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Approximately Unitarily Equivalent Morphisms and Inductive Limit C^*-Algebras*

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Abstract. It is shown that two unital $*$-homomorphisms from a commutative C^*-algebra $C(X)$ to a unital C^*-algebra B are stably approximately unitarily equivalent if and only if they have the same class in the quotient of the Kasparov group $KK(C(X), B)$ by the closure of zero. A suitable generalization of this result is used to prove a classification result for certain inductive limit C^*-algebras.

Key words: $*$-homomorphisms, unitary equivalence, Kasparov groups, C^*-algebras, real rank zero.

0. Introduction

Let X be a compact metrizable space and let B be a unital C^*-algebra. We prove that the $*$-homomorphisms from $C(X)$ to B are classified up to stable approximate unitary equivalence by K-theory invariants. This result is used to obtain a classification theorem for certain real rank zero inductive limits of homogeneous C^*-algebras. The classification of various classes of C^*-algebras of real rank zero in terms of invariants based on K-theory presupposes a passage from algebraic objects to geometric objects (see [Ell], [Li1], [EGLP1], [EG], [BrD], [G], [Rø], [LiPh]). An underlying idea of this paper is that this passage can be done by using approximate morphisms. K-theory becomes a source of approximate morphisms thanks to the realization of K-theory in terms of asymptotic morphisms, due to A. Connes and N. Higson [CH]. The main results of the paper are Theorems A and B, below.

By the universal coefficient theorem for the Kasparov KK-groups [RS], $\mathrm{Ext}(K_*(C(X)), K_{*-1}(B))$ is a subgroup of $KK(C(X), B)$. Following [Rø], we let $KL(C(X), B)$ denote the quotient of $KK(C(X), B)$ by the subgroup of pure extensions in $\mathrm{Ext}(K_*(C(X)), K_{*-1}(B))$. If $K_*(C(X))$ is isomorphic to a direct sum of cyclic groups, then $KL(C(X), B)$ coincides with $KK(C(X), B)$.

THEOREM A. *Let X be a compact metrizable space. Let B be a unital C^*-algebra and let $\varphi, \psi: C(X) \to B$ be two unital $*$-homomorphisms. The following assertions are equivalent.*

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(a) $[\varphi] = [\psi]$ *in* $KL(C(X), B)$.
(b) *For any finite subset* $F \subset C(X)$ *and any* $\varepsilon > 0$ *there exist* $k \in \mathbb{N}$, *a unitary* $u \in U_{k+1}(B)$ *and points* $x_1, \ldots, x_k$ *in* X *such that*

$$\|u \operatorname{diag}(\varphi(a), a(x_1), \ldots, a(x_k))u^* - \operatorname{diag}(\psi(a), a(x_1), \ldots, a(x_k))\| < \varepsilon$$

for all $a \in F$.

Two $*$-homomorphisms satisfying the condition (b) of Theorem A are said to be stably unitarily equivalent. If condition (b) is satisfied without the $*$-homomorphism $\eta(a) = \operatorname{diag}(a(x_1), \ldots, a(x_k))$, then we say that φ and ψ are approximately unitarily equivalent. Theorem A generalizes certain results in [Li1-3] and [EGLP]. If φ and ψ are $*$-monomorphisms and B is a purely infinite, simple C^*-algebra, then one can drop the $*$-homomorphism η from condition (b), see Theorem 1.7. The apparition of pure extensions in this context is related to the phenomenon of quasidiagonality, see [Br], [Sa], [BrD]. The author believes that future generalizations of Theorem A should be based on Voiculescu's noncommutative Weyl-Von Neumann Theorem [V].

We use a suitable version of Theorem A for asymptotic morphisms to prove the following theorem.

THEOREM B. *Let* A, B *be two simple* C^**-algebras of real rank zero. Suppose that* A *and* B *are inductive limits*

$$A = \varinjlim A_n \text{ and } B = \varinjlim B_n, \; A_n = \bigoplus_{i=1}^{k_n} M_{[n,i]}(C(X_{n,i})),$$

$$B_n = \bigoplus_{i=1}^{l_n} M_{\{n,i\}}(C(Y_{n,i})).$$

Suppose that $X_{n,i}, Y_{n,i}$ *are finite* CW *complexes,* $K^0(X_{n,i})$ *and* $K^0(Y_{n,i})$ *are torsion free and* $\sup_{n,i}\{\dim(X_{n,i}), \dim(Y_{n,i})\} < \infty$. *Then* A *is isomorphic to* B *if and only if*

$$(K_*(A), K_*(A)_+, \Sigma_*(A)) \cong (K_*(B), K_*(B)_+, \Sigma_*(B)).$$

Theorem B is proved by showing that the C^*-algebras A and B are isomorphic to certain inductive limits of subhomogeneous C^*-algebras with one-dimensional spectrum that are known to be classified by ordered K-theory [Ell]. In the process we use the semiprojectivity of the dimension-drop C^*-algebras proved by T. Loring [Lo-1]. In a remarkable recent paper [EG], Elliott and Gong classify the simple C^*-algebras of real rank zero that are inductive limits of homogeneous C^*-algebras with spectrum 3-dimensional finite CW complexes. Roughly speaking, this classification result is achieved in two stages. The first stage corresponds to passing from K-theory to homotopy and uses the connective KK-theory introduced by A. Nemethi and the author [DN]. The second stage corresponds to showing that homotopic $*$-homomorphisms are stably approximately unitarily equivalent. A similar approach is taken in [G]. In this paper, we show that these techniques can be combined with techniques of approximate morphisms to produce classification results for inductive limits of homogeneous C^*-algebras with higher dimensional spectra. A key tool of our approach is a suspension theorem in E-theory due to T. Loring and the author [DL], (see also [D2]). The idea of using homotopies of asymptotic morphisms in the study of approximate unitary equivalence of $*$-homomorphisms appears also in [LiPh]. The study of inductive limits of matrix algebras of continuous functions was proposed by E. G. Effros [Eff].

1. Approximately Unitarily Equivalent Morphisms and *K*-Theory

In this section we prove Theorem A and give an application for $*$-monomorphisms from $C(X)$ to purely infinite, simple C^*-algebras.

We need the following result from [EGLP].

THEOREM 1.1. [EGLP] *Let X be the cone over a compact metrizable space and let B be a unital C^*-algebra. For any finite subset $G \subset C(X)$, any $\varepsilon > 0$ and any unital $*$-homomorphism $\rho: C(X) \to B$, there are two unital $*$-homomorphisms with finite-dimensional image, $\eta: C(X) \to M_{s-1}(B)$ and $\xi: C(X) \to M_s(B)$, such that*

$$\|\mathrm{diag}(\rho(a), \eta(a)) - \xi(a)\| < \varepsilon$$

for all $a \in G$.

The following result is implicitly contained in [EG].

THEOREM 1.2. *Let X be a finite, connected CW complex. For any finite subset $F \subset C(X)$ and any $\varepsilon > 0$, there exist $r \in \mathbb{N}$, a unital $*$-homomorphism $\tau: C(X) \to M_{r-1}(C(X))$ and a unital $*$-homomorphism $\mu: C(X) \to M_r(C(X))$ with finite dimensional image, such that*

$$\|\mathrm{diag}(a, \tau(a)) - \mu(a)\| < \varepsilon$$

for all $a \in F$.

Proof. By [DN] there exist $R \in \mathbb{N}$ and a unital $*$-homomorphism $\sigma: C(X) \to M_{R-1}(C(X))$ such that $\psi: C(X) \to M_R(C(X))$, defined by $\psi(a) = \mathrm{diag}(a, \sigma(a))$ is homotopic to an evaluation map $a \mapsto a(x_0)1_R$. Let $\Psi: C(X) \to M_R(C(X \times [0,1]))$ be a $*$-homomorphism implementing a homotopy from ψ to the evaluation map $a \mapsto a(x_0)1_R$. Since $\Psi(a)(x,1) = a(x_0)1_R$ for all $a \in C(X)$, one can identify Ψ with a $*$-homomorphism $\Psi: C(X) \to M_R(C(\hat{X}))$ where $\hat{X} = X \times [0,1]/X \times \{1\}$ is the cone over X. Moreover, the natural inclusion

$X \cong X \times \{0\} \hookrightarrow \hat{X}$ induces a restriction $*$-homomorphism $\rho: M_R(C(\hat{X})) \to M_R(C(X))$ and ψ factors as $\psi = \rho\Psi$. Set $G = \Psi(F)$. Since $\hat{X}$ is contractible we can use Theorem 1.1 to produce unital $*$-homomorphisms

$$\eta: M_R(C(\hat{X})) \to M_{(s-1)R}(C(X)), \qquad \xi: M_R(C(\hat{X})) \to M_{sR}(C(X))$$

with finite-dimensional image such that

$$\|\mathrm{diag}(\rho(d), \eta(d)) - \xi(d)\| < \varepsilon$$

for all $d \in G$. Hence

$$\|\mathrm{diag}(\rho\Psi(a), \eta\Psi(a)) - \xi\Psi(a)\| < \varepsilon$$

for all $a \in F$. Set $r = sR$, $\tau(a) = \mathrm{diag}(\sigma(a), \eta\Psi(a))$ and $\mu(a) = \xi\Psi(a)$. Then

$$\|\mathrm{diag}(a, \tau(a)) - \mu(a)\| < \varepsilon$$

for all $a \in F$. □

DEFINITION 1.3. A C^*-algebra A is said to have property (H), if for any finite subset $F \subset A$ and any $\varepsilon > 0$, there exist $r \in \mathbb{N}$, a $*$-homomorphism $\tau: A \to M_{r-1}(A)$ and a $*$-homomorphism $\mu: A \to M_r(A)$ with finite-dimensional image such that

$$\|\mathrm{diag}(\tau(a), a) - \mu(a)\| < \varepsilon$$

for all $a \in F$.

It is clear that any finite dimensional C^*-algebra has property (H). Theorem 1.2 shows that if X is a finite CW complex, then $C(X)$ has property (H). It is not hard to see that the class of C^*-algebras with property (H) is closed under direct sums and tensor products.

For C^*-algebras C, D let $\mathrm{Map}(C, D)$ denote the set of all linear, contractive, completely positive maps from C to D. If G is a finite subset of C and $\delta > 0$ we say that $\varphi \in \mathrm{Map}(C, D)$ is *δ-multiplicative* on G if $\|\varphi(ab) - \varphi(a)\varphi(b)\| < \delta$ for all $a, b \in G$.

LEMMA 1.4. *Let A be a C^*-algebra with property (H). Let $\varepsilon > 0$ and let $F \subset A$ be a finite set. There are $\hat{\delta} > 0$ and a finite subset $\hat{F} \subset A$ such that if B is any unital C^*-algebra and $\varphi_0, \varphi_1, \ldots, \varphi_n$ is a sequence of maps in* $\mathrm{Map}(A, B)$ *such that φ_j is $\hat{\delta}$-multiplicative on $\hat{F}$ for $j = 0, \ldots, n$, then there exist $k \in \mathbb{N}$, a $*$-homomorphism $\eta: A \to M_k(B)$ with finite dimensional image and a unitary $u \in U_{k+1}(B)$ such that*

$$\|u\,\mathrm{diag}(\varphi_0(a), \eta(a))u^* - \mathrm{diag}(\varphi_n(a), \eta(a))\|$$

$$< \varepsilon + \max_{a\in F}\max_{0\leqslant j\leqslant n-1} \|\varphi_{j+1}(a) - \varphi_j(a)\|$$

for all $a \in F$.

Proof. For given $F \subset A$ and $\varepsilon > 0$, let $r \in \mathbb{N}$, τ and μ be as in Definition 1.3. Then $D = \mu(A)$ is a finite dimensional C^*-subalgebra of $M_r(A)$. By elementary perturbation theory (see [Bra]), there is a finite subset G of D containing $\mu(F)$ and there is $\delta > 0$ such that whenever E is a C^*-algebra and $\Psi \in \mathrm{Map}(D, E)$ is δ-multiplicative on G, there exists a $*$-homomorphism $\Psi' \colon D \to E$ satisfying $\|\Psi'(d) - \Psi(d)\| < \varepsilon$ for all $d \in G$.

For $s \in \mathbb{N}$ set $\varphi_{s,j} = \varphi_j \otimes \mathrm{id}_s \colon M_s(A) \to M_s(B)$. It is easily seen that one can find $\hat{\delta} > 0$ and $\hat{F} \subset A$ finite such that if $\varphi_j \in \mathrm{Map}(A, B)$ is $\hat{\delta}$-multiplicative on $\hat{F}$ then $\varphi_{r,j}$ is δ-multiplicative on G. Since $\varphi_{r,j}$ is δ-multiplicative on G, there is a $*$-homomorphism $\psi_j \colon D \to M_r(B)$ such that $\|\varphi_{r,j}(d) - \psi_j(d)\| < \varepsilon$ for all $d \in G$ and $j = 0, \dots, n$.

Define $L, L' \colon A \to M_{nr}(B)$ by

$$L = \mathrm{diag}(\varphi_{r-1,0}\tau, \varphi_0, \varphi_{r-1,1}\tau, \varphi_1, \dots, \varphi_{r-1,n-1}\tau, \varphi_{n-1}),$$

$$L' = \mathrm{diag}(\varphi_0, \varphi_{r-1,0}\tau, \varphi_1, \varphi_{r-1,1}\tau, \dots, \varphi_{n-1}, \varphi_{r-1,n-1}\tau).$$

Note that L is unitarily equivalent to L'. Thus, there is a permutation unitary $u \in U_{nr+1}(B)$ such that

$$u\,\mathrm{diag}(L', \varphi_n)u^* = \mathrm{diag}(\varphi_n, L). \tag{1}$$

Let

$$\lambda = \max_{a\in F}\max_{0\leqslant j\leqslant n-1} \|\varphi_{j+1}(a) - \varphi_j(a)\|$$

Since $\|\varphi_{j+1}(a) - \varphi_j(a)\| \leqslant \lambda$ for all $a \in F$

$$\|\mathrm{diag}(\varphi_0(a), L(a)) - \mathrm{diag}(L'(a), \varphi_n(a))\|$$

$$\leqslant \max_j \|\varphi_{j+1}(a) - \varphi_j(a)\| = \lambda. \tag{2}$$

Using (1) and (2), we obtain

$$\|u\,\mathrm{diag}(\varphi_0(a), L(a))u^* - \mathrm{diag}(\varphi_n(a), L(a))\| \leqslant \lambda \tag{3}$$

for all $a \in F$.

On the other hand,

$$\begin{aligned}
&\|L(a) - \operatorname{diag}(\varphi_{r,0}\mu(a), \dots, \varphi_{r,n-1}\mu(a))\| \\
&\quad = \|\operatorname{diag}(\varphi_{r,0}(\tau(a) \oplus a - \mu(a)), \dots, \varphi_{r,n-1}(\tau(a) \oplus a - \mu(a)))\| \\
&\quad \leqslant \|\tau(a) \oplus a - \mu(a)\| < \varepsilon
\end{aligned} \tag{4}$$

for all $a \in F$, since $\varphi_{r,j}$ are norm decreasing. Note that $\|\varphi_{r,j}\mu(a) - \psi_j\mu(a)\| < \varepsilon$ for all $a \in F$ since $\mu(F) \subset G$ and $\|\varphi_{r,j}(d) - \psi_j(d)\| < \varepsilon$ for all $d \in G$. This implies

$$\|\operatorname{diag}(\varphi_{r,0}\mu(a), \dots, \varphi_{r,n-1}\mu(a)) - \operatorname{diag}(\psi_0\mu(a), \dots, \psi_{n-1}\mu(a))\| < \varepsilon \tag{5}$$

for all $a \in F$. The $*$-homomorphism defined by $\eta = \operatorname{diag}(\psi_0\mu, \dots, \psi_{n-1}\mu)$ has finite-dimensional image. Using (4) and (5) we obtain

$$\|L(a) - \eta(a)\| < 2\varepsilon \tag{6}$$

for all $a \in F$. Combining (3) and (6), we find

$$\|u \operatorname{diag}(\varphi_0(a), \eta(a))u^* - \operatorname{diag}(\varphi_n(a), \eta(a))\| < 4\varepsilon + \lambda$$

for all $a \in F$. □

Suppose that the C^*-algebra A is unital. Suppose that the $*$-homomorphism μ from Definition 1.3 and the maps φ_j are unital. Then it easily seen that one can arrange for the $*$-homomorphism η to be unital.

The notion of asymptotic morphism due to Connes and Higson led to a geometric realization of E-theory [CH]. Let A, B be separable C^*-algebras. Roughly speaking, an asymptotic morphism from A to B is a continuous family of maps $\varphi_t\colon A \to B, t \in T = [1, \infty)$, which satisfies asymptotically the axioms for $*$-homomorphisms. A homotopy of asymptotic morphisms $\varphi_t, \psi_t\colon A \to B$ is given by an asymptotic morphism $\Phi_t\colon A \to B[0,1]$ such that $\Phi_t^{(0)} = \varphi_t$ and $\Phi_t^{(1)} = \psi_t$. Here $B[0,1]$ denotes the C^*-algebra of continuous functions from the unit interval to B. The homotopy classes of asymptotic morphisms from A to B are denoted by $[[A, B]]$ and the class of φ_t by $[[\varphi_t]]$. Two asymptotic morphisms φ_t, ψ_t are said equivalent if $\varphi_t(a) - \psi_t(a) \to 0$, as $t \to \infty$ for all $a \in A$. Equivalent asymptotic morphisms are homotopic. In this paper we deal exclusively with asymptotic morphisms from nuclear C^*-algebras. It was observed in [D] that if A is nuclear then any asymptotic morphism from A to B is equivalent to a completely positive linear asymptotic morphism. This is a consequence of the Choi–Effros theorem [CE], and it applies for homotopies of asymptotic morphisms as well. *Henceforth, throughout the paper, by an asymptotic morphism we will mean a contractive completely positive linear asymptotic morphism unless stated otherwise.* Let M_∞ denote the dense

$*$-subalgebra of the compact operators $\mathcal{K}$ obtained as the union of the C^*-algebras M_n. Using approximate units it is not hard to see that any asymptotic morphism from A to $B \otimes \mathcal{K}$ is equivalent to an asymptotic morphism $\varphi_t: A \to B \otimes M_\infty$ for which there is a function $\alpha: T \to \mathbb{N}$ such that $\varphi_t(A) \subset B \otimes M_{\alpha(t)}$. The map α is called a dominating function for φ_t. This applies also to homotopies and yields a bijection

$$[[A, B \otimes M_\infty]] \to [[A, B \otimes \mathcal{K}]].$$

We consider here only asymptotic morphisms that are dominated by functions α as above. Recall that if A is nuclear, then the Kasparov group $KK(A, B)$ is isomorphic to $[[SA, SB \otimes \mathcal{K}]]$ (see [CH]).

Let X be a finite connected CW complex with base point x_0 and let B be a unital C^*-algebra. Then by the suspension theorem of [DL], $[[C_0(X\backslash x_0), B \otimes M_\infty]] \cong KK(C_0(X\backslash x_0), B)$. Let $\varphi_t: C_0(X\backslash x_0) \to B \otimes M_\infty$ be an asymptotic morphism and let α be a dominating function for φ_t. For each $t \in T$ we let $\varphi_t^\alpha: C(X) \to B \otimes M_{\alpha(t)}$ denote the unital extension of $\varphi_t: C_0(X\backslash x_0) \to B \otimes M_{\alpha(t)}$ with $\varphi_t^\alpha(1) = 1_B \otimes 1_{\alpha(t)}$. Note that if $\alpha(t) \leqslant \beta(t)$ then $\varphi_t^\alpha = 1_B \otimes 1_{\alpha(t)}\varphi_t^\beta 1_B \otimes 1_{\alpha(t)}$.

THEOREM 1.5. *Let X be a finite connected CW complex with base point x_0 and let B be a unital C^*-algebra. Let $\varphi_t, \psi_t: C_0(X\backslash x_0) \to B \otimes M_\infty$ be two asymptotic morphisms. Suppose that $[[\varphi_t]] = [[\psi_t]]$ in $KK(C_0(X\backslash x_0), B)$. Then for any finite set $F \subset C(X)$ and any $\varepsilon > 0$ there are $t_0 \geqslant 1$ and maps $\alpha, \beta: [1, \infty) \to \mathbb{N}$ with α dominating both φ_t and ψ_t such that for any $t \geqslant t_0$ there exist a unitary $u \in U(B \otimes M_{\alpha(t)} \otimes M_{\beta(t)+1})$ and a unital $*$-homomorphism $\eta: C(X) \to B \otimes M_{\alpha(t)} \otimes M_{\beta(t)}$ with finite dimensional image such that*

$$\|u \operatorname{diag}(\varphi_t^\alpha(a), \eta(a))u^* - \operatorname{diag}(\psi_t^\alpha(a), \eta(a))\| < \varepsilon$$

for all $a \in F$.

Proof. Using the suspension theorem in E-theory of [DL], we find an asymptotic morphism $\Phi_t: C_0(X\backslash x_0) \to B[0, 1] \otimes M_\infty$ such that $\Phi_t^{(0)} = \varphi_t$ and $\Phi_t^{(1)} = \psi_t$. Let $\alpha: [1, \infty) \to \mathbb{N}$ be a function dominating φ_t, ψ_t, and Φ_t. We are going to use Lemma 1.4 and the notation from there. We find t_0 such that if $\tilde{\Phi}_t: C(X) \to (B[0, 1] \otimes M_\infty)^\sim$ is the unital extension of Φ_t, then $\tilde{\Phi}_t$ is $\hat{\delta}$-multiplicative on $\hat{F}$ for all $t \geqslant t_0$. Since the image of $\tilde{\Phi}_t$ commutes with $1_B \otimes 1_{\alpha(t)}$, it follows that $\Phi_t^\alpha: C(X) \to B[0, 1] \otimes M_{\alpha(t)}$ is $\hat{\delta}$-multiplicative on $\hat{F}$ for all $t \geqslant t_0$. Let $t \geqslant t_0$ be fixed. By uniform continuity we can find a sequence of points $s_0 = 0, \ldots, s_n = 1$ in the unit interval such that

$$\max_{a \in F} \max_{0 \leqslant j \leqslant n-1} \|\Phi_t^{(s_j),\alpha}(a) - \Phi_t^{(s_{j+1}),\alpha}(a)\| < \varepsilon.$$

By applying Lemma 1.4 for the sequence of unital maps $\Phi_t^{(s_j),\alpha}$ we find a unital $*$-homomorphism η with finite dimensional image and a unitary u such that

$$\|u \operatorname{diag}(\varphi_t^\alpha(a), \eta(a))u^* - \operatorname{diag}(\psi_t^\alpha(a), \eta(a))\| < 2\varepsilon$$

for all $a \in F$. □

In the proof of Theorem A we are going to apply Theorem 1.5 for $*$-homomorphisms. Note that if $\varphi: C(X) \to B$ is a $*$-homomorphism, whose restriction to $C_0(X \backslash x_0)$ is denoted by φ too, then $\varphi^\alpha(a) = \varphi(a) + a(x_0)(1_{\alpha(t)} - \varphi(1))$.

1.6. THE PROOF OF THEOREM A

Recall from [F] that an extension of Abelian groups

$$0 \to K \to G \xrightarrow{\pi} H \to 0$$

is called pure if its restriction to any finitely generated subgroup of H is trivial. The isomorphism classes of pure extensions form a subgroup of Ext(H, K). The universal coefficient theorem of [RS] gives an exact sequence

$$0 \to \mathrm{Ext}(K_*(A), K(B)_{*-1}) \to KK(A,B) \to \mathrm{Hom}(K_*(A), K_*(B)) \to 0$$

for A in a large class of separable nuclear C^*-algebras and any C^*-algebra B that has a countable approximate unit. The quotient of $KK(A,B)$ by the subgroup of pure extensions in Ext$(K_*(A), K_{*-1}(B))$ is denoted by $KL(A,B)$ [Rø]. The subgroup of pure extensions was studied and characterized in the setting of [BDF] in [KS].

(a) $\Rightarrow$ (b) Suppose first that X is a finite CW complex. Since $K_*(C(X))$ is finitely generated any pure extension of $K_*(C(X))$ is trivial, hence $KL(C(X), B) = KK(C(X), B)$. Let $X_1, \ldots, X_m$ be the connected components of X. Let e_i be the unit of $C(X_i)$. Using the definition of the K_0 group we find $R \in \mathbb{N}$, a $*$-homomorphism $\xi: C(X) \to M_R(\mathbb{C}1_B)$ and a unitary $v \in U_{R+1}(B)$ such that $v\,\mathrm{diag}(\varphi(e_i), \xi(e_i))v^* = \mathrm{diag}(\psi(e_i), \xi(e_i))$ for $i = 1, \ldots, m$. Note that since φ and ψ are unital, ξ can be chosen to be unital. After replacing φ, ψ by $v\,\mathrm{diag}(\varphi, \xi)v^*$ and $\mathrm{diag}(\psi, \xi)$, we may assume that $\varphi(e_i) = \psi(e_i)$, $i = 1, \ldots, m$. Let φ_i, ψ_i denote the restrictions of φ and ψ to $C(X_i)$. Then $[\varphi_i] = [\psi_i] \in KK(C(X_i), B)$. Let $\delta > 0$ and let $F_i = F \cap C(X_i)$. Using Theorem 1.5 for each i we find $k(i) \in \mathbb{N}$, a $*$-homomorphism $\eta_i: C(X_i) \to M_{k(i)}(B)$ with finite dimensional image and a unitary $u_i \in U_{k(i)+1}(B)$ such that

$$\|u_i\,\mathrm{diag}(\varphi_i(a), \eta_i(a))u_i^* - \mathrm{diag}(\psi_i(a), \eta_i(a))\| < \delta$$

for all $a \in F_i$. Let $k = k(1) + \cdots + k(m)$ and define $\eta': C(X) = \oplus_i C(X_i) \to M_k(B)$ by $\eta'(a_1, \ldots, a_m) = \mathrm{diag}(\eta_1(a_1), \ldots, \eta_m(a_m))$. By conjugating $(u_i, 1, \ldots, 1)$ by suitable permutation unitaries we find unitaries $v_i \in U_{k+1}(B)$ such that

$$\|v_i\,\mathrm{diag}(\varphi_i(a), \eta'(a))v_i^* - \mathrm{diag}(\psi_i(a), \eta'(a))\| < \delta$$

for all $a \in F_i$. Let $p_i = \mathrm{diag}(\varphi_i(e_i), \eta'(e_i)) = \mathrm{diag}(\psi_i(e_i), \eta'(e_i))$. By choosing δ small enough we can perturb v_i to unitaries $w_i \in U_{k+1}(B)$ such that $w_i p_i w_i^* = p_i$ and

$$\|w_i \,\mathrm{diag}(\varphi_i(a), \eta'(a))w_i^* - \mathrm{diag}(\psi_i(a), \eta'(a))\| < \varepsilon$$

for all $a \in F_i, i = 1, \dots, m$. Let $p = 1_k - \eta'(1)$. Let $u \in U_{k+1}(B)$ be the unitary $w_1 p_1 + \cdots + w_m p_m + \mathrm{diag}(0, p)$. Fix $x_0 \in X$ and define $\eta: C(X) \to M_k(B)$ by $\eta(a) = \eta'(a) + a(x_0)p$. Then η is a unital $*$-homomorphism with finite-dimensional image and

$$\|u \,\mathrm{diag}(\varphi(a), \eta(a))u^* - \mathrm{diag}(\psi(a), \eta(a))\| < \varepsilon$$

for all $a \in F$.

In the general case, we embed X into the Hilbert cube I^ω and write $X = \cap X_n, X_n = P_n \times I^{\omega_n}$ where P_n are finite CW complexes with $P_{n+1} \subset P_n \times I$ and $\omega_n = \{n+1, n+2, \dots\}$. Let $g_i: I^\omega \to I$ be the ith-coordinate map. By abuse of notation we let g_i also denote the restrictions of g_i to X and P_n. Without loss of generality we may assume that the set F in the statement of the Theorem is equal to $\{g_1, g_2, \dots, g_n\}$ for some n. Let now n be fixed and choose $m \geqslant n$ such that for any $y \in P_m$ there is $x \in X$ such that $|g_i(x) - g_i(y)| < \varepsilon$ for $i = 1, \dots, n$. Let $\rho_m: C(P_m) \to C(X)$ be induced by the $X \hookrightarrow P_m \times I^{\omega_m} \to P_m$ with the second arrow standing for the canonical projection. Consider the $*$-homomorphisms $\varphi\rho_m, \psi\rho_m: C(P_m) \to B$, the finite set $\{g_1, \dots, g_n\} \subset C(P_m)$ and $\varepsilon > 0$. Since the Kasparov product induces a product at the level of the KL-groups [Rø] and $KL(C(P_m), B) = KK(C(P_m), B)$, it follows that the $*$-homomorphisms $\varphi\rho_m, \psi\rho_m$ give rise to the same KK-element. By the first part of the proof we find a unital $*$-homomorphism $\eta_0: C(P_m) \to M_k(B)$ with finite-dimensional image and a unitary $u \in U_{k+1}(B)$ such that

$$\|u \,\mathrm{diag}(\varphi\rho_m(g_i), \eta_0(g_i))u^* - \mathrm{diag}(\psi\rho_m(g_i), \eta_0(g_i))\| < \varepsilon$$

for $i = 1, \dots, n$. The map η_0 can be expressed as $\eta_0(a) = \Sigma_{j=1}^{\ell} a(y_j)q_j$ with $y_j \in P_m$ and mutually orthogonal projections q_j. The integer m was chosen so that we can find points $x_j \in X$ with $|g_i(y_j) - g_i(x_j)| < \varepsilon$ for $i = 1, \dots, n$. Define $\eta: C(X) \to M_k(B)$ by $\eta(a) = \Sigma_{j=1}^{\ell} a(x_j)q_j$. Then

$$\|u \,\mathrm{diag}(\varphi(g_i), \eta(g_i))u^* - \mathrm{diag}(\psi(g_i), \eta(g_i))\| < 2\varepsilon$$

for $i = 1, \dots, n$.

So far we have proven that if $[\varphi] = [\psi] \in KL(C(X), B)$, then for any $\varepsilon > 0$ and $F \subset C(X)$ finite, there exist $k \in \mathbb{N}$, a unital $*$-homomorphism $\eta: C(X) \to M_k(B)$ with finite dimensional image and a unitary $u \in U_{k+1}(B)$ such that

$$\|u \,\mathrm{diag}(\varphi(a), \eta(a))u^* - \mathrm{diag}(\psi(a), \eta(a))\| < \varepsilon$$

for all $a \in F$. Next we show that η can be chosen of the form $\eta(a) = \mathrm{diag}(a(x_1), \ldots, a(x_k))$ with $x_1, \ldots, x_k \in X$. This is done in two steps.

First we note that if $w \in U_{k+r}(B)$, $\xi: C(X) \to M_r(B)$ is a $*$-homomorphism with finite-dimensional image, $\hat{\eta} = w\,\mathrm{diag}(\eta, \xi)w^*$, and $\hat{u} = 1 \oplus w(u \oplus 1_r)1 \oplus w^*$, then

$$\|\hat{u}\,\mathrm{diag}(\varphi(a), \hat{\eta}(a))\hat{u}^* - \mathrm{diag}(\psi(a), \hat{\eta}(a))\| < \varepsilon$$

for all $a \in F$. This remark shows that we have the freedom to change η by taking direct sums and conjugating by unitaries. The proof of (a) $\Rightarrow$ (b) is completed by using the following elementary Lemma:

LEMMA. *Let $\eta: C(X) \to M_k(B)$ be a unital $*$-homomorphism with finite dimensional image. Then there are a unital $*$-homomorphism with finite-dimensional image $\xi: C(X) \to M_r(B)$ and a unitary $w \in U_{k+r}(B)$ such that $w\,\mathrm{diag}(\eta, \xi)w^*$ is equal to a $*$-homomorphism of the form $\hat{\eta}(a) = \mathrm{diag}(a(x_1), \ldots, a(x_{k+r}))$.*

Proof. There are distinct points $x_1, \ldots, x_n$ in X and a partition of 1_k into mutually orthogonal projections $p_1, \ldots, p_n$ such that $\eta(a) = \Sigma_{i=1}^n a(x_i)p_i$ for all $a \in C(X)$. The proof is done by induction after n. Define $\xi(a) = a(x_1)p_2 + a(x_2)p_1 + \Sigma_{i=3}^n a(x_i)p_i$, $\eta'(a) = a(x_1)(p_1+p_2) + \Sigma_{i=3}^n a(x_i)p_i$, $\xi'(a) = a(x_2)(p_1 + p_2) + \Sigma_{i=3}^n a(x_i)p_i$. Then it is easily seen that $\mathrm{diag}(\eta, \xi)$ is unitarily equivalent to $\mathrm{diag}(\eta', \xi')$. The formulae for η' and ξ' involves $n-1$ distinct points each.

(b) $\Rightarrow$ (a) This is a generalization of Proposition 5.4 in [Rø]. The proof is based on an adaptation of the argument given in [Rø] and it remains valid if we replace $C(X)$ by any C^*-algebra in the class $\mathcal{N}$ of [RS]. To save notation set $A = C(X)$. The first step is to show that $K_*(\varphi) = K_*(\psi)$. This is standard and follows easily from (b) by using functional calculus and the definition of K-theory. Therefore by the universal coefficient theorem of [RS] the difference element $z = [\varphi] - [\psi]$ lies in the image of $\mathrm{Ext}(K_*(A), K_{*-1}(B))$ in $KK(A, B)$. It is shown in [Rø] that the element z is given by

$$0 \to K_*(SB) \to K_*(E) \xrightarrow{\pi_*} K_*(A) \to 0 \tag{7}$$

where E is the C^*-algebra of all pairs (f, a), where $a \in A$, $f \in B[0,1]$, $f(0) = \varphi(a)$, $f(1) = \psi(a)$. The map $\pi: E \to A$ is given by $\pi(f, a) = a$. Our aim is to show that z corresponds to a pure extension. By assumption there exist a sequence of $*$-homomorphisms $\eta_i: C(X) \to M_{n(i)}$ with finite dimensional image and a sequence of unitaries $u_i \in U_{m(i)}(B)$ where $m(i) = 1 + n(1) + \cdots + n(i)$, such that if

$$\varphi_i = \mathrm{diag}(\varphi, \eta_1, \ldots, \eta_i), \quad \psi_i = \mathrm{diag}(\psi, \eta_1, \ldots, \eta_i)$$

then

$$\lim_{i \to \infty} \|u_i \varphi_i(a) u_i^* - \psi_i(a)\| = 0 \tag{8}$$

for all $a \in C(X)$. Note that $K_*(\varphi_i) = K_*(\psi_i)$. As above we have a 'difference' extension

$$0 \to M_{m(i)}(SB) \to E_i \xrightarrow{\pi(i)} A \to 0$$

corresponding to the pair φ_i, ψ_i. There is a natural embedding $\gamma_i \colon E_i \to E_{i+1}$, $\gamma_i(f, a) = (f \oplus \eta_{i+1}(a), a)$. This yields a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & M_{m(i)}(SB) \arrow[r] \arrow[d] & E_i \arrow[r, "\pi(i)"] \arrow[d, "\gamma_i"] & A \arrow[r] \arrow[d, "id_A"] & 0 \\
0 \arrow[r] & M_{m(i+1)}(SB) \arrow[r] & E_{i+1} \arrow[r, "\pi(i+1)"] & A \arrow[r] & 0
\end{tikzcd}$$

Taking the inductive limit of these extensions we obtain an extension

$$0 \to \mathcal{K} \otimes SB \to E_\infty \xrightarrow{\pi_\infty} A \to 0$$

whose KK-theory class is equal to z (by the continuity of K-theory and the five lemma). With this construction in hands, by using (8) and reasoning as in the proof of Proposition 5.4 in [Rø] one shows that if $g \in K_*(A)$ and $ng = 0$ for some n then there is $i \geqslant 1$ and $h \in K_*(E_i)$ with $nh = 0$ and $\pi(i)_*(h) = g$. This shows that the group extension

$$0 \to K_*(B) \to K_*(E_\infty) \to K_*(A) \to 0$$

is pure. We conclude that the extension (7) is pure. □

Another proof of (b) $\Rightarrow$ (a) can be obtained by constructing a sequence $\chi_i \in \mathrm{Map}(A, E_\infty)$ such that $\pi_\infty \chi_i = \mathrm{id}_A$ and $\|\chi_i(ab) - \chi_i(a)\chi_i(b)\| \to 0$, as $i \to \infty$, for all $a, b \in A$. This goes as follows. We may assume that there is a path of unitaries $u_i(s)$, $s \in [0, 1]$, in $U_{m(i)}(B)$ with $u_i(0) = 1$ and $u_i(1) = u_i$. Let $\sigma_i \in \mathrm{Map}(A, E_i)$ be defined by

$$\sigma_i(a)(s) = \begin{cases} (u_i(2s)\varphi_i(a)u_i(2s)^*, a) & \text{for } 0 \leqslant s \leqslant 1/2 \\ ((2-2s)u_i\varphi_i(a)u_i^* + (2s-1)\psi_i(a), a) & \text{for } 1/2 \leqslant s \leqslant 1 \end{cases}$$

Let θ_i denote the embedding of E_i into E_∞. It is not hard to see that the sequence of maps $\chi_i = \theta_i \sigma_i$ has the desired properties. The sequence χ_i gives an approximate splitting of the extension E_∞. Then one shows as in [BrD] that the corresponding extension in K-theory is pure.

Suppose that the C^*-algebra B is purely infinite and simple. Then Theorem A can be modified as follows.

THEOREM 1.7. *Let X be a compact metrizable space. Let B be a purely infinite, simple, unital C^*-algebra and let $\varphi, \psi: C(X) \to B$ be two unital $*$-monomorphisms. Then $[[\varphi]] = [[\psi]]$ in $KL(C(X), B)$ if and only if φ is approximately unitarily equivalent to ψ. That is, if and only if there is a sequence of unitaries $u_n \in U(B)$ such that $\|u_n\varphi(a)u_n^* - \psi(a)\| \to 0$ for all $a \in C(X)$.*

Proof. Fix $\varepsilon > 0$ and $F \subset C(X)$ finite. Using Theorem A all we have to do is to find a unitary $w_0 \in U(B)$ such that $\|w_0\varphi(a)w_0^* - \psi(a)\| < 5\varepsilon$ for all $a \in F$. Let $k \in \mathbb{N}, \eta: C(X) \to M_k(B)$ and $u \in U_{k+1}(B)$ be provided by Theorem A. Since η is unital and has finite dimensional image there exist $L \in \mathbb{N}$, distinct points $x_1, \dots, x_L$ in X and a partition of 1_k into mutually orthogonal nonzero projections $p_1, \dots, p_L \in M_k(B)$ such that

$$\eta(a) = \sum_{i=1}^{L} a(x_i)p_i$$

for all $a \in C(X)$. Define $\gamma: C(X) \to M_{k+1}(B)$ by $\gamma = u\varphi u^*$. Set $q = \gamma(1)$ and $q_i = up_iu^*$. Then

$$u\eta(a)u^* = \sum_{i=1}^{L} a(x_i)q_i \quad \text{and} \quad q + \sum_{i=1}^{L} q_i = 1_{k+1}. \tag{9}$$

We are going to use repeatedly a result of Zhang [Zh1] asserting that a simple, purely infinite C^*-algebra has real rank zero. Since γ is a monomorphism and B has real rank zero we can apply Lemma 4.1 in [EGLP] to get nonzero, mutually orthogonal projections $d, d_1, \dots, d_L$ in $qM_{k+1}(B)q$ such that $d + d_1 + \dots + d_L = q$ and

$$\left\| \gamma(a) - d\gamma(a)d - \sum_{i=1}^{L} a(x_i)d_i \right\| < \varepsilon \tag{10}$$

for all $a \in F$. Similarly, there are nonzero, mutually orthogonal projections $c, c_1, \dots, c_L$ in B such that $c + c_1 + \dots + c_L = 1$ and

$$\left\| \psi(a) - c\psi(a)c - \sum_{i=1}^{L} a(x_i)c_i \right\| < \varepsilon \tag{10'}$$

for all $a \in F$. Since B is simple and purely infinite, d_i is equivalent to a subprojection e_i of c_i. Recall that two projections e and f in a C^*-algebra A are called equivalent if there is a partial isometry $v \in A$ such that $v^*v = e$ and $vv^* = f$. The proof of Lemma 4.1 in [EGLP] shows that (10′) remains true if we replace c_i by nonzero subprojections $e_i \leqslant c_i$ and c by $e = 1 - e_1 - \dots - e_L$. Therefore we may assume that d_i is equivalent to c_i in $M_{k+1}(B)$. Since $M_{k+1}(B)$ is simple and purely infinite, $d_i + q_i$ is equivalent to a subprojection of d_i. Similarly $c_i \oplus p_i$ is equivalent to a subprojection of c_i. Actually, we can find partial isometries $v, w \in M_{k+1}(B)$ such that $v(d_i + q_i)v^* + d'_i = d_i$ and $w(c_i \oplus p_i)w^* + c'_i = c_i$ for some nonzero projections $d'_i, c'_i, i = 1, \dots, L$. Define $T_\gamma: C(X) \to M_{2k+2}(B)$ by

$$T_\gamma(a) = \mathrm{diag}\left(d\gamma(a)d^* + \sum_{i=1}^{L} a(x_i)d_i + \sum_{i=1}^{L} a(x_i)q_i, \sum_{i=1}^{L} a(x_i)d'_i \right).$$

Using v one obtains a partial isometry $V \in M_{2k+2}(B)$ with $V^*V = 1_{k+1} + \Sigma_{i=1}^{L} d'_i$, $VV^* = q$ and such that

$$VT_\gamma(a)V^* = d\gamma(a)d + \sum_{i=1}^{L} a(x_i)d_i. \tag{11}$$

Similarly, if $T_\psi: C(X) \to M_{2k+2}(B)$ is defined by

$$T_\psi(a) = \mathrm{diag}\left(c\psi(a)c^* + \sum_{i=1}^{L} a(x_i)c_i \oplus \sum_{i=1}^{L} a(x_i)p_i, \sum_{i=1}^{L} a(x_i)c'_i \right),$$

then there is a partial isometry $W \in M_{2k+2}(B)$, $W^*W = 1_{k+1} + \Sigma_{i=1}^{L} c'_i$, $WW^* = 1_B$ such that

$$WT_\psi(a)W^* = c\psi(a)c + \sum_{i=1}^{L} a(x_i)c_i. \tag{11$'$}$$

Fix $a \in A$. From (10), (11) and (10$'$), (11$'$) we obtain

$$\|V^*\gamma(a)V - T_\gamma(a)\| < \varepsilon, \tag{12}$$

$$\|W^*\psi(a)W - T_\psi(a)\| < \varepsilon. \tag{12$'$}$$

On the other hand, using (10) and (10$'$) we obtain

$$\left\| T_\gamma(a) - \mathrm{diag}\left(u(\varphi(a) \oplus \eta(a))u^*, \sum_{i=1}^{L} a(x_i)d'_i \right) \right\| < \varepsilon, \tag{13}$$

$$\left\| T_\psi(a) - \mathrm{diag}\left(\psi(a) \oplus \eta(a), \sum_{i=1}^{L} a(x_i)c'_i \right) \right\| < \varepsilon. \tag{13$'$}$$

By construction $[d_i] = [c_i]$ and $[q_i] = [p_i]$ in $K_0(B)$. This implies $[d'_i] = [c'_i]$ in $K_0(B)$. Since these are proper projections in $M_{k+1}(B)$ we can find a unitary

$u_0 \in U_{k+1}(B)$ such that $u_0 d_i' u_0^* = c_i'$ for $i = 1, \ldots, L$ (see [Cu], [Zh2]). Let $U = 1_{k+1} \oplus u_0 \in U_{2k+2}(B)$. Then

$$\left\| U \operatorname{diag}\left(u(\varphi(a) \oplus \eta(a))u^*, \sum_{i=1}^{L} a(x_i) d_i' \right) U^* \right.$$

$$\left. - \operatorname{diag}\left(\psi(a) \oplus \eta(a), \sum_{i=1}^{L} a(x_i) c_i' \right) \right\| < \varepsilon, \tag{14}$$

since $\|u(\varphi(a) \oplus \eta(a))u^* - \psi(a) \oplus \eta(a)\| < \varepsilon$. Using (13), (13′) and (14) we obtain

$$\|U T_\gamma(a) U^* - T_\psi(a)\| < 3\varepsilon. \tag{15}$$

Then from (12), (12′) and (15)

$$\|U V^* u \varphi(a) u^* V U^* - W^* \psi(a) W\| < 5\varepsilon.$$

Finally if we set $w_0 = WUV^*u$, then

$$\|w_0 \varphi(a) w_0^* - \psi(a)\| < 5\varepsilon$$

for all $a \in F$. □

Certain special cases of Theorem 1.7 have been previously proven in [Li1-3] and [EGLP] and the above proof uses ideas from those papers. For other applications of the asymptotic morphisms in the study of $*$-homomorphisms we refer the reader to [DL], [D1] and [LiPh].

Remarks. 1.8. (a) By a result of T. Loring [Lo2], if X is a finite connected CW complex with base point x_0 and $\mathcal{O}_n$ is a Cuntz algebra with $n \geqslant 4$ even, then all the elements of $KK(C_0(X \backslash x_0), \mathcal{O}_n)$ are induced by $*$-homomorphisms $C_0(X \backslash x_0) \to \mathcal{O}_n$. By combining this result with Theorem 1.7 we obtain a bijection between $KK(C_0(X \backslash x_0), \mathcal{O}_n)$ and the set of all approximate unitary equivalence classes of unital $*$-monomorphisms from $C(X)$ to $\mathcal{O}_n$.

(b) Let Y be the dyadic solenoid and let X be the one point compactification of $(Y \backslash y_0) \times \mathbb{R}$. Let $Q(H)$ be the Calkin algebra. There are infinitely many unitary equivalence classes of unital $*$-monomorphisms from $C(X)$ to $Q(H)$. These classes are parametrized by the Brown-Douglass-Fillmore group $\mathrm{Ext}(C(X)) \cong KK(C_0(X \backslash x_0), Q(H)) \cong \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}[1/2], \mathbb{Z})$. However, since all the extensions in $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}[1/2], \mathbb{Z})$ are pure, it follows that $KL(C_0(X \backslash x_0), Q(H)) = 0$ and any two unital $*$-monomorphisms from $C(X)$ to the Calkin algebra are approximately unitarily equivalent (see [Br]).

2. Inductive Limit C^*-Algebras

Consider the following list of C^*-algebras: the scalars, $\mathbb{C}$; the circle algebra, $C(\mathbb{T})$; the unital dimension-drop interval, defined below; and all C^*-algebras arising from these by forming matrix algebras and taking finite direct sums. The collection of all these C^*-algebras will be denoted $\mathcal{D}$. The collection of all quotients of C^*-algebras in $\mathcal{D}$ will be denoted $\hat{\mathcal{D}}$. A C^*-algebra will be called an A$\mathcal{D}$-algebra if it is isomorphic to an inductive limit of C^*-algebras in $\mathcal{D}$.

By the nonunital dimension-drop interval we mean

$$\mathbb{I}_m = \{f \in C([0,1], M_m) \mid f(0) = 0, f(1) \text{ is scalar}\}.$$

The unital dimension-drop interval $\tilde{\mathbb{I}}_m$ is the unitization of $\mathbb{I}_m$. The K-theory of $\mathbb{I}_m$ is easily computed, $K_0(\mathbb{I}_m) = 0$, $K_1(\mathbb{I}_m) \cong \mathbb{Z}/m$.

LEMMA 2.1. *Let X, Y be finite, connected CW complexes and let $k, m, r \in \mathbb{N}$. Let $\gamma\colon M_k(C(X)) \to PM_m(C(Y))P$ be a unital $*$-homomorphism where P is a projection in $M_m(C(Y))$. Let $\nu\colon M_k(C(X)) \to M_{kr}(C(X))$ be a unital $*$-homomorphism of the form $\nu(a) = \mathrm{diag}(a, a(x_1), \dots, a(x_{r-1}))$, $x_i \in X$. Suppose that $\mathrm{rank}(P) > 10kr\dim(Y)$. Then there are $*$-homomorphisms γ_0, $\gamma_1\colon M_k(C(X)) \to PM_m(C(Y))P$ with orthogonal images such that γ_0 has finite dimensional image, γ_1 factors through ν and γ is homotopic to $\gamma_0 + \gamma_1$.*

Proof. The result is a consequence of Theorems 4.2.8 and 4.2.11 in [DN]. □

LEMMA 2.2. *Let H be a finite subset of $C(S^2)$, let $\varepsilon > 0$ and let X be a finite, connected CW complex. There is m_0 such that if $m \geqslant m_0$, then for any unital $*$-homomorphism $\sigma'\colon C(S^2) \to M_m(C(X))$ there exist a unital $*$-homomorphism $\sigma\colon C(S^2) \to M_m(C(X))$ and a C^*-subalgebra D of $M_m(C(X))$ isomorphic to a quotient of a direct sum of matrix algebras over $C(S^1)$, such that σ' is homotopic to σ and $\mathrm{dist}(\sigma(a), D) < \varepsilon$ for all $a \in H$.*

Proof. If X is a product of spheres, the result appears in [EGLP1]. In particular for $X = S^2$ we find n and such that the unital $*$-homomorphism $\nu'\colon C(S^2) \to M_n(C(S^2))$, $\nu'(a) = (a, a(x_0), \dots, a(x_0))$ is homotopic to a $*$-homomorphism ν with $\nu(H)$ approximately contained to within ε in a circle algebra. The more general situation we consider here is reduced to the case $X = S^2$ by using Lemma 2.1. Indeed, by Lemma 2.1 there is m_0 such that any unital $*$-homomorphism $\sigma'\colon C(S^2) \to M_m(C(X))$, $m \geqslant m_0$, is homotopic to a direct sum between a $*$-homomorphism with finite-dimensional image and a $*$-homomorphism that factors through ν. □

PROPOSITION 2.3. *Let X be a finite, connected CW complex. Let F be a finite subset of $C(X)$ and let $\varepsilon > 0$. Suppose that the K-theory group $K_0(C(X))$ is torsion free. Then there exist $r \in \mathbb{N}$, a $*$-homomorphism $\eta\colon C(X) \to M_{r-1}(C(X))$ with finite dimensional image and a C^*-subalgebra $D \subset M_r(C(X))$ with $D \in \hat{\mathcal{D}}$ such that*

$$\mathrm{dist}(a \oplus \eta(a), D) < \varepsilon$$

for all $a \in F$. The $*$-homomorphism η can be chosen of the form

$$\eta(a) = \mathrm{diag}(a(x_1), \dots, a(x_{r-1})).$$

Proof. There are $c, d \geqslant 0, m(i) \geqslant 2$ with $K_0(C(X)) \cong \mathbb{Z}^{c+1}, K_1(C(X)) \cong \mathbb{Z}^d \oplus \mathbb{Z}/m(1) \oplus \cdots \oplus \mathbb{Z}/m(k)$. Let $N = c + d + k$ and set

$$\begin{aligned}
&B_j = C_0(S^2 \backslash pt) \text{ if } 1 \leqslant j \leqslant c,\\
&B_j = C_0(S^1 \backslash pt) \text{ if } c < j \leqslant c + d,\\
&B_j = \mathbb{I}_{m(j-c-d)} \text{ if } c + d < j \leqslant N,\\
&B = B_1 \oplus \cdots \oplus B_N.
\end{aligned}$$

If $K_1(C(X))$ is torsion free, then we consider only B_j for $1 \leqslant j \leqslant c + d$. By construction, $K_*(C_0(X \backslash x_0))$ is isomorphic to $K_*(B)$. The universal coefficient theorem of [RS] shows that $C_0(X \backslash x_0)$ is KK-equivalent to B. Using the E-theory description of KK-theory of [CH] and the suspension theorem of [DL], we find an asymptotic morphism $\varphi_t \colon C_0(X \backslash x_0) \to B \otimes M_\infty$ yielding a KK-equivalence. Let $\chi \in KK(B, C_0(X \backslash x_0))$ be such that $\chi[[\varphi_t]] = [[\mathrm{id}_{C_0(X \backslash x_0)}]]$. The element χ can be written as $\chi = \chi_1 + \cdots \chi_N$ with $\chi_j \in KK(B_j, C_0(X \backslash x_0))$. It is known that every element in $KK(B_j, C_0(X \backslash x_0))$ can be realized as a $*$-homomorphism $B_j \to C_0(X \backslash x_0) \otimes M_L$ for a suitable integer L.

$$\begin{aligned}
&[C_0(S^1 \backslash pt), C_0(X \backslash x_0) \otimes M_\infty] \cong KK(C_0(S^1 \backslash pt), C_0(X \backslash x_0)) \text{ [Ros]},\\
&[C_0(S^2 \backslash pt), C_0(X \backslash x_0) \otimes M_\infty] \cong KK(C_0(S^2 \backslash pt), C_0(X \backslash x_0)) \text{ [Se], [DN]},\\
&[\mathbb{I}_m, C_0(X \backslash x_0) \otimes M_\infty] \cong KK(\mathbb{I}_m, C_0(X \backslash x_0)) \text{ [DL]}.
\end{aligned}$$

It follows that there are $*$-homomorphisms $\psi_j^0 \colon B_j \to C_0(X \backslash x_0) \otimes M_L$ with $[\psi_j^0] = \chi_j$. Let $\tilde{\psi}_j \colon \tilde{B}_j \to C(X) \otimes M_L$ be the unital extension of ψ_j^0 and let ψ_j denote the $*$-homomorphism $\psi_j = \tilde{\psi}_j \otimes \mathrm{id}_\infty : \tilde{B}_j \otimes M_\infty \to C(X) \otimes M_L \otimes M_\infty$. Define

$$\psi \colon \bigoplus_{j=1}^{N} (\tilde{B}_j \otimes M_\infty) \to C(X) \otimes M_L \otimes M_N \otimes M_\infty$$

$$\psi(b_1, \dots, b_N) = \mathrm{diag}(\psi_1(b_1), \dots, \psi_N(b_N)).$$

Form the diagram

$$\begin{tikzcd}
C_0(X \backslash x_0) \arrow[r, "\gamma"] \arrow[d, "\varphi_t"'] & C(X) \otimes M_L \otimes M_N \otimes M_\infty \\
\oplus_j (\tilde{B}_j \otimes M_\infty) \arrow[ur, "\psi"'] &
\end{tikzcd}$$

where $\gamma(a) = a \otimes e$ for a fixed one-dimensional projection e. By construction, the above diagram commutes at the level of KK-theory. Therefore there is a homotopy of asymptotic morphisms $\Phi_t : C_0(X \backslash x_0) \to C(X) \otimes M_L[0,1] \otimes M_N \otimes M_\infty$ with $\Phi_t^{(0)} = \gamma$ and $\Psi_t^{(1)} = \psi\varphi_t$.

Let $F \subset C(X)$ and ε be as in the statement of Proposition 2.3. Let $\hat{F}$ and $\hat{\delta}$ be given by Lemma 1.4. As in the proof of Theorem 1.5, let α be a dominating function for Φ_t and find t_0 such that Φ_t^α is $\hat{\delta}$-multiplicative on $\hat{F}$ for all $t \geqslant t_0$. Now let $t \geqslant t_0$ be fixed. By increasing $\alpha(t)$, we may assume that the image of φ_t is contained in $\oplus_{j=1}^N \tilde{B}_j \otimes M_{\alpha(t)}$. Moreover we may assume that φ_t^α is $\hat{\delta}$-multiplicative on $\hat{F}$. We obtain the following diagram

$$\begin{tikzcd} C(X) \arrow[r, "\gamma^\alpha"] \arrow[d, "\varphi_t^\alpha"'] & C(X) \otimes M_L \otimes M_N \otimes M_{\alpha(t)} \\ \oplus_j (\tilde{B}_j \otimes M_{\alpha(t)}) \arrow[ur, "\psi"'] & \end{tikzcd}$$

where the superscript α indicates that the corresponding maps have been unitalized as described before Theorem 1.5. The restriction of ψ to $\oplus_j (\tilde{B}_j \otimes M_{\alpha(t)})$ is denoted by ψ too. Note that $\Phi_t^{(0),\alpha} = \gamma^\alpha$ and $\Phi_t^{(1),\alpha} = \psi\varphi_t^\alpha$. The next step is to modify the above diagram so that ψ will become homotopic to a $*$-homomorphism which factors approximately through an algebra in $\hat{\mathcal{D}}$. Set $C = \oplus_{j=1}^N (\tilde{B}_j \otimes M_{\alpha(t)})$, $E = C(X) \otimes M_L \otimes M_N \otimes M_{\alpha(t)}$ and $H = \varphi_t^\alpha(F)$ (recall that t is fixed). By using Lemma 2.2 we find m, such that if $\gamma_0 : E \to M_m(E)$ is defined by $\gamma_0(c) = \operatorname{diag}(c, c(x_0), \ldots, c(x_0))$, with $x_0 \in X$, then the $*$-homomorphism $\gamma_0\psi$ is homotopic to a $*$-homomorphism $\sigma : C \to M_m(E)$ such that $\sigma(H)$ is approximately contained, to within ε, in a subalgebra D of $M_m(E)$ with $D \in \hat{\mathcal{D}}$. Actually, one applies Lemma 2.2 for the partial $*$-homomorphism $\gamma_0\psi_j$, $1 \leqslant j \leqslant c$. Let $\Psi : C \to M_m(E)[0,1]$ be a homotopy of $*$-homomorphisms with $\Psi^{(0)} = \gamma_0\psi$ and $\Psi^{(1)} = \sigma$. Next we consider the path of maps $\Gamma^{(s)}$ in $\operatorname{Map}(C(X), M_m(E))$ obtained by the juxtaposition of $\gamma_0\Phi_t^{(s),\alpha}$ with $\Psi^{(s)}\varphi_t^\alpha$. It is clear that $\gamma_0\gamma^\alpha$ is the initial point of $\Gamma^{(s)}$ and $\sigma\varphi_t^\alpha$ is the terminal point. Since Φ_t^α and φ_t^α are $\hat{\delta}$-multiplicative on $\hat{F}$, it follows that $\Gamma^{(s)}$ is $\hat{\delta}$-multiplicative on $\hat{F}$ for each s. This enables us to use Lemma 1.4 to show that $\gamma_0\gamma^\alpha$ is stably approximately unitarily equivalent to $\sigma\varphi_t^\alpha$. Namely, there exist $S \in \mathbb{N}$, a $*$-homomorphism $\eta_0 : C(X) \to M_{S-1}(M_m(E))$ with finite-dimensional image and a unitary $u \in U_S(M_m(E))$ such that

$$\|u \operatorname{diag}(\sigma\varphi_t^\alpha(a), \eta_0(a))u^* - \operatorname{diag}(\gamma_0\gamma^\alpha(a), \eta_0(a))\| < 2\varepsilon$$

for all $a \in F$. Note that $\gamma_0\gamma^\alpha$ is a unital $*$-homomorphism of the form

$$a \mapsto \operatorname{diag}(a, a(x_0), \ldots, a(x_0)) = \operatorname{diag}(a, \eta_1(a)).$$

Since $\varphi_t^\alpha(F) \subset H$ and $\sigma(H)$ is approximately contained to within ε in D, it follows that, for all $a \in F$, $\operatorname{diag}(a, \eta_1(a), \eta_0(a))$ is approximately contained to within 3ε in $u(D \oplus \operatorname{image}(\eta_0))u^* \in \hat{\mathcal{D}}$. By using the Lemma that appears in the proof of Theorem A, we see that η_0 can be taken of the form $\eta_0(a) = \operatorname{diag}(a(x_1), \ldots, a(x_{r-1}))$. □

THEOREM 2.4. *Let A be a C^*-algebra of real rank zero. Suppose that A is an inductive limit*

$$A = \varinjlim (A_n, \gamma_{m,n}), A_n = \bigoplus_{i=1}^{k_n} M_{[n,i]}(C(X_{n,i})).$$

Suppose that each $X_{n,i}$ is a finite connected CW complex, $K_0(C(X_{n,i}))$ is torsion free and $d = \sup_{n,i}\{\dim(X_{n,i})\} < \infty$. Then A is isomorphic to an $A\mathcal{D}$ algebra.

Proof. Let $e_{n,i}$ denote the unit of $A_{n,i} = M_{[n,i]}(C(X_{n,i}))$ and let $\gamma_{m,n}^{j,i}: A_{n,i} \to A_{m,j}$ be the partial $*$-homomorphisms of $\gamma_{m,n}$. The C^*-algebras in $\mathcal{D}$ are semiprojective [Lo]. By Proposition 1.2 in [DL1], (see also Theorem 3.8 in [Lo]) it suffices to show that for any finite subset $F \subset A_{n,i}$ and $\varepsilon > 0$, there is $m \geqslant n$, such that for any $1 \leqslant j \leqslant k_m$ there exist a C^*-algebra $D \in \mathcal{D}$, and a $*$-homomorphism $\sigma: D \to f_j A_{m,j} f_j$, $f_j = \gamma_{m,n}^{j,i}(e_{n,i})$, such that $\operatorname{dist}(\gamma_{m,n}^{j,i}(a), \sigma(D)) < \varepsilon$ for all $a \in F$. By Theorem 1.4.14 in [EG] we can assume that F is weakly approximately constant to within ε. To simplify notation say $A_{n,i} = M_k(C(X))$. By Proposition 2.3 there exist $r \in \mathbb{N}$ and a unital $*$-homomorphism $\nu: M_k(C(X)) \to M_{kr}(C(X))$ of the form $\nu(a) = \operatorname{diag}(a, a(x_1), \ldots a(x_{r-1}))$, and there is a unital $*$-homomorphism $\xi: D \to M_{kr}(C(X))$ with $D \in \mathcal{D}$ such that $\operatorname{dist}(\nu(a), \xi(D)) < \varepsilon$ for all $a \in F$. By Theorem 2.5 in [Su] and Lemma 2.3 in [EG] there is $m \geqslant n$ so that for each j, $1 \leqslant j \leqslant k_m$, one of the following conditions are satisfied.

(i) There is a $*$-homomorphism $\mu: A_{n,i} \to f_j A_{m,j} f_j$ with finite dimensional image such that $\|\gamma_{m,n}^{j,i}(a) - \mu(a)\| < \varepsilon$ for all $a \in F$.

(ii) $\operatorname{rank}(\gamma_{m,n}^{j,i}(e_{n,i})) = \operatorname{rank}(f_j) = kq$ where $q > 10rd$.

If we are in the first case, then we are done. Thus we may assume that (ii) holds. Define $\nu_j: M_k(C(X)) \to M_{kr}(C(X)) \oplus M_k$, $\nu_j(a) = \nu(a) \oplus a(x_0)$. We apply Lemma 2.1 for $\gamma_{m,n}^{j,i}$ and ν_j. Hence we find a $*$-homomorphism $\psi_j: M_{kr}(C(X)) \oplus M_k \to f_j A_{m,j} f_j$ such that $\psi_j \nu_j$ is homotopic to $\gamma_{m,n}^{j,i}$ as $*$-homomorphisms from $M_k(C(X))$ to $f_j A_{m,j} f_j$. Since F is approximately weakly constant to within ε and A has real rank zero, by Theorem 2.29 in [EG], there exist $s \geqslant m$ and a unitary $u \in \gamma_{sm}(f_j) A_s \gamma_{s,m}(f_j)$ such that

$$\|u\gamma_{s,m}\psi_j\nu_j(a)u^* - \gamma_{s,m}\gamma_{m,n}^{j,i}(a)\| < 70\varepsilon$$

for all $a \in F$. Define $\xi_j \colon D \oplus M_k \to M_{kr}(C(X)) \oplus M_k, \xi_j(d, \lambda) = \xi(d) \oplus \lambda$. The various maps that are involved here can be visualized on the diagram

$$\begin{tikzcd}
F \arrow[r, hook] & A_{n,i} \arrow[r, "\gamma^{j,i}_{m,n}"] \arrow[d, "\nu_j"'] & f_j A_{m,j} f_j \arrow[r, "\gamma_{s,m}"] & A_s \\
D \oplus M_k \arrow[r, "\xi_j"] & M_r(A_{n,i}) \oplus M_k \arrow[ur, "\psi_j"'] & &
\end{tikzcd}$$

Since $\operatorname{dist}(\nu_j(a), \xi_j(D \oplus M_k)) < \varepsilon$ for all $a \in F$, if we set $\sigma = u(\gamma_{s,m}\psi_j\xi_j)u^*$, then

$$\operatorname{dist}(\gamma_{s,m}\gamma^{j,i}_{m,n}(a), \sigma(D)) < 100\varepsilon.$$

This concludes the proof. □

In Theorem 2.4 one can allow $d = \infty$ under the additional assumption that A has slow dimension growth. A similar remark applies to Theorem B.

2.5 THE PROOF OF THEOREM B

Let A and B be as in the statement of Theorem B. Theorem 2.4 shows that A and B are $\mathcal{AD}$ algebras. By Theorem 7.1 in [Ell], ordered, scaled K-theory is a complete invariant for the simple $\mathcal{AD}$ algebras of real rank zero.

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