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homeomorphism in $F^{k+1}(X)$ has at most k+1 proper values it follows that both the families $(U_i)_i$ and $(V_i)_i$ cover X. Now let as look closely at each condition asked by 6.4.1.

a) We have that U_i and V_I are nonvoid. They are also connected since $X \setminus \alpha_I$ is connected (**dim** e $\geq$ 2)

b) It follows by Proposition 6.2.1 in conjunction with Corollary 6.1.11 that the vertical arrows is the following commutative diagram induce isomorphisms at π_1.

$$\begin{tikzcd}
F^k(X \setminus \alpha_I) \arrow[r] \arrow[d] & F^k(X) \arrow[d] \\
P^k(X \setminus \alpha_I) \arrow[r] \arrow[d] & P^k(X) \arrow[d] \\
P^\infty(X \setminus \alpha_I) \arrow[r] & P^\infty(X)
\end{tikzcd}$$

Now according to [15] the map

$$\pi_1(P^\infty(X \setminus \alpha_I)) \to \pi_1(P^\infty(X))$$

can be identified with the map between the one dimensional homology groups

$$H_1(X \setminus \alpha_I) \to H_1(X) \text{ (induced by } X \setminus \alpha_I \hookrightarrow X)$$

which clearly is surjective.

c) $f(U_I) \subset V_I$ since $x_0 \notin \alpha_I$

d) $X \setminus \alpha_I$ is homotopic to a space in W(n,r-1) so that

$f : U_I = F^k(X \setminus \alpha_I) \to F^{k+1}(X \setminus \alpha_I) = V_I$ is a $2[\frac{k}{3}]$-equivalence by hypothesis.

Finally the above considerations show that we can apply theorem 6.4.1 to get that $\alpha_k : F^k(X) \to F^{k+1}(X)$ is a $2[\frac{k}{3}]$-equivalence.

6.4.3. REMARK. A inspection of our proof shows that this result can be improved especially for lower values of k by imposing certain restriction on X. In any case it seems to us to be quite remarkable that there is a rather large inferior bound, which

Then $f : A \to B$ is a m-equivalence.

Proof. If $m = 1$ then the proof is acomplished by applying several times the Van Kampen theorem.

If $m \geq 2$ using the same theorem we get that $\pi_1(f)$ is an isomorphism. Let $\omega_A : \tilde{A} \to A$, and $\omega_B : \tilde{B} \to B$ be the universal covering spaces for A and B, and fix $\tilde{f} : \tilde{A} \to \tilde{B}$ a lifting for f, $\omega_B \circ \tilde{f} = f \circ \omega_A$. Let $\tilde{U}_I = \omega_A^{-1}(U_I)$, $\tilde{V}_I = \omega_B^{-1}(V_I)$ and note that $\tilde{f}(\tilde{U}_I) \subset \tilde{V}_I$. We want to prove that each $\tilde{f} : \tilde{U}_I \to \tilde{V}_I$ is a m-equivalence. Since $\pi_q(\tilde{f}|\tilde{U}_I)$ identifies with $\pi_q(f|U_I)$ for $q \geq 2$ we have only to show that $\pi_1(\tilde{f}|\tilde{U}_I)$ is an isomorphism. Using b) and the functoriality of the homotopy exact sequences for fibrations we get the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \pi_1(\tilde{U}_I) \arrow[r] \arrow[d] & \pi_1(U_I) \arrow[r] \arrow[d] & \pi_0(F) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \pi_1(\tilde{A}) \arrow[r] & \pi_1(A) \arrow[r] & \pi_0(F) \arrow[r] & 0
\end{tikzcd}$$

where F is the fibre over the base point. Note that we have $\pi_0(\tilde{U}_I) = 0$ since $\pi_1(U_I) \to \pi_1(A)$ is surjective and of course $\pi_1(\tilde{A}) = 0$. Therefore we obtain the following exact sequence

$$0 \to \pi_1(\tilde{U}_I) \to \pi_1(U_I) \to \pi_1(A) \to 0$$

and a similar sequence for V_I. We can compare this sequences by the following commutative diagram induced by f

$$\begin{tikzcd}
0 \arrow[r] & \pi_1(\tilde{U}_I) \arrow[r] \arrow[d, "\pi_1(\tilde{f})"] & \pi_1(U_I) \arrow[r] \arrow[d, "\pi_1(f)"] & \pi_1(A) \arrow[r] \arrow[d, "\pi_1(f)"] & 0 \\
0 \arrow[r] & \pi_1(\tilde{V}_I) \arrow[r] & \pi_1(V_I) \arrow[r] & \pi_1(B) \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd} \tilde{K} \arrow[r, dashed, "\tilde{p}"] \arrow[d] & \tilde{K}' \arrow[d] \\ K \arrow[r, "p"] & K \end{tikzcd}$$

such that the map $|\tilde{K}| \xrightarrow{|\tilde{p}|} |\tilde{K}'|$ can be identified with $|K|^{\sim} \xrightarrow{|p|^{\sim}} |K'|^{\sim}$ (see 6.3.8)

6.3.11. LEMMA. Let $i : Y \hookrightarrow Z$ be a NDR-pair of path connected spaces and let $m \geq 0$.

a) If i is a m-equivalence then Z/Y is m-connected.

b) Assume that both Y and Z are simply connected. If Z/Y is m-connected then i is a m-equivalence.

Proof. The Lemma is a direct consequence of the Theorems of Van-Kampen Hurewicz and Whitehead. [73].

6.3.12. PROPOSITION. $\alpha : A \to B$ is a $2[\frac{k}{3}]$-equivalence.

<u>**Proof.**</u> If $\tilde{x} \in \tilde{C}$ is such that $\omega_C(\tilde{x})$ is the base point of $C = P^{\ell(k+1)}(X)$ then $\tilde{p}_2^{-1}(\tilde{x})$ reduces to a point. If follows by Proposition 6.3.10 and Remark 6.1.3. that $\tilde{p}_2$ is a $(2[\frac{k}{3}] + 1)$-equivalence.

There is a continuous map $\tilde{s} : \tilde{C} \to \tilde{D}$ such that $\tilde{p}_2 \circ \tilde{s} = id(\tilde{C})$. (For each $\tilde{x} \in \tilde{C}$ let $y \in \tilde{p}_o^{-1}(\tilde{x})$ and define $\tilde{s}(x) = \tilde{\beta}\tilde{\alpha}(y)$). Since $\pi_q(\tilde{p}_2) \circ \pi_q(\tilde{s}) = id$ we find that $\tilde{s}$ is a $2[\frac{k}{3}]$-equivalence.

Now $\tilde{D}$ and $\tilde{C}$ was chosen such that

$$\tilde{\beta} : (\tilde{B}, \tilde{\alpha}(\tilde{A})) \to (\tilde{D}, s(\tilde{C}))$$

is a relative homeomorphism of NDR-pairs [73]. Using Lemma 6.3.11 it results that $\tilde{\alpha}$ is a $2[\frac{k}{3}]$-equivalence since $\tilde{s}$ it is so.

Finally since $\pi_1(\alpha)$ is an isomorphism (by 6.3.7), we get that α is a $2[\frac{k}{3}]$-equivalence.

6.3.13. THEOREM. The embedding $F^k(X) \to F^{k+1}(X)$ is a $2[\frac{k}{3}]$-equivalence. (Recall $X = S^1 \vee \ldots \vee S^1$).

By lemma 6.3.7. $\pi_1(\alpha)$, $\pi_1(p_o)$ and $\pi_1(p_1)$ are isomorphisms. Therefore given $\tilde{x} \in \tilde{C}$, $x \in C$ such that $\omega_C(\tilde{x}) = x$, we can use 6.3.8. to identify $p_o^{-1}(x) \to p_1^{-1}(x)$ with $\tilde{p}_o^{-1}(\tilde{x}) \to \tilde{p}_1^{-1}(\tilde{x})$ hence the later map is also a $2[\frac{k}{3}]$ equivalence.

Let $\tilde{D}$ be the space obtained by collapsing to (distinct) points all the subsets of $\tilde{B}$ of the form $\tilde{p}_1^{-1}(\tilde{x}) \cap \tilde{\alpha}(\tilde{A})$ with $\tilde{x} \in \tilde{C}$. i.e. $\tilde{D} = \tilde{B}/\sim$ where by definition $\tilde{b}_1 \sim \tilde{b}_2$ iff $\tilde{b}_1, \tilde{b}_2 \in \tilde{\alpha}(\tilde{A})$ and $\tilde{p}_1(\tilde{b}_1) = \tilde{p}_1(\tilde{b}_2)$. If $\tilde{\beta} : \tilde{B} \to \tilde{D}$ is the induced quotient map then $\tilde{p}_1$ factors through $\tilde{\beta}$ i.e. $\tilde{p}_1 = \tilde{p}_2 \circ \tilde{\beta}$, as in the following diagram

$$\begin{tikzcd}
\tilde{A} \arrow[r, "\tilde{\alpha}"] \arrow[dr, "\tilde{p}_o"'] & \tilde{B} \arrow[r, "\tilde{\beta}"] \arrow[d, "\tilde{p}_1"] & \tilde{D} \arrow[dl, "\tilde{p}_2"] \\
 & \tilde{C} &
\end{tikzcd}$$

It is easy to see that $\tilde{\beta}^{-1}\tilde{\beta}(\tilde{b})$ is equall to $\tilde{b}$ if $\tilde{b} \notin \tilde{\alpha}(A)$ and to $\tilde{\alpha}(\tilde{A}) \cap p^{-1}(p_2(\tilde{b})) = \tilde{\alpha}(\tilde{p}_o^{-1}(\tilde{p}_1(\tilde{b})))$ if $b \in \tilde{\alpha}(A)$.

6.3.10. PROPOSITION. The unique map $\tilde{p}_2 : \tilde{D} \to \tilde{C}$ satisfying $\tilde{p}_1 = \tilde{p}_2 \circ \tilde{\beta}$ is a $(2[\frac{k}{3}] + 1)$-quasifibration

Proof. We want to apply Proposition 6.1.5 for $m = 2[\frac{k}{3}]$ so that we have to check that $\tilde{p}_2$ satisfies to 6.1.4 a-d. To reach c) and d) will suffice to prove that each fiber $\tilde{p}_2^{-1}(\tilde{x})$ is $2[\frac{k}{3}]$-connected. Now it follows from the definition of $\tilde{p}_2$ that $\tilde{p}_2^{-1}(\tilde{x}) = p_1^{-1}(\tilde{x})/\tilde{\alpha}(p_o^{-1}(\tilde{x}))$. Since $\tilde{\alpha} : p_o^{-1}(x) \to p_1^{-1}(x)$ is a $2[\frac{k}{3}]$-equivalence we can apply the first part of the next lemma in order to get that $\tilde{p}_2^{-1}(x)$ is $2[\frac{k}{3}]$-connected.

In virtue of the discution from the end of 6.1.4 in order to achieved the condition a) and b) it is enough to show that the map $\tilde{p}_2 : \tilde{D} \to \tilde{C}$ can be identified with some simplicial map between simplicial complexes. In order to perform this identification one will use the following general facts about simplicial complexes.

1. Let X be a finite simplicial complex. There is a triangulation $\varphi_1 : |K_1| \to X^k$ such that for each $I \in \mathcal{I}$ (see 6.1.6) $\varphi_1^{-1}(\{\underline{x} \in X^k : I(\underline{x}) = I\})$ is the space of some subcomplex of K_1 and the action of $\mathfrak{S}_k$ on X^k is induced by some action by simplicial

$$
\begin{tikzcd}
F^{k}(X)_{\ell(k+1)} \arrow[rr, hook, "\alpha"] \arrow[dr, "p_0"'] & & F^{k+1}(X)_{\ell(k+1)} \arrow[dl, "p_1"] \\
& P^{\ell(k+1)}(X) &
\end{tikzcd}
$$

which is induced by the diagram 6.1.12 after we identify $P^{\ell(k+1)}(X)$ with its image in $P^{k+1}(X)$. Therefore α is the restriction of α_k, p_o is the restriction of $\beta_k \circ p_k$ and p_1 is the restriction of p_{k+1}. Each $\underline{x} \in X^{\ell(k+1)}$ defines a partition $I(\underline{x}) = (I_1, \ldots, I_t^*)$ of $\{1,2,\ldots,\ell(k+1)\}$ as in 6.1.6. The marked subset I_t^* corresponds to those indices i for which $x_i = x_o$. If $\underline{x}' = (\underline{x}, x_o \ldots x_o) \in X^k$ then the partition of $\{1,\ldots,k\}$ associated to $\underline{x}'$ is $I(\underline{x}) = (I_1, \ldots, I_{t-1}, \quad I_t)$ where $I_t = I_t^* \cup \{\ell(k+1)+1, \ldots, k\}$. Similarly if $\underline{x}'' = (\underline{x}', x_o) \in X^{k+1}$ then $I(\underline{x}'') = (I_1, \ldots, I_{t-1}, \quad I_t \cup \{k+1\})$. Therefore by 6.1.9 $p_o^{-1}[\underline{x}] = U(k)/U(I(\underline{x}'))$ and $p_1^{-1}[x] = U(k+1)/U(I(\underline{x}''))$. Let m_i, $1 \le i \le t$ be the cardinal of each I_i. We can identify $U(I(\underline{x}'))$ with $U(m_1) \times \ldots \times U(m_t)$ and $U(I(\underline{x}''))$ with $U(m_1) \times \ldots \times U(m_t + 1)$ in such a way that the inclusion of $U(I(\underline{x}))$ in $U(I(\underline{x}''))$ induced by $U(k) \hookrightarrow U(k+1)$ corresponds to the embedding

$$(u_1, \ldots u_{t-1}, u_t) \to \left(u_1, \ldots, u_{t-1}, \begin{pmatrix} u_t & 0 \\ 0 & 1 \end{pmatrix}\right), \text{ where } u_i \in U(m_i).$$

6.3.6. LEMMA. The map $\alpha : p_o^{-1}[\underline{x}] \to p_1^{-1}[\underline{x}]$ is a $2[\frac{k}{3}]$-equivalence.

Proof. The homotopy exact sequences associated to the fibrations

$U(k) \to p_o^{-1}[\underline{x}]$ and $U(k+1) \to p_1^{-1}[\underline{x}]$ together with the map $p_o^{-1}[\underline{x}] \to p_1^{-1}[\underline{x}]$ (which can be identified with α), induced by the embedding $U(I(x')) \hookrightarrow U(I(x''))$ give the following commutative diagram with exact rows

$$
\begin{tikzcd}[column sep=small]
\pi_q(U(m_1) \times \ldots \times U(m_t)) \arrow[r] \arrow[d] & \pi_q(U(k)) \arrow[r] \arrow[d] & \pi_q(p_o^{-1}[x]) \arrow[r] \arrow[d] & \pi_{q-1}(U(m_1) \times \ldots \times U(m_t)) \arrow[r] \arrow[d] & \pi_{q-1}(U(k)) \arrow[d] \\
\pi_q(U(m_1) \times \ldots \times U(m_t+1)) \arrow[r] & \pi_q(U(k+1)) \arrow[r] & \pi_q(p_1^{-1}[x]) \arrow[r] & \pi_{q-1}(U(m_1) \times \ldots \times U(m_t+1)) \arrow[r] & \pi_{q-1}(U(k+1))
\end{tikzcd}
$$

$(\underline{y},u) \sim (\underline{z},v) \Leftrightarrow \sigma(\underline{y}) = \underline{z}$ and $u^* v \sigma \in U(I(x))$ for some $\sigma \in \mathfrak{S}_k$. Of course if Y happens to be compact then $F^k(Y) \simeq \mathbf{Hom}_1(C(Y), M_k)$. We need later, to know $F^k((-1,1))$ is homeomorphic to $\mathbf{R}^{k^2}$. Using the notation of 6.1.6, 6.1.7 and arguments similar to those in 6.1.8. one can check that

$$[\underline{x},u] \to u\left(\sum_{r=1}^{m} \tan\left(\frac{\pi}{2}x_{i_r}\right) e(I_r(\underline{x}))\right)u^*$$

defines an homeomorphism of $F^k((-1,1))$ onto the subspace of $M_k \simeq \mathbf{C}^{k^2}$ consisting of all self-adjoint matrices which in its turn is homeomorphic to $\mathbf{R}^{k^2}$.

6.3.3. For any $\underline{a} \in A(\ell)$ let $B(\underline{a},\ell)$ denote the homogeneous spaces

$$U(k)/\,U(a_1) \times \dots \times U(a_m) \times U(k-\ell)$$

There is a well defined map $p_{\underline{a}} : F(\underline{a},\ell) \to B(\underline{a},\ell)$ which we are going to describe below. The space $B(\underline{a},\ell)$ can be identified with the space of all ordered $(m+1)$-uples $(p_1,\dots,p_m,p_o)$ of mutually orthogonal self-adjoint projections acting on $\mathbf{C}^k$ such that $\dim p_j = a_j$, $1 \le j \le m$ and $\dim p_o = k-\ell$. Given an homomorphism $\varphi \in F(\underline{a},\ell)$ we define $p_{\underline{a}}(\varphi) = (p_1,\dots,p_m,p_o)$ where for $1 \le j \le m$, p_j is equal to the sum of all spectral projections corresponding to the proper values of φ which lies in $S_j \setminus \{x_o\}$ and $p_o = 1_k - (p_1 + \dots + p_m)$ is the projection corresponding to x_o.

It is not hard to see that $p_{\underline{a}} : F(\underline{a},\ell) \to B(\underline{a},\ell)$ is a fiber bundle with fiber isomorphic to

$$F^{a_1}((-1,1)) \times \dots \times F^{a_m}((-1,1)).$$

Moreover $F(\underline{a},\ell)$ admits a canonical structure of C^∞-manifold relative to which $p_{\underline{a}} : F(\underline{a},\ell) \to B(\underline{a},\ell)$ becomes a C^∞-differentiable fiber bundle. This is due to the fact that $B(a,\ell)$ and $\displaystyle\mathop{\mathrm{X}}_{i=1}^{m} F^{a_j}((-1,1)) \simeq \mathbf{R}^{\Sigma a_j^2}$ have natural C^∞-structure relative to which the clutching maps arising from the local trivialization of the above fiber bundle are smooth.

There is a canonical section for $p_{\underline{a}}$, $s_{\underline{a}} : B(\underline{a},\ell) \to F(\underline{a},\ell)$ defined as follows. Let μ be the natural map $\mu : ([-1,1)], \{-1,1\}) \to (S^1, x_o)$, $\mu(t) = \exp(2\pi i t)$ and let $\gamma_j : [-1,1] \to X$ be given by $\gamma_j = \eta_j \circ \mu$ (6.3.1). For any $(p_1,\dots,p_m,p_o) \in B(\underline{a},\ell)$ we define $s_{\underline{a}}(p_1,\dots,p_m,p_o)$ to be the homomorphism $\varphi \in F^k(X) = \mathbf{Hom}_1(C(X), M_k)$ given by

For given $1 \le j_1 \le \dots \le j_{k+1} \le N$,
Let $t(1), t(2), \dots, t(r)$ be determined by the conditions

$$j_1 = \dots = j_{t(1)} < j_{t(1)+1} = \dots = j_{t(1)+t(2)} < \dots = j_{t(1)+\dots+t(r)}.$$

and put $J = \{j_1, \dots j_{k+1}\}$

$$J_i = \{j_s : t(1)+\dots+t(i-1)+1 \le s \le t(1)+\dots+t(i)\} \quad i \le i \le r.$$

Define

$$E(J) = [e_{j_1} \times \dots \times e_{j_{k+1}}] = \bigtimes_{q \in J} e_q / \mathfrak{S}_{k+1} \subset P^{k+1}(X)$$

$$E(J)_i = \bigtimes_{q \in J_i} e_q / \mathfrak{S}_{t(i)}$$

Then

$$E(J) = E(J_1) \times \dots \times E(J_r)$$

Note that $e_{j_1} \times \dots \times e_{j_{k+1}}$ (resp. $\bigtimes_{q \in J_i} e_q$) has a natural cone structure (as a product of balls) over the bord $\partial(e_{j_1} \times \dots \times e_{j_{k+1}})$, (resp. over $\partial(\bigtimes_{q \in J_i} e_q)$). Since the action of $\mathfrak{S}_{k+1}$ (resp. of $\mathfrak{S}_{t(i)}$) is compatible with this cone structure the quotient space $E(J)$ (resp. $E(J)_i$) has a natural cone structure over $\partial E(J)$ (resp. over $\partial E(J)_i$). But obviously $\partial E(J)_i$ can be identified with $D_{t(i),d(i)}$, where $d(i) = \dim e_{j'}$ and $j' = j_{t(1)} + \dots + j_{t(i)}$. By the general formula $\bigtimes_{i=1}^{r} \operatorname{Cone} X_i = \operatorname{Cone}\left(\mathop{*}_{i=1}^{r} X_i\right)$ where $X_1 * X_2$ denotes the join operation between spaces ([23]), we obtain that

$$E(J) = \bigtimes_{i=1}^{r} \operatorname{Cone} D_{t(i),d(i)} = \operatorname{Cone}\left(\mathop{*}_{i=1}^{r} D_{t(i),d(i)}\right)$$

In particular the bord $\partial E(J)$ can be seen by this identification as $\mathop{*}_{i=1}^{r} D_{t(i),d(i)}$.

Consequently, $P^{k+1}(X)$ is obtained from $P^k(X)$ by glueing cones of the type $\operatorname{Cone}\left(\mathop{*}_{i=1}^{r} D_{t(i),d(i)}\right)$ along their bords $\mathop{*}_{i=1}^{r} D_{t(i),d(i)}$.

6.2.5. We are now in position to prove Proposition 6.2.1. First recall [30] that if X_i are m_i-connected then $X_1 * X_2$ is $(m_1 + m_2 + 2)$-connected. Having this property it follows by 6.2.2. and 6.2.3 that

$[\underline{y},u] = [\underline{z},v]$ then there is some $\sigma \in \mathfrak{S}_k$ such that $\sigma(\underline{y}) = \underline{z}$ and $u^*v\sigma \in U(I(\underline{y}))$ (6.1.8). It follows that $\sigma(h_t(y)) = h_t(z)$ since h_t is $\mathfrak{S}_k$-equivariant and $u^*v\sigma \in U(I(\underline{y})) \subset U(I(h_t(\underline{y})))$ since $I(h_t(\underline{y})) \leq I(\underline{y})$. Hence $[h_t(\underline{y}),u] = [h_t(\underline{z}),v]$. It is clear that $p \circ \tilde{H}_t = \tilde{h}_t \circ p$ and $p^{-1}(U), \tilde{H}$ satisfy 6.1.4. b).

c) There is a commutative diagram

$$\begin{tikzcd}
p^{-1}[\underline{x}'] \arrow[r] \arrow[d, "\wr"] & p^{-1}[\underline{x}] \arrow[d, "\wr"] \\
U(k)/_{U(I(\underline{x}'))} \arrow[r] & U(k)/_{U(I(\underline{x}))}
\end{tikzcd}$$

where $x' \in V_o$ and $U(I(x')) \subset U(I(x))$ since $I(x) \leq I(x')$.

If $I,J \in \mathcal{X}$ and $J \leq I$ then $U(k)/_{U(I)} \longrightarrow U(k)/_{U(J)}$ is a 2-equivalence as noticed earlier.

d) $p^{-1}[\underline{x}] \simeq U(k)/_{U(I(x))}$ is connected.

6.1.11. COROLLARY. $p : F^k(X) \to P^k(X)$ is a 3-equivalence for any $1 \leq k \leq \infty$.

Proof. If $\underline{x} = (x_1, \ldots, x_k)$ and $x_1 = x_2 = \ldots = x_k$ then $U(I(\underline{x})) = U(k)$ so that $p^{-1}[x]$ reduces to a point. Since by Theorem 6.1.10 p is a 3-quasifibration it follows by Remark 6.1.3. that p is a 3-equivalence.

6.1.12. Note that the following diagram is commutative

$$\begin{tikzcd}
F^k(X) \arrow[r] \arrow[d, "p"'] & F^{k+1}(X) \arrow[d, "p"] \\
P^k(X) \arrow[r] & P^{k+1}(X)
\end{tikzcd}$$

6.2. THE EMBEDDINGS $P^k(X) \to P^{k+1}(X)$

If X is a finite connected CW-complex then it can be proved that the embedding $\beta_k : P^k(X) \to P^{k+1}(X)$, $\beta_k[\underline{x}] = [x_o\underline{x}]$, is a k-equivalence ($k \geq 2$). Since we do not need which is easily seen to be a 2-equivalence (consider the homotopy sequences of the two fibrations)

6.1.7. We define $\Psi : X^k \times U(k) \to \mathbf{Hom}_1(C(X),M_k)$ as follows:
if $u \in U(k)$ and $\underline{x} = (x_1,\ldots,x_k) \in X^k$, $I(\underline{x}) = (I_1(\underline{x}),\ldots,I_m(\underline{x})) \in \mathcal{I}$,
$i_r \in I_r(x)$, $1 \leq r \leq m$, then

$$\Psi(x,u)(f) = u\Big(\sum_{r=1}^{m} f(x_{i_r})e(I_r(x))\Big)u^*, \text{ for all } f \in C(X).$$

For the sake of brevity $\Psi(\underline{x},u)$ will be denoted with $[\underline{x},u]$ and sometimes with $[\underline{x},e]$ where $e = (ue(I_1)u^*,\ldots,ue(I_m)u^*)$ is the list of the spectral projections of the homomorphism $\Psi(\underline{x},u)$. Also is clear that $I(\underline{x})$ gives the multiplicities of the proper values of $\Psi(\underline{x},u)$.

6.1.8. PROPOSITION. The map $\Psi : X^k \times U(k) \to \mathbf{Hom}_1(C(X), M_k)$ is continuous and surjective. Moreover $\Psi(\underline{x},u) = \Psi(\underline{y},v)$ iff there is $\sigma \in \mathfrak{S}_k$ such that $\sigma(\underline{x}) = \underline{y}$ and $u^* v \sigma \in U(I(\underline{x}))$.

Proof. The continuity of Ψ is obvious and Ψ is surjective by the spectral theorem. Finally, equal homomorphisms must have the same proper values counted with multiplicities and the same spectral projections corresponding to equal proper values, whence the second part of the statement.

6.1.9. Let $pr : X^k \times U(k) \to X^k$ be the projection onto the first factor and let $\Psi_o : X^k \to P^k(X)$ be the canonical map $\Psi_o(\underline{x}) = [\underline{x}]$.
We define the map $p : F^k(X) \to P^k(X)$ by asking the following diagram to be commutative

$$\begin{tikzcd} X^k \times U(k) \arrow[r, "pr"] \arrow[d, "\Psi"'] & X^k \arrow[d, "\Psi_o"] \\ F^k(X) \arrow[r, "p"] & P^k(X) \end{tikzcd}$$

$h_{j,0} = \mathbf{id}(U_j)$, the image of $h_{j,1} = \{x\}$ and $h_{j,t}(x) = x$ for all $t \in I$, $j \geq 1$. (by definition $h_{j,t}(y) = h_j(y,t)$)

b) Each homotopy h_j lifts to an homotopy

$H_j : p^{-1}(U_j) \times I \to p^{-1}(U_j)$, $p \circ H_{j,t} = h_{j,t} \circ p$ for all $t \in I$,

such that $H_{j,0} = \mathbf{id}(p^{-1}(U_j))$, the image of $H_{j,1} \subset p^{-1}(x)$ and

$H_{j,t}(y) = y$ for all $y \in p^{-1}(x)$, $t \in I$, $j \geq 1$. ($H_{j,t}(y) = H_j(y,t)$)

c) If $x' \in U_j$ and $H'_{j,1}$ denotes the restriction of $H_{j,1}$ at $p^{-1}(x')$,

$H'_{j,1} : p^{-1}(x') \to p^{-1}(x)$, then for any $y' \in p^{-1}(x')$, $y = H_{j,1}(y')$,

$\pi_q(H'_{j,1}) : \pi_q(p^{-1}(x')y') \to \pi_q(p^{-1}(x),y)$

is an isomorphism for $0 \leq q \leq m-1$ and an epimorphism for $q = m$.

d) For all $x \in B$, $p^{-1}(x)$ is 0-connected.

Note that if each fibre $p^{-1}(x)$ is m-connected then c) is automatically satisfied. Moreover if there are triangulations $|K| \simeq E$ and $|L| \simeq B$ such that modulo these identifications $p : E \to B$ is induced by a simplicial proper map $K \to L$, then by standard techniques with barycentric coordinates it is easily seen that a) and b) are satisfied.

6.1.5. PROPOSITION. If $p : E \to B$ is surjective and satisfies the conditions a, b, c, d from above then p is a $(m+1)$-quasifibration.

(cf. Hilfssatz 2.10 in [14]).

Proof. For each $x \in B$ let $(U_j^x)_{j\geq 1}$, $(h_j^x)_{j\geq 1}$, $(H_j^x)_{j\geq 1}$ having the properties a-d. It is clear that the family $U = (U_j^x)_{x\in X,\ j\geq 1}$ satisfies the condition b) of Theorem 6.1.2. Moreover we shall prove that $p : p^{-1}(U_j^x) \to U_j^x$ is a $(m+1)$-quasifibration for each $x \in X$ and $j \geq 1$. Since U_j^x is ~~countable~~ controllable it is enough to check that

$$\pi_q(p^{-1}(U_j^x), p^{-1}(x'), y') = 0$$

whenever $0 \leq q \leq m$, $x' \in U_j^x$ and $y' \in p^{-1}(x')$. Now there is a commutative diagram

$$\begin{tikzcd}
\pi_q(p^{-1}(x'),y') \arrow[r] \arrow[d, "\pi_q(H'^{x}_{j,1})"] & \pi_q(p^{-1}(U_j^x),y') \arrow[r] \arrow[d, "\pi_q(H^x_{j,1})"] & \pi_q(p^{-1}(U_j^x),p^{-1}(x'),y') \arrow[r] & \pi_{q-1}(p^{-1}(x'),y') \\
\pi_q(p^{-1}(x),y) \arrow[r] & \pi_q(p^{-1}(U_j^x),y) & &
\end{tikzcd}$$

6. STABILITY PROPERTIES OF HOMOMORPHISMS

Let X be a finite connected CW-complex with base point $x_o \in X$ and let $F^k(X) = \mathbf{Hom}_1(C(X), M_k)$. There is natural embedding $\alpha_k : F^k(X) \longrightarrow F^{k+1}(X)$ given by the orthogonal sum with the morphism $f \longmapsto f(x_o)$. The main result of this section asserts that this embedding is a $2[k/3]$-homotopy equivalence for any X as above and $k \geq 3$ (see Theorem 6.4.2).

Basically the idea of the proof is the following one. As a first step it is proved that $\pi_1(F^k(X)) = \pi_1(F^{k+1}(X))$ and this is done via $\pi_1(P^k(X))$. A key fact here is the comparison theorem between $F^k(X)$ and the symmetric product $P^k(X)$. Next, the main result is proved for $X = \vee S^1$ (and this is the most difficult part). Finally the induction over the numbers of cells of dimension ≥ 2 is carried out.

6.1. A COMPARISON THEOREM BETWEEN $F^k(X)$ AND THE SYMMETRIC PRODUCT $P^k(X)$.

For $k \geq 1$, the k-fold symmetric product of X, denoted by $P^k(X)$ is defined by $P^k(X) = X^k/\mathfrak{S}_k$ where X^k denotes the k-fold cartesian product of X with itself and $\mathfrak{S}_k$ denotes the symmetric group on k objects regarded as acting on X^k by permuting the coordinates:

$$\sigma(x_1, \ldots, x_k) = (x_{\sigma(1)}, \ldots, x_{\sigma(k)}), \quad \sigma \in \mathfrak{S}_k$$

If $\underline{x} = (x_1, \ldots, x_k)$, then we shall use the notation $[\underline{x}] = [x_1, \ldots, x_k]$ for a generic element of $P^k(X)$.

There is a natural embedding $\beta_k : P^k(X) \longrightarrow P^{k+1}(X)$ given by

$$\beta_k[x_1, \ldots, x_k] = [x_o, x_1, \ldots, x_k]$$

z_o is the base point of S^1.

It is easily seen that $K_o(A(p,q)) = \mathbf{Z}[1/p]$ with the order induced by the embedding $\mathbf{Z}[1/p] \subset \mathbf{Q} \subset \mathbf{R}$ and $K_1(A(p,q)) = \mathbf{Z}[1/q]$. Let

$$A = A(2,3) \oplus A(3,2) \text{ and } B = A(2,2) \oplus A(3,3)$$

It is clear that $K_o(A) \simeq K_o(B)$ are ordered scaled groups and $K_1(A) \simeq K_1(B)$. However A is not shape equivalent to B since arithmetical reasons prevent $K_*(A)$ to be isomorphic to $K_*(B)$ as $\mathbf{Z}_2$-graded groups. To prove this observe that any isomorphisms

$$\varphi : K_o(A) = \mathbf{Z}[1/2] \oplus \mathbf{Z}[1/3] \longrightarrow \mathbf{Z}[1/2] \oplus \mathbf{Z}[1/3] = K_o(B)$$

$$\psi : K_1(A) = \mathbf{Z}[1/2] \oplus \mathbf{Z}[1/3] \longrightarrow \mathbf{Z}[1/3] \oplus \mathbf{Z}[1/2] = K_1(B)$$

are of the from

$$\varphi = \begin{pmatrix} \varphi_{11} & 0 \\ 0 & \varphi_{22} \end{pmatrix}, \quad \psi = \begin{pmatrix} 0 & \psi_{12} \\ \psi_{21} & 0 \end{pmatrix}.$$

This follows since the only morphism $\mathbf{Z}[1/p] \longrightarrow \mathbf{Z}[1/q]$ is the trivial one (provided that p,q are distinct primes). Also it is easily seen that $(a,b;x,y) \in \mathbf{Z}[1/2] \oplus \mathbf{Z}[1/3] \oplus \oplus \mathbf{Z}[1/2] \oplus \mathbf{Z}[1/3] = K_*(A)$ belongs to $K_*(A)_+$ iff $a > 0$, $b > 0$ or $a > 0$, $b = 0$, $y = 0$ or $a = 0$, $b > 0$, $x = 0$. A similar description holds for $K_*(B)$. Therefore $(\varphi,\psi)(1,0;1,0) = (\varphi_{11}(1),0,0,\psi_{21}(1))$ is not positive since $\psi_{21}(1) \neq 0$ (recall that ψ_{21} is an isomorphism).

The categories for which we succeded in shape computations are rather limited. There are two essential dificulties to be overcome in order to extend the above results to larger categories. The first one is the absence of $(KK_{+,\Sigma})$-semiprojectivity in $\mathcal{C}(n)$ even for nice algebras like $C(S^2)$ or $C(S^1 \times S^1)$. The second one is the limited power of K-theory in homotopy computations (see Section 4). For instance for $A_i = C(S^1 \times S^3) \otimes M_{n_i}$ $(i = 1, 2)$ the canonical map $[A_1,A_2] \rightarrow \mathbf{Hom}(K_*(A_1), K_*(A_2))_{+,\Sigma}$ is not surjective. Having this it is easy to construct inductive limit C^*-algebras having the same (scaled, ordered) K-theory but for which do not exist diagrams as in 5.3.5. a) (see 5.3.6 below).

If the connective K-theory would extend to a continuous theory on a larger inductive system of C^*-algebras such that $A_i \in \mathcal{C}$ but φ_{ji} are not assumed to be large in any sense. Let $A = \mathbf{lim}(A_i, \varphi_{ji})$. The associated AF-algebra r(A) defined in section 2 depends only on A. This can be proved using Propositions 5.1.4, 5.1.6 in conjunction with Proposition 2.1.3. Moreover by the results of Section 2 it follows that A can be represented as the limit of an inductive system with 3(n+3)/2-large bonding maps if and only if $K_0(A)$ has large denominators ar equivalently if and only if $K_0(r(A))$ has large denominators.

b) Let $A = \mathbf{lim}(A_i, \varphi_{ji})$ where $A_i \in \mathcal{C}_3(2n)$ but the embeddings φ_{ji} are not assumed to be large or full. Like above it can be shown that r(A) depends only on A. If r(A) is not stably isomorphic to $\mathbb{K}$ then A can be represented as the limit of an inductive system with 3(n+3)/2-full morphisms if and only if $K_0(r(A))$ is simple.

5.3.3. THEOREM. Let $\mathcal{C}$ be one of the categories $\mathcal{C}_1(n)$, $\mathcal{C}_2(2n)$, $\mathcal{C}_3(2n)$, $\mathcal{C}_4(2n-1)$. Then $\mathcal{C}$ is a shape category. Moreover for $A,B \in \mathcal{A}\mathcal{C}$ (1.2.12) the following assertions are equivalent

a) $K_*(A) \simeq K_*(B)$ as $\mathbf{Z}_2$-graded scaled ordered groups.

b) $Sh_{\mathcal{C}}(A) = Sh_{\mathcal{C}}(B)$

c) A and B have the same shape invariant in the sense of Blackadar [2].

Proof. If follows by Propositions 5.1.4, 5.1.6 and Theorems 4.3.1, 4.3.2, that the category $\mathcal{C}$ satisfies the conditions of Theorem 5.2.3. Therefore a)$\Leftrightarrow$b). The second remark of 5.2.2. gives b)$\Rightarrow$c) while c)$\Rightarrow$a) is a general fact based on the continuity of K-theory.

5.3.4. COROLLARY. Let $\mathcal{C}$ as above and $A,B \in \mathcal{A}\mathcal{C}$. The following assertions are equivalent

a) $K_*(A) \simeq K_*(B)$ as $\mathbf{Z}_2$-graded groups

b) $Sh_{\mathcal{C}}(A \otimes K) = Sh_{\mathcal{C}}(B \otimes K)$

c) $A \otimes K$ and $B \otimes K$ have the same shape invariant in the sense of Blackadar.

Proof. If $A \in \mathcal{A}\mathcal{C}$ then $A \otimes K \in \mathcal{A}\mathcal{C}$ and $\Sigma_*(A \otimes K) = K_*(A)_+$.

$$\begin{tikzcd}
K_*(A)\simeq K_*(B) \arrow[dr, Rightarrow, "(1')"'] & A\simeq B \text{ in } \mathcal{K} \arrow[r, Rightarrow, dashed] \arrow[d, Rightarrow, "(1)"] & (A_i)\simeq(B_i) \text{ in inj-}\mathcal{K} \\
& A\simeq B \text{ in } KK_{+,\Sigma} \arrow[r, Rightarrow, "(2)"] & (A_i)\simeq(B_i) \text{ in inj-}KK_{+,\Sigma} \arrow[u, Rightarrow, "(3)"']
\end{tikzcd}$$

Very informally the implications indicated by arrows hold since

(1) KK is homotopic functor

(1') This implication is a result in [37]

(2) A_i, B_j are semiprojectives in $KK_{+,\Sigma}$

(3) $KK(A_i,B_j)_{+,\Sigma} \simeq [A_i,B_j]$ or assume the weaker condition of 5.2.3.c).

Let $\mathcal{O}$ be the category of Cuntz-Krieger algebras and proper (i.e. nonunital) $*$-homomorphisms. We shall illustrate the power of our formalism by giving a short proof for a theorem of Effros and Kaminker [18], stating that two algebras in $\mathcal{A}\mathcal{O}$ are shape equivalent iff they have isomorphic K-groups. For we shall discuss each arrows in the above diagram. Let $A,B \in \mathcal{A}\mathcal{O}$ such that $K_*(A) \simeq K_*(B)$.

(1') Each $\mathcal{O}_a \in \mathcal{O}$ is nuclear. Therefore $\mathcal{A}\mathcal{O} \subset \mathcal{N}$ and we can use [37, Prop. 7.3.] to get that A is KK-equivalent to B.

(2) Since $K_*(\mathcal{O}_a)$ is finitely generated it follows by [37, Thm. 1.4. and Prop. 7.13] that the functor $KK(\mathcal{O}_a,-)$ is continuous. In our terms this means exactly that each $\mathcal{O}_a \in \mathcal{O}$ is KK-semiprojective.

(3) As a variant of the computation in [9] it is proved in [18] that $[\mathcal{O}_a, \mathcal{O}_b] \simeq KK(\mathcal{O}_a, \mathcal{O}_b)$ although the result is stated in a slightly different form.

Note that the above proof based on KK-semiprojectivity avoids the splitting principle for progroups proved in [18].

5.3. SEVERAL SHAPE CATEGORIES

Our main result concerning shape calculations is the following list of shape categories which verify the conditions of Theorem 5.2.3.

5.1.10 that $\mathcal{C}$ is a shape category.

b) If $\mathcal{C}$ is a shape category and $A,B \in \mathcal{A}\mathcal{C}$ are such $Sh_{\mathcal{C}}(A) = Sh_{\mathcal{C}}(B)$ then it follows by [2. Thm. 4.8] that A and B have the same shape in the sense of the general theory in [2].

It is interesting that one can exhibit shape categories $\mathcal{C}$ of C^*-algebras without proving that its objects are H-semiprojective in $\mathcal{C}$. As we shall see in certain cases, it is enough to look for $KK_{+,\Sigma}$-semiprojectivity. This was in fact one of the starting points of our paper. Let $\mathcal{N}$ denote the category introduced in [37] as being the smallest full subcategory of the separable nuclear C^*-algebras which contains the separable type I C^*-algebras and is closed under stable isomorphism, inductive limits, extensions and crossed products by $\mathbf{R}$ and $\mathbf{Z}$.

5.2.3. THEOREM. Let $\mathcal{C}$ be a "subcategory" of $\mathcal{N}$ satisfying the following conditions:

a) each $A \in \mathcal{C}$ is $KK_{+,\Sigma}$-semiprojective relative to $\mathcal{C}$.

b) If $A,B \in \mathcal{C}$ and $\sigma \in KK(A,B)_{+,\Sigma}$, then there is $\varphi \in \mathbf{Hom}_{\mathcal{C}}(A,B)$ such that $[\varphi]_{KK} = \sigma$

c) If $\varphi, \psi \in \mathbf{Hom}_{\mathcal{C}}(A,B)$ and $[\varphi]_{KK} = [\psi]_{KK}$ then there is an inner automorphism η of B such that $\eta \circ \psi$ belongs to $\mathcal{C}$ and φ is homotopic to $\eta \circ \psi$.

Then $\mathcal{C}$ is a shape category. Moreover if $A,B \in \mathcal{A}\mathcal{C}$ then $Sh_{\mathcal{C}}(A) = Sh_{\mathcal{C}}(B)$ iff $K_*(A) \simeq K_*(B)$ as scaled ordered groups.

Proof. Let (A_i, α_{ji}), (B_i, β_{ji}) be inductive systems in $\mathcal{C}$ and let $A = \mathbf{lim}\, A_i$, $B = \mathbf{lim}\, B_i$. In order to prove the theorem it suffices to show that the following are equivalent.

1) There is an isomorphism in $\mathbf{Hom}(K_*(A), K_*(B))_{+,\Sigma}$

2) A is isomorphic to B in the category $\mathbf{KK}_{+,\Sigma}$

3) $(A_i, [\alpha_{ji}])$ is isomorphic to $(B_i, [\beta_{ji}])$ in the category inj-$\mathcal{H}$

We have

5.1.7. Having the notion of semiprojectivity defined in 5.1.2. we can construct a "formal" shape theory following the pattern of the (shape) approaches in [29] and [17]. First we need the category inj-$\mathcal{D}$ associated with $\mathcal{D}$. The objects of inj-$\mathcal{D}$ are all the inductive systems (A_i, α_{ji}) in $\mathcal{D}$. A map of systems $\underline{\varphi} : (A_i, \alpha_{ji}) \to (B, \beta_{ji})$ consists of a sequence of integers $\varphi(1) < \varphi(2) < \ldots$ and a collection of morphisms $\varphi_i : A_i \to B_{\varphi(i)}$ in $\mathcal{D}$ such that each square of the diagram

$$\begin{tikzcd}
A_1 \arrow[r] \arrow[d, "\varphi_1"'] & A_2 \arrow[r] \arrow[d, "\varphi_2"'] & \cdots \\
B_{\varphi(1)} \arrow[r] & B_{\varphi(2)} \arrow[r] & \cdots
\end{tikzcd}$$

commutes. Two maps of systems $\underline{\varphi}, \underline{\psi} : (A_i, \alpha_{ji}) \to (B_i, \beta_{ji})$ are said to be equivalent provided for each $i > 0$ there is an $j \geq \varphi(i), \psi(i)$ such that $\beta_{j\varphi(i)} \circ \varphi_i = \beta_{j\psi(i)} \circ \psi_i$. Morphisms $\underline{\underline{\varphi}} : (A_i, \alpha_{ji}) \to (B_i, \beta_{ji})$ in inj-$\mathcal{D}$ are equivalence classes of maps of systems. Let (A_i, α_{ji}), (B_i, β_{ji}) be faithful inductive systems in $\mathcal{C}$ (i.e. we assume that all α_{ji} and β_{ji} are injective) and let $A = \lim A_i$, $B = \lim B_i$, $A, B \in \mathcal{S}$. Given an homomorphism of C^*-algebras $\varphi : A \to B$ we say that a map of inductive systems in $\mathcal{D}$

$$\underline{\varphi} : (A_i, T(\alpha_{ji})) \to (B_i, T(\beta_{ji}))$$

is associated with φ if for all $i > 0$ the diagram

$$\begin{tikzcd}
A_i \arrow[r, "T(\alpha_i)"] \arrow[d, "\varphi_i"'] & A \arrow[d, "T(\varphi)"] \\
B_{\varphi(i)} \arrow[r, "T(\beta_{\varphi(i)})"] & B
\end{tikzcd}$$

is commutative in $\mathcal{D}$.

The following three propositions are crucial for any shape theory based on semiprojectives.

They were proven in [17] for the special case of the homotopy functor H. The proofs

$\sigma \neq 0$, we have to find $i \geq 0$ and $\rho \in \mathbf{Hom}(K_*(A), K_*(B_i))_{+,\Sigma}$ such that $\sigma = K_*(\beta_i) \circ \rho$. (Here β_i is the embedding $B_i \longrightarrow B$).

Since $K_*(A)$ is finitely generated there is no problem to find $\rho = (\rho^0, \rho^1) : K_*(A) \to K_*(B_i)$ satisfying

a) $K_*(\beta_i) \circ \rho = \sigma$

As $\Sigma(r(A))$ is a finite set, $\Sigma(B) = \mathbf{lim}\, \Sigma(B_i)$ and $\sigma(\Sigma_*(A)) \subset \Sigma_*(B)$, by increasing i (if necessary), we may assume that

b) $\rho^0(\Sigma(r(A))) \subset \Sigma(B_i)$.

We will prove that if ρ satisfies the conditions a) and b) and is 2n large, then $\rho \in \mathbf{Hom}(K_*(A), K_*(B_i))_{+,\Sigma}$. In proving this is convenient to use the formalism of Section 2. Let $\rho^0 = \begin{pmatrix} \alpha & \beta \\ \delta & \gamma \end{pmatrix}$ be the matrix description of ρ^0 with respect to the decompositions

$$K_0(A) = K_0'(A) \oplus K_0(r(A)), \quad K_0(B_i) = K_0'(B_i) \oplus K_0(r(B_i)).$$

We claim that $\delta = 0$. This is equivalent to say that there is a certain map γ such that the following diagram is commutative.

$$\begin{tikzcd} K_0(A) \arrow[r, "\rho^0"] \arrow[d, "r_A"'] & K_0(B_i) \arrow[d, "r_{B_i}"] \\ K_0(r(A)) \arrow[r, "\gamma"] & K_0(r(B_i)) \end{tikzcd}$$

Since it is clear that γ exists iff $r_{B_i} \rho^0(a) = 0$ for all $a \in K_0(A)$ such that $r_A(a) = 0$, we have to deal with this last implication. Consider the comutative diagram

$$\begin{tikzcd} K_0(B_i) \arrow[r, "K_0(\beta_i)"] \arrow[d, "r_{B_i}"'] & K_0(B) \arrow[d, "r_B"] \\ K_0(r(B_i)) \arrow[r, dotted, "r(K_0(\beta_i))"] & K_0(r(B)) \end{tikzcd}$$

c) $(2n[e_k], x_k) \in K_*(B_i)_+$ for any $1 \le k \le q$ and $x_k \in \mathcal{J}_k$

(Note that $(2n[e_k], x_k) \in K_*(A)_+$ by Corollary 2.1.2 c))

A related argument show that we can assume

d) $\rho^o(\Sigma(A)) \subset \Sigma(B_i)$

since $\Sigma(A)$ is a finite subset of $K_o(A)_+$ and $\Sigma(B) = \lim \Sigma(B_i)$. Finally we may also assume that

e) ρ^o is 2n-large

since otherwise we replace ρ by $\beta_{i+n,i} \circ \rho$ which is 2n-large because $\beta_{i+n,i}$ is so and ρ^o is order preserving (see 2.1.8).

With these choices we shall prove that $\rho \in \mathbf{Hom}(K_*(A), K_*(B_i))_{+,\Sigma}$ and this will complete the proof.

As a first step we show that ρ is order preserving.

As B_i belongs to $\mathcal{C}(n)$ it has the form $B_i = \bigoplus_{j=1}^{h} D_j$ where $D_j = C(Y_j) \otimes M_{m_j}$. Writing $A = \bigoplus_{k=1}^{\ell} A_k$ we may describe ρ as a matrix of morphisms $\rho = ((\rho^o_{jk}), (\rho^1_{jk}))$ where $\rho_{jk} : K_*(A_k) \longrightarrow K_*(D_j)$. The condition b) implies that ρ^o_{jk} is order preserving and so we can consider its standard picture defined in 2.1.3-2.1.4, $\rho^o_{jk} = \begin{pmatrix} \alpha_{jk} & \beta_{jk} \\ 0 & \gamma_{jk} \end{pmatrix}$. Of course $\gamma_{jk} \ge 0$ and a simple calculation shows that if some $\gamma_{jk} = 0$ then $\alpha_{jk} = 0$, $\beta_{jk} = 0$ and $\rho^1_{jk} = 0$ for the same indices j and k. This remark is essential in what follows.

Now let $a \in K_*(A)_+$. We wish to apply Corollary 2.1.2 c) in order to prove that $\rho(a) \in K_o(B_i)_+$. To this purpose we need the following coordinate description of a and $\rho(a)$:

$$a = (a_1, \ldots, a_q),\ a_k = ((a'_k, t_k), a^1_k) \in K_o(A_k) \oplus K_o(r(A_k)) \oplus K_1(A_k),\ 1 \le k \le \ell$$

$$\rho(a) = (b_1, \ldots, b_h),\ b_j = ((b_j, s_j), b_j) \in K_o(D_j) \oplus K_o(r(D_j)) \oplus K_1(D_j),\ 1 \le j \le h.$$

Since ρ is order preserving we must have $s_j = \sum \gamma_{jk} t_k \geq 0$.

If some $s_j = 0$ then $\gamma_{jk} t_k = 0$ for each k and we will prove that $b_j = 0$

where $T(\alpha_{ji})_*(\beta) = T(\alpha_{ji}) \circ \beta$. The maps

$$T(\alpha_i)_* : \mathbf{Hom}_{\mathcal{D}}(S,A_i) \to \mathbf{Hom}_{\mathcal{D}}(S,A)$$

given by $T(\alpha_i)_*(\beta) = T(\alpha_i) \circ \beta$ induce a natural map T_* from the set theoretic inductive limit $\mathbf{lim}(\mathbf{Hom}_{\mathcal{D}}(S,A_i), T(\alpha_{ji})_*)$ to $\mathbf{Hom}_{\mathcal{D}}(S,A)$.

5.1.2. DEFINITION. A C^*-algebra $S \in \mathcal{Y}$ is called T-semiprojective relative to $\mathcal{C}$ if the natural map $T_* : \mathbf{lim}\,\mathbf{Hom}_{\mathcal{D}}(S,A_i) \to \mathbf{Hom}_{\mathcal{D}}(S, \mathbf{lim}\, A_i)$ is bijective for any $\mathcal{C}$-inductive system (A_i, α_{ji}).

Equivalently S is T-semiprojective relative to $\mathcal{C}$ if and only if the following conditions are fulfiled:

a) For every $\gamma \in \mathbf{Hom}_{\mathcal{D}}(S,A)$ there are j and $\alpha \in \mathbf{Hom}_{\mathcal{D}}(S,A_j)$ such that $T(\alpha_j) \circ \alpha = \gamma$.

b) If $\alpha, \beta \in \mathbf{Hom}_{\mathcal{D}}(S,A_i)$ and $T(\alpha_i) \circ \alpha = T(\alpha_i) \circ \beta$ then there is $j > i$ such that $T(\alpha_{ji}) \circ \alpha = T(\alpha_{ji}) \circ \beta$.

5.1.3. EXAMPLES

a) $S \in \mathcal{Y}$ is H-semiprojective relative to $\mathcal{Y}$ if and only if it is semiprojective in the sense of Effros and Kaminker. (see the examples in [17], [2], [28])

b) Let $\mathcal{C}$ be the subcategory of $\mathcal{Y}$ consisting of commutative C^*-algebras. If X is an ANR-space then C(X) is H-semiprojective relative to $\mathcal{C}$ (see [29]).

c) $C(S^2)$ and $C(S^1 \times S^1)$ are not H-semiprojective relative to $\mathcal{Y}$ (see [27]) .

d) Propositions 5.1.4 and 5.1.6 give some criteria for $KK_{+\Sigma}$-semiprojectivity. Recall from the introduction that $\mathcal{C}(n)$ denotes the category of the C^*-algebra of the form $\bigoplus_k C(X_k) \otimes M_{n_k}$ (finite sums) where X_i are finite connected CW-complexes of dimension $\leq n$. We let $\mathcal{C}'(n)$ denote the subcategory of $\mathcal{C}(n)$ having the same objects as $\mathcal{C}(n)$ but only 2-large homomorphisms.

5.1.4. PROPOSITION. Let $A = \bigoplus_{k=1}^{\ell} C(X_k) \otimes M_{n_k} \in \mathcal{C}(n)$ and assume that the semigroup $K_0(A)_+$ is finitely generated or equivalently that each $\tilde{K}^0(X_k)$ is a finite group. Then A is $KK_{+,\Sigma}$-semiprojective relative to $\mathcal{C}'(n)$.

Let $n = 2s$. By hypothesis each X_i is (2s-2)-connected. It follows that the (2s-1)-dimensional skeleton of X_i is homotopic to a wedge of (2s-1)-spheres. Since $\dim X_i \leq 2s$ this easily implies that $K^1(X_i)$ is free. Using the Universal Coefficient Theorem for KK [37], we get

$$KK(A_o,D_1) = \mathrm{Hom}(K_o(A_o),K_o(D_1)) \text{ since } K_1(A_o) \text{ is free and } K_1(D_1) = 0$$
$$KK(A_1,D_o) = \mathrm{Hom}(K_o(A_1), K_o(D_o)) \text{ since } K_1(A_1) = 0 \text{ and } K_o(A_1) \text{ are free .}$$
$$KK(A_1,D_1) = \mathrm{Hom}(K_o(A_1), K_o(D_1)) .$$

Since $\sigma \in KK(A,D)_+$ it follows by Proposition 2.1.3 that its image in $KK(A_o,D_1) = \mathrm{Hom}(K_o(A_o), K_o(D_1))$ is 0. Thus we get

$$\sigma = \begin{pmatrix} \alpha & \beta \\ 0 & \underline{k} \end{pmatrix}$$

where $\alpha = (\alpha_1, \ldots, \alpha_q)$, $\alpha_i \in KK(C_o(X_i),C_o(Y)) \simeq kk(Y,X_i)$ (see 3.4.6). From this point the proof is acomplished by analogy with the proof of Theorem 4.3.1. since β and $\underline{k}$ have the same meaning as there. However one may wonder why we have asked σ to be full. This is because in general the presence of torsion in K_o may prevent the implication $k_i = 0 \Rightarrow \alpha_i = 0$ to be true.

b) Similar to b) in 4.1.8 but use the following commutative diagram:

$$\begin{tikzcd} {[A,D]} \arrow[r] \arrow[d, "\nu"'] & KK(A,D) \arrow[d] \\ \oplus kk(Y,X_i) \arrow[r, "\sim"', "x"] & KK(A_o,D_o) \end{tikzcd}$$

5. SHAPE THEORY

In this section we use the homotopy computations from the previous section by giving shape classification results for certain inductive limits of C^*-algebras.

Proof. a) Let $A_o = \oplus_i C_o(X_i) \otimes M_{n_i}$, $A_1 = \oplus_i M_{n_i}$. We can suppose that $D = C(Y) \otimes M_m$ so that we take $D_o = C_o(Y) \otimes M_m$ and $D_1 = M_m$. Let $\sigma = (\sigma^o, \sigma^1)$ where $\sigma^o : K_o(A) \to K_o(D)$ and $\sigma^1 : K_1(A) \to K_1(D)$.

Let $\sigma^o = \begin{pmatrix} \alpha^o & \beta \\ 0 & \underline{k} \end{pmatrix}$ and $\sigma^1 = \begin{pmatrix} \alpha^1 & 0 \\ 0 & 0 \end{pmatrix}$ be the matrix description of σ^o and σ^1 corresponding to the decompositions $K_*(A) = K_*(A_o) \oplus K_*(A_1)$, $K_*(D) = K_*(D_o) \oplus K_*(D_1)$ (see 2.1.3). We have:

$$\alpha^o = (\alpha_1^o, \ldots, \alpha_q^o) : K_o(A_o) \simeq \oplus_i \tilde{K}^o(X_i) \longrightarrow \tilde{K}^o(Y) \simeq K_o(D_o)$$

$$\beta = (\beta_1, \ldots, \beta_q) : K_o(A_1) = \mathbf{Z}^q \longrightarrow \tilde{K}^o(Y), \ \beta_i \in \tilde{K}^o(Y) \ .$$

$$\underline{k} = (k_1, \ldots, k_q) : \mathbf{Z}^q \longrightarrow \mathbf{Z}$$

Since $\sigma \geq 0$ we have that if some $k_i = 0$ then $\alpha_i^o = 0$, $\alpha_i^1 = 0$ and $\beta_i = 0$ for the same index i. The condition $\sigma(\Sigma(A)) \subset \Sigma(B)$ is equivalent to $\sum_i k_i n_i \leq m$. This is easily seen using Corollary 2.1.2 b) and the fact that σ is large enough. Since σ is $3(n+3)/2$-large if $k_i \neq 0$ then $k_i \geq 3(n+3)/2$ and so $[C_o(X_i), C_o(Y) \otimes M_{k_i}] \simeq kk(Y, X_i)$ by Corollary 6.4.4. On the other hand the cohomological conditions on X_i, Y implies that $kk(Y, X_i) \simeq \mathbf{Hom}(\tilde{K}^*(X_i), \tilde{K}^*(Y))$ by the results of section 3.

Therefore we can find $\varphi_i \in \mathbf{Hom}_1(C(X_i), C(Y) \otimes M_{k_i})$ such that $K_*^{\prime}(\varphi_i) = (\alpha_i^o, \alpha_i^1)$. Define $\varphi^{\prime} \in \mathbf{Map}(Y, \mathbf{Hom}(A, M_m)) = \mathbf{Hom}(A, D)$ by

$$\varphi^{\prime}(y) = p(\varphi_1(y), \ldots, \varphi_q(y); 1) \quad \text{(see 4.1.1)}$$

Then

$$K_*(\varphi^{\prime}) = \left(\begin{pmatrix} \alpha^o & 0 \\ 0 & \underline{k} \end{pmatrix}, \begin{pmatrix} \alpha^1 & 0 \\ 0 & 0 \end{pmatrix} \right)$$

Aside $\varphi^{\prime}$, we need another morphism $\varphi^{\prime\prime}$ constructed as follows.

Let $\beta_o = -\sum_{i=1}^{q} \beta_i$. Then $(\beta_o; \beta_1, \ldots, \beta_q) \in \mathbf{Ker}\, j_* = \text{image } \varepsilon_*^c \subset \tilde{K}^o(Y)^{\tilde{q}}$. Therefore there is $\psi \in \mathbf{Map}(Y, B_m^o(\underline{k})) \simeq \mathbf{Hom}(\oplus_{i=1}^{q} M_{n_i}, C(Y) \otimes M_m)$ such that $\varepsilon_*(\psi) = (\beta_o, \beta_1, \ldots, \beta_q)$ (see 4.2.8 (b)). If ev: $A \to A_1$ is any evaluation morphism, then using 4.2.10 it can be verified that $\varphi^{\prime\prime} := \psi \circ ev : A \to B$ is such that $K_*(\varphi^{\prime\prime}) = \left(\begin{pmatrix} 0 & \beta \\ 0 & \underline{k} \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right)$ (see [13] for a similar computation).

4.2.10. Let $A = \oplus C(X_i) \otimes M_{n_i}$, $D = C(Y) \otimes M_m$ as above and let $\gamma \in \mathbf{Hom}(K_0(r(A)), K_0(r(D)))_{+,\Sigma}$. Define $[A,D]_\gamma = \{ \varphi \in [A,D] : r(\varphi) = \gamma \}$. It is clear that $[A,D]$ is the disjoint union of the $[A,D]_\gamma$. Theorem 4.2.8 computes $[A,D]_\gamma$ provided that γ is $3(n+3)/2$ - large. To make the result more clear we give below more concrete formulae for the maps $\nu, \varsigma'_*, \varepsilon_*$. The reader would have in mind the isomorphism $[Y, B_m^o(\underline{k})] \simeq [r(A), C(Y) \otimes M_m]_\gamma$, $\gamma = (k_1, \ldots, k_q)$. Thus ς'_* is just the map $[A,D]_\gamma \longrightarrow [r(A),D]_\gamma$ given by $[\varphi] \longmapsto [\varphi | r(A)]$, where $\varphi|r(A)$ denotes the restriction of φ to $r(A) = \bigoplus_{i=1}^{q} M_{n_i}$ regarded as a subalgebra of A. Let e^i be a minimal projection in M_{n_i} and for $\varphi \in \mathbf{Hom}(A,D)$ let $\varphi^i \in \mathbf{Hom}(C_0(X_i), C(Y) \otimes M_m)$ be given by the composition

$$C_0(X_i) \otimes e^i \hookrightarrow A \xrightarrow{\varphi} D.$$

With this notation ν can be identified with the map

$$[A,D]_\gamma \longrightarrow \prod_{i=1}^{q} kk(Y, X_i), \quad [\varphi] \longrightarrow ([\varphi^1], \ldots, [\varphi^q]).$$

Finally $\varepsilon_* : [A,D] \longrightarrow K^o(Y)^{\tilde{q}}$

$$\tilde{q} = \begin{cases} q & \text{if } k_o = 0 \\ q+1 & \text{if } k_o > 0 \end{cases}$$

is essentially the map $K_o(\varphi | r(A)) \in \mathbf{Hom}(K_o(r(A)), K_o(D))$. More precisely let $[\varphi] \in [A,B]_\gamma$, $\gamma = (k_1, \ldots, k_q)$ and let $x_i \oplus [k_i] \in \tilde{K}^o(Y) \oplus K^o(pt)$ be the K-theory class of the vector bundle $\varphi(e^i)$. Then we have

$$\varepsilon_*[\varphi] = (x_1, \ldots, x_q) \text{ if } k_o = 0$$

$$\varepsilon_*[\varphi] = (m - k_o - \sum_{i=1}^{q} n_i x_i, x_1, \ldots, x_q) \text{ if } k_o > 0$$

Consequently if $\varphi, \psi \in \mathbf{Hom}(A,D)$ then $K_o(\varphi | r(A)) = K_o(\psi | r(A))$ if and only if $r(\varphi) = r(\psi)$ and $\varepsilon_*(\varphi) = \varepsilon_*(\psi)$.

The following theorem is essentially a reformulation of 4.2.8.

4.2.11. THEOREM. Let $A = \bigoplus_{i=1}^{q} C(X_i) \otimes M_{n_i}$, $D = \bigoplus_{j=1}^{h} C(Y_j) \otimes M_{m_j}$ where X_i, Y_j are finite connected CW-complexes and $\mathbf{dim}(Y_j) \leq n$ for all $1 \leq j \leq h$. Let $\varphi, \psi \in \mathbf{Hom}(A,D)$

The results of a) and b) will be used several times.

Let $\varphi, \psi \in \mathbf{Map}(Y, B_m(\underline{k}))$ such that $\nu[\varphi] = \nu[\psi]$ and $\varepsilon_*[\varphi] = \varepsilon_*[\psi]$.

The condition $\nu[\varphi] = \nu[\psi]$ shows that φ and ψ have the same kk-component. Let θ' be the inverse of θ as in the proof of 4.2.7. Then there are $y_1, y_2 \in [Y_\ell, B_m(\underline{k})_\ell]$ and $z \in \prod_{i=1}^{\ell} kk(Y_i, X)$ such that $[\varphi] = \theta'(y_1, z) = i_*(y_1) + p_*(z)$ and $[\psi] = \theta'(y_2, z) = i_*(y_2) + p_*(z)$. Choose $\alpha_1, \alpha_2 \in \mathbf{Map}(Y, B^o_m(\underline{k}))$ and $\beta \in \mathbf{Map}(Y, B_m(\underline{k}))$ such that $[\alpha_i] = y_i$ and $[\beta] = p_*(z)$ and let γ^o be the map $Y \to B^o_m(\underline{k})$ which takes Y to the base point of $B^o_m(\underline{k})$. The equations $[\varphi] = \theta'(y_1, z)$ and $[\psi] = \theta'(y_2, z)$ imply that $\varphi \oplus \gamma^o$ is homotopic to $\alpha_1 \oplus \beta$ and $\psi \oplus \gamma^o$ is homotopic to $\alpha_2 \oplus \beta$ as maps from Y to $B_m(2\underline{k})$. On the other hand since $\varepsilon^*[\varphi] = \varepsilon^*[\psi]$ we must have $\varepsilon^*_o[\alpha_1] = \varepsilon^*_o[\alpha_2]$. According to the Assertion $\alpha_2 = u\alpha_1 u^*$ for some $u \in \mathbf{Map}[Y, U(m)]$. Putting the above facts together we have the following sequence of homotopies.

$$\psi \oplus \gamma^o \sim \alpha_2 \oplus \beta = (u \oplus 1_m)\, \alpha_1 \oplus \beta\, (u^* \oplus 1_m) \sim u \oplus 1(\varphi \oplus \gamma^o) u^* \oplus 1 = u\varphi u^* \oplus \gamma^o$$

By the main stability result 6.4.2 we must have $\psi \sim u\varphi u^*$. Conversely assume $\psi \sim u\varphi u^*$. As above let $\varphi \oplus \gamma^o \sim \alpha_1 \oplus \beta_1$, $\psi \oplus \gamma^o \sim \alpha_2 \oplus \beta_2$ where α_i corresponds to the kk-components and $\beta_i \in [Y, B_m(\underline{k})]$. We have

$$\alpha_2 \oplus \beta_2 \sim \psi \oplus \gamma^o \sim u \oplus 1(\varphi \oplus \gamma^o) u^* \oplus 1 \sim 1 \oplus u(\alpha_1 \oplus \beta_1) 1 \oplus u^* \sim \alpha_1 \oplus u\beta_1 u^*$$

The direct sum decomposition provided by 4.2.7 and 4.2.8 a) shows that $\alpha_1 \sim \alpha_2$ and $u\beta_1 u^* \sim \beta_2$. Consequently $\nu[\varphi] = [\alpha_1] = [\alpha_2] = \nu[\psi]$ and $\varepsilon^o_*[\varphi] = \varepsilon^o_*(u\beta_1 u^*) = \varepsilon^o_*[\beta_2] = \varepsilon^o_*[\psi]$. The proof of the Assertion relies on some general facts. Recall that if $K \hookrightarrow G$ are Lie groups then we have an exact sequence.

$$K \to G \xrightarrow{p} G/K \xrightarrow{\varepsilon^o} BK \to BG$$

The left action of G on G/K induces a left action of $\mathbf{Map}(Y, G)$ on $\mathbf{Map}(Y, G/K)$. Now the fact is that if $f^1, f^2 \in \mathbf{Map}(Y, G/K)$ then $\varepsilon^o \circ f^1$ is homotopic to $\varepsilon^o \circ f^2$ iff $f^2 = gf^1$ for some $g \in \mathbf{Map}(Y, G)$. A proof of this folklore type result is included below. The Assertion corresponds to the case $K = U(\underline{k})$, $G = U(m)$.

Let ξ^j, $j = 1, 2$ be the induced bundle over f^j of the bundle $p : G \to G/K$. Since

Proof. The construction given in 4.2.4-6 equally applies to the following diagram which commutes within homotopy:

$$\begin{tikzcd}
U(\underline{k}) \arrow[r, "j^{\circ}"] \arrow[d, equal] & U(m) \arrow[r, "p^{\circ}"] \arrow[d, "i"] & B^{\circ}_{m}(\underline{k}) \arrow[r, "\varepsilon^{\circ}"] \arrow[d, "i'"] & BU(\underline{k}) \arrow[r, "j'"] \arrow[d, equal] & BU(m) \arrow[d, equal] \\
U(\underline{k}) \arrow[r, "j"] \arrow[d, equal] & E_m(\underline{k}) \arrow[r, "p"] \arrow[d, "\rho"] & B_m(\underline{k}) \arrow[r, "\varepsilon"] \arrow[d, "\rho'"] & BU(\underline{k}) \arrow[r, "j'"] \arrow[d, equal] & BU(m) \arrow[d, equal] \\
U(\underline{k}) \arrow[r, "j^{\circ}"] & U(m) \arrow[r, "p^{\circ}"] & B^{\circ}_{m}(\underline{k}) \arrow[r, "\varepsilon^{\circ}"] & BU(\underline{k}) \arrow[r, "j'"] & BU(m)
\end{tikzcd}$$

In this way we arrive at the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{coker} j^{\circ}_{*} \arrow[r, "p^{\circ}_{*}"] \arrow[d, "i_*"] & \lim[Y, B^{\circ}_{tm}(t\underline{k})] \arrow[r, "\varepsilon^{\circ}_{*}"] \arrow[d, "i'_*"] & \ker j'_{*} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \prod kk(Y,X_i) \times \operatorname{coker} j^{\circ}_{*} \arrow[r, "p_*"] \arrow[d, "\rho_*"] & \lim[Y, B_{tm}(t\underline{k})] \arrow[r, "\varepsilon_*"] \arrow[d, "\rho'_*"] & \ker j'_{*} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \operatorname{coker} j^{\circ}_{*} \arrow[r, "p^{\circ}_{*}"] & \lim[Y, B^{\circ}_{tm}(t\underline{k})] \arrow[r, "\varepsilon^{\circ}_{*}"] & \ker j'_{*} \arrow[r] & 0
\end{tikzcd}$$

Next we define $\nu : \mathbf{lim}[Y,B_{tm}(t\underline{k})] \to \prod_{i=1}^{q} kk(Y,X_i)$

by $$\nu(x) = x - i'_* \rho'_*(x) \in p_*\Big(\prod_{i=1}^{q} kk(Y,X_i) \times \{0\}\Big).$$

(Note that $\varepsilon_* \nu(x) = 0$ and $\rho'_* \nu(x) = 0$).

Now we can define

$$\theta : \mathbf{lim}[Y,B_{tm}(t\underline{k})] \to \mathbf{lim}[Y,B^{\circ}_{tm}(t\underline{k})] \times \prod_{i=1}^{q} kk(Y,X_i)$$

by $\theta(x) = (\rho'_*(x), \nu(x))$. Using the commutativity of the above diagram it is easily seen that $\theta'(y,z) = i'_*(y) + p_*(z)$ is an inverse for θ hence θ is an isomorphism.

4.2.8. Theorem. Let Y, X_i, $1 \leq i \leq q$, be finite connected CW-complexes and let $n \geq \mathbf{dim}(Y)$. Assume that each nonzero component of $\underline{k} = (k_0; k_1, \ldots, k_q)$ is greater or

4.2.5

$$\begin{tikzcd}
{[Y,U(\underline{k})]} \arrow[r] \arrow[d, "\alpha_*"'] & {[Y,E_m(\underline{k})]} \arrow[r] \arrow[d, "\beta_*"'] & {[Y,B_m(\underline{k})]} \arrow[r] \arrow[d, "\gamma_*"'] & {[Y,BU(\underline{k})]} \arrow[r] \arrow[d, "\alpha'_*"'] & {[Y,BU(m)]} \arrow[d, "\beta'_*"'] \\
\lim[Y,U(t\underline{k})] \arrow[r, "j_*"] & \lim[Y,E_{tm}(t\underline{k})] \arrow[r, "p_*"] & \lim[Y,B_{tm}(t\underline{k})] \arrow[r, "\varepsilon_*"] & \lim[Y,BU(t\underline{k})] \arrow[r, "j'_*"] & \lim[Y,BU(tm)]
\end{tikzcd}$$

Using direct sums of unitaries, homomorphisms and respectively fiber-bundles one can introduce obvious abelian semigroup structures on each set

$$\varinjlim[Y,F_t] \text{ where } F_t = U(t\underline{k}),\ E_{tm}(t\underline{k}),\ B_{tm}(t\underline{k}),\ BU(t\underline{k}),\ BU(tm).$$

In the order to check that the addition operations given by direct sums are well defined one has to apply several times the $\begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix}$-trick in order to join by a continuous path in $\mathbf{Map}(Y,F_t)$ two objects a and a' of the form $a = b \oplus b_o \oplus c \oplus c_o$, $a' = b \oplus c \oplus b_o \oplus c_o$. Also one will notice that each map $F_t \to F_{t+1}$ is homotopic to the map $a \to a \oplus a^o$ where $a \in F_t$ and a^o is the base point of F_1.

It is also clear that the maps j_*, p_*, ε_* and j'_* preserve the direct sums so that the bottom row in the above diagram is (at least) an exact sequence of pointed semigroups.

Having in mind the definition of K-groups and kk-groups we can make the following identifications:

$$\mathbf{lim}[Y,U(t\underline{k})] = \prod_{i=0}^{q} \mathbf{lim}[Y,U(tk_i)] = \begin{cases} K^1(Y)^q & \text{if } k_o = 0 \\ K^1(Y)^{q+1} & \text{if } k_o > 0 \end{cases}$$

$$\mathbf{lim}[Y,E_{tm}(t\underline{k})] = \prod_{i=1}^{q} \mathbf{lim}[Y,F^{tk_i}(X_i)] \times \mathbf{lim}[Y,U(tm)] = \prod_{i=1}^{q} kk(Y,X_i) \times K^1(Y)$$

$$\mathbf{lim}[Y,BU(t\underline{k})] = \prod_{i=0}^{q} \mathbf{lim}[Y,BU(tk_i)] = \begin{cases} \tilde{K}^o(Y)^q & \text{if } k_o = 0 \\ \tilde{K}^o(Y)^{q+1} & \text{if } k_o > 0 \end{cases}$$

$$\mathbf{lim}[Y,BU(tm)] = \tilde{K}^o(Y)$$

Since all these are abelian groups, using 4.2.5 it follows that $\varinjlim[Y,B_{tm}(t\underline{k})]$ is also a (abelian) group. In this way we get the following exact sequence of groups:

For future purposes we need to describe j_t^o using systems of matrix units. Thus if $(e^i_{x,y})$ and $(e_{a,b})$ are the usual systems of matrix units of M_{tk_i} and M_{tm} then

$$j_t^o(\underline{w}) = \sum_{i=0}^{q} \sum_{x,y=1}^{k_i} \sum_{r=1}^{n_i} w_i^{x,y} e_{h(i-1)+rt_ik_i+x,\ h(i-1)+rt_ik_i+y}$$

where: $w_i^{x,y}$ are the components of w_i i.e. $w_i = \sum_{i=0}^{q} w_i^{x,y} e^i_{x,y}$

$$\underline{w} = (w_o, w_1, \ldots, w_q)$$

$$h(i) = \sum_{j=0}^{i} n_j k_j \quad \text{if} \quad i \geq 0 \quad (n_o := 1 \text{ if } k_o \neq 0,\ n_o := 0 \text{ if } k_o = 0)$$

$$h(-1) = 0$$

Now we are going to define the bonding maps needed for inductive limits. The first is the canonical embedding $\alpha_t : U(t\underline{k}) \longrightarrow U(s\underline{k})$,

$$\alpha_t(w_o, w_1, \ldots, w_q) = (w_o \oplus 1_{k_o},\ w_1 \oplus 1_{k_1}, \ldots, w_q \oplus 1_{k_q})$$

In order to simplify the notation we set $s = t + 1$.

As it is easily seen there is a permutation matrix $v_t \in U(sm)$ such that if we define $\beta_t^o : U(tm) \rightarrow U(sm)$ by $\beta_t^o(u) = v_t^*(u \oplus 1)v_t$ then the following diagram is commutative

$$\begin{tikzcd} U(t\underline{k}) \arrow[r, "j_t^o"] \arrow[d, "\alpha_t"'] & U(tm) \arrow[d, "\beta_t^o"] \\ U(s\underline{k}) \arrow[r, "j_s^o"'] & U(sm) \end{tikzcd}$$

To be more precise one can take v_t to be the unitary which permutes the canonical basis of $\mathbf{C}^{sm}$ according to the permutation ξ of $\{1, 2, \ldots, sm\}$ which is defined below. For each $0 \leq i \leq q$ let

$$W_i = \{x = (t+1)h(i-1) + (t+1)(r-1)k_i + a :\ 1 \leq r \leq n_i, 1 \leq a \leq tk_i\} \text{ and}$$

$$V_i = \{y = (t+1)h(i-1) + (t+1)(r-1)k_i + tk_i + b :\ 1 \leq r \leq n_i, 1 \leq b \leq k_i\}$$

These sets from a partition of $\{1, 2, \ldots, sm\}$.

we have to confine ourselves to a shorter exact sequence

$$K \xrightarrow{j} E \xrightarrow{\pi} E/K \xrightarrow{\varepsilon} BK$$

where K is embedded in E as an orbit and ε is again a classifying map for the principal K-bundle $E \to E/K$. If these spaces happen to be H-groups, then passing to homotopy classes we get exact sequences of groups rather than pointed sets.

4.2.2. The case we deal with bears some resemblance with the above situation but is more involved since we need to embed our fibration into a fibration of H-spaces. First we complete the fibration given in Proposition 4.1.3. to an exact sequence

$$U(\underline{k}) \xrightarrow{j} E_m(\underline{k}) \xrightarrow{p} B_m(\underline{k}) \xrightarrow{\varepsilon} BU(\underline{k})$$

All these spaces are pointed and the maps preserve the base points. If x_i^o is the base point of X_i, then the homomorphism $\varphi_i^o(f) = f(x_i^o).1_{k_i}$, $f \in C(X_i)$, is the base point of $\mathrm{Hom}_1(C(X_i), M_{k_i})$. Accordingly we distinguish $\underline{\varphi}^o = (\varphi_1^o, \ldots, \varphi_q^o)$ in $E(\underline{k})$, $e^o = (\underline{\varphi}^o, 1)$ in $E_m(\underline{k})$ and $b^o = p(e^o)$ in $B_m(\underline{k})$. The grup $U(\underline{k})$ is pointed by its unit 1 so that we have a corresponding base point in $BU(\underline{k})$.

Let $U(\underline{k}) \xrightarrow{j^o} U(m) \xrightarrow{p^o} B_m^o(\underline{k})$ be the fibration 4.1.3 in the special case when each space X_i reduces to a point. We define

$$i : U(m) \to E_m(\underline{k}) = E(\underline{k}) \times U(m),\ i(u) = (\underline{\varphi}^o, u) \text{ and}$$

$$\rho : E_m(\underline{k}) \to U(m),\quad \rho(\underline{\varphi}, u) = u.$$

Note that both i and ρ are $U(\underline{k})$-equivariant since $\underline{w}^* \varphi^o \underline{w} = \varphi^o$ for any $\underline{w} \in U(\underline{k})$. Therefore there are natural maps i' and ρ' induced by i and ρ such that the following diagram is commutative.

$$\begin{tikzcd}
U(\underline{k}) \arrow[r, equal] \arrow[d, "j^o"'] & U(\underline{k}) \arrow[r, equal] \arrow[d, "j"'] & U(\underline{k}) \arrow[d, "j^o"'] \\
U(m) \arrow[r, "i"] \arrow[d, "p^o"'] & E_m(\underline{k}) \arrow[r, "\rho"] \arrow[d, "p"'] & U(m) \arrow[d, "p^o"'] \\
B_m^o(\underline{k}) \arrow[r, "i'"] & B_m(\underline{k}) \arrow[r, "\rho'"] & B_m^o(\underline{k})
\end{tikzcd}$$

$$j^{o}(\underline{w}) = w_{o} \oplus (w_1 \otimes 1_{n_1}) \oplus \ldots \oplus (w_q \otimes 1_{n_q})$$

where the above description of j^{o} is given according to the unital embedding

$$M_{k_o} \oplus (M_{k_1} \otimes M_{n_1}) \oplus \ldots \oplus (M_{k_q} \otimes M_{n_q}) \hookrightarrow M_m, \; m = k_o + \sum_{i=1}^{q} k_i n_i \quad .$$

Let $(\underline{\varphi}, u)$ denote a generic element of $E_m(\underline{k})$, that is $\underline{\varphi} = (\varphi_1, \ldots, \varphi_q)$ with $\varphi_i \in \mathbf{Hom}(C(X_i), M_{k_i})$ and $u \in U(m)$. We have a right continuous action of $U(\underline{k})$ on $E_m(\underline{k})$ given by:

$$(\underline{\varphi}, u)\, \underline{w} = (w_1^* \varphi_1 w_1, \ldots, w_q^* \varphi_q w_q;\; uj^{o}(\underline{w}))$$

We also define $p : E_m(\underline{k}) \longrightarrow B_m(\underline{k})$ by

$$p(\underline{\varphi}, u) = u(O_{k_o} \oplus \varphi_1 \otimes \mathrm{id}(M_{n_1}) \oplus \ldots \oplus \varphi_q \otimes \mathrm{id}(M_{n_q}))u^*$$

The next lemma describes the homomorphisms belonging to $B_m(\underline{k})$ in simpler terms.

4.1.2. LEMMA

a) p is onto, hence $B_m(\underline{k})$ is connected

b) $p(\underline{\varphi}, u) = p(\underline{\psi}, v)$ if and only if $(\underline{\varphi}, u)\underline{w} = (\underline{\psi}, v)$ for some $\underline{w} \in U(\underline{k})$

Proof. a) If $\alpha \in B_m(k)$ then there is some $u \in U(m)$ such that $\alpha(\bigoplus_{i=1}^{q} 1_{C(X_i)} \otimes a_i) = u(O_{k_o} \oplus (1_{k_1} \otimes a_1) \oplus \ldots \oplus 1_{k_r} \otimes a_r)u^*$ for all $a_i \in M_{n_i}$. Consequently if $\alpha' = u^* \alpha\, u$ then the algebra $\alpha'(\bigoplus_{i=1}^{q} C(X_i) \otimes 1_{n_i})$ lies in the commutant of $O_{k_o} \oplus (\bigoplus_{i=1}^{q} 1_{k_i} \otimes M_{n_i})$ in M_m. Since this commutant is equal to $C = M_{k_o} \oplus (\bigoplus_{i=1}^{q} M_{k_i} \otimes 1_{n_i})$ it follows that there are $\varphi_i \in \mathbf{Hom}_1(C(X_i), M_{k_i})$, $1 \leq i \leq q$, such that $\alpha' = p(\underline{\varphi}, 1)$ whence $\alpha = p(\underline{\varphi}, u)$.

b) Let $\underline{a} = \bigoplus_{i=1}^{q} 1_{C(X_i)} \otimes a_i \in A$, $a_i \in M_{n_i}$. From $p(\underline{\varphi}, u)(\underline{a}) = p(\underline{\psi}, v)(\underline{a})$ we infer as above that u^*v is a unitary element in C so that $u^*v = j^{o}(\underline{w})$ for some $\underline{w} \in U(\underline{k})$. Therefore we have $p(\underline{\psi}, v) = p(\underline{\psi}, uj^{o}(\underline{w})) = p(\underline{\varphi}, u)$. The last equality implies $p(\underline{\psi}, j^{o}(w)) = p(\underline{\varphi}, 1)$ which means $w_i \psi_i w_i^* = \varphi_i$ for $1 \leq i \leq q$.

4.1.3. PROPOSITION. The map $p : E_m(\underline{k}) \rightarrow B_m(\underline{k})$ is a principal fiber bundle with fibre $U(\underline{k})$.

identification α_*^o is an isomorphism by the Adam's universal coefficient Theorem ([1]).

3.5.5. THEOREM. Let X,Y be finite connected CW-complexes without torsion in cohomology. There is an isomorphism α of $kk(Y,X) = [C_o(X), C_o(Y) \otimes \mathbf{K}]$ into $\mathbf{Hom}_{\mathbf{Z}}(H^*(X,\mathbf{Z}), H^*(Y,\mathbf{Z}))$. The image of α consists of all group homomorphisms which preserve both the graduation even-odd of cohomology and the filtration $F_m H^* = \bigoplus_{q \geq m} H^q$.

Proof. The theorem follows from Theorem 3.5.4 and Corollary 3.4.8.

4. HOMOTOPY COMPUTATIONS FOR LARGE HOMOMORPHISMS

In Section 3 we gave some methods for computing

$$[C_o(X), C_o(Y) \otimes \mathbf{K}] = kk(Y,X).$$

The stability results of Section 6 will imply that the natural map

$$[C_o(X), C_o(Y) \otimes M_m] \longrightarrow [C_o(X), C_o(Y) \otimes \mathbf{K}]$$

is a bijection, provided that m is large enough. Consequently in such cases we have

$$[C(X), C(Y) \otimes M_m]_1 = [C_o(X), C_o(Y) \otimes M_m] = kk(Y,X)$$

so that one can make complete computations in many concrete situations.

This Section is devoted to the more general problem of classifying up to homotopy the morphisms $A \to D$, where $A = \bigoplus_{i=1}^{q} C(X_i) \otimes M_{n_i}$ and $D = \bigoplus_{j=1}^{h} C(Y_j) \otimes M_{m_j}$ are C^*-algebras belonging to the category $\mathcal{C}(n)$ defined in Section 2. As explained below it actually suffices to compute

$$[A, C(Y) \otimes M_m] = [Y, \mathbf{Hom}(A, M_m)].$$

Briefly, our plan is as follows. First we decompose $\mathbf{Hom}(A, M_m)$ into its connected components $B_m(\underline{k})$ parametrized by certain $(q+1)$-uples of integers $\underline{k} = (k_o, \ldots, k_q)$. Each component is the base space of a certain principal fiber bundle

$$\prod_{i=0}^{q} U(k_i) \longrightarrow \prod_{i=1}^{q} \mathbf{Hom}(C(X_i), M_{k_i}) \times U(m) \longrightarrow B_m(\underline{k}).$$

Let us recall the classical algebraic-topology definition of the slant product. For $a \in k^p(Y \wedge X)$, $b \in k_q(Y)$ represented by $f : Y \wedge X \to F_p$, $g : S^{q+r} \to F_r \wedge Y$, respectively, a/b is the element represented by

$$S^{q+r} \wedge X \xrightarrow{g \wedge 1_X} F_r \wedge Y \wedge X \xrightarrow{1_{F_r} \wedge f} F_r \wedge F_p \xrightarrow{\mu} F_{n+p}$$

Our aim is to realize this slant product in terms of tensor products and compositions of $*$-homomorphisms. Namely, tensoring to the right with $\mathrm{id}(C_o(X))$ gives a map

$$i_X : k_q(Y) \longrightarrow kk_q(Y, Y \wedge X).$$

and using the product

$$k^p(Y \wedge X) \times kk_q(X, Y \wedge X) \to k^{p-q}(X)$$

(which is a special case of 3.5.2) we define

$$/\!/ : k^p(Y \wedge X) \times k_q(Y) \longrightarrow k^{p-q}(X)$$

by the rule $a /\!/ b = a.i_X(b)$

The equivalence of the two products $/$ and $/\!/$ is explained below.

We have two realizations of k_*:

i) $k_q(Y) = \varinjlim [S^{q+r}, F(S^r) \wedge Y]$ (via the spectrum $F_n = F(S^n)$) and

ii) $k_q(Y) = \varinjlim [S^{q+r}, F(S^r \wedge Y)]$ (via the kk-groups)

We have seen (3.2.2. a)) that there is a natural isomorphism T between the two theories which is induced by the maps

$$t_r : F(S^r) \wedge Y \longrightarrow F(S^r \wedge Y) \text{ given by}$$

$$t_r(\varphi \wedge y) = \varphi \otimes \varphi_y$$

where $\varphi_y \in \mathbf{Hom}(C_o(Y), \mathbf{C})$ is the evaluation map at y.

There is a similar situation concerning k^* but this time the isomorphism T is induced by the identification

$$\mathbf{Map}(Y, \mathbf{Hom}(C_o(X), \mathcal{K})) = \mathbf{Hom}(C_o(X), C_o(Y) \otimes \mathcal{K}) \text{ (see 3.2.2.b))}$$

Now the proper statement about slant product is

Proof. For a finitely generated group G let TG be its torsion part and LG its free part. In the exact sequence

$$0 \to \ker S^q \to Tk^q(X) \oplus Lk^q(X) \longrightarrow Tk^{q-2}(X) \oplus Lk^{q-2}(X) \to \operatorname{coker}(S^q) \to 0$$

$\ker S^q$ is a torsion group and coker S^q is a free group since it is a subgroup in $\widetilde{H}^{q-2}(X,\mathbf{Z})$. If we assume that $Tk^q(X) = 0$ then, as one can easily check, $\ker S^q = 0$ and $Tk^{q-2}(X) = 0$. Since $k^n(X) = \widetilde{H}^n(X,\mathbf{Z})$ and $k^{n-1}(X) = \widetilde{H}^{n-1}(X,\mathbf{Z})$, $n = \dim X$, we can use an inductive argument to prove that $\ker(S^q) = 0$ for all q. Therefore we get split extension (non natural splittings)

$$0 \to k^q(X) \xrightarrow{S} k^{q-2}(X) \rightleftarrows \widetilde{H}^{q-2}(X,\mathbf{Z}) \to 0$$

whence the statement of the Corollary. ∎

3.5. PRODUCTS

There are two fundamental operations with homomorphisms: composition and tensorization. One can use them to define various multiplicative structures on kk_*. It turns out that on this way one can reobtain <u>all</u> the products and pairings which can be introduced using the ring-spectrum structure of (F_n). We shall not develop this subject here, thus we limit our discussion to what is required in order to obtain a special Universal Coefficient Theorem for kk_*.

3.5.1. The composition of the homomorphisms induces a product

$$kk(Y,X) \times kk(Z,Y) \to kk(Z,X)$$

More precisely for $\varphi \in \mathbf{Hom}(C_o(X), C_o(Y) \otimes \mathcal{K})$ and $\psi \in \mathbf{Hom}(C_o(Y), C_o(Z) \otimes \mathcal{K})$ we define $[\varphi]\cdot[\psi] = [\psi \otimes \mathrm{id}(K) \circ \varphi]$. This product is <u>bilinear</u> and <u>associative</u>. For the bilinearity we refer to Theorem 3.1 d.) in [36]. Also the associativity is a general fact which essentially follows from the associativity of the composition. The next computations are included just in order to make things clear.

dimension $\leq p$ and $n \geq \mathbf{dim}(X) - m - 2$. Now let X be a finite connected CW-complex of dimension $p + 1$. If X_p is the p-dimensional skeleton of X, then $X/X_p =: S_{p+1}$ is a finite wedge of $(p + 1)$-spheres. The pair $X_p \hookrightarrow X$ induces the following commutative diagram

$$\begin{array}{ccccccccccc}
\to k^Y_{n+1}(S_{p+1}) & \to & k^Y_m(X_p) & \to & k^Y_m(X) & \to & k^Y_m(S_{p+1}) & \to & k^Y_{m-1}(X_p) & \to \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\to K^Y_{n+1}(S_{p+1}) & \to & K^Y_n(X_p) & \to & K^Y_n(X) & \to & K^Y_n(S_{p+1}) & \to & K^Y_{n-1}(X_p) & \to \cdots
\end{array}$$

The inductive assumptions together with the five lemma imply that $k^Y_n(X) \to K^Y_n(X)$ is an isomorphism for $n \geq p + 1 - m - 2$. ∎

3.4.6. COROLLARY. Let $X, Y \in W^c_o$ have dimension n and assume that both X and Y are $(n - 2)$-connected. Then

$$[C_o(X), C_o(Y) \otimes \mathbb{K}] \xrightarrow{\sim} KK(C_o(X), C_o(Y)).$$

Concerning the next result the referent pointed out that it can be easily deduced using the Atiyah-Hirzebruch spectral sequence or directly from the Dold's theorem which asserts that after tensoring with $\mathbb{Q}$ any generalized cohomology theory h^* becomes an ordinary cohomology theory with coefficients $h^*(\text{point}) \otimes \mathbb{Q}$. However we prefer to give a proof which is less elegant but better specifies the involved maps and requires no other identifications.

3.4.7. PROPOSITION. If X is connected, then after tensoring with $\mathbb{Q}$, the exact triangle 3.4.2. splits into short exact sequences

$$0 \to k^{q+2}(X) \otimes \mathbb{Q} \to k^q(X) \otimes \mathbb{Q} \to \tilde{H}^q(X, \mathbb{Q}) \to 0$$

Proof. Since $\mathbb{Q}$ is flat as a $\mathbf{Z}$-module the triangle 3.4.2 remains exact after tensoring with $\mathbb{Q}$. Therefore we have exact sequences

$$0 \to \mathbf{ker}(S^q \otimes 1) \to k^q(X) \otimes \mathbb{Q} \xrightarrow{S^q \otimes 1} k^{q-2}(X) \otimes \mathbb{Q} \to \mathrm{coker}(S^q \otimes 1) \to 0$$

$$0 \to \mathrm{coker}(S^{q-1} \otimes 1) \to \tilde{H}^{q-3}(X, \mathbb{Q}) \to \mathbf{ker}(S^q \otimes 1) \to 0$$

$$\longrightarrow k^{n-3}(X) \xrightarrow{\eta^*} \tilde{H}^{n-3}(X,\mathbb{Z})) \longrightarrow k^n(X) \xrightarrow{s^*} k^{n-2}(X)$$

which proves the statement.

3.4.3. COROLLARY. Assume that X is a finite connected CW-complex of dimansion n. Then

$$k_q(X) = \begin{cases} 0 & \text{if} \quad q \leq 0 \\ \tilde{H}_q(X,\mathbb{Z}) & \text{if} \quad q = 1, 2 \\ \tilde{K}_q(X) & \text{if} \quad q \geq n-1 \end{cases}$$

Complementary informations are contained in the exact sequence

$$0 \to k_{n-2}(X) \xrightarrow{s_*} k_n(X) \xrightarrow{\eta_*} \tilde{H}_n(X,\mathbb{Z}) \to k_{n-3}(X) \to \ldots \to k_1(X) \xrightarrow{s_*} k_3(X) \xrightarrow{\eta_*} \tilde{H}_3(X,\mathbb{Z}) \to 0$$

With the identifications $k_q(X) \simeq K_q(X)$ for $q \geq n-1$ the isomorphism $k_q(X) \xrightarrow[\approx]{s_*} k_{q+2}(X)$ is the Bott periodicity.

Proof. If $q \leq 0$ then $k_q(X) = \varinjlim_r [S^{q+r}, X \wedge F_r] = 0$ since F_r is (r-1)-connected and X is 0-connected and hence $X \wedge F_r$ is r-connected by [42, 3.16]. It follows from Proposition 3.4.1. that $0 = k_{-1}(X) \to k_1(X) \to \tilde{H}_1(X,\mathbb{Z}) \to k_{-2}(X) = 0$ and $0 = k_o(X) \to k_2(X) \to \tilde{H}_2(X,\mathbb{Z}) \to k_{-1}(X) = 0$ are exact sequences so that $k_q(X) \xrightarrow[\approx]{\eta_*} \tilde{H}_q(X,\mathbb{Z})$ for $q = 1, 2$. Since $\tilde{H}_q(X,\mathbb{Z}) = 0$ for $q \geq n+1$ it follows that $k_q(X) \xrightarrow[\approx]{s_*} k_{q+2}(X)$ for $q \geq n-1$. Now let X' be a N-dual of X and choose $2m > N - q$. Using [42, Cor. 7.10] and Proposition 3.4.2 we may write $k_{q+2m}(X) = k^{N-q-2m}(X') = \tilde{K}^{N-q-2m}(X') = \tilde{K}_{q+2m}(X) = \tilde{K}_q(X)$. Therefore $k_q(X) \simeq K_q(X)$ for $q \geq n-1$. A more natural proof of this isomorphism will follow from Theorem 3.4.5.

3.4.4. COROLLARY. Assume that X is a finite connected CW-complex of dimension n. Then

$$k^q(X) = \begin{cases} \tilde{K}^q(X) & \text{if} \quad q \leq 2 \\ \tilde{H}^q(X) & \text{if} \quad q = n-1, n \\ 0 & \text{if} \quad q \geq n+1. \end{cases}$$

Complementary informations are stored in the exact sequence

[42, Cor. 7.10] we have a commutative diagram

$$\begin{array}{ccccc} h_0(X, E(Y)) & \simeq & h^n(X', E(Y)) & = & k^n(X' \wedge Y) \\ \downarrow & & & & \downarrow \\ h_0(X, G(Y)) & \simeq & h^n(X', G(Y)) & = & \tilde{K}^n(X' \wedge Y) \end{array}$$

3.3.7. COROLLARY. Let $X \in W_0^c$, $Y \in W_0$ and X' an n-dual of X. There is a commutative diagram

$$\begin{array}{ccc} kk(Y, X) & \xrightarrow{\sim} & k^n(X' \wedge Y) \\ \downarrow \chi & & \downarrow \\ KK(C_0(X), C_0(Y)) & \xrightarrow{\sim} & \tilde{K}^n(X' \wedge Y) \end{array}$$

where the right vertical arrow is induced by the map of spectra $F \to BU$ and the orizontal arrows are isomorphisms induced by natural transformations.

Proof. The assertion follows from the above remarks. ■

Since any two duals are of the same stable homotopy type the above diagram does not depend on the choice of X'. The commutativity of this diagram gives a precise meaning to the assertion that kk is the connective KK (when restricted to spaces).

For a comparison result between kk and KK we refer to Theorem 3.4.5.

3.4. THE RELATION OF CONNECTIVE K-THEORY TO HOMOLOGY

The map $F_{n+2} \to F_n$ given by tensoring with S induces operations $k_n(X) \xrightarrow{S_*} k_{n+2}(X)$ and $k^{n+2}(X) \xrightarrow{S^*} k^n(X)$. If $K(\mathbf{Z}) = (K(\mathbf{Z}, n))$ is the Eilenberg-MacLane spectrum, there is a natural transformation of spectra $\eta : F \to K(\mathbf{Z})$ defined by maps $\eta_n : F_n \to K(\mathbf{Z}, n)$. To describe η_n one will use the realization of $K(\mathbf{Z}, n)$ as the infinite symmetric product of S^n, $K(\mathbf{Z}) \sim P^\infty(S^n)$, due to Dold and Thom [14]. The corresponding map $F_n \to P^\infty(S^n)$ is given in 6.1.9.

The following result is due to L. Smith [40].

$$F_{2n+2} \longrightarrow F_{2n} \rightarrow \cdots \rightarrow F_o = \Omega U$$

$$F_{2n+1} \longrightarrow F_{2n-1} \rightarrow \cdots \rightarrow F_1 = U$$

give a map of spectra $F \rightarrow BU$ which by 3.3.2 induces isomorphisms $\pi_r(F) \rightarrow \pi_r(BU)$ for all $r \geq 0$.

3.3.4. REMARK. The generalized homology theory $k_n^{S^o}(-) = kk_n(S^o,-)$ defined in Proposition 3.2.1 has the spectrum $F = (F_n)$ as was proved in Proposition 3.3.2. Therefore $k_n^{S^o}(-)$ is isomorphic to the reduced connective K-homology on W_o^c which is usually denoted by $k_n(-)$. In the same way the cohomology theory $kk_{-n+1}(-,S^1)$ is isomorphic to the reduced connective K-theory $k^n(-)$ on W_o. The formulae

$$kk_{n-1}(S^1,X) = k_n(X), \quad kk_{-n+1}(Y,S^1) = k^n(Y)$$

are important since they show that both the connective K-homology and cohomology can be realized via the homotopy of $*$-homomorphisms.

There are well known the similar formulae which relate the Kasparov KK-functor to the usual K-theory. We restrict our attention to the commutative case. Therefore starting with $KK(C_o(X), C_o(Y) \otimes \mathcal{K}) = KK(C_o(X), C_o(Y))$ one has

$$KK_{n+1}(C_o(X), C_o(S^1)) \simeq \tilde{K}_n(X) = \text{the reduced K-homology}$$

$$KK_{-n+1}(C_o(S^1), C_o(Y)) \simeq \tilde{K}^n(Y) = \text{the reduced K-theory}$$

for arbitrary $X, Y \in W_o$.

Now it becomes apparent that kk is in a certain sense the connective KK-theory. In order to give a more precise statement we need some preparations.

First of all it is clear that there is an abvious natural transformation $\chi : kk_* \rightarrow KK_*$ since even for (arbitrary separable) C^*-algebras A,B the canonical map $\mathrm{Hom}(A,B) \rightarrow KK(A,B)$ factors through $[A,B]$ Therefore χ is defined by the commutative diagram

$$\begin{tikzcd}
 & \mathrm{Hom}(C_o(X), C_o(Y) \otimes \mathcal{K}) \arrow[dl] \arrow[dr] & \\
kk(Y,X) = [C_o(X), C_o(Y) \otimes \mathcal{K}] \arrow[rr, "\chi"] & & KK(C_o(X), C_o(Y))
\end{tikzcd}$$

to define $T(X) = \varinjlim (t_r)_* : h_*(X,E(Y)) \to k'^{Y}_*(X)$. Here S' is the composite of the suspension map and the structure map $S^1 \wedge E_r(Y) \to E_{r+1}(Y)$. One can check the naturality of T and, moreover, it is clear that $T(S^o)$ is an isomorphism since $E_r(Y) \wedge S^o = \mathrm{Map}(Y,F(S^r))$. ∎

3.3. WHY kk CAN BE REGARDED AS THE CONNECTIVE KK-THEORY.

Recall that given a spectrum $E = (E_n)$ the homotopy groups of E are defined by $\pi_r(E) = \varinjlim_k [S^{r+k},E_k]$ where the arrows $[S^{r+k},E_k] \to [S^{r+k+1},E_{k+1}]$ are defined via suspensions and the structure maps $g_k : SE_k \to E_{k+1}$.

3.3.1. DEFINITION. [1] If E is a spectrum then the associated connective spectrum is a spectrum E^c together with a map of spectra $E^c \to E$ such that $\pi_r(E^c) \to \pi_r(E)$ is an isomorphism for $r \geq 0$ and $\pi_r(E^c) = 0$ for $r < 0$. E^c is uniquely determined by E up to a weak equivalence. If $h_*(-,E)$ is the homology theory defined by the spectrum E, then the homology theory $h_*(-,E^c)$ defined by the connective spectrum E^c is called the connective $h_*(-,E)$-theory. One has a similar definition for the corresponding cohomology theories. In particular these definitions work for the topological K-theory. The Ω-spectrum BU of the complex K-theory is given by he sequence

$$\Omega U,\ U,\ \Omega U,\ U, \ldots$$

where U is the infinite unitary group $U = U(\infty) = \varinjlim_n U(n)$.
Let $F = (F_n)$ be the Ω-spectrum $F_n := F(S^n) = \mathrm{Hom}(C_o(S^n),\mathcal{K})$ if $n \geq 1$ and $F_o = \Omega F_1$, with the structure maps

$$F_n \to \Omega F_{n+1}$$

given by suspending homomorphisms (see Corollary 3.1.7). Note that F is a ring spectrum with multiplication $\mu : F_n \wedge F_m \to F_{n+m}$, $\mu(\varphi,\psi) = \varphi \otimes \psi$.
Therefore $\pi_*(F)$ has a ring structure.
If is a result of G. Segal [39] that F is the connective spectrum BU^c. We include a proof of this fact and the computation of the ring $\pi_*(F)$.

3.3.2. PROPOSITION. The ring $\pi_*(F)$ is isomorphic to the polynomial ring $\mathbb{Z}[t]$

is exact.

We have to verify the above conditions for the functors k_r^Y. Now 1) it is easily checked since any homotopy f_t between f_o and f_1 induces a homotopy $F(f_t)$ between $F(f_o)$ and $F(f_1)$ (see 3.1.2). The natural transformations σ_n are induced by the natural weak equivalence $\Omega F(SX) \sim F(X)$ described in Corollary 3.1.8. The third condition follows from the first exact sequence in Proposition 3.1.10.

b) The proof is similar but one needs the second exact sequence in Proposition 3.1.10. ■

For future purposes it is useful to find the spectra of these theories, which exist by the Brown-Adams representability theorem. By definition, a spectrum E is a sequence of spaces E_n with base point, provided with structure maps, either

$$g_n : SE_n \longrightarrow E_{n+1}$$

or their adjoints

$$g'_n : E_n \longrightarrow \Omega E_{n+1}$$

A spectrum E is an Ω-spectrum if g'_n is a weak equivalence for each n. Each spectrum E defines a homology theory

$$h_n(X,E) = \varinjlim_r [S^{n+r}, E_r \wedge X]$$

and a cohomology theory

$$h^n(X,E) = \varinjlim_r [S^r X, E_{n+r}]$$

(see [42]) If E is an Ω-spectrum then $h^n(X,E) = [X,E_n]$ because of the equivalences g'_n.

We are interested in the following two spectra:

1) $E_n(Y) = \mathbf{Map}(Y,F(S^n)) = \mathbf{Hom}(C_o(S^n), C_o(Y)\otimes\mathcal{K}), \quad Y \in W_o$

2) $F_n(X) = F(S^n X) = \mathbf{Hom}(C_o(S^n X),\mathcal{K}), \quad X \in W_o^c.$

The structure maps for both these spectra are defined by taking suspensions of homomorphisms. It follows by Remark 3.1.8 b) that both $E_n(Y)$ and $F_n(X)$ are Ω-spectra.

3.2.2. PROPOSITION.

a) On the category W_o^c the generalized homology theory $k_n^Y(-)$ is given by the

$$kk(Y,X) \xrightarrow{\simeq} kk(SY, SX)$$

Consequently the groups $kk_n(Y,X)$ can be as well defined as $kk_n(Y,X) = \lim\limits_{r \to \infty} kk(S^{r+n}Y, S^rX)$ and this definition allows us to work even with non connected spaces.

b) The group operation on $kk(Y,X) = [Y,F(X)]$ given by the (infinite) loop structure of $F(X)$ coincides with that given by the orthogonal sum of homomorphisms.

c) The groups kk_n are contravariant in the first variable and covariant in the second variable. Indeed each $g \in \mathbf{Map}(Y_2,Y_1)$ induces a map $\mathbf{Map}(Y_1,F(X)) \to \mathbf{Map}(Y_2, F(X))$ and then, for each $n \in \mathbf{Z}$, a homomorphism of groups $g^* : kk_n(Y_1,X) \to kk_n(Y_2,X)$. Concerning the second variable, each $f \in \mathbf{Map}(X_1,X_2)$ induces a map $F(X_1) \to F(X_2)$ and then for each $n \in \mathbf{Z}$, a homomorphism of groups $f_* : kk_n(Y,X_1) \to kk_n(Y,X_2)$.

The next result provides us with long exact sequences for the kk-groups.

3.1.10. PROPOSITION. Let $i : A \hookrightarrow X$ be a pair in W_o^c and let $j : B \hookrightarrow Y$ be a pair in W_o. Let $p : X \to X/A$ and $q : Y \to Y/B$ denote the canonical identification maps. There are long exact sequences

a) $\quad kk_{n+1}(Y,X) \xrightarrow{p_*} kk_{n+1}(Y,X/A) \to kk_n(Y,A) \xrightarrow{i_*} kk_n(Y,X) \xrightarrow{p_*} kk_n(Y,X/A)$

b) $\quad kk_n(B,X) \xleftarrow{j^*} kk_n(Y,X) \xleftarrow{q^*} k_n(Y/B,X) \leftarrow kk_{n+1}(B,X) \xleftarrow{j^*} kk_{n+1}(Y,X) \quad n \in \mathbf{Z}$

Proof. a) Let $n \geq 0$. The sequence

$$[Y,\Omega^{n+1}F(X)] \to [Y,\Omega^{n+1}F(X/A)] \to [Y,\Omega^nF(A)] \to [Y,\Omega^nF(X)] \to [Y,\Omega^nF(X/A)]$$

is exact by theorem 3.1.5 and Proposition 3.1.6. Then by definition

$$[Y,\Omega^nF(X)] = [S^nY,F(X)] = kk_n(Y,X).$$

For $n \leq 0$ the same argument works using the isomorphism

$$kk(Y,X) \xrightarrow{\simeq} kk(SY,SX)$$

b) Let $n \geq 0$. We consider the coexact Puppe sequence associated with the pair $j : B \hookrightarrow Y$ (see [43, Chapter I, Thm. 6.22]):

$$S^nB \to S^nY \to S^n(Y/B) \to S^{n+1}B \to S^{n+1}Y$$

($Y \vee I = \{y_0\} \times I \cup Y \times \{0\}$, f = the map onto the based point of E, q = the natural quotient map) obtaining a map $\psi : Y \times I \to E$. Since $\varphi \circ q$ maps $Y \times \{1\}$ to the base point of B, ψ maps $Y \times \{1\}$ into F. This map, denoted by $\psi_1 : Y \to F$, is well defined up to homotopy. By definition one takes $\partial[\varphi] = [\psi_1]$.

When p is only a quasifibration the definition of ∂ is more involved as we have seen in the proof of Proposition 3.1.6. However for the maps $\varphi \in \mathbf{Map}(SY,B)$ which can be covered as in the above diagram the formula $\partial[\varphi] = [\psi_1]$ still holds. Indeed with the notation of 3.1.5. and ψ as above we have the following commutative diagram

$$\begin{tikzcd}
Y \vee F \arrow[rrr] \arrow[d, hook] & & & \mathrm{Cocyl}(p) \arrow[d, "p'"] \\
Y \times F \arrow[r] \arrow[rr, "\psi"] & SY \arrow[rr, "\varphi"'] & E \arrow[ur, "i"] \arrow[r, "p"] & B
\end{tikzcd}$$

which implies that $\partial'[\varphi] = j_*[\psi_1]$. By the very definition of ∂ we have $\partial'[\varphi] = j_*\partial[\varphi]$. Therefore $\partial[\varphi] = [\psi_1]$ since j_* is injective. ∎

Recall that for a C^*-algebra A the suspension of A is defined by $SA = C_0(S^1) \otimes A$. If $\varphi \in \mathbf{Hom}(A,B)$ is a $*$-homomorphism, then its suspension $S\varphi \in \mathbf{Hom}(SA,SB)$ is defined by $S\varphi = id(C_0(S^1)) \otimes \varphi$. More generally given $\varphi_i \in \mathbf{Hom}(A_i,B_i)$, $i = 1,2$, one can consider $\varphi_1 \otimes \varphi_2 \in \mathbf{Hom}(A_1 \otimes A_2, B_1 \otimes B_2)$.

3.1.8. COROLLARY. The suspension map $\varphi \to S\varphi$ induces an weak equivalence $F(X) \to \Omega F(SX)$ for every $X \in W_0^c$, (see 3.1.2).

Proof. By definition $F(X) = \mathbf{Hom}(C_0(X), \mathcal{K})$. The map $F(X) \to \Omega F(SX)$ corresponds to the suspension map

$$\mathbf{Hom}(C_0(X), \mathcal{K}) \to \mathbf{Hom}(C_0(SX), C_0(S^1) \otimes \mathcal{K})$$

via the following identifications

$$\Omega F(SX) = \mathbf{Map}(S^1, \mathbf{Hom}(C_0(SX), \mathcal{K})) = \mathbf{Hom}(C_0(SX), C_0(S^1) \otimes \mathcal{K}).$$

3.1.5. THEOREM (G. Segal [39]). Let X be a compact connected space. If A is a path connected closed subspace of X and is a neighbourhood deformation retract in X, then $F(X) \to F(X/A)$ is a quasifibration with fiber F(A).

For the purposes of homotopy theory the quasifibrations are as good as the fibrations. For instance, given a quasifibration $p: E \to B$ one can replace in the homotopy sequence of the pair $(E, F = p^{-1}(b))$, the groups $\pi_n(E,F)$ with $\pi_n(B)$, in order to obtain the homotopy exact sequence of the quasifibration:

$$\pi_{n+1}(B) \to \pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F).$$

As for fibrations even more is true.

3.1.6. PROPOSITION. If $p: E \to B$ is a quasifibration, with fiber $F = p^{-1}(b_o)$, $b_o \in B$, then for any CW-complex Y there is an exact sequence of groups ($k \geq 1$) and sets ($k = 0$):

$$[Y,\Omega^{k+1}B] \to [Y,\Omega^k F] \to [Y,\Omega^k E] \to [Y,\Omega^k B] \to [Y,\Omega^{k-1}F] \to \dots$$

Proof. Let B^I be the space of all (free) paths in B and let $Cocyl(p) = \{(u,e) \in B^I X E : u(1) = p(e)\}$ be the mapping cocylinder of p. The map $p: Cocyl(p) \to B$ given by $p'(u,e) = u(0)$ is a fibration with fiber $W(p) = \{(u,e) \in B^I X E : u(0) = b_o, u(1) = p(e)\}$. Moreover there is a commutative diagram

$$\begin{tikzcd} F \arrow[r] \arrow[d, "j"] & E \arrow[r, "p"] \arrow[d, "i"] & B \arrow[d, equal] \\ W(p) \arrow[r] & Cocyl(p) \arrow[r, "p'"] & B \end{tikzcd}$$

where i(e) := (the trivial loop at p(e),e) is a homotopy equivalence and j is induced by i. This is the standard construction used to prove that any continuous map is homotopy equivalent to a fibration (see [43, Chapter I, §7]). Now the idea of our proof is to show that j is a weak equivalence. Assume that this was proved. Then it follows by Whitehead's theorem [43, Chapter IV, Thm. 7.17] that j induces on isomorphism $[Y,F] \xrightarrow{\sim} [Y,W(p)]$ for each CW-complex Y. Let us consider the following diagram with

3.1. THE GROUPS kk_n

3.1.1. For A,B C^*-algebras, **Hom**(A,B) will denote the space of all $*$-homomorphism $A \longrightarrow B$ with the topology of pointwise-norm convergence. This is a pointed space - the base-point is the null homomorphism. We define [A,B] to be the set of homotopy classes of homomorphisms in **Hom**(A,B). If X,Y are pointed topological spaces then we define **Map**(X,Y) to be the space of all continuous base-point preserving maps of X into Y endowed with the compact-open topology. The space **Map**(S^1,X) is denoted by ΩX and is called the space of loops in X. For $f \in$ **Map**(X,Y), $\Omega f \in$ **Map**$(\Omega X, \Omega Y)$ is defined in the obvious way. If X,Y are compact then, via the Ghelfand duality, **Map**(X,Y) can be identified with **Hom**$(C_o(Y), C_o(X))$. Also $[C_o(Y), C_o(X)]$ coincides with [X,Y], the homotopy classes of maps in **Map**(X,Y). We shall use many times the obvious identification

$$\mathbf{Hom}(A, B \otimes C_o(X)) \simeq \mathbf{Map}(X, \mathbf{Hom}(A,B))$$

This gives an isomorphism

$$[A, B \otimes C_o(X)] \simeq [X, \mathbf{Hom}(A,B)].$$

Assume that B is stable, $B \simeq B \otimes K$. Then it is proved in [36] that **Hom**(A,B) is a commutative H-space with respect to the operation defined by letting $\varphi_1 + \varphi_2$ be the composite

$$A \xrightarrow{(\varphi_1,\varphi_2)} B \oplus B \hookrightarrow B \otimes M_2 \simeq B.$$

The homotopy unit is given by the null morphism. In this way [A,B] is a commutative monoid.

3.1.2. Let W_o be the category with base-vertex finite CW-complexes as objects and base-point preserving maps as morphisms. Let W_o^c be the full subcategory of W_o consisting of connected spaces.

Following Segal [39], for each $X \in W_o$, we define $F(X) = \mathbf{Hom}(C_o(X), \mathcal{K})$ and $F_D^k(X) = \mathbf{Hom}(C_o(X), M_k)$. The natural embeddings $M_k \hookrightarrow M_{k+1}$, $a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$, induce

2.2.3. PROPOSITION. Assume that $A \in \mathcal{A}\mathcal{C}(n)$ is unital. Then every state on $(K_0(A), K_0(A)_+, [1_A])$ is induced by some bounded trace state on A.

Proof. Let $A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} \ldots$, $A_i \in \mathcal{C}(n)$, be a faithful inductive system with unital embeddings such that $A = \lim A_i$. Define for each $i \geq 1$ a $*$-homomorphism $\gamma_i : r(A_i) \to r(A_{i+1})$ such that $K_0(\gamma_i) = r(K_0(\varphi_i))$. The AF-algebra $r(A)$ associated in 2.1.5 with the inductive system (A_i, φ_i) can be realized as $\lim(r(A_i), \gamma_i)$. Now let f be a state an $K_0(A)$ and let $f = f' \circ r_A$ be the factorization provided by Proposition 2.2.1. Since the statement of Proposition 2.2.3 holds for AF-algebras [3], there is some trace state σ on $r(A)$ such that f' is induced by σ. For each $i \geq 1$ we define the trace $\tau_i : A_i \to \mathbf{C}$ by $\tau_i = \sigma_i \circ ev_i$ where $ev_i : A_i \to r(A_i)$ is an evaluation map (2.1.2) and σ_i is the restriction of σ to $r(A_i)$. We shall prove that f is induced by any weak limit of the sequence (τ_i). More precisely let ω be a free ultrafilter on $\mathbf{N}$ and define a trace τ on the algebraic inductive limit of A_i by $\tau(a) = \lim_{\omega} \tau_i(a)$. Note that $\|\tau(a)\| \leq \|a\|$ since $\|\tau_i\| = \tau_i(1_{A_i}) = 1$. Therefore we can extend τ on A by continuity. Of course in general it is not true that $\tau|A_i = \tau_i$. However if e is any projection in A_i then $\tau(e) = \tau_i(e)$. To prove this equality it suffices to know that $\tau_{i+1}(e) = \tau_i(e)$ for any projection $e \in A_i$. But this follows from the following commutative diagram

$$\begin{tikzcd}
K_0(A_i) \arrow[rr, "K_0(\varphi_i)"] \arrow[d] & & K_0(A_{i+1}) \arrow[d] \\
K_0(r(A_i)) \arrow[rr, "K_0(\gamma_i)"] \arrow[dr, "f'_i"'] & & K_0(r(A_{i+1})) \arrow[dl, "f'_{i+1}"] \\
& \mathbb{R} &
\end{tikzcd}$$

where f'_i is induced by σ_i. Indeed it is easily checked that $f_i \circ (ev_i)_*([e]) = \tau_i(e)$ as a consequence of the definition of τ_i. Therefore we obtain

$$f([e]) = f'_i \circ r_{A_i}([e]) = \tau_i(e) = \lim_{i \to \infty} \tau_i(e) = \tau(e)$$

It is also clear that the above equalities hold for all projections e in $A_i \otimes \mathcal{K}$. This completes the proof.

The following corollary proves, in a special case, a conjecture in [4].

2.1.11. PROPOSITION. Let $A \in \mathcal{A}\mathcal{C}(n)$. Then $K_o(A)$ has large denominators if and only if $K_o(r(A))$ has large denominators.

Proof. One implication is trivial since $r_A : K_o(A) \to K_o(r(A))$ is a surjective morphism of ordered groups. To prove the other assume that $K_o(r(A))$ has large denominators and fix $a \in K_o(A)_+$ and $k \in \mathbf{N}$. By assumption there are $x \in K_o(r(A))_+$ and $m \in \mathbf{N}$ such that $2kx \leq r_A(a) \leq mx$. Let $b \in K_o(A)_+$ be such that $r_A(b) = x$. We wish to apply Lemma 2.1.10 in order to prove that $kb \leq a$ and $a \leq 2mb$. For the first inequality it is clear that $r_A(kb) \leq r_A(a)$ and we have to check only that $r_A(kb)$ belongs to the order ideal generated by $r_A(a - kb)$. But this is again obvious since $r_A(kb) = kx \leq r_A(a) - kx = r_A(a - kb)$. In the same way $r_A(a) \leq 2mr_A(b)$ and $r_A(a) \leq r_A(2mb - a)$ imply $a \leq 2mb$.

2.1.12. COROLLARY. Let $A \in \mathcal{A}\mathcal{C}(n)$ be such that $r(A)$ is not stably isomorphic to $\mathcal{K}$. Then $K_o(A)$ is simple if and only if $K_o(r(A))$ is simple.

Proof. One implication is again trivial. To prove the other assume that $K_o(r(A))$ is simple. Let J be a nonzero ideal in $K_o(A)$. We shall prove that J contains any given positive element $a \in K_o(A)$. Indeed if $b \in J$, $b > 0$ then $2r_A(a) \leq mr_A(b)$ for some $m \in \mathbf{N}$, since $K_o(A)$ is simple. Therefore $r_A(a) \leq r_A(mb)$ and $r_A(a)$ belongs to the order ideal generated by $r_A(mb - a)$ since $r_A(a) \leq r_A(mb - a)$. By Proposition 2.1.7 $K_o(r(A))$ has large denominators and so we can apply Lemma 2.1.10 to get $a \leq mb$. Since $b \in J$ we must have $a \in J$.

We also have the following generalization of Proposition 2.1.7.

2.1.13. COROLLARY. Let $A \in \mathcal{A}\mathcal{C}(n)$ be a simple C^*-algebra such that $r(A)$ is not stably isomorphic to $\mathcal{K}$. (e.g. A is simple, unital and has no nonzero finite dimensional representation). Then $K_o(A)$ has large denominators ,

Proof. If A is simple then $K_o(A)$ is simple. By 2.1.12 $K_o(r(A))$ is simple so that $K_o(r(A))$ has large denominators by Proposition 2.1.7. Finally we apply Proposition 2.1.11 to obtain the desired result.

$r_A : K_o(A) \to K_o(r(A))$ induced by the homomorphisms $r_{A_i} : K_o(A_i) \to K_o(r(A_i))$. Our notation is misleading since it is not clear whether the above AF-algebra depends only on A (and not on the aproximating system (A_i, γ_{ji})). This is certainly true in certain cases pointed out in section 6. In the general case we make the convention that r(A) denotes a fixed AF-algebra arising as described above from a fixed approximating system of A. Note that $(K_o(A), K_o(A)_+)$ is an ordered group since any $A \in \mathcal{AC}(n)$ is stably unital and stably finite. Also, the epimorphism $r_A : K_o(A) \to K_o(r(A))$ is order preserving and does not vanish on the nonzero elements of $K_o(A)_+$.

2.1.6. DEFINITION. ([31]) Let (G,G_+) be an ordered group. We say that G has <u>large denominators</u> if for any $a \geq 0$ and $k \in \mathbf{N}$ there are $b \in G_+$ and $m \in \mathbf{N}$ such that $kb \leq a \leq mb$.

2.1.7. PROPOSITION.([31]) If A is a simple AF-algebra, non stably isomorphic to $\mathcal{K}$, then $K_o(A)$ has large denominators.

2.1.8. Here we define the notion of m-large and m-full homomorphism.

Let $\mathbf{Z}^q, \mathbf{Z}^h$ have the standard orderings $\mathbf{Z}^q_+ = \mathbf{N}^q$, $\mathbf{Z}^h_+ = \mathbf{N}^h$ and the order units $u = (n_1, \dots, n_q)$ and $v = (m_1, \dots, m_l)$. Let $\gamma = [k_{ji}] : \mathbf{Z}^q \to \mathbf{Z}^h$ be a morphism of scaled ordered groups. This is equivalent to say that $k_{ij} \geq 0$ for all i, j and

$$t_j := m_j - \sum_{i=1}^{q} k_{ji} n_i \geq 0 \text{ for all } j.$$

The morphism γ is called <u>m-large</u> ($m \geq 0$) if satisfies the following two conditions:

a) if some $k_{ji} > 0$ then $k_{ji} \geq m$

b) if some $t_j > 0$ then $t_j \geq m$.

If, in addition, all $k_{ji} > 0$ then γ is called <u>m-full</u>. We extend the above definitions to the morphisms of (scaled) ordered groups $\sigma : K_o(A) \to K_o(B)$ with $A,B \in \mathcal{C}(n)$, by saying that σ is <u>m-large</u> (<u>m-full</u>) iff the morphism $r(\sigma)$ defined in 2.1.5 is m-large (m-full). By extension we say that a morphism $\gamma : A \to B$, $A,B \in \mathcal{C}(n)$ is <u>m-large</u> (<u>m-full</u>) iff $K_o(\gamma)$ is m-large (m-full). Note that since $\mathcal{F} \subset \mathcal{C}(n)$ the above definitions also apply to morphisms of finite dimensional C^*-algebras. The product of a m-large morphism by a p-large morphism is mp-large. Also the product of a m-full morphism by a p-full

b) If $E_1, E_2 \in Vect_k(X)$, $k \geq \langle n/2 \rangle$, and $E_1 \oplus G$ is isomorphic to $E_2 \oplus G$ for some $G \in Vect(X)$ then E_1 is isomorphic to E_2.

c) If $k \geq 0$ then the Grassmannian $G_k(k + \langle n/2 \rangle)$ classifies all vector bundles of rank k over X i.e. $Vect_k(X) \simeq [X, G_k(k + \langle n/2 \rangle)]$.

The following corollary is a direct consequence of Theorem 2.1.1. See also [3, Ex. 6.10.3].

2.1.2. COROLLARY. Let $A \in \mathcal{C}(n)$ be as above.

a) Let $A = (a_1, \ldots, a_q) \in K_o(A)$ and $r_A(a) = (s_1, \ldots s_q) \in K_o(r(A)) \simeq \mathbf{Z}^q$. If for each i, $1 \leq i \leq q$, we have either $a_i = 0$ or $s_i \geq n$ then $a \in K_o(A)_+$.

b) If $a \in K_o(A)_+$ and for each i, $1 \leq i \leq q$, the i^{th} component of $r_A(a)$ is no greater than $n_i - \langle n/2 \rangle$, then $a \in \Sigma(A)$ i.e. there is some projection in A whose K-theory class equals a.

c) Let $x = (a,b) \in K_*(A)$ with $a = (a_1, \ldots, a_q)$, $b = (b_1, \ldots, b_q)$ and $r_A(a) = (s_1, \ldots, s_q)$. If for each $1 \leq i \leq q$, one has either $a_i = b_i = 0$ or $s_i \geq n + 1$, then $x \in K_*(A)_+$ (see 1.2.7).

d) Let $x = (a,b) \in K_*(A)_+$. If for each $1 \leq i \leq q$, the i^{th} component of $r_A(a)$ is no greater that $n_i - \langle (n+1)/2 \rangle$ then $x \in \Sigma_*(A)$ (see 1.2.7).

We shall prove below that the assignment $A \mapsto r(A)$ is functorial. To this purpose, given $A, B \in \mathcal{C}(n)$ and $\sigma \in \mathbf{Hom}(K_o(A), K_o(B))_{+,\Sigma}$, it is useful to consider the matrix of σ with respect to the decomposition

$$K_o(A) = K_o'(A) \oplus K_o(r(A)), \quad K_o(B) = K_o'(B) \oplus K_o((B)).$$

2.1.3. PROPOSITION. If $\sigma : K_o(A) \to K_o(B)$ is an homomorphism of (scaled) ordered groups, then its matrix with respect to the above decompositions is triangular: $\sigma = \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix}$ and $\gamma : K_o(r(A)) \to K_o(r(B))$ is a homomorphism of (scaled) ordered groups.

Proof. If $r(A) = \bigoplus_{i=1}^{q} M_{n_i}$ then $K_o(r(A)) \simeq \mathbf{Z}^q$. The description of $r_A : K_o(A) \to K_o(r(A))$ is as follows: given $E_i \in Vect(X_i)$, $1 \leq i \leq q$, epimorphism for $q = m$. If $m = \infty$ then f is called a weak homotopy equivalence.

1.2.11. A sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ is called exact if for any space W, the sequence $[W,X] \xrightarrow{f_*} [W,Y] \xrightarrow{g_*} [W,Z]$ is an exact sequence of pointed sets i.e. $\mathrm{image}(f_*) = g_*^{-1}(\underline{0})$ where $\underline{0}$ is the trivial element in $[W,Z]$. If this happens only for CW-complexes W, then the above sequence is called weak exact.

1.2.12. In order to simplify the terminology we shall mean by "category" a mathematical entity which satisfies all the axioms of the usual categories, except possibly the axiom which posulates the existence of identities. The term "subcategory" will be used in accordance with the above convention. Now if $\mathcal{C}$ is a "subcategory" of $\mathcal{S}$ then $\mathcal{A}\mathcal{C}$ will denote the class of all C^*-algebras which can be represented as inductive limits of countable inductive systems (with injective bonding morphisms) of C^*-algebras in $\mathcal{C}$. For instance if $\mathcal{F}$ is the category of finite dimensional C^*-algebras then $\mathcal{A}\mathcal{F}$ consists of all AF-C^*-algebras [7].

For the sake of brevity we shall use the terms algebra for C^*-algebra and homomorphism, or even morphism for $*$-homomorphism.

2. ORDERED K-THEORY AND LARGE DENOMINATORS

Recall that $\mathcal{C}(n)$ denotes the category of the C^*-algebras of the form

$$\bigoplus_{i=1}^{q} C(X_i) \otimes M_{n_i}, \quad q > 0,$$

where $X_1, \dots, X_q$ are arbitrary finite connected CW-complex with $\mathbf{dim}(X_i) \leq n$. The morphisms of $\mathcal{C}(n)$ are arbitrary C^*-algebras homomorphisms. Let $\mathcal{A}\mathcal{C}(n)$ be the class of C^*-algebras defined as in 1.2.12.

The methods we shall give in the next sections for computing the homotopy classes of $*$-homomorphisms use in an essential way some stability properties of vector bundles and $*$-homomorphisms. For this reasons our calculations apply only to those morphisms in $\mathcal{C}(n)$ which are large enough in the sense of definition 2.1.8. Therefore it is natural to ask which inductive limits of C^*-algebras from $\mathcal{C}(n)$ can be written as limits continuous complex functions on X). We may identify $C_o(X)^1$ with $C(X^1)$ where X^1 denotes the one-point compactification of X. We shall use the notation $C_o(X)$ even for spaces X with base point $x_o \in X$ to mean $C_o(X) := C_o(X \setminus \{x_o\})$.

1.2.3. Let M_k denote the C^*-algebra of all $k \times k$ complex matrices and $\mathcal{K}$ the C^*-algebra of compact operators on an infinite separable Hilbert space. There are natural embeddings $M_k \hookrightarrow M_{k+1} \hookrightarrow K$ such that $\mathcal{K} = \lim M_k$.

1.2.4. If X is a compact space with base point $x_o \in X$ then the restriction map $\mathbf{Hom}_1(C(X),M_k) \to \mathbf{Hom}\,(C_o(X),M_k)$ is a homeomorphism of topological space. This is easily seen if we recall that for any $\varphi \in \mathbf{Hom}_1(C(X),M_k)$ there are $x_1,\dots,x_r \in X$ and mutually orthogonal projections $p_1,\dots,p_r \in M_k$ with $\sum p_i = 1_k$ such that $\varphi(f) = \sum_{i=1}^{n} f(x_i)p_i$ for each $f \in C(X)$. We say that $x_1,\dots,x_r$ are the proper values of φ and $p_1,\dots,p_r$ the spectral projections of φ.

1.2.5. If A is a unital $*$-algebra then U(A) stands for the unitary group of A. For neunital A let U(A) be the subgroup of all $u \in U(A^1)$ with $\mu(u) = 1_{\mathbf{C}}$.

1.2.6. If A is a C^*-algebra we denote by V(A) the semigroup of equivalence classes of projections in $A \otimes \mathcal{K}$, with orthogonal addition, and $K_*(A) = K_o(A) \oplus K_1(A)$ the K-groups of A. There is a canonical homomorphism from V(A) to $K_o(A)$. If $A \otimes \mathcal{K}$ has an approximate identity consisting of projections then $K_o(A)$ can be identified with the Grothendieck group of V(A). The image of V(A) in $K_o(A)$ is denoted by $K_o(A)_+$. We denote by $\Sigma(A)$ the subset of $K_o(A)$ corresponding to the projections of A. The triple $(K_o(A), K_o(A)_+, \Sigma(A))$ is a preordered scaled group ([3]).

1.2.7. There is a split extension

$$0 \to SA \to C(S^1) \otimes A \to A \to 0$$

where $SA = C_o(S^1) \otimes A$, which gives a natural isomorphism

$$K_o(C(S^1) \otimes A) \simeq K_o(A) \oplus K_o(SA) \simeq K_o(A) \oplus K_1(A) = K_*(A)$$

which commutes with inductive limits.

Let $A = \lim A_i$, $B = \lim B_i$ be unital C^*-algebras where $A_i, B_i \in \mathcal{C}'_j(2n)$ ($j = 2, 3$ is fixed, $n \geq 1$).

Assume that both A and B have no nonzero finite dimensional representations and also that the group $K_0(A)$ has no proper order ideals. Then the following assertions are equivalent

1) $K_0(A) \simeq K_0(B)$ as scaled ordered groups and $K_1(A) \simeq K_1(B)$.

2) There is a diagram of C^*-algebras and $*$-homomorphisms

$$\begin{tikzcd}
A_{i_1} \arrow[r] \arrow[dr] & \cdots \arrow[r] & A_{i_2} \arrow[r] \arrow[dr] & \cdots \arrow[r] & A_{i_3} \arrow[r] \arrow[dr] & \cdots \\
& B_{j_1} \arrow[ur] \arrow[r] & \cdots \arrow[r] & B_{j_2} \arrow[ur] \arrow[r] & \cdots \arrow[r] & B_{j_3} \arrow[r] & \cdots
\end{tikzcd}$$

with each triangle homotopy-commutative.

3) A is shape equivalent to B in the category of separable C^*-algebras in the sense of Blackadar [2].

The (hard) implication 1)$\Rightarrow$2) is somewhat surprising since we know by the work of Loring [27] that C^*-algebras as $C(S^1 \times S^1)$ and $C(S^2)$ are not semiprojective in the sense of Effros-Kaminker [17].

Also the general shape theory of Blackadar [2] does not give 2) even we assume A to be isomorphic to B.

In order to handle such delicate situations we introduce the notion of $KK_{+,\Sigma}$-semiprojectivity as a K-theoretical analogue of semiprojectivity. Let us briefly described the idea of the proof. Having an isomorphism $K_*(A) \simeq K_*(B)$ one first constructs a diagram as at the point 2) but in the category $KK_{+,\Sigma}$(5.1.1,b)). At this stage the formalism based on $KK_{+,\Sigma}$-semiprojectivity (of section 5) and large denominators (of section 2) are used together with some results of [37]. Then one has to replace $KK_{+,\Sigma}$-homomorphisms by actual homomorphisms. This is done by first replacing them by kk-homomorphisms using Theorems 3.4.5-3.4.6 and 3.5.2 (where the main restriction on X_k actually arise!) and then by applying the stabilization techniques of sections 4 and 6 (Theorems 4.2.8, 4.2.11, 4.3.1, 4.3.2, 6.4.2, 6.4.4.)