Unit – I - Random Variable

Introduction

In the earlier chapter, we studied the concept of Event to describe the characteristics of outcomes of an experiment. In this chapter, we introduce a new concept that will allow events to be defined in a more consistent manner; they will always be numerical. The new concept is that of a random variable and it will constitute a powerful tool in the solution of practical problems.

Random Variable:

Definition:

A real random variable is defined as a real function of the elements of a sample space S.

A random variable is represented by a capital letter such as X or Y or Z and any particular value of a random variable is represented by a lowercase letter such as x or y or z. Hence given an experiment defined by a sample space S with elements s a real number according to some rule and call X(s) a random variable.

Example: An experiment consists of rolling a die and flipping a coin. Let the random variable be a function X chosen such that

A coin head (H) outcome corresponds to positive values of X that are equal to the numbers that shown up on the die.

A coin tail (T) outcome corresponds to negative values of X that are equal in magnitude to twice the number that shows on the die.

Explanation: In the given above problem there are two experiments

1. Rolling a die
2. Flipping a coin

For rolling a die experiment the sample space is S1 = {1,2,3,4,5,6}

For flipping a coin experiment the sample space is S2 = {H,T}

As mentioned in the problem, if Head falls it corresponds to positive value of X and is equal to number that shown upon the die. Hence if Head falls, it corresponds to any number from 1 to 6 that shown upon the die.

Similarly if tail falls, it corresponds to negative values of X that is equal in magnitude to twice the number that shown on the die i.e.,{-2,-4,-6,-8,-10,-12}.

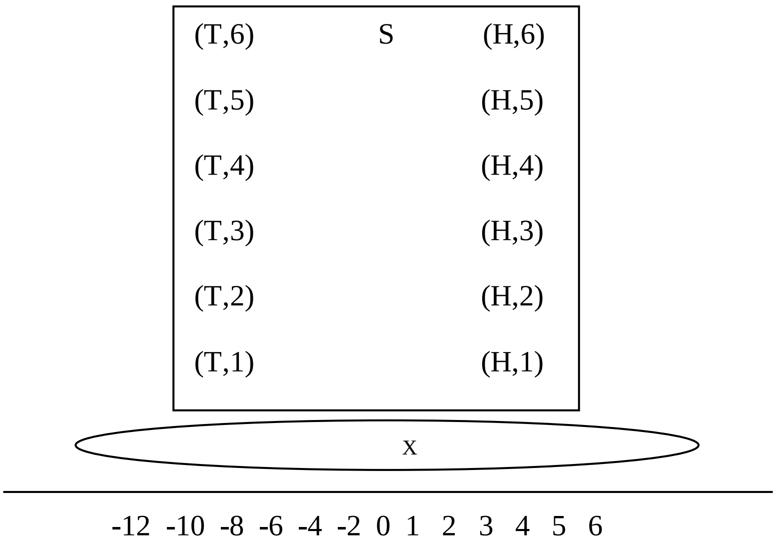


Fig 2.1 A Random Variable mapping a Sample Space

Conditions for a function to be a Random Variable:

- Random variable can be any function we wish, but however it should not be a multivalued function i.e., every point in S must correspond to only one value of the random variable.
- The Set $\{X \leq x\}$ shall be an event for any real number x, i.e., this set corresponds to those points s in the Sample space for which the random variable X(s) does not exceed the number x. It also implies that the probability of event $P\{X \leq x\}$ is equal to sum of the probabilities of all the elementary events corresponding to $\{X \leq x\}$
- The Probabilities of the events $\{X= \infty\}$ and $\{X = -\infty\}$ be zero i.e.,$P\{X= \infty\} = 0$ and $P\{X= -\infty\} = 0$

Types of Random Variables:

There are three types of Random Variables. They are

- Discrete Random Variable
- Continuous Random Variable
- Mixed Random Variable

Discrete Random Variable:

A Discrete random variable is one having only discrete values.

The sample space for a discrete random variable can be discrete, continuous or even a mixture of discrete and continuous.

Example: Consider an example of wheel of chance which has a continuous sample space, but a discrete random variable can be defined as having the value 1 for the set of outcomes $\{0 < s \leq 6\}$ and -1 for $\{6 < s \leq 12\}$. The result is a discrete random variable defined on a continuous sample space.

Continuous Random Variable:

A continuous random variable is one having continuous range of values.

It cannot be produced from a discrete sample space because of our requirement that all random variables be single valued functions of all sample space points

Similarly, a purely continuous random variable cannot result from a mixed sample space because of the presence of discrete portion of the sample space.

Example: Wheel of chance

Mixed Random Variable:

A Mixed random variable is one for which some of its values are discrete and some are continuous.

Mixed case is usually the least important type of random variable, but in some problems it has practical significance

Distribution Function:

The Probability $P\{X \leq x\}$ is the probability of the event $\{X \leq x\}$. It is a number that depends on x i.e., it is a function of x and this function is called as Cumulative probability distribution function of the random variable X and is denoted as $F_X(x)$. Hence

$$F_X(x) = P\{X \leq x\}$$

This can be just called as distribution function of X. The argument x is any real number ranging from $-\infty$ to ∞.

Properties of Distribution Function

This distribution function has some specific properties derived from the fact $F_X(x)$ is a probability. These are

1. $F_X(-\infty) = 0$
2. $F_x(\infty) = 1$
3. $0 \leq F_X(x) \leq 1$
4. $F_X(x_1) \leq F_X(x_2)$ if $x_1 < x_2$

   It states $F_X(x)$ is a non decreasing function of x
5. $P\{x_1 < X \leq x_2\} = F_X(x_2) - F_X(x_1)$

   It states that the probability that X will have values larger than some number x_1 but not exceeding another number x_2 is equal to the difference in $F_X(x)$ evaluated at the two points.
6. $F_X(x^+) = F_X(x)$

   It states that $F_X(x)$ is a function continuous from the right.

To determine whether, if some function say $F_X(x)$ could be a valid distribution function, properties 1,2,3 and 6 may be used as tests.

Example of a Continuous distribution function:

A continuous random variable will have a continuous distribution function.

Consider the wheel of chance experiment where the wheel is numbered from 0 to 12. Clearly the probability of the event $\{X \leq 0\}$ is 0 because there are no sample space points in this set. For $0 < x \leq 12$ the probability of $\{0 < X \leq x\}$ will increase linearly with x for a fair wheel. Thus $F_X(x)$ will behave as shown below, which has continuous distribution function.

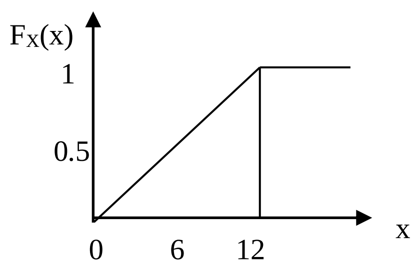


Fig 2.2 Plot of distribution function for the given continuous random variable

If 'X' is a discrete random variable, consideration of distribution function defined by $F_X(x) = P\{X \leq x\}$ must have a stair step form. The amplitude of a step will equal the probability of occurrence of the value of X where the step occurs.
If the values of X are denoted as x_i, we may write $F_X(x)$ as

$$F_X(x) = \sum_{i=1}^{N} P\{X = x_i\} u(x - x_i)$$

Where U(.) is the unit step function

As $P(x_i) = P\{X = x_i\}$ can be represented in shortened form then $F_X(x)$ for discrete can be represented as

$$F_X(x) = \sum_{i=1}^{N} P\{x_i\} u(x - x_i)$$

Example of a discrete distribution function:

Let X have the discrete values in the set {-1, -0.5, 0.7, 1.5, 3}. The corresponding probabilities are assumed to be {0.1, 0.2, 0.1, 0.4, 0.2}. Hence the plot of distribution function for the discrete random variable X is

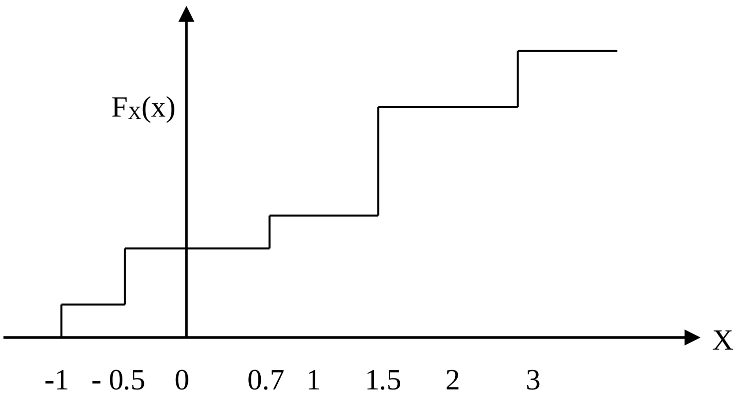


Fig 2.3 Plot of distribution function for the given discrete random variable

From the above plot, for discrete distribution function, it can observed that for $P\{X < -1\} = 0$, because there are no sample points in the set for $\{X < -1\}$.

At $x = -1$, there is an immediate jump in probability of 0.1 for $-1 < x < -0.5$, and as there are no additional sample space points in between $F_X(x)$ remains constant at the value 0.1.

At $x = -0.5$, there is another jump of 0.2 in $F_X(x)$. The process continues until all points are included. $F_X(x)$ then equals 1.0 for all x above the last point. Hence it can observed that for discrete distribution function, $F_X(x)$ has stair step waveform.

Density Function:

The Probability density function denoted by $f_X(x)$, is defined as the derivative of the distribution function.

$$f_X(x) = \frac{dF_X(x)}{dx}$$

We often call $f_X(x)$ just as density function of the random variable X

Existence of density function:

IF the derivative of $F_X(x)$ exists, then $f_X(x)$ exists and is as described above. There may however be places where $\frac{dF_X(x)}{dx}$ is not defined. For example a continuous random variable will have a continuous distribution function $F_X(x)$, but $F_X(x)$ may hove corners (points of abrupt change in slope)

Properties of density function:

1. $0 \leq f_X(x)$ for all x

   (It means density function be non negative)

2. $\int_{-\infty}^{\infty} f_X(x)dx = 1$

   (It implies that the area under curve is unity)

3. $F_X(x) = \int_{-\infty}^{x} f_X(\xi)d\xi$

   (It is just another way of writing the link between $F_X(x)$ and $f_X(x)$. We know that the derivative of distribution function gives density and hence integral of density function gives distribution function)

4. $P\{x_1 < X \leq x_2\} = \int_{x_1}^{x_2} f_X(x)dx$

   (It relates the probability that X will have values from x_1 to and including x_2 to the density function)

Properties 1 and 2 can be used as tests to verify whether the given function can be a valid probability density function or not.

Example for Continuous density function:

Consider an experiment of wheel of chance which is a continuous, the plot of density function of it is as shown below.

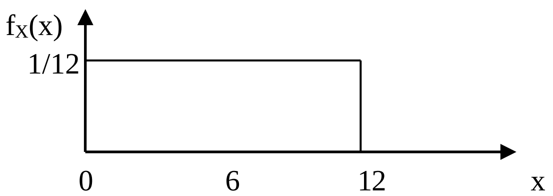


Fig 2.4 Plot of density function for the given continuous random variable

In case of a **discrete random variable**, density function can be expressed as

$$f_X(x) = \sum_{i=1}^{N} P(x_i)\delta(x - x_i)$$

i.e., the density function for a discrete random variable exists in the sense that impulse function is used to describe the derivative function of $F_X(x)$ as its stair step points.

Example of a discrete density function:

Considering the same example considered in case of discrete distribution function.

Let X have the discrete values in the set {-1, -0.5, 0.7, 1.5, 3}. The corresponding probabilities are assumed to be {0.1, 0.2, 0.1, 0.4, 0.2}. Hence the plot of density function for the discrete random variable X is

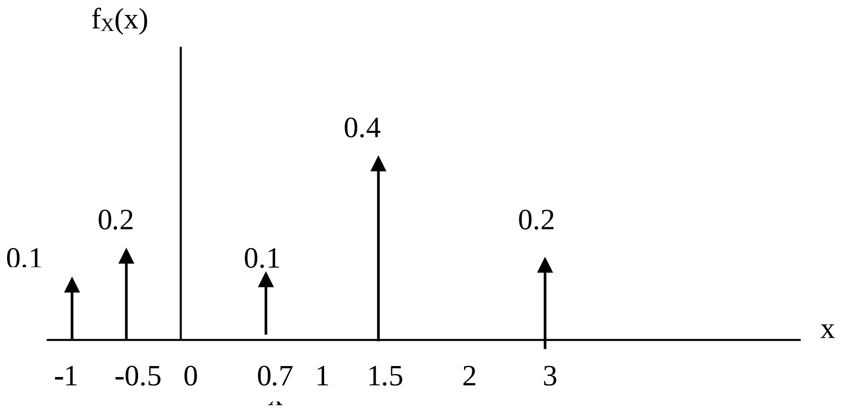


Fig 2.5 Plot of density function for the given discrete random variable

Types of Density and Distribution Functions:

There exists various Distribution and Density functions applicable for different applications will be discussed like

- Gaussian Density and Distribution
- Binomial Density and Distribution
- Poisson Density and Distribution
- Uniform Density and Distribution
- Exponential Density and Distribution
- Rayleigh Density and Distribution

Binomial and Poisson are for discrete random variables. Uniform, Exponential and Rayleigh are for continuous random variables.

Gaussian Density and Distribution Function:

(First it was derived by De Moivre and later independently by both Gauss and Laplace)

A random variable X is called Gaussian if its density function has the form

$$f(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}}\, e^{-(x-a_x)^2/\,2\sigma_x^2}$$

Where $\sigma_x > 0$ and $-\infty < a_X < \infty$ are real constants. This function is sketched as below.

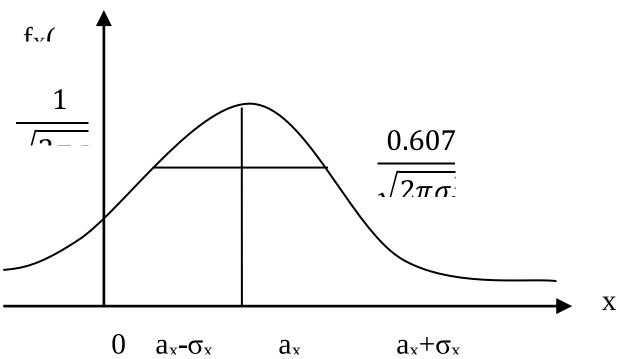


Fig 2.6 Plot of density function of a Gaussian random variable

Its maximum values $(2\pi\sigma_X^2)^{-1/2}$ occurs at x = a_X. Its spread about the point x = a_X is related to σ_X. The function decreases to 0.607 times its maximum at $x=a_X+\sigma_X$ and $x=a_X-\sigma_X$.

It is the most important of all densities and it enters into nearly all areas of science and Engineering. This importance stems from its accurate description of many practical and significant real world quantities especially when such quantities are the result of many small independent random effects acting to create the quantity of interest.

For example, the voltage across a resistor at the output of an amplifier can be random (a noise voltage) due to a random current that is the result of many contributions from other random currents at various places with in the amplifier. Random thermal agitation of electrons causes the randomness of the various currents. This type of noise is called Gaussian because the random variable representing the noise voltage has the Gaussian density.

Using Property (3) of density function, distribution function of Gaussian random variable can be calculated by integrating the Gaussian density function. The integral is

$$F_X(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}} \int_{-\infty}^{x} e^{-(\xi - a_X)^2/2\sigma_X^2}\, d\xi$$

This integral has no known closed form solution and must be evaluated by numerical or approximation methods. To make the results generally available, a set of tables of $F_X(x)$ has been developed that corresponds to specific values of ax and σ_x as parameters.

For a negative value of x we can use the relationship F (-x) = 1- F(x)

Another form of evaluating $F_X(x)$ can be evaluated using

$$F_X(x) = F\left(\frac{x - a_x}{\sigma_x}\right)$$

Example: Find the probability of the event { $X \le 5.5$} for a Gaussian random variable having $a_x = 3$ and $\sigma_x = 2$.

$$\text{Here } \left(\frac{x-a_x}{\sigma_x}\right) = \left(\frac{5.5-3}{2}\right) = 1.25$$

From the expression for $F_X(x) = F\left(\frac{x-a_x}{\sigma_x}\right)$ and the definition of $F_X(x)$

$$P\{X \le 5.5\} = F_X(5.5) = F(1.25)$$

From the table to evaluate the value of Gaussian random variable, for given $a_x = 3$ and $\sigma_x = 2$ is

$$P\{X \le 5.5\} = F(1.25) = 0.8944$$

Binomial Density and Distribution function:

The binomial density can be applied to the Bernoulli trial experiment. It applies to many games of chance, detection problems in Radar and Sonar and many experiments having only two possible outcomes on any trial like tossing a coin etc.

Binomial density function can be given as

$$f_X(x) = \sum_{k=0}^{N} \binom{N}{K} p^k (1-p)^{(N-k)} \delta(x-k) \quad 0 < p < 1, and\ N = 1,2\ , \ldots \ldots$$

The quantity $\binom{N}{K}$ is called binomial coefficient defined as $\binom{N}{k} = \frac{N!}{k!(N-k)!}$

By integrating the binomial density function, binomial distribution function can be expressed as

$$F(x) = \sum_{k=0}^{N} \binom{N}{K} p^k (1-p)^{(N-k)} u(x-k)$$

Example: Plot Binomial density and distribution function for N = 6 and p = 0.25

Using the formulae defined for Binomial density and distribution function by calculating binomial coefficients, the corresponding plots of density and distribution function can be

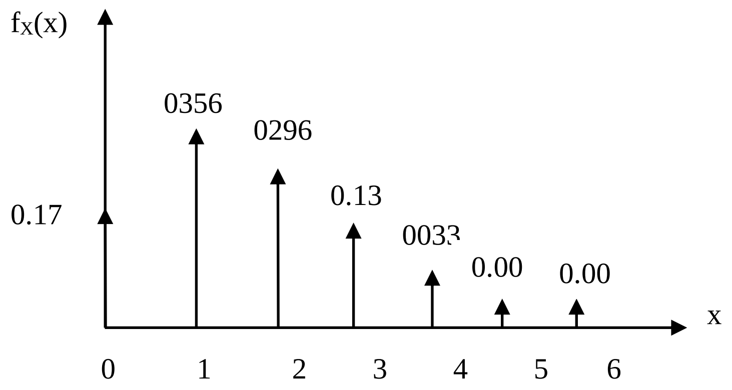


Fig 2.7 Plot of binomial density function of a given random variable

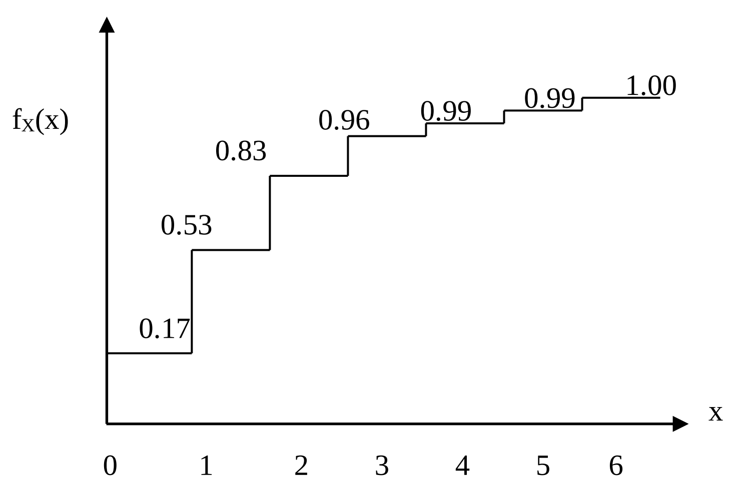


Fig 2.8 Plot of binomial distribution function of a given random variable

Poisson density and distribution function:

The Poisson random variable X has a density function given by

$$f_X(x) = e^{-b}\sum_{k=0}^{\infty}\frac{b^k}{k!}\,\delta(x-k)$$

The corresponding distribution function of a Poisson random variable can be obtained by integrating the density function and is given by

$$F_X(x) = e^{-b}\sum_{k=0}^{\infty}\frac{b^k}{k!}\,u(x-k)$$

Where $b > 0$ is a real constant.

If $N \to \infty$ and $p \to 0$, in Binomial case, in such a way that $Np = b$, a constant, the Poisson case results. If the time interval of interest has duration T and the events being counted are known to occur at an average rate λ and have a Poisson distribution, then b is given by $b = \lambda T$.

The Poisson random variable applies to a wide variety of counting type applications like, number of defective units in a sample taken from a production line, the number of telephone calls made during a period of time, the number of electrons emitted from a small section of a cathode in a given time interval etc.

Example: Assume automobile arrivals at a gasoline station are Poisson and occur at an average rate of 50 / h. The station has only one gasoline pump. If all cars are assumed to require one minute to obtain fuel, what is the probability that a waiting line will occur at the pump?

A Waiting line occurs if two or more cars arrive in any one minute interval.
The probability of this event is one minus the probability either none or one car arrives.
From the equation $b = \lambda T$, $\lambda = 50/60$ (cars/hour) and T = 1 Minute, we have $b = 5/6$.

On using equation $F_X(x) = e^{-b}\sum_{k=0}^{\infty}\frac{b^k}{k!}\,u(x-k)$,

Probability of a waiting line = $1-F_X(1) - F_X(0)$

= 1- e-(5/6) [1+(5/6)] = 0.2032

Hence, a waiting line at the petrol pump is expected for about 20.32% of the time

Uniform density and distribution function:

The uniform probability density and distribution functions are defined by

Uniform density function

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & ; a \le x \le b \\ 0 & ; else\ where \end{cases}$$

The corresponding density plot for the above expression is

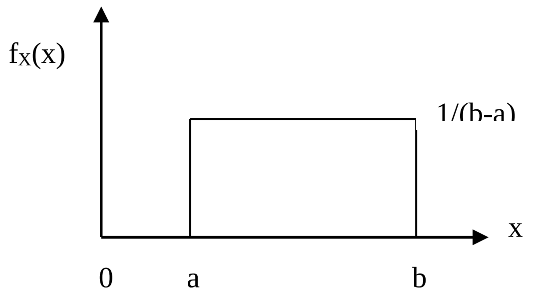


Fig 2.9 Plot of uniform density function

Uniform distribution function

$$F_X(x) = \begin{cases} 0 & ; \quad x < a \\ \dfrac{x-a}{x-b} & ; \; a \le x \le b \\ 1 & ; \quad b \le x \end{cases}$$

For real constants -∞ < a < ∞ and b > a.

Distribution plot for the above expression is

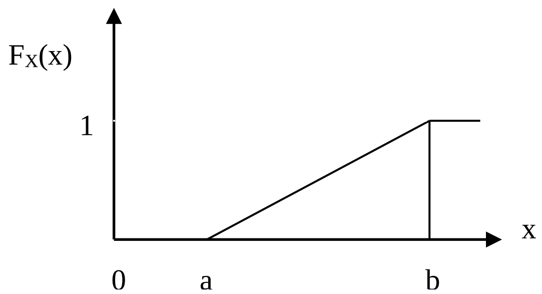


Fig 2.10 Plot of uniform distributionfunction

The uniform density finds a number of practical uses.

It is particularly important in quantization of signal samples prior to encoding in digital communication systems. Quantization amounts to rounding off the actual sample to the nearest of a large number of discrete quantum levels. The errors introduced in the round off process are uniformly distributed.

Exponential Density and Distribution Function

The exponential density and distribution functions are given by

Density funtion:

$$f_X(x) = \begin{cases} \dfrac{1}{b}\, e^{-\frac{(x-a)}{b}} & ; x > a \\ 0 & ; x < a \end{cases}$$

The corresponding density plot for the above expression is

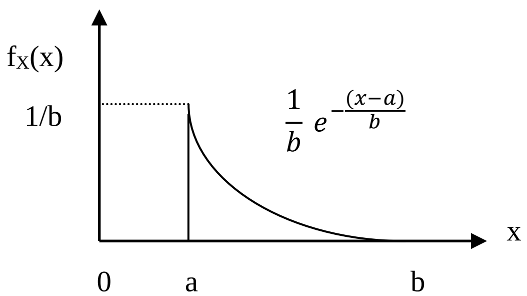


Fig 2.10 Plot of Exponential density function

Distribution funtion:

$$F_X(x) = \begin{cases} 1 - e^{-\frac{(x-a)}{b}} & ; x > a \\ 0 & ; x < a \end{cases}$$

For real numbers -∞ < a < ∞ and b > 0.

The corresponding distribution plot for the above expression is

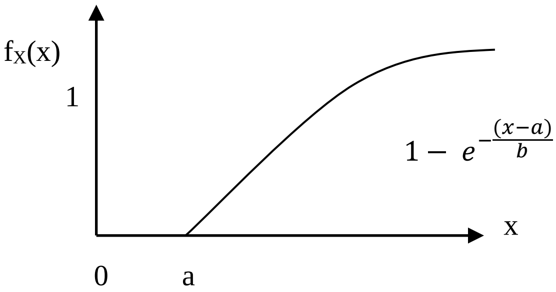


Fig 2.11 Plot of Exponential distribution function

Exponential density is useful in describing raindrop sizes when a large number of rainstorm measurements are made.
It is also known to approximately describe the fluctuations in signal strength received by radar from certain types of aircraft.

Example: The power reflected from an aircraft of complicated shape that is received by a radar can be described by an exponential random variable P. The density of P is therefore

$$f_P(p) = \begin{cases} \frac{1}{P_o} e^{-\frac{p}{p_0}} \; ; \; p > 0 \\ 0 \; ; p < 0 \end{cases}$$

Where P_o is the average amount of received power. At some given time P may have a value different from its average value and hence what is the probability that the received power is larger than the power received on the average ?

$$P\{P > P_o\} = 1 - P\{P \le P_o\} = 1 - F_p(P_o)$$

From the equation for exponential distribution function

$$P\{P > P_o\} = 1 - (1 - e^{-Po/Po}) = e^{-1} = 0.368$$

In other words, the received power is larger than its average value about 36.8 percent of time.

Rayleigh Density and Distribution function:

Rayleigh density and distribution function are given by

Rayleigh Density function

$$f_X(x) = \begin{cases} \frac{2}{b}(x-a)e^{-\frac{(x-a)^2}{b}} & ; x \ge a \\ 0 & ; x < a \end{cases}$$

The corresponding plot is

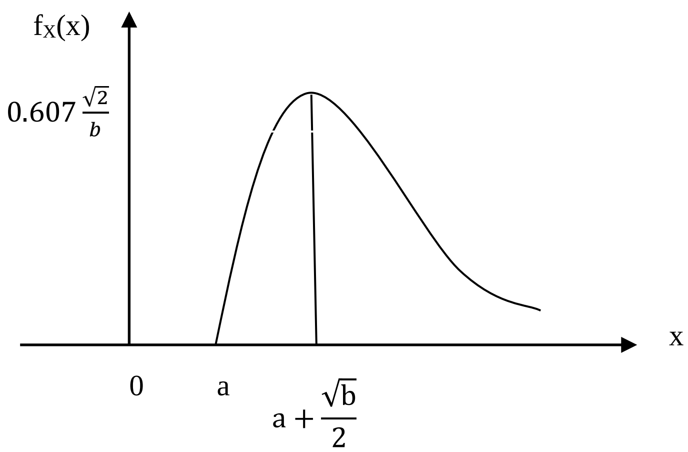
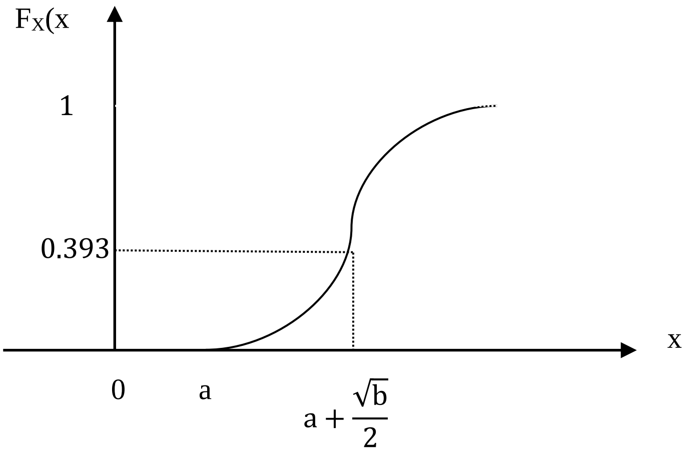


Fig 2.12 Plot of Rayleigh density function

Rayleigh Distribution function

$$F_X(x) = \begin{cases} 1 - e^{-\frac{(x-a)^2}{b}} & ; x \geq a \\ 0 & ; x < a \end{cases}$$

For real constants $-\infty < a < \infty$ and $b > 0$.
The corresponding plot is



2.13 Plot of Rayleigh distribution function

The Rayleigh density describes the envelope of one type of noise when passed through a band pass filter. It is also important in analysis of errors in various measurement systems.

Conditional Distribution and Density function:

The concept of conditional probability was discussed in unit – I. Consider two events A and B where $P(B) \neq 0$, the conditional probability of A given B is

$$P(A/B) = P(A \cap B) / P(B)$$

In this topic we extend the conditional probability concept to include random variables.

Conditional Distribution:

Let A be identified as the event and is defined as $\{X \leq x\}$ for the random variable X. The resulting probability $P\{X \leq x/B\}$ is defined as the conditional distribution function of X, which we denote $F_X(x/B)$. Thus

$$F_X\left(\frac{x}{B}\right) = P\left\{X \leq \frac{x}{B}\right\} = \frac{P\{X \leq x \cap B\}}{P(B)}$$

Where the notation $\{X \leq x \cap B\}$ to imply the joint event $\{X \leq x\} \cap B$. This joint event consists of all outcomes s such that $X(s) \leq x$ and $s \in B$.

The conditioanl distribution applies to discrete, continuous or mixed random variables.

Properties of Conditional Distribution:

All the properties of ordinary distribution function apply to conditional distribution function $F_X(x/B)$

1. $F_X(-\infty / B) = 0$
2. $F_x(\infty / B) = 1$
3. $0 \leq F_X(x/B) \leq 1$
4. $F_X(x_1/B) \leq F_X(x_2/B)$ if $x_1 < x_2$
5. $P\{x_1 < X \leq x_2/B\} = F_X(x_2/B) - F_X(x_1/B)$
6. $F_X(x^+/B) = F_X(x/B)$

Conditional Density Function:

In a manner similar to the ordinary density funtion, conditioanl density function of the random variable X is defined as the derivative of the conditional distribution function. It is denoted by $f_X(x\backslash B)$ and is given as

$$f_X(x\backslash B) = \frac{dF_X(x\backslash B)}{dx}$$

If FX(x/B) contains step discontinuities, a swhen X is a discrete or mixed random variable, we assume that impulse functions are present in fX(x/B) to account for the derivatives at the discontinuties.

Properties of Conditional Density

As conditional density is related to conditional distribution through the derivative, it satisfies the same properties as the ordinary density function. They are as follows

1. $f_X(x/B) \geq 0$
2. $\int_{-\infty}^{\infty} f_X(x/B)dx = 1$
3. $F_X(x/B) = \int_{-\infty}^{x} f_X(\xi/B)d\xi$
4. $P\{x_1 < X \leq x_2 / \mathrm{B}\} = \int_{x_1}^{x_2} f_X(x/B)dx$

Let us discuss an example to illustrate conditioanl density and distribution

Example: Two boxes have red, green and blue balls in them; the number of balls of each color is given in Table. Our experiment will be to select a box and then a ball from the selected box. One box (number 2) is slightly larger than the other causing it to be selected more frequently. Let B_2 be the event " Select the larger box" while B_1 is the event " Select the smaller box". Assume $P(B_1) = 2/10$ and $P(B_2) = 8/10$.

X_i	Ball Colour	Box		
		1	2	Total
1	Red	5	80	85
2	Green	35	60	95
3	Blue	60	10	70
	Totals	100	150	250

Let us define a discrete random variable X to have values $x_1 = 1$, $x_2 = 2$ and $x_3 = 3$ when a red, green, or blue ball is selected and let B be an event equal to eother B_1 or B_2. Hence from the Table

$$P(X = 1\backslash B = B_1) = \frac{5}{100} \qquad P(X = 1\backslash B = B_2) = \frac{80}{150}$$
$$P(X = 2\backslash B = B_1) = \frac{35}{100} \qquad P(X = 2\backslash B = B_2) = \frac{60}{150}$$
$$P(X = 3\backslash B = B_1) = \frac{60}{100} \qquad P(X = 3\backslash B = B_2) = \frac{10}{150}$$

The conditional probability density $f_X(x\backslash B_1)$ becomes

$$f_X(x\backslash B_1) = \frac{5}{100}\delta(x-1) + \frac{35}{100}\delta(x-2) + \frac{60}{100}\delta(x-3)$$

By direct integration of $f_X(x\backslash B_1)$, the conditional distribution becomes,

$$F_X(x\backslash B_1) = \frac{5}{100}u(x-1) + \frac{35}{100}u(x-2) + \frac{60}{100}u(x-3)$$

For comparison, the density and distribution of X can be found by detrmining the probabilities P(X=1), P(X=2) and P(X=3). These are found from the total probability theorem.

$$P(X = 1) = P(X = 1\backslash B_1)P(B_1) + P(X = 1\backslash B_2)P(B_2) = \frac{5}{100}\left(\frac{2}{10}\right) + \frac{80}{150}\left(\frac{8}{10}\right) = 0.437$$

$$P(X = 2) = P(X = 2\backslash B_1)P(B_1) + P(X = 2\backslash B_2)P(B_2) = \frac{35}{100}\left(\frac{2}{10}\right) + \frac{60}{150}\left(\frac{8}{10}\right) = 0.390$$

$$P(X = 3) = P(X = 3\backslash B_1)P(B_1) + P(X = 3\backslash B_2)P(B_2) = \frac{60}{100}\left(\frac{2}{10}\right) + \frac{10}{150}\left(\frac{8}{10}\right) = 0.173$$

Thus

$$f_X(x) = 0.437\delta(x-1) + 0.390\delta(x-1) + 0.173\delta(x-1)$$

and $$F_X(x) = 0.437u(x-1) + 0.390u(x-1) + 0.173u(x-1)$$

These conditional and ordianry density and distribution functions are plotted as below

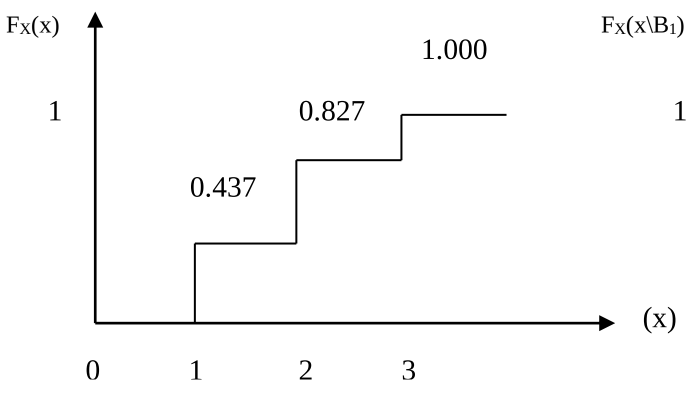


Fig 2.14 Plot of Distribution function

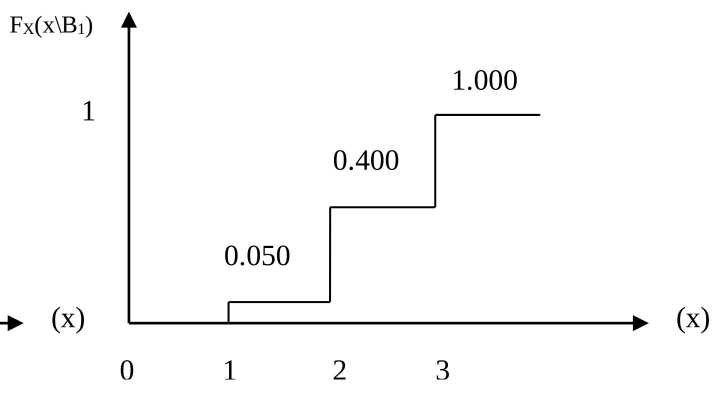


Fig 2.15 Plot of conditional Distribution function

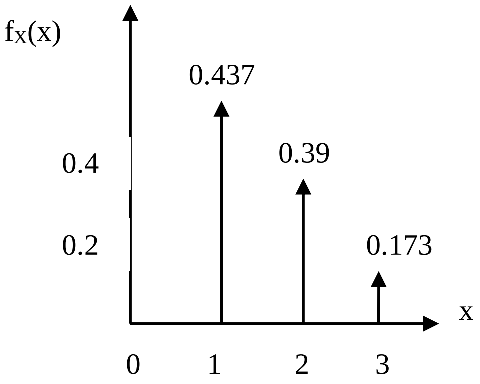


Fig 2.16 Plot of Density function

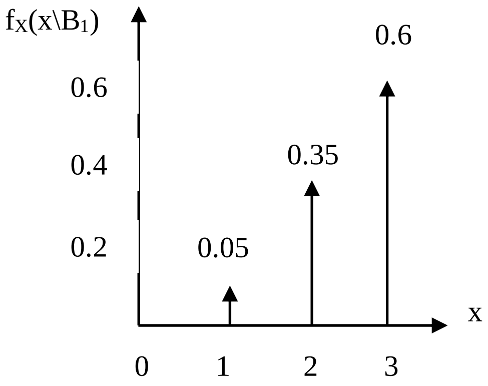


Fig 2.17 Plot of conditional density function

Methods of defining conditioning event:

In the preseeding section we discussed an example illustrating how the conditioning event B can be defined from some characteristic function of the physical experiment. There are several other ways of defining conditioning event and here we will discuss two types namely

1. Event B is defined interms of random variable X (Earlier it is defined as some random variable but now it is defined as some event)
2. Event B may be depend on some random variable other than X.

Here we will discuss type 1. Here event B is defined in terms of X and is to let B = { X ≤ b} where b is some real number -∞ < b < ∞ and by substituting B = { X ≤ b}in

$$F_X\left(\frac{x}{B}\right) = P\left\{X \le \frac{x}{B}\right\} = \frac{P\{X \le x \cap B\}}{P(B)}$$

Then we can have,

$$F_X(x\backslash X \le b) = P\{X \le x | X \le b\} = \frac{P\{X \le x \cap X \le b\}}{P\{X \le b\}} \text{ ------- (A)}$$

For all events { X ≤ b} for which P{ X ≤ b}≠ 0.

Here two cases must be considered.

1. b ≤ x
2. x < b

If b ≤ x, the event { X ≤ b} is a subset of the event{ X ≤ x}, so { X ≤ x}∩{ X ≤ b}={ X ≤ b} Therefore equation (A) becomes

$$F_X(x\backslash X \le b) = P\{X \le x | X \le b\} = \frac{P\{X \le x \cap X \le b\}}{P\{X \le b\}} = \frac{P\{X \le b\}}{P\{X \le b\}} = 1\,; \;\; b \le x \text{ ----- (B)}$$

When x < b, the event { X ≤ x} is a subset of the event{ X ≤ b}, so { X ≤ x}∩{ X ≤ b}={ X ≤ x}Therefore equation (A) becomes

$$F_X(x\backslash X \leq b) = P\{X \leq x | X \leq b\} = \frac{P\{X\leq x \cap X\leq b\}}{P\{X \leq b\}} = \frac{P\{X\leq x\}}{P\{X \leq b\}} = \frac{F_X(x)}{F_X(b)}; x < b \text{ ----- (C)}$$

By combining equations (B) and (C) , we have

$$F_X(x\backslash X \leq b) = \begin{Bmatrix} \frac{F_X(x)}{F_X(b)} ; & x < b \\ 1 \quad ; & b \leq x \end{Bmatrix} \text{ ----- (D)}$$

The conditional density function derives from the derivative of D and is given by

$$f_X(x\backslash X \leq b) = \begin{Bmatrix} \frac{f_X(x)}{F_X(b)} = \frac{f_X(x)}{\int_{-\infty}^{b} f_X(x)dx} ; & x < b \\ 0 \quad ; & b \leq x \end{Bmatrix} \text{ ------ (E)}$$

Hence from our assumptions that the conditioing event has non zero probability, we have $0 < F_X(b) \leq 1$, so the expression (D) shows that the conditional distribution function is never smaller than the ordianry distribution function

$$F_X(x\backslash X \leq b) \geq F_X(x)$$

A similar statement holds for the conditional denstiy function whenever it is non zero.

$$f_X(x\backslash X \leq b) \geq f_X(x)$$

Type 2 of defining condition event will be discussed in unit - 4