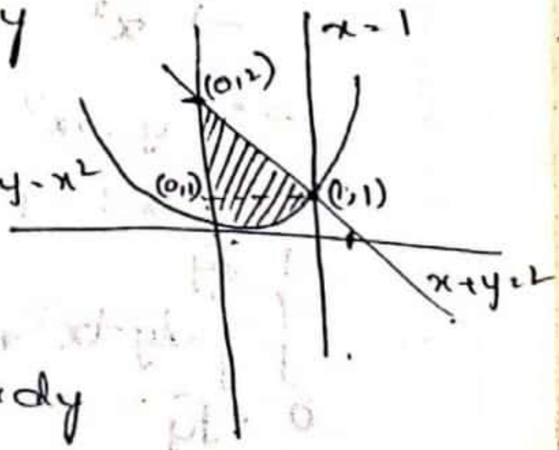
$$= \frac{1}{2}\frac{\Gamma\frac{3}{2}\,\Gamma\frac{3}{2}}{\Gamma 3} = \cancel{\frac{1}{2}\frac{\sqrt{\pi}}{2}\times\frac{\sqrt{\pi}}{2} = \frac{\pi}{8}}.$$

$$= \frac{\pi}{16}.$$

3. change the order of Integration $\int_0^1\int_{x^2}^{2-x} xy\,dx\,dy$ and hence evaluate double integral.

Sol: Given $I = \int_0^1\int_{x^2}^{2-x} xy\,dx\,dy$

$x = \quad y =$



$R:\ x:\ 0$ to 1

$y:\ x^2$ to $2-x$

$$\int_0^1\int_0^{\sqrt{y}} xy\,dx\,dy + \int_1^2\int_0^{2-y} xy\,dx\,dy$$

$$\int_0^1 y\left[\frac{x^2}{2}\right]_0^{\sqrt{y}} dy + \int_1^2 y\left[\frac{x^2}{2}\right]_0^{2-y} dy.$$

$$\int_0^1 y\left[\frac{y}{2}\right] dy + \int_1^2 y\left[\frac{(2-y)^2}{2}\right] dy$$

$$\frac{1}{2}\int_0^1 y^2\, dy + \frac{1}{2}\int_1^2 y(4+y^2-4y)\, dy$$

$$\frac{1}{2}\left[\frac{y^3}{3}\right]_0^1 + \frac{1}{2}\left[\frac{4y^2}{2}+\frac{y^4}{4}-\frac{4y^3}{3}\right]_1^2$$

$$= \frac{1}{2}\left[\frac{1}{3}\right] + \frac{1}{2}\left[8+\frac{\cancel{8}}{\cancel{3}}^{4}-\frac{32}{3}-\left[\frac{4}{2}+\frac{1}{4}-\frac{4}{3}\right]\right]$$

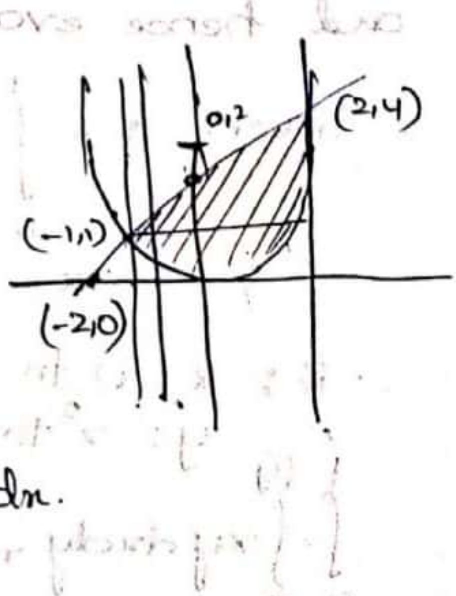
$$= \frac{1}{6} + \frac{1}{2}\left[8+\cancel{8}^{4}-\frac{32}{3}-2-\frac{1}{4}+\frac{4}{3}\right]$$

$$= \cancel{\frac{1}{6}}+4+\frac{8}{6}-\frac{32}{6}-1\cancel{-\frac{1}{6}}+\frac{4}{6}$$

$$= 3+\frac{4}{3}-\frac{16}{3}+\frac{2}{3}.$$

$$= \left[\frac{9+4-16+2}{3}\right] = \frac{3}{8}.$$

4. $I = \int_{-1}^{2}\int_{x^2}^{x+2} dy\,dx$



R: $y : x^2$ to $x+2$

$x : -1$ to 2

$$\int_0^1\int_{-\sqrt{y}}^{\sqrt{y}} dy\,dx + \int_1^4\int_{y-2}^{\sqrt{y}} dy\,dx.$$

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$$\int_0^1 [x]_{-\sqrt{y}}^{\sqrt{y}}\, dy + \int_1^4 [x]_{y-2}^{\sqrt{y}}\, dy$$

$$\int_0^1 2\sqrt{y}\, dy + \int_1^4 (\sqrt{y}-y+2)\, dy.$$

$$2\left[\frac{y^{3/2}}{3/2}\right]_0^1 + \left[\frac{y^{3/2}}{3/2} - \frac{y^2}{2} + 2y\right]_1^4$$

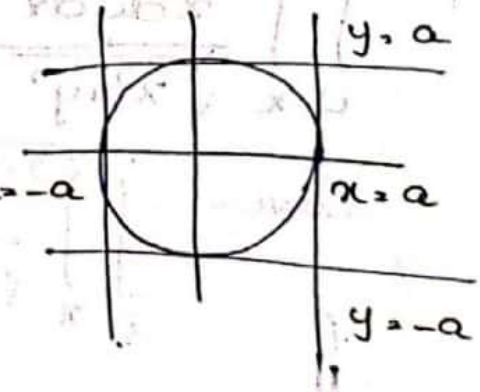
$$\frac{4}{3} + \left[\frac{2}{3}4^{3/2} - \cancel{8} + \cancel{8} - \left(\frac{1}{3/2} - \frac{1}{2} + 2\right)\right]$$

$$\frac{4}{3} + \left[\frac{2}{3}(8) - \frac{2}{3} + \frac{1}{2} - 2\right]$$

$$\frac{4}{3} + \frac{16}{3} - \frac{2}{3} + \frac{1}{2} - 2 = \cancel{\frac{4+16-2}{6}}$$

$$= \frac{8+32-4+3-12}{6} = \frac{27}{6} = \frac{9}{2}.$$

5. $$\int_{-a}^{a}\int_{0}^{\sqrt{a^2-y^2}} dx\, dy.$$



R : x : 0 to $\sqrt{a^2-y^2}$

y : $-a$ to a

R : y : 0 to $\sqrt{a^2-x^2}$

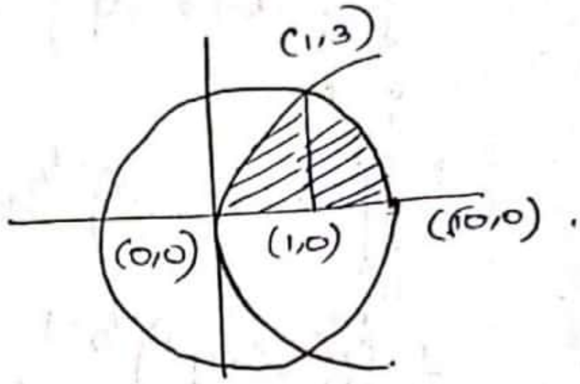
x : $-a$ to a

$$\int_{-a}^{a}\int_{0}^{\sqrt{a^2-x^2}} dy\, dx = \int_{-a}^{a} \sqrt{a^2-x^2}\, dx.$$

$$= \left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} \right]_{-a}^{a}$$

$$= \frac{\pi a^2}{2}.$$

6. $\displaystyle\int_0^3 \int_{y^2/9}^{\sqrt{10-y^2}} dy\,dx.$



$$\int_0^1 \int_0^{3\sqrt{x}} dy\,dx + \int_1^{\sqrt{10}} \int_0^{\sqrt{10-x^2}} dy\,dx$$

$$2 + \int_1^{\sqrt{10}} \left[y\right]_0^{\sqrt{10-x^2}} dx$$

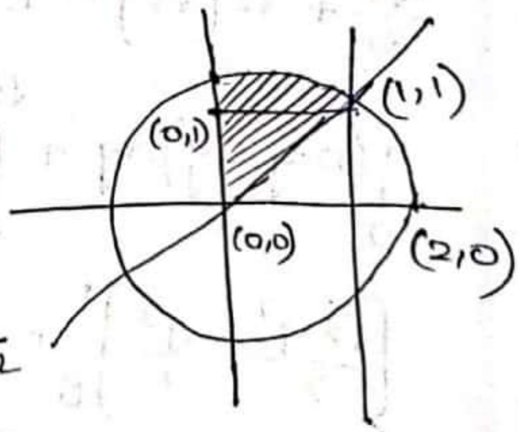
$$2 + \int_1^{\sqrt{10}} \sqrt{10-x^2}\, dx$$

$$2 + \left[\frac{x}{2}\sqrt{10-x^2} + \frac{10}{2}\sin^{-1}\left(\frac{x}{\sqrt{10}}\right)\right]_1^{\sqrt{10}}$$

$$2 + \left[\frac{\sqrt{10}}{2}(0) + \frac{5\pi}{2} - \left[\frac{1}{2}(3) + 5\sin^{-1}\left(\frac{1}{\sqrt{10}}\right)\right]\right]$$

$$2 + \left[\frac{5\pi}{2} - \frac{3}{2} - 5\sin^{-1}\left(\frac{1}{\sqrt{10}}\right)\right]$$

4. $\displaystyle\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}}\, dy\,dx.$



$$\int_0^1 \int_0^y \frac{x}{\sqrt{x^2+y^2}}\, dy\,dx + \int_1^{\sqrt{2}} \int_0^{\sqrt{2-y^2}} \frac{x}{\sqrt{x^2+y^2}}\, dy\,dx.$$

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$$\int_0^1\int_0^y x\left(x^2+y^2\right)^{-1/2}dy\,dx+\int_1^{\sqrt{2}}\int_0^{\sqrt{2-y^2}} x\left(\sqrt{x^2+y^2}\right)^{-1/2}dy\,dx$$

$$\frac{1}{2}\int_0^1\int_0^y 2x\left(x^2+y^2\right)^{-1/2}dx\,dy+\frac{1}{2}\int_1^{\sqrt{2}}\int_0^{\sqrt{2-y^2}} 2x\left(x^2+y^2\right)^{-1/2}dx\,dy$$

$$\frac{1}{2}\int_0^1\int_0^y \frac{\left(x^2+y^2\right)^{-1/2+1}}{-\frac{1}{2}+1}dy+\frac{1}{2}\int_0^{\sqrt{2}}\int_0^{\sqrt{2-y^2}}\frac{\left(x^2+y^2\right)^{-1/2+1}}{-\frac{1}{2}+1}dy.$$

$$\int_0^1\left[\left(x^2+y^2\right)^{1/2}\right]_0^y dy+\frac{1}{2}\times\frac{2}{1}\int_0^{\sqrt{2}}\left[\left(x^2+y^2\right)^{1/2}\right]_0^{\sqrt{2-y^2}}dy.$$

$$\left(\int_0^1\left(2y^2\right)^{1/2}dy+\int_0^{\sqrt{2}}\cancel{2^{1/2}dy}\,(2^{1/2}-y)\,dy.\right.$$

$$\left[\cancel{\sqrt{2}\left[\frac{y^2}{2}\right]_0^1}\right.+\sqrt{2}\left[y\right]_0^{\sqrt{2}}-\left[\frac{y^2}{2}\right]_0^{\sqrt{2}}$$

$$\cancel{\frac{\sqrt{2}}{2}}+\sqrt{2}\left[\sqrt{2}\right]-1-\sqrt{2}+1\Big]\times)\times$$

$$\int_0^1(\sqrt{2}y-y)dy+\int_0^{\sqrt{2}}(\sqrt{2}-y)dy.$$

$$\left[\frac{\sqrt{2}y^2}{2}-\frac{y^2}{2}\right]_0^1+\left[\sqrt{2}y-\frac{y^2}{2}\right]_0^{\sqrt{2}}$$

$$=1-\frac{1}{\sqrt{2}}.$$

<u>H.W:</u> $\int_0^a\int_{x^2/a}^{2a-x} xy^2\,dy\,dx$

19/2/2019 Tuesday

* Triple integrals:

→ It is extension of definite integral for fn of 3 variables. This integral can be simplified by 3 successive integrations.

Q. Evaluate these integrals!

(i) $\int_0^1 \int_0^2 \int_1^2 x^2 y z \, dz \, dy \, dx$

$= \int_0^1 \int_0^2 \int_1^2 x^2 y z \, dz \, dy \, dx.$

$= \int_0^1 \int_0^2 x^2 y \left[\frac{z^2}{2}\right]_1^2 dy \, dx.$

$= \frac{3}{2} \int_0^1 \int_0^2 x^2 y \, dy \, dx = \frac{3}{2} \int_0^1 x^2 \left[\frac{y^2}{2}\right]_0^2 dx.$

$= \frac{3}{2} \times 2 \times \frac{1}{3} = 1.$

(ii) $\int_0^1 \int_1^2 \int_2^3 xyz \, dx \, dy \, dz$

$\left[\frac{x^2}{2}\right]_2^3 \cdot \left[\frac{y^2}{2}\right]_1^2 \cdot \left[\frac{z^2}{2}\right]_0^1$

$\left[\frac{9}{2} - 2\right] \left[2 - \frac{1}{2}\right] \left[\frac{1}{2}\right] = \left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right) = \frac{15}{8}.$

(iii) $\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z} \, dx \, dy \, dz$

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$$\int_0^a\int_0^x\int_0^{x+y} e^x \cdot e^y \cdot e^z \, dz \cdot dy \cdot dx.$$

$$\int_0^a\int_0^x e^x \cdot e^y \left(e^z\right)_0^{x+y} dy \, dx.$$

$$\int_0^a\int_0^x e^x \cdot e^y \left(e^{x+y}-1\right) dy \, dx.$$

$$\int_0^a\int_0^x e^{2(x+y)} - e^{(x+y)} \, dy \, dx.$$

$$\int_0^a\int_0^x e^{2(x+y)} dy\,dx - \int_0^a\int_0^x e^x \cdot e^y \, dy\,dx.$$

$$\int_0^a\int_0^x e^{2x} \cdot e^{2y} dy\,dx - \int_0^a\int_0^x e^x \cdot e^y \, dy\,dx.$$

$$\int_0^a e^{2x}\left[\frac{e^{2y}}{2}\right]_0^x dx - \int_0^a e^x \left[e^y\right]_0^x dx$$

$$\frac{1}{2}\int_0^a e^{2x}\left[e^{2x}-1\right]dx - \int_0^a e^x\left[e^x-1\right]dx.$$

$$\frac{1}{2}\int_0^a \left(e^{4x}-e^{2x}\right)dx - \int_0^a \left(e^{2x}-e^x\right)dx.$$

$$\frac{1}{2}\left[\frac{e^{4x}}{4}-\frac{e^{2x}}{2}\right]_0^a - \left[\frac{e^{2x}}{2}-e^x\right]_0^a.$$

$$\frac{1}{2}\left[\frac{e^{4a}}{4}-\frac{e^{2a}}{2}-\frac{1}{4}+\frac{1}{2}\right] - \left[\frac{e^{2a}}{2}-e^a-\frac{1}{2}+1\right]$$

$$\frac{e^{4a}}{8}-\frac{e^{2a}}{4}-\frac{1}{8}+\frac{1}{4}-\frac{e^{2a}}{2}+e^a+\frac{1}{2}-1.$$

$$\frac{e^{4a}-6e^{2a}+8e^{a}}{8}-\frac{3}{8}=\frac{e^{4a}}{8}-\frac{3}{4}e^{2a}+e^{a}-\frac{3}{8}.$$

4. $$\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}(x+y+z)\,dx\,dy\,dz.$$

$$\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}x\,dx\,dy\,dz+\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}y\,dx\,dy\,dz+\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}z\,dx\,dy\,dz$$

$$\int_{-1}^{1}\int_{0}^{z}x\left[y\right]_{x-z}^{x+z}dx\,dz+\int_{-1}^{1}\int_{0}^{z}\left[\frac{y^2}{2}\right]_{x-z}^{x+z}dx\,dz+\int_{-1}^{1}\int_{0}^{z}z\left[y\right]_{x-z}^{x+z}dx\,dz.$$

$$=\int_{-1}^{1}\int_{0}^{z}2zx\,dx\,dz+\frac{1}{2}\int_{-1}^{1}\int_{0}^{z}4xz\,dx\,dz+\int_{-1}^{1}\int_{0}^{z}2z^2\,dx\,dz.$$

$$=\int_{-1}^{1}z\left(x^2\right)_{0}^{z}dz+\frac{4}{2}\int_{-1}^{1}\left[\frac{x^2}{2}\right]_{0}^{z}z\,dz+2\int_{-1}^{1}\cancel{\left[\frac{x^3}{3}\right]}\left[x\right]_{0}^{z}z^2\,dz$$

$$=\int_{-1}^{1}z^3dz+2\int_{-1}^{1}\frac{z^3}{2}dz+2\int_{-1}^{1}z^3dz.$$

$$=\left[\frac{z^4}{4}\right]_{-1}^{1}+2\left[\frac{z^4}{4}\right]_{-1}^{1}+2\left[\frac{z^4}{4}\right]_{-1}^{1}$$

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$$= \left[\frac{1}{4} + \frac{1}{4}\right] + \left[\frac{1}{4} - \frac{1}{4}\right] + 2\left[\frac{1}{4} - \frac{1}{4}\right] = 0$$

5. $$\int_0^{2\pi}\int_0^{b}\int_{-h}^{h} (z^2 + r^2\sin^2\theta)\, dz\, dr\, d\theta.$$

$$\int_0^{2\pi}\int_0^{b}\int_{-h}^{h} z^2\, dz\, dr\, d\theta + \int_0^{2\pi}\int_0^{b}\int_{-h}^{h} r^2\sin^2\theta\, dz\, dr\, d\theta.$$

$$\int_0^{2\pi}\int_0^{b}\left[\frac{z^3}{3}\right]_{-h}^{h} dr\, d\theta + \int_0^{2\pi}\int_0^{b}\int_{-h}^{h} r^2\sin^2\theta\, [z]_{-h}^{h}\, dr\, d\theta.$$

$$\frac{1}{3}\int_0^{2\pi}\int_0^{b} 2h^3\, dr\, d\theta + 2\int_0^{2\pi}\int_0^{b} h\, r^2\sin^2\theta\, dr\, d\theta.$$

$$\frac{2h^3}{3}\int_0^{2\pi} b\, d\theta + \cancel{2}h\int_0^{2\pi}\left[\frac{r^3}{\cancel{3}}\right]_0^{b} \sin^2\theta\, d\theta.$$

$$\frac{2h^3 b}{3}(2\pi) + \frac{2}{3}hb^3\int_0^{2\pi}\sin^2\theta\, d\theta.$$

$$\frac{4bh^3\pi}{3} + \frac{2}{3}hb^3\int_0^{2\pi}\left(\frac{1-\cos 2\theta}{2}\right)d\theta$$

$$\frac{4bh^3\pi}{3} + \frac{\cancel{2}}{3}hb^3\left[\frac{\theta}{\cancel{2}} - \sin 2\theta\right]_0^{2\pi}$$

$$\frac{4bh^3\pi}{3} + \frac{2}{3}hb^3[2\pi] = \frac{4bh^3\pi}{3} + \frac{2hb^3\pi}{3}$$

$$= \cancel{4hb\pi\left[\frac{h^2}{3} + 1\right]}.$$

$$= \frac{2}{3}hb\pi\left[2h^2 + b^2\right]$$

6 Evaluate $\iiint z^2 \, dx\,dy\,dz$ taken over the volume bounded by the surfaces

Sol: V : $x^2+y^2=a^2$

$x^2+y^2=z$ & $z=0$

z : 0 to x^2+y^2

y : $-\sqrt{a^2-x^2}$ to $\sqrt{a^2-x^2}$

x : $-a$ to a.

$$\iiint z^2 \, dx\,dy\,dz = \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_{0}^{x^2+y^2} z^2 \, dz\,dy\,dx$$

$$= \frac{1}{3}\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \left[\frac{z^3}{3}\right]_0^{x^2+y^2} dy\,dx.$$

$$= \frac{1}{3}\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} (x^2+y^2)^3 \, dy\,dx$$

$$= \frac{1}{3}\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} x^6+y^6+3x^2y^2(x^2+y^2)\, dy\,dx.$$

$$= \frac{1}{3}\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} (x^6+y^6+3x^4y^2+3x^2y^4)\, dy\,dx$$

$$= \frac{1}{3}\int_{-a}^{a} x^6 \left[y\right]_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dx + \frac{1}{3}\int_{-a}^{a}\left[\frac{y^7}{7}\right]_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dx +$$

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$$= \frac{1}{3}\int_{-a}^{a} \left[x^6(y) + \frac{y^7}{7} + 3x^4\left(\frac{y^3}{3}\right) + 3x^2\left(\frac{y^5}{5}\right)\right]_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}$$

$$= \frac{2}{3}\int_{-a}^{a} x^6 y + \frac{y^7}{7} + x^4 y^3 + \frac{3x^2}{5} y^5 \Big]_{0}^{\sqrt{a^2-x^2}} dx$$

$$= \frac{2}{3}\int_{-a}^{a} x^6\sqrt{a^2-x^2} + \frac{(a^2-x^2)^{7/2}}{7} + x^4(a^2-x^2)^{3/2}$$

$$+ \frac{3}{5}x^2(a^2-x^2)^{5/2}\, dx.$$

$$= \frac{4}{3}\int_{0}^{a} x^6\sqrt{a^2-x^2} + \frac{(a^2-x^2)^{7/2}}{7} + x^4(a^2-x^2)^{3/2} + \frac{3}{5}x^2(a^2-x^2)^{5/2}\, dx.$$

Let $x = a\sin\theta \rightarrow a^2-x^2 = a^2 - a^2\sin^2\theta$

$\Rightarrow \sqrt{a^2-x^2} = a\cos\theta$

$dx = a\cos\theta\, d\theta$

when $x = 0 \Rightarrow a\sin\theta = 0 \Rightarrow \theta = 0$

$x = a \rightarrow a\sin\theta = a \rightarrow \theta = \pi/2.$

$$I = \frac{4}{3}\int_{0}^{\pi/2} (a^6\sin^6\theta)(a\cos\theta) + (a^4\sin^4\theta)(a^3\cos^3\theta)$$

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I. ~~find the region of integration~~

$$\int_0^b \int_0^\infty \frac{\sqrt{b^2-y}}{b} \qquad \int_0^b \int_a^{b/a}$$

* change of variables (in double integrals):

Sometimes, the double integral is easy to evaluate if we change the variable suitably

Let $\phi(u,v) = x$; $y = \psi(u,v)$

be the relation b/w old variable x, y and new variable $u, v \Rightarrow J = \frac{\partial(x,y)}{\partial(u,v)}$

$$\iint_R f(x,y)\,dx\,dy = \iint_{R'} f(\phi,\psi)\,|J|\,du\,dv$$

where $J = \frac{\partial(x,y)}{\partial(u,v)}$

R and R' are the region in the xy-plane, uv-plane.

* change of variable from catesian to polar co-ordinates:

Let the cartesian coordinates and polar coordinates

$$x = r\cos\theta,\ y = r\sin\theta$$

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = r$$

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Q. ~~Change the~~

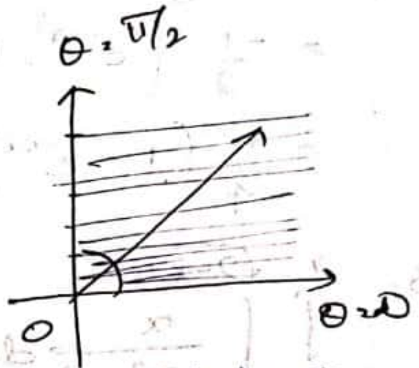
Then

$$\iint_R f(x,y)\,dx\,dy = \iint_{R'} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta.$$

R - region in xy plane

R' - polar plane

1 $\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\,dx\,dy = I.$



$x = r\cos\theta$, $y = r\sin\theta$

$J = \dfrac{\partial(x,y)}{\partial(r,\theta)} = r$

r : 0 to ∞

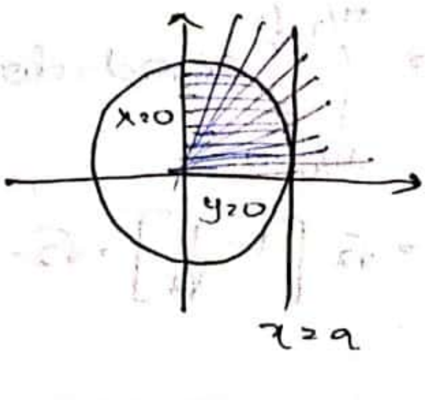
θ : 0 to $\pi/2$

$= \frac{1}{2}\int_0^{\infty} e^{-r^2}(2r\,dr)\cdot\int_0^{\pi/2} d\theta.$

$= -\frac{1}{2}\,[0-1]\left[\pi/2 - 0\right] = \pi/4$

2. Evaluate $\int_0^{a}\int_0^{\sqrt{a^2-x^2}} e^{-(x^2+y^2)}\,dx\,dy$ by changing cartesian to polar coordinates.

Sol): R be the region



x : 0 to a

y : 0 to $\sqrt{a^2-x^2}$

$x = r\cos\theta$

$a = r\cos\theta$

$\theta = \pi/2$, $\theta = \cos^{-1}(a/r)$

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$y = r\sin\theta$ $\qquad\qquad$ $\theta : 0 \text{ to } \pi/2$

$\theta = 0,\ \sin\theta = \frac{\sqrt{a^2-x^2}}{r}$ $\qquad\qquad$ $r : 0 \text{ to } a$

$$= \frac{-1}{2}\int_0^{\pi/2}\int_0^{0} e^{-r^2}(2r\,dr)\,d\theta.$$

$$= \frac{-1}{2}\int_0^{\pi/2} \left(e^{-r^2}\right)_0^a d\theta.$$

$$= \frac{-1}{2}\left[e^{-a^2}-1\right]\pi/2$$

$$= \frac{\pi}{4}\left(1-e^{-a^2}\right]$$

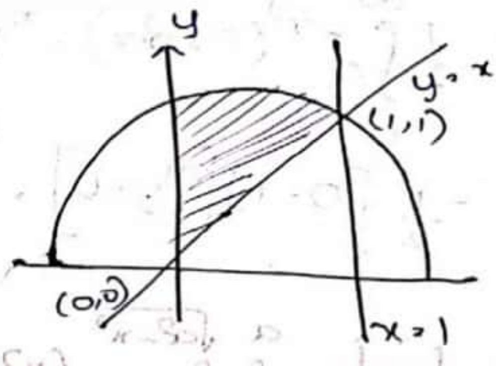
3. $$\int_0^1\int_x^{\sqrt{2-x^2}} \frac{x}{x^2+y^2}\,dy\,dx$$

$x : 0 \text{ to } 1$

$y : x \text{ to } \sqrt{2-x^2}$

$\theta \text{ to } \pi/4 \text{ to } \pi/2$

$r : 0 \text{ to } \sqrt{2}$



$$\int_{\pi/4}^{\pi/2}\int_0^{\sqrt{2}} \frac{r\cos\theta}{r^2}\,r\,dr\,d\theta$$

$$= \int_{\pi/4}^{\pi/2}\int_0^{\sqrt{2}} \cos\theta\,dr\,d\theta = \sqrt{2}\left[\sin\theta\right]_{\pi/4}^{\pi/2}$$

$$= \sqrt{2}\left[1-\frac{1}{\sqrt{2}}\right] = \sqrt{2}-1$$

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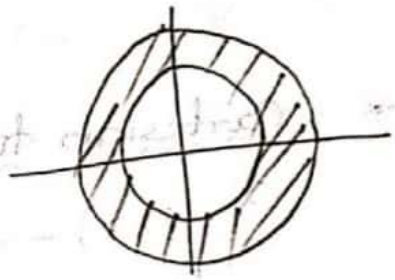
1. Solve $\int_0^{4a}\int_{y^2/4a}^{y} \frac{x^2-y^2}{x^2+y^2}\,dx\,dy$

2. Evaluate $\iint \frac{x^2y^2}{x^2+y^2}\,dx\,dy$ over the region b/w the circles $x^2+y^2=a^2$ & $x^2+y^2=b^2$ $(b>a)$

Sol: $R: x^2+y^2=a^2$ & $x^2+y^2=b^2$



θ : 0 to 2π

r : a to b

$$\int_0^{2\pi}\int_a^b \frac{r^4\cos^2\theta\sin^2\theta}{r^2}\, r\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[\frac{r^4}{4}\right]_a^b \cos^2\theta\sin^2\theta\,d\theta$$

$$= \frac{b^4-a^4}{4}\times\frac{1}{4}\int_0^{2\pi} 4\cos^2\theta\sin^2\theta\,d\theta = \frac{b^4-a^4}{16}\int_0^{2\pi}(\sin^2 2\theta)\,d\theta$$

$$= \frac{b^4-a^4}{16}\int_0^{2\pi}\frac{1-\cos 4\theta}{2}\,d\theta$$

$$= \frac{b^4-a^4}{16}\left[\left[\frac{\theta}{2}\right]_0^{2\pi} - \frac{1}{2}\left(\frac{\sin 4\theta}{4}\right)_0^{2\pi}\right]$$

$$= \frac{\pi}{16}\left[b^4-a^4\right]$$

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* **change of variables in triple integrals:**

$$(x, y, z) \longrightarrow (u, v, w)$$

$x = f_1(u, v, w)$; $y = f_2(u, v, w)$; $z = f_3(u, v, w)$

$$f\left(\frac{x, y, z}{u, v, w}\right) = J$$

$$\iiint_V F(x, y, z)\, dx\,dy\,dz = \iiint_{V_1} F(f_1, f_2, f_3)\, |J|\, du\,dv\,dz$$

* **Cartesian to spherical coordinates:**

$$(x, y, z) \longrightarrow (r, \theta, \phi) \text{ (or) } (e, \theta, \phi)$$

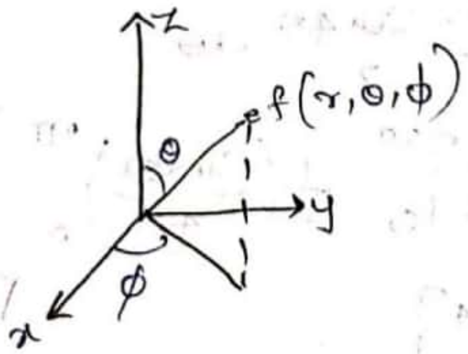
$x = r\sin\theta\cos\phi$

$y = r\sin\theta \cdot \sin\phi$ $\qquad J\left(\frac{x, y, z}{r, \theta, \phi}\right) = r^2\sin\theta.$

$z = r\cos\theta$

$$\iiint_V F(x, y, z)\, dx\,dy\,dz$$

$$= \iiint_{V_1} F(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)\, r^2\sin\theta\, dr\,d\theta\,d\phi$$



* **Cartesian to cylindrical:**

$$(x, y, z) \longrightarrow (r, \theta, z) \text{ or } (\rho, \theta, z)$$

$x = r\cos\theta$

$y = r\sin\theta$ $\qquad J\left(\frac{x,y,z}{r,\theta,z}\right) = r$

$z = z$

$$\iiint_V F(x,y,z)\,dx\,dy\,dz = \iiint_V F(r\cos\theta, r\sin\theta, z)\, r\,dr\,d\theta\,dz.$$

Q. Evaluate : $\iiint (x^2+y^2+z^2)\,dx\,dy\,dz$ taken over the volume enclosed by the sphere $x^2+y^2+z^2=1$ by transforming into spherical coordinates.

Sol: $x = r\sin\theta\cos\phi$

Let V be the volume bounded by the sphere $x^2+y^2+z^2=1$ then the limits of spherical coordinates are

$r: 0$ to 1, $\theta: 0$ to π, $\phi: 0$ to 2π

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{1} r^4\sin\theta\,d\phi\,d\theta\,dr = \frac{4\pi}{5}.$$

Q. using cylindrical coordinates, find the volume of the cylinder with base radius 'a' and height 'h'.

Sol: Region of integration is bounded by

$x^2+y^2 = a^2$ and $0 \le z \le h$

$r: 0$ to a

$\theta: 0$ to 2π

$z: 0$ to h

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$$\int_0^h \int_0^{2\pi} \int_0^a dr \cdot d\theta\, dz = \pi a^2 h$$

Q. Solve $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dx\,dy\,dz}{\sqrt{1-x^2-y^2-z^2}}$ by changing into spherical coordinates.

Sol:

$r = 0$ to 1

$\theta = 0$ to $\pi/2$ $\quad J = r^2 \sin\theta$

$\phi = 0$ to $\pi/2$

$$I = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 \frac{r^2 \sin\theta}{\sqrt{1-r^2}}\, dr\, d\theta\, d\phi$$

$\sqrt{1-r^2} = t$

$1-r^2 = t$

$1-t^2 = r^2$

$$= \int_0^1 2(1-t^2)\,dt \int_0^{\pi/2} \sin\theta\, d\theta \int_0^{\pi/2} d\phi$$

$$= -\frac{2\pi}{3}.$$

* Applications:

Area :-

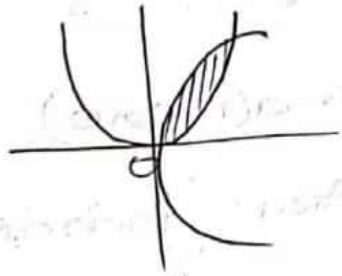
$$A = \iint_R dx\,dy = \iint_R dy\,dx = \iint_R r\,dr\,d\theta.$$

Volume: $\iiint_V dx\,dy\,dz = \iiint_V dv$

$$\iint_S z\,dx\,dy = \iint_S f(x,y)\,dx\,dy.$$

Q. Find the area enclosed by the parabolas $x^2 = y$ & $y^2 = x$.

Sol: Area (A) $= \iint_R dx\,dy$



y : (C.L) :- 0 to 1

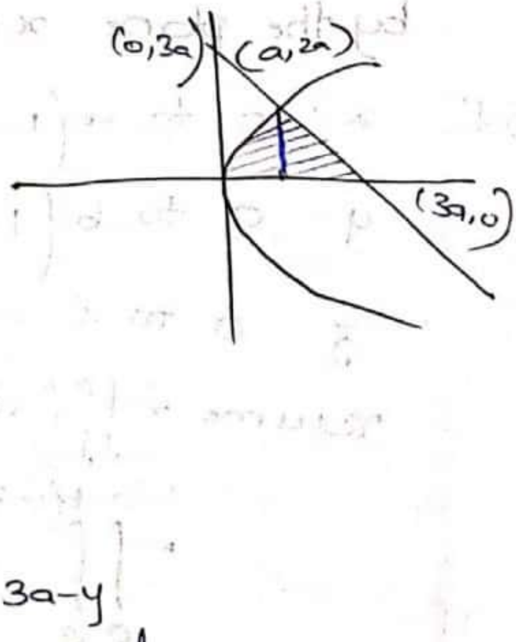
x : (V.L) :- y^2 to $\sqrt{y}$

$$\int_0^1 \int_{y^2}^{\sqrt{y}} dx\,dy = \int_0^1 (y^{1/2} - y^2)\,dy$$

$= \frac{1}{3}\,cm^2.$

Q. Find the area enclosed by the parabola $y^2 = 4ax$ & $y + x = 3a$.

Sol: $y^2 = 4ax$



$(3a-x)^2 = 4ax$

$x^2 - 10ax + 9a^2 = 0$

$x = 9a, a$

y : 0 to 2a

x : $\frac{y^2}{4a}$ to $3a - y$

$$\int_0^{2a} \int_{y^2/4a}^{3a-y} dx\,dy = \int_0^{2a} [x]_{y^2/4a}^{3a-y}\,dy$$

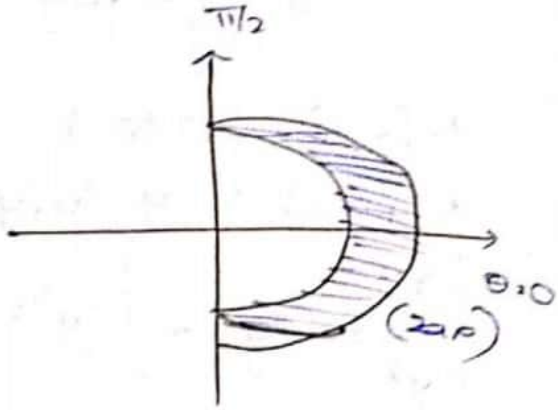
$$= \left[3ay - \frac{y^2}{2} - \frac{y^3}{12a}\right]_0^{2a}$$

$$= 6a^2 - \frac{4a^2}{2} - \frac{8a^3}{12a} = \frac{36a^2 - 12a^2 - 4a^2}{6}$$

$$= \frac{20a^2}{6} = \frac{10a^2}{3}.$$

Q. $r = a(1+\cos\theta)$

Sol: Area $= \iint_R r\,dr\,d\theta$



$\theta = \pi/2$ to $\pi/2$

$r = a$ to $a(1+\cos\theta)$

$$= \frac{a^2}{4}(\pi + 8)$$

Q: Find the volume of tetrahedron bounded by the plane $x = y = z = 0$ & $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

Sol: $x : 0$ to $a\left(1 - y/b - z/c\right)$

$y : 0$ to $b\left(1 - z/c\right)$

$z : 0$ to c

volume $= \iiint dv$

$$= \int_0^c \int_0^{b(1-z/c)} \int_0^{a(1-y/b-z/c)} dx\,dy\,dz$$

$$= \int_0^c \int_0^{b(1-z/c)} \left(a - \frac{ay}{b} - \frac{az}{c}\right) dy\,dz$$

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$$= \int_0^c \left[ay - \frac{ay^2}{2b} - \frac{azy}{c}\right]_0^{b(1-z/c)} dz$$

$$= \int_0^c \left(\frac{ab}{2} + \frac{3}{2}\frac{abz^2}{c^2} - \frac{az}{c}\right) dz$$

$$= \frac{abc}{2} + \frac{abc}{2} - \frac{ac}{2}$$

$$= abc - \frac{ac}{2}.$$

* Mass and centre of gravity

Mass of plane lamina :

→ $m = \iint dM = \rho(x,y)\,dx\,dy$

$c = (\bar{x}, \bar{y})$

$\bar{x} = \dfrac{\iint x\,dm}{\iint dm}$; $\bar{y} = \dfrac{\iint y\,dm}{\iint dm}$

→ Let $\rho(x,y)$ be the density at any point $P(x,y)$ of a plane lamina in the x-y plane in the elemental mass of elemental area around P is given by $dm = \rho(x,y)\,dx\,dy$.

→ Therefore, total mass of plane lamina is given by $\iint dm = \iint \rho(x,y)\,dx\,dy$ integrated over the area of lamina under consideration.

* Centre of mass :

→ $C(\bar{x}, \bar{y})$ of a lamina is given by $\bar{x} = \frac{\iint x\,dm}{\iint dm}$

$\bar{y} = \frac{\iint y\,dm}{\iint dm}$

→ Mass of a solid is given by

$m = \iiint dm = \iiint \rho(x,y,z)\,dx\,dy\,dz$ where $\rho(x,y,z)$ be the density of a solid at any point $P(x,y,z)$.

→ Centre of a mass of a solid is given by $C(\bar{x}, \bar{y}, \bar{z})$ where

$\bar{x} = \frac{\iiint x\,dm}{\iiint dm}$

$\bar{y} = \frac{\iiint y\,dm}{\iiint dm}$

$\bar{z} = \frac{\iiint z\,dm}{\iiint dm}$

The plane lamina is symmetrical about x-axis Therefore centroid lies on x-axis is $y = 0$.

* Note :

When we have to find centre of gravity of some area (or) mass of the lamina or solid. We assume that the lamina (or) solid has uniform density.

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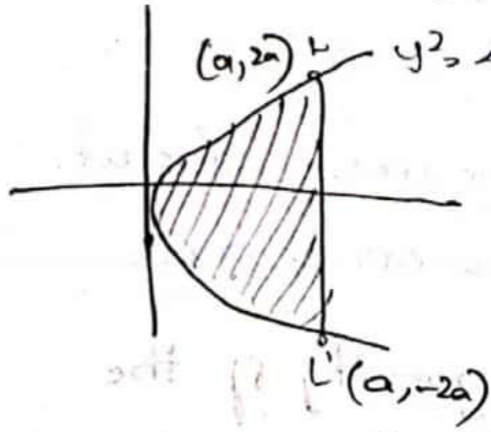
Q. Find the centroid of area enclosed by the parabola $y^2 = 4ax$ & its latus rectum.



y : (C.L) $-2a$ to $2a$

x : (V.L) : $\frac{y^2}{4a}$ to a

centroid $(\bar{x}, \bar{y})$

$$m = \iint dm = \int_{-2a}^{2a} \int_{y^2/4a}^{a} dx\, dy$$

$$= \int_{-2a}^{2a} [x]_{y^2/4a}^{a} dy = \int_{-2a}^{2a} \left[a - \frac{y^2}{4a}\right] dy$$

$$= \left[ay - \frac{y^3}{12a}\right]_{-2a}^{2a} = \frac{8a^2}{3}$$

$$\iint x\, dm = \int_{-2a}^{2a} \int_{y^2/4a}^{a} x\, dx\, dy$$

$$= \int_{-2a}^{2a} \left[\frac{x^2}{2}\right]_{y^2/4a}^{a} dy.$$

$$= \frac{1}{2} \int_{-2a}^{2a} \left(a^2 - \frac{y^4}{16a^2}\right) dy = \frac{8a^3}{5}.$$

$$\iint y\, dm = \int_{-2a}^{2a} \int_{y^2/4a}^{a} y\, dx\, dy.$$

$$= 0$$

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$$\bar{x} = \frac{\iint x\,dm}{\iint dm} = \frac{8a^3/5}{8a^2/3} = \frac{3a}{5}.$$

$\bar{y} = 0$

As the lamina is symmetric about x-axis centroid lies on x-axis.

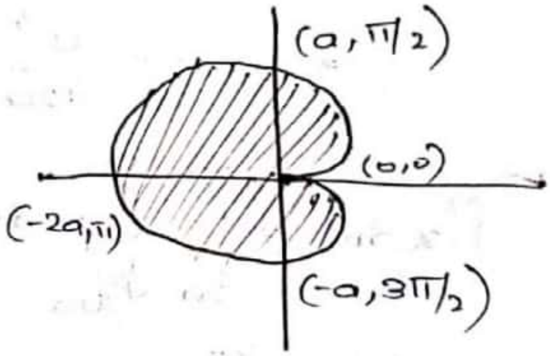
Q. Find the centre of gravity of the area of the cardioid $r = a(1-\cos\theta)$

Sol: The plane lamina which is bounded by the cardioid is symmetrical about x-axis.

Therefore, centroid of lamina lies on x-axis.

$\iint dm = m \qquad \bar{y} = 0$

Centroid $= (\bar{x}, 0)$



$$\bar{x} = \frac{\iint x\,dm}{\iint dm} = \frac{\iint r\cos\theta\,(r\,dr\,d\theta)}{\iint r\,dr\,d\theta}$$

θ : 0 to 2π

r : 0 to $a(1-\cos\theta)$

$$\int_0^{2\pi}\int_0^{a(1-\cos\theta)} r^2\cos\theta\,dr\,d\theta$$

$$= \int_0^{2\pi} \cos\theta \left(\frac{a^3(1-\cos\theta) - 0}{3} \right) d\theta.$$

$$= \frac{a^3}{3} \int_0^{2\pi} \left(2\sin^2 \theta/_2\right)^3 \cos\theta \, d\theta.$$

$$= \frac{8a^3}{3} \int_0^{2\pi} \sin^6\frac{\theta}{2} \left(1 - 2\sin^2\frac{\theta}{2}\right) d\theta$$

Let $\frac{\theta}{2} = t \Rightarrow \theta = 2t \Rightarrow d\theta = 2dt$

when $\theta = 0 \Rightarrow t = 0$

$\theta = 2\pi \Rightarrow t = \pi$

$$I = \frac{8a^3}{3} \int_0^{\pi} \sin^6 t \left(1 - 2\sin^2 t\right) 2dt.$$

$$= \frac{16a^3}{3} \int_0^{\pi} [\sin^6 t - 2\sin^8 t] \, dt.$$

$$= \frac{16a^3}{3}.(2) \int_0^{\pi/2} (\sin^6 t - 2\sin^8 t) \, dt$$

$$= \frac{32a^3}{3} \left[\frac{\sqrt{\pi}}{2} \frac{\Gamma 7/_2}{\Gamma 4} - 2 \cdot \frac{\sqrt{\pi}}{2} \frac{\Gamma 9/_2}{\Gamma 5} \right]$$

$$N = \frac{-5\pi a^3}{4}$$

$$D = \frac{3\pi a^2}{2} = \iint dm$$

$$\frac{N}{D} = \frac{-5a}{6}$$

so C.O.M $= \left(-\frac{5a}{6}, 0\right)$

<u>H.W</u>

$r = a(1 + \sin\theta)$

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$$\bar{x} = \frac{\iint r\cos\theta\,\rho\, r\, d\theta\, dr}{\iint \rho\, r\, d\theta\, dr}, \bar{y} = \frac{\iint r\sin\theta\,\rho\, r\, d\theta\, dr}{\iint \rho\, r\, d\theta\, dr}$$

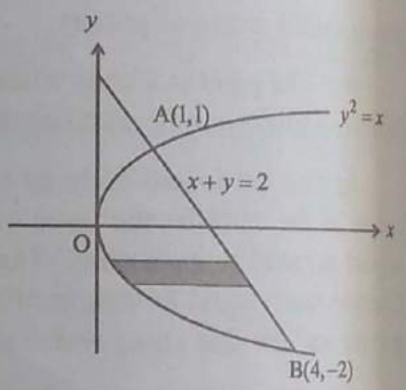
(b) Centre of Gravity of a Solid :

The centre of gravity $(\bar{x}, \bar{y}, \bar{z})$ of a solid is given by

$$\bar{x} = \frac{\iiint x\,\rho\, dx\, dy\, dz}{\iiint \rho\, dx\, dy\, dz}, \bar{y} = \frac{\iiint y\,\rho\, dx\, dy\, dz}{\iiint \rho\, dx\, dy\, dz}, \bar{z} = \frac{\iiint z\,\rho\, dx\, dy\, dz}{\iiint \rho\, dx\, dy\, dz}$$

SOLVED EXAMPLES

Example 1 : Find the centre of gravity of the area bounded by the parabola $y^2 = x$ and the line $x + y = 2$



Solution : The given parabola is $y^2 = x$ ... (1)

and the straight line is $x + y = 2 \Rightarrow y = 2 - x$ ... (2)

We solve (1) and (2) to find their points of intersection.

Substituting the value of y from (2) in (1), we get

$(2-x)^2 = x \Rightarrow 4 - 4x + x^2 = x \Rightarrow x^2 - 5x + 4 = 0$

$\Rightarrow (x-1)(x-4) = 0 \quad \therefore\ x = 1, 4$

From (2), when $x = 1, y = 1$ and when $x = 4, y = 2 - 4 = -2$.

Thus the given curves intersect at the points A(1,1) and B(4,–2).

Let $G(\bar{x}, \bar{y})$ be the centre of gravity of the lamina OAB, so that

$$\bar{x} = \frac{\iint x\,\rho\, dx\, dy}{\iint \rho\, dx\, dy}, \bar{y} = \frac{\iint y\,\rho\, dx\, dy}{\iint \rho\, dx\, dy}$$

When we have to find the centre of gravity of some area, we assume that the lamina has uniform density at any point of the lamina.

Thus $\bar{x} = \dfrac{\iint x\, dx\, dy}{\iint dx\, dy}$ and $\bar{y} = \dfrac{\iint y\, dx\, dy}{\iint dx\, dy}$

Now $\iint x\, dx\, dy = \int_{-2}^{1}\int_{y^2}^{2-y} x\, dx\, dy = \int_{-2}^{1}\left(\frac{x^2}{2}\right)_{y^2}^{2-y} dy$

$$=\frac{1}{2}\int_{-2}^{1}\left[(2-y)^2-y^4\right]dy=\frac{1}{2}\int_{-2}^{1}(4-4y+y^2-y^4)\,dy$$

$$=\frac{1}{2}\left(4y-2y^2+\frac{y^3}{3}-\frac{y^5}{5}\right)_{-2}^{1}$$

$$=\frac{1}{2}\left[\left(4-2+\frac{1}{3}-\frac{1}{5}\right)-\left(-8-8-\frac{8}{3}+\frac{32}{5}\right)\right]$$

$$=\frac{1}{2}\left(\frac{32}{15}+\frac{184}{15}\right)=\frac{36}{5}$$

and $\iint y\,dx\,dy=\int_{-2}^{1}\int_{y^2}^{2-y} y\,dx\,dy=\int_{-2}^{1} y[x]_{y^2}^{2-y}\,dy$

$$=\int_{-2}^{1} y(2-y-y^2)\,dy=\int_{-2}^{1}(2y-y^2-y^3)\,dy$$

$$=\left(y^2-\frac{y^3}{3}-\frac{y^4}{4}\right)_{-2}^{1}=\left(1-\frac{1}{3}-\frac{1}{4}\right)-\left(4+\frac{8}{3}-\frac{16}{4}\right)=\frac{-9}{4}$$

Also $\iint dx\,dy=\int_{-2}^{1}\int_{y^2}^{2-y} dx\,dy=\int_{-2}^{1}[x]_{y^2}^{2-y}\,dy=\int_{-2}^{1}(2-y-y^2)\,dy$

$$=\left(2y-\frac{y^2}{2}-\frac{y^3}{3}\right)_{-2}^{1}=\left(2-\frac{1}{2}-\frac{1}{3}\right)-\left(-4-2+\frac{8}{3}\right)=\frac{9}{2}$$

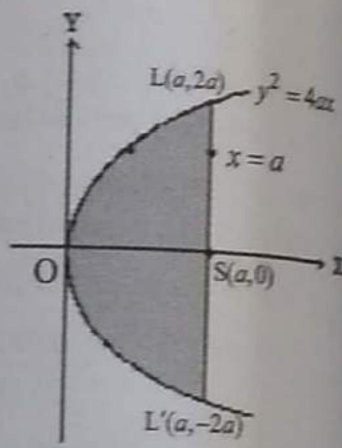
$$\therefore\ \bar{x}=\frac{\iint x\,dx\,dy}{\iint dx\,dy}=\frac{36/5}{9/2}=\frac{8}{5}$$

$$\text{and }\ \bar{y}=\frac{\iint y\,dx\,dy}{\iint dx\,dy}=\frac{-9/4}{9/2}=\frac{-1}{2}$$

Hence the required centre of gravity $=(\bar{x},\bar{y})=\left(\frac{8}{5},\frac{-1}{2}\right)$

Example 2 : Find the centroid of the area enclosed by the parabola $y^2=4ax$, the x - axis and its latus rectum.

Solution : (When we have to find the centre of gravity of some area, we assume that the lamina has uniform density at any point of the lamina)



The latus rectum of the parabola $y^2 = 4ax$ is given by $x = a$. Let $(\bar{x}, \bar{y})$ be the centroid of the region. Since the region is symmetrical about x axis, its C.F. lies on OX *i.e.* $\bar{y} = 0$.

We have $\bar{x} = \dfrac{\iint x\,dx\,dy}{\iint dx\,dy}$ ... (1)

Consider $\iint x\,dx\,dy = \iint x\,dx\,dy$

$$= \int_{y=-2a}^{2a} \int_{x=y^2/4a}^{a} x\,dx\,dy$$

$$= \int_{y=-2a}^{2a} \left(\frac{x^2}{2}\right)_{y^2/4a}^{a} dy = \int_{y=-2a}^{2a} \left[\frac{a^2}{2} - \frac{y^4}{32a^2}\right] dy$$

$$= 2\int_{y=0}^{2a} \left[\frac{a^2}{2} - \frac{y^4}{32a^2}\right] dy = 2\left[\frac{a^2 y}{2} - \frac{y^5}{160a^2}\right]_{y=0}^{2a}$$

$$= 2\left[\frac{2a^3}{2} - \frac{32a^3}{160}\right] = 2\left[a^3 - \frac{a^3}{5}\right] = \frac{8a^3}{5} \quad \text{... (2)}$$

Consider

$$\iint dxdy = \int_{y=-2a}^{2a} \int_{x=y^2/4a}^{a} dx\,dy = \int_{-2a}^{2a} \left[a - \frac{y^2}{4a}\right] dy = 2\left[ay - \frac{y^3}{12a}\right]_0^{2a}$$

$$= 2\left[2a^2 - \frac{8a^2}{12}\right] = 4a^2\left[1 - \frac{1}{3}\right] = \frac{8a^2}{3} \quad \text{... (3)}$$

Using (1), (2) and (3), we get $\bar{x} = \dfrac{8a^3/5}{8a^2/3} = \dfrac{3a}{5}$

Hence the centroid of the area under consideration $= (\bar{x}, \bar{y}) = \left(\dfrac{3a}{5}, 0\right)$

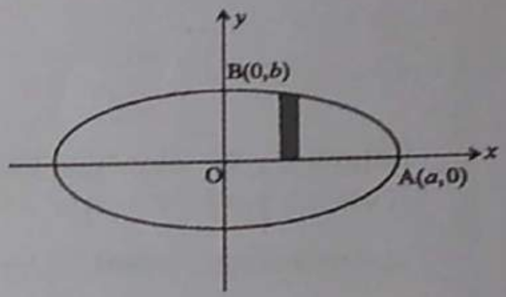
Example 3 : Find the centre of gravity of the region in the positive quadrant of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Solution : Let $G(\overline{x}, \overline{y})$ be the centre of gravity of the lamina OAB, so that

$$\overline{x} = \frac{\iint x\,dx\,dy}{\iint dx\,dy} \text{ and } \overline{y} = \frac{\iint y\,dx\,dy}{\iint dx\,dy}$$

(We assume that the lamina has uniform density at any point of the lamina)

where the integrals are taken over the area OAB so that y varies from 0 to y (to be found from the equation of the curve) and then x varies from 0 to a.



For any point on the ellipse, we have

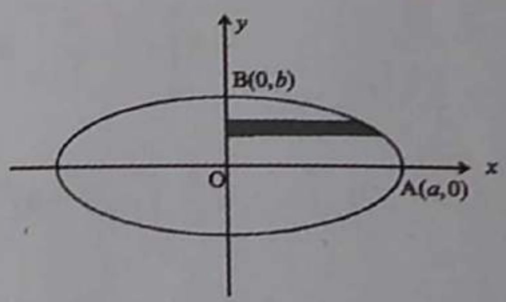
$x = a\cos\theta, y = b\sin\theta$ (parametric equations)

$$\text{Now } \iint x\,dx\,dy = \int_{x=0}^{a}\int_{y=0}^{y} x\,dy\,dx = \int_0^a x[y]_0^y\,dx$$

$$= \int_0^a xy\,dx = \int_{\pi/2}^{0} a\cos\theta . b\sin\theta\,(-a\sin\theta)\,d\theta$$

$$= a^2b\int_0^{\pi/2}\sin^2\theta\cos\theta\,d\theta = a^2b\left(\frac{\sin^3\theta}{3}\right)_0^{\pi/2} = \frac{a^2b}{3}$$

$$\iint dx\,dy = \text{Area of the ellipse in the first quadrant } = \frac{\pi ab}{4} \text{ and}$$



$$\iint y\,dx\,dy = \int_{y=0}^{b}\int_{x=0}^{x} y\,dx\,dy$$

$$= \int_0^b y[x]_0^x\,dy = \int_0^b xy\,dy$$

$$= \int_0^{\pi/2} a\cos\theta . b\sin\theta . b\cos\theta\,d\theta$$

$$= ab^2\int_0^{\pi/2}\cos^2\theta\sin\theta\,d\theta = ab^2\left(\frac{-\cos^3\theta}{3}\right)_0^{\pi/2} = \frac{ab^2}{3}$$

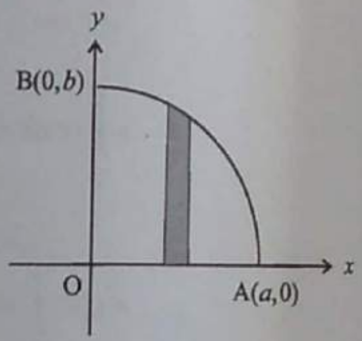
$$\therefore\ \overline{x} = \frac{a^2b/3}{\pi ab/4} = \frac{4a}{3\pi} \text{ and } \overline{y} = \frac{ab^2/3}{\pi ab/4} = \frac{4b}{3\pi}$$

Hence the required centre of gravity $= (\bar{x}, \bar{y}) = \left(\frac{4a}{3\pi}, \frac{4b}{3\pi}\right)$

Example 4 : A plane in the form of a quadrant of the ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ is of small but varying thickness, the thickness at any point being proportional to the product of the distances of that point from the axes; show that the co-ordinates of the centroid (C.G.) are $\left(\frac{8a}{15}, \frac{8b}{15}\right)$.

Solution : We know that the parametric equations of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are

$x = a\cos\theta, y = b\sin\theta$



Here density, $\rho = kxy$

Let $(\bar{x}, \bar{y})$ be the centroid of the region OAB. Then

$$\bar{x} = \frac{\iint \rho\, x\, dx\, dy}{\iint \rho\, dx\, dy}, \bar{y} = \frac{\iint \rho\, y\, dx\, dy}{\iint \rho\, dx\, dy}$$

Now $\iint \rho\, x\, dx\, dy = \iint (kxy)\, x\, dx\, dy = k\int_0^a\int_0^y x^2 y\, dx\, dy$

$$= k\int_0^a x^2 \left(\frac{y^2}{2}\right)_0^y dx = \frac{k}{2}\int_0^a x^2 y^2 dx$$

$$= \frac{k}{2}\int_{\pi/2}^{0} (a\cos\theta)^2 . (b\sin\theta)^2 . (-a\sin\theta)\, d\theta$$

$$= \frac{ka^3b^2}{2}\int_0^{\pi/2} \sin^3\theta\cos^2\theta\, d\theta$$

$$= \frac{ka^3b^2}{2} . \frac{2\times 1}{5\times 3\times 1} \left[\because \int_0^{\pi/2} \sin^m x\cos^n x dx = \frac{(m-1)(m-3)...(n-1)(n-3)}{(m+n)(m+n-2)(m+n-4)..} \times \right.$$
$$\left. (\pi/2 \text{ only if both } m \text{ and } n \text{ are even}) \right]$$

$$= \frac{ka^3b^2}{15}$$

and $\iint \rho\, y\, dx\, dy = \iint (kxy)\, y\, dx\, dy = k\int_0^a\int_0^y kxy^2 dy = k\int_0^a x\left(\frac{y^3}{3}\right)_0^y dy = \frac{k}{3}\int_0^a xy^3 dx$

$$= \frac{k}{3}\int_{\pi/2}^{0} (a\cos\theta).(b\sin\theta)^3.(-a\sin\theta)d\theta$$

$$= \frac{ka^2b^3}{3}\int_0^{\pi/2} \sin^4\theta\cos\theta d\theta$$

$$= \frac{ka^2b^3}{3}.\frac{3\times 1}{5\times 3\times 1}\left[\because \int_0^{\pi/2} \sin^m x\cos^n x dx = \frac{(m-1)(m-3)...(n-1)(n-3)}{(m+n)(m+n-2)(m+n-4)..}\times \right.$$
$$\left.(\pi/2 \text{ only if both } m \text{ and } n \text{ are even})\right]$$

$$= \frac{ka^2b^3}{15}$$

and $\iint \rho\, dx\, dy = \iint k\, x\, y\, dx\, dy = k\int_0^a\int_0^y x\, y\, dx\, dy$

$$= k\int_0^a x\left(\frac{y^2}{2}\right)_0^y dx = \frac{k}{2}\int_0^a xy^2 dx$$

$$= \frac{k}{2}\int_{\pi/2}^{0} (a\cos\theta).(b\sin\theta)^2.(-a\sin\theta)\, d\theta$$

$$= \frac{k}{2}a^2b^2\int_0^{\pi/2} \sin^3\theta.\cos\theta\, d\theta = \frac{k}{2}a^2b^2.\frac{2\times 1}{4\times 2} = \frac{ka^2b^2}{8}$$

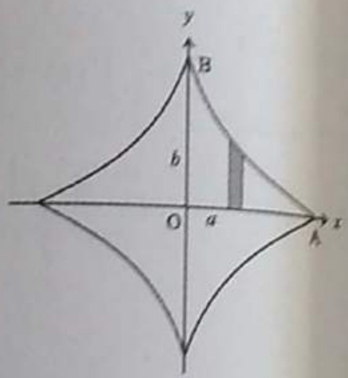
$\therefore\ \bar{x} = \dfrac{ka^3b^2/15}{ka^2b^2/8} = \dfrac{8a}{15}$ and $\bar{y} = \dfrac{ka^2b^3/15}{ka^2b^2/8} = \dfrac{8b}{15}$

Hence the centroid $= (\bar{x},\bar{y}) = \left(\dfrac{8a}{15},\dfrac{8b}{15}\right)$

Example 5 : Using double integration, find the centre of gravity (centroid) of the region in the first quadrant bounded by the curve $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1, x = 0, y = 0$ and having surface density kxy, where k is a constant.

Solution : We know that the parametric equations of the curve $\frac{x^{2/3}}{a^{2/3}} + \frac{y^{2/3}}{b^{2/3}} = 1$ are

$$x = a\cos^3\theta,\ y = b\sin^3\theta \qquad \text{... (1)}$$



Here density, $\rho = kxy$

Let $G(\overline{x}, \overline{y})$ be the centre of gravity of the lamina OAB.

Then we have

$$\overline{x} = \frac{\iint \rho\, x\, dx\, dy}{\iint \rho\, dx\, dy},\ \overline{y} = \frac{\iint \rho\, y\, dx\, dy}{\iint \rho\, dx\, dy}$$

where the integrals are taken over the area OAB so that y varies from 0 to y (to be found from the equation of the curve) and then x varies from 0 to a.

$$\text{Now } \iint \rho\, x\, dx\, dy = \int_0^a \int_0^y (kxy) \,.\, x\, dx\, dy = k \int_0^a \int_0^y x^2 y\, dx\, dy$$

$$= k\int_0^a x^2 \left(\frac{y^2}{2}\right)_0^y dx = \frac{k}{2}\int_0^a x^2 y^2 dx$$

$$= \frac{k}{2}\int_{\pi/2}^{0} (a\cos^3\theta)^2 . (b\sin^3\theta)^2 . (-3a\cos^2\theta\sin\theta)\, d\theta, \text{ using (1)}$$

$$= \frac{3ka^3b^2}{2}\int_0^{\pi/2} \sin^7\theta \,.\, \cos^8\theta\, d\theta$$

$$= \frac{3ka^3b^2}{2} . \frac{6\times4\times2\times7\times5\times3\times1}{15\times13\times11\times9\times7\times5\times3\times1} = \frac{8ka^3b^2}{15\times13\times11}$$

$$\text{and } \iint \rho\, dx\, dy = \iint k\, x\, y\, dx\, dy = k\int_0^a \int_0^y x\, y\, dx\, dy$$

$$= k\int_0^a x\left(\frac{y^2}{2}\right)_0^y dx = \frac{k}{2}\int_0^a xy^2 dx$$

$$= \frac{k}{2}\int_{\pi/2}^{0} (a\cos^3\theta).(b\sin^3\theta)^2.(-3a\cos^2\theta\sin\theta)\, d\theta, \text{ using (1)}$$

$$=\frac{3ka^2b^2}{2}\int_0^{\pi/2}\sin^7\theta\cos^5\theta\,d\theta\text{ . Apply }\int_0^{\pi/2}\sin^m x\cos^n x\,dx\text{ formula}$$

$$=\frac{3ka^2b^2}{2}\cdot\frac{6\times4\times2\times4\times2}{12\times10\times8\times6\times4\times2}=\frac{ka^2b^2}{80}$$

Further $\iint\rho\,y\,dx\,dy=\iint(kxy)y\,dx\,dy=k\int_0^a\int_0^y xy^2dx\,dy$

$$=k\int_0^a x\left(\frac{y^3}{3}\right)_0^y dx=\frac{k}{3}\int_0^a xy^3dx$$

$$=\frac{k}{3}\int_{\pi/2}^0(a\cos^3\theta).(b\sin^3\theta)^3.(-3a\cos^2\theta\sin\theta)\,d\theta$$

$$=ka^2b^3\int_0^{\pi/2}\sin^{10}\theta\cos^5\theta\,d\theta$$

$$=ka^2b^3.\frac{9\times7\times5\times3\times1\times4\times2}{15\times13\times11\times9\times7\times5\times3\times1},$$

using $\int_0^{\pi/2}\sin^m x\cos^n x\,dx$ formula

$$=\frac{8ka^2b^3}{15\times13\times11}$$

$$\therefore\ \bar{x}=\frac{8ka^3b^2/(15\times13\times11)}{ka^2b^2/80}=\frac{128a}{429}$$

and $\therefore\ \bar{y}=\dfrac{8ka^2b^3/(15\times13\times11)}{ka^2b^2/80}=\dfrac{128b}{429}$

Hence the required centre of gravity is $\left(\dfrac{128a}{429},\dfrac{128b}{429}\right)$

Example 6 : Find the centre of gravity of the region in the first quadrant bounded by the curve $x^{2/3}+y^{2/3}=a^{2/3}$

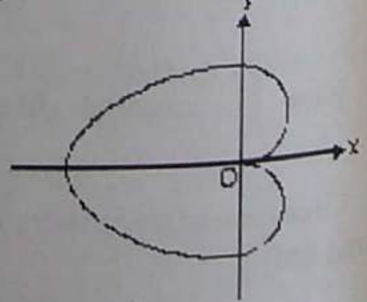
Solution : We know that the parametric equations of the given curve are

$x=a\cos^3\theta,y=a\sin^3\theta$

Example 8 : Find the centre of gravity of the area of the cardioid $r = a(1-\cos\theta)$

Solution : (When we have to find the centre of gravity of some area, we assume that the lamina has uniform density at any point of the lamina)

Let $(\bar{x}, \bar{y})$ be the centre of gravity of the lamina. The cardioid $r = a(1-\cos\theta)$ is symmetric about the initial line $\theta = 0$ (*i.e.*) x axis. Hence the centre of gravity lies on the x axis. Hence $\bar{y} = 0$.



The x coordinate of the C. G. is given by

$$\bar{x} = \frac{\iint x\,dx\,dy}{\iint dx\,dy} \qquad \text{... (1)}$$

Now $\iint x\,dx\,dy = \int_{\theta=-\pi}^{\pi}\int_{r=0}^{a(1-\cos\theta)} r\cos\theta\, r\,dr\,d\theta = \int_{\theta=-\pi}^{\pi}\left[\frac{r^3}{3}\right]_{r=0}^{a(1-\cos\theta)}\cos\theta\,d\theta$

$$= \frac{a^3}{3}\int_{\theta=-\pi}^{\pi}(1-\cos\theta)^3\cos\theta\,d\theta = \frac{a^3}{3}.2\int_{\theta=0}^{\pi}(1-\cos\theta)^3\cos\theta d\theta$$

$= \frac{2a^3}{3}\int_{\theta=0}^{\pi}(1-3\cos\theta+3\cos^2\theta-\cos^3\theta)\cos\theta\,d\theta$, using $(a-b)^3$ formula

$$= \frac{2a^3}{3}\int_{\theta=0}^{\pi}\left[-3\cos^2\theta-\cos^4\theta\right]d\theta$$

$$= -\frac{2a^3.2}{3}\int_{0}^{\pi/2}(3\cos^2\theta+\cos^4\theta)\,d\theta = -\frac{4a^3}{3}\left[3.\frac{1}{2}+\frac{3}{4}.\frac{1}{2}\right]\frac{\pi}{2}$$

$$= -\frac{5\pi a^3}{4}$$

and $\iint dx\,dy = \int_{\theta=-\pi}^{\pi}\int_{r=0}^{a(1-\cos\theta)} dr\,d\theta = \int_{\theta=-\pi}^{\pi}\left[\frac{r^2}{2}\right]_0^{a(1-\cos\theta)} d\theta$

$= \frac{a^2}{2}\int_{\theta=-\pi}^{\pi}(1-\cos\theta)^2 d\theta = a^2\int_{\theta=0}^{\pi}(1-\cos\theta)^2 d\theta$, since integrand is even function

$$= a^2\int_{\theta=0}^{\pi}(1-2\cos\theta+\cos^2\theta)\,d\theta = a^2\int_{\theta=0}^{\pi}\left(1+\frac{1+\cos 2\theta}{2}\right)d\theta$$

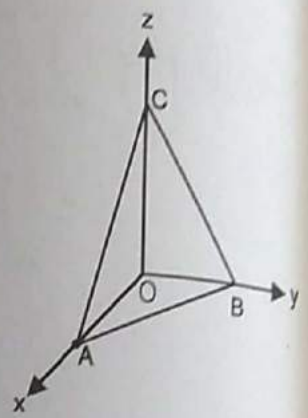
$$= a^2.\frac{3}{2}.\pi = \frac{3\pi a^2}{2}$$

$\therefore\ \bar{x} = \frac{-5\pi a^3/4}{(3\pi a^2/2)}$ [by (1)] $= -\frac{5a}{6}$

Hence $(\bar{x},\bar{y}) = \left(\frac{-5a}{6},0\right)$

Example 9 : Find the mass, centroid of the tetrahedron bounded by the coordinate planes and the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

Solution : Let ρ be the constant density of the substance.



Then elementary mass at $P = \rho\, dx\, dy\, dz$

$\therefore$ The whole mass, $M = \iiint \rho\, dx\, dy\, dz$,

the integrals embracing the whole volume OABC.

The limits for z are from $z = 0$ to $z = c\left(1 - \frac{x}{a} - \frac{y}{b}\right)$

The limits for y are from $y = 0$ to $y = b\left(1 - \frac{x}{a}\right)$ and the limits for x are from 0 to a.

$\therefore$ The required mass, $M = \rho \int\limits_{x=0}^{a} \int\limits_{y=0}^{b\left(1-\frac{x}{a}\right)} \int\limits_{z=0}^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dx\, dy\, dz$

$$= \rho \int\limits_{0}^{a} \int\limits_{0}^{b\left(1-\frac{x}{a}\right)} c\left(1 - \frac{x}{a} - \frac{y}{b}\right) dx\, dy$$

$$= \rho c \int\limits_{0}^{a} \left[\left(1 - \frac{x}{a}\right) y - \frac{y^2}{2b}\right]_{0}^{b\left(1-\frac{x}{a}\right)} dx$$

$$= \rho c \int\limits_{0}^{a} \left[b\left(1 - \frac{x}{a}\right)^2 - \frac{1}{2b}.b^2\left(1 - \frac{x}{a}\right)^2\right] dx$$

$$= \frac{\rho bc}{2} \int\limits_{0}^{a} \left(1 - \frac{x}{a}\right)^2 dx = \frac{\rho bc}{2}.\left[\frac{\left(1 - \frac{x}{a}\right)^3}{-3/a}\right]_{0}^{a}$$

$$= -\frac{\rho abc}{6}\left[(1-1)^3 - (1-0)^3\right] = \frac{\rho abc}{6} \quad \ldots (1)$$

Let G $(\bar{x}, \bar{y}, \bar{z})$ be the coordinates of the centroid. Then

$$\bar{x} = \frac{\iiint x \rho\, dx\, dy\, dz}{\iiint \rho\, dx\, dy\, dz} = \frac{\iiint x \rho\, dx\, dy\, dz}{M}$$

$$\Rightarrow M\bar{x} = \rho\int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} \int_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} x\, dz\, dy\, dx = \rho\int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} xc\left(1-\frac{x}{a}-\frac{y}{b}\right) dy\, dx$$

$$= \rho c\int_0^a \left[x\left(1-\frac{x}{a}\right)y - \frac{xy^2}{2b}\right]_0^{b\left(1-\frac{x}{a}\right)} dx$$

$$= \rho c\int_0^a \left[x\left(1-\frac{x}{a}\right).b\left(1-\frac{x}{a}\right) - \frac{x}{2b}.b^2\left(1-\frac{x}{a}\right)^2\right] dx$$

$$= \rho c\int_0^a \left[bx\left(1-\frac{x}{a}\right)^2 - \frac{bx}{2}\left(1-\frac{x}{a}\right)^2\right] dx$$

$$= \frac{\rho bc}{2}\int_0^a x\left(1-\frac{x}{a}\right)^2 dx = \frac{\rho bc}{2}\int_0^a x\left(1+\frac{x^2}{a^2}-\frac{2x}{a}\right) dx$$

$$= \frac{\rho bc}{2}\int_0^a \left(x+\frac{x^3}{a^2}-\frac{2x^2}{a}\right) dx$$

$$= \frac{\rho bc}{2}\left(\frac{x^2}{2}+\frac{1}{a^2}.\frac{x^4}{4}-\frac{2}{a}.\frac{x^3}{3}\right)_0^a$$

$$= \frac{\rho bc}{2}\left(\frac{a^2}{2}+\frac{1}{a^2}.\frac{a^4}{4}-\frac{2}{a}.\frac{a^3}{3}\right)$$

$$= \frac{\rho bc}{2}\left(\frac{a^2}{2}+\frac{a^2}{4}-\frac{2}{3}a^2\right) = \frac{\rho a^2 bc}{2}\left(\frac{1}{2}+\frac{1}{4}-\frac{2}{3}\right)$$

$$= \frac{\rho a^2 bc}{2}\left(\frac{6+3-8}{12}\right) = \frac{\rho a^2 bc}{24}$$

Thus $\bar{x} = \frac{1}{M}.\frac{\rho a^2 bc}{24} = \frac{6}{\rho abc}.\frac{\rho a^2 bc}{24} = \frac{a}{4}$ [From (1)]

Similarly, $\bar{y} = \frac{b}{4}$ and $\bar{z} = \frac{c}{4}$

Hence centroid, $G = \left(\frac{a}{4}, \frac{b}{4}, \frac{c}{4}\right)$

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UNIT-III : MULTI VARIABLE CALCULUS (Integration)

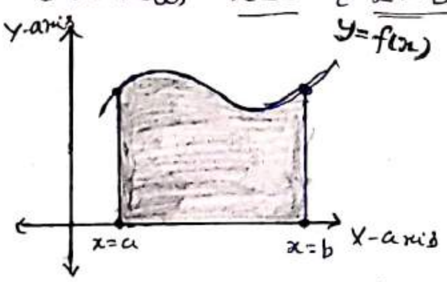
Evaluation of Double integrals (Cartesian and polar coordinates); change of order of integration (only Cartesian form); Change of variables (Cartesian to polar co-ordinates) Evaluation of Triple Integrals; Change of variables (Cartesian to spherical and cylindrical Polar co-ordinates).

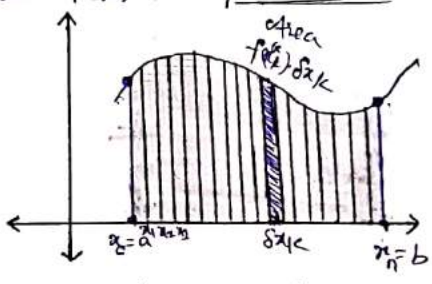
Applications: Areas and volumes, Centre of mass and gravity (constant & variable densities). Applications involving cubes, sphere and rectangular parallelopi

Introduction To Multiple Integrals:

Multiple integral is a natural extension of an ordinary definite integral to a function of two variables (Double integral) (or) three variables (Triple integral) (or) More variables. Double & Triple integrals together are called Multiple integrals.

Definite integral The definite integral $\boxed{\int_{x=a}^{b} f(x).dx}$ is physically The area under a curve $y=f(x)$, the x-axis and the two ordinates $x=a$ & $x=b$, where $f(x)$ is piece-wise continuous on $[a\,b]$





$f: R \to R$; $y = f(x)$ function of a single variable

Domain $\subseteq R$.

$a=x_0 \; x_1 \; x_2 \; x_3 \; x_4 \qquad b=x_n$

Here $\delta x_1, \delta x_2 \cdots \delta x_k \cdots \delta x_n$ are n subdivisions into which the range of integration has been divided and $x_1^*, x_2^* \cdots x_k^* \cdots x_n^*$ are the values of x resp. in $1^{st}, 2^{nd}, \ldots n^{th}$ sub intervals.

The Area Under the curve $y=f(x)$, x-axis and the two ordinates $x=a$ & $x=b$ is ~~...~~ represented and defined as

$$\int_{x=a}^{b} f(x).dx = f(x_1^*)\delta x_1 + f(x_2^*)\delta x_2 + \cdots + f(x_k^*)\delta x_k + \cdots f(x_n^*)\delta x_n$$

when $n \to \infty$, $\delta x_i \to 0$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\delta x_k$$

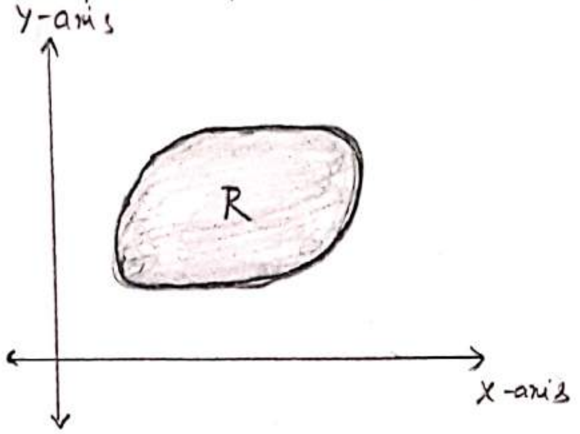
Note Definite integrals defined on Intervals $\subseteq \mathbb{R}$.

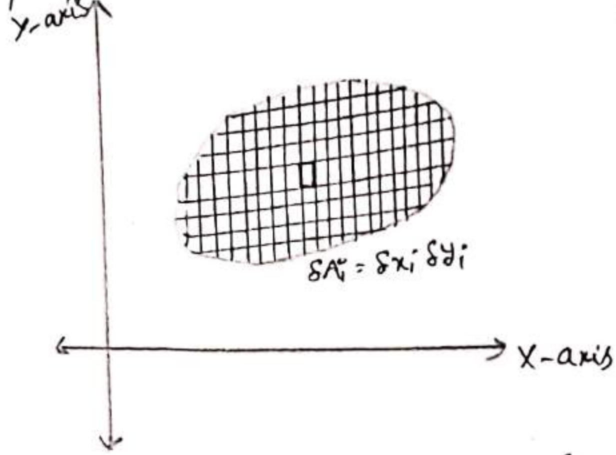
In Previous classes we learn Methods for Evaluating the definite integrals.

Double Integral: It is a extension of definite integral to a function of two variables ie $z = f(x,y)$

$f: R \times R \longrightarrow R$, Domain $\subseteq \mathbb{R}^2$.

Let $f(x,y)$ be a single valued continuous function in a simply connected, closed and bounded Region R in a two dimensional space $\mathbb{R}^2$, bounded by a simple closed curve C.





Subdivide the Region R into n elementary areas $\delta A_1, \delta A_2 \cdots \delta A_n$.
Let (x_i, y_i) be any point inside the i^{th} elementary area δA_i

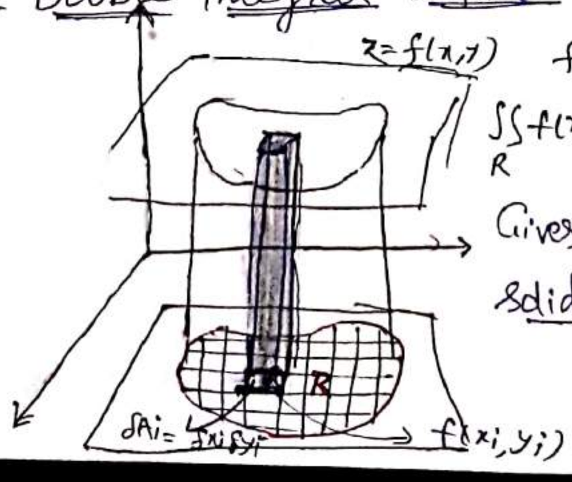
Consider the sum

$$f(x_1,y_1)\,\delta A_1 + f(x_2,y_2)\,\delta A_2 + \cdots + f(x_i,y_i)\,\delta A_i + \cdots f(x_n,y_n)\,\delta A_n = \sum_{i=1}^{n} f(x_i,y_i)\,\delta A_i$$

If the limit of the sum, if exist, as $n \to \infty$ and each sub-elementary area approaches to zero is termed as Double integral of $f(x,y)$ over the Region R. and is denoted by.

$$\boxed{\iint_R f(x,y)\,dx\,dy = \lim_{\substack{n \to \infty \\ \delta A_i \to 0}} \sum_{i=1}^{n} f(x_i,y_i)\,\delta A_i}$$

The Double integral defined over the Region R



$f(x_i,y_i)\,\delta A_i \rightarrow$ volume of the subsolid

$$\iint_R f(x,y)\cdot dx\,dy = \lim_{\substack{n \to \infty \\ \delta A_i \to 0}} \sum_{i=1}^{n} f(x_i,y_i)\,\delta A_i = V$$

Gives the volume V of the solid under the surface $z = f(x,y)$

Triple Integral: It is a extension of definite integral to a function of three variables ie $W = f(x,y,z)$ $f: R \times R \times R \to R$.
Let $f(x,y,z)$ is continuous at all Points inside and on the boundary of the Region R. The Triple integral denoted by

$$\iiint_R f(x,y,z) \cdot dx\,dy\,dz$$

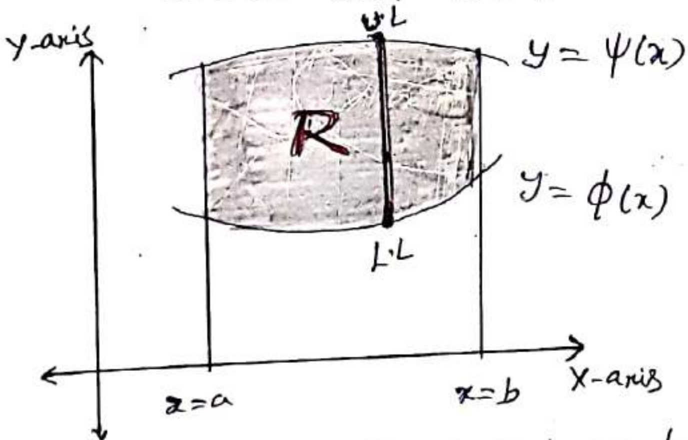
The Multiple Integral over $\mathbb{R}^n$ is written as

$$\iint_R \cdots \int f(x_1, x_2 \cdots x_n)\, dx_1\, dx_2 \cdots dx_n.$$

EVALUATION OF DOUBLE INTEGRAL

A Double Integral $\iint_R f(x,y)\,dx\,dy$, can be Evaluated by two Successive integrations.

Case(1): when the Region R is bounded by two continuous curves $y = \psi(x)$ and $y = \phi(x)$ and the two lines $x = a$ and $x = b$.



$$R = \{(x,y) \mid \phi(x) \le y \le \psi(x),\ a \le x \le b\}$$

In This case, First Integrate w.r.t y keeping x as constant and then the Resulting integral is Integrated w.r.t x within the limits $x = a$ & $x = b$.

$$\iint_R f(x,y) \cdot dx\,dy = \int_{x=a}^{x=b} \int_{y=\phi(x)}^{y=\psi(x)} f(x,y) \cdot dx\,dy = \int_{x=a}^{x=b} \left[\int_{y=\phi(x)}^{\psi(x)} f(x,y) \cdot dy \right] \cdot dx$$

keeping x as constant.

Here The order of integration is
(1) First w.r.t y, keeping x as constant
(2) second w.r.t x b/w the limits.

Problems: ① Evaluate $\int_0^1 \int_0^x dx\,dy$

$$\int_0^1 \int_0^x dx\,dy = \int_{x=0}^{1} \int_{y=0}^{y=x} dx\,dy$$

Since the limits for y depends on x
The order of integration is
1) First w.r.t y, keeping x as constant
2) Second w.r.t x b/w the limits

$$= \int_{x=0}^{1} \left[\int_{y=0}^{x} dy \right] \cdot dx$$

Keeping x as constant

$$= \int_{x=0}^{1} (y)_0^x \cdot dx = \int_{x=0}^{1} x \cdot dx = \left(\frac{x^2}{2}\right)_0^1 = \frac{1}{2}$$

② Evaluate $\int_0^1 \int_x^{x^2} xy\,dy\,dx$

The order of integration is
1) First w.r.t y, keeping x as constant
2) Second w.r.t x b/w Limits

Sol $\int_0^1 \int_x^{x^2} xy \cdot dy\,dx = \int_{x=0}^{1} \int_{y=x}^{x^2} xy \cdot dy\,dx$

$$= \int_{x=0}^{1} \left[\int_{y=x}^{x^2} xy \cdot dy \right] \cdot dx$$

Keeping x as constant

$$= \int_{x=0}^{1} x \cdot \left(\frac{y^2}{2}\right)_x^{x^2} \cdot dx = \int_{x=0}^{1} x \cdot \left[\frac{(x^2)^2}{2} - \frac{x^2}{2}\right] \cdot dx$$

$$= \int_{x=0}^{1} x \left[\frac{x^4}{2} - \frac{x^2}{2} \right] \cdot dx = \frac{1}{2} \int_{x=0}^{1} (x^5 - x^3) \cdot dx$$

$$= \frac{1}{2} \cdot \left(\frac{x^6}{6} - \frac{x^4}{4} \right)_0^1 = \frac{1}{2}\left[\frac{1}{6} - \frac{1}{4} \right] = \frac{1}{2}\left(\frac{4-6}{24} \right) = \frac{-1}{24}.$$

③ Evaluate $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dx\,dy}{1+x^2+y^2}$

Sol: $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{1}{1+x^2+y^2} \cdot dx\,dy = \int_{x=0}^{1} \int_{y=0}^{\sqrt{1+x^2}} \frac{1}{1+x^2+y^2} \cdot dx\,dy$

Since y limits depends on x, the order of integration is
1) First w.r.t y, keeping x as constant.
2) Second w.r.t x b/w the limits.

$$= \int_{x=0}^{1} \left[\int_{y=0}^{\sqrt{1+x^2}} \frac{1}{1+x^2+y^2} \cdot dy \right] \cdot dx$$

$\int \frac{1}{a^2+x^2} \cdot dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right)$

$\int \frac{1}{\sqrt{1+x^2}} \cdot dx = \log(x+\sqrt{1+x^2})$

Keeping x as constant

$$= \int_{x=0}^{1} \left[\int_{y=0}^{\sqrt{1+x^2}} \frac{1}{(\sqrt{1+x^2})^2+y^2} \cdot dy \right] \cdot dx$$

$$= \int_{x=0}^{1} \left\{ \frac{1}{\sqrt{1+x^2}} \operatorname{Tan}^{-1}\left(\frac{y}{\sqrt{1+x^2}}\right) \right\}_0^{\sqrt{1+x^2}} \cdot dx$$

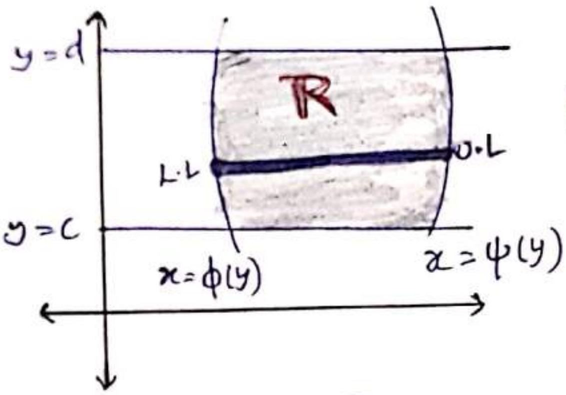
$$= \int_{x=0}^{1} \frac{1}{\sqrt{1+x^2}} \cdot \left[\operatorname{Tan}^{-1}\left(\frac{\sqrt{1+x^2}}{\sqrt{1+x^2}}\right) - \operatorname{Tan}^{-1}(0) \right] \cdot dx$$

$$= \int_{x=0}^{1} \frac{1}{\sqrt{1+x^2}} \left[\operatorname{Tan}^{-1}(1) - \operatorname{Tan}^{-1}(0) \right] \cdot dx = \int_{x=0}^{1} \frac{1}{\sqrt{1+x^2}} \left(\frac{\pi}{4} - 0\right) \cdot dx$$

$$= \frac{\pi}{4} \cdot \int_{x=0}^{1} \frac{1}{\sqrt{1+x^2}} \cdot dx = \frac{\pi}{4} \cdot \left\{ \log\left|x+\sqrt{1+x^2}\right| \right\}_0^1$$

$$= \frac{\pi}{4} \log(1+\sqrt{2})$$

Case (ii) when the Region R is bounded by two continuous curves $x = \phi(y)$ and $x = \psi(y)$ and the two lines $y = a$ & $y = b$.



$$R = \{(x,y) \mid \phi(y) \le x \le \phi(y),\ c \le y \le d\}$$

In This case, First Integrate w.r.t x, keeping y as constant and then the Resulting Integral is integrated w.r.t y within the limits $y = c$ and $y = d$.

$$\iint_R f(x,y) \cdot dx\,dy = \int_{y=c}^{y=d} \int_{x=\phi(y)}^{\psi(y)} f(x,y) \cdot dx\,dy$$

$$= \int_{y=c}^{y=d} \left[\int_{x=\phi(y)}^{\psi(y)} f(x,y) \cdot dx \right] \cdot dy$$

Keeping y as constant.

Here The order of Integration is

1) First Integrate w.r.t x, keeping y as constant
2) second Integrate w.r.t y b/w limits $y = c$ & $y = d$.

(1) Evaluate $\int_1^{\ln 8}\int_0^{\ln y} e^{x+y}\cdot dx\,dy$

Sol $\int_1^{\ln 8}\int_0^{\ln y} e^{x+y}\cdot dx\,dy = \int_{y=\phi}^{\ln 8}\int_{x=0}^{x=\ln y} e^{x}\cdot e^{y}\,dx\,dy$

Since the inner limits depends on y, First Integrate w.r.t x keeping y as constant, and the Integrate w.r.t y b/w limits

$$= \int_{y=\phi}^{\ln 8}\left[\int_{x=0}^{x=\ln y} e^{x}\cdot e^{y}\cdot dx\right]\cdot dy$$

Keeping y as constant.

$$= \int_{y=\phi}^{\ln 8} e^{y}\cdot\left(e^{x}\right)_0^{\ln y}\cdot dy = \int_{y=\phi}^{\ln 8} e^{y}\cdot\left(e^{\ln y} - e^{0}\right)\cdot dy$$

$$= \int_{y=\phi}^{\ln 8} \underset{v}{\underline{e^{y}}}\,(\underset{u}{\underline{y-1}})\cdot dy$$

$v_1 \to \int v$
$v_2 \to \int v_1$

$$\left[(y-1)\cdot e^{y} - e^{y}\right]_{\phi}^{\ln 8} \qquad \int uv = u\cdot v_1 - u'v_2 + u''v_3 - \cdots$$

$$\left[(\ln 8 - 1)e^{\ln 8} - e^{\ln 8} - \{(0 - e\}\right]$$

$= 8(\ln 8 - 1) - 8 + e$

$= 8\ln 8 - 8 - 8 + e$

$= \underline{8\ln 8 - 16 + e}$

Problems: (2) $\int_0^1 \int_0^{\sqrt{\frac{1-y^2}{2}}} \frac{dx\,dy}{\sqrt{1-x^2-y^2}}$

Sol $\int_0^1 \int_0^{\sqrt{\frac{1-y^2}{2}}} \frac{dx\,dy}{\sqrt{1-x^2-y^2}} = \int_{y=0}^{1} \int_{x=0}^{x=\sqrt{\frac{1-y^2}{2}}} \frac{dx\,dy}{\sqrt{1-x^2-y^2}}$

Since inner limits depends on y, the function is integrate first w.r.t x keeping y as constant and then w.r.t y.

$$= \int_{y=0}^{1} \left[\int_{x=0}^{\sqrt{\frac{1-y^2}{2}}} \frac{1}{\sqrt{1-x^2-y^2}} \cdot dx \right] \cdot dy \qquad \int \frac{1}{\sqrt{a^2-x^2}} \cdot dx = \sin^{-1}\left(\frac{x}{a}\right).$$

Keeping y as constant

$$= \int_{y=0}^{1} \left[\int_{x=0}^{\sqrt{\frac{1-y^2}{2}}} \frac{1}{\sqrt{\left(\sqrt{1-y^2}\right)^2 - x^2}} \cdot dx \right] \cdot dy$$

Keeping y as constant

$$= \int_{y=0}^{1} \left[\sin^{-1}\left(\frac{x}{\sqrt{1-y^2}} \right) \right]_0^{\sqrt{\frac{1-y^2}{2}}} \cdot dy$$

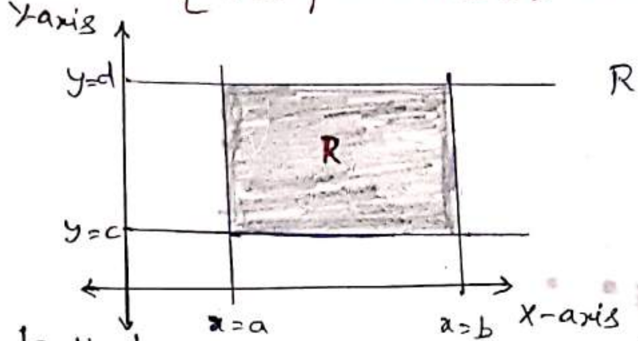
$$= \int_{y=0}^{1} \left[\sin^{-1}\left(\frac{\frac{\cancel{\sqrt{1-y^2}}}{2}}{\cancel{\sqrt{1-y^2}}} \right) - \sin^{-1}(0) \right] \cdot dy$$

$$= \int_{y=0}^{1} \left[\sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}(0) \right] \cdot dy = \int_{y=0}^{1} \frac{\pi}{6} \cdot dy$$

$$= \frac{\pi}{6} (y)_0^1 = \frac{\pi}{6}.$$

Case (iii) If all the Four limits are constants then the order of Integration can be done in either way ie Integration first w.r.t x and Later w.r.t y (or) Integration first w.r.t y and Later w.r.t x, yielding the same result., Provided the limits taken (for x & y)

$R = \{(x,y) \mid a \le x \le b;\ c \le y \le d\}$



R : Rectangular Region

$$\int_{x=a}^{b}\int_{y=c}^{y=d} f(x,y)\,dx\,dy = \int_{x=a}^{b}\left[\int_{y=c}^{y=d} f(x,y)\cdot dy\right]\cdot dx$$

Keeping x as constant

$$= \int_{y=c}^{y=d}\left[\int_{x=a}^{x=b} f(x,y)\cdot dx\right]\cdot dy$$

Keeping y as constant.

Problems $\displaystyle\int_0^1\int_0^1 \frac{dx\,dy}{\sqrt{(1-x^2)(1-y^2)}}$

Sol. $$\int_0^1\int_0^1 \frac{dx\,dy}{\sqrt{(1-x^2)(1-y^2)}} = \int_{y=0}^{1}\int_{x=0}^{1}\frac{1}{\sqrt{1-x^2}\cdot\sqrt{1-y^2}}\cdot dx\,dy$$

$$= \int_{y=0}^{1}\left[\int_{x=0}^{1}\frac{1}{\sqrt{1-x^2}\,\sqrt{1-y^2}}\cdot dx\right]\cdot dy$$

Keeping y as constant

$$= \int_{y=0}^{1}\frac{1}{\sqrt{1-y^2}}\cdot\left[\sin^{-1}(x)\right]_0^1\cdot dy = \int_{y=0}^{1}\frac{1}{\sqrt{1-y^2}}\left[\sin^{-1}(1)-\sin^{-1}(0)\right]dy$$

$$= \int_{y=0}^{1}\frac{1}{\sqrt{1-y^2}}\left(\frac{\pi}{2}-0\right)dy = \frac{\pi}{2}\cdot\left[\sin^{-1}(y)\right]_0^1 = \frac{\pi}{2}\left[\sin^{-1}(1)-\sin^{-1}(0)\right] = \frac{\pi}{2}\cdot\frac{\pi}{2} = \pi^2/4$$

(2) Evaluate $\int_0^3 \int_0^1 (x^2+3y^2)\, dy\, dx$

$$\int_0^3 \int_0^1 (x^2+3y^2)\cdot dy\, dx = \int_{x=0}^{3} \int_{y=0}^{1} (x^2+3y^2)\, dy\, dx$$

$$\int_{x=0}^{3} \left[\int_{y=0}^{1} (x^2+3y^2)\cdot dy \right] \cdot dx$$

Keeping x as constant

$$= \int_{x=0}^{3} \left(x^2\cdot y + \cancel{3}\cdot\frac{y^3}{\cancel{3}}\right)_0^1 \cdot dx = \int_{x=0}^{3} (x^2 y + y^3)_0^1 \cdot dx$$

$$= \int_{x=0}^{3} (x^2+1)\cdot dx = \left(\frac{x^3}{3}+x\right)_0^3 = \frac{3^3}{3}+3 = \underline{12}$$

(or)

$$\int_{y=0}^{1} \left[\int_{x=0}^{3} (x^2+3y^2)\cdot dx \right] \cdot dy$$

Keeping y as constants

$$= \int_{y=0}^{1} \left(\frac{x^3}{3} + 3y^2\cdot x\right)_0^3 \cdot dy = \int_{y=0}^{1} \left(\frac{3^3}{3} + 3y^2\cdot 3\right)\cdot dy$$

$$= \int_{y=0}^{1} (9+9y^2)\, dy = \left(9y + \overset{3}{\cancel{9}}\frac{y^3}{\cancel{3}}\right)_0^1$$

$$= (9y+3y^3)_0^1 = 9+3 = \underline{12}$$

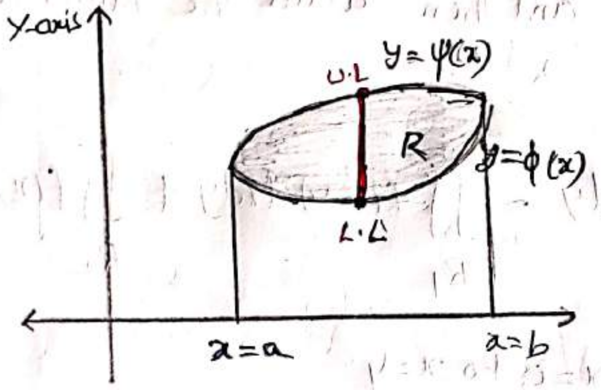
__Note__ When All 4 Limits are constants, order of Integration is immaterial, Provided limits for x & y are fixed.

Evaluation of Double integral when limits ~~of~~ of integration are not given:

If the limits of integration are not given, firstly

① Sketch the Region R, by using the curves given.

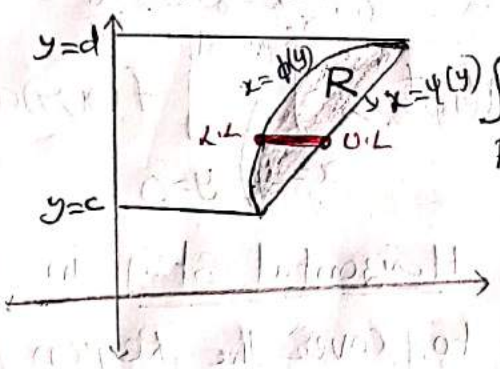
② If we fix x limits, Take a vertical strip In the given Region R, The lower end of the strip gives Lower limit for y and upper end gives upper Limit for y.



$$\iint_R f(x,y)\cdot dx\,dy$$

$$= \int_{x=a}^{b} \int_{y=\phi(x)}^{y=\psi(x)} f(x,y)\cdot dx\,dy.$$

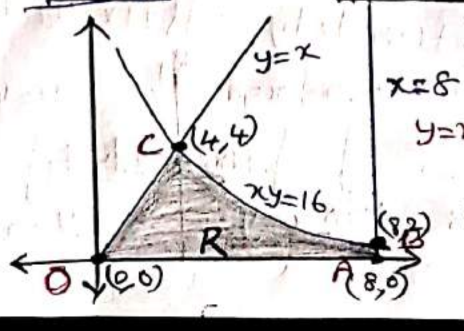
③ If we fix y limits, Take a Horizonal strip In the Region R, The Left side end gives Lower Limit for x and Right side end gives ~~Right~~ Upper limit for x



$$\iint_R f(x,y)\cdot dx\,dy$$

$$= \int_{y=c}^{y=d} \int_{x=\phi(y)}^{\psi(y)} f(x,y)\,dx\,dy.$$

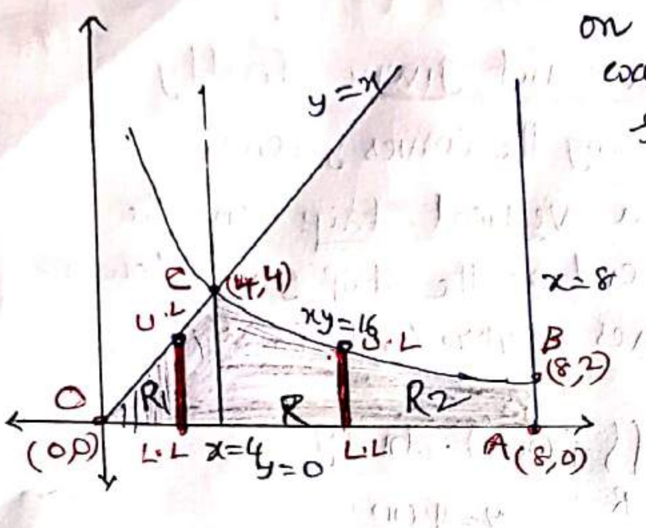
Note: If one end of the strip is varing on two (or) more curves then the Region needs to be divided into Sub Regions



R: Region in First-Quadrant $x=8$ bounded by the curves $y=x$, $xy=16$, $y=0$ & $x=8$.

For this Region R, To write limits of integration

(1) If we fix x limits, Take a vertical strip allow to slide on x-axis to cover the Region. It was observed that one end of the strip varing on two curves when strips moves along x-axis from $x=0$ to $x=8$. So we must Divide the Region R into Sub Regions and then write the limits of integration



$$\iint_R f(x,y)\,dx\,dy = \iint_{R_1 \cup R_2} f(x,y)\,dx\,dy = \iint_{R_1} f(x,y)\,dx\,dy + \iint_{R_2} f(x,y)\,dx\,dy$$

In R_1: x varies from $x=0$ to $x=4$

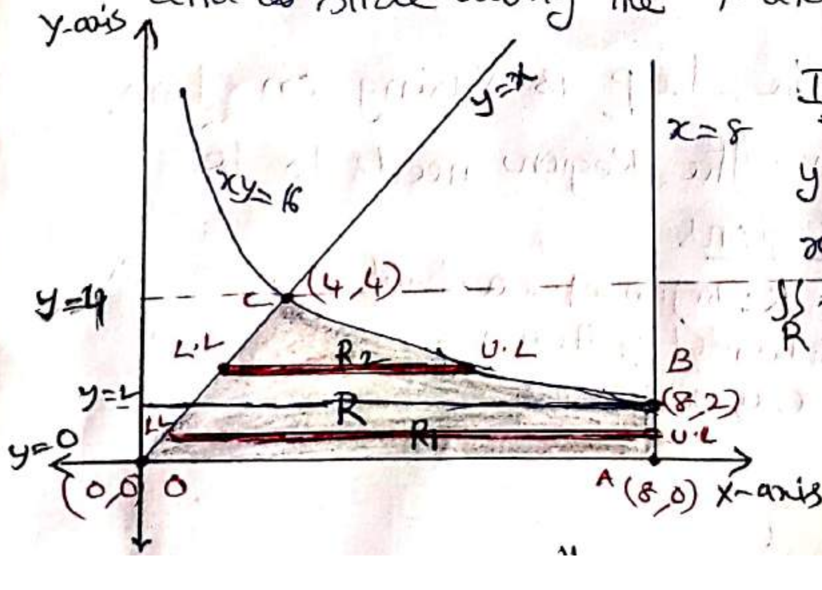
y varies from $y=0$ to $y=x$

In R_2: x varies from $x=4$ to $x=8$

y varies from $y=0$ to $y=\frac{16}{x}$.

$$\iint_R f(x,y)\,dx\,dy = \int_{x=0}^{4}\int_{y=0}^{y=x} f(x,y)\,dy\,dx + \int_{x=4}^{x=8}\int_{y=0}^{y=\frac{16}{x}} f(x,y)\,dy\,dx$$

(2) If we fix y limits, Take a Horizontal strip in R and slide along the Y-axis to cover the Region R.



In R:

y varies from $y=0$ to $y=4$

x varies from

$$\iint_R f(x,y)\,dx\,dy = \iint_{R_1} f(x,y)\,dx\,dy + \iint_{R_2} f(x,y)\,dx\,dy$$

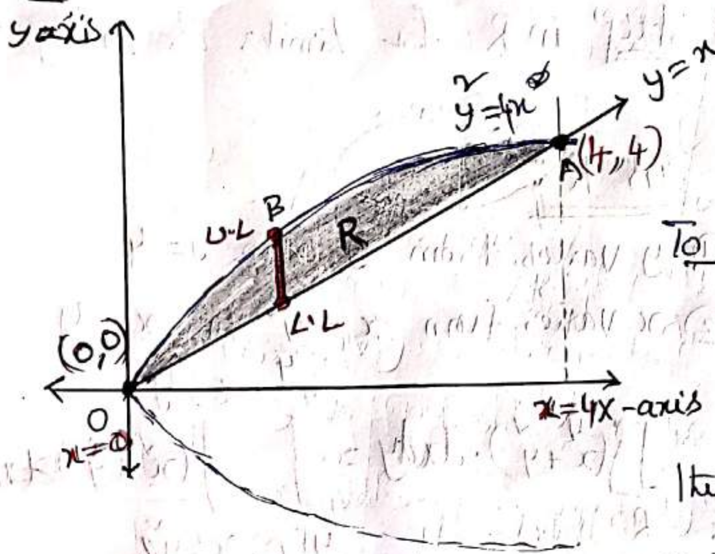
$$= \int_{y=0}^{y=2}\int_{x=y}^{x=8} f(x,y)\,dx\,dy + \int_{y=2}^{y=4}\int_{x=y}^{x=\frac{16}{y}} f(x,y)\,dx\,dy$$

Problems on Regions:

① Sketch the given Region
② Find the limits of integration for Region R and then Evaluate the Double integral.

① Evaluate $\iint_R (x^2+y^2)\,dx\,dy$, where R is Region bounded by $y=x$ and $y^2=4x$.

Sol Given R: Bounded by $y=x$ & $y^2=4x$.



By solving $y^2=4x$ & $y=x$

$y^2=4y \Rightarrow y^2-4y=0$
$\Rightarrow y(y-4)=0$
$\Rightarrow y=0,\ y=4$

Pts (0,0); (4,4)

To Evaluate

$$\iint_R (x^2+y^2)\,dx\,dy$$

The limits of integration for the Region R is given by.

(a) If we fix x limits, Take vertical strip in R, for y limits

In R: x varies from $x=0$ to $x=4$
y varies from $y=x$ to $y=\sqrt{4x}=2\sqrt{x}$

$$\iint_R (x^2+y^2)\cdot dx\,dy = \int_{x=0}^{4}\int_{y=x}^{2\sqrt{x}}(x^2+y^2)\cdot dx\,dy$$

order of integration
① First w.r.t y, keeping x as constant
② second w.r.t x b/w limits

$$= \int_{x=0}^{4}\left[\int_{y=x}^{2\sqrt{x}}(x^2+y^2)\cdot dy\right]\cdot dx$$

keeping x as constant.

$$= \int_{x=0}^{4}\left(x^2\cdot y+\frac{y^3}{3}\right)_{x}^{2\sqrt{x}}\cdot dx = \int_{x=0}^{4}\left[\left(x^2\cdot 2\sqrt{x}+\frac{(2\sqrt{x})^3}{3}\right)-\left(x^2\cdot x+\frac{x^3}{3}\right)\right]dx$$

$$= \int_{x=0}^{4}\left(2\cdot x^{\frac{5}{2}}+\frac{8}{3}\cdot x^{\frac{3}{2}}-\frac{4}{3}x^3\right)\cdot dx = \left[2\cdot\frac{x^{\frac{5}{2}+1}}{\frac{5}{2}+1}+\frac{8}{3}\cdot\frac{x^{\frac{3}{2}+1}}{\left(\frac{3}{2}+1\right)}-\frac{4}{3}\frac{x^4}{4}\right]_0^4$$

$$\left(\frac{4}{7}x^{7/2}+\frac{16}{15}x^{5/2}-\frac{1}{3}x^4\right)_0^4 = \left(\frac{4}{7}\cdot 4^{7/2}+\frac{16}{15}4^{5/2}-\frac{1}{3}\cdot 4^4\right)$$ (13)

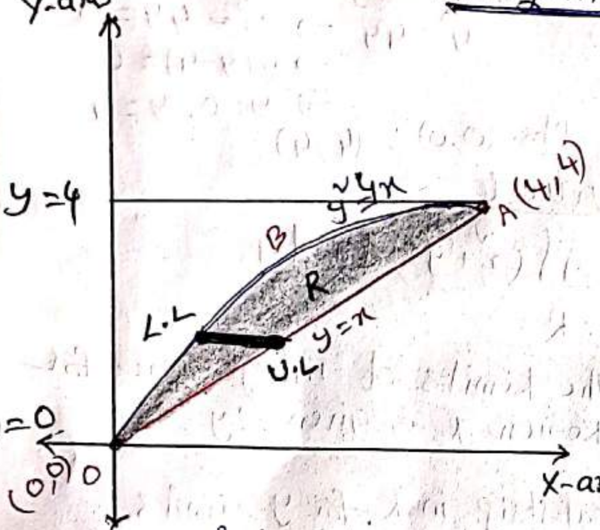
$$= \frac{4}{7}\cdot 2^7+\frac{16}{15}2^5 - \frac{256}{3} = \frac{512}{7}+\frac{512}{15}-\frac{256}{3} = \frac{7680+3584-8960}{105}$$

$3\,|\,7,15,3$; $7,5,1,1$; $3\times5\times7 = 105$

$$= \frac{2304}{105} = \frac{768}{35}$$

(d)

For the same Region in Prob (1), we we fix y Limits then take Horizontal strip in R for Limits of Integration



In R:

(1) y varies from $y=0$ to $y=4$

(2) x varies from $x=\frac{y^2}{4}$ to $x=y$

$$\iint_R (x^2+y^2)\cdot dx\,dy = \int_{y=0}^{4}\int_{x=y^2/4}^{x=y}(x^2+y^2)\,dx\,dy$$

Order of Integration: (1) First integrate w.r.t x, keeping y as constant
(2) Second w.r.t y b/w the Limits.

$$\iint_R (x^2+y^2)\,dx\,dy = \int_{y=0}^{4}\left[\int_{x=y^2/4}^{x=y}(x^2+y^2)\,dx\right]\cdot dy$$

keeping y as constant

$$=\int_{y=0}^{4}\left(\frac{x^3}{3}+y^2\cdot x\right)_{\frac{y^2}{4}}^{y}\cdot dy = \int_{y=0}^{4}\left\{\left(\frac{y^3}{3}+y^2\cdot y\right)-\left(\frac{(y^2/4)^3}{3}+y^2\cdot\frac{y^2}{4}\right)\right\}\cdot dy$$

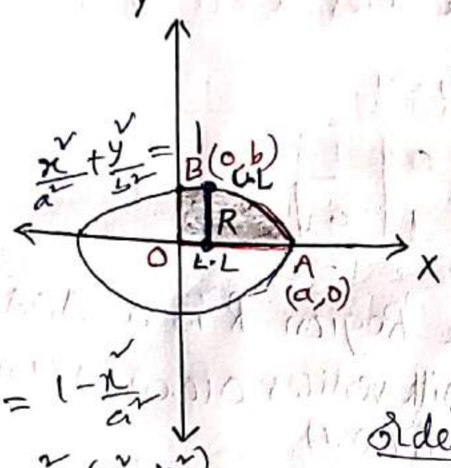
$$=\int_{y=0}^{4}\left(\frac{4}{3}\cdot y^3-\frac{y^6}{192}-\frac{y^4}{4}\right)\cdot dy = \left(\frac{4}{3}\cdot\frac{y^4}{4}-\frac{1}{192}\cdot\frac{y^7}{7}-\frac{y^5}{20}\right)_0^4$$

$$=\frac{1}{3}\cdot 4^4-\frac{1}{192}\cdot\frac{4^7}{7}-\frac{4^5}{5} = \frac{256}{3}-\frac{16384}{1344}-\frac{1024}{20}$$

$3\,|\,3,21,5$; $1,7,5$; $21,\,42$

$$= \frac{~~\ldots~~}{105} = \frac{768}{35}$$

② Evaluate $\iint_R x^3 y \, dx\,dy$, where R is the Region enclosed by the Ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in the First Quadrant.



$\frac{y^2}{b^2} = 1-\frac{x^2}{a^2}$

$y^2 = \frac{b^2}{a^2}(a^2-x^2)$

$y = \frac{b}{a}\sqrt{a^2-x^2}$

In R: x varies from $x=0$ to $x=a$

y varies from $y=0$ to $y=\frac{b}{a}\sqrt{a^2-x^2}$

$$\iint_R x^3 y \, dx\,dy = \int_{x=0}^{x=a}\int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} x^3 y \, dx\,dy$$

Order of Integration

① First w.r.t y keeping x as constant

② second w.r.t x, b/w limits.

$$\therefore \iint_R x^3 y\,dx\,dy = \int_{x=0}^{a}\left[\int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} x^3 \cdot y \cdot dy\right] \cdot dx$$

keeping x as constant.

$$= \int_{x=0}^{a} x^3 \cdot \left(\frac{y^2}{2}\right)_0^{\frac{b}{a}\sqrt{a^2-x^2}} \cdot dx$$

$$= \frac{1}{2}\int_{x=0}^{a} x^3 \left[\left(\frac{b}{a}\sqrt{a^2-x^2}\right)^2 - 0\right] \cdot dx$$

$$= \frac{1}{2}\int_{x=0}^{a} x^3 \cdot \frac{b^2}{a^2}(a^2-x^2)\,dx$$

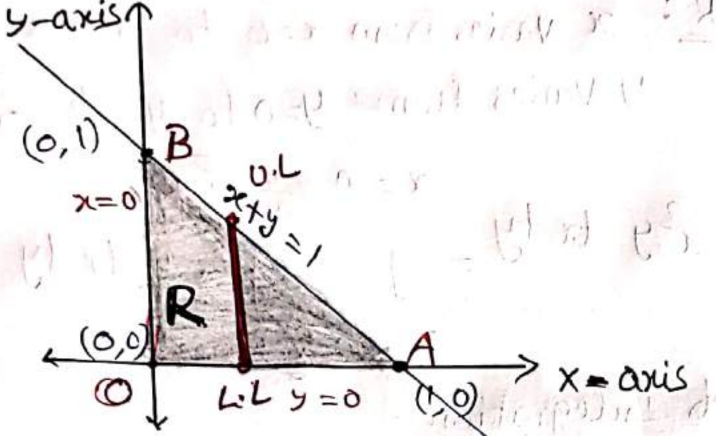
$$= \frac{b^2}{2a^2}\int_{x=0}^{a}(a^2x^3-x^5)\,dx = \frac{b^2}{2a^2}\left(a^2\cdot\frac{x^4}{4}-\frac{x^6}{6}\right)_0^a$$

$$= \frac{b^2}{2a^2}\left(a^2\cdot\frac{a^4}{4}-\frac{a^6}{6}\right) = \frac{b^2}{2a^2}\cdot a^6\left(\frac{1}{4}-\frac{1}{6}\right) = \frac{b^2}{2a^2}\cdot\frac{a^6}{12}$$

$$= \frac{a^4b^2}{24}$$

③ Evaluate $\iint e^{2x+3y}\,dx\,dy$, over the Region bounded by the Lines $x=0$, $y=0$ and $x+y=1$

Sol: The Region R is bounded by the Lines $x=0$; $y=0$; $x+y=1$



$x+y=1$

x	0	1
y	1	0

The Region R is a ΔOAB with vertices $O(0,0)$; $A(1,0)$ & $B(0,1)$

In R: ① x varies from $x=0$ to $x=1$

② y varies from $y=0$ to $y=1-x$.

$$\iint_R e^{2x+3y}\cdot dx\,dy = \int_{x=0}^{1}\int_{y=0}^{1-x} e^{2x}\cdot e^{3y}\,dx\,dy$$

Order of Integration: ① First integrate w.r.t y, keeping x as constant

② Second integrate w.r.t x b/w Limits

$$= \int_{x=0}^{1}\left[\underbrace{\int_{y=0}^{1-x} e^{2x}\cdot e^{3y}\cdot dy}_{\text{Keeping } x \text{ as constant.}}\right]\cdot dx \qquad \int e^{ax}\cdot dx = \frac{e^{ax}}{a}$$

$$= \int_{x=0}^{1} e^{2x}\cdot\left(\frac{e^{3y}}{3}\right)_0^{1-x}\cdot dx = \int_{x=0}^{1} e^{2x}\left[\frac{e^{3(1-x)}}{3} - \frac{1}{3}\right]\cdot dx$$

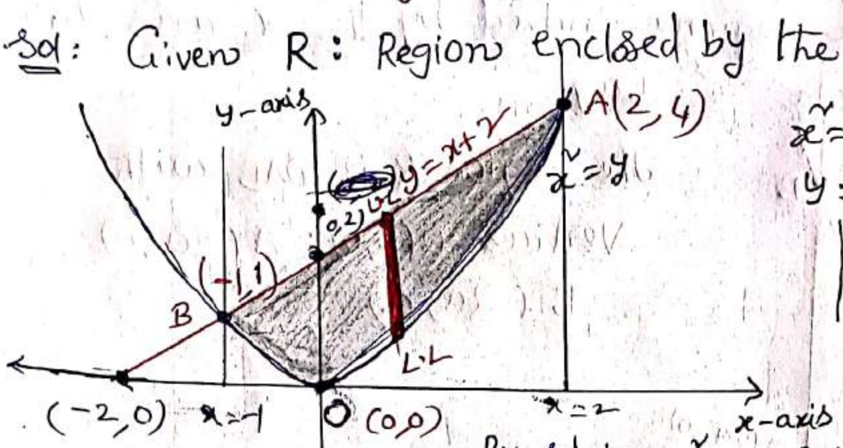
$$= \frac{1}{3}\cdot\int_{x=0}^{1}\left(e^{2x+3-3x} - e^{2x}\right)\cdot dx = \frac{1}{3}\int_{x=0}^{1}\left(e^{-x+3} - e^{2x}\right)\cdot dx$$

$$= \frac{1}{3}\cdot\left(\frac{e^{-x+3}}{-1} - \frac{e^{2x}}{2}\right)_0^1 = \frac{1}{3}\left[-e^{-1+3} - \frac{e^2}{2} - \left\{-e^3 - \frac{1}{2}\right\}\right]$$

$$= \frac{1}{3}\left[-e^2 - \frac{e^2}{2} + e^3 + \frac{1}{2}\right] = \frac{1}{3}\left[-\frac{3}{2}e^2 + e^3 + \frac{1}{2}\right] = -\frac{1}{2}e^2 + \frac{1}{3}e^3 + \frac{1}{6}$$

(4) Evaluate $\iint_R y \cdot dx\,dy$ over the region enclosed by the Parabola $x^2 = y$ and the line $y = x+2$

Sol: Given R : Region enclosed by the curves $x^2 = y$ & $y = x+2$



$x^2 = y$ is Parabola sym to y-axis

$y = x+2$ is a st. line

x	0	-2
y	2	0

By solving $x^2 = y$ & $y = x+2$

$x^2 = x+2 \Rightarrow x^2 - x - 2 = 0 \Rightarrow x^2 - 2x + x - 2 = 0$

$x(x-2) + 1(x-2) = 0 \Rightarrow (x-2)(x+1) = 0 \Rightarrow x = 2$ (or) $x = -1$

for $x = 2$, $y = 4 \Rightarrow (x,y) = (2,4)$

$x = -1$, $y = 1 \Rightarrow (x,y) = (-1,1)$

In R : x varies from $x = -1$ to $x = 2$

y varies from $y = x^2$ to $y = x+2$

$$\iint_R y \cdot dx\,dy = \int_{x=-1}^{2} \int_{y=x^2}^{x+2} y \cdot dx\,dy$$

The order of Integration

(1) first w.r.t y, keeping x as const

(2) second w.r.t x b/w Limits.

$$= \int_{x=-1}^{2} \left[\int_{y=x^2}^{x+2} y \cdot dy \right] \cdot dx = \int_{x=-1}^{2} \left(\frac{y^2}{2} \right)_{x^2}^{x+2} \cdot dx$$

Keeping y as Constant

$$= \frac{1}{2} \int_{x=-1}^{2} \left[(x+2)^2 - (x^2)^2 \right] dx = \frac{1}{2} \int_{x=-1}^{2} (x^2 + 4 + 4x - x^4) \cdot dx$$

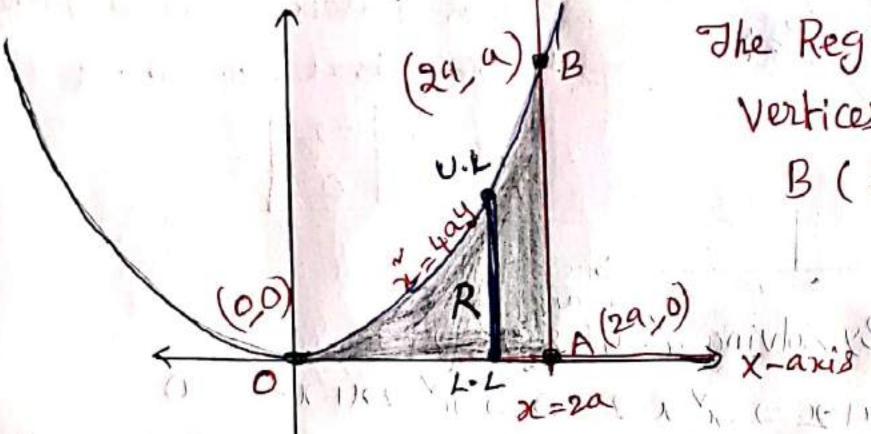
$$= \int_{x=-1}^{2} (-x^4 + x^2 + 4x + 4) \cdot dx = \left(\frac{-x^5}{5} + \frac{x^3}{3} + 4 \cdot \frac{x^2}{2} + 4x \right)_{-1}^{2}$$

$$= \left\{ -\frac{2^5}{5} + \frac{2^3}{3} + 2(2^2) + 4(2) \right\} - \left\{ -\frac{(-1)^5}{5} + \frac{(-1)^3}{3} + 2(-1)^2 + 4(-1) \right\}$$

$= 36/5$ Ans

⑤ Evaluate $\iint_R xy\,dx\,dy$ over the Region enclosed by the X-axis the line $x=2a$ and the Parabola $x^2=4ay$.

Sol: Given R is the Region enclosed by the X-axis, Line $x=2a$ and the parabola $x^2=4ay$.



The Region R is OAB with vertices O(0,0); A(2a,0) B(2a,a)

Solving $x^2=4ay$ & $x=2a$ we get $(2a)^2=4ay \Rightarrow 4a^2=4ay \Rightarrow \boxed{y=a}$

In R: x varies from $x=0$ to $x=2a$

y varies from $y=0$ to $y=\frac{x^2}{4a}$

$$\iint_R xy\,dx\,dy = \int_{x=0}^{2a}\int_{y=0}^{\frac{x^2}{4a}} xy\,dx\,dy$$

The order of integration

① First w.r.t y, keeping x as constant

② second w.r.t x b/w Limits

$$= \int_{x=0}^{2a}\left[\int_{y=0}^{\frac{x^2}{4a}} xy\cdot dy\right]\cdot dx$$

Keeping x as Constant

$$= \int_{x=0}^{2a} x\cdot\left(\frac{y^2}{2}\right)_0^{\frac{x^2}{4a}}\cdot dx = \frac{1}{2}\int_{x=0}^{2a} x\cdot\left[\left(\frac{x^2}{4a}\right)^2-0\right]\cdot dx$$

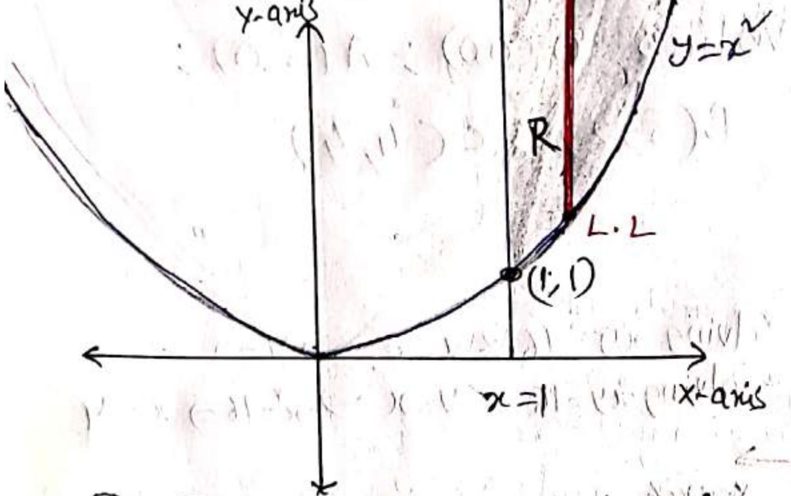
$$= \frac{1}{2}\int_{x=0}^{2a} x\cdot\frac{x^4}{16a^2}\cdot dx = \frac{1}{32a^2}\int_{x=0}^{2a} x^5\cdot dx = \frac{1}{32a^2}\left(\frac{x^6}{6}\right)_0^{2a}$$

$$= \frac{1}{32a^2}\cdot\frac{(2a)^6}{6} = \frac{\cancel{64}^{2}\cdot a^6}{\cancel{32}\times\cancel{6}_{3}\cdot a^2} = \frac{a^4}{3}$$

(6) Evaluate $\iint_R \frac{dx\,dy}{x^4+y^2}$, over the Region bounded by the $y \geq x^2$, $x \geq 1$.

Sol Given Region R is bounded by $y \geq x^2$; $x \geq 1$.

$y \geq x^2$ is the portion inside parabola



In Region R: x varies from $x=1$ to $x=\infty$

y varies from $y=x^2$ to $y=\infty$

$$\iint_R \frac{dx\,dy}{x^4+y^2} = \int_{x=1}^{\infty}\int_{y=x^2}^{\infty} \frac{1}{x^4+y^2}\cdot dx\,dy$$

The order of integration (1) First w.r.t y, keeping x as constant

(2) second w.r.t x, b/w the limits.

$$\iint_R \frac{dx\,dy}{x^4+y^2} = \int_{x=1}^{\infty}\left[\int_{y=x^2}^{\infty} \frac{1}{x^4+y^2}\cdot dy\right]\cdot dx$$

$$\boxed{\int \frac{1}{a^2+x^2}\cdot dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)}$$

Keeping x as Constant

$$= \int_{x=1}^{\infty}\left[\int_{y=x^2}^{\infty} \frac{1}{(x^2)^2+y^2}\cdot dy\right].dx$$

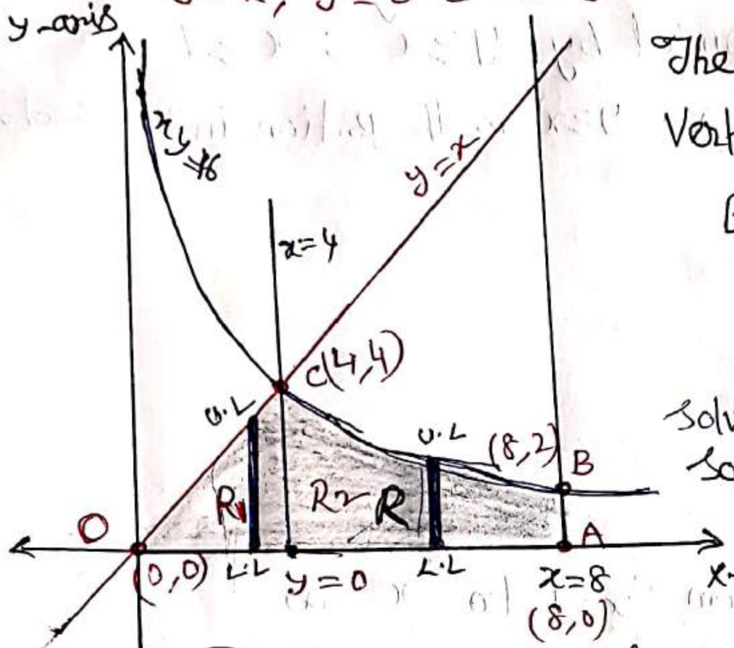
keeping x as constant

$$= \int_{x=1}^{\infty} \frac{1}{x^2}\cdot\left[\tan^{-1}\left(\frac{y}{x^2}\right)\right]_{x^2}^{\infty}\cdot dx = \int_{x=1}^{\infty} \frac{1}{x^2}\left[\tan^{-1}(\infty)-\tan^{-1}(1)\right]\cdot dx$$

$$= \int_{x=1}^{\infty} \frac{1}{x^2}\left(\frac{\pi}{2}-\frac{\pi}{4}\right)\cdot dx = \frac{\pi}{4}\cdot\int_{x=1}^{\infty} x^{-2}\cdot dx = \frac{\pi}{4}\left(\frac{-1}{x}\right)_1^{\infty}$$

$$= \frac{\pi}{4}\left[0-(-1)\right] = \frac{\pi}{4}$$

(7) Evaluate $\iint_R x^2\,dx\,dy$, over the Region in the First Quadrant enclosed by the Rectangular hyperbola $xy=16$, the lines $y=x$, $y=0$ & $x=8$.



The Region R is $OABC$ with Vertices $O(0,0)$; $A(8,0)$; $B(8,2)$; $C(4,4)$

Solving $xy=16$ & $x=8$, $\Rightarrow y=2$

Solving $xy=16$ & $y=x$ $\Rightarrow x^2=16 \Rightarrow x=4$

In R: x varies from $x=0$ to $x=8$, Take a vertical strip slide along x-axis from $x=0$ to $x=8$, we observe that one end of the strip varies on Two different Curves so the Region R must be divided into Sub Regions R_1 & R_2

Here $R = R_1 \cup R_2$

$$\iint_R x^2\,dx\,dy = \iint_{R_1} x^2\,dx\,dy + \iint_{R_2} x^2\,dx\,dy$$

In R_1 : x varies from $x=0$ to $x=4$

y varies from $y=0$ to $y=x$

In R_2 : x varies from $x=4$ to $x=8$

y varies from $y=0$ to $y=\frac{16}{x}$

$$\iint_R x^2\,dx\,dy = \int_{x=0}^{4}\int_{y=0}^{x} x^2\,dx\,dy + \int_{x=4}^{x=8}\int_{y=0}^{\frac{16}{x}} x^2\,dx\,dy$$

(20)

ⓐ $\iint_{R_1} x^2 \cdot dx\,dy = \int_{x=0}^{4} \int_{y=0}^{y=x} x^2 \cdot dx\,dy$

order of integration
① First w.r.t y, keeping x as constant
② second w.r.t x b/w the limits

$= \int_{x=0}^{4} \left[\int_{y=0}^{x} x^2 \cdot dy \right] \cdot dx = \int_{x=0}^{4} x^2 \cdot (y)_0^x \cdot dx$

keeping x as constant

$= \int_{x=0}^{4} x^2 (x-0) \cdot dx = \int_{x=0}^{4} x^3 \cdot dx = \left(\frac{x^4}{4} \right)_0^4 = \frac{4^4}{4} = \underline{\underline{64}}$

$\therefore \iint_{R_1} x^2 \cdot dx\,dy = 64$

ⓑ $\iint_{R_2} x^2 \cdot dx\,dy = \int_{x=4}^{x=8} \int_{y=0}^{y=\frac{16}{x}} x^2 \cdot dx\,dy$

order of integration
① First w.r.t y, keeping x as constant
② Second w.r.t x b/w limits

$= \int_{x=4}^{8} \left[\int_{y=0}^{\frac{16}{x}} x^2 \cdot dy \right] \cdot dx = \int_{x=4}^{8} x^2 \cdot (y)_0^{\frac{16}{x}} \cdot dx$

keeping x as constant

$= \int_{x=4}^{8} x^2 \left(\frac{16}{x} - 0 \right) \cdot dx = \int_{x=4}^{8} 16x \cdot dx = 16 \cdot \left(\frac{x^2}{2} \right)_4^8$

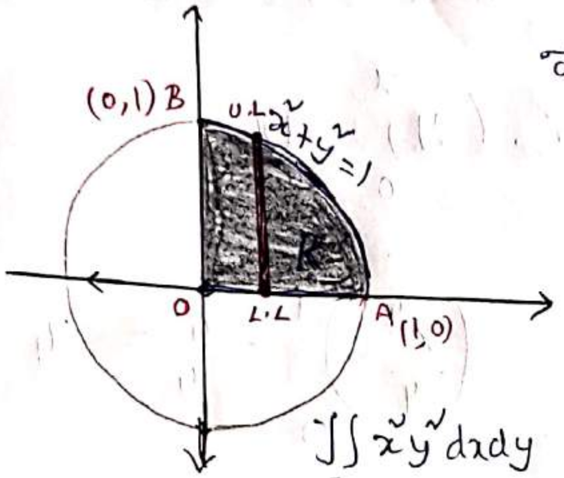
$= 8 \cdot (8^2 - 4^2) = 8(64-16) = 8(48) = \underline{\underline{384}}$

$\iint_{R} x^2 \cdot dx\,dy = \iint_{R_1} x^2 \cdot dx\,dy + \iint_{R_2} x^2 dx\,dy$

$= 64 + 384$

$\iint_{R} x^2 dx\,dy = \underline{\underline{448}}$

(8) If R is the Region bounded by the Circle $x^2+y^2=1$ in 1^{st} Quadrant, then Evaluate $\iint_R x^2y^2\,dx\,dy$.



The Region R is OAB with vertices $O(0,0)$; $A(1,0)$ & $B(0,1)$

In R

x Varies from $x=0$ to $x=1$

y Varies from $y=0$ to $y=\sqrt{1-x^2}$

$$\iint_R x^2y^2\,dx\,dy = \int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} x^2y^2\,dx\,dy$$

The order of Integration: ① First w.r.t y, keeping x as Constant ② second w.r.t x, b/w the Limits.

$$= \int_{x=0}^{1}\left[\int_{y=0}^{\sqrt{1-x^2}} x^2y^2\,dy\right]dx = \int_{x=0}^{1} x^2\left(\frac{y^3}{3}\right)_0^{\sqrt{1-x^2}} dx$$

keeping x as constant.

$$= \frac{1}{3}\int_{x=0}^{1} x^2\left[\left(\sqrt{1-x^2}\right)^3-0\right]dx = \frac{1}{3}\int_{x=0}^{1} x^2(1-x^2)^{\frac{3}{2}}dx$$

Put $x^2=t \Rightarrow 2x\,dx = dt \Rightarrow dx = \frac{1}{2x}dt = \frac{1}{2\sqrt{t}}dt$

for $x=0,\ t=0$

$x=1\ \ t=1$

$$= \frac{1}{3}\int_{t=0}^{1} t\cdot(1-t)^{\frac{3}{2}}\cdot\frac{1}{2\sqrt{t}}\,dt = \frac{1}{6}\int_{t=0}^{1} t^{1-\frac{1}{2}}(1-t)^{3/2}\,dt$$

$$\boxed{B(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx}$$

$$= \frac{1}{6}\int_0^1 t^{\frac{1}{2}}(1-t)^{\frac{3}{2}}dt = \frac{1}{6}\int_0^1 t^{\frac{3}{2}-1}\cdot(1-t)^{\frac{5}{2}-1}dt$$

$m-1=\frac{1}{2}$, $m=\frac{3}{2}$, $n-1=\frac{3}{2}$

$B(m,n)=\frac{\Gamma m\cdot\Gamma n}{\Gamma m+n}$

$$= \frac{1}{6}\cdot B\left(\frac{3}{2},\frac{5}{2}\right) = \frac{1}{6}\,\frac{\Gamma\frac{3}{2}\cdot\Gamma\frac{5}{2}}{\Gamma\frac{3}{2}+\frac{5}{2}=\Gamma 4} = \frac{1}{6}\cdot\frac{\frac{1}{2}\cdot\Gamma\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\frac{1}{2}}{3!}$$

$$= \frac{1}{6}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\frac{\sqrt{\pi}\cdot\sqrt{\pi}}{6} = \frac{\pi}{96}$$

* Change of order of Integration (Change of Limits of Integration)

Sometimes It is difficult (or) unable to Evaluate the given double Integral with the given limits of Integration. In several cases, It can be Evaluated by changing the order of Integration ie by changing the Limits of Integration.

If In the given problem, x limits are fixed and y limits as functions in x, the order of Integration is First w.r.t y keeping x as constant, and second w.r.t x b/w the limits. In order to change the order of Integration we must fix y limits, By taking horizontal strip in R write limits for x as function y, then the order of Integration becomes First w.r.t x, keeping y as constant and then second w.r.t y b/w the limits vice versa.

Problem ① Evaluate the double integral $\iint_R e^{x^2}\cdot dxdy$, where the Region R is given by

R: $2y \le x \le 2$ and $0 \le y \le 1$.

Sol Given R: $2y \le x \le 2$ and $0 \le y \le 1$

$$\iint_R e^{x^2}\cdot dxdy = \int_{y=0}^{y=1}\int_{x=2y}^{x=2} e^{x^2}\cdot dx\,dy$$

order of integration

① First w.r.t x, keeping y as constant

② second w.r.t y b/w the limits

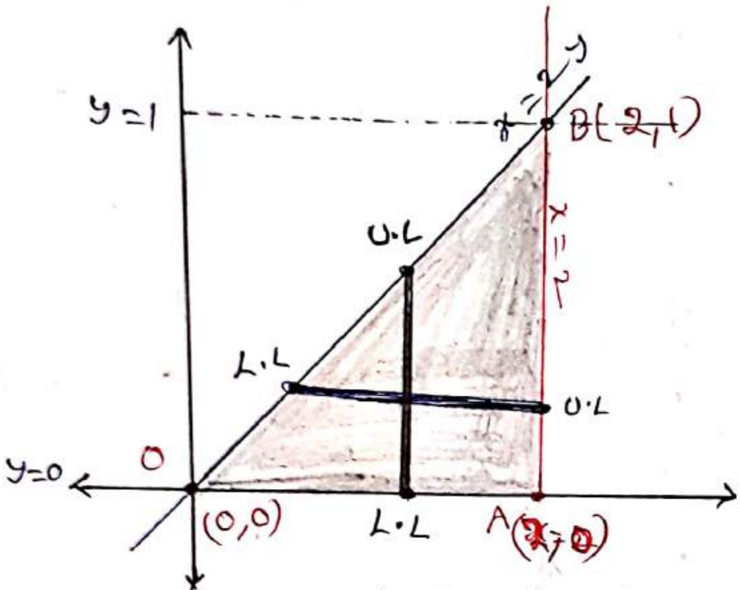
$$= \int_{y=0}^{1}\left[\int_{x=2y}^{x=2} e^{x^2}\cdot dx\right]\cdot dy$$

Keeping y as constant

Here It is not possible to Integrate the Inner Integral. So Integrating w.r.t x First is not confortable. So we wish change the order of Integration.

To change the order of Integration

* ① Draw the Region of Integration with the help of given limits of Integration
② Then By drawing vertical (or) Horizontal strip, we change the limits of Integration as per our requirment.



By solving $x=2y$ & $x=2 \Rightarrow y=1$

Here the Region of Integration R is OAB with vertices $O(0,0)$; $A(2,0)$; $B(2,1)$

In the given limits of integration y limits fixed, Horizontal strip was considered in R for the limits of x.

In order to change the order of integration, we must fix x limits, and consider veritical strip in R for the limits of y.

In R : x varies from $x=0$ to $x=2$

y varies from $y=0$ to $y=\frac{x}{2}$

$$\iint_R e^{x^2}\cdot dx\,dy = \int_{y=0}^{1}\int_{x=2y}^{x=2} e^{x^2}\cdot dx\,dy = \int_{x=0}^{x=2}\int_{y=0}^{y=\frac{x}{2}} e^{x^2}\cdot dx\,dy$$

$$= \int_{x=0}^{x=2}\left[\int_{y=0}^{\frac{x}{2}} e^{x^2}\cdot dy\right]\cdot dx$$

The order of integration

① First w.r.t y, keeping x as constant

② second w.r.t x b/w the limits

$$= \int_{x=0}^{2} e^{x^2} \cdot (y)_0^{\frac{x}{2}} \cdot dx$$

$$= \int_{x=0}^{2} e^{x^2} \cdot \frac{x}{2} \cdot dx = \frac{1}{2} \cdot \int_{x=0}^{2} e^{x^2} \cdot x \cdot dx$$

Put $x^2 = t \Rightarrow 2x\,dx = dt \Rightarrow x\,dx = \frac{1}{2} \cdot dt$

for $x = 0$; $t = 0$

$x = 2$; $t = 4$

$$= \frac{1}{2} \int_{t=0}^{4} e^{t} \cdot \frac{dt}{2} = \frac{1}{4} \cdot \left(e^{t}\right)_0^4 = \frac{1}{4}\left[e^4 - e^0\right]$$

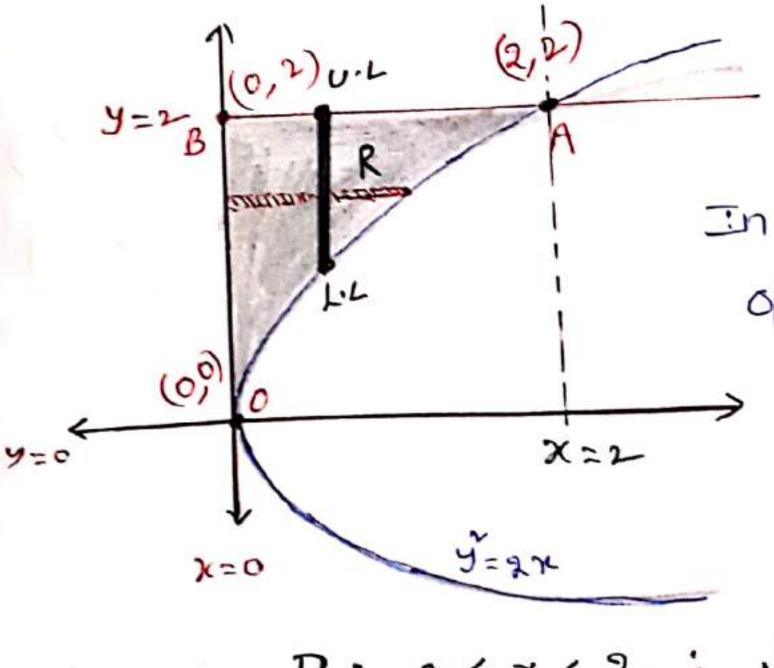
$$= \frac{1}{4} \cdot (e^4 - 1)$$

(2) Change the order of integration and Evaluate

$$\int_0^2 \int_0^{\frac{y^2}{2}} \frac{y}{\sqrt{x^2+y^2+1}} \cdot dx\,dy$$

Sol: Given limits of integration Represents the Region

$R: \; 0 \le y \le 2 \; ; \; 0 \le x \le \frac{y^2}{2}$



By solving $y^2 = 2x$ & $y = 2$

$2^2 = 2x \Rightarrow x = 2$, $(x, y) = (2, 2)$

In order to change the order of integration, we must fix limits and take vertical strip in R, To write Limits of integration

$R: \; 0 \le x \le 2 \; ; \; \sqrt{2x} \le y \le 2$

(25)

$$\iint_R f(x,y)\,dx\,dy = \int_0^2 \int_0^{y^2/2} \frac{y}{\sqrt{x^2+y^2+1}}\,dx\,dy = \int_{x=0}^{x=2} \int_{y=\sqrt{2x}}^{y=2} \frac{y}{\sqrt{x^2+y^2+1}}\,dy\,dx$$

$$= \int_{x=0}^{2} \left[\int_{y=\sqrt{2x}}^{2} \frac{y}{\sqrt{x^2+1+y^2}} \cdot dy \right] \cdot dx$$

Keeping x as constant

Put $x^2+1+y^2 = t$

Diff w.r.t y

$2y\,dy = dt$

$y \cdot dy = \frac{1}{2} \cdot dt$

for $y = \sqrt{2x}$, $t = x^2+1+2x = (x+1)^2$

$y = 2$ $\quad t = x^2+1+4 = (x^2+5)$

$$= \int_{x=0}^{2} \left[\int_{t=(x+1)^2}^{(x^2+5)} \frac{1}{\sqrt{t}} \cdot \frac{1}{2}\,dt \right] \cdot dx$$

$$= \frac{1}{2} \cdot \int_{x=0}^{2} \left[\int_{t=(x+1)^2}^{x^2+5} t^{-\frac{1}{2}} \cdot dt \right] \cdot dx$$

$$= \frac{1}{2} \int_{x=0}^{2} \left(\frac{t^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} \right)_{(x+1)^2}^{x^2+5} dx = \frac{1}{2} \times 2 \int_{x=0}^{2} \left(t^{\frac{1}{2}} \right)_{(x+1)^2}^{x^2+5} dx$$

$$= \int_{x=0}^{2} \left[\sqrt{x^2+5} - (x+1) \right] \cdot dx$$

$$\int \sqrt{x^2+a^2} \cdot dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2} \sinh^{-1}\left(\frac{x}{a}\right)$$

$$= \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2} \log\left(x + \sqrt{x^2+a^2}\right)$$

$$= \left[\frac{x}{2}\sqrt{x^2+5} + \frac{5}{2} \log\left(x+\sqrt{x^2+5}\right) - \frac{(x+1)^2}{2} \right]_0^2$$

$$= \sqrt{4+5} + \frac{5}{2} \log\left(2+\sqrt{2^2+5}\right) - \frac{(2+1)^2}{2} - \left\{ \frac{5}{2} \log\left(0+\sqrt{5}\right) - \frac{(0+1)^2}{2} \right\}$$

$$= 3 + \frac{5}{2} \log 5 - \frac{9}{2} - \left\{ \frac{5}{2} \log \sqrt{5} - \frac{1}{2} \right\}$$

$$= 3 + \frac{5}{2} \log 5 - \frac{9}{2} - \frac{5}{2} \log \sqrt{5} + \frac{1}{2} = -1 + \frac{5}{2} \log 5 - \frac{5}{2} \cdot \log 5^{\frac{1}{2}}$$

$$= -1 + \frac{5}{4} \log 5$$

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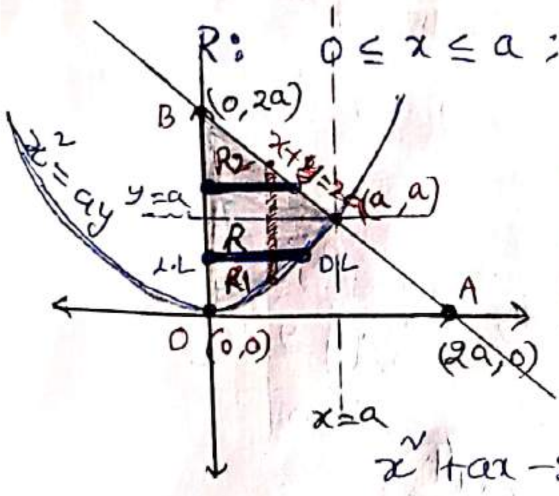
(3) By changing the order of integration, Evaluate

$$\int_0^a \int_{\frac{x^2}{a}}^{2a-x} xy\, dy\, dx, \quad a>0.$$

Sol: The given limits of integration represents the Region



R: $0 \le x \le a$; $\frac{x^2}{a} \le y \le 2a - x$.

$y = 2a - x \Rightarrow x + y = 2a$

x	0	$2a$
y	$2a$	0

By solving $x^2 = ay$ & $y = 2a - x$

$x^2 = a(2a - x) \Rightarrow x^2 = 2a^2 - ax$

$x^2 + ax - 2a^2 = 0 \Rightarrow x^2 + 2ax - ax - 2a^2 = 0$

$x(x+2a) - a(x+2a) = 0 \Rightarrow (x+2a)(x-a) = 0$

$\Rightarrow x = -2a$ & $x = a$.

for $x = -2a$; $y = 2a - (-2a) = 4a$ $\quad (x,y) = (-2a, 4a)$

$x = a$ $\quad y = 2a - a = a$ $\quad (x,y) = (a,a)$.

In order to change the order of Integration, we must fix y limits, and take Horizontal strip in R. Since one end of the horizontal strip varies on two different curves the Region R must be divided in sub Regions R_1 & R_2

$$\iint_R f(x,y)\, dx\, dy = \iint_{R_1} f(x,y)\, dx\, dy + \iint_{R_2} f(x,y)\, dx\, dy$$

In R_1: y varies from $y = 0$ to $y = a$

x varies from $x = 0$ to $x = \sqrt{ay}$

In R_2: y varies from $y = a$ to $y = 2a$

x varies from $x = 0$ to $x = 2a - y$

$$\iint_R f(x,y)\cdot dx\,dy = \int_0^a \int_{\frac{x^2}{a}}^{2a-x} xy\,dx\,dy = \iint_{R_1} xy\,dx\,dy + \iint_{R_2} xy\,dx\,dy$$

(i) $$\iint_{R_1} xy\,dx\,dy = \int_{y=0}^{a} \int_{x=0}^{\sqrt{ay}} xy\,dx\,dy$$

Order of integration
① First integrate w.r.t x, keeping y as constant
② second integrate w.r.t y b/w limits

$$= \int_{y=0}^{a} \left[\int_{x=0}^{\sqrt{ay}} xy\cdot dx\right]\cdot dy$$

keeping y as constant

$$= \int_{y=0}^{a} y\cdot\left(\frac{x^2}{2}\right)_0^{\sqrt{ay}}\cdot dy = \int_{y=0}^{a} y\cdot\frac{ay}{2}\cdot dy = \frac{a}{2}\cdot\int_{y=0}^{a} y^2\cdot dy$$

$$= \frac{a}{2}\cdot\left(\frac{y^3}{3}\right)_0^{a} = \frac{a}{2}\left(\frac{a^3}{3}\right) = \frac{a^4}{6}$$

(ii) $$\iint_{R_2} xy\,dx\,dy = \int_{y=a}^{y=2a} \int_{x=0}^{2a-y} xy\cdot dx\,dy = \int_{y=a}^{2a} \left[\int_{x=0}^{2a-y} xy\cdot dx\right]\cdot dy$$

keeping y as constant

$$= \int_{y=a}^{2a} y\cdot\left(\frac{x^2}{2}\right)_0^{2a-y}\cdot dy = \int_{y=a}^{2a} y\cdot\frac{(2a-y)^2}{2}\cdot dy$$

$$= \frac{1}{2}\int_{y=a}^{2a} y\left[4a^2+y^2-4ay\right]\cdot dy = \frac{1}{2}\int_{y=a}^{2a} \left(y^3-4ay^2+4a^2y\right)\cdot dy$$

$$= \frac{1}{2}\left(\frac{y^4}{4} - 4a\cdot\frac{y^3}{3} + 4a^2\cdot\frac{y^2}{2}\right)_a^{2a} = \frac{1}{2}\left[\frac{(2a)^4}{4} - \frac{4a}{3}(2a)^3 + 2a^2(2a)^2 - \left\{\frac{a^4}{4} + 4a\cdot\frac{a^3}{3} + 4a^2\frac{a^2}{2}\right\}\right]$$

$$= \frac{1}{2}\left[4a^4 - \frac{32}{3}a^4 + 8a^4 - \left\{\frac{a^4}{4} - \frac{4}{3}a^4 + 2a^4\right\}\right] = \frac{1}{2}\left[\left(4-\frac{32}{3}+8-\frac{1}{4}+\frac{4}{3}-2\right)a^4\right]$$

$$= \frac{1}{2}\left[\frac{5a^4}{12}\right] = \frac{5a^4}{24}$$

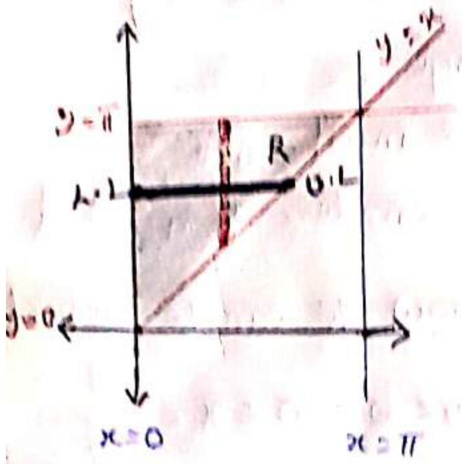
$$\iint_R f(x,y)\,dx\,dy = \iint_{R_1} + \iint_{R_2} = \frac{a^4}{6} + \frac{5a^4}{24} = \frac{9a^4}{24}$$

(11) By changing the order of integration, evaluate

$$\int_0^{\pi}\int_x^{\pi} \frac{\sin y}{y}\, dy\, dx$$

Sol: Given limits of integration represents the Region

$$R = \{(x,y) / \; 0 \le x \le \pi \;;\; x \le y \le \pi\}$$



In order to change the order of integration we must fix y limits, and consider Horizontal strip in R, gives the limits of integration as

In R : y varies from $y=0$ to $y=\pi$

x varies from $x=0$ to $x=y$

$$\iint_R f(x,y)\, dx\, dy = \int_{x=0}^{\pi}\int_{y=x}^{\pi} \frac{\sin y}{y}\, dy\, dx = \int_{y=0}^{\pi}\int_{x=0}^{x=y} \frac{\sin y}{y}\, dx\, dy.$$

The order of Integration:

1. First integrate w.r.t x, keeping y as constant.
2. Second integrate w.r.t y b/w the given limits

$$= \int_{y=0}^{\pi}\left[\int_{x=0}^{y} \frac{\sin y}{y}\, dx\right] dy = \int_{y=0}^{\pi} \frac{\sin y}{y}\,(x)_0^y\, dy$$

keeping y as constant

$$= \int_{y=0}^{\pi} \frac{\sin y}{\cancel{y}}\,\cancel{y}\, dy = (-\cos y)_0^{\pi} = -\cos\pi - (-\cos 0)$$

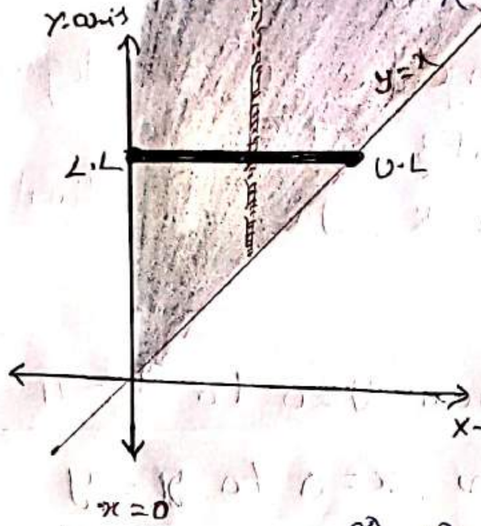
$$= -(-1) - (-1) = 1 + 1$$

$$= 2 \quad \text{Ans}$$

(5) By changing the order of integration, Evaluate

$$\int_{x=0}^{\infty}\int_{y=x}^{\infty} \frac{e^{-y}}{y} \cdot dy\,dx.$$

Sol: The given limits of integration represents the Region as, $R = \{(x,y) / 0 \le x \le \infty ; x \le y < \infty\}$



In order to change the order of Integration, we must fix y limits, and consider Horizontal strip in R. Now R can be defined as $R = \{(x,y) / 0 \le y < \infty ; 0 \le x \le y\}$

$$\iint_R f(x,y)\,dx\,dy = \int_{x=0}^{\infty}\int_{y=x}^{\infty} \frac{e^{-y}}{y} \cdot dy\,dx = \int_{y=0}^{\infty}\int_{x=0}^{y} \frac{e^{-y}}{y} \cdot dx \cdot dy$$

Here order of Integration

(1) First Integrate w.r.t x, keeping y as constant

(2) second integrate w.r.t y, b/w the given limits

$$= \int_{y=0}^{\infty}\left[\int_{x=0}^{y} \frac{e^{-y}}{y} \cdot dx\right] \cdot dy$$

keeping y as constant

$$= \int_{y=0}^{\infty} \frac{e^{-y}}{y} \cdot (x)_0^y \cdot dy$$

$$= \int_{y=0}^{\infty} \frac{e^{-y}}{\cancel{y}} \cdot \cancel{y} \cdot dy = \left(-e^{-y}\right)_0^{\infty} = -e^{-\infty} - (-e^{0})$$

$$= 0 + 1 = \underline{1} \text{ Ans!}$$

Change of variables in Double integral:

Some time the evaluation of a double integral in its present form may not be simple to evaluate.

By choice of an approximate co-ordinate system, a given integral can be Transformed into a simple integral involving the new-variables.

Case(i) The Double integral can be changed from Cartesian co-ordinates (x,y) to polar co-ordinates (r,θ) by the transformation $x = r\cos\theta$; $y = r\sin\theta$ then

$$\iint_R f(x,y)\cdot dx\,dy = \iint_{R'} f(r\cos\theta, r\sin\theta)\,|J|\,dr\,d\theta$$

where $J = \dfrac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$

$= r(\cos^2\theta + \sin^2\theta) = r.$

$$\iint_R f(x,y)\,dx\,dy = \iint_R f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

Thus To convert from Cartesian co-ordinates to Polar co-ordinates put $x = r\cos\theta$; $y = r\sin\theta$; $dx\,dy = r\,dr\,d\theta$.

Case(ii) Suppose x, y are related to u, v by the transformation
$x = x(u,v)$; $y = y(u,v)$

Let Jacobian transformation from x,y to u,v is

$J = \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \neq 0$ Then

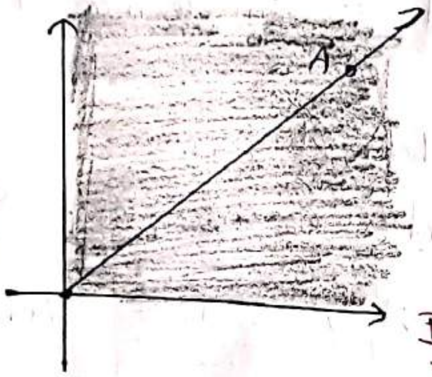
$$\iint_R f(x,y)\,dx\,dy = \iint_{R'} \phi(u,v)\,|J|\,du\,dv$$

where R is the Region in which x,y vary and R' is the Region in which u,v vary

① Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} \cdot dx\,dy$.

The Region of Integration is given by

$R = \{(x,y) \mid 0 \le x < \infty ;\ 0 \le y < \infty\}$



The Region of integration is entire First Quadrant.

In order to change into Polar Co-ordinate System, Put

$x = r\cos\theta;\ y = r\sin\theta,\ dx\,dy = r\,dr\,d\theta$

In R: θ varies from $\theta = 0$ to $\frac{\pi}{2}$

r varies from $r = 0$ to $r = \infty$

Thus

$$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} \cdot dx\,dy = \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{\infty} e^{-r^2} \cdot r \cdot dr\,d\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} \left[\frac{1}{2}\int_{r=0}^{\infty} e^{-r^2}(2r)\,dr\right] \cdot d\theta$$

Put $r^2 = t$

$2r\,dr = dt$

for $r=0,\ t=0$

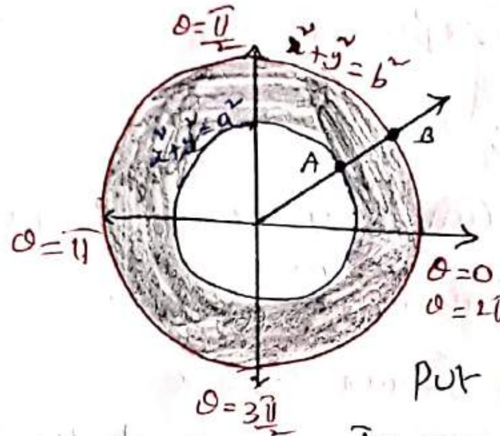
$r = \infty\ \ t = \infty$

$$= \frac{1}{2}\int_{\theta=0}^{\frac{\pi}{2}} \left(\int_{t=0}^{\infty} e^{-t} \cdot dt\right) d\theta$$

$$= \frac{1}{2}\int_{\theta=0}^{\frac{\pi}{2}} \left(-e^{-t}\right)_0^\infty \cdot d\theta = \frac{1}{2}\int_{\theta=0}^{\frac{\pi}{2}} [0-(-1)] \cdot d\theta$$

$$= \frac{1}{2}(\theta)_0^{\frac{\pi}{2}} = \frac{1}{2}\left(\frac{\pi}{2}\right) = \frac{\pi}{4}.$$

(2) By changing to Polar Co-ordinates, Evaluate $\iint_R \frac{x^2y^2}{x^2+y^2}\,dx\,dy$ over the annulus region $x^2+y^2=a^2$, $x^2+y^2=b^2$ $(b>a)$.



Given the Region of Integration R is the Annulus region bounded by $x^2+y^2=a^2$ & $x^2+y^2=b^2$ $(b>a)$.

In order to change into Polar co-ordinates put $\boxed{x=r\cos\theta;\ y=r\sin\theta;\ dx\,dy=r\cdot dr\,d\theta}$

In R

θ varies from $\underline{\theta=0 \text{ to } \theta=2\pi}$

At A, $x^2+y^2=a^2 \Rightarrow r^2=a^2 \Rightarrow r=a$

B $x^2+y^2=b^2 \Rightarrow r^2=b^2 \Rightarrow r=b$ r varies from $\underline{r=a \text{ to } r=b}$

$$\iint_R \frac{x^2y^2}{x^2+y^2}\cdot dx\,dy = \int_{\theta=0}^{2\pi}\int_{r=a}^{b} \frac{r^2\cos^2\theta\cdot r^2\sin^2\theta}{\cancel{r^2}}\cdot r\cdot dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{r=a}^{r=b} r^3 \cos^2\theta\sin^2\theta\cdot dr\,d\theta = \int_{\theta=0}^{2\pi}\left[\int_{r=a}^{b} r^3\cdot\cos^2\theta\sin^2\theta\cdot dr\right]\cdot d\theta$$

keeping θ as constant

$$= \int_{\theta=0}^{2\pi} \cos^2\theta\cdot\sin^2\theta\left(\frac{r^4}{4}\right)_a^b\cdot d\theta$$

$$= \frac{1}{4}(b^4-a^4)\int_{\theta=0}^{2\pi}\sin^2\theta\cos^2\theta\cdot d\theta = \frac{1}{16}(b^4-a^4)\int_{\theta=0}^{2\pi}(2\sin\theta\cos\theta)^2\cdot d\theta$$

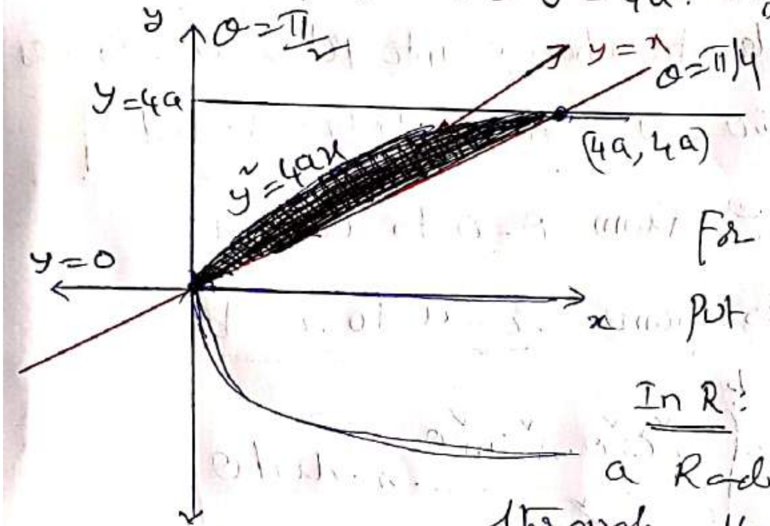
$$= \frac{1}{16}(b^4-a^4)\int_{\theta=0}^{2\pi}\sin^2 2\theta\,d\theta = \frac{1}{16}(b^4-a^4)\int_{\theta=0}^{2\pi}\frac{1-\cos 4\theta}{2}\cdot d\theta$$

$$= \frac{1}{32}(b^4-a^4)\cdot\left[\theta-\frac{\sin 4\theta}{4}\right]_0^{2\pi} = \frac{b^4-a^4}{32}\left[2\pi\right] = \frac{(b^4-a^4)\cdot\pi}{16}.$$

③ Evaluate the integral $\int_0^{4a}\int_{\frac{y^2}{4a}}^{y} \frac{x^2-y^2}{x^2+y^2}\,dx\,dy$ by Transforming to Polar co-ordinates.

The Given limits of integration represents the Region $R = \{(x,y) \mid \frac{y^2}{4a} \le x \le y;\ 0 \le y \le 4a\}$

The Region R is bounded by the curves $\frac{y^2}{4a} = x \Rightarrow y^2 = 4ax$, $x = y$; $y = 0$ & $y = 4a$. The Region is as shown



Solving $y^2 = 4ax$ & $y = x$

$y^2 = 4ay \Rightarrow y = 4a$ At $(4a, 4a)$

For Transform to polar co-ordinates put $x = r\cos\theta$; $y = r\sin\theta$; $dx\,dy = r\,dr\,d\theta$

In R: To find limits for r, draw a Radius vector OA from origin, through the Region, identify the Points where it enters, leaves the Region.

Here r varies from $r = 0$ to $\frac{4a\cos\theta}{\sin^2\theta}$

$y^2 = 4ax$

$r^2\sin^2\theta = 4a\,r\cos\theta$

$r = \frac{4a\cos\theta}{\sin^2\theta}$

θ varies from $\theta = \frac{\pi}{4}$ to $\theta = \frac{\pi}{2}$

$$\int_0^{4a}\int_{\frac{y^2}{4a}}^{y} \frac{x^2-y^2}{x^2+y^2}\,dx\,dy = \int_{\theta=\frac{\pi}{4}}^{\frac{\pi}{2}}\int_{r=0}^{r=\frac{4a\cos\theta}{\sin^2\theta}} \frac{r^2\cos^2\theta - r^2\sin^2\theta}{r^2}\,r\,dr\,d\theta$$

$$= \int_{\theta=\frac{\pi}{4}}^{\frac{\pi}{2}}\left[\int_{r=0}^{\frac{4a\cos\theta}{\sin^2\theta}} \frac{\cancel{r^2}(\cos^2\theta-\sin^2\theta)}{\cancel{r^2}}\cdot r\,dr\right]d\theta = \int_{\theta=\frac{\pi}{4}}^{\frac{\pi}{2}}(\cos^2\theta-\sin^2\theta)\left(\frac{r^2}{2}\right)_0^{\frac{4a\cos\theta}{\sin^2\theta}}\cdot d\theta$$

$$= \frac{1}{2}\int_{\theta=\frac{\pi}{4}}^{\frac{\pi}{2}}(\cos^2\theta-\sin^2\theta)\left(\frac{4a\cos\theta}{\sin^2\theta}\right)^2\cdot d\theta$$

$$= \frac{1}{2}\int_{\theta=\frac{\pi}{4}}^{\frac{\pi}{2}}(\cos^2\theta-\sin^2\theta)\,\frac{16a^2\cos^2\theta}{\sin^4\theta}\,d\theta = 8a^2\int_{\theta=\frac{\pi}{4}}^{\frac{\pi}{2}}(\cot^4\theta - \cot^2\theta)\,d\theta$$

$$= 8a^2 \int_{\theta=\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^2\theta\,(\cot^2\theta-1)\,.d\theta = 8a^2 \int_{\theta=\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^2\theta\,(\csc^2\theta-2)\,.d\theta$$

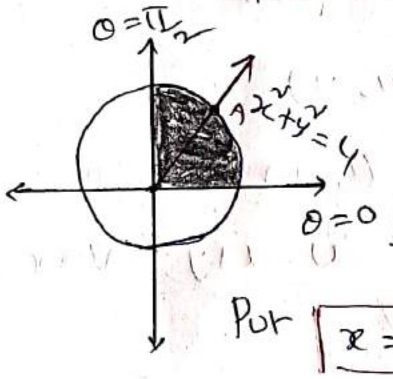
$$= 8a^2 \int_{\theta=\frac{\pi}{4}}^{\frac{\pi}{2}} (\cot^2\theta\csc^2\theta - 2\cot^2\theta)\,.d\theta = 8a^2 \int_{\theta=\frac{\pi}{4}}^{\pi/2} \cot^2\theta\csc^2\theta - 2(\csc^2\theta-1)\,.d\theta$$

$$= 8a^2 \int_{\theta=\frac{\pi}{4}}^{\pi/2} \left[\cot^2\theta\csc^2\theta - 2\csc^2\theta + 2\right].d\theta$$

$$= 8a^2\left[-\frac{\cot^3\theta}{3} + 2\cdot\cot\theta + 2\theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = 8a^2\left[2\cdot\frac{\pi}{2} - \left\{\frac{-1}{3}+2+2\cdot\frac{\pi}{4}\right\}\right]$$

$$= 8a^2\left[\pi - \frac{5}{3} - \frac{\pi}{2}\right] = 8a^2\left[\frac{\pi}{2}-\frac{5}{3}\right]$$

(4) Evaluate $\iint xy\,(x^2+y^2)^{\frac{n}{2}}\,dxdy$ over the Positive Quadrant of $x^2+y^2=4$, supposing $n+3>0$.



The given Region of integration is area enclosed by the Positive Quadrant of the Circle $x^2+y^2=4$, whose centre is $(0,0)$ and radius 2.

To Transform into polar co-ordinates

Put $\boxed{x = r\cos\theta;\ y = r\sin\theta;\ dxdy = r\cdot drd\theta}$

In R: θ varies from $\theta=0$ to $\theta=\frac{\pi}{2}$

r varies from $r=0$ to $r=2$

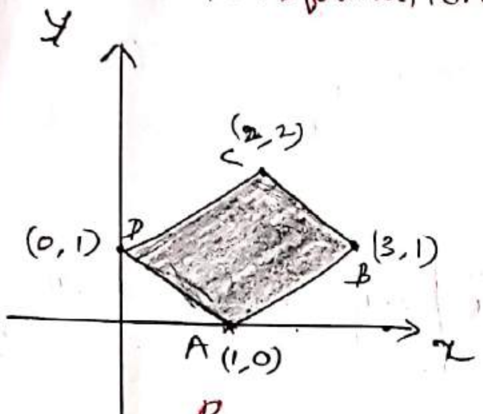
$$\iint_R xy\,(x^2+y^2)^{n/2}.dxdy = \int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{2} r\cos\theta\cdot r\sin\theta\,(r^2)^{\frac{n}{2}}.r\,drd\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{2} r^{n+3}\sin\theta\cos\theta\cdot drd\theta = \int_{\theta=0}^{\frac{\pi}{2}} \sin\theta\cos\theta\left[\int_{r=0}^{2} r^{n+3}\cdot dr\right].d\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} \sin\theta\cos\theta.\left(\frac{r^{n+4}}{n+4}\right)_0^2.d\theta = \frac{2^{n+4}}{n+4}\int_0^{\frac{\pi}{2}}\sin\theta\cos\theta.d\theta$$

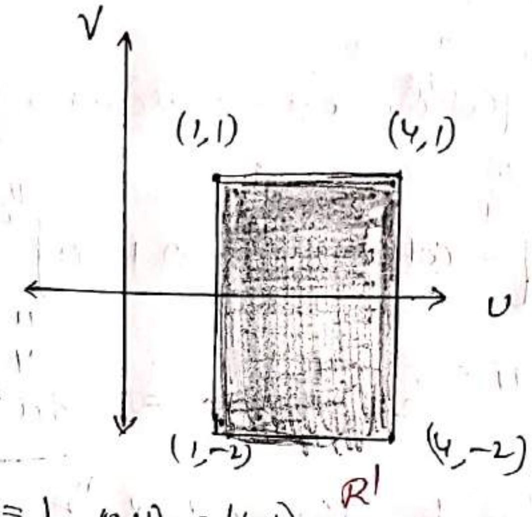
$$= \frac{2^{n+4}}{n+4}.\left(\frac{\sin^2\theta}{2}\right)_0^{\pi/2} = \frac{2^{n+4}}{n+4}\left(\frac{1}{2}\right) = \frac{2^{n+3}}{n+4},\ n+3>0$$

(5) Evaluate $\iint_R (x+y)^2\,dx\,dy$, where R is the parallelogram in the xy plane with vertices $(1,0), (3,1), (2,2), (0,1)$ using the Transformation $u = x+y$; $v = x-2y$.



R

Region in xy plane



$u = x+y$
$v = x-2y$

$A(1,0) \to u=1,\ v=1 \quad (u,v)\to(1,1)$

$B(3,1) \to u = 3+1 = 4,\ v = 3-2 = 1 \quad (u,v) \to (4,1)$

$C(2,2) \to u = 2+2 = 4,\ v = 2-4 = -2 \quad (4,-2)$

$D(0,1) \to u = 0+1 = 1,\ v = 0-2 = -2 \quad (1,-2)$

For Transformation to u,v plane $\qquad$ $u = x+y$; $v = x-2y$

$$\iint_R (x+y)^2\,dx\,dy = \iint_{R} f(u,v)\,|J|\,du\,dv$$

where $J = \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \left|\dfrac{1}{J^*}\right| = -\dfrac{1}{3} \qquad |J| = \dfrac{1}{3}.$

$$J^* = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} = -2-1 = -3$$

$$= \int_{u=1}^{4}\int_{v=-2}^{1} u^2\cdot\frac{1}{3}\,du\,dv = \frac{1}{3}\int_{u=1}^{4} u^2\left[\int_{v=-2}^{1} dv\right]du$$

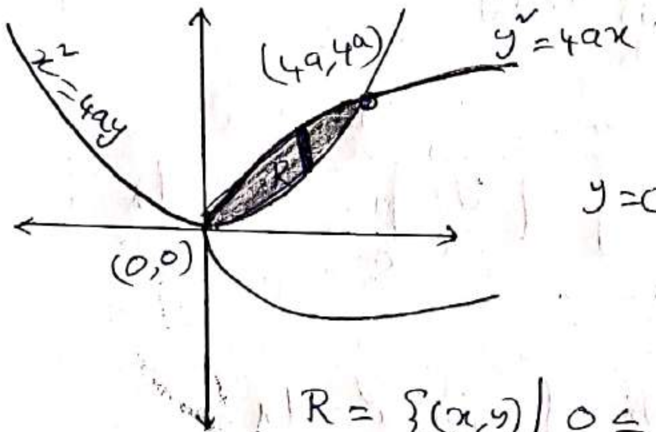
$$= \frac{1}{3}\int_{u=1}^{4} u^2\,(v)_{-2}^{1}\,du = \frac{1}{3}\int_{u=1}^{4} u^2\,[1-(-2)]\,du = \frac{1}{3}\cdot 3\int_{u=1}^{4} u^2\,du$$

$$= \left(\frac{u^3}{3}\right)_1^4 = \frac{4^3}{3} - \frac{1}{3} = \frac{63}{3} = \underline{21 \text{ Units}}$$

Area of a Region using Double integral:

The Area of Region R is given by $A = \iint_R dx\,dy$

① Find area of a region R, bounded by the curves $y^2 = 4ax$ & $x^2 = 4ay$, $a > 0$ using double integral.



By solving $x^2 = 4ay$ & $y^2 = 4ax$

$$\left(\frac{y^2}{4a}\right)^2 = 4ay \Rightarrow y^4 = 64a^3y$$

$$y(y^3 - 64a^3) = 0$$

$$y = 0 \text{ \& } y^3 = (4a)^3 \Rightarrow y = 4a$$

for $y = 0$; $x = 0$ $(x,y) = (0,0)$

$y = 4a$ $x = 4a$ (x,y) $(4a, 4a)$

$$R = \left\{(x,y) \,\middle|\, 0 \le x \le 4a ;\ \frac{x^2}{4a} \le y \le 2\sqrt{ax}\right\}$$

Area of the region enclosed by the given curves is given by

$$A = \iint_R dx\,dy$$

$$A = \int_{x=0}^{4a} \int_{y=\frac{x^2}{4a}}^{2\sqrt{ax}} dy.dx = \int_{x=0}^{4a} \left[\int_{y=\frac{x^2}{4a}}^{2\sqrt{ax}} dy\right] . dx$$

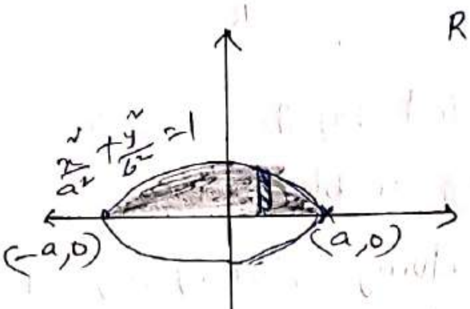
$$= \int_{x=0}^{4a} \left(y \right)_{\frac{x^2}{4a}}^{2\sqrt{ax}} . dx = \int_{x=0}^{4a} \left(2\sqrt{ax} - \frac{x^2}{4a}\right) . dx$$

$$= \left(2\sqrt{a}. \frac{x^{3/2}}{3/2} - \frac{1}{4a} . \frac{x^3}{3}\right)_0^{4a}$$

$$= \frac{4}{3}\sqrt{a}.(4a)^{3/2} - \frac{1}{4a} . \frac{(4a)^3}{3} = \frac{32}{3}a^2 - \frac{16}{3}a^2$$

$$= \frac{16}{3}a^2$$

(2) Find the area bounded by ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ above x-axis.



$R: \{(x,y) \mid -a \le x \le a;\ 0 \le y \le \frac{b}{a}\sqrt{a^2-x^2}\}$

Area of the Region is given by

$$A = \iint_R dx\,dy$$

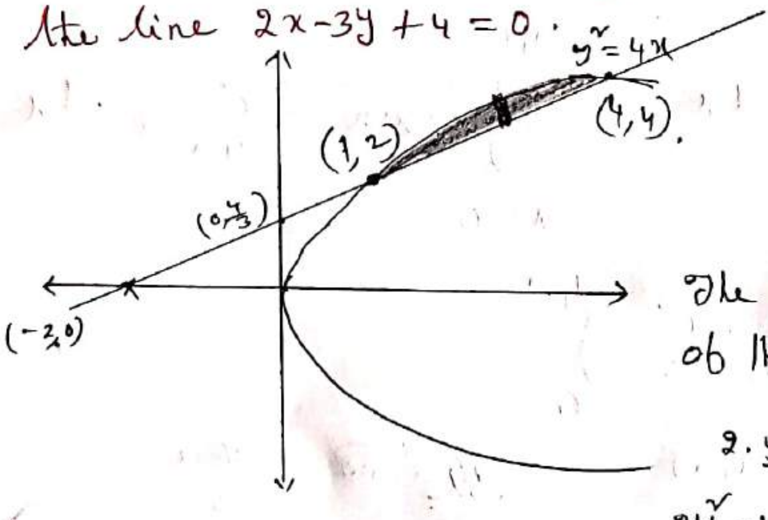
$$A = \int_{x=-a}^{a}\left[\int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} dy\right]dx = \int_{x=-a}^{a}[y]_0^{\frac{b}{a}\sqrt{a^2-x^2}}\,.dx$$

$$= \int_{x=-a}^{a}\frac{b}{a}\sqrt{a^2-x^2}\,.dx = \frac{2b}{a}\int_{x=0}^{a}\sqrt{a^2-x^2}\,.dx$$

$$= \frac{2b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$= \frac{2b}{a}\left[\frac{a^2}{2}\sin^{-1}(1)\right] = ab.\frac{\pi}{2} = \frac{\pi ab}{2}$$

(3) Find the area bounded by the Parabola $y^2=4x$ and the line $2x-3y+4=0$.



$2x-3y+4=0$

x	-2	0
y	0	$4/3$

The Points of intersection of the curves $y^2=4x$ & $2x-3y+4=0$

$2.\frac{y^2}{4}-3y+4=0$

$2y^2-12y+16=0$

$y^2-6y+8=0$

$(y-2)(y-4)=0$

$y=2$ & $y=4$

for $y=2$, $x=1$ $(x,y)=(1,2)$

$y=4$ $x=4$ $(x,y)=(4,4)$

$$R = \{(x,y) \mid 1 \le x \le 4 \ ; \ \frac{2x+4}{3} \le y \le 2\sqrt{x}\}$$

The Area enclosed by the given curves is given by

$$A = \iint_R dx\,dy = \int_{x=1}^{4}\left[\int_{y=\frac{2x+4}{3}}^{2\sqrt{x}} dy\right]dx$$

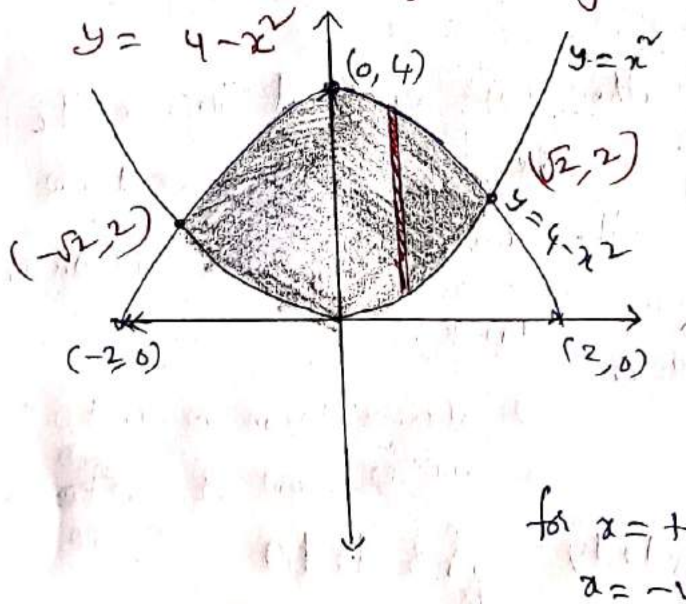
$$= \int_{x=1}^{4} (y)_{\frac{2x+4}{3}}^{2\sqrt{x}} \cdot dx = \int_{x=1}^{4}\left[2\sqrt{x} - \left(\frac{2x+4}{3}\right)\right]\cdot dx$$

$$= \left[2 \cdot \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{2}{3}\cdot\frac{x^2}{2} - \frac{4}{3}x\right]_1^4$$

$$= \frac{4}{3}(4)^{3/2} - \frac{1}{3}\cdot(4)^2 - \frac{4}{3}(4) - \left\{\cancel{\frac{4}{3}} - \frac{1}{3} - \cancel{\frac{4}{3}}\right\}$$

$$= \cancel{\frac{32}{3}} - \cancel{\frac{16}{3}} - \cancel{\frac{16}{3}} + \frac{1}{3} = \boxed{\frac{1}{3}}$$

(4) Find the area of the Region bounded by $y = x^2$, $y = 4 - x^2$



The points of intersection of the curves $y = x^2$ & $y = 4 - x^2$ are given by

$4 - x^2 = x^2 \Rightarrow 2x^2 = 4 \Rightarrow x^2 = 2$

$x = \pm\sqrt{2}$

for $x = +\sqrt{2}$, $y = 2$ $(x,y) = (\sqrt{2}, 2)$

$x = -\sqrt{2}$ $y = 2$ $(x,y) = (-\sqrt{2}, 2)$

$$R = \{(x,y) \mid -\sqrt{2} \le x \le \sqrt{2} \ ; \ x^2 \le y \le 4 - x^2\}$$

The area enclosed by the Region R is given by

$$A = \iint_R dx\,dy$$

$$A = \int_{x=-\sqrt{2}}^{\sqrt{2}} \left[\int_{y=x^2}^{4-x^2} dy \right] dx = \int_{x=-\sqrt{2}}^{\sqrt{2}} (y)_{x^2}^{4-x^2} \cdot dx$$

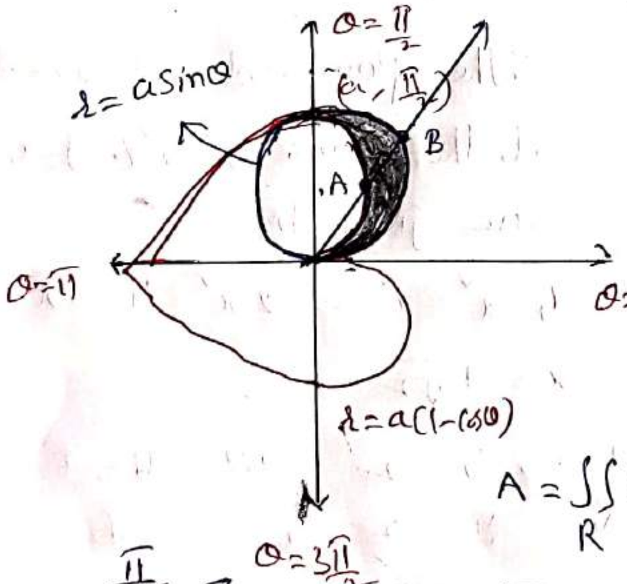
$$= \int_{x=-\sqrt{2}}^{\sqrt{2}} (4-x^2-x^2)\cdot dx = \int_{x=-\sqrt{2}}^{\sqrt{2}} (4-2x^2)\cdot dx = \left(4x - \frac{2x^3}{3}\right)_{-\sqrt{2}}^{\sqrt{2}}$$

$$= 4\sqrt{2} - \frac{2}{3}\cdot(\sqrt{2})^3 - \left\{ 4(-\sqrt{2}) - \frac{2}{3}\cdot(-\sqrt{2})^3 \right\}$$

$$= 4\sqrt{2} - \frac{2}{3}\cdot 2\sqrt{2} + 4\sqrt{2} - \frac{4}{3}\sqrt{2} = 8\sqrt{2} - \frac{8}{3}\sqrt{2}$$

$$= 8\sqrt{2}\left[1-\frac{1}{3}\right] = 8\sqrt{2}\times\frac{2}{3} = \frac{16\sqrt{2}}{3} \text{ sq. units.}$$

(5) Find by double integration, the area lying inside the circle $r = a\sin\theta$ and outside of the cardioid $r = a(1-\cos\theta)$.



The area enclosed inside the circle $r = a\sin\theta$ & outside of the cardioid is as shown in dia.

In R:

θ varies from $\theta=0$ to $\theta=\frac{\pi}{2}$

r varies from $r = a(1-\cos\theta)$ to $a\sin\theta$

$$A = \iint_R dx\,dy = \iint_R r\,dr\,d\theta$$

$$A = \int_{\theta=0}^{\pi/2} \left[\int_{r=a(1-\cos\theta)}^{a\sin\theta} r\cdot dr \right] d\theta = \int_{\theta=0}^{\pi/2} \left(\frac{r^2}{2}\right)_{a(1-\cos\theta)}^{a\sin\theta} \cdot d\theta$$

$$= \frac{1}{2}\int_{\theta=0}^{\pi/2} \left\{ [a(1-\cos\theta)]^2 + a^2\sin^2\theta \right\} \cdot d\theta = \frac{a^2}{2}\int_{\theta=0}^{\pi/2} (1+\cos^2\theta+2\cos\theta+\sin^2\theta)\,d\theta$$

$$A = -\frac{a^2}{2}\int_{\theta=0}^{\frac{\pi}{2}} (1+\cos 2\theta - 2\cos\theta)\cdot d\theta$$

$$= -\frac{a^2}{2}\left[\theta + \frac{\sin 2\theta}{2} - 2\sin\theta\right]_0^{\frac{\pi}{2}}$$

$$= -\frac{a^2}{2}\left[\frac{\pi}{2} + \frac{1}{2}\sin\pi - 2\cdot\sin\frac{\pi}{2}\right] = -\frac{a^2}{2}\left[\frac{\pi}{2} - 2\right]$$

$$= -\frac{a^2}{4}(\pi - 4) = \frac{a^2}{4}(4-\pi)$$

Evaluation of a Triple Integral:

The Triple integral $\iiint_R f(x,y,z)\,dx\,dy\,dz$ is Evaluated by 3 successive integrations.

case(i): If the Region R in $\mathbb{R}^3$ is described by

$x_1 \le x \le x_2$ where $x_1, x_2, y_1, y_2, z_1 \& z_2$ are constants

$y_1 \le y \le y_2$

$z_1 \le z \le z_2$

Then $$\iiint_R f(x,y,z)\cdot dx\,dy\,dz = \int_{z=z_1}^{z_2} \int_{y=y_1}^{y=y_2} \int_{x=x_1}^{x_2} f(x,y,z)\cdot dx\,dy\,dz$$

The order of integration depends upon the limits.

$$= \int_{z=z_1}^{z_2} \left\{ \int_{y=y_1}^{y_2} \left[\int_{x=x_1}^{x_2} f(x,y,z)\cdot dx \right]\cdot dy \right\}\cdot dz$$

Keeping y & z are constants

keeping z as constant

① Evaluate $\int_0^1 \int_0^2 \int_1^2 x^2 yz\,dx\,dy\,dz$

Sol. $$\int_0^1 \int_0^2 \int_1^2 x^2 yz\cdot dx\,dy\,dz = \int_{z=0}^{1} \int_{y=0}^{2} \int_{x=1}^{x=2} x^2 yz\cdot dx\,dy\,dz$$

order of integration ① First integrate w.r.t x, Keeping y & z as constants

② Second integrate w.r.t y, Keeping z as constants

③ Third integrate w.r.t z b/w the given limits.

$$\int_{z=0}^{1} \int_{y=0}^{2} \int_{x=1}^{2} x^2 yz\,dx\,dy\,dz = \int_{z=0}^{1} \left\{ \int_{y=0}^{2} \left[\int_{x=1}^{2} x^2 yz\cdot dx \right]\cdot dy \right\}\cdot dz$$

$$= \int_{z=0}^{1} \left\{ \int_{y=0}^{2} yz\cdot\left(\frac{x^3}{3}\right)_1^2\cdot dy \right\}\cdot dz = \int_{z=0}^{1} \int_{y=0}^{2} yz\left(\frac{7}{3}\right)\cdot dy\cdot dz$$

$$= \frac{7}{3} \int_{z=0}^{1} \left[\int_{y=0}^{2} yz\cdot dy \right]\cdot dz = \frac{7}{3} \int_{z=0}^{1} z\cdot\left(\frac{y^2}{2}\right)_0^2\cdot dz = \frac{7}{3}\cdot 2 \int_{z=0}^{1} z\cdot dz$$

$$= \frac{14}{3}\left(\frac{z^2}{2}\right)_0^1 = \frac{14}{3}\cdot\frac{1}{2} = \frac{7}{3}.$$

(2) Evaluate $\int_{-a}^{a}\int_{-c}^{c}\int_{-b}^{b}(x^2+y^2+z^2)\,dy\,dz\,dx$

Sol $\int_{-a}^{a}\int_{-c}^{c}\int_{-b}^{b}(x^2+y^2+z^2)\cdot dy\,dz\,dx = \int_{x=-a}^{a}\int_{z=-c}^{c}\int_{y=-b}^{b}(x^2+y^2+z^2)\,dy\,dz\,dx$

$$= \int_{x=-a}^{a}\int_{z=-c}^{c}\left[\int_{y=-b}^{b}(x^2+y^2+z^2)\cdot dy\right]dz\,dx$$

Keeping x & z are constants.

$$= \int_{x=-a}^{a}\int_{z=-c}^{c}\left(x^2\cdot y + \frac{y^3}{3} + z^2\cdot y\right)_{-b}^{b}\cdot dz\cdot dx$$

$$= \int_{x=-a}^{a}\int_{z=-c}^{c}\left\{(x^2+z^2)\cdot y + \frac{y^3}{3}\right\}_{-b}^{b}\cdot dz\,dx = \int_{x=-a}^{a}\int_{z=-c}^{c}\left[(x^2+z^2)b+\frac{b^3}{3} - \left\{(x^2+z^2)(-b) - \frac{b^3}{3}\right\}\right]dz\,dx$$

$$= \int_{x=-a}^{a}\int_{z=-c}^{c}\left[(x^2+z^2)\,b + \frac{b^3}{3} + (x^2+z^2)\cdot b + \frac{b^3}{3}\right]\cdot dz\cdot dx$$

$$= \int_{x=-a}^{a}\left[\int_{z=-c}^{c}\left[2b\,(x^2+z^2)+\frac{2b^3}{3}\right]\cdot dz\right]\cdot dx$$

Keeping x as constants

$$= \int_{x=-a}^{a}\left\{2b\left(x^2z+\frac{z^3}{3}\right)+\frac{2b^3}{3}\cdot z\right\}_{-c}^{c}\cdot dx$$

$$= \int_{x=-a}^{a}\left[2b\left[x^2c+\frac{c^3}{3}\right]+\frac{2b^3}{3}\cdot c - \left\{2b\left[x^2(-c)+\frac{(-c)^3}{3}\right]+\frac{2b^3}{3}(-c)\right\}\right]\cdot dx$$

$$= \int_{x=-a}^{a}\left(2bc\,x^2+\frac{2}{3}bc^3+\frac{2}{3}b^3c+2bcx^2+\frac{2bc^3}{3}+\frac{2b^3}{3}c\right)dx$$

$$= \int_{x=-a}^{a}\left(4bc\,x^2+\frac{4}{3}bc^3+\frac{4}{3}b^3c\right)\cdot dx$$

$$= \left[4bc\cdot\frac{x^3}{3} + \left(\frac{4}{3}bc^3+\frac{4}{3}b^3c\right)x\right]_{-a}^{a}$$

$$= 4bc\frac{a^3}{3} + \left(\frac{4}{3}bc^3+\frac{4}{3}b^3c\right)a - \left\{4bc\cdot\frac{(-a)^3}{3} + \left(\frac{4}{3}bc^3+\frac{4}{3}b^3c\right)(-a)\right\}$$

$$= \underline{4bc\frac{a^3}{3}} + \frac{4}{3}a\underline{b}c^3 + \frac{4}{3}ab^3c + \underline{\frac{4}{3}a^3bc} + \frac{4}{3}abc^3 + \frac{4}{3}ab^3c.$$

$$= \frac{8}{3}a^3bc + \frac{8}{3}abc^3 + \frac{8}{3}ab^3c.$$

$$= \underline{\frac{8}{3}abc\left[a^2+b^2+c^2\right]}$$

x, y, z

<u>Case (ii)</u> If the limits of z are <u>functions of (x,y)</u> and limits of y <u>are functions x</u> and <u>limits of x are constants</u>.

$$I = \int_{x=a}^{b}\int_{y=\phi(x)}^{\psi(x)}\int_{z=f(x,y)}^{g(x,y)} f(x,y,z)\,.\,dz\,dy\,dx$$

The order of integration is

① First Integrate <u>w.r.t. z</u>, keeping x & y are constants

② Second Integrate w.r.t y, keeping x as constants

③ Third Integrate w.r.t x b/w the given limits.

$$= \int_{x=a}^{b}\left\{\int_{y=\phi(x)}^{\psi(x)}\underbrace{\left[\int_{z=f(x,y)}^{g(x,y)} f(x,y,z)\,.\,dz\right]}_{\text{keeping } x \,\&\, y \text{ as Constant}}\,.\,dy\right\}.\,dx$$

$\longleftarrow$ keeping x as constants $\longrightarrow$

x, y, z

① Evaluate $\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z}\, dz\, dy\, dx$

Sol. $\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z} \cdot dz\,dy\,dx = \int_{x=0}^{a} \int_{y=0}^{y=x} \int_{z=0}^{z=x+y} e^{x+y+z} \cdot dz\,dy\,dx$

Order of Integration:

① First Integrate w.r.t z, keeping x&y as constant

② Second integrate w.r.t y, keeping x as constant

③ Third integrate w.r.t x, b/w the given limits.

$$= \int_{x=0}^{a} \int_{y=0}^{x} \left[\int_{z=0}^{x+y} e^{x+y} \cdot e^{z} \cdot dz \right] dy\,dx$$

keeping x&y are constants

$$= \int_{x=0}^{a} \int_{y=0}^{x} e^{x+y} \cdot \left(e^{z}\right)_0^{x+y} dy \cdot dx$$

$$= \int_{x=0}^{a} \int_{y=0}^{x} e^{x+y} \left(e^{x+y} - 1\right) \cdot dy\,dx$$

$$= \int_{x=0}^{a} \left[\int_{y=0}^{x} \left(e^{2(x+y)} - e^{x+y}\right) \cdot dy \right] \cdot dx$$

keeping x as constant

$$= \int_{x=0}^{a} \left[\int_{y=0}^{x} \left(e^{2x} \cdot e^{2y} - e^{x} \cdot e^{y}\right) \cdot dy \right] dx$$

$$= \int_{x=0}^{a} \left(e^{2x} \cdot \frac{e^{2y}}{2} - e^{x} \cdot e^{y}\right)_0^{x} \cdot dx$$

$$= \int_{x=0}^{a} \left\{ \left(e^{2x}\cdot \frac{e^{2x}}{2} - e^{x}\cdot e^{x}\right) - \left(\frac{e^{2x}}{2}\cdot e^{0} - e^{x}\cdot e^{0}\right)\right\} \cdot dx$$

$$= \int_{x=0}^{a} \left(\frac{e^{4x}}{2} - e^{2x} - \frac{e^{2x}}{2} + e^{x}\right)\cdot dx$$

$$= \int_{x=0}^{a} \left(\frac{e^{4x}}{2} - \frac{3}{2}e^{2x} + e^{x}\right)\cdot dx$$

$$= \left(\frac{1}{2}\cdot\frac{e^{4x}}{4} - \frac{3}{2}\cdot\frac{e^{2x}}{2} + e^{x}\right)_{0}^{a}$$

$$= \left(\frac{e^{4x}}{8} - \frac{3e^{2x}}{4} + e^{x}\right)_{0}^{a}$$

$$= \frac{e^{4a}}{8} - \frac{3}{4}e^{2a} + e^{a} - \left(\frac{e^{0}}{8} - \frac{3}{4}e^{0} + e^{0}\right)$$

$$= \frac{e^{4a}}{8} - \frac{3}{4}e^{2a} + e^{a} - \left(\frac{1}{8} - \frac{3}{4} + 1\right) \qquad \frac{1-6+8}{8}$$

$$= \frac{e^{4a}}{8} - \frac{3}{4}e^{2a} + e^{a} - \frac{3}{8}.$$

x, y, z

(2) Evaluate $\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z} (x+y+z)\, dx\, dy\, dz$

$$\int_{z=-1}^{1}\int_{x=0}^{z}\int_{y=x-z}^{y=x+z} (x+y+z)\, dx\, dy\, dz$$

The order of Integration is

1. First integrate w.r.t y, keeping x & z as constants
2. second integrate w.r.t x keeping z as constant
3. Third integrate w.r.t z, b/w the given limits.

$$\int_{z=-1}^{1}\int_{x=0}^{z}\int_{y=x-z}^{x+z}(x+y+z)\,dy\cdot dx\cdot dz$$

$$=\int_{z=-1}^{1}\int_{x=0}^{z}\left[\int_{y=x-z}^{x+z}(x+y+z)\cdot dy\right]\cdot dx\cdot dz$$

keeping x & z as constants

$$=\int_{z=-1}^{1}\int_{x=0}^{z}\left\{(x+z)y+\frac{y^2}{2}\right\}_{x-z}^{x+z}\cdot dx\,dz$$

$$=\int_{z=-1}^{1}\int_{x=0}^{z}\left[(x+z)(x+z)+\frac{(x+z)^2}{2}-\left\{(x+z)(x-z)+\frac{(x-z)^2}{2}\right\}\right]dx\,dz$$

$$=\int_{z=-1}^{1}\int_{x=0}^{z}\left[\frac{3}{2}(x+z)^2-\left(x^2-z^2+\frac{(x-z)^2}{2}\right)\right]\cdot dx\cdot dz$$

$$=\int_{z=-1}^{1}\left[\int_{x=0}^{z}\left\{\frac{3}{2}(x+z)^2-x^2+z^2-\frac{(x-z)^2}{2}\right\}\cdot dx\right]\cdot dz$$

$$=\int_{z=-1}^{1}\left\{\frac{3}{2}\cdot\frac{(x+z)^3}{3}-\frac{x^3}{3}+z^2x-\frac{(x-z)^3}{2\cdot 3}\right]_{0}^{z}\cdot dz$$

$$\int(ax+b)^n\cdot dx=\frac{(ax+b)^{n+1}}{(n+1)\cdot a}$$

$$=\int_{z=-1}^{1}\left[\frac{3}{6}\cdot(2z)^3-\frac{z^3}{3}+z^3-\left\{\frac{3}{6}\cdot z^3-\frac{(-z)^3}{6}\right\}\right]dz$$

$$=\int_{z=-1}^{1}\left(\frac{3}{6}\cdot 8z^3+\frac{2}{3}z^3-\frac{3}{6}z^3-\frac{z^3}{6}\right)\cdot dz$$

55

$$= \int_{z=-1}^{1} \left(\frac{24}{6} + \cancel{\frac{4}{6}} - \cancel{\frac{3}{6}} - \cancel{\frac{1}{6}}\right) z^3 \cdot dz = \int_{z=-1}^{1} 4 \cdot z^3 \cdot dz$$

$$= 4 \cdot \left(\frac{z^4}{4}\right)_{-1}^{1} = 1 - 1 = \underline{\underline{0.}}$$

(3) Evaluate $\int_0^4 \int_0^{2\sqrt{z}} \int_0^{\sqrt{4z-x^2}} dy\, dx\, dz$ $\quad x, y, z$

$$\int_{z=0}^{4} \int_{x=0}^{2\sqrt{z}} \int_{y=0}^{\sqrt{4z-x^2}} dy \cdot dx\, dz = \int_{z=0}^{4} \int_{x=0}^{2\sqrt{z}} \left[\int_{y=0}^{\sqrt{4z-x^2}} \cdot dy\right] dx\, dz$$

$$= \int_{z=0}^{4} \left[\int_{x=0}^{2\sqrt{z}} (y)_0^{\sqrt{4z-x^2}} \cdot dx\right] \cdot dz$$

$\int \sqrt{a^2-x^2} \cdot dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right)$

$$= \int_{z=0}^{4} \left[\int_{x=0}^{2\sqrt{z}} \sqrt{4z-x^2} \cdot dx\right] \cdot dz$$

$$= \int_{z=0}^{4} \left[\int_{x=0}^{2\sqrt{z}} \sqrt{(2\sqrt{z})^2 - x^2} \cdot dx\right] \cdot dz$$

$$= \int_{z=0}^{4} \left[\frac{x}{2}\sqrt{4z-x^2} + \frac{\cancel{4}^{2} z}{\cancel{2}} \sin^{-1}\left(\frac{x}{2\sqrt{z}}\right)\right]_0^{2\sqrt{z}} \cdot dz$$

$$= \int_{z=0}^{4} \cancel{4}2z \sin^{-1}\left(\frac{2\sqrt{z}}{2\sqrt{z}}\right) \cdot dz = \int_{z=0}^{4} 2z \cdot \sin^{-1}\left(\sin\frac{\pi}{2}\right) \cdot dz$$

$$= 2 \cdot \frac{\pi}{2} \int_{z=0}^{4} z \cdot dz = \pi \cdot \left(\frac{z^2}{2}\right)_0^4 = \pi \cdot \left(\frac{16}{2}\right)$$

$$= 8\pi$$

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Maths Assignment

unit 3: Multiple Integrals

(1) Evaluate : $\int_0^3 \int_1^2 xy(x+y)\,dx\,dy$

$$= \int_{x=0}^{3} \int_{y=1}^{2} (x^2y + xy^2)\,dy\,dx$$

$$= \int_{x=0}^{3} \left(x^2 \frac{y^2}{2} + x\, y^3/3\right)_1^2 dx$$

$$= \int_0^3 \left(2x^2 + \frac{8}{3}x - \frac{x^2}{2} - \frac{x}{3}\right) dx$$

$$= \left(\frac{2x^3}{3} + \frac{4x^2}{3} - \frac{x^3}{6} - \frac{x^2}{6}\right)_0^3$$

$$= 18 + 12 - \frac{9}{2} - \frac{3}{2} = 30 - 6 = 24$$

(2) Evaluate : $\int_0^2 \int_1^{e^x} dy\,dx$

$$= \int_{x=0}^{2} \int_{y=1}^{e^x} dy\,dx = \int_{x=0}^{2} (y)_1^{e^x}\,dx$$

$$= \int_0^2 (e^x - 1)\,dx$$

$$= (e^x - x)_0^2$$

$$= e^2 - 2 - 1$$

$$= e^2 - 3$$

③ Evaluate $\int_{-a}^{a}\int_{0}^{\sqrt{a^2-y^2}} x\,dx\,dy$

$$= \int_{y=-a}^{a}\int_{x=0}^{\sqrt{a^2-y^2}} x\,dx\,dy$$

$$= \int_{y=-a}^{a} \left(\frac{x^2}{2}\right)_0^{\sqrt{a^2-y^2}} dy$$

$$= \int_{-a}^{a} \frac{a^2-y^2}{2}\,dy$$

$$= \frac{1}{2}\left(a^2y - \frac{y^3}{3}\right)_{-a}^{a}$$

$$= \frac{1}{2}\left(a^3 - \frac{a^3}{3} + a^3 - \frac{a^3}{3}\right)$$

$$= \frac{1}{2}\left(\frac{6a^3 - 2a^3}{3}\right) = \frac{4a^3}{6}.$$

④ Evaluate $\int_{0}^{\pi}\int_{0}^{a(1-\cos\theta)} 2\pi r^2 \sin\theta\,d\theta\,dr$

$$= \int_{\theta=0}^{\pi}\int_{r=0}^{a(1-\cos\theta)} 2\pi r^2 \sin\theta\,dr\,d\theta$$

$$= \int_{\theta=0}^{\pi} 2\pi\sin\theta \left(\frac{r^3}{3}\right)_0^{a(1-\cos\theta)} d\theta$$

$$= \int_{0}^{\pi} \frac{2\pi\sin\theta}{3}\left(a^3(1-\cos\theta)^3\right) d\theta$$

$$= \frac{2}{3}\pi \int_0^{\pi} \left[\sin\theta \, a^3 \left(1 - \cos^3\theta + 3\cos^2\theta - 3\cos\theta\right)\right] d\theta$$

$$= \frac{2\pi a^3}{3} \int_0^{\pi} \left(\sin\theta - \sin\theta\cos^3\theta + 3\sin\theta\cos^2\theta - \frac{3}{2}\sin 2\theta\right) d\theta$$

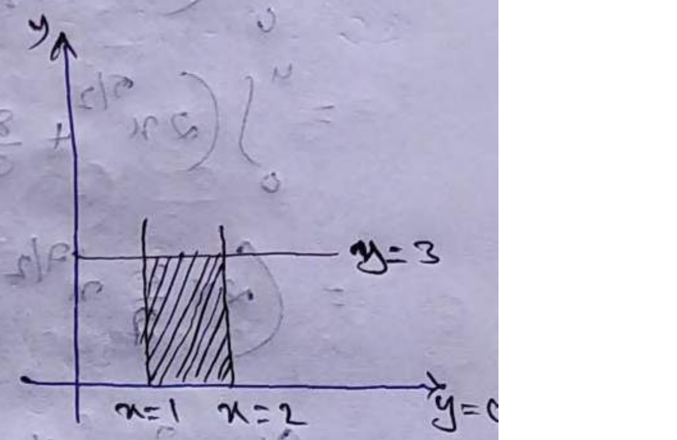
$$= \frac{2\pi a^3}{3}\left(-\cos\theta - \frac{(-\cos^4\theta)}{4} + 3\frac{(-\cos^3\theta)}{3} + \frac{3}{2}\cdot\frac{\cos 2\theta}{2}\right)_0^{\pi}$$

$$= \frac{2\pi a^3}{3}\left((1+1) + \frac{1}{4}\cancel{(1-1)} - (-1-1) + \frac{3}{4}\cancel{(1-1)}\right)$$

$$= \frac{2\pi a^3}{3}(2+2) = \frac{8\pi a^3}{3}$$

(5) Evaluate $\iint_R (4xy - y^2)\,dx\,dy$; R is the rectangle bounded by $x=1, x=2, y=0, y=3$.

$\therefore \iint_R (4xy - y^2)\,dx\,dy$



x : 1 to 2

y : 0 to 3

$$\therefore \int_{y=0}^{3}\int_{x=1}^{2} (4xy - y^2)\,dx\,dy$$

$$= \int_0^3 \left(2x^2y - xy^2\right)_1^2 dy$$

$$= \int_0^3 \left(8y - 2y^2 - 2y + y^2\right) dy = \int_0^3 (6y - y^2)\,dy$$

$$= \left(3y^2 - \frac{y^3}{3}\right)_0^3 = 27 - 9 = 18.$$

⑥ Evaluate $\iint_R x^2+y^2\,dx\,dy$, R is the region bounded by $x=y$, $y^2=4x$.

$y=x$, $y^2=4x$

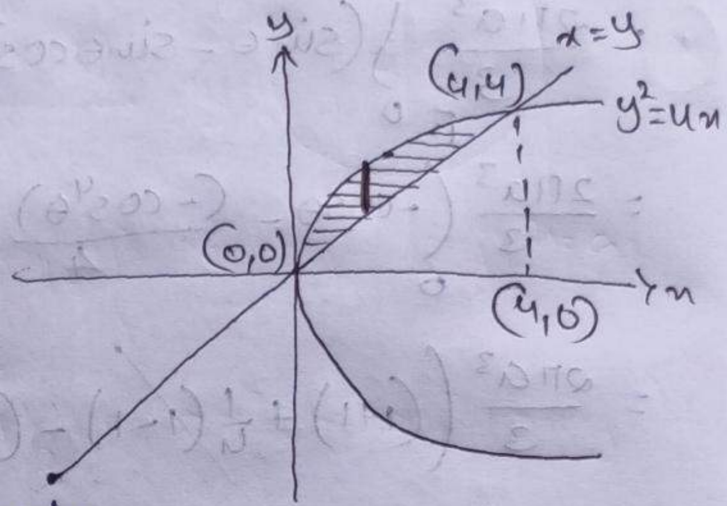
$y^2-4y=0$

$y=0,4$

$x=0,4$

$\therefore x: 0$ to 4

$y: x$ to $2\sqrt{x}$



$\therefore \iint_R x^2+y^2\,dx\,dy$

$$= \int_{x=0}^{4}\int_{y=x}^{2\sqrt{x}} (x^2+y^2)\,dy\,dx$$

$$= \int_0^4 \left(x^2y+y^3/3\right)_x^{2\sqrt{x}}\,dx$$

$$= \int_0^4 \left(2x^{5/2}+\frac{8}{3}x^{3/2}-x^3-\frac{x^3}{3}\right)dx$$

$$= \left(2\times\frac{2}{7}x^{7/2}+\frac{8}{3}\times\frac{2}{5}x^{5/2}-\frac{x^4}{4}-\frac{x^4}{12}\right)_0^4$$

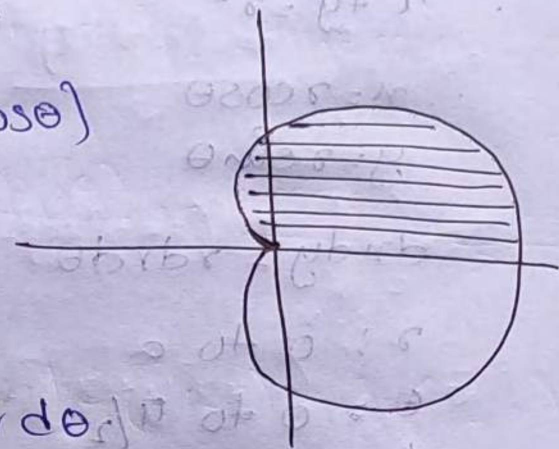
$$= \left(\frac{4}{7}\times 2^7+\frac{16}{15}(2^5)-4^3-\frac{4^3}{3}\right)_0$$

$$= 4^3\left[\frac{8}{7}+\frac{8}{15}-\frac{4}{3}\right] = 4^4\left(\frac{2}{7}+\frac{2}{15}-\frac{1}{3}\right)$$

$$= 4^4\left(\frac{44}{105}-\frac{1}{3}\right) = 4^4\left(\frac{132-105}{105\times3}\right)$$

$$= 4^4\left(\frac{27}{35\times9}\right) = \frac{768}{35}$$

(7) Evaluate $\iint r\sin\theta \, dr\, d\theta$ over the centroid $r = a(1+\cos\theta)$ above the initial value line.



$\therefore r \rightarrow 0$ to $a(1+\cos\theta)$

$\theta \rightarrow 0$ to π

$\iint r\sin\theta \, dr\, d\theta$

$$\int_{\theta=0}^{\pi}\int_{r=0}^{a(1+\cos\theta)} r\sin\theta \, dr\, d\theta$$

$$= \int_0^{\pi}\left(\frac{r^2}{2}\right)_0^{a(1+\cos\theta)} \sin\theta \, d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi}\left(1+\cos^2\theta+2\cos\theta\right)\sin\theta \, d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi}\left(\sin\theta+\sin\theta\cos^2\theta+\sin 2\theta\right)d\theta$$

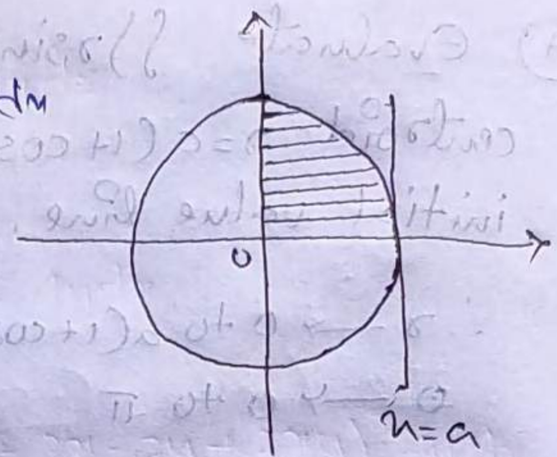
$$= \frac{a^2}{2}\left(-\cos\theta-\frac{\cos^3\theta}{3}-\frac{\cos 2\theta}{2}\right)_0^{\pi}$$

$$= -\frac{a^2}{2}\left(-1-1-\frac{1}{3}-\frac{1}{3}-\frac{1}{2}+\frac{1}{2}\right)$$

$$= \frac{a^2}{2}\left(2+\frac{2}{3}\right) = \frac{4a^2}{3} \,//$$

(8) Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\, dy\, dx$ by changing the into polar coordinates.

$\therefore \int_0^a \int_0^{\sqrt{a^2-r^2}} \sqrt{x^2+y^2}\, dy\, dx$



$\therefore x^2+y^2=r^2$

$\therefore x = r\cos\theta$

$y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

$r : 0$ to a

$\theta : 0$ to $\pi/2$

$\therefore \int_0^{\pi/2} \int_0^a (r) r\, dr\, d\theta$

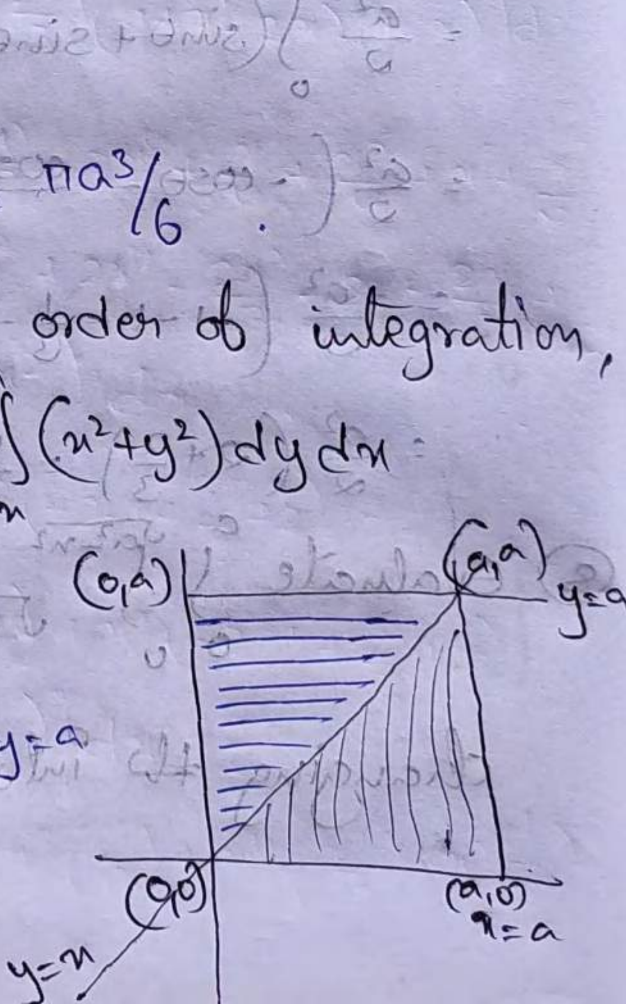
$= \int_0^{\pi/2} \int_0^a r^2\, dr\, d\theta$

$= \int_0^{\pi/2} \left(\frac{r^3}{3}\right)_0^a d\theta$

$= \frac{a^3}{3} (\theta)_0^{\pi/2} = \pi a^3/6.$

(9) By changing the order of integration, evaluate $\int_0^a \int_x^a (x^2+y^2)\, dy\, dx$

$\int_{x=0}^{a} \int_{y=x}^{a} (x^2+y^2)\, dy\, dx$



$\therefore x=y \quad x=a,\ y=a$

$\therefore x : 0$ to y

$y : 0$ to a

$$\Rightarrow \int_{y=0}^{a}\int_{x=0}^{y} (x^2+y^2)\,dx\,dy$$

$$= \int_0^a \left(\frac{x^3}{3}+xy^2\right)_0^y dy$$

$$= \int_0^a \frac{y^3}{3}+y^3\,dy$$

$$= \left(\frac{y^4}{12}+\frac{y^4}{4}\right)_0^a = \frac{a^4}{12}+\frac{a^4}{4} = \frac{a^4}{3}$$ //

(10) Evaluate : $\int_1^e\int_1^e\int_1^e (xyz)^{-1}\,dx\,dy\,dz$

$$= \int_1^e \frac{1}{x}dx \int_1^e \frac{1}{y}dy \int_1^e \frac{1}{z}dz$$

$$= (\log x)_1^e \, (\log y)_1^e \, (\log z)_1^e$$

$$= \log e \,\log e\, \log e = (\log e)^3 = 1^3 = 1.$$

(11) Evaluate $\iiint_V xyz\,dz\,dy\,dx$ where V is the domain bounded by the co-ordinate planes and the plane $x+y+z=1$.

$\therefore$ $z : 0$ to $1-x-y$

$y : 0$ to $1-x$

$x : 0$ to 1

$$\int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y} xyz\,dz\,dy\,dx$$

$$= \int_0^1 \int_0^{1-x} xy\left(\frac{z^2}{2}\right)_0^{1-x-y} dy\,dz$$

$$= \int_0^1 \int_0^{1-x} \frac{xy(1-x-y)^2}{2}\,dy\,dx$$

$$= \frac{1}{2}\int_0^1 \int_0^{1-x} xy(1+x^2+y^2-2x-2y+2xy)\,dy\,dx$$

$$= \frac{1}{2}\int_0^1 \int_0^{1-x} (xy+x^3y+xy^3-2x^2y-2xy^2+2x^2y^2)\,dy\,dx$$

$$= \frac{1}{2}\int_0^1 \left(x\frac{y^2}{2}+x^3\frac{y^2}{2}+x\frac{y^4}{4}-2x^2\frac{y^2}{2}-2x\frac{y^3}{3}+2x^2\frac{y^3}{3}\right)_0^{1-x} dx$$

$$= \frac{1}{2}\int_0^1 \left[\left(\frac{x}{2}+\frac{x^3}{2}-x^2\right)(1+x^2-2x)+\frac{x}{4}(1+x^2-2x)^2+\left(\frac{2x^2}{3}-\frac{2x}{3}\right)(1+x^2-2x)(1-x)\right]dx$$

$$= \frac{1}{2}\int_0^1 \left(\frac{x}{2}+3x^3+2x^2+\frac{x^5}{2}-2x^4+\frac{x}{4}+\frac{x^5}{4}+x^3+\frac{x^3}{2}-x^2-x^4+\frac{2x^2}{3}+2x^4-\frac{2x^5}{3}-2x-\frac{2x}{3}+\frac{2x^4}{3}-2x^3+2x^2\right)dx$$

$$= \frac{1}{2}\int_0^1 \left[x\left(\frac{1}{2}+\frac{1}{4}-\frac{2}{3}\right)+x^2\left(-2-1+\frac{2}{3}+2\right)+x^3\left(3+1+\frac{1}{2}-2-2\right)+x^4\left(-2-1+\frac{2}{3}+2\right)+x^5\left(\frac{1}{2}+\frac{1}{4}-\frac{2}{3}\right)\right]dx$$

$$= \frac{1}{2}\int_0^1 x\left(\frac{1}{12}\right)+x^2\left(\frac{-1}{3}\right)+x^3\left(\frac{1}{2}\right)+x^4\left(\frac{-1}{3}\right)+x^5\left(\frac{1}{12}\right)dx$$

$$= \frac{1}{2}\left(\frac{x^2}{24} - \frac{x^3}{9} + \frac{x^4}{8} - \frac{x^5}{5} + \frac{x^6}{72}\right)_0^1$$

$$= \frac{1}{48} - \frac{1}{18} + \frac{1}{16} - \frac{1}{30} + \frac{1}{144} = \frac{1}{720}$$

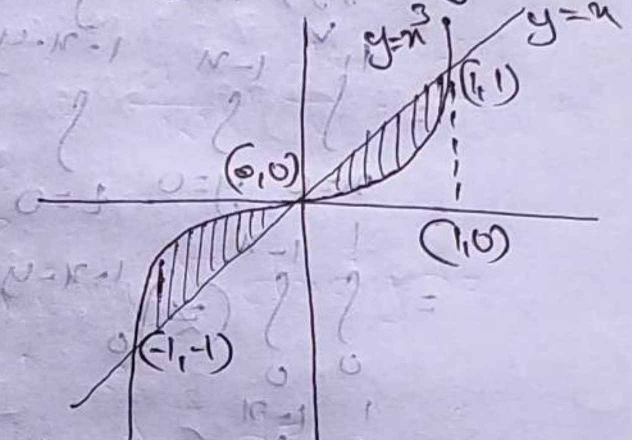
(12) Find by double integration, the area bounded by the curves $y = x^3$ and $y = x$

$y = x^3$, $y = x$

$\therefore x = \pm 1$

$\therefore x : 0$ to 1

$y : x^3$ to x



$\therefore$ Area,

$$\bar{A} = 2\int_{x=0}^{1}\int_{y=x^3}^{x} dy\,dx$$

$$= 2\int_0^1 (y)_{x^3}^{x}\,dx$$

$$= 2\int_0^1 (x - x^3)\,dx$$

$$= 2\left(\frac{x^2}{2} - \frac{x^4}{4}\right)_0^1 dx$$

$$= 2\left(\frac{1}{2} - \frac{1}{4}\right)$$

$$= 2\left(\frac{1}{4}\right) = \frac{1}{2} \text{ sq. units.}$$

(14) using double integration, find the volume of the solid bounded by the co-ordinate plane $x = 0$, $y = 0$, $z = 0$ and the plane $x + y + z = 1$.

$\therefore\ x+y+z=1$

$z: 0$ to $1-x-y$

$y: 0$ to $1-x$

$x: 0$ to 1

volume

$$V = \iiint_V dx\,dy\,dz$$

$$= \int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y} dz\,dy\,dx$$

$$= \int_0^1\int_0^{1-x} (z)_0^{1-x-y}\,dy\,dx$$

$$= \int_0^1\int_0^{1-x} (1-y-x)\,dy\,dx$$

$$= \int_0^1 \left(y-\frac{y^2}{2}-xy\right)_0^{1-x} dx$$

$$= \int_0^1 1-x-x(1-x)-\frac{1+x^2-2x}{2}\,dx$$

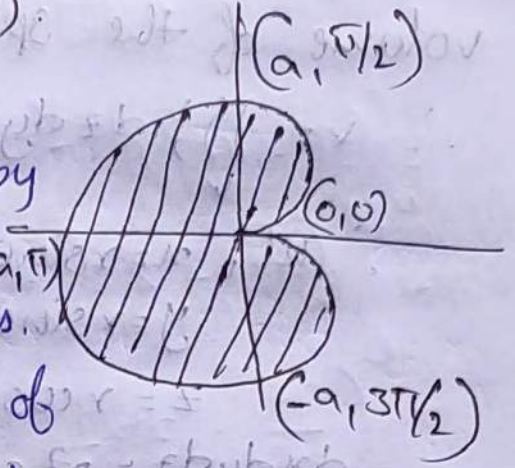
$$= \int_0^1 \left(1-2x+x^2-\frac{1}{2}(\cancel{1}+x^2-2x)\right)dx$$

$$= \left[x-x^2+\frac{x^3}{3}-\frac{1}{2}\left(x+\frac{x^3}{3}-x^2\right)\right]_0^1 dx$$

$$= \cancel{1}-\cancel{1}+\frac{1}{3}-\cancel{\frac{1}{2}}-\frac{1}{6}+\cancel{\frac{1}{2}}$$

$$= \frac{2}{6}-\frac{1}{6}=\frac{1}{6}.$$

(13) using double integral, find the area of centroid : $r = a(1+\cos\theta)$.



The plane lamina which is bounded by the centroid is symmetrical about x-axis.

Therefore, centroid of lamina lies on x-axis.

$\therefore A = \iint dx\,dy$

$dx\,dy = r\,dr\,d\theta$

$\theta : 0$ to 2π

$r : 0$ to $a(1+\cos\theta)$

$$A = \int_{\theta=0}^{2\pi} \int_{r=0}^{a(1+\cos\theta)} r\,dr\,d\theta$$

$$= \int_0^{2\pi} \left(\frac{r^2}{2}\right)_0^{a(1+\cos\theta)} d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} a^2(1+\cos^2\theta + 2\cos\theta)\,d\theta$$

$$= \frac{a^2}{2}\left(\theta + \int \frac{1+\cos 2\theta}{2}d\theta + 2\sin\theta\right)_0^{2\pi}$$

$$= \frac{a^2}{2}\left(\theta + \frac{\theta}{2} + \frac{\cancel{\sin 2\theta}}{2\times 2} + 2\cancel{\sin\theta}\right)_0^{2\pi}$$

$$= \frac{a^2}{2}(2\pi + \pi) = \frac{3\pi a^2}{2}.$$

(15) using triple integration, find the volume of the sphere $x^2+y^2+z^2=a^2$.

$$\therefore V = \iiint_V dz\,dy\,dx$$

$\therefore$ let, $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

$r = 0$ to a, $\theta = 0$ to $\pi/2$, $\phi = 0$ to $\pi/2$

$$dx\,dy\,dz = r^2 \sin\theta \, dr\, d\theta\, d\phi$$

$$\therefore \iiint_V dz\,dy\,dz = 8\int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2}\int_{r=0}^{a} r^2 \sin\theta \, dr\, d\theta\, d\phi$$

$$= 8\int_0^{\pi/2}\int_0^{\pi/2} \frac{a^3}{3}\sin\theta\, d\theta\, d\phi$$

$$= \frac{8a^3}{3}\int_0^{\pi/2} \left(-\cos\theta\right)_0^{\pi/2} d\phi$$

$$= \frac{8a^3}{3}(1)\left(\phi\right)_0^{\pi/2}$$

$$= \frac{4\pi a^3}{3}$$