G top. group $\qquad G = \mathrm{Aut}(D \otimes K)$ for example

Suppose that $\pi_0(G)$ is abelian and that the canonical map $G \xrightarrow{q} \pi_0(G)$ is split

$$\begin{tikzcd} G \arrow[r, shift left, "q"] & \pi_0(G) \arrow[l, shift left, dashed, "\sigma"] \end{tikzcd} \qquad \sigma \circ q = \mathrm{id}$$

Then the natural map

$$[X, BG] \xrightarrow{\delta_0} H^1(X, \pi_0(G))$$

is surjective and splits naturally (set theoretically)

$$B\sigma_* \circ Bq_* = \mathrm{id}$$

$$\begin{tikzcd} {[X, BG]} \arrow[r, "Bq_*"] & {[X, B\pi_0(G)]} \arrow[r, "B\sigma_*"] \arrow[d, equal] & {[X, BG]} \\ & H^1(X, \pi_0(G)) & \end{tikzcd}$$

since $\pi_0(G)$ discrete abelian

Naturality : $\quad f : X \to Y$ induces commutative diagram

$$\begin{tikzcd} H^1(X, \pi_0(G)) \arrow[r, "\sim"] & {[X, B\pi_0(G)]} \arrow[r, "\sigma_*"] & {[X, BG]} \\ H^1(Y, \pi_0(G)) \arrow[r, "\sim"] \arrow[u, "f^*"] & {[Y, B\pi_0(G)]} \arrow[r, "\sigma_*"'] \arrow[u, "f^*"] & {[Y, BG]} \arrow[u, "f^*"] \end{tikzcd}$$