UNIT 2

Part 1

UNIT – III : Operation on One Random Variable – Expectations.

Syllabus :

1. Introduction
2. Expected Value of a Random Variable
3. function of a Random variable.
4. Moments about the origin
5. Central Moments
6. Variance and Skew
7. Chebyshev's Inequality
8. Characteristic function
9. Moment Generating function
10. Transformations of a Random Variable :
    - (i) Monotonic Transformations for a continuous R.V
    - (ii) Non " " " " "
    - (iii) Transformation of a Discrete Random Variable.

Assignment - 1. , UNIT - 1

5-2008
Set-1

1. Define probability with an Axiomatic approach. (8M)

Set-2

2. Define and Explain the following with Example
   i. Sample space
   ii. Discrete Sample Space
   iii. Continuous Sample space (12M)

Set-3

3. Define an Event and when do we say the Events are mutually Exclusive. Explain with an Example (8M)

Set-4

4. Show that two Elements cannot be both mutually Exclusive and statistically independent. What is the condition for two ~~Elements~~ Events to be independent. (6M)

Nov-07
Set-1

5. Discuss Joint and conditional probability (8M)

Set-2

6. ~~When~~ ~~are~~ with an Example define and Explain the following.
   i. Equally likely Events
   ii. Exhaustive Events (6M)
   iii. Mutually Exclusive Events

Set-3

7. Define probability based on Set theory and fundamental axioms (8M)

set 4.

8. Give the classical definition of probability (4M)

Nov - 06
set - 4

9. What is Bayes' theorem ? Explain — (6M)

Nov - 05
set - 3.

10. Give the classical and axiomatic definitions of probability.

Nov - 05
set - 4.

11. State the ~~total~~ theorem on Total probability (4M)

UNIT - III : STATISTICAL ANALYSIS

§ INTRODUCTION :-

In the previous chapter, we have discussed the concept of random variables and their various properties. The random variable is used as a mathematical tool for describing the characteristics of some real, physical random phenomenon. In this chapter, we shall study some important statistical operations that may be performed on a random variable. Expectation, Variance, moments --- etc, include statistical operations. In some physical situations, the study of a Average quantities (Expected value), will be useful for the study of actual distribution systems.

(or (Expected value of Random Variable)) UNIT-2 ECE

MATHEMATICAL EXPECTATION :- (MEAN VALUE)

The averaging process, when applied to a random variable, is called the Expectation. It is denoted by $E[X]$ and is read as Expected value of X or Mean value of X. or Statistical Average of X

$E[X] = \bar{X} = \mu = m$

$E[X] = \bar{X}$

Suppose X is a discrete random variable which takes on ~~takes~~ ~~of~~ values in a finite set $x_1, x_2 \cdots x_N$, with probabilities $p(x_i) = P(X = x_i)$ for $i = 1, 2 \cdots N$, then the Expectation $E[X]$ of the discrete random variable X is given by

$$\boxed{E[X] = m = \sum_{i=1}^{N} x_i \cdot p(x_i).}$$

Thus $E[X]$ is the weighted arithmetic mean of the values x_i with the weights $p(x_i)$. physically, the Expectation $E[X]$ of a random variable X is the probability weighted average of the values of X.

If X is a continuous random variable, the concept of average can be (performed) formulated in terms of integrals as

$$E[X] = \int_{-\infty}^{\infty} x \cdot f_X(x) \, dx.$$

provided $\int_{-\infty}^{\infty} |x| f_X(x) \, dx < \infty$, where $f_X(x)$ is the probability density function of the continuous random

§ Expected value of function of a random variable :-

Mathematical Expectation of some real function "g" of the random variable X is given by

$$E[g(X)] = \sum_{i=1}^{N} g(x_i) \cdot p(x_i) \quad \because X \text{ is discrete r.v.}$$

and

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) \cdot f_X(x)\, dx \quad \because X \text{ is continuous r.v.}$$

The expected value (or mean) of random variable X is often called as First Moment m_1 of X.

§ PROPERTIES :-

1) If the probability density function $f_X(x)$ of a random variable X is symmetrical about a point "a", i.e. $f(a-x) = f(a+x)$, then $E[X] = a$

2) If the pdf $f_X(x)$ of a random variable X is an even function, i.e. $f(-x) = f(x)$, then

$$E[X] = 0$$

3) The expectation of any given linear function of X, is equal to the sum of the linear functions of the expectation of X. For example

$$E[aX+b] = a\,E[X] + b$$

4) The mathematical expectation of sum of random variables is equal to the sum of their individual expectations, provided all the expectations exist. For example, if X, Y, Z, --- T are 'n' random variables, then

$$E[X+Y+Z+\cdots+T] = E[X] + E[Y] + \cdots + E[T]$$

5) The mathematical expectation of the product of number of independent random variables is equal to the product of their individual expectations. For example, if X, Y, Z --- T are 'n' number of independent random variables, then

$$E[XYZ\cdots T] = E[X]\cdot E[Y]\cdot E[Z] \cdots E[T]$$

6) $\left| E[g(x)] \right| \leq E\left[|g(x)| \right]$

7) $E[g_1(X)] \leq E[g_2(X)]$, if $0 \leq g_1(X) \leq g_2(X)$

8) The conditional expected value is defined as

$$E(X / x \geq a) = \frac{\int_a^\infty x \cdot f_X(x)\, dx}{\int_a^\infty f_X(x)\, dx}$$

$$E(X | y \geq a) = \frac{\int_a^\infty x \cdot f_X(x)\, dx}{\int_a^\infty f_Y(y)\, dy.}$$

extent of dispersion

§ VARIANCE :- (μ_2) : second central moment

or cummulate

In many physical problems, the measure of expected value $E[X]$ of random variable X, does not completely characterize the probability distribution. It may be necessary to find the spread or dispersion about the mean values. The quantity used to measure this dispersion is called the variance of the distribution.

The variance or dispersion or spread of a random variable X is given by

$$\sigma_X^2 = E\left[(X - E(X))^2\right]$$

$$= E\left[(X-m)^2\right]$$

where 'm' represents the mean value or expected value of the random variable X. If 'X' is a ~~con~~ discrete random variable, the variance is given by

$$\sigma_X^2 = \sum_{i=1}^{N} (x_i - m)^2 p(x_i)$$

$$\text{i.e.}\quad \sigma_X^2 = \sum_{i=1}^{N} (x_i - m)^2 p(X = x_i)$$

for a continuous random variable X, the variance is written as

$$\sigma_X^2 = \int_{-\infty}^{\infty} (x-m)^2 f_X(x)\, dx.$$

of

$$\sigma_X^2 = \int_{-\infty}^{\infty} (x-m)^2 f_X(x)\,dx$$

This expression can be further simplified as

$$\sigma_X^2 = \int_{-\infty}^{\infty} (x^2 + m^2 - 2xm) f_X(x)\,dx$$

$$= \int_{-\infty}^{\infty} x^2 f_X(x)\,dx + \int_{-\infty}^{\infty} m^2 f_X(x)\,dx - \int_{-\infty}^{\infty} 2xm\, f_X(x)\,dx$$

$$= E[X^2] + E[m^2] - 2E[Xm]$$

$\boxed{\because E(m) = m}$

$\boxed{E[X] = m}$

$$= E[X^2] + m^2 - 2m\,E[X]$$

$$= E[X^2] + m^2 - 2m \cdot m$$

$$= E[X^2] + m^2 - 2m^2$$

$\Rightarrow E[X^2] - (E[X])^2$

Thus $\sigma_X^2 = E[X^2] - m^2$

$$= m_2 - m_1^2$$

In this expression, $E[X^2]$ is called the second moment, m_2 of the distribution.

The variable is also called as second central moment, of the random variable. The positive square root, σ_X of the ~~variable~~ variance is called the standard deviation of the random variable X.

It is used to measure the spread of X around

§ find the Expected Value of the number on a die when thrown.

Sol: Let X be the r.v. which represents the number on a die when thrown. i.e. X takes on the values 1, 2, 3, 4, 5 and 6. Each with equal probability ($1/6$). Therefore

x_i	1	2	3	4	5	6
$p(x_i)$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$

$$m = E[X] = \sum_{i=1}^{6} x_i \, p(x_i)$$

$$= \left(1\times\frac{1}{6}\right) + \left(2\times 1/6\right) + \left(3\times\frac{1}{6}\right) + \left(4\times\frac{1}{6}\right) + \left(5\times\frac{1}{6}\right) + \left(6\times\frac{1}{6}\right)$$

$$= \frac{1}{6}\left[1+2+3+4+5+6\right] = \frac{21}{6}$$

$$= 7/2$$

§ find the Expected value of the discrete random variable X, whose probability distribution is given by

x	1	2	3	4	5
$P(X=x)$	0.1	0.1	0.3	0.3	0.2

Sol

$$m = E[X] = \sum_{i=1}^{5} x_i \cdot P(x_i)$$

$$= (1\times 0.1) + (2\times 0.1) + (3\times 0.3) + (4\times 0.3) + (5\times 0.2)$$

$$= 0.1 + 0.2 + 0.9 + 1.2 + 1.0$$

$$m = E[X] = 3.4$$

Q. find the mean value of the random variable, whose probability density function is given by

$$f(x) = \frac{3}{5} \cdot 10^{-5} x (100-x), \quad 0 \leq x \leq 100$$

Sol

$$E[X] = \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx$$

$$= \int_0^{100} x^2 \cdot \frac{3}{5} 10^{-5} (100-x)\, dx$$

$$= \frac{3 \times 10^{-5}}{5} \int_0^{100} (100x^2 - x^3)\, dx$$

$$= \frac{3 \times 10^{-5}}{5} \left[\frac{100x^3}{3} - \frac{x^4}{4} \right]_0^{100}$$

$$= 50$$

The mean value $= 50$

Q. Find the dispersion of the discrete random Variable X whose pro. distribution is given below.

x	1	2	3	4	5
$P(X=x)$	0.1	0.1	0.3	0.3	0.2

$$m = E[X] = \sum_{i=1}^{N} x_i \cdot p(x_i)$$

$(1\times0.1)+(2\times0.1)+(3\times0.3)+(4\times0.3)+(5\times0.2) = 3.4$

$$= 3.4$$

$$\sigma_x^2 = \sum_{i=1}^{N} (x_i - m)^2 p(x_i)$$

$$= (1-3.4)^2(0.1) + (2-3.4)^2(0.1) + (3-3.4)^2(0.3) + (4-3.4)^2(0.3) + (5-3.4)^2(0.2)$$

$$= 0.576 + 0.196 + 0.048 + 0.108 + 0.512$$

$$= 1.44$$

$$\sigma_x^2 = 1.44$$

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Let X be the continuous random Variable with

pdf $= \frac{8}{x^3}$, $x>2$, find $E(w)$ where $w = x/3$

$$E(w) = \int_{-\infty}^{\infty} w \cdot f_X(x)\, dx$$

$$= \int_{2}^{\infty} \frac{x}{3} \cdot \frac{8}{x^3}\, dx$$

$$= \frac{8}{3} \cdot \left[-\frac{1}{x}\right]_2^{\infty}$$

$$= \frac{8}{3}\left[-0+\frac{1}{2}\right] = \frac{4}{3}$$

$$E[W] = 4/3$$

$\int \frac{1}{x^2} = -\frac{1}{x}$

§ PROPERTIES OF VARIANCE :-

1. The variance of a random variable ax is equal to a^2 times the variance of x.

$$\sigma_{ax}^2 = E\left[(ax - E(ax))^2\right]$$

$$= E\left[(ax - am)^2\right]$$

$$= E\left[a^2x^2 + a^2m^2 - 2a^2xm\right]$$

$$= a^2 E[x^2] + a^2m^2 - 2a^2m\,E[x]$$

$$= a^2 E[x^2] + a^2m^2 - 2a^2m^2$$

$$= a^2 E[x^2] - a^2m^2$$

$$= a^2\left[E[x^2] - m^2\right]$$

$$\underline{\sigma_{ax}^2 = a^2 \sigma_x^2}, \quad \text{where } \sigma_x^2 = E[x^2] - m^2$$

2. The variance of a random variable $(x+a)$ is equal to the variance of the random variable x itself.

$$\sigma_{x+a}^2 = \sigma_x^2$$

$$\sigma_{x+a}^2 = E\left\{\left[(x+a) - E(x+a)\right]^2\right\}$$

$$= E\left[(x + \cancel{a} - m - \cancel{a})^2\right]$$

$$= E\left[(x-m)^2\right]$$

$$\sigma_{x+a}^2 = \sigma_x^2$$

3. $$\sigma^2_{x\pm y} = \sigma_x^2 + \sigma_y^2 \pm 2\left[E[X\cdot Y] - E[X]\,E[Y] \right]$$

The quantity $\{ E[XY] - E[X]\cdot E[Y] \}$ is called the "<u>Co-Variance</u>" of the random variables X and Y.

$$\therefore \quad \sigma^2_{x\pm y} = \sigma_x^2 + \sigma_y^2 \pm \underbrace{2\cdot \mathrm{COV}(X,Y)}_{\text{Co-Variance}}$$

4 If the two random variables are independent, i.e. $E[XY] = E[X]\cdot E[Y]$, then, the random variables X and Y are Uncorrelated.

Therefore,

$$\sigma^2_{x\pm y} = \sigma_x^2 + \sigma_y^2$$

$E[X\pm Y]$
$= E(X) \pm E(Y)$
$= m_x \pm m_y$

<u>proof of 3 property</u> :-

$$\sigma^2_{x\pm y} = E\left[[X\pm Y - \underbrace{E(X\pm Y)}_{(m_x\pm m_y)}]^2 \right]$$

$$= E\left\{ (x\pm y)^2 + (m_x\pm m_y)^2 - 2(x\pm y)(m_x\pm m_y) \right\}$$

$$= E\left\{ x^2+y^2 \pm 2xy + m_x^2 + m_y^2 \pm 2m_x m_y - 2xm_x \mp 2xm_y - 2ym_x \mp 2ym_y \right\}$$

$$= E[x^2] + E[y^2] \pm E[2xy] + m_x^2 + m_y^2 \pm 2m_x m_y - 2m_x^2 \mp 2m_x m_y - 2m_y m_x \mp 2m_y^2$$

$$= E[x^2] - m_x^2 + E[y^2] - m_y^2 - 2m_y m_x \pm E(2xy)$$

Hence,

§ COVARIANCE:—

If x and y are two random variables, then the co-variance between them is defined as

$$\mathrm{Cov}(x,y) = E\left[\{x - E[x]\}\{y - E[y]\}\right]$$

$$= E\left[xy - xE[y] - yE[x] + E[x]E[y]\right]$$

$$= E[xy] - E[x]E[y] - \cancel{E[x]E[y]} + \cancel{E[x]E[y]}$$

$$\boxed{\mathrm{Cov}(x,y) = E[xy] - E[x]\cdot E[y]}$$

§ prove that $\mathrm{cov}[ax, by] = ab\,\mathrm{cov}(x,y)$

$$\mathrm{cov}[ax, by] = E\left\{\{ax - E(ax)\}\{by - E(by)\}\right\}$$

$$= \cancel{E\{abxy - axE(by) - byE(ax) + E(ax)E(by)\}}$$

$$= E\left[a\{x - E(x)\}\cdot b\{y - E(y)\}\right]$$

$$= ab\cdot E\left[\{x - E(x)\}\cdot\{y - E(y)\}\right]$$

$$= ab\cdot \mathrm{Cov}(x,y)$$

§ Given the following Table

x	-3	-2	-1	0	3	2	1
p(x)	0.05	0.1	0.3	0	0.3	0.15	0.1

Compute (i) $E[X] = 0.65$

4.3 (or) −1.7

(ii) $E(2X \pm 3)$ (iii) $E(4X+5)$: 7.6 (or) −2.4

4.3

−2.7

(iv) $E[X^2]$: 4.55

(v) $V(X)$: 4.1275

(vi) $V(2X \pm 3)$: 16.51

Ans

(i) 0.25 (ii) 3.5 or −2.5 (iii) 6 (iv) 2.95

(v) 8.1875 (vi) 12.75

2.88

ECE-2 | 24/08/10 1 period

61, 62, 63, 66, 76, 78, 82, 83, 84, 87, 90, 91, 92, 95

97, 98, 99, ~~A0~~, A2, A3, A4, A7, B1, B3, B4, B7, C0, (LE7, 8, 10, 99 supply

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Q Obtain the mean and variance of the uniform distribution $f(x) = \frac{1}{b-a}$, $a \le x \le b$

0 , otherwise

Sol:

$$\text{Mean} = E[X] = \int_a^b x \cdot f_X(x)\, dx$$

$$= \int_a^b x \cdot \frac{1}{(b-a)} \cdot dx$$

$$= \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b$$

$$= \frac{1}{b-a} \cdot \left(\frac{b^2-a^2}{2}\right) = \frac{(b+a)\cancel{(b-a)}}{2\cancel{(b-a)}}$$

$$= (a+b)/2$$

Variance $\sigma_X^2 = E[X^2] - (E[X])^2$

$$E(X^2) = \int_a^b x^2 \cdot f_X(x)\, dx$$

$$= \frac{1}{b-a}\left(\frac{b^3-a^3}{3}\right) = \frac{\cancel{(b-a)}(a^2+b^2+ab)}{\cancel{(b-a)}(3)}$$

$$= \frac{a^2+ab+b^2}{3}.$$

$$\therefore \sigma_X^2 = E[X^2] - [E(X)]^2$$

$$= \frac{a^2+ab+b^2}{3} - \frac{(a+b)^2}{4}$$

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If X and Y are two independent random variables, such that $E[X] = \alpha_1$, $E[Y] = \alpha_2$

$V[X] = \sigma_1^2$ $\quad V[Y] = \sigma_2^2$,

prove that,

$$V[XY] = \sigma_1^2\sigma_2^2 + \sigma_1^2\alpha_2^2 + \sigma_2^2\alpha_1^2.$$

<u>Sol</u> :- $\quad Var(x) = E[X^2] - (E[X])^2$

$$Var(XY) = E[X^2Y^2] - [E(XY)]^2$$

$$= E(X^2)\cdot E(Y^2) - E(XY)\cdot E(XY)$$

$$= E(X^2)\cdot E(Y^2) - E(X)E(Y)\cdot E(X)\cdot E(Y)$$

$$Var(XY) = E(X^2)\cdot E(Y^2) - [E(X)]^2[E(Y)]^2$$

$$E[X^2] = Var(x) + [E(X)]^2$$

$$= \sigma_1^2 + \alpha_1^2$$

$$E(Y^2) = Var(Y) + [E(Y)]^2$$

$$= \alpha_2^2 + \sigma_2^2$$

$$Var(XY) = (\sigma_1^2 + \alpha_1^2)(\sigma_2^2 + \alpha_2^2) - \alpha_1^2\alpha_2^2$$

$$= \sigma_1^2\sigma_2^2 + \sigma_1^2\alpha_2^2 + \sigma_2^2\alpha_1^2 + \cancel{\alpha_1^2\alpha_2^2} - \cancel{\alpha_1^2\alpha_2^2}$$

$$Var(XY) = \sigma_1^2\sigma_2^2 + \sigma_1^2\alpha_2^2 + \sigma_2^2\alpha_1^2$$

MOMENTS :-

Expected values of a function $g(x)$ of a random variable x is used for calculating the moments. There are two or three types of moments, namely,

✓ (i) Moments about the origin.

✓ (ii) Moment about the mean. or central moments

✗ (iii) Absolute moments

usually, the moments determined about the mean value are called central moments.

Moments about the origin :-

$E[(x-m)]$ $E[x]$

$E[(x-m)^2]$ $E[x^2]$

$E[(x-m)^3]$ $E[x^3]$

moment about mean — origin $(m=0)$

for each positive integer, n, $(n = 0, 1, 2, ---)$, the n^{th} moment of the continuous random variable x about the origin is given by

$$m_n = E[x^n] = \int_{-\infty}^{\infty} x^n f_x(x)\, dx.$$

If x is a discrete random variable, then

$$m_n = E[\cancel{(x-m)}^n] = \sum_{i=1}^{N} x_i^n P(x_i)$$

x_i → small

$x^n \cdot f_x(x)\, dx$ → small

UNIT 2

Part 2

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Moments about the Mean : (central moments)

For each positive integer n, $(n = 0, 1, 2 \ldots)$, the nth central moment of the continuous random variable X is defined by

$$\mu_n = E\left[(\overset{\text{capital}}{X} - m)^n\right]$$

$$\therefore \mu_n = \int_{-\infty}^{\infty} (\overset{x\ (\text{small})}{X} - m)^n \cdot f_X(x)\,dx$$

If X is a discrete random variable, then

$$\mu_n = \sum_{i=1}^{N} (\overset{x_i\ (\text{small})}{X_i} - m)^n \cdot P(x_i).$$

§ Moments Generating function :-

An important device tool that can be used to calculate the higher moments of the "Moment Generating function".

The moment generating function of a random Variable X, about the origin, whose p.d.f. is $f_X(x)$ is given by

t = real number
$-\infty < t < \infty$

$$M_X(t) = E\left[e^{tx}\right]$$

$$= \int_{-\infty}^{\infty} e^{tx} \cdot f_X(x)\, dx \quad \text{(continuous r.v)}$$

for a discrete random Variable

$$M_X(t) = \sum_{i=1}^{n} e^{tx_i}\, p(x_i)$$

Since $M_X(t)$ is used to generate moments, it is known as moment Generating function. Now, we shall study how $M_X(t)$ is used to generate the moments.

→

We know that the function e^{tx} is given by

$$e^{tx} = 1 + \frac{tx}{1!} + \frac{t^2x^2}{2!} + \frac{t^3x^3}{3!} + \cdots \frac{t^r x^r}{r!} + \cdots$$

$$M_X(t) = E\left[e^{tx}\right] = E\left[1 + \frac{tx}{1!} + \frac{t^2x^2}{2!} + \cdots \frac{t^r x^r}{r!} + \cdots\right]$$

$$= 1 + t \cdot E[X] + \frac{t^2}{2!} E[X^2] + \cdots + \frac{t^r}{r!} \cdot E[x^r] + \cdots$$

$$= 1 + t m_1 + \frac{t^2}{2!} m_2 + \cdots \frac{t^r}{r!} m_r + \cdots$$

where $m_r = E[X^r] = \int x^r \cdot f_X(x)\, dx$ for C.R.V

$= \sum x^r \cdot p(x)$ for d.R.V.

and m_r is the 'r'th moment of X about the origin, i.e. $E[X^r] = m_r$ is the co-efficients of $\frac{t^r}{r!}$ in the expansion of $M_X(t)$ in series of powers of t.

In order to find m_r, we have to differentiate the moment generating function $M_X(t)$, r times with respect to "t" and substituting $t=0$

i.e. $$m_r = \left.\frac{d^r[M_X(t)]}{dt^r}\right|_{t=0}$$

Similarly, the moment generating function of X about the point $X = a$ is defined as

$$M_a(t) = E\left[e^{t(X-a)}\right]$$

Consider $\frac{d}{dt}\left[M_X(t)\right]\Big|_{t=0} = \frac{d}{dt}\left\{1 + t \cdot E(X) + \frac{t^2}{2!} \cdot E(X^2) + \cdots\right\}\Big|_{t=0}$

$$\frac{d}{dt}\left[M_X(t)\right]\Big|_{t=0} = E(X) = m_1 \text{ (first moment)}$$

Similarly $\frac{d^2}{dt^2}\left[M_X(t)\right]\Big|_{t=0} = E(X^2) = m_2$ (second moment)

and so on --

Thus, the n^{th} moment of r.v. can be obtained as

$$m_n = \frac{d^n}{dt^n}\left[M_X(t)\right]\Big|_{t=0}$$

Properties of mgf :

Q Show that the m.g.f. of the random variable X having the p.d.f. (density)

$$f(x) = \frac{1}{3}, \quad -1 < x < 2$$
$$= 0, \quad \text{elsewhere}$$

is given by $M_X(t) = \dfrac{e^{2t} - e^{-t}}{3t}, \quad t \neq 0$

$$= 1, \quad t = 0$$

Sol:

$$M(t) = \int_{-\infty}^{\infty} e^{tx} \cdot f_X(x)\, dx$$

$$= \frac{1}{3} \int_{-1}^{2} e^{tx}\, dx$$

$$= \frac{1}{3} \left[\frac{e^{tx}}{t} \right]_{-1}^{2}$$

$$= \frac{1}{3} \left[\frac{e^{2t} - e^{-t}}{t} \right] = \frac{e^{2t} - e^{-t}}{3t}$$

$$M(t) = \frac{e^{2t} - e^{-t}}{3t}, \quad t \neq 0$$

when $t = 0$ $\quad M(t) = \frac{0}{0}$

Applying L'Hospitals rule, $M(0) = \left[\frac{2e^{2t} + e^{-t}}{3} \right]_{t=0}$

Thus, $M(t) = \frac{e^{2t} - e^{-t}}{3}, \quad t \neq 0 \qquad = \frac{2+1}{3} = 1$

§ If a random variable x has the moment generating function $M_x(t) = \frac{2}{2-t}$, determine the variance of x.

Sol

$$M_x(t) = 2/2-t$$

$$= \frac{2}{2(1-t/2)} = \left(1-\frac{t}{2}\right)^{-1}$$

$(1-x)^{-1} = 1+x+x^2+x^3+ \ldots$

$$= 1+\frac{t}{2}+\frac{t^2}{4}+ \ldots \infty$$

$$\boxed{\Rightarrow 1+\frac{t}{1!}\times\frac{1}{2}+\frac{t^2}{2!}\times 1/2 + \ldots}$$

$\therefore$ The co-efficient of $\frac{t}{1!}$ in this equation $= E[X] = 1/2$

The co-efficient of $\frac{t^2}{2!}$ is

$$E[X^2] = 1/2$$

$$\therefore \quad Var[X] = E[X^2] - [E(X)]^2$$

$$= 1/2 - 1/4 = 1/4$$

§ The random variable X can take the values 1 and -1 with equal probability. Find the moment generating function and first 4 moments about the origin.

Sol

m.g.f. $M_X(t) = E\left[e^{tx}\right]$

$$= \sum_{i=-1 \text{ and } 1} e^{tx_i} \cdot p(x_i)$$

$p(1) = 1/2$

$p(-1) = +1/2$

$$= e^{t} \cdot (1/2) + e^{-t} \cdot (1/2)$$

$$\therefore M_X(t) = \frac{e^{t} + e^{-t}}{2}$$

$$= 1/2\,(e^{t} + e^{-t})$$

$\frac{t^2}{2!} + \frac{t^2}{2!}$

$\left(\frac{t^2}{2!}\right)\frac{2}{2}$

$= \frac{t^2}{2!} \times 1$

$$e^{t} = 1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots$$

$$e^{-t} = 1 - \frac{t}{1!} + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \cdots$$

$$M_X(t) = E\left[e^{tx}\right] = 1 + m_1 t + m_2 \cdot \frac{t^2}{2!} + m_3 \cdot \frac{t^3}{3!} + m_4 \cdot \frac{t^4}{4!}$$

$\frac{t^2}{\cancel{2!}}(2/1)$

$m_1 = 0$

$\frac{1}{2!}(2t^2)$

$m_2 = 1$

$m_3 = 0$

$m_4 = 1$

Q find the characteristic function of a random variable X defined by the density function

$$f_X(x) = \begin{cases} 0, & x<0 \\ 1, & 0 \le x \le 1 \\ 0, & x>1 \end{cases}$$

Sol:-

$$\phi(t) = \int_{-\infty}^{\infty} e^{jxt} f(x)\, dx$$

$$\boxed{\int_{-\infty}^{\infty} e^{j\omega x} f_X(x)\, dx} \text{ or}$$

$$= \int_0^1 e^{jxt}\, dx$$

$$= \left[\frac{e^{jxt}}{jt}\right]_0^1 = \frac{e^{jt}}{jt} - \frac{1}{jt}$$

$$= \frac{e^{jt}-1}{jt}$$

Q. Find the characteristic function of the random variable X with density function,

$$f(x) = x/2 \quad \text{for} \quad 0 \le x \le 2$$

$$= 0 \quad , \quad \text{otherwise.}$$

Sol

$$\phi_X(\omega) = \int_{-\infty}^{\infty} e^{j\omega x} f_X(x)\, dx$$

$$= \int_0^2 \frac{x}{2} \cdot e^{j\omega x}\, dx$$

$$= 1/2 \int_0^2 x \cdot e^{j\omega x}\, dx$$

$$= 1/2 \int_0^2 x \cdot \left[\frac{d}{dx} \left(\frac{e^{j\omega x}}{j\omega} \right) \right] dx$$

$\int uv$

$u \int v - \int u' \int v$

$$= 1/2 \left[x \cdot \frac{e^{j\omega x}}{j\omega} \Big|_0^2 - \int_0^2 1 \cdot \frac{e^{j\omega x}}{j\omega} \cdot dx \right]$$

$$x\frac{e^{2j\omega}}{j\omega} - \frac{1}{j\omega}$$

$$= \cancel{\frac{1}{2}\left[2\,\frac{e^{2j\omega}}{j\omega} - 0\,\frac{1}{j\omega} \right] - \int_0^2 \frac{e^{j\omega x}}{j\omega}\, dx}$$

$$\frac{e^{2j\omega}}{j\omega}$$

$$\cancel{x \cdot \frac{e^{2j\omega}}{j\omega} - 0}$$

$$\frac{e^{2j\omega}}{j\omega} - \frac{1}{2j\omega} \left(\frac{e^{j\omega x}}{j\omega} \right|_0^2$$

$2j\omega \qquad 2j\omega$

Q find (a) co-variance and (b) co-efficient of correlation between the random variables X and Y if $E[X] = 2$, $E[Y] = 3$, $E[XY] = 10$, $E[X^2] = 9$ and $E[Y^2] = 16$.

Sol co-variance $= Cov(X,Y) = E(XY) - E(X)\cdot E(Y)$

$= 10 - (2 \times 3)$

$= 10 - 6 = \underline{4}$

$\boxed{\text{NOTE}}$:- The measure of dependence between random variables X and Y is expressed in terms of correlation coefficient or coefficient of correlation, and is given by

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \cdot \sigma_Y} = \frac{Cov(X,Y)}{\sigma_X \cdot \sigma_Y}$$

$\boxed{r(X,Y) \text{ or } \sigma_{XY}}$

• $\sigma_X^2 = E(X^2) - (E(X))^2$

$= 9 - (2)^2 = 5$

$\sigma_X = \sqrt{5}$

$\boxed{\begin{array}{l}\sigma_X : \text{standard deviation of } X \\ \sigma_Y : \text{standard deviation of } Y\end{array}}$

$\sigma_Y^2 = 16 - 9 = 7$

$\sigma_Y = \sqrt{7}$

$$\rho = \frac{4}{\sqrt{5}\cdot\sqrt{7}} = 4/\sqrt{35}$$

Q. Express 3rd moment about the mean, in terms of moments about the origin of random variable x.

Sol :

$$E\left[(x-m)^3\right]$$

$$= E\left[x^3 - m^3 - 3x^2 m + 3m^2 x\right]$$

$$= E[x^3] - m^3 - 3m\, E(x^2) + 3m^2\, E(x)$$

$$= E(x^3) - m^3 - 3m\, E(x^2) + 3m^3$$

$$= E(x^3) - 3m \cdot E(x^2) + 2m^3$$

where $m = E(x)$

$$(x-m)^3 = (x^2 + m^2 - 2xm)(x-m)$$

$$= x^3 - x^2 m + x m^2 - m^3 - 2x^2 m + 2x m^2$$

$$= x^3 - m^3 - 3x^2 m + 3x m^2$$

§ Express 3rd moment about the mean, in terms of moments about the origin of random variable X.

Sol :

$$E[(X-m)^3]$$

$$= E[X^3 - m^3 - 3X^2m + 3m^2X]$$

$$= E[X^3] - m^3 - 3m\,E(X^2) + 3m^2\,E(X)$$

$$= E(X^3) - m^3 - 3m\,E(X^2) + 3m^3$$

$$= E(X^3) - 3m\,E(X^2) + 2m^3$$

where $m = E(X)$

$$(X-m)^3 = (X^2 + m^2 - 2Xm)(X-m)$$

$$= X^3 - X^2m + Xm^2 - m^3 - 2X^2m + 2Xm^2$$

$$= X^3 - m^3 - 3X^2m + 3Xm^2$$

p. 20

find the coefficient of correlation between random variables x and y from the following data.

x	1	2	3	4	5	6	7	8	9	
~~x~~	~~0~~	~~1~~	~~2~~	~~3~~	~~4~~	~~5~~	~~6~~	~~7~~	~~8~~	~~9~~
y	9	8	10	12	11	13	14	16 ~~14~~	15 ~~18~~	~~15~~

<u>Sol</u>

coefficient of correlation $\rho_{xy} = \dfrac{\operatorname{cov}(x,y)}{\sigma_x \cdot \sigma_y}$

$$\operatorname{cov}(x,y) = E[xy] - E[x] \cdot E[y]$$

$$E(x) = \frac{1+2+3+4+5+6+7+8+9}{9} = \frac{45}{9} = 5$$

$$E(y) = \frac{9+8+10+12+11+13+14+16+15}{9}$$

$$= 12$$

$$E[xy] = \frac{\left[(1\times 9)+(2\times 8)+(3\times 10)+(4\times 12)+(5\times 11)+(6\times 13)+(7\times 14)+(8\times 16)+(9\times 15)\right]}{9}$$

$$= 66.3$$

$$\sigma_x^2 = E(x^2) - [E(x)]^2$$

$$E(x^2) = \frac{1+4+9+16+25+36+49+64+81}{9}$$

$$= 31.66$$

$$\sigma_x^2 = 31.66 - (5)^2 = 6.66$$

$$\sigma_x = 2.58$$

$$E(y^2) = \frac{81+64+100+144+121+169+196+256+225}{9}$$

$$= 150.66$$

$$\sigma_y^2 = E(y^2) - (E(y))^2$$

$$= 150.66 - (12)^2 = 6.66$$

$$\sigma_y = 2.58$$

$$Cov(x,y) = E[xy] - E(x) \cdot E(y)$$

$$= 66.3 - (5 \times 12) = 6.3$$

$$\rho_{xy} = \frac{Cov(x,y)}{\sigma_x \cdot \sigma_y} = \frac{6.3}{(2.58)(2.58)}$$

$$= 0.946$$

Let X be the outcome from rolling of one die and Y be the outcome from the rolling a second die. find $E(X/Y)$.

Sol

Since X and Y are independent

$$E\left(\frac{X}{Y}\right) = E(X)$$

$$E(X) = \sum_{i=1}^{6} x_i \, p(x_i)$$

x	$p(x)$
1	1/6
2	1/6
3	1/6
4	1/6
5	1/6
6	1/6

$$= 1\times\frac{1}{6} + 2\times\frac{1}{6} + 3\times\frac{1}{6} + 4\times 1/6 + 5\times 1/6 + 6\times 1/6$$

$$= 1/6\,(1+2+3+4+5+6) = \frac{21}{6} = 7/2$$

$$= 3.5$$

§ CHARACTERISTIC FUNCTION :-

In the previous section, we learnt that the moment generating function $M_X(t)$ is given by

$$M_X(t) = E[e^{tx}] = \int_{-\infty}^{\infty} e^{tx} \cdot f_X(x)\,dx$$

or $M_X(t) = \sum e^{tx} \cdot p(x)$

In some cases, the moment generating function does not exist, since the integral $\int_{-\infty}^{\infty} e^{tx} \cdot f_X(x)\,dx$ or $\sum e^{tx} \cdot p(x)$ does not exist for real values of t for some distributions. Under such situations, we have to look for some other tool to generate moments. Characteristic function is the best alternative for moment generating function for such cases.

The characteristic function of a random variable X is defined by

$j = \sqrt{-1}$

$-\infty < \omega < \infty$

J: Imaginary Unit

$$\phi_X(\omega) = E[e^{j\omega x}]$$

ω: is not the frequency

occasionally, the letter 't' is used instead of "ω"

for continuous random variable, $\phi_X(\omega) = E(e^{j\omega x}) =$

$$= \int_{-\infty}^{\infty} e^{j\omega x} f_X(x)\,dx$$

The Equation $\phi_x(\omega) = \int_{-\infty}^{\infty} e^{j\omega x} \cdot f_x(x)\,dx$ can be considered as the fourier transform of pdf ($f_x(x)$),

Thus the pdf can be obtained from the characteristic function as

$$pdf = f_x(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-j\omega x} \cdot \phi_x(\omega) \cdot d\omega$$

and thus can be considered as the INVERSION FOURIER TRANSFORM of $\phi_x(\omega)$.

§ PROPERTIES OF characteristic function

1) $|\phi_X(\omega)| \le 1$, i.e. The maximum magnitude of the characteristic function is Unity.

2) $\phi_X(0) = 1$ i.e. $\omega = 0$

3) If X is symmetric function, then $\phi_X(\omega) = \phi_X(-\omega)$

4) If X and Y are two independent random variables, then $\phi_{X+Y}(\omega) = \phi_X(\omega) \cdot \phi_Y(\omega)$

5) If $\phi_X(\omega)$ is the characteristic function of the random variable X, and if $Y = aX+b$,

proof needed

then $\phi_Y(\omega) = e^{j\omega b}\, \phi_X(a\omega)$

where 'a' and 'b' are constants.

6) If X and Y are random variables, having characteristic functions $\phi_X(\omega)$ and $\phi_Y(\omega)$, then X and Y are having identical distributions provided

$\phi_X(\omega) = \phi_Y(\omega)$.

7) $\phi_X(-\omega) = \overline{\phi_X(\omega)}$

8) If X is a random variable with characteristic function $\phi_X(\omega)$ and if the 'n'th moment exists, it can be determined by

$$m_n = \left(\frac{1}{j}\right)^n \left[\frac{d^n [\phi_X(\omega)]}{d\omega^n} \right]$$

The Equation $\phi_X(\omega) = \int_{-\infty}^{\infty} e^{j\omega x} \cdot f_X(x)\,dx$ can be considered as the fouriers transform of pdf $(f_X(x))$,

Thus the pdf can be obtained from the characteristic function of

$$pdf = f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-j\omega x} \cdot \phi_X(\omega) \cdot d\omega$$

and thus can be considered as the INVERSION FOURIER TRANSFORM of $\phi_X(\omega)$.

PROPERTIES OF characteristic function,

1) $|\phi_X(\omega)| \leq 1$, i.e. The maximum magnitude of the characteristic function is Unity.

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7) $\phi_X(-\omega) = \overline{\phi_X(\omega)}$

8) If X is a random variable with characteristic function $\phi_X(\omega)$ and if the 'n'th moment exists, it can be determined by

$$m_r = \left(\frac{1}{j}\right)^n \left[\frac{d^n[\phi_X(\omega)]}{d\omega^n} \right]$$

9) The function $\psi(\omega) = \log e^{|\phi_x(\omega)|}$ is called the second characteristic function.

UNIT 2

Part 3

Properties of Moment Generating function :-

1. Let x be a random variable with mgf $M_x(t)$.

Then, the mgf of $y = ax+b$ is $= e^{bt} . M_x(at)$

$$M_y(t) = E(e^{yt}) = E\left[e^{(ax+b)t}\right]$$

$$= E\left[e^{axt} . e^{bt}\right]$$

$$= e^{bt} . E\left[e^{axt}\right]$$

$$= e^{bt} . E\left[e^{x(at)}\right]$$

$$= e^{bt} . M_x(at)$$

2. If $M_x(t)$ is the mgf of r.v x, then,

the mgf of $Y = K.X$ is $= M_X(Kt)$

$$M_y(t) = E\left[e^{yt}\right]$$

$$= E\left[e^{K.x.t}\right]$$

$$= E\left[e^{x(Kt)}\right]$$

$$= M_x(Kt)$$

3. If $M_x(t)$ is the mgf of r.v. X, then, the mgf of $Y = \frac{X+a}{b}$ is $= e^{\frac{at}{b}} . M_x(t/b)$

$$M_y(t) = E\left[e^{yt}\right]$$

$$= E\left[e^{t.\left(\frac{x+a}{b}\right)}\right]$$

$$= E\left[e^{\frac{tx}{b}} . e^{\frac{at}{b}}\right]$$

$$= e^{\frac{at}{b}} . E\left[e^{x.(t/b)}\right]$$

$$= e^{\frac{at}{b}} . M_x(t/b)$$

4. If two random variables X and Y having mgfs $M_X(t)$ and $M_Y(t)$ such that $M_X(t) = M_Y(t)$, then, X and Y are said to have identical distribution i.e identical density.

To see how to generate moments from characteristic function, consider

$$\phi_X(\omega) = E\left[e^{j\omega x}\right]$$

$$= E\left[1 + \frac{j\omega x}{1!} + \frac{(j\omega x)^2}{2!} + \frac{(j\omega x)^3}{3!} + \cdots\right]$$

$$= 1 + j\omega \cdot E(x) + \frac{(j\omega)^2}{2!} \cdot E(x^2) + \frac{(j\omega)^3}{3!} E(x^3) + \cdots$$

This expansion is a polynomial in $j\omega$, where the coefficient of each term involving $j\omega$ is the moment of the random variable. ie. coefficient of $j\omega$ is the 1st moment m_1, coefficient of $\frac{(j\omega)^2}{2!}$ is the 2nd moment m_2 and so on.

characteristic function can be used to generate the moments of a random variable about origin only.

Consider, $$\frac{d}{d\omega}\left[\phi_X(\omega)\right]\bigg|_{\omega=0} = \frac{d}{d\omega}\left[1 + j\omega \cdot E(x) + \frac{(j\omega)^2}{2!} E(x^2) + \cdots\right]_{\omega=0}$$

$$\frac{d}{d\omega}\left[\phi_X(\omega)\right]\Big|_{\omega=0} = J \cdot E(X)$$

$$\therefore\ E(X) = \frac{1}{J} \cdot \frac{d}{d\omega}\left[\phi_X(\omega)\right]\Big|_{\omega=0}$$

In general, the nth moment of r.v. X can be obtained as

$$E(X^n) = \left(\frac{1}{J}\right)^n \cdot \frac{d^n}{d\omega^n}\left[\phi_X(\omega)\right]\Big|_{\omega=0}$$

Properties of Characteristic function

$Y = aX + b$

①

$$\phi_Y(\omega) = E\left[e^{j\omega Y}\right]$$

$$= E\left[e^{j\omega(aX+b)}\right]$$

$$= E\left[e^{j\omega aX} \cdot e^{j\omega b}\right]$$

$$= e^{j\omega b} \cdot E\left[e^{j(a\omega)X}\right]$$

$$= e^{j\omega b} \cdot \phi_X(a\omega)$$

②

If $Y = \cancel{aX+b}\ \frac{X+a}{b}$, $a, b \neq 0$

$$\phi_Y(\omega) = E\left[e^{j\omega\left(\frac{X+a}{b}\right)}\right]$$

$$= E\left[e^{j\frac{\omega X}{b}} \cdot e^{j\frac{\omega a}{b}}\right]$$

$$= e^{j\frac{\omega a}{b}} \cdot E\left[e^{j\frac{\omega X}{b}}\right]$$

$$= e^{j\frac{\omega a}{b}} \cdot \phi_X(\omega/b)$$

SKEW :- (μ_3)

The 3^{rd} central moment $E[(X-m)^3]$ of a random variable X is referred to as "skew" of its probability density function. It is a measure of "asymmetry" of the density function of a r.v about its mean.

→ The Normalized third central moment of a r.v X, ie $\frac{E[(X-m)^3]}{\sigma_x^3}$ is referred to as "skewness" of density function. or alternatively "coefficient of skewness".

$$\mu_3 / \sigma_x^3$$

—: Note :

$\mu_n = 0$ for all odd values of n

$\mu_0 = 1$

$\mu_1 = 0$

$\mu_3 = 0$

$\mu_5 = 0$

[illegible]

KURTOSIS:- It is a measure of peakedness of the density function. The degree of peakedness of a distribution is measured by using coefficient of kurtosis or kurtosis and is given by

$$\frac{E\left[(X-m)^4\right]}{\sigma_X^4}$$

, it is a dimensionless quantity.

CHEBYCHEV'S INEQUALITY :-

The usefull tool in some probability problems is chebychev's Inequality.

If m is the mean and σ^2 is the variance of a r.v X, then $|X-m|$ is measure of the amount by which value of X differs above or below from its mean. The upper bound for this dispersion is given by "chebychev's inequality". It is stated as

" If X is a random variable with mean 'm' and variance σ^2, then for any $k>0$,

$$P\left\{|X-m| \geq k\right\} \leq \frac{\sigma^2}{k^2}$$

The alternative form of chebyshev's inequality is stated as

$$P\left\{|X-m| \geq k\cdot\sigma\right\} \leq \frac{1}{k^2}$$

i.e.

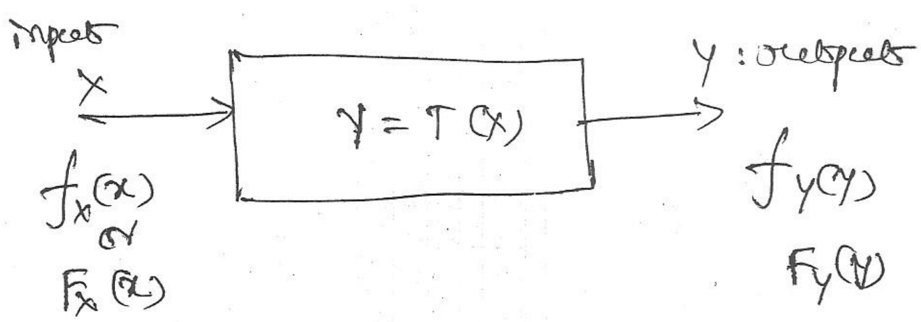
$$P\left\{|X-m| \leq k\cdot\sigma^2\right\} \geq 1-\frac{1}{k^2}$$

TRANSFORMATION OF A RANDOM VARIABLE :-

Transform (→ change) of one random variable X into a new random variable Y by means of transformation (→ Transformation).

$$\boxed{Y = T(X)}$$

Typically, the density function $f_X(x)$ or distribution function $F_X(x)$ of X is known, and the problem is to determine either the density function $f_Y(y)$ or distribution function $F_Y(y)$ of Y. The problem can be viewed as "black box" with input X, output Y, and transfer characteristic $Y = T(X)$, as illustrated below.



In general X can be a discrete, continuous or mixed random variable. In turn, the transform T can be linear, non-linear, segmented, staircase --- etc.

Clearly, there are many cases to consider in a general study, depending on the form of X and T. In this section we shall consider only three cases.

① $\underline{X}$ and $\underline{T}$ continuous and either monotonically increasing or decreasing

② $\underline{X}$ and $\underline{T}$ continuous but nonmonotonic

③ X discrete and T continuous.

Note that the transformation in all three cases is assumed continuous.

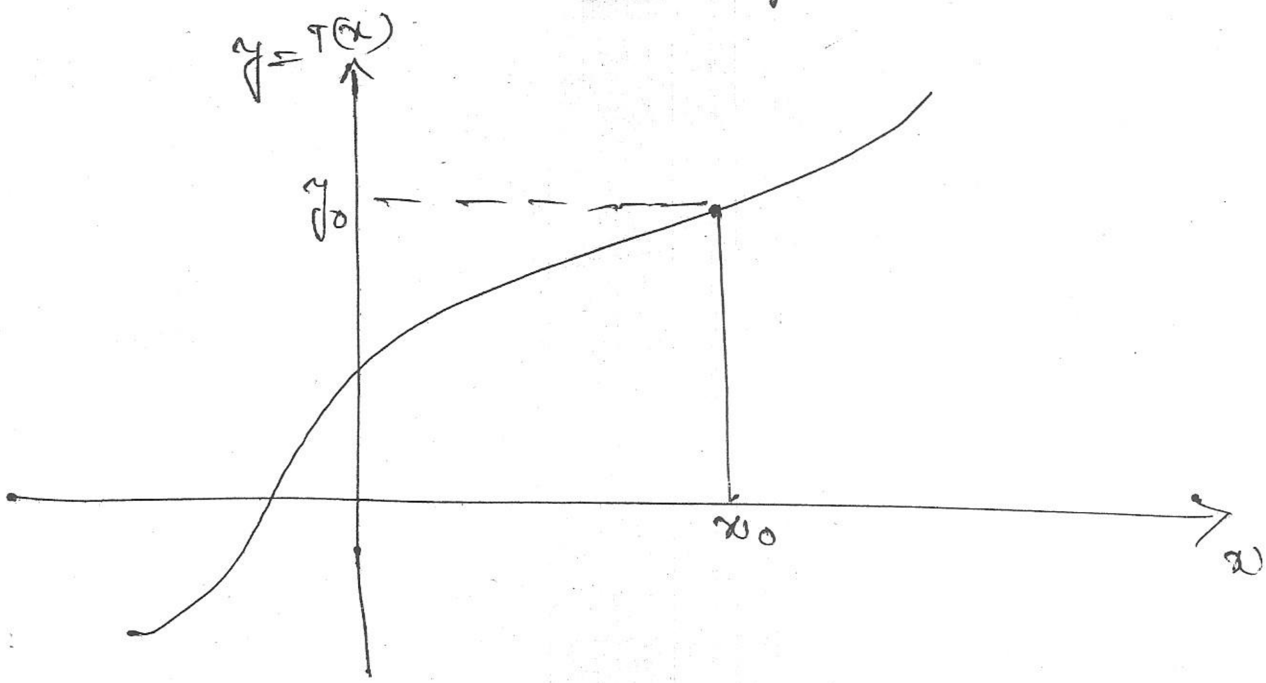
Monotonic Transformation of a Continuous Random Variable :-

A Transformation T is called monotonically increasing if $T(x_1) < T(x_2)$, for any $x_1 < x_2$.

Monotonically decreasing if ~~$T(x_1) > T(x_2)$~~ ~~$T(x_1)$~~

$T(x_1) > T(x_2)$ for any $x_1 > x_2$

Consider first the increasing transformation :



We assume that T is continuous and differentiable at ~~all~~ all values of x for which $f_X(x) \neq 0$, Let y have a particular value y_0 corresponding to the particular value x_0 of x as shown in figure. The two numbers are related by

$$y_0 = \underline{T}(x_0) \quad \text{or} \quad x_0 = \underline{T}^{-1}(y_0)$$

where T^{-1} represents the inverse transformation of T. Now the probability of the event $\{y \le y_0\}$ must equal to the probability of the event $\{x \le x_0\}$, because of the one-to-one correspondence between x and y.

Thus,

$$F_Y(y_0) = P\{Y \le y_0\} = P\{X \le x_0\}$$
$$= F_X(x_0)$$

or

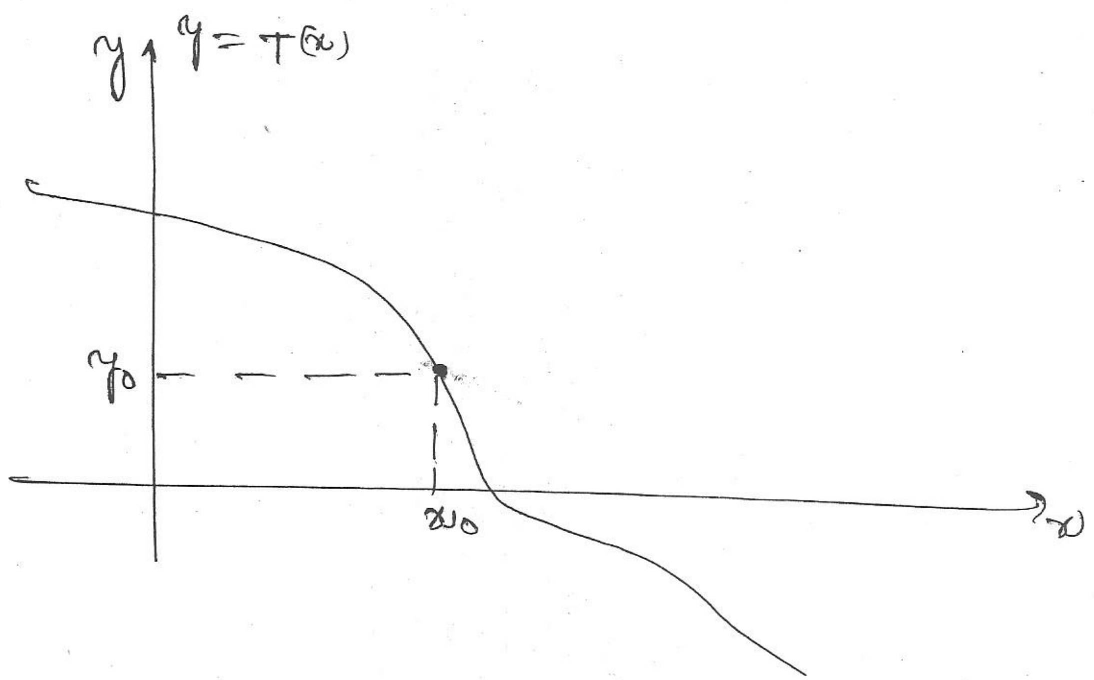
$$\int_{-\infty}^{y} f_Y(y)\, dy = \int_{-\infty}^{x = T^{-1}(y_0)} f_X(x)\, dx.$$

~~After~~, differentiate both sides w.r.t. to y_0 using Leibniz's Rule

finally $$f_Y(y_0) = f_X(x_0) \cdot \frac{dx}{dy}$$

for any valued y, $$\boxed{f_Y(y) = f_X(x) \cdot \frac{dx}{dy}}$$

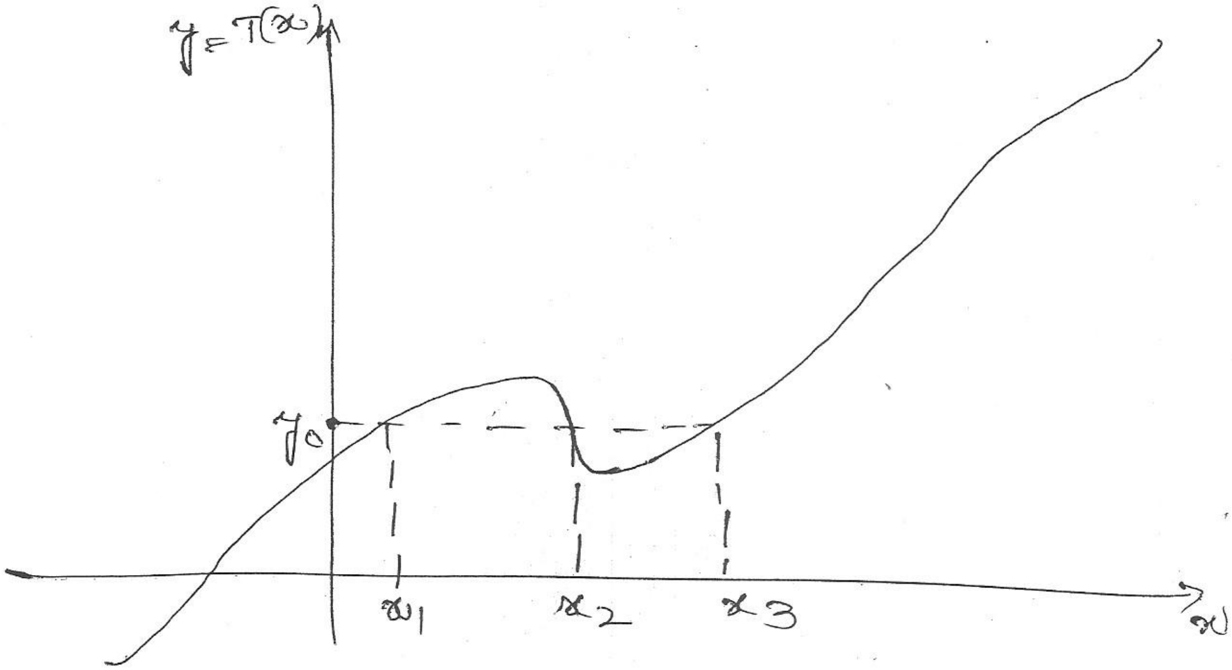
→ for the decreasing transformation :



$$f_y(y) = f_x(x) \cdot \left| \frac{dx}{dy} \right|$$

<u>Note</u> : A linear transformation of a gaussian random variable produces another gaussian random variable.

Nonmonotone transformations of a continuous Random Variable:



Here more than one interval of x that correspond to the event $\{Y \leq y_0\}$. For the value of y_0 shown in figure, the event $\{Y \leq y_0\}$ corresponds to the event $\{X \leq x_1 \text{ and } x_2 \leq X \leq x_3\}$

Thus the probability of the event $\{Y \leq y_0\}$ now equal the probability of the event $\{x \text{ values yielding } Y \leq y_0\}$, which we shall write as $\{x / Y \leq y_0\}$. In otherwords

$$F_Y(y_0) = P\{Y \leq y_0\} = P\{x / Y \leq y_0\}$$

$$= \int f_X(x)\, dx$$

density function of Y is

$$f_Y(y_0) = \frac{d}{dy_0}\left[\int_{x/Y \le y_0} f_X(x)\,dx\right]$$

or

$$f_Y(y_0) = \sum_n{}' \frac{f_X(x_n)}{\left|\left.\frac{dT(x)}{dx}\right|_{x=x_n}\right|} \rightarrow \text{(other form)}$$

x_n, $n=1,2,3\ldots$

Transformation of a discrete random variable

If X is discrete, where $y = T(x)$ is a continuous transformation, then

refer 48, 48 pages)
Peebles

$$f_X(x) = \sum_n p(x_n) \cdot \delta(x - x_n)$$

$$F_X(x) = \sum_n p(x_n) \cdot u(x - x_n)$$

where x_n, $n = 1, 2, 3 \ldots$ of X.

If the transformation is monotonic, there is a one-to-one correspondence between X and Y, so that a set $\{y_n\}$ corresponds to the set $\{x_n\}$ through the equation $y_n = T(x_n)$. The probability $p(y_n)$ equals to $p\{x_n\}$. Thus

$$f_Y(y) = \sum_n p(y_n) \cdot \delta(y - y_n)$$

$$F_Y(y) = \sum_n p(y_n) \cdot u(y - y_n)$$

where

$$y_n = T(x_n)$$

$$p(y_n) = p(x_n)$$

If T is not monotonic, the above procedure remains valid except there now exists the possibility that more than one value of x_n corresponds to a value of y_n. In such a case $P(y_n)$ will equal the sum of the probabilities of the various x_n for which $y_n = T(x_n)$.

① Let X and Y be independent random variables such that

$X = 1$ with prob $1/3$

$= 0$ " " $2/3$

and $Y = 2$ " " $3/4$

$= -3$ " " $1/4$

find ⓐ $E(3X + 2Y)$ ⓑ $E[2X^2 - Y^2]$

ⓒ $E(XY)$ ⓓ $E[X^2 \cdot Y^2]$.

Sol:

ⓐ $E(3X + 2Y)$

Let $Z = 3X + 2Y$

when $X = 1, Y = 2 \Rightarrow Z = 7$

$X = 1, Y = -3 \Rightarrow Z = -3$

$X = 0, Y = 2 \Rightarrow Z = 4$

$X = 0, Y = -3 \Rightarrow Z = -6$

The probability distribution of Z is

z_i		$P(z_i)$
-6	———	$P(X=0, Y=-3) = P(X=0) \cdot P(Y=-3) = 2/3 \cdot 1/4 = 1/6$
-3	———	$P(X=1) \cdot P(Y=-3) = 1/12$
4	———	——— $= 1/2$
7	———	——— $= 1/4$

$\therefore \quad E(3x+2y) = E(Z) = (-6 \times 1/6) + (-3 \times 1/12) + (4 \times 1/2) + (7 \times 1/4)$

$= 5/2$

(ii) Let $Z = 2x^2 - y^2$

x	y	$z = z_i$	$P(z_i)$
1	2	-2	$P(x=1) \cdot P(y=-2) = 1/4$
1	-3	-7	$1/12$
0	2	-4	$1/2$
0	-3	-9	$1/6$

$E(Z) = (-9 \times 1/6) + (-7 \times 1/12) + (-4 \times 1/2) + (-2 \times 1/4)$

$= -55/12$

(c) let $Z = XY$

x	y	z_i		$P(Z = z_i)$
1	2	2		$P(X=1, Y=2) = P(X=1) \cdot P(Y=2)$ $= 1/4$
1	-3	-3	—	$1/12$
0	2	0	—	$1/2$
0	-3	0	—	$1/6$

$$E(Z) = (2 \times 1/4) + (-3 \times 1/12) + (0 \times 1/2) + (0 \times 1/6)$$

$$= 1/2 - 1/4$$

$$= 1/4$$

(d) let $Z = X^2 Y$

x	y	z_i	$P(Z = z_i)$
1	2	2	$1/4$
1	-3	-3	$1/12$
0	2	0	$1/2$

UNIT 2

Part 4
Solved problems

Example 4.1

Assume that the joint sample space has the probabilities shown in Table 4.1. Find distribution function, $F_{XY}(x, y)$ and marginal density functions.

(XY)	(0, 0)	(1, 2)	(2, 3)	(3, 2)
$P(x, y)$	0.2	0.3	0.4	0.1

Table 4.1: Joint probabilities $P(x,y)$

Solution

We know that

$$F_{XY}(x,y) = \sum_{n}^{N}\sum_{m}^{M} P(x,y)u(x-x_n)u(y-y_m)$$

$$F_{XY}(x,y) = 0.2u(x)u(y)+0.3u(x-1)u(y-2)+0.4u(x-2)u(y-3)$$
$$+0.1u(x-3)u(y-2)$$

Marginal density functions are

$$F_X(x) = F_{X,Y}(x,\infty)$$

$$F_X(x) = 0.2u(x) + 0.3u(x-1) + 0.4u(x-2) + 0.1u(x-3)$$

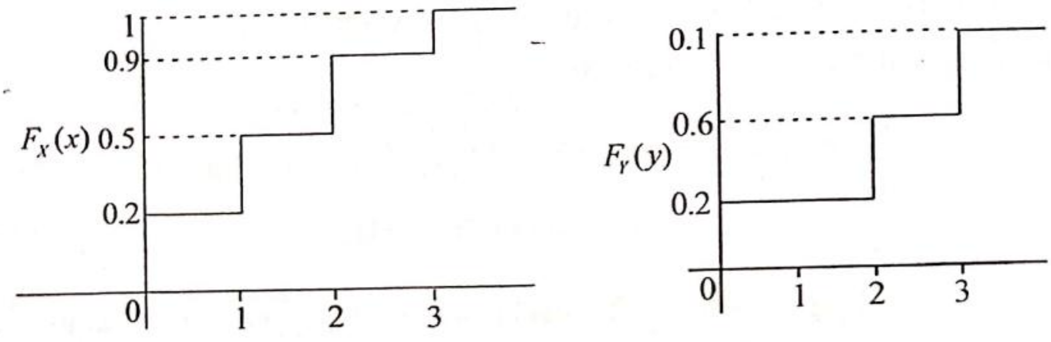
and

$$F_Y(y) = F_{XY}(\infty, y)$$

$$= 0.2u(y) + 0.3u(y-2) + 0.4u(y-3) + 0.1u(y-2)$$

$$= 0.2u(y) + 0.4u(y-2) + 0.4u(y-3)$$

Fig. 4.1 shows the plots of marginal distributions.



(*a*) (*b*)

Fig.4.1: Marginal distributions of X and Y

Example 4.3

The joint pdf is given as

$$f_{XY}(x, y) = A e^{-(2x+y)} \quad \text{for} \quad x \geq 0 \quad \text{and} \quad y \geq 0$$

Find (a) the value of A and (b) the marginal density functions

Solution

(a) Given the density function is valid, if

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x, y)\, dx\, dy = 1, \qquad x \geq 0 \quad \text{and} \qquad y \geq 0$$

$$\int_{0}^{\infty} \int_{0}^{\infty} A \cdot e^{-(2x+y)}\, dx\, dy = 1$$

$$A\left[\int_0^\infty e^{-2x}dx \int_0^\infty e^{-y}dy\right] = 1$$

$$A\left[\left(-\frac{e^{-2x}}{2}\Big|_0^\infty\right)\left(-e^{-y}\Big|_0^\infty\right)\right] = 1$$

$$\frac{A}{2}(e^{-2x})(e^{-y})\Big|_0^\infty = 1$$

$$\frac{A}{2} = 1$$

$$A = 2$$

$\therefore$ The function is

$$f_{X,Y}(x,y) = 2e^{-(2x+y)} \qquad x \geq 0,\ y \geq 0$$

(b) The Marginal density functions are

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

$$= \int_0^\infty 2e^{-(2x+y)}dy$$

$$= 2e^{-2x}\int_0^\infty e^{-y}\,dy = 2e^{-2x}(-e^{-y})\Big|_0^\infty$$

$$= 2e^{-2x}[1-0] = 2e^{-2x}$$

$$\therefore \qquad f_X(x) = 2e^{-2x}, \quad x \geq 0$$

and $$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$$

$$= \int_{-\infty}^{\infty} 2e^{-(2x+y)}\,dx$$

$$= 2e^{-y}\int_0^\infty e^{-2x}dx = e^{-y}(1-0) = e^{-y}$$

$$\therefore \qquad f_Y(y) = e^{-y}, \qquad y \geq 0$$

$\therefore$ The marginal density functions are

$$f_X(x) = 2e^{-2x}, \quad x \geq 0 \text{ and } f_Y(y) = e^{-y}, \quad y \geq 0$$

Example 4.12

Determine a constant b such that the given function is a valid joint density function.

$$f_{X,Y}(x,y) = \begin{cases} b(x^2+4y^2) & 0 \leq |x| < 1 \quad \text{and} \quad 0 \leq y < 2 \\ 0 & \text{elsewhere} \end{cases}$$

Solution

Given $\quad f_{X,Y}(x,y) = \begin{cases} b(x^2+4y^2) & 0 \leq |x| < 1 \quad \text{and} \quad 0 \leq y < 2 \\ 0 & \text{elsewhere} \end{cases}$

Since $f_{X,Y}(x,y)$ is a valid joint density function

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = 1$$

$$\int_{x=-1}^{1}\int_{y=0}^{2} b(x^2+4y^2)\,dx\,dy = 1$$

$$b\int_{-1}^{1}\left(x^2 y+\frac{4y^3}{3}\right)\Bigg|_{y=0}^{2} dx = 1$$

$$b\int_{-1}^{1}\left(\frac{32}{3}+2x^2\right)dx = 1$$

$$b\left(\frac{32x}{3}+\frac{2x^3}{3}\right)\Bigg|_{-1}^{1} = 1$$

$$2b\left[\frac{32}{3}+\frac{2}{3}\right] = 1$$

$$\frac{68b}{3} = 1 \qquad \text{or} \qquad b = \frac{3}{68}$$

Example 4.14

The joint density function is given by

$$f_{XY}(x,y) = \begin{cases} a\,x^2 y & 0 < y < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

(a) Find 'a' so that the function is a valid density function

(b) Find marginal density functions.

Solution

Given $$f_{XY}(x,y) = \begin{cases} a\,x^2 y & 0 < y < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Since it is a valid density function

Then $$\int_{y=0}^{1}\int_{x=y}^{1} a\,x^2 y\,dx\,dy = 1$$

$$\int_0^1 a y\left(\frac{x^3}{3}\right)\Bigg|_y^1 dy = 1$$

$$= \int_0^1 \frac{ay}{3}[1 - y^3]dy = 1$$

$$\frac{a}{3}\int_0^1 (y - y^4)dy = 1$$

$$\frac{a}{3}\left[\frac{y^2}{2} - \frac{y^5}{5}\right]_0^1 = 1$$

$$\frac{a}{3}\left[\frac{1}{2} - \frac{1}{5}\right] = 1$$

$$\frac{a}{3}\left[\frac{3}{10}\right] = 1$$

$$a = 10$$

The marginal density functions are

$$f_X(x) = \int_{y=x}^{1} f_{XY}(x,y)dy \qquad 0 < x < 1$$

$$f_X(x) = \int_{y=x}^{1} 10x^2 y\,dy = 10x^2\left(\frac{y^2}{2}\right)\Bigg|_x^1$$

$$= 5x^2[1 - x^2]$$

$$\therefore \qquad f_X(x) = \begin{cases} 5x^2(1-x^2) & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

and
$$f_Y(y) = \int_{x=0}^{y} f_{XY}(x, y)dx \quad 0 < y < 1$$

$$f_Y(y) = \int_{x=0}^{y} 10\,y\,x^2\,dx = 10y \cdot \frac{x^3}{3}\Bigg|_0^y$$

$$= 10y \cdot \frac{y^3}{3} = 3.33\,y^4$$

$$\therefore \qquad f_Y(y) = \begin{cases} 3.33\,y^4 & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Example 4.15

The joint distribution function of a bivariate random variable (X, Y) is given by

$$F_{X,Y}(x, y) = \begin{cases} 0 & x < 0 & y < 0 \\ 0.2 & 0 \le x < a, & 0 \le y < b \\ 0.4 & x \ge a, & 0 \le y < b \\ 0.8 & 0 \le x < a, & y \ge b \\ 1 & x \ge a, & y \ge b \end{cases}$$

Find the marginal distributions of X and Y.

Solution

Given the joint distribution of X and Y is

$$F_{XY} = \begin{cases} 0 & x < 0, & y < 0 \\ 0.2 & 0 \le x < a, & 0 \le y < b \\ 0.4 & x \ge a, & 0 \le y < b \\ 0.8 & 0 \le x < a, & y \ge b \\ 1 & x \ge a, & y \ge b \end{cases}$$

The marginal distribution functions are

$$F_X(x) = F_{XY}(x, \infty) = \begin{cases} 0 & x < 0 \\ 0.8 & 0 \le x \le a \\ 1 & x \ge a \end{cases}$$

and $$F_Y(y) = F_{XY}(\infty, y) = \begin{cases} 0 & y < 0 \\ 0.4 & 0 \le y \le b \\ 1 & y \ge b \end{cases}$$

Example 4.16

The joint pdf of a bivariate random variable (X, Y) is given by

$$f_{X,Y}(x, y) = \begin{cases} K\,xy & 0 < x < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

where k is a constant

(a) Find the value of K

(b) Are X and Y independent?

Solution

Given the joint probability density function of X and Y is

$$f_{X,Y}(x, y) = \begin{cases} K\,xy & 0 < x < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Since $f_{XY}(x, y)$ is a valid density function.

we know that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x, y)\,dx\,dy = 1$$

$$K\int_{y=0}^{1}\int_{x=0}^{y} xy\,dx\,dy = 1$$

$$K\int_0^1 y\,dy \left.\frac{x^2}{2}\right|_0^y = 1$$

$$K\int_0^1 y\,\frac{y^2}{2}\,dy = 1$$

$$K\int_0^1 \frac{y^3}{2}\,dy = K\left.\frac{y^4}{8}\right|_0^1 = 1$$

$$\frac{K}{8} = 1$$

$$\therefore \quad K = 8$$

(b) The marginal pdfs are

$$f_X(x) = \int_{y=x}^{1} f_{XY}(x, y)\,dy \qquad 0 < x < 1$$

$$f_X(x) = \int_x^1 8xy\,dy \quad = 8x \cdot \frac{y^2}{2}\bigg|_x^1$$

$$= 4x(1-x^2)$$

$$\therefore \qquad f_X(x) = \begin{cases} 4x(1-x^2) & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

and $$f_Y(y) = \int_0^y f_{XY}(x,y)\,dx = \int_0^y 8xy\,dx$$

$$= 8y \cdot \frac{x^2}{2}\bigg|_0^y = 8y \cdot \frac{y^2}{2} = 4y^3 \quad 0 < y < 1$$

$$\therefore \qquad f_Y(y) = \begin{cases} 4y^3 & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

If X and Y are independent then

$$f_X(x)\,f_Y(y) = 4x(1-x^2)4y^3$$

$$= 16xy^3(1-x^2)$$

$$\neq f_{XY}(x,y)$$

Therefore X and Y are not independent.

Example 4.17

Consider the joint pdf of random variables, X and Y is

$$f_{XY}(x,y) = \begin{cases} \frac{1}{8}(x+y) & 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

Find the (a) conditional density functions and

(b) $P((0 < Y < 1/2)/X = 1)$

Solution

Given the joint density function.

$$f_{XY}(x,y) = \begin{cases} \frac{1}{8}(x+y). & 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) The marginal density functions are

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy$$

$$= \int_0^2 \frac{1}{8}(x+y)\,dy$$

$$= \frac{1}{8}\left[xy + \frac{y^2}{2}\right]_0^2 = \frac{1}{8}[2x+2]$$

$$f_X(x) = \frac{1}{4}(x+1) \qquad 0 < x < 2$$

Similarly $f_Y(y) = \frac{1}{4}(y+1) \qquad 0 < y < 2$

The conditional density functions are

$$f_{XY}(x/y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{x+y}{2(y+1)}$$

and $f_{XY}(y/x) = \dfrac{f_{XY}(x,y)}{f_X(x)} = \dfrac{x+y}{2(x+1)}$

$$\therefore \quad f_{X,Y}(x/y) = \begin{cases} \dfrac{x+y}{2(y+1)} & 0 < x < 2, \quad 0 < y < 2 \\ 0 & \text{otherwise} \end{cases}$$

$$f_{X,Y}(y/x) = \begin{cases} \dfrac{x+y}{2(x+1)} & 0 < x < 2, \quad 0 < y < 2 \\ 0 & \text{otherwise} \end{cases}$$

(b) $P((0 < Y < 1/2)/X = 1) = \int_0^{1/2} f_{XY}(y/x=1) = \int_0^{1/2} \dfrac{1+y}{2(1+1)}\,dy$

$$= \frac{1}{4}\int_0^{1/2}(y+1)\,dy = \frac{1}{4}\left[\frac{y^2}{2} + y\right]_0^{1/2}$$

$$= \frac{1}{4}\left[\frac{1}{8} + \frac{1}{2}\right] = \frac{5}{32} = 0.156$$

Example 4.31

Given the function

$$f_{X,Y}(x,y) = \begin{cases} b(x+y)^2 & -2<x<2 \text{ and } -3<y<3 \\ 0 & \text{elsewhere} \end{cases}$$

(a) Find the constant b such that this is a valid joint density function.

(b) Determine the marginal density function $f_X(x)$ and $f_Y(y)$.

Solution

Given the function

$$f_{X,Y}(x,y) = \begin{cases} b(x+y)^2 & -2<x<2 \text{ and } -3<y<3 \\ 0. & \text{elsewhere} \end{cases}$$

(a) The function is a valid density function

Then $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = 1$

$$\int_{-2}^{2}\int_{-3}^{3} b(x+y)^2\,dx\,dy = 1$$

$$b\int_{-2}^{2}\int_{-3}^{3}(x^2+y^2+2xy)\,dx\,dy = 1$$

$$b\int_{-2}^{2}\left[x^2y+\frac{y^3}{3}+\frac{2xy^2}{2}\right]_{-3}^{3} dx = 1$$

$$b\int_{-2}^{2}[3x^2+3^2+x3^2-[-3x^2-3^2+x3^2]dx = 1$$

$$b\int_{-2}^{2}(3x^2+9+9x+3x^2+9-9x)dx = 1$$

$$6b\int_{-2}^{2}(x^2+3)dx = 1$$

$$6b\left[\frac{x^3}{3}+3x\right]_{-2}^{2} = 1$$

$$6b\left[\frac{8}{3}+6-\left(\frac{-8}{3}-6\right)\right] = 1$$

$$12b\left[\frac{8+18}{3}\right] = 1$$

$$104\,b = 1$$

or $$b = \frac{1}{104}$$

) The marginal density functions are

$$f_X(x) = \int_{-\infty}^{\infty} b(x+y)^2 dy \quad = \int_{-3}^{3}\frac{(x+y)^2}{104}dy$$

$$= \frac{1}{104}\frac{(x+y)^3}{3}\bigg|_{-3}^{3} = \frac{1}{312}[(x+3)^3-(x-3)^3]$$

$$f_X(x) = \begin{cases}\frac{1}{312}[(x+3)^3-(x-3)^3 & -2<x<2\\ 0 & \text{elsewhere}\end{cases}$$

and $$f_Y(y) = \int_{-2}^{2} b(x+y)^2 dx \quad = \int_{-2}^{2}\frac{(x+y)^2}{104}dx$$

$$= \frac{1}{312}[(y+2)^3-(y-2)^3]$$

$\therefore$ $$f_Y(y) = \begin{cases}\frac{1}{312}[(y+2)^3-(y-2)^3 & -3<y<3\\ 0 & \text{elsewhere}\end{cases}$$

UNIT 2

Part 5

MULTIPLE RANDOM VARIABLES
– UNIT – IV

Syllabus:

1. Vector Random Variables
2. Joint Distribution function
3. properties of Joint Distribution
4. Marginal Distribution functions
*5. Joint density function
6. properties of Joint density function
7. Marginal density function
8. Conditional Distribution and density – point conditioning
9. Conditional Distribution and density – Interval conditioning
10. statistical Independence
11. Sum of two Random Variables
12. Sum of several Random Variables
13. central limit Theorem (proof not Expected)
14. Unequal Distribution
15. Equal Distribution.

* : Not mentioned in JNTU syllabus

Introduction :-

In the previous Units, the concept of random variable was introduced and the statistics of the random variable are computed. In this Unit, all the above statistics will be extended for two random variables.

If only one characteristic of a random ~~variable~~ experiment is considered, it leads to the concept of a single random variable. If two characteristics of the random experiment are to be considered, this leads to the principle of two random variables.

Eg: In an Electrical Circuit, Current only can be single random variable. Both voltage and Current can be two random variables.

Vector Random Variables :-

Suppose two random variables X and Y are defined on a sample space S, where specific values of X and Y are denoted by x and y, respectively. Then any ordered pair of numbers (x, y) may be conveniently considered to be a "random point" in the xy plane.

The point may be taken as a specific value of a "Vector Random Variable" or "Random Vector."
(X, Y)

The pair (X, Y) is called "bivariate" or "two dimensional random vector."

d. Bivariate, if X & Y discrete
$\nearrow$ Discrete

c. Bivariate, if X & Y continuous.
$\downarrow$ continuous

JOINT Distribution function :-

Let $F_X(x)$ and $F_Y(y)$ represent the probability distribution functions of random variables X and Y respectively. i.e.

$$F_X(x) = P(X \leq x)$$

$$F_Y(y) = P(Y \leq y)$$

Events

The probability of the joint event $(X \leq x, Y \leq y)$ — function of x, y — is defined as the joint probability distribution function and is denoted by $F_{X,Y}(x,y)$ and

X, Y continuous

$$F_{X,Y}(x,y) = P(X \leq x, Y \leq y)$$

$$= P(A \cap B)$$

In general, for 'N' random variables X_n, $n = 1, 2, 3 \ldots N$, the joint distribution function is denoted by

$$F_{X_1, X_2 \ldots X_N}(x_1, x_2, x_3 \ldots x_N) = P[X_1 \leq x_1, X_2 \leq x_2, X_3 \leq x_3 \ldots X_N \leq x_N]$$

$$F_X(x) = \sum_{i=1}^{N} P(x_i) \cdot u(x - x_i)$$

X and Y are d. R.V

Let X have N possible values x_n and Y have M possible values y_m, then

$$F_{X,Y}(x,y) = \sum_{n=1}^{N} \sum_{m=1}^{M} P(x_n, y_m) \cdot u(x - x_n) \cdot u(y - y_m)$$

(116 pages)

Conditional Distribution and Density - point Conditioning

we know that $F_X(x/B) = P\{X \leq x/B\}$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \qquad A: X \leq x$$

$$= \frac{P\{X \leq x \cap B\}}{P(B)} \quad \text{--- (1)}$$

corresponding probability density function is given by

$$f_X(x/B) = \frac{d}{dx}\left[F_X(x/B)\right] \quad \text{--- (2)}$$

The distribution function of one random variable X conditioned by second random variable Y has some specific value y. This is called "point conditioning". Now, we will define event B by

$$B = \{y - \Delta y < Y \leq y + \Delta y\} \quad \text{--- (3)}$$

where Δy is a small quantity that we eventually let approach 0. for this Event, Eq(1) can be written as

$$F_X(x/(y-\Delta y < Y \leq y+\Delta y)) = \frac{\int_{y-\Delta y}^{y+\Delta y}\int_{-\infty}^{x} f_{X,Y}(x,y)\,dx\,dy}{\int_{y-\Delta y}^{y+\Delta y} f_Y(y)\,dy} \quad \text{--- (4)}$$

Consider two cases of Eq ④

① X and Y discrete R.V.s

② X and Y CRVs

① Both X and Y are discrete random variables.

with values x_i, $i = 1, 2, 3 --- N$ and

y_J, $J = 1, 2, 3 --- M$.

while the probabilities of these values are denoted $p(x_i)$ and $P(y_J)$, respectively. The probability of joint occurrance of x_i and y_J is $p(x_i, y_J)$. Then

$$f_Y(y) = \sum_{J=1}^{M} P(y_J) \cdot \delta(y - y_J) \quad \leftarrow ⑤$$

$$f_{X,Y}(x,y) = \sum_{i=1}^{N} \sum_{J=1}^{M} P(x_i, y_J)\, \delta(x - x_i) \cdot \delta(y - y_J) \quad \leftarrow ⑥$$

$$f_X(x) = \sum_{i=1}^{N} P(x_i) \cdot \delta(x - x_i)$$

The specific value of y of interest is y_K

with substitution of Eqs ⑤ and ⑥ into Eq ④, and

allowing $\Delta y \to 0$, we obtain

... denominator of Eq ④ will be $= 0$

conditional distribution $\rightarrow F_X(x|y=y_k) = \sum_{i=1}^{M} \frac{P(x_i, y_k)}{P(y_k)} \cdot u(x-x_i)$ — (7)

conditional density

$$f_X(x|y=y_k) = \sum_{i=1}^{M} \frac{P(x_i, y_k)}{P(y_k)} \cdot \delta(x-x_i) \quad \text{— (8)}$$

(2) If X and Y are continuous Random variables and $\Delta y \to 0$

$$F_X(x|y) = \frac{\int_{-\infty}^{x} f_{X,Y}(x,y)\,dx}{f_Y(y)}$$

and

$$f_X(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

and also

$$f_Y(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

T

* Conditional Distribution and Density – Interval Conditioning

It is sometimes convenient to define Event B in eq ① and ② in terms of random variable Y by

$$B = \{ y_a < Y \le y_b \}$$

where y_a and y_b are real numbers and we assume $P(B) = P\{ y_a < Y \le y_b \} \neq 0$, with this definition, it is readily shown that eq ① and ② become

$$F_X\{x \mid y_a < Y \le y_b\} = \frac{F_{X,Y}(x, y_b) - F_{X,Y}(x, y_a)}{F_Y(y_b) - F_Y(y_a)}$$

(*) – 1

$$= \frac{\int_{y_a}^{y_b} \int_{-\infty}^{x} f_{X,Y}(x, y)\, dx\, dy}{\int_{y_a}^{y_b} \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dx\, dy} \quad (*) - 9$$

and

$$f_X(x \mid y_a < Y \le y_b) = \frac{\int_{y_a}^{y_b} f_{X,Y}(x, y)\, dy}{\int_{y_a}^{y_b} \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dx\, dy}$$

Multiple Choice Questions - UNIT - IV

① The joint probability matrix of two r.v.s x and y is given by

$$P(x, y) = \begin{array}{c|ccc} x \backslash y & 0 & 1 & 2 \\ \hline 0 & 1/18 & 1/9 & 1/6 \\ 1 & 1/9 & 1/18 & 1/9 \\ 2 & 1/6 & 1/6 & 1/18 \end{array}$$

[a]

Then $P\left(\frac{Y=2}{X=2}\right)$

ⓐ 1/7   ⓑ 1/5   ⓒ 1/6   ⓓ 1/18

$$P\left(\frac{Y=2}{X=2}\right) = \frac{P(Y=2, X=2)}{P(X=2)}$$

$$= \frac{1/18}{1/6 + 1/6 + 1/18} = \frac{1/18}{7/18}$$

$$= \underline{1/7}$$

(2) If $f(x) = K/4x$, $x = 1,2 \cdots$ is the probability function of r.v X, then K is equal to [d]

(a) 4 (b) 2 (c) 1 (d) 3

(3) X and Y are independent random variables, with the following respective distributions

x_i	1	2	3
$p(x_i)$	0.7	0.15	0.15

y_j	2	3	4
$p(y_j)$	0.8	0.1	0.4

Then $P(X=2, Y=2)$ is [b]

(a) 0.15 (b) 0.12 (c) 0.06 (d) 0.56

(4) If the joint probability distribution of X and Y is

$f(x,y) = \frac{x+y}{30}$ for $x = 0,1,2,3$; $y = 0,1,2$ then [b]

$P(X > Y)$ is

(a) 0.5 (b) 0.6 (c) 0.3 (d) 0.4

(5) Let X be the outcome from rolling one die and Y be the outcome from rolling a second die. Then [a]

$P(X \le 3, Y > 3)$ is

(a) 0.25 (b) 0.5 (c) 0.75 (d) 0.85

(6) The joint probability matrix of X and Y is

$$P(X,Y) = \begin{array}{c|c} X/Y & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{bmatrix} 1/18 & 1/9 & 1/6 \\ 1/9 & 1/18 & 1/9 \\ 1/6 & 1/6 & 1/18 \end{bmatrix} \end{array}$$

[a]

Then $P(Y=2/X=2)$ is

(a) 1/7 (b) 1/5 (c) 1/6 (d) 1/18

(7) Two discrete random variables X and Y have

$P(X=0, Y=0) = 2/9$, $P(X=0, Y=1) = 1/9$

$P(X=1, Y=0) = 1/9$, $P(X=1, Y=1) = 5/9$

[b]

Are X and Y

(a) Independent (b) Not independent (c) $F_{X,Y}(x,y) = 1$

(8) The o/p to a communication channel is a random variable X and the o/p is another random variable Y. The joint pmf of X and Y is listed below.

$$P(X,Y) = \begin{array}{c|c} X/Y & \begin{matrix} -1 & 0 & 1 \end{matrix} \\ \begin{matrix} -1 \\ 0 \\ 1 \end{matrix} & \begin{bmatrix} 1/4 & 1/8 & 0 \\ 0 & 1/4 & 0 \\ 0 & 1/8 & 1/4 \end{bmatrix} \end{array}$$

[a]

find $P\left(\frac{Y=1}{X=1}\right)$

(a) 1 (b) 2 (c) 3 (d) 6

(9) In above problem, find $P\left(\frac{X=1}{Y=1}\right)$ [b]

(a) 3/2 (b) 2/3 (c) 1 (d) 4

§ JOINT DENSITY Function :-

Assume that the outcome of a random experiment defined by two random variables X and Y. Now, the Joint p. distribution function of these random variables is given by $F_{X,Y}(x,y)$. The Joint probability density function $f_{X,Y}(x,y)$ may be defined by the second derivative of the Joint probability distribution function. i.e

X, Y continuous

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \, \partial y}$$

for N random variables, the joint density function can be obtained by N-fold partial derivative of the N-dimensional joint distribution function i.e

$$f_{X_1,X_2,\ldots,X_N}(x_1,x_2,\ldots,x_N) = \frac{\partial^N F_{X_1,X_2,\ldots X_N}(x_1,x_2,\ldots x_N)}{\partial x_1 \, \partial x_2 \ldots \partial x_N}$$

If the Joint density function is known, the Joint probability distribution function can also be obtained as

$$F_{X_1,X_2,\ldots X_N}(x_1,x_2,\ldots x_N) = \int_{-\infty}^{x_N} \cdots \int_{-\infty}^{x_2} \int_{-\infty}^{x_1} f_{X_1,X_2,\ldots X_N}(x_1, x_2,\ldots x_N)\, dx_1\, dx_2 \ldots$$

§ properties of J. Distribution function :-

1) $F_{X,Y}(-\infty, -\infty) = 0$, $F_{X,Y}(-\infty, y) = 0$ and

$F_{X,Y}(x, -\infty) = 0$

2) $F_{X,Y}(\infty, \infty) = 1$

cdf $f_X(x)$

CDF $F_X(x)$

3) $0 \leq F_{X,Y}(x,y) \leq 1$

4) $F_{X,Y}(x,y)$ is a non-decreasing function of x and y

5) $F_{X,Y}(x_2, y_2) + F_{X,Y}(x_1, y_1) - F_{X,Y}(x_1, y_2) - F_{X,Y}(x_2, y_1) = P\{x_1 < X \leq x_2,\ y_1 < Y \leq y_2\} \geq 0$

6) $F_{X,Y}(x, \infty) = F_X(x)$ and

$F_{X,Y}(\infty, y) = F_Y(y)$

Note1: for a given function to be a valid distribution function of two random variables X and Y, it must satisfy the properties (1), (2), & (5).

Note2. In the 6th property, we see that the distribution function of one random variable can be obtained by setting the other variable to infinity in $F_{X,Y}(x,y)$. The functions $F_X(x)$ and $F_Y(y)$ obtained in this way are called "Marginal distribution functions."

(b) $$\int_0^2 \int_2^3 k \cdot e^{-(x+y)}\, dx\, dy$$

$$\int_0^2 e^{-x}\, dx \int_2^3 e^{-y}\, dy$$

$$= \left[-e^{-x}\right]_0^2 \left[-e^{-y}\right]_2^3$$

$$= \left[-e^{-2}+1\right] \; -e^{-3} - (-e^{-2})$$

$$= (1-e^{-2})(e^{-2}-e^{-3})$$

(c) $$f_x(x) = \int_{-\infty}^{\infty} f_{x,y}(x,y)\, dy$$

$$= \int_0^{\infty} e^{-(x+y)}\, dy$$

$$= e^{-x} \left[-e^{-y}\right]_0^{\infty} = e^{-x}$$

$$f_y(y) = \int_0^{\infty} e^{-(x+y)}\, dx$$

$$= e^{-y}\left[-e^{-x}\right]_0^{\infty} = e^{-y} \cdot [1] = e^{-y}$$

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$$f_{x,y}(x,y) = f_x(x) * f_y(y)$$

Q. (1998 Rep)

The Joint probability density of the random variables X and Y is $f(x,y) = K e^{-(x+y)}$, in the range $0 \le x \le \infty$, $0 \le y \le \infty$ and $f(x,y) = 0$, otherwise.

(a) find the value of the constant K

(b) find the probability $P(0 \le x \le 2,\ 2 \le y \le 3)$

(c) find marginal densities of x and y

(d) Are the random variables dependent or independent.

<u>Sol</u>

$$f(x,y) = e^{-(x+y)}, \quad 0 \le x \le \infty,\ 0 \le y \le \infty$$
$$= 0, \quad \text{otherwise}$$

(a)

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx\, dy = 1$$

$$\int_0^{\infty}\int_0^{\infty} K e^{-(x+y)}\, dx\, dy = 1$$

$$K \int_0^{\infty} e^{-x}\, dx \int_0^{\infty} e^{-y}\, dy = 1$$

$$= K \left[-e^{-x}\right]_0^{\infty} \left[-e^{-y}\right]_0^{\infty} = 1$$

$$= K[1]\,[1] = 1$$

$$\therefore K = 1$$

§ Properties of Joint density function §

A few of the important properties of the joint density function are

1) $f_{X,Y}(x,y) \geq 0$

2) $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1$$

These two properties are used to test whether some given function is a valid joint density function or not.

3) $$F_X(x) = \int_{-\infty}^{x}\int_{-\infty}^{\infty} f_{X,Y}(x_1,x_2)\,dx_1\,dx_2$$

and $$F_Y(y) = \int_{-\infty}^{y}\int_{-\infty}^{\infty} f_{X,Y}(x_1,x_2)\,dx_1\,dx_2$$

this property is used to determine the marginal distribution function

4) $$P\{x_1 < X \leq x_2,\ y_1 < Y \leq y_2\} = \int_{y_1}^{y_2}\int_{x_1}^{x_2} f_{X,Y}(x,y)\,dx\,dy.$$

5) $$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

and $$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$$

The functions $f_X(x)$ and $f_Y(y)$ determined ... called "Marginal density derivatives of the marginal distribution functions.

$$f_X(x) = \frac{d}{dx}\left[F_X(x)\right]$$

$$f_Y(y) = \frac{d}{dy}\left[F_Y(y)\right]$$

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$X \le x = A$

$\cancel{X \le y} = B$

$X \le b$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$F_X(x|B) = \frac{P(X \le x \cap B)}{P(B)}$$

$$F_{X,Y}(x|b) = \frac{P(X \le x) \cap (X \le b)}{P(X \le b)}$$

1)

$$P(A|B) = \frac{P(A\cap B)}{P(B)}$$

$$\int_{-\infty}^{\infty} F_X(x|B) = 1$$

A: $X \le x$

$$F_X(x|B) = \frac{P((X\le x)\cap B)}{P(B)}$$

$$P(A|B) = \frac{P(A\cap B)}{P(B)}$$

a) if event B is defined in terms of X

B: $X \le b$

Then

$$F_X[x/(X\le b)] = \frac{P[(X\le x)\cap(X\le b)]}{P(X\le b)}$$

b) If event B is defined in terms of other variable Y

$Y \le y$: B

* point conditioning

$$F_{X|Y}(x|y) = \frac{P[(X\le x)/(Y\le y)]}{P(Y\le y)} = P(X\le x, Y\le y)$$

$$F_{X|Y}(x|y) = \frac{F_{X,Y}(x,y)}{F_Y(y)}$$

X Y

$X\le x$ $y \le y$

$F_X(x)$ $F_Y(y)$

$F_{X,Y}(x,y)$

$f_{X,Y}(x,y)$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$F_{X|Y}(x|y) = \frac{F_{X,Y}(x,y)}{F_Y(y)} \quad (P\ Y\le y \ne 0)$$

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

UNIT 2

Part 6

1. A random variable X has a mean 3 and variance 2. Use Chebyshev's inequality to obtain an upper bound for

(a) $P\{|X-3| \geq 2\}$ (b) $P\{|X-3| \geq 1\}$

Sol: (a)

$$P\{|X-m| \geq K\} \leq \frac{\sigma^2}{K^2}$$

$$P\{|X-m| \geq 2\} \leq \frac{2}{4} = 1/2$$

∴ The upper bound is 1/2

(b) $$P\{|X-m| \geq 1\} \leq \frac{2}{1} = 2$$

∴ The upper bound is 2

① If x is the number obtained in throwing a fair die, find the probability of $|x-m| > 3$, where 'm' is the mean of x.

Sol. The probability distribution is

x_i	1	2	3	4	5	6
$p(x_i)$	1/6	1/6	1/6	1/6	1/6	1/6

$$m = E(X) = 1\times\frac{1}{6} + 2\times\frac{1}{6} + 3\times\frac{1}{6} + 4\times\frac{1}{6} + 5\times\frac{1}{6} + 6\times\frac{1}{6}$$

$$= 7/2 = 3.5$$

$$E(X^2) = (1)^2\times\frac{1}{6} + (2)^2\cdot\frac{1}{6} + (3)^2\cdot\frac{1}{6} + (4)^2\cdot 1/6 + (5)^2\cdot 1/6 + (6)^2\cdot 1/6$$

$$= \frac{91}{6}$$

$$Var(X) = E(X^2) - [E(X)]^2$$

$$= \frac{91}{6} - \frac{49}{4} = 2.9167$$

Chebyshev's inequality is stated as

$$P\{|X-m| \geq k\} \leq \frac{\sigma^2}{k^2} \quad \text{for } k>0$$

since $k=3$

$$P\{|X-3.5| \geq 3\} \leq \frac{2.9167}{(3)^2} = 0.324$$

* CENTRAL LIMIT THEOREM :- (proof not expected)

This theorem deals with the density function of a sum infinite number of random variables. It is stated as

"The density of the sum of 'n' number of independent, equally distributed random variables approaches Gaussian density in the limit 'n' tends to infinity".

Let $X_1, X_2, X_3 ---- X_N$ be the 'n' number of independent, identically distributed random variables. Let 'S' be random variable defined as $S = X_1 + X_2 + X_3 + --- + X_N$, Then

$$E(S) = E(X_1) + E(X_2) + --- + E(X_N)$$

Since all X_i's are identically distributed, all are same mean (m) and variance σ^2

$$\therefore E(S) = m + m + --- n \text{ times}$$

$$= n.m$$

Similarly, $Var(S) = Var(X_1 + X_2 + \cdots X_N)$

$$= Var(X_1) + Var(X_2) + \cdots Var(X_N)$$

$$= \sigma^2 + \sigma^2 + \cdots + \sigma^2 \text{ (n times)}$$

$$= n \cdot \sigma^2$$

The standardized random variable S', associated with S' is defined as

$$S' = \frac{S - E(S)}{\sqrt{Var(S)}} = \frac{S - n \cdot m}{\sqrt{n \cdot \sigma^2}}$$

$$= \frac{S - n \cdot m}{\sigma \cdot \sqrt{n}}$$

Central limit theorem states that 'S'' is Gaussian

— x —

Sum of Two Random Variables :-

Let X and Y be two independent random variables and their sum be W, i.e.

$W = X+Y$, Now, the probability distribution function of W is given by,

$f_X(x) = P\{X \le x\}$

$$F_W(w) = P\{W \le w\} = P\{X+Y \le w\}$$

$$= \int_{-\infty}^{\infty} f_Y(y) \int_{x=-\infty}^{w-y} f_X(x)\, dx\, dy \quad -①$$

$x+y=w$
$x = w-y$

X	Y
$F_X(x)$	$F_Y(y)$
$f_X(x)$	$f_Y(y)$
W	
$F_W(w)$	
$f_W(w)$	

Why $f_W(w) = \int_{-\infty}^{\infty} f_Y(y) \cdot f_X(w-y)\, dy$

$$f_W(w) = f_X(x) * f_Y(y)$$

Thus, it can be stated that the density function of the sum of two statistically independent random variables is the convolution of their individual density functions.

$$\boxed{f_W(w) = f_X(x) * f_Y(y).}$$

This result can be extended to any number of random variables.

By differentiating eq ①, using Leibniz's rule, we get the desired density function

* Sum of Several Random Variables :-

The Sum Y of N independent random variables $X_1, X_2, X_3 \text{-----} X_N$ is to be considered.

let $Y_1 = X_1 + X_2$

$$f_{Y_1}(y_1) = f_{X_2}(x_2) * f_{X_1}(x_1)$$

we know that X_3 will be independent of Y_1

$Y_1 = X_1 + X_2$, because of X_3 is independent of both X_1 and X_2. Then

$$Y_2 = X_3 + Y_1$$

$$\therefore f_{Y_2}(y_2) = f_{X_3}(x_3) * f_{Y_1}(y_1)$$

$$= f_{X_3}(x_3) * f_{X_2}(x_2) * f_{X_1}(x_1)$$

By continuing the process, we find the density function of $Y = X_1 + X_2 + X_3 \text{---} + X_N$ is the (N-1) fold convolution of the N independent density functions.

$$f_Y(y) = f_{X_N}(x_N) * f_{X_{(N-1)}}(x_{(N-1)}) * \text{---} * f_{X_2}(x_2) * f_{X_1}(x_1)$$

distribution and $F_Y(y) = \int f_Y(y)$

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Joint Distribution function :-

Let $F_X(x)$ and $F_Y(y)$ represent the probability distribution functions of random variables X and Y respectively. i.e.

once the events

$$F_X(x) = P(X \leq x)$$

$$F_Y(y) = P(Y \leq y)$$

→ function of x, y

The probability of the joint event $(X \leq x, Y \leq y)$ is defined as the joint probability distribution function and is denoted by $F_{X,Y}(x,y)$ and

X, Y continuous

$$F_{X,Y}(x,y) = P(X \leq x, Y \leq y)$$
$$= P(A \cap B)$$

In general, for 'N' random variables X_n, $n = 1, 2, 3 \cdots N$, the joint distribution function is denoted by

$$F_{X_1, X_2 \cdots X_N}(x_1, x_2, x_3 \cdots x_N) = P[X_1 \leq x_1, X_2 \leq x_2, X_3 \leq x_3 \cdots X_N \leq x_N]$$

$$F_X(x) = \sum_{i=1}^{N} P(x_i) \cdot u(x - x_i)$$

X and Y are d. R.V

Let X have N possible values x_n and Y have M possible values y_m, then

$$F_{X,Y}(x,y) = \sum_{n=1}^{N} \sum_{m=1}^{M} P(x_n, y_m) \cdot u(x - x_n) \cdot u(y - y_m)$$

where $P(x_n, y_m)$ is the probability of the joint event

Properties of J. Distribution function :-

1) $F_{X,Y}(-\infty,-\infty)=0$, $F_{X,Y}(-\infty,y)=0$ and

$F_{X,Y}(x,-\infty)=0$

2) $F_{X,Y}(\infty,\infty) = 1$ cdf $f_X(x)$, CDF $F_X(x)$

3) $0 \leq F_{X,Y}(x,y) \leq 1$

4) $F_{X,Y}(x,y)$ is a non-decreasing function of x and y

5) $F_{X,Y}(x_2,y_2) + F_{X,Y}(x_1,y_1) - F_{X,Y}(x_1,y_2) - F_{X,Y}(x_2,y_1) = P\{x_1 < X \leq x_2, y_1 < Y \leq y_2\} \geq$

6) $F_{X,Y}(x,\infty) = F_X(x)$ and

$F_{X,Y}(\infty,y) = F_Y(y)$

Note 1: For a given function to be a valid distribution function of two random variables X and Y, it must satisfy the properties (1), (2), & (5).

Note 2: In the 6th property, we see that the distribution function of one random variable can be obtained by setting the other variable to infinity in $F_{X,Y}(x,y)$. The functions $F_X(x)$ and $F_Y(y)$ obtained in this way are called "Marginal distribution functions."

§ JOINT DENSITY Function :-

Assume that the outcome of a random experiment defined by two random variables X and Y. Now, the Joint p. distribution function of these random variables is given by $F_{X,Y}(x,y)$. The Joint probability density function $f_{X,Y}(x,y)$ may be defined by the second derivative of the Joint probability distribution function. i.e.

X,Y continuous

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \, \partial y}.$$

For N random variables, the joint density function can be obtained by N-fold partial derivative of the N-dimensional joint distribution function i.e.

$$f_{X_1,X_2,\ldots,X_N}(x_1,x_2,\ldots,x_N) = \frac{\partial^N F_{X_1,X_2,\ldots X_N}(x_1,x_2,\ldots x_N)}{\partial x_1 \, \partial x_2 \ldots \partial x_N}$$

If the joint density function is known, the joint probability distribution function can also be obtained as

$$F_{X_1,X_2,\ldots X_N}(x_1,x_2,\ldots x_N) = \int_{-\infty}^{x_N} \cdots \int_{-\infty}^{x_2} \int_{-\infty}^{x_1} f_{X_1,X_2,\ldots X_N}(x_1,x_2,\ldots x_N)\, dx_1\, dx_2 \ldots dx_N$$

§ Properties of Joint density function :-

A few of the important properties of the joint density function are

1) $f_{X,Y}(x,y) \geq 0$

2) $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx\, dy = 1$

These two property are used to test whether some given function is a valid joint density function or not.

3) $F_X(x) = \int_{-\infty}^{x} \int_{-\infty}^{\infty} f_{X,Y}(x_1, x_2)\, dx_1\, dx_2$

and $F_Y(y) = \int_{-\infty}^{y} \int_{-\infty}^{\infty} f_{X,Y}(x_1, x_2)\, dx_1\, dx_2$

This property is used to determine the marginal distribution function

4) $P\{x_1 < X \leq x_2,\ y_1 < Y \leq y_2\} = \int_{y_1}^{y_2} \int_{x_1}^{x_2} f_{X,Y}(x,y)\, dx\, dy.$

5) $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy$

and $f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx$

The functions $f_X(x)$ and $f_Y(y)$ determined as in the fifth property are called "marginal density functions" and they can be defined as the →

UNIT 2

Part 7

Sum of Two Random Variables :-

Let X and Y be two independent random variables and their sum be W, i.e.

$W = X+Y$, Now, the probability distribution function of W is given by,

$F_X(x) = P\{X \le x\}$

$$F_W(w) = P\{W \le w\} = P\{X+Y \le w\}$$

$$= \int_{-\infty}^{\infty} f_Y(y) \int_{x=-\infty}^{w-y} f_X(x)\, dx\, dy \quad \text{—(1)}$$

$x+y=w$

$x = w-y$

X	Y
$F_X(x)$	$F_Y(y)$
$f_X(x)$	$f_Y(y)$

W
$F_W(w)$
$f_W(w)$

$$\text{ll}^{ly}\quad f_W(w) = \int_{-\infty}^{\infty} f_Y(y) \cdot f_X(w-y)\, dy$$

$$f_W(w) = f_X(x) * f_Y(y)$$

Thus, it can be stated that the density function of the sum of two statistically independent random variables is the convolution of their individual density functions.

$$\boxed{f_W(w) = f_X(x) * f_Y(y)\,.}$$

This result can be extended to any number of random variables.

---x---

By differentiating Eq (1), using Leibniz's rule, we get the desired density function

* Sum of Several Random Variables :-

The Sum Y of N independent random variables $X_1, X_2, X_3 \text{ ------ } X_N$ is to be considered.

Let $Y_1 = X_1 + X_2$

$$f_{Y_1}(y_1) = f_{X_2}(x_2) * f_{X_1}(x_1)$$

We know that X_3 will be independent of Y_1

$Y_1 = X_1 + X_2$, because of X_3 is independent of both X_1 and X_2. Then

$$Y_2 = X_3 + Y_1$$

$$\therefore f_{Y_2}(y_2) = f_{X_3}(x_3) * f_{Y_1}(y_1)$$

$$= f_{X_3}(x_3) * f_{X_2}(x_2) * f_{X_1}(x_1)$$

By continuing the process, we find the density function of $Y = X_1 + X_2 + X_3 \text{ --- } + X_N$ is the $(N-1)$ fold convolution of the N independent density functions.

$$f_Y(y) = f_{X_N}(x_N) * f_{X_{(N-1)}}(x_{(N-1)}) * \text{ --- } * f_{X_2}(x_2) * f_{X_1}(x_1)$$

distribution and $F_Y(y) = \int f_Y(y)$

STATISTICAL INDEPENDENCE :-

The two events A and B are statistically independent if (and only if)

$$P(A \cap B) = P(A) \cdot P(B) \quad \text{— (1)}$$

This condition can be used to apply to two random variables X and Y by defining the events $A = \{X \le x\}$ and $B = \{Y \le y\}$ for the real numbers x and y. Thus, X and Y are said to be statistically independent random variables if (and only if)

$$P\{X \le x, Y \le y\} = P(X \le x) \cdot P(Y \le y) \quad \text{— (2)}$$

From this expression and the definitions of distribution functions, it follows that

$$F_{X,Y}(x,y) = F_X(x) \cdot F_Y(y) \quad \text{— (3)}$$

From the definitions of density functions

$$f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y) \quad \text{— (4)}$$

The form of conditional distribution function for independent events is found by, replacing B by $\{Y \le y\}$ in the following

$$F_X(x/B) = P\{X \le x \cap B\} = \frac{P\{X \le x \cap B)}{P(B)}$$

$$F_X\{x / Y \le y\} = P(X \le x \cap Y \le y)$$

$$= \frac{P(X \le x, Y \le y)}{P(Y \le y)}$$

$$= \frac{F_{X,Y}(x,y)}{F_Y(y)} = \frac{F_X(x) \cdot \cancel{F_Y(y)}}{\cancel{F_Y(y)}} \quad \leftarrow (5)$$

$$\therefore \quad F_X(x / Y \le y) = F_X(x) \quad \leftarrow (6)$$

and also $F_Y(y / X \le x) = F_Y(y) \quad \leftarrow (7)$

|ly, the conditional density function forms,

$$f_X(x / Y \le y) = f_X(x) \quad \leftarrow (8)$$

and $f_Y(y / X \le x) = f_Y(y) \quad \leftarrow (9)$

The Joint probability density of the random variables X and Y is $f(x,y) = Ke^{-(x+y)}$, in the range $0 \leq x \leq \infty$, $0 \leq y \leq \infty$ and $f(x,y) = 0$, otherwise.

(a) find the value of the constant K

(b) find the probability ~~$p(0 \leq x \leq 2)$~~ $p(0 \leq x \leq 2,\ 2 \leq y \leq 3)$

(c) find marginal density of x and y

(d) Are the random variables dependent or independent.

Sol

$$f(x,y) = e^{-(x+y)}, \quad 0 \leq x \leq \infty,\ 0 \leq y \leq \infty$$
$$= 0, \quad \text{otherwise}$$

(a)

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{x,y}(x,y)\,dx\,dy = 1$$

$$\int_0^{\infty}\int_0^{\infty} Ke^{-(x+y)}\,dx\,dy = 1$$

$$K \cdot \int_0^{\infty} e^{-x}\,dx \int_0^{\infty} e^{-y}\,dy = 1$$

$$= K \cdot \left[-e^{-x}\right]_0^{\infty} \left[-e^{-y}\right]_0^{\infty} = 1$$

$$= K[1][1] = 1$$

$$\Rightarrow K = 1$$

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(b)

$$\int_0^2 \int_2^3 k \cdot e^{-(x+y)} \, dx\, dy$$

1

$$\int_0^2 e^{-x} dx \int_2^3 e^{-y} dy$$

$$= \left[-e^{-x}\right]_0^2 \left[-e^{-y}\right]_2^3$$

$$= \left[-e^{-2}+1\right] \; -e^{-3} - (-e^{-2})$$

$$= (1-e^{-2})(e^{-2}-e^{-3})$$

(c)

$$f_x(x) = \int_{-\infty}^{\infty} f_{x,y}(x,y)\, dy$$

$$= \int_0^{\infty} e^{-(x+y)} dy$$

$$= e^{-x} \cdot \left[-e^{-y}\right]_0^{\infty} = e^{-x}$$

$$f_y(y) = \int_0^{\infty} e^{-(x+y)} dx$$

$$= e^{-y}\left[-e^{-x}\right]_0^{\infty} = e^{-y}[1] = e^{-y}$$

$$\therefore \; f_{x,y}(x,y) = f_x(x) * f_y(y)$$

$$-(x+y) \qquad -(x+y)$$

Two r.v. are Independent

If x is a random variable, then, show that

Imp $Var(ax+b) = a^2 . Var(x)$

Sol

let $y = ax+b$

$$E[y] = E[ax+b]$$

$$E[y] = a . E[x] + b$$

$$-E[y] = -a E[x] - b$$

$$y - E[y] = ax + \cancel{b} - a E[x] - \cancel{b}$$

$$= a[x - E[x]]$$

Squaring and taking Expectation on bothsides

$$E\{[y - E[y]]^2\} = a^2 . E\{[x - E[x]]^2\}$$

$$Var(y) = a^2 \, Var(x)$$

$$\boxed{Var(ax+b) = a^2 \, Var(x)}$$

Hence proved.

Two random variables X and Y have the following probability density function

$$f(x,y) = 2-x-y \quad 0\le x\le 1,\ 0\le y\le 1,$$
$$= 0, \quad \text{otherwise}$$

find (i) marginal density functions of X and Y

(ii) Var(X) and Var(Y)

(iii) co-variance between X and Y

Sol

(i)

$$f_x(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$= \int_0^1 (2-x-y)\,dy$$

$$= \left[2y-xy-\frac{y^2}{2}\right]_0^1$$

$$= \frac{3}{2}-x$$

$$\therefore f(x) = 3/2 - x, \quad 0\le x\le 1$$
$$= 0, \quad \text{elsewhere}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$= \int_0^1 (2-x-y)\,dx$$

$$= \left(2x-\frac{x^2}{2}-xy\right)_0^1$$

$$= 3/2 - y, \quad 0\le y\le 1$$
$$= 0, \quad \text{elsewhere}$$

(ii) $\mathrm{Var}(X) = E[X^2] - (E[X])^2$

$$E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx$$

$$= \int_0^1 x\left(\tfrac{3}{2} - x\right) dx$$

$$= \left[\frac{3}{2}\cdot\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 5/12$$

$$E[X^2] = \int_0^1 x^2\left(\tfrac{3}{2} - x\right) dx$$

$$= \left[\tfrac{3}{2}\cdot\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = 1/4$$

$$\therefore \mathrm{Var}(X) = E(X^2) - [E(X)]^2$$

$$= 1/4 - (5/12)^2$$

$$= 1/4 - \frac{25}{144} = \cancel{1/16}\ \frac{36-25}{144}$$

$$= \frac{11}{144}$$

Similarly, $E[Y] = 5/12$

$E[Y^2] = 1/4$

$$\therefore \mathrm{Var}(Y) = \cancel{1/16}\ \frac{11}{144}$$

(iii)

$$Cov(X,Y) = E[XY] - E(X)\cdot E(Y)$$

$$E[XY] = \int_0^1 \int_0^1 xy\cdot(2-x-y)\,dx\,dy$$

$$= \int_0^1 \int_0^1 (2xy - x^2y - xy^2)\,dx\,dy$$

$$= \int_0^1 \left[\cancel{2}\frac{x^2}{\cancel{2}}\cdot y - \frac{x^3}{3}\cdot y - \frac{x^2}{2}y^2\right]_0^1 dy$$

$$= \int_0^1 \left[y - \frac{y}{3} - \frac{y^2}{2}\right] dy$$

$$= \int_0^1 \left[\frac{2y}{3} - \frac{y^2}{2}\right] dy$$

$$= \left[\frac{2}{3}\cdot\frac{y^2}{2} - \frac{y^3}{6}\right]_0^1$$

$$= \left(\frac{2}{3}\cdot\frac{1}{2} - 1/6\right) = \frac{2}{6} - 1/6 = 1/6$$

$$\therefore Cov(X,Y) = E[XY] - E(X)\cdot E(Y)$$

$$= 1/6 - \frac{5}{12}\times\frac{5}{12}$$

$$= \frac{1}{6} - \frac{25}{144} = \frac{24-25}{144} = -1/144$$

$$\therefore Cov(X,Y) = \frac{-1}{144}$$

$\#2$

A joint density function of random variables X and Y is given by

$$f(x,y) = 4xy\, e^{-(x^2+y^2)}, \quad x \geq 0,\ y \geq 0$$

Show that X, Y are statistically independent random variables.

Sol

$$f_{X,Y}(x,y) = f_X(x) * f_Y(y)$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy$$

$$= \int_0^{\infty} 4xy e^{-(x^2+y^2)} dy$$

$$= 4x e^{-x^2} \int_0^{\infty} y e^{-y^2} dy$$

$$y^2 = t$$

$$2y\,dy = dt$$

$$= \frac{4xe^{-x^2}}{2} \int_0^{\infty} e^{-t}\, dt$$

$$= 2xe^{-x^2}\left[-e^{-t}\right]_0^{\infty} = 2x e^{-x^2}$$

$$f_Y(y) = \int_0^{\infty} 4xy e^{-(x^2+y^2)} dx = 2y e^{-y^2}$$

$$\therefore f_{X,Y}(x,y) = f_X(x) * f_Y(y)$$

Hence X & Y are independent.

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Let x be the outcome from the rolling of the first and y be the outcome from the rolling of the second die. What is the joint probability of the event $X \leq 3$ and $Y > 3$

(1,4) (1,5) (1,6) (2,4) (2,5) (2,6)
(3,4) (3,5) (3,6)

$$p(x \leq 3,\ y > 3) = 9/36 = 1/4$$

Two random variables X and Y have a Jp. density function

$$f(x,y) = K(xy + 2x + y + 3), \quad 0 \leq x \leq 1, \ 0 \leq y \leq 1$$
$$= 0 \quad \text{elsewhere}$$

find the value of K and for which the r.v. X and Y are statistically independent)

$\frac{5}{4}\ xy +$

$$f_{X,Y}(x,y) = f_X(x) \times f_Y(y)$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy$$

$$= \int_0^1 K(xy + 2x + y + a)\, dy$$

$$= K\left[\frac{x \cdot y^2}{2} + 2xy + \frac{y^2}{2} + ay\right]_0^1$$

$$= K\left[\frac{x}{2} + 2x + \frac{1}{2} + a\right]$$

$$= K\left[\frac{5x}{2} + a + \frac{1}{2}\right] \quad \text{—(1)}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx$$

$$= \int_0^1 K(xy + 2x + y + a)\, dx$$

$$= K\left[y \cdot \frac{x^2}{2} + x^2 + xy + ax\right]_0^1$$

$$= K\left[\frac{y}{2} + 1 + y + a\right]$$

$$= K\left[\frac{3y}{2} + a + 1\right] \quad \text{—(2)}$$

$$f_{X,Y}(x,y) = (1) \times (2)$$

$$\therefore \cancel{K}\left[xy + 2x + y + a\right] = \cancel{K}\left[\frac{5x}{2} + a + \frac{1}{2}\right] \times K\left[\frac{3y}{2} + a + 1\right]$$

$$K = \frac{xy + 2x + y + a}{\left[\frac{5x}{2} + a + \frac{1}{2}\right]\left[\frac{3y}{2} + a + 1\right]}$$

Q prove the following

(a) $Cov(ax, by) = ab \cdot Cov(x, y)$

(b) $Var(ax + by) = a^2 \cdot Var(x) + b^2 \cdot Var(y) + 2ab \cdot Cov(x, y)$

<u>Sol</u>: (a) $Cov(ax, by) = E\left[\{ax - E(ax)\}\{by - E(by)\}\right]$

$= E\left[\{ax - a \cdot E(x)\}\{by - b \cdot E(y)\}\right]$

$= a \cdot b \cdot E\left[\{\{x - E(x)\}\{y - E(y)\}\right]$

$= ab \cdot Cov(x, y)$

(b) $Var(ax + by) = E\left[\{(ax + by) - E(ax + by)\}^2\right]$

$= E\left[\{ax + by - a \cdot E(x) - b \cdot E(y)\}^2\right]$

$= E\left[\{a(x - E(x)) + b \cdot (y - E(y))\}^2\right]$

$= E\left[a^2 [x - E(x)]^2 + b^2 [y - E(y)]^2 + 2ab \cdot [x - E(x)][y - E(y)]\right]$

$= a^2 \cdot E\left[\{x - E(x)\}^2\right] + b^2 E\left[\{y - E(y)\}^2\right] +$
$2ab \; E\{[x - E(x)][y - E(y)]\}$

$\Rightarrow a^2 Var(x) + b^2 \cdot Var(y) + 2ab \cdot Cov(x, y)$, Hence proved

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① The characteristic function of a r.v. X is given by $\phi_x(\omega) = \frac{1}{(1-J2\omega)^{N/2}}$.

find mean and second moment of X

Sol:

$$\text{Mean } E(X) = \frac{1}{J} \cdot \frac{d}{d\omega} \phi_x(\omega) \Big|_{\omega=0}$$

$$\frac{d}{d\omega} \phi_x(\omega) = \frac{d}{d\omega} \left[\frac{1}{(1-J2\omega)^{N/2}} \right]$$

$$= \frac{d}{d\omega} \left[(1-J2\omega)^{-N/2} \right]$$

$\frac{d}{dx}(x^n) = n \cdot x^{n-1}$

$$= +\frac{N}{2} \cdot (1-J2\omega)^{-\frac{N}{2}-1} \cdot (+2J) =$$

$$= \frac{J \cdot N}{(1-J2\omega)^{(N/2+1)}}$$

$$E(X) = \frac{1}{J} \cdot \frac{d}{d\omega} \phi_x(\omega) \Big|_{\omega=0} = N$$

$$\text{second moment } E(X^2) = \left(\frac{1}{J}\right)^2 \cdot \frac{d^2}{d\omega^2} \phi_x(\omega) \Big|_{\omega=0}$$

consider

$$\frac{d^2}{d\omega^2}\phi_y(\omega) = \frac{d}{d\omega}\left[JN\,(1-J2\omega)^{-\left(\frac{N}{2}+1\right)} \right]$$

$$= +JN.\left(1+\frac{N}{2}\right)(1-J2\omega)^{-\left(\frac{N}{2}+1\right)-1} .(+2J)$$

$\frac{N}{2}+1-1$

$$= \frac{(J)^2 . 2N\cdot\left(\frac{N}{2}+1\right)}{(1-J2\omega)^{\left(2+\frac{N}{2}\right)}}$$

$$E(Y^2) = \left(1/J\right)^2 . \frac{d^2}{d\omega^2}.\phi_y(\omega)\Big|_{\omega=0}$$

$$= 2N\left(\frac{N}{2}+1\right)$$

problems on 4th Unit

(2) (a) find a constant b (interms of 'a') so that the function $f_{XY}(x,y) = b \cdot e^{-(x+y)}$ for $0 \le x < a$, $0 < y < \infty$

$= 0$ elsewhere

is a valid joint density function.

(b) find an Expression for the joint distribution function

Sol:-

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x,y)\, dx\, dy = 1$$

$$= \int_{0}^{a} \int_{0}^{\infty} b \cdot e^{-(x+y)}\, dx\, dy = 1$$

$$= b \cdot \int_{0}^{a} e^{-x} \left[-e^{-y} \right]_{0}^{\infty} dx = 1$$

$$= b \cdot \int_{0}^{a} e^{-x}\, dx = 1$$

$$= b \left[-e^{-x} \right]_{0}^{a} = 1$$

$$= b \left(1 - e^{-a}\right) = 1$$

$$\Rightarrow b = \frac{1}{1 - e^{-a}}$$

$$F_{X,Y}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{XY}(x,y)\,dx\,dy$$

$$= \int_{0}^{x}\int_{0}^{y} \frac{1}{1-e^{-a}} \cdot e^{-x} \cdot e^{-y} \cdot dx \cdot dy$$

$$= \frac{1}{1-e^{-a}} \int_{0}^{x} e^{-x} \left(-e^{-y}\right)_0^y \, dx$$

$$= \frac{1-e^{-y}}{1-e^{-a}} \int_{0}^{x} e^{-x}\,dx$$

$$= \frac{1-e^{-y}}{1-e^{-a}} \left[-e^{-x}\right]_0^x$$

$$= \frac{(1-e^{-y})}{(1-e^{-a})}(1-e^{-x})$$

$$\therefore \quad F_{X,Y}(x,y) = 0 \quad \text{for } x<0,\ y<0$$

$$= \frac{(1-e^{-x})(1-e^{-y})}{(1-e^{-a})} \quad \text{for } 0<x<a,\ 0<y<\infty$$

$$= 1, \quad x \geq a$$

ECS-1

If the function $f(x,y) = b \cdot e^{-2x} \cdot \cos(y/2)$ for

$0 \le x \le 1,$
$0 \le y \le \pi$

$= 0,$ elsewhere

where 'b' is a positive constant, is valid joint probability density function, find 'b'.

Sol:

$$\int_0^1 \int_0^\pi b e^{-2x} \cos(y/2)\, dx\, dy = 1$$

$$= b \int_0^1 e^{-2x} \left[\int_0^\pi \cos(y/2)\, dy \right] dx = 1$$

$$= b \int_0^1 e^{-2x} \left[2 . \sin(y/2) \right]_0^\pi dx = 1$$

$$= 2b \int_0^1 e^{-2x}\, dx = 1$$

$$= 2b \left[-\frac{e^{-2x}}{2} \right]_0^1 = 1$$

$$= b\, (1 - e^{-2}) = 1$$

$$\Rightarrow b = 1/(1 - e^{-2})$$

(1) The density function

$$f_{XY}(x,y) = \frac{xy}{9}, \quad 0<x<2, \; 0<y<3$$
$$= 0, \quad \text{elsewhere}$$

(a) Show that x, y are stat. independent

(b) " " x, y are uncorrelated

Sol

$$f_X(x) = \int_{y=0}^{3} f(x,y)\, dy$$

$$= \frac{x}{9}\int_0^3 y\, dy = \frac{x}{9}\left[\frac{y^2}{2}\right]_0^3$$

$$= x/2$$

$$\therefore f_X(x) = x/2 \quad \text{for } 0<x<2$$
$$= 0 \quad \text{elsewhere}$$

$$f_Y(y) = \frac{y}{9}\int_{x=0}^{2} x \, dx = \frac{y}{9}\left[\frac{x^2}{2}\right]_0^2$$

$$= 2y/9$$

$$f_Y(y) = \frac{2y}{9}, \quad \text{for } 0<y<3$$
$$= 0, \quad \text{elsew}$$

$$f_{XY}(x,y) = f_X(x)\, f_Y(y)$$

$\therefore$ independent

(b) X, Y are uncorrelated, if $\rho_{XY} = 0$

$$\rho_{XY} = \frac{\text{Cov}(X,Y)}{\sigma_X \cdot \sigma_Y}$$

$$\text{Cov}(X,Y) = E(XY) - E(X)\,E(Y)$$

Since X, Y are independent

$$E(XY) = E(X)\,E(Y)$$

$$\Rightarrow \text{Cov}(X,Y) = 0$$

$$\therefore \rho_{XY} = 0$$

X, Y are uncorrelated

1. Two r.vs x and y are related by the expression $y = ax+b$, where a and b are any real numbers.

(a) Show that their covariance $C_{xy} = a \cdot \sigma_x^2$

(b) Show that their correlation coefficient is

$$\rho = 1, \text{ if } a>0 \text{ for any } b$$
$$= -1 \text{ if } a<0 \text{ for any } b$$

Sol

$$\text{cov}(x,y) = E(xy) - E(x) \cdot E(y)$$

$$= E[x(ax+b)] - E(x) \cdot E(ax+b)$$

$$= E[ax^2+bx] - E(x)[a \cdot E(x)+b]$$

$$= a \cdot E(x^2) + b \cdot \cancel{E(x)} - a \cdot [E(x)]^2 \cancel{-b\, E(x)}$$

$$= a \left[E(x^2) - \{E(x)\}^2\right]$$

$$= a \cdot \text{var}(x)$$

$$= a \cdot \sigma_x^2$$

(b) $$\rho_{xy} = \frac{\text{cov}(x,y)}{\sigma_x \cdot \sigma_y}$$

$$\sigma_y^2 = \text{var}(y) = \text{var}(ax+b) = a^2 \cdot \text{var}(x)$$

$$\sigma_y = a \cdot \sqrt{\text{var}(x)}$$

$$\rho_{xy} = \frac{a \cdot var(x)}{\sqrt{var(x)} \cdot \cancel{a} \cdot \sqrt{var(x)} \cdot a}$$

$$\therefore \rho_{xy} = 1$$

if $a < 0$, $cov(x, y) = -a \cdot \sigma_x^2 = -a \cdot var(x)$

$$\therefore \rho_{xy} = \frac{-a \cdot var(x)}{\sqrt{var(x)} \cdot a \cdot \sqrt{var(x)}}$$

$$= -1$$

$\therefore$ $\rho = 1$ if $a > 0$ for any b

$\rho = -1$ if $a < 0$ for any b

EXTRA PROBLEMS — U-4

① Let $f_{XY}(x,y) = x+y$ for $0 \le x \le 1, 0 \le y \le 1$

$= 0$, otherwise

find the conditional density of ⓐ x given y

ⓑ y given x

Sol: ⓐ $f(x/y) = \dfrac{f(x,y)}{f(y)}$

$$f(y) = \int_0^1 (x+y)\,dx$$

$$= \left[\frac{x^2}{2} + xy\right]_0^1 = y + 1/2$$

$\therefore f(x|y) = \dfrac{x+y}{(y+1/2)}$ for $0 \le x \le 1, 0 \le y \le 1$

$= 0$ for other x, $0 \le y \le 1$

ⓑ $f(y/x) = \dfrac{f(x,y)}{f(x)}$

$$f(x) = \int_0^1 (x+y)\,dy = \left[xy + y^2/2\right]_0^1$$

$$= x + 1/2$$

$\therefore f(y/x) = \dfrac{x+y}{(x+1/2)}$ for $0 \le x \le 1$, $0 \le y \le 1$

$= 0$, $0 \le x \le 1$, other y

(8) Given the function $f_{X,Y}(x,y) = b(x+y)^2$ for

$-2 < x < 2,$

$-3 < y < 3.$

$= 0,$ elsewhere

(a) find the constant 'b' such that this is a valid joint density function

(b) Determine the marginal density functions $f_X(x)$ & $f_Y(y)$

Small Small

Sol:

(a) $$\int_{-2}^{2}\int_{-3}^{3} b(x+y)^2 \, dx\, dy = 1$$

$$= b\int_{-2}^{2}\int_{-3}^{3}(x^2+y^2+2xy)\, dx\, dy = 1$$

$$= b\int_{-2}^{2}\left[x^2y + \frac{y^3}{3} + xy^2\right]_{-3}^{3} dx = 1$$

$$= b\int_{-2}^{2}(6x^2+18)\, dx = 1$$

$$= b\left[\frac{6x^3}{3} + 18x\right]_{-2}^{2} = 1$$

$$b = 1/104$$

(b)

$$f_x(x) = \int_{-3}^{3} f_{x,y}(x,y)\, dy \xrightarrow{} y \text{ limit only}$$

$$= \frac{1}{104}\left[x^2 y + \frac{y^3}{3} + xy^2\right]_{-3}^{3}$$

$$= \frac{1}{104}(\cancel{8}^{6}x^2 + 18) \quad \text{for } -2 < x < 2$$

$$= 0 \quad , \quad \text{else where}$$

$$f_y(y) = \int_{-2}^{2} f_{xy}(x,y)\, dx$$

$$= \frac{1}{104}\int_{-2}^{2} (x+y)^2\, dx$$

$$= \frac{1}{104}\int_{-2}^{2} (x^2 + y^2 + 2xy)\, dx$$

$$= \frac{1}{104}\left[\frac{x^3}{3} + xy^2 + x^2 y\right]_{-2}^{2}$$

$$= \frac{1}{104}\left[4y^2 + 16/3\right]$$

$$\therefore f_y(y) = \frac{1}{104}\left[4y^2 + 16/3\right] \quad \text{for } -3 < y < 3$$

$$= 0, \quad \text{else where}$$

① Two discrete Random variables X and Y have

$P(X=0, Y=0) = 2/9$, $P(X=0, Y=1) = 1/9$

$P(X=1, Y=0) = 1/9$, $P(X=1, Y=1) = 5/9$

Are X and Y independent?

<u>Sol</u>: for independency $P(X=x_i, Y=y_j) = P(X=x_i) \cdot P(Y=y_j)$

$$P(x,y) = \begin{array}{c|cc} X/Y & 0 & 1 \\ \hline 0 & 2/9 & 1/9 \\ 1 & 1/9 & 5/9 \end{array}$$

$P(X=0, Y=0) = 2/9$

$P(X=0) = 2/9 + 1/9 = 3/9$

$P(Y=0) = 2/9 + 1/9 = 3/9$

$\therefore P(X=0, Y=0) \neq P(X=0) \cdot P(Y=0)$

Hence, X and Y are not independent.

(1998 - Supply)

① The joint pdf of the random variables x and y is $f_{x,y}(x,y) = \frac{1}{4} \cdot e^{-|x|-|y|}$ for $-\infty < x < \infty$, $-\infty < y < \infty$

(a) Are x and y statistically independent

(b) find $P(x \leq 1, y \leq 0)$

Sol : (a) x and y are statistically independent, if

$$f_{x,y}(x,y) = f_x(x) \cdot f_y(y)$$

consider $f_x(x) = \int_{-\infty}^{\infty} f_{x,y}(x,y)\, dy$

$$= \int_{-\infty}^{\infty} \frac{1}{4} \cdot e^{-|x|-|y|}\, dy$$

$$= \frac{1}{4} \cdot e^{-|x|} \cdot \int_{-\infty}^{\infty} e^{-|y|}\, dy$$

$$= \frac{1}{4} \cdot e^{-|x|} \left[\int_{-\infty}^{0} e^{y}\, dy + \int_{0}^{\infty} e^{-y} \cdot dy \right]$$

$$= \frac{1}{4} \cdot e^{-|x|} [1+1] = \frac{e^{-|x|}}{2} \text{ for } -\infty < x < \infty$$

consider $f_y(y) = \int_{-\infty}^{\infty} f_{xy}(x,y)\, dx$

$$= \frac{1}{4} \cdot e^{-|y|} \cdot \int_{-\infty}^{\infty} e^{-|x|} \, dx$$

$$= \frac{1}{4} \cdot e^{-|y|} \cdot \left[\int_{-\infty}^{0} e^{x} \, dx + \int_{0}^{\infty} e^{-x} \, dx \right]$$

$$= \frac{1}{4} \cdot e^{-|y|} \, [1+1]$$

$$= \frac{e^{-|y|}}{2} \quad \text{for } -\infty < y < \infty$$

$$\therefore \quad f_X(x) \cdot f_Y(y) = \frac{1}{4} \cdot e^{-|x|-|y|}$$

$$= f_{X,Y}(x,y)$$

Hence, x and y are independent

(b) $$P(X \le 1, Y \le 0) = \int_{-\infty}^{1} \int_{-\infty}^{0} f(x,y) \, dx\, dy$$

$$= \int_{-\infty}^{1} \int_{-\infty}^{0} \frac{1}{4} e^{-|x|-|y|} \, dx \cdot dy$$

$$= \frac{1}{4} \int_{-\infty}^{1} e^{-|x|} \left[\int_{-\infty}^{0} e^{-|y|} \, dy \right] dx$$

$$= \frac{1}{4} \int_{-\infty}^{1} e^{-|x|} \left[\int_{-\infty}^{0} e^{y} \, dy \right] dx$$

$$= \frac{1}{4} \int_{-\infty}^{1} e^{-|x|} \, dx$$

$$= \frac{1}{4}\left[\int_{-\infty}^{0} e^{x} \cdot dx + \int_{0}^{1} e^{-x} \, dx\right]$$

$$= \frac{1}{4}\left[1 + \left[-e^{-x}\right]_0^1\right]$$

$$= \frac{1}{4} \cdot \left[2 - e^{-1}\right]$$

—— Y ——

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1. Let x and y be independent and identically distributed Random variables (1) UNIT-5

Each having density function

$$f_x(x) = 2 \cdot e^{-2x}, \quad x \geq 0 \quad y \geq 0$$
$$= 0, \quad \text{elsewhere}$$

find (a) $E(x+y)$ (b) $E(x^2+y^2)$ (c) $E(xy)$

Sol

$$E(x+y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x+y) \cdot f_{xy}(x,y)\, dx\, dy$$

$\therefore$ Since it is given that x and y are identically distributed independent r.v.s,

$$\therefore f_{x,y}(x,y) = f_x(x) \cdot f_y(y)$$

(1) $\therefore$

$$E(x+y) = \int_0^{\infty} \int_0^{\infty} (x+y) \cdot 2e^{-2x} \cdot 2e^{-2y}\, dx\, dy$$

$$= \int_0^{\infty} 2x \cdot e^{-2x} \cdot dx \cdot \int_0^{\infty} 2e^{-2y} dy + \int_0^{\infty} 2y \cdot e^{-2y} dy \cdot \int_0^{\infty} 2e^{-2x} \cdot dx$$

$$= 1$$

(2)

$$E(x^2+y^2) = \int_0^{\infty} \int_0^{\infty} (x^2+y^2) \cdot 2e^{-2x} \cdot 2e^{-2y} \cdot dx\,dy$$

$$= 1$$

(3)

$$E(xy) = \int_0^{\infty} \int_0^{\infty} xy \cdot 2e^{-2x} \cdot 2e^{-2y} \cdot dx\,dy$$

$$= 1/4$$

① The joint space for two random variables X and Y and corresponding probabilities are as shown in table

XY	(1,1)	(2,2)	(3,3)	(4,4)
P	0.2	0.3	0.35	0.15

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① find $F_{XY}(x,y)$

$= P\{X=1, Y=1\} + P\{X=2, Y=2\} + \cdots$

$= 0.2 + 0.3 + 0.35 + 0.15 = \underline{1}$

② $P\{X \le 2, Y \le 2\} = P(X=1, Y=1) + P(X=2, Y=2)$

$= 0.2 + 0.3 = 0.5$

③ $P\{1 < X \le 3, Y \ge 3\}$

$X = (2,3) \quad Y = 3, 4$

$= P(X=2, Y=3) + P(X=2, Y=4) + P(X=3, Y=3)$
$+ P(X=3, Y=4)$

$= 0 + 0 + 0.35 + 0 = 0.35$

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Q Let X and Y be two random variables, each taking three values −1, 0 and 1 and having the joint probability distribution shown

X \ Y	−1	0	1	Total
−1	0	0.2	0	0.2
0	0.1	0.2	0.1	0.4
1	0.1	0.2	0.1	0.4
Total	0.2	0.6	0.2	1.0

find (i) $E[X]$ and $E[Y]$

(ii) prove that X and Y are uncorrelated random variables

(iii) find $V[X]$ and $V[Y]$

Sol:- (i) $E[X] = \sum_{i=-1}^{1} x_i \, p(x_i)$

$= (-1)(0.2) + (0)(0.4) + (1)(0.4)$

$= -0.2 + 0 + 0.4 = \underline{0.2}$

$E[Y] = \sum_{j=-1}^{1} y_j \, p(y_j)$

$= (-1)(0.2) + (0)(0.6) + (1)(0.2)$

$= -0.2 + 0 + 0.2 = \underline{0}$

(ii) If x and y are uncorrelated, $E[xy] = E[x] \cdot E[y]$.

$\therefore$ Their co-variance is zero

$$E[xy] = \sum_{i,j=-1}^{1} x_i y_j P(x_i, y_j)$$

$$= (-1)(-1)(0) + (0)(-1)(0.1) + (1)(-1)(0.1) +$$
$$(-1)(0)(0.2) + (0)(0)(0.2) + (1)(0)(0.2) +$$
$$(-1)(1)(0) + (0)(1)(0.1) + (1)(1)(0.1)$$

$$= -0.1 + 0.1 = 0$$

$$E(x) \times E(y) = 0.2 \times 0 = 0$$

$E[xy] = E[x] \cdot E[y]$, the random variables x and y are uncorrelated.

(iii) $$V[x] = E[x^2] - [E(x)]^2$$

$$E(x^2) = \sum_{i=-1}^{1} x^2 P(x_i)$$

$$= (-1)^2 (0.2) + (0)^2 (0.4) + (1)^2 (0.4)$$

$$= +0.2 + 0.4 = 0.6$$

$$V[x] = 0.6 - (0.2)^2 = 0.56$$

$E(y^2) = (-1)^2(0.2) + (0)^2(0.6) + (1)^2(0.2) = 0.2 + 0.2$

$$\therefore V[y] = E(y)^2 - (E(y))^2$$

$$= 0.2 + 0.2 = 0.4$$

UNIT 2

Part 8

UNIT 2

Part 9

UNIT 2

Part 10