Approximate unitary equivalence and the topology of $\mathrm{Ext}(A, B)$

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Abstract

Let A, B be unital C*-algebras and assume that A is separable and quasidiagonal relative to B. Let $\varphi, \psi : A \to B$ be unital $*$-homomorphisms. If A is nuclear and satisfies the UCT, we prove that φ is approximately stably unitarily equivalent to ψ if and only if $\varphi_* = \psi_* : K_*(A, \mathbb{Z}/n) \to K_*(B, \mathbb{Z}/n)$ for all $n \geq 0$. We give a new proof of a result of [DE$_2$] which states that if A is separable and quasidiagonal relative to B and if $\varphi, \psi : A \to B$ have the same KK-class, then φ is approximately stably unitarily equivalent to ψ. For nuclear separable C*-algebras A, we give a KK-theoretical description of the closure of zero in $\mathrm{Ext}(A, B)$.

1 Introduction

Two representations $\gamma, \gamma' : A \to \mathbb{M}(\mathcal{K} \otimes B)$ are called *properly approximately unitarily equivalent*, written $\gamma \simeq \gamma'$, if there is a sequence of unitaries $(u_n) \in \mathbb{C}I + \mathcal{K} \otimes B$ such that

- $\lim_{n\to\infty} \|u_n \gamma(a) u_n^* - \gamma'(a)\| = 0$, for all $a \in A$
- $u_n \gamma(a) u_n^* - \gamma'(a) \in \mathcal{K} \otimes B$, for all n, and $a \in A$.

The continuous version of the above equivalence is defined as follows [DE$_2$]. Two representations $\gamma, \gamma' : A \to \mathbb{M}(\mathcal{K} \otimes B)$ are called *properly asymptotically unitarily equivalent*, written $\gamma \approxeq \gamma'$, if there is a norm-continuous path of unitaries $u : [0, \infty) \to \mathbb{C}I + \mathcal{K} \otimes B$ such that

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- $\lim_{t\to\infty} \|u_t\gamma(a)u_t^* - \gamma'(a)\| = 0$, for all $a \in A$
- $u_t\gamma(a)u_t^* - \gamma'(a) \in \mathcal{K} \otimes B$, for all $t \in [0, \infty)$, and $a \in A$.

As in [DE$_2$], the use of the word 'proper' reflects that the unitaries implementing the above equivalence relations are compact perturbations of the identity.

Let A, B be unital C*-algebras and assume that A is separable. For two unital $*$-homomorphisms $\varphi, \psi : A \to B$ consider the following conditions.

(1) $[\varphi] = [\psi]$ in $\mathrm{KK}(A, B)$.

(2) $\varphi \oplus \gamma \approxeq \psi \oplus \gamma$ for some unital representation $\gamma : A \to \mathbb{M}(\mathcal{K} \otimes B)$.

(3) $[\varphi] = [\psi]$ in Rørdam's group $\mathrm{KL}(A, B) = \mathrm{KK}(A, B)/\mathrm{Pext}(K_*(A), K_{*+1}(B))$, ([Rø]).

(4) $\varphi \oplus \gamma \simeq \psi \oplus \gamma$ for some unital representation $\gamma : A \to \mathbb{M}(\mathcal{K} \otimes B)$.

(5) φ is approximately stably unitarily equivalent to ψ (see Definition 3.7).

By a result of Eilers and the author [DE$_2$] we have (1) $\Leftrightarrow$ (2). Note that (2) $\Rightarrow$ (4) is obvious. Suppose that A is quasidiagonal relative to B. This notion was introduced by Salinas [Sa], see Definition 3.5. Then it is not hard to see that (4) $\Leftrightarrow$ (5), see Lemma 3.8.

Condition (3) is stated under the assumption that A satisfies the universal coefficient theorem (abbreviated UCT) of [RS]. In view of the universal multi-coefficient theorem (UMCT) of [DL], (3) is equivalent to

(3') $\varphi_* = \psi_* : K_*(A, \mathbb{Z}/n) \to K_*(B, \mathbb{Z}/n)$ for all $n \geq 0$.

In this paper we prove that if A is separable nuclear quasidiagonal relative to B and satisfies the UCT, then (3) $\Leftrightarrow$ (4), hence (3)$\Leftrightarrow$ (5), see Theorem 5.1. The latter equivalence was conjectured informally by Lin [L], and it was known to be true if A is abelian [D$_1$], or if A can be approximated by nuclear C*-subalgebras satisfying the UCT and having finitely generated K-theory and B is simple [L], or if A is simple and B satisfies certain conditions in unstable K-theory and has bounded exponential rank [L], [DE$_1$].

The proof of (3) $\Leftrightarrow$ (4) is based on a new proof that we give for the implication (1) $\Rightarrow$ (5) of [DE$_2$], stated here as Theorem 3.11. Unlike previous approaches, the proof does not use the theorem of Kadison and Ringrose on derivable automorphisms of C*-algebras.

The same result is used to give a KK-theoretical description of the closure of zero in $\mathrm{Ext}(A, B)$, see Theorem 4.3. This yields new proofs of several results of Schochet [Sch$_1$]-[Sch$_3$], see Corollaries 4.5–4.7. Corollary 4.7 is used in the proof of Theorem 5.1.

The results on approximate stable unitary equivalence of $*$-homomorphisms have important applications in Elliott's classification program. There are interesting situations when the maps γ_n from Definition 3.7 can be chosen to be $*$-homomorphisms. For instance if A is nuclear and residually finite dimensional then γ_n can be taken to be finite dimensional representations of A into matrices over $\mathbb{C}1_B$. Another example considered by Lin [L] and generalized in [DE$_1$] is when A is nuclear and there is a full unital embedding $\iota : A \hookrightarrow B$. In that case one can take $\gamma_n = n \cdot \iota = \iota \oplus \cdots \oplus \iota$ (n-times).

2 Some preliminaries in KK-theory

Throughout this paper, A is a separable C*-algebra and B is a σ-unital C*-algebra. We work with Hilbert B-modules E countably generated over B such as $E = H_B = H \otimes B$ where H is a separable infinite dimensional Hilbert space. We use the notation from [Kas$_1$]. Let $\mathbb{M}(\mathcal{K}\otimes B)$ denote the multiplier C*-algebra of $\mathcal{K}\otimes B$ and let $Q(\mathcal{K}\otimes B)$ denote the generalized Calkin algebra $\mathbb{M}(\mathcal{K}\otimes B)/\mathcal{K}\otimes B$. The quotient map $\mathbb{M}(\mathcal{K}\otimes B) \to Q(\mathcal{K}\otimes B)$ is denoted by π. Very often we will identify $\mathbb{M}(\mathcal{K}\otimes B)$ with $L(H_B)$. Recall that the group $\mathrm{Ext}^{-1}(A, B)$ is generated by $*$-homomorphisms from A to $Q(\mathcal{K}\otimes B)$ which admit completely positive liftings $A \to \mathbb{M}(\mathcal{K}\otimes B)$. Such a map is called a semisplit extension. The group $\mathrm{KK}^1(A, B)$ consists of equivalence classes of pairs (τ, P) where $\tau : A \to \mathbb{M}(\mathcal{K}\otimes B)$ is a representation and $P \in \mathbb{M}(\mathcal{K}\otimes B)$ is a selfadjoint projection such that $[\tau(a), P] \in \mathcal{K}\otimes B$ for all $a \in A$. Kasparov [Kas$_2$] proved that there is an isomorphism $\kappa : \mathrm{KK}^1(A, B) \to \mathrm{Ext}^{-1}(A, B)$ which maps the class of (τ, P) to the class of the extension $\pi \circ P\tau(-)P$.

Let $x \in \mathrm{KK}^1(C(S^1), \mathbb{C})$ be the element defined by the Toeplitz extension

$$0 \longrightarrow \mathcal{K} \longrightarrow T \longrightarrow C(S^1) \longrightarrow 0.$$

Recall from [Bla, 17.8.5] that there is a natural homomorphism

$$\tau_A : \mathrm{KK}^*(C(S^1), \mathbb{C}) \longrightarrow \mathrm{KK}^*(C(S^1)\otimes A, A).$$

Using the Kasparov product

$$\mathrm{KK}^1(C(S^1)\otimes A, A) \times \mathrm{KK}(A, B) \longrightarrow \mathrm{KK}^1(C(S^1)\otimes A, B)$$

we define a group homomorphism

$$\tau_A(x) \otimes - : \mathrm{KK}(A, B) \longrightarrow \mathrm{KK}^1(C(S^1)\otimes A, B).$$

This homomorphism is injective, and in fact its composition with the restriction map

$$\mathrm{KK}^1(C(S^1)\otimes A, B) \to \mathrm{KK}^1(SA, B)$$

is an isomorphism. It coincides with the homomorphism

$$\tau_A(x_0) \otimes - : \mathrm{KK}(A, B) \longrightarrow \mathrm{KK}^1(SA, B),$$

where $x_0 \in \mathrm{KK}^1(S\mathbb{C}, \mathbb{C})$ is the element defined by the reduced Toeplitz extension

$$0 \longrightarrow \mathcal{K} \longrightarrow T_0 \longrightarrow S\mathbb{C} \longrightarrow 0.$$

Up to a sign, its inverse is given by $\tau_A(y_0) \otimes - : \mathrm{KK}^1(SA, B) \longrightarrow \mathrm{KK}(A, B)$, where $y_0 \in \mathrm{KK}^1(\mathbb{C}, S\mathbb{C})$ is the class of the extension

$$0 \longrightarrow S\mathbb{C} \longrightarrow C\mathbb{C} \longrightarrow \mathbb{C} \longrightarrow 0.$$

(Recall that $x_0 \otimes y_0 = -1$ and $y_0 \otimes x_0 = -1$ by Bott periodicity, [Bla, 19.2].)

Given two unital $*$-homomorphisms $\varphi, \psi : A \to B$, we want an explicit computation of $\tau_A(x) \otimes ([\varphi] - [\psi]) \in \mathrm{KK}^1(C(S^1) \otimes A, B)$ or rather of its image in $Ext^{-1}(C(S^1) \otimes A, B)$ under Kasparov's isomorphism $\kappa : \mathrm{KK}^1(C(S^1) \otimes A, B) \to Ext^{-1}(C(S^1) \otimes A, B)$. In order to formulate the result we need some notation. Let $\mathbb{Z}_+ = \{k \in \mathbb{Z} : k \geq 0\}$, $\mathbb{Z}_- = \{k \in \mathbb{Z} : k < 0\}$, and let $p_\pm : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}_\pm)$ be the canonical projections. Let $S : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ be the bilateral shift, $Se_i = e_{i+1}$, where (e_i) is the canonical orthonormal basis of $\ell^2(\mathbb{Z})$. Define $\Phi, \Psi : A \to \mathbb{M}(\mathcal{K}(\ell^2(\mathbb{Z}_+)) \otimes B)$ by

$$\Phi = \varphi \oplus \varphi \oplus \psi \oplus \varphi \oplus \psi \oplus \cdots = \varphi \oplus (\varphi \oplus \psi)_\infty,$$

$$\Psi = \psi \oplus \varphi \oplus \psi \oplus \varphi \oplus \psi \oplus \cdots = \psi \oplus (\varphi \oplus \psi)_\infty.$$

Let $w : \ell^2(\mathbb{Z}_+) \otimes B \to \ell^2(\mathbb{Z}_+) \otimes B$ be a unitary operator defined by $w(e_0 \otimes b) = e_1 \otimes b$, $w(e_{2k-1} \otimes b) = e_{2k+1} \otimes b$ and $w(e_{2k} \otimes b) = e_{2k-2} \otimes b$ for $k \geq 1$ and $b \in B$, where (e_i) is the canonical orthonormal basis of $\ell^2(\mathbb{Z}_+)$. Note that $\Phi(a) = w\Psi(a)w^*$ for all $a \in A$. Therefore $[\Phi(a), w] \in \mathcal{K} \otimes B$ for all $a \in A$ since $\Phi(a) - \Psi(a) \in \mathcal{K} \otimes B$.

If $x \in \mathbb{M}(\mathcal{K} \otimes B)$, then $\pi(x) \in Q(\mathcal{K} \otimes B)$ will be denoted by $\dot{x}$. If $\sigma : A \to \mathbb{M}(\mathcal{K} \otimes B)$ is a map, then $\pi \circ \sigma$ will be denoted by $\dot{\sigma}$.

Proposition 2.1 *Let A, B be unital C^*-algebras with A separable and let $\varphi, \psi : A \to B$ be two unital $*$-homomorphisms. Then with notation as above,*

$$\kappa(\tau_A(x) \otimes [\varphi] - \tau_A(x) \otimes [\psi]) = [\sigma],$$

where $\sigma : C(S^1) \otimes A \to Q(\mathcal{K} \otimes B)$ is defined by $\sigma(f \otimes a) = f(\dot{w})\dot{\Phi}(a)$.

Proof. The element $\tau_A(x) \in \mathrm{KK}^1(C(S^1) \otimes A, A)$ is given by the class of the pair (τ, P_+), where $\tau : C(S^1) \otimes A \to \mathbb{M}(\mathcal{K}(\ell^2(\mathbb{Z})) \otimes A) = L(\ell^2(\mathbb{Z}) \otimes A)$, $\tau(f \otimes a) = f(S) \otimes a$, S being the bilateral shift, and $P_+ = p_+ \otimes \mathrm{id}_A$. It is then clear that $-\tau_A(x)$ is given by the class of (τ, P_-) with $P_- = p_- \otimes \mathrm{id}_A$.

Since φ is a $*$-homomorphism, we have $\tau_A(x) \otimes [\varphi] = \varphi_*(\tau_A(x))$ by [Bla][18.7.2(a)], so that $\tau_A(x) \otimes [\varphi]$ is represented by the class of the pair (τ_φ, P_+), where $\tau_\varphi : C(S^1) \otimes A \to \mathbb{M}(\mathcal{K}(\ell^2(\mathbb{Z}) \otimes B)$, $\tau_\varphi(f \otimes a) = f(S) \otimes \varphi(a)$ and $P_+ = p_+ \otimes \mathrm{id}_B : \ell^2(\mathbb{Z} \otimes B) \to \ell^2(\mathbb{Z}_+ \otimes B)$.

Similarly, $-\tau_A(x) \otimes [\psi] = [\tau, P_-] \otimes [\psi]$ is represented by (τ_ψ, P_-), where $\tau_\psi : C(S^1) \otimes A \to \mathbb{M}(\mathcal{K}(\ell^2(\mathbb{Z})) \otimes B)$, $\tau_\psi(f \otimes a) = f(S) \otimes \psi(a)$ and $P_- = p_- \otimes \mathrm{id}_B : \ell^2(\mathbb{Z} \otimes B) \to \ell^2(\mathbb{Z}_- \otimes B)$. Thus

$$\tau_A(x) \otimes ([\varphi] - [\psi]) = [\tau_\varphi, P_+] + [\tau_\psi, P_-] = [\tau_\psi \oplus \tau_\varphi, P_- \oplus P_+].$$

The image of the latter element in $Ext^{-1}(C(S^1) \otimes A, B)$ under κ is equal to $[\dot{\omega}]$, where $\omega : C(S^1) \otimes A \to L(\ell^2(\mathbb{Z}_-) \otimes B) \oplus L(\ell^2(\mathbb{Z}_+) \otimes B) \subset L(\ell^2(\mathbb{Z}) \otimes B)$ is a completely positive map defined by $\omega = P_- \tau_\psi P_- + P_+ \tau_\varphi P_+$. Therefore

$$\omega(f \otimes a) = p_- \otimes \mathrm{id}_B \left(f(S) \otimes \psi(a)\right) p_- \otimes \mathrm{id}_B + p_+ \otimes \mathrm{id}_B \left(f(S) \otimes \varphi(a)\right) p_+ \otimes \mathrm{id}_B$$

$$= p_- f(S) p_- \otimes \psi(a) + p_+ f(S) p_+ \otimes \varphi(a).$$

Since $[f(S), p_\pm] \in \mathcal{K}$, we have

$$\omega(f \otimes a) - f(S) \otimes \mathrm{id}_B \left(p_- \otimes \psi(a) + p_+ \otimes \varphi(a)\right) \in \mathcal{K}(\ell^2(\mathbb{Z})) \otimes B, \quad \forall f \in C(S^1), a \in A.$$

Therefore, if $\Phi_1 = p_- \otimes \psi + p_+ \otimes \varphi = \cdots \oplus \psi \oplus \psi \oplus \varphi \oplus \varphi \oplus \cdots$, and $\sigma_1 : C(S^1) \otimes A \to Q(\mathcal{K}(\ell^2(\mathbb{Z}) \otimes B)$ is defined by $\sigma_1(f \otimes a) = f(\pi(S \otimes \mathrm{id}_B))\dot{\Phi}_1(a)$, then $\dot{\sigma_1} = \dot{\omega}$. Let $\xi : \mathbb{Z} \to \mathbb{Z}_+$ be a bijection defined by $\xi(0) = 0$, $\xi(1) = 1$, $\xi(k) = 2k + 1$, if $k \geq 1$ and $\xi(k) = -k$ if $k < 0$. Under the corresponding identification of $\ell^2(\mathbb{Z})$ with $\ell^2(\mathbb{Z}_+)$, Φ_1, $S \otimes \mathrm{id}_B$ correspond to Φ, w, respectively, so that σ_1 corresponds to σ.

■

3 Approximate unitary equivalence

In this section we show that the conditions (1), (4) and (5) from the introduction are related as follows: (1) $\Rightarrow$ (4) $\Leftrightarrow$ (5).

Definition 3.1 *Two representations $\gamma : A \to L(E)$, $\gamma' : A \to L(E')$ are called* approximately unitarily equivalent*, written $\gamma \sim \gamma'$, if there exists a sequence of unitaries (u_n) in $L(E', E)$ such that* *(i)* $\lim_{n\to\infty} \|\gamma(a) - u_n\gamma'(a)u_n^*\| = 0$, $a \in A$,

(ii) $\gamma(a) - u_n\gamma'(a)u_n^* \in \mathcal{K}(E)$, *for all* n, $a \in A$.

Definition 3.2 *A representation* $\gamma : A \to L(H_B) = \mathbb{M}(\mathcal{K}\otimes B)$ *is called* absorbing *if* $\gamma\oplus\sigma \sim \gamma$ *for any representation* $\sigma : A \to L(E)$. *If* A *is unital, then a representation* $\gamma : A \to \mathbb{M}(\mathcal{K}\otimes B)$ *is called* unitally absorbing *if* $\gamma \oplus \sigma \sim \gamma$ *for any unital representation* $\sigma : A \to L(E)$.

Note that any absorbing representation is injective. Any two absorbing representations are approximately unitarily equivalent.

Examples 3.3 *(a) A scalar representation* $\gamma : A \longrightarrow \mathbb{M}(\mathcal{K} \otimes B)$ *is a representation which factors as*

$$A \xrightarrow{\gamma'} L(H) \xrightarrow{-\otimes 1} L(H) \otimes 1 \hookrightarrow \mathbb{M}(\mathcal{K} \otimes B).$$

Suppose that A *or* B *are nuclear,* γ' *is faithful and* $\gamma'(A) \cap \mathcal{K} = \{0\}$. *If* γ' *is unital, then* γ *is unitally absorbing. If* $\overline{\gamma'(A)H}$ *has infinite codimension in* H, *then* γ *is absorbing, [*Kas_2*].*

(b) Lin [L] showed that if A *is nuclear and separable and if* $\iota : A \to B$ *is a unital embedding, then the map* $d_\iota : A \longrightarrow \mathbb{M}(\mathcal{K} \otimes B)$ *defined by* $d_\iota(a) = 1 \otimes \iota(a)$ *is unitally absorbing, whenever either* A *or* B *is simple. A unital* $*$*-homomorphism* $\iota : A \hookrightarrow B$ *is called a* full embedding *if the linear span of* $B\iota(a)B$ *is dense in* B *for all nonzero* $a \in A$. *It was shown in [*DE_1*] that* d_ι *is unitally absorbing whenever* ι *is a full embedding.*

(c) Thomsen [Tho_2*] proved the existence of absorbing extensions* $\gamma : A \to \mathbb{M}(\mathcal{K} \otimes B)$ *for arbitrary separable C*-algebras* A *and* B.

A C*-algebra B is called *stably unital* if $B \otimes \mathcal{K}$ has a countable approximate unit consisting of projections. A subset $E \subset \mathbb{M}(\mathcal{K} \otimes B)$ is called quasidiagonal if there is a countable approximate unit (p_n) of $\mathcal{K} \otimes B$ consisting of projections such that

$$\lim_{n\to\infty} \|p_n a - a p_n\| = 0, \quad a \in E. \tag{1}$$

A representation $\gamma : A \to \mathbb{M}(\mathcal{K} \otimes B)$ is called quasidiagonal if the set $\gamma(A) \subset \mathbb{M}(\mathcal{K} \otimes B)$ is quasidiagonal.

Remark 3.4 *(a) If* $\gamma, \gamma' : A \to \mathbb{M}(\mathcal{K} \otimes B)$ *are representations with* $\gamma \sim \gamma'$ *and* γ *is quasidiagonal, then* γ' *is quasidiagonal.*

(b) If A *is unital and if* $\gamma : A \to \mathbb{M}(\mathcal{K} \otimes B)$ *is a unital, unitally absorbing representation, then* $\gamma \oplus 0_{\mathbb{M}(\mathcal{K}\otimes B)}$ *is an absorbing representation [*DE_1*, Lemma 2.17].*

(c) Suppose that $E \subset \mathbb{M}(\mathcal{K} \otimes B)$ is a quasidiagonal set and that $p \in E$ is a projection such that $px = xp = x$ for all $x \in E$. Then there is an approximate unit of projections (p_n) of $\mathcal{K} \otimes B$ such that $[p_n, x] \to 0$ for all $x \in E$ and $[p_n, p] = 0$ for all n. Indeed, since E is a quasidiagonal set, there is an approximate unit of projections (q_n) of $\mathcal{K} \otimes B$ such that $[q_n, x] \to 0$ for all $x \in E$. In particular $[q_n, p] \to 0$. Using functional calculus one can perturb each q_n to a projection $p_n \in \mathcal{K} \otimes B$, such that $\|p_n - q_n\| \to 0$, and $[p_n, p] = 0$. (see[DE$_1$, Lemma 5.1 (iii)]).

Definition 3.5 *[Sa] Let A be a separable C^*-algebra and let B be a stably unital C^*-algebra. We say that A is quasidiagonal relative to B if there exists an absorbing quasidiagonal representation $\gamma : A \to \mathbb{M}(\mathcal{K} \otimes B)$.*

By Remark 3.4 (a) we have that if a separable C*-algebra A is quasidiagonal relative to B, then any absorbing representation $\gamma : A \to \mathbb{M}(\mathcal{K} \otimes B)$ is quasidiagonal. It also follows from Remark 3.4 that a separable unital C*-algebra is quasidiagonal relative to a stably unital C*-algebra B if and only if any (or some) unital unitally absorbing representation is quasidiagonal. Note that a C*-algebra A is quasidiagonal if and only if is quasidiagonal relative to $\mathbb{C}$.

Lemma 3.6 *Let C be a unital nuclear C^*-subalgebra of a UHF algebra. Let A be a unital separable C^*-algebra and let B be a stably unital C^*-algebra. If A is quasidiagonal relative to B, then $C \otimes A$ is quasidiagonal relative to B.*

Proof. By assumption, there is a unital embedding $\jmath : C \hookrightarrow D$ where D is a unital UHF algebra $D = \overline{\cup D_i}$ with $D_i \cong M_{r(i)}(\mathbb{C})$. Let $\pi : D \to L(H)$ and $\sigma : A \to \mathbb{M}(\mathcal{K} \otimes B)$ be unital unitally absorbing representations. It is clear that $\pi \otimes \sigma : D \otimes A \to L(H) \otimes L(H_B) \subset \mathbb{M}(\mathcal{K} \otimes B)$ is quasidiagonal since both π and σ are so. Therefore $\pi\jmath \otimes \sigma$ is a quasidiagonal representation of $C \otimes A$. It remains to prove that $\pi\jmath \otimes \sigma$ is unitally absorbing. First we observe that $\pi \otimes \sigma$ is unitally absorbing since its restriction to each of the $D_i \otimes A \cong M_{r(i)}(\mathbb{C}) \otimes A$ is easily seen to be unitally absorbing, using the assumption that σ is unitally absorbing. If $\pi : A \to L(E)$ is a representation and $\varphi : A \to \mathcal{K}(F)$ is a completely positive map, we write $\varphi \prec \pi$ if there is a bounded sequence $v_i \in \mathcal{K}(F, E)$ such that

- $\lim_{i\to\infty} \|\varphi(a) - v_i^* \pi(a) v_i\| = 0$ for all $a \in A$
- $\lim_{i\to\infty} \|v_i^* \xi\| = 0$ for all $\xi \in E$.

Here E, F are countable B-modules. As a corollary of [DE$_1$, Theorem 2.13] a unital representation $\gamma : A \to \mathbb{M}(\mathcal{K} \otimes B)$ is unitally absorbing if and only if $\varphi \prec \gamma$ for any completely positive contraction $\varphi : A \to \mathcal{K}(F)$. Therefore in order to prove that $\pi\jmath \otimes \sigma$ is unitally absorbing it will suffice to show that $\varphi \prec \pi\jmath \otimes \sigma$ for any completely positive contraction $\varphi : C \otimes A \to \mathcal{K}(F)$.

Since C is nuclear, we find two sequences of completely positive contractions

$$\alpha_n : C \to M_{k(n)}(\mathbb{C}), \quad \beta_n : M_{k(n)}(\mathbb{C}) \to C$$

such that $\|\beta_n\alpha_n(c) - c\| \to 0$ for all $c \in C$. By Arveson's extension theorem, we can extend α_n to a completely positive contraction $\alpha'_n : D \to M_{k(n)}(\mathbb{C})$ such that $\alpha'_n \circ \jmath = \alpha_n$. If $E_n = \beta_n \circ \alpha'_n : D \to C$, then E_n is a completely positive contraction with $\|E_n\jmath(c) - c\| \to 0$ for all $c \in C$. Since $\pi \otimes \sigma$ is unitally absorbing, we have $\varphi \circ (E_n \otimes \mathrm{id}_A) \prec \pi \otimes \sigma$, hence

$$\varphi_n = \varphi \circ (E_n \otimes \mathrm{id}_A) \circ (\jmath \otimes \mathrm{id}_A) = \varphi \circ (E_n\jmath \otimes \mathrm{id}_A) \prec \pi\jmath \otimes \sigma.$$

Since $\|\varphi_n(x) - \varphi(x)\| \to 0$ for all $x \in C \otimes A$, it follows that $\varphi \prec \pi\jmath \otimes \sigma$.

■

Definition 3.7 *Let A, B be unital C^*-algebras and assume that A is quasidiagonal relative to B. Two unital $*$-homomorphisms $\varphi, \psi : A \to B$ are called* approximately stably unitarily equivalent *if for* any *unital unitally absorbing representation $\gamma : A \to \mathbb{M}(\mathcal{K} \otimes B)$, and* any *approximate unit of projections (p_n) of $\mathcal{K} \otimes B$ with $\|[\gamma(a), p_n]\| \to 0$, $a \in A$, if $\gamma_n(a) = p_n\gamma(a)p_n$, then there exist an increasing sequence of integers $(k(n))$ and a sequence of partial isometries (v_n) in $\mathcal{K} \otimes B$ such that $v_n^*v_n = v_nv_n^* = 1_B \oplus p_{k(n)}$ and*

$$\lim_{n\to\infty} \|v_n(\varphi(a) \oplus \gamma_{k(n)}(a))v_n^* - \psi(a) \oplus \gamma_{k(n)}(a)\| = 0, \quad a \in A. \tag{2}$$

Lemma 3.8 *Let $\varphi, \psi : A \to B$ be two unital $*$-homomorphisms. Suppose that A is separable and quasidiagonal relative to B. Then the following conditions are equivalent.*

(i) $\varphi \oplus \eta \simeq \psi \oplus \eta$ *for some unital representation* $\eta : A \to \mathbb{M}(\mathcal{K} \otimes B)$.

(ii) $\varphi \oplus \gamma \simeq \psi \oplus \gamma$ *for some unital unitally absorbing representation* $\gamma : A \to \mathbb{M}(\mathcal{K} \otimes B)$.

(iii) $\varphi \oplus \gamma \simeq \psi \oplus \gamma$ *for any unital unitally absorbing representation* $\gamma : A \to \mathbb{M}(\mathcal{K} \otimes B)$.

(iv) φ *is approximately stably unitarily equivalent to* ψ.

(v) φ *and* ψ *satisfy the conditions of Definition 3.7 for some* γ *and some* (p_n).

Proof. The implications (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (v) are obvious. By [DE$_2$, Lemma 3.4] if $\gamma \sim \gamma'$ then $\varphi \oplus \gamma \simeq \psi \oplus \gamma$ if and only if $\varphi \oplus \gamma' \simeq \psi \oplus \gamma'$. This proves that (ii) $\Rightarrow$ (iii). To prove (i) $\Rightarrow$ (ii), note that (i) $\Rightarrow$ $\varphi \oplus \eta \oplus \gamma \simeq \psi \oplus \eta \oplus \gamma$. This readily implies (ii) since $\eta \oplus \gamma \sim \gamma$ as γ is absorbing.

(iii) $\Rightarrow$ (iv) Let $\gamma : A \to \mathbb{M}(\mathcal{K} \otimes B)$ and (p_n) be as in Definition 3.7. By assumption there is a sequence of unitaries (u_n) in $\mathbb{C}I + \mathcal{K} \otimes B$ such that

$$u_n(\varphi(a) \oplus \gamma(a))u_n^* - \psi(a) \oplus \gamma(a) \to 0, \quad a \in A \tag{3}$$

as $n \to \infty$. We also have $\|p_n\gamma(a) - \gamma(a)p_n\| \to 0,\ a \in A$. After passing to a subsequence $(p_{k(n)})$ we may arrange that $\|[e_n, u_n]\| \to 0$ as $n \to \infty$, where $e_n = 1_B \oplus p_{k(n)}$. By functional calculus we find a sequence of unitaries $v_n \in e_n(\mathcal{K} \otimes B)e_n$ such that $\|e_n u_n e_n - v_n\| \to 0$ as $n \to \infty$. Compressing by e_n in (3) we obtain that

$$\|v_n(\varphi(a) \oplus \gamma_{k(n)}(a))v_n^* - \psi(a) \oplus \gamma_{k(n)}(a)\| \to 0$$

as $n \to \infty$ for all $a \in A$. This proves (iv).

(v) $\Rightarrow$ (ii) Fix $\gamma : A \to \mathbb{M}(\mathcal{K} \otimes B)$, (p_n), $(k(n))$ and (v_n) as in Definition 3.7. We want to prove that $\varphi \oplus \gamma \simeq \psi \oplus \gamma$. Let $u_n = 0_B \oplus (1 - p_{k(n)}) + v_n \in \mathbb{C}I + \mathcal{K} \otimes B$ and define $\gamma'_{k(n)}(a) = (1 - p_{k(n)})\gamma(a)(1 - p_{k(n)})$. We have

$$\|\gamma(a) - \gamma_{k(n)}(a) - \gamma'_{k(n)}(a)\| \to 0, \text{as } n \to \infty,\ a \in A, \tag{4}$$

since $[\gamma(a), p_n] \to 0$. It is now clear that (3) follows from (4) and (2). This proves that $\varphi \oplus \gamma \simeq \psi \oplus \gamma$.

■

We regard $Q(\mathcal{K} \otimes B) \oplus Q(\mathcal{K} \otimes B)$ as a unital subalgebra of $Q(\mathcal{K} \otimes B)$ is the usual way.

Lemma 3.9 *Let A be a unital separable C^*-algebra and let B be a stably unital C^*-algebra. Suppose that A is quasidiagonal relative to B and let σ be a unital semisplit extension such that $[\sigma] = 0$ in $\mathrm{Ext}^{-1}(A, B)$. Then for any unital unitally absorbing representation $\gamma : A \to \mathbb{M}(\mathcal{K} \otimes B)$ the set*

$$E_{\sigma \oplus \dot\gamma} = \{X \in \mathbb{M}(\mathcal{K} \otimes B) : \dot X \in (\sigma \oplus \dot\gamma)(A)\}$$

is quasidiagonal.

Proof. Since γ is unitally absorbing, $\gamma \oplus 0_{\mathbb{M}(\mathcal{K}\otimes B)}$ is absorbing by Remark 3.4 (b). Since $[\sigma] = 0$ in $\mathrm{Ext}^{-1}(A, B)$, $\sigma \oplus \dot\gamma \oplus 0$ is of the form $\dot u(\dot\gamma \oplus 0 \oplus 0)\dot u^*$ for some unitary $u \in \mathbb{M}(\mathcal{K} \otimes B)$. Therefore it lifts to an absorbing representation $\delta : A \to \mathbb{M}(\mathcal{K} \otimes B)$. Since A is quasidiagonal relative to B, $\delta(A)+\mathcal{K}\otimes B$ is a quasidiagonal set. Finally we observe that $E_{\sigma\oplus\dot{\gamma}}\oplus 0_{\mathbb{M}(\mathcal{K}\otimes B)}\subset\delta(A)+\mathcal{K}\otimes B$, so that $E_{\sigma\oplus\dot{\gamma}}$ is quasidiagonal by Remark 3.4(c), as it contains an element acting as a unit.

■

Lemma 3.10 *Let A, B be unital C^*-algebras with A separable. Let $\varphi,\psi:A\to B$ be two unital $*$-homomorphisms. Suppose that there exist a unital unitally absorbing representation $\eta:A\to\mathbb{M}(\mathcal{K}\otimes B)$ and a unitary $u\in\mathbb{M}(\mathcal{K}\otimes B)$ such that*

(i) $\varphi\oplus\eta=u(\psi\oplus\eta)u^*$

(ii) the set $(\varphi\oplus\eta)(A)\cup\{u\}$ *is quasidiagonal in* $\mathbb{M}(\mathcal{K}\otimes B)$.

Then φ is approximately stably equivalent to ψ.

Proof. From (ii) there exists an approximate unit of projections (e_n) of $\mathcal{K}\otimes B$ such that for all $a\in A$

$$[e_n,(\varphi\oplus\eta)(a)]\to 0,\quad [e_n,u]\to 0\quad \text{as } n\to\infty. \tag{5}$$

From (i) and (5) $[e_n,(\psi\oplus\eta)(a)]\to 0$ as $n\to\infty$. We may also arrange that, in addition to the above, $e_n\geq e_{11}\otimes 1_B\in\mathcal{K}\otimes B$, so that $e_n=e_{11}\otimes 1_B+q_n$ where (q_n) is a sequence of projections satisfying $[q_n,\eta(a)]\to 0$ for all $a\in A$ as $n\to\infty$. Compressing by e_n in (i), we obtain that for all $a\in A$

$$\|\varphi(a)\oplus q_n\eta(a)q_n-e_nue_n(\psi(a)\oplus q_n\eta(a)q_n)e_nu^*e_n\|\to 0,\quad \text{as } n\to\infty. \tag{6}$$

Since $[e_n,u]\to 0$, for large n, $u_n=e_nue_n|e_nue_n|^{-1/2}$ is a unitary in $e_n(\mathcal{K}\otimes B)e_n$ with $\|u_n-e_nue_n\|\to 0$. We then obtain from (6) that

$$\|\varphi(a)\oplus q_n\eta(a)q_n-u_n(\psi(a)\oplus q_n\eta(a)q_n)u_n^*\|\to 0,\quad \text{as } n\to\infty.$$

We conclude the proof by applying (v) $\Rightarrow$ (iv) of Lemma 3.8.

Theorem 3.11 ([DE$_2$]) *Let A, B be unital C^*-algebras. Assume that A is separable, and quasidiagonal relative to B. Let $\varphi,\psi:A\to B$ be two unital $*$-homomorphisms with $[\varphi]=[\psi]$ in $\mathrm{KK}(A,B)$. Then φ is approximately stably equivalent to ψ. Equivalently, $\varphi\oplus\gamma\simeq\psi\oplus\gamma$, for any unital unitally absorbing representation $\gamma:A\to\mathbb{M}(\mathcal{K}\otimes B)$.*

Proof. We are going to find a representation $\eta : A \to \mathbb{M}(\mathcal{K}\otimes B)$ and a unitary $u \in \mathbb{M}(\mathcal{K}\otimes B)$ such that φ, ψ, η and u satisfy the assumptions of Lemma 3.10. Recall that the canonical map $\mathbb{M}(\mathcal{K}\otimes B) \to Q(\mathcal{K}\otimes B)$ is denoted by π and that we sometimes write $\dot{a}$ for $\pi(a)$. We are using the notation from Proposition 2.1. Since $\tau_A(x)\otimes - : \mathrm{KK}(A,B) \to \mathrm{KK}^1(C(S^1)\otimes A, B)$ is a homomorphism and $[\varphi]-[\psi]=0$, it follows from Proposition 2.1 that $[\sigma]=0$ in $\mathrm{Ext}^{-1}(C(S^1)\otimes A, B)$. By assumption, A is quasidiagonal relative to B. Since $C(S^1)$ embeds unitally in a UHF algebra, it follows from Lemma 3.6 that $C(S^1)\otimes A$ is also quasidiagonal relative to B. Therefore there exists a unital, unitally absorbing quasidiagonal representation $\Delta : C(S^1)\otimes A \to \mathbb{M}(\mathcal{K}\otimes B)$. By adding a suitable representation to Δ, if necessary, we may arrange that if δ is the restriction of Δ to $1\otimes A$, $\delta(a)=\Delta(1\otimes a)$, then δ is also unitally absorbing. Let $v=\Delta(z\otimes 1)$, where z is the canonical unitary generator of $C(S^1)$. Recall that

$$\Phi = \varphi \oplus (\varphi\oplus\psi)_\infty \quad \Psi = \psi \oplus (\varphi\oplus\psi)_\infty, \quad \Phi = w\Psi w^*.$$

Therefore if we set $\eta = (\varphi\oplus\psi)_\infty \oplus \delta$, and $u = w\oplus v$, then

$$\varphi\oplus\eta = \Phi\oplus\delta, \quad \psi\oplus\eta = \Psi\oplus\delta, \quad \text{and} \;\; \varphi\oplus\eta = u(\psi\oplus\eta)u^*$$

since $\Phi = w\Psi w^*$ and δ commutes with v. Thus condition (i) of Lemma 3.10 satisfied. Note that η is unital unitally absorbing since its direct summand δ is so. It remains to verify condition (ii) of the same lemma. With our notation, that amounts to checking that the set

$$E = (\Phi\oplus\delta)(A)\cup\{u\}$$

is quasidiagonal. Note that $E \subset E_{\sigma\oplus\dot{\Delta}}$ since $\pi(\Phi\oplus\delta) = \sigma|_{1\otimes A}\oplus\dot{\delta}$ and $\dot{u} = \dot{w}\oplus\dot{v} = (\sigma\oplus\dot{\Delta})(z\otimes 1)$. Now $E_{\sigma\oplus\dot{\Delta}}$ is quasidiagonal by Lemma 3.9, since $[\sigma]=0$ in $\mathrm{Ext}^{-1}(A,B)$. Therefore its subset E is quasidiagonal as well.

■

Corollary 3.12 ([DE$_1$]) *Let A, B be unital C^*-algebras and let $\iota : A\to B$ be a full unital embedding (Example 3.3(b)). Suppose that A is nuclear and separable. Let $\varphi,\psi : A \longrightarrow B$ be two unital $*$-homomorphisms such that $[\varphi]=[\psi]$ in $\mathrm{KK}(A,B)$. Then for any finite subset $\mathcal{F}\subseteq A$ and any $\epsilon>0$, there exist n and a unitary $u\in M_{n+1}(B)$ such that*

$$\|(u(\varphi(a)\oplus n\cdot\iota(a))u^* - \psi(a)\oplus n\cdot\iota(a)\| < \epsilon.$$

for all $a\in\mathcal{F}$.

Proof. This follows from Theorem 3.11 applied for $\gamma = d_\iota$.

■

4 The closure of zero in $\mathrm{Ext}(A,B)$

In this section we give a purely KK-theoretical description of the closure of zero in $\mathrm{Ext}(A,B)$ under the assumption that A is separable and nuclear and B is σ-unital.

The following proposition is well known to the specialists. A proof is included for the sake of completeness.

Proposition 4.1 *If A is a separable C^*-algebra then $\mathrm{KK}^i(A,\mathbb{M}(\mathcal{K}\otimes B))=0$, $i=0,1$, for any C^*-algebra B.*

Proof. This is similar to the proof of [D_2, Proposition 2.2]. To simplify the notation, let $M=\mathbb{M}(\mathcal{K}\otimes B)=L(H_B)$. Since $\mathrm{KK}^1(A,M)\cong\mathrm{KK}(SA,M)$ is suffices to consider only the case $i=0$. Since M is unital, $\mathrm{KK}(A,M)\cong[qA,\mathcal{K}(E)\otimes M]$, by [Cun], where E is an infinite dimensional Hilbert space. The addition on the latter group is given by $[\varphi]+[\psi]=[\theta_{\varphi,\psi}]$, $\theta_{\varphi,\psi}(a)=w_1\varphi(a)w_1^*+w_2\psi(a)w_2^*$, where $w_i\in M(\mathcal{K}(E)\otimes M)$ are isometries with $w_1w_1^*+w_2w_2^*=1$. This definition does not depend on the particular choice of the pair w_i (see [JT, 1.3]). Let $s_1,s_2:H_B\to H_B$ be isometries with $s_1s_1^*+s_2s_2^*=1$ and set $w_i=1_E\otimes s_i:E\otimes H_B\to E\otimes H_B$. Let $s:H_B\oplus H_B\to H_B$ be defined by $s(x_1\oplus x_2)=s_1x_1+s_2x_2$.Then $1_E\otimes s:E\otimes H_B\oplus E\otimes H_B\to E\otimes H_B$ is a unitary such that $\theta_{\varphi,\psi}(a)=1_E\otimes s(\varphi(a)\oplus\psi(a))1_E\otimes s^*$.

Define $\alpha:\mathbb{M}(\mathcal{K}\otimes B)\to\mathbb{M}(\mathcal{K}\otimes\mathcal{K}\otimes B)$ by $\alpha(x)=1_H\otimes x$ and let $v=v_0\otimes\mathrm{id}_B:H\otimes H\otimes B\to H\otimes B$ for some unitary $v_0:H\otimes H\to H$.

If $\varphi:qA\to\mathcal{K}(E)\otimes M$ is a $*$-homomorphism, define $\eta:qA\to\mathcal{K}(E)\otimes M$ by

$$\eta(a)=1_E\otimes v(\mathrm{id}_{\mathcal{K}(E)}\otimes\alpha)\circ\varphi(a))1_E\otimes v^*,\quad a\in A. \tag{7}$$

We are going to show that $[\varphi]+[\eta]=[\eta]$, or equivalently $[\theta_{\varphi,\eta}]=[\eta]$, in $[qA,\mathcal{K}(E)\otimes M]$. That will show that $[\varphi]=0$ for all φ so that $[qA,\mathcal{K}(E)\otimes M]=0$. Since the unitary group of $\mathbb{M}(\mathcal{K}(E)\otimes M)$ is path connected in the strict topology [JT], it suffices to find a unitary w in $\mathbb{M}(\mathcal{K}(E)\otimes M)$ such that $w\theta_{\varphi,\eta}w^*=\eta$. Let (e_i), $i=0,1,...$ be an orthonormal basis of H and define a unitary $u_0:H\oplus H\otimes H\to H\otimes H$ by $u_0(h\oplus 0)=e_0\otimes h$, $u_0(e_i\otimes h)=e_{i+1}\otimes h$, $h\in H$. Set $u=u_0\otimes\mathrm{id}_B:H\otimes B\oplus H\otimes H\otimes B\to H\otimes H\otimes B$ and note that $1_H\otimes x=u(x\oplus 1_H\otimes x)u^*$ for all $x\in M$, hence $\alpha=u(\mathrm{id}_M\oplus\alpha)u^*$. Therefore

$$(\mathrm{id}_{\mathcal{K}(E)}\otimes\alpha)\circ\varphi(a)=1_E\otimes u(\varphi(a)\oplus(\mathrm{id}_{\mathcal{K}(E)}\otimes\alpha)\circ\varphi(a))1_E\otimes u^*,\quad a\in A. \tag{8}$$

Finally, using (7) and (8), one checks that

$$w=1_E\otimes(vu(1_{H_B}\oplus v^*)s^*)\in 1_E\otimes M\subset\mathbb{M}(\mathcal{K}(E)\otimes M)$$

is a unitary satisfying $w\theta_{\varphi,\eta}w^* = \eta$. Note that the unitary $vu(1_{H_B}\oplus v^*)s^*$ belongs to $\mathbb{M}(\mathcal{K}\otimes B)$ since it is given by a composite of B-linear unitaries

$$H_B \xrightarrow{s^*} H_B \oplus H_B \xrightarrow{1\oplus v^*} H_B \oplus (H\otimes H)_B \xrightarrow{u} (H\otimes B)_B \xrightarrow{v} H_B,$$

where $(H\otimes H)_B = H\otimes H\otimes B$.

■

For the remaining of this section we assume that A is a separable nuclear C*-algebra and B is σ-unital. In this case all extensions $\sigma : A \to Q(\mathcal{K}\otimes B)$ are semisplit by the Choi-Effros theorem, and $Ext^{-1}(A,B) = \mathrm{Ext}(A,B)$. Any $*$-homomorphism $\varphi : A \to Q(\mathcal{K}\otimes B)$ gives both and element $[\varphi]_{\mathrm{Ext}}$ of $\mathrm{Ext}(A,B)$ and element $[\varphi]_{\mathrm{KK}}$ of $\mathrm{KK}(A, Q(\mathcal{K}\otimes B))$.

Proposition 4.2 *Let A be a separable nuclear C*-algebra and let B be a σ-unital C*-algebra. Then the map $\chi : \mathrm{Ext}(A,B) \to \mathrm{KK}(A, Q(\mathcal{K}\otimes B))$, defined by $\chi[\varphi]_{\mathrm{Ext}} = [\varphi]_{\mathrm{KK}}$, is a natural isomorphism of groups.*

Proof. From the six-term exact sequence for $\mathrm{KK}(A,-)$ associated with the extension

(9) $$0 \to \mathcal{K}\otimes B \to \mathbb{M}(\mathcal{K}\otimes B) \to Q(\mathcal{K}\otimes B) \to 0$$

and from Proposition 4.1 we get an exact sequence

(10)
$$0 = \mathrm{KK}(A, \mathbb{M}(\mathcal{K}\otimes B)) \to \mathrm{KK}(A, Q(\mathcal{K}\otimes B)) \xrightarrow{\partial} \mathrm{KK}^1(A, \mathcal{K}\otimes B) \to \mathrm{KK}^1(A, \mathbb{M}(\mathcal{K}\otimes B)) = 0.$$

Thus ∂ is an isomorphism. By a theorem of Kasparov [Kas_2], there is a natural isomorphism

$$\kappa : KK^1(A,B) \longrightarrow \mathrm{Ext}(A,B) = \mathrm{Ext}(A,B)^{-1}.$$

Therefore $\partial^{-1}\circ\kappa^{-1} : \mathrm{Ext}(A,B) \to \mathrm{KK}(A, Q(\mathcal{K}\otimes B))$ is an isomorphism. We need to show that $\partial^{-1}\circ\kappa^{-1} = \chi$. The image of $[\varphi]_{\mathrm{Ext}}$ under κ^{-1} is denoted by δ_φ. Let

(11) $$0 \longrightarrow \mathcal{K}\otimes B \longrightarrow E_\varphi \longrightarrow A \longrightarrow 0$$

be the pullback of the extension (9) by $\varphi : A \to Q(\mathcal{K}\otimes B)$. Then we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{K}\otimes B & \longrightarrow & \mathbb{M}(\mathcal{K}\otimes B) & \xrightarrow{\pi} & Q(\mathcal{K}\otimes B) & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow\varphi & & \\
0 & \longrightarrow & \mathcal{K}\otimes B & \longrightarrow & E_\varphi & \longrightarrow & A & \longrightarrow & 0
\end{array}$$

By the naturality of the boundary map ∂ we obtain a commutative diagram:

$$\begin{array}{ccc}
\mathrm{KK}(A, Q(\mathcal{K}\otimes B)) & \xrightarrow{\partial} & \mathrm{KK}^1(A, \mathcal{K}\otimes B) \\
{\scriptstyle \varphi_*}\uparrow & & \| \\
\mathrm{KK}(A, A) & \xrightarrow{\partial} & \mathrm{KK}^1(A, \mathcal{K}\otimes B)
\end{array}$$

Therefore

$$\partial\varphi_*[\mathrm{id}_A]_{\mathrm{KK}} = \partial[\mathrm{id}_A]_{\mathrm{KK}}.$$

Now $\varphi_*[\mathrm{id}_A]_{\mathrm{KK}} = \partial[\varphi]_{\mathrm{KK}}$ and by [Bla, Theorem 19.5.7]

$$\partial[\mathrm{id}_A]_{\mathrm{KK}} = [\mathrm{id}_A] \otimes \delta_\varphi = \delta_\varphi = \kappa^{-1}[\varphi]_{\mathrm{Ext}}.$$

Therefore $\partial[\varphi]_{\mathrm{KK}} = \kappa^{-1}[\varphi]_{\mathrm{Ext}}$, hence $\chi = \partial^{-1} \circ \kappa^{-1}$. This completes the proof as we have seen that both ∂ and κ are natural isomorphisms.

■

Following [BDF] and [Br], Salinas [Sa] introduced a natural topology on $\mathrm{Ext}(A, B)$. This is just the quotient topology of the point-norm topology on the space of extensions $A \to Q(\mathcal{K}\otimes B)$. An extension τ is called absorbing if for any trivial extension σ, $\tau\oplus\sigma$ is unitarily equivalent to τ via a unitary liftable to $\mathbb{M}(\mathcal{K}\otimes B)$. If $\gamma : A \to \mathbb{M}(\mathcal{K}\otimes B)$ is a scalar absorbing representation (see Example 3.3(a)), then $\theta = \dot\gamma$ is an absorbing extension. Also $\tau\oplus\theta$ is absorbing for any extension τ. If $\tau : A \to Q(\mathcal{K}\otimes B)$ is an absorbing extension, and if $\theta : A \to Q(\mathcal{K}\otimes B)$ is a trivial absorbing extension, then $[\tau]$ belongs to the closure of zero in $\mathrm{Ext}^{-1}(A, B)$, denoted by $Z(A, B)$, if and only if there is a sequence of unitaries $u_n \in Q(\mathcal{K}\otimes B)$ such that $\|\tau(a) - u_n\theta(a)u_n^*\| \to 0$ for all $a \in A$, as $n \to \infty$. Since both τ and θ are absorbing, one can arrange that the unitaries u_n lift to unitaries in $\mathbb{M}(\mathcal{K}\otimes B)$. This is easily seen if we keep in mind that τ, θ are unitarily equivalent with $\tau\oplus 0$, $\theta\oplus 0$, respectively, via liftable unitaries.

To simplify notation, $Q(\mathcal{K}\otimes B)$ will be denoted by $\mathbf{Q}$. Let $d : \mathbf{Q} \to \prod_{n=1}^\infty \mathbf{Q}$ be defined by $d(x) = (x, x, x, \dots)$. Let

$$\nu : \prod_{n=1}^\infty \mathbf{Q} \to \prod_{n=1}^\infty \mathbf{Q}/\sum_{n=1}^\infty \mathbf{Q}$$

denote the quotient map and let

$$\Omega : \mathrm{KK}(A, \mathbf{Q}) \to \mathrm{KK}(A, \prod \mathbf{Q}/\sum \mathbf{Q})$$

be the map induced by $\nu\circ d$, that is $\Omega = (\nu\circ d)_*$. Fix $\theta = \dot\gamma : A \to \mathbf{Q}$ where $\gamma : A \to \mathbb{M}(\mathcal{K}\otimes B)$ is an absorbing scalar representation as in Example 3.3(a).

Theorem 4.3 *Let A, B be C^*-algebras with A nuclear and separable. Then* $\operatorname{Ker}\Omega$ *consists of the classes $[\varphi]_{\mathrm{KK}}$ of those $*$-homomorphisms $\varphi : A \to Q(\mathcal{K} \otimes B)$ for which there exists a sequence of unitaries (u_n) in $\mathrm{M}_2(Q(\mathcal{K} \otimes B))$ such that for all $a \in A$*

$$\lim_{n\to\infty} \|u_n(\varphi \oplus \theta)(a)u_n^* - (\theta \oplus \theta)(a)\| = 0. \tag{12}$$

Equivalently, if $\varphi : A \to Q(\mathcal{K} \otimes B)$ is an absorbing extension, then $[\varphi] \in \operatorname{Ker}\Omega$ if and only if there is a sequence of unitaries $v_n \in \mathbb{M}(\mathcal{K} \otimes B)$ such that $\lim_{n\to\infty} \|\varphi(a) - \dot{v}_n\theta(a)\dot{v}_n^*\| = 0$ for all $a \in A$.

Proof. If $\varphi : A \to \mathbf{Q}$ is a $*$-homomorphism and θ is as above, we let

$$\Phi = \nu \circ d \circ \varphi, \quad \Theta = \nu \circ d \circ \theta.$$

Suppose that φ and θ satisfy the condition (12) and let us show that $\Omega[\varphi]_{\mathrm{KK}} = 0$. The sequence (u_n) gives a unitary $u \in \mathrm{M}_2(\prod \mathbf{Q}/\sum \mathbf{Q})$ such that $u(\Phi \oplus \Theta)u^* = \Theta \oplus \Theta$. This clearly implies that $[\Phi] = [\Theta]$ in $\mathrm{KK}(A, \prod \mathbf{Q}/\sum \mathbf{Q})$. On the other hand

$$[\Theta]_{\mathrm{KK}} = \Omega[\theta]_{\mathrm{KK}} = \Omega\chi[\theta]_{\mathrm{Ext}} = 0,$$

since $[\theta]_{\mathrm{Ext}} = 0$ as θ lifts to a representation γ.

Conversely, assume that $\Omega[\varphi]_{\mathrm{KK}} = 0$. Let us observe that since θ is absorbing, for any integer $m \geq 1$, $m \cdot \theta = \theta \oplus \cdots \oplus \theta$ (m – *times*) is unitarily equivalent to θ. Therefore the condition (12) is equivalent to the following:

For any finite subset $\mathcal{F}$ of A and any $\epsilon > 0$, there exist $m \geq 1$ and a unitary $u \in \mathrm{M}_{m+1}(\mathbf{Q})$ such that

$$\|u(\varphi \oplus m \cdot \theta)(a)u^* - (\theta \oplus m \cdot \theta)(a)\| < \epsilon, \quad a \in \mathcal{F}. \tag{13}$$

Consequently, it suffices to prove that φ and θ satisfy (13) rather than (12). Since $\Omega[\varphi] = 0$ by assumption, and $\Omega[\theta] = 0$ as we saw above, we have $\Omega[\varphi]_{\mathrm{KK}} = \Omega[\theta]_{\mathrm{KK}}$, hence $[\Phi] = [\Theta]$ in $\mathrm{KK}(A, \prod \mathbf{Q}/\sum \mathbf{Q})$.

Let $\widetilde{A}$ denote the C*-algebra obtained by adding a unit to A. This is done even if A was unital in the first place. By replacing φ by $\varphi \oplus \theta$ if necessary, we may arrange that $1_{\mathbf{Q}} \notin \varphi(A)$. Let $\widetilde{\varphi}, \widetilde{\theta} : \widetilde{A} \to \mathbf{Q}$ be the unital extensions of φ and θ and set $\widetilde{\Phi} = \nu \circ d \circ \widetilde{\varphi}$ and $\widetilde{\Theta} = \nu \circ d \circ \widetilde{\theta}$. We have that $[\widetilde{\Phi}] = [\widetilde{\Theta}]$ in $\mathrm{KK}(\widetilde{A}, \prod \mathbf{Q}/\sum \mathbf{Q})$ since $[\Phi] = [\Theta]$ in $\mathrm{KK}(A, \prod \mathbf{Q}/\sum \mathbf{Q})$, and $\widetilde{\Phi}, \widetilde{\Theta}$ are unitizations of Φ, Θ.

Note that $\widetilde{\Theta} : A \to \prod \mathbf{Q}/\sum \mathbf{Q}$ is a full embedding since it factors as a product of unital maps

$$A \xrightarrow{\widetilde{\gamma}'} L(H) \hookrightarrow \mathbb{M}(\mathcal{K} \otimes B) \to Q(\mathcal{K} \otimes B) \longrightarrow \prod \mathbf{Q}/\sum \mathbf{Q}$$

and $\widetilde{\gamma}$ is a full embedding since $\widetilde{\gamma}'(A) \cap \mathcal{K}(H) = \{0\}$. By Corollary 3.12, for any finite subset $\mathcal{F}$ of A and any $\epsilon > 0$, there exist $m \geq 1$ and a unitary $U \in \mathrm{M}_{m+1}(\prod \mathbf{Q}/\sum \mathbf{Q})$ such that

$$\|U(\widetilde{\Phi} \oplus m \cdot \widetilde{\Theta})(a)U^* - (\widetilde{\Theta} \oplus m \cdot \widetilde{\Theta})(a)\| < \epsilon, \quad a \in \mathcal{F}. \tag{14}$$

Let $V = (v_i) \in \prod \mathbf{Q}$ be a unitary lifting of U. Then it follows from (14) that there is some large i such that

$$\|v_i(\widetilde{\varphi} \oplus m \cdot \widetilde{\theta})(a)v_i^* - (\widetilde{\theta} \oplus m \cdot \widetilde{\theta})(a)\| < \epsilon, \quad a \in \mathcal{F}. \tag{15}$$

This shows that φ and θ satisfy (13) and completes the proof.

■

Remark 4.4 *The map Ω can be integrated in a six-term exact sequence*

$$\begin{array}{ccccc}
\mathrm{KK}(A,\mathbf{Q}) & \xrightarrow{\Omega} & \mathrm{KK}(A,\prod \mathbf{Q}/\sum \mathbf{Q}) & \xrightarrow{1-\sigma} & \mathrm{KK}(A,\prod \mathbf{Q}/\sum \mathbf{Q}) \\
\uparrow & & & & \downarrow \\
\mathrm{KK}^1(A,\prod \mathbf{Q}/\sum \mathbf{Q}) & \xleftarrow{1-\sigma} & \mathrm{KK}^1(A,\prod \mathbf{Q}/\sum \mathbf{Q}) & \xleftarrow{\Omega} & \mathrm{KK}^1(A,\mathbf{Q})
\end{array}$$

which is similar to an exact sequence in E-theory found by Thomsen [Tho$_1$].

Note that Theorem 4.3 says that $\chi(Z(A,B)) = Ker\,\Omega$. Let $\alpha \in \mathrm{KK}(A, A')$ be a KK-equivalence. If $\mathbf{Q} = Q(\mathcal{K} \otimes B)$ as before, α induces a commutative diagram:

$$\begin{array}{ccccccc}
\mathrm{Ext}(A,B) & \xrightarrow{\kappa^{-1}} & \mathrm{KK}^1(A,B) & \xrightarrow{\partial^{-1}} & \mathrm{KK}(A,\mathbf{Q}) & \xrightarrow{\Omega} & \mathrm{KK}(A,\prod \mathbf{Q}/\sum \mathbf{Q}) \\
{\scriptstyle \alpha\otimes-}\downarrow & & {\scriptstyle \alpha\otimes-}\downarrow & & {\scriptstyle \alpha\otimes-}\downarrow & & {\scriptstyle \alpha\otimes-}\downarrow \\
\mathrm{Ext}(A',B) & \xrightarrow{\kappa^{-1}} & \mathrm{KK}^1(A',B) & \xrightarrow{\partial^{-1}} & \mathrm{KK}(A',\mathbf{Q}) & \xrightarrow{\Omega'} & \mathrm{KK}(A',\prod \mathbf{Q}/\sum \mathbf{Q})
\end{array}$$

This gives right away the following corollary.

Corollary 4.5 ([Sch$_1$]) *Let A, A' be separable nuclear C^*-algebras and let B be a σ-unital C^*-algebra. If $\alpha \in \mathrm{KK}(A, A')$ is a KK-equivalence, then the isomorphism $\alpha\otimes- : \mathrm{Ext}(A, B) \to \mathrm{Ext}(A', B)$ maps $Z(A, B)$ onto $Z(A', B)$.*

Corollary 4.6 ([Sch$_2$]) *Let A be a separable nuclear C^*-algebra satisfying the UCT and let B be a σ-unital C^*-algebra. Then $Z(A, B)$ is naturally isomorphic to* $\mathrm{Pext}(K_*(A), K_*(B))$.

Proof. If A satisfies the UCT, then A is KK-equivalent to a commutative C*-algebra. Therefore A satisfies the UMCT of [DL], so that there is an exact sequence

(16)
$$0 \to \mathrm{Pext}(K_*(A), K_{*+1}(Q(\mathcal{K}\otimes B)) \to \mathrm{KK}(A, Q(\mathcal{K}\otimes B)) \to \mathrm{Hom}_\Lambda(\underline{\mathrm{K}}(A), \underline{\mathrm{K}}(Q(\mathcal{K}\otimes B))) \to 0,$$

where $\underline{\mathrm{K}}(A) = \oplus_n K_*(A;\mathbb{Z}/n)$ is the total K-theory group, and Λ is a certain set of operations on $\underline{\mathrm{K}}(-)$. We regard both $Z(A,B) \cong \chi(Z(A,B))$ and $\mathrm{Pext}(K_*(A), K_*(B)) \cong \mathrm{Pext}(K_*(A), K_{*+1}(Q(\mathcal{K}\otimes B)))$ as subgroups of $\mathrm{KK}(A, Q(\mathcal{K}\otimes B))$. First we want to verify the inclusion

$$Z(A,B) \subset \mathrm{Pext}(K_*(A), K_{*+1}(Q(\mathcal{K}\otimes B))\,).$$

If θ is a trivial absorbing extension, then $[\theta]=0$ so that the map $\underline{\theta} : \underline{\mathrm{K}}(A) \to \underline{\mathrm{K}}(Q(\mathcal{K}\otimes B))$ vanishes. If $[\sigma] \in Z(A,B)$ is the class of an absorbing extension σ, then by the very definition of the topology on $\mathrm{Ext}(A,B)$, σ is approximately unitarily equivalent to θ, hence $\underline{\sigma} = \underline{\theta} = 0$. Therefore $[\sigma] = [\sigma] - [\theta] \in \mathrm{Pext}(K_*(A), K_{*+1}(Q(\mathcal{K}\otimes B))\,)$ by the UMCT (16). To prove the opposite inclusion

$$\mathrm{Pext}(K_*(A), K_{*+1}(Q(\mathcal{K}\otimes B))\,) \subset Z(A,B),$$

we note that since both subsets are invariant under KK-equivalence in the first variable, it suffices to prove the statement for any C*-algebra KK-equivalent to A. Thus, using [RS, Proposition 7.3], we may assume that $A = \cup_{i=1}^\infty A_i$ where A_i are nuclear C*-algebras satisfying the UCT and such that $K_*(A_i)$ is finitely generated for each i. In particular $\mathrm{Pext}(K_*(A_i), K_{*+1}(Q(\mathcal{K}\otimes B))\,) = 0$ for all i. Let $\sigma, \theta : A \to Q(\mathcal{K}\otimes B)$ be two absorbing extensions with θ trivial and $[\sigma] \in \mathrm{Pext}(K_*(A), K_{*+1}(Q(\mathcal{K}\otimes B))$. Then $[\sigma|_{A_i}] \in \mathrm{Pext}(K_*(A_i), K_{*+1}(Q(\mathcal{K}\otimes B)))$ vanishes, so that $[\sigma|_{A_i}] = [\theta|_{A_i}] \in \mathrm{KK}(A_i, Q(\mathcal{K}\otimes B))$ by (16). Since both $\sigma|_{A_i}$ and $\theta|_{A_i}$ are absorbing, they are unitarily equivalent. Therefore σ is approximately unitarily equivalent to θ, hence $[\sigma] \in Z(A,B)$.

■

Corollary 4.7 ([Sch$_3$]) *Let A be a separable nuclear C*-algebra satisfying the UCT, and let B be a stably-unital C*-algebra. Suppose that A is quasidiagonal relative to B. Let $\sigma : A \to Q(\mathcal{K}\otimes B)$ be an absorbing extension. Then $E_\sigma = \{y \in \mathbb{M}(\mathcal{K}\otimes B) : \dot{y} \in \sigma(A)\}$ is a quasidiagonal set if and only if $[\sigma] \in \mathrm{Pext}(K_*(A), K_*(B))$.*

Proof. This follows from Corollary 4.6 and from [Sa, Theorem 4.4] which states that if A is quasidiagonal relative to B and $\sigma : A \to Q(\mathcal{K}\otimes B)$ is an absorbing extension, then E_σ is a quasidiagonal set if and only if $[\sigma] \in Z(A,B)$. It should be noted that the proof from [Sa] also applies to the case when A is nonunital. ■

5 Approximate unitary equivalence revisited

In this section we prove the equivalences (3)⇔ (4) ⇔ (5) from the introduction.

Theorem 5.1 *Let A, B be unital C^*-algebras with A nuclear and separable. Suppose that A is quasidiagonal relative to B and let $\varphi, \psi : A \to B$ be two unital $*$-homomorphisms. The following assertions are equivalent.*

(i) $[\varphi] - [\psi] \in \mathrm{Pext}(K_*(A), K_{*+1}(B))$ *in* $\mathrm{KK}(A, B)$.

(ii) $\varphi \oplus \gamma \simeq \psi \oplus \gamma$ *for some (any) unital unitally absorbing quasidiagonal representation* $\gamma : A \to \mathbb{M}(\mathcal{K} \otimes B)$.

(iii) φ *is approximately stably unitarily equivalent to* ψ.

Proof. We have that (ii) ⇔ (iii) by Lemma 3.8.

(iii) ⇒ (i) It follows easily from (iii) that $\varphi_* = \psi_* : K_*(A, \mathbb{Z}/n) \to K_*(B, \mathbb{Z}/n)$ for all $n \geq 0$, so that (i) follows from the UMCT (16).

(i) ⇒ (iii) Let T be the group morphism defined as the composition

$$\mathrm{KK}(A, B) \xrightarrow{\tau_A(x)\otimes -} \mathrm{KK}^1(C(S^1) \otimes A, B) \xrightarrow{\partial^{-1}} \mathrm{KK}(C(S^1) \otimes A, Q(\mathcal{K} \otimes B)),$$

with $\tau_A(x)$ as in Proposition 2.1 and ∂^{-1} as in Theorem 4.3. The morphism T is clearly compatible with the UMCT (16), in the sense that it induces a commutative diagram

$$\begin{array}{ccc} \mathrm{KK}(A, B) & \longrightarrow & \mathrm{Hom}_\Lambda(\underline{\mathrm{K}}(A), \underline{\mathrm{K}}(Q(\mathcal{K} \otimes B))) \\ {\scriptstyle T}\big\downarrow & & {\scriptstyle \underline{T}}\big\downarrow \\ \mathrm{KK}(C(S^1) \otimes A, B) & \longrightarrow & \mathrm{Hom}_\Lambda(\underline{\mathrm{K}}(C(S^1) \otimes A), \underline{\mathrm{K}}(Q(\mathcal{K} \otimes B))) \end{array}$$

Here $\underline{T}(h) = \partial_*^{-1} \circ h \circ \tau$, where $\tau : \underline{\mathrm{K}}(C(S^1) \otimes A) \to \underline{\mathrm{K}}_{+1}(A)$ is induced by $\tau_A(x) \in \mathrm{KK}^1(C(S^1) \otimes A, A)$ and ∂_*^{-1} is the inverse of the isomorphism $\partial_* : \underline{\mathrm{K}}(Q(\mathcal{K} \otimes B)) \to \underline{\mathrm{K}}_{+1}(B)$. That shows that T maps $\mathrm{Pext}(K_*(A), K_{*+1}(B))$ to $\mathrm{Pext}(K_*(C(S^1) \otimes A), K_{*+1}(Q(\mathcal{K} \otimes B))$. Note that

$$T = \partial^{-1} \circ (\tau_A(x) \otimes -) = \chi \circ \kappa \circ (\tau_A(x) \otimes -).$$

Therefore by Propositions 4.2 and 2.1,

$$[\sigma]_{\mathrm{KK}} = \chi[\sigma]_{\mathrm{Ext}} = T([\varphi] - [\psi]) \in \mathrm{Pext}(K_*(C(S^1) \otimes A), K_{*+1}(Q(\mathcal{K} \otimes B))).$$

Here we use the same notation as in the proof of Theorem 3.11. Since $C(S^1) \otimes A$ is quasidiagonal relative to B and it satisfies the UCT, we have by Corollary 4.7 that $E_{\sigma\oplus\dot{\Delta}}$ is a quasidiagonal set whenever $\Delta : C(S^1) \otimes A \to \mathbb{M}(\mathcal{K} \otimes B)$ is an absorbing extension. The rest of the proof is identical with the last part of the proof of Theorem 3.11. Indeed, if $\eta = (\varphi \oplus \psi)_\infty \oplus \delta$, then the set $(\varphi \oplus \eta)(A) \cup \{u\} = (\Phi \oplus \delta)(A) \cup \{u\} \subset E_{\sigma\oplus\dot{\Delta}}$ is quasidiagonal and $\varphi \oplus \eta = u(\psi \oplus \eta)u^*$. Therefore φ is approximately stably equivalent to ψ by Lemma 3.10.

$\blacksquare$

There is a number of interesting corollaries of Theorem 5.1 where the approximate multiplicative morphisms γ_n implementing (iii) can be chosen to be $*$-homomorphisms. For instance this is the case when A is nuclear residually finite dimensional (γ_n will be finite dimensional representations) or when there is a full embedding $\iota : A \hookrightarrow B$ ($\gamma_n = k(n) \cdot \iota$ as in Corollary 3.12).

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