# CONTINUOUS FIELDS OF C*-ALGEBRAS OVER FINITE DIMENSIONAL SPACES

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Abstract. Let $X$ be a finite dimensional compact metrizable space. We study a technique which employs semiprojectivity as a tool to produce approximations of $C(X)$-algebras by $C(X)$-subalgebras with controlled complexity. The following applications are given. All unital separable continuous fields of C*-algebras over $X$ with fibers isomorphic to a fixed Cuntz algebra $\mathcal{O}_n$, $n \in \{2, 3, ..., \infty\}$ are locally trivial. They are trivial if $n = 2$ or $n = \infty$. For finite $n \geq 3$, such a field is trivial if and only if $(n-1)[1_A] = 0$ in $K_0(A)$, where $A$ is the C*-algebra of continuous sections of the field. In a more general context, we show that a separable unital continuous field over $X$ with fibers isomorphic to a $KK$-semiprojective Kirchberg C*-algebra is trivial if and only if it satisfies a global K-theoretical Fell-type condition. We also show that if each fiber of a separable continuous field of C*-algebras over $X$ is nuclear and KK-equivalent to a commutative C*-algebra, then the C*-algebra of continuous sections of the field is KK-equivalent to a commutative C*-algebra.

## 1. introduction

Gelfand's characterization of commutative C*-algebras has suggested the problem of representing non-commutative C*-algebras as sections of bundles. This led to the notion of (upper semi-) continuous fields of C*-algebras as illustrated by the work of Fell [18] and Dauns and Hofmann [15]. The continuous fields of C*-algebras (which we identify with the continuous $C(X)$-algebras) form a vast class of C*-algebras which includes the separable C*-algebras with Hausdorff primitive spectrum along with many other fundamental examples [16]. The asymptotic morphisms of Connes and Higson [8] can also be described in terms of continuous fields over $[0, 1]$ which are trivial over $[0, 1)$.

The goal of this paper is to study a technique which uses semiprojectivity as a tool to produce approximations of $C(X)$-algebras by $C(X)$-subalgebras with controlled complexity for a finite dimensional metrizable space $X$ and then to explore some of its applications. Our study is motivated by the problem of extending the classification theory of Kirchberg and Phillips [22], [32], [33] to continuous $C(X)$-algebras whose fibers are Kirchberg algebras and by the problem of describing the structure of general separable nuclear continuous $C(X)$-algebras, at least at K-theoretical level. The two problems are related, and as evidence for that, we show that any separable nuclear continuous $C(X)$-algebra is $KK_{C(X)}$-equivalent to a $C(X)$-algebra whose fibers are Kirchberg algebras (Theorem 7.4). By a Kirchberg algebra we mean a purely infinite simple nuclear separable C*-algebra [33]. Notable examples include the simple Cuntz-Krieger algebras [10]. The following theorem illustrates our results and offers further motivation for our investigation.

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**Theorem 1.1.** *Any separable unital $C(X)$-algebra $A$ over a finite dimensional metrizable compact space $X$ with fibers isomorphic to a fixed Cuntz algebra $\mathcal{O}_n$ with $n \in \{2, 3, \dots, \infty\}$ is locally trivial. If $n = 2$ or $n = \infty$, then $A \cong C(X) \otimes \mathcal{O}_n$. If $3 \leq n < \infty$, then $A$ is isomorphic to $C(X) \otimes \mathcal{O}_n$ if and only if $(n-1)[1_A] = 0$ in $K_0(A)$.*

The proof of Theorem 1.1 also yields a new proof for the triviality of $C(X)$-algebras with fibers isomorphic to the Cuntz algebra $\mathcal{O}_2$, proved by Kirchberg by different methods. We also compute the homotopy groups

$$\pi_m(\mathrm{Aut}(\mathcal{O}_n)) = \begin{cases} \mathbb{Z}/(n-1)\mathbb{Z}, & \text{if } m \text{ is odd,} \\ 0, & \text{if } m \text{ is even,} \end{cases}$$

(with the convention that $\mathbb{Z}/\infty\,\mathbb{Z} = 0$, see Theorem 6.10) and conclude that there are locally trivial unital $C(S^{2k})$-algebras $A$ with fiber $\mathcal{O}_n$ such that $A \ncong C(S^{2k}) \otimes \mathcal{O}_n$ for $n \neq 2, \infty$ and $k \geq 1$.

To put our results in perspective, let us recall that none of the general basic properties of a continuous field implies any kind of local triviality. This is emphasized in [13, Ex. 8.4] by an example of a continuous field $A$ of Kirchberg algebras over $[0, 1]$ whose fibers are abstractly mutually isomorphic, yet $A$ is not locally trivial at any point. Examples with similar properties in the realm of nonexact C*-algebras were exhibited by S. Wassermann. Our next result explains and generalizes Theorem 1.1 by showing that the *only obstruction* to local or global triviality for a continuous field of Kirchberg algebras (in a rather large class) is of purely K-theoretical nature.

**Theorem 1.2.** *Let $X$ be a finite dimensional compact metrizable space. Let $A$ be a separable unital $C(X)$-algebra the fibers of which are Kirchberg algebras and let $D$ be a unital $KK$-semiprojective Kirchberg algebra. Then $A$ is isomorphic to $C(X) \otimes D$ if and only if there is $\sigma \in KK(D, A)$ such that $K_0(\sigma)[1_D] = [1_A]$ and $\sigma_x \in KK(D, A(x))^{-1}$ for all $x \in X$. For each such $\sigma$ there is a $C(X)$-linear isomorphism $\phi : C(X) \otimes D \to A$ such that $KK(\Phi|_D) = \sigma$.*

The existence of $\sigma$ may be viewed as a KK-theoretical analog of the classical Fell's condition which implies local triviality for fields of compact operators. A key feature of this condition is that it is a priori much weaker than the condition that $A$ is $KK_{C(X)}$-equivalent to $C(X) \otimes D$. In particular it is almost trivial to verify it for the $C(X)$-algebras with fiber $\mathcal{O}_n$ and hence to derive Theorem 1.1. A separable C*-algebra $D$ is $KK$-semiprojective if and only if the functor $KK(D, -)$ is continuous. We show that any $KK$-semiprojective Kirchberg algebra must be weakly semiprojective and $KK$-stable (Theorem 3.9), and these are exactly the technical ingredients needed in the proof of Theorem 1.2. One can further simplify the matters if one assumes that $D$ is KK-equivalent to a commutative algebra, or equivalently, if $D$ satisfies the universal coefficient theorem in KK-theory (abbreviated UCT) [34]. In this case $D$ is $KK$-semiprojective if and only if $K_*(D)$ is finitely generated. As a corollary we obtain a simple K-theory criterion for triviality of fields.

**Theorem 1.3.** *Let $X$ be a finite dimensional compact metrizable space and let $A$ be a separable unital $C(X)$-algebra the fibers of which are Kirchberg algebras satisfying the UCT. Let $D$ be a unital Kirchberg algebra satisfying the UCT and such that $K_*(D)$ is finitely generated. Then $A$ is isomorphic to $C(X) \otimes D$ if and only if there is a graded group* *homomorphism $\theta : K_*(D) \to K_*(A)$ such that $\theta[1_D] = [1_A]$ and $\theta_x : K_*(D) \to K_*(A(x))$ is bijective for all $x \in X$.*

The proofs of the results stated above rely on Theorem 4.5 which plays a central role in our approach, as it produces approximations of $C(X)$-algebras by pullbacks of $n$ locally trivial $C(X)$-algebras, where $n \leq \dim(X)$. Its proof generalizes and refines a technique that was pioneered for fields over zero dimensional spaces in joint work with Pasnicu [14] and for fields over an interval in joint work with G. Elliott [13]. As another application of our approximation methods we exhibit a new permanence property for the class of nuclear C*-algebras which satisfy the UCT.

**Theorem 1.4.** *If $A$ is a separable nuclear continuous $C(X)$-algebra over a finite dimensional metrizable space such that all its fibers satisfy the UCT, then $A$ satisfies the UCT.*

A striking isomorphism result for non-simple separable nuclear purely infinite stable C*-algebras, based on a suitable generalization of Kasparov's $KK_{C(X)}$-theory, was announced by Kirchberg in [23]. In view of Kirchberg's result, the problem of finding a universal coefficient theorem for the $KK_{C(X)}$-groups becomes very important and its solution should have a huge impact in the classification of continuous fields. While the results of the present paper follow from a different approach, they could be regarded as evidence towards the existence of a UCT for $KK_{C(X)}$-groups.

In this paper we rely heavily on the classification theorem (and related results) of Kirchberg and Phillips [33], and on the work on non-simple nuclear purely infinite C*-algebras of Blanchard and Kirchberg [7], [6] and Kirchberg and Rørdam [25], [26]. The embedding theorem of Blanchard [5] is used in the proof of Theorem 1.4.

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## 2. $C(X)$-ALGEBRAS

Let $X$ be a compact Hausdorff space. A $C(X)$-algebra is a C*-algebra $A$ endowed with a $*$-monomorphism $\theta$ from $C(X)$ to the center of the multiplier algebra of $A$ such that $C(X)A$ is dense in $A$; see [21], [4]. We write $fa$ rather than $\theta(f)a$ for $f \in C(X)$ and $a \in A$. With the exception of Section 7, we will only consider unital $C(X)$-algebras with $\theta(1) = 1$. If $Y \subseteq X$ is a closed subset, we let $C(X, Y)$ denote the ideal of $C(X)$ consisting of functions vanishing on $Y$. Then $C(X, Y)A$ is a closed two-sided ideal of $A$ (by Cohen factorization). The quotient of $A$ by this ideal is a $C(Y)$-algebra denoted by $A(Y)$ and is called the restriction of $A = A(X)$ to $Y$. The quotient map is denoted by $\pi_Y : A(X) \to A(Y)$. If $Z$ is a closed subset of $Y$ we have a natural restriction map $\pi_Z^Y : A(Y) \to A(Z)$ and $\pi_Z = \pi_Z^Y \circ \pi_Y$. If $Y$ reduces to a point $x$, we write $A(x)$ for $A(\{x\})$ and $\pi_x$ for $\pi_{\{x\}}$. The C*-algebra $A(x)$ is called the fiber of $A$ at $x$. The image $\pi_x(a) \in A(x)$ of $a \in A$ is denoted by $a(x)$. A map $\eta : A \to B$ of $C(X)$-algebras induces a map $\eta_Y : A(Y) \to B(Y)$. If $Y = \emptyset$ then $A(Y)$ is interpreted as the zero algebra.

Let $A$ be a C*-algebra, $a \in A$ and $\mathcal{F}, \mathcal{G} \subseteq A$. If $\varepsilon > 0$, we write $a \in_\varepsilon \mathcal{F}$ if there is $b \in \mathcal{F}$ such that $\|a - b\| < \varepsilon$. Similarly, we write $\mathcal{F} \subset_\varepsilon \mathcal{G}$ if $a \in_\varepsilon \mathcal{G}$ for every $a \in \mathcal{F}$. The following lemma collects some basic properties of $C(X)$-algebras.

**Lemma 2.1.** *Let $X$ be a compact space, let $A$ be a $C(X)$-algebra and let $B \subset A$ be a $C(X)$-subalgebra. Let $a \in A$ and let $Y$ be a closed subset of $X$.*

(i) *The map $x \mapsto \|a(x)\|$ is upper semi-continuous.*
(ii) $\|\pi_Y(a)\| = \max\{\|\pi_x(a)\| : x \in Y\}$
(iii) *If $a(x) \in \pi_x(B)$ for all $x \in X$, then $a \in B$.*
(iv) *If $\delta > 0$ and $a(x) \in_\delta \pi_x(B)$ for all $x \in X$, then $a \in_\delta B$.*
(v) *The restriction of $\pi : A \to A(x)$ to $B$ induces an isomorphism $B(x) \cong \pi_x(B)$ for all $x \in X$.*

*Proof.* (i), (ii) are proved in [4] and (iii) follows from (iv). (iv): By assumption, for each $x \in X$, there is $b_x \in B$ such that $\|\pi_x(a - b_x)\| < \delta$. Using the upper semi-continuity of the map $x \mapsto \|\pi_x(c)\|$, $c \in A$ and (ii), we find a closed neighborhood $U_x$ of $x$ such that $\|\pi_{U_x}(a - b_x)\| < \delta$. Since $X$ is compact, there is a finite subcover $(U_{x_i})$. Let $(\alpha_i)$ be a partition of unity subordinated to this cover. Setting $b = \sum_i \alpha_i\, b_{x_i} \in B$, one checks immediately that $\|\pi_x(a - b)\| \leq \sum_i \alpha_i(x)\|\pi_x(a - b_{x_i})\| < \delta$, for all $x \in X$. Thus

$$\|a - b\| = \max\{\|\pi_x(a - b)\| : x \in X\} < \delta.$$

(v): If $\iota : B \hookrightarrow A$ is the inclusion map, then $\pi_x(B)$ coincides with the image of $\iota_x : B/C(X,x)B \to A/C(X,x)A$. Thus it suffices to check that $\iota_x$ is injective. If $\iota_x(b + C(X,x)B) = \pi_x(b) = 0$ for some $b \in B$, then $b = fa$ for some $f \in C(X,x)$ and some $a \in A$. If $(f_\lambda)$ is an approximate unit of $C(X,x)$, then $b = \lim_\lambda f_\lambda f a = \lim_\lambda f_\lambda b$ and hence $b \in C(X,x)B$. □

A $C(X)$-algebra such that the map $x \mapsto \|a(x)\|$ is continuous for all $a \in A$, is called a *continuous* $C(X)$-algebra or a C*-bundle [4], [27], [6]. A C*-algebra $A$ is a continuous $C(X)$-algebra if and only if $A$ is the C*-algebra of continuous sections of a continuous field of C*-algebras over $X$ in the sense of [16, Def. 10.3.1], (see [4], [6], [31]).

Let $\eta : B \to E$ and $\psi : D \to E$ be $*$-homomorphisms. The *pullback* of these maps is

$$B \oplus_{\eta,\psi} D = \{(b,d) \in B \oplus D : \eta(b) = \psi(d)\}.$$

We are going to use pullbacks in the context of $C(X)$-algebras. Let $X$ be a compact space and let $Y, Z$ be closed subsets of $X$ such that $X = Y \cup Z$. The following result is proved in [16, Prop. 10.1.13] for continuous $C(X)$-algebras.

**Lemma 2.2.** *If $A$ is a $C(X)$-algebra, then $A$ is isomorphic to $A(Y) \oplus_{\pi,\pi} A(Z)$, the pullback of the restriction maps $\pi^Y_{Y\cap Z} : A(Y) \to A(Y \cap Z)$ and $\pi^Z_{Y\cap Z} : A(Z) \to A(Y \cap Z)$.*

*Proof.* By the universal property of pullbacks, the maps $\pi_Y$ and $\pi_Z$ induce a map $\eta : A \to A(Y) \oplus_{\pi,\pi} A(Z)$, $\eta(a) = (\pi_Y(a), \pi_Z(a))$, which is injective by Lemma 2.1(ii). Thus it suffices to show that the range of $\eta$ is dense. Let $b, c \in A$ such that $\pi_{Y\cap Z}(b - c) = 0$ and let $\varepsilon > 0$. We shall find $a \in A$ such that $\|\eta(a) - (\pi_Y(b), \pi_Z(c))\| < \varepsilon$. By Lemma 2.1(i)-(ii), there is an open neighborhood $V$ of $Y \cap Z$ such that $\|\pi_x(b - c)\| < \varepsilon$ for all $x \in V$. Let $\{\lambda, \mu\}$ be a partition of unity on $X$ subordinated to the open cover $\{Y \cup V, Z \cup V\}$. Then $a = \lambda b + \mu c$ is an element of $A$ which has the desired property. □

Recall that if $B$ is a $C(X)$-subalgebra of $A$, then the fibers $B(x)$ of $B$ identifies with the C*-subalgebra $\pi_x(B)$ of $A(x)$ by Lemma 2.1(v). Let $B(Y) \subset A(Y)$ and $D(Z) \subset A(Z)$

be $C(X)$-subalgebras such that $\pi_x(D) \subset \pi_x(B)$ for all $x \in Y \cap Z$. Then

$$B \oplus_{Y\cap Z} D = \{a \in A : \pi_Y(a) \in B, \pi_Z(a) \in D\}$$

is a $C(X)$-subalgebra of $A$. As an immediate consequence of Lemma 2.2 we see that $B \oplus_{\pi^Z_{Y\cap Z}, \pi^Y_{Y\cap Z}} D \cong B \oplus_{Y\cap Z} D$.

**Lemma 2.3.** *Under the previous assumptions,*

$$\pi_x(B \oplus_{Y\cap Z} D) = \begin{cases} \pi_x(B), & \text{if } x \in X \setminus Z, \\ \pi_x(D), & \text{if } x \in Z, \end{cases}$$

*and there is an exact sequence of C*-algebras*

$$0 \longrightarrow \{b \in B : \pi_{Y\cap Z}(b) = 0\} \longrightarrow B \oplus_{Y\cap Z} D \xrightarrow{\pi_Z} D \longrightarrow 0 \tag{1}$$

*Proof.* Let $X \setminus Z$. The inclusion $\pi_x(B \oplus_{Y\cap Z} D) \subset \pi_x(B)$ is obvious by definition. Given $b \in B$, let us choose $f \in C(X)$ vanishing on $Z$ and such that $f(x) = 1$. Then $a = (fb, 0)$ is an element of $A$ by Lemma 2.2. Moreover $a \in B \oplus_{Y\cap Z} D$ and $\pi_x(a) = \pi_x(b)$. We have $\pi_Z(B \oplus_{Y\cap Z} D) \subset D$, by definition. Conversely, given $d \in D$, let us observe that $\pi^Z_{Y\cap Z}(d) \in \pi^Y_{Y\cap Z}(B)$ (by assumption) and hence $\pi^Z_{Y\cap Z}(d) = \pi^Y_{Y\cap Z}(b)$ for some $b \in B$. Then $a = (b, d)$ is an element of $A$ by Lemma 2.2 and $\pi_Z(a) = d$. This completes the proof for the first part of the lemma and also it shows that the map $\pi_Z$ from the sequence (1) is surjective. Its kernel is identified using Lemma 2.1(iii). □

Let $X$, $Y$, $Z$ and $A$ be as above. Let $\eta : B(Y) \hookrightarrow A(Y)$ be a $C(Y)$-linear $*$-monomorphism and let $\psi : D(Z) \hookrightarrow A(Z)$ be a $C(Z)$-linear $*$-monomorphism. Assume that

$$\pi^Z_{Y\cap Z}(\psi(D)) \subseteq \pi^Y_{Y\cap Z}(\eta(B)). \tag{2}$$

This gives a map $\eta^{-1}_{Y\cap Z}\psi_{Y\cap Z} : D(Y \cap Z) \to B(Y \cap Z)$.

**Lemma 2.4.** *(a) Under the previous assumptions, there are isomorphisms of $C(X)$-algebras:*

$$B \oplus_{\pi, \eta^{-1}\psi\pi} D \cong B \oplus_{\pi\eta, \pi\psi} D \cong \eta(B) \oplus_{Y\cap Z} \psi(D),$$

*where the latter isomorphism is given by the map $\chi : B \oplus_{\pi\eta,\pi\psi} D \to A$ induced by the pair $(\eta, \psi)$. Its components $\chi_x$ identify with $\psi_x$ for $x \in Z$ and with $\eta_x$ for $x \in X \setminus Z$.*

*(b) Condition (2) is equivalent to $\psi(D) \subset \pi_Z\big(A \oplus_Y \eta(B)\big)$.*

*(c) If $\mathcal{F}$ is a finite subset of $A$ such that $\pi_Y(\mathcal{F}) \subset_\varepsilon \eta(B)$ and $\pi_Z(\mathcal{F}) \subset_\varepsilon \psi(D)$, then $\mathcal{F} \subset_\varepsilon \eta(B) \oplus_{Y\cap Z} \psi(D) = \chi\big(B \oplus_{\pi\eta,\pi\psi} D\big)$.*

*Proof.* This is an immediate corollary of Lemmas 2.1, 2.2, 2.3. For illustration, let us verify (c). By assumption $\pi_x(\mathcal{F}) \subset_\varepsilon \eta_x(B)$ for all $x \in X \setminus Z$ and $\pi_z(\mathcal{F}) \subset_\varepsilon \psi_z(D)$ for all $z \in Z$. We deduce from Lemma 2.3 that $\pi_x(\mathcal{F}) \subset_\varepsilon \pi_x(\eta(B) \oplus_{Y\cap Z} \psi(D))$ for all $x \in X$. Therefore $\mathcal{F} \subset_\varepsilon \eta(B) \oplus_{Y\cap Z} \psi(D)$ by Lemma 2.1(iv). □

*Definition* 2.5. Let $\mathcal{C}$ be a class of unital C*-algebras. A $C(Z)$-algebra $E$ is called *elementary* (relative to the class $\mathcal{C}$) or $\mathcal{C}$-*elementary* if there is a finite partition of $X$ into disjoint non-empty closed subsets $Z_1, \dots, Z_r$ ($r \geq 1$) and there exist C*-algebras $D_1, \dots, D_r$ in $\mathcal{C}$ such that $E = \bigoplus_{i=1}^r C(Z_i) \otimes D_i$. The notion of *category* of a unital $C(X)$-algebra with respect to a class $\mathcal{C}$ is defined inductively: if $A$ is elementary relative to $\mathcal{C}$ then $\mathrm{cat}_{\mathcal{C}}(A) = 0$; $\mathrm{cat}_{\mathcal{C}}(A) \leq n$ if there are closed nonempty subsets $Y$ and $Z$ of $X$, with $X = Y \cup Z$ and there exist a unital $C(Y)$-algebra $B$, a $\mathcal{C}$-elementary unital $C(Z)$-algebra $E$ and a unital $*$-monomorphism of $C(Y \cap Z)$-algebras, $\gamma : E(Y \cap Z) \to B(Y \cap Z)$, such that $\mathrm{cat}_{\mathcal{C}}(B) \leq n-1$, and $A$ is isomorphic to

$$B \oplus_{\pi,\gamma\pi} D = \{(b,d) \in B \oplus D : \pi^{Y}_{Y\cap Z}(b) = \gamma\pi^{Z}_{Y\cap Z}(d)\}.$$

By definition $\mathrm{cat}_{\mathcal{C}}(A) = n$ if $n$ is the smallest number with the property that $\mathrm{cat}_{\mathcal{C}}(A) \leq n$. If the no such $n$ exists, then $\mathrm{cat}_{\mathcal{C}}(A) = \infty$.

*Definition* 2.6. Let $\mathcal{C}$ be a class of unital C*-algebras. Let $A$ be a unital $C(X)$-algebra. An *$n$-fibered $\mathcal{C}$-morphism into $A$* consists of $(n+1)$ unital $*$-monomorphisms $(\psi_0, \dots, \psi_n)$ with the following properties. There exist closed nonempty subsets $Y_0, \dots, Y_n$ of $X$, and $\mathcal{C}$-elementary $C(Y_i)$-algebras, $E_0, \dots, E_n$ such that each $\psi_i : E_i \to A(Y_i)$ is $C(Y_i)$-linear and

$$\pi^{Y_i}_{Y_i\cap Y_j}\psi_i(E_i) \subseteq \pi^{Y_j}_{Y_i\cap Y_j}\psi_j(E_j), \qquad \text{for all } i \leq j. \tag{3}$$

Given an $n$-fibered morphism into $A$ we have an associated $C(X)$-algebra defined as the pullback the maps $\psi_i$:

$$A(\psi_0, \dots, \psi_n) = \{(d_0, \dots d_n) : \ d_i \in E_i,\ \pi^{Y_i}_{Y_i\cap Y_j}\psi_i(d_i) = \pi^{Y_j}_{Y_i\cap Y_j}\psi_j(d_j) \text{ for all } i, j\} \tag{4}$$

and an induced $C(X)$-homomorphism

$$\eta = \eta_{(\psi_0,\dots,\psi_n)} : A(\psi_0, \dots, \psi_n) \to A \subset \bigoplus_{i=0}^{n} A(Y_i),$$

$$\eta(d_0, \dots d_n) = \big(\psi_0(d_0), \dots, \psi_n(d_n)\big).$$

There are natural projection maps $p_i : A(\psi_0, \dots, \psi_n) \to E_i$, $p_i(d_0, \dots, d_n) = d_i$. Let us set $X_k = Y_k \cup \cdots \cup Y_n$. Then, $(\psi_k, \dots \psi_n)$ is an $(n-k)$-fibered morphism into $A(X_k)$. Let $\eta_k : A(X_k)(\psi_k, \dots \psi_n) \to A(X_k)$ be the induced map and let $B_k = A(X_k)(\psi_k, \dots \psi_n)$. There exist natural $C(X_{k-1})$-isomorphisms

$$B_{k-1} \cong B_k \oplus_{\pi\eta_k,\pi\psi_{k-1}} E_{k-1} \cong B_k \oplus_{\pi,\eta_k^{-1}\psi_{k-1}\pi} E_{k-1}.$$

This shows that $\mathrm{cat}_{\mathcal{C}}(A(\psi_0, \dots, \psi_n)) \leq n$.

## 3. Semiprojectivity

Let $A$ and $B$ be C*-algebras. Two $*$-homomorphisms $\varphi, \psi : A \to B$ are approximately unitarily equivalent, written $\varphi \approx_u \psi$, if for every finite subset $\mathcal{F}$ of $A$ and any $\varepsilon > 0$ there is a unitary $u$ in C*-algebra $B^+ = B + \mathbb{C}1$ obtained by adjoining a unit to $B$, such that $\|u\varphi(a)u^* - \psi(a)\| < \varepsilon$ for all $a \in \mathcal{F}$. We say that $\varphi$ and $\psi$ are asymptotically unitarily equivalent, written $\varphi \approx_{uh} \psi$, if there is a norm continuous unitary valued map $t \to u_t \in B^+$, $t \in [0,1)$, such that $\lim_{t\to 1} \|u_t\varphi(a)u_t^* - \psi(a)\| = 0$ for all $a \in A$. We shall use several times Kirchberg's Theorem [33, Thm. 8.3.3] and the following theorem of Phillips [32].

**Theorem 3.1.** *Let $A$, $B$ be unital separable C*-algebras such that $A$ is simple and nuclear and $B \cong B \otimes \mathcal{O}_\infty$. For any $\sigma \in KK(A,B)$ such that $K_0(\sigma)[1_A] = [1_B]$ there is a unital $*$-homomorphism $\varphi : A \to B$ such that $KK(\varphi) = \sigma$. If $\psi : A \to B$ is another unital $*$-homomorphism such that $KK(\psi) = KK(\varphi)$, then $\varphi \approx_{uh} \psi$.*

Theorem 3.1 does not appear in this form in [32] but it is an immediate consequence of [32, Thm. 4.1.1]. Indeed, if $\sigma$ is given, [32, Thm. 4.1.1] gives a full $*$-homomorphism $\varphi : A \to B \otimes \mathcal{K}$ such that $KK(\varphi) = \sigma$. Let $e \in \mathcal{K}$ be a rank-one projection. Then $[\varphi(1_A)] = [1_B] = [1_B \otimes e]$ in $K_0(B)$. Since both $\varphi(1_A)$ and $1_B \otimes e$ are full projections and $B \cong B \otimes \mathcal{O}_\infty$, it follows by [32, Lemma 2.1.6] that $u\varphi(1_A)u^* = 1_B \otimes e$ for some unitary in $(B \otimes \mathcal{K})^+$. Replacing $\varphi$ by $u\varphi u^*$ we can arrange that $KK(\varphi) = \sigma$ and $\varphi$ is unital. For the second part of the theorem let us note that any unital $*$-homomorphism $\varphi : A \to B$ is full and if two unital $*$-homomorphisms $\varphi, \psi : A \to B$ are asymptotically unitarily equivalent when regarded as maps into $B \otimes \mathcal{K}$, then $\varphi \approx_{uh} \psi$ when regarded as maps into $B$, by an argument from the proof of [32, Thm. 4.1.4].

A separable unital C*-algebra $D$ is *weakly semiprojective* (see [17]) if for any finite subset $\mathcal{F} \subset D$ and any $\varepsilon > 0$, any separable C*-algebra $A$, any increasing sequence $(J_n)$ of two-sided closed ideals of $A$ with $J = \overline{\cup_n J_n}$, and any $*$-homomorphism $\varphi : D \to A/J$, there is a $*$-homomorphism $\psi : D \to A/J_n$ (for some $n$) such that $\|\pi_n\psi(c) - \varphi(c)\| < \varepsilon$ for all $c \in \mathcal{F}$ (where $\pi_n : A/J_n \to A/J$ is the natural map). If we require that there is a $*$-homomorphism $\psi : D \to A/J_n$ (for some $n$) such that $\pi_n\psi = \varphi$ then $A$ is called *semiprojective* (see [3]). We shall use (weak) semiprojectivity in the following context. Let $A$ be a $C(X)$-algebra, let $x \in X$ and let $U_n = \{y \in X : d(y,x) \leq 1/n\}$. Then $J_n = C(X, U_n)A$ is an increasing sequence of ideals of $A$ such that $J = C(X,x)A$, $A/J_n \cong A(U_n)$ and $A/J \cong A(x)$.

*Examples* 3.2. We shall consider various classes $\mathcal{C}$ consisting of unital separable weakly semiprojective simple C*-algebras. The main examples are the class of simple finite dimensional C*-algebras and the class of unital Kirchberg algebras satisfying the UCT and having finitely generated K-theory groups. The C*-algebras in the latter class are known to be weakly semiprojective by work of Neubüser [30], H. Lin [28] and Spielberg [35]. This also follows from Theorem 3.9 and Corollary 3.11 below. These C*-algebras are semiprojective if they have torsion free $K_1$-groups by a result of Spielberg [36] which extended the foundational work of Blackadar [3] and Szymanski [37].

We need the following generalizations of two results of Loring [29]; see [13].

**Proposition 3.3.** *Let $D$ be a separable semiprojective C*-algebra. For any finite subset $\mathcal{F} \subset D$ and any $\varepsilon > 0$, there exist a finite subset $\mathcal{G} \subset D$ and $\delta > 0$ with the following property. Let $\pi : A \to B$ be a surjective $*$-homomorphism, and let $\sigma : D \to B$ and $\gamma : D \to A$ be $*$-homomorphisms such that $\|\pi\gamma(d) - \sigma(d)\| < \delta$ for all $d \in \mathcal{G}$. Then there is a $*$-homomorphism $\psi : D \to A$ such that $\pi\psi = \sigma$ and $\|\gamma(c) - \psi(c)\| < \varepsilon$ for all $c \in \mathcal{F}$.*

**Proposition 3.4.** *Let $D$ be a separable semiprojective C*-algebra. For any finite subset $\mathcal{F} \subset D$ and any $\varepsilon > 0$, there exist a finite subset $\mathcal{G} \subset D$ and $\delta > 0$ with the following property. For any two $*$-homomorphisms $\varphi, \psi : D \to B$ such that $\|\varphi(d) - \psi(d)\| < \delta$ for all $d \in \mathcal{G}$, there is a homotopy $\chi : D \to B[0,1]$ of $*$-homomorphisms from $\varphi$ to $\psi$ that satisfies $\|\varphi(c) - \chi_t(c)\| < \varepsilon$ for all $c \in \mathcal{F}$.*

*Definition* 3.5. (a) A separable unital C*-algebra $D$ is called *KK-semiprojective* if for any separable C*-algebra $A$ and any sequence of increasing ideals $(J_n)$ of $A$ with $J = \overline{\cup_n J_n}$, the natural map $\varinjlim KK(D, A/J_n) \to KK(D, A/J)$ is surjective.

(b) We say that the functor $KK(D, -)$ is *continuous* if for any inductive system $B_1 \to B_2 \to ...$ of separable C*-algebras, the induced map $\varinjlim KK(D, B_i) \to KK(D, \varinjlim B_i)$ is bijective.

*Definition* 3.6. A separable C*-algebra $D$ is $KK$-stable i.e. there is a finite set $\mathcal{G} \subset D$ and there is $\delta > 0$ with the property that for any two $*$-homomorphisms $\varphi, \psi : D \to A$ such that $\|\varphi(a) - \psi(a)\| < \delta$ for all $a \in \mathcal{G}$, one has $KK(\varphi) = KK(\psi)$.

**Corollary 3.7.** *Any unital separable semiprojective C*-algebra is weakly semiprojective and $KK$-stable.*

*Proof.* This follows from Proposition 3.4. □

**Proposition 3.8.** *If a separable C*-algebra $D$ is KK-semiprojective, then $D$ is $KK$-stable.*

*Proof.* We shall prove the statement by contradiction. Let $(\mathcal{G}_n)$ be an increasing sequence of finite subsets of $D$ whose union is dense in $D$. If the statement is not true, then there are sequences of $*$-homomorphisms $\varphi_n, \psi_n : D \to A_n$ such that $\|\varphi_n(d) - \psi_n(d)\| < 1/n$ for all $d \in \mathcal{G}_n$ and yet $KK(\varphi_n) \neq KK(\psi_n)$ for all $n \geq 1$. Set $B_i = \prod_{n \geq i} A_n$ and let $\nu_i : B_i \to B_{i+1}$ be the natural projection. Let us define $\Phi_i, \Psi_i : D \to B_i$ by $\Phi_i(d) = (\varphi_i(d), \varphi_{i+1}(d), \dots)$ and $\Psi_i(d) = (\psi_i(d), \psi_{i+1}(d), \dots)$, for all $d$ in $D$. Let $B_i'$ be the separable C*-subalgebra of $B_i$ generated by the images of $\Phi_i$ and $\Psi_i$. Then $\nu_i(B_i') \subset B_{i+1}'$ and ones verifies immediately that $\varinjlim \Phi_i = \varinjlim \Psi_i : D \to \varinjlim (B_i', \nu_i)$. Since $D$ is $KK$-semiprojective, we deduce that $KK(\Phi_i) = KK(\Psi_i)$ for some $i$ and hence $KK(\varphi_n) = KK(\psi_n)$ for all $n \geq i$. This gives a contradiction. □

**Theorem 3.9.** *For a separable C*-algebra $D$ consider the following properties:*

*(i) $D$ is $KK$-semiprojective.*

*(ii) The functor $KK(D, -)$ is continuous.*

*(iii) $D$ is weakly semiprojective and $KK$-stable.*

*Then (i) ⇔ (ii). Moreover, (iii) ⇒ (i) if $D$ is nuclear and (i) ⇒ (iii) if $D$ is a unital Kirchberg algebra. Thus (i) ⇔ (ii) ⇔ (iii) for unital Kirchberg algebras.*

*Proof.* The implication (ii) ⇒ (i) is obvious. (i) ⇒ (ii): Let $(B_n, \gamma_{n,n+1})$ be an inductive system with inductive limit $B$ and let $\gamma_n : B_n \to B$ be the canonical maps. We have an induced map $\beta : \varinjlim KK(D, B_n) \to KK(D, B)$. First we show that $\beta$ is surjective. The mapping telescope construction of Larry Brown (as described in the proof of [3, Thm. 3.1]) produces an inductive system of C*-algebras $(T_n, \eta_{n,n+1})$ with inductive limit $B$ such that each $\eta_{n,n+1}$ is surjective, and each canonical map $\eta_n : T_n \to B$ is homotopic to $\gamma_n \alpha_n$ for some $*$-homomorphism $\alpha_n : T_n \to B_n$. In particular $KK(\eta_n) = KK(\gamma_n)KK(\alpha_n)$. Let $x \in KK(D, B)$. By (i) there is $n$ and $y \in KK(D, T_n)$ such that $KK(\eta_n)y = x$ and hence $KK(\gamma_n)KK(\alpha_n)y = x$. Thus $z = KK(\alpha_n)y \in KK(D, B_n)$ is a lifting of $x$. Let us show now that the map $\beta$ is injective. We shall use Cuntz' picture of $KK$-theory in terms of homotopy classes of $*$-homomorphisms: $KK(D, B) \cong [\, qD, B \otimes \mathcal{K}]$. Since $\beta$ is surjective, $qD$ is (stably) homotopy semiprojective in the sense of Effros and Kaminker [1], and as shown in their work, the surjectivity of $\beta : \varinjlim [\,qD, B_i \otimes \mathcal{K}] \to [\,qD, B \otimes \mathcal{K}]$ (for all inductive systems $(B_i)$), implies the injectivity of $\beta$ (see also [29, 15.1.3]).

(iii) $\Rightarrow$ (i): Let $A$, $(J_n)$ and $J$ be as in Definition 3.5. Using the five-lemma and the split exact sequence

$$0 \to KK(D, A) \to KK(D, A^+) \to KK(D, \mathbb{C}) \to 0,$$

we reduce the proof to the case when $A$ is unital. Let $x \in KK(D, A/J)$. By [33, Thm. 8.3.3], since $D$ is nuclear, there is a $*$-homomorphism $\varphi : D \to A/J \otimes \mathcal{O}_\infty \otimes \mathcal{K}$ such that $KK(\varphi) = x$. Since $D$ is weakly semiprojective, there is $n$ and a $*$-homomorphism $\psi : D \to A/J_n \otimes \mathcal{O}_\infty \otimes \mathcal{K}$ such that $\|\pi_n\psi(d) - \varphi(d)\| < \delta$ for all $d \in \mathcal{G}$, where $\mathcal{G}$ and $\delta$ are as in the definition of $KK$-stability. Therefore $KK(\pi_n\psi) = KK(\varphi)$ and hence $KK(\psi)$ is a lifting of $x$ to $KK(D, A/J_n)$.

(i) $\Rightarrow$ (iii): $D$ is $KK$-stable by Proposition 3.8. It remains to show that $A$ is weakly semiprojective. Let $A$, $(J_n)$ and $\pi_n : A \to A/J$ be as in the definition of weak semiprojectivity. By [3, Cor. 2.15], we may assume that $A$ and the $*$-homomorphism $\varphi : D \to A$ (that we want to lift approximately) are unital. In particular $\varphi$ is injective since $D$ is simple. Let $\varepsilon > 0$ and $\mathcal{F} \subset D$ (a finite set) be given. Let $v \in D$ be a non-unitary isometry and set $p = vv^*$ and $q = 1 - p$. Since $[q] = 0$ in $K_0(D)$, by Kirchberg's embedding theorem, there is a unital $*$-homomorphism $\theta : D \to \mathcal{O}_2 \subset qDq$ as in [33, Prop. 4.2.3]. Define $\gamma : D \to pDp$ by $\gamma(d) = vdv^*$. Since $KK(\theta) = 0$, $KK(\gamma + \theta) = KK(\gamma) = KK(\mathrm{id}_D)$. Consider the maps $\alpha = \varphi\gamma$ and $\beta = \varphi\theta$ and view them as unital $*$-homomorphisms $\alpha : D \to PA/JP$ and $\beta : D \to QA/JQ$, where $P = \varphi(p)$ and $Q = \varphi(q)$ are orthogonal projections with $P + Q = 1$. Since $\mathbb{C} \oplus \mathbb{C}$ is semiprojective, there exist $k$ and nonzero projections $P_n, Q_n \in A/J_n$ $(n \geq k)$ which are successive liftings of $P$ and $Q$ and such that $P_n + Q_n = 1$. Since $\mathcal{O}_2$ is semiprojective, and since $\beta$ factors through $\mathcal{O}_2$, there exist $m \geq k$ and a unital $*$-homomorphism $\xi : D \to Q_m A/J_m Q_m$ such that $\pi_m\xi = \beta$. Set $B_n = P_n A/J_n P_n$ and let us observe that since $\pi_n(P_n) = P$ is properly infinite, it follows by [3, Propositions 2.18 and 2.33] that $P_n$ is a properly infinite projection, for all sufficiently large $n$. Since $D$ is $KK$-semiprojective, there exist $n \geq m$ and an element $x \in KK(D, B_n)$ which lifts $KK(\alpha)$ and such that $K_0(x)[1] = [P_n]$. By [33, Thm. 8.3.3], there is a full $*$-homomorphism $\eta : D \to B_n \otimes \mathcal{K}$ such that $KK(\eta) = x$. By [33, Prop. 4.1.4], since both $\eta(1)$ and $P_n$ are full projections in $B_n \otimes \mathcal{K}$, there is a partial isometry $w \in B_n \otimes \mathcal{K}$ such that $w^*w = \eta(1)$ and $ww^* = P_n$. Replacing $\eta$ by $w\eta(-)w^*$, we may assume that $\eta : D \to B_n$ is unital. Then $KK(\pi_n\eta) = KK(\pi_n)x = KK(\alpha)$. Therefore

$$KK(\pi_n(\eta + \xi)) = KK(\alpha + \beta) = KK(\varphi(\gamma + \theta)) = KK(\varphi).$$

By [33, Thm. 8.3.3], if we set $\psi = \eta + \xi$, then $\pi_n\psi \approx_{uh} \varphi$, and so there is a unitary $u \in A/J$ such that $\|u\pi_n\psi(d)u^* - \varphi(d)\| < \varepsilon$ for all $d \in \mathcal{F}$. Since $C(\mathbb{T})$ is semiprojective, after increasing $n$ if necessary, we find a unitary $U \in A/J_n$ which lifts $u$. Then $\Phi = U\psi(-)U^*$ is an approximate lifting of $\varphi$ such that $\|\pi_n\Phi(d) - \varphi(d)\| < \varepsilon$ for all $d \in \mathcal{F}$. $\square$

**Corollary 3.10.** *Any separable nuclear semiprojective $C^*$-algebra is $KK$-semiprojective.*

*Proof.* This is very similar to the proof of the implication (iii) $\Rightarrow$ (i) of Theorem 3.9. Alternately, the statement follows from Corollary 3.7 and Theorem 3.9. $\square$

Blackadar has shown that a semiprojective Kirchberg algebra satisfying the UCT has finitely generated K-theory groups [33, Prop. 8.4.15]. This extends as follows.

**Corollary 3.11.** *Let $D$ be a nuclear separable C*-algebra satisfying the UCT. Then $D$ is $KK$-semiprojective if and only $K_*(D)$ is finitely generated.*

*Proof.* If $K_*(D)$ is finitely generated, then $D$ is $KK$-semiprojective by [34]. Conversely, assume that $D$ is $KK$-semiprojective. Since $D$ satisfies the UCT, we infer that if $G = K_i(D)$ $(i = 0, 1)$, then $G$ is semiprojective in the category of countable abelian groups, in the sense that if $H_1 \to H_2 \to \cdots$ is an inductive system of countable abelian groups with inductive limit $H$, then the natural map $\varinjlim \mathrm{Hom}(G, H_n) \to \mathrm{Hom}(G, H)$ is surjective. This implies that $G$ is finitely generated. Indeed, taking $H = G$, we see that $\mathrm{id}_G$ lifts to $\mathrm{Hom}(G, H_n)$ for some finitely generated subgroup $H_n$ of $G$ and hence $G$ is a quotient of $H_n$.

□

**Proposition 3.12.** *Let $D$ be a separable weakly semiprojective C*-algebra. For any finite subset $\mathcal{F} \subset D$ and any $\varepsilon > 0$ there exist a finite subset $\mathcal{G} \subset D$ and $\delta > 0$ such that for any C*-algebras $B \subset A$ and any $*$-homomorphism $\varphi : D \to A$ with $\varphi(\mathcal{G}) \subset_\delta B$, there is a $*$-homomorphism $\psi : D \to B$ such that $\|\varphi(a) - \psi(a)\| < \varepsilon$ for all $a \in \mathcal{F}$. If in addition $D$ is $KK$-stable, then we can choose $\mathcal{G}$ and $\delta$ such that we also have $KK(\psi) = KK(\varphi)$.*

*Proof.* This follows from [17, Thms. 3.1, 4.6]. Let us review the crux of the argument. Fix $\mathcal{F}$ and $\varepsilon$. Let $(\mathcal{G}_n)$ be an increasing sequence of finite subsets of $D$ whose union is dense in $D$. If the statement is not true, then there are sequences of C*-algebras $C_n \subset A_n$ and $*$-homomorphisms $\varphi_n : D \to A_n$ satisfying $\varphi_n(\mathcal{G}_n) \subset_{1/n} C_n$ and with the property that for any $n \geq 1$ there is no $*$-homomorphism $\psi_n : D \to C_n$ such that $\|\varphi_n(c) - \psi_n(c)\| < \varepsilon$ for all $c \in \mathcal{F}$. Set $B_i = \prod_{n \geq i} A_n$ and $E_i = \prod_{n \geq i} C_n \subset B_i$. If $\nu_i : B_i \to B_{i+1}$ is the natural projection, then $\nu_i(C_i) \subset C_{i+1}$. Let us observe that if we define $\Phi_i : D \to B_i$ by $\Phi_i(d) = (\varphi_i(d), \varphi_{i+1}(d), \dots)$, then the image of $\Phi = \varinjlim \Phi_i : D \to \varinjlim (B_i, \nu_i)$ is contained in $\varinjlim (E_i, \nu_i)$. Since $D$ is weakly semiprojective, there is $i$ and a $*$-homomorphism $\Psi_i : D \to E_i$, of the form $\Psi_i(d) = (\psi_i(d), \psi_{i+1}(d), \dots)$ such that $\|\Phi_i(c) - \Psi_i(c)\| < \varepsilon$ for all $c \in \mathcal{F}$. Therefore $\|\varphi_i(c) - \psi_i(c)\| < \varepsilon$ for all $c \in \mathcal{F}$ which gives a contradiction. □

## 4. Approximation of $C(X)$-algebras

A sequence $(A_n)$ of subalgebras of a C*-algebra $A$ is called *exhaustive* if for any finite subset $\mathcal{F}$ of $A$ and any $\varepsilon > 0$ there is $n$ such that $\mathcal{F} \subset_\varepsilon A_n$.

**Lemma 4.1.** *Let $\mathcal{C}$ be a class of separable unital simple weakly semiprojective C*-algebras. Let $X$ be a compact metrizable space and let $A$ be a unital $C(X)$-algebra. Let $\mathcal{F} \subset A$ be a finite subset and let $\varepsilon > 0$. Let $x \in X$ and assume that $A(x)$ admits an exhaustive sequence of C*-algebras isomorphic to C*-algebras in $\mathcal{C}$. Then there is a C*-algebra $D$ in the class $\mathcal{C}$ and there exist a compact neighborhood $U$ of $x$ and a unital $*$-homomorphism $\varphi : D \to A(U)$ such that $\pi_U(\mathcal{F}) \subset_\varepsilon \varphi(D)$.*

*Proof.* By hypothesis there is $D \in \mathcal{C}$ and a unital $*$-homomorphism $\iota : D \to A(x)$ such that $\pi_x(\mathcal{F}) \subset_{\varepsilon/2} \iota(D)$. Therefore if $\mathcal{F} = \{a_1, \dots, a_r\}$, then there is $\{c_1, \dots, c_r\} \subset D$

such that $\|\pi_x(a_i) - \iota(c_i)\| < \varepsilon/2$, for all $i$. Fix a metric $d$ for the topology of $X$ and set $U_n = \{y \in X : d(x, y) \le 1/n\}$. Since $D$ is weakly semiprojective, there is a unital $*$-homomorphism $\varphi : D \to A(U_n)$ (for some $n$) such that $\|\pi_x\varphi(c_i) - \iota(c_i)\| < \varepsilon/2$ for all $i = 1, ..., r$, and hence

$$\|\pi_x\varphi(c_i) - \pi_x(a_i)\| \le \|\pi_x\varphi(c_i) - \iota(c_i)\| + \|\pi_x(a_i) - \iota(c_i)\| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

By Lemma 2.1(i), after increasing $n$ and setting $U = U_n$ and $\varphi = \pi_U\varphi$, we have

$$\|\varphi(c_i) - \pi_U(a_i)\| = \|\pi_U\big(\varphi(c_i) - a_i\big)\| < \varepsilon,$$

for all $i = 1, ..., r$. This shows that $\pi_U(\mathcal{F}) \subset_\varepsilon \varphi(D)$. □

**Lemma 4.2.** *Let $X$ be a compact metrizable space and let $A$ be a unital $C(X)$-algebra the fibers of which are unital Kirchberg algebras. Let $D$ be a unital Kirchberg algebra and suppose that $\varphi : D \to A$ is a unital $*$-homomorphism such that $KK(\varphi_x) \in KK(D, A(x))^{-1}$ for some $x \in X$. Let $\mathcal{F} \subset A$ be a finite subset and let $\varepsilon > 0$. Then there is a closed neighborhood $U$ of $x$ and there exists a unital $*$-homomorphism $\psi : D \to A(U)$ satisfying $KK(\psi) = KK(\pi_U\varphi)$ and such that $\pi_U(\mathcal{F}) \subset_\varepsilon \psi(D)$.*

*Proof.* By [33, Thm. 8.4.1] there is an isomorphism $\varphi_0 : D \to A(x)$ such that $KK(\varphi_0) = KK(\varphi_x)$. Let $\mathcal{H} \subset D$ be such that $\varphi_0(\mathcal{H}) = \pi_x(\mathcal{F})$. By Theorem 3.1 there is a unitary $u_0 \in A(x)$ such that $\|\varphi_0(a) - u_0\varphi_x(a)u_0^*\| < \varepsilon$ for all $a \in \mathcal{H}$. Fix a metric $d$ for the topology of $X$ and set $U_n = \{y \in X : d(x, y) \le 1/n\}$. Since $C(\mathbb{T})$ is semiprojective, there is $n$ and a unitary $u \in A(U_n)$ such that $\pi_x(u) = u_0$. Since $\pi_x(\mathcal{F}) \subset_\varepsilon \pi_x(u\pi_{U_n}\varphi(\mathcal{H})u^*)$, there is $r \ge n$ such that $\pi_{U_r}(\mathcal{F}) \subset_\varepsilon \pi_{U_r}(u\pi_{U_n}\varphi(\mathcal{H})u^*)$, by Lemma 2.1(i). Setting $v = \pi_{U_r}(u)$, we have that $U = U_r$ and $\psi = v(\pi_U\varphi)v^*$ satisfy the conclusion of the lemma. □

The following lemma is useful for the construction of fibered morphisms.

**Lemma 4.3.** *Let $\mathcal{C}$ be a class of separable unital simple weakly semiprojective C*-algebras. Let $(D_j)_{j\in J}$ be a finite family of C*-algebras in $\mathcal{C}$. For each $j \in J$, let $\mathcal{H}_j \subset D_j$ be a finite set and let $\varepsilon > 0$. Let $\mathcal{G}_j \subset D_j$ (a finite set) and $\delta_j > 0$ be given by Proposition 3.12 applied to $D_j$, $\mathcal{H}_j$ and $\varepsilon/2$. Let $X$ be a compact metrizable space, let $(Z_j)_{j\in J}$ be a finite family of mutually disjoint nonempty closed subsets of $X$ and let $Y$ be a closed nonempty subset of $X$ such that $X = Y \cup \big(\cup_j Z_j\big)$. Let $A$ be a unital $C(X)$-algebra and let $\mathcal{F}$ be a finite subset of $A$. Let $\eta : B(Y) \hookrightarrow A(Y)$ be a unital $C(Y)$-subalgebra and let $\varphi_j : D_j \to A(Z_j)$ be unital $*$-homomorphisms satisfying*

(i) $\pi_{Z_j}(\mathcal{F}) \subset_{\varepsilon/2} \varphi_j(\mathcal{H}_j)$, *for all* $j \in J$,
(ii) $\pi_Y(\mathcal{F}) \subset_\varepsilon \eta(B)$,
(iii) $\pi^{Z_j}_{Y\cap Z_j}\varphi_j(\mathcal{G}_j) \subset_{\delta_j} \pi^Y_{Y\cap Z_j}\eta(B)$, *for all* $j \in J$.

*Then, there are $C(Z_j)$-linear unital $*$-homomorphisms $\psi_j : C(Z_j)\otimes D_j \to A(Z_j)$, satisfying*

$$\|\varphi_j(a) - \psi_j(a)\| < \varepsilon/2, \text{ for all } a \in \mathcal{H}_j, \text{ and } j \in J, \tag{5}$$

*and such that if we set $E = \oplus_j C(Z_j) \otimes D_j$, $Z = \cup_j Z_j$, and $\psi : E \to A(Z) = \oplus_j A(Z_j)$, $\psi = \oplus_j \psi_j$, then $\pi^Z_{Y\cap Z}(\psi(E)) \subseteq \pi^Y_{Y\cap Z}(\eta(B))$, $\pi_Z(\mathcal{F}) \subset_\varepsilon \psi(E)$ and*

$$\mathcal{F} \subset_\varepsilon \eta(B) \oplus_{Y\cap Z} \psi(E) = \chi(B \oplus_{\pi\eta,\pi\psi} E),$$

*where $\chi$ is the isomorphism induced by the pair $(\eta, \psi)$. If we assume that each $D_j$ is $KK$-stable, then we also have $KK(\varphi_j) = KK(\psi_j|_{D_j})$ for all $j \in J$.*

*Proof.* Let $\mathcal{F} = \{a_1, \dots, a_r\} \subset A$ be as in the statement. By (i) we find $\{c_1^{(j)}, \dots, c_r^{(j)}\} \subseteq \mathcal{H}_j$ such that $\|\varphi_j(c_i^{(j)}) - \pi_{Z_j}(a_i)\| < \varepsilon/2$ for all $i$. Consider the $C(X)$-algebra $A \oplus_Y \eta(B) \subset A$. From (iii), Lemma 2.1 (vi) and Lemma 2.3 we obtain

$$\varphi_j(\mathcal{G}_j) \subset_{\delta_j} \pi_{Z_j}(A \oplus_Y \eta(B)).$$

Applying Proposition 3.12 we perturb $\varphi_j$ to a $*$-monomorphism $\psi_j : D_j \to \pi_{Z_j}(A \oplus_Y \eta(B))$ satisfying (5), and hence such that $\|\varphi_j(c_i^{(j)}) - \psi_j(c_i^{(j)})\| < \varepsilon/2$, for all $i, j$. Therefore

$$\|\psi_j(c_i^{(j)}) - \pi_{Z_j}(a_i)\| \leq \|\psi_j(c_i^{(j)}) - \varphi_j(c_i^{(j)})\| + \|\varphi_j(c_i^{(j)}) - \pi_{Z_j}(a_i)\| < \varepsilon.$$

This shows that $\pi_{Z_j}(\mathcal{F}) \subset_\varepsilon \psi_j(D_j)$. Extending $\psi_j$ to a $C(Z_j)$-linear unital $*$-homomorphism, $\psi_j : C(Z_j) \otimes D_j \to \pi_{Z_j}(A \oplus_Y \eta(B))$, and defining $E$ and $\psi$ as in the statement and setting $Z = \cup_j Z_j$, we obtain that $\psi : E \to (A \oplus_Y \eta(B))(Z) \subset A(Z)$ satisfies

$$\pi_Z(\mathcal{F}) \subset_\varepsilon \psi(E). \tag{6}$$

The property $\psi(E) \subset (A \oplus_Y \eta(B))(Z)$ is equivalent to $\pi^Z_{Y\cap Z}(\psi(E)) \subseteq \pi^Y_{Y\cap Z}(\eta(B))$ by Lemma 2.4(b). Finally from (ii), (6) and Lemma 2.1 (iv) we get $\mathcal{F} \subset_\varepsilon \eta(B) \oplus_{Y\cap Z} \psi(E)$. $\square$

Let $A$ be a unital $C(X)$-algebra. Let $\mathcal{F} \subset A$ be a finite subset and let $\varepsilon > 0$. Let $\mathcal{C}$ be a class of separable unital simple weakly semiprojective C*-algebras. An $(\mathcal{F}, \varepsilon, \mathcal{C})$-*approximation* of $A$ is a family

$$\alpha = \{\mathcal{F}, \varepsilon, \{U_i, \varphi_i : D_i \to A(U_i), \mathcal{H}_i, \mathcal{G}_i, \delta_i\}_{i \in I}\}, \tag{7}$$

with the following properties. $(U_i)_{i\in I}$ a finite family of closed subsets of $X$, whose interiors cover $X$. $(D_i)_{i\in I}$ is a finite family of C*-algebras in $\mathcal{C}$. For each $i \in I$, $\varphi_i : D_i \to A(U_i)$ is a unital $*$-monomorphism, and $\mathcal{H}_i \subset D_i$ is a finite set such that

$$\pi_{U_i}(\mathcal{F}) \subset_{\varepsilon/2} \varphi_i(\mathcal{H}_i).$$

For each $i \in I$, $\mathcal{G}_i \subset D_i$ and $\delta_i > 0$ are given by Proposition 3.12 applied to the weakly semiprojective C*-algebra $D_i$ for the input data $\mathcal{H}_i$ and $\varepsilon/2$. If $D_i$ is $KK$-stable, then $\mathcal{G}_i$ and $\delta_i$ are chosen such that the second part of Proposition 3.12 also applies.

**Lemma 4.4.** *Let $A$ and $\mathcal{C}$ be as above. Suppose that for each $x \in X$, $A(x)$ admits an exhaustive sequence of C*-algebras isomorphic to C*-algebras in $\mathcal{C}$. Then for any finite subset $\mathcal{F}$ of $A$ and any $\varepsilon > 0$ there is an $(\mathcal{F}, \varepsilon, \mathcal{C})$-approximation of $A$. Moreover, if $A$, $D$ and $\varphi$ are as in Lemma 4.2 and $KK(\varphi_x) \in KK(D, A(x))^{-1}$ for all $x \in X$, then there is an $(\mathcal{F}, \varepsilon, D)$-approximation of $A$ such that $KK(\varphi_i) = KK(\pi_{U_i}\varphi)$ for all $i \in I$.*

*Proof.* Since $X$ is compact, this is an immediate consequence of Lemmas 4.1, 4.2 and Proposition 3.12. $\square$

It is useful to consider the following operation of restriction. Assume that $Y$ is a closed subspace of $X$, and let $(V_j)_{j\in J}$ be a finite family of closed subsets of $Y$ which refines the family $(Y \cap U_i)_{i\in I}$ and such that the interiors of $V_j's$ form a cover of $Y$. Let $\iota : J \to I$ be a map such that $V_j \subseteq Y \cap U_{\iota(j)}$. Define

$$\iota^*(\alpha) = \{\pi_Y(\mathcal{F}), \varepsilon, \{V_j, \pi_{V_j}\varphi_{\iota(j)} : D_{\iota(j)} \to A(V_j), \mathcal{H}_{\iota(j)}, \mathcal{G}_{\iota(j)}, \delta_{\iota(j)}\}_{j\in J}\}.$$

It is obvious that $\iota^*(\alpha)$ is a $(\pi_Y(\mathcal{F}), \varepsilon, \mathcal{C})$-approximation of $A(Y)$. The operation $\alpha \mapsto \iota^*(\alpha)$ is useful even in the case $X = Y$. Indeed, by applying this procedure, we can refine the cover of $X$ that appears in a given $(\mathcal{F}, \varepsilon, \mathcal{C})$-approximation of $A$.

An $(\mathcal{F}, \varepsilon, \mathcal{C})$-approximation of $A$, $\alpha = \{\mathcal{F}, \varepsilon, \{U_i, \varphi_i : D_i \to A(U_i), \mathcal{H}_i, \mathcal{G}_i, \delta_i\}_{i\in I}\}$ is subordinated to an $(\mathcal{F}', \varepsilon', \mathcal{C})$-approximation of $A$, $\alpha' = \{\mathcal{F}', \varepsilon', \{U_{i'}, \varphi_{i'} : D_{i'} \to A(U_{i'}), \mathcal{H}_{i'}, \mathcal{G}_{i'}, \delta_{i'}\}_{i'\in I"}\}$, written $\alpha \prec \alpha'$, if

(i) $\mathcal{F} \subseteq \mathcal{F}'$,
(ii) $\varphi_i(\mathcal{G}_i) \subseteq \pi_{U_i}(\mathcal{F}')$ for all $i \in I$, and
(iii) $\varepsilon' < \min\big(\{\varepsilon\} \cup \{\delta_i,\, i \in I\}\big)$.

It is clear that with notation as above, and $\iota' : I' \to J'$, we have $\iota^*(\alpha) \prec \iota'^*(\alpha')$ whenever $\alpha \prec \alpha'$ and $Y = Y'$.

The following theorem in the crucial technical result of our paper. It provides an approximation of $C(X)$-algebras by subalgebras of category $\leq \dim(X)$.

**Theorem 4.5.** *Let $\mathcal{C}$ be a class of separable simple unital weakly semiprojective C*-algebras. Let $X$ be a finite dimensional compact metrizable space and suppose that $A$ is a unital $C(X)$-algebra the fibers of which admit exhaustive sequences of C*-algebras isomorphic to C*-algebras in $\mathcal{C}$. Then for any finite set $\mathcal{F} \subset A$ and any $\varepsilon > 0$, there is a unital $n$-fibered $\mathcal{C}$-morphism into $A$, $(\psi_0, \dots, \psi_n)$, such that $\mathcal{F} \subset_\varepsilon \eta(A(\psi_0, ..., \psi_n))$, where $n \leq \dim(X)$ and $\eta$ is induced by $(\psi_0, \dots, \psi_n)$.*

*Proof.* By Lemma 4.4, for any finite set $\mathcal{F} \subset A$ and any $\varepsilon > 0$, there is an $(\mathcal{F}, \varepsilon, \mathcal{C})$-approximation of $A$. Moreover, for any finite set $\mathcal{F} \subset A$, any $\varepsilon > 0$ and any $n$, there is a sequence $\{\alpha_k : 0 \leq k \leq n\}$ of $(\mathcal{F}_k, \varepsilon_k, \mathcal{C})$-approximations of $A$ such that $(\mathcal{F}_0, \varepsilon_0) = (\mathcal{F}, \varepsilon)$ and $\alpha_k$ is subordinated to $\alpha_{k+1}$:

$$\alpha_0 \prec \alpha_1 \prec \cdots \prec \alpha_n.$$

Indeed, assume that $\alpha_k$ was constructed. Let us choose a finite set $\mathcal{F}_{k+1}$ which contains $\mathcal{F}_k$ and liftings to $A$ of all the elements in $\cup_{i_k\in I_k}\varphi_{i_k}(\mathcal{G}_{i_k})$. This choice takes care of the conditions (i) and (ii). Next we choose $\varepsilon_{k+1}$ sufficiently small such that (iii) is satisfied. Let $\alpha_{k+1}$ be an $(\mathcal{F}_{k+1}, \varepsilon_{k+1}, \mathcal{C})$-approximation of $A$ given by Lemma 4.4. Then obviously $\alpha_k \prec \alpha_{k+1}$.

Since $X$ is a compact Hausdorff space of dimension $\leq n$, by [6, Lemma 3.2], for every open cover $\mathcal{V}$ of $X$ there is a finite open cover $\mathcal{U}$ which refines $\mathcal{V}$ and such that the set $\mathcal{U}$ can be partitioned into $n+1$ subsets consisting of elements with pairwise disjoint closures. Since we can refine simultaneously the covers that appear in a finite family $\{\alpha_k : 0 \leq k \leq n\}$ of approximations while preserving subordinations, we may arrange not only that all $\alpha_k$ share the same cover $(U_i)_{\in I}$, but moreover, that the cover $(U_i)_{\in I}$ can be partitioned into $n+1$ subsets $\mathcal{U}_0, \dots, \mathcal{U}_n$ consisting of mutually disjoints elements, by [6, Lemma 3.2]. For definiteness, let us write $\mathcal{U}_k = \{U_{i_k} : i_k \in I_k\}$. Now for each $k$ we consider the closed subset of $X$

$$Y_k = \bigcup_{i_k \in I_k} U_{i_k},$$

the map $\iota_k : I_k \to I$ and the $(\pi_{Y_k}(\mathcal{F}_k), \varepsilon_k, \mathcal{C})$-approximation of $A(Y_k)$, induced by $\alpha_k$, which is of the form

$$\iota_k^*(\alpha_k) = \{\pi_{Y_k}(\mathcal{F}_k), \varepsilon, \{U_{i_k}, \varphi_{i_k} : D_{i_k} \to A(U_{i_k}), \mathcal{H}_{i_k}, \mathcal{G}_{i_k}, \delta_{i_k}\}_{i_k \in I_k}\},$$

with each $U_{i_k}$ is nonempty. We have

$$\pi_{U_{i_k}}(\mathcal{F}_k) \subset_{\varepsilon_k/2} \varphi_{i_k}(\mathcal{H}_{i_k}), \tag{8}$$

by construction. Since $\alpha_k \prec \alpha_{k+1}$ we obtain

$$\mathcal{F}_k \subseteq \mathcal{F}_{k+1}, \tag{9}$$

$$\varphi_{i_k}(\mathcal{G}_{i_k}) \subseteq \pi_{U_{i_k}}(\mathcal{F}_{k+1}), \text{ for all } i_k \in I_k, \tag{10}$$

$$\varepsilon_{k+1} < \min\big(\{\varepsilon_k\} \cup \{\delta_{i_k},\, i_k \in I_k\}.\big) \tag{11}$$

Set $X_k = Y_k \cup \cdots \cup Y_n$ and $D_k = \oplus_{i_k} C(U_{i_k}) \otimes D_{i_k}$ for $0 \le k \le n$. We construct by induction on decreasing $k$, a sequence $\psi_n, \ldots, \psi_0$ of unital $*$-monomorphisms, such that $\psi_k : D_k \to A(Y_k)$ is $C(Y_k)$-linear and such that $(\psi_k, \ldots, \psi_n)$ is an $(n-k)$-fibered morphism into $A(X_k)$. Each map

$$\psi_k = \oplus_{i_k} \psi_{i_k} : D_k \to A(Y_k) = \oplus_{i_k} A(U_{i_k})$$

will have components $\psi_{i_k} : C(U_{i_k}) \otimes D_{i_k} \to A(U_{i_k})$ whose restrictions to $D_{i_k}$ will be perturbations of $\varphi_{i_k} : D_{i_k} \to A(U_{i_k})$, $i_k \in I_k$. We will construct the maps $\psi_k$ recursively, such that if $B_k = A(X_k)(\psi_k, \ldots, \psi_n)$ and $\eta_k : B_k \to A(X_k)$ is the map induced by the $(n-k)$-fibered morphism $(\psi_k, \ldots, \psi_n)$, then

$$\pi_{X_{k+1} \cap U_{i_k}}\big(\psi_{i_k}(D_{i_k})\big) \subset \pi_{X_{k+1} \cap U_{i_k}}\big(\eta_{k+1}(B_{k+1})\big), \;\forall\, i_k \in I_k, \tag{12}$$

and

$$\pi_{X_k}(\mathcal{F}_k) \subset_{\varepsilon_k} \eta_k(B_k). \tag{13}$$

Note that (12) implies that

$$\pi_{X_{k+1} \cap Y_k}\big(\psi_k(D_k)\big) \subset \pi_{X_{k+1} \cap Y_k}\big(\eta_{k+1}(B_{k+1})\big). \tag{14}$$

For the first step of induction, $k = n$, we choose $\psi_n = \widetilde{\varphi}_n$, where $\widetilde{\varphi}_n = \oplus_{i_n} \widetilde{\varphi}_{i_n}$ and $\widetilde{\varphi}_{i_n} : C(U_{i_n}) \otimes D_{i_n} \to A(U_{i_n})$ are $C(U_{i_n})$-linear extensions of the original $\varphi_{i_n}$. Then $B_n = D_n$ and $\eta_n = \psi_n$. Assume that $\psi_n, \ldots, \psi_{k+1}$ were constructed and that they have the desired properties. We shall construct now $\psi_k$. Condition (13) formulated for $k+1$ becomes

$$\pi_{X_{k+1}}(\mathcal{F}_{k+1}) \subset_{\varepsilon_{k+1}} \eta_{k+1}(B_{k+1}). \tag{15}$$

Since $\varepsilon_{k+1} < \delta_{i_k}$, by using (10) and (15) we obtain

$$\pi_{X_{k+1} \cap U_{i_k}}\big(\varphi_{i_k}(\mathcal{G}_{i_k})\big) \subset_{\delta_{i_k}} \pi_{X_{k+1} \cap U_{i_k}}\big(\eta_{k+1}(B_{k+1})\big), \text{ for all } i_k \in I_k. \tag{16}$$

Since $\mathcal{F}_k \subseteq \mathcal{F}_{k+1}$ and $\varepsilon_{k+1} < \varepsilon_k$, we derive from (15) that

$$\pi_{X_{k+1}}(\mathcal{F}_k) \subset_{\varepsilon_k} \eta_{k+1}(B_{k+1}). \tag{17}$$

The conditions (8), (16) and (17) enable us to apply Lemma 4.3 and perturb $\varphi_{i_k}$ to a unital $*$-homomorphism $\psi_{i_k}$ satisfying (12) and (13) and

$$KK(\psi_{i_k}) = KK(\varphi_{i_k}). \tag{18}$$

if the algebras in $\mathcal{C}$ are assumed to be $KK$-stable. Then we extend each $\psi_{i_k}$ to a $C(U_{i_k})$-linear $*$-homomorphism, $\psi_{i_k} : C(U_{i_k}) \otimes D_{i_k} \to A(U_{i_k})$ and define $\psi_k = \oplus_{i_k} \psi_{i_k}$. This completes the construction of $(\psi_0, \dots, \psi_n)$. Condition (13) for $k = 0$ gives $\mathcal{F} \subset_\varepsilon \eta_0(B_0) = \eta_0(A(\psi_0, \dots, \psi_n))$. Thus $(\psi_0, \dots, \psi_n)$ satisfies the conclusion of the theorem and $cat_{\mathcal{C}}(B_0) \leq n$. □

**Theorem 4.6.** *Let $X$ be a finite dimensional compact metrizable space and let $A$ be a separable unital $C(X)$-algebra the fibers of which are Kirchberg algebras. Let $D$ be a unital $KK$-semiprojective Kirchberg algebra and suppose that there exists $\sigma \in KK(D, A)$ such that $\sigma_x \in KK(D, A(x))^{-1}$ for all $x \in X$. Then there is unital $*$-homomorphism $\varphi : D \to A$ such that $KK(\varphi) = \sigma$. Moreover, for any finite subset $\mathcal{F}$ of $A$ and any $\varepsilon > 0$, there is a unital $n$-fibered $\{D\}$-morphism into $A$, $(\psi_0, \dots, \psi_n)$, where $n \leq \dim(X)$, such that $\psi_i : C(Y_i) \otimes D \to A(Y_i)$ satisfy $KK(\psi_i) = KK(\pi_{Y_i}\widetilde{\varphi})$ for all $i = 0, \dots, n$. Moreover if $\eta : A(\psi_0, \dots, \psi_n) \to A$ is the corresponding induced map, then $\mathcal{F} \subset_\varepsilon \eta(A(\psi_0, \dots, \psi_n))$, and $KK(\eta_x)$ is a KK-equivalence for each $x \in X$.*

*Proof.* $A$ is isomorphic to $A \otimes \mathcal{O}_\infty$ by [7] and [26], as explained in [14, Lemma 3.4]. Therefore $\sigma$ lifts to a unital $*$-homomorphism $\varphi : D \to A$ by Theorem 3.1. Its $C(X)$-linear extension is denoted by $\widetilde{\varphi}$. We repeat the proof of Theorem 4.5 while using only $(\mathcal{F}_i, \varepsilon_i, D)$-approximations of $A$ provided by the second part of Lemma 4.4. The outcome will be a unital $n$-fibered $\{D\}$-morphism into $A$, $(\psi_0, \dots, \psi_n)$ such that $\mathcal{F} \subset_\varepsilon A(\psi_0, \dots, \psi_n)$. Moreover we can arrange that $KK(\psi_i) = KK(\pi_{Y_i}\widetilde{\varphi})$ for all $i = 0, \dots, n$, by (18), since $KK(\varphi_{i_k}) = KK(\pi_{U_{i_k}}\varphi)$ by Lemma 4.4. For each $x \in X$, $\eta_x = (\psi_i)_x$ for some $i$, and hence $KK(\eta_x) = KK(\varphi_x)$. □

*Remark* 4.7. Let us point out that we can strengthen the conclusion of Theorems 4.5 and 4.6 as follows. Fix a metric $d$ for the topology of $X$. Then we may arrange that there is a closed cover $\{Y_0', ..., Y_n'\}$ of $X$ and a number $\ell > 0$ such that $\{x : d(x, Y_i') \leq \ell\} \subset Y_i$ for $i = 0, ..., n$. Indeed, when we choose the finite closed cover $\mathcal{U} = (U_i)_{i \in I}$ of $X$ in the proof of Theorem 4.5 which can be partitioned into $n+1$ subsets $\mathcal{U}_0, \dots, \mathcal{U}_n$ consisting of mutually disjoints elements, as given by [6, Lemma 3.2], and which refines all the covers $\mathcal{U}(\alpha_0), ..., \mathcal{U}(\alpha_n)$ corresponding to $\alpha_0, ..., \alpha_n$, we may assume that $\mathcal{U}$ also refines the covers given by the interiors of the elements of $\mathcal{U}(\alpha_0), ..., \mathcal{U}(\alpha_n)$. Since each $U_i$ is compact and $I$ is finite, there is $\ell > 0$ such that if $V_i = \{x : d(x, U_i) \leq \ell\}$, then the cover $\mathcal{V} = (V_i)_{i \in I}$ refines all of $\mathcal{U}(\alpha_0), ..., \mathcal{U}(\alpha_n)$ and moreover the elements of $\mathcal{V}_k = \{V_i : U_i \in \mathcal{U}_k\}$, are disjoint for each $i = 0, ..., n$. We shall use the cover $\mathcal{V}$ rather than $\mathcal{U}$ in the proof of the two theorems and observe that $Y_k' = \cup_{i_k \in I_k} U_{i_k} \subset \cup_{i_k \in I_k} V_{i_k} = Y_k$ has the desired property.

## 5. Representing $C(X)$-algebras as inductive limits

Theorem 4.5 separable produces exhaustive sequences for certain algebras $C(X)$-algebras. In this section we show how to pass from an exhaustive sequence to a nested exhaustive sequence using semiprojectivity. The remainder of the paper does not depend on this section.

**Proposition 5.1.** *Let $X$ be a compact metrizable space and let $A$ be a unital $C(X)$-algebra. Let $\mathcal{C}$ be a class of separable unital simple semiprojective C*-algebras. Let $(\psi_0,\dots,\psi_n)$ be a unital $n$-fibered $\mathcal{C}$-morphism into $A$, where $\psi_i : D_i \to A(Y_i)$. Let $\mathcal{F}$ be a finite subset of $A(\psi_0,...,\psi_n)$ and let $\varepsilon > 0$. Then there are finite sets $\mathcal{G}_i \subset D_i$, and numbers $\delta_i > 0$, $i = 0,...,n$, such that for any unital $C(X)$-subalgebra $E \subset A$ which satisfies $\psi_i(\mathcal{G}_i) \subset_{\delta_i} E(Y_i)$, $i = 0,...,n$, there is a unital $n$-fibered morphism $(\psi_0',\dots,\psi_n')$ into $E$, with $\psi_i' : D_i \to E(Y_i)$, and such that*

*(i) $\|\psi_i(a)-\psi_i'(a)\| < \varepsilon$ for all $a \in p_i(\mathcal{F})$ and all $i \in \{0,...,n\}$, where $p_i : A(\psi_0,...,\psi_n) \to D_i$ is the natural projection map,*

*(ii) $(\psi_j)_x^{-1}(\psi_i)_x = (\psi_j')_x^{-1}(\psi_i')_x$ for all $x \in Y_i \cap Y_j$ and $0 \le i \le j \le n$.*

*It follows that $A(\psi_0,\dots,\psi_n) = A(\psi_0',\dots,\psi_n')$ and $\|\eta(a) - \eta'(a)\| < \varepsilon$ for all $a \in \mathcal{F}$, where $\eta$ and $\eta'$ are the maps from $A(\psi_0,...,\psi_n)$ to $A$, and respectively to $E$, induced by $(\psi_0,\dots,\psi_n)$ and $(\psi_0',\dots,\psi_n')$.*

*Proof.* We argue by induction on $n$. If $n = 0$, the statement follows from Proposition 3.12. Assume now that the statement is true for $n-1$, and let $\mathcal{F}$ and $\varepsilon$ be given. The algebra $D_0$ is of the form $D_0 = \oplus_{i_0} C(U_{i_0}) \otimes D_{i_0}$, where we use the same notation as in the proof of Theorem 4.5. Let $D = \oplus_{i_0} D_{i_0}$. Since we work with morphisms on $D_0$ whose components are $C(U_{i_0})$-linear, we may assume without any loss of generality, that $\mathcal{F}_0 := p_0(\mathcal{F}) \subset D$. Let $\mathcal{G}$ and $\delta$ be given by applying Proposition 3.3 to the C*-algebra $D$, for the input data $\mathcal{F}_0$ and $\varepsilon/2$. We may assume that $\mathcal{F}_0 \subset \mathcal{G}$ and $\delta < \varepsilon$. We will use the notation from Definition 2.6. Set $B_1 = A(X_1)(\psi_1,\dots,\psi_n)$ and let $\mathcal{H} \subset B_1$ be a finite set such that

$$\pi_{X_1}(\mathcal{F}) \subset \mathcal{H}, \quad \text{and} \tag{19}$$

$$\eta_1^{-1}\pi^{Y_0}_{X_1\cap Y_0}\psi_0(\mathcal{G}) \subset \pi^{X_1}_{X_1\cap Y_0}(\mathcal{H}). \tag{20}$$

Note that the existence of $\mathcal{H}$ is assured, since the image of $\eta_1$ restricted to $X_1 \cap Y_0$ contains the restriction to the same set of $\psi_0(D_0)$, as a consequence of (3). Let $(\psi_0,\dots,\psi_n)$ and $E$ be as above. Let $\mathcal{G}_1,...\mathcal{G}_n$ and $\delta_1,...,\delta_n$ be given by the inductive assumption for $n-1$ applied to $A(X_1)$, $(\psi_1,\dots,\psi_n)$, $\mathcal{H}$ and $\delta/2$. After this preparation, we are ready to chose $\mathcal{G}_0$ and $\delta_0$. Specifically, they are given by Proposition 3.12 applied to $D$ for the input data $\mathcal{G}$ and $\delta/2$. We need to show that $\mathcal{G}_0,\mathcal{G}_1,...\mathcal{G}_n$ and $\delta_0,\delta_1,...,\delta_n$ satisfy the statement. By the inductive assumption, there exists a unital $(n-1)$-fibered morphism $(\psi_1',\dots,\psi_n')$ into $E(X_1)$, such that

(a) $\|\psi_i(a) - \psi_i'(a)\| < \delta/2 < \varepsilon$ for all $a \in p_i(\mathcal{H})$ and all $i \in \{1,\dots,n\}$

(b) $(\psi_j)_x^{-1}(\psi_i)_x = (\psi_j')_x^{-1}(\psi_i')_x$ for all $x \in Y_i \cap Y_j$ and $1 \le i \le j$.

(c) $\|\eta_1(a) - \eta_1'(a)\| < \delta/2$ for all $a \in \mathcal{H}$.

It remains to construct $\psi_0' : D_0 \to E(Y_0)$ such that $\|\psi_0(a) - \psi_0'(a)\| < \varepsilon$ for all $a \in \mathcal{F}_0$ and such that (b) holds for $i = 0$ and $j \in \{1,\dots,n\}$. The latter condition is readily seen to be equivalent to the equation $\eta_1'\eta_1^{-1}\pi_0\psi_0 = \pi_0\psi_0'$, where $\pi_0$ denotes the restriction map to $X_1 \cap Y_0$. The setup is illustrated by the following diagram:

$$\begin{tikzcd} B_1 \arrow[r, "\pi"] \arrow[d, "\eta_1'"] & B_1(X_1 \cap Y_0) \arrow[d, "\eta_1'"] & D_0 \arrow[l, "\eta_1^{-1}\pi_0\psi_0"'] \arrow[d, dotted, "\psi_0'"] \\ E(X_1) \arrow[r, "\pi"] & E(X_1 \cap Y_0) & E(Y_0) \arrow[l, "\pi_0"'] \end{tikzcd}$$

By assumption, $\psi_0(\mathcal{G}_0) \subset_{\delta_0} E(Y_0)$. By Proposition 3.12 there is a unital $*$-homomorphism $\gamma_0 : D \to E(Y_0)$ with components $\gamma_{i_0} : D_{i_0} \to E(Y_{i_0})$ such that

$$\|\gamma_0(d) - \psi_0(d)\| < \delta/2 \tag{21}$$

for all $d \in \mathcal{G}$. Let us verify that the $C(Y_0)$-linear extension of $\gamma_0$ to $D_0$ is an approximate lifting of $\eta_1'\eta_1^{-1}\pi_0\psi_0$. If $d \in \mathcal{G}$, then by (20), $\eta_1^{-1}\pi_0\psi_0(d) = \pi_0(a)$, for some $a \in \mathcal{H}$. From (c) and (21) we have

$$\|\eta_1'\eta_1^{-1}\pi_0\psi_0(d) - \pi_0\gamma_0(d)\| \leq \|\eta_1'\pi_0(a) - \eta_1\pi_0(a)\| + \|\pi_0\psi_0(d) - \pi_0\gamma_0(d)\| < \delta$$

for all $d \in \mathcal{G}$. By Proposition 3.3, there is a $*$-homomorphism $\psi_0' : D \to E(Y_0)$ such that $\pi_0\psi_0' = \eta_1'\eta_1^{-1}\pi_0\psi_0$ and $\|\psi_0'(c) - \gamma_0(c)\| < \varepsilon/2$ for all $c \in \mathcal{F}_0$. Since $\mathcal{F}_0 \subset \mathcal{G}$ and $\delta < \varepsilon$, by combining the last estimate with (21), we obtain $\|\psi_0(a) - \psi_0'(a)\| < \varepsilon$ for all $a \in \mathcal{F}_0$. □

The following result gives an inductive limit representation for $C(X)$-algebras whose fibers are inductive limits of simple semiprojective C*-algebras. For example the fibers can be UHF-algebras or unital Kirchberg algebras $D$ satisfying the UCT and such that $K_1(D)$ is torsion free. Indeed, by [33, Prop. 8.4.13], $D$ can be written as the inductive limit of a sequence of unital Kirchberg algebras $(D_n)$ with finitely generated K-theory groups and torsion free $K_1$-groups. The algebras $D_n$ are semiprojective by [36].

**Theorem 5.2.** *Let $\mathcal{C}$ be a class of unital simple semiprojective C*-algebras. Let $X$ be a finite dimensional compact metrizable space and let $A$ be a separable unital $C(X)$-algebra such that all its fibers admit exhaustive sequences consisting C*-algebras isomorphic to C*-algebras in $\mathcal{C}$. Then $A$ is isomorphic to the inductive limit of a sequence of unital $C(X)$-algebras $A_k$ such that* $\mathrm{cat}_{\mathcal{C}}(A_k) \leq \dim(X)$.

*Proof.* By Theorem 4.5 and Proposition 5.1 we find a sequence of $n$-fibered morphisms into $A$, denoted $(\psi_0^{(k)}, ..., \psi_n^{(k)})$, with induced unital $*$-monomorphisms $\eta^{(k)} : A_k = A(\psi_0^{(k)}, ..., \psi_n^{(k)}) \to A$ with the following properties. There is a sequence of finite sets $\mathcal{F}_k \subset A_k$ and a sequence of $C(X)$-linear unital $*$-monomorphisms $\mu_k : A_k \to A_{k+1}$ such that

(i) $\|\eta^{(k+1)}\mu_k(a) - \eta^{(k)}(a)\| < 2^{-k}$ for all $a \in \mathcal{F}_k$ and all $k \geq 1$,
(ii) $\mu_k(\mathcal{F}_k) \subset \mathcal{F}_{k+1}$ for all $k \geq 1$,
(iii) $\bigcup_{j=k}^{\infty} \big(\mu_j \circ \cdots \circ \mu_k\big)^{-1}(\mathcal{F}_j)$ is dense in $A_k$ and $\bigcup_{j=k}^{\infty} \eta^{(j)}(\mathcal{F}_j)$ is dense in $A$ for all $k \geq 1$.

Arguing as in the proof of [33, Prop. 2.3.2], one verifies that

$$\varphi_k(a) = \lim_{j\to\infty} \eta^{(j)} \circ \big(\mu_j \circ \cdots \circ \mu_k\big)(a)$$

defines a sequence of $*$-monomorphisms $\varphi_k : A_k \to A$ such that $\varphi_{k+1}\mu_k = \varphi_k$ and the induced map $\varphi : \varinjlim_k (A_k, \mu_k) \to A$ is an isomorphism of $C(X)$-algebras. □

## 6. When is a $C(X)$-algebra (locally) trivial

For unital C*-algebras $A$, $B$ we endow the space $\mathrm{Hom}(A,B)$ of unital $*$-homomorphisms with the topology of the point-norm convergence. If $X$ is a compact metric space, then $\mathrm{Hom}(A, C(X)\otimes B)$ is homeomorphic to the space of continuous maps from $X$ to $\mathrm{Hom}(A,B)$ endowed with the compact-open topology. We shall identify a $*$-homomorphism $\varphi \in \mathrm{Hom}(A, C(X)\otimes B)$ with the corresponding continuous map $X \to \mathrm{Hom}(A,B)$, $x \mapsto \varphi_x$, $\varphi_x(a) = \varphi(a)(x)$ for all $x \in X$ and $a \in A$. Let $D$ be a C*-algebra and let $A$ be a $C(X)$-algebra. If $\alpha : D \to A$ is a $*$-homomorphism, let us denote by $\widetilde{\alpha} : C(X)\otimes D \to A$ its (unique) $C(X)$-linear extension and write $\widetilde{\alpha} \in \mathrm{Hom}_{C(X)}(C(X)\otimes D, A)$. For C*-algebras $D$, $B$ we shall make without further comment the following identifications

$$\mathrm{Hom}_{C(X)}(C(X)\otimes D, C(X)\otimes B) \equiv \mathrm{Hom}(D, C(X)\otimes B) \equiv C(X, \mathrm{Hom}(D,B)).$$

For a unital C*-algebra $D$ we denote by $\mathrm{End}(D)$ the unital $*$-endomorphisms of $D$ and by $\mathrm{End}(D)^0$ the path component of $\mathrm{id}_D$ in $\mathrm{End}(D)$. Let us consider

$$\mathrm{End}(D)^* = \{\gamma \in \mathrm{End}(D) : KK(\gamma) \in KK(D,D)^{-1}\}.$$

**Proposition 6.1.** *Let $X$ be a compact metrizable space and let $D$ be a unital $KK$-semiprojective Kirchberg algebra. Let $\alpha : D \to C(X)\otimes D$ be a unital $*$-homomorphism such that $KK(\alpha_x) \in KK(D,D)^{-1}$ for all $x \in X$. Then there is a unital $*$-homomorphism $\Phi : D \to C(X\times[0,1])\otimes D$ such that $\Phi_{(x,0)} = \alpha_x$ and $\Phi_{(x,t)} \in \mathrm{Aut}(D)$ for all $x \in X$ and $t \in (0,1]$. Moreover, if $\Phi_1 : D \to C(X)\otimes D$ is defined by $\Phi_1(d)(x) = \Phi_{(x,1)}(d)$, for all $d \in D$ and $x \in X$, then $\alpha \approx_{uh} \Phi_1$.*

*Proof.* Since $X$ is a metrizable compact space, $X$ is homomeorphic to the projective limit of a sequence of finite simplicial complexes $(X_i)$. Since $D$ is $KK$-semiprojective, $KK(D, \varinjlim C(X_i)\otimes D) = KK(D, C(X)\otimes D)$ by Theorem 3.9. By Theorem 3.1, there is $i$ and a unital $*$-homomorphism $\varphi : D \to C(X_i)\otimes D$ whose KK-class maps onto $KK(\alpha) \in KK(D, C(X)\otimes D)$. To summarize, we have found a finite simplicial complex $Y$, a continuous map $h : X \to Y$ and a continuous map $y \mapsto \varphi_y \in \mathrm{End}(D)$, defined on $Y$, such that the unital $*$-homomorphism $h^*\varphi : D \to C(X)\otimes D$ corresponding to the continuous map $x \mapsto \varphi_{h(x)}$ satisfies $KK(h^*\varphi) = KK(\alpha)$. We may arrange that $h(X)$ intersects all the path components of $Y$, by dropping the path components which are not intersected. Since $\alpha_x \in \mathrm{End}(D)^*$ by hypothesis, and since $KK(\alpha_x) = KK(\varphi_{h(x)})$, we infer that $\varphi_y \in \mathrm{End}(D)^*$ for all $y \in Y$. We shall find two continuous maps $y \mapsto \psi_y \in \mathrm{End}(D)^*$, $y \mapsto \theta_y \in \mathrm{Aut}(D)$ defined on $Y$, such that the maps $y \mapsto \psi_y\theta_y\varphi_y$ and $y \mapsto \theta_y\varphi_y\psi_y$ are homotopic to the constant function that maps $Y$ to $\mathrm{id}_D$. It is clear that it suffices to deal with each path component of $Y$ separately, so that for this part of the proof we may assume that $Y$ is connected. Fix a point $z \in Y$. By [33, Thm. 8.4.1] there is $\nu \in \mathrm{Aut}(D)$ such that $KK(\nu^{-1}) = KK(\varphi_z)$ and hence $KK(\nu\varphi_z) = KK(\mathrm{id}_D)$. By Theorem 3.1, there is a unitary $u \in D$ such that $u\nu\alpha_z(-)u^*$ is homotopic to $\mathrm{id}_D$. Let us set $\theta = u\nu(-)u^* \in \mathrm{Aut}(D)$ and observe that $\theta\varphi_z \in \mathrm{End}(D)^0$. Since $Y$ is path connected, it follows that the entire image of the map $y \mapsto \theta\varphi_y$ is contained in $\mathrm{End}(D)^0$. Since $\mathrm{End}(D)^0$ is a path connected H-space with unit element, it follows by [38, Thm. 2.4, p462] that the homotopy classes $[Y, \mathrm{End}(D)^0]$ (with no condition on basepoints, since the action of the fundamental group $\pi_1(\mathrm{End}(D)^0, \mathrm{id}_D)$ is trivial by [19, p422]) form a group under the natural multiplication.

This proves the existence of the maps $\psi$ and $\theta$; $\theta$ is constant on each path component of $Y$. Composing with $h$ we obtain the maps $x \mapsto \theta_{h(x)}\varphi_{h(x)}\psi_{h(x)}$ and $x \mapsto \psi_{h(x)}\theta_{h(x)}\varphi_{h(x)}$ are homotopic to the constant function that maps $X$ to $\mathrm{id}_D$. By the homotopy invariance of $KK$-theory we obtain that

$$KK(\widetilde{h^*\theta}\widetilde{h^*\varphi}h^*\psi) = KK(\widetilde{h^*\psi}\widetilde{h^*\theta}h^*\varphi) = KK(\iota_D),$$

where $\widetilde{h^*\theta}$, $\widetilde{h^*\varphi}$ and $\widetilde{h^*\psi}$ denote the $C(X)$-linear extensions of the corresponding maps and $\iota_D : D \to C(X) \otimes D$ is defined by $\iota_D(d) = d$ for all $d \in D$. Set $\Theta = h^*\theta$ and $\Psi = h^*\psi$ and recall that $KK(h^*\varphi) = KK(\alpha)$ and hence $KK(\widetilde{h^*\varphi}) = KK(\widetilde{\alpha})$. Therefore

$$KK(\widetilde{\Theta}\widetilde{\alpha}\Psi) = KK(\widetilde{\Psi}\widetilde{\Theta}\alpha) = KK(\iota_D).$$

By Thm. 3.1, $\widetilde{\Theta}\widetilde{\alpha}\Psi \approx_u \iota_D$ and $\widetilde{\Psi}\widetilde{\Theta}\alpha \approx_u \iota_D$. Therefore $\widetilde{\Theta}\widetilde{\alpha}\widetilde{\Psi} \approx_u \mathrm{id}_{C(X)\otimes D}$ and $\widetilde{\Psi}\widetilde{\Theta}\widetilde{\alpha} \approx_u \mathrm{id}_{C(X)\otimes D}$. By [33, Cor. 2.3.4], there is an isomorphism $\Omega : C(X) \otimes D \to C(X) \otimes D$ such that $\Omega \approx_u \widetilde{\Theta}\widetilde{\alpha}$. Hence, if we set $\Gamma = \widetilde{\Theta}^{-1}\Omega$, then $\Gamma \approx_u \widetilde{\alpha}$. In particular $\Gamma$ is $C(X)$-linear and $\Gamma_x \in \mathrm{Aut}(D)$ for all $x \in X$. Replacing $\Gamma$ by $u\Gamma(\cdot)u^*$ for some unitary $u \in C(X)\otimes D$ we can arrange that $\Gamma|_D$ is close to $\alpha$. Therefore $KK(\Gamma|_D) = KK(\alpha)$ since $D$ is KK-stable. By Thm. 3.1 there is a continuous map $(0, 1] \to U(C(X) \otimes D)$, $t \mapsto u_t$, with the property that

$$\lim_{t\to 0} \|u_t\Gamma(a)u_t^* - \alpha(a)\| = 0, \text{ for all } \quad a \in D.$$

Therefore the equation

$$\Phi(x,t) = \begin{cases} u_t(x)\Gamma_x u_t(x)^*, & \text{if } t \in (0,1], \\ \alpha_x, & \text{if } t = 0, \end{cases}$$

defines a continuous map $\Phi : X \times [0, 1] \to \mathrm{End}(D)^*$ which extends $\alpha$ and such that $\Phi(X \times (0, 1]) \subset \mathrm{Aut}(D)$. Since $\alpha$ is homotopic to $\Phi_1$, we have that $\alpha \approx_{uh} \Phi_1$ by Theorem 3.1. □

**Proposition 6.2.** *Let $X$ be a compact metrizable space and let $D$ be a unital $KK$-semiprojective Kirchberg algebra. Let $Y$ be a closed subset of $X$. Assume that a map $\gamma : Y \to \mathrm{End}(D)^*$ extends to a continuous map $\alpha : X \to \mathrm{End}(D)^*$. Then there is a continuous extension $\eta : X \to \mathrm{End}(D)^*$ of $\gamma$, such that $\eta(X \setminus Y) \subset \mathrm{Aut}(D)$.*

*Proof.* Since the map $x \to \alpha_x$ takes values in $\mathrm{End}(D)^*$, by Proposition 6.1 there exists a continuous map $\Phi : X \times [0, 1] \to \mathrm{End}(D)^*$ which extends $\alpha$ and such that $\Phi(X \times (0, 1]) \subset \mathrm{Aut}(D)$. Let $d$ be a metric for the topology of $X$ such that $\mathrm{diam}(X) \leq 1$. The equation $\eta(x) = \Phi(x, d(x, Y))$ defines a map on $X$ that satisfies the conclusion of the proposition. □

**Proposition 6.3.** *Let $X$ be a compact metrizable space and let $D$ be a unital $KK$-semiprojective Kirchberg algebra. Let $Y$ be a closed subset of $X$. Let $\alpha : Y \times [0, 1] \cup X \times \{0\} \to \mathrm{End}(D)$ be a continuous map such that $\alpha_{(x,0)} \in \mathrm{End}(D)^*$ for all $x \in X$. Suppose that there is an open set $V$ in $X$ which contains $Y$ and such that $\alpha$ extends to a continuous map $\alpha_V : V \times [0, 1] \cup X \times \{0\} \to \mathrm{End}(D)$. Then there is $\eta : X \times [0, 1] \to \mathrm{End}(D)^*$ such that $\eta$ extends $\alpha$ and $\eta_{(x,t)} \in \mathrm{Aut}(D)$ for all $x \in X \setminus Y$ and $t \in (0, 1]$.*

*Proof.* By Proposition 6.2 it suffices to find a continuous map $\widehat{\alpha} : X \times [0,1] \to \mathrm{End}(D)^*$ which extends $\alpha$. Fix a metric $d$ for the topology of $X$ and define $\lambda : X \to [0,1]$ by $\lambda(x) = d(x, X \setminus V)\big(d(x, X \setminus V) + d(x,Y)\big)^{-1}$. Let us define $\widehat{\alpha} : X \times [0,1] \to \mathrm{End}(D)^*$ by $\widehat{\alpha}_{(x,t)} = \alpha_U(x, \lambda(x)t)$ and observe that $\widehat{\alpha}$ extends $\alpha$. Finally, since $\widehat{\alpha}_{(x,t)}$ is homotopic to $\widehat{\alpha}_{(x,0)} = \alpha_{(x,0)}$, we conclude that the image of $\widehat{\alpha}$ in contained in $\mathrm{End}(D)^*$. $\square$

**Proposition 6.4.** *Let $X$ be a compact metrizable space and let $D$ be a unital $KK$-semiprojective Kirchberg algebra. Let $A$ be a separable unital $C(X)$-algebra which locally isomorphic to $C(X) \otimes D$. Suppose that there is a unital $*$-homomorphism $\varphi : D \to A$ such that $KK(\varphi_x) \in KK(D, A(x))^{-1}$ for all $x \in X$. Then there is a unital isomorphism of $C(X)$-algebras $\psi : C(X) \otimes D \to A$ such that $\psi$ is homotopic to $\widetilde{\varphi}$, the $C(X)$-linear extension of $\varphi$.*

*Proof.* Let $d$ be a metric for the topology of $X$. We denote by $B(x,r)$ the closed ball of radius $r$ centered at $x$. Using the compactness of $X$ and the local triviality of $A$, we find points $x_1, \dots, x_m \in X$ and numbers $r_1, \dots, r_m > 0$ such that if we set $V_i = B(x_i, 2r_i)$, then there are $C(V_i)$-linear isomorphisms $\nu_i : A(V_i) \to C(V_i) \otimes D$, $i = 1, \dots, m$. The morphism $\varphi_i = \nu_i \pi_{V_i} \varphi : C(V_i) \otimes D \to C(V_i) \otimes D$ corresponds a map $\varphi_i : V_i \to \mathrm{End}(D)$ which takes values in $\mathrm{End}(D)^*$. For $i, k \in \{1, \dots, m\}$ let us consider the sets: $V_i^{(k)} = B(x_i, 2r_i - r_i(k-1)/m)$,

$$S_i = V_1^{(i+1)} \cup V_2^{(i)} \cup \cdots \cup V_i^{(2)},$$
$$T_i = V_1^{(i)} \cup V_2^{(i-1)} \cup \cdots \cup V_i^{(1)}.$$

Let us observe that

$$V_i^{(n)} \subset V_i^{(n-1)} \subset \cdots \subset V_i^{(1)} = V_i,$$
$$S_i \subset T_i, \quad S_i \cup V_{i+1} = T_{i+1},$$

that $T_i$ is a neighborhood of $S_i$ and that $T_n = X$. This array of sets is needed in order to assure the existence of the local extension required by Proposition 6.3. We shall construct inductively a sequence of homotopies $H_i : D \to A(T_i) \otimes C[0,1]$, with components $(H_i)_{(x,t)} : D \to A(x)$, such that the restriction of $H_{i+1} : D \to A(T_{i+1}) \otimes C[0,1]$ to $S_i$ is equal to the restriction of $H_i$ to $S_i$, i.e. $(H_{i+1})_{(x,t)} = (H_i)_{(x,t)}$ for all $x \in S_i$ and $t \in [0,1]$. Moreover, we shall also have that $(H_i)_{(x,0)} = \varphi_x$ and that $(H_i)_{(x,1)}$ is an isomorphism for each $x \in T_i$, $i = 0, \dots, m$. We start with $i = 1$ and regard $\varphi_1 = \nu_1 \pi_{V_1} \varphi : D \to C(V_1) \otimes D$ as a continuous map $x \to (\varphi_1)_x = (\nu_1)_x \varphi_x \in \mathrm{End}(D)^*$ defined on $V_1$. By Proposition 6.2 applied for $Y = V_1$, $X = V_1 \times [0,1]$, $\gamma = \varphi_1$ and $\alpha_{(x,t)} = (\varphi_1)_x$ there is a homotopy $\eta_1 : V_1 \times [0,1] \to End(D)^*$ such that $(\eta_1)_{(x,0)} = (\varphi_1)_x$ and $(\eta_1)_{(x,1)} \in \mathrm{Aut}(D)$. We set $(H_1)_{(x,t)} = (\nu_1)_x^{-1} \circ (\eta_1)_{(x,t)}$. Suppose now that $H_1, \dots, H_i$ were constructed. We shall construct $H_{i+1}$ by restricting $H_i$ to $S_i \times [0,1]$, and then by extending this restriction to $T_{i+1} \times [0,1] = (S_i \cup V_{i+1}) \times [0,1]$. Clearly it suffices to extend $H_i$ from $(S_i \cap V_{i+1}) \times [0,1]$ to $V_{i+1} \times [0,1]$ and then set $(H_{i+1})_{(x,t)} = (H_i)_{(x,t)}$ for $x \in S_i \setminus V_{i+1}$ and $t \in [0,1]$. To this purpose we define a continuous map $\alpha : (T_i \cap V_{i+1}) \times [0,1] \cup V_{i+1} \times \{0\} \to \mathrm{End}(D)$ by $\alpha_{(x,t)} = (\nu_{i+1})_x \circ (H_i)_{(x,t)}$ for $x \in T_i \cap V_{i+1}$ and $t \in [0,1]$ and $\alpha_{(x,0)} = (\varphi_{i+1})_x$ for $x \in V_{i+1}$. Since $T_i \cap V_{i+1}$ is a neighborhood of $S_i \cap V_{i+1}$ in $V_{i+1}$ and since $(\varphi_{i+1})_x \in \mathrm{End}(D)^*$ for all $x \in V_{i+1}$, we can apply Proposition 6.3 to obtain a continuous map $\eta_{i+1} : V_{i+1} \times [0,1] \to$

$\mathrm{End}(D)^*$ such that $\eta_{i+1}$ extends the restriction of $\alpha$ to $(S_i \cap V_{i+1}) \times [0,1] \cup V_{i+1} \times \{0\}$ and $(\eta_{i+1})_{(x,1)} \in \mathrm{Aut}(D)$ for all $x \in V_{i+1}$. We conclude the construction of $H_{i+1}$ by defining $(H_{i+1})_{(x,t)} = (\nu_{i+1})_x^{-1} \circ (\eta_{i+1})_{(x,t)}$ for $x \in V_{i+1}$ and $t \in [0,1]$. Finally we observe that since $T_n = X$, $H_n : D \to C[0,1] \otimes A$ is a homotopy from $\varphi$ to some unital $*$-homomorphism $\psi$ such that $\psi_x \in \mathrm{Aut}(D)$ for all $x \in X$. Therefore $\widetilde{\psi} : C(X) \otimes D \to A$ is an isomorphism of $C(X)$-algebras homotopic to $\widetilde{\varphi}$. □

**Lemma 6.5.** *Let $X$ be a compact metrizable space and let $D$ be a unital $KK$-semiprojective Kirchberg algebra. Let $Y$, $Z$ be closed subsets of $X$ such that $X = Y \cup Z$ and $Y \cap Z \neq \emptyset$. Let $\gamma : D \to C(Y \cap Z) \otimes D$ be a unital $*$-homomorphism. Assume that there is a unital $*$-homomorphism $\alpha : D \to C(Y) \otimes D$ such that $\alpha_x \in KK(D,D)^{-1}$ for all $x \in Y$ and such that $\alpha_x = \gamma_x$ for all $x \in Y \cap Z$. Then the pullback $C(Y) \otimes D \oplus_{\pi_{Y\cap Z}, \widetilde{\gamma}\pi_{Y\cap Z}} C(Z) \otimes D$ is isomorphic to $C(X) \otimes D$.*

*Proof.* By Prop. 6.2, there is a unital $*$-homomorphism $\eta : D \to C(Y) \otimes D$ such that $\eta_x = \gamma_x$ for $x \in Y \cap Z$ and such that $\eta_x \in \mathrm{Aut}(D)$ for $x \in Y \setminus Z$. One checks immediately that the pair $\widetilde{\eta}$, $\mathrm{id}_{C(Z)\otimes D}$ defines a $C(X)$-linear isomorphism $\theta : C(X) \otimes D = C(Y) \otimes D \oplus_{\pi_{Y\cap Z}, \pi_{Y\cap Z}} C(Z) \otimes D \to C(Y) \otimes D \oplus_{\pi_{Y\cap Z}, \widetilde{\gamma}\pi_{Y\cap Z}} C(Z) \otimes D$:

$$\begin{tikzcd}
C(Y) \otimes D \arrow[r, "\pi"] \arrow[d, "\widetilde{\eta}"'] & C(Y \cap Z) \otimes D \arrow[d, "\widetilde{\gamma}"] & C(Z) \otimes D \arrow[l, "\pi"'] \arrow[d, equal] \\
C(Y) \otimes D \arrow[r, "\pi"] & C(Y \cap Z) \otimes D & C(Z) \otimes D \arrow[l, "\widetilde{\gamma}\pi"']
\end{tikzcd}$$

□

**Lemma 6.6.** *Let $X$ be a compact metrizable space and let $D$ be a unital $KK$-semiprojective Kirchberg algebra. Let $Y, Y'$ and $Z, Z'$ be closed subsets of $X$ such that $Y'$ is a neighborhood of $Y$, $Z'$ is a neighborhood of $Z$, $X = Y \cup Z$ and $Y \cap Z \neq \emptyset$. Let $A$ be a unital separable $C(X)$-algebra. Let $B$ be a unital $C(Y')$-algebra locally isomorphic to $C(Y') \otimes D$ and let $E$ be a unital $C(Z')$-algebra locally isomorphic to $C(Z') \otimes D$. Let $\varphi : B \to A(Y')$ and $\psi : E \to A(Z')$ unital morphisms of $C(X)$-algebras such that $\psi_x(E(x)) \subset \varphi_x(B(x))$ for all $x \in Y' \cap Z'$. Suppose that $KK(\varphi_x) \in KK(B(x), A(x))^{-1}$ and that $KK(\psi_x) \in KK(E(x), A(x))^{-1}$ for all $x \in Y' \cap Z'$. Then $B(X) \oplus_{\pi_{Y\cap Z}\varphi, \pi_{Y\cap Z}\psi} E(Y)$ is locally isomorphic to $C(X) \otimes D$. Moreover, if $\chi : B(X) \oplus_{\pi_{Y\cap Z}\varphi, \pi_{Y\cap Z}\psi} E(Y) \to A$ is the morphism induced by the pair $\varphi, \psi$, then $KK(\chi_x)$ is a KK-equivalence for all $x \in X$.*

*Proof.* Since we are dealing with a local property, we may assume that $B = C(Y') \otimes D$ and $E = C(Z') \otimes D$. If $\alpha : Y' \cap Z' \to \mathrm{End}(D)$ is defined by $\alpha_x = \varphi_x^{-1}\psi_x$, then $\alpha_x \in \mathrm{End}(D)^*$. The restriction of $\alpha$ to $Y \cap Z$ is denoted by $\gamma$. Let us denote by $H$ the $C(X)$-algebra

$$C(Y) \oplus_{\pi\varphi, \pi\psi} C(Z) \cong C(Y) \oplus_{\pi, \pi\widetilde{\gamma}} C(Z),$$

(where $\pi$ stands for $\pi_{Y\cap Z}$). We must show that $H$ is locally trivial. Let $x \in X$. If $x \notin Y \cap Z$, then there is a closed neighborhood $U$ of $x$ which does not intersect $Y \cap Z$, and hence the restriction of $H$ to $U$ is isomorphic to $C(U) \otimes D$, as it follows immediately from the definition of $H$. It remains to consider the case when $x \in Y \cap Z$. Let us observe that

$V = Y' \cap Z'$ is a neighborhood of $Y \cap Z$ in $X$. The restriction of $H$ to $V$ is isomorphic to

$$C(Y \cap V) \oplus_{\pi\varphi, \pi\psi} C(Z \cap V) \cong C(Y \cap V) \oplus_{\pi, \pi\widetilde{\gamma}} C(Z \cap V),$$

(where $\pi$ stands for $\pi_{Y \cap Z \cap V}$). Since $\gamma : Y \cap Z \cap V \to \mathrm{End}(D)^*$ admits a continuous extension $\alpha : Y \cap V \to \mathrm{End}(D)^*$, it follows that $H(V)$ is isomorphic to $C(V) \otimes D$ by Lemma 6.5. Since $\chi_x$ identifies with $\varphi_x$ if $x \in X \setminus Z$ and with $\psi_x$ if $x \in Z$, $KK(\chi_x)$ is a KK-equivalence. □

**Proposition 6.7.** *Let $X$, $A$, $D$ and $\sigma$ be as in Theorem 4.6. For any finite subset $\mathcal{F}$ of $A$ and any $\varepsilon > 0$ there is a unital $C(X)$-algebra $B$ which is locally isomorphic to $C(X) \otimes D$ and there exists a unital $C(X)$-linear $*$-monomorphism $\eta : B \to A$ such that $\mathcal{F} \subset_\varepsilon \eta(B)$ and $KK(\eta_x) \in KK(B(x), A(x))^{-1}$ for all $x \in X$.*

*Proof.* Let $\varphi$ and $\psi_i : C(Y_i) \to A(Y_i)$, $i = 0, ..., n$ be as in the conclusion of Theorem 4.6, but strengthen as in Remark 4.7. In particular we have

$$\pi_{Y_i}(\mathcal{F}) \subset_\varepsilon \psi_i(C(Y_i) \otimes D), \tag{22}$$

and $KK((\psi_i)_x) = KK(\varphi_x) \in KK(D, A(x))^{-1}$ for all $x \in Y_i$ and $i = 0, \dots, n$. By Remark 4.7, for each $i \in \{0, ..., n\}$, we can find closed subsets $Y_i^{(n)} \subset Y_i^{(n-1)} \subset ... \subset Y_i^{(0)} = Y_i$ such that $Y_i^{(j-1)}$ is a neighborhood of $Y_i^{(j)}$ for all $j = 1, ..., n$, and such that the family $Y_0^{(n)}, ..., Y_n^{(n)}$ covers $X$. Let $\pi_i^{(k)} : A(X) \to A\big(Y_{i-1}^{(k)} \cap \big(Y_i^{(k)} \cup ... \cup Y_n^{(k)}\big)\big)$ be the restriction map. Define

$$B_{n-1} = C(Y_n^{(1)}) \otimes D \oplus_{\pi_n^{(1)}\psi_n, \pi_n^{(1)}\psi_{n-1}} C(Y_{n-1}^{(1)}) \otimes D.$$

By Lemma 6.6, $B_{n-1}$ is locally isomorphic to $C(Y_n^{(1)} \cup Y_{n-1}^{(1)}) \otimes D$ since $\psi_n$ extends to $C(Y_n^{(0)}) \otimes D$, $\psi_{n-1}$ extends to $C(Y_{n-1}^{(0)}) \otimes D$, the fiberwise components of these maps are KK-equivalences and $Y_i^{(0)}$ is a closed neighborhood of $Y_i^{(1)}$. The map $\eta_{n-1} : B_{n-1} \to A(Y_n^{(1)} \cup Y_{n-1}^{(1)})$ induced by the pair $\psi_n, \psi_{n-1}$ is such that $KK((\eta_{n-1})_x)$ is a KK-equivalence for each $x \in Y_n^{(1)} \cup Y_{n-1}^{(1)}$. Moreover

$$\pi_{Y_n^{(1)} \cup Y_{n-1}^{(1)}}(\mathcal{F}) \subset_\varepsilon \eta_{n-1}(B_{n-1}), \tag{23}$$

by (22) and Lemma 2.1(iv). By applying Lemma 6.6 again, a similar reasoning, shows that if we define

$$B_{n-2} = B_{n-1}(Y_n^{(2)} \cup Y_{n-1}^{(2)}) \oplus_{\pi_{n-1}^{(2)}\eta_{n-1}, \pi_{n-1}^{(2)}\psi_{n-2}} C(Y_{n-2}^{(2)}) \otimes D,$$

then $B_{n-2}$ is locally isomorphic to $C(Y_n^{(2)} \cup Y_{n-1}^{(2)} \cup Y_{n-2}^{(2)}) \otimes D$, (since both $\eta_{n-1}$ and $\psi_{n-2}$ extend to locally trivial fields supported on larger neighborhoods) and the map $\eta_{n-2} : B_{n-2} \to A(Y_n^{(2)} \cup Y_{n-1}^{(2)} \cup Y_{n-2}^{(2)})$, induced by the pair $\eta_{n-1}, \psi_{n-2}$ is such that its fiberwise components are KK-equivalences. Moreover $\pi_{Y_n^{(2)} \cup Y_{n-1}^{(2)} \cup Y_{n-2}^{(2)}}(\mathcal{F}) \subset_\varepsilon \eta_{n-2}(B_{n-2})$ by (22), (23) and Lemma 2.1(iv). Arguing similarly, after $n$-steps we obtain a unital $C(X)$-algebra $B_0$ which is locally isomorphic to $C(X) \otimes D$ and a unital $C(X)$-linear map $\eta_0 : B_0 \hookrightarrow A(Y_n^{(0)} \cup ... \cup Y_0^{(0)}) = A(X)$ such that $\mathcal{F} \subset_\varepsilon \eta_0(B_0)$ and $KK((\eta_0)_x) \in KK(B_0(x), A(x))^{-1}$ for all $x \in X$. □

*Proof of Theorem 1.2.*

By Theorem 4.6 there is a unital $*$-homomorphism $\varphi : D \to A$ such that $KK(\varphi) = \sigma$. Since $A$ is separable, by Proposition 6.4 there is also a sequence of unital $C(X)$-algebras $(A_k)_{k=1}^{\infty}$ and a sequence of unital $C(X)$-linear $*$-monomorphisms $(\eta^{(k)})_{k=1}^{\infty}$, $\eta^{(k)} : A_k \to A$, such that $A_k$ is locally isomorphic to $C(X) \otimes D$, $KK(\eta_x^{(k)})$ is a KK-equivalence for each $x \in X$ and $(\eta^{(k)}(A_k))_{k=1}^{\infty}$ is an exhaustive sequence of $C(X)$-subalgebras of $A$. Since $D$ is weakly semiprojective and $KK$-stable, after passing to a subsequence of $(A_k)_{k=1}^{\infty}$ if necessary, we find unital $*$-homomorphisms $\varphi^{(k)} : D \to A_k$ such that $\eta^{(k)}\varphi^{(k)}$ is close to $\varphi$ and hence $KK(\eta^{(k)}\varphi^{(k)}) = KK(\varphi)$ for all $k \geq 1$. Since both $KK(\eta_x^{(k)})$ and $KK(\varphi_x)$ are KK-equivalences, we deduce that $KK(\varphi_x^{(k)}) \in KK(D, A_k(x))^{-1}$ for all $x \in X$. Let us set $B = C(X) \otimes D$. Since $A_k$ is locally isomorphic to $B$ by Proposition 6.7, we may apply Proposition 6.4 to find an isomorphism of $C(X)$-algebras $\Phi^{(k)} : B \to A_k$ whose restriction to $D$ is homotopic to $\varphi^{(k)}$. Therefore if we set $\theta^{(k)} = \eta^{(k)}\Phi^{(k)}$, then $\theta^{(k)}$ is a unital $C(X)$-linear $*$-monomorphism from $B$ to $A$ such that $KK(\theta^{(k)}) = KK(\widetilde{\varphi})$ and $(\theta^{(k)}(B))_{k=1}^{\infty}$ is an exhaustive sequence of $C(X)$-subalgebras of $A$. Using again the weak semiprojectivity and the KK-stability of $D$, after passing to a subsequence of $(\theta^{(k)})_{k=1}^{\infty}$ we construct a sequence of finite sets $\mathcal{F}_k \subset B$ and a sequence of $C(X)$-linear unital $*$-monomorphisms $\mu_k : B \to B$ such that

(i) $KK(\theta^{(k+1)}\mu_k) = KK(\theta^{(k)})$ for all $k \geq 1$,
(ii) $\|\theta^{(k+1)}\mu_k(a) - \theta^{(k)}(a)\| < 2^{-k}$ for all $a \in \mathcal{F}_k$ and all $k \geq 1$,
(iii) $\mu_k(\mathcal{F}_k) \subset \mathcal{F}_{k+1}$ for all $k \geq 1$,
(iv) $\bigcup_{j=k}^{\infty} \big(\mu_j \circ \cdots \circ \mu_k\big)^{-1}(\mathcal{F}_j)$ is dense in $B$ and $\bigcup_{j=k}^{\infty} \theta^{(j)}(\mathcal{F}_j)$ is dense in $A$ for all $k \geq 1$.

Arguing as in the proof of [33, Prop. 2.3.2], one verifies that

$$\Delta_k(a) = \lim_{j\to\infty} \theta^{(j)} \circ \big(\mu_j \circ \cdots \circ \mu_k\big)(a)$$

defines a sequence of $*$-monomorphisms $\Delta_k : A_k \to A$ such that $\Delta_{k+1}\mu_k = \Delta_k$ and the induced map $\Delta : \varinjlim_k (B, \mu_k) \to A$ is an isomorphism of $C(X)$-algebras. Let us show that $\varinjlim_k (B, \mu_k)$ is isomorphic to $B$. To this purpose, in view of Elliott's intertwining argument, it suffices to show that each map $\mu_k$ is approximately unitarily equivalent to a $C(X)$-linear automorphism of $B$. Since each $KK(\theta_x^{(k)}) = KK(\varphi_x)$ for all $x \in X$ and since $\varphi_x$ is a KK-equivalence, we deduce from (i) that $KK((\mu_k)_x) = KK(\mathrm{id}_D)$ for each $x \in X$. By Proposition 6.1, this property implies that each map $\mu_k$ is approximately unitarily equivalent to a $C(X)$-linear automorphism of $B$. We have a found an isomorphism of $C(X)$-algebras $\Delta : B \to A$. Let us show that we can arrange that $KK(\Delta) = KK(\widetilde{\varphi})$. Since $KK((\Delta^{-1}\varphi)_x) \in KK(D, D)^{-1}$, we can apply Proposition 6.1 to find $\Phi_1 \in \mathrm{Aut}_{C(X)}(B)$ such that $KK(\Phi_1) = KK(\Delta^{-1}\widetilde{\varphi})$. Then $\Phi = \Delta\Phi_1 : B \to A$ is an isomorphism such that $KK(\Phi) = KK(\widetilde{\varphi})$.

*Proof of Theorem 1.3.* Since $D$ satisfies the UCT, $\theta$ lifts to some $\sigma \in KK(D, A)$. Since $K_*(\sigma_x) = K_*(\pi_x)\theta = \theta_x$ is bijective and since $A(x)$ satisfies the UCT, it follows that $\sigma_x \in KK(D, A(x))^{-1}$ by [34]. We conclude by applying Theorem 1.2.

**Theorem 6.8.** *Let $X$ be a finite dimensional compact metrizable space, and let $A$ be a separable unital $C(X)$-algebra with all fibers isomorphic to the Cuntz algebra $\mathcal{O}_\infty$. Then $A \cong C(X) \otimes \mathcal{O}_\infty$.*

*Proof.* It is known that $\mathcal{O}_\infty$ is semiprojective and it satisfies the UCT. Moreover $K_0(\mathcal{O}_\infty) \cong \mathbb{Z}$ is generated by the class of $1_{\mathcal{O}_\infty}$ and $K_1(\mathcal{O}_\infty) = 0$. Therefore the morphism $\theta : K_0(\mathcal{O}_\infty) \to K_0(A)$ defined by $\theta(k[1_{\mathcal{O}_\infty}]) = k[1_A]$ satisfies the assumptions of Theorem 1.3. □

**Theorem 6.9.** *Any separable unital $C(X)$-algebra $A$ over a finite dimensional metrizable compact space with fibers isomorphic to a fixed Cuntz algebra $\mathcal{O}_n$ is locally trivial. $A$ is isomorphic to $C(X) \otimes \mathcal{O}_n$ if and only if $(n-1)[1_A] = 0$ in $K_0(A)$. This is always the case if $n = 2$.*

*Proof.* We prove first the second part of the theorem. One implication follows easily since $(n-1)K_0(C(X) \otimes \mathcal{O}_n) = 0$. Conversely, assume that $(n-1)[1_A] = 0$. It is known that $\mathcal{O}_n$ is semiprojective and it satisfies the UCT. Moreover $K_0(\mathcal{O}_n) \cong \mathbb{Z}/(n-1)\mathbb{Z}$ is generated by the class of $1_{\mathcal{O}_n}$ and $K_1(\mathcal{O}_\infty) = 0$. Thus there is a morphism of groups $\theta : K_0(\mathcal{O}_n) \to K_0(A)$ which maps $[1_{\mathcal{O}_n}]$ to $[1_A]$ and which satisfies the assumptions of Theorem 1.3. Therefore $A \cong C(X) \otimes \mathcal{O}_n$. By the second part of the theorem, in order to prove the first part, it suffices to show that for any $x \in X$ there is a closed neighborhood $V$ of $x$ such that $(n-1)[1_{A(V)}] = 0$ in $K_0(A(V))$. Let $V_k = \{y \in X : d(y,x) \leq 1/k\}$. Then $K_0(\mathcal{O}_n) \cong K_0(A(x)) \cong \varinjlim_k K_0(A(V_k))$, and hence $(n-1)[1_{A(V_k)}] = 0$ for some $k$ and hence $A(V_k) \cong C(V_k) \otimes \mathcal{O}_n$.

It remains to show that if $n = 2$ then we always have $[1_A] = 0$ in $K_0(A)$. By what was already proven we know that $A$ is locally isomorphic to $C(X) \otimes \mathcal{O}_2$. It follows that $K_0(A) = 0$ by the Meyer-Vietoris sequence. □

As a corollary of Theorem 6.8 we have that $[X, \mathrm{Aut}(\mathcal{O}_\infty)]$ reduces to a point. Let $v_1, \dots, v_n$ be the canonical generators of $\mathcal{O}_n$, $2 \leq n < \infty$. The homotopy groups of the endomorphisms of the stable Cuntz-Krieger algebras were computed in [9].

**Theorem 6.10.** *If $X$ is a finite dimensional metric space, then there is a bijection $[X, \mathrm{Aut}(\mathcal{O}_n)] \to K_1(C(X) \otimes \mathcal{O}_n)$. The homotopy group $\pi_k(\mathrm{Aut}(\mathcal{O}_n))$ is isomorphic to $\mathbb{Z}/(n-1)\mathbb{Z}$ if $k$ is odd and it vanishes if $k$ is even. In particular $\pi_1(\mathrm{Aut}(\mathcal{O}_n))$ is generated by the class of the canonical action of $\mathbb{T}$ on $\mathcal{O}_n$, $\lambda_z(v_i) = zv_i$.*

*Proof.* Since $\mathcal{O}_n$ satisfies the UCT, we deduce that $\mathrm{End}(\mathcal{O}_n)^* = \mathrm{End}(\mathcal{O}_n)$. An immediate application of Proposition 6.1 shows that the natural map $\mathrm{Aut}(\mathcal{O}_n) \hookrightarrow \mathrm{End}(\mathcal{O}_n)$ induces an isomorphism of groups $[X, \mathrm{Aut}(\mathcal{O}_n)] \cong [X, \mathrm{End}(\mathcal{O}_n)]$. Let $\iota : \mathcal{O}_n \to C(X) \otimes \mathcal{O}_n$ be defined by $\iota(v_i) = 1_{C(X)} \otimes v_i$, $i = 1, ..., n$. The map $\psi \mapsto u(\psi) = \psi(v_1)\iota(v_1)^* + \cdots + \psi(v_n)\iota(v_n)^*$ is known to be a homeomorphism from $\mathrm{Hom}(\mathcal{O}_n, C(X) \otimes \mathcal{O}_n)$ to the unitary group of $C(X) \otimes \mathcal{O}_n$. Its inverse maps a unitary $u$ to the $*$-homomorphism $\psi$ uniquely defined by $\psi(v_i) = uv_i$, $i = 1, ..., n$. Therefore

$$[X, \mathrm{Aut}(\mathcal{O}_n)] \cong [X, \mathrm{End}(\mathcal{O}_n)] \cong \pi_0(U(C(X) \otimes \mathcal{O}_n)) \cong K_1(C(X) \otimes \mathcal{O}_n).$$

The last isomorphism holds since $U(B)/U(B)_0 \cong K_1(B)$ if $B \cong B \otimes \mathcal{O}_\infty$, by [32, Lemma 2.1.7]. One verifies easily that if $\varphi \in \mathrm{Hom}(\mathcal{O}_n, C(X) \otimes \mathcal{O}_n)$, then $u(\widetilde{\psi}\varphi) =$

$\widetilde{\psi}(u(\varphi))u(\psi)$. Therefore the bijection $\chi : [X, \mathrm{End}(D)] \to K_1(C(X) \otimes \mathcal{O}_n)$ is an isomorphism of groups whenever $K_1(\widetilde{\psi}) = \mathrm{id}$ for all $\psi \in \mathrm{Hom}(\mathcal{O}_n, C(X) \otimes \mathcal{O}_n)$. Using the $C(X)$-linearity of $\widetilde{\psi}$ one observes that this happens if the $n-1$ torsion of $K_0(C(X))$ reduces to $\{0\}$, since in that case the map $K_1(C(X)) \to K_1(C(X) \otimes \mathcal{O}_n)$ is surjective by the Künneth formula. □

**Corollary 6.11.** *For any integers $n \geq 3$, $k \geq 1$ there are exactly $(n-1)$ nonisomorphic separable unital $C(S^{2k})$-algebras with all fibers isomorphic to $\mathcal{O}_n$.*

*Proof.* This follows from Theorem 6.9 and Proposition 6.10, since the locally trivial principal $H$-bundles over the sphere $S^m$ are parameterized by $\pi_{m-1}(H)$ if $H$ is a path connected group [20, Cor. 8.4]. We apply this result for $H = \mathrm{Aut}(\mathcal{O}_n)$. □

## 7. Continuous fields and the Universal Coefficient Theorem

Kirchberg has shown that any nuclear separable C*-algebra is KK-equivalent to a Kirchberg algebra [33, Prop. 8.4.5]. This inspired us to extend the result in the context of continuous fields and $KK_{C(X)}$-theory (see Theorem 7.4). The main application of this result is Theorem 1.4 which exhibits a new permanence property of the nuclear C*-algebras satisfying the UCT. We need the following lemma.

**Lemma 7.1** ([24, Lemma 1.2])**.** *Let $X$ be a compact Hausdorff space and let $A$ be a continuous $C(X)$-algebra. There is a split short exact sequence of $C(X)$-algebras*

$$0 \longrightarrow A \longrightarrow A^+ \underset{\alpha}{\rightleftarrows} C(X) \longrightarrow 0 \tag{24}$$

*where $A^+$ is unital, $\alpha$ is $C(X)$-linear and $\alpha(1) = 1$.*

Consider the category of separable $C(X)$-algebras such that the morphisms from $A$ to $B$, are the elements of $KK_{C(X)}(A, B)$ with composition given by the Kasparov product. The isomorphisms in this category are the KK-invertible elements denoted by $KK_{C(X)}(A, B)^{-1}$. Two $C(X)$-algebras are $KK_{C(X)}$-equivalent if they are isomorphic objects in this category. In the sequel we shall use twice the following elementary observation (valid in any category). If composition with $\gamma \in KK_{C(X)}(A, B)$ induces a bijection $KK_{C(X)}(B, C) \to KK_{C(X)}(A, C)$ for $C = A$ and $C = B$, then $\gamma \in KK_{C(X)}(A, B)^{-1}$.

**Lemma 7.2.** *Let $A$ be a separable nuclear continuous $C(X)$-algebra. Then there exist a separable nuclear unital continuous $C(X)$-algebra $A^\flat$ and two $C(X)$-linear $*$-monomorphisms $\alpha : C(X) \otimes \mathcal{O}_2 \to A^\flat$, and $\jmath : A \to A^\flat$ such that $\alpha$ is unital and $[\jmath] \in KK_{C(X)}(A, A^\flat)^{-1}$.*

*Proof.* Let $p \in \mathcal{O}_\infty$ be a non-zero projection with $[p] = 0$ in $K_0(\mathcal{O}_\infty)$. Then there is a unital $*$-homomorphism $\mathcal{O}_2 \to p\mathcal{O}_\infty p$ which induces a $C(X)$-linear unital $*$-monomorphism $\mu : C(X) \otimes \mathcal{O}_2 \to C(X) \otimes p\mathcal{O}_\infty p$. We tensor the exact sequence (7.1) by $p\mathcal{O}_\infty p$ and then take the pullback by $\mu$. This gives a split exact sequence of unital $C(X)$-algebras:

$$\begin{tikzcd} 0 \arrow[r] & A \otimes p\mathcal{O}_\infty p \arrow[r] & A^+ \otimes p\mathcal{O}_\infty p \arrow[r, shift left] & C(X) \otimes p\mathcal{O}_\infty p \arrow[l, shift left, "\alpha"] \arrow[r] & 0 \\ 0 \arrow[r] & A \otimes p\mathcal{O}_\infty p \arrow[u, equal] \arrow[r, "\jmath"] & A^\flat \arrow[u] \arrow[r, shift left] & C(X) \otimes \mathcal{O}_2 \arrow[u, "\mu"] \arrow[l, shift left, "\alpha"] \arrow[r] & 0 \end{tikzcd} \tag{25}$$

The map $A^\flat \to A^+ \otimes p\mathcal{O}_\infty p$ is a unital $C(X)$-linear $*$-monomorphism, so that $A^\flat$ is a continuous $C(X)$-algebra. It is nuclear being an extension of nuclear C*-algebras. It follows by [21], [2, Thm. 5.4] that for any separable $C(X)$-algebra $B$ we have an exact sequence of groups

$$0 \to KK_{C(X)}(C(X)\otimes \mathcal{O}_2, B) \longrightarrow KK_{C(X)}(A^\flat, B) \xrightarrow{j^*} KK_{C(X)}(A\otimes p\mathcal{O}_\infty p, B) \to 0$$

Since the class identity map of $C(X)\otimes\mathcal{O}_2$ vanishes in $KK_{C(X)}$, $KK_{C(X)}(C(X)\otimes \mathcal{O}_2, B) = 0$ for any separable $C(X)$-algebra $B$. Therefore $j^*$ is a $KK_{C(X)}$-equivalence. We conclude the proof by observing that map, $A \to A\otimes p\mathcal{O}_\infty p$, $a \mapsto a\otimes p$, is also a $KK_{C(X)}$-equivalence. ☐

**Proposition 7.3.** *Let $(A_i, \varphi_i)$ be an inductive system of separable nuclear $C(X)$-algebras with injective connecting maps. If $\varphi_i \in KK_{C(X)}(A_i, A_{i+1})^{-1}$ for all $i$, and $\Phi : A_1 \to \varinjlim(A_i,\varphi_i) = A_\infty$ is the induced map, then $\Phi \in KK_{C(X)}(A_1, A_\infty)^{-1}$.*

*Proof.* We use Milnor's $\varprojlim^1$-sequence for $KK_{C(X)}$-theory. Its proof is essentially identical to the proof of the corresponding sequence for regular KK-theory (argue as in [34] using [2]).
(26)

$$0 \longrightarrow \varprojlim{}^1 KK_{C(X)}(A_i, B) \longrightarrow KK_{C(X)}(A_\infty, B) \longrightarrow \varprojlim KK_{C(X)}(A_i, B) \longrightarrow 0$$

Since $\varprojlim^1 \big(G_i, \lambda_i\big) = 0$ and $G_1 \cong \varprojlim \big(G_i, \lambda_i\big)$ for any sequence of abelian groups $(G_i)_{i=1}^\infty$ and group isomorphisms $\lambda_i : G_i \to G_{i+1}$, we obtain from (26) that for any separable $C(X)$ algebra $B$, $\Phi$ induces a bijection $KK_{C(X)}(A_\infty, B) \to \mathrm{KK}_{C(X)}(A, B)$. This implies that $[\Phi] \in KK(A, A_\infty)^{-1}$. ☐

We need the following $C(X)$-equivariant construction which parallels a construction of Kirchberg. We follow the exposition from [33]. A similar deformation technique has appeared earlier in [11].

**Theorem 7.4.** *Let $A$ be a separable nuclear continuous $C(X)$-algebra. Then there exist a separable nuclear continuous unital $C(X)$-algebra $A^\sharp$ whose fibers are Kirchberg C*-algebras and a $C(X)$-linear $*$-monomorphism $\Phi : A \to A^\sharp$ such that $\Phi$ is a $KK_{C(X)}$-equivalence. For any $x \in X$ the map $\Phi_x : A(x) \hookrightarrow A^\sharp(x)$ is a $KK$-equivalence.*

*Proof.* By Proposition 7.2 we may assume that there is a unital $C(X)$-linear $*$-monomorphism $\alpha : C(X)\otimes\mathcal{O}_2 \to A$. By [5, Thm. 2.5], there is a unital $C(X)$-linear $*$-monomorphism $\beta : A \to C(X)\otimes \mathcal{O}_2$. Let $s_1, s_2$ be the images in $A$ under the map $\alpha$ of the canonical generators of $v_1, v_2 \in \mathcal{O}_2 \subset C(X)\otimes\mathcal{O}_2$. Let $\theta = \alpha\circ\beta : A \to A$ and define a $C(X)$-linear unital monomorphism $\varphi : A \to A$ by $\varphi(a) = s_1 a s_1^* + s_2\theta(a)s_2^*$. The unital $*$-homomorphism $\varphi_x : A(x) \to A(x)$, $\varphi_x(a) = s_1(x)a(x)s_1(x)^* + s_2(x)\theta_x(a)s_2(x)^*$, induced by $\varphi$, satisfies $\pi_x\varphi_x = \varphi\pi_x$. Moreover we have a factorization $\theta_x = \alpha_x\circ\beta_x$ :, and hence $\theta_x$ factors through $\mathcal{O}_2$. Let $A^\sharp$ be the inductive limit of the inductive system

$$A \xrightarrow{\varphi} A \xrightarrow{\varphi} A \xrightarrow{\varphi} \cdots$$

and let $\Phi : A \to A^\sharp$ be the induced map. We have a commutative diagram

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "\pi_x"'] & A \arrow[r, "\varphi"] \arrow[d, "\pi_x"] & A \arrow[r, "\varphi"] \arrow[d, "\pi_x"] & \cdots \arrow[r] & A^\sharp \arrow[d, "\pi_x"] \\ A(x) \arrow[r, "\varphi_x"] & A(x) \arrow[r, "\varphi_x"] & A(x) \arrow[r, "\varphi_x"] & \cdots \arrow[r] & A^\sharp(x) \end{tikzcd}$$

By the proof of [33, Prop. 8.4.5] the C*-algebra $A^\sharp(x)$ is a unital Kirchberg algebra for any $x \in X$. It remains to prove that the map $\Phi : A \to A^\sharp$ induces a $KK_{C(X)}$-equivalence. In view of Proposition 7.3, it suffices to verify that $[\varphi] = [id] \in KK_{C(X)}(A, A)^{-1}$. However this follows from the equation $\varphi(a) = s_1 a s_1^* + s_2 \theta(a) s_2^*$, as $[\theta] = 0$, since it factors through $C(X) \otimes \mathcal{O}_2$. □

Let $\mathcal{C}$ denote the class of all unital Kirchberg algebras satisfying the UCT.

**Lemma 7.5.** *Let $X$ be a compact metrizable space and let $A$ be a $C(X)$-algebra such that $\mathrm{cat}_{\mathcal{C}}(A) = n < \infty$. Then $A$ satisfies the UCT.*

*Proof.* We shall prove by induction on $n$ that if $\mathrm{cat}_{\mathcal{C}}(A) \leq n$, then $A$ satisfy the UCT. If $n = 0$, then $A \cong \oplus_i C(Z_i) \otimes D_i$ and all its ideals satisfy the UCT since each $D_i$ is simple and satisfies the UCT. By a result of [34], if two out of three separable nuclear C*-algebras in a short exact sequence satisfy the UCT, then all three of them satisfy the UCT. For the inductive step we use the exact sequence (1), where $B$ is elementary and $\mathrm{cat}_{\mathcal{C}}(D) \leq n-1$. □

**Theorem 7.6.** [12] *Let $A$ be a nuclear separable C*-algebra. Assume that for any finite set $\mathcal{F} \subset A$ and any $\varepsilon > 0$ there is a C*-subalgebra $B$ of $A$ satisfying the UCT and such that $\mathcal{F} \subset_\varepsilon B$. Then $A$ satisfies the UCT.*

*Proof.* For the convenience of the reader, assuming that $B$ is nuclear, we sketch an alternate proof to the one in [12]. It is just this case that is needed in the sequel. By assumption, there is an exhaustive sequence $(A_n)$ consisting of nuclear separable C*-algebras of $A$. We may assume that $A$ is unital and its unit is contained in each $A_n$. After replacing the pair $A_n \subseteq A$ by $A_n \otimes p\mathcal{O}_\infty p \subseteq A \otimes p\mathcal{O}_\infty p$, we observe that the construction $A \mapsto A^\sharp$ is functorial with respect to subalgebras. Thus we obtain an exhaustive sequence of Kirchberg subalgebras $A_n^\sharp$ of $A^\sharp$ such that each $A_n^\sharp$ is KK-equivalent to $A_n$ and hence it satisfies the UCT. Since each $A_n^\sharp$ can be written as an inductive limit of Kirchberg algebras satisfying the UCT and having finitely generated K-theory groups and since the latter algebras are weakly semiprojective, we have exhibited an exhaustive sequence for $A^\sharp$, $(B_n)$, such that $B_n$ are weakly semiprojective C*-algebras satisfying the UCT. By a familiar perturbation argument $A^\sharp$ is isomorphic to the inductive limit of a subsequence $(B_{i_n})$ of $(B_n)$. Therefore $A^\sharp$ and hence $A$ satisfy the UCT. □

*Proof of Theorem 1.4.* By Theorem 7.4 we may assume that the fibers of $A$ are Kirchberg C*-algebras satisfying the UCT. By Theorem 4.5, $A$ admits an exhaustive sequence $(A_k)$ of finite type $C(X)$-algebras. Each $A_k$ satisfies the UCT by Lemma 7.5. We conclude the proof by applying Theorem 7.6.

Let us note that the above proof only requires a weaker version of Theorem 7.4 which asserts that $\Phi : A \to A^\sharp$ and each $\Phi_x$ are KK-equivalences. Its proof requires only the usual $\varprojlim^1$-sequence for $KK$-theory.

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