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                    Frontdoor Adjustment
                                               1207
46
<u>47</u>
```

<u>1</u> 2  $\frac{3}{4}$ <u>5</u> <u>6</u>  $\underline{7}$ 8 12 13

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**Bibliography** 1223

38.9 Further Reading

9 <u>10</u> 11

<u>14</u>  $\underline{15}$ <u>16</u>

<u>17</u> <u>18</u> <u>19</u>

<u>21</u> <u>22</u>  $\underline{23}$  $\underline{24}$ 

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<u>25</u>  $\underline{26}$  $\underline{27}$ 

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31 32 <u>33</u> <u>34</u>  $\underline{35}$ 

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Draft of "Probabilistic Machine Learning: Advanced Topics" by Kevin Murphy. February 27, 2022