Example 23 : Find the analytic function $f(z) = u(x, y) + i\,v(x, y)$

if $u - v = \dfrac{\cos x + \sin x - e^{-y}}{2\cos x - e^{y} - e^{-y}}$ and $f\left(\dfrac{\pi}{2}\right) = 0$.

(or) If $f(z) = u + iv$ is an analytic function and $u - v = \dfrac{\cos x + \sin x - e^{-y}}{2\cos x - e^{y} - e^{-y}}$, find $f(z)$ subject to the condition $f(\pi/2) = 0$. **[JNTU Nov. 2006, Nov. 2008S, (K) Nov. 2010 (Set No. 1)]**

Solution : We have $f(z) = u + iv$ ... (1)

$\therefore \quad i f(z) = iu - v$ ... (2)

(1) + (2) gives $(1+i) f(z) = (u - v) + i(u + v)$ ... (3)

Putting $(1+i)\, f(z) = F(z)$, $u - v = U, u + v = V$, (3) becomes $F(z) = U + iV$

It is given that $U = u - v = \dfrac{\cos x + \sin x - e^{-y}}{2\cos x - e^{y} - e^{-y}} = \dfrac{\cos x + \sin x - e^{-y}}{2\,(\cos x - \cosh y)}$

$$\therefore \quad \frac{\partial U}{\partial x} = \frac{(\cos x - \cosh y)\,(-\sin x + \cos x) - (\cos x + \sin x - e^{-y})\,(-\sin x)}{2\,(\cos x - \cosh y)^2}$$

$$\text{and} \quad \frac{\partial U}{\partial y} = \frac{(\cos x - \cosh y)\, e^{-y} - (\cos x + \sin x - e^{-y})\,(-\sinh y)}{2\,(\cos x - \cosh y)^2}$$

$$\text{Now} \quad F'(z) = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x} = \frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y}$$

$$= \frac{1}{2(\cos x - \cosh y)^2}[(\cos x - \cosh y)\,(-\sin x + \cos x) + \sin x\,(\cos x + \sin x - e^{-y})$$

$$-i\,[(\cos x - \cosh y)\, e^{-y} + (\cos x + \sin x - e^{-y})\sinh y]$$

By Milne – Thomson method, we express $F'(z)$ in terms of z by putting $x = z$ and $y = 0$.

$$\therefore \quad F'(z) = \frac{(\cos z - 1)(-\sin z - \cos z) + \sin z(\cos z + \sin z - 1) - i[(\cos z - 1) + 0]}{2(\cos z - 1)^2}$$

$$= \frac{\cos z\,(\cos z - 1) + \sin^2 z - i\,(\cos z - 1)}{2\,(\cos z - 1)^2} = \frac{(1 - \cos z) - i\,(\cos z - 1)}{2\,(\cos z - 1)^2} = \frac{-1 - i}{2\,(\cos z - 1)}$$

$$\text{i.e.} \quad (1+i)\, f'(z) = \frac{-(1+i)}{2\,(\cos z - 1)}$$

$$\text{or} \quad f'(z) = -\frac{1}{2\,(\cos z - 1)} = -\frac{1}{2\left(1 - 2\sin^2 \frac{z}{2} - 1\right)} = \frac{1}{4}\operatorname{cosec}^2\left(\frac{z}{2}\right)$$

Integrating with respect to z, we get

$$f(z) = \frac{1}{4}\int \operatorname{cosec}^2\left(\frac{z}{2}\right) dz + c = -\frac{1}{2}\cot\left(\frac{z}{2}\right) + c$$

Example 24 : Find $f(z)=u+iv$ given that $u+v=\dfrac{\sin 2x}{\cosh 2y-\cos 2x}$.

[JNTU 2003S, 2008S (Set

Solution : Given $f(z)=u+iv$ ... (1) $\quad \therefore\ if(z)=iu-v$ ... (2)

(1) + (2) gives, $(1+i)\,f(z)=(u-v)+i\,(u+v)$

Letting $(1+i)\,f(z)=F(z), u-v=U$ and $u+v=V$, we obtain $F(z)=U+iV$

It is given that $V=u+v=\dfrac{\sin 2x}{\cosh 2y-\cos 2x}$

$$\therefore\ \frac{\partial V}{\partial x}=\frac{(\cosh 2y-\cos 2x)\,(2\cos 2x)-\sin 2x\,(2\sin 2x)}{(\cosh 2y-\cos 2x)^2}$$

and $$\frac{\partial V}{\partial y}=\sin 2x.\frac{\partial}{\partial y}\left(\frac{1}{\cosh 2y-\cos 2x}\right)=\frac{-2\sin 2x\sinh y}{(\cosh 2y-\cos 2x)^2}$$

Now $$F'(z)=\frac{\partial U}{\partial x}+i\frac{\partial V}{\partial x}=\frac{\partial V}{\partial y}+i\frac{\partial V}{\partial x}$$

$$=\frac{(-2\sin 2x\sinh y)+i\left[2\cos 2x\,(\cosh 2y-\cos 2x)-2\sin^2 2x\right]}{(\cosh 2y-\cos 2x)^2}$$

By Milne – Thomson method, we express $F'(z)$ in terms of z by putting $x=z$ and

$$\therefore\ F'(z)=\frac{i\,[2\cos 2z\,(1-\cos 2z)-2\sin^2 2z]}{(1-\cos 2z)^2}=\frac{i\,2\,(\cos 2z-1)}{(1-\cos 2z)^2}$$

$$=\frac{2i}{\cos 2z-1}=\frac{2i}{-2\sin^2 z}=-i\,\text{cosec}^2 z$$

Integrating, $F(z)=i\cot z+c$

i.e. $(1+i)\,f(z)=i\cot z+c$ or $f(z)=\dfrac{i}{1+i}\cot z+\dfrac{c}{1+i}=\dfrac{i\,(1-i)}{2}\cot z+c_1$

$$\therefore\ f(z)=\left(\frac{1+i}{2}\right)\cot z+c_1$$

Example 25 : Find a and b if $f(z)=(x^2-2xy+ay^2)+i\,(bx^2-y^2 \ldots$ is analytic, find $f\,(z)$ in terms of z.

Solution : Let ...

Hence $a = -1$ and $b = 1$.

Now $f'(z) = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x} = 2x - 2y + i(2bx + 2y) = 2[(x-y) + i(x+y)]$ $\quad (\because \; b = 1)$

By Milne-Thomson method, $f'(z)$ is expressed in terms of z by replacing x by z and y b

Hence $f'(z) = 2(z + iz) = 2z(1+i)$

Integrating, $f(z) = 2(1+i)\dfrac{z^2}{2} + C = (1+i)z^2 + C$, where C is a complex constant.

Example 26 : Find an analytic function $f(z)$ such that $\text{Re}[f'(z)] = 3x^2 - 4y - 3y^2$ ar
$1+i) = 0$.

[JNTU 2003, (A) Dec. 2009 (Set No. 3

Solution : Since $f(z)$ is analytic, therefore, $f'(z)$ is also analytic.

Let $f'(z) = U + iV$. Then $U = 3x^2 - 4y - 3y^2$.

$\therefore \quad U_x = 6x$ and $U_y = -4 - 6y$

Since U and V satisfy Cauchy – Riemann equations,

$\therefore \quad U_x = 6x = V_y$

Integrating with respect to 'y', we get, $V = 6xy + c_1(x)$ ... (1)

Now $\dfrac{\partial V}{\partial x} = V_x = 6y + \dfrac{dc_1}{dx}$

Since $V_x = -U_y$, we have , $6y + \dfrac{dc_1}{dx} = 4 + 6y$

$\Rightarrow \quad c_1(x) = 4x + c_2$ ... (2) where c_2 is an arbitrary constant.

From (1) and (2), we have $V = 6xy + 4x + c_2$

$\therefore \quad f'(z) = U + iV = (3x^2 - 4y - 3y^2) + i(6xy + 4x + c_2)$

By Milne-Thomson method, $f'(z)$ is expressed in terms of z by replacing x by z and y b

Hence $f'(z) = 3z^2 + i\,4z + c_2$

Integrating, $f(z) = 3\dfrac{z^3}{3} + i\,4\cdot\dfrac{z^2}{2} + c_2 z + c_3 = z^3 + 2iz^2 + c_2 z + c_3$ ... (3)

Given $f(1+i) = 0 \Rightarrow 0 = (1+i)^3 + 2i(1+i)^2 + c_2(1+i) + c_3$

Thus $f(z) = z^3 + 2iz^2 + c_2 z - c_2(1+i) - 6 + 2i$ [by (3)]

Solution : We have by Cauchy–Riemann equations in polar coordinates

$$r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta} = -2r^2 \sin 2\theta + r \sin\theta \qquad ...(1)$$

and $$-\frac{1}{r}\frac{\partial u}{\partial \theta} = \frac{\partial v}{\partial r} = 2r \cos 2\theta - \cos\theta \qquad ...(2)$$

$\therefore$ From (1), we get $\frac{\partial u}{\partial r} = -2r \sin 2\theta + \sin\theta$

Integrating with respect to r, we get

$$u = -r^2 \sin 2\theta + r \sin\theta + \phi(\theta) \text{ where } \phi(\theta) \text{ is an arbitrary function} \qquad ...(3)$$

Differentiating u w.r.t. θ, we get

$$\frac{\partial u}{\partial \theta} = -2r^2 \cos 2\theta + r\cos\theta + \phi'(\theta) \qquad ...(4)$$

From (2) and (4), we get

$$-2r^2 \cos 2\theta + r\cos\theta = \frac{\partial u}{\partial \theta} = -2r^2 \cos 2\theta + r\cos\theta + \phi'(\theta)$$

$\therefore \quad \phi'(\theta) = 0 \Rightarrow \phi(\theta) = c$

Thus $u = -r^2 \sin 2\theta + r\sin\theta + c$ [From (3)]

Hence, $f(z) = u + iv = r^2(-\sin 2\theta + i\cos 2\theta) + r(\sin\theta - i\cos\theta) + c + 2i$

$$= i\left(r^2 e^{2i\theta} - r e^{i\theta}\right) + c + 2i$$

Example 15 : Find the analytic function $f(z) = u(r,\theta) + iv(r,\theta)$ such that

$v(r,\theta) = \left(r - \frac{1}{r}\right)\sin\theta, r \neq 0$

Solution : Given $v = \left(r - \frac{1}{r}\sin\theta\right) \Rightarrow \frac{\partial v}{\partial \theta} = \left(r - \frac{1}{r}\right)\cos\theta$ and $\frac{\partial v}{\partial r} = \left(1 + \frac{1}{r^2}\right)\sin\theta$

By Cauchy-Riemann equations,

$$r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta} = \left(r - \frac{1}{r}\right)\cos\theta \Rightarrow \frac{\partial u}{\partial r} = \left(1 - \frac{1}{r^2}\right)\cos\theta \qquad ...(1)$$

Integrating w.r. to r, $u = \left(r + \frac{1}{r}\right)\cos\theta + k(\theta)$ where $k(\theta)$ is a constant

Diff. w.r. to θ, we get

$$\frac{\partial u}{\partial \theta} = \left(r + \frac{1}{r}\right)(-\sin\theta) + k'(\theta) \qquad ...(2)$$

Also $$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r} = (-r)\left(1 + \frac{1}{r^2}\right)\sin\theta = \left(-r - \frac{1}{r}\right)\sin\theta \qquad ...(3)$$

Comparing (2) & (3), $\left(r + \frac{1}{r}\right)(-\sin\theta) + k'(\theta) = \left(-r - \frac{1}{r}\right)\sin\theta$