Unit-V

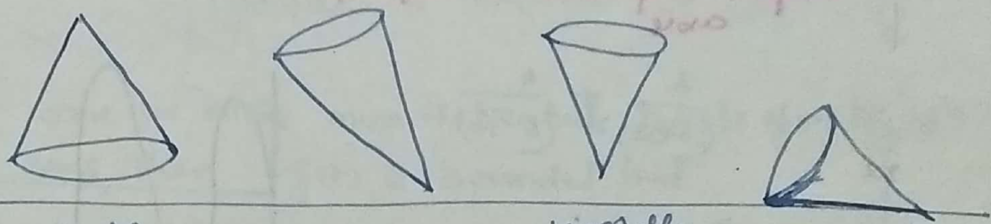
Concept of Stability :-

The LTI System will be stable if:

i) Bounded i/p has bounded o/p

ii) when No i/p applied, the the o/p tends zero irrespective of initial conditions. thus stability is called asymptotic stability.

- Stability depends only on parameters of the system and not on the i/p or driving function of the system (LTI)

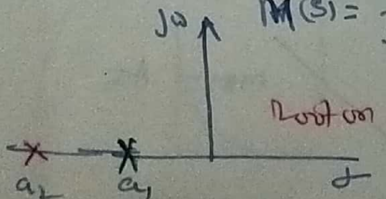


Stable — unstable — neutral

$$M(s) = \frac{C(s)}{R(s)} \quad \text{——} (1)$$

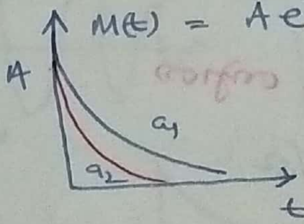
Response of various types of roots

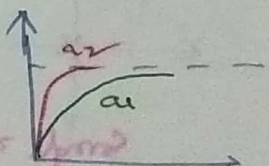
$M(s) = \frac{A}{s+a}$



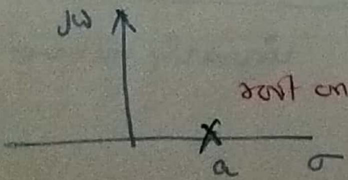
Root on negative axis

$M(t) = Ae^{-at}$



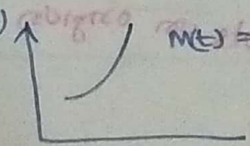


exponential decreasing



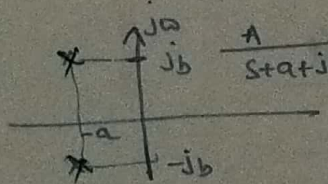
root on Positive axes

$M(t) = Ae^{at}$



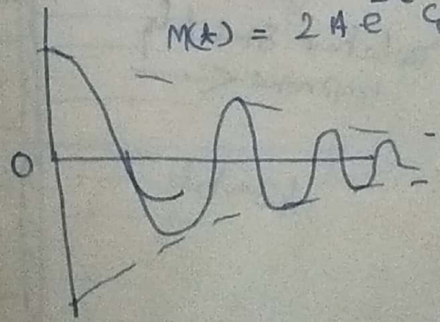
exponential increasing

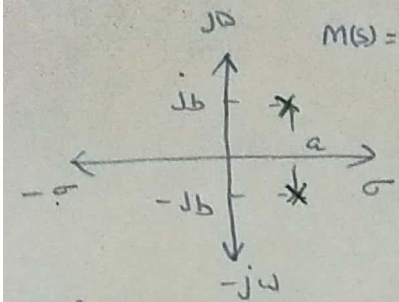
$\frac{A}{s+a+jb} + \frac{A}{s+a-jb}$



Complex conjugate roots at LHS

$M(t) = 2Ae^{-at}\cos bt$

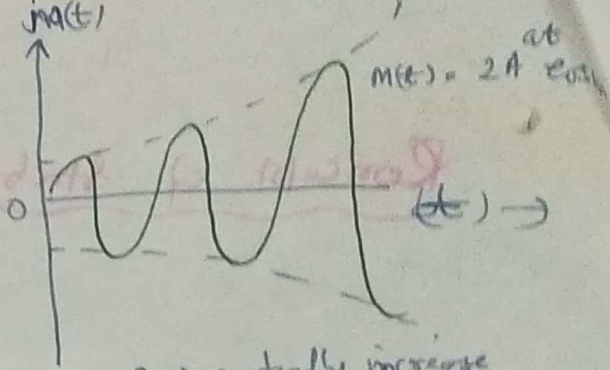




$$M(s) = \frac{A}{s-a+jb} + \frac{A}{s-a-jb}$$

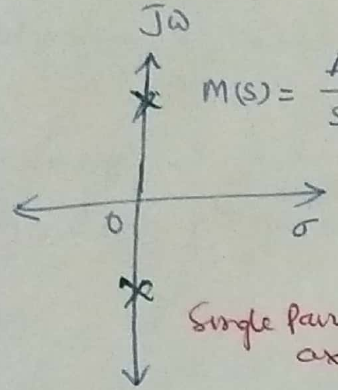
$$m(t) = 2Ae^{at}\cos bt$$

Complex conjugate roots at RHS



$M(t) = 2Ae^{at}\cos$

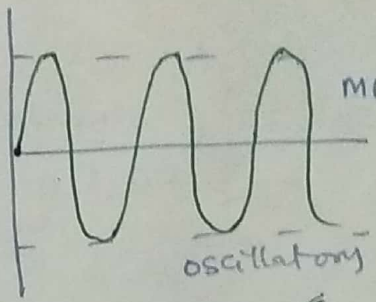
exponentially increase sinusoidal



$$M(s) = \frac{A}{s+jb} + \frac{A}{s-jb}$$

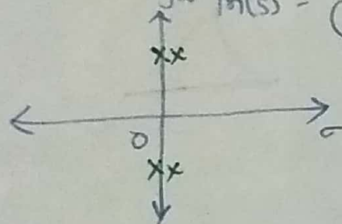
$$m(t) = 2A\cos bt$$

Single pair of roots on imaginary axis



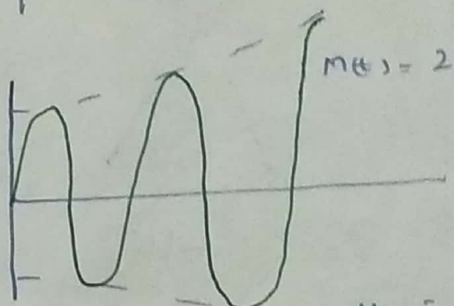
$M(t) = 2A\cos bt$

oscillatory



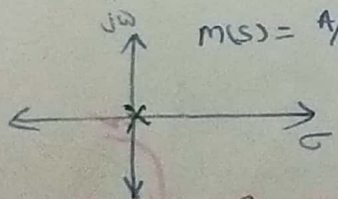
$$M(s) = \frac{A}{(s+jb)^2} + \frac{A}{(s-jb)^2}$$

Double pairs of roots on imaginary axis



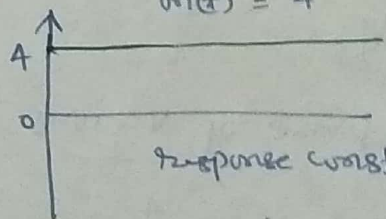
$M(t) = 2At\cos bt$

exponentially increase sinusoidal



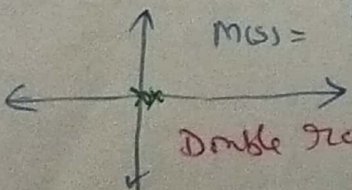
$$M(s) = A/s$$

Single root on origin



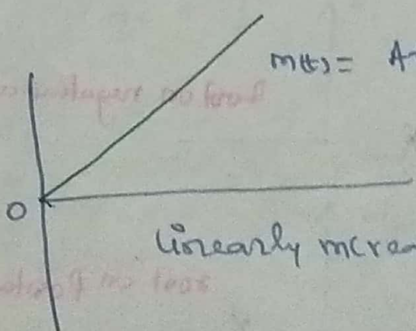
$m(t) = A$

response constant



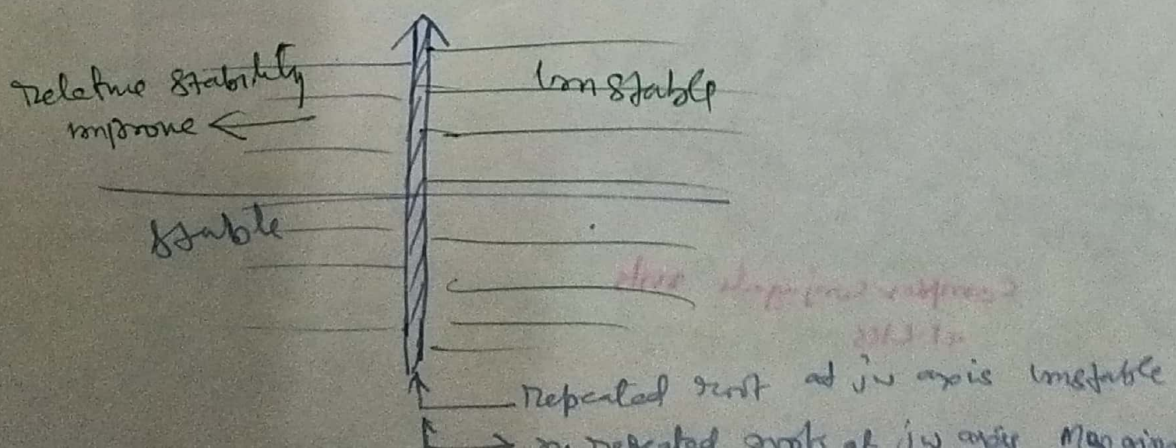
$$M(s) = A/s^2$$

Double roots on origin



$m(t) = At$

linearly increase with time



04/2

→ If all the roots of the characteristics eqn have negative real parts, then Impulse response is bounded and eventually decreases to zero therefore

$$\int_0^\infty |g(\tau)|\,d\tau$$ is finite and system BIBO

– If the roots of the characteristics eqn has Positive real part, $g(t)$ is unbounded and

$$\int_0^\infty |g(\tau)|\,d\tau$$ is infinite → System is unstable

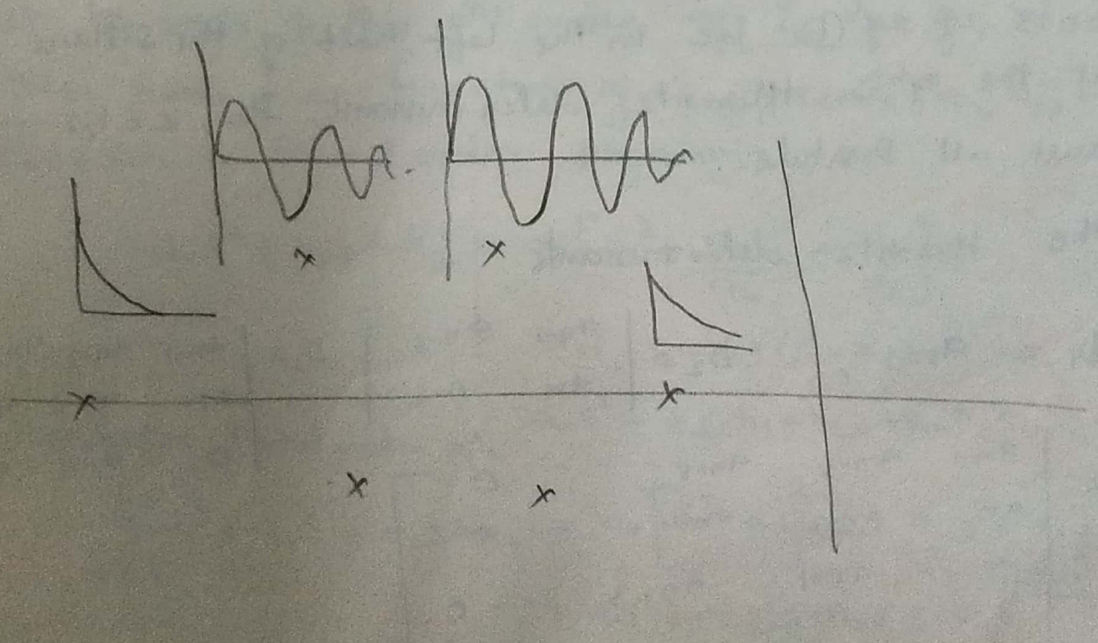
– If characteristics eqn ~~roots~~ has repeated roots on $j\omega$ axis, $g(t)$ is unbounded and

$$\int_0^\infty |g(\tau)|\,d\tau$$ is infinite → System is unstable

– If one or more non Repeated roots of ch eqn are on $j\omega$ axis, Then $g(t)$ is bounded but

$$\int_0^\infty |g(\tau)|\,d\tau$$ is infinite, The system is unstable (oscillatory system)

or, marginally/critically stable



Routh Hurwitz Criterion :-

It is an algebraic method that provides information on the absolute stability of LTI System that has algebraic equation with constant coefficients. The criterion tests whether any of roots of the characteristic eqn lie in the RHS of S-Plane. The number of roots that lies on the jω-axis and in the right half Plane is also indicated.

Let eqn of LTI SISO System

$$q(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 = 0 \qquad \text{—(1)}$$

where all the coefficient are real. In order that eqn (1) not have roots with positive real parts, it is necessary and sufficient that the following ~~hold~~ conditions hold :

1. All the coefficients of the eqn have the same sign
2. None of the coefficients vanishes

Hurwitz Criterion :-

The necessary and sufficient condition that all roots of eqn (1) lie in the left half of the S-Plane is that the eqn's Hurwitz determinant, D_K, $K = 1, 2 \cdots n$ must all positive.

The Hurwitz determinants

$$D_1 = a_{n-1}, \quad D_2 = \begin{vmatrix} a_{n-1} & a_{n-3} \\ a_n & a_{n-2} \end{vmatrix} \quad D_3 = \begin{vmatrix} a_{n-1} & a_{n-3} & a_{n-5} \\ a_n & a_{n-2} & a_{n-4} \\ 0 & a_{n-1} & a_{n-3} \end{vmatrix}$$

$$D_n = \begin{vmatrix} a_{n-1} & a_{n-3} & a_{n-5} & \cdots & 0 \\ a_n & a_{n-2} & a_{n-4} & \cdots & 0 \\ 0 & a_{n-1} & a_{n-3} & \cdots & 0 \\ \vdots & \vdots & & & \\ 0 & 0 & \cdots & \cdots & 0 \end{vmatrix}$$

where coefficients with indices larger than n or negative should be replaced by zero.

Routh simplified the process by introducing a Tabulation

V4/2-2

Necessary conditions for stability:

The necessary (but not sufficient) conditions for stability of linear time variant, single-input single output systems described by characteristics eqn.

$$q(s) = a_0 s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n = 0 \; ; \; a_0 > 0$$

is that

(1) All the coefficients of its char eqn be real and have same sign.

(2) Non of the coefficients should be zero.

However these conditions are not sufficients, because it is quite possible that an equation with all its coefficients non zero and of the same sign may not have all the roots in the left half of the s-plane.

~~Let these the roots of the~~

Consider an nth order ch eqn

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots a_{n-1} s + a_n = 0$$

Let the roots of this nth order eqn be $s = r_1, r_2 \dots r_n$

these roots are function of coefficients $a_0, a_1, \dots a_n$

Consider a second order polynomial:

$$a_0 s^2 + a_1 s + a_2 = a_0 \left[s^2 + \frac{a_1}{a_0} s + \frac{a_2}{a_0} \right]$$

$$= a_0 (s - r_1)(s - r_2)$$

$$= a_0 s^2 - a_0 (r_1 + r_2) + a_0 r_1 r_2$$

Consider third order eqn

$$a_0 s^3 + a_1 s^2 + a_2 s + a_3 = a_0 \left[s^3 + \frac{a_1 s^2}{a_0} + \frac{a_2 s}{a_0} + \frac{a_3}{a_0} \right]$$

$$= a_0 (s - r_1)(s - r_2)(s - r_3)$$

$$= a_0 s^3 - a_0 (r_1 + r_2 + r_3) s^2 + a_0 (r_1 r_2 + r_1 r_3 + r_2 r_3) s$$
$$+ a_0 r_1 r_2 r_3$$

on expanding this expension to the nth order polynomial we get,

$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots a_{n-1} s + a_n$

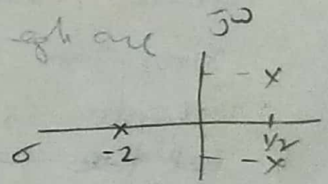
$= a_0 s^n - a_0(\text{Sum of all roots}) + a_0(\text{Sum of product of 2 roots taken at a time})\, s^{n-2} - a_0(\text{Sum of the product of 3 roots taken at time})\, s^{n-3} + \cdots$

$+ \cdots + a_0(-1)^n$ Product of all the n roots

If all the roots of a polynomial are real and are in the LH of S-plane, then all r_i in the above eqn are real and negative. Therefore all polynomial coefficients are positive.

If at least one root is in the right half of S-plane. Then some of the coefficients will be negative. also, it can be observed that if all the roots are in left half of the S-plane, no coefficient can be zero.

<u>example</u> $s^3 + s^2 + 2s + 8 = 0$, roots of these ch eqn are



$(s+2)\left[\left(s - \frac{1}{2} + j\frac{\sqrt{15}}{2}\right)\left(s - \frac{1}{2} - j\frac{\sqrt{15}}{2}\right)\right]$

the coefficient of this eqn are positive, but two roots lies in RH of S plane and so, system is unstable.

one or more coefficients of ch eqn can be zero or negative only if:

1. one or more roots have positive real parts/and
2. one or more roots are at the origin or/and
3. one or more ~~roots~~ pairs of conjugate roots on imaginary axis

The absence or negativeness of any of coefficients of the q(s) (with $a_0 > 0$) indicates that the system is either unstable or at most limitedly stable.

Positiveness of the coefficients of the ch eqn ensures that negativeness of real roots as well as the negativeness of the real parts of complex roots of system of 1st and second order but not for higher order. So, for third and higher order systems, they may be unstable all coefficients are positive and none is missing.

~~Routh~~ method in place of Hurwitz determinants 04/3

Routh's Tabulation

The simplified Hurwitz criterion is now called Routh-Hurwitz criterion, is arrange the coefficients of eq^n ① into two rows. the first row consists of the first, third, fifth .... coefficient and second row consists of second, ~~fourt~~ fourth, sixth ...... coefficients, all counting from highest term,

Example

$$q(s) = a_0 s^8 + a_1 s^7 + a_2 s^6 + a_3 s^5 + a_4 s^4 + a_5 s^3 + a_6 s^2 + a_7 s^1 + a_8 = 0$$

Routh array

s^8	a_0	a_2	a_4	a_6	a_8
s^7	a_1	a_3	a_5	a_7	0
s^6	$\frac{a_1 a_2 - a_0 a_3}{a_1} = b_1$	$\frac{a_1 a_4 - a_0 a_5}{a_1} = b_2$	$\frac{a_1 a_6 - a_0 a_7}{a_1} = b_3$	$\frac{a_1 a_8 - a_0 \times 0}{a_1} = a_8$
s^5	$\frac{b_1 a_3 - a_1 b_2}{b_1} = c_1$	$\frac{b_1 a_5 - a_1 b_3}{b_1} = c_2$	$\frac{b_1 a_7 - a_1 a_8}{b_1} = c_3$	0
s^4	$\frac{c_1 b_2 - b_1 c_2}{c_1} = d_1$	$\frac{c_1 b_3 - b_1 c_3}{c_1} = d_2$	$\frac{c_1 a_8 - b_1 \times 0}{c_1} = a_8$	0
s^3	$\frac{d_1 c_2 - c_1 d_2}{c_1} = e_1$	$\frac{d_1 c_3 - c_1 a_8}{c_1} = e_2$	0	
s^2	$\frac{e_1 d_2 - d_1 e_2}{e_1} = f_1$	$\frac{e_1 a_8 - d_1 \times 0}{e_1} = a_8$		
s^1	$\frac{f_1 e_2 - e_1 a_8}{e_1} = g_1$	0		
s^0	$\frac{g_1 a_8 - f_1 \times 0}{g_1} = a_8$			

The column of s's on the left side is used for identification purposes. The reference column keeps track of the calculations, and last row of the Routh table should always be the s^0 row.

In Processes of ~~generating~~ Routh array missing terms are (replaced) regarded as zero. also all the elements of any row can ~~not~~ be divided by a positive constant during the Process of simplify the computational work.

Routh Stability criterion:

The roots of the equation are all in the left half of S-Plane if all the elements of the first column of the Routh's tabulation are of the same Sign. The number of changes of Signs in the elements of the first column equals the number of roots with Positive real parts or in the right-half-S Plane.

example

$$2s^4 + s^3 + 3s^2 + 5s + 10 = 0$$

s^4	2	3	10
s^3	1	5	0
s^2	$\frac{3-10}{1} = -7$	$\frac{1\times10-0}{1} = 10$	
s^1	$\frac{-7\times5-1\times10}{-7} = 6.43$	0	
s^0	10		

Sign change (between s^3 and s^2)

Sign change (between s^2 and s^1)

$$\boxed{\begin{aligned} s &= -1.0055 \pm j0.9331 \\ s &= 0.7555 \pm j1.444 \end{aligned}}$$

Since there are two ~~Sign~~ changes in Sign in the first column of tabulation, The equation has two roots in The RHS of S-Plane

System is unstable

Special Cases when Routh's tabulation terminate prematurely (generally such system are unstable)

- The 1st element in any one row of Routh's tabulation is zero but the others are not
- The elements in one row of Routh's tabulation are all zero.

If any element of 1st Column is zero replace by small Positive number ε and proceed with Routh tabulation.

Note! If above two condition appear generally System are unstable.

04/4

Q. $s^4 + s^3 + 2s^2 + 2s + 3 = 0$

s^4	1	2	3
s^3	1	2	0
s^2	~~0~~ ε	3	
s^1	$\frac{2\varepsilon - 3}{\varepsilon} = \frac{-3}{\varepsilon}$	0	
s^0	3		

Sign Change ~~replace~~

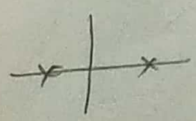
replaced s^2 coefficient of ε

There are two sign change means two roots are in RHS of s-Plane.

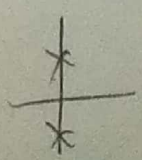
$$\boxed{s = -0.09057 \pm j0.902, \quad s = 0.4057 \pm j1.2928}$$

It should be noted that the ε-method may not give correct results if the eqn has pure imaginary roots

→ If all the elements in one row are zeros before tabulation is properly terminated, it indicates that one or more following condition may exist

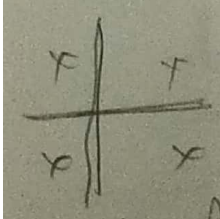


① The eqn has at least one pair of real roots with equal magnetude but opposite sign.



② The eqn has one or more pairs of imaginary roots

③ The eqn has pairs of complex-conjugate roots forming symmetry about the origin of s-Plane (e.g. $s = -1 \pm j1$, $s = 1 \pm j1$)



If rows of zero appears, then

Localion of even Polynomial

① Form the auxileary eqn $A(s) = 0$, by use the coefficients from the row ~~above~~ just (above) preceding the row of zeros

② Take the derivative of auxiliary w.r.t s this gives $\frac{dA(s)}{ds} = 0$

③ Replace the row of zeros with coefficient of $\frac{dA(s)}{ds} = 0$

④ Continue with Routh's tabulation in the usual manner with newly formed row of coefficients replacing the row of zeros

⑤ Interpret the changes of sign if any, of the coefficient in usual manner

Q $s^5 + 4s^4 + 8s^3 + 8s^2 + 7s + 4 = 0$

Routh tabulation

s^5	1	8	7
s^4	4	8	4
s^3	$\frac{8\times4 - 8\times1}{4} = 6$	6	0
s^2	$\frac{8\times6 - 4\times6}{6} = 4,$	4	
s^1	0	0	

s^5	1	8	7
s^4	4	8	4
s^3	6	6	0
s^2	4	4	
s^1	0	0	
s^1	8	0	(new) coefficient of $\frac{dA(s)}{ds}$
s^0	4		

Since all row of zeros appears, Then auxiliary eqn A(s)

$A(s) = 4s^2 + 4 = 0$

$$\frac{dA(s)}{ds} = 8s + 0$$

The coefficient are 8 and 0, replace the zeros in s^1 row of the original tabulation.

Since there is no sign changes in first column of entire Routh's tabulation, eqn does not have any root in the RHS of S-plane so system is stable

Q $s^3 + 3Ks^2 + (K+2)s + 4 = 0$, Find range of K so that system is stable

s^3	1	$K+2$
s^2	$3K$	4
s^1	$\frac{3K(K+2) - 4}{3K}$	0
s^0	4	

From the s^2 row, the condition for stability is $K > 0$, and from s^1 row, the condition for stability is

$3K^2 + 6K - 4 > 0$

or, $K < -2.528$ or $K > 0.528$

When the conditions of $K > 0$, and $K > 0.528$ are compared. It is apparent that the later requirement is more stringent, Thus closed-loop system to be stable K must satisfy $K > 0.528$

The requirement $K < -2.528$ is disregarded since K can't be negative.

Limitation of Routh-Hurwitz Criterion

① It is valid if and only if the characteristic eqn is algebraic with real coefficients. If any coefficient is complex, exponential or sinusoidal function Routh criterion can't applied.

② It is valid only for the determination of the roots of the characteristics eqn with respect to the left-half or right half of the S-plane. The stability boundary is jω axis of S-plane.

③ It gives information of absolute stability of a system. It does not suggest how to improve the relative stability or how to stabilize an unstable system.

④ It can only tell how many roots are in RHS (Right-half) of S-plane but can't give exact location. Also it can't tell whether the roots are real or complex

⑤ This stability criterion applies to polynomials with only a finite number of terms

Q. $s^5 + s^4 + 2s^3 + 2s^2 + 3s + 5$

s^5	1	2	3
s^4	1	2	5
s^3	0 $\left[\frac{1\times2-1\times2}{1}=0\right]$	-2	0
s^3	ε	-2	0
s^2	$\frac{2\epsilon+2}{\epsilon}$	5	
s^1	$\frac{-(5\epsilon^2+4\epsilon+4)}{2\epsilon+2}$		
s^0	5		

when $\epsilon \to 0$

Two sign change

Two roots lies at Right half of S-plane System is unstable.

Alternate method → above eqn change $s \to 1/z$

$$\left(\frac{1}{z}\right)^5 + \left(\frac{1}{z}\right)^4 + 2\left(\frac{1}{z}\right)^3 + 2\left(\frac{1}{z}\right)^2 + 3\left(\frac{1}{z}\right) + 5$$

or, $5z^5 + 3z^4 + 2z^3 + 2z^2 + z + 1 = 0$

z^5	5	2	1
z^4	3	2	1
z^3	$-1.33\left(\frac{3\times2-5\times2}{3}\right)$	-0.66	0
z^2	0.5	1	
z^1	2	0	
z^0	1		

Two sign changes means two roots of equation lies in the Right half of s-Plane and System is unstable.

$Q(s) = \quad s^3 + s^2 + 2s + 8 = 0$ Test through RH

s^3	1	2
s^2	1	8
s^1	$\frac{2-8}{2} = -3$, 1	0
s^0	$\frac{-3\times8-0}{-3} = 8$ 2	

Two Sign changes means two roots lies in RHS of s-Plane.