CONTINUOUS FIELDS OF C*-ALGEBRAS OVER FINITE DIMENSIONAL SPACES

MARIUS DADARLAT

Abstract. Let X be a finite dimensional compact metrizable space. We prove that all unital continuous fields of C*-algebras over X with fibers isomorphic to the Cuntz algebra $\mathcal{O}_\infty$ are trivial. In a more general context, assuming that X is locally contractible, we show that if A is a unital continuous field over X with fibers Kirchberg C*-algebras satisfying the universal coefficient theorem in KK-theory (UCT) and having finitely generated K-theory groups, then A is isomorphic to a locally trivial field if and only if the K-theory presheaf associated to A is locally trivial. Using certain approximation and deformation techniques, we show that the C*-algebra of sections associated to a continuous field of separable C*-algebras over X satisfies UCT, provided that each fiber is nuclear and satisfies the UCT.

1. introduction

Gelfand's theorem identifies the unital commutative C*-algebras with the algebras of complex valued continuous functions on their primitive spectrum. This suggested the problem of representing non-commutative C*-algebras as sections of bundles [15], [13], [12], [16]. Fell showed that any separable C*-algebra A with Hausdorff primitive spectrum X is isomorphic to the algebra of sections associated to a continuous field over X whose fibers are the primitive quotients of A [15]. Dauns and Hofmann [12] proved that any unital C*-algebra can be represented as the algebra of sections of an upper semi-continuous field of C*-algebras over the primitive spectrum of its center. In view of these results the continuous fields of C*-algebras have become fundamental objects in the theory of C*-algebras.

The structure of a continuous field of C*-algebras can be rather subtle even over spaces such as the unit interval or the one point compactification of the natural numbers. For instance the asymptotic morphisms of Connes and Higson [7] can be described in terms of continuous fields over $[0, 1]$ which are locally trivial over the $[0, 1)$. Perhaps, what makes the subject of continuous fields even more interesting, is the fact that in general a continuous field may fail to be locally trivial at any point (as illustrated for instance by the C*-algebra of the discrete Heisenberg group $\mathbb{H}_3$). The challenge is to develop tools which are adequate for describing how the fibers of continuous fields are glued together.

Date: August 25.

An important progress in the study of continuous fields is represented by the subtriviality theorem of Blanchard [4]: any nuclear separable continuous field of C*-algebras over a compact metrizable space X embeds as a subfield of the trivial field $C(X)\otimes\mathcal{O}_2$, where $\mathcal{O}_2$ is the Cuntz algebra. Its proofs builds on fundamental results of Kirchberg. In fact, the remarkable progress in the classification theory of purely infinite simple nuclear separable C*-algebras (called henceforth Kirchberg algebras) in terms of K-theory [19], [28], [29] has suggested an analogous classification program for non-simple purely infinite nuclear C*-algebras. The structure of these algebras was studied in [22],[23], [6], [4]. A number of striking classification results based on a suitable generalization of Kasparov's $KK_{C(X)}$-theory were announced by Kirchberg in [20]. An effective application of Kirchberg's classification theory to continuous fields seems to require a (yet missing) suitable universal coefficient theorem for the $KK_{C(X)}$-groups.

In this paper we follow a rather different approach in which the continuous fields of Kirchberg algebras over a space X are approximated by subfields with controlled complexity, see Theorems 2.12 and 2.15. This approach was pioneered for fields over zero dimensional spaces in [11]. The case when X is an interval is already quite involved and it was studied in our joint work with G. Elliott [9] where a certain classification result is also given. In the present paper, inspired in part by [6], we extend and refine the methods of [9] to the case of finite dimensional spectra. As it was the case in [9], we shall rely heavily on the classification theorem (and related results) of Kirchberg and Phillips [29], and on the work on non-simple nuclear purely infinite C*-algebras of Blanchard and Kirchberg [6] and Rørdam and Kirchberg [22], [23]. The results of Spielberg [32] on the semiprojectivity of Kirchberg algebras and the results of Spielberg [31] and Lin [25] on the weak semiprojectivity of Kirchberg algebras also play an important role.

Let us discuss here some consequences of our results. We show that in many instances the obstruction to local triviality of a continuous field of Kirchberg algebras is essentially of K-theoretical nature.

Theorem 1.1. *Let X be a finite dimensional locally contractible compact metrizable space and let A be a unital continuous $C(X)$-algebra. Assume that all the fibers of A are Kirchberg algebras with finitely generated K-theory groups and satisfying the UCT. Then A is locally trivial if and only if the K-theory presheaf invariant of A is locally trivial.*

A refined version of this result, Theorem 3.6, allows us to derive the following:

Theorem 1.2. *All unital $C(X)$-algebras A over a finite dimensional locally contractible metrizable space with fibers isomorphic to the Cuntz algebra $\mathcal{O}_n$ ($0\leq n\leq\infty$) are locally trivial.*

If $n=2$ or $n=\infty$ in the above statement, then A is trivial for any finite dimensional metrizable space X, without further restrictions. Our analysis of continuous fields of Kirchberg algebras leads to a new permanence property of the class of nuclear C*-algebras satisfying the UCT.

Theorem 1.3. *If A is a separable continuous $C(X)$-algebra over a finite dimensional metrizable space such that each fiber is nuclear and satisfies the UCT, then A satisfies the UCT.*

The author is grateful to Chris Phillips for useful discussions on the homotopy theory of the automorphism groups of Kirchberg algebras.

2. Approximation of continuous fields

Let X be a compact Hausdorff space. A $C(X)$-algebra is a C*-algebra A endowed with a $*$-monomorphism θ from $C(X)$ to the center of the multiplier algebra of A such that $C(X)A$ is dense in A; see [18], [3]. With the exception of Section 4, we will only consider unital $C(X)$-algebras with $\theta(1) = 1$. If $Y \subseteq X$ is a closed subset, we let $C(X, Y)$ denote the ideal of $C(X)$ consisting of functions vanishing on Y. Then $C(X, Y)A$ is a closed two-sided ideal of A. The quotient of A by this ideal is a $C(Y)$-algebra denoted by $A(Y)$ and is called the restriction of $A = A(X)$ to Y. The quotient map is denoted by $\pi_Y : A(X) \to A(Y)$. If Z is a closed subset of Y we have a natural restriction map $\pi_Z^Y : A(Y) \to A(Z)$ and $\pi_Z = \pi_Z^Y \circ \pi_Y$. If Y reduces to a point x, we write $A(x)$ for $A(\{x\})$ and π_x for $\pi_{\{x\}}$. The C*-algebra $A(x)$ is called the fiber of A at x. The image $\pi_x(a) \in A(x)$ of $a \in A$ is denoted by $a(x)$. If $Y = \emptyset$ then $A(Y)$ is interpreted as the zero algebra.

Let A be a C*-algebra, $a \in A$ and $\mathcal{F}, \mathcal{G} \subseteq A$. If $\varepsilon > 0$, we write $a \in_\varepsilon \mathcal{F}$ if there is $b \in \mathcal{F}$ such that $\|a - b\| < \varepsilon$. Similarly, we write $\mathcal{F} \subset_\varepsilon \mathcal{G}$ if $a \in_\varepsilon \mathcal{G}$ for every $a \in \mathcal{F}$. The following lemma collects some useful known properties.

Lemma 2.1. *Let X be a compact metrizable space and let A be a $C(X)$-algebra and let $B \subset A$ be a $C(X)$-subalgebra. Let $a \in A$ and let Y be a closed nonempty subset of X.*

(i) *The map $x \mapsto \|a(x)\|$ is upper semi-continuous.*
(ii) $\|\pi_Y(a)\| = \max\{\|\pi_x(a)\| : x \in Y\}$
(iii) *If $a(x) \in \pi_x(B)$ for all $x \in X$, then $a \in B$.*
(iv) *If $\delta > 0$ and $a(x) \in_\delta \pi_x(B)$ for all $x \in X$, then $a \in_\delta B$.*

Proof. (i), (ii) and (iii) are proved in [3]. Since B is closed, (iv) $\Rightarrow$ (iii). Thus it suffices to prove (iv). By assumption, for each $x \in X$, there is $b_x \in B$ such that $\|\pi_x(a - b_x)\| < \delta$. Using the upper semi-continuity of the map $x \mapsto \|\pi_x(c)\|$, $c \in A$ and (ii), we find a closed neighborhood U_x of x such that $\|\pi_{U_x}(a - b_x)\| < \delta$. Since X is compact, there is a finite subcover (U_{x_i}). Let (α_i) be a partition of unity subordinated to this cover. Setting $b = \sum_i \alpha_i b_{x_i} \in B$, one checks immediately that $\|\pi_x(a - b)\| \le \sum_i \alpha_i(x) \|\pi_x(a - b_{x_i})\| < \delta$, for all $x \in X$. Thus

$$\|a - b\| = \max\{\|\pi_x(a - b)\| : x \in X\} < \delta.$$

□

A $C(X)$-algebra such that the map $x \mapsto \|a(x)\|$ is continuous for all $a \in A$, is called a *continuous* $C(X)$-algebra or a C*-bundle [3], [24]. For any $C(X)$-algebra A, the map $\pi : A \to \prod_{x\in X} A(x)$ defined by $\pi(a)(x) = \pi_x(a)$ is a $*$-monomorphism and its image Γ satisfies the following properties:

(i) $\Gamma \subset \prod_{x\in X} A(x)$ is a C*-subalgebra.
(ii) $\{s(x) : s \in \Gamma\}$ is dense in $A(x)$.
(iii) The map $x \mapsto \|s(x)\|$ is upper semi-continuous for any $s \in \Gamma$.
(iv) If $C(X)\Gamma \subseteq \Gamma$.
(v) Γ is closed under local uniform approximation, i.e. if $t \in \prod_{x\in X} A(x)$ has the property that for any $\varepsilon > 0$ and any $x_0 \in X$ there is $s \in \Gamma$ and a neighborhood V of x_0 such that $\|t(x) - s(x)\| < \varepsilon$ for all $x \in V$, then $t \in \Gamma$.

The conditions (iv) and (v) are equivalent in the presence of (i). Conversely, if Γ satisfies the conditions (i)-(v), then Γ is a $C(X)$-algebra. Thus the $C(X)$-algebras correspond to C*-algebras of sections in upper semi-continuous fields of C*-algebras. Similarly, a C*-algebra A is a continuous $C(X)$-algebra if and only if A is the C*-algebra of continuous sections of a continuous field of C*-algebras over X in the sense of [13, Def. 10.3.1], i.e. A satisfies the conditions (i)–(v), except that in (iii) one requires continuity of the map $x \mapsto \|s(x)\|$. (see [3], [5], [27]).

Let $\eta : B \to E$ and $\psi : D \to E$ be $*$-homomorphisms. The pullback of these maps is

$$B \oplus_{\eta,\psi} D = \{(b,d) \in B \oplus D : \eta(b) = \psi(d)\}.$$

We are going to use pullbacks in the context of $C(X)$-algebras. Let X be a compact metrizable space and let $A = A(X)$ be a $C(X)$-algebra. Let Y, Z be closed nonempty subsets of X with $X = Y \cup Z$.

Lemma 2.2. *A is isomorphic to $A(Y) \oplus_{\pi,\pi} A(Z)$, the pullback of the restriction maps $\pi^Y_{Y\cap Z} : A(Y) \to A(Y \cap Z)$ and $\pi^Z_{Y\cap Z} : A(Z) \to A(Y \cap Z)$.*

Proof. This is proved in [13, Prop. 10.1.13] for continuous $C(X)$-algebras but the same proof works for general $C(X)$-algebras. Let us briefly review the argument since the result is crucial for our paper. By the universal property of the pullbacks, the maps π_Y and π_Z induce a map $\eta : A \to A(Y) \oplus_{\pi,\pi} A(Z)$, $\eta(a) = (\pi_Y(a), \pi_Z(a))$, which is injective by Lemma 2.1(ii). Let $b \in A(Y)$ and $c \in A(Z)$ be such that $\pi^Y_{Y\cap Z}(b) = \pi^Z_{Y\cap Z}(c)$. To prove surjectivity of η we need to find $a \in A$ such that $\pi_Y(a) = b$ and $\pi_Z(a) = c$. Define $t \in \prod_{x\in X} A(x)$ by $t(x) = b(x)$ if $x \in Y$ and $t(x) = c(x)$ if $x \in Z$. To complete the proof it suffices to show that t admits local uniform approximations as in the property (v) from above. Let $x_0 \in X$ and $\varepsilon > 0$. Let $b_1, c_1 \in A$ be liftings of b and c. If $x_0 \in Y \setminus Z$ then $\|b_1(x) - t(x)\| = 0$ on the open set $Y \setminus Z$. Similarly, if $x_0 \in Z \setminus Y$ then $\|c_1(x) - t(x)\| = 0$ on the open set $Z \setminus Y$. Finally if $x_0 \in Y \cap Z$, then $\|(b_1 - c_1)(x_0)\| = 0$, hence by Lemma 2.1(i), $\|(b_1 - c_1)(x)\| < \varepsilon$

for all x in some neighborhood V of x_0. It follows that $\|b_1(x) - t(x)\| < \varepsilon$ for all $x \in V$ since $t(x) \in \{b_1(x), c_1(x)\}$ for all $x \in V$. $\square$

If B is a $C(X)$-subalgebra of A, then the fibers $B(x)$ of B identifies with the C*-subalgebra $\pi_x(B)$ of $A(x)$. Let $B(Y) \subset A(Y)$ and $D(Z) \subset A(Z)$ be $C(X)$-subalgebras such that $D(x) \subset B(x)$ for all $x \in Y \cap Z$. Then

$$B \oplus_{Y\cap Z} D = \{a \in A(X) : a(y) \in B(y), \text{if } y \in Y \text{ and } a(z) \in D(z), \text{if } z \in Z\}$$

defines a $C(X)$-subalgebra of A. Using Lemma 2.2, it is easily seen that its fibers are

$$(B \oplus_{Y\cap Z} D)(x) = \begin{cases} B(x), & x \in X \setminus Z \\ D(x), & x \in Z. \end{cases}$$

Let X, Y, Z and $A(X)$ be as above. Let $\eta : B(Y) \hookrightarrow A(Y)$ be a $C(Y)$-linear $*$-monomorphism and let $\psi : D(Z) \hookrightarrow A(Z)$ be a $C(Z)$-linear $*$-monomorphism. Assume that

$$\pi^Z_{Y\cap Z}(\psi(D)) \subseteq \pi^Y_{Y\cap Z}(\eta(B)). \tag{1}$$

This gives a map $\eta^{-1}\pi^Z_{Y\cap Z}\psi : D(Z) \to B(Y \cap Z)$.

Lemma 2.3. *(a) Under the previous assumptions, there are isomorphisms of $C(X)$-algebras:*

$$B \oplus_{\pi\eta,\pi\psi} D \cong B \oplus_{\pi,\eta^{-1}\pi\psi} D \cong \eta(B) \oplus_{Y\cap Z} \psi(D),$$

where the latter isomorphism denoted by Φ is induced by the pair (η, ψ).

(b) Condition (1) *is equivalent to $\psi(D) \subset \big(A \oplus_Y \eta(B)\big)(Z)$. Moreover*

$$\big(A \oplus_Y \eta(B)\big) \oplus_Z \psi(D) \cong \eta(B) \oplus_{Y\cap Z} \psi(D).$$

(c) Assume that $\mathcal{F}$ is a finite subset of A such that $\pi_Y(\mathcal{F}) \subset_\varepsilon \eta(B)$ and $\pi_Z(\mathcal{F}) \subset_\varepsilon \psi(D)$. Then $\mathcal{F} \subset_\varepsilon \Phi\big(B \oplus_{\pi,\eta^{-1}\pi\psi} D\big) = \eta(B) \oplus_{Y\cap Z} \psi(D)$.

Proof. This is an immediate corollary of Lemma 2.1 and Lemma 2.2. $\square$

A separable unital C*-algebra D is *weakly semiprojective* (see [14]) if for any finite subset $\mathcal{F} \subset D$ and any $\varepsilon > 0$, any C*-algebra B , any increasing sequence (J_n) of two-sided closed ideals of B and any $*$-homomorphism $\iota : D \to B/J$ (where J is the closure of $\cup_n J_n$) there is a $*$-homomorphism $\varphi : D \to B/J_n$ (for some n) such that $\|\pi_n\varphi(c) - \iota(c)\| < \varepsilon$ for all $c \in \mathcal{F}$ (where $\pi_n : B/J_n \to B/J$ is the natural map). If we require that there is φ such that $\pi_n\varphi = \iota$ then A is *semiprojective* (see [2]). We shall use (weak) semiprojectivity in the following context. Let B be a $C(X)$-algebra, let $x \in X$ and let $U_n = \{y \in X : d(y, x) \leq 1/n\}$. Then $J_n = C(X, U_n)B$ is an increasing sequence of ideals of B such that $B/J_n \cong B(U_n)$ and $B/J \cong B(x)$. All the C*-algebras and the morphisms appearing in the next three propositions will be assumed to be unital.

Proposition 2.4. [14, Thms. 3.1, 4.6] *Let D be a separable weakly semiprojective C*-algebra. For any finite subset $\mathcal{F} \subset D$ and any $\varepsilon > 0$ there exist a finite subset $\mathcal{G} \subset D$ and $\delta > 0$ such that for any C*-algebras $B \subset A$ and any $*$-homomorphism $\varphi : D \to A$ with $\varphi(\mathcal{G}) \subset_\delta B$, there is a $*$-homomorphism $\psi : D \to B$ such that $\|\varphi(a) - \psi(a)\| < \varepsilon$ for all $a \in \mathcal{F}$. If $K_*(D)$ is finitely generated, then we can choose $\mathcal{G}$ and δ such that $K_*(\psi) = K_*(\varphi)$.*

We need the following generalizations of two results of Loring [26]; see [9].

Proposition 2.5. *Let D be a separable semiprojective C*-algebra. For any finite subset $\mathcal{F} \subset D$ and any $\varepsilon > 0$, there exist a finite subset $\mathcal{G} \subset D$ and $\delta > 0$ with the following property. Let $\pi : A \to B$ be a surjective $*$-homomorphism, and let $\sigma : D \to B$ and $\gamma : D \to A$ be $*$-homomorphisms such that $\|\pi\gamma(d) - \sigma(d)\| < \delta$ for all $d \in \mathcal{G}$. Then there is a $*$-homomorphism $\psi : D \to A$ such that $\pi\psi = \sigma$ and $\|\gamma(c) - \psi(c)\| < \varepsilon$ for all $c \in \mathcal{F}$.*

Examples 2.6. We will consider various classes $\mathcal{C}$ consisting of unital separable weakly semiprojective simple C*-algebras. The main examples are the class of simple finite dimensional C*-algebras and the class of unital Kirchberg algebras satisfying the UCT and having finitely generated K-theory groups. The C*-algebras in the latter class are known to be weakly semiprojective by work of H. Lin [25] and Spielberg [31]. They are semiprojective if they have torsion free K_1-groups by a result of Spielberg [32].

Definition 2.7. Let $\mathcal{C}$ be a class of unital separable simple C*-algebras. A unital $C(Z)$-algebra D is called *elementary* (relative to the class $\mathcal{C}$) if there is a finite partition of X into closed disjoint non-empty subsets $Z_1, \dots, Z_r$ ($r \geq 1$) together with C*-algebras $D_i \in \mathcal{C}$ such that A is isomorphic $\oplus_i C(Z_i) \otimes D_i$. The notion of *type* of a unital $C(X)$-algebra with respect to a class $\mathcal{C}$ is defined inductively: $\mathrm{type}_{\mathcal{C}}(A) = 0$ if A is elementary relative to $\mathcal{C}$ and $\mathrm{type}_{\mathcal{C}}(A) \leq n$ if there are closed nonempty subsets Y and Z of X, with $X = Y \cup Z$ and there exist a unital $C(Y)$-algebra B and a unital $C(Z)$-algebra D together with a unital $*$-monomorphism of $C(Y \cap Z)$-algebras, $\gamma : D(Y \cap Z) \to B(Y \cap Z)$, such that $\mathrm{type}_{\mathcal{C}}(B) \leq n - 1$, $\mathrm{type}_{\mathcal{C}}(D) = 0$ and A is isomorphic to

$$B \oplus_{\pi, \gamma\pi} D = \{(b, d) \in B \oplus D : \pi^Y_{Y \cap Z}(b) = \gamma\pi^Z_{Y \cap Z}(d)\}.$$

By definition $\mathrm{type}_{\mathcal{C}}(A) = n$ if n is the smallest number with the property that $\mathrm{type}_{\mathcal{C}}(A) \leq n$.

Definition 2.8. Let $\mathcal{C}$ be a class of unital C*-algebras. Let A be a unital $C(X)$-algebra. An *n-fibered morphism into A with base $\mathcal{C}$* consists of $(n+1)$ unital $*$-monomorphisms $(\psi_0, \dots, \psi_n)$ with the following properties. There exist closed non-empty subsets of X, $Y_0, \dots, Y_n$ and elementary $C(Y_i)$-algebras, $D_0, \dots, D_n$ such that each $\psi_i : D_i \to A(Y_i)$ is $C(Y_i)$-linear and

$$\pi^{Y_i}_{Y_i\cap Y_j}\psi_i(D_i) \subseteq \pi^{Y_j}_{Y_i\cap Y_j}\psi_j(D_j), \quad \text{for all } i \le j. \tag{2}$$

Given an n-fibered morphism into A we have an associated $C(X)$-algebra defined as a certain pullback involving the maps ψ_i:

$$A(\psi_0,\dots,\psi_n) = \{(d_0,\dots d_n) \,:\, d_i \in D_i,\, \pi^{Y_i}_{Y_i\cap Y_j}\psi_i(d_i) = \pi^{Y_j}_{Y_i\cap Y_j}\psi_j(d_j) \text{ for all } i,j\} \tag{3}$$

and an induced $C(X)$-homomorphism

$$\eta = \eta_{(\psi_0,\dots,\psi_n)} : A(\psi_0,\dots,\psi_n) \to A \subset \bigoplus_{i=0}^{n} A(Y_i),$$

$$\eta(d_0,\dots d_n) = \big(\psi_0(d_0),\dots,\psi_n(d_n)\big).$$

There are natural projection maps $p_i : A(\psi_0,\dots,\psi_n) \to D_i$, $p_i(d_0,\dots,d_n) = d_i$. Let us set $X_k = Y_k \cup \dots \cup Y_n$. Then, $(\psi_k,\dots\psi_n)$ is an $(n-k)$-fibered morphism into $A(X_k)$. Let $\eta_k : A(X_k)(\psi_k,\dots\psi_n) \to A(X_k)$ be the induced map and let $B_k = A(X_k)(\psi_k,\dots\psi_n)$. Then, there exist natural $C(X_{k-1})$-isomorphisms

$$B_k \oplus_{\pi\eta_k,\pi\psi_{k-1}} D_{k-1} \cong B_k \oplus_{\pi,\eta_k^{-1}\pi\psi_{k-1}} D_{k-1} \cong B_{k-1}. \tag{4}$$

Therefore if $A(\psi_0,\dots,\psi_n)$ is the pullback of an n-fibered morphism into A, then

$$\mathrm{type}_{\mathcal{C}}\big(A(\psi_0,\dots,\psi_n)\big) \le n. \tag{5}$$

An n-fibered morphism into A, $(\psi_0,\dots,\psi_n)$ (defined as above), is called *open* if there is an n-fibered morphism into A, $(\psi_0',\dots,\psi_n')$, with components $\psi_i' : D_i' \to C(Y_i')$ such that Y_i' is a neighborhood of Y_i, $D_i'(Y_i) = D_i$, and $\pi^{Y_i'}_{Y_i}\psi_i' = \psi_i$ for all $i = 0,\dots,n$. One should think of an open n-fibered morphism as a kind of germ of n-fibered morphisms.

The following lemma is useful for the construction of fibered morphisms.

Lemma 2.9. *Let $\mathcal{C}$ be a class of separable unital simple weakly semiprojective C*-algebras. Let $(D_j)_{j\in J}$ be a finite family of C*-algebras in $\mathcal{C}$. For each $j \in J$, let $\mathcal{H}_j \subset D_j$ be a finite set and let $\varepsilon > 0$. Let $\mathcal{G}_j \subset D_j$ (a finite set) and $\delta_j > 0$ be given by Proposition 2.4 applied to D_j, $\mathcal{H}_j$ and $\varepsilon/2$. Let X be a compact metrizable space, let $(Z_j)_{j\in J}$ be a finite family of mutually disjoint nonempty closed subsets of X and let Y be a closed nonempty subset of X such that $X = Y \cup \big(\cup_j Z_j\big)$. Let A be a unital $C(X)$-algebra and let $\mathcal{F}$ be a finite subset of A. Let $\eta : B(Y) \hookrightarrow A(Y)$ be a unital $C(Y)$-subalgebra and let $\varphi_j : D_j \to A(Z_j)$ be unital $*$-homomorphisms satisfying*

(i) $\pi_{Z_j}(\mathcal{F}) \subset_{\varepsilon/2} \varphi_j(\mathcal{H}_j)$, *for all* $j \in J$,
(ii) $\pi_Y(\mathcal{F}) \subset_{\varepsilon} \eta(B)$,
(iii) $\pi^{Z_j}_{Y\cap Z_j}\varphi_j(\mathcal{G}_j) \subset_{\delta_j} \pi^{Y}_{Y\cap Z_j}\eta(B)$, *for all* $j \in J$.

Then, there are $C(Z_j)$-linear unital $*$-homomorphisms $\psi_j : C(Z_j) \otimes D_j \to A(Z_j)$, satisfying

$$\|\varphi_j(a) - \psi_j(a)\| < \varepsilon/2, \text{ for all } a \in \mathcal{H}_j, \text{ and } j \in J, \tag{6}$$

and such that if we set $D = \oplus_j C(Z_j) \otimes D_j$, $Z = \cup_j Z_j$, and $\psi : D \to A(Z) = \oplus_j A(Z_j)$ has components ψ_j, then $\pi^Z_{Y\cap Z}(\psi(D)) \subseteq \pi^Y_{Y\cap Z}(\eta(B))$ and $\pi_Z(\mathcal{F}) \subset_\varepsilon \psi(D)$. Moreover

$$\mathcal{F} \subset_\varepsilon \eta(B) \oplus_{Y\cap Z} \psi(D) \cong B \oplus_{\pi, \eta^{-1}\pi\psi} D$$

where the last isomorphism is induced by the pair (η, ψ).

Proof. Let $\mathcal{F} = \{a_1, \ldots, a_r\} \subset A$ be as in the statement. By (i) we find $\{c^{(j)}_1, \ldots, c^{(j)}_r\} \subseteq \mathcal{H}_j$ such that $\|\varphi_j(c^{(j)}_i) - \pi_{Z_j}(a_i)\| < \varepsilon/2$ for all i. Consider the $C(X)$-algebra $A \oplus_Y \eta(B) \subset A$. From (iii) and Lemma 2.1 (vi) we obtain

$$\varphi_j(\mathcal{G}_j) \subset_{\delta_j} (A \oplus_Y \eta(B))(Z_j).$$

Applying Proposition 2.4 we perturb φ_j to a $*$-monomorphism $\psi_j : D_j \to (A \oplus_Y \eta(B))(Z_j)$ satisfying (6), and hence such that $\|\varphi_j(c^{(j)}_i) - \psi_j(c^{(j)}_i)\| < \varepsilon/2$, for all i, j. Therefore

$$\|\psi_j(c^{(j)}_i) - \pi_{Z_j}(a_i)\| \le \|\psi_j(c^{(j)}_i) - \varphi_j(c^{(j)}_i)\| + \|\varphi_j(c^{(j)}_i) - \pi_{Z_j}(a_i)\| < \varepsilon.$$

This shows that $\pi_{Z_j}(\mathcal{F}) \subset_\varepsilon \psi_j(D_j)$. Extending ψ_j to a $C(Z_j)$-linear unital $*$-homomorphism, $\psi_j : C(Z_j) \otimes D_j \to (A \oplus_Y \eta(B))(Z_j)$, and defining D and ψ as in the statement and setting $Z = \cup_j Z_j$, we obtain that $\psi : D \to (A \oplus_Y \eta(B))(Z) \subset A(Z)$ satisfies

$$\pi_Z(\mathcal{F}) \subset_\varepsilon \psi(D). \tag{7}$$

The property $\psi(D) \subset (A \oplus_Y \eta(B))(Z)$ is equivalent to $\pi^Z_{Y\cap Z}(\psi(D)) \subseteq \pi^Y_{Y\cap Z}(\eta(B))$ by Lemma 2.3(b). Finally from (ii), (7) and Lemma 2.1 (iv) we get $\mathcal{F} \subset_\varepsilon \eta(B) \oplus_{Y\cap Z} \psi(D)$. $\square$

Lemma 2.10. *Let $\mathcal{C}$ be a class of separable unital simple weakly semiprojective C*-algebras. Let X be a compact metrizable space and let A be a unital $C(X)$-algebra. Let $\mathcal{F} \subset A$ be a finite subset and let $\varepsilon > 0$. Let $x \in X$ and assume that $A(x)$ is isomorphic to the inductive limit of a sequence of C*-algebras in $\mathcal{C}$ with unital connecting maps. Then there is $D \in \mathcal{C}$, and there exist a compact neighborhood U of x and a unital $*$-homomorphism $\varphi : D \to A(U)$ such that $\pi_U(\mathcal{F}) \subset_\varepsilon \varphi(D)$.*

Proof. By hypothesis we find $D \in \mathcal{C}$ and a unital $*$-homomorphism $\iota : D \to A(x)$ such that

$$\pi_x(\mathcal{F}) \subset_{\varepsilon/2} \iota(D).$$

If $\mathcal{F} = \{a_1, \ldots, a_r\}$, there is $\{c_1, \ldots, c_r\} \subset D$ such that $\|\pi_x(a_i) - \iota(c_i)\| < \varepsilon/2$, for all i. Let $U_n = \{y \in X : d(x, y) \le 1/n\}$. Since D is weakly semiprojective there is a unital $*$-homomorphism $\varphi : D \to A(U)$ (with $U = U_n$ for some n) such that $\|\pi_x\varphi(c_i) - \iota(c_i)\| < \varepsilon/2$ for all i. Therefore for all $1 \le i \le r$

$$\|\pi_x\varphi(c_i) - \pi_x(a_i)\| \le \|\pi_x\varphi(c_i) - \iota(c_i)\| + \|\pi_x(a_i) - \iota(c_i)\| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Using Lemma 2.1, after replacing U with U_N for some large enough N and setting $\varphi = \pi_U\varphi$, we have for all i, $1 \le i \le r$

$$\|\varphi(c_i) - \pi_U(a_i)\| = \|\pi_U\big(\varphi(c_i) - a_i\big)\| < \varepsilon.$$

This shows that $\pi_U(\mathcal{F}) \subset_\varepsilon \varphi(D)$. □

Lemma 2.11. *Let $\mathcal{C}$ be a class of unital Kirchberg algebras satisfying the UCT and having finitely generated K-theory groups. Let X be a compact metrizable space and let A be a unital $C(X)$-algebra. Let $\mathcal{F} \subset A$ be a finite subset and let $\varepsilon > 0$. Let $x \in X$ and assume that $A(x) \in \mathcal{C}$. Assume furthermore that there are a C*-algebra $D \in \mathcal{C}$, a closed neighborhood V of x and a morphism of groups $\theta : K_*(D) \to K_*(A(V))$ such that $\theta[1_D] = [1_{A(V)}]$ and $K_*(\pi_x) \circ \theta : K_*(D) \to K_*(A(x))$ is bijective. Then there are a closed neighborhood U of x and a unital $*$-homomorphism $\varphi : D \to A(U)$ such that $\pi_U(\mathcal{F}) \subset_\varepsilon \varphi(D)$ and $K_*(\varphi) = K_*(\pi_U^V) \circ \theta$.*

Proof. Let $\mathcal{F}$ and ε be given. The C*-algebra $A(V)$ is isomorphic to $A(V) \otimes \mathcal{O}_\infty$ by [6] and [23], as explained in [11, Lemma 3.4]. Since D satisfies the UCT we can lift θ first to an element of $KK(D, A(V))$, and then to a unital $*$-homomorphism $\psi : D \to A(V)$ by [29, Thm. 8.2.1]. Since $K_*(\pi_x\psi) = K_*(\pi_x) \circ \theta$, we deduce that $K_*(\pi_x\psi)$ is bijective and maps $[1_D]$ to $[1_{A(x)}]$. By [29, Thm. 8.4.1], since both D and $A(x)$ satisfy the UCT, there is a $*$-isomorphism $\iota : D \to A(x)$ such that $\pi_x\psi$ is approximately unitarily equivalent to ι. Therefore, there is a unitary $v \in A(x)$ such that $\pi_x(\mathcal{F}) \subset_\varepsilon v\pi_x\psi(D)v^*$. Arguing as in the proof of Lemma 2.10, we find a closed neighborhood U of x contained in V and a lifting of v to a unitary $u \in A(U)$ such that $\pi_U(\mathcal{F}) \subset_\varepsilon u\pi_U^V\psi(D)u^*$. Therefore, $\varphi = u\pi_U^V\psi u^*$ satisfies the desired conclusions. □

Let $\mathcal{C}$ be a class of separable unital simple weakly semiprojective C*-algebras. Let A be a unital $C(X)$-algebra with all fibers isomorphic to inductive limits of sequences of C*-algebras in $\mathcal{C}$. Let $\mathcal{F} \subset A$ be a finite subset and let $\varepsilon > 0$. Using Lemma 2.10 and the compactness of X, we find a finite family of closed subsets of X, $(U_i)_{i\in I}$, whose interiors cover X, and a family of unital $*$-homomorphisms $\varphi_i : D_i \to A(U_i)$, $D_i \in \mathcal{C}$, such that

$$\pi_{U_i}(\mathcal{F}) \subset_{\varepsilon/2} \varphi_i(\mathcal{H}_i),$$

for some finite sets $\mathcal{H}_i \subset D_i$. Let $\mathcal{G}_i \subset D_i$ and $\delta_i > 0$ be given by Proposition 2.4 applied to the weakly semiprojective C*-algebra D_i for the input data $\mathcal{H}_i$ and $\varepsilon/2$. The outcome of this construction, denoted by

$$\alpha = \{\mathcal{F}, \varepsilon, \{U_i, \varphi_i : D_i \to A(U_i), \mathcal{H}_i, \mathcal{G}_i, \delta_i\}_{i\in I}\},$$

is called an $(\mathcal{F},\varepsilon)$-*approximation* of A (relative to $\mathcal{C}$). It is useful to consider the following operation of restriction. Assume that Y is a closed subspace of X, and let $(V_j)_{j\in J}$ be a finite family of closed subsets of Y that refines the family $(Y\cap U_i)_{i\in I}$ and such that the interiors of $V_j's$ form a cover of Y. Let $\iota: J\to I$ be a map such that $V_j\subseteq Y\cap U_{\iota(j)}$. Define

$$\iota^*(\alpha)=\{\pi_Y(\mathcal{F}),\varepsilon,\{V_j,\pi_{V_j}\varphi_{\iota(j)}:D_{\iota(j)}\to A(V_j),\mathcal{H}_{\iota(j)},\mathcal{G}_{\iota(j)},\delta_{\iota(j)}\}_{j\in J}\}.$$

It is obvious that $\iota^*(\alpha)$ is a $(\pi_Y(\mathcal{F}),\varepsilon)$-approximation of $A(Y)$. The operation $\alpha\mapsto\iota^*(\alpha)$ is useful even in the case $X=Y$. Indeed, by applying this procedure, we can refine the cover of X that appears in a given $(\mathcal{F},\varepsilon)$-approximation of A.

An $(\mathcal{F},\varepsilon)$-approximation of A, $\alpha=\{\mathcal{F},\varepsilon,\{U_i,\varphi_i:D_i\to A(U_i),\mathcal{H}_i,\mathcal{G}_i,\delta_i\}_{i\in I}\}$ is subordinated to an $(\mathcal{F}',\varepsilon')$-approximation of A, $\alpha'=\{\mathcal{F}',\varepsilon',\{U_{i'},\varphi_{i'}:D_{i'}\to A(U_{i'}),\mathcal{H}_{i'},\mathcal{G}_{i'},\delta_{i'}\}_{i'\in I"}\}$, written $\alpha\prec\alpha'$, if

(i) $\mathcal{F}\subseteq\mathcal{F}'$,
(ii) $\varphi_i(\mathcal{G}_i)\subseteq\pi_{U_i}(\mathcal{F}')$ for all $i\in I$, and
(iii) $\varepsilon'<\min\big(\{\varepsilon\}\cup\{\delta_i,\, i\in I\}\big)$.

It is clear that with notation as above, and $\iota':I'\to J'$, we have $\iota^*(\alpha)\prec\iota'^*(\alpha')$ whenever $\alpha\prec\alpha'$ and $Y=Y'$.

Theorem 2.12. *Let $\mathcal{C}$ be a class of separable simple unital weakly semiprojective C^*-algebras. Let X be a finite dimensional compact metrizable space and let A be a unital $C(X)$-algebra with all fibers isomorphic to inductive limits of sequences of C^*-algebras in $\mathcal{C}$ with unital connecting maps. Let $\mathcal{F}\subset A$ be a finite subset and let $\varepsilon>0$. Then, there is a unital $C(X)$-subalgebra B of A such that $\mathcal{F}\subset_\varepsilon B$ and* $\mathrm{type}_{\mathcal{C}}(B)\le n$, *where $n=\dim(X)$. Moreover, B is isomorphic to the pullback of an open n-fibered morphism with base $\mathcal{C}$ into A.*

Proof. For any finite set $\mathcal{F}\subset A$ and any $\varepsilon>0$, Lemma 2.10 and Proposition 2.4 shows that there is an $(\mathcal{F},\varepsilon)$-approximation of A. Moreover, for any finite set $\mathcal{F}\subset A$, any $\varepsilon>0$ and any n, there is a sequence $\{\alpha_k: 0\le k\le n\}$ of $(\mathcal{F}_k,\varepsilon_k)$-approximations of A such that $(\mathcal{F}_0,\varepsilon_0)=(\mathcal{F},\varepsilon)$ and α_k is subordinated to α_{k+1}:

$$\alpha_0\prec\alpha_1\prec\cdots\prec\alpha_n.$$

Indeed, assume that α_k was constructed. To construct α_{k+1}, we consider a finite set $\mathcal{F}_{k+1}$ which contains $\mathcal{F}_k$ and liftings to $A(X)$ of all the elements in $\cup_{i_k\in I_k}\varphi_{i_k}(\mathcal{G}_{i_k})$. That choice takes care simultaneously of the conditions (i) and (ii) so that $\alpha_k\prec\alpha_{k+1}$ provided that we choose ε_{k+1} sufficiently small.

Since X is a compact Hausdorff space of dimension $\le n$, by [5, Lemma 3.2], for every open cover $\mathcal{V}$ of X there is a finite open cover $\mathcal{U}$ which refines $\mathcal{V}$ and such that the set $\mathcal{U}$ can be partitioned into $n+1$ subsets consisting of elements with pairwise disjoint closures. Since we can refine simultaneously the covers that appear in a finite family of $(\mathcal{F},\varepsilon)$-approximations while preserving subordinations, we may arrange not only that all α_k share the same cover $(U_i)_{\in I}$, but moreover, that the cover $(U_i)_{\in I}$ can be partitioned into $n+1$ subsets $\mathcal{U}_0, \dots, \mathcal{U}_n$ consisting of mutually disjoints elements, by [5, Lemma 3.2]. For definiteness, let $\mathcal{U}_k = \{U_{i_k} : i_k \in I_k\}$. Now for each k we consider the closed subset of X

$$Y_k = \bigcup_{i_k \in I_k} U_{i_k},$$

the map $\iota_k : I_k \to I$ and the $(\pi_{Y_k}(\mathcal{F}_k), \varepsilon_k)$-approximation of $A(Y_k)$, induced by α_k, which is of the form

$$\iota_k^*(\alpha_k) = \{\pi_{Y_k}(\mathcal{F}_k), \varepsilon, \{U_{i_k}, \varphi_{i_k} : D_{i_k} \to A(U_{i_k}), \mathcal{H}_{i_k}, \mathcal{G}_{i_k}, \delta_{i_k}\}_{i_k \in I_k}\}.$$

We have

$$\pi_{U_{i_k}}(\mathcal{F}_k) \subset_{\varepsilon_k/2} \varphi_{i_k}(\mathcal{H}_{i_k}), \tag{8}$$

by construction. Since $\alpha_k \prec \alpha_{k+1}$ we obtain

$$\mathcal{F}_k \subseteq \mathcal{F}_{k+1}, \tag{9}$$

$$\varphi_{i_k}(\mathcal{G}_{i_k}) \subseteq \pi_{U_{i_k}}(\mathcal{F}_{k+1}), \text{ for all } i_k \in I_k, \tag{10}$$

$$\varepsilon_{k+1} < \min\big(\{\varepsilon_k\} \cup \{\delta_{i_k},\, i_k \in I_k\}.\big) \tag{11}$$

Set $X_k = Y_k \cup \dots \cup Y_n$ and $D_k = \oplus_{i_k} C(U_{i_k}) \otimes D_{i_k}$ for $0 \le k \le n$. We construct by induction on decreasing k, a sequence $\psi_n, \dots, \psi_0$ of unital $*$-monomorphisms, such that $\psi_k : D_k \to A(Y_k)$ is $C(Y_k)$-linear and such that $(\psi_k, \dots, \psi_n)$ is an $(n-k)$-fibered morphism into $A(X_k)$. Each map

$$\psi_k = \oplus_{i_k} \psi_{i_k} : D_k \to A(Y_k) = \oplus_{i_k} A(U_{i_k})$$

will have components $\psi_{i_k} : C(U_{i_k}) \otimes D_{i_k} \to A(U_{i_k})$ whose restrictions to D_{i_k} will be perturbations of $\varphi_{i_k} : D_{i_k} \to A(U_{i_k})$, $i_k \in I_k$. We will construct the maps ψ_k recursively, such that if $B_k = A(X_k)(\psi_k, \dots, \psi_n)$ and $\eta_k : B_k \to A(X_k)$ is the map induced by the $(n-k)$-fibered morphism $(\psi_k, \dots, \psi_n)$, then

$$\pi_{X_{k+1} \cap U_{i_k}}\big(\psi_{i_k}(D_{i_k})\big) \subset \pi_{X_{k+1} \cap U_{i_k}}\big(\eta_{k+1}(B_{k+1})\big),\ \forall\, i_k \in I_k, \tag{12}$$

and

$$\pi_{X_k}(\mathcal{F}_k) \subset_{\varepsilon_k} \eta_k(B_k). \tag{13}$$

Note that (12) implies that

$$\pi_{X_{k+1} \cap Y_k}\big(\psi_k(D_k)\big) \subset \pi_{X_{k+1} \cap Y_k}\big(\eta_{k+1}(B_{k+1})\big). \tag{14}$$

For the first step of induction, $k = n$, we choose $\psi_n = \widetilde{\varphi}_n$, where $\widetilde{\varphi}_n = \oplus_{i_n} \widetilde{\varphi}_{i_n}$ and $\widetilde{\varphi}_{i_n} : C(U_{i_n}) \otimes D_{i_n} \to A(U_{i_n})$ are $C(U_{i_n})$-linear extensions of the original φ_{i_n}. Then $B_n = D_n$ and $\eta_n = \psi_n$. Assume now that $\psi_n, \dots, \psi_{k+1}$ were constructed and that they have the desired properties. Condition (13) formulated for $k+1$ becomes

$$\pi_{X_{k+1}}(\mathcal{F}_{k+1}) \subset_{\varepsilon_{k+1}} \eta_{k+1}(B_{k+1}). \tag{15}$$

Since $\varepsilon_{k+1} < \delta_{i_k}$, by using (10) and (15) we obtain

$$\pi_{X_{k+1}\cap U_{i_k}}\big(\varphi_{i_k}(\mathcal{G}_{i_k})\big) \subset_{\delta_{i_k}} \pi_{X_{k+1}\cap U_{i_k}}\big(\eta_{k+1}(B_{k+1})\big), \text{ for all } i_k \in I_k. \tag{16}$$

Since $\mathcal{F}_k \subseteq \mathcal{F}_{k+1}$ and $\varepsilon_{k+1} < \varepsilon_k$, we derive from (15) that

$$\pi_{X_{k+1}}(\mathcal{F}_k) \subset_{\varepsilon_k} \eta_{k+1}(B_{k+1}). \tag{17}$$

The conditions (8), (16) and (17) enable us to apply Lemma 2.9 and perturb φ_{i_k} to a unital $*$-homomorphism ψ_{i_k} satisfying (12) and (13). Then we extend each ψ_{i_k} to a $C(U_{i_k})$-linear $*$-homomorphism, $\psi_{i_k} : C(U_{i_k}) \otimes D_{i_k} \to A(U_{i_k})$ and define $\psi_k = \oplus_{i_k} \psi_{i_k}$. This completes the construction of $(\psi_0, \dots, \psi_n)$. Condition (13) for $k = 0$ gives $\mathcal{F} \subset_\varepsilon \eta_0(B_0) = \eta_0(A(\psi_0, \dots, \psi_n))$. Therefore $B = \eta_0(B_0)$ satisfies the conclusion of the theorem, except that we must show that the n-fibered morphism $(\psi_0, \dots, \psi_n)$ can be arranged to be open. First let us note that it suffices to work in the above arguments with a cover $\mathcal{U} = (U_i)_{i\in I}$ for which there is another cover $\mathcal{U}' = (U_i')_{i\in I}$ such that, U_i' is closed and U_i is a neighborhood of U_i', for all $i \in I$. Indeed, in this case let us set $Y_k' = \bigcup_{i_k\in I_k} U_{i_k}'$, $D_k' = \bigoplus_{i_k} C(U_{i_k}') \otimes D_{i_k}$ and observe that since $\pi_{Y_k}(\mathcal{F}_k) \subset_{\varepsilon_k} \psi_k(D_k)$ by (13), we must have $\pi_{Y_k'}(\mathcal{F}_k) \subset_{\varepsilon_k} \pi^{Y_k}_{Y_k'}\psi_k(D_k)$. Therefore, if we define $\psi_k' : D_k \to A(Y_k')$ by $\psi_k' = \pi^{Y_k}_{Y_k'}\psi_k$, then $\pi_{Y_k'}(\mathcal{F}_k) \subset_{\varepsilon_k} \psi_k'(D_k')$, and hence $\mathcal{F} \subset_\varepsilon \eta'(A(\psi_0', \dots, \psi_n'))$ where η' is induced by the n-fibered morphism $(\psi_0', \dots, \psi_n')$. It is clear that the n-fibered morphism $(\psi_0', \dots, \psi_n')$ is open since it is a restriction of the n-fibered morphism $(\psi_0, \dots, \psi_n)$. Finally, let us explain how to construct the cover $\mathcal{U}$ for which there is a subcover $\mathcal{U}'$ as above, and such that $\mathcal{U}$ refines a given open cover $\mathcal{W}$. By [5, Lemma 3.2] there is a finite family of closed subsets of X, $\mathcal{V} = (V_i)_{i\in I}$, whose interiors cover X, which refines $\mathcal{W}$, and such that the set $\mathcal{V}$ can be partitioned into $n + 1$ subsets $\mathcal{V}_0, ..., \mathcal{V}_n$ consisting of pairwise disjoint elements. Let d be a metric for the topology of X. Since each V_i is compact, there is a number $\ell > 0$, such that if we set $U_i = \{x \in X : d(x, V_i) \le \ell\}$ and $\mathcal{U}_k = \{U_i : V_i \in \mathcal{V}_k\}$, then the elements of $\mathcal{U}_0, ..., \mathcal{U}_n$ are still pairwise disjoint and $\bigcup_i \mathcal{U}_i$ is still a refinement of $\mathcal{V}$. The cover $\mathcal{U}'$ is defined to consist of $U_i' = \{x \in X : d(x, V_i) \le \ell/2\}$. $\square$

Remark 2.13. With a subsequent application in mind, let us record that we have the following additional control on the perturbations ψ_{i_k} of φ_{i_k}. If the K-theory groups of D_{i_k} are finitely generated, then we can arrange that $K_*(\psi_{i_k}) = K_*(\varphi_{i_k})$ for all $i_k \in I_k$ and $k = 0, ..., n$.

Proposition 2.14. *Let X be a compact metrizable space and let A be a unital $C(X)$-algebra. Let $\mathcal{C}$ be a class of separable unital simple semiprojective C^*-algebras. Let $(\psi_0, \dots, \psi_n)$ be a unital n-fibered morphism into A with base $\mathcal{C}$, where $\psi_i : D_i \to A(Y_i)$. Let $\mathcal{F}$ be a finite subset of $A(\psi_0, ..., \psi_n)$ and let $\varepsilon > 0$. Then there are finite sets $\mathcal{G}_i \subset D_i$, and numbers $\delta_i > 0$, $i = 0, ..., n$, such that for any unital $C(X)$-subalgebra $E \subset A$ which satisfies $\psi_i(\mathcal{G}_i) \subset_{\delta_i} E(Y_i)$, $i = 0, ..., n$, there is a unital n-fibered morphism $(\psi_0', \dots, \psi_n')$ into E, with $\psi_i' : D_i \to E(Y_i)$, and such that* *(i)* $\|\psi_i(a) - \psi_i'(a)\| < \varepsilon$ *for all* $a \in p_i(\mathcal{F})$ *and all* $i \in \{0, ..., n\}$*, where* $p_i : A(\psi_0, ..., \psi_n) \to D_i$ *is the natural projection map,*

(ii) $(\psi_j)_x^{-1}(\psi_i)_x = (\psi_j')_x^{-1}(\psi_i')_x$ *for all* $x \in Y_i \cap Y_j$ *and* $0 \le i \le j \le n$*.*

It follows that $A(\psi_0, \dots, \psi_n) = A(\psi_0', \dots, \psi_n')$ *and* $\|\eta(a) - \eta'(a)\| < \varepsilon$ *for all* $a \in \mathcal{F}$*, where* η *and* η' *are the maps from* $A(\psi_0, ..., \psi_n)$ *to* A*, and respectively to* E*, induced by* $(\psi_0, \dots, \psi_n)$ *and* $(\psi_0', \dots, \psi_n')$*.*

Proof. We argue by induction on n. If $n = 0$, the statement follows from Proposition 2.4. Assume now that the statement is true for $n-1$, and let $\mathcal{F}$ and ε be given. The algebra D_0 is of the form $D_0 = \oplus_{i_0} C(U_{i_0}) \otimes D_{i_0}$, where we use the same notation as in the proof of Theorem 2.12. Let $D = \oplus_{i_0} D_{i_0}$. Since we work with morphisms on D_0 whose components are $C(U_{i_0})$-linear, we may assume without any loss of generality, that $\mathcal{F}_0 := p_0(\mathcal{F}) \subset D$. Let $\mathcal{G}$ and δ be given by applying Proposition 2.5 to the C*-algebra D, for the input data $\mathcal{F}_0$ and $\varepsilon/2$. We may assume that $\mathcal{F}_0 \subset \mathcal{G}$ and $\delta < \varepsilon$. We will use the notation from Definition 2.8. Set $B_1 = A(X_1)(\psi_1, \dots, \psi_n)$ and let $\mathcal{H} \subset B_1$ be a finite set such that

$$\pi_{X_1}(\mathcal{F}) \subset \mathcal{H}, \quad \text{and} \tag{18}$$

$$\eta_1^{-1} \pi_{X_1 \cap Y_0}^{Y_0} \psi_0(\mathcal{G}) \subset \pi_{X_1 \cap Y_0}^{X_1}(\mathcal{H}). \tag{19}$$

Note that the existence of $\mathcal{H}$ is assured, since the image of η_1 restricted to $X_1 \cap Y_0$ contains the restriction to the same set of $\psi_0(D_0)$, as a consequence of (2). Let $(\psi_0, \dots, \psi_n)$ and E be as above. Let $\mathcal{G}_1, ...\mathcal{G}_n$ and $\delta_1, ..., \delta_n$ be given by the inductive assumption for $n-1$ applied to $A(X_1)$, $(\psi_1, \dots, \psi_n)$, $\mathcal{H}$ and $\delta/2$. After this preparation, we are ready to chose $\mathcal{G}_0$ and δ_0. Specifically, they are given by Proposition 2.4 applied to D for the input data $\mathcal{G}$ and $\delta/2$. We need to show that $\mathcal{G}_0, \mathcal{G}_1, ...\mathcal{G}_n$ and $\delta_0, \delta_1, ..., \delta_n$ satisfy the statement. By the inductive assumption, there exists a unital $(n-1)$-fibered morphism $(\psi_1', \dots, \psi_n')$ into $E(X_1)$, such that

(a) $\|\psi_i(a) - \psi_i'(a)\| < \delta/2 < \varepsilon$ for all $a \in p_i(\mathcal{H})$ and all $i \in \{1, \dots, n\}$

(b) $(\psi_j)_x^{-1}(\psi_i)_x = (\psi_j')_x^{-1}(\psi_i')_x$ for all $x \in Y_i \cap Y_j$ and $1 \le i \le j$.

(c) $\|\eta_1(a) - \eta_1'(a)\| < \delta/2$ for all $a \in \mathcal{H}$.

It remains to construct $\psi_0' : D_0 \to E(Y_0)$ such that $\|\psi_0(a) - \psi_0'(a)\| < \varepsilon$ for all $a \in \mathcal{F}_0$ and such that (b) holds for $i = 0$ and $j \in \{1, \dots, n\}$. The latter condition is readily seen to be equivalent to the equation $\eta_1' \eta_1^{-1} \pi_0 \psi_0 = \pi_0 \psi_0'$, where π_0 denotes the restriction map to $X_1 \cap Y_0$. The setup is illustrated by the following diagram:

$$\begin{tikzcd}
B_1 \arrow[r, "\pi"] \arrow[d, "\eta_1'"] & B_1(X_1 \cap Y_0) \arrow[d, "\eta_1'"] & D_0 \arrow[l, "\eta_1^{-1}\pi_0\psi_0"'] \arrow[d, dotted, "\psi_0'"] \\
E(X_1) \arrow[r, "\pi"] & E(X_1 \cap Y_0) & E(Y_0) \arrow[l, "\pi_0"']
\end{tikzcd}$$

By assumption, $\psi_0(\mathcal{G}_0) \subset_{\delta_0} E(Y_0)$. By Proposition 2.4 there is a unital $*$-homomorphism $\gamma_0 : D \to E(Y_0)$ with components $\gamma_{i_0} : D_{i_0} \to E(Y_{i_0})$ such that

$$\|\gamma_0(d) - \psi_0(d)\| < \delta/2 \tag{20}$$

for all $d \in \mathcal{G}$. Let us verify that the $C(Y_0)$-linear extension of γ_0 to D_0 is an approximate lifting of $\eta_1'\eta_1^{-1}\pi_0\psi_0$. If $d \in \mathcal{G}$, then by (19), $\eta_1^{-1}\pi_0\psi_0(d) = \pi_0(a)$, for some $a \in \mathcal{H}$. From (c) and (20) we have

$$\|\eta_1'\eta_1^{-1}\pi_0\psi_0(d) - \pi_0\gamma_0(d)\| \leq \|\eta_1'\pi_0(a) - \eta_1\pi_0(a)\| + \|\pi_0\psi_0(d) - \pi_0\gamma_0(d)\| < \delta$$

for all $d \in \mathcal{G}$. By Proposition 2.5, there is a $*$-homomorphism $\psi_0' : D \to E(Y_0)$ such that $\pi_0\psi_0' = \eta_1'\eta_1^{-1}\pi_0\psi_0$ and $\|\psi_0'(c) - \gamma_0(c)\| < \varepsilon/2$ for all $c \in \mathcal{F}_0$. Since $\mathcal{F}_0 \subset \mathcal{G}$ and $\delta < \varepsilon$, by combining the last estimate with (20), we obtain $\|\psi_0(a) - \psi_0'(a)\| < \varepsilon$ for all $a \in \mathcal{F}_0$. $\square$

The following result gives an inductive limit representation for $C(X)$-algebras whose fibers are inductive limits of simple semiprojective C*-algebras. For example the fibers can be UHF-algebras or unital Kirchberg algebras D satisfying the UCT and such that $K_1(D)$ is torsion free. Indeed, by [29, Prop. 8.4.13], D can be written as the inductive limit of a sequence of unital Kirchberg algebras (D_n) with finitely generated K-theory groups and torsion free K_1-groups. The algebras D_n are semiprojective by [32].

Theorem 2.15. *Let $\mathcal{C}$ be a class of unital simple semiprojective C*-algebras. Let X be a finite dimensional compact metrizable space and let A be a separable unital $C(X)$-algebra such that all its fibers are isomorphic to inductive limits of sequences of C*-algebras in $\mathcal{C}$ with unital connecting maps. Then A is isomorphic to the inductive limit of a sequence of $C(X)$-algebras A_k with unital connecting maps and such that* $\mathrm{type}_{\mathcal{C}}(A_k) \leq \dim(X)$.

Proof. By Theorem 2.12 and Proposition 2.14 we find a sequence of n-fibered morphisms into A, denoted $(\psi_0^{(k)}, ..., \psi_n^{(k)})$, with induced unital $*$-monomorphisms $\eta^{(k)} : A_k = A(\psi_0^{(k)}, ..., \psi_n^{(k)})) \to A$ with the following properties. There is a sequence of finite sets $\mathcal{F}_k \subset A_k$ and a sequence of $C(X)$-linear unital $*$-homomorphisms $\mu_k : A_k \to A_{k+1}$ such that

(i) $\|\eta^{(k+1)}\mu_k(a) - \eta^{(k)}(a)\| < 2^{-k}$ for all $a \in \mathcal{F}_k$ and all $k \geq 1$,
(ii) $\mu_k(\mathcal{F}_k) \subset \mathcal{F}_{k+1}$ for all $k \geq 1$,
(iii) $\bigcup_{j=k}^{\infty} \big(\mu_j \circ \cdots \circ \mu_k\big)^{-1}(\mathcal{F}_j)$ is dense in A_k and $\bigcup_{j=k}^{\infty} \eta^{(j)}(\mathcal{F}_j)$ is dense in A for all $k \geq 1$.

Arguing as in the proof of [29, Prop. 2.3.2], one verifies that the sequence $(\eta^{(k)})$ induces an isomorphism of $C(X)$-algebras $\eta : \varinjlim_k (A_k, \mu_k) \to A$. $\square$

3. Locally trivial $C(X)$-algebras

Let A and B be C*-algebras. Two $*$-homomorphisms $\varphi, \psi : A \to B$ are approximately unitarily equivalent, written $\varphi \approx_u \psi$, if for every finite subset $\mathcal{F}$ of A and any $\varepsilon > 0$ there is a unitary u in the multiplier algebra $M(B)$ of B, such that $\|u\varphi(a)u^* - \psi(a)\| < \varepsilon$ for all $a \in \mathcal{F}$. We say that φ and ψ are asymptotically unitarily equivalent, written $\varphi \approx_{uh} \psi$, if there is a norm continuous unitary valued map $t \to u_t \in M(B)$, $t \in [0,1)$, such that $\lim_{t\to 1} \|u_t\varphi(a)u_t^* - \psi(a)\| = 0$ for all $a \in A$. Let D be a C*-algebra and let A be a $C(X)$-algebra. If $\alpha : D \to A$ is a $*$-homomorphism, let us denote by $\widetilde{\alpha} : C(X) \otimes D \to A$ its $C(X)$-linear extension.

Lemma 3.1. *Let X be a compact metrizable space and let Y, Z be closed subsets of X such that $X = Y \cup Z$ and $Y \cap Z \neq \emptyset$. Let D be a unital Kirchberg algebra satisfying the UCT and such that $K_*(D)$ is finitely generated and let $\gamma : D \to C(Y \cap Z) \otimes D$ be a unital $*$-homomorphism. Assume that there are unital $*$-homomorphisms $\alpha, \beta : D \to C(Y) \otimes D$ such that $[\widetilde{\alpha}\beta] = [\widetilde{\beta}\alpha] = [\iota_D]$ in $KK(D, C(Y) \otimes D)$ (where $\iota_D : D \to C(Y) \otimes D$ is the natural inclusion map) and such that α is a lifting of γ, i.e. $\pi^Y_{Y\cap Z}\alpha = \gamma$. Then the pullback $C(Y) \otimes D \oplus_{\pi,\widetilde{\gamma}\pi} C(Z) \otimes D$ is isomorphic to $C(X) \otimes D$.*

Proof. By Phillips' classification theorem [29, Thm. 8.2.1], $\widetilde{\alpha}\beta \approx_u \iota_D$ and $\widetilde{\beta}\alpha \approx_u \iota_D$. Therefore $\widetilde{\alpha}\widetilde{\beta} \approx_u \mathrm{id}_{C(Y)\otimes D}$ and $\widetilde{\beta}\widetilde{\alpha} \approx_u \mathrm{id}_{C(Y)\otimes D}$. Applying [29, Cor. 2.3.4], we find a $*$-isomorphism $\Gamma : C(Y) \otimes D \to C(Y) \otimes D$ such that $\Gamma \approx_u \alpha$. In particular Γ is $C(Y)$-linear. Since $K_*(D)$ is finitely generated and D satisfies the UCT, we obtain that $[\Gamma|_D] = [\widetilde{\alpha}|_D] = [\alpha] \in KK(D, C(Y) \otimes D)$ hence $\Gamma|_D \approx_{uh} \alpha$. The last equivalence is implemented by a continuous map $(0,1] \to U(C(Y) \otimes D)$, $t \mapsto u_t$, with the property that

$$\lim_{t\to 0} \|u_t\Gamma(a)u_t^* - \widetilde{\alpha}(a)\| = 0, \text{ for all} \quad a \in C(Y) \otimes D.$$

Define $\eta : C(Y) \otimes D \to C(Y) \otimes D$ by

$$\eta_y(a(y)) = \begin{cases} v(y)\Gamma_y(a(y))v(y)^*, & \text{if } y \in Y \setminus (Y \cap Z), \\ \gamma_y(a(y)), & \text{if } y \in Y \cap Z, \end{cases}$$

where $v(y) = u_{d(y,Y\cap Z)}(y) \in U(D)$, for a fixed metric d on X such that $\mathrm{diam}(X) \leq 1$. One checks immediately that the triple η,, $\widetilde{\gamma}$, $\mathrm{id}_{C(Z)\otimes D}$ defines a $C(X)$-linear isomorphism $C(X) \otimes D = C(Y) \otimes D \oplus_{\pi,\pi} C(Z) \otimes D \to C(Y) \otimes D \oplus_{\pi,\widetilde{\gamma}\pi} C(Z) \otimes D$:

$$\begin{array}{ccccc} C(Y)\otimes D & \xrightarrow{\pi} & C(Y\cap Z)\otimes D & \xleftarrow{\pi} & C(Z)\otimes D \\ \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \widetilde{\gamma}} & & \| \\ C(Y)\otimes D & \xrightarrow{\pi} & C(Y\cap Z)\otimes D & \xleftarrow{\widetilde{\gamma}\pi} & C(Z)\otimes D \end{array}$$

$\square$

Definition 3.2. A morphism of C*-bundles $\eta : B \to A$ is said to be fiberwise K_*-bijective if the map $K_*(\eta_x) : K_*(B(x)) \to K_*(A(x))$ is bijective for all $x \in X$. Let D be a C*-algebra. A $*$-homomorphism $\varphi : D \to A$ is said to be fiberwise K_*-bijective if its $C(X)$-linear extension $\widetilde{\varphi} : C(X) \otimes D \to A$ has this property. An n-fibered morphism $(\psi_0, \dots, \psi_n)$ into A is said to be fiberwise K_*-bijective if each ψ_i is so. In this case the induced morphism $\eta : A(\psi_0, \dots, \psi_n) \to A$ is also fiberwise K_*-bijective, since each η_x equals $\psi_{i,x}$ for some $0 \le i \le n$.

Lemma 3.3. *Let X be a compact metrizable space and let Y, Z be closed subsets of X such that $X = Y \cup Z$ and $Y \cap Z \neq \emptyset$. Let D be a unital Kirchberg algebra satisfying the UCT and such that $K_*(D)$ is finitely generated. Let $\gamma : C(Y \cap Z) \otimes D \to C(Y \cap Z) \otimes D$ be a unital morphism of $C(Y \cap Z)$-algebras. Assume that γ is fiberwise K_*-bijective and that γ extends to a unital morphism of $C(V)$-algebras $\gamma' : C(V) \otimes D \to C(V) \otimes D$ for some closed neighborhood V of $Y \cap Z$. Then $C(Y) \otimes D \oplus_{\pi_{Y\cap Z}, \gamma\pi_{Y\cap Z}} C(Z) \otimes D$ is locally isomorphic to $C(X) \otimes D$.*

Proof. Let us denote by B the $C(X)$-algebra $C(Y) \otimes D \oplus_{\pi_{Y\cap Z}, \gamma\pi_{Y\cap Z}} C(Z) \otimes D$. We must show that B is locally trivial. Let $x \in X$. If $x \notin Y \cap Z$, then there is a closed neighborhood W of x which does not intersect $Y \cap Z$, and hence the restriction of B to W is isomorphic to $C(W) \otimes D$, as it follows immediately from the definition of B. It remains to consider the case when $x \in Y \cap Z$. By assumption, the map $K_*(\gamma_x)$ is bijective. It follows that the unital $*$-homomorphism $\gamma_x : D \to D$ induces a KK-equivalence, since D satisfies the UCT. Let h be the multiplicative inverse of $[\gamma_x]$ in the ring $KK(D, D)$. Let us observe that $W_n = \{w \in V : d(w, x) \le 1/n\}$ is a closed neighborhood of x. Since $\varinjlim_n KK(D, C(W_n) \otimes D) = KK(D, D)$, by [29, Thm. 8.2.1] there is a unital $*$-homomorphism $\beta : D \to C(W_N) \otimes D$ (for some N) such that $[\beta_x] = [\pi_x \beta] = h$. Therefore $[\gamma_x][\beta_x] = [\beta_x][\gamma_x] = [\mathrm{id}_D]$. By [29, Thm. 8.2.1] we obtain

$$\gamma_x \beta_x \approx_u \mathrm{id}_D \ \text{ and } \beta_x \gamma_x \approx_u \mathrm{id}_D. \tag{21}$$

Let us set $\gamma_n = \pi_{W_n} \gamma'$ and $\beta_n = \pi_{W_n} \beta$ $(n \ge N)$. The maps $w \mapsto \pi_w \gamma'(d)$ and $w \mapsto \pi_w \beta(d)$ are continuous for $d \in D$ fixed. Using (21), for any finite subset $\mathcal{F}$ of D and any $\varepsilon > 0$, we can find $n \ge N$ and unitaries $u, v \in C(W_n) \otimes D$ such that $\|u\, \iota_n(d) u^* - \widetilde{\gamma}_n \beta_n(d)\| < \varepsilon$ and $\|v\, \iota_n(d) v^* - \widetilde{\beta}_n \gamma_n(d)\| < \varepsilon$ for all $d \in \mathcal{F}$, where $\iota_n : D \to C(W_n) \otimes D$ is the natural inclusion map. Since D satisfies the UCT and since $K_*(D)$ is finitely generated, there is n such that $[\widetilde{\gamma}_n \beta_n] = [\iota_n] = [\widetilde{\beta}_n \gamma_n] \in KK(D, C(W_n) \otimes D)$; see [10]. Moreover γ_n is a lifting of $\pi^{Y\cap Z}_{Y\cap Z\cap W_n} \gamma : D \to C(Y \cap Z \cap W_n) \otimes D$. By Lemma 3.1, $B(W_n) = C(Y \cap W_n) \otimes D \oplus_{\pi, \gamma\pi} C(W_n \cap Z) \otimes D$ is isomorphic to $C(W_n) \otimes D$. Since W_n is a closed neighborhood of x, this concludes the proof. □

Corollary 3.4. *Let X be a compact metrizable space and let Y, Y' and Z, Z' be closed subsets of X such that Y' is a neighborhood of Y, Z' is a neighborhood of*

Z, $X = Y \cup Z$ and $Y \cap Z \neq \emptyset$. Let A be a unital separable $C(X)$-algebra. Let D be a unital Kirchberg algebra satisfying the UCT and such that $K_*(D)$ is finitely generated. Let B be a unital $C(Y')$-algebra locally isomorphic to $C(Y') \otimes D$ and let E be a unital $C(Z')$-algebra locally isomorphic to $C(Z') \otimes D$. Let $\varphi : B \to A(Y')$ and $\psi : E \to A(Z')$ be fiberwise K_*-bijective unital morphisms of $C(X)$-algebras such that $\psi_x(E(x)) \subset \varphi_x(B(x))$ for all $x \in Y' \cap Z'$. Then $B(Y) \oplus_{\pi_{Y\cap Z}\psi, \pi_{Y\cap Z}\varphi} E(Z)$ is locally isomorphic to $C(X) \otimes D$. Moreover, the morphism $\eta : B(Y) \oplus_{\pi_{Y\cap Z}\psi, \pi_{Y\cap Z}\varphi} E(Z) \to A(X)$ induced by the pair φ, ψ is fiberwise K_*-bijective.

Proof. Since the properties from the conclusion of the corollary are local in nature and since the pullback construction for $C(X)$-algebras commutes with the operation of restriction, we may assume that $B = C(Y') \otimes D$ and $E = C(Z') \otimes D$. Let us define $\gamma' : C(Y' \cap Z') \otimes D \to C(Y' \cap Z') \otimes D$ by $\gamma'_x = \varphi_x^{-1}\psi_x$, $x \in Y' \cap Z'$ and let γ denote the restriction of γ' to $C(Y \cap Z) \otimes D$. Then γ is fiberwise K_*-bijective and

$$B(Y) \oplus_{\pi_{Y\cap Z}\psi, \pi_{Y\cap Z}\varphi} E(Z) \cong C(Y) \otimes D \oplus_{\pi_{Y\cap Z}, \gamma\pi_{Y\cap Z}} C(Z) \otimes D$$

is locally isomorphic to $C(X) \otimes D$ by Lemma 3.3. The last part of the statement is obvious, since $\eta_x = \psi_x$ if $x \in Z$ and $\eta_x = \varphi_x$ if $x \in Y \setminus Z$. □

Proposition 3.5. *Let $\mathcal{C}$ be a class of unital Kirchberg algebras satisfying the UCT and having finitely generated K-theory groups. Let X be a finite dimensional compact metrizable space, and let A be a unital $C(X)$-algebra with all fibers in $\mathcal{C}$. Let $\mathcal{F} \subset A$ be a finite subset and let $\varepsilon > 0$. Assume that for each $x \in X$, there exist $D \in \mathcal{C}$, a closed neighborhood V of x and a morphism of groups $\theta : K_*(D) \to K_*(A(V))$ such that $\theta[1_D] = [1_{A(V)}]$ and such that $K_*(\pi_y) \circ \theta : K_*(D) \to K_*(A(y))$ is bijective for all $y \in V$. Then there is a unital $C(X)$-subalgebra B of A such that $\mathcal{F} \subset_\varepsilon B$, B is locally isomorphic to $C(X) \otimes D$, and the inclusion morphism $B \hookrightarrow A$ is fiberwise K_*-bijective.*

Proof. By Lemma 2.11, there are $(\mathcal{F}, \varepsilon)$-approximations of A whose components $\varphi_i : D \to A(U_i)$ induce bijective maps $K_*(\pi_x\varphi_i) : K_*(D) \to K_*(A(x))$, for all $x \in U_i$ and $i \in I$. We repeat the proof of Theorem 2.12 by working with $(\mathcal{F}, \varepsilon)$-approximations satisfying this additional property. As an outcome, we obtain an open n-fibered morphism into A, $(\psi_0, \ldots, \psi_n)$, where $\psi_i : C(Y_i) \otimes D \to A(Y_i)$ are such that $\pi_{Y_i}(\mathcal{F}) \subset_\varepsilon \psi_i(C(Y_i) \otimes D)$, and $n = \dim(X)$. Moreover, by Remark 2.13, the morphisms ψ_i (which are small controlled perturbations of the morphisms φ_i) induce the same maps on K-theory. Thus we can arrange that the maps ψ_i are fiberwise K_*-bijective. Since the n-fibered morphism $(\psi_0, \ldots, \psi_n)$ is open and fiberwise K_*-bijective, we can find for each $i \in \{0, ..., n\}$, closed subsets of X, $Y_i = Y_i^{(n)} \subset Y_i^{(n-1)} \subset ... \subset Y_i^{(0)}$ such that $Y_i^{(j-1)}$ is a neighborhood of $Y_i^{(j)}$ for all j, and such that $(\psi_0, \ldots, \psi_n)$ extends to an fiberwise K_*-bijective n-fibered morphism into A, denoted in the same way, and with components $\psi_i : C(Y_i^{(0)}) \otimes D \to A(Y_i^{(0)})$ that satisfy

$$\pi_{Y_i^{(0)}}(\mathcal{F}) \subset_\varepsilon \psi_i(C(Y_i^{(0)}) \otimes D). \tag{22}$$

Let $\pi_i^{(k)} : A(X) \to A\big(Y_{i-1}^{(k)} \cap \big(Y_i^{(k)} \cup ... \cup Y_n^{(k)}\big)\big)$ denote the restriction map. Define

$$B_{n-1} = C(Y_n^{(1)}) \otimes D \oplus_{\pi_n^{(1)}\psi_n, \pi_n^{(1)}\psi_{n-1}} C(Y_{n-1}^{(1)}) \otimes D.$$

By Corollary 3.3, B_{n-1} is locally isomorphic to $C(Y_n^{(1)} \cup Y_{n-1}^{(1)}) \otimes D$ since ψ_n extends to $C(Y_n^{(0)}) \otimes D$ and ψ_{n-1} extends to $C(Y_{n-1}^{(0)}) \otimes D$ and $Y_i^{(0)}$ is a closed neighborhood of $Y_i^{(1)}$. The map $\eta_{n-1} : B_{n-1} \to A(Y_n^{(1)} \cup Y_{n-1}^{(1)})$, induced by the pair ψ_n, ψ_{n-1}, is fiberwise K_*-bijective, since $\pi_x \eta_{n-1}$ is equal to either $\pi_x \psi_n$ or $\pi_x \psi_{n-1}$, for each x. Moreover

$$\pi_{Y_n^{(1)} \cup Y_{n-1}^{(1)}}(\mathcal{F}) \subset_\varepsilon \eta_{n-1}(B_{n-1}), \tag{23}$$

by (22) and Lemma 2.1. By applying Corollary 3.3 again, we see that if we define

$$B_{n-2} = B_{n-1}(Y_n^{(2)} \cup Y_{n-1}^{(2)}) \oplus_{\pi_{n-1}^{(2)}\eta_{n-1}, \pi_{n-1}^{(2)}\psi_{n-2}} C(Y_{n-2}^{(2)}) \otimes D,$$

then B_{n-2} is locally isomorphic to $C(Y_n^{(2)} \cup Y_{n-1}^{(2)} \cup Y_{n-2}^{(2)}) \otimes D$, (since both η_{n-1} and ψ_{n-2} extend to locally trivial fields supported on larger neighborhoods) and the map $\eta_{n-2} : B_{n-2} \to A(Y_n^{(2)} \cup Y_{n-1}^{(2)} \cup Y_{n-2}^{(2)})$, induced by the pair η_{n-1}, ψ_{n-2}, is fiberwise K_*-bijective, since $\pi_x \eta_{n-2}$ is equal to either $\pi_x \eta_{n-1}$ or $\pi_x \psi_{n-2}$, for each x. Moreover $\pi_{Y_n^{(2)} \cup Y_{n-1}^{(2)} \cup Y_{n-2}^{(2)}}(\mathcal{F}) \subset_\varepsilon \eta_{n-2}(B_{n-2})$ by (22), (23) and Lemma 2.1. Arguing similarly, after n-steps we obtain a unital $C(X)$-algebra B_0 locally isomorphic to $C(X) \otimes D$, and a unital $C(X)$-linear map $\eta_0 : B_0 \hookrightarrow A(Y_n^{(0)} \cup ... \cup Y_0^{(0)}) = A(X)$ which is fiberwise K_*-bijective, and such that $\mathcal{F} \subset_\varepsilon \eta_0(B_0)$. $\square$

Let A be a unital $C(X)$-algebra. The K-theory presheaf of A is a functor from the category of closed nondegenerate balls U of X, with morphisms inclusion maps $i_U^V : V \hookrightarrow U$, to $\mathbb{Z}/2\mathbb{Z}$-graded abelian groups. It takes U to $K_*(A(U))$ and $i_U^V : V \hookrightarrow U$ to $K_*(\pi_V^U) : K_*(A(U)) \to K_*(A(V))$. Its usefulness in the study of $C(X)$-algebras was illustrated in [9]. A sequence (A_n) of subsets of a C*-algebra A is called *exhaustive* if for any finite subset $\mathcal{F}$ of A and any $\varepsilon > 0$ there is n such that $\mathcal{F} \subset_\varepsilon A_n$.

Theorem 3.6. *Let X be a finite dimensional compact metrizable space. Let D be a unital C*-algebra and let A be a separable unital continuous $C(X)$-algebra. Assume that D and all the fibers of A are Kirchberg algebras with finitely generated K-theory groups and satisfying the UCT. Assume that for each $x \in X$, there are a closed neighborhood V of x and a morphism of groups $\theta_V : K_*(D) \to K_*(A(V))$ such that $\theta_V[1_D] = [1_{A(V)}]$ and $K_*(\pi_y) \circ \theta_V : K_*(D) \to K_*(A(y))$ is bijective for all $y \in V$. (This condition is satisfied if we assume that the K-theory presheaf invariant of A is locally isomorphic to the K-theory presheaf associated to $C(X) \otimes D$). Then:*

(a) A admits an exhausting sequence of unital $C(X)$-subalgebras locally isomorphic to $C(X) \otimes D$.

(b) If D is semiprojective, then A is isomorphic to an inductive limit of unital $C(X)$-algebras locally isomorphic to $C(X) \otimes D$.

(c) If X is contractible, then $A \cong C(X) \otimes D$.

(d) If X is locally contractible, then A is locally isomorphic to $C(X) \otimes D$.

Proof. (a) This is proved in Proposition 3.5.

(b) If D is semiprojective, one uses Proposition 2.14 and Theorem 2.15, in order to pass from an exhaustive sequence of $C(X)$-subalgebras of A, given by part (a), to a nested subsequence whose union is dense in A. It suffices to verify that if B is a unital $C(X)$-algebra which is locally isomorphic to $C(X) \otimes D$, then $B \cong B(\psi_0, \dots, \psi_n)$ for some n-fibered morphism into B with base $\{D\}$. Indeed Proposition 2.14 applies to $C(X)$-subalgebras of A of this type. Since B is locally trivial, there is a finite family of $*$-isomorphisms $\varphi_i : C(V_i) \otimes D \to B(V_i)$, where $(V_i)_{i \in I}$ are closed subsets of X whose interiors cover X. By [5, Lemma 3.2] there is a finite family of closed subsets of X, $\mathcal{U} = (U_j)_{j \in J}$, whose interiors cover X, which refines $\mathcal{V}$, and such that the set $\mathcal{U}$ can be partitioned into $n+1$ subsets consisting of pairwise disjoint elements. Thus after restricting each morphism φ_i to the sets U_j contained in V_i and changing notation, we may assume that $\mathcal{V}$ can be can be partitioned into $n+1$ subsets $\mathcal{V}_0, \dots, \mathcal{V}_n$ consisting of pairwise disjoint elements. In this case we set $\psi_k = \bigoplus_{V_i \subset \mathcal{V}_k} \varphi_i$.

Since (d) is a consequence of (c), it remains to prove (c). By (a), A admits an exhaustive sequence (A_k, η_k) where A_k are $C(X)$-algebras locally isomorphic to $C(X) \otimes D$ and each $\eta_k : A_k \to A$ is fiberwise K_*-bijective. Therefore, each A_k is given by an $\mathrm{Aut}(D)$-principal bundle. Since any principal bundle over a contractible metrizable space is trivial [17], it follows that $A_k \cong C(X) \otimes D$ for all k. Let $\mathcal{F}_k$ be an increasing sequence of finite subsets of A with union dense in A. Thus, there is a sequence of morphisms of $C(X)$-algebras, $\phi_k : C(X) \otimes D \to A$, such that each ϕ_k is fiberwise K_*-bijective and $\mathcal{F}_k \subset_{\varepsilon_k} B_k$ where $B_k = \phi_k\big(C(X) \otimes D\big)$ and $\varepsilon_k \to 0$. Using the weak semiprojectivity of D, after passing to a subsequence of (ϕ_k) if necessary, we may perturb ϕ_k, without changing its K-theory class, to ensure that $\phi_k(D) \subset B_{k+1}$, and hence $B_k \subset B_{k+1}$ for all k. This shows that A is isomorphic to the inductive limit of a sequence

$$C(X) \otimes D \xrightarrow{\theta_1} C(X) \otimes D \xrightarrow{\theta_2} \cdots$$

where the maps θ_n are $C(X)$-linear unital $*$-monomorphisms such that $K_*(\theta_k)$ are bijective. By [29, Thm. 8.2.4], each θ_k is approximately unitarily equivalent to a $C(X)$-linear automorphism of $C(X) \otimes D$. It follows that A is isomorphic to $C(X) \otimes D$ by Elliott's intertwining argument. $\square$

As an immediate corollary we derive the following result which generalizes the case $X = [0,1]$ proved in [9].

Theorem 3.7. *Let X be a finite dimensional compact metrizable space, and let A be a separable unital $C(X)$-algebra with all fibers isomorphic to $\mathcal{O}_n$ for a fixed $n \in \{2, 3, ..., \infty\}$. Then:*

(a) A is isomorphic to an inductive limit of unital $C(X)$-algebras which are locally isomorphic to $C(X) \otimes \mathcal{O}_n$.

(b) If X is contractible, then $A \cong C(X) \otimes \mathcal{O}_n$.

(c) If X is locally contractible, then A is locally isomorphic to $C(X) \otimes \mathcal{O}_n$.

Proof. Assume first that $n = \infty$. Since $K_0(\mathcal{O}_\infty) \cong \mathbb{Z}$ is generated by the class of the unit of $\mathcal{O}_\infty$, and $K_1(\mathcal{O}_\infty) = 0$, the group homomorphism $\theta : K_0(\mathcal{O}_\infty) \to K_0(A(X))$, defined by $\theta[1] = [1]$ satisfies the hypothesis of Theorem 3.6, since the all maps $\pi_y : A \to A(y)$ are unital.

Assume now that $2 \leq n < \infty$. Since $K_0(\mathcal{O}_n) \cong \mathbb{Z}/(n-1)\mathbb{Z}$ is generated by the class of the unit of $\mathcal{O}_n$ and $K_1(\mathcal{O}_n) = 0$, in view of Theorem 3.6, it suffices to show that for each $x \in X$ there is a closed neighborhood V of x such that $(n-1)[1_{A(V)}] = 0$ in $K_0(A(V))$. Indeed, in that case, the group homomorphism $\theta : K_0(\mathcal{O}_n) \to K_0(A(V))$, defined by $\theta[1] = [1]$, has the required properties. Let $V_k = \{y \in X : d(y, x) \leq 1/k\}$. Then $K_0(\mathcal{O}_n) \cong K_0(A(x)) \cong \varinjlim_k K_0(A(V_k))$, and hence $(n-1)[1_{A(V_k)}] = 0$ for some k. □

Lemma 3.8. *Let B be a unital separable C^*-algebra such that $B \cong B \otimes \mathcal{O}_\infty$. For any unital $*$-homomorphism $\varphi : \mathcal{O}_\infty \to B$ and any unitary $u \in U(B)$, there is a sequence (v_n) of unitaries in B such that each v_n is homotopic to u and $\lim_{n\to\infty} \|[\varphi(a), v_n]\| = 0$ for all $a \in \mathcal{O}_\infty$.*

Proof. It suffices to prove the lemma with B replaced by $B \otimes \mathcal{O}_\infty$. Let φ and u be given. We need to show that for any finite set $\mathcal{G} \subset \mathcal{O}_\infty$ and any $\delta > 0$, there is a unitary $v \in U(B \otimes \mathcal{O}_\infty)$ homotopic to u such that $\|[\varphi(a), v]\| < \delta$ for all $a \in \mathcal{G}$. Let $\psi : \mathcal{O}_\infty \to B \otimes \mathcal{O}_\infty$ be defined by $\psi(a) = 1_B \otimes a$. Since $B \cong B \otimes \mathcal{O}_\infty$, by [28, Lemma 2.1.7] there is a unitary $w \in B \otimes 1_{\mathcal{O}_\infty}$ which is homotopic to u in $U(B \otimes \mathcal{O}_\infty)$. By [29, Thm. 8.2.1], any two unital $*$-homomorphisms from $\mathcal{O}_\infty$ to $B \otimes \mathcal{O}_\infty$ are approximately unitarily equivalent. Therefore, there is a unitary $h \in U(B \otimes \mathcal{O}_\infty)$ such that $\|\varphi(a) - h\psi(a)h^*\| < \delta/2$ for all $a \in \mathcal{G}$. Let us set $v = hwh^*$ and observe that since v commutes with $h\psi(a)h^*$ for all $a \in \mathcal{O}_\infty$, $\|[v, \varphi(a)]\| < \delta$ for all $a \in \mathcal{G}$. Finally we apply [28, Lemma 2.1.7] to conclude that v is homotopic to u since they have the same class in $K_1(B \otimes \mathcal{O}_\infty)$. □

Proposition 3.9. *Let A be a unital separable C^*-algebra such that $A \cong A \otimes \mathcal{O}_\infty$. Let $\pi : A \to B$ be a unital surjective $*$-homomorphism to some C^*-algebra B. Then any unital $*$-homomorphism $\varphi : \mathcal{O}_\infty \to B$ lifts to a unital $*$-homomorphism $\Phi : \mathcal{O}_\infty \to A$.*

Proof. Since $\mathcal{O}_\infty$ is semiprojective, by Proposition 2.5 it suffices to show that for any finite subset $\mathcal{G} \subset \mathcal{O}_\infty$ and any $\delta > 0$, there is a unital $*$-homomorphism $\Psi : \mathcal{O}_\infty \to A$

such that $\|\pi\Psi(a) - \varphi(a)\| < \delta$ for all $a \in \mathcal{G}$. Since A is unital and $A \cong A \otimes \mathcal{O}_\infty$, there is a unital $*$-homomorphism $\psi : \mathcal{O}_\infty \to A$. The unital $*$-homomorphisms $\pi\psi$ and φ have the same class in $KK(\mathcal{O}_\infty, B)$. Since $B \cong B \otimes \mathcal{O}_\infty$, it follows by [29, Thm. 8.2.1] that $\pi\psi \approx_u \varphi$. Therefore, there is a unitary $u \in U(B)$ such that $\|\pi\psi(a) - u\varphi(a)u^*\| < \delta/2$ for all $a \in \mathcal{G}$. By Lemma 3.8, there is a unitary $v \in U(B)$ homotopic to u^* such that $\|\varphi(a) - v\varphi(a)v^*\| < \delta/2$ for all $a \in \mathcal{G}$. Let us set $w = uv$ and observe that

$$\|\pi\psi(a) - w\varphi(a)w^*\| \leq \|\pi\psi(a) - u\varphi(a)u^*\| + \|u\varphi(a)u^* - uv\varphi(a)v^*u^*\| < \delta/2 + \delta/2 = \delta$$

for all $a \in \mathcal{G}$. Since w is homotopic to 1_B, it lifts to a unitary $W \in U(A)$. We conclude by observing that $\Psi(a) = W^*\psi(a)W$, $a \in \mathcal{O}_\infty$, is a $*$-homomorphism that satisfies the desired approximate lifting condition. □

Theorem 3.10. *Let X be a finite dimensional compact metrizable space, and let A be a separable unital $C(X)$-algebra with all fibers isomorphic to $\mathcal{O}_\infty$. Then $A \cong C(X) \otimes \mathcal{O}_\infty$. A similar result holds for $\mathcal{O}_2$ with a similar (simpler) proof.*

Proof. First, let us prove the statement under the additional assumption that A is a finite type $C(X)$-algebra. We argue by induction on the type of A. All we need to observe is that if we take $D = \mathcal{O}_\infty$ in Lemma 3.1, then for any unital $*$-homomorphism $\gamma : \mathcal{O}_\infty \to C(Y \cap Z) \otimes \mathcal{O}_\infty$ there exist α and β satisfying the properties from the statement of Lemma 3.1. Indeed, by Proposition 3.9, there is a unital $*$-homomorphism $\alpha : \mathcal{O}_\infty \to C(Y) \otimes \mathcal{O}_\infty$ such that $\pi^Y_{Y\cap Z}\alpha = \gamma$. Let $\beta : \mathcal{O}_\infty \to C(Y) \otimes \mathcal{O}_\infty$ be any unital $*$-homomorphism. By Proposition 3.1 it follows that the pullback $C(Y) \otimes \mathcal{O}_\infty \oplus_{\pi,\tilde{\gamma}\pi} C(Z) \otimes \mathcal{O}_\infty$ is isomorphic to $C(X) \otimes \mathcal{O}_\infty$. By Theorem 2.15, applied for the class $\mathcal{C}$ consisting of $\mathcal{O}_\infty$ alone, and by the first part of the proof, A is isomorphic to the inductive limit of a sequence

$$C(X) \otimes \mathcal{O}_\infty \xrightarrow{\theta_1} C(X) \otimes \mathcal{O}_\infty \xrightarrow{\theta_2} \cdots$$

where the maps θ_n are $C(X)$-linear unital $*$-monomorphisms. Applying the UCT, we see that any two unital $*$-homomorphisms from $\mathcal{O}_\infty \to C(X) \otimes \mathcal{O}_\infty$ have the same KK-theory class. By [29, Thm. 8.2.4], each θ_k is approximately unitarily equivalent to the identity map of $C(X) \otimes \mathcal{O}_\infty$. It follows that A is isomorphic to $C(X) \otimes \mathcal{O}_\infty$ by Elliott's intertwining argument. □

4. Continuous fields and the Universal Coefficient Theorem

Kirchberg has shown that any nuclear separable C*-algebra is KK-equivalent to a Kirchberg algebra [29, Prop. 8.4.5]. This inspired us to extend the result in the context of continuous fields and $KK_{C(X)}$-theory (see Theorem 4.4). The main application of this result is Theorem 4.7 which exhibits a new permanence property of the nuclear C*-algebras satisfying the UCT. We need the following lemma of [21, Lemma 1.2].

Lemma 4.1. *Let A be a continuous field of C^*-algebras with compact spectrum X. Then there is a split short exact sequence of continuous fields*

$$0 \longrightarrow A \longrightarrow A^+ \underset{\alpha}{\overset{}{\rightleftarrows}} C(X) \longrightarrow 0 \tag{24}$$

where each fiber $A^+(x)$ is the unitization of $A(x)$ for every $x \in X$, and such that the continuous sections of A^+ are the sections of the form $a(x) = a_0(x) + f(x) \cdot 1_{A^+(x)}$ for a_0 a continuous section of A and $f \in C(X)$. The splitting α is defined by $\alpha(f)(x) = f(x) \cdot 1_{A^+(x)}$.

Consider the category of separable $C(X)$-algebras such that the morphisms from A to B, are the elements of $KK_{C(X)}(A, B)$ with composition given by the Kasparov product. The isomorphisms in this category are the KK-invertible elements denoted by $KK_{C(X)}(A, B)^{-1}$. Two $C(X)$-algebras are $KK_{C(X)}$-equivalent if they are isomorphic objects in this category. In the sequel we shall we twice the following elementary observation valid in any category. If composition with $\gamma \in KK_{C(X)}(A, B)$ induces a bijection $KK_{C(X)}(B, C) \to KK_{C(X)}(A, C)$ for $C = A$ and $C = B$, then $\gamma \in KK_{C(X)}(A, B)^{-1}$.

Lemma 4.2. *Let A be a separable nuclear continuous $C(X)$-algebra. Then there exist a separable nuclear unital continuous $C(X)$-algebra $A^\flat$ and two $C(X)$-linear $*$-monomorphisms $\alpha : C(X) \otimes \mathcal{O}_2 \to A^\flat$, and $\jmath : A \to A^\flat$ such that α is unital and $[\jmath] \in KK_{C(X)}(A, A^\flat)^{-1}$.*

Proof. Let $p \in \mathcal{O}_\infty$ be a non-zero projection with $[p] = 0$ in $K_0(\mathcal{O}_\infty)$. Then there is a unital $*$-homomorphism $\mathcal{O}_2 \to p\mathcal{O}_\infty p$ which induces a $C(X)$-linear unital $*$-monomorphism $\mu : C(X) \otimes \mathcal{O}_2 \to C(X) \otimes p\mathcal{O}_\infty p$. We tensor the exact sequence (4.1) by $p\mathcal{O}_\infty p$ and then take the pullback by μ. This gives a split exact sequence of unital $C(X)$-algebras:

$$\begin{tikzcd}
0 \arrow[r] & A \otimes p\mathcal{O}_\infty p \arrow[r] & A^+ \otimes p\mathcal{O}_\infty p \arrow[r, shift left] & C(X) \otimes p\mathcal{O}_\infty p \arrow[l, shift left, "\alpha"] \arrow[r] & 0 \\
0 \arrow[r] & A \otimes p\mathcal{O}_\infty p \arrow[u, equal] \arrow[r, "j"] & A^\flat \arrow[u] \arrow[r, shift left] & C(X) \otimes \mathcal{O}_2 \arrow[u, "\mu"'] \arrow[l, shift left, "\alpha"] \arrow[r] & 0
\end{tikzcd} \tag{25}$$

The map $A^\flat \to A^+ \otimes p\mathcal{O}_\infty p$ is a unital $C(X)$-linear $*$-monomorphism, so that $A^\flat$ is a continuous $C(X)$-algebra. It is nuclear being an extension of nuclear C*-algebras. It follows by [18], [1, Thm. 5.4] that for any separable $C(X)$-algebra B we have an exact sequence of groups

$$0 \to KK_{C(X)}(C(X) \otimes \mathcal{O}_2, B) \longrightarrow KK_{C(X)}(A^\flat, B) \xrightarrow{j^*} KK_{C(X)}(A \otimes p\mathcal{O}_\infty p, B) \to 0$$

Since the class identity map of $C(X) \otimes \mathcal{O}_2$ vanishes in $KK_{C(X)}$, one verifies immediately that $KK_{C(X)}(C(X) \otimes \mathcal{O}_2, B) = 0$ for any separable $C(X)$-algebra B.

Therefore j^* is a $KK_{C(X)}$-equivalence. We conclude the proof by observing that map, $A \to A \otimes p\mathcal{O}_\infty p$, $a \mapsto a \otimes p$, is also a $KK_{C(X)}$-equivalence. □

Proposition 4.3. *Let (A_i, φ_i) be an inductive system of separable nuclear $C(X)$-algebras with injective connecting maps. If $\varphi_i \in KK_{C(X)}(A_i, A_{i+1})^{-1}$ for all i, and $\Phi : A_1 \to \varinjlim(A_i, \varphi_i) = A_\infty$ is the induced map, then $\Phi \in KK_{C(X)}(A_1, A_\infty)^{-1}$.*

Proof. We use Milnor's $\varprojlim^1$-sequence for $KK_{C(X)}$-theory. Its proof is essentially identical to the proof of the corresponding sequence for regular KK-theory (argue as in [30] using [1]).

(26)

$$0 \longrightarrow \varprojlim{}^1 KK_{C(X)}(A_i, B) \longrightarrow KK_{C(X)}(A_\infty, B) \longrightarrow \varprojlim KK_{C(X)}(A_i, B) \longrightarrow 0$$

Since $\varprojlim^1 \big(G_i, \lambda_i\big) = 0$ and $G_1 \cong \varprojlim \big(G_i, \lambda_i\big)$ for any sequence of abelian groups $(G_i)_{i=1}^\infty$ and group isomorphisms $\lambda_i : G_i \to G_{i+1}$, we obtain from (26) that for any separable $C(X)$ algebra B, Φ induces a bijection $KK_{C(X)}(A_\infty, B) \to \mathrm{KK}_{C(X)}(A, B)$. This implies that $[\Phi] \in KK(A, A_\infty)^{-1}$. □

We need the following $C(X)$-equivariant construction which parallels a construction of Kirchberg.

Theorem 4.4. *Let A be a separable nuclear continuous $C(X)$-algebra. Then there exist a separable nuclear continuous $C(X)$-algebra B whose fibers are Kirchberg C*-algebras and a $C(X)$-linear $*$-monomorphism $\Phi : A \to B$ such that Φ is a $KK_{C(X)}$-equivalence. For any $x \in X$ the map $\Phi_x : A(x) \hookrightarrow B(x)$ is a KK-equivalence.*

Proof. By Proposition 4.2 we may assume that there is a unital $C(X)$-linear $*$-monomorphism $\alpha : C(X) \otimes \mathcal{O}_2 \to A$. By [4, Thm. 2.5], there is a unital $C(X)$-linear $*$-monomorphism $\beta : A \to C(X) \otimes \mathcal{O}_2$. Let s_1, s_2 be the images in A under the map α of the canonical generators of $v_1, v_2 \in \mathcal{O}_2 \subset C(X) \otimes \mathcal{O}_2$. Let $\theta = \alpha \circ \beta : A \to A$ and define $\varphi : A \to A$ by $\varphi(a) = s_1 a s_1^* + s_2 \theta(a) s_2^*$. The map φ is $C(X)$-linear. For any $x \in X$, it induces a unital $*$-homomorphism $\varphi_x : A(x) \to A(x)$, $\varphi_x(a) = s_1(x) a(x) s_1(x)^* + s_2(x) \theta_x(a) s_2(x)^*$ such that the following diagram is commutative.

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "\pi_x"'] & A \arrow[d, "\pi_x"] \\ A(x) \arrow[r, "\varphi_x"] & A(x) \end{tikzcd}$$

Moreover we have a factorization $\varphi_x = \alpha_x \circ \beta_x$:

$$A(x) \xrightarrow{\beta_x} \mathcal{O}_2 \xrightarrow{\alpha_x} A(x)$$

Let B be the inductive limit of the inductive system

$$A \xrightarrow{\varphi} A \xrightarrow{\varphi} A \xrightarrow{\varphi} \cdots$$

and let $\Phi : A \to B$ be the induced map. We have a commutative diagram

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "\pi_x"'] & A \arrow[r, "\varphi"] \arrow[d, "\pi_x"] & A \arrow[r, "\varphi"] \arrow[d, "\pi_x"] & \cdots \arrow[r] & B \arrow[d, "\pi_x"] \\ A(x) \arrow[r, "\varphi_x"] & A(x) \arrow[r, "\varphi_x"] & A(x) \arrow[r, "\varphi_x"] & \cdots \arrow[r] & B(x) \end{tikzcd}$$

By the proof of [29, Prop. 8.4.5] the C*-algebra $B(x)$ is a unital Kirchberg algebra for any $x \in X$. It remains to prove that the map $\Phi : A \to B$ induces a $KK_{C(X)}$-equivalence. In view of Proposition 4.3, it suffices to verify that $[\varphi] = [id] \in KK_{C(X)}(A, A)^{-1}$. However this follows from the equation $\varphi(a) = s_1 a s_1^* + s_2\theta(a)s_2^*$, as $[\theta] = 0$, since it factors through $C(X) \otimes \mathcal{O}_2$. $\square$

Let $\mathcal{C}$ denote the class of all unital Kirchberg algebras satisfying the UCT.

Lemma 4.5. *Let X be a compact metrizable space and let A be a $C(X)$-algebra such that* $\mathrm{type}_{\mathcal{C}}(A) = n < \infty$. *Then A satisfies the UCT.*

Proof. We use the notation of Definitions 2.6–2.8 and prove by induction on n that if $\mathrm{type}_{\mathcal{C}}(A) \le n$, then all the ideals and the quotients of A satisfy the UCT. This is clear for $n = 0$ since in that case $A \cong \oplus_i C(Z_i) \otimes D_i$ and D_i satisfy the UCT. By a result of [30], if two out of three separable nuclear C*-algebras A, B, C in a short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

satisfy the UCT, then all three of them satisfy the UCT. It is then clear that for the inductive step one shall use the exact sequence

$$0 \longrightarrow A(X \cap Z, Y_0 \cap Z) \longrightarrow A(X \cap Z) \longrightarrow A(Y_0 \cap Z) \longrightarrow 0$$

where $A(X, Z)$ denotes the ideal of $A(X)$ consisting of elements a such that $\pi_Z(a) = 0$, and $Z \subset Y$ is closed and nonempty. The argument works because $\mathrm{type}_{\mathcal{C}}(A(Y_0 \cap Z)) = 0$ and

$$A(X \cap Z, Y_0 \cap Z) \cong A(X_1, X_1 \cap Y_0 \cap Z),$$

since $X = Y_0 \cap X_1$, hence $\mathrm{type}_{\mathcal{C}}\big(A(X \cap Z, Y_0 \cap Z)\big) = \mathrm{type}_{\mathcal{C}}\big(A(X_1, X_1 \cap Y_0 \cap Z)\big) \le n - 1$.

$\square$

Theorem 4.6. [8] *Let A be a nuclear separable C*-algebra. Assume that for any finite set $\mathcal{F} \subset A$ and any $\varepsilon > 0$ there is a C*-subalgebra B of A satisfying the UCT and such that $\mathcal{F} \subset_\varepsilon B$. Then A satisfies the UCT.*

Proof. For the convenience of the reader, assuming that B is nuclear, we sketch an alternate proof to the one in [8]. It is just this case that is needed in the sequel. By assumption there is an exhausting sequence (A_n) of nuclear separable C*-algebras of A. We may assume that A is unital and its unit is contained in each A_n. After replacing $A_n \subseteq A$ by $A_n \otimes p\mathcal{O}_\infty p \subseteq A \otimes p\mathcal{O}_\infty p$ we observe that the construction $A \mapsto A^\flat$ is functorial with respect to subalgebras. Thus we obtain an exhausting sequence of Kirchberg subalgebras $A_n^\flat$ of $A^\flat$ such that each $A_n^\flat$ is KK-equivalent to A_n and hence it satisfies the UCT. Since each $A_n^\flat$ can be written as an inductive limit of Kirchberg algebras satisfying the UCT and having finitely generated K-theory groups and since the latter algebras are weakly semiprojective, we have exhibited an exhaustive sequence for $A^\flat$, (B_n), such that B_n are weakly semiprojective algebras satisfying the UCT. By a familiar perturbation argument $A^\flat$ is isomorphic to the inductive limit of a subsequence (B_{i_n}) of (B_n). Therefore $A^\flat$ and hence A satisfy the UCT. $\square$

Theorem 4.7. *Let X be a finite dimensional compact metrizable space. Let A be a separable continuous $C(X)$-algebra. Assume that each fiber of A is nuclear and satisfies the UCT. Then A satisfies the UCT.*

Proof. By Theorem 4.4 we may assume that fibers of A are Kirchberg C*-algebras satisfying the UCT. By Theorem 2.12, A admits an exhaustive sequence (A_k) of finite type $C(X)$-algebras. Each A_k satisfies the UCT by Lemma 4.5. We conclude by applying Theorem 4.6. $\square$

References

[1] A. Bauval. $RKK(X)$-nucléarité (d'après G. Skandalis). *K-Theory*, 13(1):23–40, 1998.

[2] B. Blackadar. Shape theory for C^*-algebras. *Math. Scand.*, 56(2):249–275, 1985.

[3] É. Blanchard. Déformations de C^*-algèbres de Hopf. *Bull. Soc. Math. France*, 124(1):141–215, 1996.

[4] É. Blanchard. Subtriviality of continuous fields of nuclear C^*-algebras. *J. Reine Angew. Math.*, 489:133–149, 1997.

[5] É. Blanchard and E. Kirchberg. Global Glimm halving for C*-bundles. *J. Oper. Theory*, 52:385–420, 2004.

[6] É. Blanchard and E. Kirchberg. Non-simple purely infinite C*-algebras: the Hausdorff case. *J. Funct. Anal.*, 207:461–513, 2004.

[7] A. Connes and N. Higson. Déformations, morphismes asymptotiques et K-théorie bivariante. *C. R. Acad. Sci. Paris Sér. I Math.*, 311(2):101–106, 1990.

[8] M. Dadarlat. Some remarks on the universal coefficient theorem in KK-theory. In *Operator algebras and mathematical physics (Constanţa, 2001)*, pages 65–74. Theta, Bucharest, 2003.

[9] M. Dadarlat and G. A. Elliott. One-parameter Continuous Fields of Kirchberg algebras. Preprint 2005.

[10] M. Dadarlat and T. A. Loring. A universal multicoefficient theorem for the Kasparov groups. *Duke Math. J.*, 84(2):355–377, 1996.

[11] M. Dadarlat and C. Pasnicu. Continuous fields of Kirchberg C*-algebras. to appear in J. Funct. Anal. 2005.

[12] J. Dauns and K. H. Hofmann. *Representation of rings by sections*. Memoirs of the American Mathematical Society, No. 83. American Mathematical Society, Providence, R.I., 1968.

[13] J. Dixmier. *C^*-algebras*. North Holland, Amsterdam, 1982.

[14] S. Eilers and T. A. Loring. Computing contingencies for stable relations. *Internat. J. Math.*, 10(3):301–326, 1999.

[15] J. M. G. Fell. The structure of algebras of operator fields. *Acta Math.*, 106:233–280, 1961.

[16] J. M. G. Fell and R. S. Doran. *Representations of *-algebras, locally compact groups, and Banach *-algebraic bundles. Vol. 1-2*, volume 125 of *Pure and Applied Mathematics*. Academic Press Inc., Boston, MA, 1988. Basic representation theory of groups and algebras.

[17] D. Husemoller. *Fibre Bundles*. Number 20 in Graduate Texts in Mathematics. Springer Verlag, New York, 3rd. edition, 1966, 1994.

[18] G. G. Kasparov. Equivariant KK-theory and the Novikov conjecture. *Invent. Math.*, 91(1):147–201, 1988.

[19] E. Kirchberg. The classification of purely infinite C^*-algebras using Kasparov's theory. preprint, 1994.

[20] E. Kirchberg. Das nicht-kommutative Michael-auswahlprinzip und die klassifikation nicht-einfacher algebren. In *C^*-algebras*, pages 92–141, Berlin, 2000. Springer. (Münster, 1999).

[21] E. Kirchberg and N. C. Phillips. Embedding of continuous fields of c^*-algebras in the cuntz algebra $\wr_2$. *J. Reine Angew. Math.*, 525:55–94, 2000.

[22] E. Kirchberg and M. Rørdam. Non-simple purely infinite C^*-algebras. *Amer. J. Math.*, 122(3):637–666, 2000.

[23] E. Kirchberg and M. Rørdam. Infinite non-simple C^*-algebras: absorbing the Cuntz algebra O_∞. *Advances in Math.*, 167(2):195–264, 2002.

[24] E. Kirchberg and S. Wassermann. Operations on continuous bundles of C^*-algebras. *Math. Ann.*, 303(4):677–697, 1995.

[25] H. Lin. Weak semiprojectivity in purely infinite C*-algebras. preprint 2002.

[26] T. Loring. *Lifting Solutions to Perturbing Problems in C^*-Algebras*, volume 8 of *Fields Institute Monographs*. Amer. Math. Soc., Providence, Rhode Island, 1997.

[27] M. Nilsen. C^*-bundles and $C_0(X)$-algebras. *Indiana Univ. Math. J.*, 45(2):463–477, 1996.

[28] N. C. Phillips. A classification theorem for nuclear purely infinite simple C^*-algebras. *Documenta Math.*, (5):49–114, 2000.

[29] M. Rørdam. *Classification of nuclear, simple C^*-algebras*, volume 126 of *Encyclopaedia Math. Sci.* Springer, Berlin, 2002.

[30] J. Rosenberg and C. Schochet. The Künneth theorem and the universal coefficient theorem for Kasparov's generalized K-functor. *Duke Math. J.*, 55(2):431–474, 1987.

[31] J. S. Spielberg. Weak semiprojectivity for purely infinite C*-algebras. arXiv:math.OA/0206185.

[32] J. S. Spielberg. Semiprojectivity for certain purely infinite C^*-algebras. preprint, 2001.

Department of Mathematics, Purdue University, West Lafayette IN 47907, U.S.A.
E-mail address: mdd@math.purdue.edu