SHAPE THEORY AND CONNECTIVE K-THEORY

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1.1. INTRODUCTION

E.G. Effros [16] posed the problem of finding suitable invariants for studying inductive limits of the form

$$C(X_1)\otimes A_1 \to C(X_2)\otimes A_2 \to \dots$$

where the X_i are CW-complexes and the A_i are finite dimensional C^*-algebras. In the present paper we study this problem from the viewpoint of homotopy theory and shape theory. Our algebraic models for shape invariants are based on ordered K-theory.

The material is organized as follows:

1. Introduction and preliminaries
2. Ordered K-theory and large denominators
3. Connective KK-theory for spaces
4. Homotopy computations for large homomorphisms
5. Shape theory
6. A connectivity result

We consider the category $\mathcal{C}(n)$ whose objects are C^*-algebras of the form

$$\bigoplus_k C(X_k)\otimes M_{m_k} \quad \text{(finite sums)}$$

where the X_k are finite connected CW-complexes of dimension $\leq n$. The set **Hom**(A,B) of morphisms in $\mathcal{C}(n)$ from A to B consists of all $*$-homomorphisms of C^*-algebras (including the nonunital ones).

Our main object of study will be the inductive limits of C^*-algebras from $\mathcal{C}(n)$. This requires a satisfactory classification of the morphisms of $\mathcal{C}(n)$. Now it is clear that the space **Hom**(A,B) with $A,B \in \mathcal{C}(n)$ is too big. Therefore, as in commutative topology, This will imply that

$$kk(Y,X) = [C_o(X), C_o(Y)\otimes M_m] = [C(X), C(Y)\otimes M_m]_1$$

provided that the dimension of Y is less than $\nu(m)$.

It is worth noting that this connectivity result extends the stability properties of vector bundles to cocycles of connective K-theory i.e. to $*$-homomorphisms. The proof is quite intricate and we have defered it to section 6.

The next problem with homomorphisms is how to reduce the study of $[\oplus_i C(X_i)\otimes M_{n_i}, \oplus_j C(Y_j)\otimes M_{m_j}]$ to the study of the simpler homotopy classes of the form $[C(X_i), C(Y_j)\otimes M_{k_j}]$. This completely nontrivial problem is discussed in section 4. Since we use techniques based on stability properties of vector bundles and homomorphisms, our classification results applies only to those $*$-homomorphisms which are large in the sense that they "amplify" many times - with respect to the dimension of each Y_i - each matrix subalgebra M_{n_i} of $\oplus_i C(X_i)\otimes M_{n_i}$. This is a natural restriction if we want to obtain purely algebraic but complete invariants for the homotopy classes of $*$-homomorphisms. The precise definition of large homomorphisms is given in section 2. The main topic of section 2 is to give an intrinsic characterization of those inductive limits $A = \lim A_i$ with $A_i \in \mathcal{C}(n)$, which can be written as limits of inductive systems with all the bonding homomorphisms large. This is accomplished using the notion of ordered group with large denominators introduced in [31].

Having a rather satisfactory homotopy classification of $*$-homomorphisms we pass to the question of how this local invariants can be patched together to yield an invariant for both the diagram $A_1 \to A_2 \to \dots$ and the inductive limit $A = \lim A_i$. This is the shape problem to which we devote section 5.

Let us state informally some special cases of our results concerning shape classifications. Let $\mathcal{C}_3^1(2n)$ be the category of the C^*-algebras of the form $\oplus_k C(X_k)\otimes M_{m_k}$ where the X_k are (2n - 2)-connected finite CW-complexes of dimension $\leq$ 2n with $\tilde{K}^o(X_k)$ torsion groups. Let $\mathcal{C}_2^1(2n)$ be the category of the C^*-algebras of the form $C(X)\otimes M_m$ where X ranges over the (2n - 2) connected finite CW-complexes of dimension $\leq$ 2n.

Related ideas are used to give a short proof (5.2.4) of a theorem of Effros and Kaminker concerning shape classification for inductive limits of Cuntz - Krieger algebras.

One may conclude from our computations the important role played by the connective K-theory in problems concerning homotopy theory. This is mainly due to the fact that it detects phenomena which are not seen by ordinary K-theory. What is however missing is a suitable continuous extension of connective K-theory to the category of C^*-algebras, which, among other things, would give a rather satisfactory shape invariant. We hope to discuss this problem in a future paper.

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1.2. PRELIMINARIES AND NOTATION

In this section we shall fix some notation and make some conventions and definitions to be used in the sequel.

1.2.1. Let $\mathcal{T}$ denote the category of all separable C^*-algebras and $*$-homomrphisms. If $A,B \in \mathcal{T}$ then we shall denote by **Hom**(A,B) (resp. $\mathbf{Hom}_1(A,B)$) the space of all (resp. all unital) $*$-homomorphisms $A \to B$ with the topology of pointwise--norm convergence. Accordingly we define $[A,B]$ (resp. $[A,B]_1$) to be the set of homotopy classes of homomorphisms in **Hom**(A,B) (resp., $\mathbf{Hom}_1(A,B)$).

For unital $A \in \mathcal{T}$ let 1_A denote its unit.

1.2.2. Given $A \in \mathcal{T}$ nonunital, let

$$0 \to A \to A^1 \xrightarrow{\mu} \mathbf{C} \to 0$$

be the unital extension of A. If X is a locally compact space (resp. compact), let $C_0(X)$ (resp. C(X)) denote the continuous complex functions vanishing at infinity on X (resp. all

We define $K_*(A)_+$ to be the image of $K_o(C(S^1)\otimes A)_+$ in $K_*(A)$ under this isomorphism. It is clear that $K_*(A)_+$ is a subset of $(K_o(A)_+\setminus\{0\})\oplus K_1(A)\cup\{(0,0)\}$. In general it is a proper subset (e.g. for $A = C(S^1)\oplus C(S^1)$). In a similar manner we define $\Sigma_*(A)\subset K_*(A)_+$ as corresponding to $\Sigma(C(S^1)\otimes A)$ under the above isomorphism. It is important to note that

$$K_*(\lim A_i)_+ = \lim K_*(A_i)_+ \text{ and } \Sigma_*(\lim A_i) = \lim \Sigma_*(A_i).$$

We also define $\mathbf{Hom}(K_*(A), K_*(B))_{+,\Sigma}$ to be the set of all homomorphisms of $\mathbf{Z}_2$-graded groups $K_*(A)\to K_*(B)$ which take $K_*(A)_+$ into $K_*(B)_+$ and $\Sigma_*(A)$ into $\Sigma_*(B)$.

1.2.8. For $A,B\in\mathcal{Y}$ we shall consider the Kasparov groups $KK_n(A,B)$ ([26]). As a special case of the Kasparov product we have the pairing

$$KK_*(\mathbf{C},A)\otimes KK_*(A,B)\to KK_*(\mathbf{C},B)$$

which gives a natural map

$$\gamma : KK(A,B)\to \mathbf{Hom}(K_*(A),K_*(B)) \qquad \text{(see [37]).}$$

It is useful to make the following notation

$$KK(A,B)_{+,\Sigma} = \{x\in KK(A,B) : \gamma(x)\in \mathbf{Hom}(K_*(A), K_*(B))_{+,\Sigma}\}$$

Notice that if $x\in KK(A,B)_{+,\Sigma}$ and $y\in KK(B,C)_{+,\Sigma}$ then $xy\in KK(A,C)_{+,\Sigma}$. As it follows from [37, Proposition 7.3] the map

$$KK(A,B)_{+,\Sigma}\to \mathbf{Hom}(K_*(A), K_*(B))_{+,\Sigma} \text{ is surjective}$$

for a large class of C^*-algebras.

1.2.9. For a compact space X let $\mathrm{Vect}_k(X)$ denote the set of isomorphism classes of complex vector bundles of rank k on X. In $\mathrm{Vect}_k(X)$ we have a distinguished element [k] - the class of the trivial bundle of rank k. Let $\mathrm{Vect}(X) = \bigcup_k \mathrm{Vect}_k(X)$. We shall freely identify Vect(X) with the monoid of equivalence classes of idempotents in $C(X)\otimes\mathcal{K}$ i.e. with V(C(X)), and $K^o(X)$ with $K_o(C(X))$.

1.2.10. Recall that a map $f:(X,x_o)\to(Y,y_o)$ is a ~~...~~ m-homotopy equivalence $(0\le m\le\infty)$ if $f_*:\pi_q(X,x_o)\to\pi_q(Y,y_o)$ is an isomorphism for $0\le q\le m-1$ and an of inductive systems with arbitrary large bonding morphisms and how can they be characterized in an intrinsic manner. The first part of this section is devoted to these and related questions. The answers we offer are given in terms of K-theory groups. They are based on the notion of ordered group with large denominators introduced by Nistor [31] in order to settle similar questions for AF-algebras.

The second part of this section deals with the states of the ordered group $K_0(A)$ for $A \in \mathcal{AC}(n)$.

2.1. THE DIMENSION MAP

The main technical tool of this subsection is the dimension map associated with each description of $A \in \mathcal{AC}(n)$ as the limit of an inductive system of C^*-algebras from $\mathcal{C}(n)$. We first introduce the dimension map for C^*-algebras in $\mathcal{C}(n)$.

For $A \in \mathcal{C}(n)$ of the form $A = \bigoplus_{i=1}^{q} C(X_i) \otimes M_{n_i}$, we define $r(A) = \bigoplus_{i=1}^{q} M_{n_i}$. Let $x_i \in X_i$, $1 \leq i \leq q$. The evaluation map $A \to r(A)$ given by $(f_i) \mapsto (f_i(x_i))$, $f_i \in C(X_i) \otimes M_{n_i}$, induces a split extension of groups

$$0 \to K_0'(A) \xrightarrow{\iota_A} K_0(A) \underset{\longleftarrow}{\xrightarrow{r_A}} K_0(r(A)) \to 0$$

where $K_0'(A) := \mathbf{ker}(r_A)$. Notice that r_A does not depend on the choice of x_i in X_i since each X_i is connected. If $r(A) = \mathbf{C}^q$, $n_i = 1$, $1 \leq i \leq q$, then the above extension reduces to

$$0 \to K'(X) \to K^0(X) \to H^0(X, \mathbf{Z}) \to 0$$

where $X = X_1 \sqcup \ldots \sqcup X_q$, see [25], and this justifies our notation for $\mathbf{ker}(r_A)$.

We need the following stability properties of complex vector bundles (see [23, Ch.8, Thms. 1.2, 1.5 and 2.6]). For $x \in \mathbf{R}$ let $\langle x \rangle$ denote the smallest integer t with $x \leq t$.

2.1.1. THEOREM. Let X be a CW-complex of dimension $\leq n$.

a) If $E \in \mathrm{Vect}_k(X)$, $k \geq \langle (n - 1)/2 \rangle$, then there is $F \in \mathrm{Vect}_{\langle (n-1)/2 \rangle}(X)$ such that E is isomorphic to $F \oplus [k - \langle (n - 1/2) \rangle]$.

$r_A([E_1],\dots,[E_q]) = (\mathrm{rank}(E_1),\dots,\mathrm{rank}(E_q))$ (recall that each X_i is connected).

Now let $\sigma = \begin{pmatrix} \alpha & \beta \\ \delta & \gamma \end{pmatrix}$ and $x \in K_0(A)$. By Corollary 2.1.2 a), $kx + n[1_A] \in K_0(A)_+$ for all $k \in \mathbf{Z}$. Since σ is an order preserving homomorphism we infer that $k\delta(x) + \gamma((n[1_A]) \in K_0(r(B))_+$ for all $k \in \mathbf{Z}$. This implies $\delta(x) = 0$.

2.1.4. If we further decompose

$$\alpha = (\alpha_{ji})_{\substack{1\le j\le h \\ 1\le h\le q}} : \bigoplus_{i=1}^{q} K_0'(A_i) \longrightarrow \bigoplus_{j=1}^{h} K_0'(B_j)$$

$$\beta = (\beta_{ji}) \quad : \bigoplus_{i=1}^{q} K_0(r(A_i)) \longrightarrow \bigoplus_{j=1}^{h} K_0'(B_j)$$

$$\gamma = (\gamma_{ji}) \quad : \bigoplus_{i=1}^{q} K_0(r(A_i)) \longrightarrow \bigoplus_{j=1}^{h} K_0(r(B_j))$$

where $A = \bigoplus_{i=1}^{q} A_i$, $B = \bigoplus_{j=1}^{h} B_j$, $A_i = C(X_i)\otimes M_{n_i}$, $B_j = C(Y_j)\otimes M_{m_j}$

then $\begin{pmatrix} (\alpha_{ji}) & (\beta_{ji}) \\ 0 & (\gamma_{ji}) \end{pmatrix}$ will be called the <u>standard picture</u> of σ.

2.1.5. The homomorphism γ associated in 2.1.3 to σ will be denoted by $r(\sigma)$. Proposition 2.1.3 shows that σ induces a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_0'(A) & \longrightarrow & K_0(A) & \longrightarrow & K_0(r(A)) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle r(\sigma)} & & \\
0 & \longrightarrow & K_0'(B) & \longrightarrow & K_0(B) & \longrightarrow & K_0(r(B)) & \longrightarrow & 0
\end{array}$$

Moreover it follows that the correspondence $\sigma \mapsto r(\sigma)$ (and even $\sigma \mapsto \alpha$) is functorial. For $A \in \mathcal{C}(n)$ the homomorphism $r_A : K_0(A) \to K_0(r(A))$ will be called the <u>dimension map</u> associated to A.

If $A \in \mathcal{A}\mathcal{C}(n)$ is the limit of the inductive system (A_i, φ_{ji}), $A_i \in \mathcal{C}(n)$, we shall denote by $r(A)$ the unique (up to isomorphism) AF-algebra determined by the scaled dimension group $\lim(K_0(r(A_i)), r(\varphi_{ji})_*)$. There is also a surjective homomorphism

morphism is a mp-full morphism.

2.1.9. REMARK. a) Let A be an AF-algebra and assume that $K_0(A)$ has large denominators. Then any faithful inductive system of finite dimensional C^*-algebras $A_1 \to A_2 \to \dots$ with $\lim A_i = A$ can be refined to an inductive system $A_{i_1} \to A_{i_2} \to \dots$ with arbitrary large embeddings. The converse it is also true, (see [3]).

b) If A is as in Proposition 2.1.7, then the inductive system (A_i) can be refined to an inductive system with m-full embeddings for arbitrary large $m \in \mathbf{N}$ (see [15] and [31]).

The C^*-algebras in $\mathcal{A}\mathcal{C}(n)$ have similar properties (2.1.14).

2.1.10. LEMMA. Let $A \in \mathcal{A}\mathcal{C}(n)$ and $a,b \in K_0(A)_+$ such that $r_A(a) \leq r_A(b)$ and $r_A(a)$ belongs to the order ideal generated by $r_A(b-a)$ in $K_0(r(A))$. If $K_0(r(A))$ has large denominators then $a \leq b$.

Proof. Let $A = \lim A_i$ with $A_i \in \mathcal{C}(n)$. Then $K_0(A) = \lim K_0(A_i)$ and $K_0(r(A)) = \lim K_0(r(A_i))$ in the category of ordered groups. The idea of the proof is to show that $b - a \in K_0(A)$ comes from some element of some group $K_0(A_j)$ satisfying the hypotheses of Corollary 2.1.2 a).

Let $x = r_A(a)$, $y = r_A(b)$, $z = r_A(b-a) \in K_0(r(A))_+$.

We have $0 \leq x \leq y$, $y = x + z$ and x belongs to the order ideal generated by z. It follows that y and z generate the same order ideal. On the other hand since $K_0(r(A))$ has large denominators there are $w \in K_0(r(A))_+$ and $m \in \mathbf{N}$ such that $nw \leq z \leq mw$. Combining the above data we obtain that y and w generate the same order ideal in $K_0(r(A))$. Therefore we can find some $j \in \mathbf{N}$ such that $y,z \in K_0(r(A_j))_+$ and they have the same support i.e. their coordinates in $\mathbf{Z}^q \simeq K_0(r(A_j))$ vanish simultaneously. Also we can assume j large enough such that $a,b \in K_0(A_j)_+$ and $y - x - nw \in K_0(r(A_j))_+$. Let $r_{A_j}(a) = (x_1,\dots,x_q)$, $r_{A_j}(b) = (y_1,\dots,y_q)$, $w = (w_1,\dots w_q) \in \mathbf{Z}^q$. We must have $y_i - x_i \geq nw_i$, $1 \leq i \leq q$, and moreover if some $w_i = 0$ then $y_i = x_i = 0$ since y and w have the some support. Therefore we may apply Corollary 2.1.2 a) to get $b - a \in K_0(A_j)_+$.

2.1.14. REMARK. Propositions 2.1.11-12 and Corollary 2.1.13 exhibit classes of C^*-algebras in $\mathcal{AC}(n)$ which display properties which are analogous with those in 2.1.9. For instance if $A \in \mathcal{AC}(n)$ and $K_0(A)$ has large denominators then any faithful inductive system of algebras in $\mathcal{C}(n)$, $A_1 \to A_2 \to \dots$, such that $A = \lim A_i$, can be refined to an inductive system $A_{i_1} \to A_{i_2} \to \dots$ with arbitrary large embeddings. The converse is also true. If A is as in Corollary 2.1.12 and $K_0(A)$ is simple then these embeddings can be chosen m-full, for arbitrary large $m \in \mathbf{N}$.

2.2. SOME PROPERTIES RELATED TO LARGE DENOMINATORS

The results of this subsection are not used later in the paper but we find them enough interesting to be included here.

2.2.1. PROPOSITION. Let $A \in \mathcal{AC}(n)$ and suppose that $K_0(A)$ has large denominators. Then

a) A has cancellation .

b) $\pi_i(U(A)) \simeq K_{i+1}(A)$, $i \geq 0$.

c) $\Sigma(A)$ is a generating, hereditary and directed subset of $K_0(A)_+$ (see [15] for definitions).

Proof. Write $A = \lim A_j$ with $A_j \in \mathcal{C}(n)$ and n-large embeddings $A_j \to A_{j+1}$. Having in mind Theorem 2.1.1 and the fact that $U(k) \to U(k+1)$ is a 2k-equivalence the proof goes along standard arguments. See [4] and [35] for related situations.

2.2.2. PROPOSITION. If $A \in \mathcal{AC}(n)$, then for every state f on $(K_0(A), K_0(A)_+)$ there is a unique state f' on $(K_0(r(A)), K_0(r(A))_+)$ such that $f = f' \circ r_A$.

Proof. It is enough to consider the case $A \in \mathcal{C}(n)$, when the proof is similar to the proof of Proposition 2.1.3. Indeed, as in 2.1.3, if f is regarded as a map from $K_0^1(A) \oplus K_0(r(A))$ to $\mathbf{R}$ then the positivity of f implies that f vanish on $K_0^1(A)$. Hence f factors through r_A. The uniqueness holds since r_A is onto.

2.2.4. COROLLARY. Let $A \in \mathcal{AT}(n)$ be a unital simple C^*-algebra and let p,q be projections in A. If $\tau(p) < \tau(q)$ for every trace state τ of A, then $upu^* < q$ for some unitary $u \in U(A)$.

Proof. It is useful to consider the following two cases.

a) r(A) is stably isomorphic to $\mathbb{K}$.

b) r(A) is not stably isomorphic to $\mathbb{K}$,

Since A is a unital simple C^*-algebra in $\mathcal{AT}(n)$ it can be proved that the first case can occurs only if A is isomophic to some matrix algebra M_k. (The proof is easy and we omit it). As our corollary is trivial for $A \cong M_k$ we turn our attention to the second case. Assuming b) we can apply Corollary 2.1.13 to find that $K_o(A)$ has large denominators.

On the other hand if $\tau(p) < \tau(q)$ for every trace state τ of A, then it follows (by Proposition 2.2.2) that $f' \circ r_A([p]) < f' \circ r_A([q])$ for every state f' of $K_o(r(A))$. Now $K_o(r(A))$ is a simple group which is also unperforated. Therefore it follows by [3, Thm. 6.8.5] that $K_o(r(A))$ has the strict ordering from its states (see also [15]). This implies $r_A([p]) < r_A([q])$ in $K_o(r(A))$. From Lemma 2.1.10 we derive $[p] < [q]$ in $K_o(A)$. (Note that $K_o(A)$ has no proper order ideals since A is simple). After conjugating with suitable unitaries we may assume that p,q belong to some A_j and moreover that $[p] < [q]$ in $K_o(A_j)$. Since $K_o(A)$ has large denominators by Theorem 2.1.1 it follows that p is homotopic in A_{j+k} (for k large enough) to some projection $p_1 < q$. But homotopic projections are unitarily equivalent.

3. CONNECTIVE KK-THEORY (FOR SPACES)

In this section we develop methods for computing the homotopy classes of $*$-homomorphisms $C_o(X) \to C_o(Y) \otimes \mathbb{K}$. We will introduce a bivariant functor kk(Y,X) which coresponds to such homotopy classes and which, as explained in the subsection 3.3, defines the natural connective theory associated with Kasparov KK-theory. As noticed in the Introductin or our results heavily depend on [39].

embeddings $F_o^k(X) \hookrightarrow F_o^{k+1}(X)$.

It is proved in [39] that the canonical map $\varinjlim_k F_o^k(X) \hookrightarrow F(X)$ is a homotopy equivalence, where the former space has the inductive limit topology. For $X = S^1$ this shows that the infinite unitary group $U(\infty)$ is homotopy equivalent to $U(\mathbb{K})$.

It is clear that each map $f : X \to Y$ induces a map $F(f) : F(X) \to F(Y)$. If f_1 is homotopic to f_2, then $F(f_1)$ is homotopic to $F(f_2)$. If X is connected, then $F(X)$ is connected.

3.1.3. PROPOSITION. If $X \in W_o^c$, $Y \in W_o$, then $[C_o(X), C_o(Y) \otimes \mathbb{K}]$ is an abelian group with respect to the direct sum of homomorphisms.

Proof. If $X \in W_o^c$, then $F(X)$ is a path connected H-space and therefore we can apply [43, Chapter 10, Theorem 2.4.] to get that $[Y,F(X)] \simeq [C_o(X), C_o(Y) \otimes \mathbb{K}]$ is a group. ∎

Another proof of this proposition will be available after we shall see that $F(X)$ is an infinite loop space. More precisely

$$F(X) \sim \Omega F(SX) \sim \ldots \sim \Omega^k F(S^k X) \sim \ldots \qquad \text{(see Corollary 3.1.7)}$$

3.1.4. DEFINITION. If $X \in W_o^c$ and $Y \in W_o$ (see 3.1.2) then we define $kk(Y,X)$ to be the group $[C_o(X), C_o(Y) \otimes \mathbb{K}]$. More generally for $n \in \mathbf{Z}$ we set

$$kk_n(Y,X) = \begin{cases} kk(S^n Y, X) & \text{if } n \geq 0 \\ \\ kk(Y, S^{-n} X) & \text{if } n < 0 \end{cases}$$

This definition is extended for $X,Y \in W_o$ in 3.1.9. a).

Our next purpose is to find exact sequences for these groups. This requires the notion of quasifibration introduced by Dold and Thom [14].

Recall that a continuous map $p : E \to B$ between topological Hausdorff spaces is called <u>quasifibration</u> if for all points $b \in B$ and $e \in p^{-1}(b)$, the induced maps

$$p_* : \pi_q(E, p^{-1}(b), e) \to \pi_q(B, b)$$

are isomorphisms for all $q \geq 0$.

commutative squares

$$\begin{tikzcd} {[Y,\Omega^{k+1}B]} \arrow[r, dashed, "\partial"] \arrow[d, equal] & {[Y,\Omega^k F]} \arrow[r] \arrow[d, "j_*"] & {[Y,\Omega^k E]} \arrow[r] \arrow[d, "i_*"] & {[Y,\Omega^k B]} \arrow[r, dashed] \arrow[d, equal] & {[Y,\Omega^{k-1}F]} \arrow[d] \\ {[Y,\Omega^{k+1}B]} \arrow[r, "\partial'"] & {[Y,\Omega^k W(p)]} \arrow[r] & {[Y,\Omega^k \mathrm{Cocyl}(p)]} \arrow[r] & {[Y,\Omega^k B]} \arrow[r] & {[Y,\Omega^{k-1}W(p)]} \end{tikzcd}$$

The bottom sequence is exact since p' is a fibration [43, Chapter I, Theorem 6.11*]. Therefore if the boundary maps ∂ are defined such that $\partial = j_*^{-1}\circ\partial'$ then the upper sequence is also exact. Finally let us prove that $j : F \to W(p)$ is a weak homotopy equivalence. Since the homotopy exact sequence of a quasifibration is natural in the sense of category theory we can use the first diagram in the proof to obtain the following commutative diagram

$$\begin{tikzcd} \pi_{k+1}(B) \arrow[r] \arrow[d, equal] & \pi_k(F) \arrow[r] \arrow[d, "j_*"] & \pi_k(E) \arrow[r] \arrow[d, "i_*"] & \pi_k(B) \arrow[r] \arrow[d, equal] & \pi_{k-1}(F) \arrow[d] \\ \pi_{k+1}(B) \arrow[r] & \pi_k(W(p)) \arrow[r] & \pi_k(\mathrm{Cocyl}(p)) \arrow[r] & \pi_k(B) \arrow[r] & \pi_{k-1}(W(p)) \end{tikzcd}$$

Using the five lemma it follows that $j_* : \pi_k(F) \to \pi_k(W(p))$ is an isomorphism for each $k \geq 0$.. ■

3.1.7. Suppose that $p : E \to B$ is a fibration with fiber F. The boundary map ∂ in the sequence below

$$\begin{tikzcd} {[Y,\Omega B]} \arrow[r, "\partial"] \arrow[d, equal] & {[Y,F]} \arrow[r] & {[Y,E]} \arrow[r] & {} \\ {[SY,B]} \arrow[ur] & & & \end{tikzcd}$$

is defined using the homotopy covering property of p. More precisely given a map $\varphi : SY \to B$ we solve the homotopy lifting problem with initial data

$$\begin{tikzcd} Y\vee I \arrow[r, "f"] \arrow[d, hook] & E \arrow[d, "p"] \\ Y\times I \arrow[ur, "\psi"] \arrow[r] & SY \arrow[r, "\varphi"] & B \end{tikzcd}$$

Let $Y \in W_o$. We shall prove that the induced map

$$S_* : [Y,F(X)] \longrightarrow [Y,\Omega F(SX)]$$

is a bijection.

To this end we consider the quasifibration $p: F(CX) \to F(SX)$ with fiber $F(X)$, arising from the pair $X \subset CX$ = the cone over X, by Theorem 3.1.5. By Proposition 3.1.6 there is an exact sequence (of groups and pointed sets)

$$[Y,\Omega F(CX)] \to [Y,\Omega F(SX)] \xrightarrow{\partial} [Y,F(X)] \to [Y,F(CX)]$$

Since $F(CX)$ is contractible (see the last part of 3.1.2) we find that the boundary map ∂ is a bijection. In what follows we check that S_* is a right inverse for ∂ and this will complete the proof of our Corollary. Let $\varphi \in \mathbf{Map}(Y,F(X))$ and let us think $S\varphi$ as a map $SY \to F(SX)$. In the same way

$$\psi = id(C_o(I)) \otimes \varphi \in \mathbf{Hom}(C_o(X \wedge I), C_o(Y \wedge I) \otimes \mathbb{K})$$

is regarded as a map $CY \to F(CX)$. Here $I = [0,1]$ is pointed by 0 and the smash products $X \wedge I$, $Y \wedge I$ are identified with CX and CY respectively. Let $X \times I \to CI$ and $X \times I \to SX$ be the natural quotient maps.

Let $Y \vee I \to F(CX)$ be the map onto the null morphism. It easily checked that the following diagram is commutative

$$\begin{tikzcd}
Y\vee I \arrow[rrr] \arrow[dd, hook] & & & F(CX) \arrow[dd, "p"] \\
& & CY \arrow[ur, "\psi"] & \\
Y\times I \arrow[urr] \arrow[r] & SY \arrow[rr, "S\varphi"] & & F(SX)
\end{tikzcd}$$

Using 3.1.7 it follows that $\partial[S\varphi] = [\psi_1]$ where ψ_1 is the restriction of ψ to $Y \times \{1\} \subset CY$ and image$(\psi_1) \subset F(X) = p^{-1}(0)$. Finally note that $\psi_1 : Y \to F(X)$ can be identified with φ. This proves that $\partial[S\varphi] = [\varphi]$ and hence S_* is a right inverse for ∂. ■

3.1.9. REMARKS

a) If $X \in W_o^c$, $Y \in W_o$, then the suspension map induces an isomorphism of groups

Therefore there is an exact sequence

$$[S^nB,F(X)] \longleftarrow [S^nY,F(X)] \longleftarrow [S^n(Y/B),\ F(X)] \longleftarrow [S^{n+1}B,F(X)] \longleftarrow [S^{n+1}Y,F(X)].$$

It follows from the definition of the kk-groups that this is just the required exact sequence. For $n \leq 0$ one uses again the suspension isomorphism. ∎

3.2. SOME GENERALIZED HOMOLOGY (COHOMOLOGY) THEORIES AND THEIR SPECTRA

In this subsection we look at $kk_n\ (Y,X)$ in the spirit of [42].

3.2.1. PROPOSITION.

a) For any fixed space $Y \in W_o$, the correspondence

$$X \longrightarrow k_n^Y(X) := kk_n(Y,X), \quad n \in \mathbf{Z},$$

defines a generalized (reduced) homology theory on te category W_o^c, (see 3.1.2).

b) For any fixed space $X \in W_o^c$, the correspondence

$$Y \longrightarrow k_X^n(Y) := kk_{-n}(Y,X), \quad n \in \mathbf{Z}$$

defines a generalized (reduced) cohomology theory on the category W_o, (see 3.1.2).

Proof. a) Recall that a generalized reduced homology theory on W_o is a sequence of covariant functors

$$h_n : W_o \longrightarrow \{\text{abelian groups}\}$$

together with a sequence of natural transformations

$$\sigma_n : h_n \longrightarrow h_{n+1} \circ S,\ S = \text{suspension}$$

verifying the following conditions:

1) If $f_o, f_1 \in W_o$ are homotopic maps, then $h_n(f_o) = h_n(f_1)$

2) If $X \in W_o$ then $\sigma_n(X) : h_n(X) \xrightarrow{\sim} h_{n+1}(SX)$

3) If $i : A \hookrightarrow X$ is a pair in W_o, and if $p : X \longrightarrow X/A$ is the identification map, then the sequence

$$h_n(A) \xrightarrow{h_n(i)} h_n(X) \xrightarrow{h_n(p)} h_n(X/A)$$

Ω-spectrum $E_n(Y)$.

b) On the category W_o the generalized cohomology theory $k_X^n(-)$ is given by the Ω-spectrum $F_n(X)$.

Proof. b) The assertion is immediate since by definition

$$k_X^n(Y) = kk_{-n}(Y,X) = [Y,F(S^nX)]$$

a) First we extend the homology theory $k_n^Y(-)$ on the category W_o by setting $k_n'^Y(X) = kk(S^{n+1}Y,SX) = k_{n+1}^Y(SX)$. By Remark 3.1.8 we have $k_n'^Y(X)\ \ k_r^Y(X)$ whenever X is connected. Let $h_n(-, E(Y))$ be the generalized homology defined on W_o by the spectrum $(E_n(Y))$. We want to prove that $h_n(-,E(Y))$ is isomorphic to $k_n'^Y(-)$. This will follow once we define a natural transformation of homology theories

$$T : h_*(-,E(Y)) \rightarrow k_*'^Y(-)$$

which induces an isomorphism on coefficients i.e.

$$T : h_*(S^o,E(Y) \simeq k_*'^Y(S^o).$$

For $X \in W_o$, let $T(X)$ be induced by the maps

$$\begin{array}{cccc} t_r : & \mathrm{Hom}(C_o(S^r), C_o(Y)\otimes\mathcal{K}) \wedge X & \longrightarrow & \mathrm{Hom}(C_o(S^rX), C_o(Y)\otimes\mathcal{K}) \\ & \wr\wr & & \wr\wr \\ & E_r(Y)\wedge X & & \mathrm{Map}(Y, F(S^rX)) \end{array}$$

where $t_r(\varphi \wedge x) = \varphi \otimes \varphi_x$ and $\varphi_x \in \mathbf{Hom}(C_o(X),\mathbf{C})$ is the homomorphism of evaluation at $x \in X$. More precisely we use the following commutative diagram

$$\begin{array}{ccc} [S^{n+r}, E_r(Y)\wedge X] & \xrightarrow{\quad s' \quad} & [S^{n+r+1}, E_{r+1}(Y)\wedge X] \\ \big\downarrow (t_r)_* & & \big\downarrow (t_{r+1})_* \\ [S^{n+r}, \mathrm{Map}(Y,F(S^rX))] & & [S^{n+r+1}, \mathrm{Map}(Y,F(S^{r+1}X))] \\ \big\downarrow \wr & & \big\downarrow \wr \\ kk(S^{n+r}Y, S^rX) & \xrightarrow{\quad \sim \quad} & kk(S^{n+r+1}Y, S^{r+1}X) \end{array}$$

with $\deg(t) = 2$; t corresponds to the generator of $\pi_3(U) \simeq \mathbf{Z}$ regarded as a homomorphism $S \in \mathbf{Hom}(C_0(S^1), C_0(S^3) \otimes K)$.

Proof. Note that each $\varphi \in \mathbf{Hom}(C_0(S^1), \mathcal{K})$ is given by some unitary $u \in U(K) = \{1 + x : x \in \mathcal{K},\ x^*x + x + x^* = 0\}$ and $U(\mathcal{K}) \sim U$. Therefore $F_1 \simeq U$. Let $r \geq 0$. By definition $\pi_r(F) = \lim_{n\to\infty}[S^{r+n}, F_n] = \lim_{n\to\infty}[S^{r+1}, \Omega^{n-1}F_n]$. Since F is an Ω-spectrum we find

$$\pi_r(F) = [S^{r+1}, U] = \begin{cases} \mathbf{Z} \text{ for } r \text{ even} \\ \\ 0 \text{ for } r \text{ odd} \end{cases}$$

For $r < 0$ $\pi_r(F) = \lim_{n\to\infty}[S^{r+n}, F_n] = [S^0, F_r] = 0$ since F_r is connected Thus $\pi_*(F) \simeq \mathbf{Z}[t]$ as groups. In order to determine the ring structure on $\pi_*(F)$ we observe that for

$$[\varphi] \in \pi_p(F) = [(C_0(S^1), C_0(S^{p+1}) \otimes \mathcal{K}] \quad \text{and}$$
$$[\psi] \in \pi_q(F) = [C_0(S^1), C_0(S^{q+1}) \otimes \mathcal{K}]$$

the element $\mu_*([\varphi],[\psi]) \in \pi_{p+q}(F) = [C_0(S^2), C_0(S^{p+q+2}) \otimes \mathcal{K}]$ is represented by

$$C_0(S^1) \otimes C_0(S^1) \xrightarrow{\varphi\otimes 1} C_0(S^{p+1}) \otimes \mathcal{K} \otimes C_0(S^1) \xrightarrow{1\otimes\psi} C_0(S^{p+1}) \otimes \mathcal{K} \otimes C_0(S^{q+1}) \otimes \mathcal{K}.$$

Therefore using the fact that the K_*-functor induces a bijection

$$[C_0(S^m), C_0(S^n) \otimes \mathcal{K}] \rightarrow \mathbf{Hom}(K_*(C_0(S^m)), K_*(C_0(S^n)))$$

for $n \geq m \geq 0$, the product

$$\mu_{2p,2q} : \pi_{2p}(F) \times \pi_{2q}(F) \rightarrow \pi_{2(p+q)}(F)$$

can be identified with the composition of group homomorphisms

$$\mathbf{Hom}(\mathbf{Z},\mathbf{Z}) \times \mathbf{Hom}(\mathbf{Z},\mathbf{Z}) \longrightarrow \mathbf{Hom}(\mathbf{Z},\mathbf{Z})$$

3.3.3. PROPOSITION (G. SEGAL). $F = (F_n)$ is the connective spectrum associated with the spectrum of complex K-theory.

Proof. Use the morphism S from Proposition 3.3.2 to define the maps $F_{n+2} \rightarrow \Omega^3 F_{n+3} = F_n$ by the rule $\varphi \mapsto \varphi \otimes S$. The compositions :

We shall see (3.3.6) that χ is induced by some map at the level of spectra.

3.3.5. PROPOSITION. On the category W_o the generalized homology theory $X \to KK_n(C_o(X),\ C_o(Y))$ is isomorphic to the generalized homology theory $X \to h_n(X,G(Y))$ defined by the spectrum $G_n(Y) = \mathbf{Map}(Y,\Omega^{n+1}U)$.

Proof. This result is contained in [26, §6, Thm.4] since $G_n(Y)$ is weak homotopy equivalent to the space $\mathbf{Map}(Y,\tilde{\mathcal{F}}_n)$ defined there, but for our purposes we prefer the description with $G_n(Y)$.

Thus we define $\tilde{T}(X): h_n(X,G(Y)) \to KK_n(C_o(X),C_o(Y))$, as beeing induced by the maps:

$[S^{n+r}, X \wedge G_n(Y)] \xrightarrow{(t_r)_*} [C_o(S^1X), C_o(S^{2n+r+1}Y) \otimes \mathcal{K}] \to KK(C_o(X),C_o(S^r(Y))$ via the identifications $U \sim U(\mathcal{K})$ and

$$G_n(Y) = \mathbf{Map}(Y,\Omega^{n+1}U) \simeq \mathbf{Hom}(C_o(S^1),\ C_o(S^{n+1}Y) \otimes \mathcal{K})$$

The map t_r figuring above is the same as the map t_r defined in the proof of Proposition 3.2.2. Moreover the proof is carried out in the same manner. ∎

3.3.6. REMARKS

a) There is a commutative diagram

$$\begin{tikzcd}
kk_n(Y,X) \arrow[d, "\chi"] & h_n(X,E(Y)) \arrow[l, "T"', "\sim"] \arrow[d, "\xi_*"] \\
KK(C_o(X),C_o(Y)) & h_n(X,G(Y)) \arrow[l, "\tilde{T}"', "\sim"]
\end{tikzcd}$$

where $h_n(X,E(Y))$ and T were defined in the proof of Proposition 3.2.2 and ξ is the map of spectra

$$\xi : \mathbf{Map}(Y,F_n) \to \mathbf{Map}(Y,\Omega^{n+1}U)$$

induced by the map $F_n \to BU_n \sim \Omega^{n+1}U$ given in the proof of Proposition 3.3.3.

b) Let X' be an n-dual of X, for example a compact deformation retract of the complement of X embedded in S^{n+1}. By the naturality of the duality result stated in

3.4.1. PROPOSITION. For any finite CW-complex X, there is a natural exact triangle

$$\begin{tikzcd} k_*(X) \arrow[rr, "S_*"] & & k_*(X) \arrow[dl, "\eta_*"] \\ & \tilde{H}_*(X,\mathbb{Z}) \arrow[ul, "\partial_*"] & \end{tikzcd}$$

with **deg** $S_* = 2$, **deg** $\eta_* = 0$ and **deg** $\partial_* = -3$.

We need also the dual triangle.

3.4.2. PROPOSITION. For any finite CW-complex X, there is a natural exact triangle

$$\begin{tikzcd} k^*(X) \arrow[rr, "S^*"] & & k^*(X) \arrow[dl, "\eta^*"] \\ & \tilde{H}^*(X,\mathbb{Z}) \arrow[ul, "\delta_*"] & \end{tikzcd}$$

with **deg** $S^* = -2$, **deg** $\eta^* = 0$ and **deg** $\delta_* = 3$.

Proof. Let W_n be the mapping fibre of $F_n \to F_{n-2}$ i.e. $W_n = \{(u,e) \in F^I_{n-2} \times F_n : u(0) = 0,\ u(1) = \text{the image of } e \text{ under the map } F_n \to F_{n-2}\}$. Then the Puppe exact sequence ([43, Chapter I, Theorem 6.11*])

$$\Omega^{q+1}F_{n-2} \to \Omega^q W_n \to \Omega^q F_n \to \Omega^q F_{n-2}$$

together with Proposition 3.3.2. imply

$$\pi_q(W_n) = \begin{cases} \mathbf{Z} & \text{if } q = n-3 \\ 0 & \text{otherwise} \end{cases}$$

Consequently W_n is a $K(\mathbf{Z}, n-3)$-space and we get an exact sequence.

$$F_{n-3} \longrightarrow K(\mathbf{Z}, n-3) \longrightarrow F_n \longrightarrow F_{n-2}.$$

Passing to homotopy classes $[X,-]$ we get the exact sequence

$$0 \to k^3(X) \xrightarrow{S^*} k^1(X) \overset{\eta^*}{\rightleftarrows} \check{H}^1(X,\mathbb{Z}) \to k^4(X) \to \cdots \to k^n(X) \xrightarrow{S^*} k^{n-2}(X) \xrightarrow{\eta^*} \tilde{H}^{n-2}(X,\mathbb{Z}) \to 0$$

With the identifications $k^q(X) \simeq \widetilde{K}^q(X)$ for $q \leq 2$ the isomorphism $k^q(X) \overset{S^*}{\underset{\sim}{\to}} k^{q-2}(X)$ is the Bott periodicity.

Proof. If $q \geq n+1$, then $k^q(X) = [X,F_q] = 0$, since F_q is $(q-1)$-connected. If $0 \leq q \leq 1$, then $k^q(X) = [X,F_q] = \widetilde{K}^q(X)$, since $F_o = \Omega U$, $F_1 = U$. As $\widetilde{H}^q(X,\mathbb{Z}) = 0$ for $q \leq 0$, it follows from Proposition 3.4.2 that $k^2(X) \xrightarrow{\sim} k^o(X) = \widetilde{K}^o(X)$. If $q < 0$, then

$$k^q(X) = \varinjlim_r [S^rX, F_{q+r}] = [S^{-q}X, F_o] = \widetilde{K}^o(S^{-q}X) \simeq \widetilde{K}^q(X).$$

Finally let us explain why we have a split extension

$$0 \to k^3(X) \xrightarrow{S^*} k^1(X) \overset{\eta^*}{\rightleftarrows} H^1(X,\mathbb{Z}) \to 0$$

If we take $K(\mathbb{Z},1) \simeq S^1$ and $F_1 \simeq U$ then the map $\eta : U \to S^1$ is the determinant map and the inclusion $S^1 = U(1) \hookrightarrow U$ gives a multiplicative left inverse for η. ∎

3.4.5. THEOREM. Let X,Y be a finite CW-complexes and suppose that X is 0-connected and Y is m-connected ($m \geq -1$, if Y is not connected then $m = -1$). Then the canonical map

$$\chi : kk_n(Y,X) \to KK_n(C_o(X),C_o(Y))$$

is an isomorphism for $n \geq \mathbf{dim}(X) - m - 2$.

Proof. We shall compare two homology theories

$$k_n^Y(X) = kk_n(Y,X) \text{ and } K_n^Y(X) = KK_n(C_o(X),C_o(Y))$$

If $X = S^q$ with $1 \leq q \leq n+m+2$ then $k_n^Y(S^q) = kk_n(Y,S^q) = k^{q-n}(Y)$ and $K_n^Y(S^q) = \widetilde{K}^{q-n}(Y)$. Using $H^j(Y,\mathbb{Z}) = 0$ for $j \leq m$ we infer from the exact triangle in Proposition 3.4.2 that $k^{j+2}(Y) \overset{S^*}{\underset{\sim}{\to}} k^j(Y)$ for $j \leq m$. Since, for negative j, $k^j(Y) \simeq \widetilde{K}^j(Y)$, it follows that $k^{q-n}(Y) \xrightarrow{\sim} \widetilde{K}^{q-n}(Y)$ for all $q \leq n+m+2$ and moreover this isomorphism can be identified with χ. It is clear that the same assertion is true if we substitute S^q with a finite wedge of q-spheres. Having this, the proof is carried out by induction on the dimension of X. If $\mathbf{dim}(X) = 1$, then X is homotopic to a wedge of circles, hence the assertion is true. Therefore assume that χ is an isomorphism for each connected X of

This shows that if $a_q = \dim_{\mathbb{Q}} k^q(X)\otimes \mathbb{Q}$, $b_q = \dim_{\mathbb{Q}} H^q(X,\mathbb{Q})$, $d_q = \dim_{\mathbb{Q}} \ker(S^q \otimes 1)$, $c_q = \dim_{\mathbb{Q}} \operatorname{coker}(S^q \otimes 1)$, then

$$d_q - a_q + a_{q-2} - c_q = 0 \quad \text{and} \quad c_{q-1} - b_{q-3} + d_q = 0$$

According to 3.4.4 $a_q = b_q = 0$ for $q \geq n+1$, $n = \dim(X)$, and $d_q = 0$ for $q \leq 3$. Also recall that $k^2(X) \simeq \widetilde{K}^2(X)$.

Using the isomorphism $\widetilde{K}^o(X)\otimes \mathbb{Q} \simeq \widetilde{H}^{\text{even}}(X,\mathbb{Q})$ given by the Chern character we get

$$a_2 = \sum_{q\geq 1} b_{2q} = \sum_{q\geq 1} (c_{2q+2} + d_{2q+3}).$$

On the other hand from $a_q - a_{q-2} = d_q - c_q$ we infer

$$-a_2 = \sum_{q\geq 2} (a_{2q} - a_{2q-2}) = \sum_{q\geq 2} (d_{2q} - c_{2q})$$

Therefore we must have

$$\sum_{q\geq 1} (c_{2q+2} + d_{2q+3}) = \sum_{q\geq 2} (c_{2q} - d_{2q})$$

This is possible only if $d_q = 0$ for all $q \geq 4$. Since we have already seen that $d_q = 0$ for $q \leq 3$ it results that $\mathbf{Ker}(S^q \otimes 1) = 0$ for all $q \in \mathbb{Z}$. ∎

The above Proposition shows that the image of the map $\widetilde{H}^*(X,\mathbb{Z}) \longrightarrow k^{*+3}(X)$ is always a torsion group. In the case when $H^*(X,\mathbb{Z})$ is torsion free the $k^*(S^o) = \mathbb{Z}[t]$ - module structure of $k^*(X)$ is completely described by the following

3.4.8. COROLLARY. Let X be a finite connected CW-complex such that $H^*(X,\mathbb{Z})$ is torsion free. Then for each $q \in \mathbb{Z}$ there is an isomorphism $k^q(X) \to \bigoplus_{j\geq 0} H^{q+2j}(X,\mathbb{Z})$ such that the following diagram commutes

$$\begin{tikzcd}
k^{q+2}(X) \arrow[r, "S"] \arrow[d] & k^q(X) \arrow[d] \\
\bigoplus_{j\geq 1} \widetilde{H}^{q+2j}(X,\mathbb{Z}) \arrow[r, hook] & \bigoplus_{j\geq 0} \widetilde{H}^{q+2j}(X,\mathbb{Z})
\end{tikzcd}$$

Let A_i, $1 \leq i \leq 3$ be C^*-algebras and $\varphi_i \in \mathbf{Hom}(A_i, A_{i+1} \otimes \mathcal{K})$, $1 \leq i \leq 3$. Let 1 denote the identity morphism of K. One has to check that

$$\varphi_3 \otimes 1 \otimes 1 \circ (\varphi_2 \otimes 1 \circ \varphi_1) = (\varphi_3 \otimes 1 \circ \varphi_2) \otimes 1 \circ \varphi_1,$$

but this follows from the equality

$$(\varphi_3 \otimes 1 \otimes 1) \circ (\varphi_2 \otimes 1) = (\varphi_3 \otimes 1 \circ \varphi_2) \otimes 1$$

which is straightforward.

3.5.2. Using the natural isomorphism $kk(SY,SX) \simeq kk(Y,X)$ we can extend the above product to a well defined product

$$kk_n(Y,X) \times kk_m(Z,Y) \to kk_{n+m}(Z,X)$$

In particular we get the following facts:

a) $k^*(X) = \bigoplus_{q \in \mathbf{Z}} k^q(X)$ is endowed with a structure of left $k^*(S^o)$ - module which coincides with that given by the operation $S^* : k^{q+2}(X) \to k^q(X)$.

b) The product $k^*(X) \times kk_*(Y,X) \to k^*(Y)$ induces morphisms $\alpha_r : kk_{-r}(Y,X) \to \mathbf{Hom}^r(k^*(X), k^*(Y))$ which actually map into $\mathbf{Hom}^r_{k^*(S^o)}(k^*(X), k^*(Y))$ since the product is associative. Here we use the notation $\mathbf{Hom}^r_{k^*(S^o)}(k^*(X), k^*(Y)$ for the group of $k^*(S^o)$-morphisms $\varphi : k^*(X) \to k^*(Y)$ of degree r (i.e. $\varphi(k^q(X)) \subset k^{q+r}(Y)$ for all $q \in \mathbf{Z}$).

We want to prove that α_r is an isomorphism provided that $k^*(X)$ is a free $k^*(S^o)$-module. The following discussion on products will enable us to use Adam's universal coefficient theorem [1].

3.5.3. Recall that (F_n) is a ring-spectrum with multiplication $\mu : F_n \wedge F_m \to F_{n+m}$ given by $\mu(\varphi,\psi) = \varphi \otimes \psi$. This allows one to construct four basic external products: an external product in k_* an external product in k^*, and two slant products. Among them we are especially interested in the slant product

$$/ : k^p(Y \wedge X) \times k_q(Y) \to k^{p-q}(X)$$

$$T(a/b) = T(a) /\!\!/ T(b)$$

This equality follows from the identity.

$$((t_r \circ g) \otimes id(C_o(X)) \otimes id(\mathcal{K})) \circ (id(C_o(S^r)) \circ f)(s.x) = \varphi \otimes f(y \wedge x)$$

where $s \in S^{q+r}$, $x \in X$, $g(s) = \varphi \wedge y$.

It is worth noting that all the other products defined in terms of spectra admit similar realizations using the product 3.5.2.

With this preparation we are able to prove the following special Universal Coefficient Theorem.

3.5.4. THEOREM. Let $X \in W_o^c$, $Y \in W_o$ and assume that $H^*(X,\mathbf{Z})$ is torsion free. Then the map

$$\alpha_r : kk_{-r}(Y,X) \to \mathbf{Hom}^r_{k^*(S^o)}(k^*(X), k^*(Y)), \; r \in \mathbf{Z}$$

is an isomorphism of groups.

Proof. We have seen that $k^*_X(-) = kk_{-*}(-,X)$ is a cohomology theory (see 3.2.1). The same is true for

$$h^*_X(-) := \mathbf{Hom}^*_{k^*(S^o)}(k^*(X), k^*(-))$$

since by Corollary 3.4.8 $k^*(X)$ is a free $k^*(S^o)$-module.

Since $\alpha_* : k^*_X(-) \to h^*_X(-)$ is a natural transformation of cohomology theories all we need is to prove that it induces an isomorphism on coefficients.

$$\text{i.e.} \quad \alpha^o_* : k_{-*}(X) \xrightarrow{\simeq} \mathbf{Hom}^*_{k^*(S^o)}(k^*(X), k^*(S^o)).$$

By the very definition of α^o_* we have that

$$\alpha^o_*(b)(a) = a /\!\!/ b$$

Here $/\!\!/$ is a special case of $/\!\!/$ in 3.5.3:

$$/\!\!/ : k^p(X) \times k_q(X) \to k^{p-q}(S^o)$$

According to the discussion in 3.5.3 this pairing can be identified with $/$ which, in this special case, is just the Kronecker pairing for connective K-theory. With this so that the associated Puppe sequence will give some information on $[Y,B_m(\underline{k})]$. In order to reach a group structure on $[Y,B_m(\underline{k})]$ we embed the above bundle into a bundle of H-spaces. Using certain stability results for vector bundles and homomorphisms it is shown that this procedure does not affect the homotopy in small dimensions. This technique enables us to obtain complete algebraic invariants (ranging in kk and K-groups) for the homotopy classes of those homomorphisms from A to D which are $3(n+3)/2$-large in the sense of 2.1.8 (see 4.2.8 and 4.2.1).

4.1. SOME FIBERINGS

4.1.1. Let $D_j = C(Y_j) \otimes M_{m_j}$. Since **Hom**(A,D) is the disjoint union of the **Hom**(A,D_j)'s for $1 \leq j \leq h$ it suffices to consider the case when $D = C(Y) \otimes M_m$.

Let $B_m = \mathbf{Hom}(A,M_m) = \mathbf{Hom}(\bigoplus_{i=1}^{q} C(X_i) \otimes M_{n_i},\ M_m)$ with X_i and n_i fixed throughout this Section. Using the notation of Section 2, each $\varphi \in B_m$ induces a homomorphism of scaled ordered groups

$$r(\varphi) \in \mathbf{Hom}(K_o(\bigoplus_{i=1}^{q} M_{n_i}),\ K_o(M_m))_{+,\Sigma} \subset \mathbf{Hom}(\mathbf{Z}^q,\mathbf{Z})$$

given by some integer vector $(k_1,\ldots,k_q)$. In fact k_i is the multiplicity of the embedding $(\varphi|M_{n_i}) : M_{n_i} \longrightarrow M_m$ hence $\sum_{i=1}^{q} k_i n_i \leq m$. If $k_o = m - \sum_{i=1}^{q} k_i n_i$, then φ is unital if and only if $k_o = 0$. Let $\underline{k} = (k_o;\ k_1,\ldots,k_q)$ and let $B_m(\underline{k})$ be the set of those φ in B_m with $r(\varphi) = (k_1,\ldots,k_q)$. $B_m(\underline{k})$ is a closed and open subspace of B_m and, as we shall see in 4.1.2, the $B_m(\underline{k})$ are exactly the connected components of B_m. Assume that $k_j = 0$ for some $j \geq 1$. Then $B_m = \mathbf{Hom}(\bigoplus_{i \neq j} C(X_i) \otimes M_{n_i},\ M_m)$. This fact allows us to make all the computations under the assumption that $k_j > 0$ for $1 \leq j \leq q$. We need some more notation and definitions. Let

$$U(\underline{k}) = U(k_o) \times U(k_1) \times \ldots \times U(k_q);\ \text{for } k_o = 0,\ U(k_o) = \text{a point.}$$

$$E(\underline{k}) = \mathbf{Hom}(C(X_1), M_{k_1}) \times \ldots \times \mathbf{Hom}(C(X_q), M_{k_q})$$

$$E_m(\underline{k}) = E(\underline{k}) \times U(m)$$

$$j^o : U(\underline{k}) \longrightarrow U(m),\ \underline{w} = (w_o, w_1, \ldots, w_q) \in U(\underline{k})$$

Proof. Since $U(\underline{k})$ is a Lie group freely acting on the compact space $E_m(\underline{k})$ it follows by a Theorem of Gleason [22] that the quotient map onto the orbit space

$$E_m(\underline{k}) \to E_m(\underline{k})/U(\underline{k})$$

is a principal fibre bundle with fibre $U(\underline{k})$. By Lemma 4.1.2 p can be identified with this map. ∎

4.1.4. In order to discuss the unitary equivalence relation for homomorphisms it is useful to consider the continuous left action of $U(m)$ on $E_m(\underline{k})$ given by

$$v\cdot(\underline{\varphi},u) = (\underline{\varphi},vu), \quad \underline{\varphi} \in E(\underline{k}),\ u,v \in U(m).$$

Since this action commutes with the right action of $U(\underline{k})$on $E_m(\underline{k})$, we get an action of $U(m)$ of $B_m(\underline{k})$ which is easily identified to

$$U(m) \times B_m(\underline{k}) \to B_m(\underline{k}), \quad (v,\psi) \mapsto v\psi v^*$$

since $p(v\,(\underline{\varphi},u)) = v\,p(\underline{\varphi},u)v^*$ for all $\underline{\varphi} \in E(\underline{k})$, $u,v \in U(m)$. In the case when each X_i is a point, the action of $U(m)$ on $B_m(\underline{k})$ corresponds to the action of $U(m)$ on the homogeneous space $U(m)/U(\underline{k})$.

4.2. EXACT SEQUENCES

4.2.1. Given a fibration $F \to E \to B$ the Puppe sequence

$$\to [SY,B] \to [Y,F] \to [Y,E] \to [Y,B]$$

is not always suficient for the computatin of $[Y,B]$ (without some additional structure). Letting aside the algebraic structure this is essentially due to the fact we don't know how to extend the above sequence to the right. However this can be done under certain favorable circumstances. For instance if $K \subset G$ are Lie groups, then the fibration $K \to G \to G/K$ can be extended to a homotopy exact sequence

$$K \overset{j}{\hookrightarrow} G \xrightarrow{\pi} G/K \xrightarrow{\varepsilon} BK \xrightarrow{j'} BG$$

where BK, BG are the classifying spaces for the principal K,G-bundles, j' is the map naturally induced by the inclusion $j: K \hookrightarrow G$ and ε is a classifying map for the K-bundle $G \to G/K$. In the more general case when K acts freely on a compact space E

Using the naturality of the augumented Puppe sequence we derive the following diagram

$$\begin{tikzcd} B^{o}_{m}(\underline{k}) \arrow[r, "i'"] \arrow[d, "\varepsilon^{o}"'] & B_{m}(\underline{k}) \arrow[r, "\rho'"] \arrow[d, "\varepsilon"] & B^{o}_{m}(\underline{k}) \arrow[d, "\varepsilon^{o}"] \\ BU(\underline{k}) \arrow[r, equal] & BU(\underline{k}) \arrow[r, equal] & BU(\underline{k}) \end{tikzcd}$$

which commutes within homotopy. Therefore

$$\varepsilon \circ i' \sim \varepsilon^{o} \text{ and } \varepsilon^{o} \circ \rho' \sim \varepsilon$$

These factorizations allows us to prove the following

4.2.3. PROPOSITION. There is an exact sequence

$$U(\underline{k}) \xrightarrow{j} E_m(\underline{k}) \xrightarrow{\rho} B_m(\underline{k}) \xrightarrow{\varepsilon} BU(\underline{k}) \xrightarrow{j'} BU(m)$$

where j' is naturally induces by $j^{o} : U(\underline{k}) \longrightarrow U(m)$.

Proof. Given a space Y we have to check the exactness of the following sequence $[Y,B_m(\underline{k})] \xrightarrow{\varepsilon_*} [Y,BU(\underline{k})] \xrightarrow{j'_*} [Y,BU(m)]$ of pointed sets. Of course we shall use the exact sequence

$$[Y,B^{o}_{m}(\underline{k})] \xrightarrow{\varepsilon^{o}_{*}} [Y,BU(\underline{k})] \xrightarrow{j'_{*}} [Y,BU(m)].$$

First observe that $j'_* \circ \varepsilon_* = j'_* \circ (\varepsilon^{o}_{*} \circ \rho'_*) = 0$ since $j'_* \circ \varepsilon^{o}_{*} = 0$.

Now if $g \in \mathbf{Map}(Y,BU(\underline{k}))$ is such that $j' \circ g$ is null homotopic, then there is $f \in \mathbf{Map}(Y,B^{o}_{m}(\underline{k}))$ such that $\varepsilon^{o} \circ f$ is homotopic to g. Therefore $h = i' \circ f \in \mathbf{Map}(Y,B_m(\underline{k}))$ is such that $\varepsilon \circ h = (\varepsilon \circ i') \circ f \sim \varepsilon \circ f \sim g$.

4.2.4. We have reached the sequence 4.2.3 but it is not entirely satisfactory since it gives us only exact sequences of pointed sets. However after we pass to inductive limits in 4.2.3 natural group structures will be available. It what follows we shall describe this construction.

For any positive integer t let $t\underline{k} = (tk_o; tk_1, \ldots, tk_q)$ and let $j^{o}_{t} : U(t\underline{k}) \longrightarrow U(tm)$ be the corresponding embedding defined in 4.4.1 i.e. $j^{o}_{t}(w_o, w_1, \ldots, w_q) =$
$= w_o \oplus (w_1 \otimes 1_{n_1}) \oplus \ldots \oplus (w_q \otimes 1_{n_q})$.

If x and y are generic elements of W_i and V_i as above we put

$$\xi(x) = th(i-1) + t(r-1)k_i + a$$

$$\xi(y) = th(q) + h(i-1) + (r-1)k_i + b$$

Now define the second bonding map

$$\beta_t : E_{tm}(t\underline{k}) \to E_{sm}(s\underline{k})$$

by the rule

$$\beta_t(\underline{\Upsilon},u) = (\underline{\Upsilon} \oplus \underline{\Upsilon}^o,\ \beta_t^o(u))$$

where $\underline{\Upsilon} = (\Upsilon_1, \dots, \Upsilon_q) \in E(t\underline{k})$, $\underline{\Upsilon}^o = (\Upsilon_1^o, \dots \Upsilon_q^o) \in E(\underline{k})$

$$\underline{\Upsilon} \oplus \underline{\Upsilon}^o = (\Upsilon_1 \oplus \Upsilon_1^o, \dots, \Upsilon_q \oplus \Upsilon_q^o) \in E((t+1)\underline{k})$$

(see also the notation introduced in 4.1.1. and 4.2.2.). Since β_t makes commutative the above diagram it follows that β_t is $U(t\underline{k})$ - equivariant in the sense that

$$\beta_t((\underline{\Upsilon},u),\underline{w}) = \beta_t(\underline{\Upsilon},u))\,\alpha_t(\underline{w}).$$

Therefore β_t naturally induces a map $\gamma_t : B_{tm}(t\underline{k}) \to B_{sm}(s\underline{k})$. Also we consider the maps $\alpha_t' : BU(t\underline{k}) \to BU(s\underline{k})$ and $\beta_t' : BU(tm) \to BU(sm)$, naturally induced by the group homomorphisms α_t and β_t, so that the following diagram commutes within homotopy.

$$\begin{tikzcd}
U(t\underline{k}) \arrow[r] \arrow[d, "\alpha_t"'] & E_{tm}(t\underline{k}) \arrow[r] \arrow[d, "\beta_t"'] & B_{tm}(t\underline{k}) \arrow[r] \arrow[d, "\gamma_t"'] & BU(t\underline{k}) \arrow[r] \arrow[d, "\alpha_t'"'] & BU(tm) \arrow[d, "\beta_t'"'] \\
U(s\underline{k}) \arrow[r] & E_{sm}(s\underline{k}) \arrow[r] & B_{sm}(s\underline{k}) \arrow[r] & BU(s\underline{k}) \arrow[r] & BU(sm)
\end{tikzcd}$$

Given a space Y we pass to homotopy classes [Y,-] in the above diagram and then we take inductive limits. Since taking inductive limits preserves the exactness we arrive at the following commutative diagram with exact rows:

4.2.6.

$$K^1(Y)^{\tilde q} \xrightarrow{j_*} \prod_{i=1}^{q} kk(Y,X_i) \times K^1(Y) \xrightarrow{\rho_*} \mathbf{lim}[Y,B_{tm}(t\underline{k})] \xrightarrow{\varepsilon_*} \tilde K^o(Y)^{\tilde q} \xrightarrow{j'_*} \tilde K^o(Y)$$

where $\tilde q = q$ if $k_o = 0$ and $\tilde q = q+1$ if $k_o > 0$.

The homomorphisms j_* and j'_* can be easily described if we recall the definition of $j : U(\underline{k}) \to E(\underline{k}) \times U(m)$ namely

$$j(\underline{w}) = (\underline{\varphi}^o, w_o \oplus (w_1 \otimes 1_{n_1}) \oplus \ldots \oplus (w_q \otimes 1_{n_q})).$$

Therefore if $\underline{k_o = 0}$ then

$j_*(y_1,\ldots,y_q) = (0, n_1y_1 + \ldots + n_qy_q)$, (the null component corresponding to $\prod_{i=1}^{q} kk(Y,X_i)$)

$j'_*(x_1,\ldots,x_q) = n_1x_1 + \cdots + n_qx_q$

$$\mathbf{coker}\, j_* = \prod_{i=1}^{q} kk(Y,X_i) \times (K^1(Y)/\sum_{i=1}^{q} n_iK^1(Y))$$

$$\mathbf{ker}\, j'_* = \{(x_1,\ldots,x_q) \in \tilde K^o(Y)^q : \sum_{i=1}^{q} n_ix_i = 0\}$$

For $\underline{k_o > 0}$
$$j_*(y_o,y_1,\ldots,y_q) = (0, y_o + n_1y_1 + \ldots + n_qy_q)$$

$$j'_*(x_o,x_1,\ldots,x_q) = x_o + n_1x_1 + \ldots + n_qx_q$$

$$\mathbf{coker}\, j_* = \prod_{i=1}^{q} kk(Y,X_i)$$

$$\mathbf{ker}\, j'_* = \{(x_o,x_1,\ldots,x_q) \in \tilde K^o(Y)^{q+1} : x_o + \sum_{i=1}^{q} n_ix_i = 0\}$$

For spaces X_i reducing to a point, the sequence 4.2.6 becomes

$$K^1(Y)^{\tilde q} \xrightarrow{j^o_*} K^1(Y) \xrightarrow{\rho^o_*} \mathbf{lim}[Y,B^o_{tm}(t\underline{k})] \xrightarrow{\varepsilon^o_*} \tilde K^o(Y)^{\tilde q} \xrightarrow{j'_*} \tilde K^o(Y)$$

which gives the middle term up to an extension:

$$0 \to \mathbf{coker}\, j^o_* \to \mathbf{lim}[Y,B^o_{tm}(t\underline{k})] \to \mathbf{ker}\, j'_* \to 0$$

4.2.7. PROPOSITION. There is a natural isomorphism

$$\theta : \mathbf{lim}[Y,B_{tm}(t\underline{k})] \longrightarrow \mathbf{lim}[Y,B^o_{tm}(t\underline{k})] \times \prod_{i=1}^{q} kk(Y,X_i)$$

equal than $3(n + 3)/2$. Then

a) There is an isomorphism

$$\theta = (\rho'_*, \nu) : [Y, B_m(\underline{k})] \to [Y, B^o_m(\underline{k})] \times \prod_{i=1}^{q} kk(Y_i, X)$$

b) There is an exact sequence of abelian groups

$$0 \longrightarrow \mathrm{coker}(j_*) \longrightarrow [Y, B^o_m(\underline{k})] \longrightarrow \mathbf{ker}(j'_*) \longrightarrow 0$$

c) If $\varphi, \psi \in \mathbf{Map}(Y, B_m(\underline{k}))$ then $\nu[\varphi] = \nu[\psi]$ and $\varepsilon_*[\varphi] = \varepsilon_*[\psi]$ if and only if there is some $u \in \mathbf{Map}(Y, U(m))$ such that $[\varphi] = [u\psi u^*]$.

Proof. a) The natural embedding $U(s) \hookrightarrow U(s+1)$ and $BU(s) \hookrightarrow BU(s+1)$ are 2s-equivalences [23]. Moreover it follows by Theorem 6.4.2 that the embedding $E(\underline{k}) \hookrightarrow E(2\underline{k})$ is a $(n+1)$-equivalence. Using these facts it follows from the last diagram in 4.2.4. (via a Five Lemma argument) that the map

$$\gamma_t : B_m(t\underline{k}) \longrightarrow B_m((t+1)\underline{k}) \text{ is a } (n+1)\text{-equivalence for any } t \geq 1.$$

Therefore by Whitehead theorem the map

$$\gamma_* : [Y, B_m(\underline{k})] \to \mathbf{lim}[Y, B_{tm}(t\underline{k})]$$

is a bijection since $\mathbf{dim}(Y) \leq n$. This map is used to transfer the group structure on $[Y, B_m(\underline{k})]$ and we shall identify x with $\gamma_*(x)$ for every x in $[Y, B_m(\underline{k})]$. Accordingly the maps ρ'_*, ν and ε_* may be seen as maps from $[Y, B_m(\underline{k})]$. Finally after these identifications the assertion follows from 4.2.7.

b) similar to a)

c) The proof is divided into two parts.

In the first part we prove the statement assuming that the following assertion is true:

Assertion. If $\varphi, \psi \in \mathbf{Map}(Y, B^o_m(\underline{k}))$ then $\varepsilon^o_*(\varphi) = \varepsilon^o_*(\psi)$ if and only if $\psi = u^*\varphi u$ for some $u \in \mathbf{Map}(Y, U(m))$.

In the second part we prove the above assertion.

$\varepsilon^{o}\circ f^{j}$ is a clasifying map for ξ^{j}, ξ^{1} is isomorphic to ξ^{2} iff $\varepsilon^{o}\circ f^{1}\sim\varepsilon^{o}\circ f^{2}$. Therefore we have to prove that ξ^{1} is isomorphic to ξ^{2} iff $f^{2}=g\cdot f^{1}$ for some $g\in \mathbf{Map}(Y,G/K)$. To prove this equivalence we shall work with G-cocycles (=systems of transition functions for principal G-bundles) (see [23]).

First assume $\xi^{1}\simeq\xi^{2}$. Then there is an open cover $(U_\alpha)_\alpha$ of Y and for each α, $g^{j}_{\alpha}\in\mathbf{Map}(U_\alpha,G)$, $j=1,2$, such that g^{j}_{α} lifts $f^{j}|U_\alpha$ i.e. $p(g^{j}_{\alpha}(x))=f^{j}(x)$ for all $x\in U_\alpha$. For any α,β with $U_\alpha\cap U_\beta\neq\emptyset$ define $h^{j}_{\alpha\beta}\in\mathbf{Map}(U_\alpha\cap U_\beta,K)$ by $h^{j}_{\alpha\beta}(x)=g^{j}_{\alpha}(x)^{-1}g^{j}_{\beta}(x)$, $x\in U_\alpha\cap U_\beta$.

The system $(U_\alpha, h^{j}_{\alpha\beta})$ is called a G-cocycle associated with the G-bundle ξ^{j}. There is an equivalence relation in the set of G-cocycles such that two bundles are isomorphic if and only if any two G-cocycles associated with them are equivalent ([23]). The equivalence relation for G-cocycles takes a simpler form when they corresponds to the same covering of the base space. Thus in our case $\xi^{1}\sim\xi^{2}$ iff there exist $h_\alpha\in\mathbf{Map}(U_\alpha,G)$ such that $h^{2}_{\alpha\beta}(x)=h^{-1}_{\alpha}(x)h^{1}_{\alpha\beta}(x)h_\beta(x)$ for $x\in U_\alpha\cap U_\beta$. This implies

$$g^{2}_{\alpha}(x)h_\alpha(x)g^{1}_{\alpha}(x)^{-1}=g^{2}_{\beta}(x)h_\beta(x)g^{1}_{\beta}(x)^{-1}\text{ for } x\in U_\alpha\cap U_\beta .$$

Therefore we can define $g\in\mathbf{Map}(Y,G)$ by

$$(g|U_\alpha)(x):=g^{2}_{\alpha}(x)h_\alpha(x)g^{1}_{\alpha}(x)^{-1},$$

and is easily seen that $f^{2}=g\,f^{1}$.

The converse is almost contained in the above arguments.

If $f^{2}=g\,f^{1}$ and $g^{1}_{\alpha}\in\mathbf{Map}(U_\alpha,G)$ is chosen as above then we may take $g^{2}_{\alpha}:=(g|U_\alpha)\,g^{1}_{\alpha}$. Therefore $h^{2}_{\alpha\beta}(x)=g^{1}_{\alpha}(x)^{-1}\underbrace{g(x)}_{g(x)^{-1}}g^{1}_{\beta}(x)=h^{1}_{\alpha\beta}(x)$ for $x\in U_\alpha\cap U_\beta$ and so ξ^{1} is isomorphic to ξ^{2}.

4.2.9. REMARK. Define $B_\infty=\mathbf{lim}\,B_{tm}(t\underline{k})$. Since B_∞ is a weak H space it follows that the action of $\pi_1(B_\infty)$ on $[Y,B_\infty]$ is trivial. As the embedding $B_m(\underline{k})\to B_\infty$ is a $(n+1)$-equivalence it is easily seen that $\pi_1(B_m(\underline{k}))$ acts trivially on $[Y,B_m(\underline{k})]$. Consequently $[Y,B_m(\underline{k})]$ coincides with the free homotopy classes $[Y,B_m(\underline{k})]_{free}=[A,C(Y)\otimes M_m]_{\underline{k}}$.

be $3(n+3)/2$-large. Then

a) $[\varphi] = [\psi]$ if and only if $[\varphi^{j,i}] = [\psi^{j,i}]$ in $kk(Y_j, X_i)$ for all i, j and $[\varphi / r(A)] = [\psi / r(A)]$ in $[r(A), D]$

b) $[\varphi] = [u \psi u^*]$ for some unitary $u \in U(D)$ if and only if $[\varphi^{j,i}] = [\psi^{j,i}]$ in $kk(Y_j, X_i)$ for all i, j and $K_0(\varphi | r(A)) = K_0(\psi | r(A))$.

(For $\varphi \in \mathbf{Hom}(A,D)$, $\varphi^{j,i}$ stands for the composition

$$C_0(X_i) \otimes e^i \hookrightarrow A \xrightarrow{\varphi} D \longrightarrow C(Y) \otimes M_{m_j})$$

4.3. HOMOTOPY AND K-THEORY

Throughout this section we let A, D stand for two fixed C^*-algebras in $\mathcal{C}(n)$

$$A = \bigoplus_{i=1}^{\ell} C(X_i) \otimes M_{n_i}, \; D = \bigoplus_{j=1}^{h} C(Y_j) \otimes M_{m_j}.$$

We gave in 4.2 complete invariants based on kk and K-theory for the large morphisms belonging to [A,D]. In order to use our computations for shape classifications it is useful to point out some cases when K-theory suffices for computing [A,D].

This is done by comparing kk with KK. A related problem is to describe the image of [A,D] in KK[A,D] using order concepts.

Let $\mathcal{X}(n)$ be the class of finite connected CW-complexes of dimension $\leq n$ whose total cohomology is torsion free and supported in two dimensions having distinct parity. Thus for given $X \in \mathcal{X}(n)$ there are $p, q \in \mathbf{N}$ depending on X such that $H^{even}(X, \mathbf{Z}) = H^p(X, \mathbf{Z})$, $H^{odd}(X, \mathbf{Z}) = H^q(X, \mathbf{Z})$ and both these groups are torsion free.

4.3.1. THEOREM. Let A,D as above and assume that X_i, $Y_j \in \mathcal{X}(n)$ for all i, j. Then

a) If $\sigma \in \mathbf{Hom}(K_*(A), K_*(D))_{+,\Sigma}$ is $3(n+3)/2$-large then there is some $\varphi \in \mathbf{Hom}(A,D)$ such that $K_*(\varphi) = \sigma$.

b) If $\varphi_1, \varphi_2 \in \mathbf{Hom}(A,D)$ are $3(n+3)/2$-large and $K_*(\varphi_1) = K_*(\varphi_2)$ then there is $u \in U(D)$ such that

$$\varphi_1 \text{ is homotopic to } u \varphi_2 u^*$$

Finally we choose $[\varphi]\in[Y,B_m(\underline{k})]$ such that its image in $[Y,B_{2m}(2\underline{k})]$ is equal to $[\varphi'\oplus\varphi'']$. The above computations show that $K_*(\varphi)=\sigma$.

b) Let $K_*(\varphi_i)=\begin{pmatrix}\alpha(\varphi_i) & \beta(\varphi_i)\\ 0 & r(\varphi_i)\end{pmatrix}: K_*(A_0)\oplus K_*(A_1)\to K_*(D_0)\oplus K_*(D_1)$

We have $\beta(\varphi_1)=\beta(\varphi_2)$ and $r(\varphi_1)=r(\varphi_2)$ since $K_*(\varphi_1)=K_*(\varphi_2)$.

The following commutative diagram

$$\begin{tikzcd}
{[A,D]} \arrow[r] \arrow[d, "\nu"'] & \mathrm{Hom}(K_*(A),K_*(D)) \arrow[d, "\alpha"] \\
\oplus kk(Y,X_i) \arrow[r, "\sim"] & \mathrm{Hom}(K_*(A_0),K_*(D_0))
\end{tikzcd}$$

shows that $\alpha(\varphi_1)=\alpha(\varphi_2)$ implies $\nu(\varphi_1)=\nu(\varphi_2)$. Now we can apply Theorem 4.2.11 (b) in order to derive the desired conclusion.

The following result concerns spaces which may have torsion in K_0 but we have to make some restrictions.

4.3.2. THEOREM. Let $A=\bigoplus_{i=1}^{\ell} C(X_i)\otimes M_{n_i}$, $D=\bigoplus_{j=1}^{h} C(Y_j)\otimes M_{m_j}$ where X_i,Y_j are (n-2)-connected finite CW-complexes of dimension $\leq n$ and n is even.

a) If $\sigma\in KK(A,D)_{+,\Sigma}$ is 3(n+3)/2-full (2.1.8) then there is some $\varphi\in$ **Hom**(A,D) such that $[\varphi]_{KK}=\sigma$.

b) If $\varphi_1,\varphi_2\in$ **Hom**(A,D) are 3(n+3)/2-full and $[\varphi_1]_{KK}=[\varphi_2]_{KK}$ then φ_1 is homotopic to $u\varphi_2u^*$ for some $u\in U(D)$.

($\sigma\in KK(A,D)_{+,\Sigma}$ is called m-full if its image in $\mathrm{Hom}(K_0(A),K_0(B))_{+,\Sigma}$ is m-full.)

Proof. a) Let A_0, A_1, D_0, D_1 be as in the proof of 4.3.1. We shall analyse the components of σ corresponding to the decomposition

$$KK(A,B)=\begin{pmatrix} KK(A_0,D_0) & KK(A_1,D_0)\\ KK(A_0,D_1) & KK(A,D_1)\end{pmatrix}$$

5.1. SEMIPROJECTIVITY.

In this subsection we extend the notion of semiprojectivity for C^*-algebras introduced by Effros and Kaminker [17] to a more general setting which allow us to develop a satisfactory shape theory for a larger class of C^*-algebras.

Let $\mathcal{S}$ denote the category of separable C^*-algebras. We start with a covariant functor $T : \mathcal{S} \to \mathcal{D}$, with values in a category $\mathcal{D}$ having the same objects as $\mathcal{S}$, such that $T(A) = A$ for each C^*-algebra A in $\mathcal{S}$. We have in mind two basical examples.

5.1.1. EXAMPLES.

a) Let $\mathcal{H}$ be the category of separable C^*-algebras and homotopy classes of homomorphisms, and let $H : \mathcal{S} \to \mathcal{H}$ be the homotopy functor i.e. H preserves the objects and takes the homomorphisms into their classes of homotopy equivalence : $\varphi \mapsto H(\varphi) = [\varphi]$.

b) Let $KK_{+,\Sigma}$ be the category whose objects are separable C^*-algebras, and for which the morphisms from A to B are elements of $KK(A,B)_{+,\Sigma}$(1.2.8). The law of composition is the Kasparov product. (cf. [21]). There is a canonical functor $KK_{+,\Sigma} : \mathcal{S} \to KK_{+,\Sigma}$ since each $\varphi \in \mathrm{Hom}_{\mathcal{S}}(A,B)$ defines in a natural way an element $[\varphi]_{KK} \in KK(A,B)_{+,\Sigma}$ and this assignment preserves the products ([26]).

Let $\mathcal{C}$ denote a fixed subcategory of $\mathcal{S}$. By a $\mathcal{C}$-inductive system (A_i, α_{ji}) we shall mean a diagram of objects and morphisms in $\mathcal{C}$

$$A_1 \xrightarrow{\alpha_{21}} A_2 \xrightarrow{\alpha_{32}} \cdots$$

By definition for $i < j$ $\alpha_{ji} := \alpha_{j,j-1} \circ \cdots \circ \alpha_{i+1,i}$.

Of course the above definition makes sense in any category.

If (A_i, α_{ji}) denote a $\mathcal{C}$-inductive system and $A = \varinjlim A_i \in \mathcal{S}$ is the associated inductive limit C^*-algebra then we have canonical maps in $\mathcal{S}$, : $\alpha_i : A_i \longrightarrow A$ such that $\alpha_j \circ \alpha_{ji} = \alpha_i$.

For each $S \in \mathcal{S}$ we have an inductive system of sets

$$(\mathrm{Hom}_{\mathcal{D}}(S,A_i),\ \ T(\alpha_{ji})_*)$$

Proof.

Let $B = \varinjlim(B_i, \beta_{ji})$ with $B_i, \beta_{ji} \in \mathscr{C}^1(n)$. We have to prove that

$$KK(A,B)_{+,\Sigma} = \varinjlim KK(A,B_i)_{+,\Sigma}$$

As we shall see below all the difficulty comes from the ordering.

Indeed since $K_0(A)$ is finitely generated and $K_*(B) = \lim K_*(B_i)$ we have

$$\mathbf{Hom}(K_*(A), K_*(B)) = \lim \mathbf{Hom}(K_*(A), K_*(B_i))$$

Moreover since A is nuclear it follows by [37, Thm. 1.4 and Prop. 7.13] that

$$KK(A,B) = \lim KK(A,B_i)$$

Now recall from 1.2.8 that $KK(A,B)_{+,\Sigma} = \{x \in KK(A,B) : \gamma(x) \in \mathbf{Hom}(K_*(A),K_*(B))_{+,\Sigma}\}$.

Therefore the canonical map $\lim KK(A,B_i)_{+,\Sigma} \to KK(A,B)_{+,\Sigma}$ is injective. Also it is easy to show (using set-theoretic arguments related to inductive limits and the naturality of γ), that the surjectivity of the above map reduces to the surjectivity of the canonical map

$$\lim \mathbf{Hom}(K_0(A), K_0(B_i))_{+,\Sigma} \longrightarrow \mathrm{Hom}(K_0(A), K_0(B))_{+,\Sigma}$$

Therefore what we need is to prove that this last map is onto. To begin with, we need some notation.

Let A_k denote the subalgebra $C(X_k) \otimes M_{n_k}$, let e_k be a minimal projection in M_{n_k} and let J_k be a finite set of generators for $K_1(A_k)$. Given $\sigma = (\sigma^0, \sigma^1) \in \mathbf{Hom}(K_*(A), K_*(B))_{+,\Sigma}$, $\sigma \neq 0$, we can find $i \geq 0$ and a morphism of $\mathbb{Z}_2$-graded groups $\rho = (\rho^0, \rho^1) : K_*(A) \longrightarrow K_*(B_i)$ such that

a) $K_*(\beta_i) \circ \rho = \sigma$

where β_i is the embedding $B_i \longrightarrow B$.

Note that if $x \in K_*(A)_+$ is given we can replace ρ by $\beta_{ji} \circ \rho$, for large enough j, in order to get $\rho(x) \in K_*(B_j)_+$. This is possible since $K_*(B)_+ = \lim K_*(B_i)_+$. Consequently by increasing i we may assume that ρ also satisfies

b) $\rho^0(K_0(A)_+) \subset K_0(B_i)_+$ (recall that $K_0(A)$ is finitely generated).

and

Let $I = \{k : \gamma_{jk} = 0\}$ and $J = \{k : t_k = 0\}$. We have already seen that $\alpha_{jk} = 0$, $\beta_{jk} = 0$ and $\rho^1_{jk} = 0$ for each $k \in I$. Also it is clear that $a'_k = 0$ and $a^1_k = 0$ for each $k \in J$ since $a \geq 0$. Therefore

$$b_j = \sum_k \left(\begin{pmatrix} \alpha_{jk} & \beta_{jk} \\ 0 & \gamma_{jk} \end{pmatrix} \begin{pmatrix} a'_k \\ t_k \end{pmatrix}, \rho^1_{jk}(a^1_k) \right) = 0$$

If $s_j > 0$ then $\gamma_{jk} t_k \neq 0$ for the same k and so $s_j \geq 2n$ since $\gamma_{jk} \geq 2n$ because ρ^o is 2n-large. Thus we may apply Corollary 2.1.2 c) to get $\rho(a) \geq 0$. It remains to show that $\rho(\Sigma_*(A)) \subset \Sigma_*(B)$. We shall use the point d) of the same Corollary. The condition d) satisfied by ρ^o implies that $o \leq s_j \leq m_j$. There are three cases to be considered for each index j:

(i) if $s_j = 0$ then $b_j = 0$ since $b_j \geq 0$.

(ii) if $0 < s_j < m_j$ then $s_j \leq m_j - 2n \leq m_j - \langle (n+1)/2 \rangle$ since ρ^o is 2n-large.

(iii) if $s_j = m_j$, or equivalently $\sum_{k=1}^{q} \gamma_{jk} t_k = m_j$, then

$t_k = n_k$ for each $k \in J = \{p : \gamma_{jp} \neq 0\}$, since otherwise $\sum_{k=1}^{q} \gamma_{jk} n_k > m_j$ which contradicts $\rho^o[1_A] \in \Sigma(B_i)$.

As $a_k \in \Sigma_*(A_k)$ we must have $a_k = [1_{A_k}]$ for each $k \in J$.

Therefore b_j coincides with the j^{th} component of $\rho^o[1_A]$ and $b_j \in \Sigma_*(D_j)$ by condition d).

5.1.5. One cannot drop the assumption on $K_o(A)_+$ made in 5.1.4. For instance it follows by [27] that $C(S^2)$ and $C(S^1 \times S^1)$ are not $KK_{+,\Sigma}$- semiprojective in $\mathcal{C}'(2)$.

Let $\mathcal{C}''(n)$ be the subcategory of $\mathcal{C}(n)$ with objects of the form $C(X) \otimes M_m$ and 2-large (possibly nonunital) $*$-homomorphisms.

5.1.6. PROPOSITION. Any C^*-algebra $A \in \mathcal{C}''(n)$ is $KK_{+,\Sigma}$-semiprojective in $\mathcal{C}''(n)$.

Proof. Let $A = C(X) \otimes M_m$, $B_i = C(Y_i) \otimes M_{m_i} \in \mathcal{C}''(n)$ and let $B = \varinjlim(B_i, \beta_{ji})$ each β_{ji} being 2-large. Like in the proof of 5.1.4 for a given $\sigma \in \mathbf{Hom}(K_*(A), K_*(B))_{+\Sigma}$, where $r(B)$ is the matroid C^*-algebra arising from the inductive system (B_i, β_{ji}) as described in 2.1.5. Since $r(K_o(\beta_i))$ is injective the equalities

$$r(K_o(\beta_i))r_{B_i}\rho^o = r_B K_o(\beta_i)\rho^o = r_B \sigma^o$$

shows that our claim reduces to the implication

$$r_A(a) = 0 \Rightarrow f(a) = 0, \text{ where by definition } f = r_B \sigma^o.$$

Now if $r_A(a) = 0$ then $a \in K'_o(A)$ and so $ka + (n+1)[1_A] \in K_o(A)_+$ for each $k \in \mathbf{Z}$. Since the morphism f is order preserving we have $k\,f(a) + (n+1)\,f([1_A]) \in K_o(r(B))_+$ for any $k \in \mathbf{Z}$ and this is possible only if $f(a) = 0$. To derive the last implication one can use an embedding of $(K_o(r(B)), K_o(r(B))_+)$ in $(\mathbf{R},\mathbf{R}_+)$. Thus we get $\rho^o = \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix}$. Let us observe that $\gamma > 0$. Indeed if γ would be less or equal to zero then b) would imply $\gamma = 0$, $\beta = 0$ and so $\rho^o[1_A] = 0$. But this gives a contradiction since $\sigma^o[1_A] = 0$ would imply $\sigma^o = 0$ because σ^o is order preserving.

Once we know that $\gamma > 0$ we replace ρ by $\beta_{i+n,i} \circ \rho$ to reach

c) ρ is 2n-large

From now on the proof is similar to the last part of the proof of 5.1.4. Like there we need coordinate descriptions of a and $\rho(a)$.

$$a = ((a',t)a^1) \in K_o(A) \oplus K_o(r(A)) \oplus K_1(A)$$

$$\rho(a) = ((\alpha(a') + \beta(t), \gamma t), \rho^1(a^1)) \in K_o(B_i) \oplus K_o(r(B_i)) \oplus K_1(B_i).$$

If a $K_*(A)_+$ and $a \neq 0$ then $\gamma t \geqslant 2n \geq n+1$ and so $\rho(a) \in K_*(B_i)_+$ by Corollary 2.1.1 c). Therefore we checked that ρ is order preserving. To complete the proof we must show that $\rho(\Sigma_*(A)) \subset \Sigma_*(B_i)$. If $a \in \Sigma_*(A)$, $a \neq 0$ then $t \leq m$ and $\gamma t \leq m_i$ since $\gamma t = r_{B_i}\rho^o(a',t) \in \Sigma(r(B_i))$ by condition b).

Next we shall consider two cases:

i) if $\gamma t = m_i$ then $t = m$ since $t \leq m$ and $\gamma m \leq m_i$.

As $a \in \Sigma_*(A)$, $A = C(X) \otimes M_m$ it follows that $a = [1_A]$ and so $\rho(a) \in \Sigma(B)$ by b).

ii) if $\gamma t < m_i$ then $\gamma t \leq m_i - 2n \leq m_i - \langle(n+1)/2\rangle$ since γ is 2n-large. Therefore $\rho(a) \in \Sigma_*(B_i)$ by Corollary 2.1.2. d).

from there are set theoretical (in their essence) and can be easily modified to work in our more general setting.

5.1.8. PROPOSITION. Let $A = \lim A_i$, $B = \lim B_i$ as above and assume that the A_i are T-semiprojective relative to $\mathcal{C}$. Then any $*$-homomorphism $\varphi : A \longrightarrow B$ has an associated system map $\underline{\varphi}$.

5.1.9. PROPOSITION. Suppose that A,B are as above and that $\varphi, \psi : A \longrightarrow B$ have associated inductive systems maps $\underline{\varphi}, \underline{\psi} : (A_i, T(\alpha_{ji})) \longrightarrow (B_i, T(\beta_{ji}))$. If $T(\varphi) = T(\psi)$ then $\underline{\varphi}$ and $\underline{\psi}$ are equivalent and therefore they define the same morphism in inj-$\mathcal{D}$.

5.1.10. PROPOSITION. Suppose that $A = \lim A_i$ and $B = \lim B_i$ where A_i and B_i are T-semiprojective relative to $\mathcal{C}$. If A is isomorphic to B in $\mathcal{D}$ then $(A_i, T(\alpha_{ji}))$ is isomorphic to $(B_i, T(\beta_{ji}))$ in inj-$\mathcal{D}$.

5.2. SHAPE THEORY

If $\mathcal{C}$ is a subcategory of $\mathcal{S}$ recall that $\mathcal{A}\mathcal{C}$ denotes the class of all C^*-algebras in $\mathcal{S}$ which can be written as inductive limits of faithful inductive systems from $\mathcal{C}$.

The next definition points out the crucial property which must be enjoy by a category $\mathcal{C}$ of C^*-algebras in order to give sense for shape invariants.

5.2.1. DEFINITION. a) $\mathcal{C}$ is called a <u>shape category</u> if and only if the following implication holds.

If (A_i, α_{ji}) and (B_i, β_{ji}) are faithful inductive systems in $\mathcal{C}$, which have homotopic inductive limits: $\lim A_i \sim \lim B_i$, then $(A_i, [\alpha_{ji}])$ is isomorphic to $(B_i, [\beta_{ji}])$ in the category inj-$\mathcal{H}$(see 5.1.1. a) and 5.1.7).

b) Let $\mathcal{C}$ be a shape category and $A = \lim(A_i, \alpha_{ij}) \in \mathcal{A}\mathcal{C}$. <u>The shape invariant of A</u> denoted by $Sh_{\mathcal{C}}(A)$ is the class of isomorphism of $(A_i, [\alpha_{ji}])$ in inj-$\mathcal{H}$.

5.2.2. REMARKS.

a) If the objects of $\mathcal{C}$ are H-semiprojective in $\mathcal{C}$ then it follows by Proposition

1) $\Leftrightarrow$ 2) by Proposition 7.3 in [37] which applies since $A, B \in \mathcal{N}$.

3) $\Rightarrow$ 1) is immediate as noticed in [2].

It remains to prove 2) $\Rightarrow$ 3). By hypothesis A_i and B_i are $KK_{+,\Sigma}$-semiprojective. Using Proposition 5.1.10 it follows from 2) that $(A_i, [\alpha_{ji}]_{KK})$ is isomorphic to $(B_i, [\beta_{ji}]_{KK})$ in inj-$KK_{+,\Sigma}$. Such an isomorphism gives a diagram in $KK_{+,\Sigma}$

$$\begin{tikzcd}
A_{i_1} \arrow[r] \arrow[dr, "\sigma_1"'] & \cdots \arrow[r] & A_{i_2} \arrow[r] \arrow[dr, "\sigma_2"] & \cdots \arrow[r] & A_{i_3} \arrow[r] & \cdots \\
 & B_{j_1} \arrow[r] \arrow[ur, "\tau_1"] & \cdots \arrow[r] & B_{j_2} \arrow[r] \arrow[ur, "\tau_2"] & \cdots \arrow[r] & {}
\end{tikzcd}$$

where the triangles commutes. Using condition b) we may assume that there are suitable homomorphisms φ_k, ψ_k, $k \geq 1$, in $\mathcal{C}$, such that

$$\sigma_k = [\varphi_k]_{KK} \text{ and } \tau_k = [\psi_k]_{KK}$$

Therefore $[\psi_k \circ \varphi_k]_{KK} = [\alpha_{i_{k+1} i_k}]_{KK}$ and $[\varphi_{k+1} \circ \psi_k]_{KK} = [\beta_{j_{k+1}, j_k}]_{KK}$.
Based on condition c) we can find inductively two sequences of inner automorphisms $\gamma_k \in \mathbf{Aut}(B_{j_k})$ and $\delta_k \in \mathbf{Aut}(A_{i_{k+1}})$ such that if $\varphi'_k = \gamma_k \circ \varphi_k$ and $\psi'_k = \delta_k \circ \psi_k$ then $\psi'_k \circ \varphi'_k$ is homotopic to $\alpha_{i_{k+1} i_k}$ and $\varphi'_{k+1} \circ \psi_k$ is homotopic to $\beta_{j_{k+1} j_k}$. This means exactly that $(A_i, [\alpha_{ji}])$ and $(B_i, [\beta_{ji}])$ are isomorphic in inj-$\mathcal{H}$.

5.2.4. So for we have been very formalistic. That's why some comments on our technique for approaching shape problems are perhaps in order. Now the fundamental problem is to relate in inj-$\mathcal{H}$ two inductive systems whose limits are homotopic. More generally what can be said if the limits have the same K-theory groups (including order and scale if neccesary)? The main dificulty in solving this problem within the homotopy category $\mathcal{H}$ comes from the possible absence of H-semiprojectivity. The point is that for certain categories $\mathcal{C} \subset \mathcal{S}$ we can overcome this difficulty by taking a devious route provided by the $KK_{+,\Sigma}$-functor. Diagrammaticaly this is as follows. Let $A = \mathbf{lim}\, A_i$, $B = \mathbf{lim}\, B_i$.

5.3.1. Recall that W_o^c denotes the category of finite connected CW-complexes. For each $n \geq 1$ we define the folowing four classes of spaces included in W_o^c.

$\mathcal{X}_1(n)$ consists of all spaces $X \in W_o^c$ of dimension $\leq n$ for which there are $p,q \in \mathbf{N}$, depending on X, such that $H^{even}(X,\mathbf{Z}) = H^p(X,\mathbf{Z})$, $H^{odd}(X,\mathbf{Z}) = H^q(X,\mathbf{Z})$ and these groups are torsion free.

Note that $S^p \vee S^q \in \mathcal{X}_1(n)$ provided that $p,q \leq n$ and $p - q \equiv 1 \pmod 2$.

$\mathcal{X}_2(2n)$ consists of all spaces $X \in W_o^c$ of dimension $\leq 2n$ which are $(2n-2)$-connected. Note that any finite connected CW-complex of dimension ≤ 2 belongs to $\mathcal{X}_2(2)$.

$\mathcal{X}_3(2n)$ consists of those spaces $X \in \mathcal{X}_2(2n)$ with the property that $H^{2n}(X,\mathbf{Z})$ is a torsion group. Note that $\mathcal{X}_3(2)$ contains the non-orientable manifolds of dimension 2.

$\mathcal{X}_4(2n-1)$ consists of finite wedges of $(2n-1)$-spheres. Starting with the above classes of spaces we define the "categories" of C^*-algebras $\mathcal{C}_1(n)$, $\mathcal{C}_2(2n)$, $\mathcal{C}_3(2n)$, $\mathcal{C}_4(2n-1)$ as follows.

The objects of $\mathcal{C}_1(n)$ have the generic form $C(X) \otimes M_m$ where $X \in \mathcal{X}_1(n)$ and $m \in \mathbf{N}$. The set $\mathbf{Hom}_{\mathcal{C}_1(n)}(A,B)$ of morphisms from A to B consists of all $3(n+3)/2$-large $*$-homomorphisms (2.1.8).

The objects of $\mathcal{C}_2(2n)$ have the generic form $C(X) \otimes M_m$ where $X \in \mathcal{X}_2(2n)$ and $m \in \mathbf{N}$. The set $\mathbf{Hom}_{\mathcal{C}_2(2n)}(A,B)$ consists of all $3(n+3)/2$-large $*$-homomorphisms from A to B.

The objects of $\mathcal{C}_3(2n)$ have the generic form $\bigoplus_{i=1}^{q} C(X_i) \otimes M_{n_i}$ where $X_i \in \mathcal{C}_3(2n)$ and $n_i, q \in \mathbf{N}$. The set $\mathbf{Hom}_{\mathcal{C}_3(2n)}(A,B)$ of morphisms from A to B consists of all $3(n+3)/2$-full $*$-homomorphisms (2.1.8).

The objects of $\mathcal{C}_4(2n-1)$ have the generic form $\bigoplus_{i=1}^{q} C(X_i) \otimes M_{n_i}$ where $X_i \in \mathcal{X}_4(2n-1)$ and $n_i, q \in \mathbf{N}$. The set $\mathbf{Hom}_{\mathcal{C}_4(2n-1)}(A,B)$ consists of all $3(n+3)/2$-large $*$-homomorphisms.

5.3.2. REMARKS

a) Let $\mathcal{C}$ be one of the categories $\mathcal{C}_1(n)$, $\mathcal{C}_2(2n)$, $\mathcal{C}_4(2n-1)$. Let (A_i, φ_{ji}) be an

5.3.5. REMARKS

a) Let $\mathscr{C}$ be a shape category

Let $A = \mathbf{lim}(A_i, \alpha_{ji})$, $B = \mathbf{lim}(B_i, \beta_{ji})$ with $A_i, B_i, \alpha_{ji}, \beta_{ji} \in \mathscr{C}$. Then $Sh_{\mathscr{C}}(A) = Sh_{\mathscr{C}}(B)$ means exactly that there is a diagram

$$\begin{tikzcd}
A_{i_1} \arrow[rr, "\cdots"] \arrow[dr] & & A_{i_2} \arrow[rr, "\cdots"] \arrow[dr] & & A_{i_3} \arrow[r, "\cdots"] \arrow[dr] & \cdots \\
& B_{j_1} \arrow[ur] \arrow[rr, "\cdots"] & & B_{j_2} \arrow[ur] \arrow[rr, "\cdots"] & & B_{j_3}
\end{tikzcd}$$

such that each triangle commutes within homotopy.

The existence of such a diagram is not automatically assured even if $A = B$ unless $\mathscr{C}$ is a shape category or has other related properties since it is not evident that inductive systems having the same limit are related in inj-$\mathscr{H}$.

b) If $A \in A\mathscr{C}$ for $\mathscr{C} = \mathscr{C}_1(n), \mathscr{C}_2(2n), \mathscr{C}_3(2n)$ then $K_0(A)$ is simple (as an ordered group) and it can be proved that

$$K_*(A)_+ = \{(0,0)\} \cup \left((K_0(A)_+ \setminus \{0\}) \oplus K_1(A)\right).$$

Consequently the assertion a) of Theorem 5.3.2 is equivalent to

a') $K_0(A) \simeq K_0(B)$ as scaled ordered groups and $K_1(A) \simeq K_1(B)$.

However for $\mathscr{C} = \mathscr{C}_4(2n-1)$ a) cannot be replaced by a') as it is shown below.

c) For $p,q \geq 2$ let $A(p,q)$ be the C^*-algebra arising as the inductive limit of

$$\longrightarrow C(S^1) \otimes M_{p^n} \xrightarrow{\varphi_n} C(S^1) \otimes M_{p^{n+1}} \longrightarrow \cdots$$

where the embedding φ_n is given by the rule

$$\varphi_n(f)(z) = \begin{pmatrix} f(z^2) & & & \\ & f(z_0) & & \\ & & \ddots & \\ & & & f(z_0) \end{pmatrix}$$

$f \in C(S^1) \otimes M_{p^n}$; $z \in S^1$ and category of C^*-algebras one would have a powerful tool in shape problems.

5.3.6. Let $A_i = C(S^1 \vee S^3) \times M_{9^i}$. Using Theorems $\overset{4.2.11}{3.5.5}$ it is easily seen that the image of the map

$$[A_i, A_{i+1}] \longrightarrow \mathbf{Hom}(K_1(A_i), K_1(A_{i+1})) \simeq M_2(\mathbf{Z})$$

is exactly $\quad \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} ; \ a, b, c \in \mathbf{Z} \right\}$

Let $u = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $v = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$, $x = vu = \begin{pmatrix} 3 & -1 \\ 0 & 2 \end{pmatrix}$, $y = uv = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$.

Choose $\varphi_i, \psi_i \in \mathbf{Hom}(A_i, A_{i+1})$ such that $K_1(\varphi_i) = x$ and $K_1(\psi_i) = y$ and define $A = \lim(A_i, \varphi_i)$, $B = \lim(A_i, \psi_i)$. It is clear that $K_0(A) \simeq K_0(B)$ as ordered scaled group. Moreover the commutative diagram

$$\begin{tikzcd}
K_1(A_1) \arrow[rr, "x"] \arrow[dr, "u"'] & & K_1(A_2) \arrow[rr, "x"] \arrow[dr, "u"] & & K_1(A_3) \arrow[rr, "x"] \arrow[dr, "u"] & & {} \\
& K_1(A_2) \arrow[rr, "y"'] \arrow[ur, "v"] & & K_1(A_3) \arrow[rr, "y"'] \arrow[ur, "v"] & & K_1(A_4) &
\end{tikzcd}$$

shows that $K_1(A) \simeq K_1(B)$.

However the inductive systems (A_i, φ_i) and (A_i, ψ_i) are not isomorphic in inj-$\mathcal{H}$. Indeed if there would exist a homotopy commutative diagram as in 5.3.5 a) then passing to K_1-groups we should find a commutative diagram of the form

$$\begin{tikzcd}
K_1(A_{i_1}) \arrow[rr, "x^{n_1}"] \arrow[dr, "z_1"'] & & K_1(A_{i_2}) \arrow[rr, "x^{n_2}"] \arrow[dr, "z_3"] & & K_1(A_{i_3}) \arrow[dr] & \\
& K_1(A_{j_1}) \arrow[rr, "y^{m_1}"'] \arrow[ur, "z_2"] & & K_1(A_{j_2}) \arrow[rr, "y^{m_2}"'] \arrow[ur, "z_4"] & & {}
\end{tikzcd}$$

with $z_i = \begin{pmatrix} a_i & b_i \\ 0 & c_i \end{pmatrix} \in M_2(\mathbf{Z})$ and $n_i, m_i \geq 2$. But the equalities $z_2 z_1 = x^{n_1}$, $z_3 z_2 = y^{m_1}$, $z_4 z_3 = x^{n_2}$ can not hold simultaneously since there do not exist $c_1, c_2, c_3 \in \mathbf{Z}$ such that $c_2 c_1 = 2^{n_1}$, $c_3 c_2 = 3^{m_1}$, $c_4 c_3 = 2^{n_2}$.

which is used to define $P^{\infty}(X) = \lim P^{k}(X)$

Dold and Thom [14] have introduced the notion of quasifibration which enable them to prove that $\pi_n(P^{\infty}(X)) = \tilde{H}_n(X,\mathbf{Z})$.

We shall study maps which are not quasifibrations but which have similar properties up to some dimension. Therefore it is natural to make the following

6.1.1. DEFINITION. A continuous map $p : E \longrightarrow B$ between topological Hausdorff spaces is called m-quasifibration, $(0, \leq m \leq \infty)$, if for all points $b \in B$ and $e \in p^{-1}(b)$ the induced maps $p_* : \pi_q(E,p^{-1}(b),e) \rightarrow \pi_q(B,b)$ are isomorphisms for $0 \leq q \leq m-1$ and epimorphisms for $q = m$. (For $m = \infty$ one obtain the definition of the quasifibration).

A careful inspection of the proof of Satz 2.2 in [14] shows that one has the following

6.1.2. THEOREM. Let $0 \leq m \leq \infty$, $p : E \rightarrow B$ continuous and $\mathcal{U} = (U_i)_{i \in L}$ an open covering of B such that

e) For each $i \in L$, $p : p^{-1}(U_i) \rightarrow U_i$ is a m-quasifibration

b) Each nonvoid intersection $U_i \cap U_j$ can be written as a union of elements in $\mathcal{U}$.

Then p is a m-quasifibration.

6.1.3. REMARK. If $p : E \rightarrow B$ is a m-quasifibration then it follows from the homotopy sequence of the pair $p^{-1}(b) \subset E$ that there is an exact sequence:

$$\pi_m(p^{-1}(b)) \rightarrow \pi_m(E) \rightarrow \pi_m(E,p^{-1}(B)) \rightarrow \pi_{m-1}(p^{-1}(b)) \rightarrow \pi_{m-1}(E) \rightarrow \pi_{m-1}(B) \rightarrow$$
$$\rightarrow \pi_{m-2}(p^{-1}(b)) \rightarrow \dots$$

Therefore if $p : E \rightarrow B$ is a m-quasifibration with connected E and $p^{-1}(b)$ is m-connected for some $b \in B$, then p is a m-equivalence.

In this section we shall meet several times continuous maps $p : E \rightarrow B$ which are surjective and satisfy the following conditions

6.1.4. a) B is locally contractible, i.e. each $x \in B$ has a fundamental system of open neighbourhoods $(U_j)_{j \geq 1}$ together with continuous homotopies

$h_j : U_j \times I \rightarrow U_j$ such that

The first left vertical arrow is an isomorphism for $q \leq m-1$ and an isomorphism for $q=m$ by 6.1.4. c). The second vertical arrow is an isomorphism for all q since $p^{-1}(U_j^x)$ is 0-connected. The bottom orizontal arrow is an isomorphism by 6.1.4 b). Since the upper sequence is exact it follows that $\pi_q(p^{-1}(U_j^x), p^{-1}(x'),y') = 0$ for all $0 \leq q \leq m$.

6.1.6. We need also the following formalism which enable us to describe the stratification of $\mathbf{Hom}_1(C(X), M_k)$ given by the multiplicities of the proper values of the homomorphisms.

Let $\mathcal{L}$ be the set of all disjoint partitions of the set $A = \{1,2,\dots,k\}$. A generic element $I \in \mathcal{L}$ is described by $I = (I_1,\dots,I_m)$ so that A is the union of I_j and $I_i \cap I_j = \phi$ whenever $i \neq j$. $\mathcal{L}$ becomes a lattice with the order:

$J \leq I$ iff the partition I is finer than J ,

The symmetric groups $\mathfrak{S}_k$ acts by order preserving automorphisms on $(\mathcal{L},\leq)$ by the rule:

$$\sigma(I) = \big(\sigma(I_1),\dots,\sigma(I_m)\big), \quad \sigma\in\mathfrak{S}_k.$$

If $\underline{x} = (x_1,\dots,x_k) \in X^k$ then $I(\underline{x})$ denotes the partition of A which corresponds to the following equivalence relation:

if $i,j \in A$ then $i \sim j$ iff $x_i = x_j$. That is i and j are contained in the same I_r iff $x_i = x_j$.

Let $e_1,\dots,e_k$ be the canonical minimal projections in M_k; $e_ie_j = \delta_{ij}e_i$ and $e_1+\dots+e_k = 1_k$. For $I = (I_1,\dots,I_m) \in \mathcal{L}$ we define

$$e(I_r) = \sum_{i\in I_r} e_i$$

Each $\sigma \in \mathfrak{S}_k$ gives an isometric endomorphism of $\mathbf{C}^k$

$$\sigma(\lambda_1,\dots,\lambda_k) = (\lambda_{\sigma(1)},\dots,\lambda_{\sigma(k)})$$

If σ is regarded as an element of $U(k)$ we have $\sigma^* e_i \sigma = e_{\sigma(i)}$ hence $\sigma^* e(I_r)\sigma = e(\sigma(I_r))$. Let $A(I)$ be the C^*-algebra of those elements in M_k which commute with all the projections $e(I_1),\dots,e(I_n)$. We let $\mathbf{U}(I)$ denote the unitary group of $A(I)$. If $J \leq I$ then $U(I) \subset U(J)$. Therefore there is a natural map

$$U(k)/U(I) \longrightarrow U(k)/U(J)$$

That is $p[\underline{x},u] = [\underline{x}]$ and this formula is correct by Proposition 6.1.8. Note that $p^{-1}[\underline{x}] \cong U(k)/U(I(\underline{x}))$.

Since $F^k(X)$ is homeomorphic to $F^k_o(X) = \mathbf{Hom}(C_o(X), M_k)$, p will induce a map $\eta : \mathbf{Hom}(C_o(X), M_k) \longrightarrow P^k(X)$ which naturally extends to $\eta : \underrightarrow{\mathbf{lim}}\mathbf{Hom}(C_o(X)\, M_k) \longrightarrow$ $\longrightarrow \underrightarrow{\mathbf{lim}}P^k(X) = P^{\infty}(X)$.

Now the natural embedding $\underrightarrow{\lim}\, F^k_o(X) \longrightarrow F(X) = \mathbf{Hom}\,(C_o(X), \mathbb{K})$ is a homotopy equivalence so that up to homotopy we have a map $\eta : \mathbf{Hom}(C_o(X), \mathbb{K}) \longrightarrow P^{\infty}(X)$.

6.1.10. THEOREM. The map $p : F^k(X) \longrightarrow P^k(X)$ is a 3-quasifibration for any $1 \leq k \leq \infty$.

Proof. We shall prove that p verifies the hypotheses of Proposition 6.1.4. for $m = 2$. Let $\underline{x} = (x_1, \ldots, x_k) \in X^k$ and choose $V_o = V_1 \times \ldots \times V_k$, (depending on $\underline{x}$), such that V_i are open neighbourhoods of x_i small enough to assure that $x_i \neq x_j$ iff $V_i \cap V_j = \phi$, and $x_i = x_j$ iff $V_i = V_j$. These choices imply that for any $\sigma_1, \sigma_2 \in \mathfrak{S}_k$, $\sigma_1(V_o) \cap \sigma_2(V_o) \neq \phi$ iff $\cancel{\sigma_1 = \sigma_2}$ $\sigma_1(I(\underline{x})) = \sigma_2(I(\underline{x}))$. Moreover if $\underline{y} \in \sigma(V_o)$ for some $\sigma \in \mathfrak{S}_k$, then $I(\sigma(x)) \leq I(y)$. Since X is locally contractible, shrinking the V_i if necessary, we can find $h^o : V_o \times I \longrightarrow I$, $h^o = (h_1, \ldots, h_k)$, where $h_i : V_i \times I \longrightarrow V_i$ with $h_i = h_j \Leftrightarrow V_i = V_j$ are such that

$$h^o_o = id(V_o),\text{ the image of } h^o_1 = \{x\} \text{ and } h^o_t(x) = x \text{ for all } t \in I.$$

Starting with h^o we define for each $\sigma \in \mathfrak{S}_k$, $h^o_\sigma : \sigma(V_o) \times I \longrightarrow \sigma(V_o)$ by $h^o_\sigma = (h_{\sigma(1)}, \ldots, h_{\sigma(k)})$. Finally we let V be the union of all $\sigma(V_o)$ over $\sigma \in \mathfrak{S}_k$ and define the homotopy $h : V \times I \longrightarrow V$ such $h|\sigma(V_o) = h^o_\sigma$. By construction $h_t(\sigma(\underline{y})) = \sigma(h_t(\underline{y}))$ and $I(h_t(\underline{y})) \leq I(\underline{y})$ for $\underline{y} \in V$ and $t \in I$.

After these preparations we can reach the condition 6.1.4. a), b), c), d).

a) Let V as above and let $U = \psi_o(V)$ be the corresponding open neighbourhood of $[\underline{x}]$ in $P^k(X)$. We define $\tilde{h} : U \times I \longrightarrow U$ by $\tilde{h}_t([\underline{y}]) = [h_t(\underline{y})]$ and the couple $(U, \tilde{h})$ satisfies 6.1.4. a).

b) Since V is $\mathfrak{S}_k$-equivariant $p^{-1}(U) = \psi(V \times U(k))$. We define $\tilde{H} : p^{-1}(U) \times I \longrightarrow p^{-1}(U)$ by $\tilde{H}_t[\underline{y},u] = [h_t(\underline{y}), u]$. Let us check that $\tilde{H}$ is well defined. If this result in full generality we shall prove in this paragraph only the following weaker result.

6.2.1. PROPOSITION. If $k \geq 2$ then $\pi_1(P^k(X)) \longrightarrow \pi_1(P^{k+1}(X))$ is an isomorphism. This proposition can be regarded as a first step to the main connectivity theorem since by Corollary 6.1.11. $\pi_1(P^k(X)) \simeq \pi_1(F^k(X))$. The proof of 6.2.1 requires certain preliminaries.

6.2.2. For $\ell, s \geq 1$ let $T_{\ell,s}$ denote the set of all $(a_1, \ldots, a_\ell) \in ([-1,1]^s)^\ell$ such that some a_i has at least one coordinate equal to ± 1. We define $D_{\ell,s}$ to be the image of $T_{\ell,s}$ in $P^\ell([-1,1]^s)$, i.e. $D_{\ell,s} = T_{\ell,s}/\mathfrak{S}_\ell$. Note that $D_{1,1} \simeq S^0$ and $D_{\ell,s}$ is connected if $(\ell,s) \neq (1,1)$.

6.2.3. LEMMA. If $(\ell,s) \neq (1,1),(1,2)$, then $D_{\ell,s}$ is simply connected.

Proof. $T_{\ell,s}$ can be identified to $S^{\ell s-1}$ and the quotient map $\gamma : S^{\ell s-1} \to D_{\ell,s}$ is a 1-quasifibration. Since there are points $x \in D_{\ell,s}$ such that $\gamma^{-1}(x)$ is a singleton it follows that $\gamma_* : \pi_1(S^{\ell s-1}) \longrightarrow \pi_1(D_{\ell,s})$ is onto.

6.2.4. Now we are going to describe $P^{k+1}(X) \setminus P^k(X)$ using the cell structure of X. Let $X = e_1 \cup e_2 \cup \ldots \cup e_N$ be a cell decomposition of X with $\dim e_i \leq \dim e_{i+1}$. Since we are interested into homotopy questions, we may assume that X has a single vertex e_1 and so $\dim e_2 \geq 1$. Recal that if $(x_1, \ldots, x_k) \in X^k$ then its class in $P^k(X)$ is denote by $[x_1, \ldots, x_k]$.

Now $P^{k+1}(X)$ has a decoposition

$$P^{k+1}(X) = P^k(X) \cup \bigcup_{\substack{j_1,\ldots,j_{k+1} \\ 1 \leq j_i \leq N}} [e_{j_1} \times e_{j_2} \times \ldots \times e_{j_{k+1}}]$$

Since $[e_{j_1} \times \ldots \times e_{j_{k+1}}] = [e_{j_{\sigma(1)}} \times \ldots \times e_{j_{\sigma(k+1)}}]$ for all $\sigma \in \mathfrak{S}_{k+1}$ we have

$$\bigcup_{\substack{j_1,\ldots,j_{k+1} \\ 1 \leq j_i \leq N}} [e_{j_1} \times \ldots \times e_{j_{k+1}}] = \bigcup_{1 \leq j_1 \leq \ldots \leq j_{k+1} \leq N} [e_{j_1} \times \ldots \times e_{j_{k+1}}]$$

$\mathfrak{D}\, E(J) = \mathop{\ast}\limits_{i=1}^{r} D_{l(i),d(i)}$ is simply connected provided that $k \geq 2$. Therefore $\mathfrak{D}\, E(J)$ is homotopy equivalent to a CW-complex with a single vertex and no cells of dimension one [30]. This shows that up to homotopy $F^{k+1}(X)$ is obtained by attaching to $F^k(X)$ cells of dimension ≥ 3 and this does not change the fundamental group.

6.3. THE EMBEDDING $F^k(S^1 \vee \ldots \vee S^1) \to F^{k+1}(S^1 \vee \ldots \vee S^1)$.

Throughout this paragraph X will denote a finite cluster of standard circles. The main result here is that the embedding

$$F^k(X) \to F^{k+1}(X)$$

is a $2[\frac{k}{3}]$-equivalence (not depending on the number of the circles entering in X).

6.3.1. Let $X = S_1 \vee \ldots \vee S_m$ where each $S_i \simeq S^1$ and let $\eta_i : S^1 \to X$ denote the inclusion onto the i^{th} factor. In order to analyse $F^k(X)$ it is useful to consider the following filtration

$$F^k(X)_0 \subset \ldots \subset F^k(X)_\ell \subset F^k(X)_{\ell+1} \subset \ldots \subset F^k(X) \quad 0 \leq \ell \leq k$$

where $F^k(X)_\ell$ consists of those $[\underline{x},e] \in F^k(X)$ for which $\underline{x} = (x_1, \ldots, x_k)$ has at most ℓ coordinates x_i which are not equal to x_0. Here the base point x_0 is choose to be the common point of the circles S_i in X.

For $0 \leq \ell \leq k$ let $A(\ell)$ denote the set of all ordered multi-indices $\underline{a} = (a_1, \ldots, a_m)$ such that $a_i \in \mathbb{N} \setminus \{0\}$ and $\sum a_i = \ell$. If $\underline{a} \in A(\ell)$ then we shall denote by $F(\underline{a},\ell)$ the set of those homomorphisms in $F_k(X)$ which for any $1 \leq i \leq m$ have exactly a_i proper values (counted with multiplicities) belonging to $S_i \setminus \{x_0\}$.

It is easily seen that

$$F^k(X)_\ell \setminus F^k(X)_{\ell-1} = \bigcup_{\underline{a} \in A(\ell)} F(\underline{a},\ell)$$

gives the decomposition of $F^k(X)_\ell \setminus F^k(X)_{\ell-1}$ into its connected components.

6.3.2. Let $k \geq 1$. It is useful to define $F^k(Y)$ even for noncompact spaces Y. Having in mind Proposition 6.1.8 we shall define $F^k(Y)$ to be the space $Y^k \times U(k)$ factorized to the equivalence relation:

$$\gamma(f) = f(x_o)p_o + \sum_{j=0}^{m} f(\gamma_j(0))p_i \text{ for any } f \in C(X).$$

The image of $s_{\underline{a}}$ can be identified with a smooth submanifold of codimension $\sum_{i=1}^{m} a_i^2$ in $F(\underline{a},\ell)$ denoted by $\operatorname{im} s_{\underline{a}}$.

The previous facts enable us to prove the following

6.3.4. PROPOSITION. The embedding $F^k(X)_{\ell-1} \to F^k(X)_\ell$ is a $(\ell-1)$-equivalence, $k \geq 2$, $\ell \geq 1$.

Proof. The embedding

$$F^k(X)_{\ell-1} = F^k(X)_\ell \setminus \bigcup_{\underline{a}\in A(\ell)} F(\underline{a},\ell) \hookrightarrow F^k(X)_\ell \setminus \bigcup_{a\in A(\ell)} \operatorname{im} s_{\underline{a}}$$

admits a deformation retract so it is a homotopy equivalence. To have a good image for this deformation retract one can look at the following topologically equivalent situation. Consider a hermitian vector bundle E together with a continuous map $f : SE = \{v \in E : \|v\| = 1\} \to Y$ and the inclusion map $SE \hookrightarrow BE = \{v \in E : \|v\| \leq 1\}$. Then there is a deformation retract for the embedding of Y in $Y \cup_f BE - \{\text{zero section}\}$ which acts on BE-$\{$the zero section$\}$ by pushing out the vectors $(t,v) \longmapsto v/(t\|v\|+1-t)$.

Consequently it is enough to look at the map

$$F^k(X)_\ell \setminus \bigcup_{a\in A(\ell)} \operatorname{im} s_{\underline{a}} \hookrightarrow F^k(X)_\ell$$

Now each $\operatorname{im} s_{\underline{a}}$ has an open neighbourhood $V_{\underline{a}}$ with smooth differential structure such that $\operatorname{im} s_{\underline{a}}$ has codimension $\sum a_j^2 \geq \sum a_j = \ell$ in $V_{\underline{a}}$. Therefore we may use the aproximation theorem of continuous maps by differentiable maps and the Transversality theorem [20] to see that the above inclusion is a $(\ell-1)$-equivalence

6.3.5. let $\ell(k) = [\frac{2k}{3}]$. By proposition 6.3.4 each of the following inclusions

$$F^k(X)_{\ell(k)} \subset F^k(X)_{\ell(k)+1} \subset \dots \subset F^k(X)_{k-1} \subset F^k(X)$$

is an $\ell(k)$-equivalence so that $F^k(X)_{\ell(k)} \hookrightarrow F^k(X)$ is an $\ell(k)$-equivalence. We shall consider the following commutative diagram

Since the embedding $U(k) \to U(k+1)$ is a $2k$-equivalence and $k \geq m_t \geq k - \ell(k+1)$ we can apply the five lemma to obtain that $p_0^{-1}[\underline{x}] \to p_1^{-1}[\underline{x}]$ is a $2\,(k-\ell(k+1))$-equivalence. The proof is complete since $k - \ell(k+1) = k - [(2(k+1))/3] = [\frac{k}{3}]$.

6.3.7. LEMMA. If $k \geq 2$ then each map in the following diagram

$$\begin{tikzcd} \pi_1\left(F^k(X)_{\ell(k+1)}\right) \arrow[rr, "\alpha_*"] \arrow[dr, "p_{0*}"'] & & \pi_1\left(F^{k+1}(X)_{\ell(k+1)}\right) \arrow[dl, "p_{1*}"] \\ & \pi_1\left(P^{\ell(k+1)}(X)\right) & \end{tikzcd}$$

is an isomorphism.

Proof. The result follows from Corollary 6.1.11 and Propositions 6.2.1, 6.3.5,

6.3.8. Let $f : (A,a) \to (B,b)$ be a continuous map between topological spaces such that $f_* : \pi_1(A,a) \to \pi_1(B,b)$ is an isomorphism. Assume that A,B are semi-locally simply connected, connected and locally path-connected and let $\omega_A : (\tilde{A},\tilde{a}) \to (A,a)$, $\omega_B : (\tilde{B},\tilde{b}) \to (B,b)$ be realizations of their universal covering spaces. If $\tilde{f} : (\tilde{A},\tilde{a}) \to (\tilde{B},\tilde{b})$ is the unique continuous lifting of f, $\omega_B \circ \tilde{f} = f \circ \omega_A$, then it is easy to check that for every $\tilde{x} \in \tilde{B}$ the restriction of ω_A to $\tilde{f}^{-1}(\tilde{x})$ gives an homeomorphism between $\tilde{f}^{-1}(\tilde{x})$ and $f^{-1}(\omega_B(\tilde{x}))$.

6.3.9. In order to simplify notation we let $A = F^k(X)_{\ell(k+1)}$, $B = F^{k+1}(X)_{\ell(k+1)}$, $C = P^{\ell(k+1)}(X)$. Consequently the diagram from 6.3.5. becomes

$$\begin{tikzcd} A \arrow[rr, "\alpha"] \arrow[dr, "p_0"'] & & B \arrow[dl, "p_1"] \\ & C & \end{tikzcd}$$

If we pass at the universal covering spaces as in 6.3.8 we get the following commutative diagram

$$\begin{tikzcd} \tilde{A} \arrow[rr, "\tilde{\alpha}"] \arrow[dr, "\tilde{p}_0"'] & & \tilde{B} \arrow[dl, "\tilde{p}_1"] \\ & C & \end{tikzcd}$$

maps of $\mathfrak{S}_k$ on K_1.

2. There is a triangulation $\varphi_2 : |K_2| \longrightarrow U(k)$ such that the subspaces U(I) (see 6.1.6) correspond to some simplicial subcomplexes and the action of $\mathfrak{S}_k$ on U(k), $(\sigma,u) \longmapsto \sigma u \sigma^*$, is induced by some action by simplicial maps of $\mathfrak{S}_k$ on K_2 (see A.Verona, Stratified mappings-structure and triangulability, Lecture Notes in Math. No.

3. Let $p : K \longrightarrow K_o$ be a simplicial map between finite simplicial complexes and let $L \subset K$ be a simplicial subcomplex such that $p(L)$ = some vertex of K_o. Then there are subdivisions K' of K and K'_o of K_o and a simplicial complex K_L such that there exists a commutative diagram

$$\begin{tikzcd} K' \arrow[rr, "p'"] \arrow[dr] & & K'_o \\ & K_L \arrow[ur, "p_L"'] & \end{tikzcd}$$

(p' induced by p)

with the property that the map $|p_L| : |K_L| \longrightarrow |K'_o|$ can be identified with $|p|\hat{} : |K| /_{|L|} \longrightarrow |K_o|$ ($|p|\hat{}$ is induced by $|p|$).

4. Let $p : K \longrightarrow K_o$ be a simplicial map between finite simplicial complexes and let G be a finite group which acts on K by simplicial maps such that $p(g.v) = p(v)$ for each vertex $v \in K$ and $g \in G$. Then there are suitable divisions K' of K and K'_o of K and a simplicial complex K_G such that there exists a commutative diagram of simplicial complexes:

$$\begin{tikzcd} K' \arrow[rr, "p'"] \arrow[dr] & & K'_o \\ & K_G \arrow[ur, "p_G"'] & \end{tikzcd}$$

with the property that the map $|p_G| : |K_G| \longrightarrow |K'_o|$ can be identified with $|p|\hat{} : |K| /G \longrightarrow |K_o|$; ($|p|\hat{}$ is induced by $|p|$).

5. Let $p : K \longrightarrow K'$ be a simplicial map between finite simplicial complexes. There is a commutative diagram of simplicial complexes

Proof. Consider the following commutative diagram

$$\begin{tikzcd}
F^{k+1}(X)_{\ell(k+1)} \arrow[r, "{[2\frac{(k+1)}{3}]}"] & F^{k+1}(X) \\
F^{k}(X)_{\ell(k+1)} \arrow[u, hook, "{2[\frac{k}{3}]}"] \arrow[r, "{[2\frac{(k+1)}{3}]}"] & F^{k}(X) \arrow[u, hook]
\end{tikzcd}$$

in which the orizontal arrows are $[(2(k+1)/3)]$-equivalences as it was noticed in 6.3.5. The left arrow is a $2[\frac{k}{3}]$-equivalence by Proposition 6.3.12. Now the result follows since $[2(k+1)/3)] \geq 2[\frac{k}{3}]$

6.4. THE MAIN CONNECTIVITY THEOREM

The last step towards the central result of this section is the following Mayer-Vietoris type result.

6.4.1. THEOREM. Let A,B be connected, locally connected and semilocally simply connected spaces and $f : A \to B$ a continuous map. Let $(U_i)_i$, $(V_i)_i$, $1 \leq i \leq r$ be open coverings of A and respective B and define

$$U_I = \bigcap_{i\in I} U_i \text{ and } V_I = \bigcap_{i\in I} V_i \text{ for } I \subset \{1,2,\dots,r\}$$

Suppose that there is $m \geq 1$ such that for each nonvoid $I \subset \{1,2,\dots,r\}$ the following conditions are fulfiled:

a) U_I and V_I are nonvoid and connected

b) The embeddings $U_I \hookrightarrow A$ and $V_I \hookrightarrow B$ are 1-equivalences

c) $f(U_I) \subset V_I$

d) $f : U_I \to V_I$ is a m-equivalence

in order to obtain that $\pi_1(\tilde{f}\,|\,\tilde{U}_I)$ is an isomorphism. Once we know that $\tilde{f}\,|\,\tilde{U}_I$ is a m-equivalence we can apply the Whitehead theorem to find that $\tilde{f}_* : H_q(\tilde{U}_I) \to H_q(\tilde{U}_I)$ is an isomorphism for $q < m$ and an epimorphism for $q = m$. Now the usual Mayer-Vietoris argument gives the same conclusions for $\tilde{f}_* : H_q(\tilde{A}) \to H_q(\tilde{B})$. Since $\tilde{A}, \tilde{B}$ are simply connected we can apply the converse of Whitehead theorem to get that $\tilde{f} : \tilde{A} \to \tilde{B}$ is a m-equivalence. Since we know that $\pi_1(f)$ is an isomorphism this shows that $f : A \to B$ is an m-equivalence

6.4.2. THEOREM. The natural embedding

$$\alpha_k : \mathbf{Hom}_1(C(X), M_k) \to \mathbf{Hom}_1(C(X), M_{k+1}), \ k \geq 3,$$

is a $2[\frac{k}{3}]$-homotopy equivalence for any finite connected CW-complex X.

Proof. For $n \geq 1$ and $r \geq 0$ define $W(n,r)$ as follows

$W(n,0)$ is the class of all finite connected CW-complexes of dimension $n-1$.

$W(n,r)$ is the class of all finite connected CW-complexes of dimension n having exactly $r \geq 1$ cells of dimension n.

If $X \in W(2,0)$ then $\mathbf{dim}(X) = 1$ hence X is homotopic to a finite cluster of circles. Therefore $\alpha_k : F^k(X) \to F^{k+1}(X)$ is a $2[\frac{k}{3}]$-equivalence by Theorem 6.3.13. The theorem will be proved if we will show the following implication which allow an inductive argument.

If $\alpha_k : F^k(X) \to F^{k+1}(X)$ is a $2[\frac{k}{3}]$-equivalence for any space in $W(n,r-1)$, $r \geq 1$, then the same is true for any space in $W(n,r)$. Therefore let us fix $n \geq 2$, $r \geq 1$ and $X \in W(n,r)$. Let e be a cell of dimension n in X and choose $(k+2)$-distinct points in the interior of e, no one of them being equal to x_0. For any nonvoid $I \subset \{1,2,\ldots,k+2\}$ let $\alpha_I = \{x_i : i \in I\}$ and define U_I(resp. V_I) to be the set of all homeomorphisms in $F^k(X)$ (resp. $F^{k+1}(X)$) which have no proper values in α_I or equivalently $U_I = F^k(X \setminus \alpha_I)$ and $V_I = F^{k+1}(X \setminus \alpha_I)$. Note that $U_I = \bigcap_{i \in I} U_i$ and $V_I = \bigcap_{i \in I} V_i$ and let $U_i = U_{\{i\}}$ $V_i = V_{\{i\}}$. We want to show that U_i, V_i and f satisfy the hypotheses of Theorem 6.4.3. First of all it is clear that U_i, V_i are open and each (intersection U_I, V_I is nonvoid. Since any