CONSTRUCTIONS PRESERVING HILBERT SPACE UNIFORM EMBEDDABILITY OF DISCRETE GROUPS

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ABSTRACT. Uniform embeddability (in a Hilbert space), introduced by Gromov, is a geometric property of metric spaces. As applied to countable discrete groups, it has important consequences to the Novikov conjecture. Exactness, introduced and studied extensively by Kirchberg-Wassermann, is a functional analytic property of locally compact groups. Recently it has become apparent that as properties of countable discrete groups uniform embeddability and exactness are closely related. We further develop the parallel between these classes by proving that the class of uniformly embeddable groups shares a number of permanence properties with the class of exact groups. In particular, we prove that it is closed under direct and free products (with and without amalgam), inductive limits and certain extensions.

1. INTRODUCTION

Gromov introduced the notion of uniform embeddability of metric spaces and suggested that finitely generated discrete groups that are uniformly embeddable in a Hilbert space, when viewed as metric spaces, might satisfy the Novikov Conjecture [12, 10]. Yu proved that this is indeed the case [20, 18].

Kirchberg-Wassermann defined the notion of exactness of a locally compact group in terms of the behavior of its reduced crossed product functor. Subsequently, they developed the main properties of exact groups. In particular they showed that in the case of a countable discrete group, exactness can be reformulated entirely in terms of the reduced C^*-algebra [14], that is, that exactness is a property of the harmonic analysis of the left regular representation of such a group.

The starting point of this work is the startling fact that for countable discrete groups, uniform embeddability (in a Hilbert space, a geometric property) and exactness (an analytic property) are closely related. The first indications of the relationship between uniform

embeddability and exactness are found in the work of Guentner-Kaminker [13]; these preliminary steps were quickly expanded by Ozawa [16], Anantharaman-Delaroche [2] and others.

We are concerned with uniformly embeddable groups. Our main results are outlined in the following theorem (more precise statements follow in later sections) which summarizes the basic permanence properties of the class of uniformly embeddable groups. Observe that the properties described are all shared by the class of countable discrete exact groups [15]. Indeed, in each case it is possible to give a unified account of the results for uniform embeddability and exactness; in some cases our methods provide alternate proofs of the known results concerning exactness.

Theorem. *The class of countable discrete groups that are uniformly embeddable in a Hilbert space is closed under subgroups and products, direct limits, free products with amalgam, and extensions by exact groups.* □

The fact that subgroups and products of uniformly embeddable groups are again uniformly embeddable is elementary and quite well-known; they are included in the statement for completeness.

The other properties are more difficult to establish. It is possible to construct a uniform embedding of a free product (without amalgam) directly from uniform embeddings of the factors [5]. On the other hand, the corresponding result for free products with amalgam is considerably more difficult in view of the fact that the common subgroup of the amalgam can introduce considerable distortion into the product. Our proof is based on a suitable adaptation of an argument given by Tu in his work on Property A [19]; although we are not able to verify a number of assertions concerning the metric defined in Section 9 in Tu's paper we are able to adapt his arguments to the present context. The proof we give works equally well for countable exact groups, and is unrelated to Dykema's original proof that the class of countable exact groups is closed under free products with amalgam [9, 8]. Again, in the case without amalgam a considerably simpler proof of this fact is now available [5].

The general problem of uniform embeddability of extensions is intriguing. Our proof that the class of uniformly embeddable groups is closed under extensions by exact groups is inspired by the argument of Anantharaman-Delaroche-Renault showing that the class of countable exact groups is closed under extensions [3]. It is unknown whether the class of uniformly embeddable groups is closed under general extensions; even the case of a central extension of $\mathbb{Z}$ by a uniformly embeddable group remains open. At present, the behavior with respect to extensions provides the best possibility of distinguishing the classes of uniformly embeddable and exact groups.

We draw two immediate corollaries. Since they are peripheral to our study we will not establish notation or provide the relevant definitions; rather, we provide references.

Corollary. *The class of countable discrete groups that are uniformly embeddable in a Hilbert space is closed under the formation of HNN extensions.*

Proof. An HNN extension is built from free products with amalgam, direct limits, and a semi-direct product by $\mathbb{Z}$, which is exact [17, 4]. (See [6] for related results.) $\qquad\square$

Corollary. *The fundamental group of a graph of countable discrete groups is uniformly embeddable in a Hilbert space if and only if each of the groups is uniformly embeddable in a Hilbert space.*

Proof. Each constituent group is a subgroup of the fundamental group. Conversely, the fundamental group of a graph of groups is built from free products with amalgam, HNN extensions and direct limits [17, 4]. (See [1] for related remarks.) $\qquad\square$

2. BACKGROUND

Let X and Y be metric spaces, with metrics d_X and d_Y, respectively. A function $F : X \to Y$ is a *uniform embedding* if there exist non-decreasing functions $\rho_\pm : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{t \to \infty} \rho_\pm(t) = \infty$ and such that

$$\rho_-(d_X(x, x')) \leq d_Y(F(x), F(x')) \leq \rho_+(d_X(x, x')), \quad \text{for all } x, x' \in X. \tag{1}$$

The space X is *uniformly embeddable* if there exists a uniform embedding F of X into a Hilbert space $\mathcal{H}$. Uniform embeddability in a real Hilbert space is equivalent to uniform embeddability in a complex Hilbert space; henceforth we shall deal only with real Hilbert spaces. Obviously, if X is countable we may assume that the Hilbert space is separable.

A metric space X is *locally finite* if for every $x \in X$ and $R > 0$ the metric ball with center x and radius R is finite. In this case the metric is called *proper*. A locally finite metric space is discrete (as a topological space in the metric topology). In the case of locally finite metric spaces there are a number of equivalent formulations of uniform embeddability [7, 13]; to these we add the following simple extension, which applies to *any* metric space and which will be our fundamental criterion for uniform embeddability.

Proposition 2.1. *Let X be a metric space. Then X is uniformly embeddable if and only if for every $R > 0$ and $\varepsilon > 0$ there exists a Hilbert space valued map $\xi : X \to \mathcal{H}$, $(\xi_x)_{x \in X}$, such that $\|\xi_x\| = 1$, for all $x \in X$ and such that*

(i) $\sup\{\|\xi_x - \xi_{x'}\| : d(x, x') \leq R, \ x, x' \in X\} \leq \varepsilon$

(ii) $\lim_{S\to\infty} \sup\{|\langle \xi_x, \xi_{x'}\rangle| : d(x, x') \geq S, \ x, x' \in X\} = 0.$

These conditions may be replaced by

(iii) $\sup\{|1 - \langle \xi_x, \xi_{x'}\rangle| : d(x, x') \leq R, \ x, x' \in X\} \leq \varepsilon$

(iv) $\lim_{S\to\infty} \inf\{\|\xi_x - \xi_{x'}\| : d(x, x') \geq S, \ x, x' \in X\} = 2,$

respectively.

Remark. We refer to (i) and (iii) collectively as the *convergence condition*; similarly, we refer to (ii) and (iv) collectively as the *support condition.*

Proof. The interchangeability of (i) $\Leftrightarrow$ (iii) and (ii) $\Leftrightarrow$ (iv) follows from the simple observation that for unit vectors $\xi, \eta \in \mathcal{H}$ we have $\|\xi - \eta\|^2 = 2 - 2\langle \xi, \eta\rangle$.

Assume that X is uniformly embeddable and let $F : X \to \mathcal{H}$ be a uniform embedding in a real Hilbert space $\mathcal{H}$. Let

$$\mathrm{Exp}(\mathcal{H}) = \mathbb{R} \oplus \mathcal{H} \oplus (\mathcal{H} \otimes \mathcal{H}) \oplus (\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}) \oplus \cdots$$

and define $\mathrm{Exp} : \mathcal{H} \to \mathrm{Exp}(\mathcal{H})$ by

$$\mathrm{Exp}(\zeta) = 1 \oplus \zeta \oplus \left(\frac{1}{\sqrt{2!}} \zeta \otimes \zeta\right) \oplus \left(\frac{1}{\sqrt{3!}} \zeta \otimes \zeta \otimes \zeta\right) \oplus \cdots.$$

Note that $\langle \mathrm{Exp}(\zeta), \mathrm{Exp}(\zeta')\rangle = e^{\langle \zeta, \zeta'\rangle}$, for all $\zeta, \zeta' \in \mathcal{H}$. For $t > 0$ define

$$\xi_x = e^{-t\|F(x)\|^2} \ \mathrm{Exp}(\sqrt{2t}\, F(x)).$$

It is easily verified that $\langle \xi_x, \xi_{x'}\rangle = e^{-t\|F(x)-F(x')\|^2}$. Consequently, for all $x, x' \in X$ we have $\|\xi_x\| = 1$, and

$$e^{-t\rho_+(d(x,x'))^2} \leq \langle \xi_x, \xi_{x'}\rangle \leq e^{-t\rho_-(d(x,x'))^2}. \tag{2}$$

Letting $t = \varepsilon(1 + \rho_+(R)^2)^{-1}$ it is easy to verify the conditions (iii) and (ii) above.

Conversely, assume that X satisfies the conditions in the second part of the statement. There exists a sequence of maps $\eta_n : X \to \mathcal{H}_n$ and a sequence of numbers $S_0 = 0 < S_1 < S_2 < \dots$ increasing to infinity such that for every $n \geq 1$ and every $x, x' \in X$

(i) $\|\eta_n(x)\| = 1$,

(ii) $\|\eta_n(x) - \eta_n(x')\| \leq 1/n$, provided $d(x, x') \leq \sqrt{n}$

(iii) $\|\eta_n(x) - \eta_n(x')\| \geq 1$, provided $d(x, x') \geq S_n$.

Choose a base point $x_0 \in X$ and define $F : X \to \oplus_{n=1}^\infty \mathcal{H}_n$ by

$$F(x) = \frac{1}{2}\left(\eta_1(x) - \eta_1(x_0) \oplus \eta_2(x) - \eta_2(x_0) \oplus \cdots\right).$$

It is not hard to verify that F is well defined and

$$\rho_-(d(x,x')) \leq \|F(x) - F(x')\| \leq d(x,x') + 1, \quad \text{for all } x, x' \in X,$$

where $\rho_- = \frac{1}{2} \sum_{n=1}^{\infty} \sqrt{n-1} \, \chi_{[S_{n-1},S_n)}$, and the $\chi_{[S_{n-1},S_n)}$ are the characteristic functions of the sets $[S_{n-1}, S_n)$.

Indeed, let $x, x' \in X$. If n is such that $\sqrt{n-1} \leq d(x,x') < \sqrt{n}$ we have

$$
\begin{aligned}
\|F(x) - F(x')\|^2 &= \frac{1}{4} \sum_{i \leq n-1} \|\eta_i(x) - \eta_i(x')\|^2 + \frac{1}{4} \sum_{i \geq n} \|\eta_i(x) - \eta_i(x')\|^2 \\
&\leq (n-1) + \frac{1}{4} \sum_{i \geq n} \frac{1}{i^2} \leq d(x,x')^2 + 1.
\end{aligned}
$$

Similarly, if n is such that $S_{n-1} \leq d(x,x') < S_n$ we have

$$\|F(x) - F(x')\|^2 \geq \frac{1}{4} \sum_{i \leq n-1} \|\eta_i(x) - \eta_i(x')\|^2 \geq \frac{n-1}{4} = \rho_-(d(x,x'))^2.$$

$\square$

Remark. Straightforward modifications of the above argument produce a uniform embedding F satisfying the sharper estimate $\|F(x) - F(x')\| \leq d(x,x') + \delta$, for an arbitrarily chosen $\delta > 0$; simply replace the $1/n$ in (ii) by $\delta/2^n$.

Two metrics d and d' on the set X are *coarsely equivalent* if for every $R > 0$ there exists an $S > 0$ such that the d–metric ball with center x and radius R is contained in the d'–metric ball with center x and radius S; and conversely. Equivalently, two metrics on X are coarsely equivalent if the identity map $X \to X$ is a uniform embedding.

Proposition 2.2. *Let d and d' be coarsely equivalent metrics on X. Then X is uniformly embeddable with respect to d if and only if it is uniformly embeddable with respect to d'.*

Proof. If two maps are uniform embeddings, so is their composition. $\square$

Remark. Below we require only the fact that if $Y \to \mathcal{H}$ and $X \to Y$ are uniform embeddings then the composite $X \to \mathcal{H}$ is a uniform embedding. In other words, we do not need to know that X and Y are coarsely equivalent to conclude the uniform embeddability of X from that of Y.

Let Γ be a countable discrete group. A *length function* on Γ is a non-negative, real-valued function l satisfying, for all a and $b \in \Gamma$,

(i) $l(ab) \leq l(a) + l(b)$,

(ii) $l(a^{-1}) = l(a)$,

(iii) $l(a) = 0$ if and only if $a = 1$,

A length function l is *proper* if for all $C > 0$ the subset $l^{-1}([0, C]) \subset \Gamma$ is finite. One can construct an integer-valued proper length function on Γ as follows. Let S be a symmetric set of generators of Γ. Let $l_0 : S \to \mathbb{N}$ be a proper function satisfying (ii)–(iii) above. Then

$$l(a) = \inf\{l_0(a_1) + \cdots + l_0(a_n) : a = a_1 \cdots a_n, \, a_i \in S\}$$

is a proper length function on Γ. Given a length function l we define a metric d by $d(a, b) = l(a^{-1}b)$. A metric constructed in this way from a length function is *left invariant* in the sense that $d(ca, cb) = d(a, b)$, for all a, b and $c \in \Gamma$. Conversely, every left invariant metric arises in this way from a length function. A length function is proper if and only if the corresponding (left invariant) metric has bounded geometry. Recall that a metric space X has *bounded geometry* if for every $R > 0$, there is a uniform bound on the number of elements in the balls of radius R in X.

We require the following well-known proposition (compare [19, Lemma 2.1]).

Proposition 2.3. *Let Γ be a countable discrete group and let d and d' be metrics on Γ associated to proper length functions l and l', respectively. Then d and d' are coarsely equivalent.*

Proof. Since the metrics are left invariant it suffices to consider the containment, as in the definition, of balls centered at the identity element. By symmetry it suffices to show that for every $R > 0$ there exists an $S > 0$ such that for all $a \in \Gamma$ if $l(a) < R$ then $l'(a) < S$. But, since l is proper this is obvious. □

As a consequence of the previous two propositions, uniform embeddability of a countable discrete group Γ is independent of the particular proper length function used to define its metric. Consequently, *we systematically omit reference to a specific length function or metric in the statements of all results* and say simply Γ *is uniformly embeddable* to mean that Γ is uniformly embeddable in a Hilbert space for some (equivalently all) left invariant proper metric(s).

Finally, we draw two simple consequences. An action of a discrete group on a locally finite metric space X is *proper* if for every bounded subset $B \subset X$ the set $\{a \in \Gamma : a \cdot B \cap B \neq \emptyset\}$ is finite. Equivalently, for every $x \in X$ and $R \geq 0$ the set $\{a \in \Gamma : a \cdot x \in B_R(x)\}$ is finite. Observe that a free action of a discrete group on a locally finite metric space is proper.

Corollary 2.4. *Let Γ be a countable discrete group equipped with a left invariant proper metric. Let X be a locally finite metric space equipped with a free isometric action of Γ. Then the inclusion $\Gamma \to X$ as an orbit is a uniform embedding.*

Proof. Let $x_0 \in X$. Since the action of Γ on X is by isometries, $l(a) = d_X(a \cdot x_0, x_0)$ defines a length function on Γ. Since the action is free and X is locally finite, l is a proper length function. Let d be the left invariant metric associated to l. According to the previous proposition, the original metric on Γ is coarsely equivalent to d, which is precisely what was to be proved. $\square$

Corollary 2.5. *Let X and Γ be as in the statement of the previous corollary. If X is uniformly embeddable then so is Γ.* $\square$

Property A is a condition on metric spaces introduced by Yu [20]. We will work with the following characterization of Property A.

Proposition 2.6 ([19]). *A discrete metric space X with bounded geometry has Property A if and only if for every $R > 0$ and $\varepsilon > 0$ there exists an $S > 0$ and a Hilbert space valued function $\xi : X \to \mathcal{H}$ such that for all x, $x' \in X$ we have $\|\xi_x\| = 1$, and*

(i) $d(x, x') \leq R \Rightarrow \|\xi_x - \xi_{x'}\| \leq \varepsilon$
(ii) $d(x, x') \geq S \Rightarrow \langle \xi_x, \xi_{x'} \rangle = 0$

Equivalently, for every $R > 0$ and $\varepsilon > 0$ there exists an $S > 0$ and $\xi : X \to l^2(X)$ such that for all x, $x' \in X$ we have $\|\xi_x\| = 1$, (i) as above and

(iii) $\operatorname{supp} \xi_x \subset B_S(x)$. $\square$

Remark. As in the case of uniform embeddability we refer to (i) as the *convergence condition* and to (ii) and (iii) collectively as the *support condition*.

It is clear from the proposition that Property A is invariant under coarse equivalence. We refer the reader to [15] for an introduction to exact groups. Our interest is Prop. 2.6 is motivated by the following result inspired by [13].

Theorem 2.7 ([16], and also [2]). *A countable discrete group Γ is exact if and only if Γ has Property A with respect to some (every) left-invariant proper metric.* $\square$

In analogy with Cor. 2.5 we have the following result, in which we do *not* assume that X itself has Property A.

Corollary 2.8. *Let Γ be a countable discrete group. Assume that Γ acts freely and isometrically on a locally finite metric space X satisfying the convergence and support conditions of Prop. 2.6. Then Γ is exact.*

Proof. Include $\Gamma \subset X$ as an orbit. The convergence and support conditions of Prop. 2.6 pass from X to the subspace Γ. Since Γ has bounded geometry Prop. 2.6 applies. $\qquad\square$

3. Limits

We begin to establish the closure properties of the class of countable discrete uniformly embeddable groups. In this section we treat direct limits, our main result being the following proposition.

Proposition 3.1. *Let Γ be the limit of a directed system of countable discrete groups $G_1 \to G_2 \to G_3 \to \cdots$ in which the the maps $G_n \to G_{n+1}$ are injective. If each of the groups G_n is uniformly embeddable then so is Γ.*

Proof. The proof is based on a method of extending Hilbert space valued functions from a subgroup to an ambient group. Specifically, let $\xi : G \to \mathcal{H}$ be a Hilbert space valued function on a subgroup G of a countable discrete group Γ. Choose and fix a family of coset representatives $x \in X \subset \Gamma$ for Γ/G; having done so each element $a \in \Gamma$ can be uniquely expressed as a product $x_a \, g_a$, where $x_a \in X$ and $g_a \in G$. The extension $\widehat{\xi}$ of ξ is defined by

$$\widehat{\xi} : \Gamma \to \mathcal{H} \otimes l^2(\Gamma/G) \cong \mathcal{H} \otimes l^2(X), \quad \widehat{\xi}_a = \xi_{g_a} \otimes \delta_{aG} \cong \xi_{g_a} \otimes \delta_{x_a}.$$

Let now Γ be a direct limit as in the statement of the theorem. Equip Γ with a proper length function l_Γ and associated metric d_Γ. Metrize each of the subgroups G_n as subspaces of Γ; the metric and length function on G_n are simply the restriction of d_Γ and l_Γ. Observe that for $a, b \in \Gamma$ we have

$$a^{-1}b \in G_n \Leftrightarrow aG_n = bG_n \Leftrightarrow x_a = x_b \in \Gamma,$$

and that in this case,

$$d_{G_n}(g_a, g_b) = d_\Gamma(g_a, g_b) = d_\Gamma(x_a \, g_a, x_b \, g_b) = d_\Gamma(a, b). \tag{3}$$

We show that Γ satisfies the convergence and support conditions of Prop. 2.1. Let $\varepsilon > 0$, and $R > 0$ be given. Obtain n such that if $a \in \Gamma$ has $l_\Gamma(a) \leq R$ then $a \in G_n$; this is possible because the length function l_Γ on Γ is assumed to be proper. According to the criterion for uniform embeddability obtain a Hilbert space valued function $\xi : G_n \to \mathcal{H}$ and such that for all $g, h \in G_n$ we have $\|\xi_g\| = 1$ and

(i) if $d_{G_n}(g, h) \leq R$ then $\|\xi_g - \xi_h\| < \varepsilon$,

(ii) $\forall \hat{\varepsilon} > 0 \ \exists S > 0$ such that if $d_{G_n}(g, h) \geq S$ then $|\langle \xi_g, \xi_h \rangle| < \hat{\varepsilon}$.

Let $\widehat{\xi}$ be the extension of ξ to Γ defined above. Clearly, $\|\widehat{\xi}_a\| = 1$ for all $a \in \Gamma$ and it remains to verify the conditions of Prop. 2.1. Let $a, b \in \Gamma$. For the convergence condition assume that $l_\Gamma(a^{-1}b) = d_\Gamma(a, b) \leq R$. By our choice of n we have $a^{-1}b \in G_n$ and, according to (3), $d_{G_n}(g_a, g_b) = d_\Gamma(a, b) \leq R$. Therefore,

$$\|\widehat{\xi}_a - \widehat{\xi}_b\| = \|\xi_{g_a} \otimes \delta_{aG_n} - \xi_{g_b} \otimes \delta_{bG_n}\| = \|\xi_{g_a} - \xi_{g_b}\| < \varepsilon.$$

For the support condition let $\hat{\varepsilon} > 0$, obtain S as in (ii) above and assume $d_\Gamma(a, b) \geq S$. According to

$$\langle \widehat{\xi}_a, \widehat{\xi}_b \rangle = \langle \xi_{g_a}, \xi_{g_b} \rangle \langle \delta_{aG_n}, \delta_{bG_n} \rangle = \begin{cases} \langle \xi_{g_a}, \xi_{g_b} \rangle, & \text{if } aG_n = bG_n \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

we may assume $a^{-1}b \in G_n$. But then, according to (3) again, $d_{G_n}(g_a, g_b) = d_\Gamma(a, b)$ and $|\langle \widehat{\xi}_a, \widehat{\xi}_b \rangle| < \hat{\varepsilon}$. $\qquad\square$

Remark. The previous argument is easily adjusted to yield a new proof of the fact that a direct limit, as in the statement of the proposition, of exact groups is again exact [15]. Indeed, under the assumption of exactness, employing Prop. 2.6 instead of Prop. 2.1 we replace (ii) by

$$\exists S > 0 \text{ such that } d_{G_n}(g, h) \geq S \Rightarrow \langle \xi_g, \xi_h \rangle = 0.$$

Using (4) and the surrounding discussion conclude that this property is shared by $\widehat{\xi}$.

4. Extensions

Let $1 \to H \to \Gamma \to G \to 1$ be an extension of countable discrete groups. We study uniform embeddability of Γ under various hypotheses on H and G. Our primary result, of which our other results are consequences, is the following theorem.

Theorem 4.1. *Let Γ be an extension of H by G as above. If H is uniformly embeddable and G is exact then Γ is uniformly embeddable.*

As corollaries we mention the following two results about semi-direct products.

Corollary 4.2. *Let G and H be countable discrete groups. Let $\alpha : G \to \operatorname{Aut}(H)$ be an action of G on H. If both H and G are uniformly embeddable and $\alpha(G) \subset \operatorname{Aut}(H)$ is exact then the semi-direct product $\Gamma = H \rtimes G$ is uniformly embeddable.*

Proof. The semi-direct product Γ is the set of pairs $(s,x) \in H \times G$, with product $(s,x)(t,y) = (s\alpha_x(t), xy)$. The assignment

$$(s,x) \mapsto ((s, \alpha_x), x) : \Gamma \to (H \rtimes \alpha(G)) \times G.$$

defines an injective homomorphism.

By the theorem the semi-direct product $H \rtimes \alpha(G)$ is uniformly embeddable, as is $(H \rtimes \alpha(G)) \times G$. In particular, Γ is a subgroup of the uniformly embeddable group and is therefore uniformly embeddable. $\square$

Corollary 4.3. *Let $\Gamma = \mathbb{Z}^n \rtimes G$ be a semi-direct product. If G is uniformly embeddable then Γ is uniformly embeddable.*

Proof. Apply the previous corollary using the fact that $GL_n(\mathbb{Z})$ is exact [15]. $\square$

Remark. Central extensions are more difficult to analyze than semi-direct products. Our theorem applies to central extensions in which the quotient is exact. Gersten has shown that if a central extension of $\mathbb{Z}$ is described by a bounded cocycle then Γ is quasi-isometric to the product $\mathbb{Z} \times G$ [11]. Consequently, if G is uniformly embeddable so is Γ. Beyond these two results little is known.

As remarked earlier, the property of uniform embeddability of a countable discrete group is independent of the proper length function. Consequently, we are free to choose these for our groups G, H and Γ in a convenient manner, which we do as follows. Let l_Γ be a proper length function on Γ. Define length functions on H and G according to

$$l_H(s) \quad = \quad l_\Gamma(s), \quad \text{for all } s \in H, \tag{5}$$

$$l_G(x) \quad = \quad \min\{\, l_\Gamma(a) : a \in \Gamma \text{ and } \dot{a} = x \,\}, \quad \text{for all } x \in G, \tag{6}$$

where we have introduced the notation $a \mapsto \dot{a}$ for the quotient map $\Gamma \to G$. It is easily verified that the minimum in the definition of l_G is attained, and that l_G is a proper length function on G; that l_H is a proper length function is immediate. Denote the associated left invariant metrics by d_Γ, d_G and d_H, respectively. Observe that the inclusion $H \hookrightarrow \Gamma$ is an isometry, and the quotient map $\Gamma \to G$ is contractive. Finally, choose a set theoretic section σ of the quotient map $\Gamma \to G$ with the property that

$$l_\Gamma(\sigma(x)) = l_G(x), \quad \text{for all } x \in G \tag{7}$$

and define $\eta : \Gamma \times G \to H$ by

$$\eta(a,x) = \sigma(x)^{-1} a \sigma(\dot{a}^{-1}x), \quad \text{for all } a \in \Gamma, x \in G. \tag{8}$$

Lemma 4.4. *Let a, $b \in \Gamma$, $x \in G$. We have*

$$d_\Gamma(a,b) \quad \leq \quad d_G(x,\dot{a}) + d_G(x,\dot{b}) + d_H(\eta(a,x),\eta(b,x)) \tag{9}$$

$$d_H(\eta(a,x),\eta(b,x)) \quad \leq \quad d_G(x,\dot{a}) + d_G(x,\dot{b}) + d_\Gamma(a,b). \tag{10}$$

Proof. Let a, b and $x \in G$ be as in the statement. From the definition (8) of η we obtain,

$$\eta(a,x)^{-1}\eta(b,x) = \sigma(\dot{a}^{-1}x)^{-1}\,a^{-1}b\,\sigma(\dot{b}^{-1}x).$$

The desired inequalities follow easily from this equality, together with (5), (7) and the sub-additivity of length functions. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proof of Thm. 4.1. We prove that Γ satisfies the conditions of Prop. 2.1. Let $\varepsilon > 0$ and $R > 0$ be given. We show that there exists an $f : \Gamma \to l^2(G,\mathcal{H})$ such that $\|f(a)\| = 1$ for all $a \in \Gamma$, and that satisfies the convergence and support properties:

(i) if $d_\Gamma(a,b) \leq R$ then $|1 - \langle\, f(a), f(b)\,\rangle| < \varepsilon$

(ii) $\forall \hat{\varepsilon} > 0\ \exists S > 0$ such that if $d_\Gamma(a,b) \geq S$ then $|\langle\, f(a), f(b)\,\rangle| < \hat{\varepsilon}$.

Adapting an argument of Anantharaman-Delaroche [3] our strategy is to use a g satisfying the conditions of Prop. 2.6 to average an h satisfying those of Prop. 2.1 to produce f. Since G is exact there exists, according to Prop. 2.6, a $g : G \to l^2(G)$ and an $S_G > 0$ such that $\|g(x)\| = 1$ for all $x \in G$ and such that

(i$_g$) $|1 - \langle\, g(x), g(y)\,\rangle| < \varepsilon/2$, provided $d_G(x,y) \leq R$,

(ii$_g$) $\operatorname{supp} g(x) \subset B_{S_G}(x)$, for all $x \in G$.

We shall without comment view g as a function on $G \times G$ whenever convenient. Since H is uniformly embeddable there exists, according to Prop. 2.1, an $h : H \to \mathcal{H}$ such that $\|h(s)\| = 1$ for all $s \in H$ and such that

(i$_h$) $|1 - \langle\, h(s), h(t)\,\rangle| < \varepsilon/2$, provided $d_H(s,t) \leq 2S_G + R$,

(ii$_h$) $\forall \hat{\varepsilon} > 0\ \exists S_H > 0$ such that if $d_H(s,t) \geq S_H$ then $|\langle\, h(s), h(t)\,\rangle| < \hat{\varepsilon}$.

Having chosen g and h define $f : \Gamma \to l^2(G,\mathcal{H})$ by

$$f(a)(x) = g(\dot{a},x)h(\eta(a,x)), \quad \text{for all } a \in \Gamma,\, x \in G.$$

Note that $f(a) \in l^2(G,\mathcal{H})$. It is elementary to verify that $\|f(a)\| = 1$, for all $a \in \Gamma$. We verify the remaining properties.

For the convergence property let $a, b \in \Gamma$ with $d_\Gamma(a, b) \leq R$. Consider

$$|1 - \langle f(a), f(b) \rangle| \leq \begin{cases} \left| \sum_{x \in G} \{ 1 - \langle h(\eta(a,x), h(\eta(b,x)) \rangle \} g(\dot a, x) g(\dot b, x) \right| + \\ |1 - \langle g(\dot a), g(\dot b) \rangle|. \end{cases} \tag{11}$$

Since the map $\Gamma \to G$ is contractive we have $d_G(\dot a, \dot b) \leq R$ so that by (i_g) the second term in (11) is bounded by $\varepsilon/2$. Observe that, according to (ii_g) the sum in the first term is over $x \in B_{S_G}(\dot a) \cap B_{S_G}(\dot b)$. Recalling that $\|g(\dot a)\| = \|g(\dot b)\| = 1$ we therefore bound the first term by

$$\sup\{ |1 - \langle h(\eta(a,x)), h(\eta(b,x)) \rangle| : x \in B_{S_G}(\dot a) \cap B_{S_G}(\dot b)\}.$$

From (10) we see that for $x \in B_{S_G}(\dot a) \cap B_{S_G}(\dot b)$ we have $d_H(\eta(a,x), \eta(b,x)) \leq 2S_G + R$ so that by (i_h) this supremum, and consequently the first term in (11), is bounded by $\varepsilon/2$.

For the support property let $\hat\varepsilon > 0$ be given and obtain S_H as in (ii_h). Let $a, b \in \Gamma$ be such that $d_\Gamma(a, b) \geq 2S_G + S_H$. Then,

$$\begin{aligned} |\langle f(a), f(b) \rangle| &= \left| \sum_{x \in G} g(\dot a, x) g(\dot b, x) \langle h(\eta(a,x), h(\eta(b,x)) \rangle \right| \\ &\leq \sum_x |g(\dot a, x) g(\dot b, x)| \, |\langle h(\eta(a,x), h(\eta(b,x)) \rangle| \\ &\leq \sup\{ |\langle h(\eta(a,x), h(\eta(b,x)) \rangle| : x \in B_{S_G}(\dot a) \cap B_{S_G}(\dot b) \}, \end{aligned}$$

where, again we use the fact that $\|g(\dot a)\| = \|g(\dot b)\| = 1$ to obtain the second inequality and note that by (ii_g) the sums are in fact over $x \in B_{S_G}(\dot a) \cap B_{S_G}(\dot b)$. From (9) we see that for such x we have $d_H(\eta(a,x), \eta(b,x)) \geq d_\Gamma(a, b) - d_G(x, \dot a) - d_G(x, \dot b) \geq S_H$ so that by (ii_h) the supremum is indeed bounded by $\hat\varepsilon$. $\qquad\square$

5. Free Products

The main result of this section is the following theorem.

Theorem 5.1. *Let A and B be countable discrete groups and let C be a common subgroup. If both A and B are uniformly embeddable then the amalgamated free product $A *_C B$ is uniformly embeddable.*

Our strategy for proving the theorem is to construct a locally finite metric space X which is uniformly embeddable, and on which Γ acts freely by isometries. The construction of X is based on the notion of a tree of metric spaces, which we now recall.

A *tree* T consists of two sets, a set V of vertices and a set E of edges, together with two *endpoint maps* $E \to V$ associating to each edge its endpoints. Every two vertices are connected by a unique geodesic edge path, that is, a path without backtracking.

A *tree of spaces* $\mathcal{X}$ (with base the tree T) consists of a family of metric spaces $\{ X_v, X_e \}$ indexed by the vertices $v \in V$ and edges $e \in E$ of T together with maps $\sigma_{e,v} : X_e \to X_v$ whenever v is an endpoint of e. The $\sigma_{e,v}$ are the *structural maps* of $\mathcal{X}$. We will assume, although this is not strictly necessary, that the metrics on the vertex and edge spaces are integer-valued.

The *total space* X of the tree of spaces $\mathcal{X}$ is the metric space defined as follows. The underlying set of X is the disjoint union of the vertex spaces X_v; the metric on X is the metric envelope d of the partial metric $\widehat{d}$ (see the appendix) defined by

$$\widehat{d}(x,y) = \begin{cases} d(x,y), & \exists v \in V \text{ such that } x,\, y \in X_v \\ 1, & \exists e \in E \text{ and } z \in X_e \text{ such that } x = \sigma_{e,v}(z),\, y = \sigma_{e,w}(z), \end{cases}$$

for all (x,y) in the domain

$$\mathcal{D} = \{ (x,y) : x,\, y \in X_v,\, v \in V \} \cup \{ (\sigma_{e,v}(z), \sigma_{e,w}(z)) : z \in X_e,\, e \in E \ \}.$$

Observe that $\mathcal{D}$ is ample (see the appendix); this follows from our assumption that the underlying tree is connected, and that $\widehat{d}$ is defined for all pairs (x,y) where x and y are in the same vertex space.

We call (x,y) an *adjacency* if there exists an edge e, with endpoints v and w, and an element $z \in X_e$, such that $\sigma_{e,v}(z) = x$ and $\sigma_{e,w}(z) = y$. Using this terminology, the partial metric can be described as being the given metric on each vertex space and 1 on adjacencies.

Example 5.2. Consider the case in which the vertex spaces are metric graphs (or rather the set of vertices of a graph, equipped with the graph metric) and the edges spaces are singletons. The total space X is itself a metric graph. Indeed, it is the disjoint union of the graphs, together with additional edges coming from the underlying tree. Precisely, the edges of X are first, the edges in the individual X_v and second, the edges e of the underlying tree; the endpoints of such an edge e are the images of the maps $X_e \to X_v$ to the endpoints v of e in T.

Remark. The generality of the definition above is mandated by the fact that when considering amalgamated free products, we will encounter vertex spaces that are not metric graphs.

Theorem 5.3. *Let $\mathcal{X}$ be a tree of metric spaces in which the structural maps are isometries. If the vertex spaces X_v are uniformly embeddable in a Hilbert space (with common distortion bound) then the total space X is uniformly embeddable in a Hilbert space.*

Remark. Quite explicitly, the hypothesis is that there exist maps $\rho_\pm$ as in (1) such that for every vertex $v \in V$ there exists $F_v : X_v \to \mathcal{H}$ satisfying

$$\rho_-(d_{X_v}(x, x')) \leq \|F_v(x) - F_v(x')\| \leq \rho_+(d_{X_v}(x, x')), \quad \text{for all } x, x' \in X_v.$$

In other words, the same distortion bounds may be used for every vertex space.

Remark. If there exist only finitely many distinct isometry types of vertex spaces (as will be the case for the amalgamated free product) this simply means that every vertex space is uniformly embeddable.

Remark. When we prove the theorem we will use the fact that the existence of a common distortion bound on the uniform embeddings of the vertex spaces implies that the estimates in the fundamental criterion for embeddability are uniform. This follows from the proof of Prop. 2.1, specifically from the estimate (2).

Assuming Thm. 5.3 we prepare for the proof Thm. 5.1 by associating a tree of metric spaces $\mathcal{X}_\Gamma$ to the amalgamated free product $\Gamma = A *_C B$. The tree T is the Bass-Serre tree of Γ [17, 4]. Precisely, the vertex and edge sets of T are given by

$$
\begin{aligned}
V &= \Gamma/A \cup \Gamma/B \\
E &= \Gamma/C,
\end{aligned}
$$

respectively; the endpoint maps are the quotient maps $\Gamma/C \to \Gamma/A$ and $\Gamma/C \to \Gamma/B$. In other words, the endpoints of an edge (a C-coset) are the vertices (one A and one B-coset) that contain it.

We associate a metric space to each vertex and edge of T as follows. Equip Γ with an integer-valued proper length function and associated metric. Let $v \in V$ be a vertex and assume that $v \in \Gamma/A$. In particular, v is an A-coset in Γ; denote by X_v this coset itself, metrized as a subspace of Γ. Proceed similarly for vertices $v \in \Gamma/B$. Let $e \in E$ be an edge. In particular, e is a C-coset in Γ; denote by X_e this coset itself, again metrized as a subspace of Γ.

The structural maps are defined as follows. Let the vertex v be an endpoint of the edge e. Inclusion of cosets (subsets of Γ) provides the structural map $\sigma_{e,v} : X_e \to X_v$.

Remark. The metric space X_v is *not* (isometric to) the graph metric space of the Cayley graph of A or B, or even of the restriction of the Cayley graph of Γ to the subset A or B. Similar remarks apply the edge space X_e. Indeed, *it is important that we metrize X_v and X_e as subspaces of Γ and not via their identification with A or B and C using the given metrics on these groups.*

Let X_Γ be the total space of $\mathcal{X}_\Gamma$. The group Γ acts on X_Γ by left multiplication. Precisely, if $x \in X_v$ and $a \in \Gamma$ we define $a \cdot x \in X_{a \cdot v}$, using ordinary multiplication in Γ, by virtue of the fact that $x \in v \subset \Gamma$. This action preserves adjacencies and, since in addition the metric on Γ is left invariant, it preserves the partial metric. According to Prop. 7.2 of the appendix Γ acts by isometries on the total space X_Γ.

Proposition 5.4. *Let A and B be countable discrete groups and let C be a common subgroup. Let $\Gamma = A *_C B$ be the amalgamated free product. Let $\mathcal{X}_\Gamma$ be the tree of metric spaces associated to Γ and let X_Γ be its total space. We have:*

 (i) the structural maps of $\mathcal{X}_\Gamma$ are isometries,
 (ii) every vertex space of $\mathcal{X}_\Gamma$ is isometric to one of A or B (which are metrized with the subspace metric via the inclusions $A, B \subset \Gamma$).
 (iii) The action of Γ by isometries on X_Γ is free.
 (iv) X_Γ is locally finite.

Proof. All of the assertions but (iv) follow from the previous discussion. The condition (iv), while apparent, it will be derived formally in Prop. 5.6. $\qquad\square$

Proof of Thm. 5.1. According to the previous proposition the tree of metric spaces $\mathcal{X}_\Gamma$ associated to the amalgamated free product $\Gamma = A *_C B$ satisfies the hypothesis of Thm. 5.3, according to which the total space X_Γ is uniformly embeddable. Again according to the previous proposition Γ acts freely by isometries on X_Γ. Hence, by Cor. 2.5, Γ is uniformly embeddable. $\qquad\square$

We have reduced our main theorem concerning amalgamated free products, Thm. 5.1, to a theorem concerning trees of metric spaces, Thm. 5.3. We now turn attention to the proof of that theorem. We suitably adapt the method employed by Tu in his study of Property A for discrete metric spaces [19].

For a tree of metric spaces $\mathcal{X}$ the inclusions $X_v \to X$ of vertex spaces into the total space are, as a general rule, *not* isometric; the presence of "shortcuts" in neighboring X_w may cause the distance in X between two vertices $x, y \in X_v$ to be considerably smaller than the

distance between them in X_v itself. Nevertheless, for our $\mathcal{X}_\Gamma$ these inclusions are isometries, a fact that may be traced back to the manner in which the vertex and edge spaces of $\mathcal{X}_\Gamma$ are metrized as subspaces of Γ which circumvents any distortion that may have otherwise been introduced by the amalgamating subgroup. The following proposition has no analog in Tu's work [19]; nevertheless, we require it in order to complete our arguments.

Proposition 5.5. *Let $\mathcal{X} = \{ X_v, X_e \}$ be a tree of metric spaces in which the structural maps $X_e \to X_v$ are isometries. Let X be the total space of $\mathcal{X}$. Then*

> *(i) the inclusions $X_v \to X$ are isometries, and*
> *(ii) if (x, y) is an adjacency then $d(x, y) = 1$.*

Further, for all $x \in X_v$, $y \in X_w$

$$d(x, y) = d_T(v, w) + \inf\{\, d(x_0, x_1) + d(x_2, x_3) + \cdots + d(x_{p-1}, x_p) \,\}, \qquad (12)$$

where d_T is the distance on the tree T and the infimum is taken over all sequences $x_0, \ldots, x_p$ where $p = 2d_T(v, w) + 1$, and

> *(i) $x = x_0$, $y = x_p$,*
> *(ii) (x_{2k-1}, x_{2k}) is an adjacency for $k = 1, \ldots, d_T(v, w)$,*
> *(iii) x_{2k}, $x_{2k+1} \in X_{v_k}$, for $k = 0, \ldots, d_T(v, w)$, and*

$v = v_0, \ldots, v_{d_T(v,w)} = w$ are the vertices along the unique geodesic path in T from v to w.

Remark. Formula (12) is Tu's formula [19, Section 9] with slightly different notation.

Proof. Let $v, w \in V$ and let $x \in X_v$ and $y \in X_w$. A *reduced path* from x to y is a sequence of elements $x = x_0, x_1, \ldots, x_{2n}, x_{2n+1}$ of X for which there exist vertices $v_0, \ldots, v_n \in V$ such that

> (i) $v_j \neq v_{j+1}$, for $j = 1, \ldots, n$, and
> (ii) x_{2k}, $x_{2k+1} \in X_{v_k}$, for $k = 0, 1, \ldots, n$.

Observe that by appropriately inserting and deleting x's we may alter a path, without increasing its length, so as to obtain a reduced path (use the triangle inequality for the metrics on the individual X_v). Consequently, when computing distances in X it suffices to consider reduced paths.

Let $x = x_0, \ldots, x_{2n+1} = y$ be a reduced path from x to y. According to the definitions of a reduced path and of the domain $\mathcal{D}$ of the partial metric we obtain sequences of

> (i) vertices $v = v_0, \ldots, v_n = w$ in V,
> (ii) edges $e_1, \ldots, e_n$ in E, and

(iii) elements $z_1, \ldots, z_n$ of the edge spaces $X_{e_1}, \ldots, X_{e_n}$,

satisfying

    (i) $x_{2k}, x_{2k+1} \in X_{v_k}$, for $k = 0, \ldots, n$,

    (ii) the endpoints of e_k are v_{k-1} and v_k, for $k = 1, \ldots, n$, and

    (iii) $\sigma_{e_k, v_{k-1}}(z_k) = x_{2k-1}$ and $\sigma_{e_k, v_k}(z_k) = x_{2k}$, for $k = 1, \ldots, n$

(The sequences are uniquely determined by these conditions.) The given reduced path $x = x_0, \ldots, x_p = y$ in X *lies over* the edge path $e_1, \ldots, e_n$ in T.

We show that in the definition of d it suffices to consider reduced paths lying over the unique geodesic edge path in T from v to w; the assertions of the proposition follow easily from this fact. Let $x = x_0, \ldots, x_{2n+1} = y$ lie over the non-geodesic path $e_1, \ldots, e_n$ in T. We show that by successive elimination of certain x_i we obtain a shorter path lying over the geodesic path in T. Indeed, there exists $i \in 1, \ldots, n-1$ such that e_i and e_{i+1} have the same endpoints, that is, such that $v_{i-1} = v_{i+1}$. We have

$$\left. \begin{array}{c} x_{2i-2} \\ x_{2i-1} \end{array} \right\} \in X_{v_{i-1}}, \quad \left. \begin{array}{c} x_{2i} \\ x_{2i+1} \end{array} \right\} \in X_{v_i}, \quad \left. \begin{array}{c} x_{2i+2} \\ x_{2i+3} \end{array} \right\} \in X_{v_{i+1}} = X_{v_{i-1}}.$$

We eliminate $x_{2i-1}, x_{2i}, x_{2i+1}, x_{2i+2}$ from the given path and claim that the resulting path

    (i) is shorter than $x_0, \ldots, x_{2n+1}$, and

    (ii) lies over an edge path with fewer backtracks than $e_1, \ldots, e_n$.

Of these (ii) is obvious; the new path lies over the edge path $e_1, \ldots, e_{i-1}, e_{i+2}, \ldots, e_n$ obtained by eliminating e_i and e_{i+1}. Further, (i) follows from the fact that the structural maps are isometries. In particular we have

$$d(x_{2i-1}, x_{2i+2}) = d(z_i, z_{i+1}) = d(x_{2i}, x_{2i+1}),$$

from which follows

$$d(x_{2i-2}, x_{2i+3}) \leq d(x_{2i-2}, x_{2i-1}) + d(x_{2i-1}, x_{2i+2}) + d(x_{2i+2}, x_{2i+3})$$

$$= d(x_{2i-2}, x_{2i-1}) + d(x_{2i}, x_{2i+1}) + d(x_{2i+2}, x_{2i+3}). \quad \square$$

Proposition 5.6. *Let X be the total space of a be a tree of metric spaces $\mathcal{X}$ in which the structural maps are isometries. Assume that each vertex space X_v is locally finite and there is a uniform bound for the number of adjacencies of each point in X. Then X is locally finite.*

Proof. In view of formula (12) a finite radius ball in X can intersect only finitely many vertex spaces. $\quad \square$

6. Proof of Theorem 5.3

We may enlarge the tree of metric spaces (if necessary) such that the underlying tree T will contain an infinite geodesic ω starting at some basepoint. For every $v \in V$ let $\alpha(v) \in V$ be such that the edge $[v, \alpha(v)]$ points towards ω. For each $v \in V$ let $Y_v = \sigma_{e,v}(X_e) \subset X_v$ and $f_v = \sigma_{e,\alpha(v)} \circ \sigma_{e,v}^{-1} : Y_v \to X_{\alpha(v)}$ where $e = [v, \alpha(v)]$. It follows immediately from Prop. 5.5 that each f_v is isometric.

Using the same proposition, and the notation just introduced, we rewrite the distance formula (12) as follows. Let $x_0 \in X_v$ and $x_0' \in X_{v'}$. There exist a unique pair of nonnegative integers k, ℓ such that $\alpha^k(v) = \alpha^\ell(v')$ and $d_T(v, v') = k + \ell$, where again d_T denotes the distance in the tree T. By symmetry we may assume that $k \geq \ell$. If $k \geq 1$ and $\ell \geq 1$ then

$$d(x_0, x_0') = k + \ell + \inf \left\{ \begin{array}{l} d(x_0, y_0) + \sum_{i=0}^{k-2} d(f_{\alpha^i(v)}(y_i), y_{i+1}) \\ \qquad + d(f_{\alpha^{k-1}(v)}(y_{k-1}), f_{\alpha^{\ell-1}(v')}(y_{\ell-1}')) \\ \qquad + \sum_{j=0}^{\ell-2} d(f_{\alpha^j(v')}(y_j'), y_{j+1}') + d(y_0', x_0') \end{array} \right\}, \qquad (13)$$

subject to the constraints that $y_i \in Y_{\alpha^i(v)}$ and $y_j' \in Y_{\alpha^j(v')}$. If $k \geq 1$ and $\ell = 0$ then

$$d(x_0, x_0') = k + \inf \left\{ d(x_0, y_0) + \sum_{i=0}^{k-2} d(f_{\alpha^i(v)}(y_i), y_{i+1}) + d(f_{\alpha^{k-1}(v)}(y_{k-1}), x_0') \right\}, \qquad (14)$$

subject to similar constraints.

An *n-chain* is a sequence $\mathbf{x} = (x_0, x_1, \ldots, x_{n-1})$ such that for each $0 \leq k \leq n-2$ there exists $\bar{x}_k \in Y_{\alpha^k(v)}$ satisfying $d(x_k, \bar{x}_k) < d(x_k, Y_{\alpha^k(v)}) + 1$ and $f_{\alpha^k(v)}(\bar{x}_k) = x_{k+1}$. If $x_0 \in X_v$ the n-chain *starts in* X_v. Note that for any $n \geq 1$, $v \in V$ and $x_0 \in X_v$ there exists an n-chain whose initial element is x_0.

Lemma 6.1. *Let $x_0 \in X_v$ and $x_0' \in X_{v'}$, $v \neq v'$ with $d(x_0, x_0') < R$ and let k and ℓ be as in (13)–(14) with $k \geq \ell$. If $\ell = 0$, there exists a chain $(x_0, x_1, \ldots, x_k)$ such that*

$$\max \left\{ \left(\sup_{0 \leq i \leq k-1} d(x_i, x_{i+1}) \right), d(x_k, x_0') \right\} < 2^R R.$$

If $\ell \geq 1$, there exist chains $(x_0, x_1, \ldots, x_k)$, $(x_0', x_1', \ldots, x_\ell')$ and $z, z' \in X_{\alpha^k(v)} = X_{\alpha^\ell(v')}$ such that

$$\max \left\{ \left(\sup_{0 \leq i \leq k-1} d(x_i, x_{i+1}) \right), \left(\sup_{0 \leq j \leq \ell-1} d(x_j', x_{j+1}') \right), d(x_k, z), d(x_\ell', z'), d(z, z') \right\} < 2^R R.$$

Remark. In the proof of the lemma, and at a number of subsequent points, we require the fact that if $x \in X_v$ and $y \in Y_v$ then

$$d(x, f_v(y)) = d(x, y) + 1.$$

This follows from Thm. 5.5. Indeed, since $(y, f_v(y))$ is an adjacency, $d(x, f_v(y)) \leq d(x, y) + 1$. For the reverse inequality, let $\varepsilon > 0$ and obtain $x' \in Y_v$ such that $d(x, f_v(y)) + \varepsilon \geq 1 + d(x, x') + d(f_v(x'), f_v(y))$. Since f_v is an isometry, $d(x, x') + d(f_v(x'), f_v(y)) \geq d(x, y)$ and we are done.

Proof. <u>*Case* $\ell = 0$</u>: We have $k \geq 1$ since $v \neq v'$ and $k \geq \ell$. Since $d(x_0, x_0') < R$, by (14) there is $y_0 \in Y_v$ such that $k + d(x_0, y_0) < R$. The sequence $x_1, \ldots, x_k$ is constructed inductively. Let $\bar{x}_0 \in Y_v$ be such that $d(x_0, \bar{x}_0) \leq d(x_0, y_0)$ and $d(x_0, \bar{x}_0) < d(x_0, Y_v) + 1$ and define $x_1 = f_v(\bar{x}_0)$. Then

$$d(x_0, x_1) = d(x_0, \bar{x}_0) + 1 \leq d(x_0, y_0) + 1 < R.$$

Thus $d(x_0, x_1) < R$ and $d(x_1, x_0') \leq d(x_0, x_0') + d(x_0, x_1) < 2R$. Repeating the same argument for the pair of points x_1, x_0' (if $k \geq 2$) we find $x_2 \in X_{\alpha^2(v)}$ with (x_0, x_1, x_2) being a chain, $d(x_1, x_2) < 2R$ and $d(x_2, x_0') < 2^2 R$. Continuing in the same way we obtain a chain $(x_0, x_1, \ldots, x_k)$ such that $d(x_{i-1}, x_i) < 2^{i-1} R$ and $d(x_i, x_0') < 2^i R$, $1 \leq i \leq k$. Since $k \leq R$ this completes the proof of (a).

<u>*Case* $\ell \geq 1$</u>: Let $y_0, \ldots, y_{k-1}$ with $y_i \in Y_{\alpha^i(v)}$ and $y_0', \ldots, y_{\ell-1}'$ with $y_j' \in Y_{\alpha^j(v')}$ be sequences such that the expression whose infimum is taken in (13) is less than $R - k - \ell$. Let $z' = f_{\alpha^{\ell-1}(v')}(y_{\ell-1}')$ and $z = f_{\alpha^{k-1}(v)}(y_{k-1})$. Then we have $d(z, z') < R$, $d(x_0, z) < R$ and $d(x_0', z') < R$. The proof is completed by applying part (a) to the pairs x_0, z and x_0', z'. $\qquad \square$

Given an n-chain $\mathbf{x} = (x_0, x_1, \ldots, x_{n-1})$ starting in X_v define, for $0 \leq k \leq n-1$, $\delta_k = d(x_k, Y_{\alpha^k(v)})$, and $\theta_k = \delta_0 \vee \cdots \vee \delta_k$, where we introduce the notation $a \vee b = \max\{a, b\}$. For future notational convenience define $\theta_{-1} = 0$. Note that if $\mathbf{x}' = (x_0', x_1', \ldots, x_{n-1}')$ is another n-chain starting in X_v then

$$d(x_{k-1}, x_k) \leq \delta_{k-1} + 2, \tag{15}$$

$$|\delta_k - \delta_k'| = |d(x_k, Y_{\alpha^k(v)}) - d(x_k', Y_{\alpha^k(v)})| \leq d(x_k, x_k'), \quad 0 \leq k \leq n-1. \tag{16}$$

Lemma 6.2. *Let* $\mathbf{x} = (x_0, x_1, \ldots, x_{n-1})$ *and* $\mathbf{x}' = (x_0', x_1', \ldots, x_{n-1}')$ *be n-chains starting in* X_v. *Then*

$$|d(x_k, x_k') - d(x_0, x_0')| \leq 2k(\theta_{k-1} \vee \theta_{k-1}') + 2k, \quad 0 \leq k \leq n-1. \tag{17}$$

Proof. Since the statement is obvious in the case $k = 0$ we assume $k \geq 1$. Let $\bar{x}_k$ and $\bar{x}_k'$ be as in the definition of n-chains. Observe that

$$d(\bar{x}_k, \bar{x}_k') \leq d(\bar{x}_k, x_k) + d(\bar{x}_k', x_k') + d(x_k, x_k') \leq d(x_k, x_k') + 2(\theta_k \vee \theta_k') + 2,$$
$$d(\bar{x}_k, \bar{x}_k') \geq d(x_k, x_k') - d(\bar{x}_k, x_k) - d(\bar{x}_k', x_k') \geq d(x_k, x_k') - 2(\theta_k \vee \theta_k') - 2.$$

Since $f_{\alpha^k(v)}$ is an isometry we have $d(x_{k+1}, x_{k+1}') = d(\bar{x}_k, \bar{x}_k')$ and the lemma follows from these inequalities by induction. $\qquad\square$

Given an n-chain $\mathbf{x} = (x_0, x_1, \ldots, x_{n-1})$ and $N > 0$ we define, for $0 \leq k \leq n-1$,

$$a_k = \left(1 - \frac{\ln(1 + \theta_k)}{N}\right)_+, \tag{18}$$

$$c_k = \sqrt{a_0 \cdots a_{k-1} \left(1 + (n - k - 1)(1 - a_k)\right)}, \tag{19}$$

where we introduce the notation $a_+ = \max\{a, 0\}$. Note that $c_0 = \sqrt{1 + (n-1)(1 - a_0)}$. One checks immediately that $a_0 \geq a_1 \geq \cdots \geq a_{n-1}$,

$$c_0^2 + \cdots + c_{n-1}^2 = n, \quad 0 \leq c_{n-1} \leq 1, \tag{20}$$

$$\delta_k \geq e^N \Longrightarrow \theta_k \geq e^N \Longrightarrow a_k = 0 \Longrightarrow c_{k+1} = \cdots = c_{n-1} = 0. \tag{21}$$

The coefficients c_k were introduced by Tu [19] (actually we work with the square root of Tu's coefficients). Since the maps f_v are isometries, it is possible in our case to define a_k explicitly as in (18). Both a_k and c_k should be regarded as functions $a_k^{\mathbf{x}}$ and $c_k^{\mathbf{x}}$ of n-chains $\mathbf{x} = (x_0, x_1, \ldots, x_{n-1})$. We will often write a_k' and c_k' instead of $a_k^{\mathbf{x}'}$ and $c_k^{\mathbf{x}'}$.

Lemma 6.3. *Let* $\mathbf{x} = (x_0, x_1, \ldots, x_{n-1})$ *and* $\mathbf{x}' = (x_0', x_1', \ldots, x_{n-1}')$ *be n-chains starting in* X_v. *If* $\omega = \max_{0 \leq i \leq n-1} |a_i - a_i'|$ *then*
  (i) $|c_k^2 - c_k'^2| \leq \left(\frac{n+1}{2}\right)^2 \omega$,
  (ii) $|c_k - c_k'| \leq \frac{n+1}{\sqrt{2}} \sqrt{\omega}$, *and*
  (iii) $\sum_{k=0}^{n-1} \frac{1}{n} |c_k - c_k'|^2 \leq \frac{(n+1)^2}{2} \omega$.

Proof. This is an exercise. For (i) $\Rightarrow$ (ii) use the inequality $|\sqrt{a} - \sqrt{b}| \leq \sqrt{2|a-b|}$. $\qquad\square$

The following continuity property of the coefficients a_k is a minor variation of a formula in Tu's paper [19, formula 9.2].

Lemma 6.4. *Let* $\mathbf{x} = (x_0, x_1, \ldots, x_{n-1})$ *and* $\mathbf{x}' = (x_0', x_1', \ldots, x_{n-1}')$ *be n-chains starting in* X_v *and assume* $d(x_0, x_0') \leq n$. *Then*

$$\max_{0 \leq k \leq n-1} |a_k - a_k'| \leq \frac{7n^2}{N} \tag{22}$$

Proof. Denote $\lambda = \ln 7n$. We are going to show that

$$
\begin{aligned}
|a_0 - a_0'| &\leq \frac{n}{N} \\
|a_k - a_k'| &\leq |a_{k-1} - a_{k-1}'| + \frac{\lambda}{N}.
\end{aligned}
\tag{23}
$$

From these we immediately obtain $|a_k - a_k'| \leq \dfrac{n(1 + \lambda)}{N} \leq \dfrac{7n^2}{N}$, concluding the proof. Of these inequalities, the first is straightforward. Indeed, using (16) and the fact that the map $t \mapsto (1 - \frac{1}{N} \ln(1 + t))_+$ is $\frac{1}{N}$-Lipschitz we have

$$|a_0 - a_0'| \leq \frac{1}{N}|\delta_0 - \delta_0'| \leq \frac{1}{N} d(x_0, x_0') \leq \frac{n}{N}.$$

To prove the second inequality denote $f(t) = 4nt + 6n$. According to (16) and Lemma 6.2, we have

$$|\delta_k - \delta_k'| \leq d(x_k, x_k') \leq \frac{1}{2} f(\theta_{k-1} \vee \theta_{k-1}'). \tag{24}$$

Denote $\psi(t) = \ln(1 + t)$, so that we have for all $t \geq 0$

$$\psi(t) \leq \psi(f(t)) \leq \psi(t) + \lambda. \tag{25}$$

Case $\theta_k \vee \theta_k' \leq f(\theta_{k-1} \vee \theta_{k-1}')$: By symmetry we may assume that $\theta_{k-1} \geq \theta_{k-1}'$. Thus $\theta_{k-1}' \leq \theta_k' \leq f(\theta_{k-1})$ and $\theta_{k-1}' \leq \theta_{k-1} \leq \theta_k \leq f(\theta_{k-1})$. Using (25) we obtain

$$
\begin{aligned}
\psi(\theta_{k-1}') &\leq \psi(\theta_k') \leq \psi(f(\theta_{k-1})) \leq \psi(\theta_{k-1}) + \lambda, \\
\psi(\theta_{k-1}') &\leq \psi(\theta_k) \leq \psi(f(\theta_{k-1})) \leq \psi(\theta_{k-1}) + \lambda.
\end{aligned}
$$

Now, (23) follows immediately from these inequalities, together with the fact that for real numbers $a \leq s, t \leq b + \lambda$ we have

$$|(1 - s/N)_+ - (1 - t/N)_+| \leq |(1 - b/N)_+ - (1 - a/N)_+| + \lambda.$$

Case $\theta_k \vee \theta_k' \geq f(\theta_{k-1} \vee \theta_{k-1}')$: By symmetry we may assume that $\theta_k \geq \theta_k'$. Then $\theta_k \geq f(\theta_{k-1}) \geq \theta_{k-1}$ hence $\theta_k = \delta_k$. Therefore using (24)

$$\theta_k \leq |\theta_k - \theta_k'| + \theta_k' \leq |\delta_k - \delta_k'| + \theta_k' \leq \frac{1}{2} f(\theta_{k-1} \vee \theta_{k-1}') + \theta_k' \leq \frac{1}{2}\theta_k + \theta_k',$$

hence $\theta_k \leq 2\theta'_k \leq f(\theta'_k)$. From $\theta'_k \leq \theta_k \leq f(\theta'_k)$ and (25) we obtain

$$\psi(\theta'_k) \leq \psi(\theta_k) \leq \psi(f(\theta'_k)) \leq \psi(\theta'_k) + \lambda.$$

From this inequality, and the fact that the map $t \mapsto (1 - t/N)_+$ is $\frac{1}{N}$-Lipschitz we obtain $|a_k - a'_k| \leq \frac{\lambda}{N}$, which implies (23). $\qquad\square$

Definition. *Given $R > 0$ and $\varepsilon > 0$ choose and fix $n \in \mathbb{N}$ such that*

$$\left(\frac{\ln n + 9}{4n}\right)^{\frac{1}{2}} < \frac{\varepsilon}{3(R+3)}, \quad n > 2^R R, \tag{26}$$

and $N \in \mathbb{N}$ such that

$$\frac{6n(n+1)}{\sqrt{N}} < \frac{\varepsilon}{3(R+3)}. \tag{27}$$

Having done so apply the fundamental criterion for uniform embeddability, Prop. 2.1, to choose and fix a family $(\xi_x)_{x \in X}$ of unit vectors in a Hilbert space $\mathcal{H}_X = \oplus_{v \in V} \mathcal{H}_v$ with $\xi_x \in \mathcal{H}_v$ if $x \in X_v$ and such that

$$\sup\{\|\xi_y - \xi_{y'}\| : d(y, y') \leq n(2e^N + 3), y, y' \in X_v, v \in V\} < \frac{\varepsilon}{3(R+3)}, \tag{28}$$

$$\lim_{S \to \infty} \sup\{|\langle \xi_y, \xi_y \rangle| : d(y, y') \leq S, y, y' \in X_v, v \in V\} = 0. \tag{29}$$

(See the remarks after Thm. 5.3 for comments on why this is possible.) Finally, for every n-chain $\mathbf{x} = (x_0, x_1, \ldots, x_{n-1})$ in X define the unit vector $\eta^{\mathbf{x}} \in \mathcal{H}_X$ by

$$\eta^{\mathbf{x}} = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} c_k \, \xi_{x_k}, \tag{30}$$

where, of course, the c_k's are defined according to (19) and depend on the chain $\mathbf{x}$.

Lemma 6.5. *Let $\mathbf{x} = (x_0, x_1, \ldots, x_{n-1})$ and $\mathbf{x}' = (x'_0, x'_1, \ldots, x'_{n-1})$ be n-chains starting in X_v. If $d(x_0, x'_0) \leq n$, then $\|\eta^{\mathbf{x}} - \eta^{\mathbf{x}'}\| \leq \frac{2\varepsilon}{3(R+3)}$.*

Proof. Let $I = \{k : c_k \neq 0, c'_k \neq 0\}$ and $J = \{k : c_k = 0, c'_k \neq 0\}$. If $1 \leq k \in I$, then $a_{k-1} = \left(1 - \frac{1}{N} \ln(1 + \theta_{k-1})\right)_+ \neq 0$ hence $\theta_{k-1} \leq e^N$. Similarly, $\theta'_{k-1} \leq e^N$. Hence from (17) we have

$$d(x_k, x'_k) \leq 2ne^N + 2n + d(x_0, x'_0) \leq n(2e^N + 3). \tag{31}$$

Consider

$$\|\eta^{\mathbf{x}} - \eta^{\mathbf{x}'}\| = \left\| \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} c_k\, \xi_{x_k} - \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} c_k'\, \xi_{x_k'} \right\|$$

$$\leq \begin{cases} \left\| \dfrac{1}{\sqrt{n}} \sum_{k=0}^{n-1} (c_k - c_k')\, \xi_{x_k} \right\| + \left\| \dfrac{1}{\sqrt{n}} \sum_{k \in J} (c_k' - c_k)\, (\xi_{x_k} - \xi_{x_k'}) \right\| \\[2em] \qquad\qquad\qquad + \left\| \dfrac{1}{\sqrt{n}} \sum_{k \in I} c_k'\, (\xi_{x_k} - \xi_{x_k'}) \right\|. \end{cases}$$

Observe that the ξ_{x_k}'s and $\xi_{x_k'}$'s are in orthogonal components of $\mathcal{H}_X$. We bound the third term on the right using (20), (31) and (28) as follows:

$$\frac{1}{\sqrt{n}} \left(\sum_{k \in I} (c_k')^2\, \|\xi_{x_k} - \xi_{x_k'}\|^2 \right)^{1/2} \leq \frac{1}{\sqrt{n}} \left(\sum_{k=0}^{n-1} (c_k')^2 \right)^{1/2} \cdot \left(\sup_{k \in I} \|\xi_{x_k} - \xi_{x_k'}\| \right)$$

$$\leq \sup_{\Delta} \|\xi_y - \xi_{y'}\| \leq \frac{\varepsilon}{3(R+3)},$$

where $\Delta = \{ (y, y') : d(y, y') \leq n(2e^N + 3),\ y, y' \in X_v,\ v \in V \}$. We bound the sum of the first two terms on the right, by

$$3 \left(\sum_{k=0}^{n-1} \frac{1}{n} |c_k - c_k'|^2 \right)^{\frac{1}{2}} \leq \frac{3(n+1)}{\sqrt{2}} \max_{0 \leq i \leq n-1} |a_i - a_i'|^{\frac{1}{2}} \leq \frac{3(n+1)}{\sqrt{2}} \left(\frac{7n^2}{N} \right)^{\frac{1}{2}} \leq \frac{\varepsilon}{3(R+3)},$$

where the inequalities are from Lemma 6.3, (22) and (27), respectively. Combining these observations we obtain the result. $\qquad\square$

Lemma 6.6. *Let* $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ *and* $\mathbf{x}' = (x_0, x_1', \ldots, x_{n-1}')$ *be n-chains with* $x_0 \in X_v$ *and* $x_1 \in X_{\alpha(v)}$. *If* (x_0, x_1) *is a 2-chain and* $d(x_0, x_1) \leq n$, *then* $\|\eta^{\mathbf{x}} - \eta^{\mathbf{x}'}\| \leq \frac{\varepsilon}{R+3}$.

Proof. Let $\bar{x}_0 = f_v^{-1}(x_1)$ and set $\bar{\mathbf{x}} = (\bar{x}_0, x_1, \ldots, x_{n-1})$. According to the definition (30) of $\eta^{\mathbf{x}}$ we have

$$\eta^{\mathbf{x}} = \eta^{(x_1, x_2, \ldots, x_n)} = \frac{1}{\sqrt{n}} \left(\sum_{k=1}^{n-1} d_k \xi_{x_k} + d_n \xi_{x_n} \right),$$

$$\eta^{\bar{\mathbf{x}}} = \eta^{(\bar{x}_0, x_1, \ldots, x_{n-1})} = \frac{1}{\sqrt{n}} \left(\bar{d}_0 \xi_{\bar{x}_0} + \sum_{k=1}^{n-1} \bar{d}_k \xi_{x_k} \right),$$

where

$$d_k = c_{k-1}(x_1, \ldots, x_k) = \sqrt{\bar{a}_1 \cdots \bar{a}_{k-1}(1 + (n-k)(1 - \bar{a}_k))},$$

$$\bar{d}_k = c_k(\bar{x}_0, x_1, \ldots, x_k) = \sqrt{\bar{a}_0 \bar{a}_1 \cdots \bar{a}_{k-1}(1 + (n-k-1)(1 - \bar{a}_k))},$$

with $\bar{a}_0 = 1$, $\bar{d}_0 = 1$ and $\bar{a}_k = a_k(\bar{x}_0, x_1, \ldots, x_k) = \min\left\{\left(1 - \frac{\ln(1+\delta_i)}{N}\right)_+ : 1 \le i \le k\right\}$. For $1 \le k \le n-1$ we have either $1 - \bar{a}_k = 0$, hence $d_k - \bar{d}_k = 0$, or

$$d_k - \bar{d}_k = \frac{\sqrt{\bar{a}_1 \cdots \bar{a}_{k-1}}(1 - \bar{a}_k)}{\sqrt{1 + (n-k)(1 - \bar{a}_k)} + \sqrt{1 + (n-k-1)(1 - \bar{a}_k)}}$$

$$\le \frac{(1 - \bar{a}_k)}{2\sqrt{(n-k)(1 - \bar{a}_k)}} \le \frac{1}{2\sqrt{n-k}}.$$

Applying the above expressions for $\eta^{\mathbf{x}}$ and $\eta^{\bar{\mathbf{x}}}$ with this inequality we estimate

$$\|\eta^{\bar{\mathbf{x}}} - \eta^{\mathbf{x}}\| = \frac{1}{\sqrt{n}} \left\| (\bar{d}_0 \xi_{\bar{x}_0} + \sum_{k=1}^{n-1} (\bar{d}_k - d_k)\xi_{x_k} - d_n \xi_{x_n} \right\|$$

$$\le \left(\frac{2}{n} + \frac{1}{n}\sum_{k=1}^{n-1}(d_k - \bar{d}_k)^2\right)^{\frac{1}{2}}$$

$$\le \left(\frac{2}{n} + \frac{1}{4n}\left(\frac{1}{n-1} + \frac{1}{n-2} + \cdots + 1\right)\right)^{\frac{1}{2}}$$

$$\le \left(\frac{\ln n + 9}{4n}\right)^{\frac{1}{2}} \le \frac{\varepsilon}{3(R+3)},$$

where the final inequality comes from the choice (26) of n. Apply Lemma 6.5 to the chains $\mathbf{x}'$ and $\mathbf{x}$, noting that $d(x_0, \bar{x}_0) = d(x_0, x_1) - 1 \le n$, and use the previous inequality to conclude

$$\|\eta^{\mathbf{x}'} - \eta^{\mathbf{x}}\| \le \|\eta^{\mathbf{x}'} - \eta^{\bar{\mathbf{x}}}\| + \|\eta^{\bar{\mathbf{x}}} - \eta^{\mathbf{x}}\| \le \frac{2\varepsilon}{3(R+3)} + \frac{\varepsilon}{3(R+3)} = \frac{\varepsilon}{R+3}. \quad \square$$

Lemma 6.7. *Let* $\mathbf{x} = (x_0, x_1, \ldots, x_{n-1})$ *and* $\mathbf{x}' = (x'_0, x'_1, \ldots, x'_{n-1})$ *be n-chains in* X. *Then*

$$|\langle \eta^{\mathbf{x}}, \eta^{\mathbf{x}'} \rangle| \le \sup\left\{|\langle \xi_y, \xi_{y'}\rangle| \ : \ y, y' \in X_v, v \in V, d(y, y') \ge d(x_0, x'_0) - 6n(e^N + 1)\right\}.$$

Proof. Let k and ℓ be as in (13). Considering symmetry we assume that $k \ge \ell$. Also, if $k \ge n$, then $\langle \eta^{\mathbf{x}}, \eta^{\mathbf{x}'}\rangle = 0$ so we assume that $k < n$. With these assumptions

$$\langle \eta^{\mathbf{x}}, \eta^{\mathbf{x}'} \rangle = \frac{1}{n} \sum_{i=0}^{n-k-1} c_{k+i} c'_{\ell+i} \langle \xi_{x_{k+i}}, \xi_{x'_{\ell+i}} \rangle. \tag{32}$$

Making use of (21), conclude that if $\theta_{k-1} \vee \theta'_{\ell-1} \geq e^N$ then all the products $c_{k+i}c'_{\ell+i}$ in (32) are zero and we are done. Thus we assume that $\theta_{k-1} \vee \theta'_{\ell-1} \leq e^N$. Let m be the largest number $0 \leq m \leq n - k - 1$ with the property that $\theta_{k+m-1} \vee \theta'_{\ell+m-1} \leq e^N$. By (17)

$$d(x_{k+i}, x'_{\ell+i}) \geq d(x_k, x'_\ell) - 2n(e^N + 1)$$

for all $0 \leq i \leq m$. On the other hand using (15)

$$\begin{aligned}
d(x_k, x'_\ell) &\geq d(x_0, x'_0) - \sum_{i=0}^{k-1} d(x_i, x_{i+1}) - \sum_{j=0}^{\ell-1} d(x'_j, x'_{j+1}) \\
&\geq d(x_0, x'_0) - \sum_{i=0}^{k-1} (\delta_i + 2) - \sum_{j=0}^{\ell-1} (\delta'_j + 2) \\
&\geq d(x_0, x'_0) - (k + \ell)(2(\theta_{k-1} \vee \theta'_{\ell-1}) + 2) \\
&\geq d(x_0, x'_0) - 4n(e^N + 1).
\end{aligned}$$

Combining these two inequalities we obtain

$$d(x_{k+i}, x'_{\ell+i}) \geq d(x_0, x'_0) - 6n(e^N + 1), \quad \text{for all} \quad 0 \leq i \leq m.$$

Finally, arguing again on the basis of (21) as above, conclude that the terms in (32) for $i > m$ are zero. Thus, applying (20) and the previous inequality, we see that

$$|\langle \eta^{\mathbf{x}}, \eta^{\mathbf{x}'} \rangle| \leq \left(\frac{1}{n} \sum_{i=0}^{m} c_{k+i}c'_{\ell+i} \right) \cdot \sup_{\Omega} |\langle \xi_y, \xi_{y'} \rangle| \leq \sup_{\Omega} |\langle \xi_y, \xi_{y'} \rangle|,$$

where $\Omega = \{(y, y') : y, y' \in X_v, v \in V, d(y, y') \geq d(x_0, x'_0) - 6n(e^N + 1)\}$. $\square$

Conclusion of proof of Theorem 5.3. Given $R > 0$ and $\varepsilon > 0$ let n, N, and $(\xi_x)_{x \in X}$ be constructed as in the Definition. For each $x_0 \in X$ choose and fix an n-chain $\mathbf{x} = (x_0, x_1, \ldots, x_{n-1})$ and consider the corresponding vector $\eta^{\mathbf{x}}$. We are going to show that the map $x_0 \longmapsto \eta^{(x_0, x_1, \ldots, x_{n-1})}$ satisfies the convergence and support conditions of Prop. 2.1.

The support condition follows immediately from Lemma 6.7 in conjunction with (29).

For the convergence condition we show that for any $x_0, x'_0 \in X$ with $d(x_0, x'_0) < R$ and any two n-chains $\mathbf{x}$ and $\mathbf{x}'$ starting at x_0 and x'_0, respectively, we have $\|\eta^{\mathbf{x}} - \eta^{\mathbf{x}'}\| \leq \varepsilon$. Let k and ℓ be as in (13)–(14). We may assume that $k \geq \ell$. Furthermore we treat only the case $\ell \geq 1$, the case $\ell = 0$ being similar. Let $\mathbf{x}(i)$, $\mathbf{x}'(j)$, $\mathbf{z}$ and $\mathbf{z}'$ be n-chains whose initial elements are the points x_i, x'_j, z, z' given by Lemma 6.1, $0 \leq i \leq k$, $0 \leq j \leq \ell$. Applying Lemmas 6.5 and

6.6 repeatedly (note that since $n > 2^R R$ this is possible by Lemma 6.1), with the convention that all empty sums are zero, we have

$$\|\eta^{\mathbf{x}} - \eta^{\mathbf{x}'}\| \leq \begin{cases} \displaystyle\sum_{i=0}^{k-1} \|\eta^{\mathbf{x}(i)} - \eta^{\mathbf{x}(i+1)}\| + \|\eta^{\mathbf{x}(k)} - \eta^{\mathbf{z}}\| + \|\eta^{\mathbf{z}} - \eta^{\mathbf{z}'}\| \\ \\ \qquad\qquad + \|\eta^{\mathbf{z}'} - \eta^{\mathbf{x}'(\ell)}\| + \displaystyle\sum_{j=0}^{\ell-1} \|\eta^{\mathbf{x}'(j)} - \eta^{\mathbf{x}'(j+1)}\| \end{cases}$$

$$\leq \quad \frac{(k+\ell+3)\varepsilon}{R+3} \leq \varepsilon. \quad \square$$

Remark. In the proof of Thm. 5.3 we showed that for every $R > 0$ and every $\varepsilon > 0$ there is a family $(\eta^{\mathbf{x}})_{x_0 \in X}$ of unit vectors in $\mathcal{H}_X$ such that

$$\sup\{\|\eta^{\mathbf{x}} - \eta^{\mathbf{x}'}\| : d(x_0, x_0') < R\} \leq \varepsilon,$$

$$|\langle \eta^{\mathbf{x}}, \eta^{\mathbf{x}'} \rangle| \leq \sup\{|\langle \xi_y, \xi_{y'} \rangle| \mid y, y' \in X_v,\, v \in V,\, d(y, y') \geq d(x_0, x_0') - 6n(e^N + 1)\}.$$

Let us argue that, in addition to Thm. 5.3, these estimates prove that $A *_C B$ is exact whenever A and B are exact [8, 9, 19]. Indeed, if we base the construction (30) of $\eta^{\mathbf{x}}$ on a family $(\xi_x)_{x \in X}$ satisfying the support condition (ii) of Prop. 2.6 uniformly with respect to $v \in V$, then it follows from the second inequality that $\langle \eta^{\mathbf{x}}, \eta^{\mathbf{x}'} \rangle = 0$ whenever $d(x_0, x_0') \geq S + 6n(e^N + 1)$. In particular, X satisfies the convergence and support conditions of Prop. 2.6 and $A *_C B$ is exact by Cor. 2.8.

7. Appendix

We collect several elementary results on the construction of metrics required for our treatment of amalgamated free products. Let X be a set. A *partial metric* on X is an integer-valued function $\widehat{d} : \mathcal{D} \to \mathbb{R}_+$ defined on a domain $\mathcal{D} \subset X \times X$ satisfying, for all $x, y \in X$

(i) $\widehat{d}(x, y) = \widehat{d}(y, x)$
(ii) $\widehat{d}(x, y) = 0$ if and only if $x = y$

In these statements it is assumed that all relevant pairs belong to the domain $\mathcal{D}$ of $\widehat{d}$ and that the domain $\mathcal{D}$ is symmetric and contains the diagonal.

We now associate a metric to a partial metric. The construction is analogous to that of path metrics; intuitively the distance between two points is the length of the shortest path between them. A *path* from x to $y \in X$ is a sequence $x = x_0, x_1, \ldots, x_n = y$ such that for every $1 \leq j \leq n$ we have $(x_{i-1}, x_i) \in \mathcal{D}$. The *length* of a path is $\sum_{i=1}^{n} \widehat{d}(x_{i-1}, x_i)$. The domain $\mathcal{D}$ is *ample* if for every $x, y \in \mathcal{D}$ there exists a path from x to y.

Proposition 7.1. *Let $\widehat{d}$ be a partial metric on X, defined on an ample domain $\mathcal{D}$. Define, for all $x, y \in X$*

$$d(x,y) = \inf\{\text{ length of paths from } x \text{ to } y\,\}.$$

Then d is an integer-valued metric on X. □

The metric d defined in the proposition is the *metric envelope* of $\widehat{d}$.

Example. If X is the vertex set of a connected graph and $\widehat{d}$ is constant 1 on the domain of all pairs (x, y) that represent edges then d is the path metric; $d(x, y)$ is the smallest number of edges on a path from x to y.

Remark. If $(x, y) \in \mathcal{D}$ then $d(x, y) = \widehat{d}(x, y)$ if and only if the length of every path from x to y is greater than or equal to $\widehat{d}(x, y)$. If $\mathcal{D} = X \times X$ and $\widehat{d}$ is a metric then $d = \widehat{d}$.

Proposition 7.2. *Let $\widehat{d}$ be a partial metric on X and let $\varphi : X \to X$ be a bijection with the property that*

(i) $(x, y) \in \mathcal{D}$ if and only if $(\varphi(x), \varphi(y)) \in \mathcal{D}$, and
(ii) $\widehat{d}(\varphi(x), \varphi(y)) = \widehat{d}(x, y)$ for all $(x, y) \in \mathcal{D}$

Then φ is an isometry for the metric envelope.

Proof. It suffices to show that

$$d(\varphi(x), \varphi(y)) \leq d(x,y), \quad \text{for all } x,\, y \in X.$$

Let $\varepsilon > 0$ be given. Let $(x_0, \ldots, x_n)$ be a path in X from x to y such that $\sum_{j=1}^n \widehat{d}(x_{j-1}, x_j) < d(x, y) + \varepsilon$. Then, $(\varphi(x_0), \ldots, \varphi(x_n))$ is a path from $\varphi(x)$ to $\varphi(y)$ and we have

$$d(\varphi(x), \varphi(y)) \leq \sum_{i=1}^n \widehat{d}(\varphi(x_{i-1}), \varphi(x_i)) = \sum_{j=1}^n \widehat{d}(x_{j-1}, x_j) < d(x, y) + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we are done. □

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