Assignment - 4

① Define Power Density Spectrum.

The power spectral density (PSD) of the signal describes the power present in the signal as a function of frequency per unit frequency.

Denoted by $S_{XX}(\omega) = \underset{T \to \infty}{Lt} \dfrac{E[|X_T(\omega)|^2]}{2T}$

② State the wienner - kincher relationship.

The average of auto correlation function and the power spectral density from a fourier transform pair i.e.

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)^{-j\omega\tau} d\tau$$

$$R_{XX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) e^{j\omega\tau} d\omega$$

$$R_{XX}(\tau) \xleftrightarrow{FT} S_{XX}(\omega)$$

This is called wienner kincher relationship.

③ Prove that $S_{XX}(\omega) = S_{XX}(-\omega)$

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau$$

$$S_{XX}(-\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{j\omega\tau} d\tau$$ [Replace τ by $-\tau$]

$$S_{XX}(-\omega) = \int_{-\infty}^{\infty} R_{XX}(-\tau)^{-j\omega\tau} d\tau \rightarrow ①$$

$R_{xx}(\tau) = E[x(t)x(t+\tau)]$

$R_{xx}(-\tau) = E[x(t)x(t-\tau)]$

let, $t-\tau = u \Rightarrow t = u+\tau$

$R_{xx}(-\tau) = E[x(u+\tau)x(u)]$

$R_{xx}(-\tau) = R_{xx}(\tau)$

substitute in eq. ①

$$S_{xx}(-\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-j\omega\tau} d\tau$$

$\therefore S_{xx}(-\omega) = S_{xx}(\omega)$.

④ show that cross PSD and cross correlation function forms a fourier transform pair.

w.k.T. $S_{xy}(\omega) = \underset{T\to\infty}{Lt} \dfrac{E[X_T^*(\omega) Y_T(\omega)]}{2T}$

$$X_T^*(\omega) = \int_{-T}^{T} x(t_1) e^{j\omega t_1} dt_1$$

$$Y_T(\omega) = \int_{-T}^{T} y(t_2) e^{-j\omega t_2} dt_2$$

$$S_{xy}(\omega) = \underset{T\to\infty}{Lt} \frac{1}{2T} E\left[\int_{-T}^{T} x(t_1) e^{j\omega t_1} dt_1 \cdot \int_{-T}^{T} y(t_2) e^{-j\omega t_2} dt_2\right]$$

$$= \underset{T\to\infty}{Lt} \frac{1}{2T} \int_{-T}^{T}\int_{-T}^{T} E[x(t_1)y(t_2)] e^{-j\omega(t_2-t_1)} dt_1 dt_2$$

$$= \underset{T\to\infty}{Lt} \frac{1}{2T} \int_{-T}^{T}\int_{-T}^{T} R_{xy}(t_1,t_2) e^{-j\omega(t_2-t_1)} dt_1 dt_2$$

IFT of $S_{xy}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xy}(\omega) e^{j\omega t}\, d\omega$

$$F^{-1}[S_{xy}(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{xy}(t_1,t_2)\,\delta(\tau - t_2 + t_1)\, dt_1\, dt_2$$

since $\delta(\tau - t_2 + t_1) = 1$ if $\tau - t_2 + t_1 = 0$

$t_2 = \tau + t_1$

$$F^{-1}[S_{xy}(\omega)] = \underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T} R_{xy}(t_1, t_2+\tau)\, dt_1$$

let $t_1 = t$, $dt_1 = dt$

$$F^{-1}[S_{xy}(\omega)] = \underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T} R_{xy}(t, t+\tau)\, dt$$

If $x(t)$ & $y(t)$ are jointly w.s.s. then

$$R_{xy}(t, t+\tau) = R_{xy}(\tau)$$

$$\therefore F^{-1}[S_{xy}(\omega)] = R_{xy}(\tau)$$

$$\therefore R_{xy}(\tau) \xleftrightarrow{F.T.} S_{xy}(\omega).$$

(5) state and prove wiener kinchen relationship.

Time average of auto correlation function and the power spectral density forms a fourier transform pair i.e.

$$R_{xx}(\tau) \xleftrightarrow{F.T.} S_{xx}(\omega)$$

This is wiener - kinchen relationship.

Proof: w.k.t. $S_{xx}(\omega) = \underset{T\to\infty}{Lt} \frac{E[|X_T(\omega)|^2]}{2T}$

$$S_{XX}(\omega) = \underset{T\to\infty}{Lt} \frac{E[X_T(\omega) X_T^*(\omega)]}{2T}$$

$$X_T(\omega) = \int_{-T}^{T} x(t_1) e^{-j\omega t_1} dt_1$$

$$X_T^*(\omega) = \int_{-T}^{T} x(t_2) e^{j\omega t_2} dt_2$$

$$S_{XX}(\omega) = \underset{T\to\infty}{Lt} \frac{1}{2T} E\left[\int_{-T}^{T} x(t_1) e^{-j\omega t_1} dt_1 \int_{-T}^{T} x(t_2) e^{j\omega t_2} dt_2\right]$$

$$= \underset{T\to\infty}{Lt} \frac{1}{2T} E\left[\int_{-T}^{T}\int_{-T}^{T} x(t_1) x(t_2) e^{-j\omega(t_2-t_1)} dt_1 dt_2\right]$$

$$= \underset{T\to\infty}{Lt} \frac{1}{2T} \int_{-T}^{T}\int_{-T}^{T} R_{XX}(t_1,t_2) e^{-j\omega(t_2-t_1)} dt_1 dt_2$$

IFT of $S_{XX}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega) e^{j\omega t}$

$$F^{-1}[S_{XX}(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{XX}(t_1,t_2) e^{-j\omega(t_2-t_1)} e^{j\omega\tau} d\omega \cdot dt_1 \cdot dt_2$$

$$= \underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{XX}(t_1,t_2) \cdot \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega(\tau-t_2+t_1)} d\omega\, dt_1 dt_2$$

we have $F(\delta(t)) = 1$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} 1 \cdot e^{j\omega t} d\omega = \delta(t)$$

$$\Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega(\tau-t_2+t_1)} d\omega = \delta(\tau-t_2+t_1)$$

$$F^{-1}[S_{XX}(\omega)] = \underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{XX}(t_1,t_2)\, \delta(\tau-t_2+t_1) dt_1 dt_2$$

$\delta(\tau - t_2 + t_1) = 1$ at $\tau - t_2 + t_1 = 0$

i.e. $t_2 = \tau + t_1$

$$F^{-1}[S_{xx}(\omega)] = \underset{T\to\infty}{Lt} \frac{1}{2T} \int_{-T}^{T} R_{xx}(t_1, \tau + t_1)\, dt_1$$

let, $t_1 = t$, $dt_1 = dt$

$$F^{-1}[S_{xx}(\omega)] = \underset{T\to\infty}{Lt} \frac{1}{2T} \int_{-T}^{T} R_{xx}(t, \tau + t)\, dt$$

$\because R_{xx}(t, t+\tau)$ is independent of time.

$$R_{xx}(t, t+\tau) = R_{xx}(\tau)$$

$$F^{-1}[S_{xx}(\omega)] = R_{xx}(\tau)$$

$$\therefore R_{xx}(\tau) \xleftrightarrow{FT} S_{xx}(\omega).$$

⑥ Determine the auto correlation function and PSD of the random process $x(t) = A\cos(\omega_0 t + \theta)$ where θ is a random variable over the ensemble and is uniformly distributed over the range $(0, 2\pi)$

Given $x(t) = A \cdot \cos(\omega_0 t + \theta)$

$$f_\theta(\theta) = \frac{1}{2\pi}, \quad 0 \le \theta \le 2\pi$$

$$R_{xx}(\tau) = E[x(t)\, x(t+\tau)]$$

$$= E[A\cos(\omega_0 t + \theta)\, A\cos(\omega_0(t+\tau) + \theta)]$$

$$= \frac{A^2}{2} E[\cos(\omega_0 t + \theta)\cos(\omega_0(t+\tau) + \theta)]$$

$$= \frac{A^2}{2} E\left[\cos(2\omega_0 t + \omega_0\tau + 2\theta) + \cos\omega_0\tau\right]$$

$$= \frac{A^2}{2}\left[E(\cos\omega_0\tau) + E(\cos(2\omega_0 t + \omega_0\tau) + 2\theta)\right]$$

$$R_{xx}(\tau) = \frac{A^2}{2}\int_0^{2\pi} \frac{\cos\omega_0\tau}{2\pi}\, d\theta + \frac{A^2}{2}\int_0^{2\pi} \frac{1}{2\pi}\cos(2\omega_0 t + \omega_0\tau + 2\theta)\, d\theta$$

$$= \frac{A^2}{2}\cos\omega_0\tau + 0 = \frac{A^2}{2}\cos\omega_0\tau$$

$$S_{xx}(\omega) = FT\left[R_{xx}(\tau)\right] = F.T.\left[\frac{A^2}{2}\cos\omega_0\tau\right]$$

$$= \frac{A^2}{2}\int_{-\infty}^{\infty} \cos\omega_0\tau\, e^{-j\omega\tau}\, d\tau$$

$$= \frac{A^2}{2}\int_{-\infty}^{\infty}\left(\frac{e^{+j\omega_0\tau} + e^{-j\omega_0\tau}}{2}\right) e^{-j\omega\tau}\, d\tau$$

$$= \frac{A^2}{4}\left(\int_{-\infty}^{\infty} e^{-j(\omega-\omega_0)}\, d\tau + \int_{-\infty}^{\infty} e^{-j(\omega+\omega_0)\tau}\, d\tau\right)$$

$$= \frac{A^2}{2}\pi\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right].$$

(7) Find the auto correlation function of random process whose PSD is $\frac{16}{4+\omega^2}$.

Given, $S_{xx}(\omega) = \frac{16}{4+\omega^2}$

w.k.T. $S_{xx}(\omega) \xleftrightarrow{F.T.} R_{xx}(\tau)$

w.k.t. $e^{-a(t)} \xleftrightarrow{F.T.} \frac{2a}{a^2+\omega^2}$.

$$\frac{16}{4+\omega^2} = 4\,\frac{2(2)}{4+\omega^2} \longleftrightarrow 4e^{-2(t)}$$

$$\therefore R_{xx}(\tau) = 4e^{-2(\tau)}.$$

(8) A Random process has the PSD function $S_{xx}(\omega) = \frac{6\omega^2}{1+\omega^2}$. Find the P_{avg} in the process.

$$S_{xx}(\omega) = \frac{6\omega^2}{1+\omega^2} = 6 - \frac{6}{1+\omega^2}$$

$$S_{xx}(\omega) \xleftrightarrow{FT} R_{xx}(\tau)$$

$$R_{xx}(\tau) = 6\delta(\tau) - 3e^{-2(\tau)}$$

$$P_{avg} = R_{xx}(0) = 6\delta(0) - 3e^{(0)} = 6-3 = 3$$