THE C*-ALGEBRA OF A VECTOR BUNDLE

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Abstract. We prove that the Cuntz-Pimsner algebra O_E of a vector bundle E of rank ≥ 2 over a compact metrizable space X is determined up to an isomorphism of $C(X)$-algebras by the ideal $(1-[E])K^0(X)$ of the K-theory ring $K^0(X)$. Moreover, if E and F are vector bundles of rank ≥ 2, then a unital embedding of $C(X)$-algebras $O_E \subset O_F$ exists if and only if $1-[E]$ is divisible by $1-[F]$ in the ring $K^0(X)$. We introduce related, but more computable K-theory and cohomology invariants for O_E and study their completeness. As an application we classify the unital separable continuous fields with fibers isomorphic to the Cuntz algebra O_{m+1} over a finite connected CW complex X of dimension $d \leq 2m+3$ provided that the cohomology of X has no m-torsion.

1. Introduction

Let $E \in \mathrm{Vect}(X)$ be a locally trivial complex vector bundle over a compact Hausdorff space X. If we endow E with a hermitian metric, then the space $\Gamma(E)$ of all continuous sections of E becomes a finitely generated projective Hilbert $C(X)$-module, whose isomorphism class does not depend on the choice of the metric. Since the action of $C(X)$ is central, $\Gamma(E)$ is naturally a Hilbert $C(X)$-bimodule. Let O_E denote the Cuntz-Pimsner algebra associated to $\Gamma(E)$ as defined in [16]. Since $\Gamma(E)$ is projective, O_E is isomorphic to the Doplicher-Roberts algebra of $\Gamma(E)$, see [7]. Let us recall that if $\mathcal{E}$ is the Hilbert $C(X)$-module $\oplus_{n\geq 0}\Gamma(E)^{\otimes n}$, then O_E is obtained as the quotient of the Toeplitz (or tensor) C*-algebra T_E generated by the multiplication operators $T_\xi : \mathcal{E} \to \mathcal{E}$, $T_\xi(\eta) = \xi \otimes \eta$, $\xi \in \Gamma(E)$, $\eta \in \mathcal{E}$, by the ideal of "compact operators" $K(\mathcal{E})$. If X is a point, then $E \cong \mathbb{C}^n$ for some $n \geq 1$, and O_E is isomorphic to the Cuntz algebra O_n, with the convention that $O_1 = C(\mathbb{T})$. In the general case, O_E is a locally trivial unital $C(X)$-algebra (continuous field) whose fiber at x is isomorphic to the Cuntz algebra $O_{n(x)}$, where $n(x)$ is the rank of the fiber E_x of E, see [19, Prop. 2].

The motivation for this paper comes from an informal question of Cuntz: What are the invariants of E captured by the $C(X)$-algebra O_E? In other words, how are E and F related if there is a $C(X)$-linear $*$-isomorphism $O_E \cong O_F$. We have shown in [5] that if X has finite covering dimension, then all separable unital $C(X)$-algebras with fibers isomorphic to a fixed Cuntz algebra O_n, $n \geq 2$, are automatically locally trivial. Thus it

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is also natural to ask which of these algebras are isomorphic to Cuntz-Pimsner algebras associated to a vector bundle of constant rank n.

If E is a line bundle, then O_E is commutative with spectrum homeomorphic to the circle bundle of E, see [20]. One verifies that if $E, F \in Vect_1(X)$ and X is path-connected, then $O_E \cong O_F$ as $C(X)$-algebras if and only if $E \cong F$ or $E \cong \bar{F}$, where $\bar{F}$ is the conjugate of F, see Proposition 4.3. In view of this property, we shall only consider vector bundles of rank ≥ 2. In the first part of the paper we answer the isomorphism question for O_E.

Theorem 1.1. *Let X be a compact metrizable space and let $E, F \in \mathrm{Vect}(X)$ be complex vector bundles of rank ≥ 2. Then O_E embeds as a unital $C(X)$-subalgebra of O_F if and only if there is $h \in K^0(X)$ such that $1 - [E] = (1 - [F])h$. Moreover, $O_E \cong O_F$ as $C(X)$-algebras if and only if there is h as above of virtual rank one.*

Thus the principal ideal $(1 - [E])K^0(X)$ determines O_E up to isomorphism and an inclusion of principal ideals $(1 - [E])K^0(X) \subset (1 - [F])K^0(X)$ corresponds to unital embeddings $O_E \subset O_F$. In particular if $E \in \mathrm{Vect}_{m+1}(X)$, then $O_E \cong C(X) \otimes O_{m+1}$ if and only if $[E] - 1$ is divisible by $m \geq 1$.

Let $\widetilde{K}^0(X) = \ker(K^0(X) \xrightarrow{rank} H^0(X, \mathbb{Z}))$ be the subgroup of $K^0(X)$ corresponding to elements of virtual rank zero, and set $[\widetilde{E}] := [E] - \mathrm{rank}(E) \in \widetilde{K}^0(X)$. We denote by $H^*(X, \mathbb{Z})$ the Čech cohomology. Using the nilpotency of $\widetilde{K}^0(X)$ we derive the following:

Theorem 1.2. *Let X be a compact metrizable space of finite dimension n. Suppose that $\mathrm{Tor}(K^0(X), \mathbb{Z}/m) = 0$. If $E, F \in Vect_{m+1}(X)$, then $O_E \cong O_F$ as $C(X)$-algebras if and only if $([\widetilde{E}] - [\widetilde{F}])\left(\sum_{k=1}^{n}(-1)^{k-1}m^{n-k}[\widetilde{F}]^{k-1}\right)$ is divisible by m^n in $\widetilde{K}^0(X)$.*

In view of Theorem 1.1 it is natural to seek explicit and computable invariants (e.g. characteristic classes) of a vector bundle E that depend only on the principal ideal $(1 - [E])K^0(X)$ and hence which are invariants of O_E.

For each $m \geq 1$, consider the sequence of polynomials $p_n \in \mathbb{Z}[x]$,

$$p_n(x) = \ell(n)\, m^n \, \log\left(1 + \frac{x}{m}\right)_{[n]} = \sum_{k=1}^{n}(-1)^{k-1}\frac{\ell(n)}{k}m^{n-k}x^k, \tag{1}$$

where $\ell(n)$ denotes the least common multiple of the numbers $\{1, 2, ..., n\}$ and the index $[n]$ indicates that the formal series of the natural logarithm is truncated after its nth term.

Theorem 1.3. *Let X be a finite CW complex of dimension d and let $E, F \in \mathrm{Vect}_{m+1}(X)$. If $O_E \cong O_F$ as $C(X)$-algebras, then $p_{\lfloor d/2 \rfloor}([\widetilde{E}]) - p_{\lfloor d/2 \rfloor}([\widetilde{F}])$ is divisible by $m^{\lfloor d/2 \rfloor}$ in $\widetilde{K}^0(X)$.*

For $x \in \mathbb{R}$, we set $\lfloor x \rfloor := \max\{k \in \mathbb{Z} : k \leq x\}$ and $\lceil x \rceil := \min\{k \in \mathbb{Z} : k \geq x\}$. Theorem 1.3 extends to finite dimensional compact metrizable spaces: if $n \geq 1$ is an integer such that $\widetilde{K}^0(X)^{n+1} = \{0\}$, then $p_n([\widetilde{E}]) - p_n([\widetilde{F}])$ is divisible by m^n in $\widetilde{K}^0(X)$

whenever $O_E \cong O_F$ as $C(X)$-algebras. The same conclusion holds for infinite dimensional spaces X but in that case n depends on E and F.

Concerning the completeness of the above invariant we have the following:

Theorem 1.4. *Let X be a finite CW complex of dimension d. Suppose that m and $\lfloor d/2 \rfloor !$ are relatively prime and that* $\mathrm{Tor}(H^*(X,\mathbb{Z}),\mathbb{Z}/m) = 0$. *If $E, F \in Vect_{m+1}(X)$, then $O_E \cong O_F$ as $C(X)$-algebras if and only if $p_{\lfloor d/2 \rfloor}([\widetilde{E}]) - p_{\lfloor d/2 \rfloor}([\widetilde{F}])$ is divisible by $m^{\lfloor d/2 \rfloor}$ in $\widetilde{K}^0(X)$.*

The condition that m and $\lfloor d/2 \rfloor !$ are relatively prime is necessary. To show this, we take $m = 2$ and let X be the complex projective space $\mathbb{C}\mathrm{P}^2$. Then $K^0(X)$ is isomorphic to the polynomial ring $\mathbb{Z}[x]$ with $x^3 = 0$, [11]. Let E and F be bundles with K-theory classes $[E] = 3 + 3x$ and $[F] = 3 + x$. Then $[\widetilde{E}] = 3x$ and $[\widetilde{F}] = x$, so that $p_2(3x) - p_2(x) = 8(x - x^2)$ is divisible by 4 and yet Theorem 1.2 shows that $O_E \ncong O_F$, since $([\widetilde{E}] - [\widetilde{F}])(2 - [\widetilde{F}]) = 4x - 2x^2$ is not divisible by 4. The vanishing of m-torsion is also necessary in both Theorems 1.3 and 1.4 as it is seen by taking $m = 2$ and $X = \mathbb{R}\mathrm{P}^2 \vee \mathbb{C}\mathrm{P}^2$, where $\mathbb{R}\mathrm{P}^2$ is the real projective space. Indeed, let $E, F \in \mathrm{Vect}_3(X)$ be such that F is trivial and $[\widetilde{E}]|_{\mathbb{R}\mathrm{P}^2} = z$ is the generator of $\widetilde{K}^0(\mathbb{R}\mathrm{P}^2) = \mathbb{Z}/2$ and $[\widetilde{E}]|_{\mathbb{C}\mathrm{P}^2} = 2x + 2x^2$. Then $([\widetilde{E}]-[\widetilde{F}])(2-[\widetilde{F}]) = (z+2x+2x^2)(2) = 4x+4x^2$ is divisible by 4 and yet $O_E \ncong C(X) \otimes O_3$ by Theorem 1.1 since $[E] - 1 = 2 + z + 2x + 2x^2$ is not divisible by 2.

Next we exhibit characteristic classes of E which are invariants of O_E. For each $n \geq 1$ consider the polynomial $q_n \in \mathbb{Z}[x_1, ..., x_n]$:

$$(2) \quad q_n = \sum_{k_1+2k_2+\cdots+nk_n=n} (-1)^{k_1+\cdots+k_n-1} m^{n-(k_1+\cdots+k_n)} \frac{n!\,(k_1+\cdots+k_n-1)!}{1!^{k_1}\cdots n!^{k_n}\, k_1!\cdots k_n!}\, x_1^{k_1}\cdots x_n^{k_n}.$$

Thus $q_1(x_1) = x_1$, $q_2(x_1,x_2) = mx_2 - x_1^2$, $q_3(x_1,x_2,x_3) = m^2x_3 - 3mx_1x_2 + 2x_1^3$, etc. Let ch_n be the integral characteristic classes that appear in the Chern character, $ch = \sum_{n\geq 0} \frac{1}{n!} ch_n$.

Theorem 1.5. *Let X be a compact metrizable space and let $E, F \in \mathrm{Vect}_{m+1}(X)$. If $O_E \cong O_F$ as $C(X)$-algebras, then $q_n(ch_1(E), ..., ch_n(E)) - q_n(ch_1(F), ..., ch_n(F))$ is divisible by m^n in $H^{2n}(X,\mathbb{Z})$, for all $n \geq 1$.*

Reducing mod m^n it follows that the sequence $q_n(\dot{ch}_1(E), ..., \dot{ch}_n(E)) \in H^{2n}(X,\mathbb{Z}/m^n)$, $n \geq 1$, is an invariant of the $C(X)$-algebra O_E.

Let us denote by $\mathcal{O}_{m+1}(X)$ the set of isomorphism classes of unital separable $C(X)$-algebras with all fibers isomorphic to O_{m+1}. In the second part of the paper we study the range of the map $\mathrm{Vect}_{m+1}(X) \to \mathcal{O}_{m+1}(X)$. This relies on the computation of the homotopy groups of $\mathrm{Aut}(O_{m+1})$ of [5]. If T is a set, we denote by $|T|$ its cardinality.

Theorem 1.6. *Let X be a finite CW complex of dimension d. Suppose that $m \geq \lceil (d-3)/2 \rceil$ and* $\mathrm{Tor}(H^*(X,\mathbb{Z}),\mathbb{Z}/m) = 0$. *Then each element of $\mathcal{O}_{m+1}(X)$ is isomorphic to O_E for some E in $Vect_{m+1}(X)$. Moreover $|\mathcal{O}_{m+1}(X)| = |\widetilde{K}^0(X) \otimes \mathbb{Z}/m| = |\widetilde{H}^{even}(X,\mathbb{Z}/m)|$.*

The hypotheses of Theorem 1.6 are necessary. Indeed, to see that the condition $m \geq \lceil (d-3)/2 \rceil$ is necessary even in the absence of torsion, we note that $Vect_3(S^8) = \{*\}$ since $\pi_7(U(3)) = 0$ by [12], whereas $\mathcal{O}_3(S^8) \cong \pi_7(\mathrm{Aut}(O_3)) \cong \mathbb{Z}/2$ by [5]. To see that the condition on torsion is necessary when $m \geq \lceil (d-3)/2 \rceil$, we note that if $X = \mathbb{R}\mathrm{P}^2$, then $Vect_3(SX) = \{*\}$ since $\widetilde{K}^0(SX) \cong K^1(X) = \{0\}$ and $\dim(SX) = 3$, whereas $\mathcal{O}_3(SX) \cong K^1(X, \mathbb{Z}/2) \cong \mathbb{Z}/2$ by [5].

The study of the map $Vect_{m+1}(X) \to \mathcal{O}_{m+1}(X)$ simplifies considerably if X is a suspension as explained in Theorem 7.1 from Section 7.

In Section 2 we prove Theorems 1.1 - 1.2. Theorem 1.3 is proved in Section 3 and Theorem 1.4 is proved in Section 6. The proofs of Theorem 1.5 and Theorems 1.6 are given in Section 4 and respectively Section 5.

Cuntz-Pimsner algebras come with a natural $\mathbb{T}$-action and hence with a $\mathbb{Z}$-grading. The question studied by Vasselli in [19] of when O_E and O_F are isomorphic as $\mathbb{Z}$-graded $C(X)$-algebras is not directly related to the questions addressed in this paper. I would like to thank Ezio Vasselli for making me aware of the isomorphism $O_E \cong O_{\bar{E}}$ for line bundles, see [20].

2. When is O_E isomorphic to O_F?

In this section we prove Theorems 1.1-1.2 and discuss the case of line bundles.

Proof. (of Theorem 1.1) We identify $K_0(C(X))$ with $K^0(X)$. Let ι_E denote the canonical unital inclusion $C(X) \to O_E$. By [16], the K-theory group $K_0(O_E)$ fits into an exact sequence

$$K_0(C(X)) \xrightarrow{1-[E]} K_0(C(X)) \xrightarrow{(\iota_E)_*} K_0(O_E),$$

where $1-[E]$ corresponds to the multiplication map by the element $1-[E]$. Therefore $\ker(\iota_E)_* = (1-[E])K^0(X)$. Suppose that $\phi : O_E \to O_F$ is a $C(X)$-linear unital $*$-homomorphism. Then $\phi \circ \iota_E = \iota_F$ and hence $\ker(\iota_E)_* \subset \ker(\iota_F)_*$. It follows that $(1-[E])K^0(X) \subset (1-[F])K^0(X)$ and hence $1-[E] = (1-[F])h$ for some $h \in K^0(X)$. If ϕ is an isomorphism, we deduce similarly that $1-[F] = (1-[E])k$ for some $k \in K^0(X)$. In that case $rank(E_x) = rank(F_x)$ for each $x \in X$ and h must have constant virtual rank equal to one.

Conversely, suppose that there is $h \in K^0(X)$ such that $(1-[E]) = (1-[F])h$. We have $O_E \otimes O_\infty \cong O_E$ and $O_F \otimes O_\infty \cong O_F$ by [3]. We are going to show the existence of a unital $C(X)$-linear embedding $O_E \subset O_F$ by producing an element $\chi \in KK_X(O_E, O_F)$ which maps $[1_{O_E}]$ to $[1_{O_F}]$ and then appeal to [13]. If the virtual rank of h is equal to one and hence h is invertible in the ring $K^0(X)$, we show that χ is a KK_X-equivalence and that will imply that O_E is isomorphic to O_F. Since the operation of suspension is an isomorphism in KK_X, it suffices to show that there is $\eta \in KK_X(SO_E, SO_F)$, respectively $\eta \in KK_X(SO_E, SO_F)^{-1}$, such that $\eta \circ [S\iota_E] = [S\iota_F]$.

Let us recall that the mapping cone of a $*$-homomorphism $\alpha : A \to B$ is

$$C_\alpha = \{(f,a) \in C([0,1],B) \oplus A : f(0) = \alpha(a), f(1) = 0\}.$$

If α is a morphism of continuous $C(X)$-algebras, then the natural extension

$$0 \longrightarrow SB \xrightarrow{\lambda} C_\alpha \xrightarrow{p} A \longrightarrow 0,$$

where $\lambda(f) = (f,0)$ and $p(f,a) = a$, is an extension of continuous $C(X)$-algebras.

Let $\mathrm{KK}(X)$ denote the additive category with objects separable $C(X)$-algebras and morphisms from A to B given by the group $KK_X(A,B)$. Nest and Meyer have shown that $\mathrm{KK}(X)$ is a triangulated category [14]. In particular, for any diagram of separable $C(X)$-algebras and $C(X)$-linear $*$-homomorphisms

$$\begin{tikzcd} SB \arrow[r, "\lambda"] \arrow[d, "S\varphi"'] & C_\alpha \arrow[r, "p"] \arrow[d, dashed, "\gamma"] & A \arrow[r, "\alpha"] \arrow[d, "\psi"'] & B \arrow[d, "\varphi"'] \\ SB' \arrow[r, "\lambda'"] & C_{\alpha'} \arrow[r, "p"] & A' \arrow[r, "\alpha'"] & B' \end{tikzcd}$$

such that the right square commutes in $\mathrm{KK}(X)$, there is $\gamma \in KK_X(C_\alpha, C_{\alpha'})$ that makes to remaining squares commute. If the right square commutes up to homotopy of $C(X)$-linear $*$-homomorphisms, then γ can be chosen to be a $C(X)$-linear $*$-homomorphism, see [18, Prop. 2.9]. The general case is proved in a similar way, see [14, Appendix A]. Let us note that if φ and ψ are KK_X-equivalences, so is γ by the exactness of the Puppe sequence and the five-lemma.

We need another general observation. If

$$0 \longrightarrow J \xrightarrow{j} B \xrightarrow{\pi} B/J \longrightarrow 0$$

is an extension of separable continuous $C(X)$-algebras, then there is a surjective $C(X)$-linear $*$-homomorphism $\mu : C_j \to SB/J$, $\mu(f,b) = \pi \circ f$, and hence an extension of separable continuous $C(X)$-algebras

$$0 \longrightarrow CJ \longrightarrow C_j \xrightarrow{\mu} SB/J \longrightarrow 0.$$

If B/J is nuclear then it is $C(X)$-nuclear and since CJ is KK_X-contractible it follows that μ must be a KK_X-equivalence, see [2].

Let us recall that O_E is defined by the extension $0 \longrightarrow \mathcal{K}(\mathcal{E}) \xrightarrow{j_E} T_E \xrightarrow{\pi_E} 0.$

By Theorem 4.4 and Lemma 4.7 of [16] there is a commutative diagram in $\mathrm{KK}(X)$:

$$\begin{tikzcd} \mathcal{K}(\mathcal{E}) \arrow[r, "{[j_E]}"] \arrow[d, "{[\mathcal{E}]}"'] & T_E \\ C(X) \arrow[r, "{[id]-[E]}"] & C(X) \arrow[u, "{[i_E]}"'] \end{tikzcd}$$

where both vertical maps are KK_X-equivalences. Here i_E is the canonical unital inclusion, $[\mathcal{E}]$ is the class in $KK_X(\mathcal{K}(\mathcal{E}), C(X))$ defined by the bimodule $\mathcal{E}$ that implements the strong Morita equivalence between $\mathcal{K}(\mathcal{E})$ and $C(X)$ and $[E]$ is the class in $KK_X(C(X), C(X))$ defined by the finitely generated $C(X)$-module $\Gamma(E)$. Note that $[id] - [E] \in KK_X(C(X), C(X))$ induces the multiplication map $K^0(X) \xrightarrow{1-[E]} K^0(X)$. Pimsner's statements refer to ordinary KK-theory but his constructions and arguments are natural and preserve the $C(X)$-structure.

After tensoring the C*-algebras in the above diagram by O_∞ we can realize $[\mathcal{E}]^{-1}$ and $[id]-[E]$ as KK_X classes of $C(X)$-linear $*$-homomorphisms ψ_E and respectively α_E. This is easily seen by using the identification $KK_X(C(X), B) \cong KK(\mathbb{C}, B)$ for B a $C(X)$-algebra and noting that $\mathcal{K}(\mathcal{E})$ contains a full projection since $\mathcal{E}$ is isomorphic to $C(X) \otimes \ell^2(\mathbb{N})$ by Kuiper's theorem. Thus we obtain a diagram

$$\begin{tikzcd}
\mathcal{K}(\mathcal{E}) \otimes O_\infty \arrow[r, "j_E \otimes id"] & T_E \otimes O_\infty \\
C(X) \otimes O_\infty \arrow[u, "\psi_E"] \arrow[r, "\alpha_E"] & C(X) \otimes O_\infty \arrow[u, "i_E \otimes id"']
\end{tikzcd}$$

that commutes in KK(X). We construct a similar diagram for the bundle F. Let $H : C(X) \otimes O_\infty \to C(X) \otimes O_\infty$ be a $C(X)$-linear $*$-homomorphism which sends $[1_{C(X)\otimes O_\infty}]$ to $h \in K^0(X) \cong K_0(C(X) \otimes O_\infty)$. Note that H is a KK_X-equivalence whenever h is invertible in the ring $K^0(X)$. Since $1 - [E] = (1 - [F])h$ by assumption, the diagram

$$\begin{tikzcd}
C(X) \otimes O_\infty \arrow[r, "\alpha_E"] \arrow[d, "H"'] & C(X) \otimes O_\infty \arrow[d, equal] \\
C(X) \otimes O_\infty \arrow[r, "\alpha_F"] & C(X) \otimes O_\infty
\end{tikzcd}$$

commutes in the category $\mathrm{KK}(X)$. The proof of the theorem is based on the following commutative diagram in $\mathrm{KK}(X)$:

$$
\begin{tikzcd}
 & SO_E \otimes O_\infty & & \\
ST_E \otimes O_\infty \arrow[r, "\lambda_E"] & C_{j_E} \otimes O_\infty \arrow[u, "\mu_E \otimes id"'] \arrow[r] & \mathcal{K}(\mathcal{E}) \otimes O_\infty \arrow[r, "j_E \otimes id"] & T_E \otimes O_\infty \\
SC(X) \otimes O_\infty \arrow[u, "Si_E \otimes id"] \arrow[r] \arrow[d, equal] & C_{\alpha_E} \arrow[u, dashed, "\gamma_E"'] \arrow[r] \arrow[d, dashed, "\gamma"'] & C(X) \otimes O_\infty \arrow[u, "\psi_E"'] \arrow[r, "\alpha_E"] \arrow[d, "H"'] & C(X) \otimes O_\infty \arrow[u, "i_E \otimes id"'] \arrow[d, equal] \\
SC(X) \otimes O_\infty \arrow[d, "Si_F \otimes id"'] \arrow[r] & C_{\alpha_F} \arrow[r] \arrow[d, dashed, "\gamma_F"'] & C(X) \otimes O_\infty \arrow[r, "\alpha_F"] \arrow[d, "\psi_F"'] & C(X) \otimes O_\infty \arrow[d, "i_F \otimes id"'] \\
ST_F \otimes O_\infty \arrow[r, "\lambda_F"] & C_{j_F} \otimes O_\infty \arrow[r] \arrow[d, "\mu_F \otimes id"'] & \mathcal{K}(\mathcal{F}) \otimes O_\infty \arrow[r, "j_F \otimes id"] & T_F \otimes O_\infty \\
 & SO_F \otimes O_\infty & &
\end{tikzcd}
$$

The elements γ, γ_E and γ_F are constructed as explained above since the category $\mathrm{KK}(X)$ is triangulated. Moreover γ_E and γ_F are KK_X-equivalences since they are induced by the KK_X-equivalences i_E, ψ_E and i_F, ψ_F. Arguing similarly, we see that γ is a KK_X-equivalence whenever h is invertible in the ring $K^0(X)$. The morphisms μ_E and μ_F are associated to the Toeplitz extensions $0 \to \mathcal{K}(\mathcal{E}) \to T_E \to O_E \to 0$ and respectively $0 \to \mathcal{K}(\mathcal{F}) \to T_F \to O_F \to 0$ and they are also KK_X-equivalences as we argued earlier in the proof. Let us note that $(\mu_E \otimes id) \circ \lambda_E \circ (Si_E \otimes id_{O_\infty}) = S\iota_E \otimes id_{O_\infty}$ and $(\mu_F \otimes id) \circ \lambda_F \circ (Si_F \otimes id_{O_\infty}) = S\iota_F \otimes id_{O_\infty}$ by a simple direct verification. In this way we are able to find an element

$$\eta := [\mu_F \otimes id] \circ \gamma_F \circ \gamma \circ \gamma_E^{-1} \circ [\mu_E \otimes id]^{-1} \in KK_X(SO_E \otimes O_\infty, SO_F \otimes O_\infty) \cong KK_X(SO_E, SO_F)$$

such that the following diagram commutes in $\mathrm{KK}(X)$.

$$
\begin{tikzcd}
 & SC(X) \arrow[dl, "S\iota_E"'] \arrow[dr, "S\iota_F"] & \\
SO_E \arrow[rr, "\eta"] & & SO_F
\end{tikzcd}
$$

Unsuspending, we find $\chi \in KK_X(O_E, O_F)$ which maps $[1_{O_E}]$ to $[1_{O_F}]$, since $\chi \circ [\iota_E] = [\iota_F]$. If h is invertible in $K^0(X)$, then χ is a KK_X-equivalence. It follows that $O_E \cong O_F$ as $C(X)$-algebras by applying Kirchberg's isomorphism theorem [13]. If χ is just a KK_X-element which preserves the classes of the units, we invoke again [13] in order to lift χ to a unital $C(X)$-linear embedding $O_E \subset O_F$. $\square$

Set $T_n(b) := \sum_{k=1}^{n}(-1)^{k-1}\, m^{n-k}\, b^{k-1}$. Then $(m+b)T_n(b) = m^n - (-b)^n$.

Lemma 2.1. *Let R be a commutative ring such that $R^{n+1} = \{0\}$ for some $n \geq 1$ and let $a, b \in R$. If there is $h \in R$ such that $a = b + mh + bh$, then $(a-b)T_n(b) = m^n h$. Conversely, if $(a-b)T_n(b) = m^n h$ for some $h \in R$, then $m^n(a - b - mh - bh) = 0$.*

Proof. Suppose that $a-b = (m+b)h$ for some $h \in R$. Then $(a-b)T_n(b) = (m^n-(-b)^n)h = m^n h - (-b)^n h$. Since $(-b)^n h \in R^{n+1}$ must vanish it follows that $(a-b)T_n(b) = m^n h$. Conversely, suppose that $(a-b)T_n(b) = m^n h$ for some $h \in R$. Then $(a-b)T_n(b)(m+b) = m^n(m+b)h$ and hence $(a-b)(m^n - (-b)^n) = m^n(m+b)h$. But $(a-b)(-b)^n = 0$ since $R^{n+1} = \{0\}$. Therefore $m^n(a-b-mh-bh) = 0$. □

We are now prepared to prove Theorem 1.2.

Proof. Since $\dim(X) \leq n$, we can embed X in $\mathbb{R}^{2n+1}$ and then find a decreasing sequence X_i of polyhedra whose intersection is X. We have $\widetilde{K}^0(X_i)^{n+1} = \{0\}$ since $\dim(X_i) \leq 2n+1$ (see the next section for further discussion). It follows that $\widetilde{K}^0(X)^{n+1} = \{0\}$ since $\widetilde{K}^0(X) \cong \varinjlim \widetilde{K}^0(X_i)$. Let us write $[E]-1 = m + a$ and $[F]-1 = m+b$ where $a = [\widetilde{E}]$ and $b = [\widetilde{F}] \in \widetilde{K}^0(X)$. By Theorem 1.1, $O_E \cong O_F$ if and only if $[E]-1 = ([F]-1)(1+h)$ for some $h \in \widetilde{K}^0(X)$ and hence if and only if $a = b + mb + bh$ for some $h \in \widetilde{K}^0(X)$. With this observation we conclude the proof by applying Lemma 2.1. □

For a hermitian bundle E we denote by $\bar{E}$ the conjugate bundle, by E_0 the set of all nonzero elements in E and by $S(E)$ the unit sphere bundle of E.

Proposition 2.2. *Let E and F be hermitian line bundles over a path-connected compact metrizable space X. Then $O_E \cong O_F$ as $C(X)$-algebras if and only if either $E \cong F$ or $E \cong \bar{F}$.*

Proof. Vasselli has shown that for a line bundle E, $O_E \cong C(S(E))$ as $C(X)$-algebras [20]. Therefore it suffices to show that there is a homeomorphism of sphere-bundles $\phi : S(E) \to S(F)$ if and only if either $E \cong F$ or $E \cong \bar{F}$. The isomorphism $O_E \cong O_{\bar{E}}$ was noted in [20]. One can argue as follows. The conjugate bundle $\bar{E}$ has the same underlying real vector bundle as E but with opposite complex structure; the identity map $E \to \bar{E}$ is conjugate linear. If we endow $\bar{E}$ with the conjugate hermitian metric it follows that the identity map is fiberwise norm-preserving and hence it identifies $S(E)$ with $S(\bar{E})$.

Conversely, suppose that there is a homeomorphism of sphere-bundles $\phi : S(E) \to S(F)$. By homogeneity we can extend ϕ to a fiber-preserving homeomorphism $\Phi : E \to F$ such that $\Phi(E_0) \subset F_0$. Let $p_E : E \to X$ be the projection map and let i_E be the inclusion map $(E, \emptyset) \subset (E, E_0)$. Let us recall that the underlying real vector bundle $E_{\mathbb{R}}$ has a canonical preferred orientation which yields a Thom class $u_E \in H^2(E, E_0, \mathbb{Z})$, see [15, ch.9, ch.14]. Since X is path connected, $\mathbb{Z} \cong H^0(X,\mathbb{Z}) \cong H^2(E, E_0, \mathbb{Z}) = \mathbb{Z}u_E$ by the Thom isomorphism. The first Chern class $c_1(E)$ is equal to the Euler class $e(E_{\mathbb{R}}) =$

$(p_E^*)^{-1}i_E^*(u_E)$. The map Φ induces a commutative diagram

$$\begin{tikzcd}
\mathbb{Z}u_E = H^2(E,E_0,\mathbb{Z}) \arrow[r, "i_E^*"] & H^2(E,\mathbb{Z}) & H^2(X,\mathbb{Z}) \arrow[l, "p_E^*"'] \\
\mathbb{Z}u_F = H^2(F,F_0,\mathbb{Z}) \arrow[r, "i_F^*"] \arrow[u, "\Phi^*"] & H^2(F,\mathbb{Z}) \arrow[u, "\Phi^*"] & H^2(X,\mathbb{Z}) \arrow[l, "p_F^*"'] \arrow[u, equal]
\end{tikzcd}$$

Since Φ is a homeomorphism, $\Phi^*(u_F) = \pm u_E$ and hence $c_1(E) = \pm c_1(F)$. It follows that either $E \cong F$ or $E \cong \bar{F}$. □

3. K-theory invariants of O_E

In this section we construct a sequence $(\mu_n(E))_n$ of K-theory invariants of O_E. The class of the trivial bundle of rank r is denoted by $r \in K^0(X)$. All the elements of the ring $\widetilde{K}^0(X)$ are nilpotent [11].

Recall that for $E \in \mathrm{Vect}_{m+1}(X)$ we denote by $[\widetilde{E}]$ the $\widetilde{K}^0(X)$-component $[E]-(m+1)$ of $[E]$. We introduce the following equivalence relation on $\widetilde{K}^0(X)$: $a \sim b$, if and only if $a = b+mh+bh$ for some $h \in \widetilde{K}^0(X)$. Rewriting $a = b+mh+bh$ as $m+a = (m+b)(1+h)$ in $K^0(X)$, it becomes obvious that $\sim$ is an equivalence relation since $1+h$ in invertible in the ring $K^0(X)$ with inverse $1+\sum_{k\geq 1}(-1)^k h^k$. Moreover, if $E, F \in \mathrm{Vect}_{m+1}(X)$, then $(1-[E])K^0(X) = (1-[F])K^0(X)$ if and only if $[\widetilde{E}] \sim [\widetilde{F}]$.

Note that with our new notation, Theorem 1.1 shows that $O_E \cong O_F \Rightarrow [\widetilde{E}] \sim [\widetilde{F}]$. In other words the equivalence class of $[\widetilde{E}]$ in $\widetilde{K}^0(X)/\sim$ is an invariant of O_E. In order to obtain more computable invariants, for each $m \geq 1$, we use the sequence of polynomials $p_n \in \mathbb{Z}[x]$ introduced in (1). It is immediate that

$$p_{n+1}(x) = \frac{\ell(n+1)}{\ell(n)} m p_n(x) + (-1)^n \frac{\ell(n+1)}{n+1} x^{n+1}. \tag{3}$$

The first five polynomials in the sequence are:

$p_1(x) = x,$
$p_2(x) = 2mx - x^2,$
$p_3(x) = 6m^2x - 3mx^2 + 2x^3,$
$p_4(x) = 12m^3x - 6m^2x^2 + 4mx^3 - 3x^4,$
$p_5(x) = 60m^4x - 30m^3x^2 + 20m^2x^3 - 15mx^4 + 12x^5.$

Lemma 3.1. *For any $n \geq 1$ there are polynomials $u_n, s_{n+1} \in \mathbb{Z}[x,y]$ and $v_n \in \mathbb{Z}[x]$ such that each monomial of s_{n+1} has total degree $\geq n+1$ and*

(i) $p_n(x+y) = p_n(x) + p_n(y) + xy\, u_n(x,y)$,
(ii) $p_n(x+my+xy) = p_n(x) + m^n\, v_n(y) + s_{n+1}(x,y)$.

Proof. It follows from the binomial formula that for any polynomial $p \in \mathbb{Z}[x]$ with $p(0)=0$ there is a polynomial $u \in \mathbb{Z}[x,y]$ such that $p(x+y) = p(x) + p(y) + xy\, u(x,y)$. This proves (i). Let us now prove (ii). Set $V_n(x) = \sum_{k=1}^{n}(-1)^{k-1}x^k/k = \log(1+x)_{[n]}$. Then $p_n(x) = \ell(n)m^n V_n(x/m)$. Each monomial in x and y that appears in expansion of the series $\sum_{k\geq n+1}(-1)^{k-1}(x+y+xy)^k/k$ has total degree $\geq n+1$. Therefore, the equality of formal series $\log(1+x+y+xy) = \log(1+x)+\log(1+y)$ shows that in the reduced form of the polynomial $r_{n+1}(x,y) := V_n(x+y+xy) - V_n(x) - V_n(y)$ all the monomials have total degree $\geq n+1$. It follows that

$$\begin{aligned} p_n(x+my+xy) &= \ell(n)m^n V_n(x/m + y + x/m\cdot y) \\ &= \ell(n)m^n V_n(x/m) + \ell(n)m^n V_n(y) + \ell(n)m^n r_{n+1}(x/m, y) \\ &= p_n(x) + m^n \cdot v_n(y) + s_{n+1}(x,y), \end{aligned}$$

where $s_{n+1}(x,y) := \ell(n)m^n r_{n+1}(x/m,y)$ and $v_n(y) := \ell(n)V_n(y)$. Since both p_n and v_n have integer coefficients, so must have s_{n+1}. $\square$

Let X be a finite CW complex of dimension d with skeleton decomposition

$$\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_d = X.$$

Consider the induced filtration of $K^*(X)$

$$K_q^*(X) = \ker\big(K^*(X) \to K^*(X_{q-1})\big) = \mathrm{image}\big(K^*(X, X_{q-1}) \to K^*(X)\big).$$

One has $\{0\} = K_{d+1}^*(X) \subset K_d^*(X) \subset \cdots \subset K_1^*(X) \subset K_0^*(X) = K^*(X)$ and

$$K_q^*(X)K_r^*(X) \subset K_{q+r}^*(X).$$

Since the map $K^0(X_{2i+1}) \to K^0(X_{2i})$ is injective, it follows that $K_{2i+1}^0(X) = K_{2i+2}^0(X)$. We will use only the even components of this filtration corresponding to $K^0(X)$, namely

$$\{0\} = K^0_{2\lfloor d/2\rfloor+2}(X) \subset K^0_{2\lfloor d/2\rfloor}(X) \subset \cdots \subset K_2^0(X) \subset K_0^0(X) = K^0(X). \tag{4}$$

Since $\widetilde{K}^0(X) = K_1^0(X) = K_2^0(X)$ we have

$$\widetilde{K}^0(X)^j \subset K_{2j}^0(X). \tag{5}$$

In particular, $\widetilde{K}^0(X)^{\lfloor d/2\rfloor+1} \subset K_{d+1}^0(X)$ and hence $\widetilde{K}^0(X)^{\lfloor d/2\rfloor+1} = \{0\}$.

Definition 3.2. *Let X be a finite CW complex of dimension d. For each $n \geq 1$ we define the map*

$$\mu_n : \mathrm{Vect}_{m+1}(X) \to \widetilde{K}^0(X)/K_{2n+2}^0(X),$$

by $\mu_n(E) = \pi_n(p_n([\widetilde{E}]))$ where $\pi_n : K^0(X) \to \widetilde{K}^0(X)/K^0_{2n+2}(X)$ is the natural quotient map. For $n \geq \lfloor d/2\rfloor$, $\mu_n(E) = p_n([\widetilde{E}]) \in \widetilde{K}^0(X)$ since $K_{2n+2}^0(X) = \{0\}$.

Theorem 3.3. *Let X be a finite CW complex and let $E, F \in \mathrm{Vect}_{m+1}(X)$. If O_E and O_F are isomorphic as $C(X)$-algebras, then $\mu_n(E) - \mu_n(F)$ is divisible by m^n, for $n \geq 1$.*

Proof. Set $a = [\widetilde{E}]$ and $b = [\widetilde{F}]$. If $O_E \cong O_F$, then by Theorem 1.1 there is $h \in \widetilde{K}^0(X)$ such that $a = b + mh + bh$. Then, by Lemma 3.1 (ii)

$$p_n(a) = p_n(b + mh + bh) = p_n(b) + m^n v_n(h) + s_{n+1}(b, h),$$

and $s_{n+1}(b, h) \in \widetilde{K}^0(X)^{n+1} \subset K^0_{2n+2}(X)$ by (5). Thus $\mu_n(E) - \mu_n(F) = m^n \pi_n(v_n(h))$. □

As a corollary, we derive Theorem 1.3, restated here as follows:

Corollary 3.4. *Let X be a finite CW complex of dimension d and let $E, F \in \mathrm{Vect}_{m+1}(X)$. If $O_E \cong O_F$ as $C(X)$-algebras, then $p_{\lfloor d/2 \rfloor}([\widetilde{E}]) - p_{\lfloor d/2 \rfloor}([\widetilde{E}])$ is divisible by $m^{\lfloor d/2 \rfloor}$ in $\widetilde{K}^0(X)$.*

Proof. If $n \geq \lfloor d/2 \rfloor$, then $K^0_{2n+2}(X) = \{0\}$ and so $p_n(a) - p_n(b) \in m^n \widetilde{K}^0(X)$. □

Remark 3.5. *Let us note that $\mu_{\lfloor d/2 \rfloor}(E)$ determines $\mu_{\lfloor d/2 \rfloor + k}(E)$ for $k \geq 1$. Indeed, letting $n = \lfloor d/2 \rfloor$ it follows from (3) that*

$$\mu_{n+k}(E) = \frac{\ell(n+k)}{\ell(n)} m^k \mu_n(E),$$

since $\widetilde{K}^0(X)^{n+k} = \{0\}$. Let us note that $\mu_{\lfloor d/2 \rfloor}(E)$ is also related to the lower order invariants. Indeed, it follows immediately from (3) and (5) that if $1 \leq j \leq \lfloor d/2 \rfloor$, then

$$\frac{\ell(j)}{\ell(j-1)}\, m\, \mu_{j-1}(E) = \pi_{j,j-1}(\mu_j(E)),$$

where $\pi_{j,j-1}$ stands for the quotient map $\widetilde{K}^0(X)/K^0_{2j+2}(X) \to \widetilde{K}^0(X)/K^0_{2j}(X)$. From this, with $n = \lfloor d/2 \rfloor$, we obtain

$$\frac{\ell(n)}{\ell(n-j)}\, m^j\, \mu_{n-j}(E) = \pi_{n-j}(\mu_n(E)).$$

Assuming that $\mathrm{Tor}(K^0_{2j}(X)/K^0_{2j+2}(X), \mathbb{Z}/m) = 0$ for all $j \geq 1$, and that m and $\lfloor d/2 \rfloor!$ are relatively prime, it follows that if $\mu_{\lfloor d/2 \rfloor}(E) - \mu_{\lfloor d/2 \rfloor}(F)$ is divisible by $m^{\lfloor d/2 \rfloor}$, then $\mu_j(E) - \mu_j(F)$ is divisible by m^j for all $j \geq 1$.

The groups $\widetilde{K}^0(X)/K^0_{2j}(X)$ are homotopy invariants of X, and they are actually independent of the CW structure [1]. Let $k^j(X)$ denote the reduced connective K-theory of X and let $\beta : k^{j+2}(X) \to k^j(X)$ be the Bott operation. One can identify $k^{2j+2}(X)$ with $K^0(X, X_{2j})$ in such a way that β corresponds to the map $K^0(X, X_{2j}) \to K^0(X, X_{2j-2})$. Thus the image of $\beta^{j+1} : k^{2j+2}(X) \to k^0(X) \cong \widetilde{K}^0(X)$ coincides with $K^0_{2j}(X)$, and hence $\mu_j(E)$ can be viewed as an element of $k^0(X)/\beta^{j+1}k^{2j+2}(X)$.

4. Cohomology invariants of O_E

Let us recall that $V_n(x) = \log(1+x)_{[n]}$ and consider the polynomials

$$W_n(x) = \frac{n!}{\ell(n)} p_n(x) = n! m^n \log\left(1 + \frac{x}{m}\right)_{[n]} = n! m^n V_n(\frac{x}{m}) = \sum_{r=1}^{n} (-1)^{r-1} m^{n-r} \frac{n!}{r} x^r.$$

For a polynomial P in variables x_1,...x_n, we assign to the variable x_k the weight k and denote by $P(x_1, ..., x_n)_{\langle n\rangle}$ the sum of all monomials of P of total weight n. For example if $P(x_1, x_2, x_3) = (x_1 + \frac{x_2}{2} + \frac{x_3}{3})^2$, then $P(x_1, x_2, x_3)_{\langle 3\rangle} = x_1 x_2$. Consider the polynomials

$$\begin{aligned} q_n(x_1, ..., x_n) &= W_n\left(\frac{x_1}{1!} + ... + \frac{x_n}{n!}\right)_{\langle n\rangle} = \sum_{r=1}^{n} (-1)^{r-1} m^{n-r} \frac{n!}{r} \left(\frac{x_1}{1!} + ... + \frac{x_n}{n!}\right)^r_{\langle n\rangle} \\ &= \sum_{r=1}^{n} (-1)^{r-1} m^{n-r} \sum_{\substack{k_1+...+k_n = r,\\ k_1+2k_2+...+nk_n = n}} \frac{n!\, r!}{1!^{k_1} \cdots n!^{k_n}\, k_1! \cdots k_n!\, r} x_1^{k_1} \cdots x_n^{k_n}. \end{aligned}$$

Thus

$$q_n(x_1, ..., x_n) = \sum_{k_1+2k_2+...+nk_n = n} (-1)^{k_1+\cdots+k_n-1} m^{n-(k_1+\cdots+k_n)} \frac{n!\,(k_1+\cdots+k_n-1)!}{1!^{k_1} \cdots n!^{k_n}\, k_1! \cdots k_n!} x_1^{k_1} \cdots x_n^{k_n}.$$

Consider also the polynomials r_n obtained from q_n by taking $m = 1$, i.e.

$$r_n(x_1, ..., x_n) = n! V_n\left(\frac{x_1}{1!} + ... + \frac{x_n}{n!}\right)_{\langle n\rangle} = \sum_{r=1}^{n} (-1)^{r-1} \frac{n!}{r} \left(\frac{x_1}{1!} + ... + \frac{x_n}{n!}\right)^r_{\langle n\rangle}.$$

Lemma 4.1. *The polynomials $q_n(x_1, ..., x_n)$ and $r_n(x_1, ..., x_n)$ have integer coefficients.*

Proof. We have a factorization

$$\frac{n!\,(k_1+\cdots+k_n-1)!}{1!^{k_1} \cdots n!^{k_n}\, k_1! \cdots k_n!} = \frac{(k_1+2k_2+\cdots+nk_n)!}{(k_1)!(2k_2)! \cdots (nk_n)!} a(1,k_1) \cdots a(n,k_n)(k_1+\cdots+k_n-1)!,$$

where $a(j,k) = \frac{(jk)!}{(j!)^k\, k!}$. It follows that the coefficient of $x_1^{k_1} \cdots x_n^{k_n}$ is an integer since it involves a multinomial coefficient and numbers $a(j,k)$ which are easily seen to be integers by using the recurrence formula $a(j,k) = \binom{jk-1}{j-1} a(j,k-1)$ where $a(j,1) = 1$. $\square$

Let us recall from [11] that the components of the Chern character $ch(E) = \sum_{k\geq 0} s_k(E)/k!$ involve integral stable characteristic classes s_k, also denoted by ch_k. The classes s_k have two important properties. If one sets $s_0(E) = \mathrm{rank}(E)$, then for $k \geq 0$:

$$s_k(E \oplus F) = s_k(E) + s_k(F)$$

$$s_k(E \otimes F) = \sum_{i+j=k} \frac{k!}{i!j!} s_i(E) s_j(F). \tag{6}$$

We are now ready to prove Theorem 1.5, restated here for the convenience of the reader:

Theorem 4.2. *Let X be a compact metrizable space and let $E, F \in \mathrm{Vect}_{m+1}(X)$. If $O_E \cong O_F$ as $C(X)$-algebras, then $q_n(s_1(E), ..., s_n(E)) - q_n(s_1(F), ..., s_n(F))$ is divisible by m^n in $H^{2n}(X, \mathbb{Z})$, for each $n \geq 1$.*

Proof. Multiplying by $n!/\ell(n)$ in Lemma 3.1, (ii), we obtain

$$W_n(y + mh + yh) - W_n(y) = m^n n!\, V_n(h) + \frac{n!}{\ell(n)} s_{n+1}(y, h). \tag{7}$$

Recall that s_{n+1} is a polynomial with all monomials of degree at least $n+1$. Let us make in (7) the substitutions

$$y = \frac{y_1}{1!} + \cdots + \frac{y_n}{n!}, \quad h = \frac{h_1}{1!} + \cdots + \frac{h_n}{n!},$$

where y_k and h_k have weight k. With these substitutions we have

$$W_n(y)_{\langle n \rangle} = q_n(y_1, ..., y_n), \quad n!\, V_n(h)_{\langle n \rangle} = r_n(h_1, ..., h_n),$$

whereas $\frac{n!}{\ell(n)} s_{n+1}(y, h)_{\langle n \rangle} = 0$. By grouping the terms of $y + mh + yh$ of the same weight, we have

$$y + mh + yh = \frac{y_1 + mh_1}{1!} + \frac{y_2 + mh_2 + 2y_1 h_1}{2!} + \cdots + \frac{y_n + mh_n + \sum_{i+j=n} \frac{n!}{i!j!} y_i h_j}{n!},$$

and hence

$$W_n(y + mh + yh)_{\langle n \rangle} = q_n(y_1 + mh_1,\, y_2 + mh_2 + 2y_1 h_1\,, \ldots, y_n + mh_n + \sum_{i+j=n} \frac{n!}{i!j!} y_i h_j)$$

Thus, the equation $W_n(y + mh + yh)_{\langle n \rangle} - W_n(y)_{\langle n \rangle} = n!\, V_n(h)_{\langle n \rangle}$ implies that

$$q_n(y_1 + mh_1,\, y_2 + mh_2 + 2y_1 h_1,\, \ldots, y_n + mh_n + \sum_{i+j=n} \frac{n!}{i!j!} y_i h_j) - q_n(y_1, ..., y_n)$$

is equal to $m^n r_n(h_1, ..., h_n)$.

Suppose now $O_E \cong O_F$. Then, by Theorem 1.1, $[\widetilde{E}] = [\widetilde{F}] + mH + [\widetilde{F}]H$ for some $H \in \widetilde{K}^0(X)$. Using (6) it follows that for $k \geq 1$

$$s_k(E) = s_k(F) + m\, s_k(H) + \sum_{i+j=k} \frac{k!}{i!j!} s_i(F) s_j(H),$$

and hence

$$q_n(s_1(E), ..., s_n(E)) - q_n(s_1(F), ..., s_n(F)) = m^n\, r_n(s_1(H), ..., s_n(H)).$$

□

It follows by Theorem 4.2, that the image of $q_n(s_1(E), ..., s_n(E))$ in $H^{2n}(X, \mathbb{Z}/m^n)$ is an invariant of O_E for each $n \geq 1$. The first four invariants in this sequence are:

$\dot{s}_1(E) \in H^2(X; \mathbb{Z}/m)$
$m\dot{s}_2(E) - \dot{s}_1(E)^2 \in H^4(X; \mathbb{Z}/m^2)$

$m^2\dot{s}_3(E) - 3m\dot{s}_1(E)\dot{s}_2(E) + 2\dot{s}_1(E)^3 \in H^6(X;\mathbb{Z}/m^3)$
$m^3\dot{s}_4(E) - m^2(3\dot{s}_2(E)^2 + 4\dot{s}_1(E)\dot{s}_3(E)) + 12m\dot{s}_1(E)^2\dot{s}_2(E) - 6\dot{s}_1(E)^4 \in H^8(X;\mathbb{Z}/m^4)$

The classes $s_k(E)$ are related to the Chern classes $c_k(E)$ via the Newton polynomials:

$$s_k(E) = Q_k(c_1(E), ..., c_k(E)) \in H^{2k}(X;\mathbb{Z}),$$

which express the symmetric power sum functions in terms of elementary symmetric functions σ_i. The first four Newton polynomials are

$Q_1(\sigma_1) = \sigma_1,$
$Q_2(\sigma_1, \sigma_2) = \sigma_1^2 - 2\sigma_2,$
$Q_3(\sigma_1, \sigma_2, \sigma_3) = \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3,$
$Q_1(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = \sigma_1^4 - 4\sigma_1^2\sigma_2 + 4\sigma_1\sigma_3 + 2\sigma_2^2 - 4\sigma_4.$

By expressing s_k in terms of Chern classes we obtain a sequence of characteristic classes of E which are invariants of O_E. The first four classes in the sequence are:

(1) $\dot{c}_1(E) \in H^2(X;\mathbb{Z}/m)$
(2) $(m-1)\dot{c}_1(E)^2 - 2m\dot{c}_2(E) \in H^4(X;\mathbb{Z}/m^2)$
(3) $(m^2 - 3m + 2)\dot{c}_1(E)^3 - (3m^2 - 6m)\dot{c}_1(E)c_2(E) + 3m^2\dot{c}_3(E) \in H^6(X;\mathbb{Z}/m^3)$
(4) $(m^3 - 7m^2 + 12m - 6)\dot{c}_1(E)^4 - (4m^3 - 24m^2 + 24m)\dot{c}_1(E)^2\dot{c}_2(E) +$
$(4m^3 - 12m^2)\dot{c}_1(E)\dot{c}_3(E) + (2m^3 - 12m^2)\dot{c}_2(E)^2 - 4m^3\dot{c}_4(E) \in H^8(X;\mathbb{Z}/m^4)$

Here we denote by $\dot{c}_k(E)$ the image of the Chern class $c_k(E)$ under the coefficient map $H^{2k}(X,\mathbb{Z}) \to H^{2k}(X,\mathbb{Z}/m^k)$.

Corollary 4.3. *Let X be a compact metrizable space and let L and L' be two line bundles over X. If $O_{m+L} \cong O_{m+L'}$ as $C(X)$-algebras, then $q_n(1, ..., 1)(c_1(L)^n - c_1(L')^n)$ is divisible by m^n for all $n \geq 1$.*

Proof. If L is a line bundle, then $s_k(L) = c_1(L)^k$. Since all monomials of q_n have weight n, $q_n(s_1(L), ..., s_n(L)) = q_n(1, ..., 1)c_1(L)^n$. □

Let us note that there is a more direct way to derive cohomology invariants for O_E.

Proposition 4.4. *Let X be a compact metrizable space. Then $s_n(p_n([\widetilde{E}]))$ is an element of $H^{2n}(X,\mathbb{Z})$ whose image in $H^{2n}(X,\mathbb{Z}/m^n)$ is an invariant of O_E, $n \geq 1$.*

Proof. Suppose that $O_E \cong O_F$ as $C(X)$-algebras. Then, we saw in the proof of the Theorem 3.3 that $p_n([\widetilde{E}]) - p_n([\widetilde{F}]) = m^n c + d$ for some $c \in \widetilde{K}^0(X)$ and $d \in \widetilde{K}^0(X)^{n+1}$. From the multiplicative properties of the s_n-classes (6), one deduces that s_n vanishes on $\widetilde{K}^0(X)^{n+1}$. Therefore $s_n(p_n([\widetilde{E}])) - s_n(p_n([\widetilde{F}])) = m^n s_n(c)$. □

We note that this is not really a new invariant, since it is not hard to prove that

$$s_n(p_n([\widetilde{E}]) = \ell(n)q_n(s_1(E), ..., s_n(E)).$$

Theorem 4.2 shows that we can remove the factor $\ell(n)$ and hence obtain a finer invariant.

5. Proof of Theorem 1.6

Recall that $\mathcal{O}_{m+1}(X)$ denotes the set of isomorphism classes of unital separable $C(X)$-algebras with all fibers isomorphic to O_{m+1}. These $C(X)$-algebras are automatically locally trivial if X is finite dimensional.

For a discrete abelian group G and $n \geq 1$ let $K(G, n)$ be an Eilenberg-MacLane space. It is a connected CW complex Y having just one nontrivial homotopy group $\pi_n(Y) \cong G$. A $K(G, n)$ space is unique up to homotopy equivalence. For a CW complex X, there is an isomorphism $H^n(X, G) \cong [X, K(G, n)]$.

Let us recall from [9] that if Y is a connected CW complex with $\pi_1(Y)$ acting trivially on $\pi_n(Y)$ for $n \geq 1$, then Y admits a Postnikov tower

$$\cdots \to Y_n \to Y_{n-1} \to \cdots \to Y_2 \to Y_1 = K(\pi_1(Y), 1).$$

Each space Y_n carries the homotopy groups of Y up to level n. More precisely, there exist compatible maps $Y \to Y_n$ that induce isomorphisms $\pi_i(Y) \to \pi(Y_n)$ for $i \leq n$ and $\pi_i(Y_n) = 0$ for $i > n$. Each map $Y_n \to Y_{n-1}$ is a fibration with fiber $K(\pi_n(Y), n)$. Thus Y_n can be thought as a twisted product of Y_{n-1} by $K(\pi_n(Y), n)$. The space Y is weakly homotopy equivalent to the projective limit $\varprojlim Y_n$.

Proposition 5.1. *Let X be a finite connected CW complex and let $m \geq 1$ be an integer. Then $|\mathcal{O}_{m+1}(X)| \leq |\widetilde{H}^{even}(X, \mathbb{Z}/m)|$.*

Proof. Let Y be a CW complex weakly homotopy equivalent to the classifying space $B\mathrm{Aut}(O_{m+1})$ of principal $\mathrm{Aut}(O_{m+1})$-bundles. Then, there are bijections

$$\mathcal{O}_{m+1}(X) \cong [X, B\mathrm{Aut}(O_{m+1})] \cong [X, Y].$$

The homotopy groups of $\mathrm{Aut}(O_{m+1})$ were computed in [4]. That calculation gives $\pi_{2k-1}(Y) = 0$ and $\pi_{2k}(Y) = \mathbb{Z}/m$, $k \geq 1$. Consequently, the Postnikov tower of Y reduces to its even terms

$$\cdots \to Y_{2k} \to Y_{2k-2} \to \cdots \to Y_2 = K(\mathbb{Z}/m, 2).$$

The homotopy sequence of the fibration $K(\mathbb{Z}/m, 2k) \to Y_{2k} \to Y_{2k-2}$ gives for all choices of the base points an exact sequence of sets

$$[X, K(\mathbb{Z}/m, 2k)] \to [X, Y_{2k}] \to [X, Y_{2k-2}].$$

This shows that

$$|[X, Y_{2k}]| \leq |[X, Y_{2k-2}]| \cdot |H^{2k}(X, \mathbb{Z}/m)|.$$

By Whitehead's theorem, if $n > \dim(X)/2$, then the map $Y \to Y_{2n}$ induces a bijection $[X, Y] \cong [X, Y_{2n}]$. It follows that

$$|[X, Y]| \leq \prod_{1 \leq k \leq n} |H^{2k}(X, \mathbb{Z}/m)| = |\widetilde{H}^{even}(X, \mathbb{Z}/m)|. \qquad \square$$

We consider a commutative ring R which admits a filtration by ideals

$$\cdots \subset R_{k+1} \subset R_k \subset \cdots \subset R_1 = R$$

with the property that $R_q R_k \subset R_{q+k}$ and there is n such that $R_{n+1} = \{0\}$. On R we consider the following equivalence relation: $a \sim b$ if there is $h \in R$ such that $a = b+mh+bh$. Let us denote by $R/\!\sim$ the set of equivalence classes.

Lemma 5.2. *Let R be a filtered commutative ring with $R_{n+1} = \{0\}$. Suppose that* $\mathrm{Tor}(R_k/R_{k+1}, \mathbb{Z}/m) = 0$ *for all $k \geq 1$. Then* $|R/\!\sim| = |R \otimes \mathbb{Z}/m| = \prod_{k\geq 1} |R_k/R_{k+1} \otimes \mathbb{Z}/m|$.

Proof. Using the exact sequence for Tor, we observe first that $\mathrm{Tor}(R_1/R_k, \mathbb{Z}/m) = 0$ for all $k \geq 1$ and hence if $h \in R$ satisfies $mh \in R_k$ for some k, then $h \in R_k$. Using the exact sequences

$$0 \to R_{k+1} \otimes \mathbb{Z}/m \to R_k \otimes \mathbb{Z}/m \to R_k/R_{k+1} \otimes \mathbb{Z}/m \to 0$$

we see that $|R \otimes \mathbb{Z}/m| = \prod_{k\geq 1} |R_k/R_{k+1} \otimes \mathbb{Z}/m|$. For each k choose a finite subset $A_k \subset R_k$ such that the quotient map $\pi_k : R_k \to R_k/R_{k+1} \otimes \mathbb{Z}/m$ induces a bijective map $\pi_k : A_k \to R_k/R_{k+1} \otimes \mathbb{Z}/m$. Consider the map $\eta : A_1 \times A_2 \times \cdots \times A_n \to R$ defined by $\eta(a_1, \dots, a_n) = a_1 + \cdots + a_n$. To prove the proposition it suffices to show that η induces a bijection of $\bar{\eta} : A_1 \times A_2 \times \cdots \times A_n \to R/\!\sim$. First we verify that $\bar{\eta}$ is injective. Let $a_k, b_k \in A_k$, $1 \leq k \leq n$ and assume that

$$a_1 + \cdots + a_n \sim b_1 + \cdots + b_n.$$

We must show that $a_k = b_k$ for all k. Set $r_k = a_k + \cdots + a_n$ and $s_k = b_k + \cdots + b_n$. Then $r_k, s_k \in R_k$. Since $a_1 + r_2 \sim b_1 + s_2$ there is $h_1 \in R_1$ such that

$$a_1 + r_2 = b_1 + s_2 + mh_1 + b_1 h_1 + s_2 h_1$$

and hence $a_1 - b_1 - mh_1 \in R_2$. Therefore $\pi_1(a_1) = \pi_1(b_1)$ and so $a_1 = b_1$. Arguing by induction, suppose that we have shown that $a_i = b_i$ for all $i \leq k-1$. Set $w = a_1 + \cdots + a_{k-1}$. By assumption $w + r_k \sim w + s_k$ and hence there is $h \in R_1$ such that

$$w + r_k = w + s_k + mh + wh + s_k h. \tag{8}$$

Let us notice that if $h \in R_i$ for some $i \leq k-1$, then equation (8) shows that $mh = (r_k - s_k) - wh - s_k h \in R_k \cup R_{i+1} = R_{i+1}$ and hence $h \in R_{i+1}$. This shows that in fact $h = h_k \in R_k$. From equation (8) we obtain

$$a_k - b_k - mh_k = s_{k+1} - r_{k+1} + (w + s_k)h_k \in R_{k+1}.$$

This shows that $\pi_k(a_k) = \pi_k(b_k)$ and hence $a_k = b_k$.

It remains to verify that $\bar{\eta}$ is surjective. In other words for any given $x_1 \in R_1$ we must find $a_k \in A_k$, $1 \leq k \leq n$, such that $x_1 \sim a_1 + \cdots + a_n$. We do this by induction, showing that for each $k \geq 1$ there exist $a_i \in A_i$, $1 \leq i \leq k$ and $x_{k+1} \in R_{k+1}$ such that

$$x_1 \sim a_1 + \cdots + a_k + x_{k+1},$$

and observe that $x_{n+1} = 0$ since $R_{n+1} = \{0\}$.

By the definition of A_1 there is $a_1 \in A_1$ such that $\pi_1(a_1) = \pi_1(x_1) \in R_1/R_2 \otimes \mathbb{Z}/m$. Therefore there exist $h_1 \in R_1$ and $y_2 \in R_2$ such that $a_1 = x_1 + mh_1 + y_2$. Setting $x_2 = x_1h_1 - y_2 \in R_2$ we obtain

$$x_1 \sim x_1 + mh_1 + x_1h_1 = a_1 + x_2.$$

Suppose now that we found $a_i \in A_i$, $1 \le i \le k-1$ and $x_k \in R_k$ such that

$$x_1 \sim a_1 + \cdots + a_{k-1} + x_k.$$

Let $a_k \in A_k$ be such that $\pi_k(a_k) = \pi_k(x_k)$. Then there exist $h_k \in R_k$ and $y_{k+1} \in R_{k+1}$ such that $a_k = x_k + mh_k + y_{k+1}$. Thus

$$\begin{aligned} a_1 + \cdots + a_{k-1} + x_k &\sim a_1 + \cdots + a_{k-1} + x_k + mh_k + (a_1 + \cdots + a_{k-1} + x_k)h_k \\ &= a_1 + \cdots + a_{k-1} + a_k + x_{k+1} \end{aligned}$$

where $x_{k+1} = (a_1 + \cdots + a_{k-1} + x_k)h_k - y_{k+1} \in R_{k+1}$. □

Theorem 5.3. *Let X be a finite connected CW complex of dimension d and let $m \ge 1$ be an integer. Suppose that* $\mathrm{Tor}(H^*(X,\mathbb{Z}), \mathbb{Z}/m) = 0$ *and that $m \ge \lceil (d-3)/2 \rceil$. Then:*

(i) *Any separable unital $C(X)$-algebra with fiber O_{m+1} is isomorphic to O_E for some $E \in Vect_{m+1}(X)$.*
(ii) *If $E, F \in Vect_{m+1}(X)$, then $O_E \cong O_F$ as $C(X)$-algebras if and only if there is $h \in K^0(X)$ such that $1 - [E] = (1 - [F])h$.*
(iii) *The cardinality of $\mathcal{O}_{m+1}(X)$ is equal to $|\widetilde{K}^0(X) \otimes \mathbb{Z}/m| = |\widetilde{H}^{even}(X, \mathbb{Z}/m)|$.*

Proof. Part (ii) is already contained in Theorem 1.1 but we state it again nevertheless since a new proof of the implication $(1-[E])\widetilde{K}^0(X) = (1-[F])\widetilde{K}^0(X) \Rightarrow O_E \cong O_F$ is given here under the assumptions from the statement of the theorem. Recall that we defined an equivalence relation on $\widetilde{K}^0(X)$ by $a \sim b$ if and only if $a = b + mh + bh$ for some $h \in \widetilde{K}^0(X)$. Let $\gamma : Vect_{m+1}(X) \to \widetilde{K}^0(X)/\!\sim$ be the map which takes E to the equivalence class of $[\widetilde{E}] = [E] - m - 1$. We saw in Section 3 that $1 - [E] = (1 - [F])k$ for some $k \in K^0(X)$ if and only if $[\widetilde{E}] \sim [\widetilde{F}]$ in $\widetilde{K}^0(X)$, i.e. $\gamma(E) = \gamma(F)$. Let $\omega : Vect_{m+1}(X) \to \mathcal{O}_{m+1}(X)$ the map which takes E to the isomorphism class of the $C(X)$-algebra O_E. We shall construct a bijective map χ such that the diagram

$$\begin{tikzcd} Vect_{m+1}(X) \arrow[r, "\omega"] \arrow[d, two heads, "\gamma"'] & \mathcal{O}_{m+1}(X) \arrow[dl, dashed, "\chi"] \\ \widetilde{K}^0(X)/\!\sim & \end{tikzcd}$$

is commutative. Let us note that in order to prove the parts (i), (ii) and (iii) of the theorem it suffices to verify the following four conditions.

(a) γ is surjective;

(b) If $\omega(E) = \omega(F)$ then $\gamma(E) = \gamma(F)$;
(c) $|\mathcal{O}_{m+1}(X)| \leq |\widetilde{H}^{even}(X, \mathbb{Z}/m)|$;
(d) $|\widetilde{K}^0(X)/\sim| = |\widetilde{K}^0(X) \otimes \mathbb{Z}/m| = |\widetilde{H}^{even}(X, \mathbb{Z}/m)|$.

Indeed, from (a) and (b) we see that there is a well-defined surjective map $\chi : \text{image}(\omega) \to \widetilde{K}^0(X)/\sim$, given by $\chi(\omega(E)) = \gamma(E)$ and hence $|\widetilde{K}^0(X)/\sim| \leq |\text{image}(\omega)| \leq |\mathcal{O}_{m+1}(X)|$. On the other hand from (c) and (d) we deduce that $|\mathcal{O}_{m+1}(X)| \leq |\widetilde{K}^0(X)/\sim|$. Altogether this implies that ω is surjective and χ is bijective.

It remains to verify the four conditions from above. If $m \geq \lceil (d-3)/2 \rceil$, then the map $Vect_{m+1}(X) \to \widetilde{K}^0(X)$, $E \mapsto [E]-m-1$ is surjective by [10, Thm. 1.2] and this implies (a). Condition (b) follows from Theorem 1.1 and condition (c) was proved in Proposition 5.1. It remains to verify condition (d) using the assumption that $\mathrm{Tor}(H^*(X,\mathbb{Z}), \mathbb{Z}/m) = 0$. The first step is to use a known argument to deduce the absence of m-torsion in the K-theory of X and its skeleton filtration. We will then appeal to Lemma 5.2 to conclude the proof.

Let p be a prime which divides m. Then $\mathrm{Tor}(H^*(X,\mathbb{Z}), \mathbb{Z}/p) = 0$ by assumption. Let $\mathbb{Z}_{(p)}$ denote $\mathbb{Z}$ localized at p, i.e. the subring of $\mathbb{Q}$ consisting of all fractions with denominator prime to p. Let (E_r, d_r) be the Atiyah-Hirzebruch spectral sequence $H^*(X,\mathbb{Z}) \Rightarrow K^*(X)$. Recall that $E_2^{s,t} = H^s(X, K^t(pt))$ and $E_\infty^{s,t} = K_s^{s+t}(X)/K_{s+1}^{s+t}(X)$, see [1]. Since $\mathbb{Z}_{(p)}$ is torsion free, it follows from the universal coefficient theorem that the spectral sequence $(E_r \otimes \mathbb{Z}_{(p)}, d_r \otimes 1)$ is convergent to $K^*(X) \otimes \mathbb{Z}_{(p)}$. On the other hand since all the differentials d_r are torsion operators by [1, 2.4] and since $H^*(X,\mathbb{Z})$ has no p-torsion, it follows that $d_r \otimes 1 = 0$ for all $r \geq 2$ and hence $E_2 \otimes \mathbb{Z}_{(p)} = E_\infty \otimes \mathbb{Z}_{(p)}$. Therefore for all $q \geq 0$

$$H^{2q}(X,\mathbb{Z}) \otimes \mathbb{Z}_{(p)} \cong \big(K^0_{2q}(X)/K^0_{2q+2}(X)\big) \otimes \mathbb{Z}_{(p)}. \tag{9}$$

Since $\mathrm{Tor}(G \otimes \mathbb{Z}_{(p)}, \mathbb{Z}/p) \cong \mathrm{Tor}(G, \mathbb{Z}/p)$ for all finitely generated abelian groups G, it follows that $\mathrm{Tor}(K^0_{2q}(X)/K^0_{2q+2}(X), \mathbb{Z}/p) = 0$ for any prime p that divides m. Therefore for all $q \geq 0$ we have

$$\mathrm{Tor}(K^0_{2q}(X)/K^0_{2q+2}(X), \mathbb{Z}/m) = 0.$$

This enables us to apply Lemma 5.2 for the ring $R = \widetilde{K}^0(X) = K^0_2(X)$ filtered by the ideals $R_q = \widetilde{K}^0_{2q}(X)$ to obtain that

$$|\widetilde{K}^0(X)/\sim| = |\widetilde{K}^0(X) \otimes \mathbb{Z}/m| = \prod_{q \geq 1} |K^0_{2q}(X)/K^0_{2q+2}(X) \otimes \mathbb{Z}/m|.$$

Since $\mathrm{Tor}(H^*(X,\mathbb{Z}), \mathbb{Z}/m) = 0$, we have $H^{2q}(X, \mathbb{Z}/m) \cong H^{2q}(X,\mathbb{Z}) \otimes \mathbb{Z}/m$. From equation (9) we deduce that

$$H^{2q}(X,\mathbb{Z}) \otimes \mathbb{Z}/m \cong \big(K^0_{2q}(X)/K^0_{2q+2}(X)\big) \otimes \mathbb{Z}/m.$$

This completes the proof (d). $\square$

6. Proof of Theorem 1.4

Lemma 6.1. *Let R be a filtered commutative ring with $R_{n+1} = \{0\}$. Suppose that* $\mathrm{Tor}(R_k/R_{k+1}, \mathbb{Z}/m) = 0$ *for all $k \geq 1$ and that m and $n!$ are relatively prime. If $a, b \in R$ and $p_n(a) - p_n(b) \in m^n R$, then $a \sim b$.*

Proof. We prove this by induction on n. Suppose first that $n = 1$. Then $p_1(a) - p_1(b) = a - b \in mR$ by assumption and so $a = b + mh$ for some $h \in R$. Since $R_2 = \{0\}$ by assumption, $bh = 0$ and so $a = b + mh + bh$, i.e. $a \sim b$. Suppose now that the statement is true for a given n for all filtered rings R as in the statement. Let R be now a filtered ring such that $R_{n+2} = \{0\}$, m and $(n+1)!$ are relatively prime and $\mathrm{Tor}(R_k/R_{k+1}, \mathbb{Z}/m) = 0$ for all $k \geq 1$. Consider the ring $S := R/R_{n+1}$ with filtration $S_k = R_k/R_{n+1}$, $S_{n+1} = \{0\}$, and the quotient map $\pi : R \to S$. Let $a, b \in R$ satisfy $p_{n+1}(a) - p_{n+1}(b) \in m^{n+1}R$. Since

$$p_{n+1}(x) = \frac{\ell(n+1)}{\ell(n)} m p_n(x) + (-1)^n \frac{\ell(n+1)}{n+1} x^{n+1}.$$

we deduce that $\frac{\ell(n+1)}{\ell(n)} m\big(p_n(\pi(a)) - p_n(\pi(b))\big) \in m^{n+1}S$. Since $Tor(S, \mathbb{Z}/m) = 0$ and $(n+1)!$ and m are relatively prime it follows that $p_n(\pi(a)) - p_n(\pi(b)) \in m^n S$. Since $S_{n+1} = 0$ we obtain by the inductive hypothesis that $\pi(a) \sim \pi(b)$ in S and hence $a = b + mh + bh + r_{n+1}$ for some $h \in R$ and $r_{n+1} \in R_{n+1}$. We have that $(b + mh + bh) \cdot r_{n+1} = 0$ and $r_{n+1}^2 = 0$ as these are elements of $R^{n+2} \subset R_{n+2} = \{0\}$. Therefore, by Lemma 3.1(i),

$$\begin{aligned} p_{n+1}(a) = p_{n+1}(b + mh + bh + r_{n+1}) &= p_{n+1}(b + mh + bh) + p_{n+1}(r_{n+1}) \\ &= p_{n+1}(b + mh + bh) + \ell(n+1) m^n r_{n+1} \end{aligned}$$

On the other hand, $p_{n+1}(b + mh + bh) = p_{n+1}(b) + m^{n+1} v_{n+1}(h)$ by Lemma 3.1(ii), since $R_{n+2} = \{0\}$. Therefore

$$p_{n+1}(a) - p_{n+1}(b) = m^{n+1} v_{n+1}(h) + \ell(n+1) m^n r_{n+1}.$$

Since $p_{n+1}(a) - p_{n+1}(b) \in m^{n+1}R$ by assumption, we obtain that $\ell(n+1) m^n r_{n+1} \in m^{n+1}R$. Since $\mathrm{Tor}(R, \mathbb{Z}/m) = 0$ we deduce that $\ell(n+1) r_{n+1} \in mR$ and hence that $r_{n+1} = m\, h_{n+1}$ for some $h_{n+1} \in R$ since m is relatively prime to $\ell(n+1)$. We must have that in fact $h_{n+1} \in R_{n+1}$ since $\mathrm{Tor}(R/R_{n+1}, \mathbb{Z}/m) = 0$ and hence $b\, h_{n+1} = 0$. It follows that $a \sim b$ since we can now rewrite $a = b + mh + bh + r_{n+1}$ as $a = b + m(h + h_{n+1}) + b(h + h_{n+1})$. □

We are now in position to prove Theorem 1.4.

Proof. By Theorem 1.1 it suffices to show that $[\widetilde{E}] \sim [\widetilde{F}]$ whenever $p_{\lfloor d/2 \rfloor}([\widetilde{E}]) - p_{\lfloor d/2 \rfloor}([\widetilde{F}])$ is divisible by $m^{\lfloor d/2 \rfloor}$. We have seen in the proof of Theorem 5.3 that if $\mathrm{Tor}(H^*(X, \mathbb{Z}), \mathbb{Z}/m) = 0$ then $\mathrm{Tor}(K^0_{2q}(X)/K^0_{2q+2}(X), \mathbb{Z}/m) = 0$. Therefore the desired implication follows from Lemma 6.1 applied to the ring $\widetilde{K}^0(X)$ filtered as in (4) with $n = \lfloor d/2 \rfloor$. □

7. SUSPENSIONS

In this final part of the paper we study $\mathcal{O}_{m+1}(SX)$ and the image of the map $\mathrm{Vect}(SX) \to \mathcal{O}_{m+1}(SX)$. We shall use the universal coefficient exact sequence

$$0 \to K^1(X) \otimes \mathbb{Z}/m \xrightarrow{\bar{\rho}} K^1(X, \mathbb{Z}/m) \xrightarrow{\beta} \mathrm{Tor}(K^0(X), \mathbb{Z}/m) \to 0,$$

where β is the Bockstein operation and $\bar{\rho}$ is induced by the coefficient map ρ.

Theorem 7.1. *Let X be a compact metrizable space and let $m \geq 1$.*

(i) *There is a bijection $\gamma : \mathcal{O}_{m+1}(SX) \to K_1(C(X) \otimes O_{m+1}) \cong K^1(X, \mathbb{Z}/m)$.*

(ii) *If $E, F \in Vect_{m+1}(SX)$, then $O_E \cong O_F$ as $C(SX)$-algebras if and only if $[E] - [F] \in mK^0(SX)$.*

(iii) *If $A \in \mathcal{O}_{m+1}(SX)$ and $\beta(\gamma(A)) \neq 0$, then A is not isomorphic to O_E for any $E \in Vect_{m+1}(SX)$.*

Proof. Part (i) is proved in [5]. We revisit the argument from [5] as it is needed for the proof of the other two parts. Let $v_1, ..., v_{m+1}$ be the canonical generators of O_{m+1}. There is natural a map $\gamma_0 : \mathrm{Aut}(O_{m+1}) \to U(O_{m+1})$ which maps an automorphism φ to the unitary $\sum_{j=1}^{m+1} \varphi(v_j)v_j^*$. We showed in [5, Thm. 7.4] that γ_0 induces a bijection of homotopy classes $[X, \mathrm{Aut}(O_{m+1})] \to [X, U(O_{m+1})]$. By [17] there is a $*$-isomorphism $\nu : O_{m+1} \to M_{m+1}(O_{m+1})$. We have bijections $\mathcal{O}_{m+1}(SX) \cong [SX, B\mathrm{Aut}(O_{m+1})] \cong [X, \mathrm{Aut}(O_{m+1})]$ and

$$[X, \mathrm{Aut}(O_{m+1})] \xrightarrow{(\gamma_0)_*} [X, U(O_{m+1})] \xrightarrow{\nu_*} [X, U(M_{m+1}(O_{m+1}))] \cong K_1(C(X) \otimes \mathcal{O}_{m+1}) \ .$$

The composition of these maps defines the bijection γ from (i). We are now prepared to prove (ii) and (iii). Consider the monomorphism of groups $\alpha : U(m+1) \to \mathrm{Aut}(O_{m+1})$ introduced in [8]. If $u \in U(m+1)$ has components u_{ij}, then $\alpha_u(v_j) = \sum_{i=1}^{m+1} u_{ij}v_i$. The map α induces a map $BU(m+1) \to B\mathrm{Aut}(O_{m+1})$ which in its turn induces the natural map $\alpha_* : Vect_{m+1}(Y) \to \mathcal{O}_{m+1}(Y)$ that we are studying. Let η be the composition of the maps from the diagram

$$U(m+1) \xrightarrow{\alpha} \mathrm{Aut}(O_{m+1}) \xrightarrow{\gamma_0} U(O_{m+1}) \xrightarrow{\nu} U(M_{m+1}(O_{m+1})).$$

An easy calculation shows that $\eta(u) = \sum_{i,j=1}^{m+1} u_{ij}\nu(v_iv_j^*)$, where u_{ij} are the components of the unitary u. Let us observe that η is induced by a unital $*$-homomorphism $\bar{\eta} : M_{m+1}(\mathbb{C}) \to M_{m+1}(O_{m+1})$. It follows that there is a unitary $w \in M_{m+1}(O_{m+1})$ such that $w\bar{\eta}(a)w^* = a \otimes 1_{O_{m+1}}$, for all $a \in M_{m+1}(\mathbb{C})$. This implies that η will induce the coefficient map $\rho : K^1(X) \to K^1(X, \mathbb{Z}/m)$.

We a commutative diagram

$$\begin{tikzcd}
Vect_{m+1}(SX) \arrow[r] \arrow[d, "\alpha_*"'] & {[SX, BU(m+1)]} \arrow[r] \arrow[d] & {[X, U(m+1)]} \arrow[r, equal] \arrow[d, "\alpha_*"'] & {[X, U(M_{m+1}(\mathbb{C}))]} \arrow[d, "\eta_*"] \\
\mathcal{O}_{m+1}(SX) \arrow[r] & {[SX, B\mathrm{Aut}(O_{m+1})]} \arrow[r] & {[X, \mathrm{Aut}(O_{m+1})]} \arrow[r, "(\nu\gamma_0)_*"] & {[X, U(M_{m+1}(O_{m+1}))]}
\end{tikzcd}$$

and hence a commutative diagram

$$\begin{tikzcd}
 & Vect_{m+1}(SX) \arrow[r, "\alpha_*"] \arrow[d] & \mathcal{O}_{m+1}(SX) \arrow[d, "\gamma"] & \\
K^1(X) \arrow[r, "\times m"] & K^1(X) \arrow[r, "\rho"] & K^1(X, \mathbb{Z}/m) \arrow[r, "\beta"] & \mathrm{Tor}(K^0(X), \mathbb{Z}/m) \to 0
\end{tikzcd}$$

Now both (*ii*) and (*iii*) follow from the commutativity of the diagram above and the exactness of its bottom row. □

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