Analog & Digital Communications

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UNIT- V

Digital Modulation Techniques

&

Baseband Transmission and Optimal Reception of Digital Signal

Contents

Digital Modulation Techniques

ASK- Modulator, Coherent ASK Detector, FSK-modulator, Non- Coherent FSK Detector, BPSK-Modulator, Coherent BPSK Detection. Principles of QPSK, Differential PSK and QAM.

Baseband Transmission and Optimal Reception of Digital Signal:

A Baseband Signal Receiver, Probability of Error, Optimum Receiver, Coherent Reception, ISI, Eye Diagrams.

Binary Modulation schemes

- ASK
- FSK
- PSK

M- ary Modulation schemes

- QPSK
- QAM
- 8PSK,16 PSK,32PSK…….

General

$$v(t) = A \, \mathrm{Sin}(2\pi \, fc \; t + \Theta)$$

ASK FSK PSK

QAM

AMPLITUDE-SHIFT KEYING (ASK)

AMPLITUDE-SHIFT KEYING

- **Mathematically, amplitude-shift keying is**

$$v_{(ask)}(t) = [1 + v_m(t)]\left[\frac{A}{2}\cos(\omega_c t)\right]$$

- **Modulating signal $v_m(t)$ *is* normalized where**

  + 1 V = logic 1

  - 1 V = logic 0

-

- **Therefore for a logic 1 input, $v_m(t)$ = +1 V, Equation *reduces to***

$$v_{(ask)}(t) = [1 + 1]\left[\frac{A}{2}\cos(\omega_c t)\right]$$

$$= A\cos(\omega_c t)$$

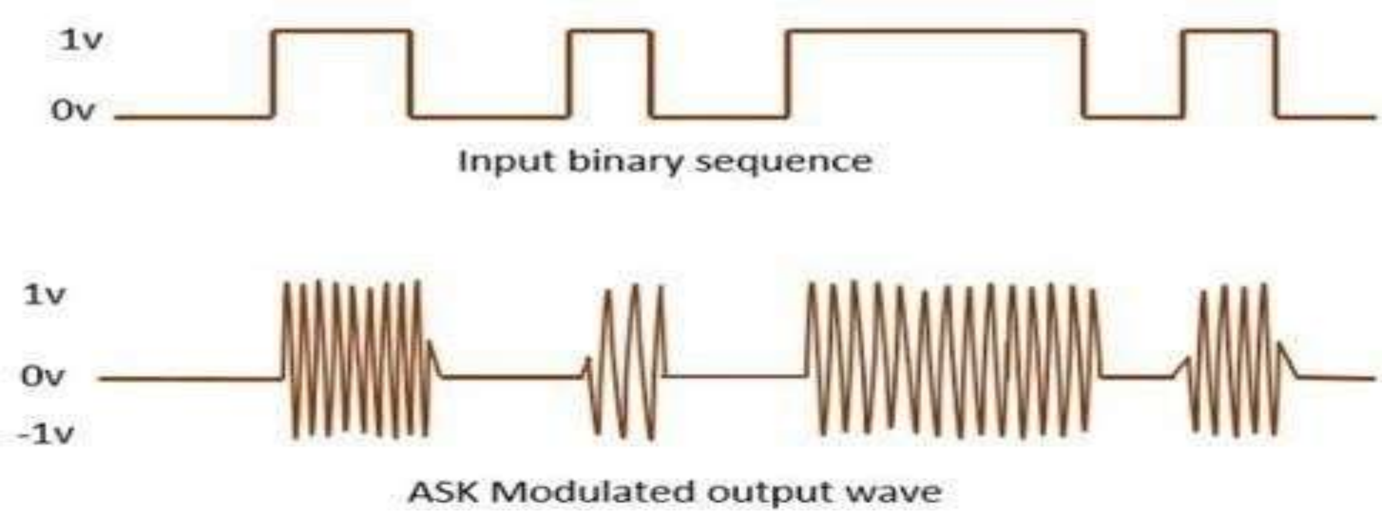
- **And for a logic 0 input, $v_m(t)$ = -1 V, Equation *reduces to***

$$v_{(ask)}(t) = [1 - 1]\left[\frac{A}{2}\cos(\omega_c t)\right]$$

- **Thus, the modulated wave *is either A cos(w_ct) or 0.***

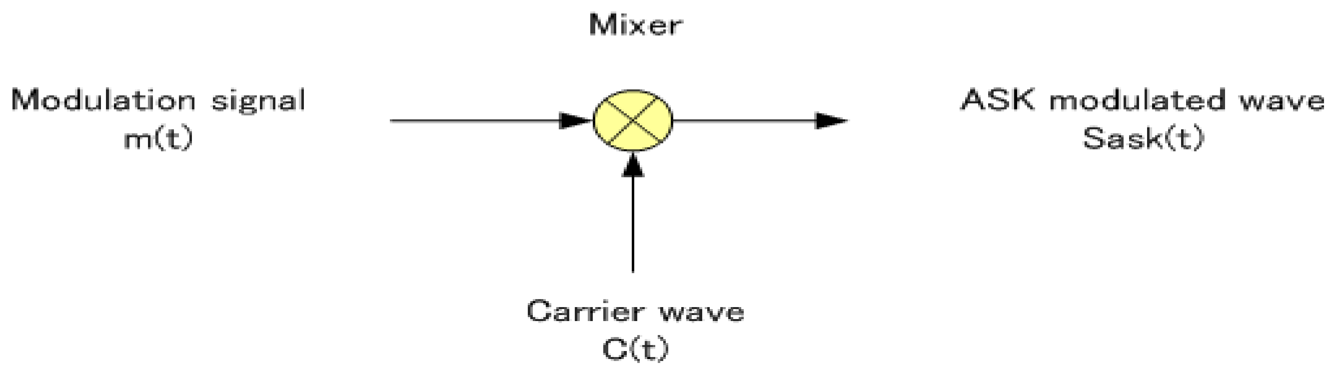
- **Hence, the carrier is either *"on“ or "off" which is why* amplitude-shift keying is sometimes referred to as *on-off keying(OOK).***

ASK



Hence, the carrier is either *"on" or "off" which is why* amplitude-shift keying is sometimes referred to as *on- off keying(OOK).*

ASK Modulator



s(t)= $A\text{Cos}2\pi f_c t$ where, $P_s = \left(\frac{A}{\sqrt{2}}\right)^2$

s(t)= $\sqrt{2P_s}\text{Cos}2\pi f_c t$

s(t)= b(t) $\sqrt{2P_s}\text{Cos}2\pi f_c t$ b(t)= 1 for Logic-1

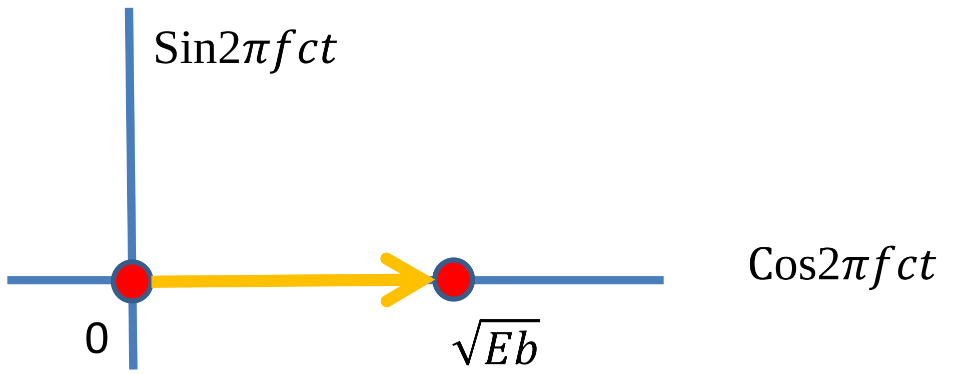
= 0 for Logic-0

s(t)= b(t) $\sqrt{P_s T_b}\ \sqrt{\frac{2}{T_b}}\text{Cos}2\pi f_c t$

s(t)= $\sqrt{P_s T_b}\ \ \varphi(t)$; where, $\varphi(t) = \sqrt{\frac{2}{T_b}}\text{Cos}2\pi f_c t$

s(t)= $\sqrt{E_b}\ \ \varphi(t)$

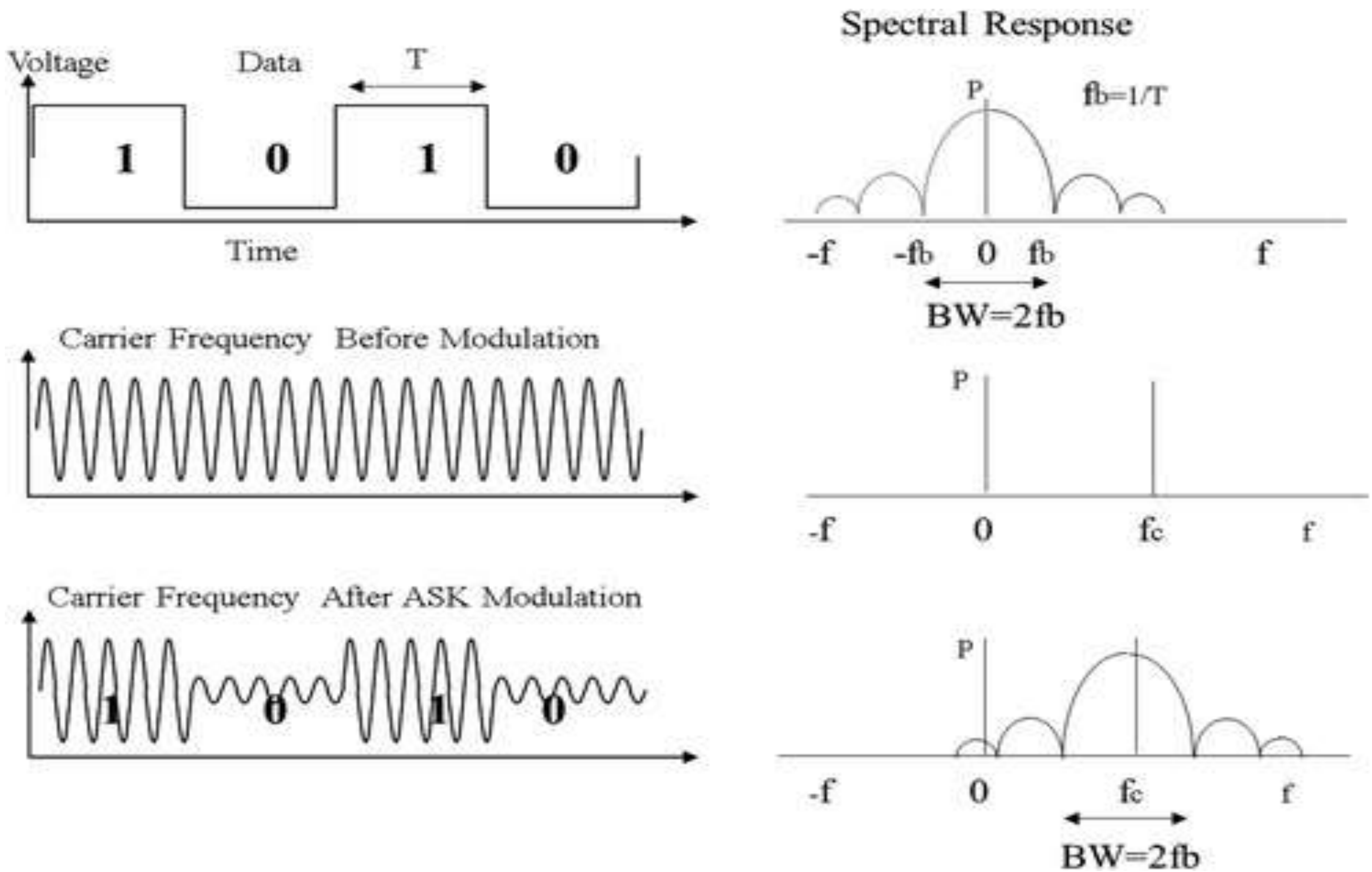
- Phasor Diagram



- Constellation Diagram

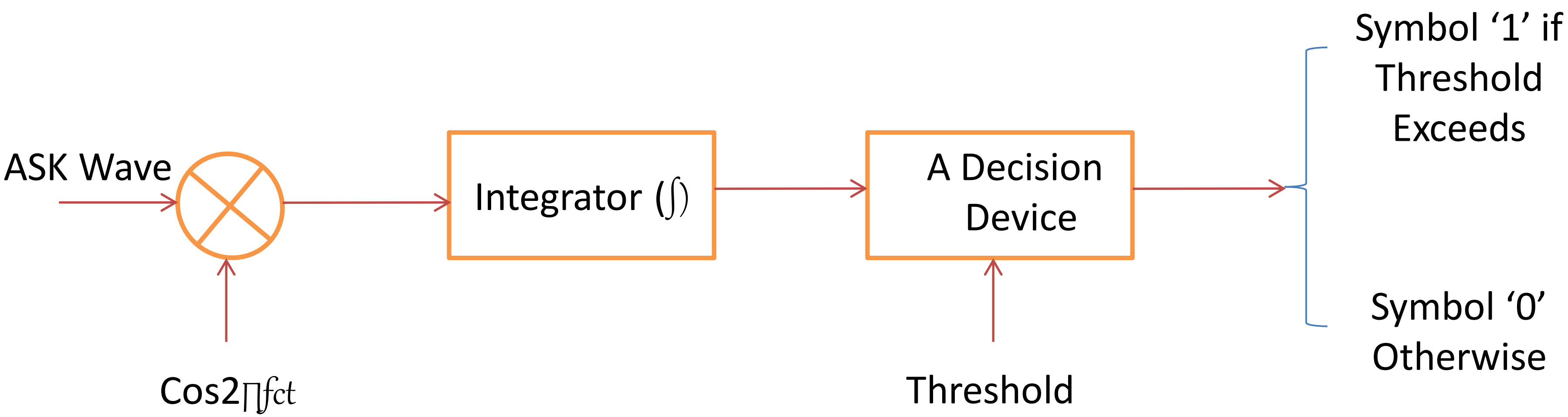
Distance (d) = $\sqrt{Eb}$

Spectrum of ASK



Hence, Bandwidth of ASK is 2fb

Detection of ASK



FREQUENCY SHIFT KEYING (FSK)

FREQUENCY-SHIFT KEYING

- **FSK is a form of constant-amplitude angle modulation similar to standard frequency modulation (FM) except the modulating signal is a binary signal**
- **FSK is sometimes called *binary FSK (BFSK).***
- **The general expression for FSK is**

$$v_{fsk}(t) = V_c \cos\{2\pi[f_c + v_m(t)\,\Delta f]t\}$$

- The modulating signal is a normalized binary waveform where a logic 1 = + 1 V and a logic 0 = -1 V.

- Thus, for a logic l input, we can write

$$v_{fsk}(t) = V_c \cos[2\pi(f_c + \Delta f)t]$$

- For a logic 0 input, Vm(t) = -1,

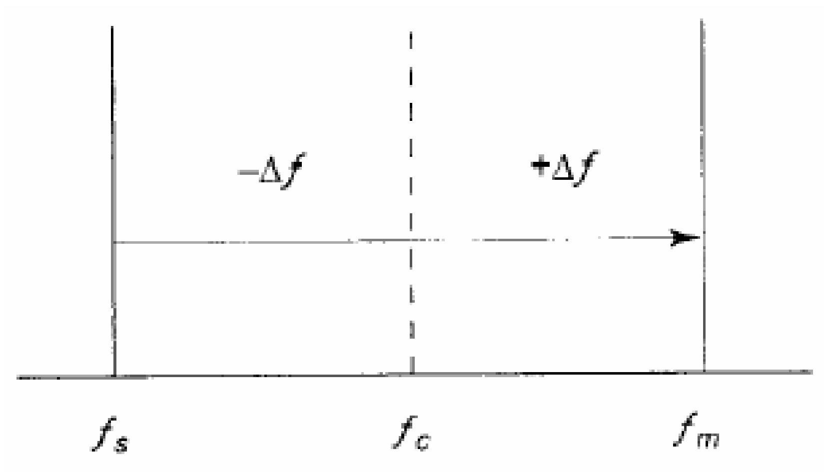
$$v_{fsk}(t) = V_c \cos[2\pi(f_c - \Delta f)t]$$

- Thus With binary FSK, the carrier center frequency (f_c) is shifted (deviated) up and down in the frequency domain by the binary input signal

•As the binary input signal changes from a logic 0 to a logic 1 and vice versa, the output frequency shifts between two frequencies

(a) Mark, or logic 1 frequency (fm),
(b) Space, or logic 0 frequency (fs).

*The mark and space frequencies are separated from the carrier frequency by the peak frequency Δf deviation and from each other by 2Δf .



s(t)= $A\text{Cos}2\pi f_c t$ where, $P_s = \left(\frac{A}{\sqrt{2}}\right)^2$

s(t)= $\sqrt{2P_s}\text{Cos}2\pi f_c t$

s(t)= b(t) $\sqrt{2P_s}\text{Cos}2\pi f_c t$ b(t)= 1 for Logic-1

= 0 for Logic-0

s(t)= b(t) $\sqrt{P_s T_b}\sqrt{\frac{2}{T_b}}\text{Cos}2\pi f_c t$

s(t)= $\sqrt{P_s T_b}\,\emptyset_1$(t); where, $\emptyset_1$(t) = $\sqrt{\frac{2}{T_b}}\text{Cos}2\pi f_c t$

s(t)= $\sqrt{E_b}\,\emptyset_1$(t)

If b(t) =1; $S_H(t) = \sqrt{2P_s}\text{Cos}(2\pi f_c + \Omega)t$ – Logic ‘1’

If b(t) =0; $S_L(t) = \sqrt{2P_s}\text{Cos}(2\pi f_c - \Omega)t$ – Logic ‘0’

Hence, There is an Increase or Decrease of Frequency by Ω.

b(t)	d(t)	P_H(t)	P_L(t)
1	1V	+1V	OV
0	-1V	0V	1V

The Equation can now be Written as

$$S(t) = \sqrt{2P_s}\text{Cos}[(2\pi f_c + d(t)\Omega)]t$$

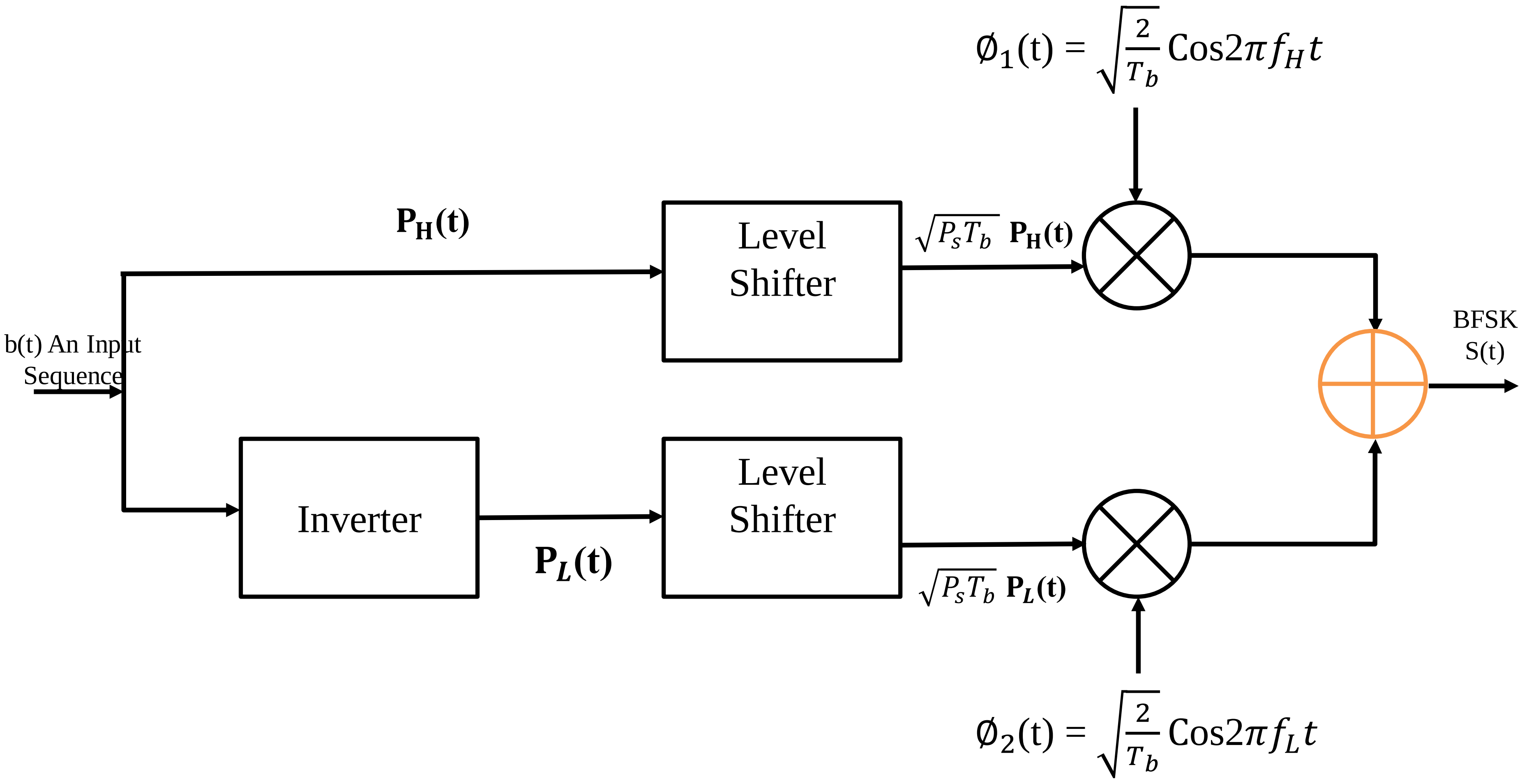
- Here, $f_H = f_c + \frac{\Omega}{2\pi}$ for, Symbol-1

  $f_L = f_c - \frac{\Omega}{2\pi}$ for, Symbol-0

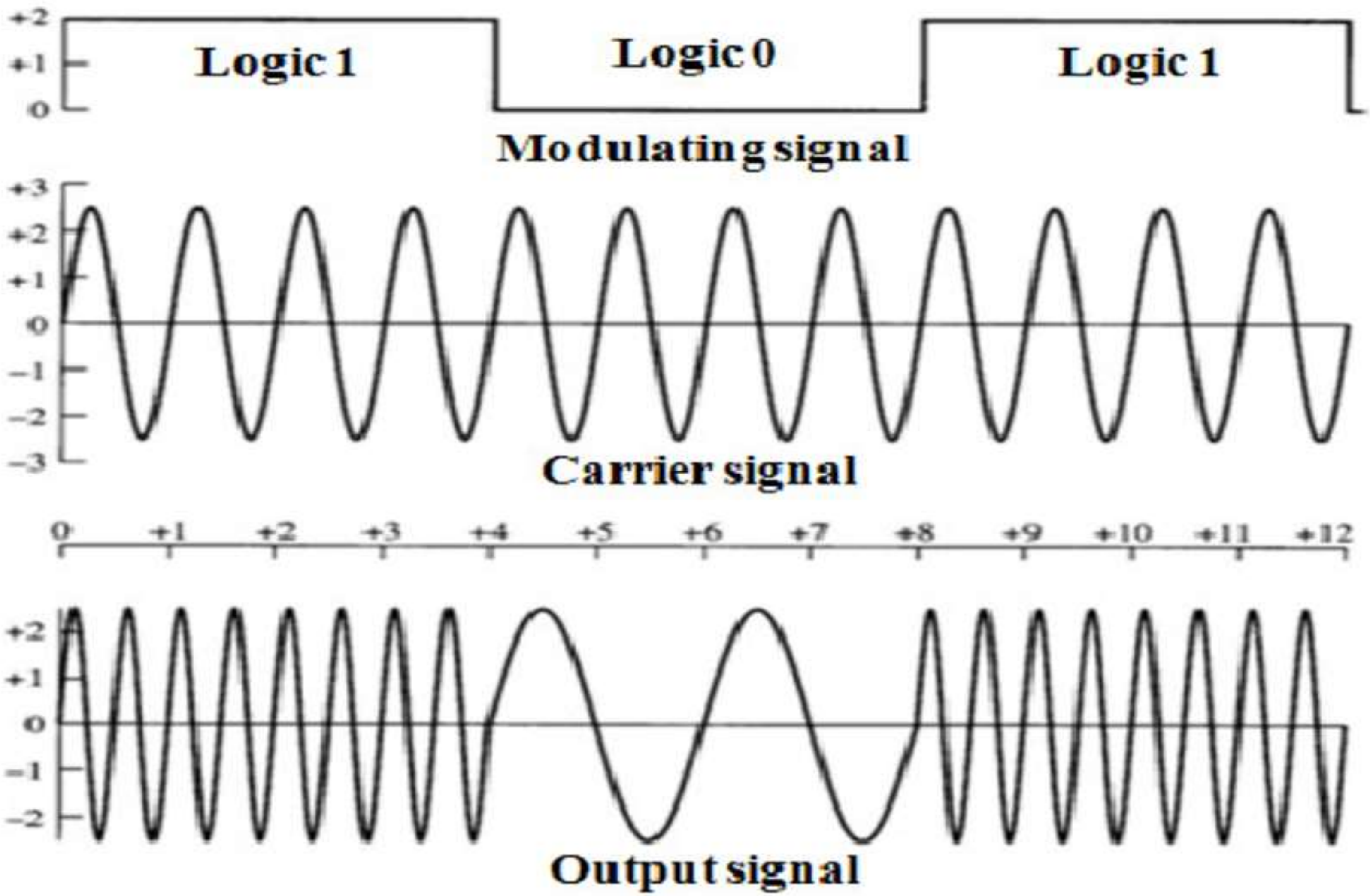
The General Equation for Both Symbols ‘1’ & ‘0’

$$s(t) = \sqrt{2P}P_H(t)Cos2\pi f_H t + \sqrt{2P}P_L(t)Cos2\pi f_L t$$

Generation of FSK



FSK Waveform



Spectrum of BFSK

The BFSK Waveform is,

$$S_{BFSK}(t) = \sqrt{2P_s}\, P_H(t)\, \mathrm{Cos}2\pi\, f_H t + \sqrt{2P_s}\, P_L(t)\, \mathrm{Cos}2\pi\, f_L t$$

$$S_{BPSK}(t) = b(t)\sqrt{2P}Cos2\pi f_c t$$

In BPSK Signal b (t) is Bipolar Signal whereas in BFSK $P_H(t)$& $P_L(t)$ are Unipolar Let us Convert into Bipolar Signal.

$$P_H(t) = \frac{1}{2} + \frac{1}{2}\, P_H'(t)$$

$$P_L(t) = \frac{1}{2} + \frac{1}{2}\, P_L'(t)$$

Here $P_H'(t)$& $P_L'(t)$ are Bipolar Signals

Substituting these Equations in 1, we get

$$s(t) = \sqrt{2P}P_H(t)Cos2\pi f_H t + \sqrt{2P}P_L(t)Cos2\pi f_L t$$

$$= \sqrt{2P}[\frac{1}{2} + \frac{1}{2}\ P'_H(t)]Cos2\pi f_H t + \sqrt{2P}[\frac{1}{2} + \frac{1}{2}\ P'_L(t)]Cos2\pi f_L t$$

$$= \sqrt{\frac{P_S}{2}}\,Cos2\pi f_H t + \sqrt{\frac{P_S}{2}}\,Cos2\pi f_L t + \sqrt{\frac{P_S}{2}}\,P'_H(t)Cos2\pi f_H t + \sqrt{\frac{P_S}{2}}\,P'_L(t)Cos2\pi f_L t$$

$$S(f) = \sqrt{\frac{P_S}{2}}\ \delta[(f - f_H) + \delta(f - f_L)] + \frac{P_S T_b}{2}\left\{\left(\frac{Sin(\pi f_H T_b)}{\pi f_H T_b}\right)^2\right\} + \frac{P_S T_b}{2}\left\{\left(\frac{Sin(\pi f_L T_b)}{\pi f_L T_b}\right)^2\right\}$$

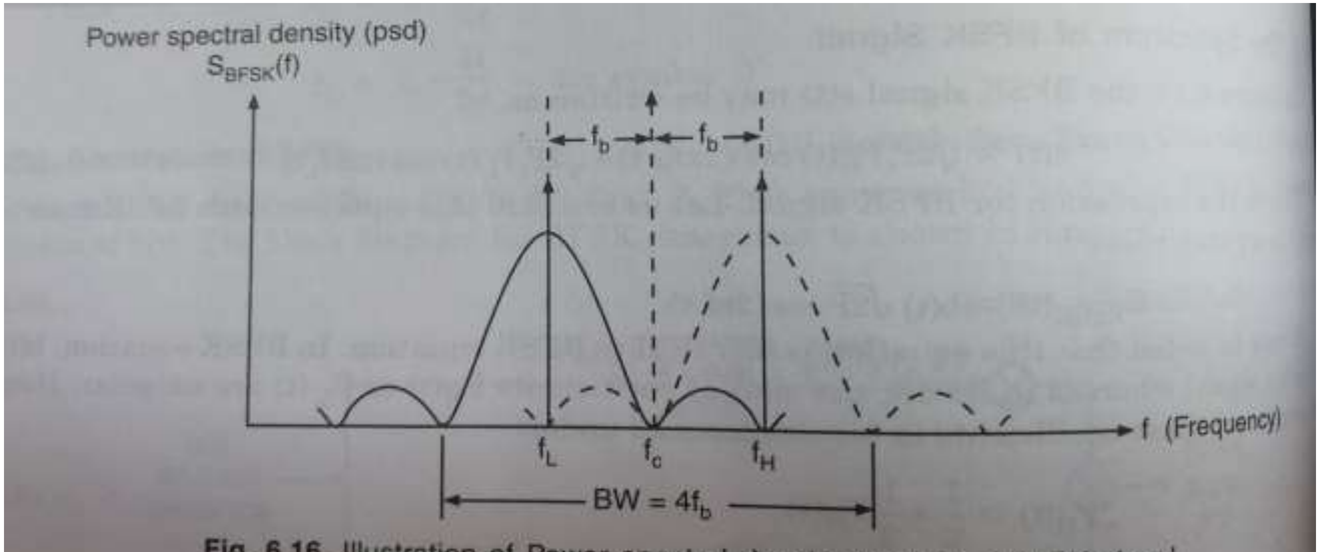


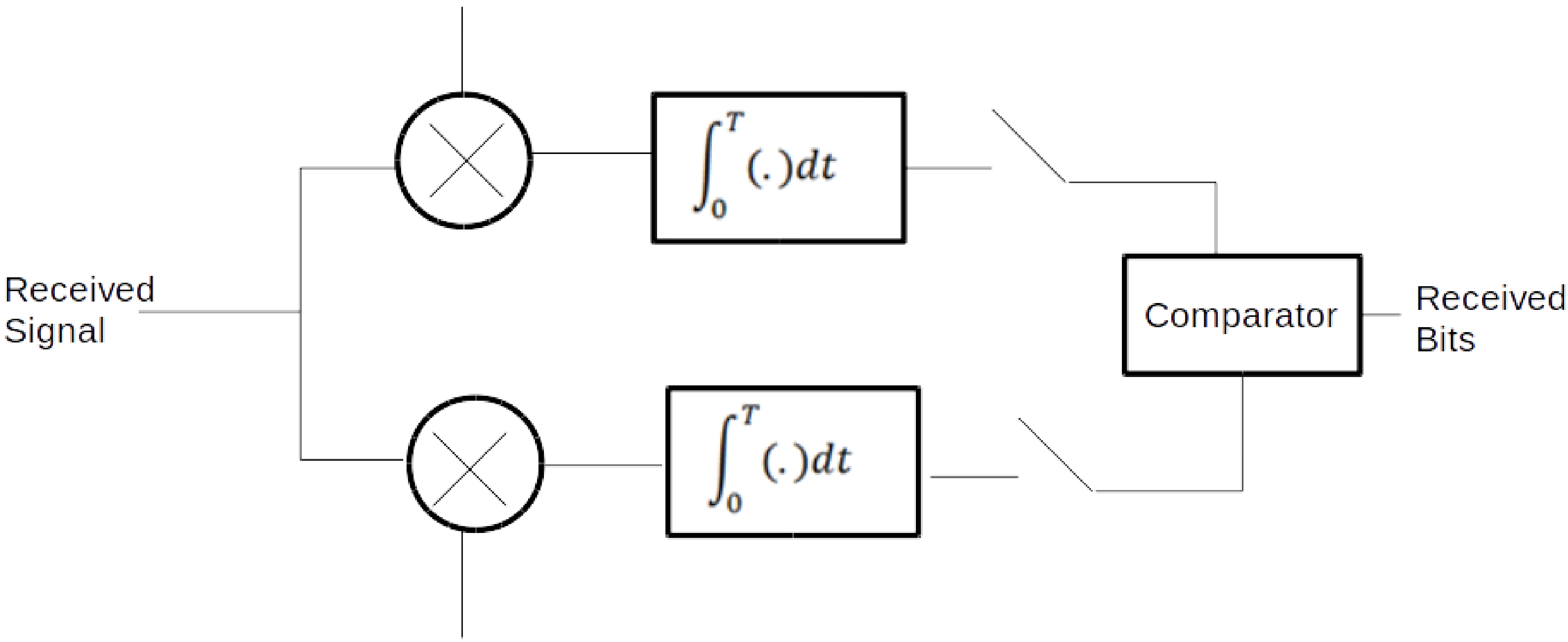
Fig. 6.16. Illustration of Power spectral density (psd) of a BFSK signal

BW= 2 f_b

FSK Demodulator

$$\psi_1 = \sqrt{\frac{2}{T}}\cos(2\pi f_1 t)$$

Here, $f_1 = f_H$



$$\psi_2 = \sqrt{\frac{2}{T}}\cos(2\pi f_2 t)$$

Here, $f_2 = f_L$

Geometrical Representation of BFSK

Let ,

$$f_H = mf_b$$

$$f_L = nf_b$$

$$f_b = \frac{1}{T_b}$$

$$\emptyset_1(\mathrm{t}) = \sqrt{\frac{2}{T_b}}\,\mathrm{Cos}2\pi m f_b t$$

$$\emptyset_2(\mathrm{t}) = \sqrt{\frac{2}{T_b}}\,\mathrm{Cos}2\pi n f_b t$$

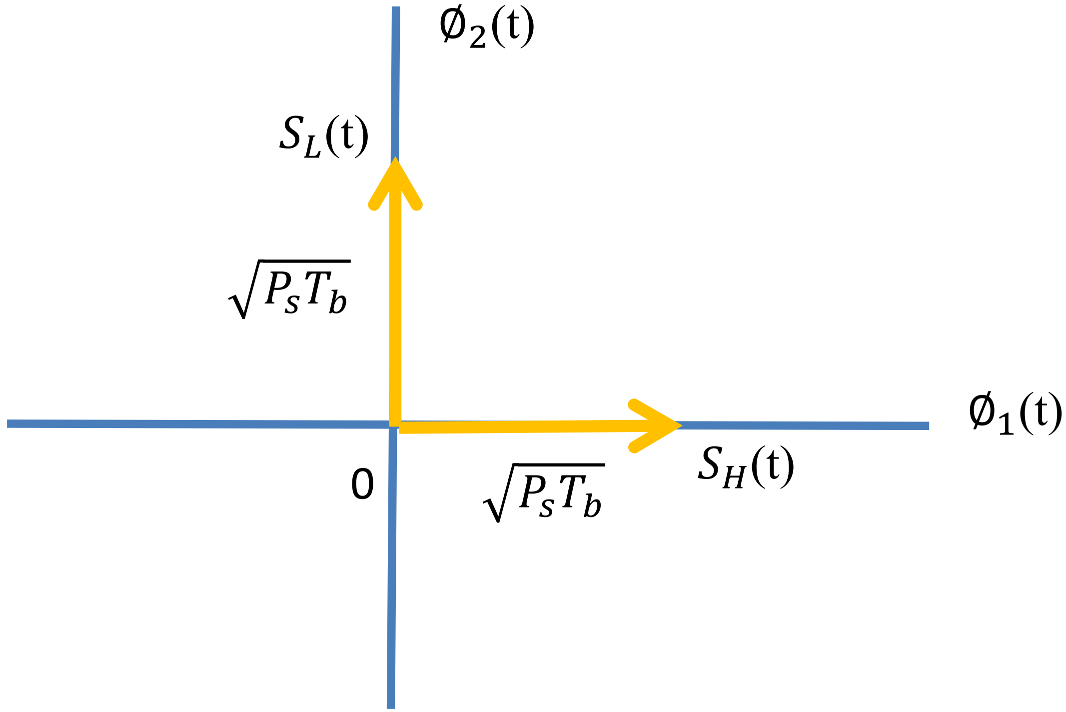
The Carriers $\emptyset_1$(t) & $\emptyset_2$(t) are Orthogonal over the Interval T_b

$$S_H(\mathrm{t}) = \sqrt{P_s T_b}\sqrt{\frac{2}{T_b}}\,\mathrm{Cos}2\pi f_H t \quad \& \quad S_L(\mathrm{t}) = \sqrt{P_s T_b}\sqrt{\frac{2}{T_b}}\,\mathrm{Cos}2\pi f_L t$$

$$f_H = f_c + \frac{\Omega}{2\pi} \quad ; \quad f_L = f_c - \frac{\Omega}{2\pi}$$

$$S_H(\mathrm{t}) = \sqrt{P_s T_b}\sqrt{\frac{2}{T_b}}\,\mathrm{Cos}2\pi f_H t \;=\; \sqrt{P_s T_b}\,\emptyset_1(\mathrm{t})$$

$$S_L(\mathrm{t}) = \sqrt{P_s T_b}\sqrt{\frac{2}{T_b}}\,\mathrm{Cos}2\pi f_L t = \;\sqrt{P_s T_b}\,\emptyset_2(\mathrm{t})$$



$$d^2 = \left(\sqrt{P_s T_b}\,\right)^2 + \left(\sqrt{P_s T_b}\,\right)^2$$

$$d^2 = 2\;P_s T_b \;= \sqrt{2E_b}$$

PHASE SHIFT KEYING (BPSK)

s(t)= $A\text{Cos}2\pi f_c t$ where, $P_s = \left(\frac{A}{\sqrt{2}}\right)^2$

S_1(t)= $\sqrt{2P_s}\text{Cos}2\pi f_c t$ For Logic-1

S_2(t)= $\sqrt{2P_s}\text{Cos}(2\pi f_c t + \pi)$

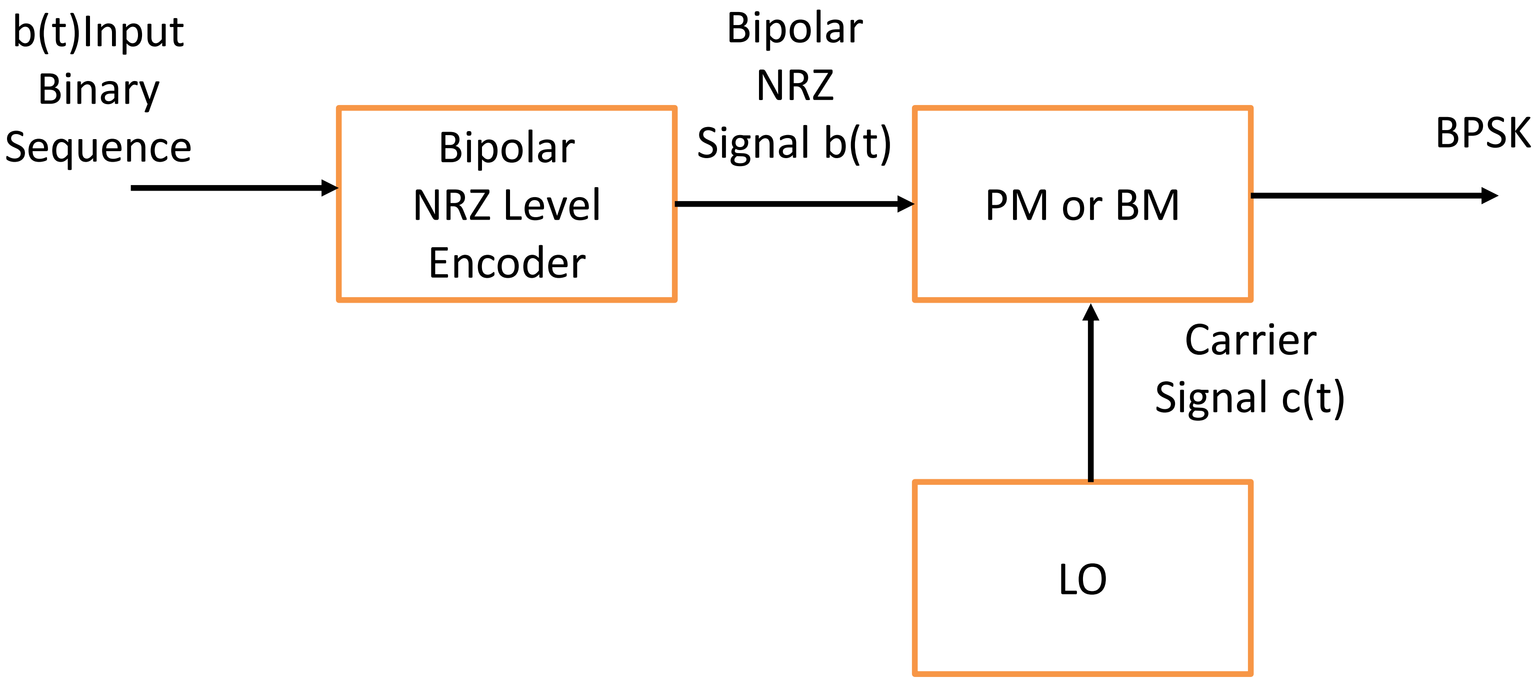
$= -\sqrt{2P_s}\text{Cos}2\pi f_c t$ For Logic-0

s(t)= b(t) $\sqrt{2P_s}\text{Cos}2\pi f_c t$

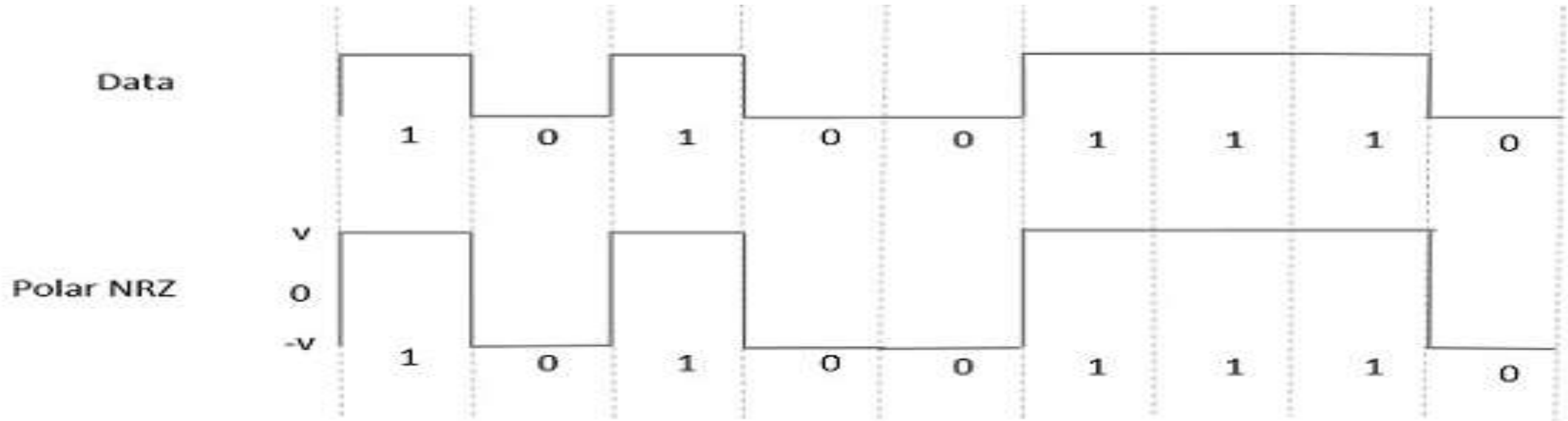
b(t) = 1 for Symbol '1'

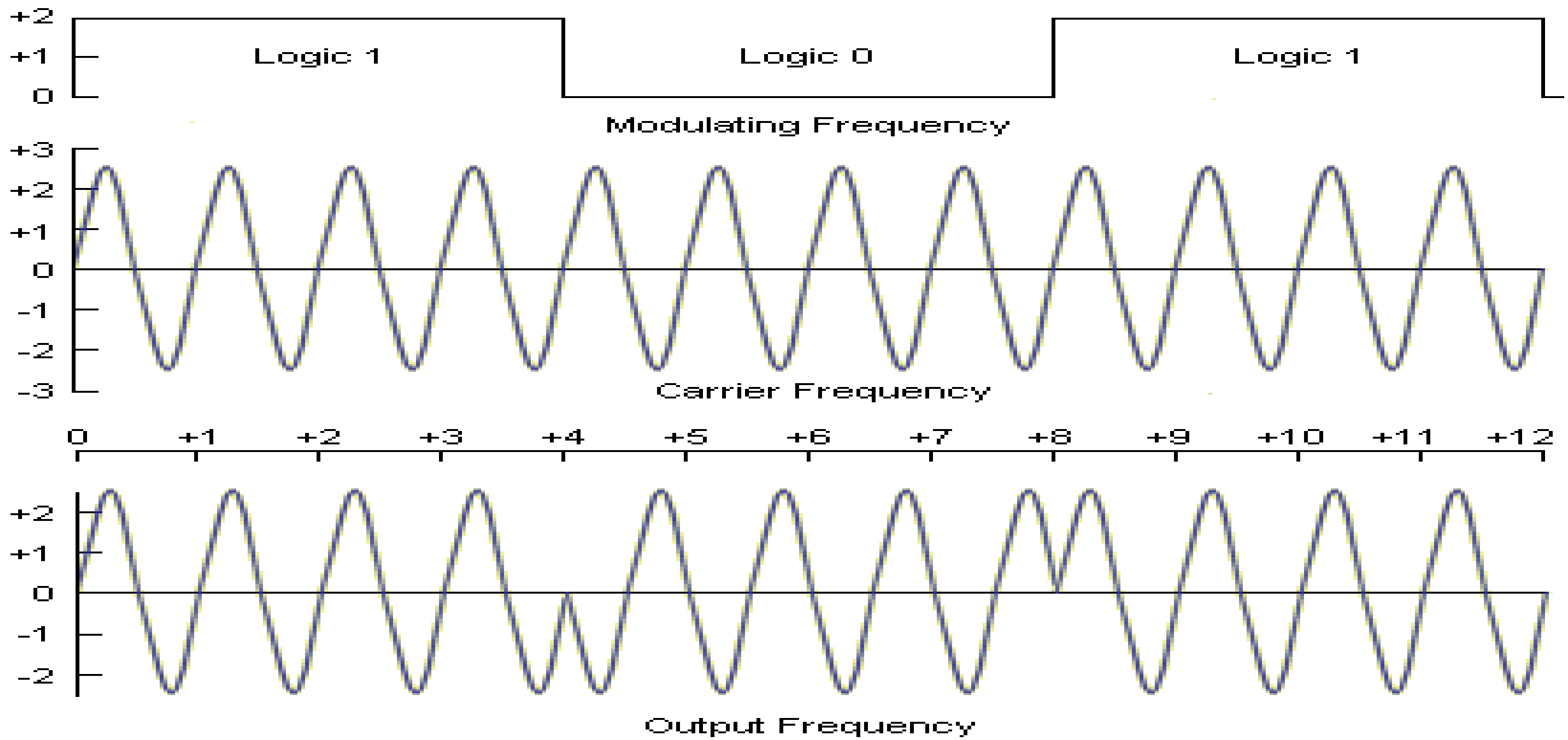
b(t) = -1 for Symbol '0'

Generation of BPSK



S.No	Input Data	Bipolar NRZ Signal b(t)	BPSK Output
1	1	+1V	$\sqrt{2P_s}Cos2\pi f_c t$
2	0	-1V	$-\sqrt{2P_s}Cos2\pi f_c t$


Data
1 0 1 0 0 1 1 1 0
Polar NRZ
v
0
-v
1 0 1 0 0 1 1 1 0


+2
+1
0
Logic 1
Logic 0
Logic 1
Modulating Frequency
+3
+2
+1
0
-1
-2
-3
Carrier Frequency
0 +1 +2 +3 +4 +5 +6 +7 +8 +9 +10 +11 +12
+2
+1
0
-1
-2
Output Frequency

s(t)= $A\text{Cos}2\pi f_c t$ where, $P_s = \left(\frac{A}{\sqrt{2}}\right)^2$

s(t) = b(t) $\sqrt{P_s T_b}\,\sqrt{\frac{2}{T_b}}\,\text{Cos}2\pi f_c t$

S_1(t)= $\sqrt{2P_s}\text{Cos}2\pi f_c t$ For Logic-1

S_2(t)= $\sqrt{2P_s}\text{Cos}(2\pi f_c t + \pi)$

$= -\sqrt{2P_s}\text{Cos}2\pi f_c t$ For Logic-0

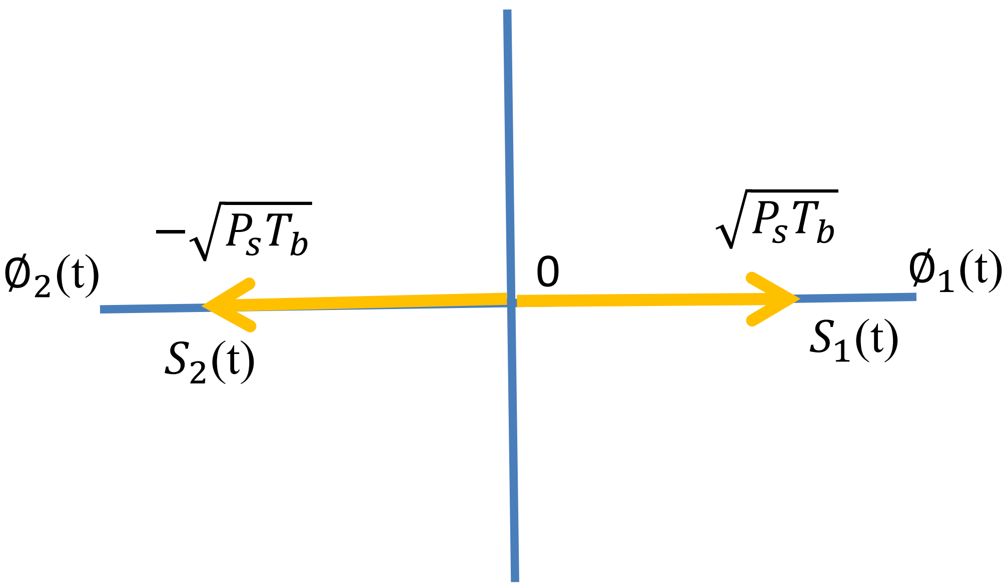
S_1(t)= $\sqrt{P_s T_b}\ \emptyset_1$(t) $\emptyset_1$(t) = $\sqrt{\frac{2}{T_b}}\,\text{Cos}2\pi f_c t$

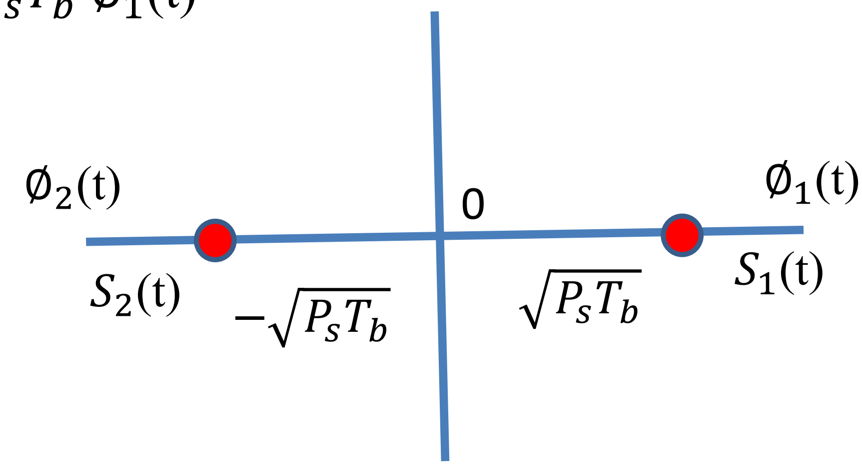
S_2(t)= $-\sqrt{P_s T_b}\,\sqrt{\frac{2}{T_b}}\,\text{Cos}2\pi f_c t$ For Logic-0

$= -\sqrt{P_s T_b}\ \emptyset_1$(t) $\emptyset_1$(t) = $\sqrt{\frac{2}{T_b}}\,\text{Cos}2\pi f_c t$

$$S_1(\mathrm{t}) = \sqrt{P_s T_b}\sqrt{\frac{2}{T_b}}\,\mathrm{Cos}2\pi f_c t \;=\sqrt{P_s T_b}\,\phi_1(\mathrm{t})$$

$$S_2(\mathrm{t}) = -\sqrt{P_s T_b}\sqrt{\frac{2}{T_b}}\,\mathrm{Cos}2\pi f_c t = \;- \sqrt{P_s T_b}\,\phi_1(\mathrm{t})$$



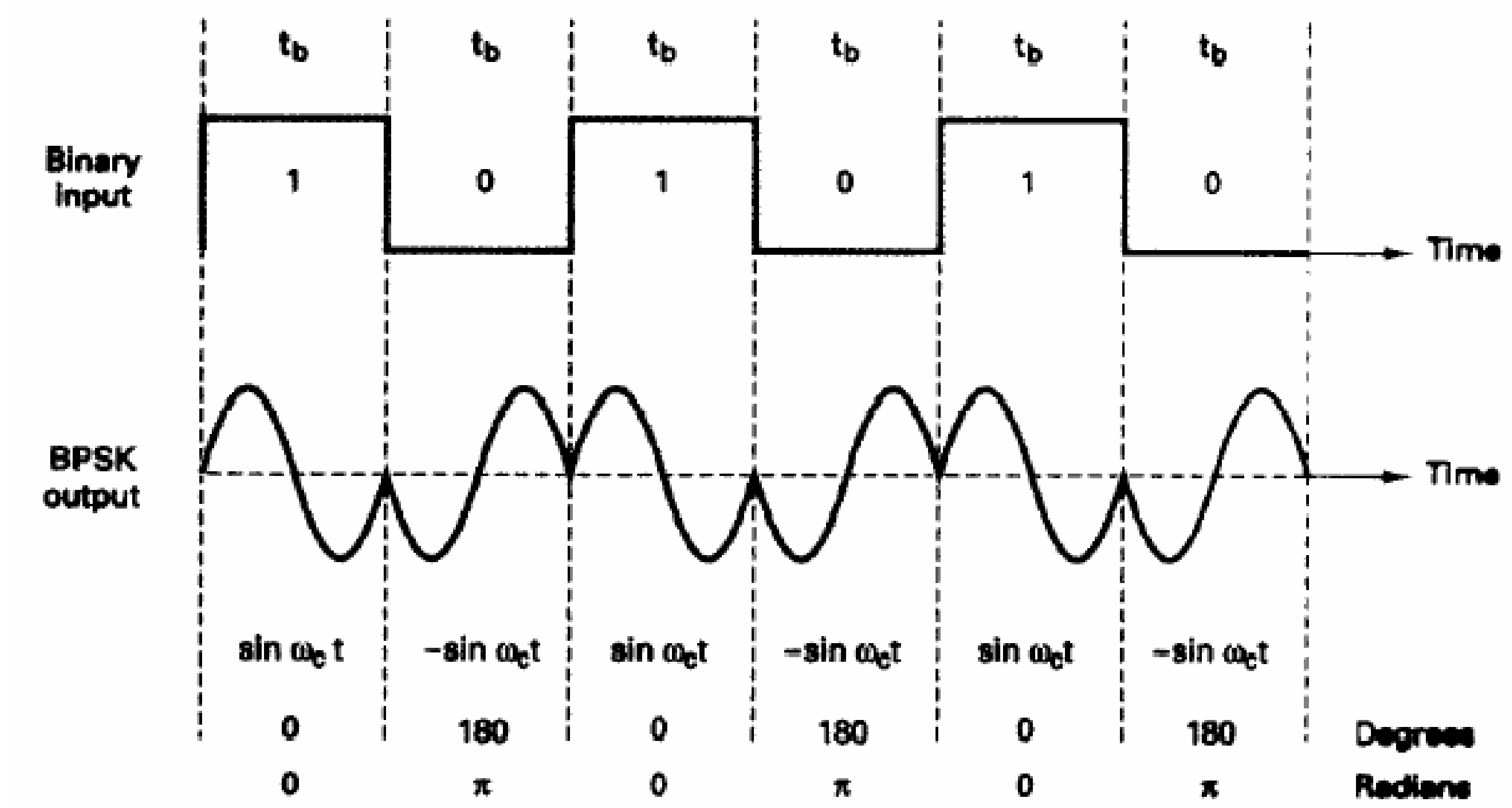


Binary Input	Output Phase
Logic-1	0 Degrees
Logic-0	180 Degrees

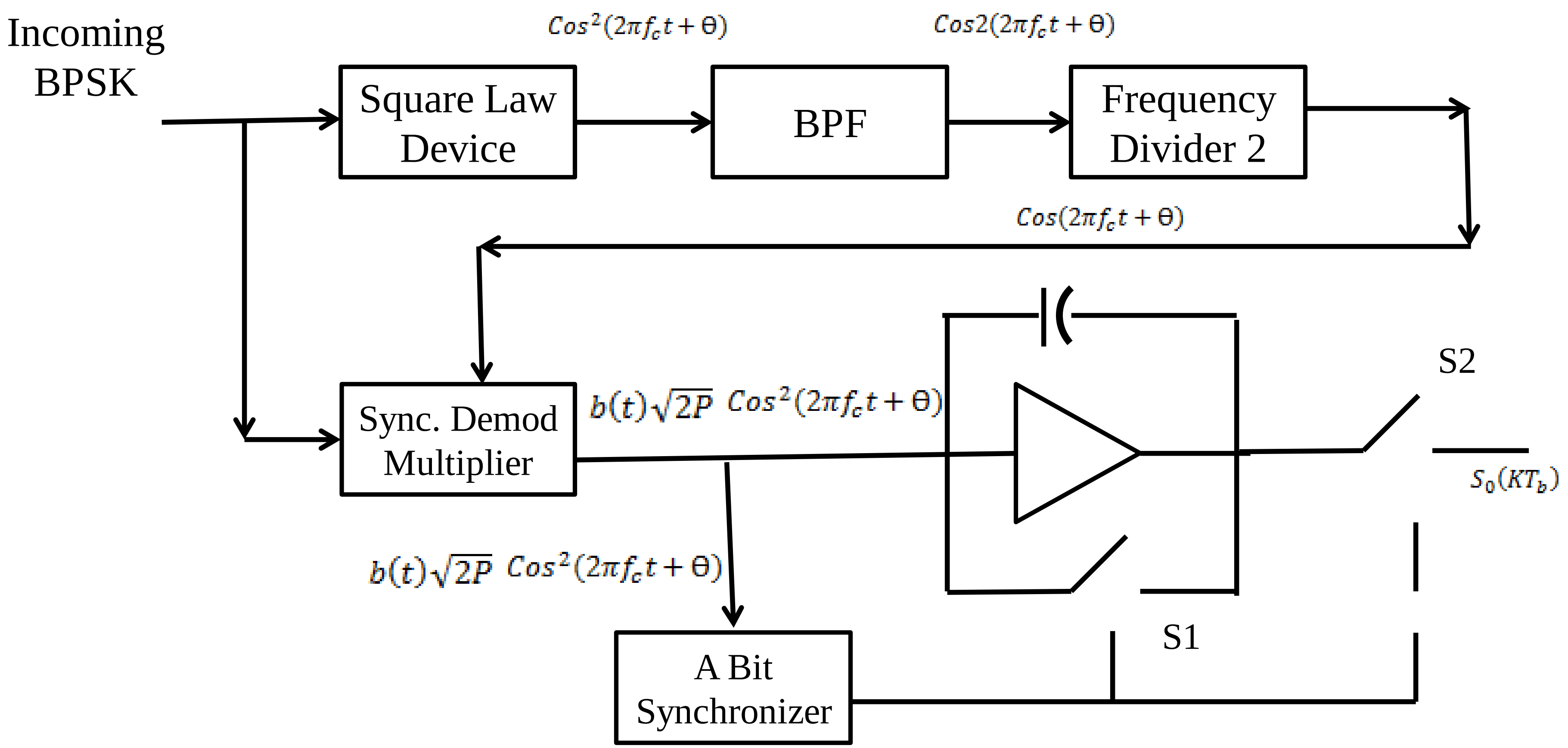
$$d = \left(\sqrt{P_s T_b}\,\right)^2 + \left(\sqrt{P_s T_b}\,\right)^2$$

$$d = 2\,P_s T_b \;= 2\,E_b$$

Output phase vs Time relationship for a BPSK modulator



Demodulation of BPSK



Demodulation of BPSK

$$S_{BPSK}(t) = b(t)\sqrt{2P}Cos(2\pi f_c t + \Theta)$$

In Coherent Detection carrier is separated at the receiver, by passing through square law detector and the output is $Cos^2(2\pi f_c t + \Theta)$ we are neglecting the Amplitude for the ease of calculations, and we are interested in carrier of the signal.

$$Cos^2(2\pi f_c t + \Theta) = \frac{1 + Cos2(2\pi f_c t + \Theta)}{2} = \frac{1}{2} + \frac{1}{2}\ Cos2(2\pi f_c t + \Theta)$$

Here, $\frac{1}{2}$ is a DC Level and the above equation is passed through BPF centered at $2f_c$ and it removes DC also which gives the output

$$Cos2(2\pi f_c t + \Theta)$$

The Above Equation is passed through Frequency divider by 2 and the output is

$$Cos(2\pi f_c t + \Theta)$$

The Synchronous demodulator multiplies the incoming signal with Frequency divider output as

$$b(t)\sqrt{2P}Cos(2\pi f_c t + \Theta)* Cos2(2\pi f_c t + \Theta)=b(t)\sqrt{2P}Cos^2(2\pi f_c t + \Theta)$$

$$= b(t)\sqrt{2P}\left(\frac{1}{2} + \frac{1}{2}\ Cos2(2\pi f_c t + \Theta)\right)$$

$$= b(t)\ \sqrt{\frac{P_s}{2}}\,(1 + Cos2(2\pi f_c t + \Theta))$$

Kth Bit Interval, we can write the Output signal as,

$$S_0(KT_b) = b(KT_b)\ \sqrt{\frac{P_s}{2}}\int_{(K-1)T_b}^{KT_b}(1 + Cos2(2\pi f_c t + \Theta))dt$$

Here the Integration is performed over the interval T_b (one bit period). We can write the above equation as

$$S_0(KT_b) = b(KT_b)\sqrt{\frac{P_s}{2}}\left[\int_{(K-1)T_b}^{KT_b} 1\,dt \;+\; \int_{(K-1)T_b}^{KT_b} (Cos2(2\pi f_c t + \Theta))dt\right]$$

$\int_{(K-1)T_b}^{KT_b} (Cos2(2\pi f_c t + \Theta))dt = 0$ it is the average value of Sine/Cos, and its value is zero and we get,

$$S_0(KT_b) = b(KT_b)\sqrt{\frac{P_s}{2}}\left[\int_{(K-1)T_b}^{KT_b} 1\,dt\right]$$

$$= b(KT_b)\sqrt{\frac{P_s}{2}}\,[KT_b - (K-1)T_b\,]$$

$$= b(KT_b)\sqrt{\frac{P_s}{2}}\;T_b$$

$$\boldsymbol{S_0(KT_b)\ \alpha\ b(KT_b)}$$

Spectrum of BPSK

We Know b (t) is a NRZ binary waveform. In this waveform there are rectangular pulses of $\pm V_b$. It we assume each pulse is centered about $\pm \frac{T_b}{2}$. The Fourier Transform of that pulse is

$$X(f) = V_b T_b \left(\frac{Sin(\pi f T_b)}{\pi f T_b}\right)$$

For large number of Positive and Negative pulses are expressed as

$S(f) = \frac{\overline{|X(f)|}^{2}}{T_s}$ where $|X(f)|$ is the average value of X (f) due to all pulses in b (t). T_s is the symbol duration which is equal to T_b

$$S(f) = \frac{{V_b}^2 {T_b}^2}{T_s} \left(\frac{Sin(\pi f T_b)}{\pi f T_b}\right)^2 \quad \text{Since } T_s = T_b$$

$$S(f) = {V_b}^2 T_b \left(\frac{Sin(\pi f T_b)}{\pi f T_b}\right)^2$$

This gives the Power spectral density of Baseband signal b (t). The signal is then modulated by the carrier and the above equation becomes

$$S_{BPSK}(t) = \frac{V_b{}^2 T_b}{2}\left[\left\{\left(\frac{Sin(\pi(f-f_c)T_b)}{\pi(f-f_b)T_b}\right)^2\right\} + \left\{\left(\frac{Sin(\pi(f+f_c)T_b)}{\pi(f-f_b)T_b}\right)^2\right\}\right]$$

The Above equation gives the Power spectral density of the BPSK modulated signal having Amplitude equal to $\pm P$

$$S_{BPSK}(t) = \frac{PT_b}{2}\left[\left\{\left(\frac{Sin(\pi(f-f_c)T_b)}{\pi(f-f_b)T_b}\right)^2\right\} + \left\{\left(\frac{Sin(\pi(f+f_c)T_b)}{\pi(f-f_b)T_b}\right)^2\right\}\right]$$

Quaternary Phase-Shift Keying (QPSK)

M- ary Encoding

$N = \log_2 M$

M = Number of Conditions, Levels, or Combinations Possible with N Bits.

N = Number of bits necessary.

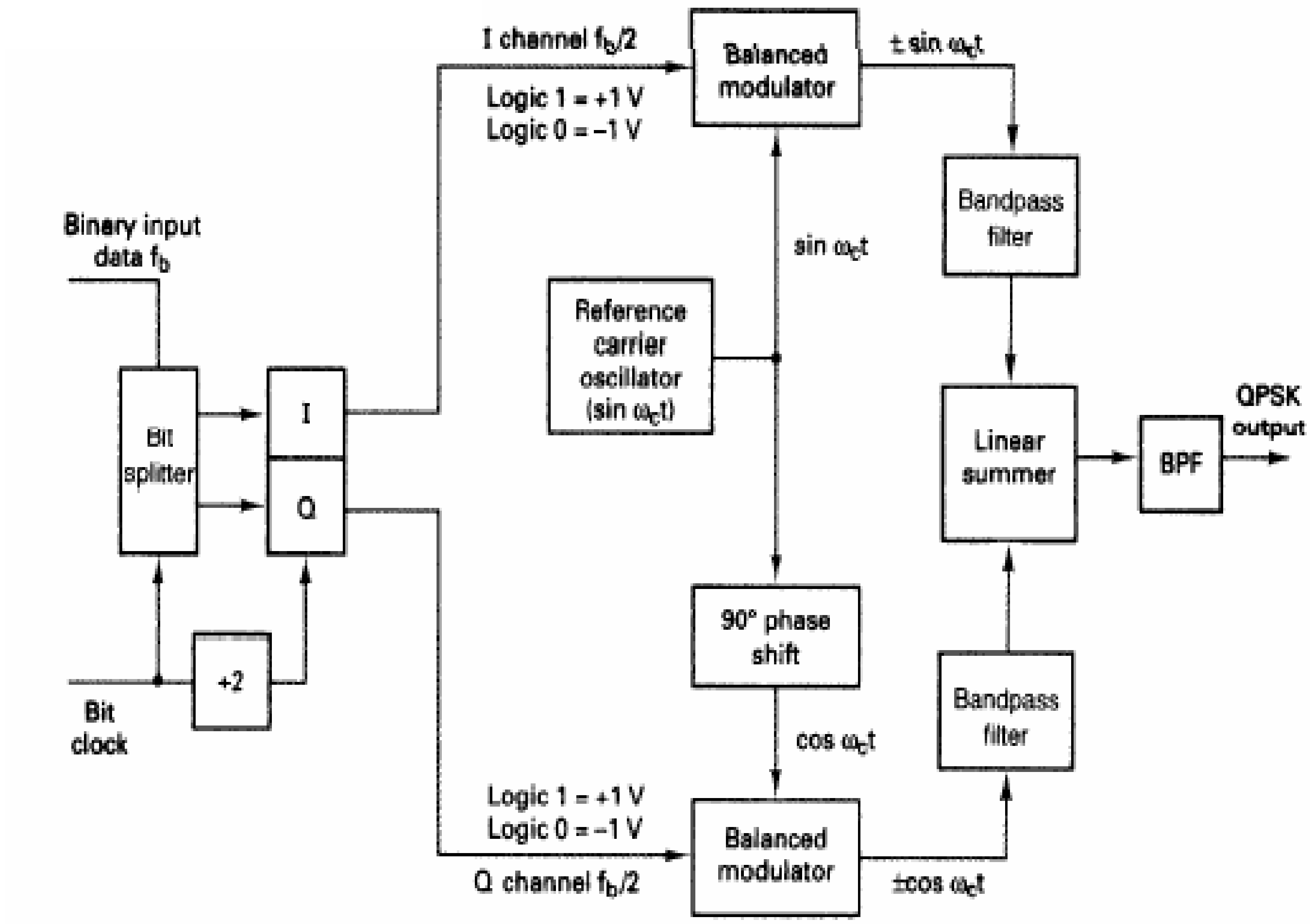
$M = 2^N$

$BW = f_b / \log 2 M$

Quaternary Phase-Shift Keying(QPSK)

- **QPSK is an M-ary encoding scheme where N = *2 and M= 4***
- **Therefore, with QPSK, the binary input data are combined into groups of two bits, called *dibits*.**
- **Each dibit code Generates one of the four possible output phases (+45°, +135°, -45°, and -135°).**

QPSK transmitter.



- **For a logic 1 = + 1 and a logic 0= - 1 , two phases are possible at the output of the I modulator ($+\sin w_c t$ and $-\sin w_c t$)**
- **Similarly two phases are possible at the output of the Q balanced modulator ($+\cos w_c t$), and ($-\cos w_c t$).**
- **For input 11 , Q=I=1, the two inputs to the I balanced modulator are +1 and $\sin w_c t$, and The two inputs to the Q balanced modulator are +1 and $\cos w_c t$.**
- **Outputs are**

  (a)I balanced modulator $=(+1)(\sin w_c t) = +1 \sin w_c t$

  (b)Q balanced modulator $=(+1)(\cos w_c t) = +1 \cos w_c t$

Output of the linear summer for 1,1 input dibits is

$$= sin\, w_c t + \cos w_c t$$

$$= \left\{ sin\; w_c\, t + \; sin(90_{\,0} + w_c t) \right\}$$

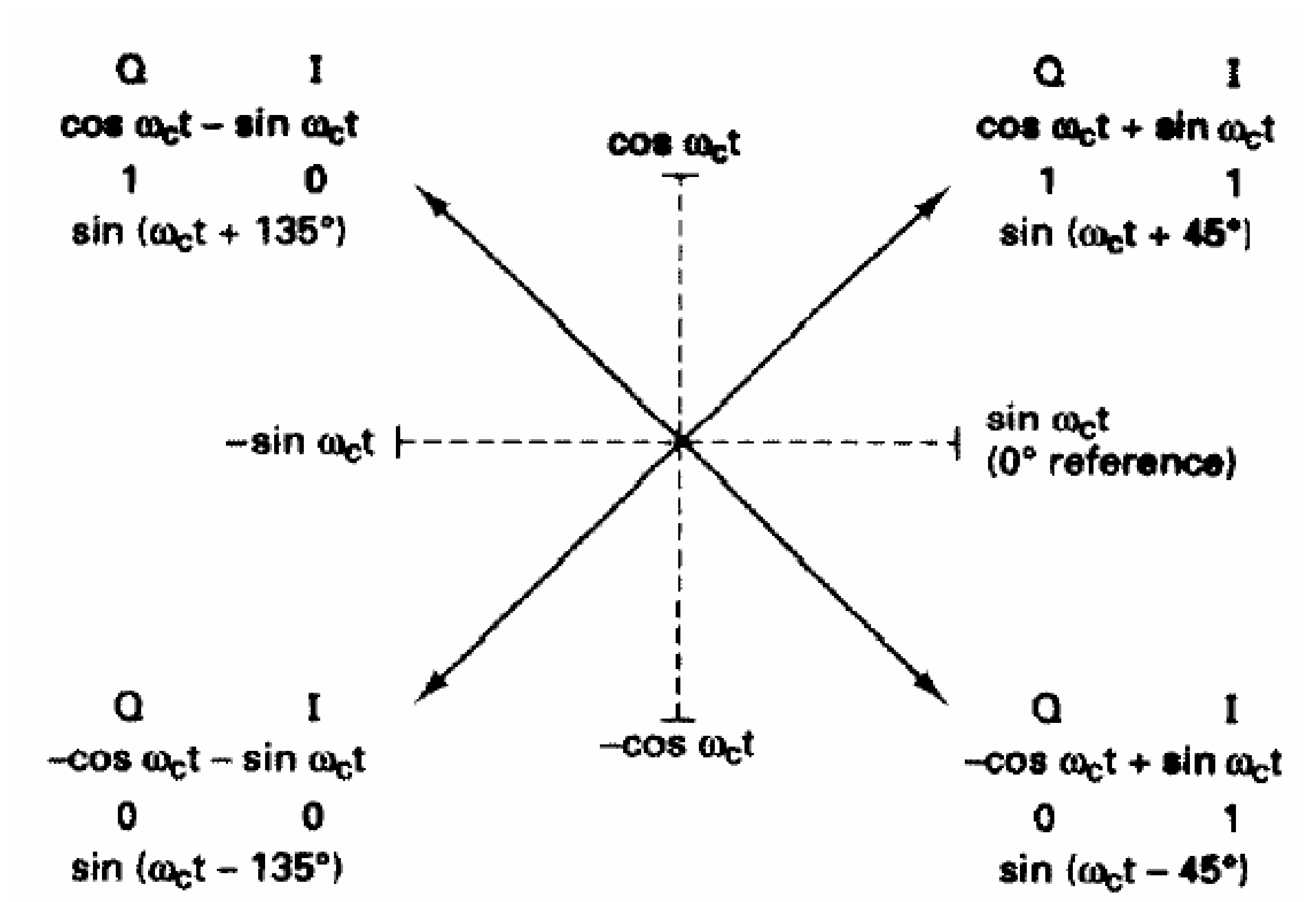
$$= 1.414 \sin(w\; t + 45^o\}$$

(Similarly it can be calculated for all other input combinations)

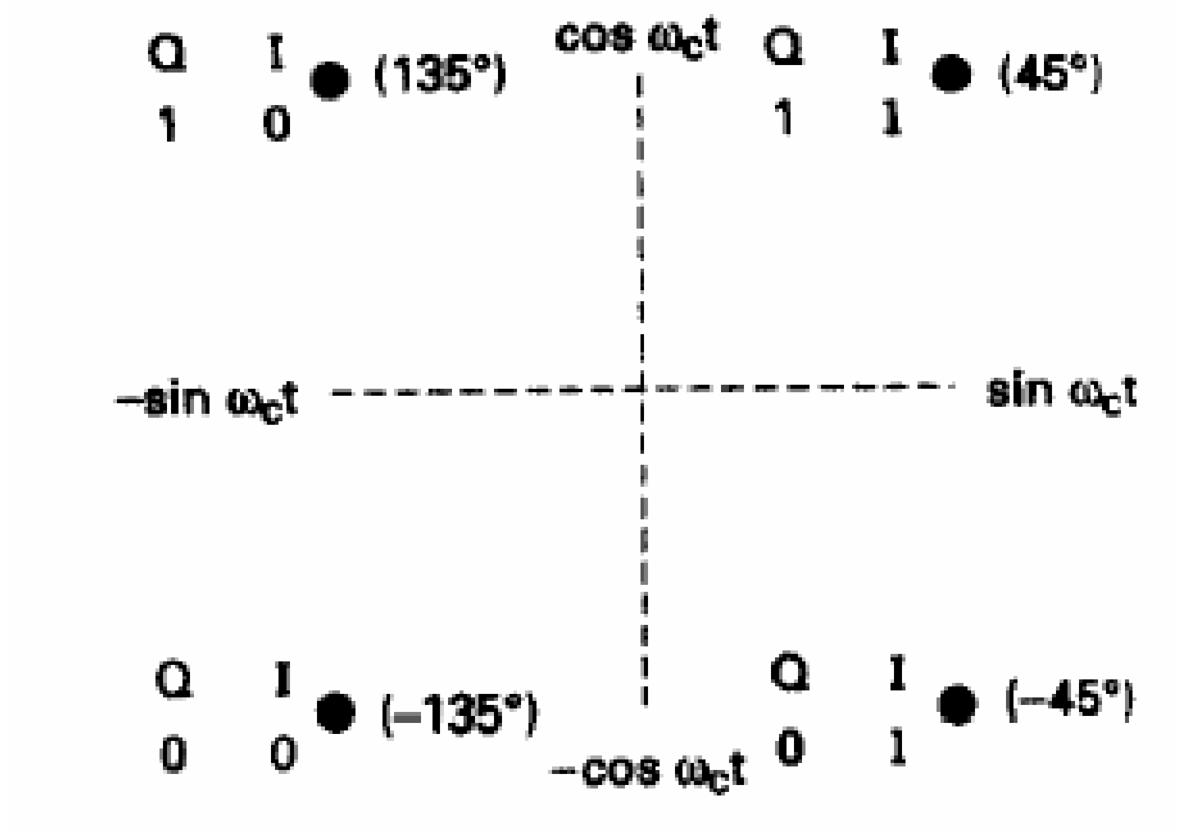
Truth table

Binary input		QPSK output phase
Q	I	
0	0	−135°
0	1	−45°
1	0	+135°
1	1	+45°

Phasor diagram



Constellation diagram



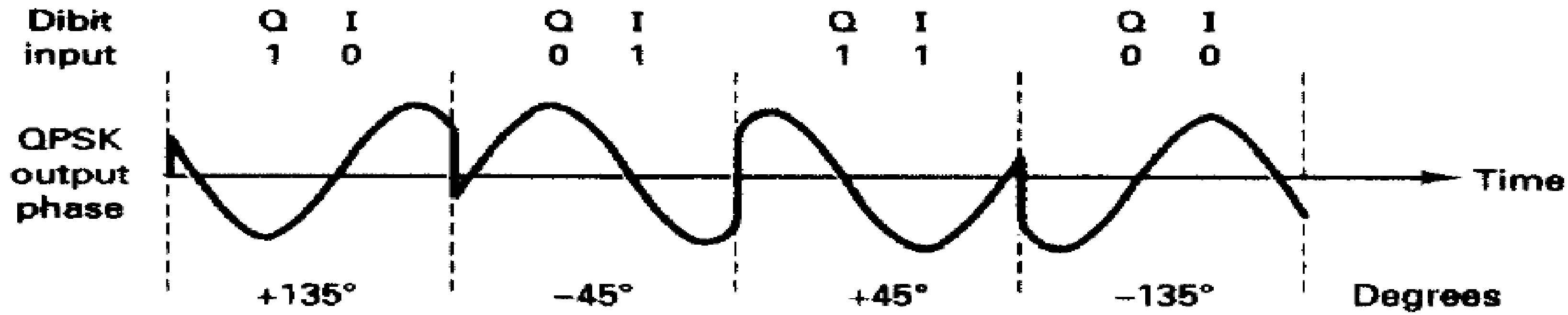
(How to remember: For 1 +ve will go and for 0 –ve will go)

I is for sine and Q for cosine

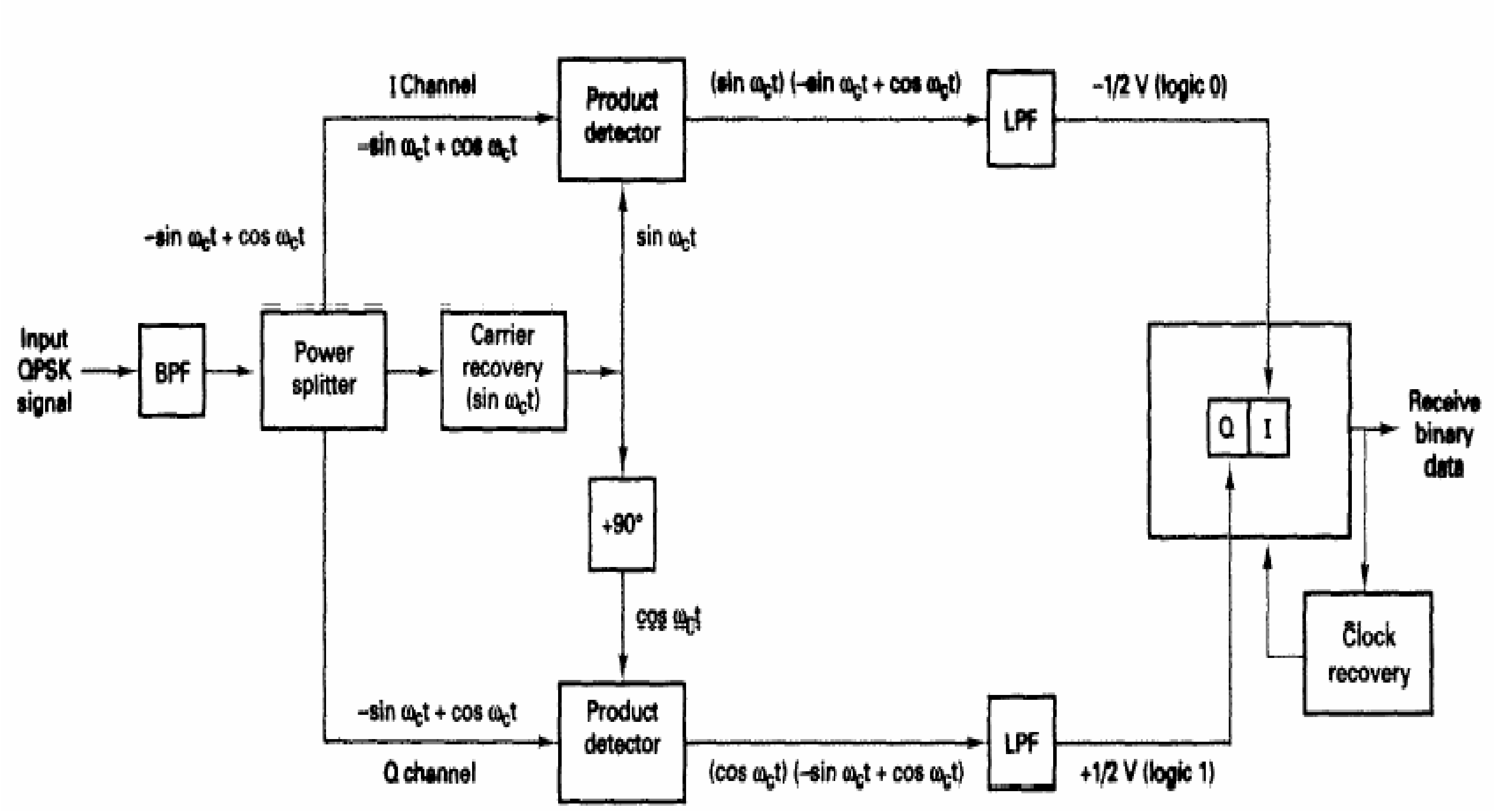
Salient features

- **Each of the four possible output phasors has exactly the same amplitude. Therefore, the binary information must be encoded entirely in the phase of the output signal.**
- **The angular separation between any two adjacent phasors in QPSK is 90 Degrees .**
- **Thus, a QPSK signal can undergo almost a +45° or -45° shift in phase during transmission and still retain the correct encoded information when demodulated at the receiver.**

Output Phase Vs Time relationship for a QPSK



QPSK receiver



Output of the I product detector

(For input I=0 and Q=1)

$$I = \underbrace{(-\sin \omega_c t + \cos \omega_c t)}_{\text{QPSK input signal}} \underbrace{(\sin \omega_c t)}_{\text{carrier}}$$

$$= (-\sin \omega_c t)(\sin \omega_c t) + (\cos \omega_c t)(\sin \omega_c t)$$

$$= -\sin^2 \omega_c t + (\cos \omega_c t)(\sin \omega_c t)$$

$$= -\frac{1}{2}(1 - \cos 2\omega_c t) + \frac{1}{2}\sin(\omega_c + \omega_c)t + \frac{1}{2}\sin(\omega_c - \omega_c)t$$

$$I = -\frac{1}{2} + \underbrace{\frac{1}{2}\cos 2\omega_c t}_{\text{(filtered out)}} + \underbrace{\frac{1}{2}\sin 2\omega_c t}_{\text{(filtered out)}} + \underbrace{\frac{1}{2}\sin 0}_{\text{(equals 0)}}$$

$$= -\frac{1}{2}\text{V (logic 0)}$$

Output of the Q product detector

$$Q = \underbrace{(-\sin \omega_c t + \cos \omega_c t)}_{\text{QPSK input signal}}\underbrace{(\cos \omega_c t)}_{\text{carrier}}$$

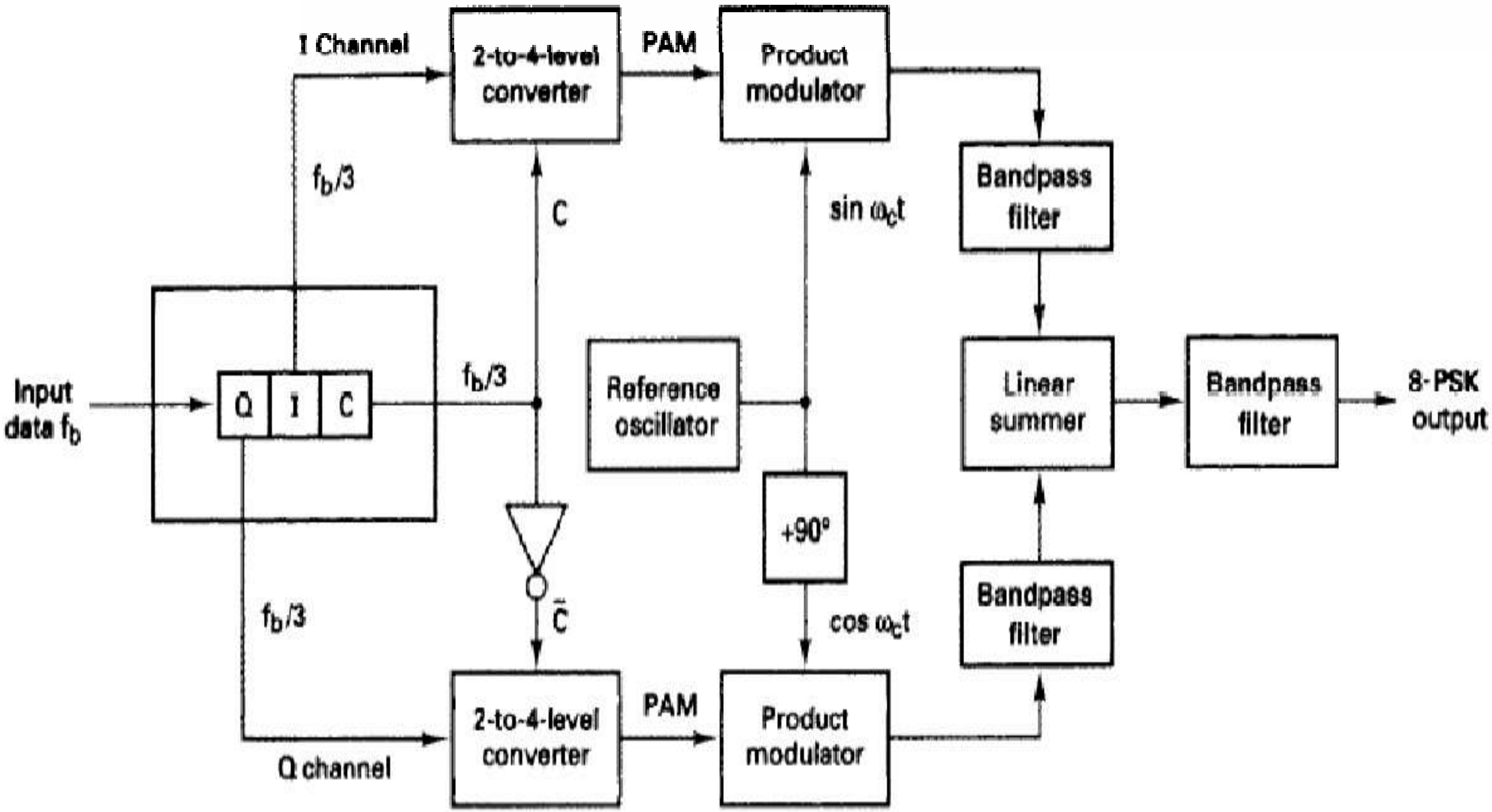
$$= \cos^2 \omega_c t - (\sin \omega_c t)(\cos \omega_c t)$$

$$= \frac{1}{2}(1 + \cos 2\omega_c t) - \frac{1}{2}\sin(\omega_c + \omega_c)t - \frac{1}{2}\sin(\omega_c - \omega_c)t$$

$$Q = \frac{1}{2} + \overset{\text{(filtered out)}}{\frac{1}{2}\cos 2\omega_c t} - \overset{\text{(equals 0)}}{\frac{1}{2}\sin 2\omega_c t} - \frac{1}{2}\sin 0$$

$$= \frac{1}{2}\text{V(logic 1)}$$

8-PSK transmitter



<u>How to find angles</u> For 111 input

Output= 1.307sinwt+0.541 coswt

Angle= tan^-1(.541/1.307) in 1st quadrangle

=67.5°

Truth Table

I	C	Output
0	0	−0.541 V
0	1	−1.307 V
1	0	+0.541 V
1	1	+1.307 V

Q	$\bar{C}$	Output
0	1	−1.307 V
0	0	−0.541 V
1	1	+1.307 V
1	0	+0.541 V

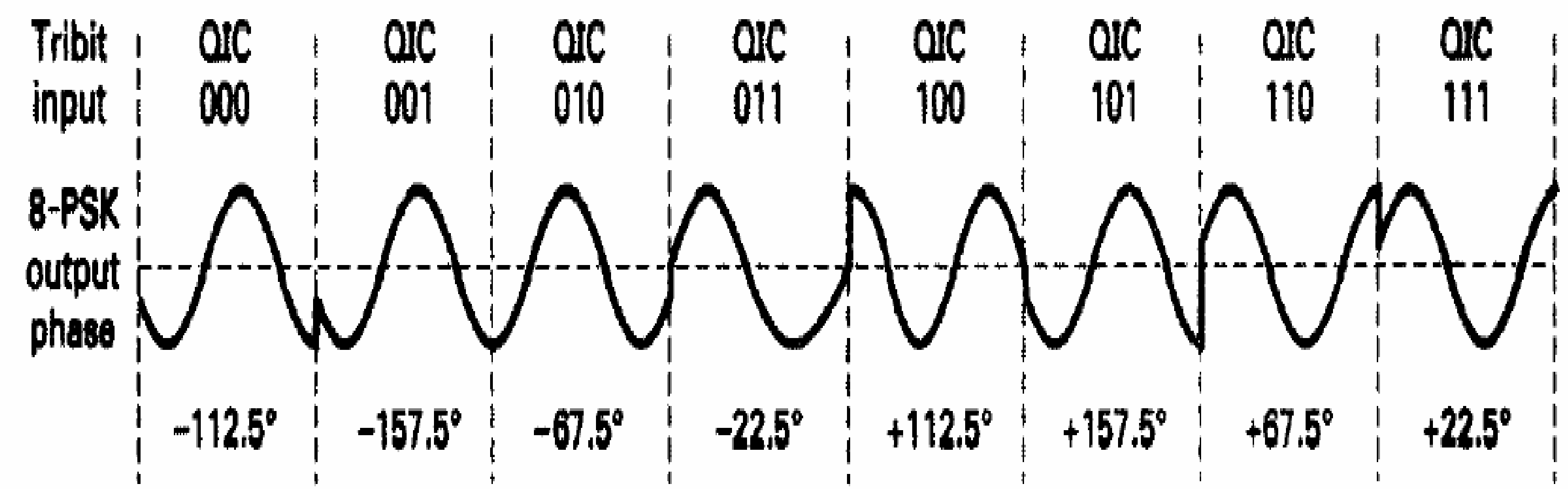
Output Phases

Note

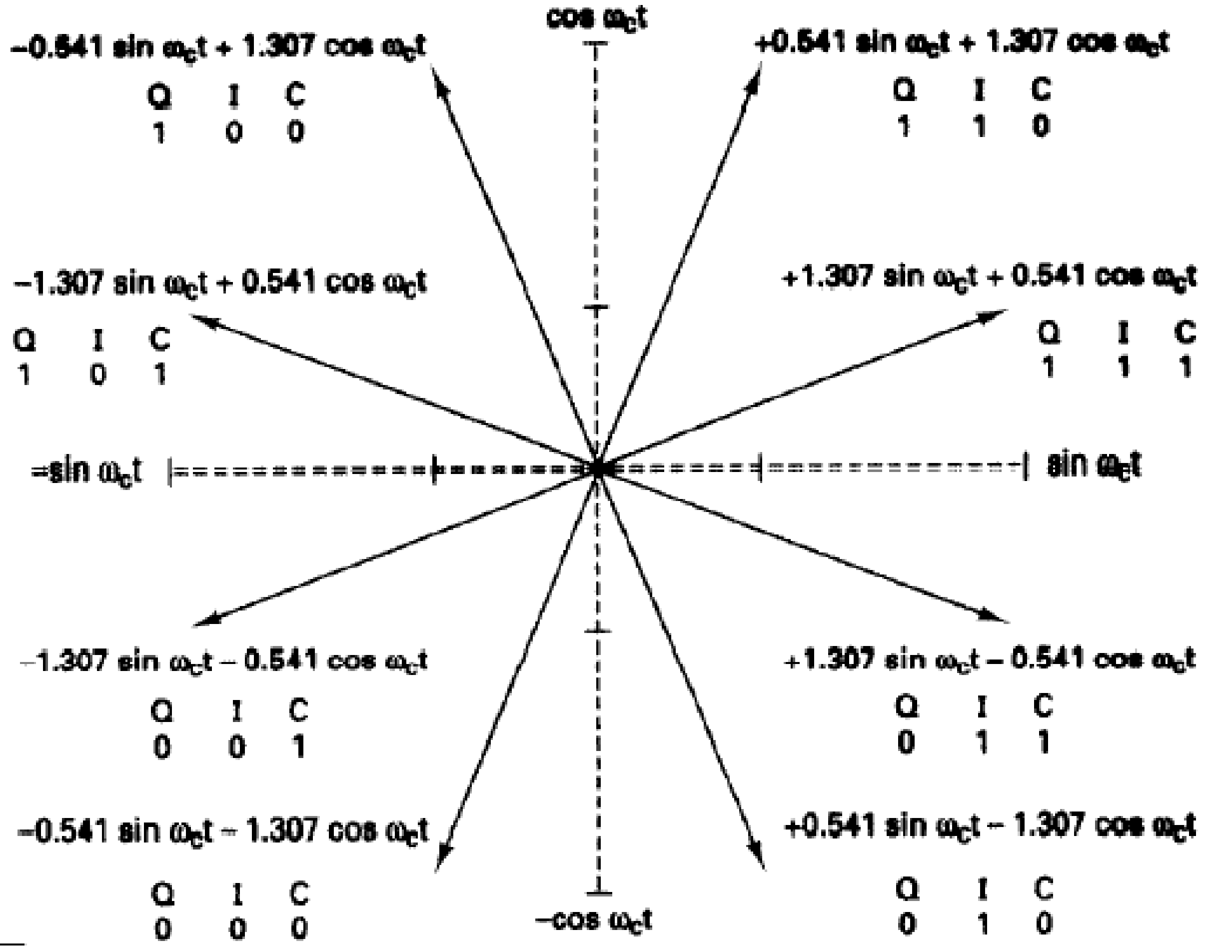
Phases are +/-(22.5°+45°)

Binary input			8-PSK output phase
Q	I	C	
0	0	0	−112.5°
0	0	1	−157.5°
0	1	0	−67.5°
0	1	1	−22.5°
1	0	0	+112.5°
1	0	1	+157.5°
1	1	0	+67.5°
1	1	1	+22.5°

Output phase-versus-time relationship for an 8-PSK modulator

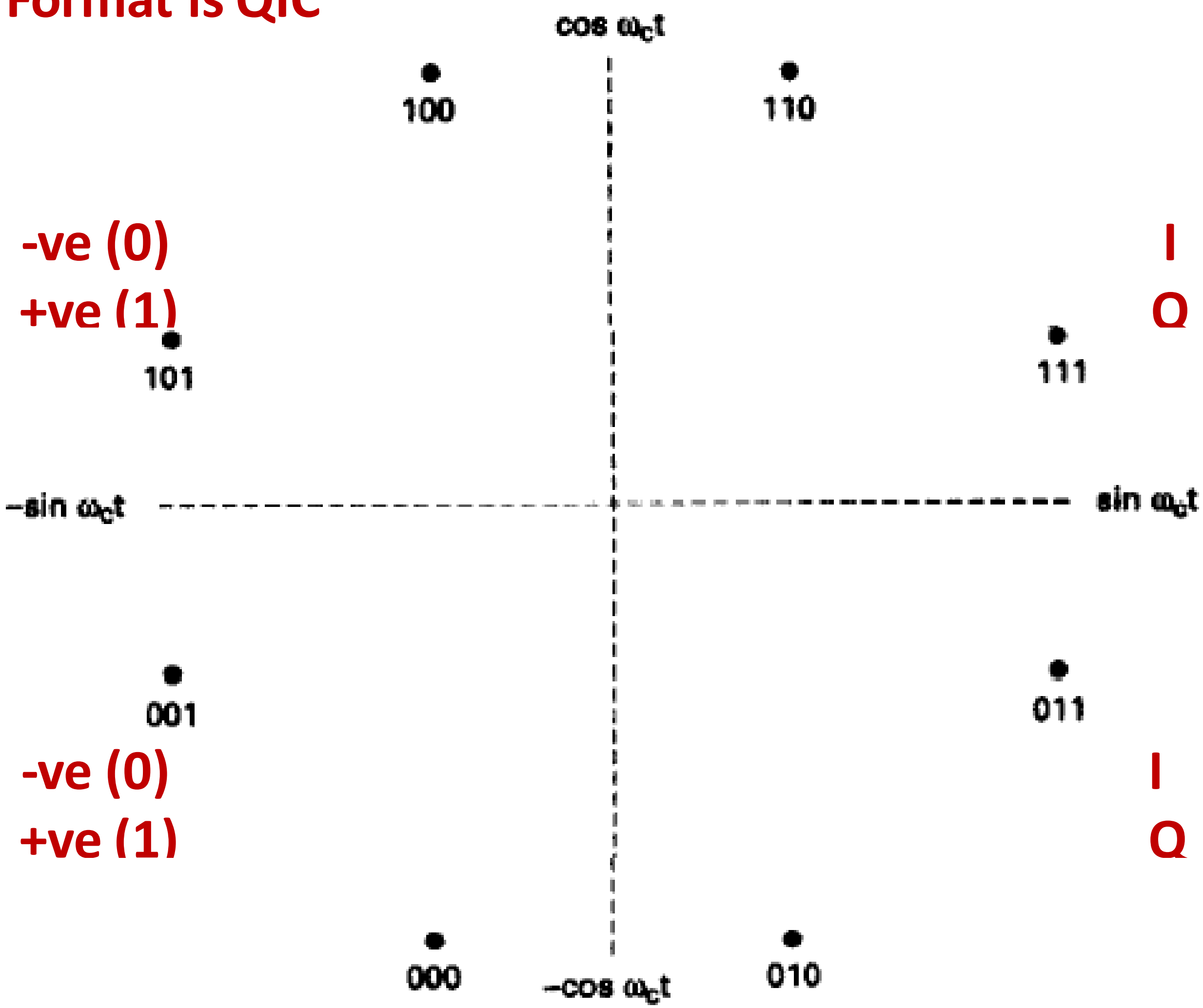


Phasor Diagram



Constellation Diagram: 8 PSK

Note: Format is QIC



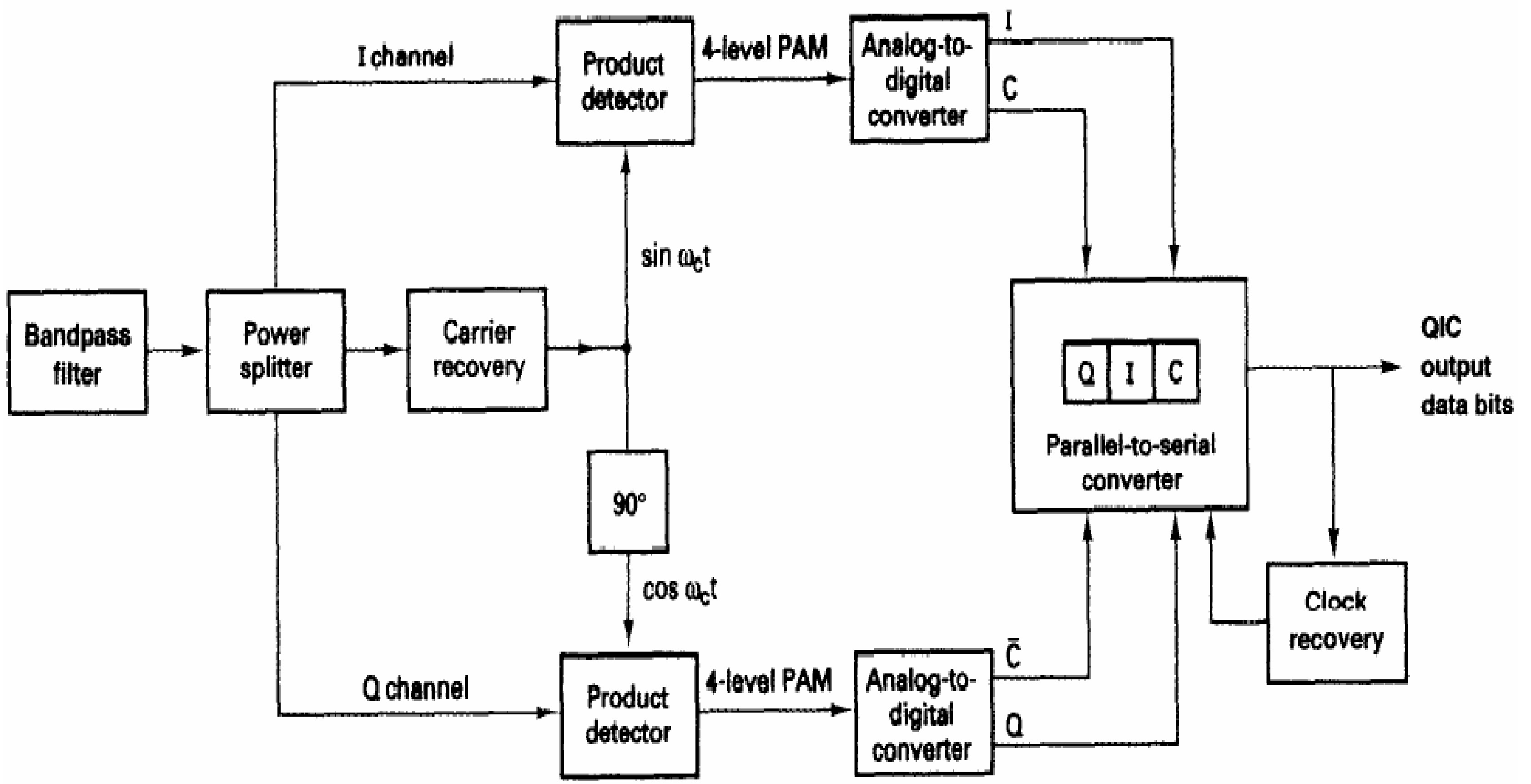
I -ve (0)
Q +ve (1)

I +ve (1)
Q +ve (1)

I -ve (0)
Q +ve (1)

I +ve
Q -ve

8-PSK receiver.

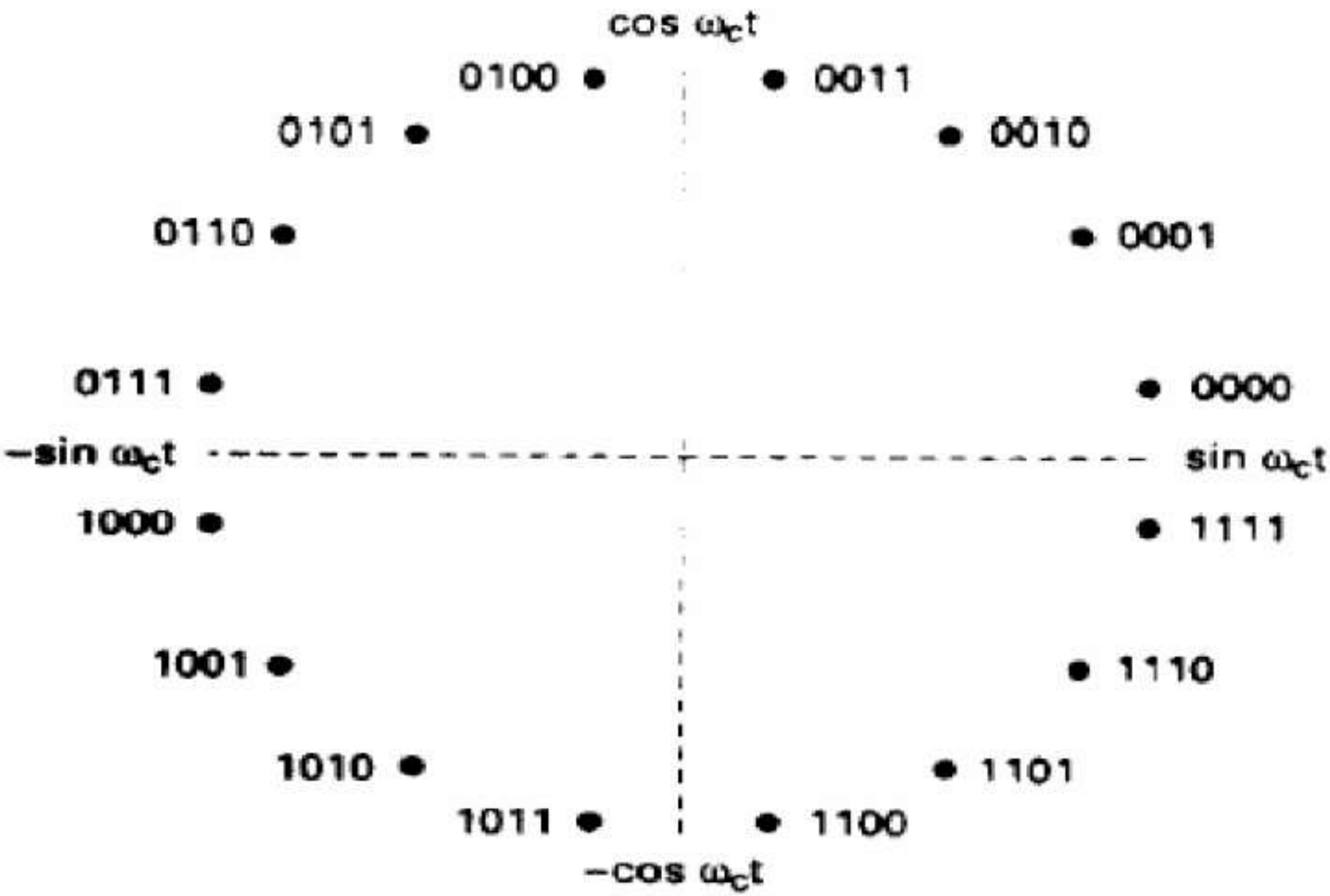


16-PSK

Truth table

Bit code	Phase	Bit code	Phase
0000	11.25°	1000	191.25°
0001	33.75°	1001	213.75°
0010	56.25°	1010	236.25°
0011	78.75°	1011	258.75°
0100	101.25°	1100	281.25°
0101	123.75°	1101	303.75°
0110	146.25°	1110	326.25°
0111	168.75°	1111	348.75°

Constellation diagram



16 PSK Performance

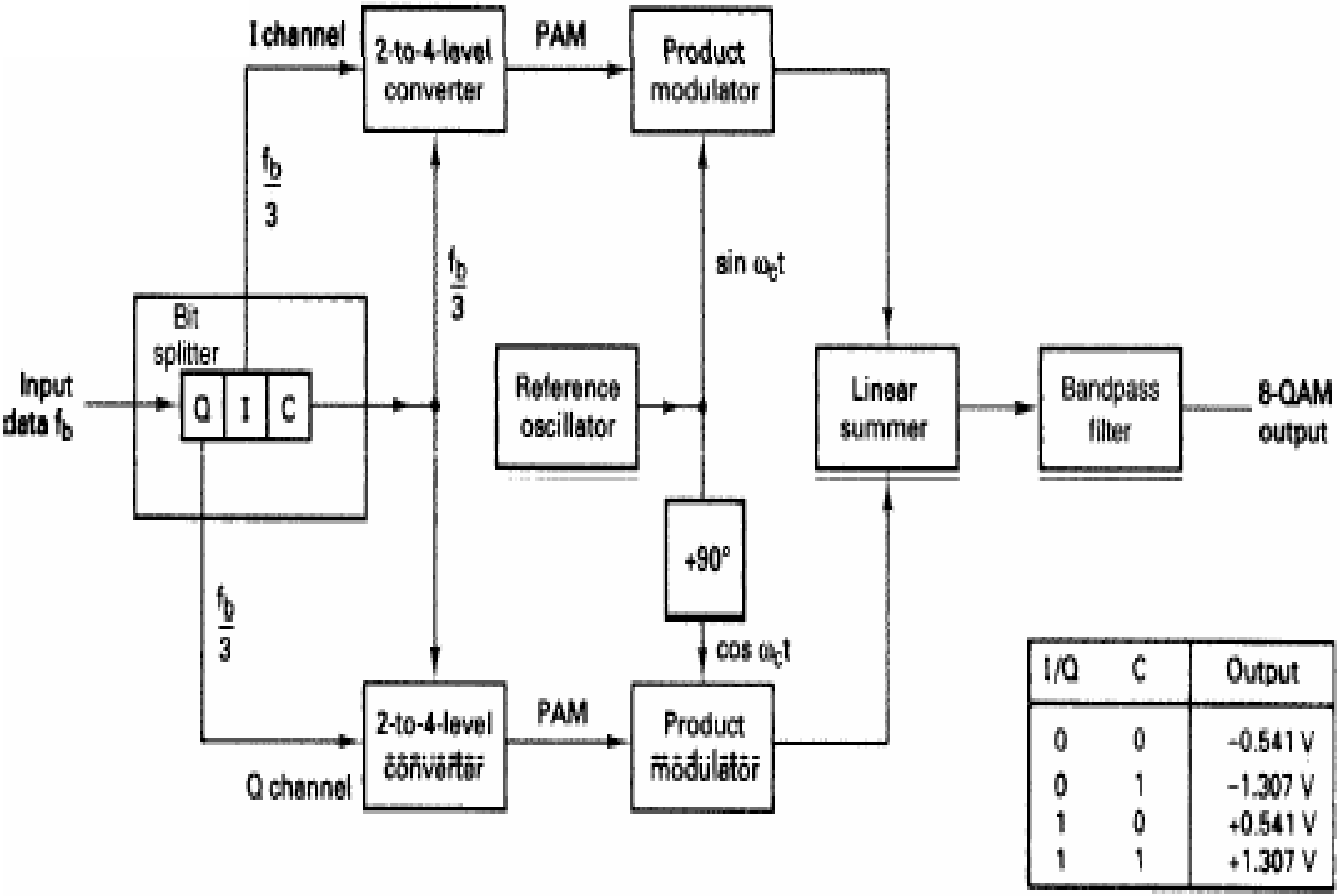
- With 16-PSK, the angular separation between adjacent output phases is only 22.5° (360 / 16).

- Therefore, 16-PSK can undergo only a 11.25° phase shift during transmission and still retain its integrity.

QUADRATURE – AMPLITUDE MODULATION

- 8-QAM is an M-ary encoding technique where M = 8.
- Unlike 8-PSK, the output signal from an 8- QAM modulator is not a constant-amplitude signal.

8-QAM modulator



I/Q	C	Output
0	0	−0.541 V
0	1	−1.307 V
1	0	+0.541 V
1	1	+1.307 V

8-QAM modulator

- **The incoming data are divided into groups of three bits (tribits): the I, Q, and C bit streams.**

- **Each stream has a bit rate equal to one-third of the incoming data rate.**

- **The I and Q bits determine the polarity of the PAM signal at the output of the 2-to-4-level converters**

- **The C channel determines the magnitude.**

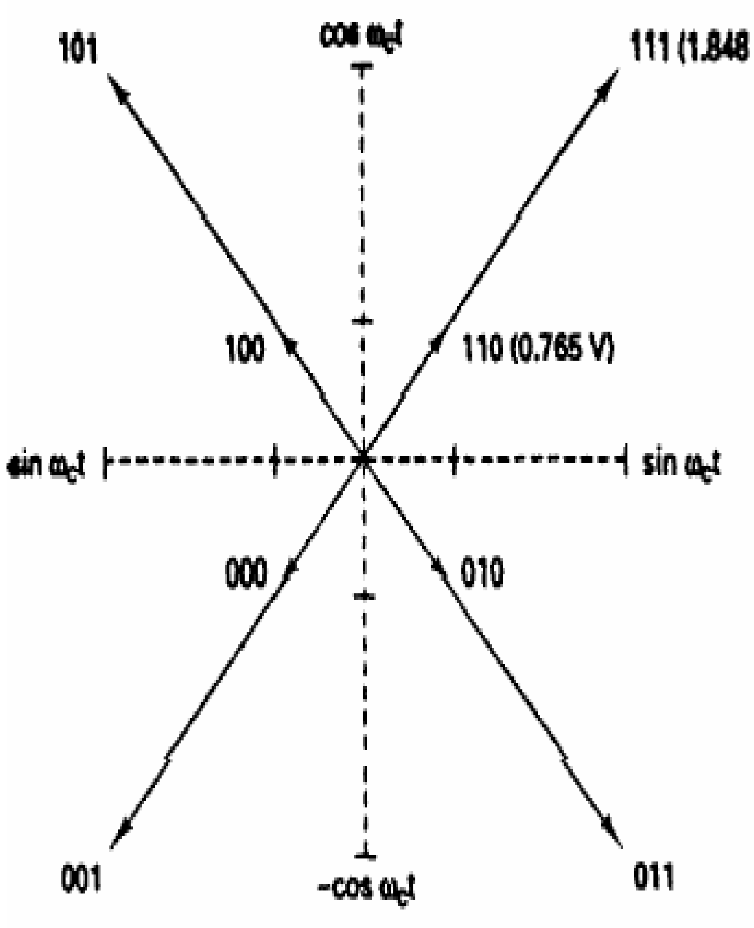
- **Because the c bit is fed un-inverted to both the i and the q channel 2-to-4-level converters, the magnitudes of the I and Q PAM signals are always equal.**

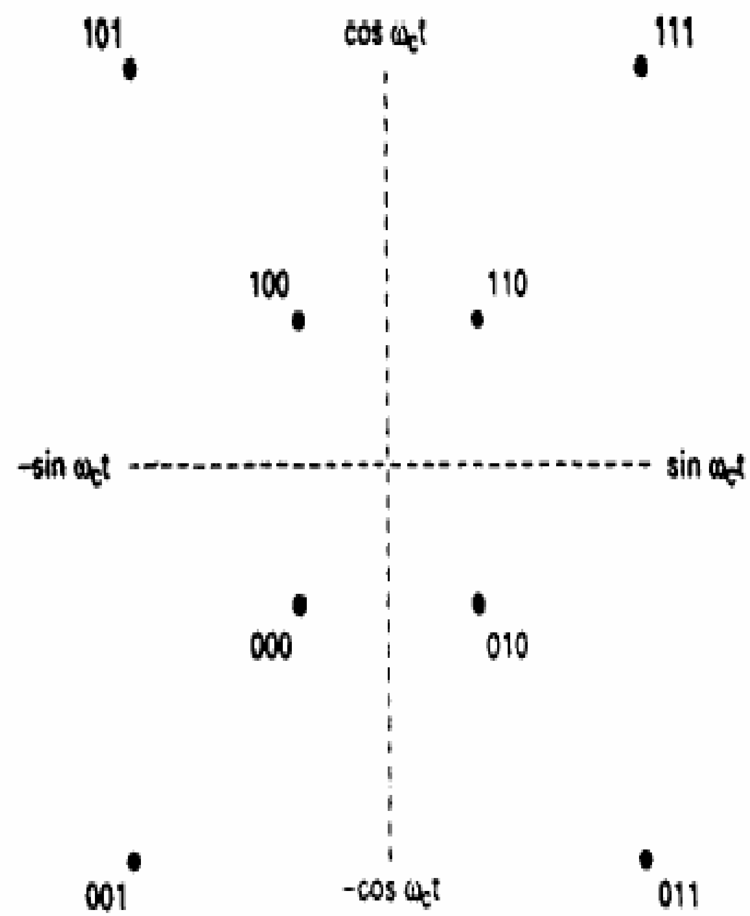
- **Their polarities depend on the logic condition of the i and q bits**

Truth Table

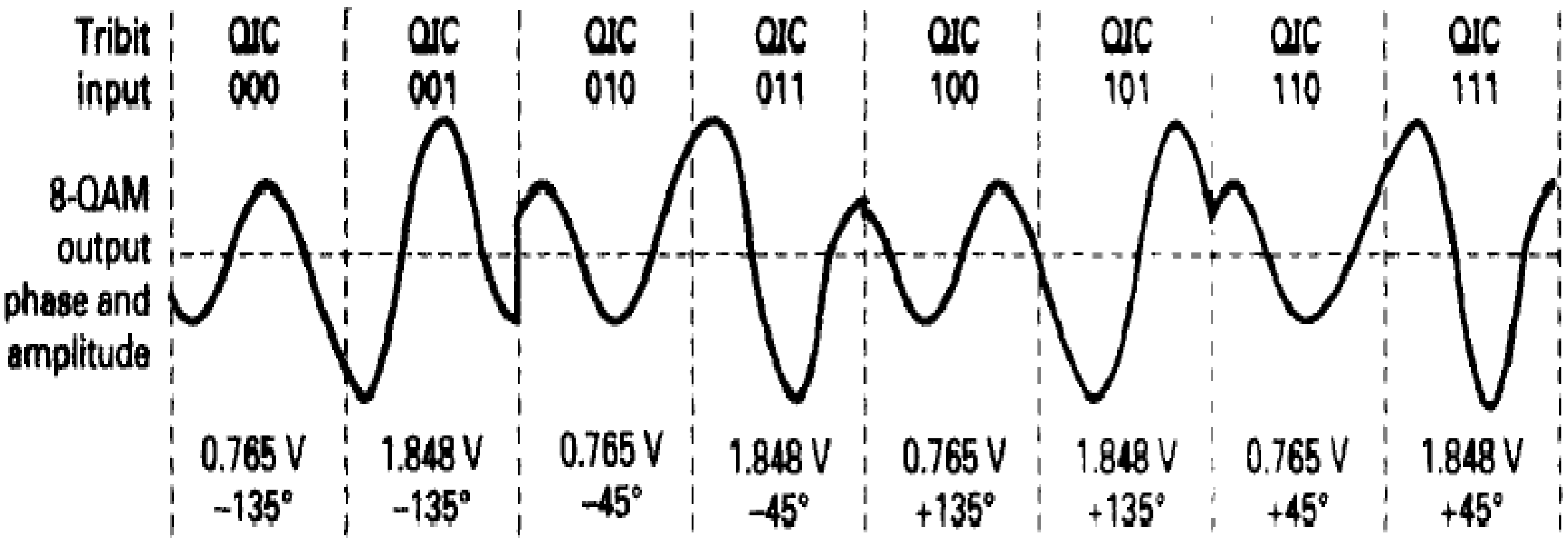
Binary input			8-QAM output	
Q	I	C	Amplitude	Phase
0	0	0	0.765 V	−135°
0	0	1	1.848 V	−135°
0	1	0	0.765 V	−45°
0	1	1	1.848 V	−45°
1	0	0	0.765 V	+135°
1	0	1	1.848 V	+135°
1	1	0	0.765 V	+45°
1	1	1	1.848 V	+45°

Phasor and constellation





Output phase and amplitude vs time relationship for 8 QAM



Bandwidth considerations of 8-QAM.

- N=3
- Thus the minimum bandwidth required for 8- QAM is $f_b / 3$, *the* same as in 8-PSK.

8-QAM receiver.

An 8-QAM receiver is almost identical to the 8-PSK receiver

16-QAM

- As with the 16-PSK, *16-QAM is an M-ary system where M* =16.
- The input data are acted on in groups of four (2^*4 = 16)*.
- *As* with 8-QAM,both the phase and the amplitude of the transmit carrier are varied.

QAM transmitter

- The input binary data are divided into four channels: I, I', Q, and Q'.
- The bit rate in each channel is equal to one-fourth of the input bit rate (fb/4).
- The I and Q bits determine the polarity at the output of the 2-to-4-level converters
    - logic 1 = positive
    - logic 0 =negative
- The I' and Q' bits determine the magnitude
    - logic 1 = 0.821V
    - logic 0 = 0.22 V

Truth Table

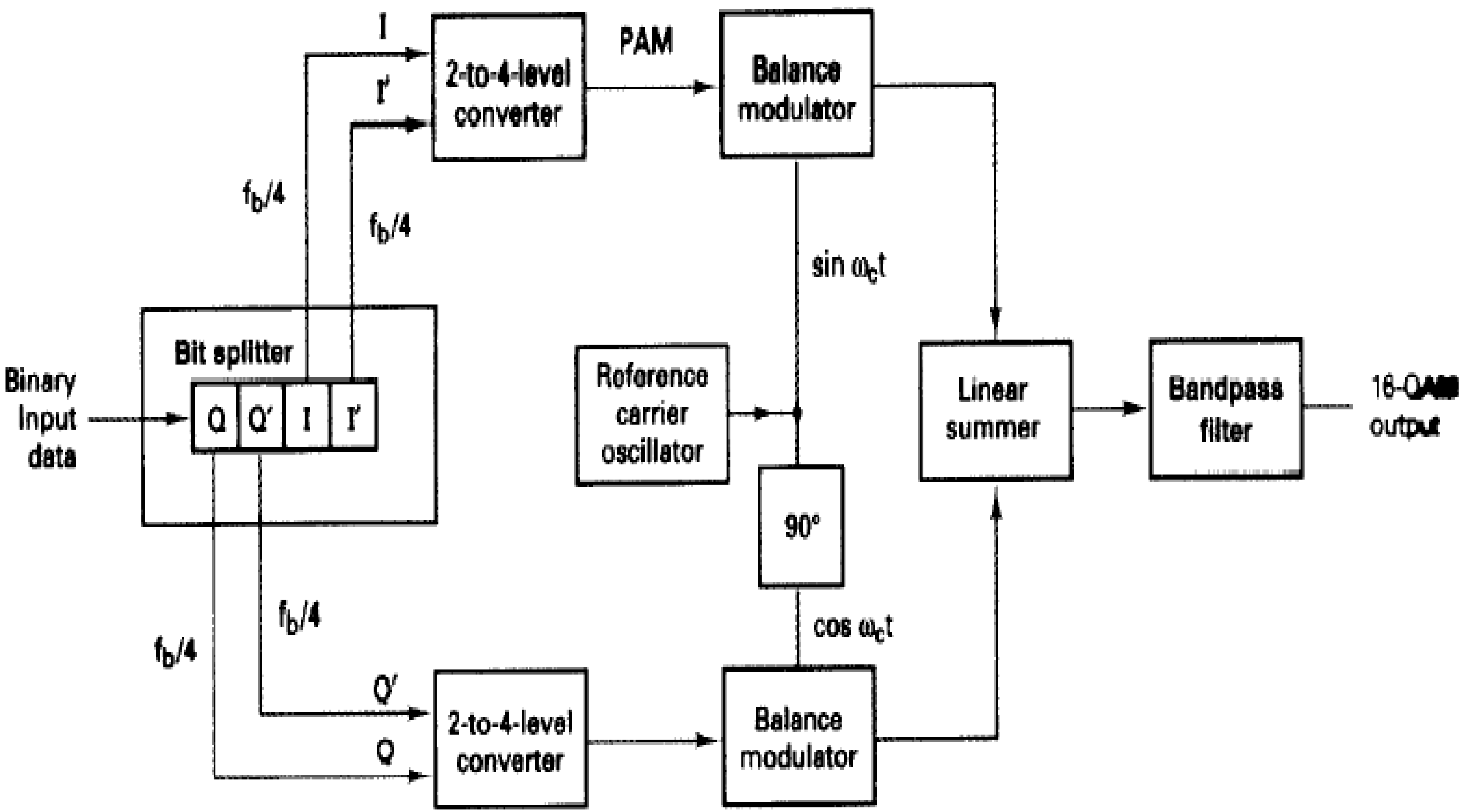
I	I'	Output
0	0	−0.22 V
0	1	−0.821 V
1	0	+0.22 V
1	1	+0.821 V

Q	Q'	Output
0	0	−0.22 V
0	1	−0.821 V
1	0	+0.22 V
1	1	+0.821 V

QAM Transmitter

I and Q, 1 = +ve logic 0 =-ve

I’ and Q’ logic 1 = 0.821V, logic 0 = 0.22 V



Computation of values of amplitude and phases

- The outputs from the I and Q channel product modulators are combined in the linear summer and produce a modulated output
- For a quadbit input of I= 1, I' = 0, Q = 1, and Q' = 0 i.e.(1010) logic 1 for I and Q gives +ve value and logic 0 for I’ and Q’ gives 0.22 V

•Thus summer output = 0.22 sin wc t + 0.22cos wc t

=0.22{sinwct + sin(90 + wct)}

=0.22[2{sin(wct + 45). cos45}

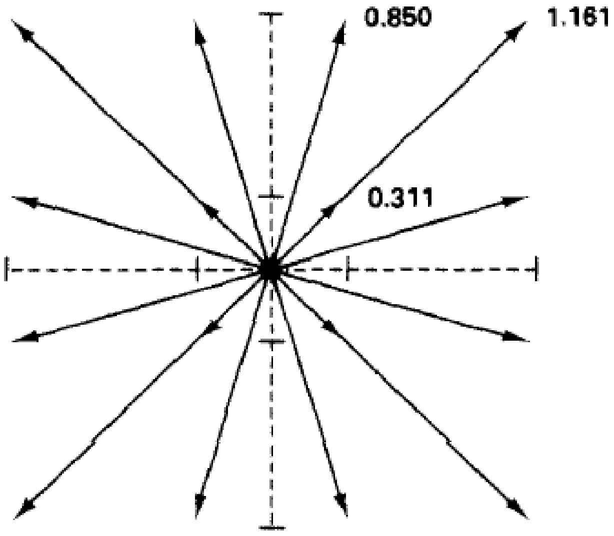
=0.311sin(wct+45)

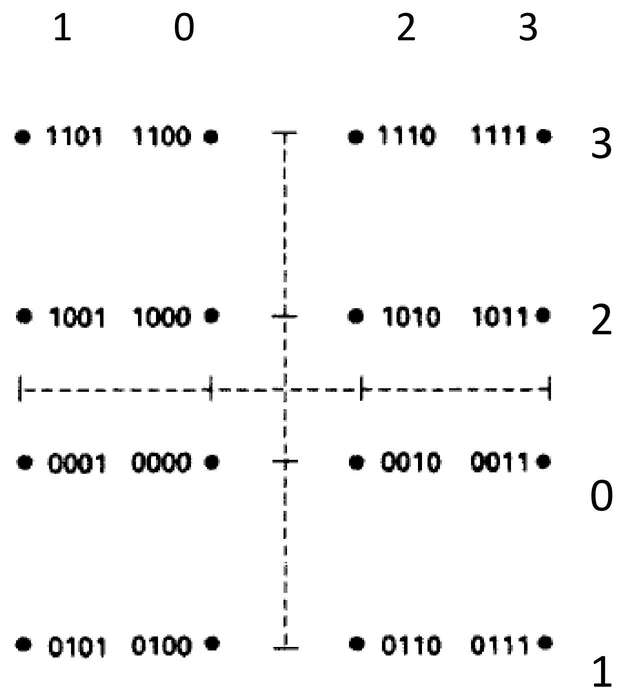
- Similarly other values of amplitude and phase can be computed

Values of amplitude and phases

Binary input				16-QAM output	
Q	Q′	I	I′		
0	0	0	0	0.311 V	−135°
0	0	0	1	0.850 V	−165°
0	0	1	0	0.311 V	−45°
0	0	1	1	0.850 V	−15°
0	1	0	0	0.850 V	−105°
0	1	0	1	1.161 V	−135°
0	1	1	0	0.850 V	−75°
0	1	1	1	1.161 V	−45°
1	0	0	0	0.311 V	135°
1	0	0	1	0.850 V	165°
1	0	1	0	0.311 V	45°
1	0	1	1	0.850 V	15°
1	1	0	0	0.850 V	105°
1	1	0	1	1.161 V	135°
1	1	1	0	0.850 V	75°
1	1	1	1	1.161 V	45°

Phasor and constellation diagram





How to remember

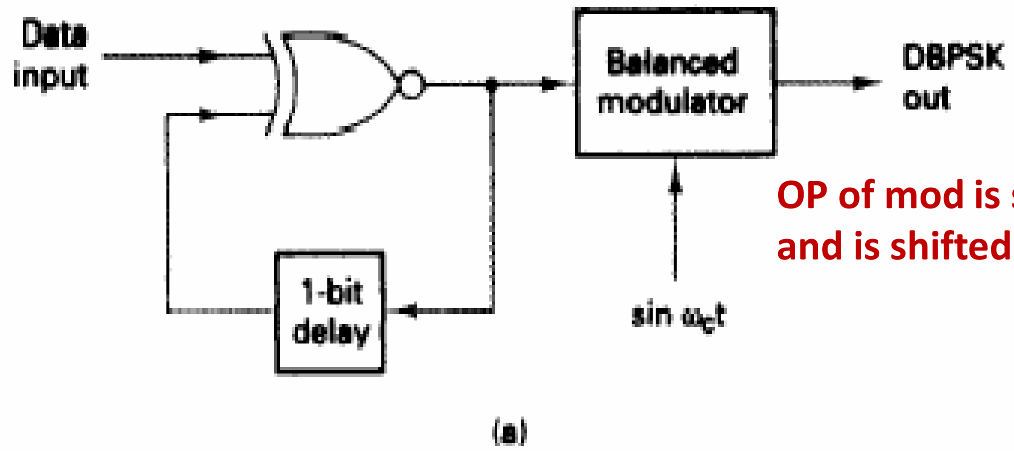
Table 2-2 ASK, FSK, PSK and QAM summary

Modulation	Encoding Scheme	Outputs Possible	Minimum Bandwidth	Baud	Bη
ASK	Single bit	2	f_b	f_b	1
FSK	Single bit	2	f_b	f_b	1
BPSK	Single bit	2	f_b	f_b	1
QPSK	Dibits	4	$f_b/2$	$f_b/2$	2
8-PSK	Tribits	8	$f_b/3$	$f_b/3$	3
8-QAM	Tribits	8	$f_b/3$	$f_b/3$	3
16-PSK	Quadbits	16	$f_b/4$	$f_b/4$	4
16-QAM	Quadbits	16	$f_b/4$	$f_b/4$	4
32-PSK	Five bits	32	$f_b/5$	$f_b/5$	5
64-QAM	Six bits	64	$f_b/6$	$f_b/6$	6

Note: f_b indicates a magnitude equal to the input bit rate.

DIFFERENTIAL PHASE-SHIFT KEYING

Is an alternative form of digital modulation where the binary input information is contained in the difference between two successive signalling elements rather than the absolute phase.

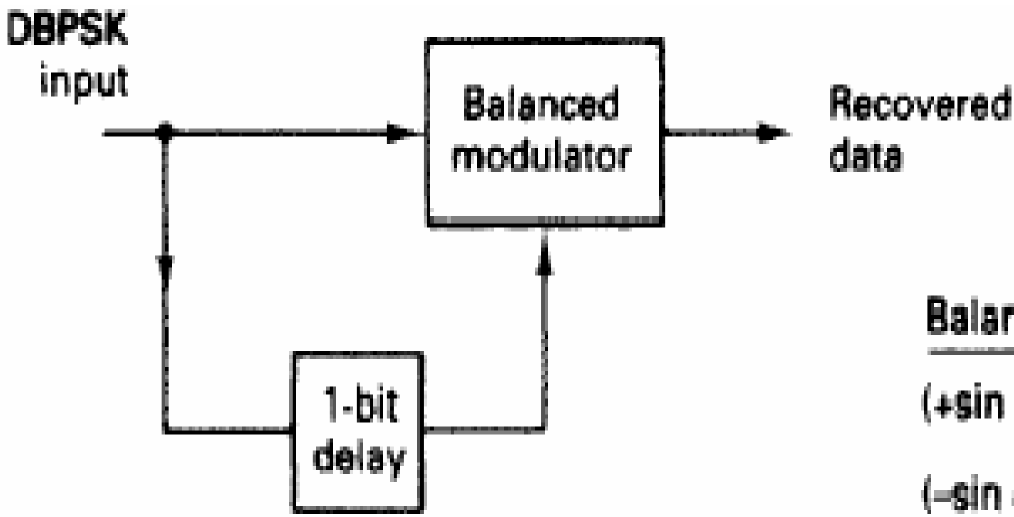


OP of mod is same for input as '1' and is shifted by 180° for input '0'

Input Data		1	0	1	1	1	0	0	0	1	1	0	1
	0 (reference bit)	0	1	1	1	1	0	1	0	0	0	1	1
		180°	0°	0°	0°	0°	180°	0°	180°	180°	180°	0°	0°

(initial reference bit is assumed a logic 0, If the initial reference bit is assumed a logic 1, the output from the XNOR circuit is simply the complement of that shown)

Demodulation



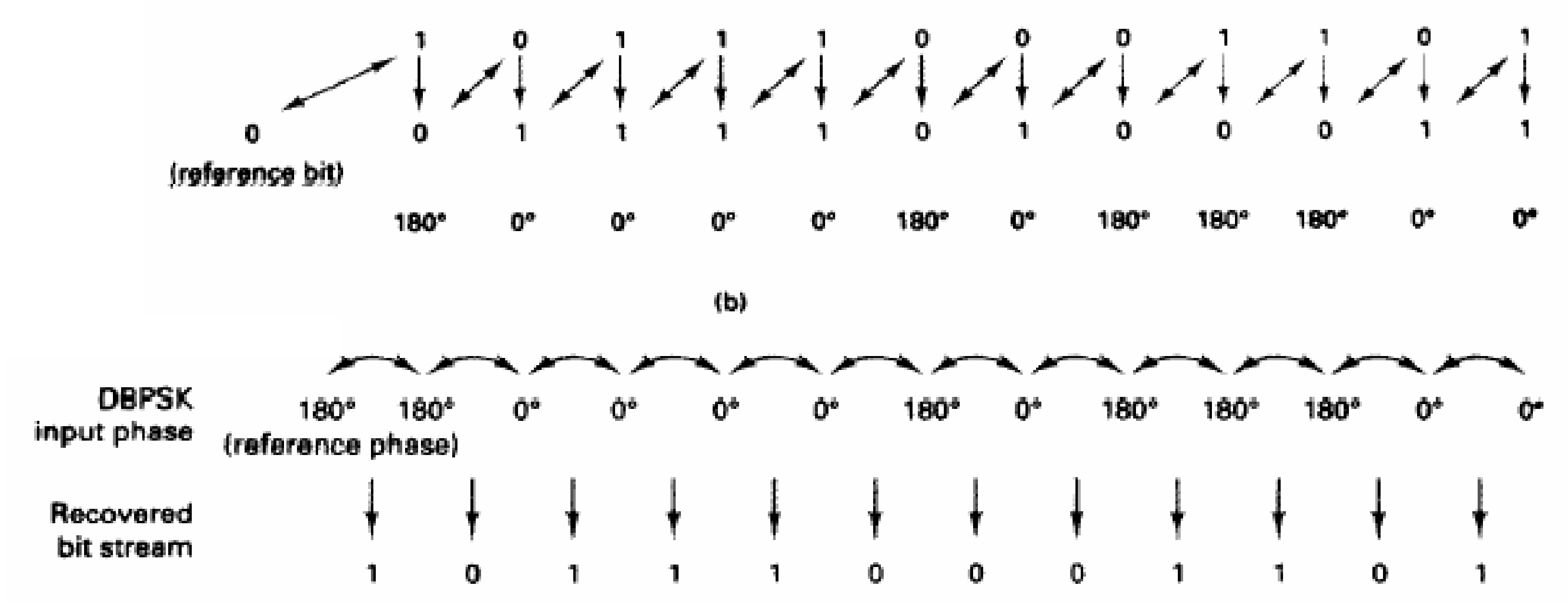
Balanced modulator output

$$(+\sin \omega_c t)\,(+\sin \omega_c t) = +\frac{1}{2} - \frac{1}{2}\cos 2\omega_c t$$

$$(-\sin \omega_c t)\,(-\sin \omega_c t) = +\frac{1}{2} - \frac{1}{2}\cos 2\omega_c t$$

$$(-\sin \omega_c t)\,(+\sin \omega_c t) = -\frac{1}{2} + \frac{1}{2}\cos 2\omega_c t$$

Demodulation



Change of phase indicates 0, same phase indicates 1

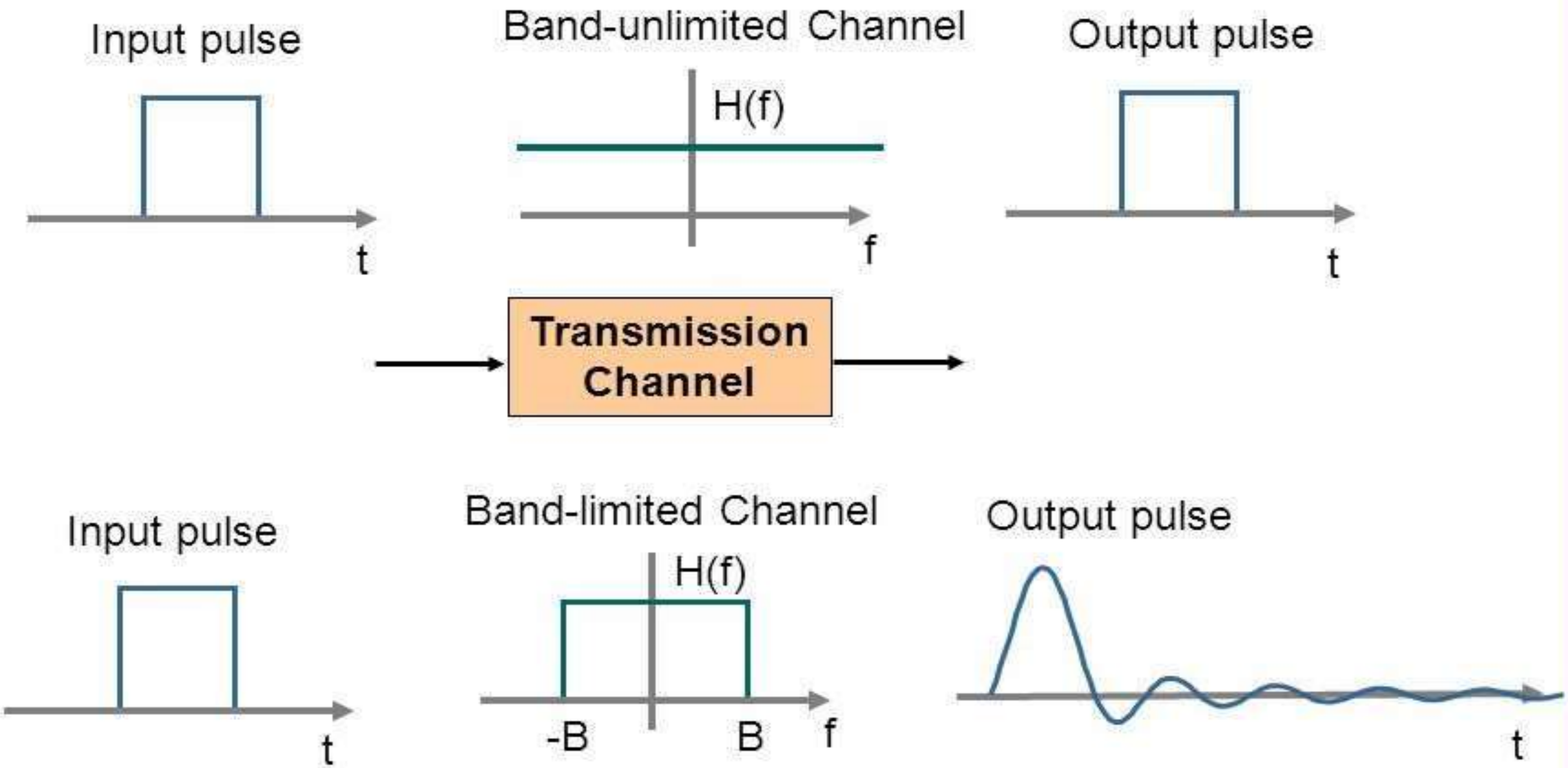
Baseband Transmission and Optimal Reception of Digital Signal

ISI and Eye diagram

Steps in designing the receiver

- Find optimum solution for receiver design with the following goals:
  1. Maximize SNR
  2. Minimize ISI
- Steps in design:
  - Model the received signal
  - Find separate solutions for each of the goals.

Bandlimited Channels: Inter-Symbol Interference (ISI)



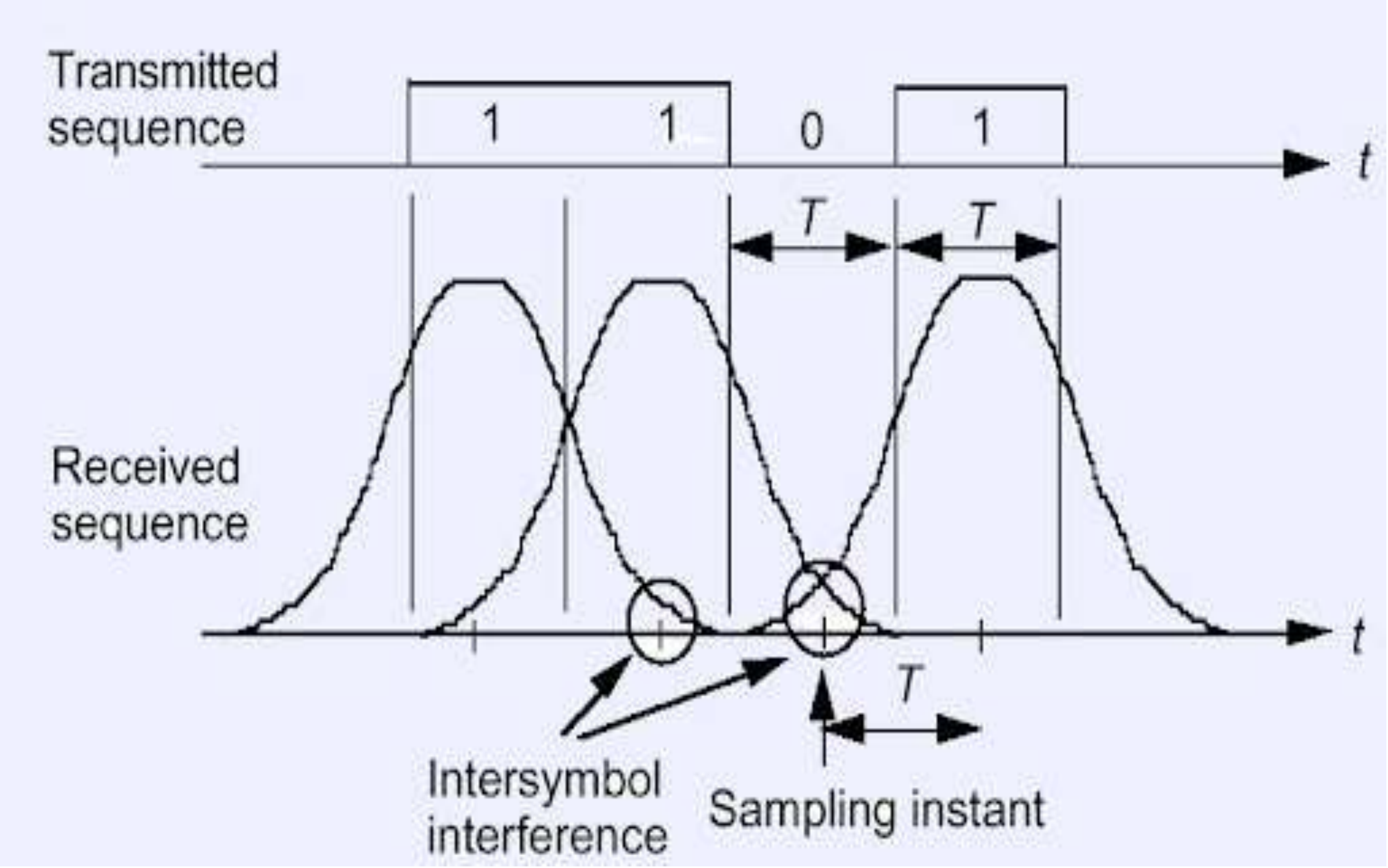
Intersymbol Interference (ISI)

- If the transmission channel is bandlimited, then high frequency components will be cut off
  - Hence, the pulses will spread out
  - If the pulse spread out into the ***adjacent symbol*** **periods**, then it is said that intersymbol interference (ISI) has occurred

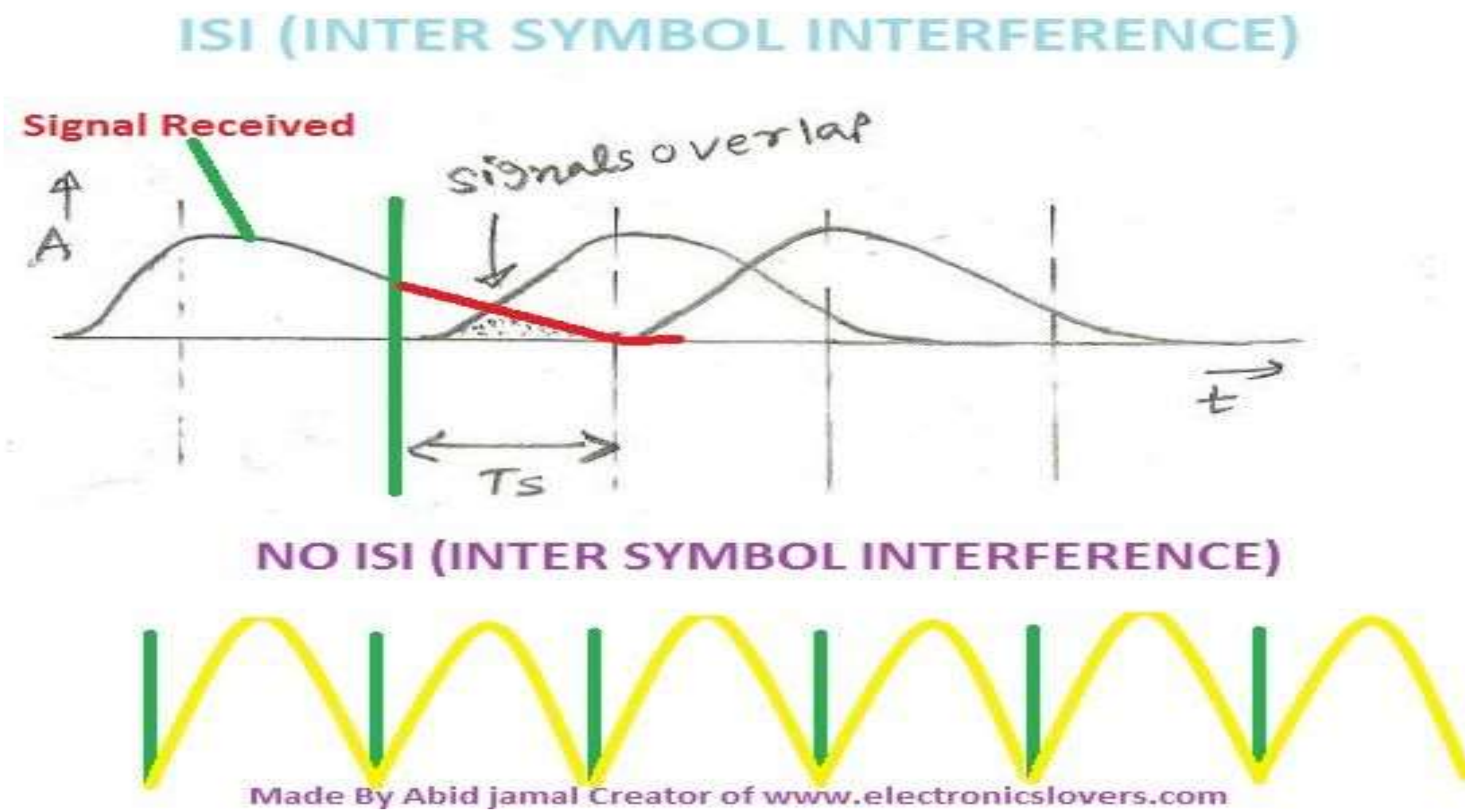
Intersymbol Interference (ISI)

- Intersymbol interference (ISI) occurs when a pulse spreads out in such a way that it interferes with adjacent pulses at the ***sample instant***
- **Causes**
  - Channel induced distortion which spreads or disperses the pulses
  - Multipath effects (echo)

Steps in designing the receiver



Steps in designing the receiver



- In all digital systems, the pulses are transmitted from the transmitter to receiver.
- But the channels carrying these pulses never have enough bandwidth required to ensure the preservation of the shape of the pulse.
- Such channels are called as the **band limited channels**.
- **The band limited channel tends to spread the narrow pulses**. This produces the secondary lobes as shown.
- The **secondary lobes** are called as **ringing tails.**

Pulse Shaping: Intersymbol Interference (ISI)

- **The process of residual signals at the receiver output due to other symbols interfering with the required symbol is known as intersymbol interference.**
- **Spreading of a pulse beyond its allotted time period T_b (pulse width) will tend to interfere with adjacent pulses.**
- In digital transmission, ISI arises due to the **dispersive nature** of a communications channel. In a band limited PCM channel, the received pulse waveform is distorted and may extend to the next time slot. This may result in error in the determination of received bits.
- **ISI is an undesirable phenomenon that results into degradations in the performance of digital communication systems.**

Intersymbol Interference

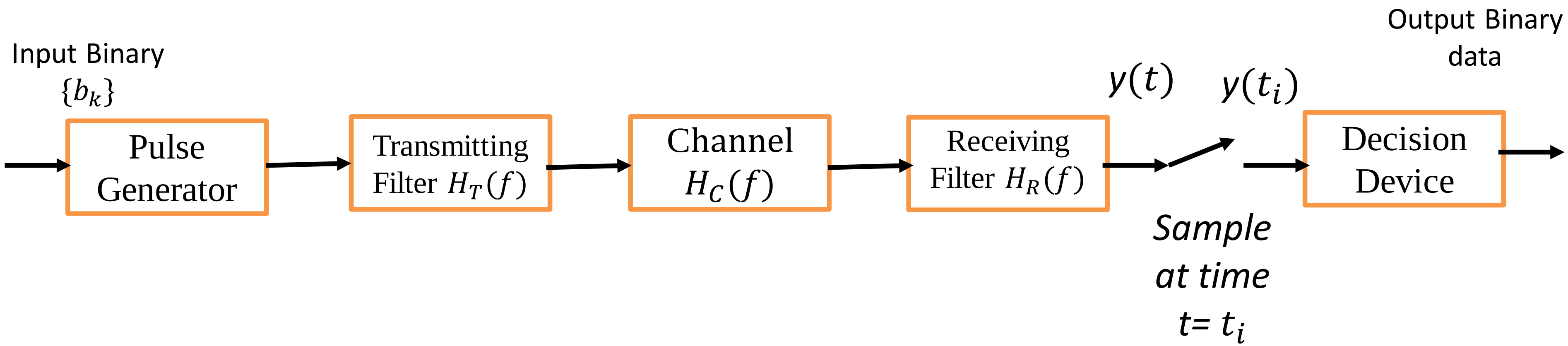


Fig: Baseband Data Transmission system

- The input signal consists of a binary data sequence $\{b_k\}$ with a bit duration of T_b seconds.
- The sequence is applied to a pulse generator, producing the discrete PAM signal.

$$x(t) = \sum_{k=-\infty}^{\infty} a_k g(t - k\,T_b)$$

$g(t)$ is the shaping pulse that is normalized and can be written as $g(0) = 1$

$$a_k = \begin{cases} +a & \textit{if the input bit } b_k \textit{ is represented by symbol } "1" \\ -a & \textit{if the input bit } b_k \textit{ is represented by symbol } "0" \end{cases}$$

The Receiving filter output may be written as

$$y(t) = \sum_{k=-\infty}^{\infty} A_k p(t - k\,T_b) + n(t)$$

Where, A_k is the Amplitude, the pulse $p(t)$ is normalized such that $p(0) = 1$

The pulse $A_k p(t)$ is the response of the cascade connection of transmitting filter, Channel and the Receiving filter which is produced by the pulse $a_k g(t)$ applied to the input of cascade connection. Therefore we may relate $p(t)$ to $g(t)$ in Frequency domain as follows

$$A_k p(f) = a_k g(f)\, H_T(f)\, H_C(f)\, H_R(f)$$

The frequency filter output $y(t)$ is sampled at time $t = t_i$ *yielding to get* $y(t_i)$

$$y(t_i) = \sum_{k=-\infty}^{\infty} A_k\, p[(i-k)T_b) + n(t_i)$$

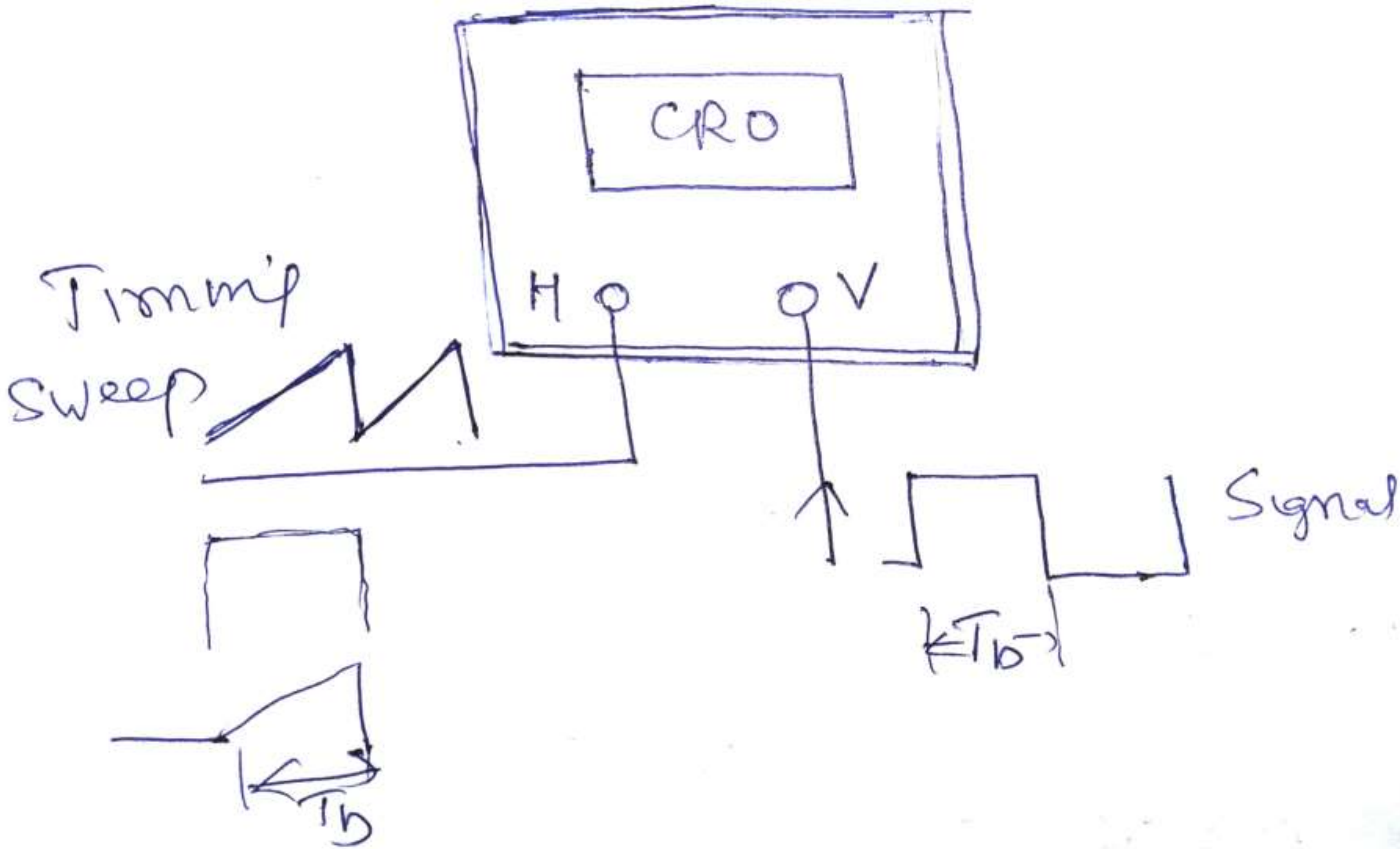
$$y(t_i) = A_i + \sum_{\substack{k=-\infty \\ i\neq k}}^{\infty} A_k\, p[(i-k)T_b) + n(t_i)$$

The **first term** is produced by the ith transmitted bit. The **second term** represents the residual effect of all other transmitted bits on the decoding of the ith bit; this residual effect is called ISI.

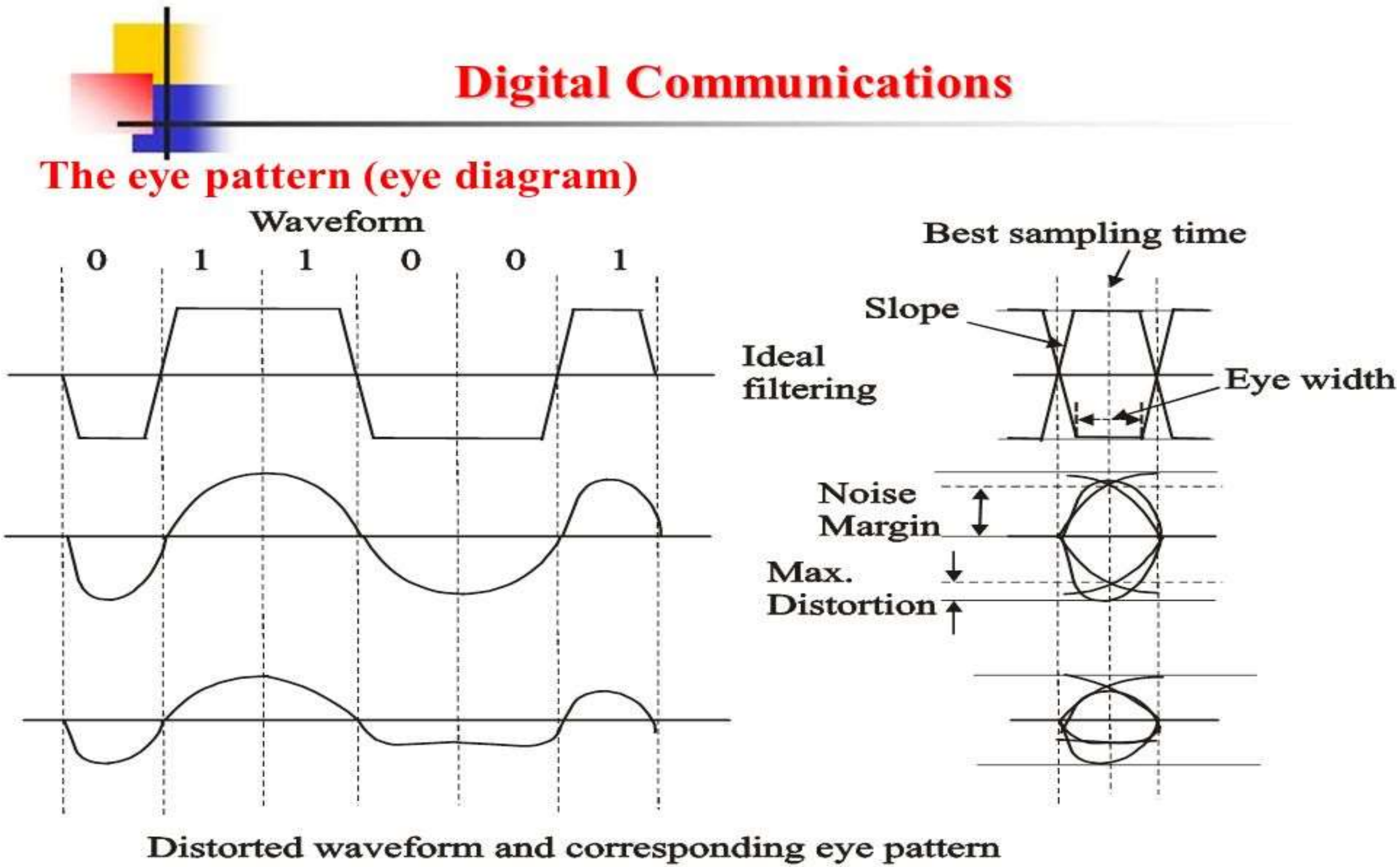
ISI arises because of imperfections in the overall frequency response of the system. The frequency components constituting the input pulses are differently attenuated and differently delayed by the system. Consequently pulse is dispersed at the output.

EYE DIAGRAM

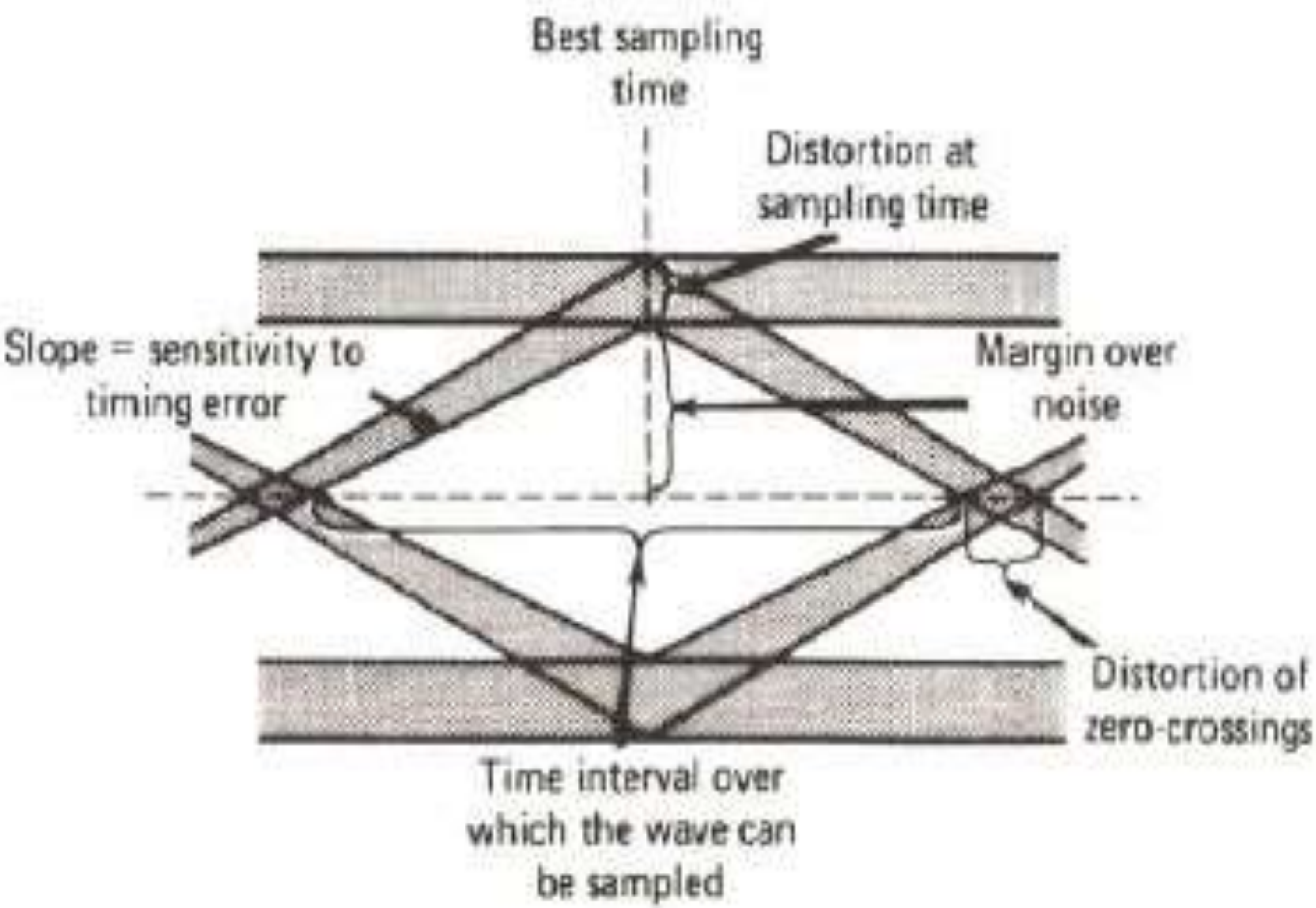
- The amount of ISI and noise in a digital communication system can be viewed on an oscilloscope.
- For PAM signals, we can display the received signal *y(t) on the* vertical input with the horizontal sweep rate set at $1/T$.
- The resulting oscilloscope display is called an *eye pattern*.
- Eye diagram is a means of evaluating the quality of a received "digital waveform"
  - By quality is meant the ability to correctly recover symbols and timing
  - The received signal could be examined at the input to a digital receiver or at some stage within the receiver before the decision stage


CRO
H
V
Timing
sweep
Signal
T_D
T_D

Eye diagram

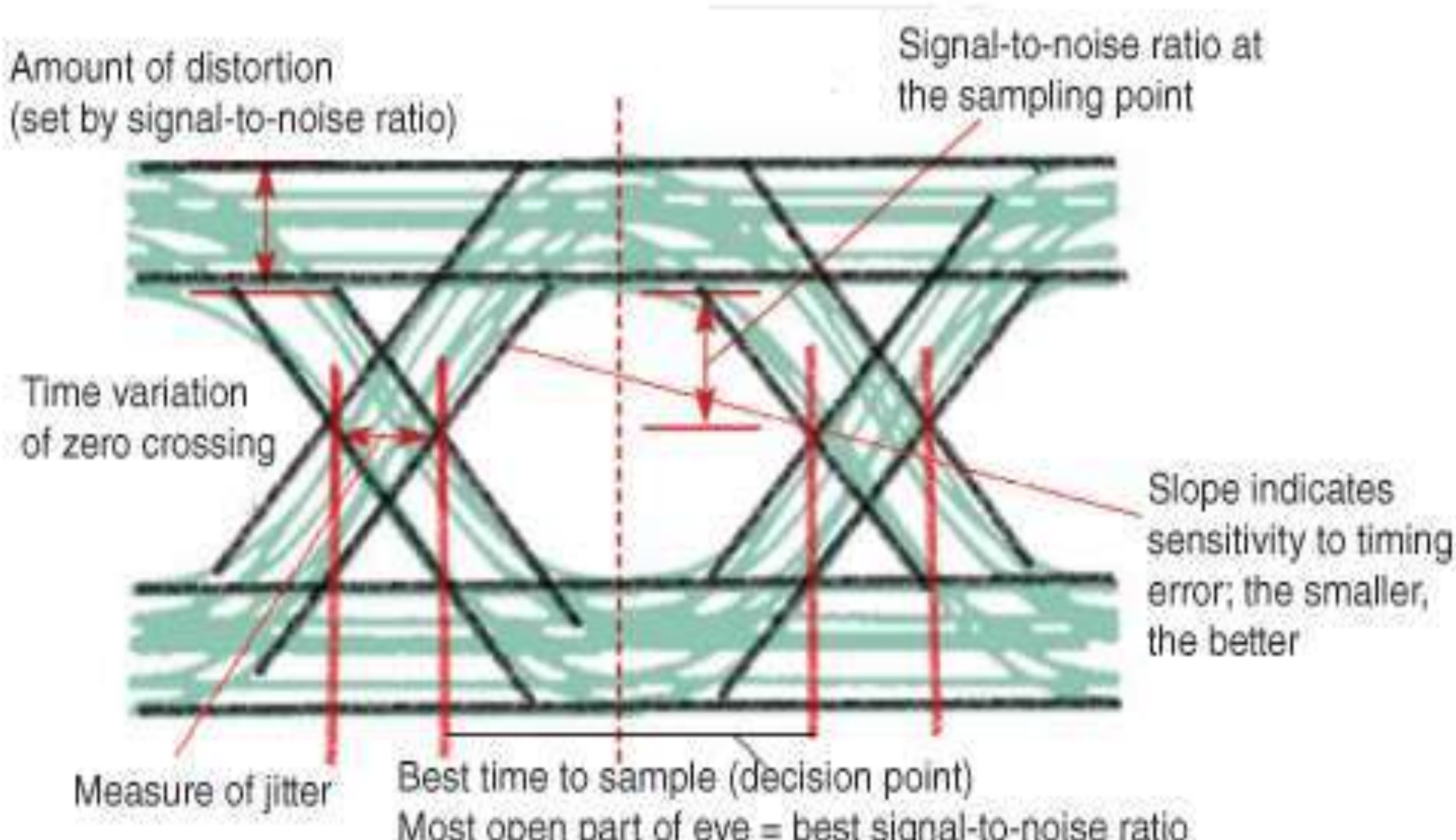

Digital Communications
The eye pattern (eye diagram)
Waveform
0
1
1
0
0
1
Ideal
filtering
Best sampling time
Slope
Eye width
Noise
Margin
Max.
Distortion
Distorted waveform and corresponding eye pattern

Eye diagram



Best sampling time
Distortion at sampling time
Slope = sensitivity to timing error
Margin over noise
Distortion of zero-crossings
Time interval over which the wave can be sampled

Eye diagram



Amount of distortion
(set by signal-to-noise ratio)
Signal-to-noise ratio at
the sampling point
Time variation
of zero crossing
Slope indicates
sensitivity to timing
error; the smaller,
the better
Measure of jitter
Best time to sample (decision point)
Most open part of eye = best signal-to-noise ratio

Eye Pattern or Eye Diagram

Eye diagram is a **simple and convenient engineering tool** applied on received signals **for studying the effects of**

- ***ISI***
- ***Accuracy of timing extraction***
- ***Noise immunity***
- ***Determining the bit error rate (BER)***

- Eye diagram provides information about the ***<u>state of the channel</u>*** and the ***<u>quality of the received pulse</u>***.
- This information is useful for faithful detection of received signal and ***<u>determination of overall performance of digital communication system.</u>***

Detection Techniques

1)Integrator & Dump Filter

2)Optimum Filter

3)Matched Filter

4)Correlator or Coherent Receiver

7.4 EFFECT OF NOISE ON THE TRANSMITTED SIGNAL

The data transmission systems used for the transmission of a sequence of binary digits, 0s and 1s as discussed earlier. These digits can be represented by different patterns such as unipolar RZ and NRZ, bipolar RZ and NRZ, AMI and Manchester. In general the binary digits are encoded in such a way that a 1 is represented by $x_1(t)$ and a 0 is represented by signal $x_2(t)$, where $x_1(t)$ and $x_2(t)$,where $x_1(t)$ and $x_2(t)$ each have a duration T. The resulting signal may be transmitted directly or used to modulate a carrier. The received signal is corrupted by noise. Hence, there is a probability that the receiver will make an error in deciding whether a 1 or a 0 was transmitted. In this chapter, let us make calculations of such error probabilities and discuss methods to minimize them. We consider a binary sequence 1 0 1 1 being transmitted. While travelling from the transmitter to receiver, noise gets added to it. Thus, the signal received by a receiver is corrupted by noise as shown in figure 7.9. The first transmitted bit is represented by voltage + A volts which sustains over the time t_1 to t_2, i.e., over one bit interval. Noise has been superimposed on this signal. In order to make a judgement of whether a 1 or a 0 is received, the receiver, samples the received signal once in every bit interval. In the first bit interval of figure 7.9. if the sampling happens to take place at instant ($t_1 + \Delta t$), then the receiver will decide that a 0 has been received thus introducing an error. In order to reduce the probability of error the sampling instant in each interval should be selected in such a way that the signal amplitude is maximum at the instant of sampling.

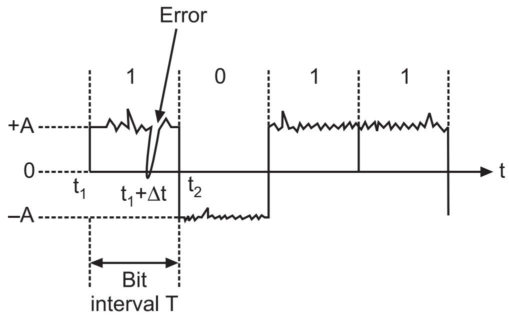


Fig. 7.9. Signal corrupted by noise

In the following sections, we shall discuss various types of detection methods to detect the baseband signals.

7.4.1 Requirements of Detection Technique

(i) A detection technique must have minimum probability of error.
(ii) It should sample (check) the received signal in every bit interval at the instant when the signal has maximum possible amplitude.
(iii) The detection method must maximize the signal to noise ratio, by amplifying the signal and attenuating the noise.
(iv) One of the detection techniques used is integrate and dump circuit.

7.4.2. Detection Techniques

(i) Integrate and dump receiver (filter) (ii) Optimum filter
(iii) Matched filter (iv) Correlator or coherent receiver

7.5 INTEGRATE AND DUMP FILTER (i.e., RECEIVER)

Let us consider a very simple and basic detector circuit for the detection of digital signals. Figure 7.10 illustrates the circuit of such type of detector which is called an integrate and dump filter.

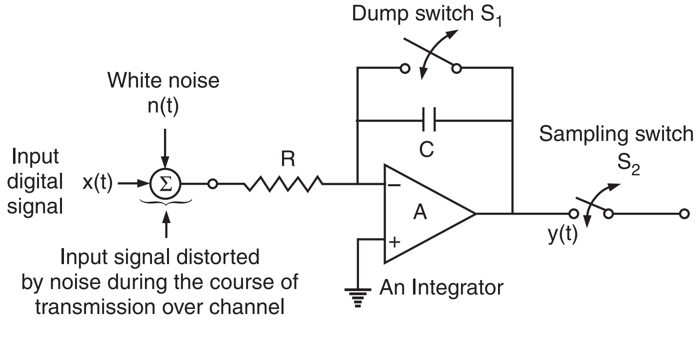


Fig. 7.10. Integrate and dump filter

Here, the digital signal x(t) is distorted by white noise n(t) during the transmission over channel.

This noisy signal [x(t) + n(t)] is applied to the input of integrate and dump filter. The capacitor is discharged fully at the beginning of the bit interval. This is achieved by temporarily closing switch S_2 at the beginning of the bit interval. The integrator then integrates noisy input signal over one bit period. This integrated signal is shown as y(t) in figure 7.10. For the square pulse input, the output of the integrator would be a triangular pulse as shown in figure 7.10.

Figure 7.11(b) illustrates the waveform of y(t). At the end of bit period i.e., at t = T, the magnitude of y(t) attains its maximum amplitude. Hence, the value of y(t) is sampled at the end of bit period. It can be further proved that the signal to noise ratio is maximum at the end of bit period (i.e., at t = T). Depending upon the value of y(T), the decision is taken. The dump switch S_2 is then closed momentarily to discharge the capacitor to receive the next bit. Therefore, the integrator integrates (i.e. generates output) which is independent of the value of previous bit. This reveals the fact that the detection in integrate and dump filter is unaffected by the values of previous bits. In figure 7.11(b), the output of integrator would decrease after t > T.

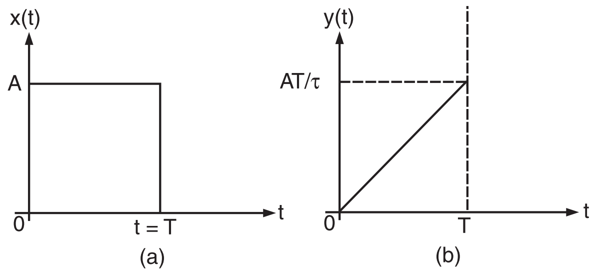


Fig. 7.11. (a) Input pulse to the integrator (Here, we have assumed that the noise is absent). This pulse represents binary '1'. The width of pulse is 'T'
(b) Output of integrator. The initial output is zero. At t = T, the output of integrator is y(t) = AT/t.

7.5.1. Calculation of Signal to Noise Ratio for the Integrate and Dump Filter

We know that the output of the integrator may be written as,

$$y(t) = \frac{1}{RC}\int_0^T [x(t) + n(t)]\, dt \qquad ...(7.5)$$

Here, the integration is performed over one bit period i.e., from 0 to T. The noisy signal [x(t) + n(t)] is the input to the integrator. Equation (7.5) may be modified as

$$y(t) = \frac{1}{RC}\int_0^T x(t)\, dt + \frac{1}{RC}\int_0^T n(t)\, dt \qquad ...(7.6)$$

or $\qquad y(t) = x_0(t) + n_0(t)$

where $x_0(t)$ is the output signal voltage and $n_0(t)$ is the output noise voltage.

Now, let us consider output signal voltage which is

$$x_0(t) = \frac{1}{RC}\int_0^T x(t)\, dt \qquad ...(7.7)$$

Because the value of x(t) = A from 0 to T, the above equation can be written as,

$$x_0(t) = \frac{1}{RC}\int_0^T A\, dt = \frac{A}{RC}\int_0^T 1 \cdot dt$$

Integrating, we get

$$x(t) = \frac{A}{RC}[t]_0^T \quad \text{or} \quad x_0(t) = \frac{AT}{RC} \qquad ...(7.8)$$

Let the time constant be RC = τ.

Hence, equation (7.8) will become,

$$x_0(t) = \frac{AT}{\tau} \qquad ...(7.9)$$

The normalized signal power in standard resistance of 1 Ω would be,

$$\text{Output signal power} = \frac{x_0^2(t)}{1\Omega}$$

Putting the value of $x_0(t)$, we get

$$\text{Output signal power} = \frac{A^2T^2}{\tau^2} \qquad ...(7.10)$$

Now, let us find the noise power. However, before doing so, we have to calculate the transfer function of the integrator. Recall that a network which performs integration operation has the transfer function equal to $\frac{1}{j\omega RC}$ *. Now, since a delay of t = T in time domain is equivalent to $e^{-j\omega T}$ in frequency domain, therefore, the network performing integration over the period of T can be represented by the transfer function as under:

$$H(f) = \frac{1 - e^{-j\omega T}}{j\omega RC} \qquad ...(7.11)$$

For $\omega = 2\pi f$ and $RC = \tau$, equation (7.11) takes the form

$$H(f) = \frac{1 - e^{-j2\pi fT}}{j2\pi f\tau}$$

Simplifying, we get

$$H(f) = \frac{1 - [\cos(2\pi fT) - j\sin(2\pi fT)]^{**}}{j2\pi f\tau}$$

Separating the real and imaginary parts, we have

$$H(f) = \frac{\sin(2\pi fT)}{2\pi f\tau} - j\frac{1 - \cos(2\pi fT)}{2\pi f\tau}$$

Then, the magnitude of above transfer function will become

$$|H(f)|^2 = \frac{\sin^2(2\pi fT) + 1 - 2\cos(2\pi fT) + \cos^2(2\pi fT)}{(2\pi f\tau)^2}$$

Simplifying the last equation, we obtain,

$$|H(f)|^2 = \frac{\sin^2(\pi fT)}{(\pi f\tau)^2} \qquad ...(7.12)$$

Now, the average power of the output noise signal $n_0(t)$ may be obtained by integrating its power density spectrum.

This means that

$$\text{Power, } P = \int_{-\infty}^{\infty} S(f)\, df$$

For a standard 1 Ω resistance, the noise power would be $\frac{\overline{n_0^2(t)}}{1\,\Omega} = \overline{n_0^2(t)}$.

Here, mean square value of noise signal is considered since it is a random signal. i.e.,

$$\text{Noise power, } \quad \overline{n_0^2(t)} = \int_{-\infty}^{\infty} S_{no}(f)\, df \qquad ...(7.13)$$

* In fact, this can be very easily verified by taking the example of an circuit. The reason is that an RC low pass filter also basically performs integration operation.

**Using Euler's identity $e^{j\theta} = \cos\theta + j\sin\theta$.

Again, we know that the input and output power spectral densities are related as,

$$S_{no}(f) = |H(f)|^2 S_{ni}(f) \quad ...(7.14)$$

where $H(f)$ = transfer function of filter,

$S_{no}(f)$ = power spectral density (psd) of output noise

$S_{ni}(f)$ = power spectral density (psd) of input noise.

Again, let us assume that white noise is present, therefore, the power spectral density (psd) of this noise would be,

$$S_{ni}(f) = \frac{N_0}{2} \quad ...(7.15)$$

Substituting this value in equation (7.14), we get

$$S_{no}(f) = |H(f)|^2 \cdot \frac{N_0}{2}$$

Substituting this value in equation (7.13), we have

$$\overline{n_0^2(t)} = \int_{-\infty}^{\infty} |H(f)|^2 \cdot \frac{N_0}{2} df$$

Substituting the value of $|H(f)|^2$ from equation (7.12) in above equation, we obtain

$$\overline{n_0^2(t)} = \int_{-\infty}^{\infty} \frac{\sin^2(\pi fT)}{(\pi f\tau)^2} \cdot \frac{N_0}{2} df = \frac{N_0}{2} \int_{-\infty}^{\infty} \frac{\sin^2(\pi fT)}{(\pi f\tau)^2} df \quad ...(7.16)$$

Now, let us substitute $\pi f\tau = x$

so that $dx = \pi\tau\, df$ or $df = \frac{1}{\pi\tau} dx$

Further, since $f = \frac{x}{\pi\tau}$

therefore, $\pi fT = \frac{xT}{\tau}$

With all these substitutions, equation (7.16) becomes,

$$\overline{n_0^2(t)} = \frac{N_0}{2} \int_{-\infty}^{\infty} \frac{\sin^2\left(\frac{xT}{\tau}\right)}{x^2} \cdot \frac{1}{\pi\tau} dx$$

Rearranging above equation, we get

$$\overline{n_0^2(t)} = \frac{N_0}{2} \int_{-\infty}^{\infty} \frac{\sin^2\left(\frac{xT}{\tau}\right)}{\left(\frac{xT}{\tau}\right)} \cdot \left(\frac{T}{\tau}\right)^2 \cdot \frac{1}{\pi\tau} dx = \frac{N_0}{2} \cdot \frac{T^2}{\pi\tau^3} \int_{-\infty}^{\infty} \frac{\sin^2\left(\frac{xT}{\tau}\right)}{\left(\frac{xT}{\tau}\right)} \cdot dx$$

Now, let $\frac{xT}{\tau} = u$

so that $dx = \frac{\tau}{T} du$ and limits will remain unchanged, therefore, above equation will become,

$$\overline{n_0^2(t)} = \frac{N_0}{2} \cdot \frac{T^2}{\pi\tau^3} \int_{-\infty}^{\infty} \frac{\sin^2 u}{u^2} \cdot \frac{\tau}{T} du = \frac{N_0 T}{2\pi\tau^2} \int_{-\infty}^{\infty} \left(\frac{\sin u}{u}\right)^2 du$$

Because, here the function $\frac{\sin u}{u}$ is squared, the above equation can be written as,

$$\overline{n_0^2(t)} = \frac{N_0T}{2\pi\tau^2}\cdot 2\int_0^{\infty}\left(\frac{\sin u}{u}\right)^2 du = \frac{N_0T}{2\pi\tau^2}\cdot 2\cdot\frac{\pi}{2} = \frac{N_0T}{2\tau^2} \quad ...(7.17)$$

This relation describes the noise power at the output. The signal to noise power ratio at output of integrator can be obtained as,

$$\text{Signal to noise ratio, } \left(\frac{S}{N}\right)_0 = \frac{\text{Signal power}}{\text{Noise power}}$$

Substituting the values of signal power from equation (7.10) and noise power from equation (7.17), we get

$$\left(\frac{S}{N}\right)_0 = \frac{\dfrac{A^2T^2}{\tau^2}}{\dfrac{N_0T}{2\tau^2}} = \frac{2A^2T}{N_0} = \frac{A^2T}{N_0/2}$$

Hence, we conclude signal to noise ratio of integrate and dump receiver:

$$\left(\frac{S}{N}\right)_0 = \frac{A^2T}{N_0/2} \quad ...(7.18)$$

This signal to noise ratio is also known as figure of merit.

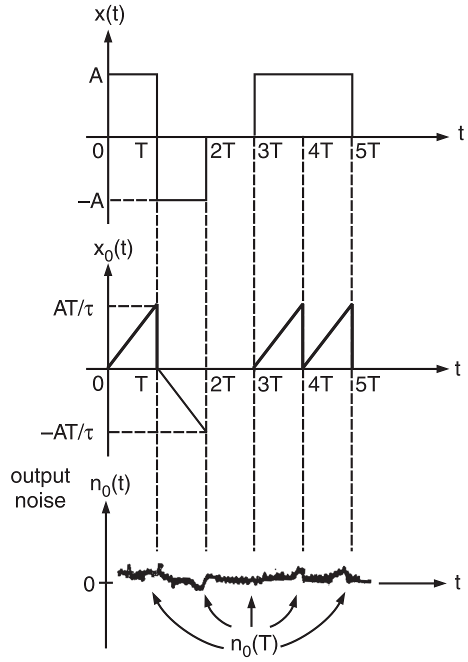


Fig. 7.12. (a) Input signal to the integrate and dump filter, (b) Output signal of the integrate and dump filter, (c) Output noise of the integrate and dump filter

Few points about signal to noise ratio:

(i) The result in equation (7.18) shows that signal to noise ratio improves in proportion to sampling period T. It is also increased as signal amplitude 'A' is increased.

(ii) Because noise has Gaussian distribution and zero mean value at any time, therefore, the output of integrator also increases by a very small margin at the end of bit interval.

Figure 7.12 shows the waveform of $x_0(t)$ and $n_0(t)$ for the input signal x(t).

Figure 7.12 shows the waveforms of integrate and dump filter receiver for the rectangular pulses input.

It may observed that the output signal voltage reaches to the value of $\pm\frac{AT}{\tau}$ at the sampling instant. This is the maximum signal voltage. But the noise voltage $n_0(t)$ does not increase in same proportion. This is due to the fact that the noise has zero average value and Gaussian distribution.

EXAMPLE 7.2. Show that for an integrate and dump filter receiver, the maximum signal to noise ratio is expressed as,

$$\left(\frac{S}{N}\right)_0 = \frac{2E}{N_0}$$

Given that the input signal x(t) is rectangular pulses of amplitudes ± A and duration T.

Solution: We know that the energy of the signal x(t) is given by the relation

$$E = \int_{-\infty}^{\infty} x^2(t)\,dt \quad ...(i)$$

Given that x(t) is a rectangular pulse of amplitude ± A and duration, T_1. Therefore, equation (i) can be written as

$$E = \int_0^T (\pm A)^2\, dt = A^2 \int_0^T dt = A^2[t]_0^T = A^2 T \quad ...(ii)$$

We know that the signal to noise ratio of integrate and dump filter receiver is given as,

$$\left(\frac{S}{N}\right)_0 = \frac{A^2T}{N_0/2} \quad ...(iii)$$

Substituting, $A^2T = E$ from equation (ii) in equation (iii), we get

$$\left(\frac{S}{N}\right)_0 = \frac{E}{N_0/2} = \frac{2E}{N_0}$$

Since, output is maximum at sampling instant, T, therefore, the above value is maximum i.e.,

$$\left(\frac{S}{N}\right)_{0\max} = \frac{2E}{N_0}$$

Hence Proved

7.5.2. Computation of Probability of Error for Integrate and Dump Filter Receiver

As discussed earlier that a noise interference may lead to wrong decision at the receiver end. As a matter of fact, the probability of error denoted by P_e is a good measure of performance of the detector.

We know that the output of the integrator is expressed as,

$$y(t) = x_0(t) + n_0(t)$$

For the positive pulse of amplitude A, $x_0(t)$ is given as,

$$x_0(t) = \frac{AT}{\tau} \quad \text{for} \quad x(t) = A$$

Similarly, for the input pulse of amplitude – A, $x_0(t)$ is given by,

$$x_0(t) = -\frac{AT}{\tau} \quad \text{for} \quad x(t) = -A \quad ...(7.19)$$

Therefore, the output y(t) may be written as

$$y(t) = \frac{AT}{\tau} + n_0(t) \quad \text{for} \quad x(t) = A \quad ...(7.20)$$

Similarly, $$y(t) = -\frac{AT}{\tau} + n_0(t) \quad \text{for} \quad x(t) = -A \quad ...(7.21)$$

Let us consider that x(t) = – A. Further, if noise $n_0(t)$ is greater than $\frac{AT}{\tau}$, then the output y(t) would be positive according to equation (7.20). After that the receiver will decide in favour of symbol + A, which is wrong decision. This means that an error is introduced. Similarly, let us consider that x(t) = + A. If noise $n_0(t) > -\frac{AT}{\tau}$ then, the output y(t) will be negative according to equation (7.21). This leads to decision in favour of – A, which is erroneous. Based on the above discussion, we can make conclusions about probability of error in the form of a Table 7.1.

Table 7.1. Probability of error in integrate and dump filter receiver

S. No.	Input x(t)	Value of $n_0(t)$ for error in the output	The probability of error P_e
1.	– A	An error will be introduced if $n_0(t) > \frac{AT}{\tau}$	In this case, the probability of error may be obtained by calculating probability that $n_0(t) > \frac{AT}{\tau}$
2.	+ A	An error will be introduced if $n_0(t) < -\frac{AT}{\tau}$	In this case, the probability of error may be obtained by calculating probability that, $n_0(t) < -\frac{AT}{\tau}$

As shown in Table 7.1, the error will be introduced depending upon probability that $n_0(t)$ takes a particular value. These probabilities may be obtained from PDF of $n_0(t)$. Recall that the probability density function (PDF) of the Gaussian distributed function is expressed by standard relation as under:

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-m)^2/2\sigma^2} \quad ...(7.22)$$

where, $f_X(x)$ = the PDF of random function x.

m = the mean value

and σ = the standard deviation.

Here, because we have to evaluate PDF for white Gaussian noise, therefore, we have

$$x = n_0(t)$$

Since this noise has zero mean value i.e., m = 0, therefore equation (7.22) may be written as,

$$f_X\{n_0(t)\} = \frac{1}{\sigma\sqrt{2\pi}} e^{-[n_0(t)]^2/2\sigma^2} \quad ...(7.23)$$

The standard deviation σ is expressed as,

$$\sigma = [\text{mean square value} - \text{square of mean value}]^{1/2}$$

This means that $\sigma_x = \left[\overline{X^2} - m_x^2\right]^{1/2}$

Further, we have mean square value $\overline{X^2} = \overline{n_0^2(t)} = \frac{N_0 T}{2\tau^2}$

Also, mean value $m_x = 0$ for this noise.

Therefore, we have

$$\sigma = \left[\overline{n_0^2(t)}\right]^{1/2} = \sqrt{\frac{N_0 T}{2\tau^2}} \quad ...(7.24)$$

Hence equation (7.23), can be written as,

$$f_X\{n_0(t)\} = \frac{1}{\sqrt{\frac{N_0 T}{2\tau^2}}\sqrt{2\pi}} \cdot e^{-[n_0(t)]^2 \Big/ 2\left(\frac{N_0 T}{2\tau^2}\right)}$$

At this stage, it may be noted that $n_0(t)$ is the function like 'x'. It is a random variable and we are evaluating its PDF.

On simplifying above equation, we get

$$f_X\{n_0(t)\} = \frac{\tau}{\sqrt{\pi N_0 T}} e^{-[n_0(t)]^2 \Big/ \left(\frac{N_0 T}{\tau^2}\right)} \quad ...(7.25)$$

This equation describes PDF of white Gaussian noise.
Figure 7.13 shows the graphical representation of this PDF.
From the property of PDF, we know that,

$$P\left[n_0(t) > \frac{AT}{\tau}\right] = \int_{\frac{AT}{\tau}}^{\infty} f_X[n_0(t)]\, d[n_0(t)] \quad ...(7.26)$$

This equation gives the probability that $n_0(t)$ takes values greater than $\frac{AT}{\tau}$.

The above integration gives the area under the curve from $\frac{AT}{\tau}$ onwards. It has been illustrated by shaded region in figure 7.13. Similarly, the probability that $n_0(t)$ attains value less than $\frac{AT}{\tau}$ is given by area under the curve from $-\frac{AT}{\tau}$ onwards on left side. This portion has also been shown shaded in figure 7.13.

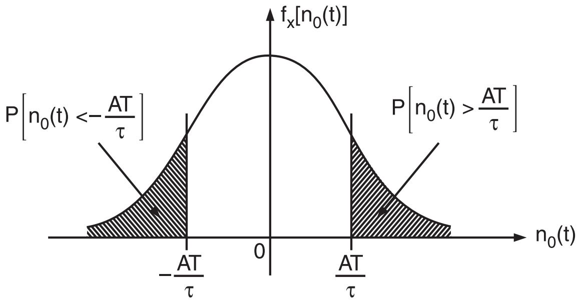


Fig. 7.13. Illustration of PDF of white Gaussian noise having zero mean

Since the PDF curve is symmetric, therefore, we can write.

$$P\left[n_0(t) > \frac{AT}{\tau}\right] = P\left[n_0(t) < -\frac{AT}{\tau}\right] \quad ...(7.27)$$

We know that the above probabilities represent error probability. Because, occurrence of – A or + A is mutually exclusive, therefore, the probability of error is given by either of the two in above equation i.e.,

$$P_e = P\left[n_0(t) > \frac{AT}{\tau}\right] = P\left[n_0(t) < -\frac{AT}{\tau}\right]$$

Substituting value of $P\left[n_0(t) > \frac{AT}{\tau}\right]$ in above equation from equation (7.26), we get

$$P_e = P\left[n_0(t) > \frac{AT}{\tau}\right] = \int_{\frac{AT}{\tau}}^{\infty} f_X[n_0(t)]\, d\,[n_0(t)]$$

Again, substituting value of $f_X[n_0(t)]$ from equation (7.25), in above equation, we get

$$P_e = \int_{\frac{AT}{\tau}}^{\infty} \frac{\tau}{\sqrt{\pi N_0 T}} e^{-[n_0(t)]^2} \Big/ \left(\frac{N_0 T}{\tau^2}\right) d[n_0(t)] \quad ...(7.28)$$

No man ever yet became great by imitation. – **Samuel Johnson**

$$\frac{[n_0(t)]^2}{\frac{N_0T}{\tau^2}} = y^2$$

so that
$$\frac{n_0(t)}{\sqrt{N_0T/\tau}} = y$$

Thus,
$$d[n_0(t)] = \frac{\sqrt{N_0T}}{\tau}\,dy$$

when $n_0(t) \to \infty$, then $y \to \infty$

when
$$n_0(t) = \frac{AT}{\tau}$$

then
$$y = \frac{AT/\tau}{\sqrt{N_0T/\tau}} = \sqrt{\frac{A^2T}{N_0}}$$

With all these substitutions, equation (7.28) takes the form

$$P_e = \int_{\sqrt{\frac{A^2T}{N_0}}}^{\infty} \frac{\tau}{\sqrt{\pi N_0 T}}\, e^{-y^2} \cdot \frac{\sqrt{N_0T}}{\tau}\,dy = \frac{1}{\sqrt{\pi}} \int_{\sqrt{\frac{A^2T}{N_0}}}^{\infty} e^{-y^2}\,dy$$

This equation can be rearranged as under:

$$P_e = \frac{1}{2}\cdot\left[\frac{2}{\sqrt{\pi}} \int_{\sqrt{\frac{A^2T}{N_0}}}^{\infty} e^{-y^2}\,dy\right] \qquad ...(7.29)$$

The integration inside brackets may be evaluated with the help of complementary error function i.e.,

$$\frac{2}{\sqrt{\pi}} \int_{u}^{\infty} e^{-y^2}\,dy = \text{erfc}\,(u)$$

Recall that this is a standard result and generally evaluated using numerical methods. Thus, with the help of this definition, equation (7.29) becomes,

$$P_e = \frac{1}{2}\,\text{erfc}\sqrt{\frac{A^2T}{N_0}} \qquad ...(7.30)$$

This equation describes the probability of error P_e of the integrate and dump filter receiver. Now, since $A^2T = E$, i.e., energy of the bit, therefore, we have

$$P_e = \frac{1}{2}\,\text{erfc}\sqrt{\frac{E}{N_0}} \qquad ...(7.31)$$

✍ **Important point:** *It may be noted that basically 'erfc' is the monotonically decreasing function. Therefore, P_e falls rapidly as the ratio $\frac{E}{N_0}$ increases. Hence, the maximum value of P_e is $\frac{1}{2}$ when $\frac{E}{N_0}$ is very very small. This means that even if the signal is lost entirely in the noise even then, the probability of error P_e will be $\frac{1}{2}$. Thus, the receiver would make incorrect decisions for half number of times.*

7.6 THE OPTIMUM FILTER (i.e., OPTIMUM RECEIVER)

In fact, the goal of the receiving filter is to recover a baseband pulse with the best possible signal-to-noise ratio (SNR), free of any ISI. The optimum receiving filter for accomplishing this is called a **matched filter** or **correlator**.

In the previous article, we discussed integrate and Dump receiver. But, at this point, let us think whether integrate and dump filter is an optimum* filter for the purpose of minimizing the probability of errors P_e. For this purpose, we shall discuss a generalized filter to receive binary coded signals. It is known as optimum filter. Therefore, let us consider the generalized Gaussian noise which is having zero mean. In the last section, we assumed 'white' Gaussian noise of zero mean.

Let the received signal be a binary waveform. Let us again assume that the polar NRZ signal is used to represent binary 1's and 0's. i.e.,

for binary '1',

$$x_1(t) = +A \quad \text{for one bit period T}$$

and for binary '0' ,

$$x_2(t) = -A \quad \text{for one bit period T}$$

Hence, the input signal x(t) will be either $x_1(t)$ or $x_2(t)$ depending upon the polarity of the NRZ signal. Figure 7.14 shows the block diagram of a receiver for such a binary coded signal.

As shown in this figure, the noise n(t) is added to the signal x(t) in the channel during the transmission.

Thus, input to the optimum filter is [x(t) + n(t)] i.e.,

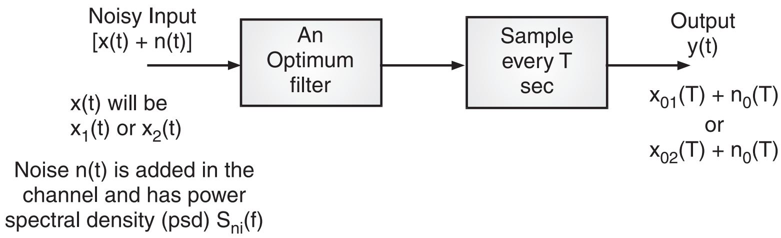


Fig. 7.14. Block diagram of receiver for binary coded signal

Input to the receiver = x(t) + n(t)

and output from the receiver = $x_{01}(T) + n_0(T)$ or $x_{02}(T) + n_0(T)$

Also, in the absence of the noise n(t), the output of the receiver will be,

$$y(T) = x_{01}(T) \quad \text{if} \quad x(t) = x_1(t)$$

and

$$y(T) = x_{02}(T) \quad \text{if} \quad x(t) = x_2(t)$$

Hence, in the absence of noise, decisions are taken clearly. However, if noise is present then, we select $x_1(t)$ if y(T) is closer to $x_{01}(T)$ than $x_{02}(T)$ and we select $x_2(t)$ if y(T) is closer to $x_{02}(T)$ than $x_{01}(T)$.

Therefore, the decision boundary will be in the middle of $x_{01}(T)$ and $x_{02}(T)$.

It is expressed as,

$$\text{Decision boundary} = \frac{x_{01}(T) + x_{02}(T)}{2} \quad ...(7.32)$$

7.6.1. Calculation of Probability of Error (P_e) for Optimum Filter

The probability of error (P_e) may be evaluated on the same basis as in the last section. Here, we shall consider generalized Gaussian noise. Let us consider that $x_2(t)$ was transmitted, but $x_{01}(T)$ is greater than $x_{02}(T)$. If noise $n_0(T)$ is positive and larger in magnitude compared to the

* Here, the word optimum means to have maximum advantages.

voltage difference $\frac{1}{2}[x_{01}(T) + x_{02}(T)] - x_{02}(T)$, then, in this case, the incorrect decision will be taken. This means that the error will be generated if,

$$n_0(T) \geq \frac{x_{01}(T) + x_{02}(T)}{2} - x_{02}(T) \geq \frac{x_{01}(T) - x_{02}(T)}{2} \qquad ...(7.33)$$

We have obtained the probability density function (PDF) for $n_0(t)$ in the previous section. It is given as

$$f_X[n_0(t)] = \frac{1}{\sigma\sqrt{2\pi}} e^{-[n_0(t)]^2/2\sigma^2} \qquad ....(7.34)$$

Here, $n_0(t)$ is the random function whose PDF is given by above equation. Also, σ is its standard deviation and the function has zero mean value.

Hence, to evaluate the probability of error, we must integrate the area under the PDF curve from $n_0(T) \geq \frac{x_{01}(T) - x_{02}(T)}{2}$. This portion of the curve has been shown shaded in figure 7.15.

Therefore, we have

$$P_e = P\left[n_0(T) \geq \frac{x_{01}(T) - x_{02}(T)}{2}\right] = \int_{\frac{x_{01}(T) - x_{02}(T)}{2}}^{\infty} f_X[n_0(t)]\, d[n_0(t)]$$

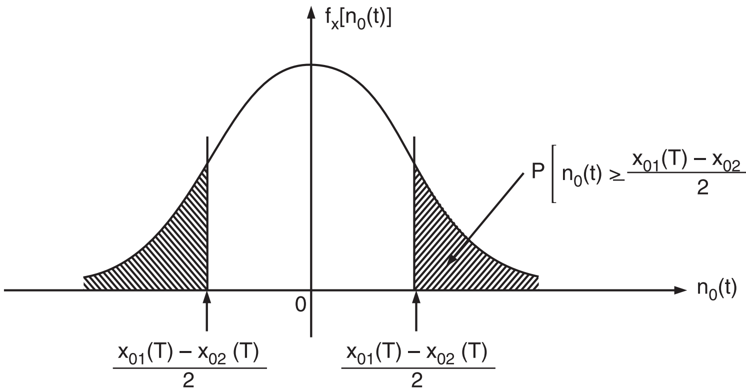


Fig. 7.15. Evaluation of P_e for optimum filter receiver.

Using equation (7.34), the last equation becomes,

$$P_e = \int_{\frac{x_{01}(T) - x_{02}(T)}{2}}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-[n_0(t)]^2/2\sigma^2} \qquad ...(7.35)$$

Let us substitute $\frac{[n_0(t)]^2}{2\sigma^2} = y^2$

so that $n_0(t) = \sigma\sqrt{2}\, y$

or $d\,[n_0(t)] = \sigma\sqrt{2}\, dy$

when $n_0(t) = \infty$, then $y = \infty$

A good teacher must know how to arouse the interest of the pupil in the field of study for which he is responsible. **– S. Radhakrishnan** and then $$n_o(t) = \frac{x_{01}(T) - x_{02}(T)}{2}$$

then, we have $$y = \frac{x_{01}(T) - x_{02}(T)}{2\sqrt{2}\,\sigma}$$

Substituting all these values in equation (7.34), we obtain,

$$P_e = \int_{\frac{x_{01}(T) - x_{02}(T)}{2\sqrt{2}\,\sigma}}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-y^2} \cdot \sigma\sqrt{2}\, dy = \frac{1}{\sqrt{\pi}} \int_{\frac{x_{01}(T) - x_{02}(T)}{2\sqrt{2}\,\sigma}}^{\infty} e^{-y^2} dy$$

Let us rearrange this equation as under:

$$P_e = \frac{1}{2}\left[\frac{2}{\sqrt{\pi}} \int_{\frac{x_{01}(T) - x_{02}(T)}{2\sqrt{2}\,\sigma}}^{\infty} e^{-y^2} dy\right] \quad ...(7.36)$$

To solve this integration, let us use the following standard result.

$$\frac{2}{\sqrt{\pi}} \int_{u}^{\infty} e^{-y^2} dy = \text{erfc}\,(u)$$

Then, equation (7.36) becomes

$$P_e = \frac{1}{2}\,\text{erfc}\left[\frac{x_{01}(T) - x_{02}(T)}{2\sqrt{2}\,\sigma}\right] \quad ...(7.37)$$

Important point: *This is the required expression for error probability P_e of optimum filter. It may be noted that the 'erfc' function is the monotonically decreasing function. Hence, P_e decreases as the difference $x_{01}(T) - x_{02}(T)$ becomes greater and the rms noise voltage σ becomes smaller. The optimum filter has to maximize the ratio $\frac{x_{01}(T) - x_{02}(T)}{\sigma}$ in such a manner that the probability of error P_e is maximum.*

7.6.2. Evaluation of Transfer Function for the Optimum Filter

Now, let us find the transfer function of the optimum filter in such a way that it will maximize the ratio $\frac{x_{01}(T) - x_{02}(T)}{\sigma}$. For this, let us represent the difference signal $x_{01}(T) - x_{02}(T)$ by $x_0(T)$. Thus, we have

$$x_0(T) = x_{01}(T) - x_{02}(T) \quad ...(7.38)$$

This means that the optimum filter has to maximize the ratio $\frac{x_0(T)}{\sigma}$. Now, let us derive the transfer function of the optimum filter such that the square of the ratio $\frac{x_0(T)}{\sigma}$ is maximized. The square of this ratio is,

$$\left(\frac{S}{N}\right)_0 = \frac{x_0^2(T)}{\sigma^2}$$

where $\left(\frac{S}{N}\right)_0$ is known as the signal to noise power ratio. Thus, we are maximizing square of

$\left[\frac{x_0(T)}{\sigma}\right]$ for mathematical convenience. In this equation, $x_0^2(T)$ is the normalized signal power in $1\ \Omega$ load.

Further, $\sigma^2 = \overline{n_0^2(T)} = E[n_0^2(T)]$ is normalized noise power because mean value of noise is zero*. Thus, the optimum filter has to maximize the ratio

$$\left(\frac{S}{N}\right)_0 = \frac{x_0^2(T)}{\overline{n_0^2(T)}} \text{ or } \frac{x_0^2(T)}{E[n_0^2(T)]} \text{ or } \frac{x_0^2(T)}{\sigma^2} \qquad ...(7.39)$$

Again, let $X_0(f)$ be Fourier transform of $x_0(t)$. If $X(f)$ is the Fourier transform of input difference signal $x(t)$ [i.e., $x(t) = x_1(t) - x_2(t)$], then we have,

$$X_0(f) = H(f)\, X(f) \qquad ...(7.40)$$

Here, $H(f)$ is the transfer function of optimum filter. Also, $x_0(T)$ is found by taking inverse Fourier transform of $X_0(f)$ i.e.,

$$x_0(T) = \text{IFT}\,[X_0(f)] = \int_{-\infty}^{\infty} X_0(f)\, e^{j2\pi fT} df$$

We know that

$$x_0(T) = \int_{-\infty}^{\infty} H(f)\, X(f)\, e^{j2\pi fT}\, df \qquad ...(7.41)$$

Also, the input noise to the optimum filter is $n(t)$. Let, its power spectral density (psd) be $S_{ni}(f)$. The output noise of optimum filter is $n_0(t)$. Let its power spectral density be $S_{n_0}(f)$.

We know that the input and output power spectral densities of noise are related as,

$$S_{n_0}(f) = |H(f)|^2\, S_{n_i}(f) \qquad ...(7.42)$$

The normalized noise power can be obtained by integrating the power spectral density.

Thus, we have

$$\sigma^2 = \int_{-\infty}^{\infty} S_{n_0}(f)\, df^{**} = \int_{-\infty}^{\infty} |H(f)|^2\, S_{ni}(f)\, df \qquad ...(7.43)$$

Substituting for $x_0(T)$ from equation (7.41) and σ^2 from equation (7.43) in equation (7.39), we get,

$$\left(\frac{S}{N}\right)_0 = \frac{x_0^2(T)}{\sigma^2} = \frac{\left|\int_{-\infty}^{\infty} H(f)\, X(f)\, e^{j2\pi fT} df\right|^2}{\int_{-\infty}^{\infty} |H(f)|^2\, S_{ni}(f)\, df} \qquad ...(7.44)$$

Here, let us use the well known Schwarz's inequality which states that

$$\left|\int_{-\infty}^{\infty} \theta_1(x)\, \theta_2(x)\, dx\right|^2 \le \int_{-\infty}^{\infty} |\theta_1(x)|^2\, dx \int_{-\infty}^{\infty} |\theta_2(x)|^2\, dx \qquad ...(7.45)$$

Let $\theta_1(f) = \sqrt{S_{ni}(f)}\, H(f)$ and $\theta_2(f) = \frac{1}{\sqrt{S_{ni}(f)}} X(f) e^{j2\pi fT}$

With all these substitutions, the equation (7.44) becomes,

---

* $\because \sigma^2 = \overline{n_0^2(T)} - (\text{mean value})^2$.

** By property of psd.

$$\left(\frac{S}{N}\right)_0 = \frac{\left|\int_{-\infty}^{\infty} \theta_1(f)\cdot\theta_2(f)\,df\right|^2}{\int_{-\infty}^{\infty} |\theta_1(f)|^2\,df}$$

Applying Schwarz's inequality of equation (7.45) to the numerator of above equation, we have

$$\left(\frac{S}{N}\right)_0 \le \frac{\int_{-\infty}^{\infty} |\theta_1(f)|^2\,df \int_{-\infty}^{\infty} |\theta_2(f)|^2\,df}{\int_{-\infty}^{\infty} |\theta(f)|^2\,df}$$

$$\left(\frac{S}{N}\right)_0 \le \int_{-\infty}^{\infty} |\theta_2(f)|^2\,df \le \int_{-\infty}^{\infty} \frac{1}{S_{ni}(f)} |X(f)e^{j2\pi fT}|^2\,df$$

Now, substituting value of $\theta_2(f)$ in the above equation, we have

$$\left|X(f)e^{j2\pi fT}\right|^2 = |X(f)|^2 \qquad [\text{Since}, |e^{j2\pi fT}| = 1]$$

Thus, we have

$$\left(\frac{S}{N}\right)_0 \le \int_{-\infty}^{\infty} \frac{|X(f)|^2}{S_{ni}(f)}\,df \qquad ...(7.46)$$

The signal to noise power ratio expressed by equation (7.46) will be maximum when we consider equality.

In other words, we must have

$$\left(\frac{S}{N}\right)_{0\,max} = \int_{-\infty}^{\infty} \frac{|X(f)|^2}{S_{ni}(f)}\,df$$

This is possible only when equality applies in Schwarz's inequality i.e., in equation (7.45),

$$\left|\int_{-\infty}^{\infty} \theta_1(x)\,\theta_2(x)\,dx\right|^2 = \int_{-\infty}^{\infty} |\theta_1(x)|^2\,dx \int_{-\infty}^{\infty} |\theta_2(x)|^2\,dx$$

This equality is possible only if we have $\theta_1(f) = k\theta^*{}_2(f)$

Substituting the values of $\theta_1(f)$ and $\theta_2(f)$, we obtain,

$$\sqrt{S_{ni}(f)}\,H(f) = k\cdot\frac{1}{\sqrt{S_{ni}(f)}} X^*(f)\,e^{-j2\pi fT}$$

Therefore, when this equation is satisfied, the signal to noise ratio $\left(\frac{S}{N}\right)_0$ will be maximum. In this case, the transfer function H(f) can be obtained from above equation as under:

$$H(f) = k\cdot\frac{X^*(f)}{S_{ni}(f)} e^{-j2\pi fT}$$

It may be noted that at the beginning of this derivation, we stated that

$$\left(\frac{S}{N}\right)_0 = \frac{x_0^2(T)}{\sigma^2} = \left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]^2$$

Now we can have the following conclusion:

The optimum filter which minimizes the probability of error (P_e) has to maximize the ratio $\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]^2$*. To maximize this ratio, the filter has the transfer function given by following expression,*

$$H(f) = k \cdot \frac{X^*(f)}{S_{ni}(f)} e^{-j2\pi fT} \quad \text{...(7.47)}$$

and the maximized ratio is expressed as,

$$\left(\frac{S}{N}\right)_{0\,max} = \left[\frac{x_0^2(T)}{\sigma^2}\right]_{max} = \left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]^2_{max}$$

or
$$\left(\frac{S}{N}\right)_{0\,max} = \int_{-\infty}^{\infty} \frac{|X(f)|^2}{S_{ni}(f)} df \quad \text{...(7.48)}$$

7.7 MATCHED FILTER

Basically, a matched filter is a linear filter designed to provide the maximum signal-to-noise power ratio at its output for a given transmitted symbol waveform. The mathematical operation of a matched filter (MF) is convolution; a signal is convolved with the impulse response of a filter.

In previous section, we have discussed optimum filter. For this filter, we have considered generalized Gaussian noise. When this noise is white Gaussian noise, then the optimum filter is known as matched filter. Recall that for the white Gaussian noise, the power spectral density is given as,

$$S_{ni}(f) = \frac{N_0}{2} \quad \text{...(7.49)}$$

7.7.1. Calculation of Impulse Response for the Matched Filter

The transfer function of the optimum filter is expressed as,

$$H(f) = k \cdot \frac{X^*(f)}{S_{ni}(f)} e^{-j2\pi fT} \quad \text{...(7.50)}$$

In this equation, if we substitute $S_{ni}(f) = \frac{N_0}{2}$ i.e., psd of white noise, then H(f) becomes transfer function of matched filter.

This means that transfer function of matched filter, $H(f) = k \cdot \frac{X^*(f)}{\frac{N_0}{2}} e^{-j2\pi fT}$

or
$$H(f) = \frac{2k}{N_0} X^*(f)\, e^{-j2\pi fT} \quad \text{...(7.51)}$$

From the property of Fourier transform, we know that

$$X^*(f) = X(-f) \quad ...(7.52)$$

Using this property, we can write equation (7.51) as under:

$$H(f) = \frac{2k}{N_0} X(-f)\, e^{-j2\pi fT}$$

The impulse response of a matched filter can be evaluated by taking inverse Fourier transform of above equation.

Thus, we have

$$h(t) = \text{IFT}[H(f)] = \text{IFT}\left[\frac{2k}{N_0} X(-f)\, e^{-j2\pi fT}\right] \quad ...(7.53)$$

The inverse Fourier transform of $X(-f)$ is $x(-t)$ and $e^{-j2\pi fT}$ represents time shift of T seconds. Hence, we have

$$\text{FT}[x(-t)] = X(-f)$$

and $$\text{FT}[x(T-t)] = X(-f)\, e^{-j2\pi fT}$$

With the help of all these properties of Fourier transform, the equation (7.53) takes the form

$$h(t) = \frac{2k}{N_0} x(T-t) \quad ...(7.54)$$

Note that we have considered $x(t) = x_1(t) - x_2(t)$, therefore, equation (7.54) will become,

$$h(t) = \frac{2k}{N_0}[x_1(T-t) - x_2(T-t)] \quad ...(7.55)$$

Thus, two equations (7.54) and (7.55) give the required impulse response of the matched filter.

EXAMPLE 7.3. A polar NRZ waveform has to be received with the help of a matched filter. Here, binary 1 is represented by a rectangular positive pulse. Also, binary zero is represented by a rectangular negative pulse. Determine the impulse response of the matched filter. Also, sketch it.

Solution: Let $x_1(t)$ represent the positive rectangular pulse with duration T as shown in figure 7.16(a).

Let $x_2(t)$ represent a negative rectangular pulse with duration T as shown in figure 7.16(b)

Thus, we have

$$x_1(t) = +A \quad \text{for} \quad 0 \le t \le T$$

and $$x_2(t) = -A \quad \text{for} \quad 0 \le t \le T \quad ...(i)$$

Let us calculate the difference signal x(t) i.e.,

$$x(t) = x_1(t) - x_2(t) \quad \text{for } 0 \le t \le T$$

or $$x(t) = A - (-A)$$

or $$x(t) = 2A \quad \text{for} \quad 0 \le t \le T \quad ...(ii)$$

This difference signal has been shown in figure 7.16(c). The time reversed version of x(t) will be,

$$x(-t) = 2A \quad \text{for} \quad -T \le t \le 0 \quad ...(iii)$$

It may be noted that that in the above equation, the time reference is inverted. This signal has been shown in figure 7.16(d).

Now let us delay $x(-t)$ by T seconds.

i.e., $$x(T-t) = 2A \quad \text{for} \quad 0 \le t \le T \quad ...(iv)$$

The signal has been shown in figure 7.16 (e)

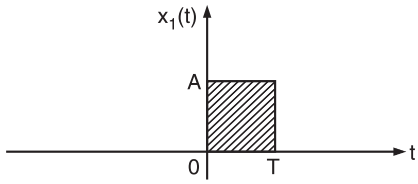
Impulse response of a matched filter is expressed as,

$$h(t) = \frac{2k}{N_0} x(T-t)$$

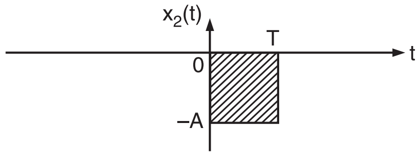
Substituting $x(T-t)$ from equation (iv), we get

$$h(t) = \frac{2k}{N_0} \cdot 2A \quad \text{for} \quad 0 \le t \le T$$

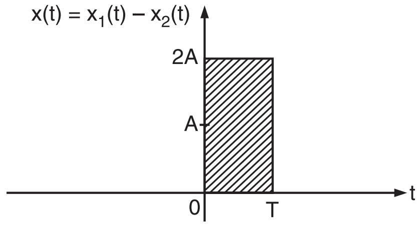
or $$h(t) = \frac{4Ak}{N_0} \quad \text{for} \quad 0 \le t \le T$$



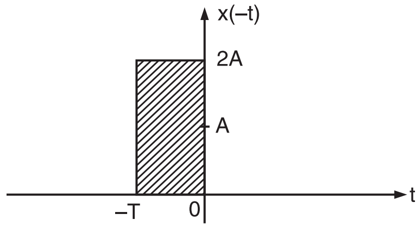
(a) The signal pulse $x_1(t)$ representing binary '1'. It is given by equation (i)



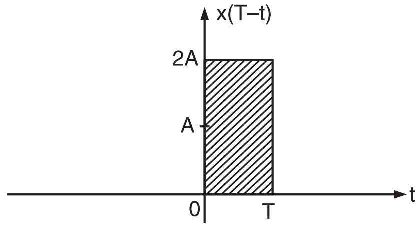
(b) The signal pulse $x_2(t)$ representing binary '0'. It is given by equation (i)



(c) The difference of $x_1(t)$ and $x_2(t)$. It has the amplitude of 2A. It is given by equation (ii)



(d) The difference signal which has been folded in time. It also has amplitude equal to 2A.
It is given by equation (ii)



(e) The difference signal which has been folded in time. It also has amplitude equal to 2A.
It is given by equation (iii)

Fig. 7.16.

Figure 7.17 shows the sketch of this impulse response.

It may be noted that with $\frac{2k}{N_0} = 1$, this figure will be same as figure 7.16(e) and figure 7.16(c).

This proves that the shape of the impulse response of the matched filter is similar (or matched) to the shape of the input signal x(t). ***Hence, it is known as matched filter.***

7.7.2. Calculation of Probability of Error (P_e) for the Matched Filter*

To evaluate the probability of error for matched filter, let us again start with optimum filter and we shall consider the special case of white Gaussian noise.

We know that error probability of optimum filter is expressed as,

$$P_e = \frac{1}{2} \operatorname{erfc}\left[\frac{x_{01}(T) - x_{02}(T)}{2\sqrt{2}\,\sigma}\right] \quad ...(7.56)$$

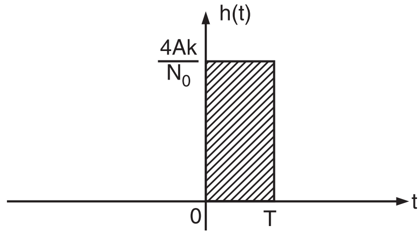


Fig. 7.17. Impulse response of a matched filter for a rectangular pulse input

In this equation, we have

$$\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]^2_{max} = \int_{-\infty}^{\infty} \frac{|X(f)|^2}{S_{ni}(f)} df$$

In this equation, let us substitute $S_{ni}(f)$ for white Gaussian noise.

Thus, we have

$$\text{psd of white noise} = S_{ni}(f) = \frac{N_0}{2}$$

Hence,

$$\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]^2_{max} = \int_{-\infty}^{\infty} \frac{|X(f)|^2}{\frac{N_0}{2}} df = \frac{2}{N_0} \int_{-\infty}^{\infty} |X(f)|^2 df \quad ...(7.57)$$

Also, Parseval's power theorem states that,

$$\int_{-\infty}^{\infty} |X(f)|^2 df = \int_{-\infty}^{\infty} x^2(t) dt = \int_0^T x^2(t) dt \quad ...(7.58)$$

In the last integral we have taken limits from 0 to T because x(t) exists from 0 to T only. We know that $x(t) = x_1(t) - x_2(t)$.

Hence, above equation becomes,

$$\int_{-\infty}^{\infty} |X(f)|^2 df = \int_0^T [x_1(t) - x_2(t)]^2 dt$$

or
$$\int_{-\infty}^{\infty} |X(f)|^2 df = \int_0^T [x_1^2(t) + x_2^2(t) - 2x_1(t)\,x_2(t)]\,dt$$

or
$$\int_{-\infty}^{\infty} |X(f)|^2 df = \int_0^T x_1^2(t)\,dt + \int_0^T x_2^2(t)\,dt - 2\int_0^T x_1(t)\,x_2(t)\,dt \quad ...(7.59)$$

where, $\int_0^T x_1^2(t)\,dt = E_1$ i.e., energy of $x_1(t)$ by standard relations.

and $\int_0^T x_2^2(t)\,dt = E_2$ i.e., energy of $x_2(t)$ by standard relations.

* The matched filter is optimum in the sense that it maximizes the signal-to-noise ratio (SNR) at the decision-making instant.

and $\int_0^T x_1(t)\, x_2(t) = E_{12}$ represents energy due to autocorrelation between $x_1(t)$ and $x_2(t)$.

Now, if we choose $x_1(t) = -\, x_2(t)$, then these energies will be equal, i.e.,

$$E_1 = E_2 = -\, E_{12} = E \quad \text{...(7.60)}$$

Substituting all these values in equation (7.60), we get

$$\int_{-\infty}^{\infty} |X(f)|^2\, df = [E + E - 2(-\, E)] = 4E \quad \text{...(7.61)}$$

Substituting this value of $\int_{-\infty}^{\infty} |X(f)|^2\, df$ in equation (7.57), we get

$$\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]_{max}^2 = \frac{2}{N_0} \cdot 4E = \frac{8E}{N_0} \quad \text{...(7.62)}$$

Therefore,

$$\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]_{max} = 2\sqrt{2} \cdot \sqrt{\frac{E}{N_0}}$$

Substituting this value of $\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]$ in equation (7.56), we get probability of error of matched filter as under:

$$P_e = \frac{1}{2}\,\text{erfc}\sqrt{\frac{E}{N_0}} \quad \text{...(7.63)}$$

✍ **Important Point:** *Here, erfc is the monotonically decreasing function. Therefore, when we substitute* $\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]_{max}$ *as the argument of erfc, we obtain minimum value of probability of error* P_e*. This means that the above equation really provides minimum probability of error of matched filter.*

Thus, minimum error probability of matched filter:

$$P_e = \frac{1}{2}\,\text{erfc}\sqrt{\frac{E}{N_0}} \quad \text{...(7.64)}$$

7.7.3. Few Points about Error Probability of Error, P_e of Matched Filter

From the equation (7.64), we can draw the following important conclusions:

(i) It is obvious from equation (7.64) that the error probability depends only upon the signal energy E. It does not depend on the shape of the signal.

(ii) We have obtained equation (7.64) by considering that $x_1(t) = -\, x_2(t)$. Also, in previous sections we have derived error probability for integrate and dump filter where we have taken $x_1(t) = +\, A$ and $x_2(t) = -\, A$.

In other words, $x_1(t) = -\, x_2(t) = A$ and we obtained error probability of integrator as,

$$P_e = \frac{1}{2}\,\text{erfc}\sqrt{\frac{E}{N_0}} \quad \text{...(7.65)}$$

He who establishes his argument by noise and command shows that his reason is weak.

– Michel de Montaigne

This result is same as error probability of matched filter given by equation (7.65).

Hence, *Integrator is equivalent to matched filter when $x_1(t) = -x_2(t) = A$ or in other words, we can say that for a rectangular bipolar pulse input, the integrate and dump filter is same as a matched filter.*

EXAMPLE 7.4. Prove that the maximum signal to noise ratio for the matched filter is found to be,

$$\left(\frac{S}{N}\right)_{0\max} = \frac{2E}{N_0}$$

Solution: To derive this equation, let us consider the signal to noise power ratio of optimum filter which is given as,

$$\left(\frac{S}{N}\right)_{0\max} = \int_{-\infty}^{\infty} \frac{|X(f)|^2}{S_{ni}(f)} df \qquad \text{...(i)}$$

We know that when white noise is present, then optimum filter is known as matched filter.

PSD of white noise is $S_{ni}(f) = \frac{N_0}{2}$. Substituting this value in equation (i), we get

$$\left(\frac{S}{N}\right)_{0\max} = \int_{-\infty}^{\infty} \frac{|X(f)|^2}{\frac{N_0}{2}} df = \frac{2}{N_0} \int_{-\infty}^{\infty} |X(f)|^2 df \qquad \text{...(ii)}$$

Also, Rayleigh's energy theorem states that,

$$\int_{-\infty}^{\infty} |X(f)|^2 df = \int_{-\infty}^{\infty} x^2(t)\, dt = E \text{ (energy)} \qquad \text{...(iii)}$$

With this result, equation (ii) becomes

$$\left(\frac{S}{N}\right)_{0\max} = \frac{2}{N_0} \cdot E$$

Hence, maximum signal to noise power ratio of matched filter will be

$$\left(\frac{S}{N}\right)_{0\max} = \frac{2E}{N_0} \qquad \text{...(iv)}$$

Now, this equation can be rearranged as under:

$$\left(\frac{S}{N}\right)_{0\max} = \frac{E}{\frac{N_0}{2}}$$

Here, E is energy of signal x(t) and $\frac{N_0}{2}$ is power spectral density (psd) of white noise.

Thus, we have

$$\left(\frac{S}{N}\right)_{0\max} = \frac{\text{Energy of the signal } x(t)}{\text{psd of white noise}}$$

Hence Proved

EXAMPLE 7.5. Prove that for a matched filter, the maximum signal component occurs at t = T (i.e., sampling instant) and has magnitude equal to E, i.e., energy of the signal x(t).

Solution: We know that the Fourier transform of the output signal $x_0(t)$ is expressed as,

$$X_0(f) = X(f)\, H(f) \qquad \text{...(i)}$$

where X(f) = Fourier transform of input signal x(t)

and H(f) = Transfer function of matched filter

The jealous man knows nothing, suspects a great deal and fears everything. – **Curt Goetz** where $E = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|S(\omega)^2\right| d\omega$, energy of input signal.

EXAMPLE 7.14. Compute the matched filter output over (0, T) to the pulse waveform

$$s(t) = \begin{cases} e^{-t} & 0 \le t \le T \\ 0 & \text{otherwise} \end{cases}$$

Solution: For the given s(t), the impulse response of the matched filter is,

$$h(t) = s(T - t).$$

or $$h(t) = e^{-(T - t)}$$

Now, output $$z(t) = s(t) \otimes h(t)$$

or $$z(t) = \int_{-\infty}^{\infty} e^{-\tau} . e^{-(T - 1 + \tau)} . d\tau$$

or $$z(t) = -\frac{e^{-T}}{2} . \left[e^{t - 2\tau}\right]_0^t = e^{-T} . \frac{e^t - e^{-t}}{2} = e^{-T} \sin ht \qquad \textbf{Ans.}$$

7.8 THE CORRELATOR : COHERENT RECEPTION

The mathematical operation of a correlator is correlation; a signal is correlated with a replica of itself.

In fact, the term 'matched filter' is often used synonymously with 'correlator'.

Key Point: It is important to note that the correlator output and the matched filter output are the same only at time t = T.

In this article, we shall discuss a little different type of receiver which is known as **correlator**. Figure 7.27 shows the block diagram of this correlator.

In figure 7.27, f(t) represents input noisy signal,

i.e., $$f(t) = x(t) + n(t).$$

The signal f(t) is multiplied to the locally generated replica of input signal x(t). Then, result of multiplication f(t) · x(t) is integrated. The output of the integrator is sampled at t = T (i.e., end of one symbol period). Then based on this sampled value, decision is made. This is how the correlator works. It is known as correlator because it correlates the received signal f(t) with a stored replica of the known signal x(t). In the block diagram of figure 7.27, the product f(t) x(t) is integrated over one symbol period, i.e., T.

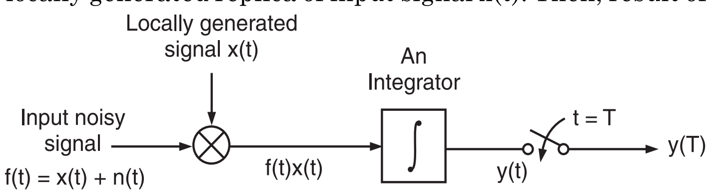


Fig. 7.27. Block diagram of a correlator

Thus, output y(t) will be,

$$y(t) = \int_0^T f(t)\, x(t)\, dt$$

At t = T, this equation will become,

A consistent man believes in destiny, a capricious man in chance. – **Benjamin Disraeli** output of correlator,

$$y(T) = \int_0^T f(t)\, x(t)\, dt \qquad ...(7.66)$$

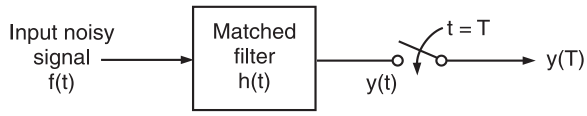


Fig. 7.28. Block diagram of a matched filter receiver

Now consider the matched filter as shown in figure 7.28.

In this block diagram, it may be observed that the matched filter does not need locally generated replica of input signal x(t). The output of the matched filter is obtained by convolution of input f(t) and its impulse response h(t).

This means that

$$y(t) = f(t) \otimes h(t) = \int_{-\infty}^{\infty} f(\tau) \cdot h(t-\tau)\, d\tau \qquad ...(7.67)$$

We know that the impulse response h(t) of the matched filter is given by,

$$h(t) = \frac{2k}{N_0} x(T-t) \qquad ...(7.68)$$

Therefore, $\quad h(t-\tau) = \dfrac{2k}{N_0} x(T-t+\tau)$

Substituting this value of $h(t-\tau)$ in equation (7.67), we get

$$y(t) = \int_{-\infty}^{\infty} f(\tau) \frac{2k}{N_0} x(T-t+\tau)\, d\tau$$

Because the integration is performed over one bit period, therefore, we can change integration limits from 0 to T.

Hence, we write

$$y(t) = \frac{2k}{N_0} \int_0^T f(\tau)\, x(T-t+\tau)\, d\tau$$

At t = T, the last equation becomes,

$$y(T) = \frac{2k}{N_0} \int_0^T f(\tau)\, x(T-T+\tau)\, d\tau = \frac{2k}{N_0} \int_0^T f(\tau)\, x(\tau)\, d\tau$$

Now, let us substitute $\tau = t$ just for convenience for notation, then we have output of matched filter

$$y(T) = \frac{2k}{N_0} \int_0^T f(t)\, x(t)\, dt \qquad ...(7.69)$$

Important point: *This equation (i.e., 7.69) gives the output of matched filter. It may be observed that this equation and equation (7.66) (which gives output of correlator) are identical. In equation (7.69) the constant $\frac{2k}{N_0}$ is present which can be normalized to 1. The similarity between equation (7.66) and equation (7.69) shows that the matched filter and correlator provides same output.*

7.12 PROBABILITY OF ERROR (P_e)

The receiver or detector is supposed to make a correct decision about which symbol is transmitted by a transmitter. But it is not always possible for a practical receiver to make a correct decision. The wrong decision of receiver is called as an error and the probability of such error is called as error probability. The error probability is denoted by P_e. All the errors should be made to reduce the error probability to its minimum possible value. The error probability P_e is also called as the average probability of symbol error. A system with smaller value of P_e is better than the system having larger value of P_e.

7.12.1 Relation between P_e and Euclidean Distance

In previous chapter we have defined the term Euclidean distance (d). The error probability depends on the Euclidean distance. The error probability P_e decreases with increase in the Euclidean distance.

7.13. INTRODUCTION TO THE ERROR PERFORMANCE

The performance of digital communication system can be measured and compared on the basis of their error probability P_e. We are going to take help of the expressions derived for the probability of error of the matched filters or the optimum filter. Probability of error P_e is the probability of the detector making an incorrect decision. The error probability is of two types:

(i) Probability of bit error (P_B)

(ii) Probability of symbol error

Probability of bit error is used for the binary communication system whereas the probability of symbol error is used for the communication system with the M-ary communication systems. In the following sections, we shall obtain the bit error probabilities of the binary systems such as, binary ASK, binary PSK, differentially encoded binary PSK (DEPSK), binary FSK etc., and compare their error performance. The detection of these signal can be done by using the coherent detection (using optimum or matched filters) or using the non-coherent detection, using a bank of filters (for FSK).

7.14 ERROR PROBABILITY OF ASK

We know that the ASK (amplitude shift keying) signal is represented as:

Binary 1: $x_1(t) = \sqrt{2\,P_s}\cos\omega_c t$

Binary 0: $x_2(t) = 0$

where P_s = Average normalized signal power

$$P_s = \frac{A^2}{2}$$

Now, let us obtain the expression for probability of error using the matched filter. We have already derived the expression of error probability with optimum filter as

$$P_e = \frac{1}{2}\,\mathrm{erfc}\left[\frac{x_{01}(T) - x_{02}(T)}{2\sqrt{2}\,\sigma}\right] \quad \text{... (7.76)}$$

We have also derived the expression for maximum output signal to noise ratio of an optimum filter as,

$$\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]^2_{max} = \int_{-\infty}^{\infty} \frac{|X(f)|^2}{S_{ni}(f)}\,df \quad \text{... (7.77)}$$

The value of psd for a white noise input is $S_{ni}(f) = N_0/2$. Substituting this into equation (7.77), we get the expression for maximum signal to noise ratio of a matched filter as:

$$\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]^2_{max} = \frac{2}{N_0}\int_{-\infty}^{\infty} |X(f)|^2 \, df \quad \dots (7.78)$$

According to the Rayleigh's energy theorem,

$$\int_{-\infty}^{\infty} |X(f)|^2 \, df = E = \int_{-\infty}^{\infty} x^2(t) \, dt \quad \dots (7.79)$$

Substituting this value into equation (7.78), we get

$$\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]^2_{max} = \frac{2}{N_0}\int_{-\infty}^{\infty} x^2(t) \, dt \quad \dots (7.80)$$

But as the signal x(t) is present only over a bit interval T, the limits of integration will change from 0 to T.

Therefore, $$\left[\frac{x_{o1}(T) - x_{o2}(T)}{\sigma}\right]^2_{max} = \frac{2}{N_0}\int_{0}^{T} x^2(t) \, dt \quad \dots (7.81)$$

where $x(t) = x_1(t) - x_2(t)$

But in case of ASK, $x_2(t) = 0$. Therefore $x(t) = x_1(t) = \sqrt{2 P_s} \cos \omega_c t$

We have $$\left[\frac{x_{o1}(T) - x_{o2}(T)}{\sigma}\right]^2_{max} = \frac{2}{N_0}\int_{0}^{T} 2 P_s \cos^2 \omega_c t \, dt \quad \dots (7.82)$$

or $$\left[\frac{x_{o1}(T) - x_{o2}(T)}{\sigma}\right]^2_{max} = \frac{4 P_s}{N_0}\int_{0}^{T} \cos^2 \omega_c t \, dt \quad \dots (7.83)$$

But, $$\cos^2 \omega_c t = \frac{1 + \cos 2\omega_c t}{2}$$

or $$\left[\frac{x_{o1}(T) - x_{o2}(T)}{\sigma}\right]^2_{max} = \frac{4 P_s}{N_0}\int_{0}^{T} \frac{1 + \cos 2\omega_c t}{2}$$

or $$\left[\frac{x_{o1}(T) - x_{o2}(T)}{\sigma}\right]^2_{max} = \frac{2 P_s}{N_0}\left\{\int_{0}^{T} 1 \, dt + \int_{0}^{T} \cos 2\omega_c t \, dt\right\}$$

or $$\left[\frac{x_{o1}(T) - x_{o2}(T)}{\sigma}\right]^2_{max} = \frac{2 P_s}{N_0}\left\{[t]_0^T + \frac{1}{2\omega_c}[\sin 2\omega_c t]_0^T\right\}$$

or $$\left[\frac{x_{o1}(T) - x_{o2}(T)}{\sigma}\right]^2_{max} = \frac{2 P_s}{N_0}\left\{T + \frac{\sin 2\omega_c T}{2\omega_c}\right\} \quad \dots (7.84)$$

Let us consider the second term on the right hand side of equation (7.84) i.e.,

$$\sin 2\omega_c T = \sin(2 \times 2 \times \pi f_c T)$$
$$= \sin 4\pi f_c T \quad \dots (7.85)$$

We assume that the frequency of the carrier signal (f_c) is selected such that there are k number of complete cycles of the carrier during one bit duration T, as shown in figure 7.30.

In figure 7.30, we have k = 2.

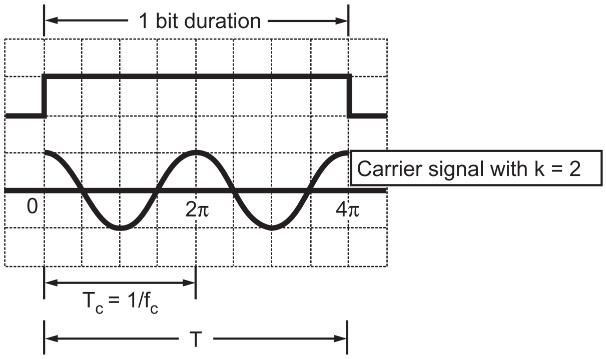


Fig. 7.30. Relation between bit period T and carrier frequency

From figure 7.30, it is obvious that,

$$\text{one bit period} \quad T = 2\,T_c = \frac{2}{f_c} \quad \text{as } T_c = \frac{1}{f_c}$$

$$\text{or} \qquad f_c\,T = 2 \qquad \text{... (7.86)}$$

Here, k = 2, therefore, equation (7.86) can be written in general as under:

$$f_c\,T = k \qquad \text{... (7.87)}$$

Substituting this value into equation (7.85), we get,

$$\sin 2\omega_c\,T = \sin 4\pi k \qquad \text{where } k = 1, 2 \qquad \text{.... (7.88)}$$

$$\text{or} \qquad \sin 2\,\omega_c\,T = \sin 4\pi, \sin 8\,\pi, \sin 12\,\pi\text{...etc., for } k = 1, 2, 3, \text{.....}$$

$$\text{or} \qquad \sin 2\,\omega_c\,T = 0 \text{ for all values of } k.$$

Therefore, equation (7.84) gets modified as under:

$$\left[\frac{x_{o1}(T) - x_{o2}(T)}{\sigma}\right]^2_{max} = \frac{2P_s\,T}{N_0} \qquad \text{... (7.89)}$$

Taking square root of both the sides, we get

$$\left[\frac{x_{o1}(T) - x_{o2}(T)}{\sigma}\right]_{max} = \sqrt{\frac{2\,P_s\,T}{N_0}} \qquad \text{... (7.90)}$$

Now, let us substitute equation (7.90) into equation (7.76) to obtain the error probability for ASK as under:

$$P_e = \frac{1}{2}\,\text{erfc}\left[\frac{\sqrt{\frac{2\,P_s\,T}{N_0}}}{2\sqrt{2}}\right]$$

$$\text{or} \qquad P_e = \frac{1}{2}\,\text{erfc}\left[\sqrt{\frac{P_s\,T}{4\,N_0}}\right] \qquad \text{... (7.91)}$$

This is the expression for the bit error probability denoted by P_B.

$$P_B = \frac{1}{2}\,\text{erfc}\left[\sqrt{\frac{P_s\,T}{4\,N_0}}\right]$$

But, P_s T = E = Energy of the signal. (Energy per bit)

Therefore, $$P_B = \frac{1}{2}\,\text{erfc}\left[\sqrt{E/4\,N_0}\right]$$

This is the expression for bit error probability of ASK using the matched filter.

✍ **Important point:** The complementary error function erfc(x) is a monotonically decreasing function of x. This means that as x increases, the value of erfc x decreases. Hence, the bit error probability P_B decreases with increase in the ratio E/4 N_0. Thus, the error probability depends only on the signal energy and not its shape or any other parameter.

7.15. ERROR PROBABILITY OF BPSK (WITH COHERENT DETECTION)

The various steps to follow in order to obtain the expression for the error probability, are exactly same as those followed to obtain error probability of an ASK system.

We know that the BPSK signal is represented as follows:

Binary 1: $x_1(t) = -\sqrt{2\,P_s}\;\cos\omega_c t$

Binary 2: $x_2(t) = \sqrt{2\,P_s}$

Therefore, $x_2(t) = -\,x_1(t)$

We have to use the matched filter for detection of BPSK signal. The expression for error probability of an optimum filter is given by

$$P_e = \frac{1}{2}\,\text{erfc}\left[\frac{x_{o1}(T) - x_{o2}(T)}{2\sqrt{2}\,\sigma}\right] \quad \text{... (7.92)}$$

The expression for the signal to noise ratio of a matched filter is given by,

$$\left[\frac{x_{o1}(T) - x_{o2}(T)}{\sigma}\right] = \frac{2}{N_0}\int_{-\infty}^{\infty} |X(f)|^2\,df \quad \text{... (7.93)}$$

Using the Rayleigh's energy theorem, we have

$$\int_{-\infty}^{\infty} |X(f)|^2\,df = \int_{-\infty}^{+\infty} x^2(t)dt = \int_0^T x^2(t)\,dt \quad \text{... (7.94)}$$

The limits of integration of the last term in equation (7.94) are 0 to T because x(t) is present only over one bit interval T. Substituting equation (7.94) into equation (7.93), we get,

$$\left[\frac{x_{01}(\tau) - x_{o2}(T)}{\sigma}\right]^2_{max} = \frac{2}{N_0}\int_0^T x^2(t)dt \quad \text{...(7.95)}$$

But, $x(t) = x_1(t) - x_2(t)$

and for BPSK, $x_2(t) = -\,x_1(t)$

Therefore, $x(t) = 2x_1(t) = 2\sqrt{2\,P_s}\,\cos\omega_c\,t$

Substituting this value of x(t) into equation (7.95), we get

$$\left[\frac{x_{o1}(T) - x_{o2}(T)}{\sigma}\right]^2_{max} = \frac{2}{N_0}\int_0^T 8\,P_s\cos^2\omega_c\,t\,dt$$

$$= \frac{16\,P_s}{N_0}\int_0^T \cos^2 \omega_c\, t\, dt \quad \text{... (7.96)}$$

But $$\cos^2 \omega_c t = \frac{1+\cos 2\omega_c t}{2}$$

Substituting this value of $\cos^2 \omega_c t$ into equation (7.96), we obtain

$$\left[\frac{x_{o1}(T)-x_{o2}(T)}{\sigma}\right]^2_{max} = \frac{16\,P_s}{N_0}\int_0^T \frac{1+\cos 2\,\omega_c\, t}{2}$$

or $$\left[\frac{x_{o1}(T)-x_{o2}(T)}{\sigma}\right]^2_{max} = \frac{8\,P_s}{N_0}\left[\int_0^T 1\, dt + \int_0^T \cos 2\,\omega_c\, t\, dt\right]$$

or $$\left[\frac{x_{o1}(T)-x_{o2}(T)}{\sigma}\right]^2_{max} = \frac{8\,P_s}{N_0}\left\{[t]_0^T + \frac{1}{2\,\omega_c}(\sin 2\,\omega_c\, t)_0^T\right\}$$

or $$\left[\frac{x_{o1}(T)-x_{o2}(T)}{\sigma}\right]^2_{max} = \frac{8\,P_s}{N_0}\left\{T + \frac{\sin 2\,\omega_c\, T}{2\,\omega_c}\right\} \quad \text{...(7.97)}$$

The value of second term in the RHS of equation (7.97) is zero,

Therefore, $$\left[\frac{x_{o1}\,(T)-x_{o2}(T)}{\sigma}\right]^2_{max} = \frac{8\,P_s T}{N_0} \quad \text{... (7.98)}$$

But P_s T = Energy E.

Therefore, $$\left[\frac{x_{o1}(T)-x_{o2}(T)}{\sigma}\right]^2_{max} = \frac{8\,E}{N_0}$$

Taking the square root of both sides, we have

Therefore, $$\left[\frac{x_{o1}(T)-x_{o2}(T)}{\sigma}\right]_{max} = \sqrt{\frac{8E}{N_0}} \quad \text{... (7.99)}$$

Substituting this expression into equation (7.99), we get the error probability for BPSK as under:

$$P_e = \frac{1}{2}\,\text{erfc}\left[\frac{1}{2\sqrt{2}}\cdot\sqrt{\frac{8E}{N_0}}\right]$$

Therefore, $$P_e = \frac{1}{2}\,\text{erfc}\left[\sqrt{\frac{E}{N_0}}\right] \quad \text{... (7.100)}$$

This is the expression for error probability of BPSK with matched filter receiver.

This is the expression for bit error probability P_B.

It indicates that the probability of error depends on the energy contents of the signal E. It does not depend on the shape of the signal. As the energy increases, value of erfc function will decrease and the probability of error will also reduce.

This result can be expressed in terms of the Q functions as under:

$$P_B = Q\sqrt{\frac{2E}{N_0}} \quad \text{... (7.101)}$$

This is because the relation between erfc and Q function is,

$$Q(x) = \frac{1}{2}\,\text{erfc}\,(x/\sqrt{2}) \text{ and erfc}\,(x) = 2\,Q\left(x\sqrt{2}\right)$$

Important Point: The erfc function is a monotonic decreasing function. Therefore the value of erfc $\left[\sqrt{E/N_0}\right]$ will be less than erfc $\left[\sqrt{E/4N_0}\right]$. Hence the error probability with BPSK technique is less than with the ASK technique. Hence BPSK system will be superior in performance as compared to the ASK system.

EXAMPLE 7.15. Obtain the error probability for a BPSK system having a bit rate of 1 Mbits/s. The receiver receives the waveforms $S_1(t) = A\cos\omega_c t$ and $S_2(t) = -A\cos\omega_c t$. The received signals are coherently detected using a matched filter. If A = 10 mV and single sided noise power spectral density is $N_0 = 10^{-11}$ W/Hz. Assume that the signal power and energy per bit are normalized.

Solution: We know that
$$A = \sqrt{\frac{2\,E_b}{T}} = 10 \times 10^{-3}\text{ V}$$

But T = one bit period = $\dfrac{1}{1\times 10^6} = 1\ \mu s$

Therefore,
$$E_b = \frac{A^2 T}{2} = \frac{(10\times 10^{-3})^2 \times 1\times 10^{-6}}{2}$$

or,
$$E_b = 5 \times 10^{-11}\text{ Joules}$$

This is the energy per bit.

Therefore,
$$\sqrt{\frac{E_b}{N_0}} = \sqrt{\frac{5\times 10^{-11}}{1\times 10^{-11}}} = 2.24$$

Therefore, bit error probability
$$P_B = \frac{1}{2}\,\text{erfc}\left[\sqrt{\frac{E_b}{N_0}}\right] = \frac{1}{2}\,\text{erfc}\,|2.24| = \frac{1}{4}\times 0.00041$$

$$P_B = 2.05 \times 10^{-4}$$ **Ans.**

7.15.1 Probability of Bit Error for Coherently Detected Differentially Encoded BPSK (DEBPSK)

We have discussed DEBPSK system eariler. Sometimes a signal experiences an inversion while travelling through a networks. (Such as the telephone network). The conventional BPSK system cannot identify the phase reversal but the DEBPSK can. The type of detection used for DEBPSK also is the coherent detection using the matched filter. The probability of bit error for coherently detected, differentially encoded DEBPSK is given by,

$$P_B = 2Q\left(\sqrt{\frac{2\,E}{N_0}}\right)\left[1 - Q\left(\sqrt{\frac{2\,E}{N_0}}\right)\right] \quad \text{... (7.102)}$$

But
$$Q(x) = \frac{1}{2}\,\text{erfc}\left(x/\sqrt{2}\right)$$

Therefore,
$$P_B = 2\times\frac{1}{2}\,\text{erfc}\left(\sqrt{\frac{2\,E}{2\,N_0}}\right)\left[1 - \frac{1}{2}\,\text{erfc}\left(\sqrt{\frac{2\,E}{2\,N_0}}\right)\right]$$

$$P_B = \text{erfc}\left(\sqrt{E/N_0}\right)\left[1 - \frac{1}{2}\,\text{erfc}\left(\sqrt{E/N_0}\right)\right] \quad \text{... (7.103)}$$

This expression shows that error performance has degraded as compared to that of coherently detected BPSK. This is because of the differential encoding. The differential encoding results in a pair of errors at the receiver, due to a single error introduced in the system.

7.16 BIT ERROR PROBABILITY OF COHERENTLY DETECTED BFSK

Here, also, let us assume that a matched filter is being used for the detection of BFSK signal. In BFSK (binary frequency shift keying), the received signal is as follows:

Binary 1: $x_1(t) = A \cos (\omega_c + \Omega)\, t$

Binary 2: $x_2(t) = A \cos (\omega_c - \Omega)\, t$

We know that one way of synthesizing a matched filter is to construct a correlation receiver system, because the correlation receiver will give exactly the same performance as a matched filter provided that the locally generated waveform is $[x_1(t) - x_2(t)]$

Therefore, local signal: $x_1(t) - x_2(t) = A \cos (\omega_c + \Omega)\, t - A \cos (\omega_c - \Omega)t$ ... (7.104)

But, $A = \sqrt{2\,P_s}$

Therefore, $x_1\,(t) - x_2\,(t) = \sqrt{2\,P_s}\;[\cos (\omega_c + \Omega)\, t - \cos (\omega_c - \Omega)t]$ ... (7.105)

We know that the error probability of a matched filter is given by,

$$P_e = \frac{1}{2}\,\text{erfc}\left[\frac{x_{o1}(T) - x_{o2}(T)}{2\sqrt{2}\,\sigma}\right] \quad \text{... (7.106)}$$

And for a matched filter detection, the maximum signal to noise ratio is given by,

$$\left[\frac{x_{o1}(T) - x_{o2}(T)}{\sigma}\right]^2_{max} = \frac{2}{N_0}\int_0^T x^2\,(t)\,dt$$

But $x(t) = x_1(t) - x_2(t)$

or, $x(t) = \sqrt{2\,P_s}\;[\cos (\omega_c + \Omega)\, t - \cos (\omega_c - \Omega)t]$ [From equation (7.105)]

Substituting this value of x(t) into equation (7.95), we get,

$$\left[\frac{x_{o1}(T) - x_{o2}(T)}{\sigma}\right]^2_{max} = \frac{2}{N_0}\int_0^T \left\{\sqrt{2\,P_s}\,[\cos(\omega_c + \Omega)\,t - \cos(\omega_c - \Omega)t]\right\}^2 dt \quad \text{... (7.107)}$$

But we do not have value of $x^2(t)$. i.e., the RHS of the above equation.

To obtain the value of $x^2(t)$, we have

$$x^2(t) = \left\{\sqrt{2\,P_s}\,[\cos\,(\omega_c + \Omega)\,t - \cos(\omega_c - \Omega)t]\right\}^2$$

$$= 2\,P_s\,[\cos\,(\omega_c + \Omega)\,t - \cos\,(\omega_c - \Omega)t]^2 \quad \text{... (7.108)}$$

Let us use the following standard trigonometric identity which states that,

$\cos\,(A + B) - \cos\,(A - B) = - 2 \sin A \sin B$

Applying this identity to equation (7.108), we get,

$$x^2(t) = 2\,P_s\,[-\,2 \sin(\omega_c t)\,\sin\,(\Omega t)]^2$$

$$x^2(t) = 2\,P_s\,[4 \sin^2\,(\omega_c t)\,\sin^2\,(\Omega t)]$$

Therefore, $x^2(t) = 2\,P_s\,[2 \sin^2\,(\omega_c t) \times 2 \sin^2\,(\Omega t)]$ ... (7.109)

We know that $2 \sin^2 \theta = 1 - \cos 2\,\theta$

Therefore, $x^2(t) = 2\,P_s\,[(1 - \cos 2\,\omega_c t)\,(1 - \cos\,2\Omega t)]$

$$x^2(t) = 2\,P_s\,[\,1 - \cos\,2\Omega t - \cos\,2\omega_c t + \cos\,2\,\omega_c t\,\cos\Omega t] \quad \text{...(7.110)}$$

In this equation, we use: $\cos A \cos B = \frac{1}{2}[\cos\,(A - B) + \cos\,(A + B)]$ to get,

$$x^2(t) = 2\,P_s\left[1 - \cos 2\,\Omega t - \cos 2\omega_c t + \frac{1}{2}\cos 2(\omega_c - \Omega)t + \frac{1}{2}\cos 2(\omega_c + \Omega)t\right] \quad \text{...(7.111)}$$

Thus, we have obtained the expression for $x^2(t)$

Now, we obtain the value of $\int_0^T x^2(t)\,dt$.

Taking integration of the sides of equation (7.111), we get,

$$\int_0^T x^2(t)\,dt = \int_0^T 2P_s\left\{1-\cos 2\,\Omega t-\cos 2\omega_c t+\frac{1}{2}[\cos 2(\omega_c-\Omega)t+\cos 2\,(\omega_c+\Omega)\,t]\right\}dt$$

or $$\int_0^T x^2(t)\,dt = 2P_s\left\{\int_0^T 1\,dt-\int_0^T \cos 2\,\Omega t\,dt-\int_0^T \cos 2\,\omega_c\,t\,dt+\frac{1}{2}\int_0^T \cos 2\,(\omega_c-\Omega)\,t\,dt+\frac{1}{2}\int_0^T \cos 2\,\omega_c+\Omega)t\,dt\right\}$$

or $$\int_0^T x^2(t)\,dt = 2P_s\left\{T-\frac{\sin 2\Omega\,T}{2\Omega}-\frac{\sin 2\,\omega_c\,T}{2\omega_c}+\frac{1}{2}\frac{\sin 2(\omega_c-\Omega)T}{2(\omega_c-\Omega)}+\frac{1}{2}\frac{\sin 2(\omega_c+\Omega)}{2\,(\omega_c+\Omega)}\right\} \quad \text{... (7.112)}$$

Therefore,

$$\int_0^T x^2(t)\,dt = 2\,P_s\,T\left\{1-\frac{\sin 2\,\Omega T}{2\Omega T}-\frac{\sin 2\,\omega_c T}{2\omega_c\,T}+\frac{1}{2}\frac{\sin 2(\omega_c-\Omega)T}{2(\omega_c-\Omega)T}+\frac{1}{2}\frac{\sin 2(\omega_c+\Omega)T}{2\,(\omega_c+\Omega)T}\right\} \quad \text{...(7.113)}$$

If we assume that the offset angular frequency Ω is very small as compared to the carrier angular frequency ω_c, then the last three terms in equation (7.113) each will have a form (sin 2 ω_cT)/2 ω_cT. This ratio approaches zero as the value of $\omega_c t$ increases.

Generally, ω_c T >> 1 therefore the last three terms in RHS of equation (7.113) can be neglected. Therefore, equation (7.113) gets modified to.

$$\int_0^T x^2(t)\,dt = 2\,P_s\,T\left[1-\frac{\sin 2\Omega T}{2\,\Omega T}\right] \quad \text{... (7.114)}$$

Thus, we have obtained the expression for $\int_0^T x^2(t)\,dt$. Now, substituting it into equation (7.95), we get

$$\left[\frac{x_{o1}(T)-x_{o2}(T)}{\sigma}\right]^2_{max} = \frac{2}{N_0}\times 2\,P_s T\left[1-\frac{\sin 2\Omega T}{2\,\Omega T}\right]$$

or $$\left[\frac{x_{o1}(T)-x_{o2}(T)}{\sigma}\right]^2_{max} = \frac{4\,P_s\,T}{N_0}\left[1-\frac{\sin 2\,\Omega T}{2\,\Omega T}\right] \quad \text{... (7.115)}$$

The value of the quantity $\left[\frac{x_{o1}(T)-x_{o2}(T)}{\sigma}\right]^2_{max}$ in equation (7.115) attains its largest value when Ω is selected so that $2\,\Omega T = \frac{3\pi}{2}$. Substituting this value of Ω into equation (7.115), we get,

$$\left[\frac{x_{o1}(T)-x_{o2}(T)}{\sigma}\right]^2_{max} = \frac{4\,P_s\,T}{N_0}\left[1-\frac{\sin(3\pi/2)}{(3\pi/2)}\right]$$

or $$\left[\frac{x_{o1}(T)-x_{o2}(T)}{\sigma}\right]^2_{max} = \frac{4\,P_s T}{N_0}\left[1-\frac{(-1)}{(3\pi/2)}\right] = 4.84\frac{P_s T}{N_0} \quad \text{... (7.116)}$$

Taking square root of both the sides, we get

$$\left[\frac{x_{o1}(T)-x_{o2}(T)}{\sigma}\right]_{max} = \sqrt{\frac{4.84\,P_s\,T}{N_0}} \qquad \text{... (7.117)}$$

Substituting this value of $\left[\frac{x_{o1}(T)-x_{o2}(T)}{\sigma}\right]_{max}$ into equation (7.92), we obtain the error probability as under:

$$P_e = \frac{1}{2}\,\text{erfc}\left[\frac{1}{2\sqrt{2}}\cdot\sqrt{\frac{4.84\,P_s\,T}{N_0}}\right]$$

Therefore, $$P_e = \frac{1}{2}\,\text{erfc}\left[\sqrt{\frac{0.6\,P_s\,T}{N_0}}\right]$$

But P_s T = Energy E of one bit.

Therefore, $$P_e = \frac{1}{2}\,\text{erfc}\left[\sqrt{\frac{0.6\,E}{N_0}}\right] \qquad \text{.. (7.118)}$$

This is the required expression for bit error probability of BFSK when matched filter is used for detection (P_B). Let us compare this equation with the probability of error obtained for BPSK equation. As the erfc is a monotonic decreasing function, the error probability for BFSK system is higher than that of a BPSK system. This happens because in BPSK, $x_2(t) = -x_1(t)$ and in BFSK this conditions is not satisfied.

7.17 BIT ERROR PROBABILITY FOR NON-COHERENTLY DETECTED BINARY ORTHOGONAL FSK

Now, let us consider the BFSK system in which the non-coherent detection using the bandpass filter is done. The bit error probability of such a system is given by,

$$P_B = \frac{1}{2}\exp\left[\frac{-A^2}{4\,N_0\,W_f}\right] \qquad \text{... (7.119)}$$

where A = Peak signal amplitude

$N_0/2$ = PSD of white noise

W_f = Filter bandwidth

Equation (7.119) states that the error performance of the non-coherent BFSK system is dependent on the bandwidth of the filter W_f. The error probability decreases with decreases in the filter bandwidth W_f. This expression is valid only when the intersymbol interference (ISI) is negligible.

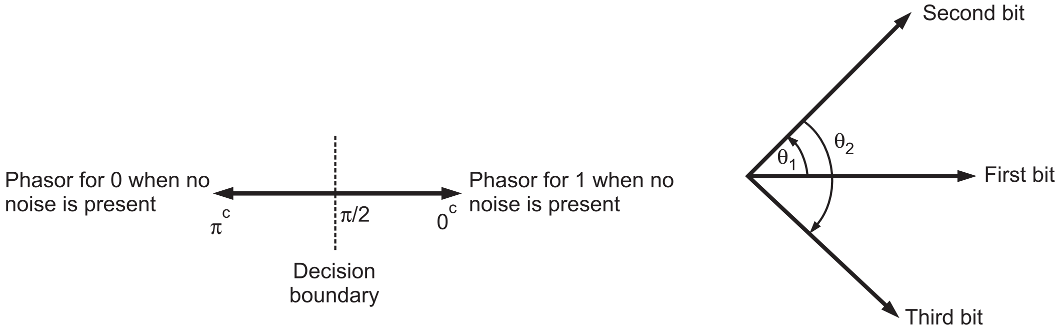
7.18 ERROR PROBABILITY OF DPSK SYSTEM

To obtain the error probability of a differential PSK (DPSK) system, we shall show the suboptimum nature of DPSK by, considering the system in terms of the phasor diagrams shown in figure 7.31.

1. Phasors when no noise is present

In figure 7.31(a), the phasor shows that when no noise is present the phase of received signal is either 0 or π. Therefore a decision boundary is located at angle π/2. The receiver decides that a 1 was transmitted if the phase difference between the consecutive bits differs by less than π/2 and that a 0 was transmitted if the phase difference between the consecutive bits differs by more than π/2.

Phase difference between consecutive bits	*Decision at the receiver*
Less than $\pi/2$	1 is transmitted
More than $\pi/2$	0 is transmitted.



(a) Phasor when no noise is present (b) Phasor when noise is present

Fig. 7.31. Reception of DPSK

2. Phasors when Noise is Present

Figure 7.31(b) shows the phasors for three consecutive received bits. Each bit was transmitted as 1, but because of noise, each phasor is perturbed from the x-axis as shown in figure 7.31(b). The DPSK receiver compares bit 2 with bit 1 and makes a decision that bit 2 was 1 because θ_1 is less than 90°. The DPSK receiver then compares bit 3 with bit 2 and decides that bit 3 was a 0 because the angle θ_2 is greater than 90°. Here, the receiver has made an error in the decision-making. This error is due to the fact that the DPSK receiver uses only one previous bit as a reference. Instead if all the previous positive bits were somehow averaged and if this average is used as a stable reference, then the error described earlier would not have occurred at all, because then the system would have been BPSK and not DPSK. Thus, DPSK is sub-optimum and results in a higher probability of error than in BPSK where we have a stable reference phase. The bit error probability of DPSK is given by:

$$P_B = \frac{1}{2} e^{-E/N_0}$$

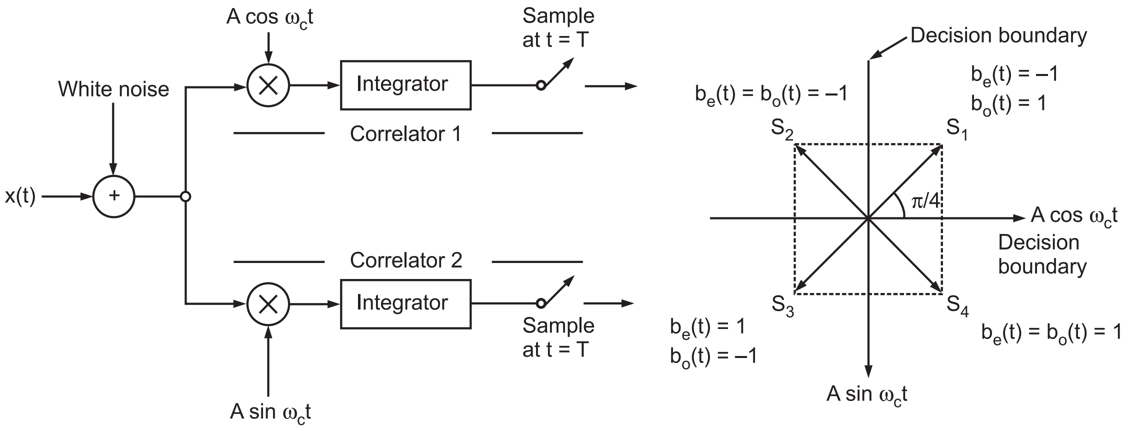
where E = Energy per bit.

7.19 PROBABILITY OF ERROR OF QPSK SYSTEM

In order to understand the error probability, let us reproduce the QPSK receiver and the phasor diagram of QPSK, as shown in figure 7.32(a) and (b) respectively. From figure 7.32(a), it is obvious that two correlators are required and the locally generated reference waveforms are $A \cos \omega_c t$ and $\sin \omega_c t$. The received signal plus noise is passed through these correlators to generate the even and odd bit sequences $b_e(t)$ and $b_o(t)$ as explained earlier. These bit sequences are then added together to obtain the required message signal as shown in figure 7.32(b).

Let us observe figure 7.32(b) carefully. The reference waveform of correlator 1, i.e., $A \cos \omega_c t$ is at an angle $\phi = 45°$ to the axis of orientation of all the four possible signals. Therefore, the axis $A \cos \omega_c t$ can be treated as the decision boundary. Note that this decision boundary is at $\phi = 45°$ therefore, any correlator of the two present can make a mistake if a phase shift of 45° or $\pi/4$ radians occurs in the corresponding carrier. Therefore, the probability that correlator 1 or correlator 2 make an error is given by,

$$P_1'(e) = P_2'(e) = \frac{1}{2} \operatorname{erfc} \sqrt{\frac{E}{N_0} \cos^2 \phi} \quad \text{... (7.120)}$$



(a) A correlation receiver for QPSK (b) Signal space representation of QPSK

Fig. 7.32.

Substituting $\phi = 45°$ in equation (7.120), we obtain

$$P_1'(e) = P_2'(e) = \frac{1}{2}\,\text{erfc}\sqrt{\frac{E}{N_0}(\cos 45)^2}$$

But, $(\cos 45)^2 = 1/2$

$$\therefore \qquad P_1'(e) = P_2'(e) = \frac{1}{2}\,\text{erfc}\,\sqrt{E/2\,N_0} \qquad \text{... (7.121)}$$

The probability "P(c)" that the QPSK receiver will correctly identify the transmitted signal is equal to the product of the individual probabilities of correct identification of the two correlators.

Therefore, probability of correct reception

$$P(c) = P_1'(c) \times P_2'(c) \qquad \text{... (7.122)}$$

where $P_1'(c)$ = Probability that correlator 1 receives signal correctly.

$P_2'(c)$ = Probability that correlator 2 receives signal correctly.

But $P_1'(c) = 1 - P_1'(e)$ and $P_2'(c) = 1 - P_2'(e)$. Substituting these values into equation (7.122) we get.

$$P(c) = \left[1 - P_1'(e)\right] \times \left[1 - P_2'(e)\right]$$

Therefore, $$P(c) = 1 - P'_1(e) - P'_2(e) + P'_1(e)\,P'_2(e) \qquad \text{... (7.123)}$$

Substitute $P'_1(e) = P'_2(e) = P'(e)$ in equation (7.123), we get

$$P(c) = 1 - 2\,P'(e) + (P'(e))^2 \qquad \text{.. (7.124)}$$

Normally, $P'(e)$ is very small. Therefore $P'_2(e)$ is still smaller. Hence, we can neglect the third term on RHS of equation (7.124) to get,

$$P(c) = 1 - 2\,P'(e) \qquad \text{... (7.125)}$$

Hence, the error probability of a QPSK system is given by,

$$P(c) = 1 - P(c) = 1 - [1 - 2\,P'(e)]$$

Therefore, $$P(c) = 2\,P'(e) \qquad \text{...(7.126)}$$

But $P'(e) = P'_1(e) = P'_2\ (e) = \frac{1}{2}\,\text{erfc}\,\sqrt{E/2N_0}$ From equation (7.126)

Therefore, error probability of QPSK system = $P(e) = 2 \times \frac{1}{2}\,\text{erfc}\,\sqrt{E/2N_0}$

Therefore, $$P(e) = \text{erfc}\,\sqrt{E/2N_0} \quad \text{... (7.127)}$$

This is the required expression. In equation (7.127) E corresponds to the energy of each symbol (s_1, s_2, ...etc.). As each symbol is two bit duration long.

$$E = 2\,E_b$$

Substituting this into equation (7.127), we get,

$$P(e) = \text{erfc}\,\sqrt{E_b/N_0} \quad \text{... (7.128)}$$

7.20 PERFORMANCE COMPARISON OF ERROR PERFORMANCE FOR DIFFERENT TYPES OF BINARY SYSTEMS

Figure 7.33 shows the bit error probability for various types of binary systems. The bit error probabilities of various binary systems have been listed in Table 7.2 for the sake of comparison.

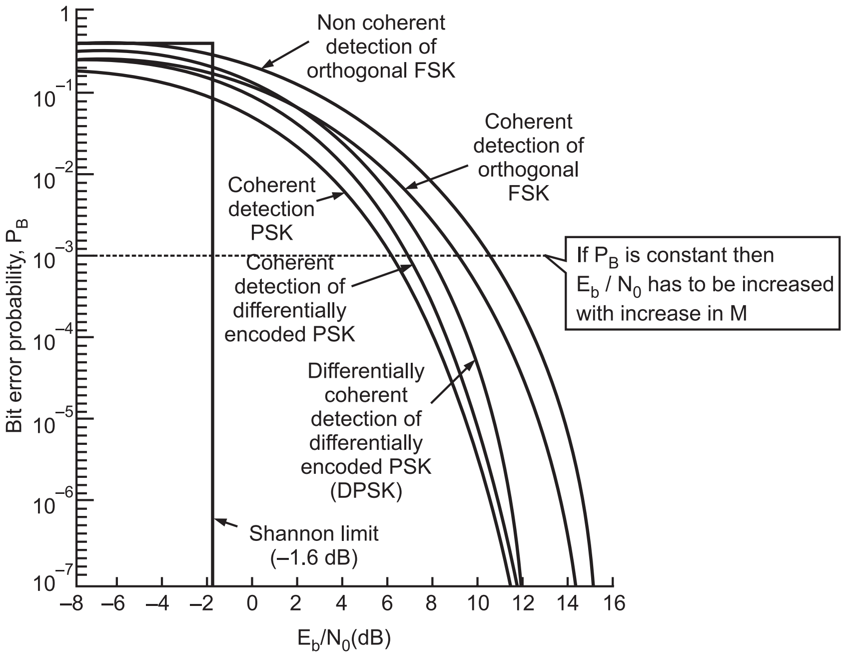


Fig. 7.33. Illustration of bit error probability for several types of binary systems

Table 7.2. Bit error probabilities of various binary system

S.No.	Name of modulation system	Expression for bit error probability P_B
1.	Amplitude shift keying (coherent detection)	$\frac{1}{2}\text{erfc}\left[\sqrt{E/4N_0}\right]$
2.	BPSK (coherent)	$\frac{1}{2}\text{erfc}\left[\sqrt{E/N_0}\right]$
3.	Differentially encoded PSK (coherent)	$\text{erfc}\left[\sqrt{E/N_0}\right]\left[1+\frac{1}{2}\text{erfc}\left(\sqrt{E/N_0}\right)\right]$
4.	BFSK (coherent)	$P_e = \frac{1}{2}\text{erfc}\left[\sqrt{\frac{0.6\,E}{N_0}}\right]$
5.	BFSK (non-coherent)	$\frac{1}{2}\exp\left[-\frac{A^2}{4\,N_0\,W_f}\right]$
6.	Differential PSK (coherent)	$\frac{1}{2}\exp\left[-E/N_0\right]$
7.	QPSK (coherent)	$\text{erfc}\left[\sqrt{E_b/N_0}\right]$

EXAMPLE 7.16. An FSK system transmits binary data at the rate of 2.5 × 10^6 bits per second. During the course of transmission, Gaussian noise of zero mean of power spectral density 10^{-20} Watts/Hz is added to the signal. In the absence of noise, the amplitude of received sinusoidal wave for digit 1 or 0 is 1 microvolt. Determine the average probability of symbol error, assuming coherent detection.

Solution: Given that

(i) Bit rate = 25 × 10^6 bits/sec.

Therefore,
$$T_b = \frac{1}{\text{bit rate}} = \frac{1}{2.5\times10^6}$$

or $T_b = 0.4 \times 10^{-6}$ sec or 0.4 μs **Ans.**

(ii) Power spectral density of noise $= \frac{N_0}{2} = 10^{-20}$ Watt/Hz.

or $N_0 = 2 \times 10^{-20}$ **Ans.**

(iii) Amplitude of the signal = A = 1 μV = 1 × 10^{-6}

Therefore, normalized power $P_s = \frac{A^2}{2} = \frac{1\times10^{-12}}{2}$

The error probability of FSK with coherent detection is given by equation (7.118) as under:

$$P_e = \frac{1}{2}\text{erfc}\left[\frac{0.6\,E_b}{N_0}\right]^{1/2}$$

But $E_b = P_s\,T_b$

or
$$P_e = \frac{1}{2}\text{erfc}\left[\frac{0.6\,P_s\,T_b}{N_0}\right]^{1/2}$$

Substituting the values, we get

$$P_e = \frac{1}{2}\text{erfc}\left[\frac{0.6\times1\times10^{-12}\times1}{2\times2\times10^{-12}\times2.5\times10^6}\right]^{1/2} = \frac{1}{2}\text{erfc}\,[7.745]^{1/2}$$

Therefore, $P_e = \frac{1}{2}\,\text{erfc}\,[2.78]$

or $P_e \equiv 1.5 \times 10^{-4}$ **Ans.**

EXAMPLE 7.17. A received signal is either + 2V or – 2V hold for a time T. The signal is corrupted by white Gaussian noise of power spectral density 10^{-4} volt^2/Hz. If the signal is processed by an integrator and dump receiver, what is the minimum time T during which a signal must be sustained if the probability of error is not exceed 10^{-4}?

Solution: The probability of error of integrate and dump receiver is given as:

$$P_e = \frac{1}{2}\text{erfc}\left[\frac{A^2T}{N_0}\right]^{1/2}$$

It has been given that $\frac{N_0}{2} = 10^{-4}$

Therefore, $N_0 = 2 \times 10^{-4}$ and $A = 2$

The error probability should not exceed 10^{-4}

Therefore, $P_e \le 10^{-4}$

Therefore, $$10^{-4} \le \frac{1}{2}\text{erfc}\left[\frac{A^2T}{N_0}\right]^{1/2}$$

or $$10^{-4} \le \frac{1}{2}\text{erfc}\left[\frac{2^2 \times T}{2 \times 10^{-4}}\right]^{1/2}$$

Using the sign of equality we get,

$$\text{erfc}\left[\frac{4\,T_{min}}{2 \times 10^{-4}}\right]^{1/2} = 2 \times 10^{-4}$$

Using the table of erfc we get,

$$\left[\frac{4\,T_{min}}{2 \times 10^{-4}}\right]^{1/2} \cong 2.6$$

Therefore, $$\frac{4\,T_{min}}{2 \times 10^{-4}} = 6.76$$

or $T_{min} = 0.338 \times 10^{-3}$ sec. or 0.338 msec. **Ans.**

Thus the minimum time for which the signal should be extended is 0.338 msec.

7.21 SYMBOL ERROR PERFORMANCE FOR M-ARY SYSTEM (M > 2)

In the preceeding sections, we have discussed about the bit error probability of various binary communication systems. Now we will learn about the error performance of M-ary system. The symbol error probability is denoted by $P_e(M)$.

7.21.1 Probability of Symbol Error for MPSK

If the energy to noise ratio of an M-ary PSK signal is large, then, the symbol error probability using the coherent PSK is given by,

$$P_e\,(M) = 2Q\left[\sqrt{\frac{2\,E_s}{N_0}}\sin\frac{\pi}{M}\right] \quad \text{... (7.129)}$$

But $$Q(x) = \frac{1}{2}\,\text{erfc}\left(x/\sqrt{2}\right)$$

Therefore, $$P_e(M) = 2\times\frac{1}{2}\,\text{erfc}\left[\frac{1}{\sqrt{2}}\times\sqrt{\frac{2\,E_s}{N_0}}\sin\frac{\pi}{M}\right]$$

or $$P_e(M) = \text{erfc}\left[\sqrt{E/N_0}\sin\frac{\pi}{M}\right] \qquad \ldots (7.130)$$

In equations (7.129) and (7.130), we get

P_e (M) = Probability of symbol error

E = Energy per symbol = E_b ($\log_2$ M)

M = Number of messages being transmitted = 2^N

N = Number of bits per symbol

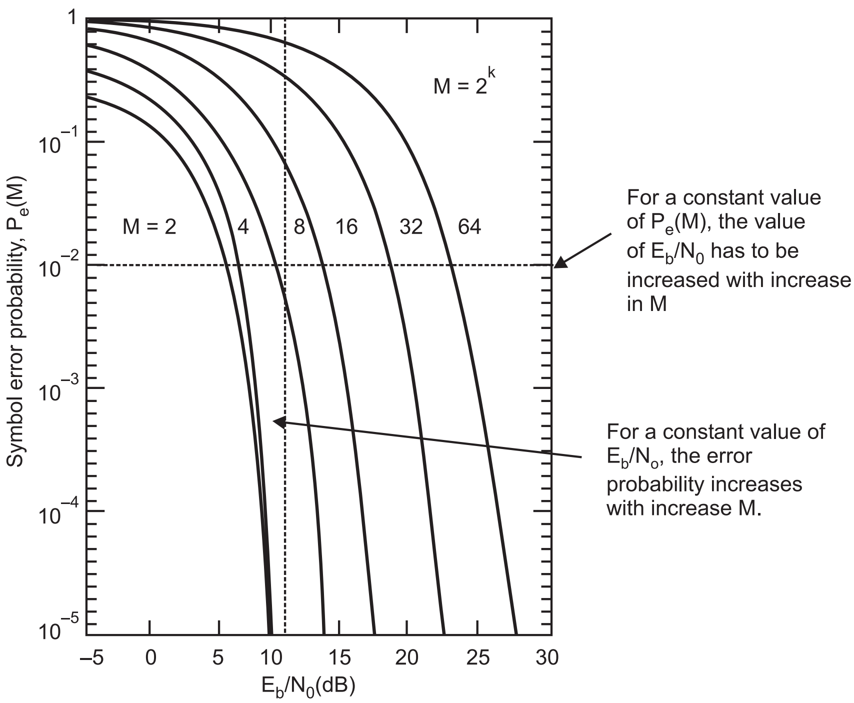


Fig. 7.34. Symbol error performance for M-ary PSK system using coherent detection

Equations (7.129) and (7.130) indicate that the probability of symbol error depends on the ratio $\sqrt{E_s/N_0}$ and M. The error probability decreases i.e., error performance improves with increases in the value of $\sqrt{E_s/N_0}$ but error probability increases with increase in the value of M. Figure 7.34 shows the graph of symbol error probability versus (E_s/N_0) with coherent detection.

Important Point: *Figure 7.34 indicates that in order to obtain the same symbol error, we have to increase the value of signal energy to noise ratio* $\sqrt{E_s/N_0}$ *as the value of M goes on increasing. If the ratio* $\sqrt{E_s/N_0}$ *is maintained constant then with increases in the value of M, the error probability goes on increasing.*

Thus, the error performance of BPSK is superior to the M-ary PSK system with M > 2.

7.22 PROBABILITY OF SYMBOL ERROR FOR MFSK (COHERENT DETECTION)

In M-ary FSK signal, M orthogonal carrier waves are used to represent M different symbol or message. The symbol error probability $P_e(M)$ for M equally likely orthogonal signals have an upper bound given by:

$$P_e(M) = (M - 1)\, Q\left[\sqrt{E_s/N_0}\right] \quad \text{... (7.131)}$$

But $$Q(x) = \frac{1}{2}\,\text{erfc}\left[x/\sqrt{2}\right]$$

Therefore, $$P_e(M) = (M - 1) \times \frac{1}{2}\,\text{erfc}\left[\sqrt{E_s/2N_0}\right] \quad \text{.. (7.132)}$$

where M = Number of symbol or message

E_s = Energy per symbol = E_b ($\log_2$ M)

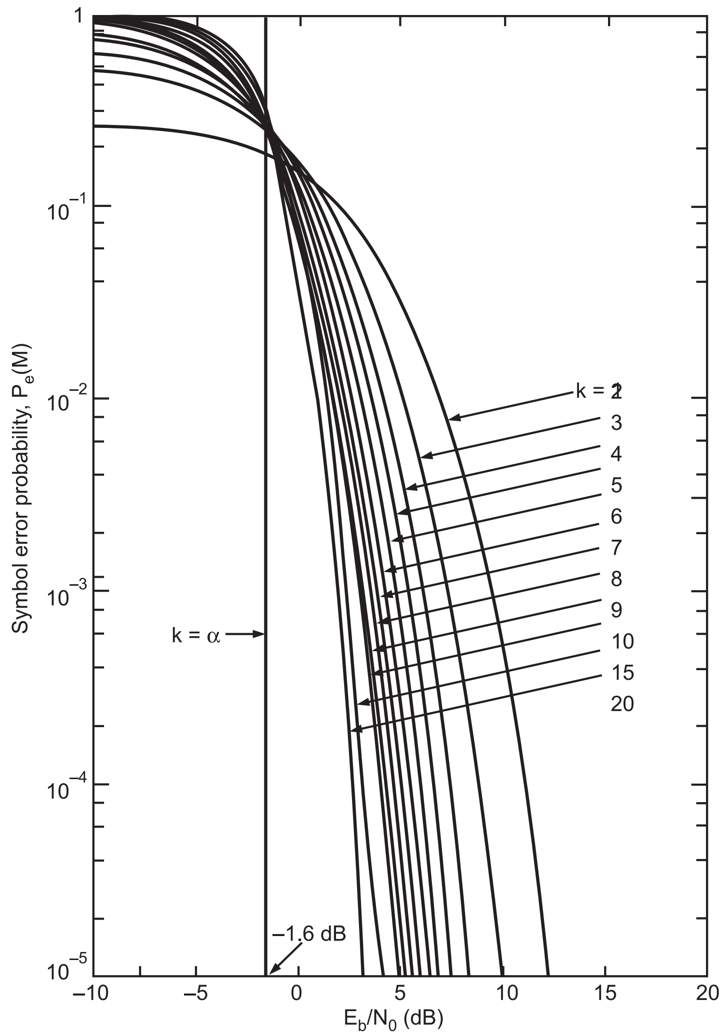


Fig. 7.35. Symbol error probability for coherently detected M-ary FSK

Equations (7.131) and (7.132) indicate that the symbol error probability increases (error performance becomes poor) with increase in the value of M. But, the error probability decreases (performance improves) with increase in the ratio $\sqrt{E_s/N_0}$. Figure 7.35 shows a graph of symbol error probability verus (E_s/N_0) using the coherent detection.

Important point: *Figure 7.35 indicates that to obtain the same symbol error we have to increase the value of signal energy to noise ratio E_s/N_0 as M is increased. If the ratio E_s/N_0 is maintained constant then with increase in the value of M, the error probability goes on increasing.*

7.22.1 Performance Comparison of MPSK and MFSK System

If the ratio E_s/N_0 and M are same for the MPSK and MFSK system then the symbol error probability of MPSK system is less than that of MFSK system. Hence, the error performance of MPSK system is better than that of MFSK system.

7.22.2 Symbol Error Probability of MFSK by Non-Coherent Detection

The symbol error probability of M-equally likely MFSK signal is given by,

$$P_e(m) < \frac{(M-1)}{2} \exp\left(-\frac{E_s}{2\,N_0}\right) \quad \text{... (7.133)}$$

7.23 BIT ERROR PROBABILITY VERSUS SYMBOL ERROR PROBABILITY

The relation between the probability of bit error (P_B) and the probability of symbol error P_E for a set of M-orthogonal signals is given by

$$\frac{P_B}{P_E} = \frac{2^{(N-1)}}{2^N - 1} = \frac{2^N}{2\left[2^N - 1\right]}$$

But $2^N = M$

Therefore,
$$\frac{P_B}{P_E} = \frac{(M/2)}{(M-1)} \quad \text{... (7.134)}$$

If the number of bits per symbol, i.e., N is very large (approaches ∞) then

$$\lim_{N\to\infty} \frac{P_B}{P_E} = \lim_{N\to\infty} \frac{M}{2(M-1)} = \lim_{N\to\infty} \frac{2^N}{2(2^N - 1)}$$

as $N \to \infty$, $2^N \to \infty$. Hence, $2^N - 1 \cong 2^N$

or
$$\lim_{N\to\infty} \frac{P_B}{P_E} = \frac{1}{2} \quad \text{... (7.135)}$$

Thus, for symbols with a large value of N, the bit error probability is twice as high as the symbol error probability.

EXAMPLE 7.18. A transmitter transmits symbols with 3-bits per symbol. Calculate the ratio of bit error probability P_B to the symbol error probability P_E.

Solution: With three bits per symbol (N = 3), there will be 8 possible transmitted symbols $M = 2^3 = 8$ as follows:

1. 8–Possible transmitted symbols

	0	0	0
	0	0	1
	0	1	0
	0	1	1
	1	0	0
Transmitted symbol	1	0	1
	1	1	0
	1	1	1
			Bit Position

Fig. 7.36.

2. Bit error probability (P_B)

Let us consider the bit position box in figure 8.36. Let us obtain the number of ways to cause error to binary 1. In the bit position box of the given figure, there are 4-ones and 4-zeros. We can consider any other bit position, the same thing is observed. The number of ways to cause error to binary-1 in any bit position is 2^{N-1} i.e., $2^2 = 4$.

3. Symbol error probability (P_E)

The number of ways a symbol error can made is $2^N - 1 = 2^3 - 1 = 7$. Hence, the final relationship P_B/P_E for orthogonal signalling is obtained by calculating the value of the ratio 2^{N-1} and $2^N - 1$.

Therefore, $$\frac{P_B}{P_E} = \frac{2^{N-1}}{2^N - 1} = \frac{4}{7}$$

7.24 EFFECTS OF INTERSYMBOL INTERFERENCE

Earlier we have discussed the concept of intersymbol interference. In practice ISI must be accounted because it is a second source of interference. The ISI is generated due to dispersed pulse (in the time domain) due to the use of bandlimiting filters. The ISI can be observed at various points in the system such as transmitter output, in the communication channel, or at the receiver input. The effect of this additional interference is the degradation of error performance (increase in the error probability) for coherent as well as the non-coherent reception.

7.25 ERROR PROBABILITY OF BINARY PCM

As we know that the PCM signal is a train of binary 0s and 1s. Therefore, let us assume that the binary 0 is represented by 0 volts and a binary 1 is represented by + A volts and let the bit duration be T sec.

$$x_1(t) = A \quad \text{for } 0 \le t \le T \quad \text{for binary 1}$$

$$\text{and} \quad x_2(t) = 0 \quad \text{for } 0 \le t \le T \quad \text{for binary 0} \qquad \text{...(7.136)}$$

Assuming that an optimum filter is used for detection, we can write the expression for its error probability as,

$$P_e = \frac{1}{2}\,\text{erfc}\left[\frac{x_{o1}(T) - x_{o2}T)}{2\sqrt{2}\,\sigma}\right] \qquad \text{... (7.137)}$$

$$\left[\frac{x_{o1}(T) - x_{o2}(T)}{2\sqrt{2}\,\sigma}\right]^2_{max} = \int_{-\infty}^{\infty} \frac{|X|(f)|^2}{S_{ni}(f)}\,df \qquad \text{... (7.138)}$$

Assuming that the input noise is white gaussian noise, substituting $S_{ni}(f) = N_0/2$, in equation (7.138), we get

$$\left[\frac{x_{o1}(T) - x_{o2}(T)}{2\sqrt{2}\,\sigma}\right]^2_{max} = \frac{2}{N_0}\int_{-\infty}^{\infty} |X(f)|^2\, df \qquad \text{... (7.139)}$$

But, according to Parseval's theorem, we have $\int_{-\infty}^{\infty} |X(f)|^2 = P = \int_{-\infty}^{\infty} x^2(t)\,dt$

Therefore,
$$\left[\frac{x_{o1}(T) - x_{o2}(T)}{2\sqrt{2}\,\sigma}\right]^2_{max} = \frac{2}{N_0}\int_{-\infty}^{\infty} x^2(t)\, dt \qquad \text{... (7.140)}$$

As the input signal $x(t) = x_1(t) - x_2(t) = x_1(t)$ as $x_2(t) = 0$

$$\left[\frac{x_{o1}(T) - x_{o2}(T)}{2\sqrt{2}\,\sigma}\right]^2_{max} = \frac{2}{N_0}\int_{-\infty}^{\infty} x_1^2(t).\, dt \qquad \text{... (7.141)}$$

$$= \frac{2}{N_0}\int_0^T A^2\, dt \qquad [\text{Since } x_1(t) = A, \text{ for } 0 \le t \le T]$$

$$= \frac{2A^2T}{N_0}$$

Taking square root of both the sides, we get

$$\left[\frac{x_{o1}(T) - x_{o2}(T)}{\sigma}\right]_{max} = \sqrt{\frac{2A^2T}{N_0}} \qquad \text{...(7.142)}$$

Substituting this into equation (7.137), we get

$$P_e = \frac{2}{1}\,\text{erfc}\left[\frac{1}{2\sqrt{2}} \times \sqrt{\frac{2A^2T}{N_0}}\right]$$

Therefore,
$$P_e = \frac{1}{2}\,\text{erfc}\left[\sqrt{\frac{A^2T}{4N_0}}\right] \qquad \text{... (7.143)}$$

This is the expression for error probability of PCM.

By substituting $A^2T = E$ in the above expression, we obtain,

$$P_e = \frac{1}{2}\,\text{erfc}\left[\sqrt{\frac{E}{4N_0}}\right] \qquad \text{... (7.144)}$$

EXAMPLE 7.19. A signal is either $s_1(t) = A\cos(2\pi f_0 t)$ or $s_2(t) = 0$ for an interval $T = \frac{n}{f_0}$ with 'n' an integer. The signal is corrupted by white noise with PSD = $\frac{N_0}{2}$. Find the transfer function of the matched filter for this signal. Write an expression for the probability of error P_e.

Solution: The given signal is an BASK signal. It has been given that,

$$x_1(t) = A\cos(2\pi f_0 t) = \sqrt{2P_s}\cos(2\pi f_0 t)$$

and
$$x_2(t) = 0$$

$$A = \sqrt{2P_s} \quad \text{or} \quad P_s = \frac{A^2}{2}$$

We have already derived the expressions for the transfer function and error probability. They are as under:

$$H(f) = \frac{2k}{N_0} X^*(f) e^{-j2\pi fT}$$

and
$$P_e = \frac{1}{2} \operatorname{erfc} \sqrt{\frac{E}{4 N_0}}$$

where
$$X(f) = \text{F.T. of } x(t) \text{ and } E = P_s T = \frac{A^2}{2} T$$

Expression for transfer function

$$H(f) = \frac{2k}{N_0} X^*(f) e^{-j2\pi fT}$$

$$x(t) = x_1(t) - x_2(t) = s_1(t) - s_2(t) = A \cos (2\pi f_0 t)$$

Therefore,
$$X(f) = \text{FT } [x(t)] = \text{FT } [A \cos 2\pi f_0 t]$$

$$X(f) = \frac{A}{2} \{\delta(f - f_0) + \delta(f + f_0)\}$$

As the imaginary part in the above expression is zero.

$$X^*(f) = X(f) = \left\{\frac{A}{2} \delta(f - f_0) + \delta(f + f_0)\right\}$$

Substituting, we get,
$$H(f) = \frac{2\,k}{N_0} \times \frac{A}{2} \left\{\delta(f - f_0) + \delta(f + f_0) e^{-j2\pi fT}\right\}$$

$$H(f) = \frac{kA}{N_0} \{\delta(f - f_0) + \delta(f + f_0)\} e^{-j2\pi fT}$$

This is the desired result.

Expression for P_e

$$P_e = \frac{1}{2} \operatorname{erfc} \sqrt{\frac{E}{4 N_0}}$$

EXAMPLE 7.20. It is given that the binary data is transmitted at a rate of 10^6 bits/second over a channel having a bandwidth 3 MHz. Assume that the noise PSD at the receiver is $\frac{N_0}{2} = 10^{-10}$ watts/Hz. Find the average carrier required at the receiver input for coherent PSK and DPSK-signaling schemes to maintain a probability of error $P_e = 10^{-4}$.

Solution: We know that the average error probability of PSK system is given by

$$P_e = \frac{1}{2} \operatorname{erfc} \sqrt{\frac{E}{N_0}}$$

But
$$P_e = 10^{-4} \quad \text{(given)}$$

Therefore,
$$10^{-4} = \frac{1}{2} \operatorname{erfc} \sqrt{\frac{E}{N_0}}$$

or
$$2 \times 10^{-4} = \operatorname{erfc} \sqrt{\frac{E}{N_0}}$$

But
$$\operatorname{erf}(u) = 1 - \operatorname{erfc}(u)$$

Therefore, $1 - 2 \times 10^{-4} = 1 - \text{erfc}\sqrt{\frac{E}{N_0}}$

or $0.998 = \text{erf}\sqrt{\frac{E}{N_0}}$

Further, we have

$$\text{erf}\,(2.6) \cong 0.998$$

Therefore, $\sqrt{\frac{E}{N_0}} = 2.6$

$$\frac{E}{N_0} = 6.76$$

or $E = 6.76\,N_0$

But $N_0 = 2 \times 10^{-10}$

Therefore, $E = 6.67 \times 2 \times 10^{-10} = 1.352 \times 10^{-9}$ Joules

But $E = PT$ and $T = \frac{1}{\text{Bit rate}} = \frac{1}{10^6}$

Therefore, $P = \frac{E}{T} = 1.352 \times 10^{-10} \times 10^6 = 1.352$ mW **Ans.**

The is the required carrier power.

Now, the error probability of DPSK system is given by,

$$P_e = \frac{1}{2}e^{-E_b/N_0}$$

Therefore, $10^{-4} = \frac{1}{2}e^{-E_b/N_0}$

or $\frac{-E_b}{N_0} = -8.5171$

or $\frac{E_b}{N_0} = 8.5171,$

or $E_b = 8.5171\,N_0$

or $E_b = 8.5171 \times 2 \times 10^{-10} = 1.7 \times 10^{-9}$ Joules

Hence, power will be $P = \frac{E_b}{T_b} = 1.7 \times 10^{-9} \times 10^6 = 1.7$ mW **Ans.**

Important point: *For the same value of P_e, the DPSK needs to transmit more power than that transmitted by a PSK system.*

EXAMPLE 7.21. An on-off binary system uses the pulse waveforms,

$$s_1(t) = \begin{cases} s_1(t) = A\sin\left(\frac{\pi t}{T}\right), & 0 \le t \le T \\ s_2(t) = 0, & \text{otherwise} \end{cases}$$

Let A = 0.2 mV and T = 2 μs. Additive white Gaussian noise with power spectral density $\frac{\eta}{2} = 10^{-15}$ Watts/Hz is added to the signal. Determine the probability of error when $P(s_1) = P(s_2) = \frac{1}{2}$. Take $Q\left(\sqrt{10}\right) = 7.83 \times 10^{-4}$.

Solution: The given system is an ASK system.

Therefore, $s_1(t) = A \sin(\pi t/T)$

and $s_2(t) = 0$

We know that the error probability of an ASK system is given by

$$P_e = \frac{1}{2}\,\text{erfc}\left\{\frac{x_{01}(T) - x_{02}(T)}{2\sqrt{2}\,\sigma}\right\} \qquad \text{... (i)}$$

But, $$\left\{\frac{x_{01}(T) - x_{02}(T)}{2\sqrt{2}\,\sigma}\right\}^2_{max} = \frac{2}{N_0}\int_0^T x^2(t)$$

where $x(t) = x_1(t) - x_2(t) = s_1(t) - s_2(t)$

$$x(t) = A \sin(\pi t./T)$$

Therefore, $$\left\{\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right\}^2_{max} = \frac{2}{N_0}\int_0^T A^2 \sin^2(\pi t/T)$$

But $$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

Therefore, $$\left\{\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right\}^2_{max} = \frac{2}{N_0}\int_0^T \frac{A^2}{2}\left[1 - \cos\left(\frac{2\pi t}{T}\right)\right]dt$$

$$= \frac{2}{N_0}\int_0^T A^2 dt - \frac{2}{N_0}\int_0^T \frac{A^2}{2}\cos\frac{2\pi t}{T}dt$$

$$= \frac{A^2}{N_0}[t]_0^T - \frac{A^2}{N_0}\left[\sin\frac{2\pi t}{T}\right]_0^T = \frac{A^2 T}{N_0} - 0$$

or $$\left\{\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right\}^2_{max} = \frac{A^2 T}{N_0} \qquad \text{...(ii)}$$

Substituting equation (ii) into (i), we get,

$$P_e = \frac{1}{2}\,\text{erfc}\left\{\frac{x_{01}(T) - x_{02}(T)}{2\sqrt{2}\,\sigma}\right\} = \frac{1}{2}\,\text{erfc}\left\{\frac{1}{2\sqrt{2}}\sqrt{\frac{A^2 T}{N_0}}\right\} \qquad \text{... (iii)}$$

We know that the relation between erfc and Q function is given by

$$Q(x) = \frac{1}{2}\,\text{erfc}\left(\frac{x}{\sqrt{2}}\right)$$

Hence, we rearrange equation (iii) as under:

$$P_e = \frac{1}{2}\,\text{erfc}\left\{\frac{\sqrt{A^2\,T/4\,N_0}}{\sqrt{2}}\right\}$$

Therefore, $$P_e = Q\left\{\sqrt{\frac{A^2\,T}{4\,N_0}}\right\} \qquad \text{... (iv)}$$

But, $A = 0.2 \times 10^{-3}$ V, $T = 2 \times 10^{-6}$ sec, $\frac{N_0}{2} = 10^{-15}$ W/Hz.

Therefore, $$P_e = Q\left\{\sqrt{\frac{(0.2\times10^{-3})^2\times2\times10^{-6}}{4\times2\times10^{-15}}}\right\} = Q\left[\sqrt{10}\right]$$

or $P_e = 7.83 \times 10^{-4}$ **Ans.**

EXAMPLE 7.22. Binary data is transmitted over a microwave link of a rate of 10^6 bits/sec and the PSD of noise at the receiver input is 10^{-10} watts/Hz. Find the average carrier power required to maintain an average probability of error $P_e \le 10^{-4}$ for coherent binary FSK. What is the required channel bandwidth?

Solution: Given: Bit rate = 10^6 bits/s

$$\frac{N_0}{2} = 10^{-10} \text{ W/Hz}$$

Therefore, $$N_0 = 2 \times 10^{-10} \text{ W/Hz}$$

$$P_e \le 10^{-4}, \quad P_s = ?$$

The error probability of FSK with coherent detection is given by,

$$P_e = \frac{1}{2}\text{erf}_c\left[\frac{0.6\,E_b}{N_0}\right]^{1/2}$$

But $$E_b = P_s\,T_b$$

Therefore, $$P_e = \frac{1}{2}\text{erf}_c\left[\frac{0.6\,P_s\,T_b}{N_0}\right]^{1/2}$$

But $$T_b = \frac{1}{\text{bit rate}}$$

Therefore, $$10^{-4} = \frac{1}{2}\text{erfc}\left[\frac{0.6\,P_s}{2\times10^{-10}\times10^6}\right]^{1/2}$$

or $$2 \times 10^{-4} = \text{erf}_c\,[3000\,P_s]^{1/2}$$

But, $1 - \text{erfc}(u) = \text{erf}(u)$

Hence, $$1 - 2 \times 10^{-4} = 1 - \text{erfc}\,[3000\,P_s]^{1/2}$$

or $$0.9998 = \text{erf}\,[3000\,P_s]$$

We have,

$$0.9998 = \text{erf}\,[2.8]$$

or $$3000\,P_s = 2.8$$

$$P_s = \frac{2.8}{3\times10^3} = 0.933 \text{ mW} \qquad \textbf{Ans.}$$

EXAMPLE 7.23. Error probability of 10^{-5} is desired and a channel bandwidth of 20 kHz is available for 16 QAM or QPSK system. Calculate the value of E_b/η required for each of these systems.

Solution: Given that: $P_e = 10^{-5}$, B = 20 kHz, Bit Rate = 80 kbps

We know that the error probability of a QPSK system is given by

$$P_e = \text{erfc}\sqrt{E_b/N_0}$$

Therefore, $$10^{-5} = \text{erfc}\sqrt{E_b/N_0}$$

But erf (u) = 1 – erfc (u)

Therefore, $$1 - 10^{-5} = 1 - \text{erfc}\sqrt{E_b/N_0} = \text{erf}\sqrt{E_b/N_0}$$

or $$0.99999 = \text{erf}\sqrt{E_b/N_0}$$

Then, we have

$$\sqrt{E_b/N_0} = 3$$

or $$E_b/N_0 = 9 \qquad \textbf{Ans.}$$