BOUNDARY CONDITIONS

* So far, we considered the Electric field $\vec{E}$ in a homogeneous medium.

* If field $\vec{E}$ exists in a region consisting of <u>two different media</u>, the condition that the field must satisfy at the interface seperating the media is called <u>boundary Conditions</u>.

* These conditions are helpful in determining the field on one side of the boundary if the field on the other side is known.

1) DiElectric (ε_{r_1}) and DiElectric (ε_{r_2}).

2) Conductor and DiElectric

3) Conductor and freespace

→ To determine the boundary conditions, we use Maxwell's Equ.

$$\left\{ \begin{aligned} &\oint_L \vec{E}\cdot\vec{dl} = 0 \\ \text{and} \quad &\oint_S \vec{D}\cdot\vec{ds} = Q_{enc} \end{aligned} \right\}$$

E = Electric field intensity

D = Electric ~~field~~ flux Density

→ Also, decompose the Electric field intensity $\vec{E}$ into two Orthogonal components.

$$\{ \vec{E} = \vec{E}_t + \vec{E}_n \}$$

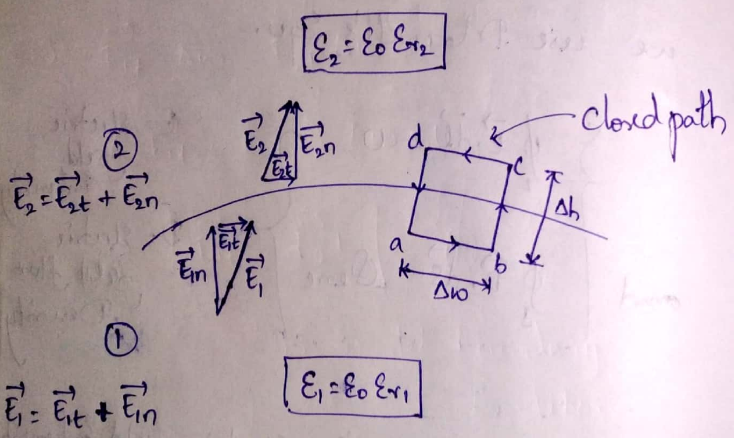
$\vec{E}_t$ = tangential Component of $\vec{E}$

$\vec{E}_n$ = Normal " " $\vec{E}$

Similarly, Electric flux density $\vec{D}$

$$\left\{ \vec{D} = \vec{D}_t + \vec{D}_n \right\}$$

Dielectric - Dielectric Boundary Conditions



Consider two dielectric mediums with permitivities ε_1 & ε_2 such that

$$\varepsilon_1 = \varepsilon_0 \varepsilon_{r_1}$$

$$\varepsilon_2 = \varepsilon_0 \varepsilon_{r_2}$$

Let $\vec{E_1}$ and $\vec{E_2}$ be the field strengths of medium 1 and 2 respectively. Let $\vec{E}_{1t}$ and $\vec{E}_{2t}$ be the tangential components of Electric fields and $\vec{E}_{1n}$ and $\vec{E}_{2n}$ be normal component of Electric fields in mediums 1 & 2 respectively.

$\therefore$ we can write

$$\vec{E_1} = \vec{E}_{1t} + \vec{E}_{1n}$$

$$\vec{E_2} = \vec{E}_{2t} + \vec{E}_{2n}$$

Consider a closed path A-B-C-D-A with Δw and Δh as length and height of considered path w.r.t boundary

Now if we apply maxwell's 2nd Equ to closed path, we get

$$\oint_L \vec{E} \cdot d\vec{l} = 0.$$

$$\int_A^B \vec{E}\cdot\vec{dl} + \int_B^C \vec{E}\cdot\vec{dl} + \int_C^D \vec{E}\cdot\vec{dl} + \int_D^A \vec{E}\cdot\vec{dl} = 0$$

$$\Rightarrow \vec{E}_{1t}\cdot\Delta w + \cancel{\vec{E}_{1n}\frac{\Delta h}{2}} + \cancel{\vec{E}_{2n}\frac{\Delta h}{2}} - \vec{E}_{2t}\,\Delta w - \cancel{\vec{E}_{2n}\frac{\Delta h}{2}} - \cancel{\vec{E}_{1n}\frac{\Delta h}{2}} = 0$$

$$\Rightarrow \vec{E}_{1t}\cdot\Delta w - \vec{E}_{2t}\,\Delta w = 0$$

$$\Rightarrow (\vec{E}_{1t} - \vec{E}_{2t})\,\Delta w = 0$$

$$\Rightarrow \boxed{\vec{E}_{1t} = \vec{E}_{2t}} \quad \text{—} (1)$$

This is 1st boundary Condition.

$\because \vec{D} = \varepsilon\vec{E} \Rightarrow \vec{D}_{1t} = \varepsilon_1\vec{E}_{1t}$

$$\vec{E}_{1t} = \frac{\vec{D}_{1t}}{\varepsilon_1} \quad \& \quad \vec{E}_{2t} = \frac{\vec{D}_{2t}}{\varepsilon_2}$$

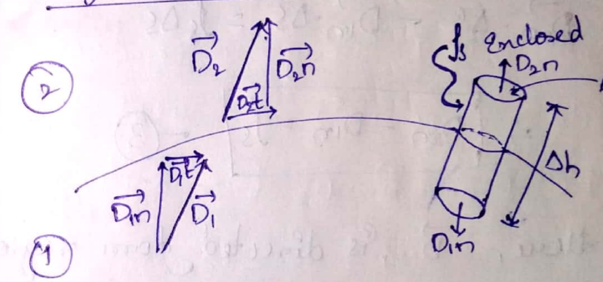
$\therefore$ from (1)

$$\boxed{\frac{D_{1t}}{\varepsilon_1} = \frac{D_{2t}}{\varepsilon_2}} \quad \text{—} (2)$$

from (1) & (2)

Tangential Components of Electric field intensity $\vec{E}$ are continous along boundary while tangential components of Electric flux density are discontinous along boundary.

To get normal Components



→ Consider, cylindrical gaussian Surface. In which Apply Maxwell 1st Equ.

$$\boxed{\oint_S \vec{D}\cdot\vec{ds} = Q_{enc}}$$

ρ_s is surface Charge density and it's Cross sectional Area is ΔS.

Cancel due to Symmetry

$$\int_{top} \vec{D}\cdot d\vec{s} + \int_{bottom} \vec{D}\cdot d\vec{s} + \cancel{\int_{side} \vec{D}\cdot d\vec{s}} = Q_{enc}$$

$$\Rightarrow \int_{top} \vec{D}\cdot d\vec{s} + \int_{bottom} \vec{D}\cdot d\vec{s} = Q_{enc}$$

$$\Rightarrow \vec{D_{2n}}\cdot \Delta S - \vec{D_{1n}}\cdot \Delta S = \rho_s \Delta S$$

$$\Rightarrow \boxed{D_{2n} - D_{1n} = \rho_s} \quad —(3)$$

Here, $\vec{D}$ is directed from region 1 to region 2.

NOTE: (1) If no free charges at interface

i.e $\rho_s = 0$

So, Eqn (3) becomes

$$\boxed{D_{1n} = D_{2n}} \quad —(4)$$

Normal component of $\vec{D}$ is continuous across the interface.

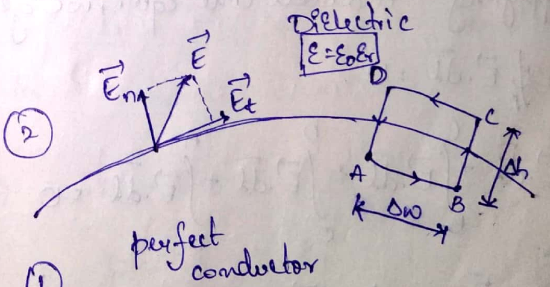
(2) Since, $\vec{D} = \varepsilon\vec{E}$

$$\boxed{\varepsilon_1 E_{1n} = \varepsilon_2 E_{2n}} \quad —(5)$$

Normal component of $\vec{E}$ is discontinuous at the boundary/interface.

Conductor - DiElectric Boundary Conditions

Here we take two regions conductor and dielectric.



$\boxed{\vec{E}=0}$ $\left(\begin{matrix}\sigma \to \infty \\ \rho_c \to 0\end{matrix}\right)$ $\left[\begin{matrix}\sigma \\ \downarrow \\ \text{Conductivity}\end{matrix}\right]$

[Ex:- Copper & Silver]

~~Apply Maxwell~~

In this case we assume a perfect conducting medium i.e $\rho_v = 0$ and conductivity tends to ∞

$$\sigma \to \infty, \; \rho_v = 0.$$

And w.k.T along the conductor field components are zeros i.e $\vec{E}$ and $\vec{D} = 0$ at conductor boundary.

Applying maxwell line Equ., we get

$$\oint_L \vec{E} \cdot \vec{dl} = 0.$$

$$\int_A^B \vec{E} \cdot \vec{dl} + \int_B^C \vec{E} \cdot \vec{dl} + \int_C^D \vec{E} \cdot \vec{dl} + \int_D^A \vec{E} \cdot \vec{dl} = 0$$

$$\Rightarrow 0 \cdot \Delta w + 0 \cdot \frac{\Delta h}{2} + \vec{E_n} \cdot \frac{\Delta h}{2} - \vec{E_t} \Delta w - \vec{E_n} \cdot \frac{\Delta h}{2} - 0 \cdot \frac{\Delta h}{2} = 0.$$

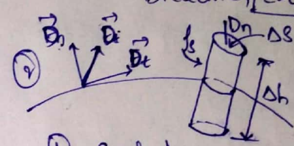
$$\Rightarrow 0 + 0 + \cancel{\vec{E_n} \cdot \frac{\Delta h}{2}} - \vec{E_t} \Delta w - \cancel{\vec{E_n} \cdot \frac{\Delta h}{2}} - 0 = 0.$$

$$\Rightarrow \vec{E_t} \cdot \Delta w = 0$$

or $\boxed{\vec{E_t} = 0}$ — (1) $\boxed{\frac{D_t}{\varepsilon} = 0}$

To get normal component

Dielectric, $\boxed{\varepsilon = \varepsilon_0 \varepsilon_r}$



(1) Conductor

$\boxed{\vec{E} = 0 \text{ (or) } \vec{D} = 0}$

Apply Maxwell Surface Equ.

$$\boxed{\oint_S \vec{D} \cdot \vec{ds} = Q_{enc}}$$

Cancel due to Symmetry

$$\int_{top} \vec{D} \cdot ds + \int_{bottom} \vec{D} \cdot \vec{ds} + \int_{Surface} \cancel{\vec{D} \cdot \vec{ds}} = Q_{enc}$$

$$\Rightarrow D_n \cdot \Delta S - 0 \cdot \Delta S = \rho_s \Delta S$$

$$\Rightarrow D_n \Delta S = \rho_s \Delta S$$

$$\Rightarrow \boxed{D_n = \rho_s} \quad \text{— (2)}$$

As, we know

$$\vec{D} = \varepsilon \vec{E}$$

$$\Rightarrow \boxed{\varepsilon \vec{E_n} = \rho_s}$$

∴ The following Conclusions canbe made out of these boundary conditions.

(1) No Electric field Exist in conductor

(2) As $\vec{E}$ is 0 then potential difference b/w any two points in conductor is 0. So conductor is an Equi potential body.

(3) Electric field is External to conductor & it must be normal to the surface.

MAGNETIC BOUNDARY CONDITIONS

$\oint_s \vec{B}\cdot\vec{ds} = 0$ —— Gauss's Law of Magnetic field

$\oint_l \vec{H}\cdot\vec{dl} = I$ —— Ampere's Circuit Law

Boundary Conditions b/w two Magnetic Media:

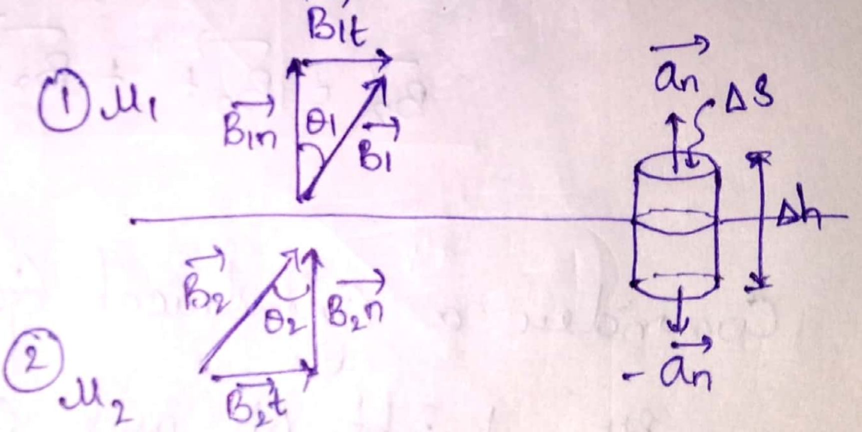


Fig: Boundary b/w two Magnetic media 1 & 2.

Consider two magnetic mediums with permeability μ_1 & μ_2.

Let $\vec{B_1}$ & $\vec{B_2}$ be the magnetic flux density of medium 1 & 2, ~~and~~

Let $\vec{B_1 t}$ & $\vec{B_2 t}$ be the tangential Components of Magnetic flux densities $\vec{B_1}$ & $\vec{B_2}$, And $\vec{B_1 n}$ & $\vec{B_2 n}$ be the normal components. ~~of~~

$\therefore$ we can write

$$\vec{B_1} = \vec{B_1 t} + \vec{B_1 n}$$

$$\vec{B_2} = \vec{B_2 t} + \vec{B_2 n}$$

Consider a Cylindrical Gaussian Surface with height Δh and Cross Section Area of ΔS, ~~where hd~~

Now by apply Maxwell's closed surface Integration.

$$\oint_S \vec{B} \cdot \vec{ds} = 0$$

Cancel due to Symmetrical surface

$$\Rightarrow \int_{top} \vec{B} \cdot \vec{ds} + \int_{bottom} \vec{B} \cdot \vec{ds} + \cancel{\int_{side} \vec{B} \cdot \vec{ds}} = 0$$

$$\Rightarrow B_{1n} \Delta S - B_{2n} \Delta S + 0 = 0$$

$$\Rightarrow \boxed{B_{1n} = B_{2n}} \quad \text{--- (1)}$$

$\therefore$ Normal Component of $\vec{B}$ is continous at the boundary.

from (1).

$$\because \vec{B} = \mu \vec{H}$$

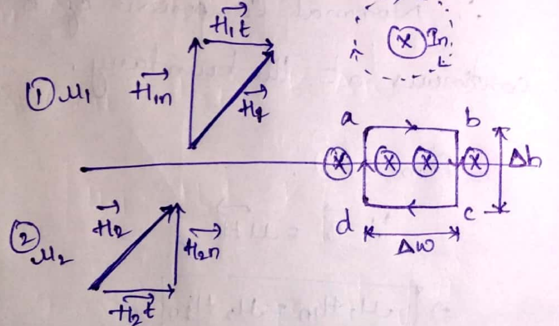
$$\Rightarrow \boxed{\mu_1 H_{1n} = \mu_2 H_{2n}}$$

$$\boxed{\frac{H_{1n}}{H_{2n}} = \frac{\mu_2}{\mu_1}}$$

$\therefore$ Normal Component of $\vec{H}$ is discontinous at the boundary.

To find tangential Component

Consider a two magnetic media with permeability μ_1 & μ_2 of media 1 & 2 respectively.



Let $\vec{H_1}$ & $\vec{H_2}$ be the magnetic field densities of media 1 & 2 respectively.

Let $\vec{H_{1t}}$ & $\vec{H_{2t}}$ be the tangential Components and $\vec{H_{1n}}$ & $\vec{H_{2n}}$ be the normal components of Magnetic field densities $\vec{H_1}$ & $\vec{H_2}$.

$$\therefore \vec{H_1} = \vec{H_{1t}} + \vec{H_{1n}}$$

$$\vec{H_2} = \vec{H_{2t}} + \vec{H_{2n}}$$

Consider a closed path A-B-C-D-A with length Δw and height Δh.

Now apply Maxwell's closed Line Integration.

$$\oint_L \vec{H} \cdot \vec{dl} = I$$

$$\Rightarrow \int_a^b \vec{H} \cdot \vec{dl} + \int_b^c \vec{H} \cdot \vec{dl} + \int_c^d \vec{H} \cdot \vec{dl} + \int_d^a \vec{H} \cdot \vec{dl} = I$$

$$\Rightarrow H_{1t} \cdot \Delta w + \left[- H_{1n} \cdot \frac{\Delta h}{2} - H_{2n} \frac{\Delta h}{2} \right] + \left[- H_{2t} \cdot \Delta w \right]$$

$$+ \left[H_{2n} \cdot \frac{\Delta h}{2} + H_{1n} \frac{\Delta h}{2} \right] = I$$

$$\Rightarrow H_{1t}\cdot \Delta w - \cancel{H_{1n}\frac{\Delta h}{2}} - \cancel{H_{2n}\frac{\Delta h}{2}} - H_{2t}\cdot \Delta w$$

$$+ \cancel{H_{2n}\cdot\frac{\Delta h}{2}} + \cancel{H_{1n}\frac{\Delta h}{2}} = K\cdot \Delta w$$

$\because K = \frac{I}{b}$

$\hookrightarrow$ ⊥r distance

$$\Rightarrow H_{1t}\cdot \Delta w - H_{2t}\cdot \Delta w = K\cdot \Delta w$$

$$\Rightarrow \Delta w(H_{1t} - H_{2t}) = K\cdot \Delta w$$

$$\Rightarrow \boxed{H_{1t} - H_{2t} = K} \quad \text{--- (2)}$$

$\therefore$ Tangential Component of $\vec{H}$ is discontinous ~~at the boundary~~.

<u>NOTE</u>: If boundary is free of Current, or media are not conductors.

i.e $\vec{K} = 0$.

So, Equ. (2) becomes

$$\boxed{H_{1t} = H_{2t}} \quad \text{--- (3)}$$

$\therefore$ Tangential Component of $\vec{H}$ is continous at the boundary.

from (3).

$\because B = \mu H$

$\Rightarrow H = \frac{B}{\mu}$

$$\boxed{\frac{B_{1t}}{\mu_1} = \frac{B_{2t}}{\mu_2}}$$

$\therefore$ Tangential Component of $\vec{B}$ is discontinous at the boundary.