FUZZY LOGIC & EXPERT SYSTEMS: AN INTRODUCTION WITH BIOMEDICAL APPLICATIONS

Presented to

The Biomedical Computing Interest Group at the National Institutes of Health, Bethesda, MD

Prepared by

Amy J. O'Brien

Digital System Resources (DSR)
12450 Fair Lakes Circle, Suite 500
Fairfax, VA 22033
703-234-4474 • aobrien@dsrnet.com

November 2002

Digital System Resources®, Inc.

Overview

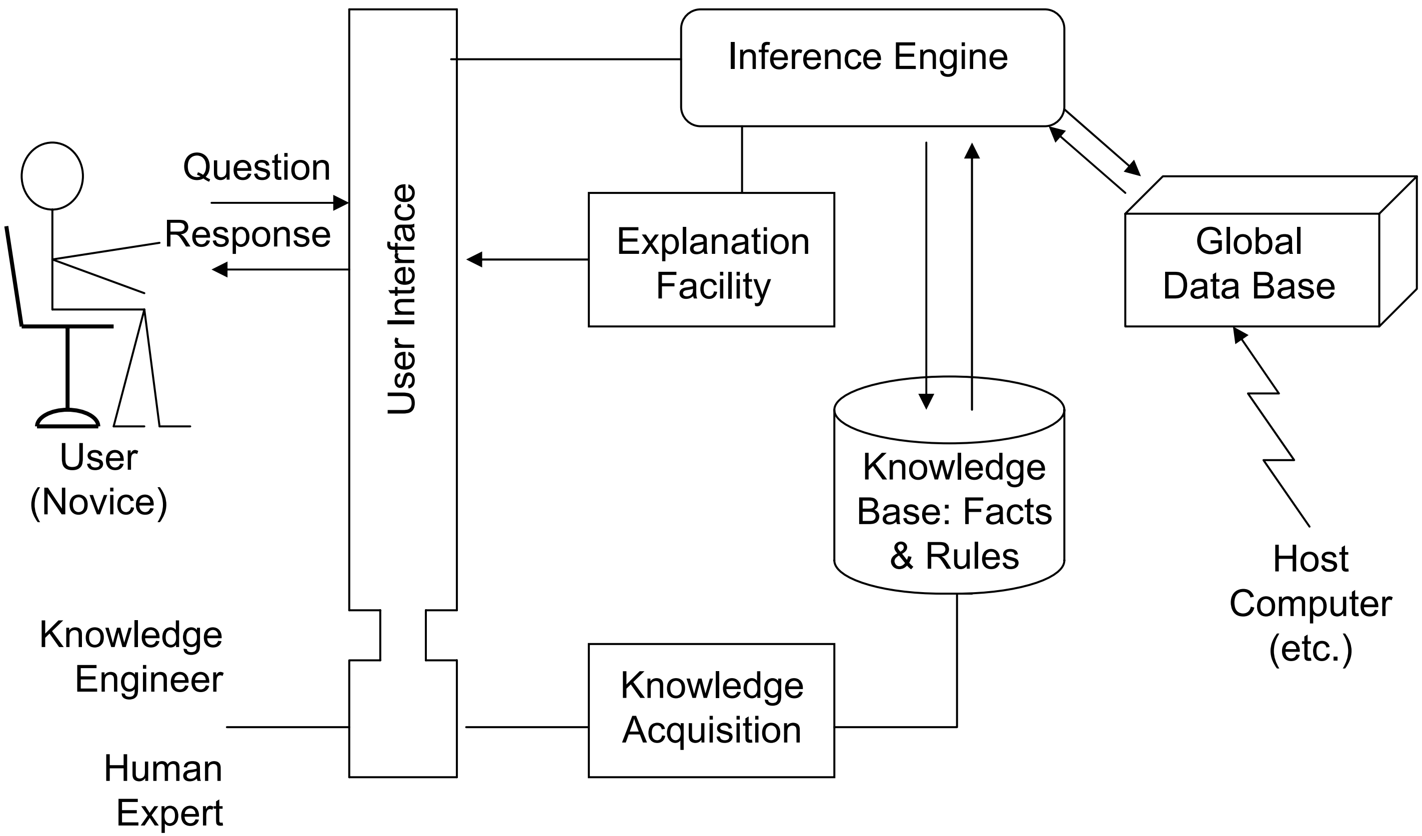
- Definitions
- Conceptual expert system diagram
- Fuzzy sets—where have they gotten us?
- Just how popular is fuzzy logic?
- A brief example of fuzzy logic performance
- Design considerations: methods and validation
- Introduction to fuzzy logic and reasoning
- Conceptual fuzzy system applications
- Conclusions

Definitions

- **Intelligent System (IS)**: design approach which attempts to incorporate human intelligence via architecture or processing algorithm
- **Expert System (ES)**: type of IS which incorporates knowledge of human expert regarding specific problem into rules for inference—needs well-defined problem
- **Neural Network (NN)**: brain-inspired IS processing architecture using non-linear processing elements based on neurons--very robust to nonlinearity & vague / imprecise data but incapable of incorporating human expert knowledge directly in the design (can only be done via supervised training)
- **Fuzzy Logic**: unlike bi-state logic (0 or 1, black or white), allows multi-state partial membership of an element in a set (shades of gray)—robust to vague / imprecise / missing data and nonlinearity
- **Fuzzy Expert System (FES)**: an ES implemented in fuzzy logic—incorporates human expert knowledge directly but incapable of learning / adapting on the fly
- **Neuro-Fuzzy Expert System (NFS)**: combines adaptability of NN with ability of FES to incorporate human knowledge—best of both / very robust to imperfect data

Diagram of an Expert System

(Adapted from [1], p. 5)



Fuzzy sets—where have they gotten us?

Unlike a traditional set, a fuzzy set has no hard boundaries: the transition from "belonging" to "not belonging" is gradual (characterized by a *membership function*). This invites the description of physical systems in imprecise linguistic terms (e.g., "the water is hot").

In 1965[2], Lotfi A. Zadeh introduced the concept of fuzzy sets. From this concept, a plethora of concepts and applications arose, including

Concepts

- Fuzzy relations
- Fuzzy measures
- Possibility theory and fuzzy arithmetic
- Fuzzy logic—the basis for fuzzy reasoning and inference

Applications

- Fuzzy clustering & pattern recognition
- Fuzzy databases
- Fuzzy programming
- Fuzzy pattern recognition
- ***Fuzzy reasoning & inference—the basis for fuzzy experts and controllers***
- Fuzzy – neural network hybrids (neurofuzzy, fuzzy-neural)

Just how popular is fuzzy logic? 1 of 3

- **Fuzzy systems can accomplish what humans can't:**

  "To me, the most impressive accomplishment is a fuzzy system built by Michio Sugeno of the Tokyo Institute of Technology. It can stabilize a helicopter that has lost a rotor blade. No human pilot can manage that—and no mathematical model, either."
  -- C. V. Negoita (Professor, City University of NY; author; & fuzzy systems specialist)

- **Fuzzy systems can perform some tasks more easily than humans:**

  "This [R-50 helicopter] unit is normally flown by remote control via joysticks. Yamaha told us that to master flying using the joystick takes several months and requires that the operator be in constant visual contact with the R-50."
  -- D. K. Kahaner (Asian Technology Information Program)

- **Fuzzy logic has numerous commercial applications, including***

  Anti-lock brake control based on car/wheel speed/acceleration (Nissan)
  Chemical mixer control based on plant conditions (Fuji Electric)
  Health management system: >500 rules to track employees' health/fitness (Omron)
  Plasma etch time/strategy control (Mitsubishi Electric)
  Autofocus adjustment for still cameras (Canon, Minolta)

*Kosko B (1993) Fuzzy Thinking: The New Science of Fuzzy Logic, Hyperion

Just how popular is fuzzy logic? 2 of 3

- **Fuzzy logic has numerous biomedical applications in the 2002 literature alone:**

  Fuzzy controller for patient-controlled analgesia in shock-wave lithotripsy
  -- Shieh JS et al. (*Med Biol Eng Comput*)

  Fuzzy inference classifier to improve diagnostic value of lung cancer tumor markers
  -- Schneider J *et al*. (*Int J Clin Oncol*)

  Fuzzy intervals to refine data acquisition for rheumatoid arthritis consultant system
  -- Leitich H *et al*. (*Artif Intell Med*)

  Elucidation of quantitative structure-activity relationships in drug design/toxicology
  -- Mekenyan O (*Curr Pharm Des*)

  Two forms of fuzzy clustering for ophthalmologic MRI tissue differentiation
  -- Yang MS *et al*. (*Magn Reson Imaging*)

  Neurofuzzy clusterer & waveform/fuzzy classifier to identify GI reflux from impedance
  -- al-Zaben A and Chandraskar V (*Biomed Sci Instrum*)

  Neurofuzzy rules to deal with uncertain and missing data in brain glioma malignancy
  -- Ye CZ *et al*. (*Med Biol Eng Comput*)

Just how popular is fuzzy logic? 3 of 3

- **Fuzzy logic has other appealing biomedical applications (to be discussed in greater detail later in the presentation):**

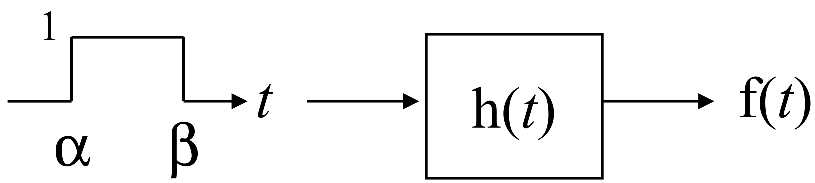
  Gait Analysis Stance Detector – Concept
  A fuzzy inference system that implements traditional algorithmic if-then-else statements in fuzzy rules to combat oscillation around thresholds

  Outpatient Pressure Ulcer Protocol Assistant – Concept
  A fuzzy expert system that supports systematic medicine by providing consistent treatment recommendations in cases of imprecise and/or missing input data

  Fuzzy Hill-Based Muscle Model / Clinician Assistant
  <u>Model</u> – a fuzzy inference/expert system that implements high-order dynamical muscle model differential equations using fuzzy rules that also incorporate empirical muscle behavior; tantamount to a sculpted EMG-to-force estimator
  <u>Clinician Assistant</u> – a fuzzy expert system to aid in diagnosis of pathology using estimated forces produced by processing patient EMG and musculotendon length measurements through the fuzzy muscle model

So, fuzzy logic is popular … but how well does it perform?

It is possible to reinvent relationships within integro-differential calculus using fuzzy logic. Consider this example:



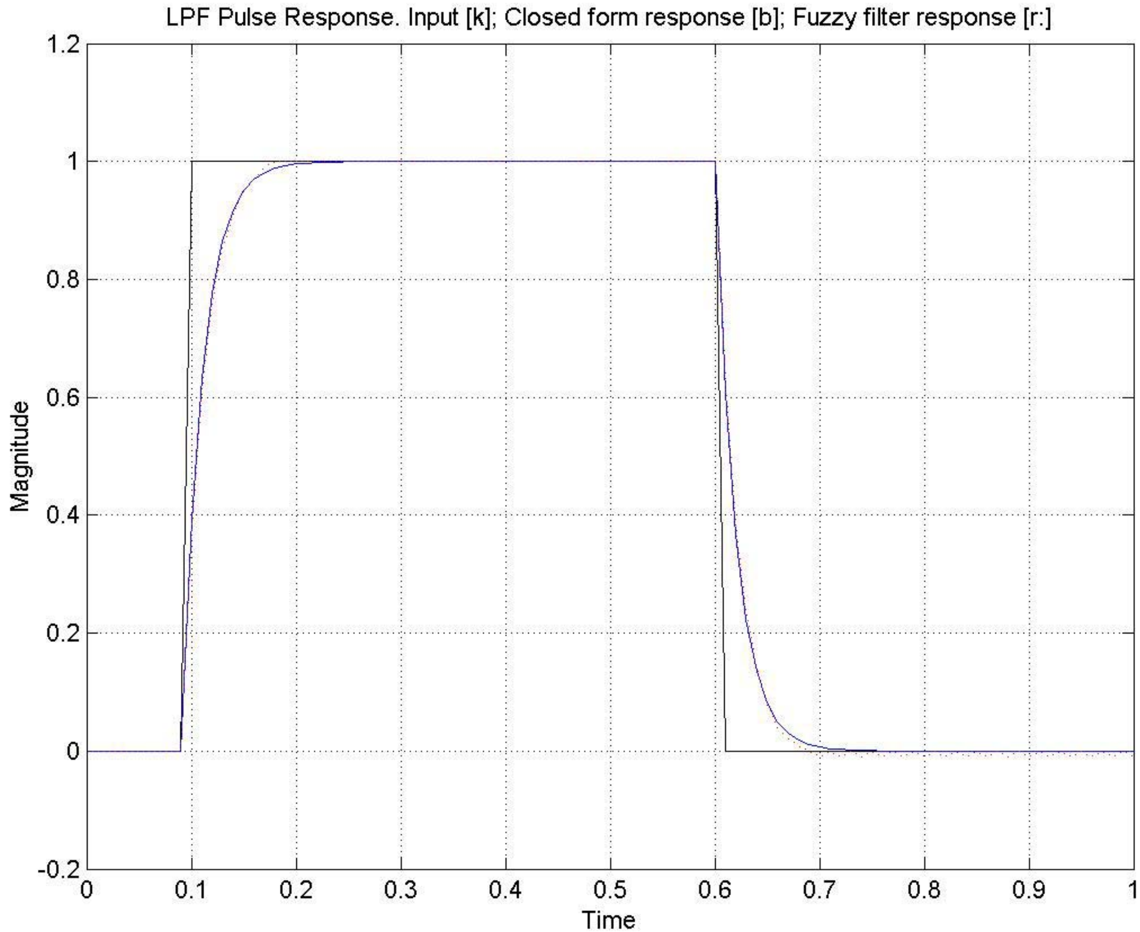
A low pass filter with the time-domain transfer function $h(t) = (e^{-t/\tau})/\tau$ receives a unit-magnitude pulse input starting at $t = \alpha$ and ending at $t = \beta$. The closed form time-domain pulse response can be expressed as $f(t)$:

$$f(t) = u(t - \alpha) - u(t - \beta) - e^{-(t - \alpha)/\tau} + e^{-(t - \beta)/\tau}$$

where $u(\cdot)$ denotes the unit step function and τ is the filter's time constant.

The filter $h(t)$ can be realized by a fuzzy inference system comprising a single input, a single output, and a single rule. How well does it capture the behavior of the ideal closed form response?

Pretty darn well …☺



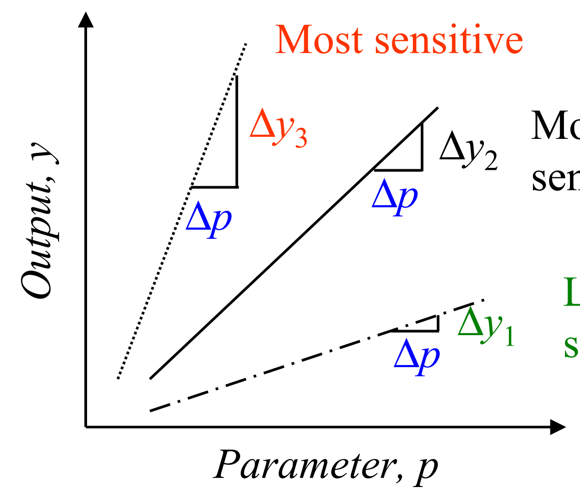
tau = 0.02; sample interval = 0.01s

This fuzzy filter is so simple that any desired number can be daisy-chained to implement higher-order fuzzy filters. Also, the sample interval is on the order of τ, which suggests potentially reduced sampling rates. How cool is that?

Design Considerations: Methods and Validation

Fuzzy system design methods are similar to traditional methods. The greatest difference is in the freedom and capability fuzzy offers.

- Use fuzzy to re-implement an existing non-fuzzy system or model:
  - Existing design provides framework for fuzzy design
  - Validate by comparing performance
- Use traditional methods & intuition, adding fuzzy as a design tool:
  - Identify task (e.g., clustering, classification, model, expert system)
  - Build a prototype (usually software)
  - Validate with tuning data
    - * Smaller set than for neural net (unless neuro-fuzzy design)
    - * Refine design as needed:
      - MF inflection parameters
      - Choice of fuzzy math
      - MF shape parameters



Parameter sensitivity analysis is helpful to evaluate any system design.

Intro to Fuzzy Logic & Reasoning, 1

The majority of this tutorial focuses on using fuzzy logic to perform fuzzy inference and reasoning—the key component of fuzzy expert systems and many controllers. To begin, consider the following example.

Suppose …

- You pick a tomato from your garden.
- Your expert gardener gives you a rule:

  "If a tomato is red, then it is ripe."

- You realize that your tomato is only "sort of" red.

What do you infer from this?

Intro to Fuzzy Logic & Reasoning, 2

Hopefully, you infer that your tomato is only "sort of" ripe.

- Fuzzy logic has allowed you to *assign partial membership* of the tomato in the set "red" along a continuum from 0 ("not red at all") to 1 ("completely and undeniably red").

- From this partial membership in the set "red," you are able to *infer partial membership* of the tomato in the set "ripe"—a classic example of fuzzy inference.

Humans perform fuzzy inferences every day.

Intro to Fuzzy Logic & Reasoning, 3

Two of the most commonly used forms of fuzzy inference are named after their developers: Mamdani and Sugeno (yes, there are other forms).

- In Mamdani fuzzy inference, inputs and outputs both are described in terms of fuzzy membership continua.
- In a Sugeno fuzzy model, the inputs are described via fuzzy membership continua, but outputs are calculated as mathematical functions of the inputs. Here is a typical Sugeno rule:

  "If x is A and y is B, then z is $f(x,y)$" where f is typically a polynomial.
- Fuzzy membership continua may be continuous or discrete.
- Continuous / discrete membership continua may be mixed & matched as needed – this allows virtual comparison of apples and oranges.

Intro to Fuzzy Logic & Reasoning, 4

Intuitively, Mamdani fuzzy inference can easily be done on machine:

1. Fuzzify inputs by mapping from numerical domain onto linguistic fuzzy membership continua, useful regions of which are denoted as *membership functions*

   Calculate membership of each input measurement in each input membership function—obtain antecedent values

2. Operate on the fuzzified inputs' membership values using rules obtained from human expert—apply antecedent values to rules & obtain consequent areas

3. Collect all rules' consequent areas into an *aggregate*

4. Defuzzify: calculate a single-valued output estimate (the "best representative" point within the aggregate)

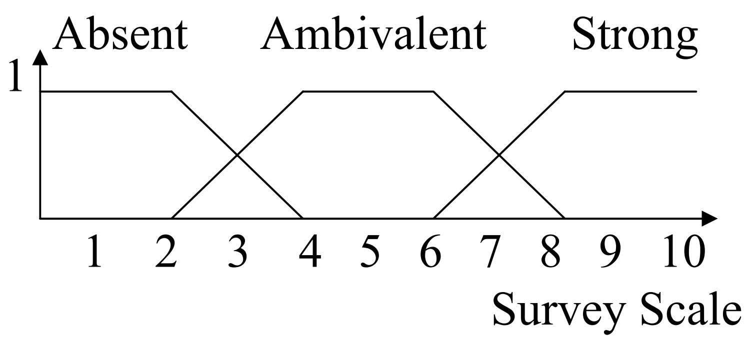
Intro to Fuzzy Logic & Reasoning, 5

1. Fuzzify inputs.

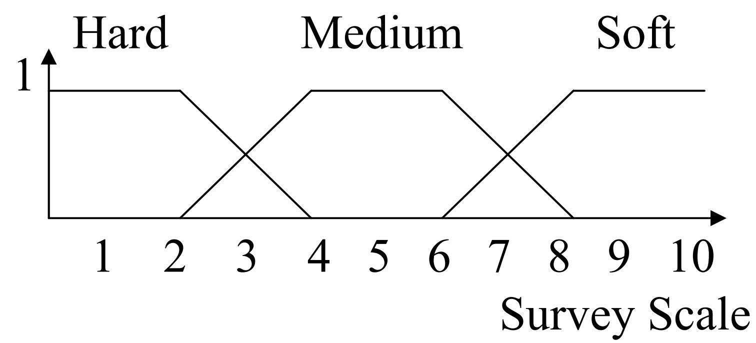
Fuzzification is the method by which physical measurements are assigned linguistic meaning.

Reconsider the tomato example. Suppose, however, that in addition to redness, firmness will also be considered as an input to determine the tomato's ripeness. Here is one approach toward fuzzifying these inputs:

Redness



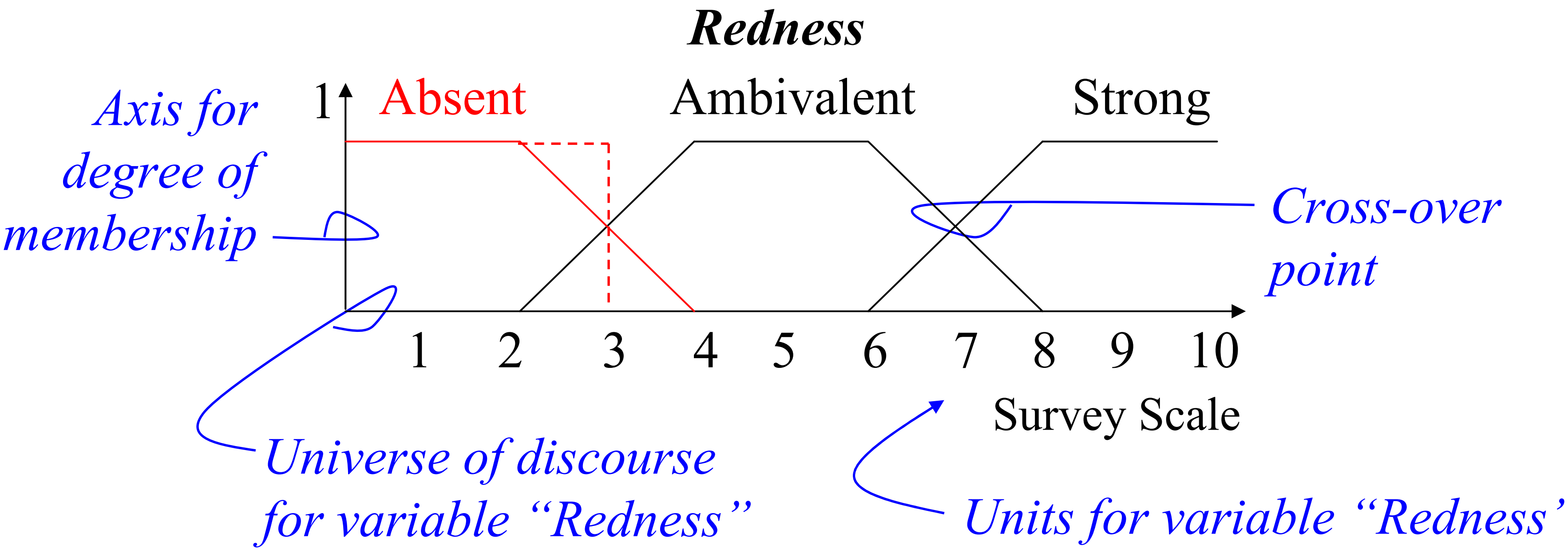
Firmness



This example demonstrates fuzzy inference system design: selecting relevant variables; choosing form of fuzzy inference; partitioning the input and output spaces (can be >1-D) into appropriate linguistic terms; and designing rules.

Intro to Fuzzy Logic & Reasoning, 6

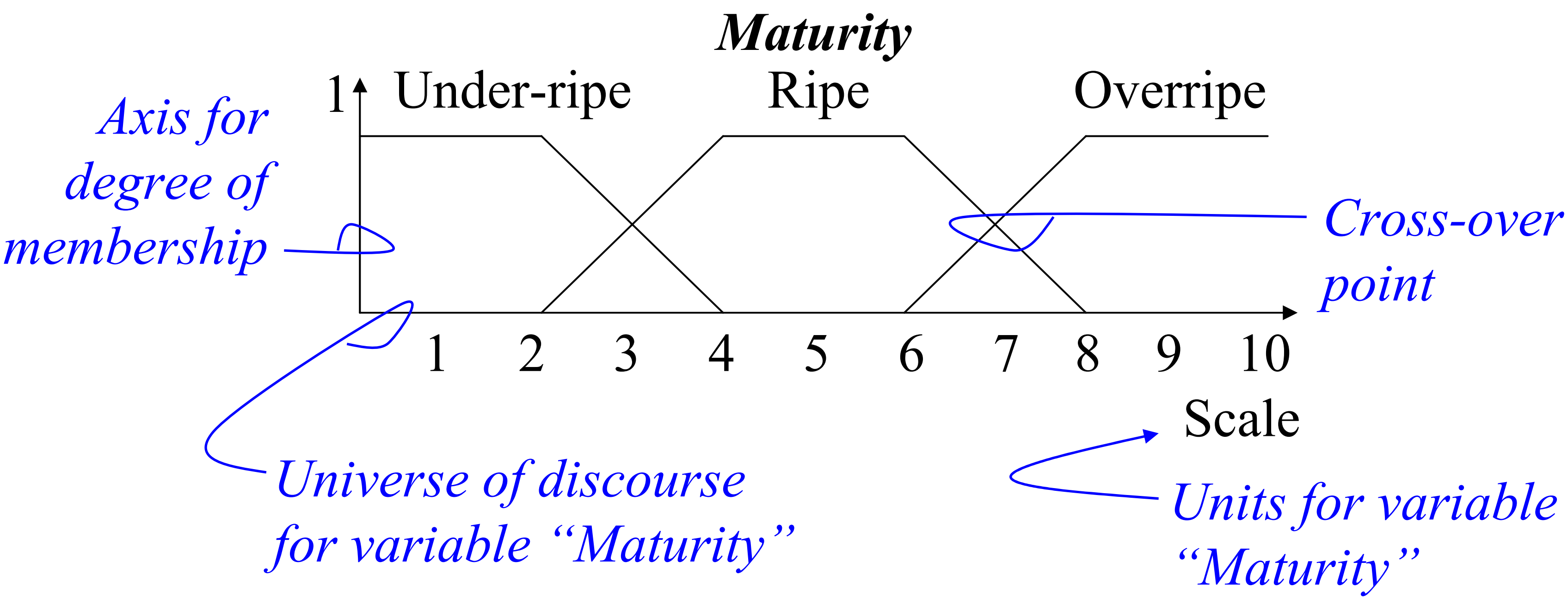
More on fuzzification – anatomy of a plot:



The item in red is a *membership function (**MF**)*: a portion of variable's universe of discourse, or range, designated for its usefulness to the problem. The shape (trapezoidal here) could be anything from rectangular (shown dashed – reverts to bi-state, a special case of fuzzy) to triangular to sigmoidal to gaussian to whatever the designer desires (or can program).

Intro to Fuzzy Logic & Reasoning, 7

Last remark about fuzzification – it's not just for input variables:



For Mamdani fuzzy inference, MFs must be designed for all input and output variables.

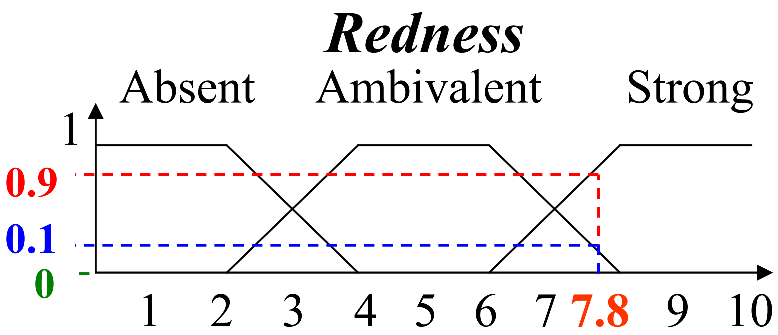
(It is just coincidence that all input and output variables for this example have units of scale.)

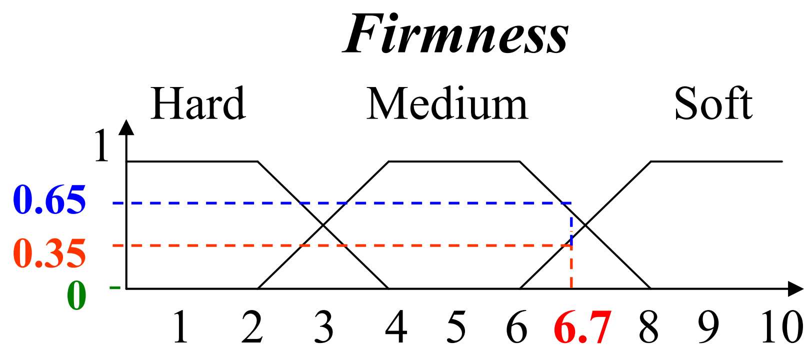
Intro to Fuzzy Logic & Reasoning, 8

… Calculate membership of input measurements—obtain antecedent values.

Suppose an individual tomato has been optically and tactilely evaluated as having, on a scale of 1 to 10, a redness of 7.8 and a firmness of 6.7.

How do these input measurements interact with membership functions?





Plot each input measurement on the variable's universe of discourse & read upward: vertical intercepts along MFs give measurements' membership values. Here are results for the inputs (7.8, 6.7):

$\mu_{Absent}(7.8) = 0$ $\quad\mu_{Ambivalent}(7.8) = 0.1$ $\quad\mu_{Strong}(7.8) = 0.9$

$\mu_{Hard}(6.7) = 0$ $\quad\mu_{Medium}(6.7) = 0.65$ $\quad\mu_{Soft}(6.7) = 0.35$

where "μ" denotes membership.

Intro to Fuzzy Logic & Reasoning, 9

2. Operate on fuzzified inputs using human expert rules—apply antecedent values to rules to obtain consequent areas.

Suppose the following rulebase is used with the input pair (7.8, 6.7):

1. If "redness" is "strong" & "firmness" is "medium," then "maturity" is "ripe"
2. If "redness" is "strong" & "firmness" is "soft," then "maturity" is "overripe"
3. If "redness" is "absent" or "firmness" is "hard," then "maturity" is "under-ripe"

For this rulebase, to what degree of maturity does the input ordered pair (7.8, 6.7) correspond?

Before considering this, it is instructive to understand the anatomy of a rule …

Intro to Fuzzy Logic & Reasoning, 10

Anatomy of a rule:

1. If "redness" is "strong" and "firmness" is "medium," then "maturity" is "ripe"

Antecedent #1 *Antecedent #2* *Consequent*

Compound antecedent

The degree to which the input data point (measurement) satisfies the antecedent—the antecedent's value—dictates the degree to which the consequent will be true … meaning, the degree to which the antecedent is true dictates the amplitude of the rule's output.

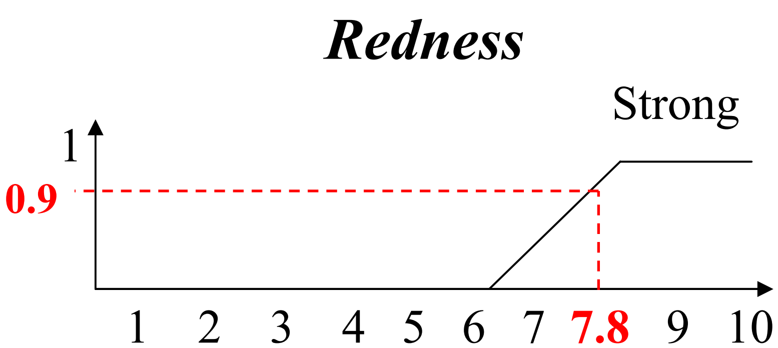
Another way to say it: the degree to which the antecedent is true determines how strongly the rule fires.

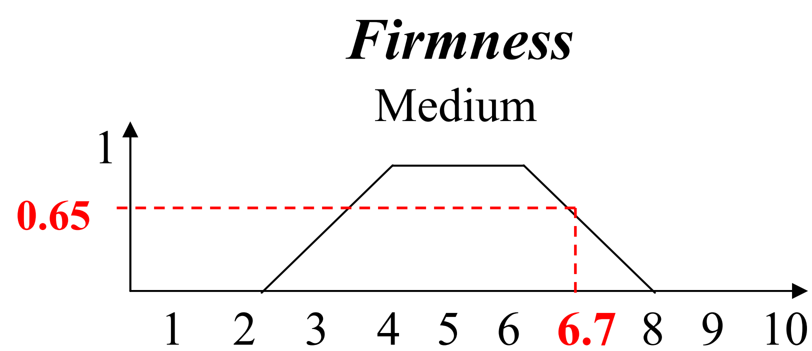
Now to apply this to the example …

Recall that the inputs are (7.8, 6.7). The rule

1. If “redness” is “strong” and “firmness” is “medium,” then “maturity” is “ripe”

is only interested in μ_{Strong} for redness and μ_{Medium} for firmness. So, the key results for the inputs (7.8, 6.7), as found in Step #1, are $\mu_{\text{Strong}}(7.8) = 0.9$ and $\mu_{\text{Medium}}(6.7) = 0.65$.

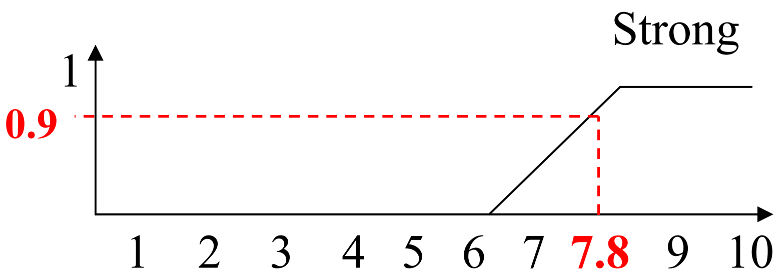




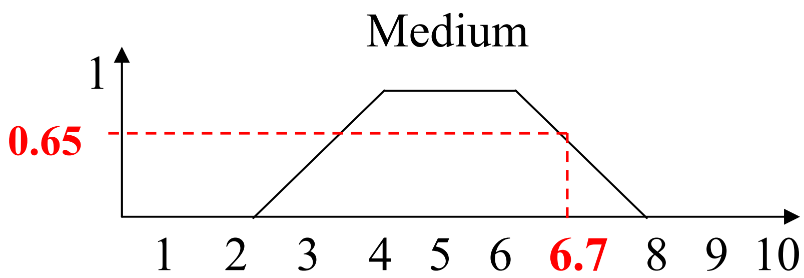
These two membership values must be ANDed together to form the compound antecedent’s value.

Intro to Fuzzy Logic & Reasoning, 12

Redness



Firmness



How does one AND $\mu_{Strong}(7.8)$ and $\mu_{Medium}(6.7)$ together mathematically?

There are several approaches, all well-discussed in the literature. Two of the most common are *min* and algebraic *prod*uct:

min (0.9, 0.65) = 0.65
prod(0.9, 0.65) = 0.59

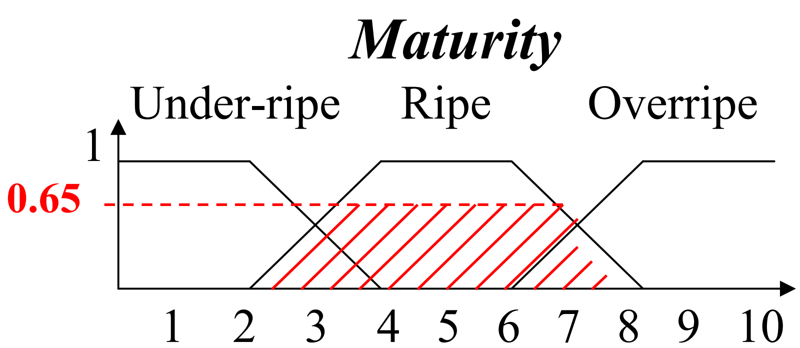
The difference between these two is significant!

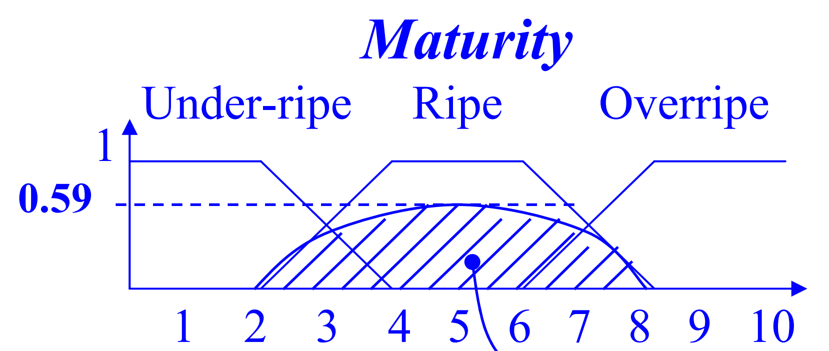
Fuzzy inference is sensitive to choice of fuzzy math—one of many degrees of freedom exploited by fuzzy system designers.

Yes, we're still on Step 2., "Operate on fuzzified inputs using human expert rules" … we want the degree of maturity corresponding to the input ordered pair (redness, firmness) = (7.8, 6.7) for the rule

1. If "redness" is "strong" & "firmness" is "medium," then "maturity" is "ripe"

Assuming *min* as the AND method, we now know that the inputs (7.8, 6.7) give the compound antecedent a value of 0.65. What does that do to the consequent area?





Using prod *as the AND method would have yielded a smoother result:*

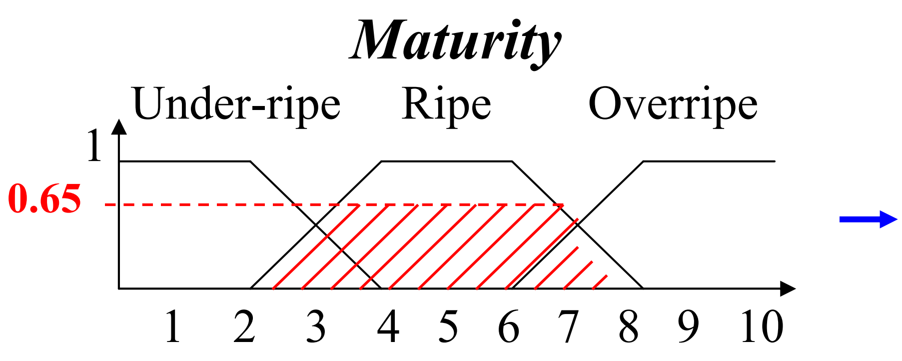
Intro to Fuzzy Logic & Reasoning, 14

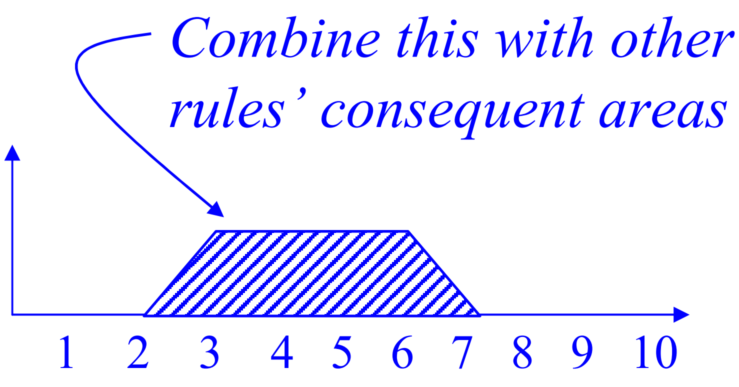
For the rule

1. If “redness” is “strong” & “firmness” is “medium,” then “maturity” is “ripe”

the area hatched in red on the diagram on the left represents the consequent area for this rule.

This consequent area will have to be combined with consequent areas for the other two rules before a final pronouncement on the tomato’s maturity can be made.



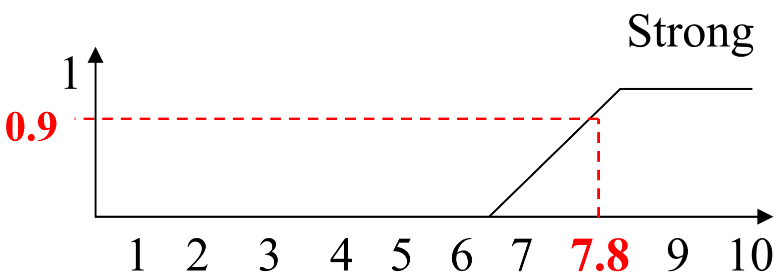


Intro to Fuzzy Logic & Reasoning, 15

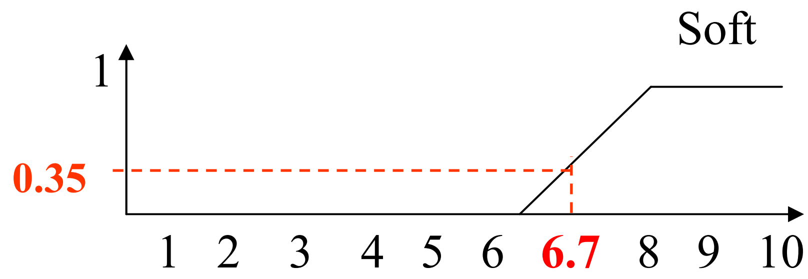
The next rule may be considered for the input ordered pair (redness, firmness) = (7.8, 6.7):

2. If "redness" is "strong" & "firmness" is "soft," then "maturity" is "overripe"

Redness



Firmness

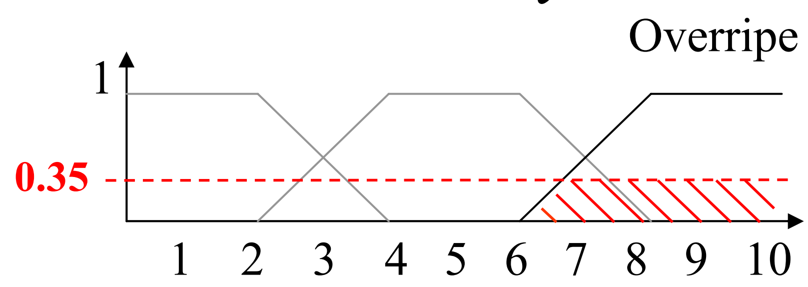


As with Rule 1, this rule has a compound antecedent. Using the *min* AND method gives the antecedent's value:

min(0.9, 0.35) = 0.35

Again, the area hatched in red must be combined with other rules' consequent areas.

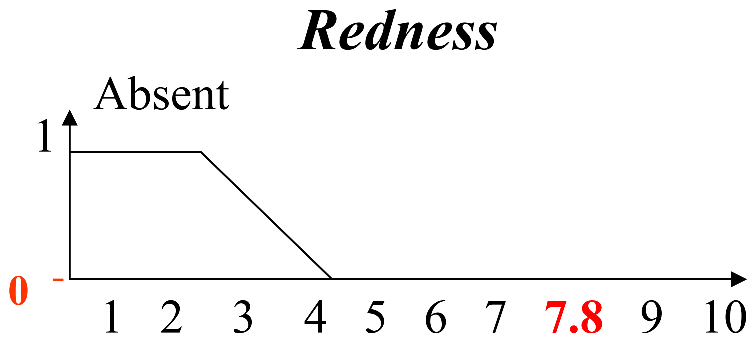
Maturity

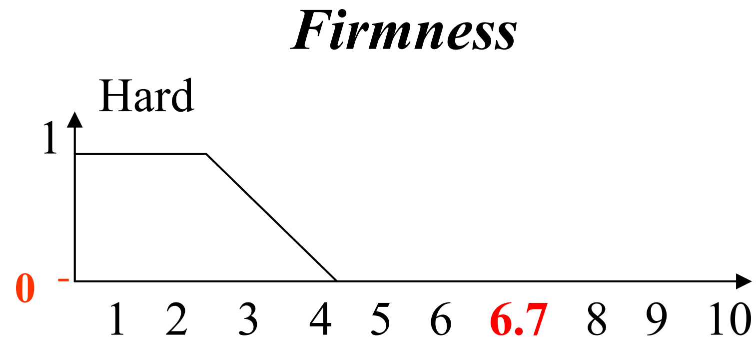


Intro to Fuzzy Logic & Reasoning, 16

Now, the last rule may be considered for (redness, firmness) = (7.8, 6.7):

3. If "redness" is "absent" or "firmness" is "hard," then "maturity" is "under-ripe"

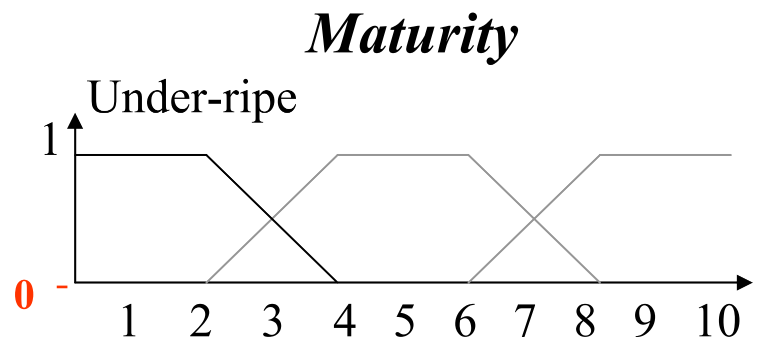




There is a compound antecedent here, too, but it's an OR. As with AND, there are many methods, but ORs are typically implemented with *Max* or algebraic *Sum*:

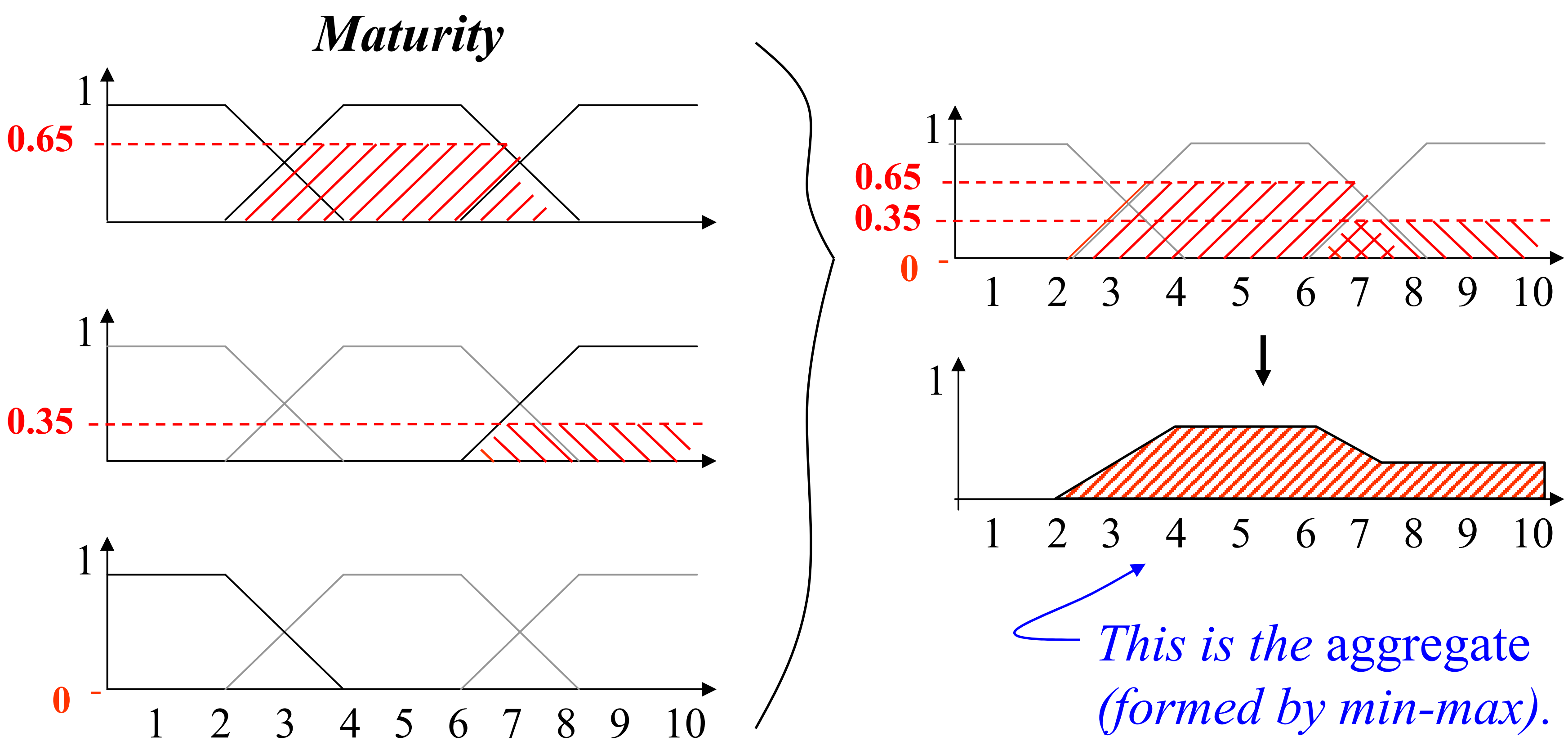
Max(0, 0) = 0

Again, the area hatched in red must be combined with other rules' consequent areas. ☺



3. Collect all rules' consequent areas into an aggregate.

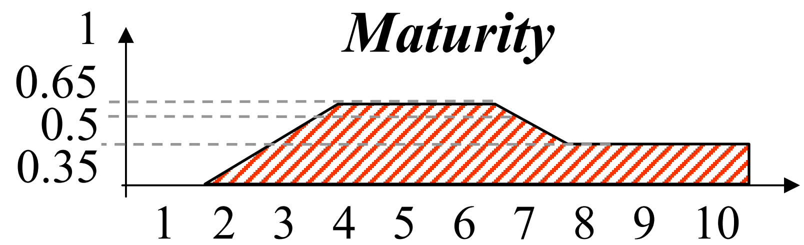
While there are other well-documented methods, aggregation of all the rules' consequent areas is typically done by *Max*:



4. Defuzzify: calculate a single-valued output estimate (the "best representative" point within the aggregate).

As with other fuzzy math operations, there are many methods of defuzzification. One common approach involves calculating the centroid of area (analogous to the centroid of mass):

UofD	Aggr	Aggr*UofD
1	0	0
2	0	0
3	0.5	1.5
4	0.65	2.6
5	0.65	3.25
6	0.65	3.9
7	0.5	3.5
8	0.35	2.8
9	0.35	3.15
10	0.35	3.5



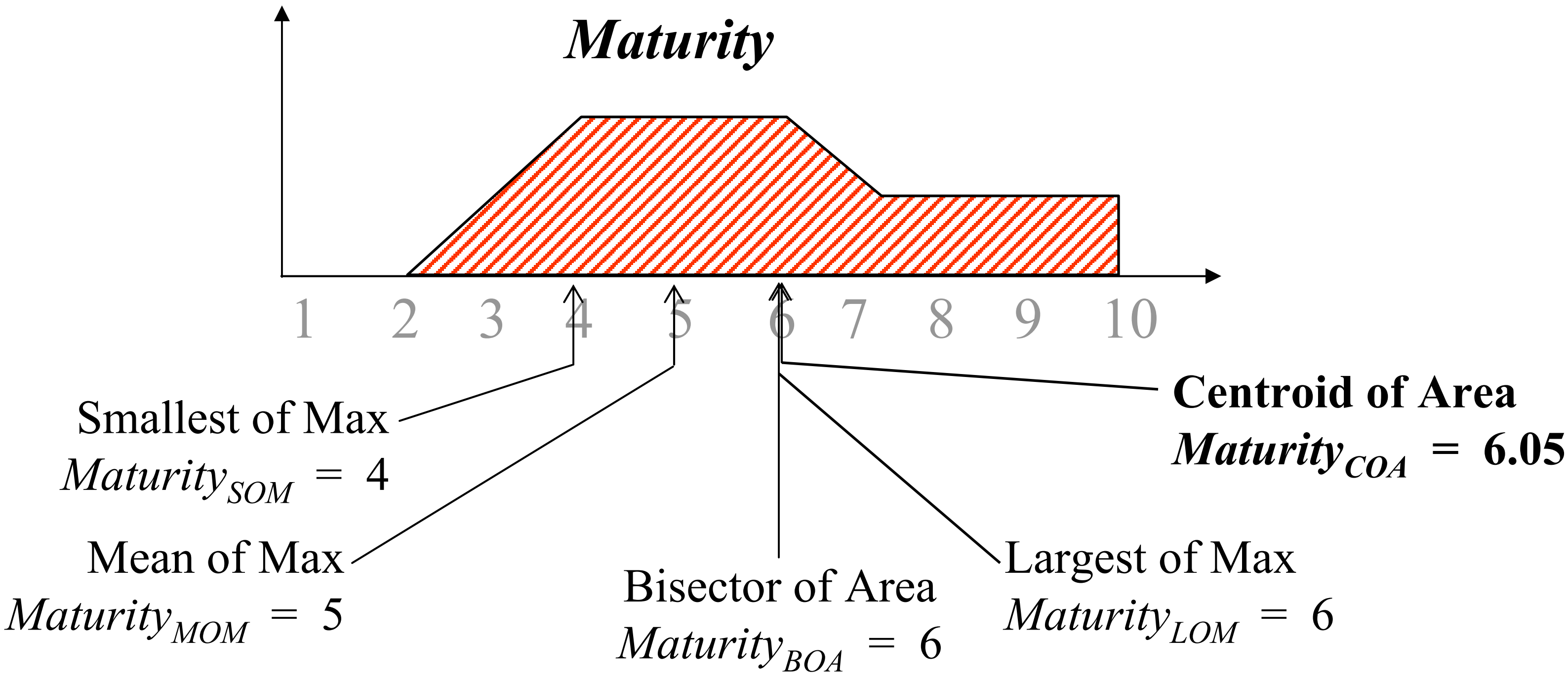
$$\textit{Maturity}_{COA} = \frac{\Sigma_i \text{ Aggr}(i) * \text{UofD}(i)}{\Sigma_i \text{ Aggr}(i)}$$

$$\textit{Maturity}_{COA} = 24.2 / 4$$

$$\textit{Maturity}_{COA} \overset{\checkmark}{=} \mathbf{6.05}$$

Intro to Fuzzy Logic & Reasoning, 19

It is interesting to compare defuzzification by centroid of area with other methods …



… yet another demonstration that

- Fuzzy inference is sensitive to choice of fuzzy math
- FES designers have many degrees of freedom—sometimes too many!

Intro to Fuzzy Logic & Reasoning, 20

So far, this entire discussion has focused on Mamdani fuzzy inference, which uses membership functions for its inputs as well as its outputs.

What about Sugeno fuzzy inference (as introduced, oh, on *Intro to Fuzzy Logic & Reasoning* slide #3 … as if I'd let you forget)?

- Simple Sugeno—1 rule:
  If "redness" is "strong" & "firmness" is " medium,"
  then $\boldsymbol{maturity}_{\textbf{Sugeno}} = 0.5(\text{redness} + \text{firmness}) = 0.5(7.8 + 6.7) \overset{\checkmark}{=} \mathbf{7.25}$

- Previous 3-rule rulebase modified for Sugeno:
  1. If "redness" is "strong" & "firmness" is "medium,"
     then $\boldsymbol{maturity}_{\textbf{Sugeno-rule1}} = 0.5(\text{redness} + \text{firmness}) = 0.5(7.8 + 6.7) = 7.25$
  2. If "redness" is "strong" & "firmness" is "soft,"
     then $\boldsymbol{maturity}_{\textbf{Sugeno-rule2}} = 0.33{*}\text{redness} + 0.67{*}\text{firmness} = 7.06$
  3. If "redness" is "absent" or "firmness" is "hard,"
     then $\boldsymbol{maturity}_{\textbf{Sugeno-rule3}} = 0.25{*}\text{redness} + 0.75{*}\text{firmness} = 6.98$

$$\therefore\ \boldsymbol{maturity}_{\textbf{Sugeno}} = \Sigma_i \boldsymbol{maturity}_{\textbf{Sugeno-rule}_\mathbf{i}}/3 = (7.25 + 7.06 + 6.98)\ /\ 3 \overset{\checkmark}{=} \mathbf{7.1}$$

Intro to Fuzzy Logic & Reasoning, LAST SLIDE of *this* section ☺

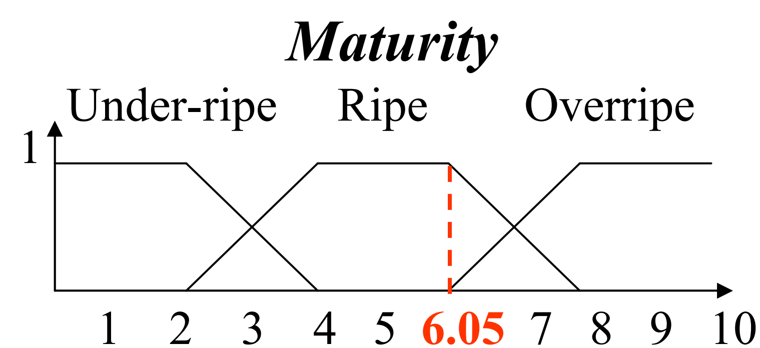
So, for a fuzzy expert tomato picker that was given three rules

1. If "redness" is "strong" & "firmness" is "medium," then "maturity" is "ripe"
2. If "redness" is "strong" & "firmness" is "soft," then "maturity" is "overripe"
3. If "redness" is "absent" or "firmness" is "hard," then "maturity" is "under-ripe"

with a tomato having the input ordered pair (redness, firmness) = (7.8, 6.7)

using Mamdani inference with *min* to implement AND, *Max* to implement OR and aggregation, and centroid of area to defuzzify, it is possible to conclude that the given tomato has a maturity of **6.05** on a scale of 1 to 10.

So, it's a good time to pick the tomato.



That's nice, but how about some real-world applications?

What's all this fuzzy stuff good for?

Fuzzy logic is very common in control and expert systems (ubiquitous in Japanese technology—everything from toasters to automobiles).

The greatest difference between a fuzzy expert and a fuzzy controller is that, in a fuzzy controller, the output from the fuzzy system drives something. Both can use fuzzy inference to perform their functions.

Fuzzy systems are applicable in defense …

- Radar & Sonar Target ID / Classification; Fuzzy Trackers

… commercially,

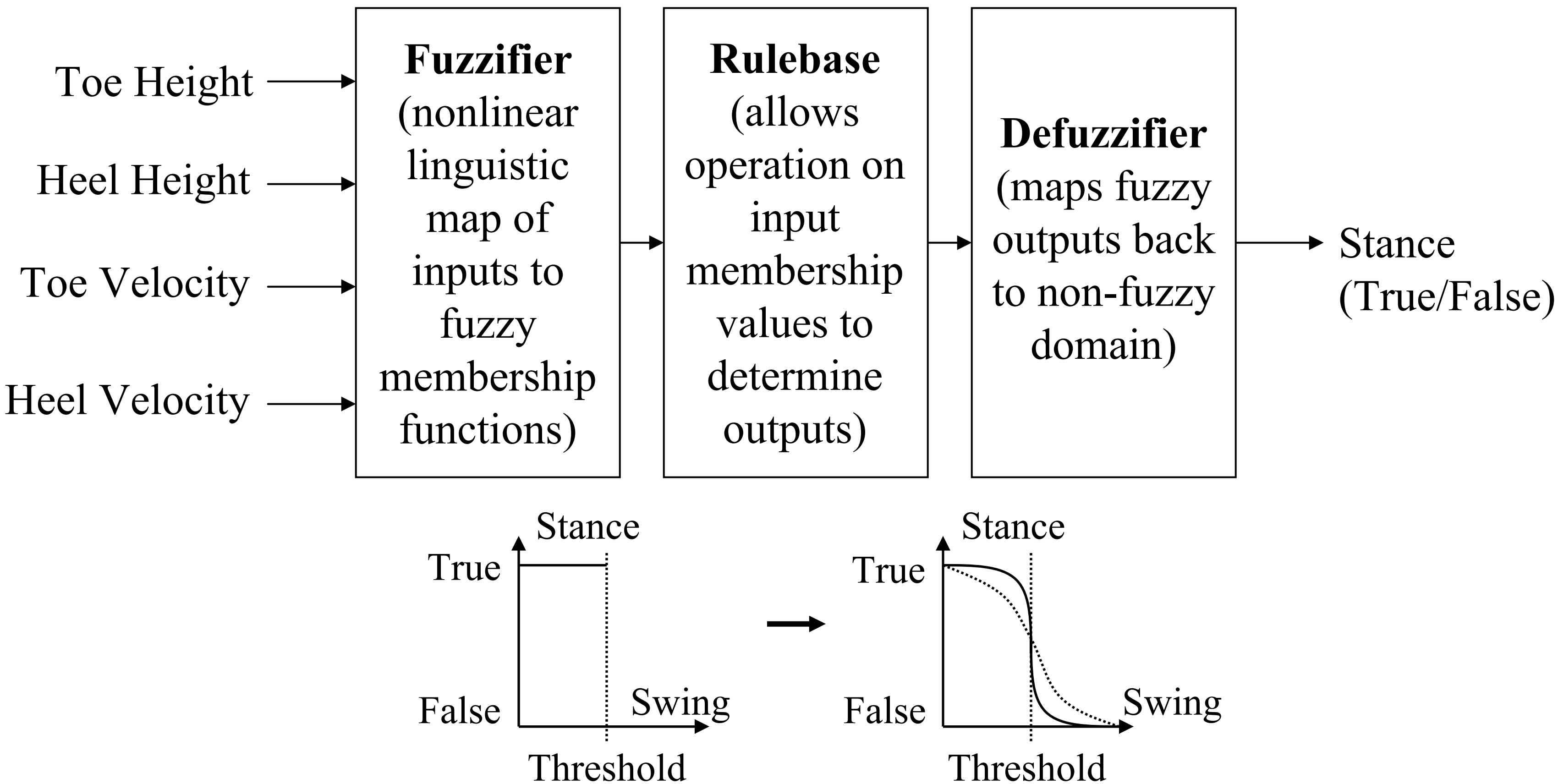
- Oil Debris / Hydraulic Machine Condition Assessment

… and biomedically:

- Gait Analysis Stance Detector - Concept
- Outpatient Pressure Ulcer Protocol Assistant - Concept
- Fuzzy Hill Muscle Model / Clinician Assistant

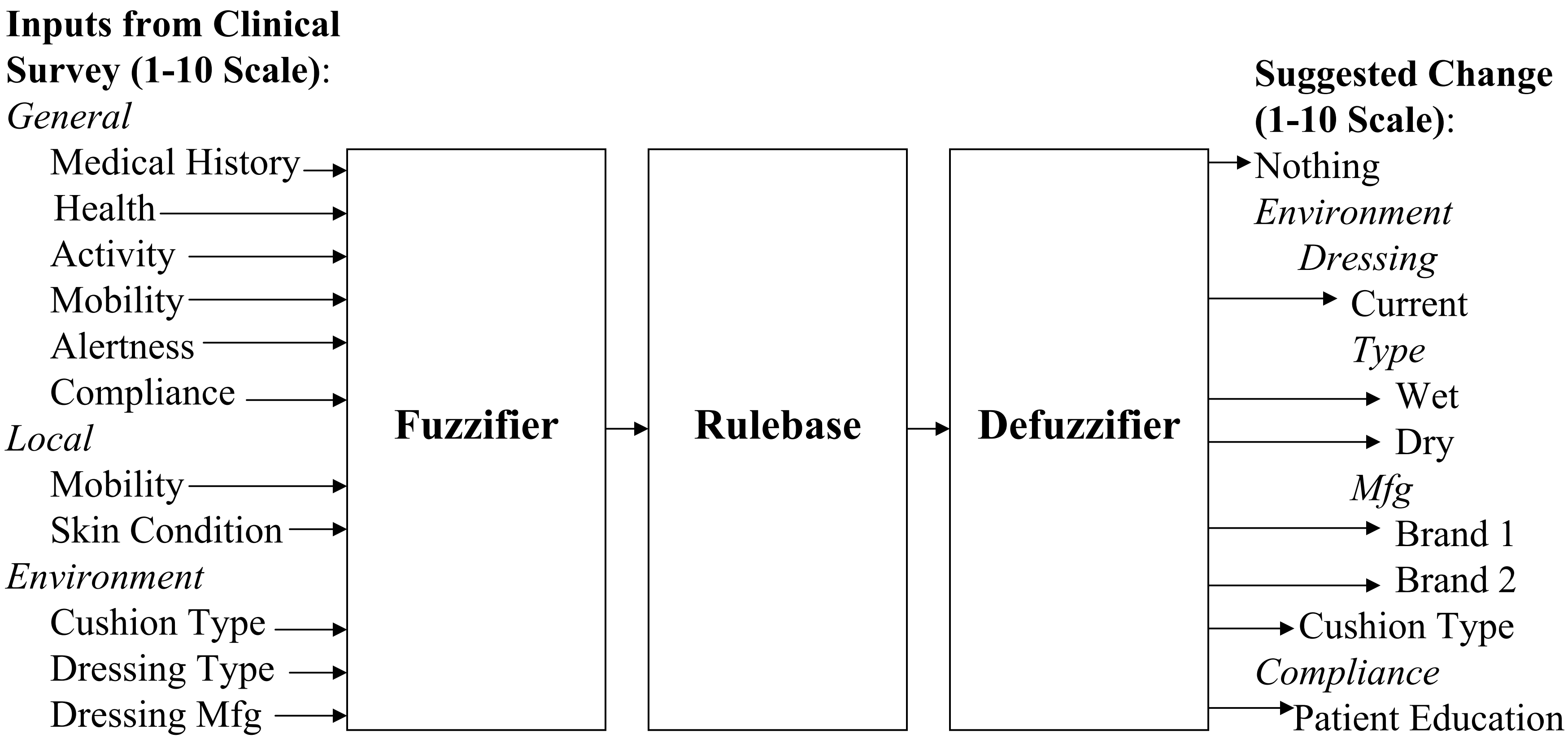
Fuzzy Systems Applications, 1 of 4

- Gait Analysis Stance Detector - Concept
  Task: given measurements of toe and heel velocities and heights (distances off the ground), estimate whether the foot is in stance or not
  Benefit: soft threshold is more robust than hard (won't oscillate)



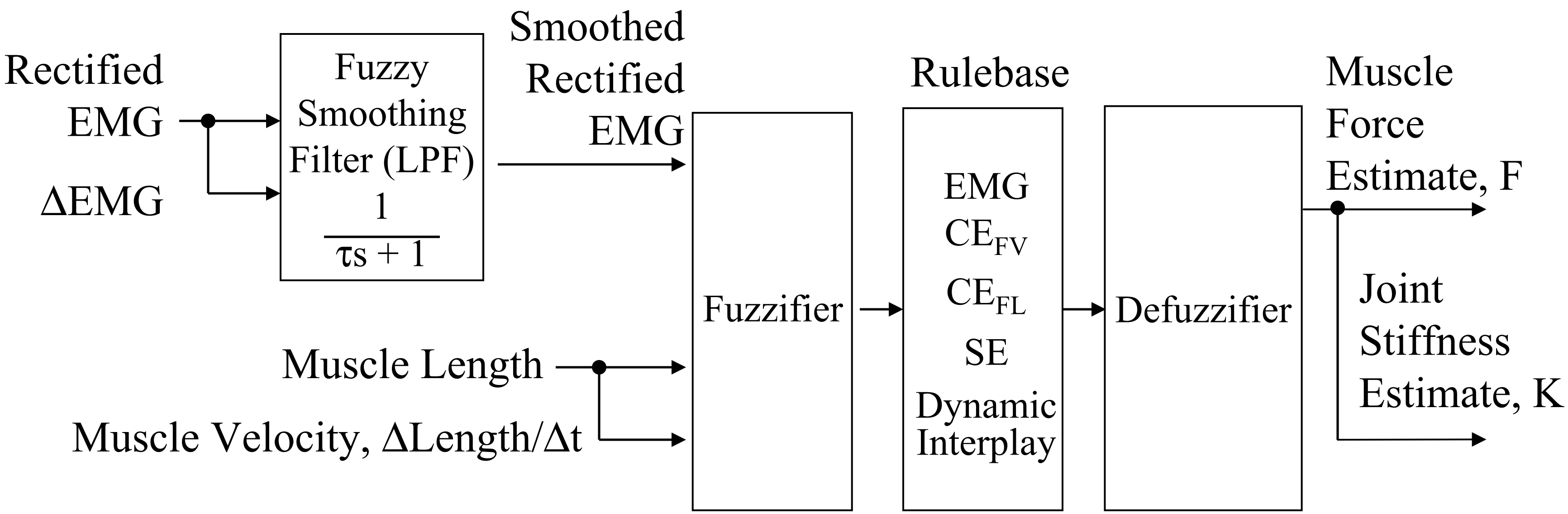
Fuzzy Systems Applications, 2 of 4

- Outpatient Pressure Ulcer Protocol Assistant - Concept
  Task: given clinical survey results as inputs, suggest pressure ulcer prophylaxis or treatment
  Benefit: systematic approach which can be improved with experience



Fuzzy Systems Applications, 3 of 4

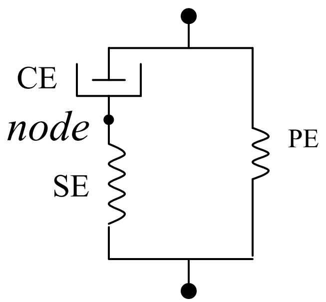
- Fuzzy Hill Muscle Model / Clinician Assistant
  Task: given muscle inputs for a multiple muscle system, estimate muscle force / joint stiffness; classify as normal / pathological

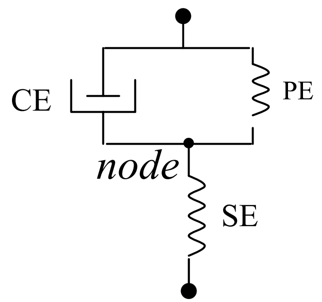


Models like this one will be used to construct musculoskeletal models, which can then be evaluated against other models in the literature. The ultimate goal is to produce a clinical tool (e.g., spasticity estimation).

Fuzzy Systems Applications, 4 of 4

- Fuzzy Hill Muscle Model, continued





– CE is the contractile element; it represents muscle's contractile tissue (viscous dashpot)
– SE, the series elasticity, represents connective tissue (nonlinear spring)
– PE, the parallel elasticity, represents boundary conditions (nonlinear spring)

Rules for the fuzzy model:

– If EMG is high, estimated force is high (nonlinearly)
– If CE is lengthening (CE velocity is positive), estimated force is high
– If CE length is optimal (near rest length), estimated force is high

Other considerations:

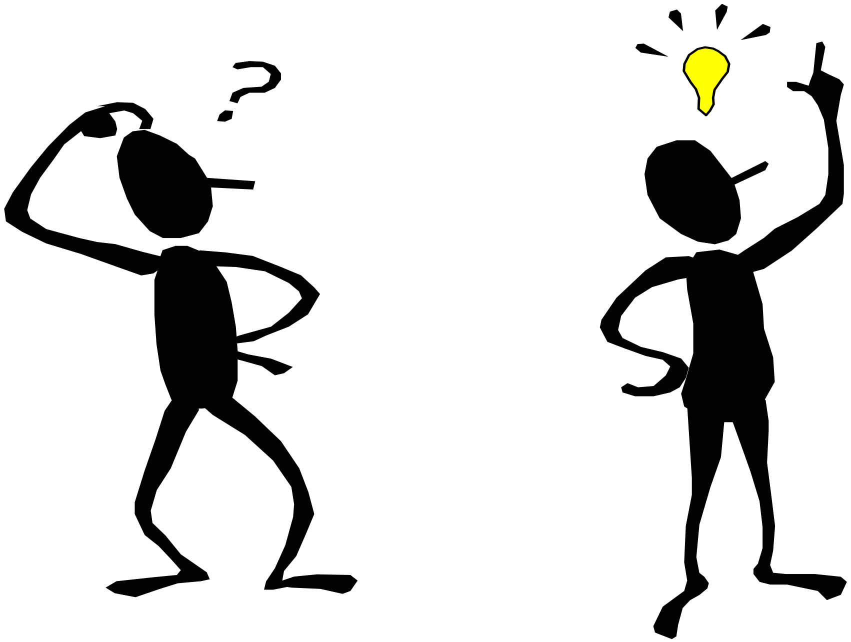
– CE viscosity (sluggishness moving the node; SE may need to stretch)
– Since they're in series, CE and SE must produce the same force
– The CE and SE lengths must sum to total musculotendon unit length

Conclusions

- Fuzzy systems can perform some tasks humans can’t; they can perform some tasks more easily than humans
- Fuzzy and neurofuzzy systems have many and varied commercial applications, especially as controllers, classifiers, trackers, and experts
- Fuzzy and neurofuzzy systems are in widespread use biomedically as controllers, classifiers, and medical advisement systems
- Relationships from classical mathematics can be implemented in fuzzy logic
- High-order dynamical models can be implemented in fuzzy logic via fuzzy inference and expert systems

Discussion

Questions ... Suggestions ... Comments ... Ideas ...?



References

1. Jang JSR, Sun CT, & Mizutani E (1997) Neuro-Fuzzy and Soft Computing: a Computational Approach to Learning and Machine Intelligence, Prentice Hall, Upper Saddle River, NJ

2. Zadeh LA (1965) Fuzzy Sets. *Information and Control* **8**:338-353

3. Lin CT & Lee CSG (1996) Neural Fuzzy Systems: a Neuro-Fuzzy Synergism to Intelligent Systems, Prentice Hall, Upper Saddle River, NJ

4. Mamdani EH and Assilan A (1975) An experiment in linguistic synthesis with a fuzzy logic controller. *International Journal of Man-Machine Studies* 7(**1**):1-13

5. Hill, A.V. (1938) The heat of shortening and the dynamic constants of muscle *Proc. Roy. Soc.* **B126**:136-19

6. Jang JSR & Gulley N (1997) MATLAB Fuzzy Logic Toolbox, The MathWorks, Natick, MA

Wow—bonus material! ☺ Fuzzy for Math Buffs, 1 of 3

Probability and possibility (fuzzy) theory are analogous but different. Here are some interesting observations and relationships:

- Fuzzy arithmetic is really just interval analysis/arithmetic
- Possibility theory relates to fuzzy set theory by defining possibility distribution as a fuzzy restriction—an elastic constraint—on values a fuzzy variable can take on[3]
- Fuzzy logic's "membership function" acts as the "conditional possibility distribution" of possibility theory
- High possibility does not imply high probability; low possibility does not imply low probability

Fuzzy for Math Buffs, 2 of 3

- Probability, possibility, and necessity of events are related.

  Suppose that an event is described as A where event A' is the negation of A.

  Then, denoting the probability of an event as P, the possibility of an event as Π, and the necessity of an event as N, the following three relationships hold:

$$P(A) + P(A') = 1$$
$$\Pi(A) + \Pi(A') \geq 1$$
$$N(A) + N(A') \leq 1$$

  For special cases, possibility and necessity limit probability:

$$N(A) \leq P(A) \leq \Pi(A)$$

Fuzzy for Math Buffs, 3 of 3

A heuristic (caveat emptor!) relationship between possibility and probability is offered by Lotfi Zadeh in this "possibility/probability consistency principle":

if a variable X can take the the values $x_1, x_2, \ldots x_n$ with respective possibility distribution $\Pi_x = (\pi_1, \pi_2, \ldots \pi_n)$ and probability distribution $P_x = (p_1, p_2, \ldots p_n)$, then the degree of consistency of the probability distribution P_x with the possibility distribution Π_x is expressed by

$$\gamma = \pi_1 p_1 + \pi_2 p_2 + \ldots + \pi_n p_n = \sum \pi_i p_i \text{ where } i = 1 \text{ to } n$$

where "+" denotes arithmetic sum. This principle is NOT a law! What makes it useful is that one often knows a variable's possibility distribution but not its probability distribution: the principle basically says not to assign high probability to a value of X for which there is low possibility (Zadeh cited in [3], p. 90).