

Strongly Self-Absorbing C^* -algebras which contain a nontrivial projection

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Abstract. It is shown that a strongly self-absorbing C^* -algebra is of real rank zero and absorbs the Jiang-Su algebra if it contains a non-trivial projection. We also consider cases where the UCT is automatic for strongly self-absorbing C^* -algebras, and K -theoretical ways of characterizing when Kirchberg algebras are strongly self-absorbing.

1. INTRODUCTION

Strongly self-absorbing C^* -algebras were first systematically studied by Toms and Winter in [11]. The classification program of Elliott had prior to that been seen to work out particularly well for those (separable, nuclear) C^* -algebras that tensorially absorb one of the Cuntz algebras \mathcal{O}_2 , \mathcal{O}_∞ , or the Jiang-Su algebra \mathcal{Z} . More precisely, thanks to deep theorems of Kirchberg, the classification of separable, nuclear, stable C^* -algebras that absorb the Cuntz algebra \mathcal{O}_2 is complete (the invariant is the primitive ideal space); and separable, nuclear, stable C^* -algebras that absorb the Cuntz algebra \mathcal{O}_∞ are classified by an ideal related KK -theory. The situation for separable, nuclear C^* -algebras that absorb the Jiang-Su algebra is at present very promising (see for example [13]) but not as complete as in the purely infinite case.

The C^* -algebras \mathcal{O}_2 , \mathcal{O}_∞ and \mathcal{Z} are all examples of strongly self-absorbing C^* -algebras. They are in [11] defined to be those unital separable C^* -algebras $D \neq \mathbb{C}$ for which there is an isomorphism $D \rightarrow D \otimes D$ that is approximately unitarily equivalent to the $*$ -homomorphism $d \mapsto d \otimes 1$. Strongly self-absorbing C^* -algebras are automatically simple and nuclear, and they have at most one tracial state. It is shown in [11] that if D is a strongly self-absorbing C^* -algebra in the UCT class, then it has the same K -theory as one of the C^* -algebras in the following list: \mathcal{Z} , UHF-algebras of infinite type, \mathcal{O}_∞ , \mathcal{O}_∞ tensor a UHF-algebra of infinite type, or \mathcal{O}_2 . It is an open problem if nuclear C^* -algebras always satisfy the UCT (and also if strongly self-absorbing C^* -algebras enjoy this property); and it is an intriguing problem, very much related to the Elliott classification program, if the list above exhausts all strongly self-absorbing

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C^* -algebras. Should the latter be the case, then it would in particular follow that every strongly self-absorbing C^* -algebra absorbs the Jiang-Su algebra \mathcal{Z} . By the Kirchberg-Phillips classification theorem, a strongly self-absorbing Kirchberg algebra belongs to the list above if and only if it belongs to the UCT class. Let us also remind the reader that a strongly self-absorbing C^* -algebra is a Kirchberg algebra if and only if it is not stably finite (or, equivalently, if and only if it is traceless).

In Section 2 of this paper we show that every strongly self-absorbing C^* -algebra which contains a non-trivial projection is of real rank zero and absorbs the Jiang-Su algebra. In Section 3 we consider K -theoretical conditions on strongly self-absorbing Kirchberg algebras. One such condition (phrased at the level of K -homology) characterizes the Kirchberg algebra \mathcal{O}_∞ , and other results in Section 3 give K -theoretical characterizations on when a Kirchberg algebra is strongly self-absorbing.

2. STRONGLY SELF-ABSORBING C^* -ALGEBRAS WITH A NON-TRIVIAL PROJECTION

In this section we show that any strongly self-absorbing C^* -algebra that contains a non-trivial projection is automatically approximately divisible, of real rank zero, and absorbs the Jiang-Su algebra \mathcal{Z} .

Lemma 2.1. *There is a unital $*$ -homomorphism from $M_3 \oplus M_2$ into a unital C^* -algebra A if and only if A contains projections e, e' such that $e \perp e'$, $e \sim e'$, and $1 - e - e' \precsim e$.*

Proof. It is easy to see that such projections e and e' exist in $M_3 \oplus M_2$ and hence in any unital C^* -algebra A that is the target of a unital $*$ -homomorphism from $M_3 \oplus M_2$.

Assume now that such projections e and e' exist. Let $v \in A$ be a partial isometry such that $v^*v = e$ and $vv^* = e'$. Put $f_0 = 1 - e - e'$. Find a subprojection f_1 of e which is equivalent to f_0 , and put $f_2 = vf_1v^*$. Put $g_1 = e - f_1$ and put $g_2 = e' - f_2 = vg_1v^*$. The projections f_0, f_1, f_2, g_1, g_2 then satisfy

$$1 = f_0 + f_1 + f_2 + g_1 + g_2, \quad f_0 \sim f_1 \sim f_2, \quad g_1 \sim g_2.$$

Extending the sets $\{f_0, f_1, f_2\}$ and $\{g_1, g_2\}$ to sets of matrix units for M_3 and M_2 , respectively, yields a unital $*$ -homomorphism from $M_3 \oplus M_2$ into A . (If the f_j 's are zero or if the g_j 's are zero, then this $*$ -homomorphism will fail to be injective, and will instead give a unital embedding of M_2 or M_3 into A .) \square

If D is any unital (nuclear¹) C^* -algebra then we let $D^{\otimes n}$ denote the n -fold tensor product $D \otimes D \otimes \cdots \otimes D$ (with n tensor factors), and we let $D^{\otimes \infty}$

¹We shall here exclusively be concerned with nuclear C^* -algebras, where the tensor product is unique; otherwise we must specify a tensor product, for example the minimal one.

denote the infinite tensor product $\bigotimes_{n=1}^{\infty} D$. The latter is the inductive limit of the sequence

$$D \rightarrow D^{\otimes 2} \rightarrow D^{\otimes 3} \rightarrow D^{\otimes 4} \rightarrow \dots,$$

(with connecting mappings $d \mapsto d \otimes 1_D$). We shall view D as a (unital) sub- C^* -algebra of $D^{\otimes n}$, $D^{\otimes n}$ as a sub- C^* -algebra of $D^{\otimes m}$ (if $n \leq m$), and finally D and $D^{\otimes n}$ are viewed as subalgebras of $D^{\otimes \infty}$.

If $x \in D^{\otimes n}$, then $x^{\otimes k}$ will denote the k -fold tensor product

$$x^{\otimes k} = x \otimes x \otimes x \otimes \dots \otimes x \in D^{\otimes kn}.$$

The proof of the lemma below resembles the proof of [9, Lemma 6.4].

Lemma 2.2. *Let D be a strongly self-absorbing C^* -algebra, and let p be a projection in D . Consider the following projections in $D \otimes D$,*

$$e_1 = p \otimes (1 - p), \quad e'_1 = (1 - p) \otimes p, \quad f = p \otimes p + (1 - p) \otimes (1 - p).$$

For each natural number n consider also the following projections in $D^{\otimes 2(n+1)}$,

$$e_{n+1} = f^{\otimes n} \otimes p \otimes (1 - p), \quad e'_{n+1} = f^{\otimes n} \otimes (1 - p) \otimes p.$$

It follows that the projections $e_1, e_2, \dots, e'_1, e'_2, \dots$ are pairwise orthogonal in $D^{\otimes \infty}$, and that $e_j \sim e'_j$. Moreover, for each natural number n , set

$$E_n = e_1 + e_2 + \dots + e_n, \quad E'_n = e'_1 + e'_2 + \dots + e'_n.$$

Then $E_n \perp E'_n$, $E_n \sim E'_n$, and

$$(1) \quad 1 - E_n - E'_n = f^{\otimes n}.$$

Proof. The equivalence $e_j \sim e'_j$ comes from the fact that the flip automorphism $a \otimes b \mapsto b \otimes a$ on $D \otimes D$ is approximately inner when D is strongly self-absorbing. The projections $e_1, e_2, \dots, e'_1, e'_2, \dots$ are pairwise orthogonal by construction. The only thing left to prove is (1). We prove this by induction after n , and note first that (1) for $n = 1$ follows from the fact that $e_1 + e'_1 + f = 1$. Suppose that (1) holds for some $n \geq 1$. Then

$$\begin{aligned} & 1 - E_{n+1} - E'_{n+1} \\ &= 1 - E_n - E'_n - e_{n+1} - e'_{n+1} \\ &= f^{\otimes n} \otimes 1_D \otimes 1_D - f^{\otimes n} \otimes p \otimes (1 - p) - f^{\otimes n} \otimes (1 - p) \otimes p \\ &= f^{\otimes (n+1)}. \end{aligned}$$

□

Lemma 2.3. *Let D be a strongly self-absorbing C^* -algebra and let p be a projection in D such that $p \neq 1$. Then there exists a natural number n such that $p^{\otimes n} \lesssim 1 - p^{\otimes n}$ in $D^{\otimes n}$.*

Proof. To simplify the notation we express our calculations in terms of the monoid $V(D)$ of Murray-von Neumann equivalence classes of projections in D and in matrix algebras over D . Let $[e] \in V(D)$ denote the equivalence class containing the projection e in (a matrix algebra over) D .

Since D is simple and $p \neq 1$ there is a natural number n such that $n[1-p] \geq [p]$. It follows that

$$\begin{aligned} & [1 - p^{\otimes n}] \\ & \geq [(1-p) \otimes p \otimes \cdots \otimes p] + [p \otimes (1-p) \otimes \cdots \otimes p] + [p \otimes p \otimes \cdots \otimes (1-p)] \\ & = n[(1-p) \otimes p \otimes \cdots \otimes p] \\ & \geq [p \otimes p \otimes p \otimes \cdots \otimes p] = [p^{\otimes n}], \end{aligned}$$

where the equality between the second and third expression holds because the flip on a strongly self-absorbing C^* -algebra is approximately inner. \square

Lemma 2.4. *Let D be a strongly self-absorbing C^* -algebra, let p be a projection in $D^{\otimes k}$, and let e be a projection in $D^{\otimes l}$ for some natural numbers k and l . Assume that $p \neq 1$ and that $e \neq 0$. It follows that there exists a natural number n such that $p^{\otimes n} \precsim e$ in $D^{\otimes \infty}$.*

Proof. Let d be a natural number such that $dk \geq l$. Upon replacing p with $p^{\otimes d}$, e with $e \otimes 1_D^{\otimes (dk-l)}$, and D with $D^{\otimes dk}$ we can assume that p and e both belong to D . Use Lemma 2.3 to find m such that $p^{\otimes m} \precsim 1-p^{\otimes m}$. By replacing p with $p^{\otimes m}$, e with $e \otimes 1_D^{\otimes (m-1)}$, and D with $D^{\otimes m}$ we can assume that p and e both belong to D and that $p \precsim 1-p$.

Now, $p \sim q \leq 1-p$ for some projection q in D . In the language of the monoid $V(D)$ we have

$$[1_D^{\otimes k}] \geq [(p+q)^{\otimes k}] = 2^k [p^{\otimes k}]$$

for any natural number k . Using simplicity of D we can choose n such that $2^{n-1}[e] \geq [p]$. Then

$$[e] = [e \otimes 1_D^{\otimes (n-1)}] \geq 2^{n-1}[e \otimes p^{\otimes (n-1)}] \geq [p^{\otimes n}],$$

in $V(D^{\otimes n})$ as desired, where we in the first identity have used that the embedding of D into $D^{\otimes n}$ maps e onto $e \otimes 1_D^{\otimes (n-1)}$. \square

Theorem 2.5. *Let D be a strongly self-absorbing C^* -algebra. Then the following three conditions are equivalent:*

- (1) D contains a non-trivial projection (i.e., a projection other than 0 and 1).
- (2) D is approximately divisible.
- (3) D is of real rank zero.

If any of the three equivalent conditions are satisfied, then D absorbs the Jiang-Su algebra, i.e., $D \cong D \otimes \mathcal{Z}$.

Proof. (i) \Rightarrow (ii). If D is strongly self-absorbing, then there is an asymptotically central sequence of embeddings of D into itself, i.e., a sequence $\rho_k: D \rightarrow D$ of unital $*$ -homomorphisms such that $\|\rho_k(x)y - y\rho_k(x)\| \rightarrow 0$ as $k \rightarrow \infty$ for all $x, y \in D$.

Identify D with $D_0^{\otimes \infty}$ where $D_0 \cong D$. Take a non-trivial projection p in D_0 . For each natural number n let $E_n, E'_n \in D_0^{\otimes 2n}$ be as in Lemma 2.2

(corresponding to our non-trivial projection p). Then $e_n \neq 0$, $E_n \neq 0$, and so $0 \neq f^{\otimes n} \neq 1$. Use (1) and Lemma 2.4 to find n such that $1 - E_n - E'_n \precsim p \otimes (1 - p) \leq E_n$. It then follows from Lemma 2.1 that there is an injective unital $*$ -homomorphism from $M_3 \otimes M_2$ into $D_0^{\otimes 2n} \subseteq D$. Composing this unital $*$ -homomorphism with the unital $*$ -homomorphism ρ_k yields an asymptotically central sequence of unital $*$ -homomorphisms from $M_3 \otimes M_2$ into D . This shows that D is approximately divisible.

(ii) \Rightarrow (iii). It is shown in [2] that a simple approximately divisible C^* -algebra is of real rank zero if and only if projections in the C^* -algebra separate the quasitraces. As quasitraces on an exact C^* -algebra are traces, [7], a result that applies to our case since strongly self-absorbing C^* -algebras are nuclear and hence exact, and since a strongly self-absorbing C^* -algebra has at most one tracial state, quasitraces are automatically separated by just one projection, say the unit.

(iii) \Rightarrow (i). This is trivial. The only C^* -algebra of real rank zero that does not have a non-trivial projection is \mathbb{C} , the algebra of complex numbers. This C^* -algebra is not strongly self-absorbing by convention.

Finally, any simple approximately divisible C^* -algebra is \mathcal{Z} -absorbing, cf. [12]. \square

Lemma 2.6. *Let D be a strongly self-absorbing C^* -algebra. Then $K_0(D)$ has a natural structure of a commutative unital ring with unit $[1_D]$. If τ is a unital trace on D , then τ induces a morphism of unital rings $\tau_*: K_0(D) \rightarrow \mathbb{R}$.*

Proof. Fix an isomorphism $\gamma: D \otimes D \rightarrow D$. The multiplication on $K_0(D)$ is defined by composing $\gamma_*: K_0(D \otimes D) \rightarrow K_0(D)$ with the canonical map $K_0(D) \otimes K_0(D) \rightarrow K_0(D \otimes D)$. Since any two unital $*$ -homomorphisms from $D \otimes D$ to D are approximately unitarily equivalent, the above multiplication is well-defined and commutative. We leave the rest of proof for the reader, but note that if D has a unital trace, then $\tau \otimes \tau$ is the unique unital trace of $D \otimes D$. \square

Proposition 2.7. *Let D be a strongly self-absorbing C^* -algebra. Suppose that D is quasidiagonal and that $K_0(D)$ is torsion free. Then either $K_0(D) \cong \mathbb{Z}$ or there is a UHF algebra B of infinite type such that $K_0(D) \cong K_0(B)$. If, in addition, D is assumed to contain a nontrivial projection, then $D \otimes B \cong D$, where B is as above.*

Proof. Since D is quasidiagonal it embeds unitaly in the universal UHF algebra $B_{\mathbb{Q}}$ and $D \otimes B_{\mathbb{Q}} \cong B_{\mathbb{Q}}$, as explained in [5, Rem. 3.10]. The restriction of the unital trace of $B_{\mathbb{Q}}$ to D is denoted by τ . Thus we have an exact sequence

$$0 \longrightarrow H \longrightarrow K_0(D) \xrightarrow{\tau_*} \tau_* K_0(D) \longrightarrow 0$$

where H is the kernel of τ_* . Since $\mathbb{Z} \subseteq \tau_* K_0(D) \subseteq \mathbb{Q}$, and $K_0(D) \otimes \mathbb{Q} \cong \mathbb{Q}$, the map $\tau_* \otimes \text{id}_{\mathbb{Q}}: K_0(D) \otimes \mathbb{Q} \rightarrow \tau_* K_0(D) \otimes \mathbb{Q}$ is an isomorphism. Therefore $H \otimes \mathbb{Q} = 0$ and so H is a torsion subgroup of $K_0(D)$. But we assumed that

$K_0(D)$ is torsion free and hence $H = \{0\}$ and $\tau_*: K_0(D) \rightarrow \tau_*K_0(D) \subseteq \mathbb{Q}$ is an isomorphism of unital rings. The unital subrings of \mathbb{Q} are easily determined and well-known. They are parametrized by arbitrary sets P of prime numbers. For each P the corresponding ring R_P consists of rational numbers r/s with r and s relatively prime and such that all prime factors of s are in P . If $P = \emptyset$ then $R_P = \mathbb{Z}$, otherwise R_P is isomorphic to the K_0 -ring associated to a UHF algebra B of infinite type.

Suppose now that D contains a nontrivial projection. By Theorem 2.5, D has real rank zero and absorbs the Jiang-Su algebra \mathcal{Z} . In particular, $K_0(D)$ is not \mathbb{Z} and is hence isomorphic (as a scaled abelian group) to $K_0(B)$ for some UHF-algebra B of infinite type. It follows from [8] that D has stable rank one and that $K_0(D)$ is weakly unperforated. Moreover, by [1, Sect. 6.9], $K_0(D)$ has the strict order induced by τ_* . The isomorphism $K_0(B) \cong K_0(D)$ of scaled abelian groups is therefore an order isomorphism, and by the properties of D established above we can conclude that B embeds unittally into D , whence $D \otimes B \cong D$. \square

Corollary 2.8. *Let D be a strongly self-absorbing C^* -algebra with torsion free K_0 -group. Suppose that D contains a non-trivial projection and that D embeds unittally into the UHF algebra M_{p^∞} for some prime number p . Then $D \cong M_{p^\infty}$.*

Proof. By Proposition 2.7 there is a prime q such that M_{q^∞} is contained unittally in D and hence in M_{p^∞} . From this we deduce that $q = p$. Finally since $M_{p^\infty} \subseteq D \subseteq M_{p^\infty}$ we conclude that $D \cong M_{p^\infty}$. \square

3. STRONGLY SELF-ABSORBING ALGEBRAS AND K-THEORY

The class of strongly self-absorbing Kirchberg algebras satisfying the UCT was completely described in [11]. In this section we give properties and characterizations of strongly self-absorbing Kirchberg algebras which can be derived without assuming the UCT. For a unital C^* -algebra D we denote by ν_D the unital $*$ -homomorphism $\mathbb{C} \rightarrow D$. When the C^* -algebra D is clear from context we will write ν instead of ν_D .

Proposition 3.1. *Let D be a strongly self-absorbing C^* -algebra. If D is not finite and the unital $*$ -homomorphism $\mathbb{C} \rightarrow D$ induces a surjection $K^0(D) \rightarrow K^0(\mathbb{C})$, then $D \cong \mathcal{O}_\infty$.*

Proof. By [11, Prop. 5.12], two strongly self-absorbing C^* -algebras are isomorphic if and only if they embed unittally into each other. Thus it suffices to show the existence of unital $*$ -homomorphisms $\mathcal{O}_\infty \rightarrow D$ and $D \rightarrow \mathcal{O}_\infty$. Since D is not finite, it must be a Kirchberg algebra, see [11, Sec. 1], and hence \mathcal{O}_∞ embeds unittally in D by [10, Prop. 4.2.3]. It remains to show that D embeds unittally in \mathcal{O}_∞ .

By assumption, the map $\nu^*: KK(D, \mathbb{C}) \rightarrow KK(\mathbb{C}, \mathbb{C})$ is surjective. By multiplying with the KK -equivalence class given by the unital morphism

$\mathbb{C} \rightarrow \mathcal{O}_\infty$, we obtain that the map $\nu^*: KK(D, \mathcal{O}_\infty) \rightarrow KK(\mathbb{C}, \mathcal{O}_\infty)$ is surjective. If $\varphi: D \rightarrow \mathcal{O}_\infty \otimes \mathcal{K}$ is a $*$ -homomorphism, then, after identifying $KK(\mathbb{C}, \mathcal{O}_\infty) \cong K_0(\mathcal{O}_\infty)$, the map ν^* sends $[\varphi]$ to the class $[\varphi(1_D)] \in K_0(\mathcal{O}_\infty)$. By [10, Thm. 8.3.3] each element of $KK(D, \mathcal{O}_\infty)$ is represented by a $*$ -homomorphism. Therefore, by the surjectivity of ν^* , there is a $*$ -homomorphism $\varphi: D \rightarrow \mathcal{O}_\infty \otimes \mathcal{K}$ such that $[\varphi(1_D)] = [1_{\mathcal{O}_\infty}]$. Since these are both full projections, by [10, Prop. 4.1.4] there is a partial isometry $v \in \mathcal{O}_\infty \otimes \mathcal{K}$ such that $v^*v = \varphi(1_D)$ and $vv^* = 1_{\mathcal{O}_\infty}$. Then $v\varphi v^*$ is a unital embedding $D \rightarrow \mathcal{O}_\infty$. \square

Remark 3.2. Note that the isomorphism $D \cong \mathcal{O}_\infty$ was obtained without assuming that D satisfies the UCT. Let us argue that assumptions of Proposition 3.1 are natural. Let A and B be unital C^* -algebras and let $\nu: \mathbb{C} \rightarrow A$ and $\nu: \mathbb{C} \rightarrow B$ be the corresponding unital $*$ -homomorphisms. The condition that there is a morphism of pointed groups $(K_0(A), [1_A]) \rightarrow (K_0(B), [1_B])$ can be viewed as the condition that the diagram

$$\begin{array}{ccc} A & & B \\ & \swarrow \nu & \searrow \nu \\ & \mathbb{C} & \end{array}$$

can be completed to a commutative diagram after passing to K -theory:

$$\begin{array}{ccc} K_0(A) & \dashrightarrow & K_0(B) \\ & \swarrow \nu_* & \searrow \nu_* \\ & K_0(\mathbb{C}) & \end{array}$$

It would then be completely natural to use K -homology instead of K -theory and ask that the first diagram can be completed to a commutative diagram after passing to K -homology.

$$\begin{array}{ccc} K^0(A) & \dashleftarrow & K^0(B) \\ & \swarrow \nu^* & \searrow \nu^* \\ & K^0(\mathbb{C}) & \end{array}$$

Now let us observe that the condition, imposed in Proposition 3.1, that $\nu^*: K^0(D) \rightarrow K^0(\mathbb{C})$ is surjective clearly is equivalent to the existence of a commutative diagram

$$\begin{array}{ccc} K^0(\mathcal{O}_\infty) & \xleftarrow{\alpha} & K^0(D) \\ & \swarrow \nu^* & \searrow \nu^* \\ & K^0(\mathbb{C}) & \end{array}$$

where α is a surjective morphism.

If D satisfies the UCT, then the condition above can be translated in terms of K -theory as follows. Since the commutative diagram

$$\begin{array}{ccc} K^0(D) & \longrightarrow & \text{Hom}(K_0(D), \mathbb{Z}) \\ \nu^* \downarrow & & \downarrow \\ K^0(\mathbb{C}) & \longrightarrow & \text{Hom}(K_0(\mathbb{C}), \mathbb{Z}) \end{array}$$

has surjective horizontal arrows, the assumption on K -homology in Proposition 3.1 is equivalent for the existence a group homomorphism $K_0(D) \rightarrow \mathbb{Z}$ which maps $[1_D]$ to 1. This is obviously equivalent to the condition that $[1_D]$ is an infinite order element of $K_0(D)$ and that the subgroup that it generates, $\mathbb{Z}[1_D]$, is a direct summand of $K_0(D)$.

Our next goal is to show that for a unital Kirchberg algebra the property of being strongly self-absorbing is purely a KK -theoretical condition. Let

$$C_\nu = \{f: [0, 1] \rightarrow D \mid f(0) \in \mathbb{C}1_D, \quad f(1) = 0\}$$

be the mapping cone of the unital $*$ -homomorphism $\nu: \mathbb{C} \rightarrow D$.

Proposition 3.3. *Let D be a unital Kirchberg algebra. Then D is strongly self-absorbing if and only if $C_\nu \otimes D$ is KK -equivalent to zero.*

Proof. We begin with a general observation. For a $*$ -homomorphism $\varphi: A \rightarrow B$ of separable C^* -algebras and any separable C^* -algebra C , there is an exact Puppe sequence in KK -theory ([1, Thm. 19.4.3]):

$$\begin{array}{ccccc} KK(B, C) & \xrightarrow{\varphi^*} & KK(A, C) & \longrightarrow & KK(C_\varphi, C) \\ \uparrow & & & & \downarrow \\ KK_1(C_\varphi, C) & \longleftarrow & KK_1(A, C) & \xleftarrow{\varphi_*} & KK_1(B, C) \end{array}$$

It is apparent that $[\varphi] \in KK(A, B)^{-1}$ if and only if composition with $[\varphi] \in KK(A, B)$ induces a bijection $\varphi^*: KK(B, C) \rightarrow KK(A, C)$ for any separable C^* -algebra C , or equivalently, for just $C = A$ and $C = B$. Therefore, by the exactness of the Puppe sequence, we see that that φ induces a KK -equivalence if and only if its mapping cone C^* -algebra C_φ is KK -contractible.

By applying this observation to the unital $*$ -homomorphism $\nu \otimes \text{id}_D: D \rightarrow D \otimes D$ we deduce that $\nu \otimes \text{id}_D$ induces a KK -equivalence if and only if its mapping cone $C_{\nu \otimes \text{id}_D} \cong C_\nu \otimes D$ is KK -contractible. Suppose now that D is a strongly self-absorbing Kirchberg algebra. Then $\nu \otimes \text{id}_D$ is asymptotically unitarily equivalent to an isomorphism by [5, Thm. 2.2] and hence $\nu \otimes \text{id}_D$ induces a KK -equivalence. Conversely, if $\nu \otimes \text{id}_D$ induces a KK -equivalence, then $\nu \otimes \text{id}_D$ is asymptotically unitarily equivalent to an isomorphism $D \rightarrow D \otimes D$ by [10, Thm. 8.3.3] and hence D is strongly self-absorbing. \square

We have the following result related to Proposition 3.3.

Proposition 3.4. *Let D be a unital Kirchberg algebra such that $D \cong D \otimes D$. The following assertions are equivalent:*

- (1) D is strongly self-absorbing.
- (2) $KK(C_\nu, SD) = 0$.
- (3) $KK(C_\nu, D \otimes A) = 0$ for all separable C^* -algebras A .
- (4) The map $KK(D, D \otimes A) \rightarrow KK(\mathbb{C}, D \otimes A)$ is bijective for all separable C^* -algebras A .

Proof. (iii) \Leftrightarrow (iv). This equivalence is verified by using the Puppe sequence associated to $\nu: \mathbb{C} \rightarrow D$, arguing as in the proof of Proposition 3.3.

(i) \Rightarrow (iv). This implication is proved in [5, Thm. 3.4].

(iii) \Rightarrow (ii). This follows by taking $A = S\mathbb{C}$ in (iii).

(ii) \Rightarrow (i). Fix an isomorphism $\gamma: D \rightarrow D \otimes D$. Since $KK_1(C_\nu, D \otimes D) = 0$ by hypothesis, it follows from the Puppe sequence that the map $\nu^*: KK(D, D \otimes D) \rightarrow KK(\mathbb{C}, D \otimes D)$ is injective. Therefore γ and $\nu \otimes \text{id}_D$ induce the same class in $KK(D, D \otimes D)$ since they are both unital. It follows that $\nu \otimes \text{id}_D$ is asymptotically unitarily equivalent to γ and so D is strongly self-absorbing. \square

Corollary 3.5. *Let D be a unital Kirchberg algebra such that $D \cong D \otimes D$. Then D is strongly self-absorbing if and only if $\pi_2 \text{Aut}(D) = 0$.*

Proof. Since $\pi_2 \text{Aut}(D) \cong KK(C_\nu, SD)$ by [5, Cor. 3.1], the conclusion follows from Proposition 3.4. \square

It was shown in [4, Prop. 4.1] that if a unital Kirchberg algebra satisfies the UCT, then D is strongly self-absorbing if and only if the homotopy classes $[X, \text{Aut}(D)]$ reduces to a singleton for any path connected compact metrizable space X .

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