ONE-PARAMETER CONTINUOUS FIELDS OF KIRCHBERG ALGEBRAS

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Abstract. We prove that all unital separable continuous fields of C*-algebras over $[0,1]$ with fibers isomorphic to the Cuntz algebra $\mathcal{O}_n$ $(2 \leq n \leq \infty)$ are trivial. More generally, we show that if A is a separable, unital or stable, continuous field over $[0,1]$ of Kirchberg C*-algebras satisfying the UCT and having finitely generated K-theory groups, then A is isomorphic to a trivial field if and only if the associated K-theory presheaf is trivial. For fixed $d \in \{0,1\}$ we also show that, under the additional assumption that the fibers have torsion free K_d-group and trivial K_{d+1}-group, the K_d-sheaf is a complete invariant for separable stable continuous fields of Kirchberg algebras.

1. Introduction

A separable nuclear purely infinite simple C*-algebra is called a Kirchberg algebra. In this paper we shall consider separable unital or stable continuous fields of Kirchberg algebras over the unit interval. We shall prove an approximation result, Theorem 6.1, and an inductive limit representation, Theorem 6.2, for these fields. These results lead us to Theorem 7.3 (a triviality result for unital $\mathcal{O}_n$-fields), Theorem 7.5 (a triviality result for fields whose associated K-theory presheaf is isomorphic to the presheaf of a trivial field), and Theorem 8.2 (a classification result based on the K_d-theory sheaf). The fields classified by Theorem 8.2 may have a rather complicated structure and can fail to be locally trivial at any point in $[0,1]$, as illustrated by Example 8.4.

We shall rely heavily on the classification theorem (and related results) of Kirchberg and Phillips [24], and on the work on non-simple nuclear purely infinite C*-algebras of Blanchard and Kirchberg [6] and Rørdam and Kirchberg [15], [16]. The results of Blackadar [2] and Spielberg [28] on the semiprojectivity of Kirchberg algebras and the results of Spielberg [27] and Lin [19] on the weak semiprojectivity of Kirchberg algebras also play an important role.

There are two key ideas on which we base our approach. The first is to approximate continuous fields by what we shall call elementary fields — i.e., fields which are locally trivial at all but finitely many points; see Section 6. The second is to introduce the notion of fibered morphism of fields, a natural blow-up construction based on the usual notion of morphism of fields; see Section 5.

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The triviality of $\mathcal{O}_2$-stable continuous fields was announced by Kirchberg [14] in a vastly more general context. An isomorphism theorem for continuous fields of Kirchberg algebras over zero dimensional spaces was given in [10].

One word concerning the terminology: in many of our statements, we will refer to the C*-algebra of continuous sections associated to a continuous field of C*-algebras over a compact space X as a continuous C*-bundle over X. This is consistent with the terminology used in [4], [17] and [5].

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2. Semiprojective algebras

Recall that a separable C*-algebra D is *weakly semiprojective* if for any finite subset $\mathcal{F} \subset D$ and any $\varepsilon > 0$, any C*-algebra B, any increasing sequence (J_n) of (closed, two-sided) ideals of B and any $*$-homomorphism $\iota : D \to B/J$ (where J is the closure of $\bigcup_n J_n$) there is a $*$-homomorphism $\varphi : D \to B/J_n$ (for some n) such that $\|\pi_n\varphi(c) - \iota(c)\| < \varepsilon$ for all $c \in \mathcal{F}$ (where $\pi_n : B/J_n \to B/J$ is the natural map). If we assume that there is φ such that $\pi_n\varphi = \iota$ then A is *semiprojective*. We shall use (weak) semiprojectivity in the following context (see Section 3 for terminology). Let B be a continuous C*-bundle over a compact Hausdorff space X, let $x \in X$ and consider the sets $U_n = \{y \in X : d(y, x) \leq 1/n\}$. Then $J_n = C_{U_n}(X)B$ is an increasing sequence of ideals of B such that $B/J_n \cong B(U_n)$ and $B/J \cong B(x)$. Here $C_{U_n}(X)$ denotes the ideal of $C(X)$ consisting of all continuous functions that vanish on U_n.

Let us recall that all unital Kirchberg algebras A with $K_*(A)$ finitely generated and satisfying the UCT are weakly semiprojective by a result of Spielberg [27] and Lin [19]. If moreover $K_1(A)$ is torsion free, then A is semiprojective by a result of Spielberg [28]. Blackadar showed that a Kirchberg algebra A is semiprojective if and only if $A \otimes \mathcal{K}$ is semiprojective [2]. Similarly, a Kirchberg algebra A is weakly semiprojective if and only if $A \otimes \mathcal{K}$ is weakly semiprojective [8]. Let A be a C*-algebra, $a \in A$ and $\mathcal{F}, \mathcal{G} \subseteq A$. If $\varepsilon > 0$, we write $a \in_\varepsilon \mathcal{F}$ if there is $b \in \mathcal{F}$ such that $\|a - b\| < \varepsilon$. Similarly, we write $\mathcal{F} \subset_\varepsilon \mathcal{G}$ if $a \in_\varepsilon \mathcal{G}$ for every $a \in \mathcal{F}$.

Proposition 2.1. [12, Thms. 3.1, 4.6] *Let D be a separable weakly semiprojective C*-algebra. For any finite subset $\mathcal{F} \subset D$ and any $\varepsilon > 0$ there exist a finite subset $\mathcal{G} \subset D$ and $\delta > 0$ such that for any C*-algebra $B \subset A$ and any $*$-homomorphism $\varphi : D \to A$ with $\varphi(\mathcal{G}) \subset_\delta B$, there is a $*$-homomorphism $\psi : D \to B$ such that $\|\varphi(a) - \psi(a)\| < \varepsilon$ for all $a \in \mathcal{F}$.*

The following two results appear in Loring's book [20], except that he makes a certain assumption that is not necessary. Specifically, the assumption that the semiprojective algebras are finitely presented can be removed. While the proofs of the sharpened results do not really require new ideas, the consequences are quite useful. Although it is not clear whether or not $\mathcal{O}_\infty$ or other semiprojective Kirchberg algebras are finitely presented in Loring's sense, Propositions 2.2 and 2.3 now allows us to deal with arbitrary semiprojective Kirchberg algebras.

Proposition 2.2. *Let A be a separable semiprojective C*-algebra and let (x_i) be a sequence dense in the unit ball of A. For any $n \geq 1$ and $\varepsilon > 0$ there are $m \geq 1$ and $\delta > 0$ such that for any diagram*

$$\begin{tikzcd} & B \arrow[d, "\pi"] \\ A \arrow[ur, "\psi"] \arrow[r, "\sigma"] & B/J \end{tikzcd}$$

of C-algebras and $*$-homomorphisms (J an ideal of B and π the quotient map) such that $\|\pi\psi(x_i) - \sigma(x_i)\| < \delta$ for all $1 \leq i \leq m$, there is a $*$-homomorphism $\varphi : A \to B$ such that $\pi\varphi = \sigma$ and $\|\varphi(x_i) - \psi(x_i)\| < \varepsilon$ for all $1 \leq i \leq n$.*

Proposition 2.3. *Let A be a separable semiprojective C*-algebra and let (x_i) be a sequence dense in the unit ball of A. For any $n \geq 1$ and $\varepsilon > 0$ there are $m \geq 1$ and $\delta > 0$ such that for any C*-algebra B and any two $*$-homomorphisms $\varphi, \psi : A \to B$ such that $\|\varphi(x_i) - \psi(x_i)\| < \delta$ for all $1 \leq i \leq m$, there is a homotopy $\chi_t : A \to B$, $t \in [0,1]$, of $*$-homomorphisms from φ to ψ that satisfies*

$$\|\varphi(x_i) - \chi_t(x_i)\| < \varepsilon, \text{ for all } 1 \leq i \leq n \text{ and all } t \in [0,1].$$

Propositions 2.2 and 2.3 remain valid if one requires that all C*-algebras and morphisms are unital.

Proposition 2.4. *Let A be a semiprojective Kirchberg algebra. Let B be a separable C*-algebra with a full projection such that $B \cong B \otimes \mathcal{O}_\infty \otimes \mathcal{K}$, let J be a proper ideal of B and let $\pi : B \to B/J$ denote the quotient map. Let $\sigma : A \to B/J$ be a full $*$-homomorphism. Assume that there is $\alpha \in KK(A,B)$ such that $[\pi]\alpha = [\sigma]$ in $KK(A, B/J)$. Then there is a full $*$-homomorphism $\varphi : A \to B$ such that $[\varphi] = \alpha$ and $\pi\varphi = \sigma$.*

Proof. Let (x_i) be a sequence dense in the unit ball of A. By Proposition 2.3 there are $n \geq 1$ and $\varepsilon > 0$ such that any two $*$-homomorphisms $\varphi, \psi_1 : A \to B$ satisfying $\|\varphi(x_i) - \psi_1(x_i)\| < \varepsilon$, for all $1 \leq i \leq n$, are homotopic. Let m and δ be as in Proposition 2.2. By [24, Thm. 8.2.1] there is a full $*$-homomorphism $\psi : A \to B$ such that $[\psi] = \alpha$. Since $[\pi\psi] = [\sigma]$ in $KK(A, B/J)$ and both maps are full, it follows from [24, Thm. 8.2.1] that there is a unitary $v \in M(B/J)$ such that

$$\|\pi\psi(x_i) - v^*\sigma(x_i)v\| < \delta,$$

for all $1 \leq i \leq m$. Since B is stable, so is B/J [24]. Hence the unitary group of $M(B/J)$ is path connected. It follows that v lifts to a unitary $u \in M(B)$. The map $\psi_1 = u\psi u^*$ is an approximate lifting of σ in the sense that $\|\pi\psi_1(x_i) - \sigma(x_i)\| < \delta$ for all $1 \leq i \leq m$. By Proposition 2.2 there is a $*$-homomorphism $\varphi : A \to B$ lifting σ and such that $\|\varphi(x_i) - \psi_1(x_i)\| < \varepsilon$ for all $1 \leq i \leq n$. By Proposition 2.3, φ is homotopic to ψ_1 and hence $[\varphi] = [\psi_1] = [\psi] = \alpha$. Moreover φ is full, since for every nonzero projection $e \in A$, $\varphi(e)$ is homotopic to the full projection $u\psi(e)u^*$. □

The following result was proved by H. Lin [18] in the case that A is a separable nuclear unital C*-algebra. While the result extends to exact C*-algebras (as stated below; the proof is similar) the nuclear case is all we shall need in the present paper.

Theorem 2.5. *Let A be a separable exact unital C^*-algebra and let B be a unital C^*-algebra. Let $\varphi, \psi : A \to B$ be two unital nuclear full $*$-monomorphisms with $[\varphi] = [\psi]$ in $KK_{nuc}(A, B)$. Let $\jmath : B \to B \otimes \mathcal{O}_\infty$ be defined by $\jmath(b) = b \otimes 1$. Then $\jmath \circ \varphi$ is approximately unitarily equivalent to $\jmath \circ \psi$.*

Let $\mathcal{G} \subset A$ and let $\delta > 0$. A map $\varphi : A \to B$ is called $(\mathcal{G}, \delta)$-multiplicative if $\|\varphi(a)\varphi(b) - \varphi(ab)\| < \delta$ for all $a, b \in \mathcal{G}$.

Theorem 2.6. *Let A be a unital Kirchberg C^*-algebra. Suppose that $K_*(A)$ is finitely generated and that A satisfies the UCT. Then for any finite subset $\mathcal{F}$ of A and any $\varepsilon, \varepsilon' > 0$ there are a finite subset $\mathcal{G}$ of A and $\delta > 0$ with the following property. For any unital Kirchberg C^*-algebra B, any $(\mathcal{G}, \delta)$-multiplicative unital completely positive map $\varphi : A \to B$ and any unitary $w \in B$ with spectrum δ-dense in $\mathbb{T}$ and satisfying*

$$\|[\varphi(a), w]\| < \delta, \quad \textit{for all } a \in \mathcal{G},$$

there is a unital full $*$-homomorphism $\phi : C(\mathbb{T}) \otimes A \to B$ such that

$$\|\phi(1 \otimes a) - \varphi(a)\| < \varepsilon \quad \textit{for all } a \in \mathcal{F}, \textit{ and}$$
$$\|\phi(z \otimes 1) - w\| < \varepsilon.$$

If in addition $KK(SA, B) = 0$, then there is a continuous path $(w_t)_{t \in [0,1]}$ of unitaries in B with $w_0 = 1$, $w_1 = w$ and

$$\|[\varphi(a), w_t]\| < \varepsilon', \quad \textit{for all } a \in \mathcal{F} \textit{ and all } t \in [0, 1].$$

For the second part of the theorem the condition that the spectrum of w is δ-dense in $\mathbb{T}$ is not needed.

Proof. Suppose, to obtain a contradiction, that there are a finite subset $\mathcal{F}$ of the unit ball of A and $\varepsilon > 0$ for which no $\mathcal{G}$ and δ can be found satisfying the conclusion of the theorem. Choose a sequence (a_n) dense in the unit ball of A. In particular, for each n, the finite set $\mathcal{G}_n = \{a_1, \ldots, a_n\}$ and the tolerance $\delta_n = 1/n$ will not do. In other words, for each n there exist a unital Kirchberg algebra B_n, a unital completely positive map $\varphi_n : A \to B_n$ which is $(\mathcal{G}_n, \delta_n)$-multiplicative, and a unitary w_n with $\|[w_n, \varphi_n(a_i)]\| < \delta_n$ for $1 \leq i \leq n$, with the spectrum of w_n δ_n-dense in $\mathbb{T}$, such that the pair (φ_n, w_n) cannot be approximated as in the statement. The sequence (φ_n) defines a unital $*$-homomorphism $\varphi_\infty : A \to \ell^\infty(B_n)/c_0(B_n)$. The sequence (w_n) defines a unitary $w_\infty \in \ell^\infty(B_n)/c_0(B_n)$ which commutes with the image of φ_∞. Thus we obtain a $*$-homomorphism $\Phi_\infty : C(\mathbb{T}) \otimes A \to \ell^\infty(B_n)/c_0(B_n)$. One verifies immediately that Φ_∞ is a unital full $*$-monomorphism. Indeed, if I is a nonzero ideal of $C(\mathbb{T}) \otimes A$, let us show that $\Phi_\infty(I)$ is contained in no proper ideal of $\ell^\infty(B_n)/c_0(B_n)$. Since A is simple, $I = C_F(\mathbb{T}) \otimes A$ for some proper closed subset F of $\mathbb{T}$, where $C_F(\mathbb{T}) = \{f \in C(\mathbb{T}) : f|_F = 0\}$. Let $g \in C_F(\mathbb{T})$ with $0 \leq g \leq 1$ be such that $g(t) = 1$ on some arc c disjoint from F. It suffices to show that $\Phi_\infty(g \otimes 1)$ is full in $\ell^\infty(B_n)/c_0(B_n)$. Note that $(g(w_n))$ is a lifting of $\Phi_\infty(g \otimes 1)$ to $\ell^\infty(B_n)$. Since there is n_0 such that the spectrum of w_n intersects the arc c for $n \geq n_0$, we have $\|g(w_n)\| = 1$ for $n \geq n_0$. Since B_n is unital simple and purely infinite, by [24, Lemma 4.1.7] we find

$b_n \in B_n$ of norm at most two with $b_n g(w_n) b_n^* = 1_{B_n}$, for all $n \geq n_0$. Thus $(g(w_n))$ is full in $\ell^\infty(B_n)$ and hence so is its image in $\ell^\infty(B_n)/c_0(B_n)$.

Since the extension

$$0 \to c_0(B_n) \to \ell^\infty(B_n) \to \ell^\infty(B_n)/c_0(B_n) \to 0 \tag{1}$$

is quasidiagonal, the boundary map $\partial : \underline{\mathrm{K}}(\ell^\infty(B_n)/c_0(B_n)) \to \underline{\mathrm{K}}(c_0(B_n))$ vanishes by [7, Thm. 8]. By the UMCT of [9], if D is a separable C*-algebra that satisfies the UCT and such that $K_*(D)$ is finitely generated, then there is a natural isomorphism

$$KK(D, E) \cong \mathrm{Hom}_\Lambda(\underline{\mathrm{K}}(D), \underline{\mathrm{K}}(E)),$$

for any σ-unital C*-algebra E. Under this isomorphism, the boundary map

$$KK(D, \ell^\infty(B_n)/c_0(B_n)) \to KK(SD, c_0(B_n))$$

corresponds to composition with ∂, so that it also vanishes. Therefore, by the six-term exact sequence in KK-theory, the map

$$KK(D, \ell^\infty(B_n)) \to KK(D, \ell^\infty(B_n)/c_0(B_n))$$

is surjective. Using Kirchberg's theorem [24, Thm. 8.3.3], and the hypothesis that B_n are Kirchberg algebras, we verify immediately that the natural map $\prod_n \mathrm{Hom}(D, B_n) \to \mathrm{Hom}(D, \ell^\infty(B_n))$ induces a surjection $\prod_n KK(D, B_n) \to KK(D, \ell^\infty(B_n))$. In view of the above discussion, we obtain a surjective map

$$\prod_n KK(C(\mathbb{T}) \otimes A, B_n) \to KK(C(\mathbb{T}) \otimes A, \ell^\infty(B_n)/c_0(B_n)).$$

By applying [24, Thm. 8.3.3] again, we find a sequence of unital $*$-monomorphisms $\phi_n' : C(\mathbb{T}) \otimes A \to B_n$ which induces a unital full $*$-monomorphism $\Phi_\infty' : C(\mathbb{T}) \otimes A \to \ell^\infty(B_n)/c_0(B_n)$ and which has the same KK-theory class as Φ_∞. Therefore, by Theorem 2.5, $\jmath \circ \Phi_\infty, \jmath \circ \Phi_\infty' : C(\mathbb{T}) \otimes A \to (\ell^\infty(B_n)/c_0(B_n)) \otimes \mathcal{O}_\infty$ are approximately unitarily equivalent. In particular, there is a unitary $v \in (\ell^\infty(B_n)/c_0(B_n)) \otimes \mathcal{O}_\infty$ such that

$$\|v\big(\jmath \circ \Phi_\infty(a)\big)v^* - \jmath \circ \Phi_\infty'(a)\| < \varepsilon, \tag{2}$$

for all $a \in \{1 \otimes b : b \in \mathcal{F}\} \cup \{z \otimes 1\}$, where z denotes the identity map of $\mathbb{T}$. Since the extension (1) is quasidiagonal, we can lift v to a unitary in $\ell^\infty(B_n) \otimes \mathcal{O}_\infty$ given by a sequence of unitaries $v_n \in B_n \otimes \mathcal{O}_\infty$. By a result of Kirchberg (see also [24, Thm. 8.4.1]), for each n, there is an isomorphism $\nu_n : B_n \otimes \mathcal{O}_\infty \to B_n$, whose composition with the inclusion map $\jmath_n : B_n \to B_n \otimes \mathcal{O}_\infty$ is approximately unitarily equivalent to id_{B_n}. Using this in conjunction with (2), we find unitaries $u_n \in B_n$ such that

$$\limsup_{n\to\infty} \|u_n \left(\phi_n'(1 \otimes a)\right) u_n^* - \varphi_n(a)\| < \varepsilon,$$

$$\limsup_{n\to\infty} \|u_n \left(\phi_n'(z \otimes 1)\right) u_n^* - w_n\| < \varepsilon,$$

for all a in $\mathcal{F}$. Letting $\phi_n(-) = u_n\,(\phi_n'(-))\,u_n^*$, we have

$$\limsup_{n\to\infty} \|\phi_n(1 \otimes a) - \varphi_n(a)\| < \varepsilon,$$

$$\limsup_{n\to\infty} \|\phi_n(z \otimes 1) - w_n\| < \varepsilon,$$

which produces a contradiction.

Consider now the second part of the theorem. Once again we will assume that the conclusion is false for some finite set $\mathcal{F} \subset A$ and $\varepsilon' > 0$ and will seek a contradiction. Let $\alpha : [0, +\infty) \to [0, +\infty)$ be a continuous function with $\alpha(0) = 0$ which will be specified later. There is $\varepsilon > 0$ such that $\alpha([0, \varepsilon]) \subset [0, \varepsilon')$. Let (φ_n), $\mathcal{G}_n$, δ_n and (w_n) be as in the first part of the proof, except that we make no assumptions on the spectrum K_n of the unitary w_n. After passing to a subsequence of (w_n) we may assume that the sequence of compact subsets (K_n) converges in the Gromov-Hausdorff distance to a nonempty compact subset K of $\mathbb{T}$ by [23, Prop. 7.2]. If $K = \mathbb{T}$, then we apply the first part of the theorem to $\varphi = \varphi_n$ and $w = w_n$ for some sufficiently large n. Thus, we obtain a unital $*$-monomorphism $\phi : C(\mathbb{T}) \otimes A \to B$ such that

$$\|\phi(1 \otimes a) - \varphi(a)\| < \varepsilon \quad \text{for all } a \in \mathcal{F}, \text{ and } \|\phi(z \otimes 1) - w\| < \varepsilon.$$

Observe that if $KK(SA, B) = 0$, then the canonical injection

$$KK(A, B) \to KK(C(\mathbb{T}) \otimes A, B)$$

is bijective. Thus, if $\nu : B \otimes \mathcal{O}_\infty \to B$ is an isomorphism as above, then the class of $[\phi]$ in $KK(C(\mathbb{T}) \otimes A, B)$ is equal to the class of some unital $*$-monomorphism ϕ' of the form $\nu \circ (\theta' \otimes \psi')$ for some $*$-homomorphisms $\psi' : A \to B$ and $\theta' : C(\mathbb{T}) \to \mathcal{O}_\infty$. By Phillips's theorem [24, 8.2.1], ϕ is approximately unitarily equivalent to ϕ'. Without any loss of generality, we may assume that ϕ is unitarily equivalent to ϕ'. Set $\phi(z \otimes 1) = u_1$. Since the unitary group of $\mathcal{O}_\infty$ is path connected, we find a continuous path of unitaries $(u_t)_{t \in [0,1]}$ in $\mathcal{O}_\infty$ joining u_1 with 1, satisfying $[\phi(1 \otimes a), u_t] = 0$ for all $a \in A$ and $t \in [0, 1]$. In follows that $\|[\varphi(a), u_t]\| < 2\varepsilon$ for all $a \in A$ and $t \in [0, 1]$. If $v_t = (1 - t)w + tu_1$, then $v_t|v_t|^{-1/2}$ is a path of unitaries in B joining w to u_1. Then the juxtaposition of this path with u_t gives a path w_t from w to 1 with

$$\|[\varphi(a), w_t]\| < \alpha(\varepsilon), \quad \text{for all } a \in \mathcal{F}, \text{ and } t \in [0, 1].$$

Here α is a universal non-negative continuous function with $\alpha(0) = 0$ (recall that $\|a\| \leq 1$ for all $a \in \mathcal{F}$). Since $\alpha(\varepsilon) < \varepsilon'$ we ran into a contradiction.

It remains to argue the case when K is a proper subset of $\mathbb{T}$. After dropping finitely many w_n's we may assume that there is some fixed closed neighborhood V of K in $\mathbb{T}$ such that $V \neq \mathbb{T}$ and each K_n is contained in V. By functional calculus we find a sequence of selfadjoint elements $h_n \in B_n$ such that $\lim_{n \to \infty} \|[\varphi_n(a), h_n]\| = 0$ for all $a \in A$, $w_n = \exp(ih_n)$ and $\sup_n \|h_n\| < \infty$. Let us set $v_n(t) = \exp(ith_n)$. One shows immediately that for some sufficiently large n,

$$\|[\varphi_n(a), v_n(t)]\| < \varepsilon', \quad \text{for all } a \in \mathcal{F} \text{ and all } t \in [0, 1],$$

which gives a contradiction, since $v_n(0) = 1$ and $v_n(1) = w_n$. □

Corollary 2.7. *Let A be a unital Kirchberg C*-algebra. Suppose that $K_*(A)$ is finitely generated and that A satisfies the UCT. Then for any finite subset $\mathcal{F}$ of $A \otimes \mathcal{K}$ and any $\varepsilon > 0$ there are a finite subset $\mathcal{G}$ of $A \otimes \mathcal{K}$ and $\delta > 0$ with the following property.*

For any unital Kirchberg C-algebra B with $KK(SA, B) = 0$, any full $*$-homomorphism $\varphi : A \otimes \mathcal{K} \to B \otimes \mathcal{K}$ and any unitary $w \in M(B \otimes \mathcal{K})$ satisfying*

$$\|[\varphi(a), w]\| < \delta, \quad \textit{for all } a \in \mathcal{G},$$

there is a continuous path $(w_t)_{t\in[0,1]}$ of unitaries in $M(B \otimes \mathcal{K})$ with $w_0 = w$, $w_1 = 1$ and

$$\|[\varphi(a), w_t]\| < \varepsilon, \quad \textit{for all } a \in \mathcal{F} \textit{ and all } t \in [0, 1].$$

Proof. We may assume that $\mathcal{F} \subset A \otimes e\mathcal{K}e$ for some projection $e \in \mathcal{K}$. Let $\delta' > 0$ and $\mathcal{G} \subset A \otimes e\mathcal{K}e$ (with $1_A \otimes e \in \mathcal{G}$) be given by the second part of Theorem 2.6 applied to $A \otimes e\mathcal{K}e$, $\mathcal{F}$ and ε. Let us choose $\delta > 0$ small enough such that if $\|[\varphi(a), w]\| < \delta$ for all $a \in \mathcal{G}$, then there is a unitary $v \in M(B\otimes\mathcal{K})$ such that v commutes with $f = \varphi(1_A\otimes e)$ and v is sufficiently close to w so that $\|[\varphi(a), v]\| < \delta'$ for all $a \in \mathcal{G}$ and there is a continuous path of unitaries (ω_t) from w to v such that $\|[\varphi(a), \omega_t]\| < \varepsilon$ for all $a \in \mathcal{F}$. In particular $\|[\varphi(a), fvf]\| < \delta'$ for all $a \in \mathcal{G}$. By Theorem 2.6, there is a continuous path (y_t) of unitaries in $f(B \otimes \mathcal{K})f$ joining fvf to f such that $\|[\varphi(a), y_t]\| < \varepsilon$ for all $a \in \mathcal{F}$. Let us argue that there is a continuous path (z_t) of unitaries in $(1 - f)M(B \otimes \mathcal{K})(1 - f)$ joining $(1 - f)v(1 - f)$ to $1 - f$. Indeed, since $f \in B \otimes \mathcal{K}$, $(1 - f)H_B \cong H_B$ by Kasparov's absorption theorem [13] (where H_B is the Hilbert B-module $B \oplus B \oplus \cdots$),

$$(1 - f)M(B \otimes \mathcal{K})(1 - f) \cong L((1 - f)H_B) \cong L(H_B) \cong M(B \otimes \mathcal{K}).$$

Since the unitary group of $M(B \otimes \mathcal{K})$ is path connected, we have verified the existence of the path (z_t). Finally let us observe that the juxtaposition of the paths (ω_t) and $(y_t + z_t)$ gives a continuous path of unitaries (w_t) in $M(B \otimes \mathcal{K})$ such that the path $t \mapsto w_t$ has the desired properties. □

3. C*-bundles

Let X be a compact Hausdorff space. A C*-bundle A over X is a C*-algebra A endowed with a unital $*$-monomorphism from $C(X)$ to the center of the multiplier C*-algebra $M(A)$ of A. If $Y \subseteq X$ is a closed subset, we let $C_Y(X)$ denote the ideal of $C(X)$ consisting of functions vanishing on Y. Then $C_Y(X)A$ is a closed two-sided ideal of A. The quotient of A by this ideal is a C*-bundle denoted by $A(Y)$ and called the restriction of $A = A(X)$ to Y. The quotient map is denoted by $\pi_Y : A(X) \to A(Y)$. If Z is a closed subset of Y we have a natural restriction map $\pi^Y_Z : A(Y) \to A(Z)$ and $\pi_Z = \pi^Y_Z \circ \pi_Y$. If Y reduces to a point x, we write $A(x)$ for $A(\{x\})$ and π_x for $\pi_{\{x\}}$. The C*-algebra $A(x)$ is called the fiber of A at x. The image $\pi_x(a) \in A(x)$ of $a \in A$ is denoted by $a(x)$. For any $a \in A$, the map $x \mapsto \|a(x)\|$ is upper semi-continuous. If the map $x \mapsto \|a(x)\|$ is continuous for all $a \in A$, then A is called a continuous C*-bundle. If $\varphi : A \to B$ is a morphism of $C(X)$-bundles and Y is a closed subset of X, then the induced map $A(Y) \to B(Y)$ is denoted by φ_Y.

Lemma 3.1. *Let X be a compact Hausdorff space and let A be a continuous C*-bundle over X. If Y, Z are closed subsets of X, then $M(A(Y \cup Z))$ is isomorphic to the pullback of the two restriction maps $M(A(Y)) \to M(A(Y \cap Z))$ and $M(A(Z)) \to M(A(Y \cap Z))$.*

Proof. The pullback of the diagram $A(Y) \xrightarrow{\pi} A(Y \cap Z) \xleftarrow{\pi} A(Z)$, is isomorphic to $A(Y \cup Z)$ by [11, Prop. 10.1.13]. The statement follows now from the description of the multiplier algebra of a pullback given by [21, Prop. 7.2]. Alternatively, one can derive the statement from [1, Thm. 3.3] which identifies $M(A)$ with the set of sections $(s(x))_{x\in X}$, $s(x)\in M(A(x))$ which are bounded and strictly continuous. □

Proposition 3.2. *Let A be a stable Kirchberg algebra. Suppose that $K_*(A)$ is finitely generated and that A satisfies the UCT. Then for any finite subset $\mathcal{F}$ of A and any $\varepsilon>0$ there are a finite subset $\mathcal{G}$ of A and $\delta>0$ with the following property. Let B be a stable continuous C^*-bundle of Kirchberg algebras over $[\alpha,\beta]$ and let $\phi: A\to B$ be a full $*$-homomorphism. Let $z\in[\alpha,\beta]$ and let $w\in M(B(z))$ be a unitary such that*

$$\|[\pi_z\phi(a),w]\|<\delta,\quad \textit{for all } a\in\mathcal{G}.$$

Assume that $KK(SA,B(z))=0$. Then there is a neighborhood $[z_1,z_2]$ of z and there is a unitary $W\in M(B)$ such that

$$\|[\phi(a),W]\|<\varepsilon,\quad \textit{for all } a\in\mathcal{F},$$

$$W(z)=w,\ \textit{and } W(x)=1 \textit{ for all } x\in[\alpha,\beta]\setminus[z_1,z_2].$$

Moreover one may arrange that $[z_1,z_2]$ is contained in a given neighborhood of z.

Proof. If $\mathcal{G}$ and δ are as in Corollary 2.7, then there is a continuous path $(w_t)_{t\in[0,1]}$ of unitaries in $M(B(z))$ with $w_0=1$, $w_1=w$ and

$$\|[\pi_z\phi(a),w_t]\|<\varepsilon,\quad \text{for all } a\in\mathcal{F}.$$

Let $[z_1,z_2]$ be a neighborhood of z. Since $w_1=1$, the path $(w_t)_{t\in[0,1]}$ lifts to a continuous path of unitaries in $M(B[z_1,z_2])$, denoted by $(\Omega_t)_{t\in[0,1]}$, such that $\Omega_1=1$. After passing to a smaller neighborhood of z if necessary, we may arrange that

$$\|[\pi_{[z_1,z_2]}\phi(a),\Omega_t]\|<\varepsilon,\quad \text{for all } a\in\mathcal{F}\,\text{and}\,t\in[0,1].$$

Let $h:[z_1,z_2]\to[0,1]$ be a continuous map such that $h(z)=1$ and h vanishes on $\{z_1,z_2\}\setminus\{z\}$. Then the map

$$x\mapsto\begin{cases}\Omega_{h(x)}(x), & \text{if } x\in[z_1,z_2],\\ 1, & \text{if } x\in[\alpha,\beta]\setminus[z_1,z_2],\end{cases}$$

defines a unitary W in $M(B)$ (by Lemma 3.1) which satisfies the conclusion of the proposition. □

Corollary 3.3. *Let A be a stable Kirchberg algebra satisfying the UCT and such that $K_*(A)$ is finitely generated. Let $Z=[z_1,z_2]$ be an interval with $z_1<z_2$. Then for any finite subset $\mathcal{F}$ of $C(Z)\otimes A$ and any $\varepsilon>0$ there exist a finite subset $\mathcal{G}$ of A and $\delta>0$ with the following property. Let B be a stable continuous C^*-bundle of Kirchberg algebras over Z and let $\phi,\psi: C(Z)\otimes A\to B$ be two injective $*$-homomorphisms which are $C(Z)$-linear. Let $u_i\in M(B(z_i))$, $i=1,2$, and let $v\in M(B)$ be unitaries such that*

$$\|u_i\phi_{z_i}(a)u_i^*-\psi_{z_i}(a)\|<\delta,\quad \textit{for all } a\in\mathcal{G} \textit{ and } i=1,2, \tag{3}$$

$$\|v\phi(a)v^*-\psi(a)\|<\delta,\quad \textit{for all } a\in\mathcal{G}. \tag{4}$$

Assume that $KK(SA, B(z_i)) = 0$, $i = 1, 2$. Then there is a unitary $u \in M(B)$ such that $u(z_i) = u_i$, $i = 1, 2$, and

$$\|u\phi(a)u^* - \psi(a)\| < \varepsilon, \quad \textit{for all } a \in \mathcal{F}.$$

Proof. Since both ϕ and ψ are $C(Z)$-linear it suffices to prove the statement for $\varepsilon > 0$ and $\mathcal{F} \subset A$. Choose $\mathcal{G}$ and δ as in Proposition 3.2 such that $\delta < \varepsilon$. Assume that (3) and (4) are satisfied for $\delta/2$. Then

$$\|[v(z_i)^* u_i, \phi_{z_i}(a)]\| < \delta, \quad \text{for all } a \in \mathcal{G}. \tag{5}$$

By applying Proposition 3.2 to both ends of Z, we find a unitary $w \in M(B)$ such that $w(z_i) = v(z_i)^* u_i$, $i = 1, 2$, and

$$\|[\phi(a), w]\| < \varepsilon, \quad \text{for all } a \in \mathcal{F}.$$

Consider the unitary $u = vw \in M(B)$. We have $u(z_i) = u_i$, $i = 1, 2$, and

$$\|u\phi(a)u^* - \psi(a)\| \leq \|v\big(w\phi(a)w^* - \phi(a)\big)v^*]\| + \|v\phi(a)v^* - \psi(a)\| < \varepsilon + \delta < 2\varepsilon$$

for all $a \in \mathcal{F}$, as desired. □

4. Invariants

Let X be the unit interval. Let $\mathcal{U}$ be set of all closed subintervals of X of positive length. We regard $\mathcal{U}$ as a category with morphisms given by inclusions. A presheaf on $\mathcal{U}$ consists of the following data:

(a) An assignment to each $U \in \mathcal{U}$ of a set $\mathcal{S}(U)$.

(b) A collection of mappings (called restriction homomorphisms) $r^U_V : \mathcal{S}(U) \to \mathcal{S}(V)$ for each pair U, V in $\mathcal{U}$ such that $V \subset U$, satisfying

(1) $r^U_U = \mathrm{id}_U$ (the identity map),

(2) for $W \subset V \subset U$, $r^U_W = r^V_W r^U_V$.

If $\mathcal{S}$ and $\mathcal{S}'$ are presheaves over $\mathcal{U}$, then a morphism of presheaves $\alpha \in \mathrm{Hom}(\mathcal{S}, \mathcal{S}')$ is a collection of maps $\alpha_U : \mathcal{S}(U) \to \mathcal{S}'(U)$ for each $U \in \mathcal{U}$ such that for $V \subset U$ the following diagram commutes:

$$\begin{tikzcd} \mathcal{S}(U) \arrow[r, "\alpha_U"] \arrow[d, "r^U_V"'] & \mathcal{S}'(U) \arrow[d, "r^U_V"] \\ \mathcal{S}(V) \arrow[r, "\alpha_V"'] & \mathcal{S}'(V) \end{tikzcd} \tag{6}$$

In other words, $\mathcal{S}$ is a contravariant functor from the category $\mathcal{U}$ to the category of sets and α is a natural transformation of functors. In our examples, $\mathcal{S}$ will take values in a category that has some algebraic structure (e.g., abelian semigroups, groups, ordered groups, etc.) Naturally, we shall require that the restriction maps and the morphisms preserve the algebraic structure. Our main example is the presheaf $\mathcal{S} = \mathbb{K}_d(A)$ associated to a continuous C*-bundle A on X:

$$\mathcal{S}(U) = \mathbb{K}_d(A)(U) = K_d(A(U)), \quad r^U_V = K_d(\pi^U_V),$$

where $d \in \{0,1\}$ is fixed and $\pi^U_V : A(U) \to A(V)$ is the natural quotient map. A morphism of C*-bundles over X, $\varphi : A \to B$ induces a morphism of presheaves $\mathbb{K}_d(\varphi) \in \mathrm{Hom}(\mathbb{K}_d(A), \mathbb{K}_d(B))$. Let us observe that we can extend $\mathcal{S}$ to singletons $V = \{x\}$ by setting $\mathcal{S}(V) = K_d(A(x))$ and $r^U_V = K_d(\pi^U_x)$. The stalk of $\mathcal{S}$ at $x \in X$ is the direct limit $\mathcal{S}_x = \varinjlim_{x \in \mathring{U}} \mathcal{S}(U)$ with respect to the restriction maps (r^U_V) where U runs through those elements of $\mathcal{U}$ which contain x in their interior. Since $A(x) = \varinjlim_{x\in\mathring{U}} A(U)$, we can identify $\mathcal{S}(\{x\})$ with $\mathcal{S}_x$, by continuity of K-theory. Consequently, if B is a C*-bundle over X and $\alpha \in \mathrm{Hom}(\mathbb{K}_d(A), \mathbb{K}_d(B))$, then for any $V = \{x\} \subset U$ we have a commutative diagram as in (6), where $\alpha_{\{x\}} : \mathcal{S}(\{x\}) \to \mathcal{S}'(\{x\})$ corresponds to the map $\mathcal{S}_x \to \mathcal{S}'_x$ induced by α, and $\mathcal{S}' = \mathbb{K}_d(B)$. Since a map $\alpha_{\{x\}}$ which makes the diagram (6) commutative for all U with $x \in U$ and $V = \{x\}$ is uniquely determined by a morphism of presheaves $\alpha : \mathbb{K}_d(A) \to \mathbb{K}_d(B)$, we see that for any morphism of C*-bundles $\varphi : A \to B$, $\mathbb{K}_d(\varphi)_{\{x\}} = K_d(\varphi_x)$.

We are going to verify that the presheaf $\mathcal{S} = \mathbb{K}_d(A)$ on $\mathcal{U}$ satisfies the following properties which are similar to those of a sheaf. For every $U \in \mathcal{U}$ and any collection $U_1, ..., U_n$ of elements of $\mathcal{U}$ with $U = \bigcup_i U_i$,

(i) If $s, t \in \mathcal{S}(U)$ and $r^U_{U_i}(s) = r^U_{U_i}(t)$ for all i, then $s = t$.

(ii) If $s_i \in \mathcal{S}(U_i)$ for all i and if for $U_i \cap U_j \neq \emptyset$ we have $r^{U_i}_{U_i\cap U_j}(s_i) = r^{U_j}_{U_i \cap U_j}(s_j)$ then there exists $s \in \mathcal{S}(U)$ such that $r^U_{U_i}(s) = s_i$ for all i.

Every sheaf $\mathcal{S}$ is isomorphic to the sheaf of sections of an etale bundle $\pi : E_{\mathcal{S}} \to X$ [29]. This is also the case with $\mathcal{S} = \mathbb{K}_d(A)$. There is a natural map $r^U_x : \mathcal{S}(U) \to \mathcal{S}_x$. If $s \in \mathcal{S}(U)$ and $x \in \mathring{U}$, then $s_x := r^U_x(s)$ is called the germ of s at x. Consider the disjoint union

$$E_{\mathcal{S}} = \bigsqcup_{x \in X} \mathcal{S}_x$$

and denote by $\pi : E_{\mathcal{S}} \to X$ the natural projection which maps $\mathcal{S}_x$ to x. For each $s \in \mathcal{S}(U)$ consider the section $\hat{s} : \mathring{U} \to E_{\mathcal{S}}$ defined by $\hat{s}(x) = s_x$. The family of sets

$$\{\hat{s}(\mathring{U}) : U \in \mathcal{U}, s \in \mathcal{S}(U)\}$$

forms a basis for a topology on $E_{\mathcal{S}}$. This topology glues together all the stalks $(\mathcal{S}_x)_{x\in X}$ forming the space $E_{\mathcal{S}}$. The bundle $\pi : E_{\mathcal{S}} \to X$ is called etale since the continuous surjection π is a local homeomorphism. For $U \in \mathcal{U}$ let $\Gamma(U, E_{\mathcal{S}})$ denote the set of continuous sections from U to $E_{\mathcal{S}}$. Using the conditions (i) and (ii) (that we verify next) and the continuity of the K-theory one shows (as in the proof of [29, Thm. 2.2, Chap. II]) that the natural map $\mathcal{S}(U) \to \Gamma(U, E_{\mathcal{S}})$, $s \mapsto \hat{s}$, is an isomorphism. Thus we are justified in calling $\mathbb{K}_d(A)$ a sheaf.

Proposition 4.1. *Let A be a continuous C*-bundle over $X = [0,1]$. Fix $d \in \{0,1\}$ and assume that $K_{d+1}(A(x)) = 0$ for all fibers of A. Then $\mathbb{K}_d(A)(U) = K_d(A(U))$ is a sheaf and the natural map $\mathbb{K}_d(A)(U) \to \Gamma(U, E_{\mathbb{K}_d(A)})$ is an isomorphism of sheaves.*

Proof. Let us give the proof in the case $d = 0$, the case $d = 1$ being similar. We need to verify the conditions (i) and (ii) from above. It is clear that, given (U_i), it suffices to verify

(i) and (ii) for some cover of U which refines the cover (U_i). Therefore we may assume that $U_i \cap U_{i+1}$ reduces to a point, and that $U_i \cap U_j = \emptyset$ if $|i-j| > 1$. Set $X_i = U_1 \cup \cdots \cup U_i$ and note that $K_1(A(X_i \cap U_{i+1})) = K_1(A(U_i \cap U_{i+1})) = 0$ by assumption. If Y, Z are closed subsets of U, one has the Mayer-Vietoris exact sequence ([26]):

$$\begin{tikzcd} K_0(A(Y)) \oplus K_0(A(Z)) \arrow[r, "\Delta"] & K_0(A(Y \cap Z)) \arrow[r] & K_1(A(Y \cup Z)) \arrow[d] \\ K_0(A(Y \cup Z)) \arrow[u, "\rho"] & K_1(A(Y \cap Z)) \arrow[l] & K_1(A(Y)) \oplus K_1(A(Z)) \arrow[l] \end{tikzcd}$$

This yields the exact sequence

$$0 \longrightarrow K_0(A(X_{i+1})) \xrightarrow{\rho} K_0(A(X_i)) \oplus K_0(A(U_{i+1})) \xrightarrow{\Delta} K_0(A(X_i \cap U_{i+1})), \tag{7}$$

where ρ is the natural restriction map and $\Delta = K_0(\pi^{X_i}_{X_i \cap U_{i+1}}) - K_0(\pi^{U_{i+1}}_{X_i \cap U_{i+1}})$.

One easily verifies (i) and (ii) by induction using the exactness of the sequence (7), at its first term for (i) and at its second term for (ii). □

5. Fibered morphisms of C*-bundles

Let $\mathcal{C}$ be a class of Kirchberg algebras and set $X = [0,1]$. Let us use the term *admissible* cover of X to mean closed intervals $Y_i = [a_{2i}, a_{2i+1}]$ and $Z_j = [a_{2j+1}, a_{2j+2}]$ where $0 = a_0 < a_1 < \cdots < a_{2m+1} = 1$. Let us set $Y = [a_0, a_1] \cup [a_2, a_3] \cup ...$, $Z = [a_1, a_2] \cup [a_3, a_4] \cup ....$; thus $Y \cap Z = \{a_1, a_2, ..., a_{2m}\}$. For the sake of brevity let us refer to the above cover by $\{Y, Z\}$. Consider diagrams $\mathcal{D}$ of the form

$$D \xrightarrow{\pi} F \xleftarrow{\eta} E \tag{8}$$

based on an admissible cover $\{Y, Z\}$ of X. Let us assume that D, E, F are locally trivial C*-bundles over Y, Z and $Y \cap Z$ respectively with fibers in $\mathcal{C}$. More precisely, $D = \bigoplus_{i=0}^{m} C(Y_i, D_i)$ and $E = \bigoplus_{j=0}^{m-1} C(Z_j, E_j)$ where $D_i, E_j \in \mathcal{C}$. Let us also assume that F is the restriction of D to $Y \cap Z$, so that

$$F = D(Y \cap Z) = \bigoplus_{Y_i \cap Z_j \neq \emptyset} C(Y_i \cap Z_j, D_i) = \bigoplus_{i=0}^{m-1} (D_i \oplus D_{i+1}).$$

The map $\pi : D \to F$ is the restriction $\pi_{Y \cap Z} : D(Y) \to D(Y \cap Z)$ and the map $\eta : E \to F$ is obtained as the composition

$$E(Z) \xrightarrow{\pi_{Y \cap Z}} E(Y \cap Z) \xrightarrow{\gamma} D(Y \cap Z) = F,$$

where γ is a monomorphism of C*-bundles. It is useful to denote the components of γ by $\gamma_{i,i} = \gamma_{a_{2i+1}} : E_i \to D_i$ and $\gamma_{i,i+1} = \gamma_{a_{2i+2}} : E_i \to D_{i+1}$, and the corresponding components of η by $\eta_{i,i} : C(Z_i, E_i) \to D_i$ and $\eta_{i,i+1} : C(Z_i, E_i) \to D_{i+1}$.

The pullback of $\mathcal{D}$ is the continuous C*-bundle over X

$$P_{\mathcal{D}} = \{(d, e) \in D \oplus E \, : \, \pi(d) = \eta(e)\}.$$

Its fibers are isomorphic to E_j on Z_j and to D_i on $Y_i \setminus Z$.

Let us call a diagram $\mathcal{D}$ satisfying all the conditions described above *admissible*. Let us call a continuous C*-bundle A *elementary* if there is an admissible diagram $\mathcal{D}$ such that $A \cong P_{\mathcal{D}}$. For a fixed isomorphism $\iota : A \to P_{\mathcal{D}}$ there is a commutative diagram

$$\begin{tikzcd} A(Y) \arrow[r, "\pi"] \arrow[d, "\iota_Y"] & A(Y\cap Z) \arrow[d, "\iota_{Y\cap Z}"] & A(Z) \arrow[l, "\pi"'] \arrow[d, "\iota_Z"] \\ D \arrow[r, "\pi"] & F & E \arrow[l, "\eta"'] \end{tikzcd} \tag{9}$$

with the vertical maps monomorphisms of C*-bundles, such that the induced $*$-homomorphism $A \to P_{\mathcal{D}}$ can be identified with ι. An admissible diagram $\mathcal{D}$ comes with a closed cover of X, namely $\{Y, Z\}$. If A is a continuous C*-bundle over X, it is convenient to denote by $\mathcal{D}A$ the (not necessarily admissible) diagram

$$A(Y) \xrightarrow{\pi} A(Y\cap Z) \xleftarrow{\pi} A(Z)\,,$$

whose pullback is isomorphic to A by [11, Prop. 10.1.13]. Moreover if $\varphi : A \to B$ is a morphism of C*-bundles we denote by $\mathcal{D}\varphi$ the corresponding morphism of diagrams $\mathcal{D}A \to \mathcal{D}B$, with components $\varphi_Y, \varphi_{Y\cap Z}, \varphi_Z$. Thus, a continuous C*-bundle A is elementary if there is an admissible diagram $\mathcal{D}$ and a unital morphism of diagrams $\iota : \mathcal{D}A \to \mathcal{D}$ which induces a $*$-isomorphism $A \to P_{\mathcal{D}}$. In that case let us say that $\iota : \mathcal{D}A \to \mathcal{D}$ is a *fibered presentation* of A.

Let A, B be continuous C*-bundles over X such that A is elementary. A *fibered* morphism ϕ from A to B consists of a fibered presentation of A, $\iota : \mathcal{D}A \to \mathcal{D}$, together with injective morphisms of C*-bundles $\phi_Y, \phi_{Y\cap Z}, \phi_Z$ such that the following diagram is commutative:

$$\begin{tikzcd} A(Y) \arrow[r, "\pi"] \arrow[d, "\iota_Y"] & A(Y\cap Z) \arrow[d, "\iota_{Y\cap Z}"] & A(Z) \arrow[l, "\pi"'] \arrow[d, "\iota_Z"] \\ D \arrow[r, "\pi"] \arrow[d, "\phi_Y"] & F \arrow[d, "\phi_{Y\cap Z}"] & E \arrow[l, "\eta"'] \arrow[d, "\phi_Z"] \\ B(Y) \arrow[r, "\pi"] & B(Y\cap Z) & B(Z). \arrow[l, "\pi"'] \end{tikzcd}$$

Equivalently, a fibered morphism from A to B is given by a triple $(\iota, \mathcal{D}, \phi)$,

$$\mathcal{D}A \xrightarrow{\iota} \mathcal{D} \xrightarrow{\phi} \mathcal{D}B,$$

where $\iota : \mathcal{D}A \to \mathcal{D}$ is a fibered presentation of A and $\phi = (\phi_Y, \phi_{Y\cap Z}, \phi_Z)$. To simplify notation we will write ϕ for $(\iota, \mathcal{D}, \phi)$. Note that a fibered morphism ϕ induces a morphism of C*-bundles $\widehat{\phi} : A \to B$. A morphism φ of C*-bundles is called *elementary* if it is induced by a fibered morphism ϕ, i.e $\varphi = \widehat{\phi}$. The set of all fibered morphisms from A to B, corresponding to a given fibered presentation $\iota : \mathcal{D}A \to \mathcal{D}$ of A will be denoted by $\mathrm{Hom}_{\mathcal{D}}(A, B)$. There is a natural composition of fibered morphisms

$$\mathrm{Hom}_{\mathcal{D}}(A, B) \times \mathrm{Hom}_{\mathcal{D}'}(B, C) \to \mathrm{Hom}_{\mathcal{D}}(A, C),$$

$(\phi,\psi)\mapsto\psi\circ\phi=\mathcal{D}(\widehat{\psi})\circ\phi$. In other words the components of $\psi\circ\phi$ are $(\psi\circ\phi)_Y=\widehat{\psi}_Y\circ\phi_Y$, $(\psi\circ\phi)_{Y\cap Z}=\widehat{\psi}_{Y\cap Z}\circ\phi_{Y\cap Z}$ and $(\psi\circ\phi)_Z=\widehat{\psi}_Z\circ\phi_Z$.

Two fibered morphisms $\phi,\psi\in\mathrm{Hom}_{\mathcal{D}}(A,B)$ are *approximately unitarily equivalent* if there is a sequence of unitaries (u_n) in $M(B)$ such that the sequence $(\pi_Y(u_n))$ induces an approximate unitary equivalence between ϕ_Y and ψ_Y and the sequence $(\pi_Z(u_n))$ induces an approximate unitary equivalence between ϕ_Z and ψ_Z. It follows that $\phi_{Y\cap Z}$ and $\psi_{Y\cap Z}$ are also approximately unitarily equivalent since they are restrictions of ϕ_Y and ψ_Y, respectively. It is obvious that if ϕ is approximately unitarily equivalent to ψ, then $\widehat{\phi}$ is approximately unitarily equivalent to $\widehat{\psi}$.

Fix $d\in\{0,1\}$. By a *fibered* K_d-morphism from A to B, corresponding to a given fibered presentation $\iota:\mathcal{D}A\to\mathcal{D}$ of A, let us mean a triple of maps $\alpha=(\alpha_Y,\alpha_{Y\cap Z},\alpha_Z)$, where α_Y has components $\alpha_i:K_d(D(Y_i))\to K_d(B(Y_i))$, α_Z has components $\alpha_j:K_d(E(Z_j))\to K_d(B(Z_j))$, and $\alpha_{Y\cap Z}$ has components $\alpha_{i,j}:K_d(D(Y_i\cap Z_j))\to K_d(B(Y_i\cap Z_j))$, such that the following diagram is commutative.

$$\begin{tikzcd}
K_d(A(Y)) \arrow[r, "\pi_*"] \arrow[d, "\iota_*"] & K_d(A(Y\cap Z)) \arrow[d, "\iota_*"] & K_d(A(Z)) \arrow[l, "\pi_*"'] \arrow[d, "\iota_*"] \\
K_d(D) \arrow[r, "\pi_*"] \arrow[d, "\alpha_Y"] & K_d(F) \arrow[d, "\alpha_{Y\cap Z}"] & K_d(E) \arrow[l, "\eta_*"'] \arrow[d, "\alpha_Z"] \\
K_d(B(Y)) \arrow[r, "\pi_*"] & K_d(B(Y\cap Z)) & K_d(B(Z)) \arrow[l, "\pi_*"']
\end{tikzcd}$$

Let us summarize the above diagram by the notation

$$K_d(\mathcal{D}A)\xrightarrow{K_d(\iota)}K_d(\mathcal{D})\xrightarrow{\alpha}K_d(\mathcal{D}B). \tag{10}$$

Note that $\alpha_{Y\cap Z}(K_d(D(Y_i\cap Z_j)))\subseteq K_d(B(Y_i\cap Z_j))$ and

$$\alpha_Y(K_d(D(Y_i)))\subseteq K_d(B(Y_i)),\ \alpha_Z(K_d(E(Z_j)))\subseteq K_d(B(Z_j)),$$

by definition. The set of fibered K_d-morphisms from A to B corresponding to a given fibered presentation $\iota:\mathcal{D}A\to\mathcal{D}$ of A is denoted by $\mathrm{Hom}(K_d(\mathcal{D}),K_d(\mathcal{D}B))$. If B and H are C*-bundles over X and $\mathcal{D}$ is as above let us denote by $\mathrm{Hom}(K_d(\mathcal{D}B),K_d(\mathcal{D}H))$ the families of group morphisms $\beta_Y=(\beta_{Y_i})$, $\beta_{Y\cap Z}=(\beta_{Y_i\cap Z_j})$, $\beta_Z=(\beta_{Z_j})$ which make the following diagram commutative:

$$\begin{tikzcd}
K_d(B(Y)) \arrow[r, "\pi"] \arrow[d, "\beta_Y"] & K_d(B(Y\cap Z)) \arrow[d, "\beta_{Y\cap Z}"] & K_d(B(Z)) \arrow[l, "\pi"'] \arrow[d, "\beta_Z"] \\
K_d(H(Y)) \arrow[r, "\pi"] & K_d(H(Y\cap Z)) & K_d(H(Z)) \arrow[l, "\pi"']
\end{tikzcd}$$

Note that despite the conspicuous notation, the role of $\mathcal{D}$ here is limited to giving an admissible cover. One has an obvious composition $(\alpha,\beta)\mapsto\beta\circ\alpha$,

$$\mathrm{Hom}(K_d(\mathcal{D}),K_d(\mathcal{D}B))\times\mathrm{Hom}(K_d(\mathcal{D}B),K_d(\mathcal{D}H))\to\mathrm{Hom}(K_d(\mathcal{D}),K_d(\mathcal{D}H)).$$

There is a natural restriction map

$$\mathrm{Hom}(\mathbb{K}_d(B), \mathbb{K}_d(H)) \to \mathrm{Hom}(K_d(\mathcal{D}B), K_d(\mathcal{D}H)),$$

$\beta \mapsto \mathcal{D}\beta$ which depends only on the admissible cover $\{Y, Z\}$ of $[0,1]$ that appears in the definition of $\mathcal{D}$. This map takes the family of maps $\beta = (\beta_U)$ to the subfamily $\mathcal{D}\beta$ consisting of those β_U with U of the form Y_i, Z_j or $Y_i \cap Z_j$. If $U = \{x\}$, β_U stands for $\beta_x : K_d(B(x)) \to K_d(H(x))$.

If D is a C*-algebra and B is a C*-bundle over X, then any morphism of groups $\alpha : K_d(D) \to K_d(B)$ induces a morphism of presheaves $\widehat{\alpha} : \mathbb{K}_d(C(X, D)) \to \mathbb{K}_d(B)$, where for a closed subinterval $U \subset X$, $\widehat{\alpha}_U : K_d(C(U, D)) \cong K_d(D) \to K_d(B(U))$ is defined by $\widehat{\alpha}_U = (\pi_U)_* \alpha$. This observation extends to elementary C*-bundles:

Proposition 5.1. *Let A, B and H be continuous C*-bundles over $[0,1]$ whose fibers have vanishing K_{d+1}-groups. Assume moreover that A is an elementary C*-bundle given by a fibered presentation $\iota : \mathcal{D}A \to \mathcal{D}$. Then, there is a map $\mathrm{Hom}(K_d(\mathcal{D}), K_d(\mathcal{D}B)) \to \mathrm{Hom}(\mathbb{K}_d(A), \mathbb{K}_d(B))$, $\alpha \mapsto \widehat{\alpha}$, such that*

(a) $\widehat{K_d(\phi)} = \mathbb{K}_d(\widehat{\phi})$ for all $\phi \in \mathrm{Hom}_{\mathcal{D}}(A, B)$,

(b) $((\mathcal{D}\beta) \circ \alpha)\widehat{\ } = \beta \circ \widehat{\alpha}$ for all $\alpha \in \mathrm{Hom}(K_d(\mathcal{D}), K_d(\mathcal{D}B))$ and $\beta \in \mathrm{Hom}(\mathbb{K}_d(B), \mathbb{K}_d(H))$.

Proof. Let $\alpha \in \mathrm{Hom}(K_d(\mathcal{D}), K_d(\mathcal{D}B))$. For each closed subinterval U of $[0,1]$ we shall construct a morphism of groups $\widehat{\alpha}_U : K_d(A(U)) \to K_d(B(U))$ as follows.

Let us consider the diagram $\mathcal{D}_U$

$$D(Y \cap U) \xrightarrow{\pi} F(Y \cap Z \cap U) \xleftarrow{\eta} E(Z \cap U)$$

and note that it enjoys all the essential properties of an admissible diagram. The only possible differences are that $Y_i \cap U$ or $Z_j \cap U$ can reduce to boundary points of U (for at most two indices) and that the cover $\{Y \cap U, Z \cap U\}$ of U may begin or end by a subinterval of the form $Z_j \cap U$ rather than $Y_i \cap U$. We also consider the diagram $\mathcal{D}_U B$

$$B(Y \cap U) \xrightarrow{\pi} B(Y \cap Z \cap U) \xleftarrow{\pi} B(Z \cap U)\ .$$

Let $I(U)$ and $J(U)$ consists of those indices i and j such that $Y_i \cap U \neq \emptyset$ and respectively $Z_j \cap U \neq \emptyset$. The proof relies on three observations. First, we note that $A(U)$ is a pullback of $\mathcal{D}_U$. Second, we note that since $K_d(D(Y_i \cap U)) \cong K_d(D(Y_i)) \cong K_d(D_i)$ if $i \in I(U)$ and $K_d(E(Z_j \cap U)) \cong K_d(E(Z_j)) \cong K_d(E_j)$ if $j \in J(U)$, the diagram $K_d(\mathcal{D}_U)$ can be naturally identified with the following full subdiagram of $K_d(\mathcal{D})$:

$$\bigoplus_{i \in I(U)} K_d(D(Y_i)) \xrightarrow{\pi} \bigoplus_{i \in I(U), j \in J(U)} K_d(F(Y_i \cap Z_j)) \xleftarrow{\eta} \bigoplus_{j \in J(U)} K_d(E(Z_j))\ ,$$

denoted by $K_d(\mathcal{D})_U$. Third, we observe that $K_d(B(U))$ is isomorphic to the pullback of the diagram $K_d(\mathcal{D}_U B)$, as a consequence of the Mayer-Vietoris exact sequence and the assumption that all the fibers of B have vanishing K_{d+1}-groups. Let us denote by $K_d(\mathcal{D}B)_U$ the following full diagram of $K_d(\mathcal{D}B)$:

$$\bigoplus_{i \in I(U)} K_d(B(Y_i)) \xrightarrow{\pi} \bigoplus_{i \in I(U), j \in J(U)} K_d(B(Y_i \cap Z_j)) \xleftarrow{\pi} \bigoplus_{j \in J(U)} K_d(B(Z_j))\ .$$

The first observation gives a morphism of groups from $K_d(A(U))$ to $K_d(P_{\mathcal{D}_U})$ and hence to $P_{K_d(\mathcal{D}_U)}$, the pullback of the diagram $K_d(\mathcal{D}_U)$. The second observation, allows us to restrict α to a map of diagrams $K_d(\mathcal{D}_U) \cong K_d(\mathcal{D})_U \to K_d(\mathcal{D}B)_U \to K_d(\mathcal{D}_U B)$ (as explained in more detail below), giving a map $P_{K_d(\mathcal{D}_U)} \to P_{K_d(\mathcal{D}_U B)}$. By the third observation we can identify $P_{K_d(\mathcal{D}_U B)}$ with $K_d(B(U))$. Then we define $\widehat{\alpha}_U : K_d(A(U)) \to K_d(B(U))$ as the composition of the three maps from above. Finally it is easy to verify that the family $\widehat{\alpha} = (\widehat{\alpha}_U)_{U\in\mathcal{U}}$ is a morphism of sheaves and that the properties (a) and (b) are satisfied.

Let us verify the properties (a) and (b) using an explicit calculation of $\widehat{\alpha} : \mathbb{K}_d(A) \to \mathbb{K}_d(B)$. While this calculation is not really needed, we think that it is useful to compute the basic invariant $\mathbb{K}_d(A)$ at least for elementary C*-bundles. Since the fibers of A have vanishing K_{d+1}-groups, we may identify $K_d(A(U))$ with $P_{K_d(\mathcal{D}_U)}$ by the Mayer-Vietoris exact sequence. Thus

$$K_d(A(U)) \cong \{(d_i, e_j) \in \big(\bigoplus_{i\in I(U)} K_d(D_i)\big) \oplus \big(\bigoplus_{j\in J(U)} K_d(E_j)\big) \ : \ K_d(\gamma_{ji})(e_j) = d_i\}$$

where the condition $K_d(\gamma_{ji})(e_j) = d_i$ is required for $j \in \{i-1, i\}$. To simplify notation we let (d_i, e_j) stand for $((d_i)_{i\in I(U)}, (e_j)_{j\in J(U)})$. One can rewrite the calculation of $K_d(A(U))$ in the following equivalent form. If $J(U) = \emptyset$ then $I(U) = \{i\}$ for some i and $K_d(A(U)) \cong K_d(D_i)$. If $J(U) = \{s, ..., s+k\}$, then $K_d(A(U)) \cong K_d(E_s)$ if $k = 0$ and

$$K_d(A(U)) \cong \{(e_j) \in \big(\bigoplus_{j\in J(U)} K_d(E_j)\big) : K_d(\gamma_{j-1,j})(e_{j-1}) = K_d(\gamma_{j,j})(e_j),\ j = s{+}1, ..., s{+}k\}$$

if $k \geq 1$. Let $x \in K_d(A(U))$ be viewed as an element of the pullback of the diagram $K_d(\mathcal{D}_U)$, $x = (d_i, e_j)$. Then $\widehat{\alpha}_U(x) = \big(K_d(\pi^{Y_i}_{Y_i\cap U})\alpha_i(d_i), K_d(\pi^{Z_j}_{Z_j\cap U})\alpha_j(e_j)\big)$ viewed as an element of the pullback of the diagram $K_d(\mathcal{D}_U B)$, where i runs in $I(U)$ and j runs in $J(U)$. To verify (a) we need to show that $\widehat{K_d(\phi)}_U = K_d(\widehat{\phi}_U)$:

$$\begin{aligned}
&\widehat{K_d(\phi)}_U(x)\\
&= \big(K_d(\pi^{Y_i}_{Y_i\cap U})K_d(\phi_{Y_i})(d_i), K_d(\pi^{Z_j}_{Z_j\cap U})K_d(\phi_{Z_j})(e_j)\big)\\
&= \big(K_d(\phi_{Y_i\cap U})(d_i), K_d(\phi_{Z_j\cap U})(e_j)\big)\\
&= K_d(\widehat{\phi}_U)(x).
\end{aligned}$$

To verify (b) we need to show that $\widehat{((\mathcal{D}\beta)\circ\alpha)}_U = \beta_U \circ \widehat{\alpha}_U$. With notation as above we have

$$\begin{aligned}
&\widehat{((\mathcal{D}\beta)\circ\alpha)}_U(x)\\
&= \big(K_d(\pi^{Y_i}_{Y_i\cap U})\beta_{Y_i}\alpha_i(d_i), K_d(\pi^{Z_j}_{Z_j\cap U})\beta_{Z_j}\alpha_j(e_j)\big)\\
&= \big(\beta_{Y_i\cap U}K_d(\pi^{Y_i}_{Y_i\cap U})\alpha_i(d_i), \beta_{Z_j\cap U}K_d(\pi^{Z_j}_{Z_j\cap U})\alpha_j(e_j)\big)\\
&= \beta_U \circ \widehat{\alpha}_U(x).
\end{aligned}$$

□

6. Approximation and inductive limit results

The results in this section are useful for describing the structure of continuous C*-bundles of Kirchberg algebras. The following result applies to C*-bundles whose fibers are Kirchberg algebras satisfying the UCT. Indeed, such Kirchberg algebras are isomorphic to inductive limits of weakly semiprojective Kirchberg algebras (see [24]).

Theorem 6.1. *Let $\mathcal{C}$ be a class of unital weakly semiprojective Kirchberg algebras. Let A be a unital continuous C*-bundle over the unit interval such that all of its fibers are inductive limits of sequences of algebras in $\mathcal{C}$. For any finite subset $\mathcal{F} \subset A$ and any $\varepsilon > 0$ there are an elementary continuous C*-bundle A_1 with fibers in $\mathcal{C}$ and an elementary unital morphism of C*-bundles $\widehat{\Phi} : A_1 \to A$ such that $\mathcal{F} \subset_\varepsilon \widehat{\Phi}(A_1)$. A similar result is valid if one assumes that all the C*-algebras in $\mathcal{C}$ are stable rather than unital.*

Proof. We give the proof of the unital case. The stable case is entirely similar. Let $\mathcal{F}$ and ε be as in the statement. We must find points $0 = a_0 < a_1 < \cdots < a_{2m+1} = 1$ and C*-algebras $D_i, E_j \in \mathcal{C}$ ($0 \le i \le m$, $0 \le j \le m-1$) such that if we set $Y_i = [a_{2i}, a_{2i+1}]$, $Z_j = [a_{2j+1}, a_{2j+2}]$ and $D = \bigoplus_i C(Y_i, D_i)$, $E = \bigoplus_j C(Z_j, E_j)$, $Y = \bigcup_i Y_i$, $Z = \bigcup_j Z_j$, then there are C*-bundle monomorphisms $\varphi : D \to A(Y)$, $\psi : E \to A(Z)$ such that

$$\pi^Z_{Y\cap Z}\big(\psi(E)\big) \subset \pi^Y_{Y\cap Z}\big(\varphi(D)\big), \tag{11}$$

$$\pi_Y(\mathcal{F}) \subset_\varepsilon \varphi(D), \quad \pi_Z(\mathcal{F}) \subset_\varepsilon \psi(E). \tag{12}$$

Let A_1 be the pullback of the maps $\pi^Y_{Y\cap Z}\varphi$ and $\pi^Z_{Y\cap Z}\psi$,

$$A_1 = \{(d,e) \in D \oplus E \,:\, \pi^Y_{Y\cap Z}\varphi(d) = \pi^Z_{Y\cap Z}\psi(e)\}.$$

If $\mathcal{D}$ is defined by

$$D \xrightarrow{\pi} F \xleftarrow{\eta} E \tag{13}$$

where $F = D(Y\cap Z)$ and η is obtained as the composition

$$E(Z) \xrightarrow{\pi_{Y\cap Z}} E(Y\cap Z) \xrightarrow{\gamma} D(Y\cap Z) = F,$$

where $\gamma(e) = (\varphi^{-1}\psi)|_{Y\cap Z}(e)$, then $A_1 \cong P_{\mathcal{D}}$. Letting φ_F be the restriction of φ to F, we obtain a commutative diagram

$$\begin{tikzcd} D \arrow[r, "\pi"] \arrow[d, "\varphi"] & F \arrow[d, "\varphi_F"] & E \arrow[l, "\eta"'] \arrow[d, "\psi"] \\ A(Y) \arrow[r, "\pi"] & A(Y\cap Z) & A(Z) \arrow[l, "\pi"'] \end{tikzcd}$$

and hence a unital fibered morphism $\Phi \in \mathrm{Hom}_{\mathcal{D}}(A_1, A)$. Using a partition of unity one verifies immediately that the induced $*$-homomorphism $\widehat{\Phi} : A_1 \to A$ satisfies $\mathcal{F} \subset_\varepsilon \widehat{\Phi}(A_1)$ as a consequence of (12). Therefore it remains to construct D, E, φ and ψ with properties as above. By hypothesis, if $x \in [0,1]$, then $A(x)$ is an inductive limit of a sequence of C*-algebras in $\mathcal{C}$ with unital connecting maps. Therefore there are $E_x \in \mathcal{C}$ and a unital $*$-homomorphism $\imath_x : E_x \to A(x)$ such that $\pi_x(\mathcal{F}) \subset_{\varepsilon/2} \imath_x(E_x)$. Let $\mathcal{H}$ be a finite subset of E_x such that for each $a \in \mathcal{F}$ there is $h_a \in \mathcal{H}$ satisfying $\|\imath_x(h_a) - \pi_x(a)\| < \varepsilon/2$. Since E_x is weakly semiprojective, there are a closed neighborhood U_x of x and a $*$-homomorphism $\eta_x : E_x \to A(U_x)$ such that $\|\pi_x\eta_x(h) - \imath_x(h)\| < \varepsilon/2$ for all $h \in \mathcal{H}$. Therefore, $\|\pi_x(\eta_x(h_a) - \pi_{U_x}(a))\| = \|\pi_x\eta_x(h_a) - \pi_x(a)\| < \varepsilon$ for all $a \in \mathcal{F}$. Using the semicontinuity of the norm for C*-bundles, after passing to a smaller neighborhood if necessary, we may arrange that η_x is unital and $\|\eta_x(h_a) - \pi_{U_x}(a)\| < \varepsilon$ for all $a \in \mathcal{F}$. In particular, $\pi_{U_x}(\mathcal{F}) \subset_\varepsilon \eta_x(E_x)$. By compactness of $[0,1]$, there are points $0 = y_0 < y_1 < \cdots < y_m = 1$, C*-algebras $E_j \in \mathcal{C}$ $(0 \le j \le m-1)$, unital $*$-homomorphisms $\eta_j : E_j \to A(U_j)$, where $U_j = [y_j, y_{j+1}]$, and finite sets $\mathcal{F}_j \subset E_j$ such that

$$\pi_{U_j}(\mathcal{F}) \subset_{\varepsilon/2} \eta_j(\mathcal{F}_j) \tag{14}$$

for all $0 \le j \le m-1$. Let $\mathcal{G}_j \subset E_j$ and δ_j be given by Proposition 2.1 applied to E_j for the input data $\mathcal{F}_j$ and $\varepsilon/2$. Choose $\delta > 0$ such that $\delta < \min\{\delta_0, \dots, \delta_{m-1}\}$. By repeating the reasoning from above for the fibers $A(y_i)$ we obtain mutually disjoint closed intervals $Y_i = [a_{2i}, a_{2i+1}]$ $(0 \le i \le m)$, such that Y_i is a neighborhood of y_i and there are C*-algebras $D_i \in \mathcal{C}$ and unital $*$-homomorphisms $\varphi_i : D_i \to A(Y_i)$ such that $\pi_{Y_i}(\mathcal{F}) \subset_\varepsilon \varphi_i(D_i)$ and $\pi_{U_j \cap Y_i}(\eta_j(\mathcal{G}_j)) \subset_\delta \varphi_i(D_i)$ for all i, j for which $U_j \cap Y_i$ is nonempty, i.e., for which $j \in \{i-1, i\}$. Consider the C*-subbundle of A

$$B = \{a \in A \,:\, \pi_x(a) \in \pi_x(\varphi_i(D_i)), \text{ whenever}\, x \in Y_i,\, 0 \le i \le m\}.$$

By construction we have $\eta_j(\mathcal{G}_j) \subset_\delta B(U_j)$ for all j. Since each E_j is weakly semiprojective, by Proposition 2.1 we can perturb η_j to a unital $*$-homomorphism $\psi_j : E_j \to B(U_j)$ such that

$$\|\psi_j(a) - \eta_j(a)\| < \varepsilon/2 \tag{15}$$

for all $a \in \mathcal{F}_j$. Hence if we set $Z_j = [a_{2j+1}, a_{2j+2}] \subset U_j$, then the sets in the family (Z_j) are mutually disjoint and $\pi_{Z_j \cap Y_i}\psi_j(E_j) \subset \pi_{Z_j \cap Y_i}\varphi_i(D_i)$ (whenever $Z_j \cap Y_i \neq \emptyset$). Extend $\varphi_i : D_i \to A(Y_i)$ and $\psi_j : E_j \to A(Z_j)$ to maps of continuous C*-bundles and define φ and ψ as above. Then φ and ψ satisfy (11). Moreover, φ satisfies (12) since $\pi_{Y_i}(\mathcal{F}) \subset_\varepsilon \varphi_i(D_i)$, and ψ satisfies (12) as a consequence of (14) and (15). □

Let A be a separable C*-algebra. A sequence (A_k) of C*-subalgebras of A is called exhaustive if for any finite subset $\mathcal{F}$ of A and any $\varepsilon > 0$ there is k such that $\mathcal{F} \subset_\varepsilon A_k$. If we further assume that each A_k is weakly semiprojective, then A is isomorphic to the inductive limit of a subsequence $(A_{k(n)})$ of (A_k) where the connecting maps are perturbations of the inclusion maps $A_{k(n)} \hookrightarrow A$; see [20, 15.2.2]. Let us now show that the elementary continuous C*-bundles with semiprojective fibers satisfy a similar weak semiprojectivity property in the category of fibered morphisms. The following result applies to C*-bundles whose fibers are Kirchberg algebras satisfying the UCT and having torsion free K_1-groups. Indeed, such Kirchberg algebras are isomorphic to inductive limits of semiprojective Kirchberg algebras (see [24]).

Theorem 6.2. *Let $\mathcal{C}$ be a class of unital semiprojective Kirchberg algebras. Let A be a separable unital continuous C*-bundle over the unit interval such that all of its fibers are inductive limits of sequences of algebras in $\mathcal{C}$. There exists an inductive system $(A_k, \widehat{\Phi}_{k,k+1})$*

consisting of elementary continuous C-bundles with fibers in $\mathcal{C}$ and elementary morphisms of C*-bundles $\widehat{\Phi}_{k,k+1} \in \mathrm{Hom}(A_k, A_{k+1})$ such that $\varinjlim(A_k, \widehat{\Phi}_{k,k+1}) \cong A$. A similar result is valid if one assumes that all the C*-algebras in $\mathcal{C}$ are stable rather than unital.*

Proof. We give the proof of the unital case. The stable case is entirely similar. By Theorem 6.1 there is a sequence of admissible diagrams $(\mathcal{D}_k)$ and fibered unital morphisms $\Phi_{k,\infty} \in \mathrm{Hom}_{\mathcal{D}_k}(A_k, A)$, where $A_k = P_{\mathcal{D}_k}$, such that if we set $B_k = \widehat{\Phi}_{k,\infty}(A_k)$, then the sequence (B_k) is exhaustive for A. Arguing as in the proof of [20, 15.2.2], we see that it suffices to prove a natural weak semiprojectivity property for elementary bundles and fibered morphisms that we describe below. Using that property we then perturb $\Phi_{k,\infty}$ to a fibered morphism $\Phi_{k,n} \in \mathrm{Hom}_{\mathcal{D}_k}(A_k, B_n)$ for some large n that depends on k and then we set $\Phi_{k,k+1} = \mathcal{D}_k(\widehat{\Phi}_{n,\infty})^{-1} \circ \Phi_{k,n}$ to conclude the proof.

Let $\mathcal{D}$ be an admissible diagram (with components D_i, E_j semiprojective Kirchberg algebras and based on a cover $\{Y, Z\}$ of X as above) and let

$$\begin{array}{ccccc} D & \xrightarrow{\pi} & F & \xleftarrow{\eta} & E \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi_F} & & \downarrow{\scriptstyle \psi} \\ A(Y) & \xrightarrow{\pi} & A(Y \cap Z) & \xleftarrow{\pi} & A(Z) \end{array}$$

be a commutative diagram with the vertical maps unital morphisms of C*-bundles. Then, we assert that for any finite sets $\mathcal{F}_D \subset D$, $\mathcal{F}_E \subset E$ and any $\varepsilon > 0$ there are finite sets $\mathcal{G}_D \subset D$, $\mathcal{G}_E \subset E$ and $\delta > 0$ such that for any C*-subbundle $B \subset A$ with $\varphi(\mathcal{G}_D) \subset_\delta B(Y)$ and $\psi(\mathcal{G}_E) \subset_\delta B(Z)$, there is a commutative diagram

$$\begin{array}{ccccc} D & \xrightarrow{\pi} & F & \xleftarrow{\eta} & E \\ \downarrow{\scriptstyle \varphi'} & & \downarrow{\scriptstyle \varphi'_F} & & \downarrow{\scriptstyle \psi'} \\ B(Y) & \xrightarrow{\pi} & B(Y \cap Z) & \xleftarrow{\pi} & B(Z) \end{array} \tag{16}$$

with the vertical maps unital morphisms of C*-bundles, such that $\|\varphi(d) - \varphi'(d)\| < \varepsilon$ for all $d \in \mathcal{F}_D$ and $\|\psi(e) - \psi'(e)\| < \varepsilon$ for all $e \in \mathcal{F}_E$. Let us outline the proof of the above assertion. Using the semiprojectivity of D_i and E_j, for given $\mathcal{F}_D$, $\mathcal{F}_E$ and ε we can find $\mathcal{G}_D$, $\mathcal{G}_E$ and δ such that φ and ψ perturb to φ' and ψ', by Proposition 2.1. By defining φ'_F to be the restriction of φ' to F, we arrange that the left square of the diagram (16) is commutative, whereas we only have that the right square is approximately commutative. However, since the degree of approximate commutativity of the right square can be controlled by choosing $\mathcal{G}_D$, $\mathcal{G}_E$ and δ appropriately, we may invoke Proposition 2.2 to perturb ψ' further to a $*$-homomorphism which makes the right square of (16) commutative and which approximates ψ as desired. □

7. Unital C*-bundles of Cuntz algebras

Proposition 7.1. *Let A be an elementary continuous C*-bundle given by an admissible diagram as in (9) such that all its maps are unital and $D_i = E_j = B$ for all i and j where*

B is a unital Kirchberg algebra. Assume that all the components of γ are KK-equivalences. Then A is isomorphic to the trivial C^*-bundle $C(X) \otimes B$.

Proof. Since X is contractible it suffices to prove that A is locally trivial. Therefore we may assume that the spaces Y and Z are closed intervals with $Y \cap Z$ consisting of a single point and that $D = C(Y) \otimes B$ and $E = C(Z) \otimes B$. Then γ is a unital $*$-homomorphism $\gamma : B \to B$ which induces a KK-equivalence. For the sake of simplicity, let us assume that $Y = [0, 1]$ and $Z = [1, 2]$.

Since $[\gamma] \in KK(B, B)^{-1}$, by the Kirchberg-Phillips theorem [24, Thm. 8.4.1 and Cor. 8.4.10], there is an automorphism $\theta : B \to B$ and a continuous unitary-valued map $t \mapsto u(t) \in B$, with $t \in [0, 1)$, such that

$$\lim_{t\to 1} \|u(t)\theta(b)u(t)^* - \gamma(b)\| = 0,$$

for all $b \in B$. By hypothesis A is isomorphic to

$$\{(g, h) \in C[0, 1] \otimes B \oplus C[1, 2] \otimes B \,:\, g(1) = \gamma(h(1))\}.$$

One verifies immediately that the equation $\iota(f) = (g, h)$, where

$$g(t) = \begin{cases} u(t)\theta(f(t))u(t)^*, & \text{if } 0 \le t < 1, \\ \gamma(f(1)), & \text{if } t = 1, \end{cases}$$

and $h(t) = f(t)$ for $1 \le t \le 2$, defines an isomorphism $\iota : C[0, 2] \otimes B \to A$. □

Proposition 7.2. *Let $n \in \{2, 3, ..., \infty\}$ be fixed. If A is an elementary unital continuous C^*-bundle over $[0, 1]$ with all fibers isomorphic to $\mathcal{O}_n$, then A is $*$-isomorphic to $C[0, 1] \otimes \mathcal{O}_n$.*

Proof. Since $K_*(\mathcal{O}_n) = K_0(\mathcal{O}_n)$ is cyclic and generated by the class of the unit, any unital $*$-homomorphism $\gamma : \mathcal{O}_n \to \mathcal{O}_n$ induces an automorphism of $K_*(\mathcal{O}_n)$ and therefore $[\gamma] \in KK(\mathcal{O}_n, \mathcal{O}_n)^{-1}$, since $\mathcal{O}_n$ satisfies the UCT. Therefore A is trivial by Proposition 7.1. □

Theorem 7.3. *Let $n \in \{2, 3, ..., \infty\}$ be fixed. Any separable unital continuous C^*-bundle over an interval, or over a circle, with all fibers isomorphic to $\mathcal{O}_n$ is trivial.*

Proof. Let A be as in the statement. It suffices to prove that A is locally trivial. Indeed, if that is the case, then A is given by a principal $\mathrm{Aut}(\mathcal{O}_n)$-bundle. On the other hand $\mathrm{Aut}(\mathcal{O}_n)$ is path connected [22, Thm. 4.1.4]. Since all principal G-bundles over the circle are trivial whenever the structure group G is path-connected, our assertion is justified.

To prove that A is locally trivial it suffices to prove that A is trivial if its spectrum is an interval. By Theorem 6.2 (applied with $\mathcal{C} = \{\mathcal{O}_n\}$) and Proposition 7.1, A is isomorphic to the limit of an inductive system (A_i, ϕ_i) where $A_i = C[0, 1] \otimes \mathcal{O}_n$ and $\phi_i : A_i \to A_{i+1}$ are unital $*$-homomorphisms. We assert that ϕ_i is approximately unitarily equivalent to the identity map of $C[0, 1] \otimes \mathcal{O}_n$. Indeed, since ϕ_i is $C[0, 1]$-linear, it suffices to verify that the restriction of ϕ_i to $\mathcal{O}_n$ is approximately unitarily equivalent to the unital $*$-homomorphism which maps $\mathcal{O}_n$ onto the constant functions in $C[0, 1] \otimes \mathcal{O}_n$. This holds by [24, Thm. 8.2.1]. We conclude that $A \cong C[0, 1] \otimes \mathcal{O}_n$ by Elliott's intertwining argument [24, Cor. 2.3.3]. □

We need the following results of Blanchard, Kirchberg and Rørdam:

Theorem 7.4. *Let A be a separable continuous C*-bundle over a finite dimensional compact space with fibers Kirchberg algebras. If A is either stable or unital, then $A\otimes\mathcal{O}_\infty\cong A$. A is stable if and only if each fiber is stable.*

Proof. By [6, Cor. 5.11], $A\otimes\mathcal{O}_\infty\otimes\mathcal{K}\cong A\otimes\mathcal{K}$, and hence $A\otimes\mathcal{K}$ is strongly purely infinite by [16, Thm. 9.1]. It follows that A is strongly purely infinite by [15, Prop. 5.11]. Therefore $A\otimes\mathcal{O}_\infty\cong A$ by [16, Thm. 8.6]. The last part of the statement is a result from [25]. □

We give now a trivialization result for C*-bundles A in terms of the K-theory presheaf $\mathbb{K}_*(A)$, where for a closed interval V of $[0,1]$, $\mathbb{K}_*(A)(V)$ is the graded group $K_*(A(V))$. For unital C*-bundles we require that the morphisms of presheaves preserve the class of unit.

Theorem 7.5. *Let A be a separable unital continuous C*-bundle over $[0,1]$ the fibers of which are Kirchberg algebras satisfying the UCT. Assume that the K-theory presheaf of A is (unitally) isomorphic to the K-theory presheaf of $C[0,1]\otimes D$, for some unital Kirchberg algebra D with finitely generated K-theory groups and satisfying the UCT. Then $A\cong C[0,1]\otimes D$. A similar result is valid if one assumes that all the fibers of A are stable rather than unital.*

Proof. We give only the proof of the unital case as the stable case is entirely similar. By assumption, for every closed nondegenerate subinterval U of $X=[0,1]$ there is an isomorphism $\alpha_U : K_*(D)=K_*(C(U,D))\to K_*(A(U))$ such that the family (α_U) defines a morphism of presheaves. We assert that for any finite set $\mathcal{F}\subset A$ and $\varepsilon>0$ there is a unital morphism of C*-bundles $\phi : C(X,D)\to A$ such that $K_*(\phi_x) : K_*(D)\to K_*(A(x))$ is bijective for each $x\in X$ and $\mathcal{F}\subset_\varepsilon \phi\big(C(X,D)\big)$. Let us show how this assertion implies the theorem. Since $K_*(C(X,D))=K_*(D)$ is finitely generated, there exists a finite set $\mathcal{G}\subset D\subset C(X,D)$ and a number $\delta>0$ such that if ϕ and ψ are two unital $*$-homomorphisms defined on $C(X,D)$ and satisfying $\|\phi(a)-\psi(a)\|<\delta$ for all $a\in\mathcal{G}$, then $K_*(\phi)=K_*(\psi)$. Let (ε_k) be a sequence of numbers such that $0<\varepsilon_k<\delta$ and $\sum_k\varepsilon_k<\infty$. Using the assertion, and the weak semiprojectivity of D, we construct inductively two sequences of unital morphisms of C*-bundles, $\phi_k : C(X,D)\to A$, $\theta_k : C(X,D)\to C(X,D)$, $k=1,2,\dots$, and a sequence of finite sets $\mathcal{F}_k\subset C(X,D)$ that contain $\mathcal{G}$, such that

(i) $K_*((\phi_k)_x)$ is bijective for all $x\in X$ and $k\geq 1$;
(ii) $\|\phi_{k+1}\theta_k(a)-\phi_k(a)\|<\varepsilon_k$ for all $a\in\mathcal{F}_k$ and all $k\geq 1$;
(iii) $\theta_k(\mathcal{F}_k)\subset\mathcal{F}_{k+1}$ for all $k\geq 1$;
(iv) $\bigcup_{j=k}^{\infty}\big(\theta_j\circ\cdots\circ\theta_k\big)^{-1}(\mathcal{F}_{j+1})$ is dense in $C(X,D)$ and $\bigcup_{j=k}^{\infty}\phi_j(\mathcal{F}_j)$ is dense in A for all k.

Arguing as in the proof of [24, Prop. 2.3.2], one verifies that the sequence (ϕ_k) induces an isomorphism of unital C*-bundles $\varinjlim_k(C(X,D),\theta_k)\to A$. Let us show that each $K_*(\theta_k)$ is bijective. It suffices to check that $K_*((\theta_k)_x)$ is bijective for some x. Since $\mathcal{G}\subset\mathcal{F}_k$ and $\varepsilon_k<\delta$, we deduce from (ii) that $\|(\phi_{k+1}\theta_k)_x(a)-(\phi_k)_x(a)\|<\delta$ for all $a\in\mathcal{G}$ and hence $K_*((\phi_{k+1})_x)K_*((\theta_k)_x)=K_*((\phi_k)_x)$ for all $x\in X$. Consequently $K_*((\theta_k)_x)$

is bijective since $K_*((\phi_k)_x)$ is so for all k, by (i). Since D satisfies the UCT, θ_k is a KK-equivalence. By [24, Thm. 8.2.1], each θ_k is approximately unitarily equivalent to a $C(X)$-linear automorphism of $C(X, D)$ of the form $\mathrm{id}_{C(X)} \otimes \sigma_k$ where $\sigma_k \in \mathrm{Aut}(D)$. Hence A is isomorphic to $C(X, D)$ by Elliott's intertwining argument.

Let us prove now the assertion made at the beginning of the proof. The first part of the argument is essentially a repetition of the proof of Theorem 6.1 together with the observation that one can arrange that the components of the fibered morphism $\Phi \in \mathrm{Hom}_{\mathcal{D}}(A_1, A)$ have the property that $K_*(\phi_{Y_i}) = \alpha_{Y_i}$ and $K_*(\phi_{Z_j}) = \alpha_{Z_j}$.

We have that $A \otimes \mathcal{O}_\infty \cong A$ by Theorem 7.4. Since A absorbs $\mathcal{O}_\infty$ and D satisfies the UCT, we can apply Phillips's theorem [24, Thm. 8.2.1] to lift α_X to a unital $*$-homomorphism $\eta : D \to A$. If U is a closed subinterval of X, set $\eta_U = \pi_U \eta$. Since the horizontal maps in the commutative diagram

$$\begin{tikzcd} K_*(C(X) \otimes D) \arrow[r, "\alpha_X"] \arrow[d] & K_*(A(X)) \arrow[d] \\ K_*(C(U) \otimes D) \arrow[r, "\alpha_U"] & K_*(A(U)) \end{tikzcd}$$

are bijections by hypothesis, the restriction map $K_*(A(X)) \to K_*(A(U))$ is also a bijection. Let $x \in X$ and consider the sets $U_n = \{y \in X : |y - x| \le 1/n\}$. We deduce that $K_*(A(X)) \to K_*(A(x)) = \varinjlim K_*(A(U_n))$ is also a bijection, and hence $\eta_x : D \to A(x)$ induces a bijection $K_*(D) \to K_*(A(x))$ and $K_0(\eta_x)[1] = [1]$. By the Kirchberg-Phillips classification theorem, η_x is approximately unitarily equivalent to an isomorphism. Therefore there is a unitary $u_x \in A(x)$ such that $\pi_x(\mathcal{F}) \subset_{\varepsilon/2} u_x \eta_x(D) u_x^*$. Using the continuity of the norm for sections of continuous C*-bundles, we find a closed neighborhood U_x of x and a unitary $u_{U_x} \in A(U_x)$ such that $\pi_{U_x}(\mathcal{F}) \subset_{\varepsilon/2} u_{U_x} \eta_{U_x}(D) u_{U_x}^*$. We also have $K_*(\eta_{U_x}) = \alpha_{U_x}$ by construction. In this manner we obtain a family of maps $\eta_j : D \to A(U_j)$, $0 \le j \le m$, as in the proof of Theorem 6.1, with the additional property that $K_*(\eta_j) = \alpha_{U_j}$. Let $\varphi_i : D \to A(Y_i)$ be constructed as in the proof of Theorem 6.1, with the modifications described above, so that we can arrange to have $K_*(\varphi_i) = \alpha_{Y_i}$. Arguing as there, we perturb η_j to ψ_j such that $\|\psi_j(a) - \eta_j(a)\| < \varepsilon/2$ for all $a \in \mathcal{F}_j$. By choosing $\mathcal{F}_j$ appropriately this last condition also implies that $K_*(\psi_j) = K_*(\eta_j)$ and hence after restricting both ψ_j and η_j to $Z_j \subset U_j$, we have $K_*(\psi_j) = \alpha_{Z_j}$. Consider the maps $\phi_{Y_i} : C(Y_i) \otimes D \to A(Y_i)$ and $\phi_{Z_j} : C(Z_j) \otimes D \to A(Z_j)$ obtained by extending φ_i and ψ_j to morphisms of C*-bundles. They satisfy the conclusion of the Theorem 6.1 and moreover they have the property that $K_*(\phi_U) = \alpha_U$ for any closed interval $U \subset Y_i$ or $U \subset Z_j$. This implies immediately that the components of γ induce the identity map on $K_*(D)$. Let A_1 be the pullback of the first row of the commutative diagram

$$\begin{tikzcd} C(Y) \otimes D \arrow[r, "\pi"] \arrow[d, "\phi_X"] & C(Y \cap Z) \otimes D \arrow[d, "\phi_{Y \cap Z}"] & C(Z) \otimes D \arrow[l, "\eta"'] \arrow[d, "\phi_Z"] \\ A(Y) \arrow[r, "\pi"] & A(Y \cap Z) & A(Z) \arrow[l, "\pi"'] \end{tikzcd}$$

This diagram defines a fibered morphism which induces a unital $*$-homomorphism $\widehat{\phi}$: $A_1 \to A$ such that $K_*(\widehat{\phi}_x)$ is bijective for all $x \in X$ and $\mathcal{F} \subset_\varepsilon \widehat{\phi}(A_1)$. By Proposition 7.1, $A_1 \cong C(X, D)$. □

8. Classification results

For the remainder of the paper we fix $d \in \{0, 1\}$. When employing the terminology of Sections 5 and 6 we shall restrict ourselves to the class $\mathcal{C} = \mathcal{C}_d$ of stable Kirchberg algebras satisfying the UCT, with K_d-group finitely generated and torsion free and K_{d+1}-group equal to zero. If B is a stable Kirchberg C*-algebra satisfying the UCT and such that $K_d(B)$ is torsion free and $K_{d+1}(B) = 0$, then B can be written as an inductive limit of a sequence of C*-algebras in $\mathcal{C}_d$ [24]. By the UCT, $KK(SA, B) = 0$ for all $A \in \mathcal{C}_d$ and B as above. The following theorem gives an existence and uniqueness result for fibered morphisms.

Theorem 8.1. *Let A and B be continuous C*-bundles over $[0, 1]$ such that the fibers of A are in $\mathcal{C}_d$ and the fibers of B are nonzero inductive limits of sequences of C*-algebras in $\mathcal{C}_d$. Suppose that A is elementary with fibered presentation $\iota : \mathcal{D}A \to \mathcal{D}$. For any K_d-fibered morphism $\alpha \in \mathrm{Hom}(K_d(\mathcal{D}), K_d(\mathcal{D}B))$ there is a fibered monomorphism $\phi \in \mathrm{Hom}_{\mathcal{D}}(A, B)$ such that $K_d(\phi) = \alpha$. If $\psi \in \mathrm{Hom}_{\mathcal{D}}(A, B)$ is another fibered monomorphism satisfying $K_d(\psi) = \alpha$, then ϕ is approximately unitarily equivalent to ψ, and hence $\widehat{\phi}$ is approximately unitarily equivalent to $\widehat{\psi}$.*

Proof. We need to find $\phi \in \mathrm{Hom}(\mathcal{D}, \mathcal{D}B)$ such that $K_d(\phi_Y) = \alpha_Y$, $K_d(\phi_Z) = \alpha_Z$, and $K_d(\phi_{Y\cap Z}) = \alpha_{Y\cap Z}$. By assumption we have $\alpha_Y(K_d(D(Y_i))) \subseteq K_d(B(Y_i))$, $\alpha_Z(K_d(E(Z_j))) \subseteq K_d(B(Z_j))$ and $\alpha_{Y\cap Z}(K_d(D(Y_i\cap Z_j))) \subseteq K_d(B(Y_i\cap Z_j))$. Let α_i^D and α_j^E denote the corresponding components of α_Y and α_Z. Similarly let $\alpha_{i,j} : K_d(D_i) \to K_d(B(Y_i \cap Z_j))$ denote the components of $\alpha_{Y\cap Z}$ when $Y_i \cap Z_j \neq \emptyset$. We have $B \cong B \otimes \mathcal{O}_\infty \otimes \mathcal{K}$ by Theorem 7.4 and B contains a properly infinite full projection by [6, Prop. 5.6]. By [24, Thm. 8.2.1] there are $*$-monomorphisms $\phi_{i,j} : D_i \to B(Y_i \cap Z_j)$, such that $K_d(\phi_{i,j}) = \alpha_{i,j}$. By Proposition 2.4, for each $i = 0, \ldots, m$ there is a $*$-monomorphism $\phi_i^D : D_i \to B(Y_i)$, which we then extend to a monomorphism of C*-bundles $\phi_i^D : C(Y_i) \otimes D_i \to B(Y_i)$, with $K_d(\phi_i^D) = \alpha_i^D$ and such that ϕ_i^D lifts simultaneously the maps $\phi_{i,i-1} \circ \pi_{Y_i\cap Z_{i-1}}$ and $\phi_{i,i} \circ \pi_{Y_i\cap Z_i}$. Similarly, by applying Proposition 2.4 again, for each $j = 0, ..., m-1$ there is a monomorphism of C*-bundles $\phi_j^E : C(Z_j) \otimes E_j \to B(Z_j)$ which lifts simultaneously the maps $\phi_{j,j} \circ \eta_{j,j}$ and $\phi_{j+1,j} \circ \eta_{j,j+1}$, such that $K_d(\phi_j^E) = \alpha_j^E$. Then $\phi_Y = (\phi_i^D)$, $\phi_Z = (\phi_j^E)$ and $\phi_{Y\cap Z} = (\phi_{i,j})$ is the desired lifting of α.

We must now address the degree of uniqueness of a lifting. Let ϕ and ψ be as in the statement, with $K_d(\phi) = K_d(\psi) = \alpha$ and components ϕ_i^D, ψ_i^D, ϕ_j^E, ψ_j^E, $\phi_{i,j}, \psi_{i,j}$. By [24, Thm. 8.2.1], ϕ_i^D is approximately unitarily equivalent to ψ_i^D and similarly, ϕ_j^E is approximately unitarily equivalent to ψ_j^E. This yields unitaries $u_i \in M(B(Y_i))$ and $v_j \in M(B(Z_j))$ which approximately intertwine the corresponding pairs of $*$-monomorphisms. While the restrictions of these unitaries to $Y_i \cap Z_j$ are not necessarily equal, we may use Corollary 3.3 to replace each unitary v_j by a unitary which agrees with u_i on $Y_i \cap Z_j$ for

$i = j, j+1$. This procedure yields unitaries $u^{(n)} \in M(B)$ implementing an approximate unitary equivalence between ϕ and ψ. More explicitly, let us focus on a fixed component $Z_i = [a_{2i+1}, a_{2i+2}]$. Its neighbors are Y_i and Y_{i+1}. If we let $\approx_u$ denote approximate unitary equivalence for $*$-homomorphisms, from the discussion above we have: $\phi_i^D \approx_u \psi_i^D$ (implemented by a sequence $(u_i^{(n)})_n$), $\phi_{i+1}^D \approx_u \psi_{i+1}^D$ (implemented by $(u_{i+1}^{(n)})_n$), and $\phi_i^E \approx_u \psi_i^E$ (implemented by $(v_i^{(n)})$). After restricting to the endpoints of Z_i and composing with $\eta_{i,i}$ and $\eta_{i,i+1}$ (in the first two equivalences), we obtain $(\phi_i^E)_{a_{2i+1}} \approx_u (\psi_i^E)_{a_{2i+1}}$ implemented by $\big(\pi_{a_{2i+1}}(u_i^{(n)})\big)_n$, and $(\phi_i^E)_{a_{2i+2}} \approx_u (\psi_i^E)_{a_{2i+2}}$ implemented by $\big(\pi_{a_{2i+2}}(u_{i+1}^{(n)})\big)_n$. This enables us to apply Corollary 3.3 to $\phi_i^E, \psi_i^E : C(Z_i) \otimes E_i \to B(Z_i)$ and replace the sequence $(v_i^{(n)})$ by a sequence of unitaries $(w_i^{(n)})$ in $M(B(Z_i))$ which still implements $\phi_i^E \approx_u \psi_i^E$ and such that $\pi_{a_{2i+1}}(w_i^{(n)}) = \pi_{a_{2i+1}}(u_i^{(n)})$ and $\pi_{a_{2i+2}}(w_i^{(n)}) = \pi_{a_{2i+2}}(u_{i+1}^{(n)})$. The unitaries $u_i^{(n)}$ and $w_i^{(n)}$ then glue together to a unitary $u^{(n)} \in M(B)$, by Lemma 3.1, and the sequence $(u^{(n)})_n$ gives an approximate unitary equivalence between ϕ and ψ. □

Recall that if A is a C*-bundle over $X = [0,1]$, we are working with the K_d-theory sheaf

$$\mathbb{K}_d(A)(U) = K_d(A(U)),$$

defined on the category of closed subintervals U of $[0,1]$.

We are now ready to prove the main isomorphism result of the paper.

Theorem 8.2. *Fix $d \in \{0,1\}$. Let A, B be separable continuous C*-bundles over $[0,1]$ the fibers of which are stable Kirchberg algebras satisfying the UCT, with torsion free K_d-groups and vanishing K_{d+1}-groups. Then any morphism $\alpha : \mathbb{K}_d(A) \to \mathbb{K}_d(B)$ of K_d-sheaves lifts to an injective morphism of C*-bundles $\Phi : A \to B$, which is unique up to approximate unitary equivalence. If α is an isomorphism of sheaves, then we may arrange that Φ is an isomorphism of C*-bundles.*

Proof. By Theorem 6.2 there exist a sequence (A_k) of elementary C*-bundles with fibers in $\mathcal{C}_d$ such that $A_k = P_{\mathcal{D}_k}$ for admissible diagrams $\mathcal{D}_k$ and injective fibered morphisms $\phi_{k,k+1} \in \mathrm{Hom}_{\mathcal{D}_k}(A_k, A_{k+1})$, $\phi_{k,\infty} \in \mathrm{Hom}_{\mathcal{D}_k}(A_k, A)$ such that $\widehat{\phi}_{k,\infty} = \widehat{\phi}_{k+1,\infty} \circ \widehat{\phi}_{k,k+1}$. Moreover, these morphisms induce an isomorphism $A \cong \varinjlim(A_k, \widehat{\phi}_{k,k+1})$. In the first part of the proof we establish the following uniqueness result. If $\Phi, \Psi : A \to B$ are monomorphisms of C*-bundles such that $\mathbb{K}_d(\Phi) = \mathbb{K}_d(\Psi)$ then $\Phi \approx_u \Psi$. It suffices to show that $\Phi \circ \widehat{\phi}_{k,\infty} \approx_u \Psi \circ \widehat{\phi}_{k,\infty}$ for each $k \geq 1$. This would follow provided that we show that $\mathcal{D}_k\Phi \circ \phi_{k,\infty} \approx_u \mathcal{D}_k\Psi \circ \phi_{k,\infty}$. These maps are illustrated in the diagram

$$\mathcal{D}_k \xrightarrow{\phi_{k,\infty}} \mathcal{D}_k A \underset{\mathcal{D}_k\Psi}{\overset{\mathcal{D}_k\Phi}{\rightrightarrows}} \mathcal{D}_k B.$$

By Theorem 8.1 it suffices to show that $K_d(\mathcal{D}_k\Phi \circ \phi_{k,\infty}) = K_d(\mathcal{D}_k\Psi \circ \phi_{k,\infty})$. But this is verified using the functoriality of K_d and the assumption that $\mathbb{K}_d(\Phi) = \mathbb{K}_d(\Psi)$:

$$K_d(\mathcal{D}_k\Phi \circ \phi_{k,\infty}) = \mathcal{D}_k\mathbb{K}_d(\Phi) \circ K_d(\phi_{k,\infty}) = \mathcal{D}_k\mathbb{K}_d(\Psi) \circ K_d(\phi_{k,\infty}) = K_d(\mathcal{D}_k\Psi \circ \phi_{k,\infty}).$$

In the second part of the proof we shall lift a morphism of sheaves $\alpha : \mathbb{K}_d(A) \to \mathbb{K}_d(B)$ to a monomorphism of C*-bundles $A \to B$. To that purpose we are going to construct a sequence (φ_k) of fibered monomorphisms $\varphi_k \in \mathrm{Hom}_{\mathcal{D}_k}(A_k, B)$ such that $\widehat{\varphi}_{k+1} \circ \widehat{\phi}_{k,k+1} \approx_u \widehat{\varphi}_k$ and $\mathbb{K}_d(\widehat{\varphi}_k) = \alpha \circ \mathbb{K}_d(\widehat{\phi}_{k,\infty})$. The conclusion of the theorem will then follow by applying Elliott's intertwining argument [24, Sec. 2.3]. Let $\alpha_k \in \mathrm{Hom}(K_d(\mathcal{D}_k), K_d(\mathcal{D}_k B))$ be defined by the commutative diagram

$$\begin{tikzcd} & K_d(\mathcal{D}_k B) \\ K_d(\mathcal{D}_k) \arrow[ur, "\alpha_k"] \arrow[r, "K_d(\phi_{k,\infty})"'] & K_d(\mathcal{D}_k A) \arrow[u, "\mathcal{D}_k \alpha"'] \end{tikzcd}$$

By Theorem 8.1, for each $k \geq 1$ there is a fibered monomorphism $\varphi_k \in \mathrm{Hom}_{\mathcal{D}_k}(A_k, B)$ which lifts α_k, i.e $K_d(\varphi_k) = \alpha_k$. By Proposition 5.1

$$\alpha \circ \mathbb{K}_d(\widehat{\phi}_{k,\infty}) = \alpha \circ \widehat{K_d(\phi_{k,\infty})} = (\mathcal{D}_k \alpha \circ K_d(\phi_{k,\infty}))\widehat{\ } = \widehat{K_d(\varphi_k)} = \mathbb{K}_d(\widehat{\varphi}_k),$$

and hence the diagram

$$\begin{tikzcd} & \mathbb{K}_d(B) \\ \mathbb{K}_d(A_k) \arrow[ur, "\mathbb{K}_d(\widehat{\varphi}_k)"] \arrow[r, "\mathbb{K}_d(\widehat{\phi}_{k,\infty})"'] & \mathbb{K}_d(A) \arrow[u, "\alpha"'] \end{tikzcd}$$

is commutative for each $k \geq 1$. Therefore the left triangle of the diagram

$$\begin{tikzcd} & & \mathbb{K}_d(B) \\ \mathbb{K}_d(A_k) \arrow[urr, "\mathbb{K}_d(\widehat{\varphi}_k)"] \arrow[r, "\mathbb{K}_d(\widehat{\phi}_{k,k+1})"'] & \mathbb{K}_d(A_{k+1}) \arrow[ur, "\mathbb{K}_d(\widehat{\varphi}_{k+1})"'] \arrow[r, "\mathbb{K}_d(\widehat{\phi}_{k+1,\infty})"'] & \mathbb{K}_d(A) \arrow[u, "\alpha"'] \end{tikzcd}$$

is commutative. By the uniqueness result established in the first part of the proof we obtain that $\widehat{\varphi}_{k+1} \circ \widehat{\phi}_{k,k+1} \approx_u \widehat{\varphi}_k$. $\square$

Corollary 8.3. *Let A, B be separable unital continuous C^*-bundles over $[0,1]$ the fibers of which are Kirchberg algebras satisfying the UCT, with torsion free K_0-groups and vanishing K_1-groups. Suppose that there is an isomorphism of K_0-sheaves $\alpha : \mathbb{K}_0(A) \to \mathbb{K}_0(B)$ such that $\alpha[1_A] = [1_B]$. Then $A \cong B$.*

Proof. By the previous theorem there is an isomorphism $\varphi : A \otimes \mathcal{K} \to B \otimes \mathcal{K}$ such that $K_0(\varphi)[1_A \otimes e] = [1_B \otimes e]$ for a rank one projection $e \in \mathcal{K}$. Since both $\varphi(1_A \otimes e)$ and $1_B \otimes e$ are properly infinite and full projections in $B \otimes \mathcal{K}$, we may arrange that $\varphi(1_A \otimes e) = 1_B \otimes e$ after conjugating φ by a suitable unitary in $M(B \otimes \mathcal{K})$, by [24, Lemma 4.1.4]. It follows that φ induces an isomorphism $A \cong A \otimes e \to B \otimes e \cong B$. $\square$

Example 8.4. In order to illustrate the possible complexity of the continuous C*-bundles classified by Corollary 8.3, we construct now a continuous C*-bundle A satisfying all the assumptions of that theorem and such that for any closed nondegenerate subinterval I of $[0,1]$, $A(I)$ is not isomorphic to a trivial continuous C*-bundle, even though all the fibers of A are mutually isomorphic.

Let D be a unital Kirchberg algebra satisfying the UCT and such that $K_0(D)=\mathbb{Z}\oplus\mathbb{Z}$, $[1_A]=(1,0)$ and $K_1(D)=0$. Let $\gamma: D\to D$ be a unital $*$-homomorphism such that $K_0(\gamma)(0,1)=(0,0)$. Let (x_n) be a sequence of numbers dense in $[0,1]$. For each n consider the continuous C*-bundle over $[0,1]$

$$D_n=\{f\in C([0,1],D)\,:\,f(x_n)\in\gamma(D)\}.$$

Let us define $A_1=D_1$, $A_{n+1}=A_n\otimes_{C[0,1]}D_{n+1}$, and $A=\varinjlim(A_n,\theta_n)$, where $\theta: A_n\to A_{n+1}$ is defined by $\theta_n(a)=a\otimes 1$. In other words A is the infinite tensor product over $C[0,1]$ of $C[0,1]$-modules $A=\bigotimes_{n=1}^{\infty}D_n$. It is a continuous C*-bundle of Kirchberg algebras over $[0,1]$ (see [3]) which satisfies the assumptions of Theorem 8.2. It is not hard to see that, while all its fibers are isomorphic to $\bigotimes_{n=1}^{\infty}D$, A is nowhere locally trivial. Let us verify the latter assertion. This is done by showing that for any closed nondegenerate subinterval I of $[0,1]$ and any $x\in I$, the evaluation map $A(I)\to A(x)$ induces a non-injective map $K_0(A(I))\to K_0(A(x))$. Such a situation cannot occur for trivial continuous C*-bundles. Fix I as above and observe that $D_n(I)=C(I,D)$ if $x_n\notin I$ and that the map $\phi_n: D\to D_n(I)$, defined by $\phi_n(d)(x)=\gamma(d)$, $x\in I$, induces an isomorphism $K_0(D)\to K_0(D_n(I))$ if $x_n\in I$. Moreover, if $x_n\in I$ and if $x\in I\setminus\{x_n\}$, then the projection $\pi_x: D_n(I)\to D_n(x)\cong D$ induces a map $K_0(\pi_x)$ which can be identified with $K_0(\gamma)$ and hence it is not injective. Using the Künneth theorem, one verifies that the inclusion $\theta_n: A_n\to A_{n+1}\cong A_n\otimes_{C[0,1]}D_{n+1}$ induces an injective map $K_0(A_n(I))\to K_0(A_{n+1}(I))$ for any I as above. Therefore the map $\eta: K_0(A_n(I))\to K_0(A(I))$ is injective. For $x\in I$, let us consider the commutative diagram induced by evaluating at x:

$$\begin{array}{ccc} K_0(A_n(I)) & \xrightarrow{\eta} & K_0(A(I)) \\ \downarrow{\scriptstyle K_0(\pi_x^{(n)})} & & \downarrow{\scriptstyle K_0(\pi_x^{(\infty)})} \\ K_0(A_n(x)) & \longrightarrow & K_0(A(x)) \end{array} .$$

Fix $x\in I$. Since the sequence (x_n) is dense in $[0,1]$, there is n such that $x_n\in I$ and $x\neq x_n$. Using the Künneth theorem again, one verifies that the map $K_0(\pi_x^{(n)})$ is not injective if $x\in I\setminus\{x_n\}$, since it can be identified with

$$K_0(\pi_x^{(n-1)})\otimes K_0(\pi_x): K_0(A_{n-1}(I))\otimes K_0(D_n(I))\to K_0(A_{n-1}(x))\otimes K_0(D_n(x)),$$

and as seen earlier $K_0(\pi_x)$ is not injective. Since η is injective, it follows that $K_0(\pi_x^{(\infty)})$ is not injective and hence $A(I)$ cannot be isomorphic to $C(I,\bigotimes_{n=1}^{\infty}D)$.

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