CONTINUOUS FIELDS OF C*-ALGEBRAS OVER FINITE DIMENSIONAL SPACES

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1. Introduction

Gelfand's characterization of commutative C*-algebras has suggested the problem of representing non-commutative C*-algebras as sections of bundles. A theorem of Fell [18] and Tomiyama and Takesaki [37] asserts that a unital C*-algebra all of whose irreducible representations are of the same finite dimension n is given by the continuous sections of a locally trivial bundle over a compact Hausdorff space with fiber $M_n(\mathbb{C})$. It is well-known that the picture changes dramatically for C*-algebras which admit infinite dimensional irreducible representations. Let B be a simple C*-algebra such that $B \cong B \otimes M_n(\mathbb{C})$. For instance B is the compact operators $\mathcal{K}$ or the Cuntz algebra $\mathcal{O}_n$. Then the tensor product of $\{f \in C[0,1] \otimes M_n(\mathbb{C}) : f(1) \text{ is diagonal}\}$ and B is a C*-algebra whose primitive spectrum is not Hausdorff even though all its primitive quotients are isomorphic to B.

What if the primitive spectrum X of a separable C*-algebra A is Hausdorff? Then, by a result of Fell [18], A is isomorphic to the C*-algebra of continuous sections vanishing at infinity of a continuous field of simple C*-algebras over X. In particular A is a continuous $C(X)$-algebra in the sense of Kasparov [20]. This description is very satisfactory, since as explained in [6, Remark 6.7], the continuous fields of C*-algebras are in natural correspondence with the bundles of C*-algebras in the sense of topology. Nevertheless, only a tiny fraction of the continuous fields of C*-algebras correspond to locally trivial bundles.

In this paper we prove automatic and conditional local/global trivialization results for continuous fields of Kirchberg algebras. By a Kirchberg algebra we mean a purely infinite simple nuclear separable C*-algebra [31]. Notable examples include the simple Cuntz-Krieger algebras [10]. The following theorem illustrates of our results.

Theorem 1.1. *A separable unital $C(X)$-algebra A over a finite dimensional compact Hausdorff space X all of whose fibers are isomorphic to the same Cuntz algebra $\mathcal{O}_n$, $n \in \{2, 3, \dots, \infty\}$, is locally trivial. If $n = 2$ or $n = \infty$, then $A \cong C(X) \otimes \mathcal{O}_n$. If $3 \le n < \infty$, then A is isomorphic to $C(X) \otimes \mathcal{O}_n$ if and only if $(n-1)[1_A] = 0$ in $K_0(A)$.*

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The case $X = [0,1]$ of Theorem 1.1 was proved in a joint paper with G. Elliott [13].

We compute the homotopy classes

$$[X, \mathrm{Aut}(\mathcal{O}_n)] \cong \begin{cases} K_1(C(X) \otimes \mathcal{O}_n) & \text{if } 2 \le n < \infty, \\ \{*\} & \text{if } n = \infty, \end{cases}$$

(see Theorem 7.3) and hence classify the unital separable $C(SX)$-algebras A with fiber $\mathcal{O}_n$ over the suspension SX of a finite dimensional metrizable Hausdorff space X.

To put our results in perspective, let us recall that none of the general basic properties of a continuous field implies any kind of local triviality. An example of a continuous field of Kirchberg algebras over $[0,1]$ which is not locally trivial at any point even though all of its fibers are mutually isomorphic is exhibited in [13, Ex. 8.4]. Examples of nonexact continuous fields with similar properties were found by S. Wassermann [38].

It is very interesting that the *only obstruction* to local or global triviality for a continuous field of Kirchberg algebras is of purely K-theoretical nature.

Theorem 1.2. *Let A be a separable C*-algebra whose primitive spectrum X is compact Hausdorff and of finite dimension. Suppose that each primitive quotient $A(x)$ of A is nuclear, purely infinite and stable. Then A is isomorphic to $C(X) \otimes D$, for some KK-semiprojective stable Kirchberg algebra D, if and only if there is $\sigma \in KK(D, A)$ such that $\sigma_x \in KK(D, A(x))^{-1}$ for all $x \in X$. For any such σ there is an isomorphism of $C(X)$-algebras $\Phi : C(X) \otimes D \to A$ such that $KK(\Phi|_D) = \sigma$.*

We have an entirely result similar result covering the unital case: Theorem 7.2. The required existence of σ is a KK-theoretical analog of the classical condition of Fell that appears in the trivialization theorem of Dixmier and Douady [15] of continuous fields with fibers isomorphic to the compact operators. An important feature of our condition is that it is a priori much weaker than the condition that A is $KK_{C(X)}$-equivalent to $C(X) \otimes D$. In particular, we do not need to worry at all about the potentially hard issue of constructing elements in $KK_{C(X)}(A, C(X) \otimes D)$. To illustrate this point, let us note that it is almost trivial to verify that the local existence of σ is automatic for unital $C(X)$-algebras with fiber $\mathcal{O}_n$ and hence to derive Theorem 1.1. A (unital) C*-algebra D has the *automatic local triviality property* (in the unital sense) if any separable (unital) $C(X)$-algebra over a finite dimensional compact Hausdorff space X all of whose fibers are isomorphic to D is locally trivial. The automatic triviality property is defined similarly..

Theorem 1.3. *(*Automatic local triviality*) A separable continuous $C(X)$-algebra over a finite dimensional compact Hausdorff space X all of whose fibers are isomorphic to $\mathcal{O}_2 \otimes \mathcal{K}$ is isomorphic to $C(X) \otimes \mathcal{O}_2 \otimes \mathcal{K}$. The C*-algebra $\mathcal{O}_2 \otimes \mathcal{K}$ is the only Kirchberg algebra satisfying the automatic local triviality property (with no assumptions on units).*

Theorem 1.4. *(*Automatic local triviality in the unital sense*) A unital KK-semiprojective Kirchberg algebra D has the automatic local triviality property in the unital sense if and only if all unital $*$-endomorphisms of D are KK-equivalences. In that case, if A is a separable unital $C(X)$-algebra over a finite dimensional compact Hausdorff space X all of whose fibers are isomorphic to D, then $A \cong C(X) \otimes D$ if and only if there is $\sigma \in KK(D, A)$ such that $K_0(\sigma)[1_D] = [1_A]$.*

A separable C*-algebra D is KK-semiprojective if the functor $KK(D, -)$ is continuous. The class of KK-semiprojective C*-algebras includes the nuclear semiprojective C*-algebras and also the C*-algebras which satisfy the Universal Coefficient Theorem in KK-theory (abbreviated UCT [33]) and whose K-theory groups are finitely generated.

It is natural to ask if there are other unital Kirchberg algebras besides the Cuntz algebras which have the automatic (local) triviality property in the unital sense. Let (s, k, m, q) be a quadruple of integers satisfying the following conditions: $s \in \{0, 1\}$, $k \in \{s, 2s, 3s, \dots\}$, $0 \le q < m$ and $qp_1 ... p_n$ divides both m and k whenever $m \ge 2$ and $m = p_1^{r_1} \cdots p_n^{r_n}$ is the factorization of m into distinct primes. For each quadruple (s, k, m, q) as above, let us choose a unital Kirchberg algebra $D = D(s, k, m, q)$ which satisfies the UCT and such that $(K_0(D), [1_D]) = (s\mathbb{Z} \oplus \mathbb{Z}/m, k \oplus \bar{q})$ and $K_1(D) = 0$. Note that $D(s, k, m, q)$ is not stably isomorphic to a Cuntz algebra unless $(m-1)s = 0$. In that case, $D(0, 0, m, q) \cong M_q(\mathcal{O}_{m+1})$ (where $M_0(\mathcal{O}_2) \equiv \mathcal{O}_2$) and $D(1, k, 1, 0) \cong M_k(\mathcal{O}_\infty)$.

Theorem 1.5. *(*Automatic local triviality in the unital sense – the UCT case*) The series $D(s, k, m, q)$ (respectively $\mathcal{O}_2, \mathcal{O}_\infty$) is a complete list, up to isomorphism, of all unital Kirchberg algebras which satisfy the UCT and have finitely generated K-theory groups and the automatic local triviality (respectively automatic triviality) property in the unital sense. A separable unital $C(X)$-algebra A over a finite dimensional compact Hausdorff space X all of whose fibers are isomorphic to the same C*-algebra $D(s, k, m, q)$ is trivial if and only if $[1_A] \in k\, K_0(A) + q \operatorname{Tor}(K_0(A), \mathbb{Z}/m)$.*

We use semiprojectivity (in various flavors) to approximate and represent continuous $C(X)$-algebras as inductive limits of fibered products of n locally trivial $C(X)$-subalgebras where $n \le \dim(X) < \infty$. This clarifies the local structure of many $C(X)$-algebras (see Theorem 5.2) and gives a new understanding of the K-theory of separable continuous $C(X)$-algebras with arbitrary nuclear fibers. Indeed any such an algebra is $KK_{C(X)}$-equivalent to a continuous $C(X)$-algebra whose fibers are Kirchberg algebras provided that $\dim(X) < \infty$ (see Theorem 8.4). This result in conjunction with Theorem 4.6 leads to a new permanence property for the class of nuclear C*-algebras which satisfy the UCT:

Theorem 1.6. *A separable nuclear continuous $C(X)$-algebra over a finite dimensional locally compact space satisfies the UCT if all its fibers satisfy the UCT.*

A remarkable isomorphism result for separable nuclear strongly purely infinite stable C*-algebras was announced (with a sketched proof) by Kirchberg in [22]: two such C*-algebras A and B with the same primitive spectrum X are isomorphic if and only if they are KK_X-equivalent. This is always the case after tensoring with $\mathcal{O}_2$. However the problem of recognizing when A and B are KK_X-equivalent is open even for very simple spaces X such as the unit interval or non-Hausdorff spaces with more than two points.

The proof of Theorem 4.6 (one of our main results) generalizes and refines a technique that was pioneered for fields over zero dimensional spaces in joint work with Pasnicu [14] and for fields over an interval in joint work with G. Elliott [13]. We shall rely heavily on the classification theorem (and related results) of Kirchberg [21] and Phillips [30], and on the work on non-simple nuclear purely infinite C*-algebras of Blanchard and Kirchberg [7], [6] and Kirchberg and Rørdam [23], [24]. The subtriviality theorem of Blanchard [5] is used in the proof of Theorem 1.6. A basic reference for Kirchberg algebras is [31].

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2. $C(X)$-algebras

Let X be a locally compact Hausdorff space. A $C(X)$-algebra is a C*-algebra A endowed with a $*$-homomorphism θ from $C_0(X)$ to the center $ZM(A)$ of the multiplier algebra $M(A)$ of A such that $C_0(X)A$ is dense in A; see [20], [4]. We write fa rather than $\theta(f)a$ for $f \in C_0(X)$ and $a \in A$. If $Y \subseteq X$ is a closed set, we let $C_0(X, Y)$ denote the ideal of $C_0(X)$ consisting of functions vanishing on Y. Then $C_0(X, Y)A$ is a closed two-sided ideal of A (by Cohen factorization). The quotient of A by this ideal is a $C(X)$-algebra denoted by $A(Y)$ and is called the restriction of $A = A(X)$ to Y. The quotient map is denoted by $\pi_Y : A(X) \to A(Y)$. If Z is a closed subset of Y we have a natural restriction map $\pi^Y_Z : A(Y) \to A(Z)$ and $\pi_Z = \pi^Y_Z \circ \pi_Y$. If Y reduces to a point x, we write $A(x)$ for $A(\{x\})$ and π_x for $\pi_{\{x\}}$. The C*-algebra $A(x)$ is called the fiber of A at x. The image $\pi_x(a) \in A(x)$ of $a \in A$ is denoted by $a(x)$. A morphism of $C(X)$-algebras $\eta : A \to B$ induces a morphism $\eta_Y : A(Y) \to B(Y)$. If $Y = \emptyset$ then $A(Y)$ is interpreted as the zero algebra. If $A(x) \neq 0$ for x in a dense subset of X, then θ is injective. If X is compact, then $\theta(1) = 1_{M(A)}$. Let A be a C*-algebra, $a \in A$ and $\mathcal{F}, \mathcal{G} \subseteq A$. Throughout the paper we will assume that X is a compact Hausdorff space unless stated otherwise. If $\varepsilon > 0$, we write $a \in_\varepsilon \mathcal{F}$ if there is $b \in \mathcal{F}$ such that $\|a - b\| < \varepsilon$. Similarly, we write $\mathcal{F} \subset_\varepsilon \mathcal{G}$ if $a \in_\varepsilon \mathcal{G}$ for every $a \in \mathcal{F}$. The following lemma collects some basic properties of $C(X)$-algebras.

Lemma 2.1. *Let A be a $C(X)$-algebra and let $B \subset A$ be a $C(X)$-subalgebra. Let $a \in A$ and let Y be a closed subset of X.*

(i) *The map $x \mapsto \|a(x)\|$ is upper semi-continuous.*

(ii) $\|\pi_Y(a)\| = \max\{\|\pi_x(a)\| : x \in Y\}$
(iii) *If* $a(x) \in \pi_x(B)$ *for all* $x \in X$, *then* $a \in B$.
(iv) *If* $\delta > 0$ *and* $a(x) \in_\delta \pi_x(B)$ *for all* $x \in X$, *then* $a \in_\delta B$.
(v) *The restriction of* $\pi_x : A \to A(x)$ *to* B *induces an isomorphism* $B(x) \cong \pi_x(B)$ *for all* $x \in X$.

Proof. (i), (ii) are proved in [4] and (iii) follows from (iv). (iv): By assumption, for each $x \in X$, there is $b_x \in B$ such that $\|\pi_x(a - b_x)\| < \delta$. Using (i) and (ii), we find a closed neighborhood U_x of x such that $\|\pi_{U_x}(a - b_x)\| < \delta$. Since X is compact, there is a finite subcover (U_{x_i}). Let (α_i) be a partition of unity subordinated to this cover. Setting $b = \sum_i \alpha_i b_{x_i} \in B$, one checks immediately that $\|\pi_x(a - b)\| \le \sum_i \alpha_i(x)\|\pi_x(a - b_{x_i})\| < \delta$, for all $x \in X$. Thus $\|a - b\| < \delta$ by (ii). (v): If $\iota : B \hookrightarrow A$ is the inclusion map, then $\pi_x(B)$ coincides with the image of $\iota_x : B/C(X, x)B \to A/C(X, x)A$. Thus it suffices to check that ι_x is injective. If $\iota_x(b + C(X, x)B) = \pi_x(b) = 0$ for some $b \in B$, then $b = fa$ for some $f \in C(X, x)$ and some $a \in A$. If (f_λ) is an approximate unit of $C(X, x)$, then $b = \lim_\lambda f_\lambda fa = \lim_\lambda f_\lambda b$ and hence $b \in C(X, x)B$. □

A $C(X)$-algebra such that the map $x \mapsto \|a(x)\|$ is continuous for all $a \in A$ is called a *continuous* $C(X)$-algebra or a C*-bundle [4], [25], [6]. A C*-algebra A is a continuous $C(X)$-algebra if and only if A is the C*-algebra of continuous sections of a continuous field of C*-algebras over X in the sense of [15, Def. 10.3.1], (see [4], [6], [29]).

Lemma 2.2. *Let A be a separable continuous $C(X)$-algebra over a locally compact Hausdorff space X. If all the fibers of A are nonzero, then X has a countable basis of open sets. Thus the compact subspaces of X are metrizable.*

Proof. Since A is separable, its primitive spectrum $\mathrm{Prim}(A)$ has a countable basis of open sets by [15, 3.3.4]. The continuous map $\eta : \mathrm{Prim}(A) \to X$ (induced by $\theta : C_0(X) \to ZM(A) \cong C_b(\mathrm{Prim}(A))$) is open since the $C(X)$-algebra A is continuous and surjective since $A(x) \neq 0$ for all $x \in X$ (see [6, p. 388] and [29, Prop. 2.1, Thm. 2.3]). □

Lemma 2.3. *Let X be a compact metrizable space. A $C(X)$-algebra A all of whose fibers are nonzero and simple is continuous if and only if there is $e \in A$ such that $\|e(x)\| \ge 1$ for all $x \in X$.*

Proof. By Lemma 2.1(i) it suffices to prove that $\liminf_{n\to\infty} \|a(x_n)\| \ge \|a(x_0)\|$ for any $a \in A$ and any sequence (x_n) converging to x_0 in X. Set $D = A(x_0)$ and let e be as in the statement. Let $\psi : D \to A$ be a set-theoretical lifting of id_D such that $\|\psi(d)\| = \|d\|$ for all $d \in D$. Then $\lim_{n\to\infty} \|\pi_{x_n}\psi(a(x_0)) - a(x_n)\| = 0$ for all $a \in A$, by Lemma 2.1(i). By applying this to e, since $\|e(x_n)\| \ge 1$, we see that $\liminf_{n\to\infty} \|\pi_{x_n}\psi(e(x_0))\| \ge 1$. Since D is a simple C*-algebra, if $\varphi_n : D \to B_n$ is a sequence of contractive maps such that $\lim_{n\to\infty} \|\varphi_n(\lambda c + d) - \lambda\varphi_n(c) - \varphi_n(d)\| = 0$, $\lim_{n\to\infty} \|\varphi_n(cd) - \varphi_n(c)\varphi_n(d)\| = 0$, $\lim_{n\to\infty} \|\varphi_n(c^*) - \varphi_n(c)^*\| = 0$, for all $c, d \in D$, $\lambda \in \mathbb{C}$, and $\liminf_{n\to\infty} \|\varphi_n(c)\| > 0$ for some $c \in D$, then $\lim_{n\to\infty} \|\varphi_n(c)\| = \|c\|$ for all $c \in D$. In particular this observation applies to $\varphi_n = \pi_{x_n}\psi$ by Lemma 2.1(i). Therefore

$$\liminf_{n\to\infty} \|a(x_n)\| \geq \liminf_{n\to\infty} \big(\|\pi_{x_n}\psi(a(x_0))\| - \|\pi_{x_n}\psi(a(x_0)) - a(x_n)\|\big) = \|a(x_0)\|.$$

Conversely, if A is continuous, take e to be a large multiple of some full element of A. □

Let $\eta : B \to A$ and $\psi : E \to A$ be $*$-homomorphisms. The *pullback* of these maps is

$$B \oplus_{\eta,\psi} E = \{(b, e) \in B \oplus E : \eta(b) = \psi(e)\}.$$

We are going to use pullbacks in the context of $C(X)$-algebras. Let X be a compact space and let Y, Z be closed subsets of X such that $X = Y \cup Z$. The following result is proved in [15, Prop. 10.1.13] for continuous $C(X)$-algebras.

Lemma 2.4. *If A is a $C(X)$-algebra, then A is isomorphic to $A(Y) \oplus_{\pi,\pi} A(Z)$, the pullback of the restriction maps $\pi^Y_{Y\cap Z} : A(Y) \to A(Y \cap Z)$ and $\pi^Z_{Y\cap Z} : A(Z) \to A(Y \cap Z)$.*

Proof. By the universal property of pullbacks, the maps π_Y and π_Z induce a map $\eta : A \to A(Y) \oplus_{\pi,\pi} A(Z)$, $\eta(a) = (\pi_Y(a), \pi_Z(a))$, which is injective by Lemma 2.1(ii). Thus it suffices to show that the range of η is dense. Let $b, c \in A$ be such that $\pi_{Y\cap Z}(b-c) = 0$ and let $\varepsilon > 0$. We shall find $a \in A$ such that $\|\eta(a) - (\pi_Y(b), \pi_Z(c))\| < \varepsilon$. By Lemma 2.1(i), there is an open neighborhood V of $Y \cap Z$ such that $\|\pi_x(b - c)\| < \varepsilon$ for all $x \in V$. Let $\{\lambda, \mu\}$ be a partition of unity on X subordinated to the open cover $\{Y \cup V, Z \cup V\}$. Then $a = \lambda b + \mu c$ is an element of A which has the desired property. □

Let $B \subset A(Y)$ and $E \subset A(Z)$ be $C(X)$-subalgebras such that $\pi^Z_{Y\cap Z}(E) \subseteq \pi^Y_{Y\cap Z}(B)$. As an immediate consequence of Lemma 2.4 we see that the pullback $B \oplus_{\pi^Z_{Y\cap Z},\pi^Y_{Y\cap Z}} E$ is isomorphic to the $C(X)$-subalgebra $B \oplus_{Y\cap Z} E$ of A defined as

$$B \oplus_{Y\cap Z} E = \{a \in A : \pi_Y(a) \in B, \pi_Z(a) \in E\}.$$

Lemma 2.5. *The fibers of $B \oplus_{Y\cap Z} E$ are given by*

$$\pi_x(B \oplus_{Y\cap Z} E) = \begin{cases} \pi_x(B), & \text{if } x \in X \setminus Z, \\ \pi_x(E), & \text{if } x \in Z, \end{cases}$$

and there is an exact sequence of C-algebras*

$$0 \longrightarrow \{b \in B : \pi_{Y\cap Z}(b) = 0\} \longrightarrow B \oplus_{Y\cap Z} E \xrightarrow{\pi_Z} E \longrightarrow 0 \tag{1}$$

Proof. Let $x \in X \setminus Z$. The inclusion $\pi_x(B \oplus_{Y\cap Z} E) \subset \pi_x(B)$ is obvious by definition. Given $b \in B$, let us choose $f \in C(X)$ vanishing on Z and such that $f(x) = 1$. Then $a = (fb, 0)$ is an element of A by Lemma 2.4. Moreover $a \in B \oplus_{Y\cap Z} E$ and $\pi_x(a) = \pi_x(b)$. We have $\pi_Z(B \oplus_{Y\cap Z} E) \subset E$, by definition. Conversely, given $e \in E$, let us observe that $\pi^Z_{Y\cap Z}(e) \in \pi^Y_{Y\cap Z}(B)$ (by assumption) and hence $\pi^Z_{Y\cap Z}(e) = \pi^Y_{Y\cap Z}(b)$ for some $b \in B$. Then $a = (b, e)$ is an element of A by Lemma 2.4 and $\pi_Z(a) = e$. This completes the proof for the first part of the lemma and also it shows that the map π_Z from the sequence (1) is surjective. Its kernel is identified using Lemma 2.1(iii). □

Let X, Y, Z and A be as above. Let $\eta : B \hookrightarrow A(Y)$ be a $C(Y)$-linear $*$-monomorphism and let $\psi : E \hookrightarrow A(Z)$ be a $C(Z)$-linear $*$-monomorphism. Assume that

$$\pi^Z_{Y\cap Z}(\psi(E)) \subseteq \pi^Y_{Y\cap Z}(\eta(B)). \tag{2}$$

This gives a map $\gamma = \eta^{-1}_{Y\cap Z}\psi_{Y\cap Z} : E(Y \cap Z) \to B(Y \cap Z)$. To simplify notation we let π stand for both $\pi^Y_{Y\cap Z}$ and $\pi^Z_{Y\cap Z}$ in the following lemma.

Lemma 2.6. *(a) There are isomorphisms of $C(X)$-algebras*

$$B \oplus_{\pi,\gamma\pi} E \cong B \oplus_{\pi\eta,\pi\psi} E \cong \eta(B) \oplus_{Y\cap Z} \psi(E),$$

where the second isomorphism is given by the map $\chi : B \oplus_{\pi\eta,\pi\psi} E \to A$ induced by the pair (η, ψ). Its components χ_x can be identified with ψ_x for $x \in Z$ and with η_x for $x \in X \setminus Z$.

(b) Condition (2) *is equivalent to $\psi(E) \subset \pi_Z\big(A \oplus_Y \eta(B)\big)$.*

(c) If $\mathcal{F}$ is a finite subset of A such that $\pi_Y(\mathcal{F}) \subset_\varepsilon \eta(B)$ and $\pi_Z(\mathcal{F}) \subset_\varepsilon \psi(E)$, then $\mathcal{F} \subset_\varepsilon \eta(B) \oplus_{Y\cap Z} \psi(E) = \chi\big(B \oplus_{\pi\eta,\pi\psi} E\big)$.

Proof. This is an immediate corollary of Lemmas 2.1, 2.4, 2.5. For illustration, let us verify (c). By assumption $\pi_x(\mathcal{F}) \subset_\varepsilon \eta_x(B)$ for all $x \in X \setminus Z$ and $\pi_z(\mathcal{F}) \subset_\varepsilon \psi_z(E)$ for all $z \in Z$. We deduce from Lemma 2.5 that $\pi_x(\mathcal{F}) \subset_\varepsilon \pi_x(\eta(B) \oplus_{Y\cap Z} \psi(E))$ for all $x \in X$. Therefore $\mathcal{F} \subset_\varepsilon \eta(B) \oplus_{Y\cap Z} \psi(E)$ by Lemma 2.1(iv). □

Definition 2.7. Let $\mathcal{C}$ be a class of C*-algebras. A $C(Z)$-algebra E is called *$\mathcal{C}$-elementary* if there is a finite partition of Z into closed subsets $Z_1, \dots, Z_r$ ($r \geq 1$) and there exist C*-algebras $D_1, \dots, D_r$ in $\mathcal{C}$ such that $E \cong \bigoplus_{i=1}^r C(Z_i) \otimes D_i$. The notion of *category* of a $C(X)$-algebra with respect to a class $\mathcal{C}$ is defined inductively: if A is $\mathcal{C}$-elementary then $\mathrm{cat}_{\mathcal{C}}(A) = 0$; $\mathrm{cat}_{\mathcal{C}}(A) \leq n$ if there are closed subsets Y and Z of X with $X = Y \cup Z$ and there exist a $C(Y)$-algebra B such that $\mathrm{cat}_{\mathcal{C}}(B) \leq n-1$, a $\mathcal{C}$-elementary $C(Z)$-algebra E and a $*$-monomorphism of $C(Y \cap Z)$-algebras $\gamma : E(Y \cap Z) \to B(Y \cap Z)$ such that A is isomorphic to

$$B \oplus_{\pi,\gamma\pi} E = \{(b, d) \in B \oplus E : \pi^Y_{Y\cap Z}(b) = \gamma\pi^Z_{Y\cap Z}(d)\}.$$

By definition $\mathrm{cat}_{\mathcal{C}}(A) = n$ if n is the smallest number with the property that $\mathrm{cat}_{\mathcal{C}}(A) \leq n$. If no such n exists, then $\mathrm{cat}_{\mathcal{C}}(A) = \infty$.

Definition 2.8. Let $\mathcal{C}$ be a class of C*-algebras and let A be a $C(X)$-algebra. An *n-fibered $\mathcal{C}$-monomorphism* $(\psi_0, \ldots, \psi_n)$ *into* A consists of $(n+1)$ $*$-monomorphisms of $C(X)$-algebras $\psi_i : E_i \to A(Y_i)$, where $Y_0, \ldots, Y_n$ is a closed cover of X, each E_i is a $\mathcal{C}$-elementary $C(Y_i)$-algebra and

$$\pi^{Y_i}_{Y_i \cap Y_j} \psi_i(E_i) \subseteq \pi^{Y_j}_{Y_i \cap Y_j} \psi_j(E_j), \quad \text{for all } i \leq j. \tag{3}$$

Given an n-fibered morphism into A we have an associated *continuous* $C(X)$-algebra defined as the fibered product (or pullback) of the $*$-monomorphisms ψ_i:

$$A(\psi_0, \ldots, \psi_n) = \{(d_0, \ldots d_n) :\ d_i \in E_i,\ \pi^{Y_i}_{Y_i \cap Y_j} \psi_i(d_i) = \pi^{Y_j}_{Y_i \cap Y_j} \psi_j(d_j) \text{ for all } i, j\} \tag{4}$$

and an induced $C(X)$-monomorphism (defined by using Lemma 2.4)

$$\eta = \eta_{(\psi_0, \ldots, \psi_n)} : A(\psi_0, \ldots, \psi_n) \to A \subset \bigoplus_{i=0}^{n} A(Y_i),$$

$$\eta(d_0, \ldots d_n) = \big(\psi_0(d_0), \ldots, \psi_n(d_n)\big).$$

There are natural coordinate maps $p_i : A(\psi_0, \ldots, \psi_n) \to E_i$, $p_i(d_0, \ldots, d_n) = d_i$. Let us set $X_k = Y_k \cup \cdots \cup Y_n$. Then, $(\psi_k, \ldots \psi_n)$ is an $(n-k)$-fibered $\mathcal{C}$-monomorphism into $A(X_k)$. Let $\eta_k : A(X_k)(\psi_k, \ldots \psi_n) \to A(X_k)$ be the induced map and set $B_k = A(X_k)(\psi_k, \ldots \psi_n)$. Let us note that $B_0 = A(\psi_0, \ldots, \psi_n)$ and that there are natural $C(X_{k-1})$-isomorphisms

$$B_{k-1} \cong B_k \oplus_{\pi\eta_k, \pi\psi_{k-1}} E_{k-1} \cong B_k \oplus_{\pi, \gamma_k \pi} E_{k-1}. \tag{5}$$

where π stands for $\pi_{X_k \cap Y_{k-1}}$ and $\gamma_k : E_{k-1}(X_k \cap Y_{k-1}) \to B_k(X_k \cap Y_{k-1})$ is defined by $(\gamma_k)_x = (\eta_k)^{-1}_x (\psi_{k-1})_x$, for all $x \in X_k \cap Y_{k-1}$. In particular, this decomposition shows that $\mathrm{cat}_{\mathcal{C}}(A(\psi_0, \ldots, \psi_n)) \leq n$.

Lemma 2.9. *Suppose that the class $\mathcal{C}$ from Definition 2.7 consists of stable Kirchberg algebras. If A is a $C(X)$-algebra over a compact metrizable space X such that* $\mathrm{cat}_{\mathcal{C}}(A) <$ ∞*, then A contains a full properly infinite projection and $A \cong A \otimes \mathcal{O}_\infty \otimes \mathcal{K}$.*

Proof. We prove this by induction on $n = \mathrm{cat}_{\mathcal{C}}(A)$. The case $n = 0$ is immediate since $D \cong D \otimes \mathcal{O}_\infty$ for any Kirchberg algebra D [21]. Let $A = B \oplus_{\pi, \gamma\pi} E$ where B, E and γ are as in Definition 2.7 with $\mathrm{cat}_{\mathcal{C}}(B) = n-1$ and $\mathrm{cat}_{\mathcal{C}}(E) = 0$. Let us consider the exact sequence $0 \to J \to A \to E \to 0$, where $J = \{b \in B : \pi_{Y \cap Z}(b) = 0\}$. Since J is an ideal of $B \cong B \otimes \mathcal{O}_\infty \otimes \mathcal{K}$, J absorbs $\mathcal{O}_\infty \otimes \mathcal{K}$ by [24, Prop. 8.5]. Since both E and J are stable and purely infinite, it follows that A is stable by [32, Prop. 6.12] and purely infinite by [24, Prop. 3.5]. Since A has Hausdorff primitive spectrum, A is strongly purely infinite by [7, Thm. 5.8]. It follows that $A \cong A \otimes \mathcal{O}_\infty$ by [24, Thm. 9.1]. Finally A

contains a full properly infinite projection since there is a full embedding of $\mathcal{O}_2$ into A by [7, Prop. 5.6]. □

3. Semiprojectivity

In this section we study the notion of KK-semiprojectivity. The main result is Theorem 3.11. Let A and B be C*-algebras. Two $*$-homomorphisms $\varphi, \psi : A \to B$ are approximately unitarily equivalent, written $\varphi \approx_u \psi$, if there is a sequence of unitaries (u_n) in the C*-algebra $B^+ = B + \mathbb{C}1$ obtained by adjoining a unit to B, such that $\lim_{n\to\infty} \|u_n\varphi(a)u_n^* - \psi(a)\| = 0$ for all $a \in A$. We say that φ and ψ are asymptotically unitarily equivalent, written $\varphi \approx_{uh} \psi$, if there is a norm continuous unitary valued map $t \to u_t \in B^+$, $t \in [0, 1)$, such that $\lim_{t\to 1} \|u_t\varphi(a)u_t^* - \psi(a)\| = 0$ for all $a \in A$. A $*$-homomorphism $\varphi : D \to A$ is full if $\varphi(d)$ is not contained in any proper two-sided closed ideal of A if $d \in D$ is nonzero.

We shall use several times Kirchberg's Theorem [31, Thm. 8.3.3] and the following theorem of Phillips [30].

Theorem 3.1. *Let A and B be separable C*-algebras such that A is simple and nuclear, $B \cong B \otimes \mathcal{O}_\infty$, and there exist full projections $p \in A$ and $q \in B$. For any $\sigma \in KK(A, B)$ there is a full $*$-homomorphism $\varphi : A \to B$ such that $KK(\varphi) = \sigma$. If $K_0(\sigma)[p] = [q]$ then we may arrange that $\varphi(p) = q$. If $\psi : A \to B$ is another $*$-homomorphism such that $KK(\psi) = KK(\varphi)$ and $\psi(p) = q$, then $\varphi \approx_{uh} \psi$ via a path of unitaries $t \mapsto u_t \in U(qBq)$.*

Theorem 3.1 does not appear in this form in [30] but it is an immediate consequence of [30, Thm. 4.1.1]. Since $pAp\otimes\mathcal{K} \cong A\otimes\mathcal{K}$ and $qBq\otimes\mathcal{K} \cong B\otimes\mathcal{K}$ by [8], and $qBq\otimes\mathcal{O}_\infty \cong qBq$ by [24, Prop. 8.5], it suffices to discuss the case when p and q are the units of A and B. If σ is given, [30, Thm. 4.1.1] yields a full $*$-homomorphism $\varphi : A \to B \otimes \mathcal{K}$ such that $KK(\varphi) = \sigma$. Let $e \in \mathcal{K}$ be a rank-one projection and suppose that $[\varphi(1_A)] = [1_B \otimes e]$ in $K_0(B)$. Since both $\varphi(1_A)$ and $1_B \otimes e$ are full projections and $B \cong B \otimes \mathcal{O}_\infty$, it follows by [30, Lemma 2.1.8] that $u\varphi(1_A)u^* = 1_B \otimes e$ for some unitary in $(B \otimes \mathcal{K})^+$. Replacing φ by $u\,\varphi\,u^*$ we can arrange that $KK(\varphi) = \sigma$ and φ is unital. For the second part of the theorem let us note that any unital $*$-homomorphism $\varphi : A \to B$ is full and if two unital $*$-homomorphisms $\varphi, \psi : A \to B$ are asymptotically unitarily equivalent when regarded as maps into $B \otimes \mathcal{K}$, then $\varphi \approx_{uh} \psi$ when regarded as maps into B, by an argument from the proof of [30, Thm. 4.1.4].

A separable nonzero C*-algebra D is *semiprojective* [2] if for any separable C*-algebra A and any increasing sequence of two-sided closed ideals (J_n) of A with $J = \overline{\cup_n J_n}$, the natural map $\varinjlim \mathrm{Hom}(D, A/J_n) \to \mathrm{Hom}(D, A/J)$ (induced by $\pi_n : A/J_n \to A/J$) is surjective. If we weaken this condition and require only that the above map has dense range, where

$\mathrm{Hom}(D, A/J)$ is given the point-norm topology, then D is called *weakly semiprojective* [17]. These definitions do not change if we drop the separability of A. We shall use (weak) semiprojectivity in the following context. Let A be a $C(X)$-algebra (with X metrizable), let $x \in X$ and set $U_n = \{y \in X : d(y,x) \leq 1/n\}$. Then $J_n = C(X, U_n)A$ is an increasing sequence of ideals of A such that $J = C(X,x)A$, $A/J_n \cong A(U_n)$ and $A/J \cong A(x)$.

Examples 3.2. (Weakly semiprojective C*-algebras) Any finite dimensional C*-algebra is semiprojective. A Kirchberg algebra D satisfying the UCT and having finitely generated K-theory groups is weakly semiprojective by work of Neubüser [28], H. Lin [26] and Spielberg [34]. This also follows from Theorem 3.11 and Proposition 3.13 below. If in addition $K_1(D)$ is torsion free, then D is semiprojective as proved by Spielberg [35] who extended the foundational work of Blackadar [2] and Szymanski [36].

The following generalizations of two results of Loring [27] are used in section 5; see [13].

Proposition 3.3. *Let D be a separable semiprojective C*-algebra. For any finite set $\mathcal{F} \subset D$ and any $\varepsilon > 0$, there exist a finite set $\mathcal{G} \subset D$ and $\delta > 0$ with the following property. Let $\pi : A \to B$ be a surjective $*$-homomorphism, and let $\varphi : D \to B$ and $\gamma : D \to A$ be $*$-homomorphisms such that $\|\pi\gamma(d) - \varphi(d)\| < \delta$ for all $d \in \mathcal{G}$. Then there is a $*$-homomorphism $\psi : D \to A$ such that $\pi\psi = \varphi$ and $\|\gamma(c) - \psi(c)\| < \varepsilon$ for all $c \in \mathcal{F}$.*

Proposition 3.4. *Let D be a separable semiprojective C*-algebra. For any finite set $\mathcal{F} \subset D$ and any $\varepsilon > 0$, there exist a finite set $\mathcal{G} \subset D$ and $\delta > 0$ with the following property. For any two $*$-homomorphisms $\varphi, \psi : D \to B$ such that $\|\varphi(d) - \psi(d)\| < \delta$ for all $d \in \mathcal{G}$, there is a homotopy $\Phi \in \mathrm{Hom}(D, C[0,1] \otimes B)$ such that $\Phi_0 = \varphi$ to $\Phi_1 = \psi$ and $\|\varphi(c) - \Phi_t(c)\| < \varepsilon$ for all $c \in \mathcal{F}$ and $t \in [0,1]$.*

Definition 3.5. A separable C*-algebra D is KK-stable if there is a finite set $\mathcal{G} \subset D$ and there is $\delta > 0$ with the property that for any two $*$-homomorphisms $\varphi, \psi : D \to A$ such that $\|\varphi(a) - \psi(a)\| < \delta$ for all $a \in \mathcal{G}$, one has $KK(\varphi) = KK(\psi)$.

Corollary 3.6. *Any semiprojective C*-algebra is weakly semiprojective and KK-stable.*

Proof. This follows from Proposition 3.4. □

Proposition 3.7. *Let D be a separable weakly semiprojective C*-algebra. For any finite set $\mathcal{F} \subset D$ and any $\varepsilon > 0$ there exist a finite set $\mathcal{G} \subset D$ and $\delta > 0$ such that for any C*-algebras $B \subset A$ and any $*$-homomorphism $\varphi : D \to A$ with $\varphi(\mathcal{G}) \subset_\delta B$, there is a $*$-homomorphism $\psi : D \to B$ such that $\|\varphi(c) - \psi(c)\| < \varepsilon$ for all $c \in \mathcal{F}$. If in addition D is KK-stable, then we can choose $\mathcal{G}$ and δ such that we also have $KK(\psi) = KK(\varphi)$.*

Proof. This follows from [17, Thms. 3.1, 4.6]. Since the result is essential to us we include a short proof. Fix $\mathcal{F}$ and ε. Let $(\mathcal{G}_n)$ be an increasing sequence of finite subsets of D

whose union is dense in D. If the statement is not true, then there are sequences of C*-algebras $C_n \subset A_n$ and $*$-homomorphisms $\varphi_n : D \to A_n$ satisfying $\varphi_n(\mathcal{G}_n) \subset_{1/n} C_n$ and with the property that for any $n \geq 1$ there is no $*$-homomorphism $\psi_n : D \to C_n$ such that $\|\varphi_n(c) - \psi_n(c)\| < \varepsilon$ for all $c \in \mathcal{F}$. Set $B_i = \prod_{n \geq i} A_n$ and $E_i = \prod_{n \geq i} C_n \subset B_i$. If $\nu_i : B_i \to B_{i+1}$ is the natural projection, then $\nu_i(E_i) = E_{i+1}$. Let us observe that if we define $\Phi_i : D \to B_i$ by $\Phi_i(d) = (\varphi_i(d), \varphi_{i+1}(d), \dots)$, then the image of $\Phi = \varinjlim \Phi_i : D \to \varinjlim (B_i, \nu_i)$ is contained in $\varinjlim (E_i, \nu_i)$. Since D is weakly semiprojective, there is i and a $*$-homomorphism $\Psi_i : D \to E_i$, of the form $\Psi_i(d) = (\psi_i(d), \psi_{i+1}(d), \dots)$ such that $\|\Phi_i(c) - \Psi_i(c)\| < \varepsilon$ for all $c \in \mathcal{F}$. Therefore $\|\varphi_i(c) - \psi_i(c)\| < \varepsilon$ for all $c \in \mathcal{F}$ which gives a contradiction. $\square$

Definition 3.8. (a) A separable C*-algebra D is *KK-semiprojective* if for any separable C*-algebra A and any increasing sequence of two-sided closed ideals (J_n) of A with $J = \overline{\cup_n J_n}$, the natural map $\varinjlim KK(D, A/J_n) \to KK(D, A/J)$ is surjective.

(b) We say that the functor $KK(D, -)$ is *continuous* if for any inductive system $B_1 \to B_2 \to \dots$ of separable C*-algebras, the induced map $\varinjlim KK(D, B_n) \to KK(D, \varinjlim B_n)$ is bijective.

Proposition 3.9. *Any separable KK-semiprojective C*-algebra is KK-stable.*

Proof. We shall prove the statement by contradiction. Let D be separable KK-semiprojective C*-algebra. Let $(\mathcal{G}_n)$ be an increasing sequence of finite subsets of D whose union is dense in D. If the statement is not true, then there are sequences of $*$-homomorphisms $\varphi_n, \psi_n : D \to A_n$ such that $\|\varphi_n(d) - \psi_n(d)\| < 1/n$ for all $d \in \mathcal{G}_n$ and yet $KK(\varphi_n) \neq KK(\psi_n)$ for all $n \geq 1$. Set $B_i = \prod_{n \geq i} A_n$ and let $\nu_i : B_i \to B_{i+1}$ be the natural projection. Let us define $\Phi_i, \Psi_i : D \to B_i$ by $\Phi_i(d) = (\varphi_i(d), \varphi_{i+1}(d), \dots)$ and $\Psi_i(d) = (\psi_i(d), \psi_{i+1}(d), \dots)$, for all d in D. Let B_i' be the separable C*-subalgebra of B_i generated by the images of Φ_i and Ψ_i. Then $\nu_i(B_i') = B_{i+1}'$ and one verifies immediately that $\varinjlim \Phi_i = \varinjlim \Psi_i : D \to \varinjlim (B_i', \nu_i)$. Since D is KK-semiprojective, we must have $KK(\Phi_i) = KK(\Psi_i)$ for some i and hence $KK(\varphi_n) = KK(\psi_n)$ for all $n \geq i$. This gives a contradiction. $\square$

Proposition 3.10. *A unital Kirchberg algebra D is KK-stable if and only if $D \otimes \mathcal{K}$ is KK-stable. D is weakly semiprojective if and only if $D \otimes \mathcal{K}$ is weakly semiprojective.*

Proof. Since $KK(D, A) \cong KK(D, A \otimes \mathcal{K}) \cong KK(D \otimes \mathcal{K}, A \otimes \mathcal{K})$ the first part of the proposition is immediate. Suppose now that $D \otimes \mathcal{K}$ is weakly semiprojective. Then D is weakly semiprojective as shown in the proof of [34, Thm. 2.2]. Conversely, assume that D is weakly semiprojective. It suffices to find $\alpha \in \mathrm{Hom}(D \otimes \mathcal{K}, D)$ and a sequence (β_n) in $\mathrm{Hom}(D, D \otimes \mathcal{K})$ such that $\beta_n \alpha$ converges to $\mathrm{id}_{D \otimes \mathcal{K}}$ in the point-norm topology. Let s_i be the canonical generators of $\mathcal{O}_\infty$. If (e_{ij}) is a system of matrix units for $\mathcal{K}$, then $\lambda(e_{ij}) = s_i s_j^*$

defines a $*$-homomorphism $\mathcal{K} \to \mathcal{O}_\infty$ such that $KK(\lambda) \in KK(\mathcal{K}, \mathcal{O}_\infty)^{-1}$. Therefore, by composing $\mathrm{id}_D \otimes \lambda$ with some isomorphism $D \otimes \mathcal{O}_\infty \cong D$ (given by [31, Thm. 7.6.6]) we obtain a $*$-monomorphism $\alpha : D \otimes \mathcal{K} \to D$ which induces a KK-equivalence. Let $\beta : D \to D \otimes \mathcal{K}$ be defined by $\beta(d) = d \otimes e_{11}$. Then $\beta\alpha \in \mathrm{End}(D \otimes \mathcal{K})$ induces a KK-equivalence and hence after replacing β by $\theta\beta$ for some automorphism θ of $D \otimes \mathcal{K}$, we may arrange that $KK(\beta\alpha) = KK(\mathrm{id}_D)$. By Theorem 3.1, $\beta\alpha \approx_u \mathrm{id}_{D\otimes\mathcal{K}}$, so that there is a sequence of unitaries $u_n \in (D \otimes \mathcal{K})^+$ such that $u_n \beta\alpha(-)u_n^*$ converges to $\mathrm{id}_{D\otimes\mathcal{K}}$. $\square$

Theorem 3.11. *For a separable C*-algebra D consider the following properties:*

(i) D is KK-semiprojective.

(ii) The functor $KK(D, -)$ is continuous.

(iii) D is weakly semiprojective and KK-stable.

Then (i) $\Leftrightarrow$ (ii). Moreover, (iii) $\Rightarrow$ (i) if D is nuclear and (i) $\Rightarrow$ (iii) if D is a Kirchberg algebra. Thus (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) for any Kirchberg algebra D.

Proof. The implication (ii) $\Rightarrow$ (i) is obvious. (i) $\Rightarrow$ (ii): Let $(B_n, \gamma_{n,m})$ be an inductive system with inductive limit B and let $\gamma_n : B_n \to B$ be the canonical maps. We have an induced map $\beta : \varinjlim KK(D, B_n) \to KK(D, B)$. First we show that β is surjective. The mapping telescope construction of L. G. Brown (as described in the proof of [2, Thm. 3.1]) produces an inductive system of C*-algebras $(T_n, \eta_{n,m})$ with inductive limit B such that each $\eta_{n,n+1}$ is surjective, and each canonical map $\eta_n : T_n \to B$ is homotopic to $\gamma_n\alpha_n$ for some $*$-homomorphism $\alpha_n : T_n \to B_n$. In particular $KK(\eta_n) = KK(\gamma_n)KK(\alpha_n)$. Let $x \in KK(D, B)$. By (i) there are n and $y \in KK(D, T_n)$ such that $KK(\eta_n)y = x$ and hence $KK(\gamma_n)KK(\alpha_n)y = x$. Thus $z = KK(\alpha_n)y \in KK(D, B_n)$ is a lifting of x. Let us show now that the map β is injective. Let x be an element in the kernel of the map $KK(D, B_n) \to KK(D, B)$. Consider the commutative diagram whose exact rows are portions of the Puppe sequence in KK-theory [3, Thm. 19.4.3] and with vertical maps induced by $\gamma_m : B_m \to B$, $m \geq n$.

$$\begin{tikzcd} KK(D, C_{\gamma_n}) \arrow[r] & KK(D, B_n) \arrow[r] & KK(D, B) \\ KK(D, C_{\gamma_{n,m}}) \arrow[r] \arrow[u] & KK(D, B_n) \arrow[r] \arrow[u, equal] & KK(D, B_m) \arrow[u] \end{tikzcd}$$

By exactness, x is the image of some element $y \in KK(D, C_{\gamma_n})$. Since $C_{\gamma_n} = \varinjlim C_{\gamma_{n,m}}$, the map $\varinjlim KK(D, C_{\gamma_{n,m}}) \to KK(D, C_{\gamma_n})$ is surjective by the first part of the proof. Therefore there is $m \geq n$ such that y lifts to some $z \in KK(D, C_{\gamma_{n,m}})$. The image of z in $KK(D, B_m)$ equals $KK(\gamma_{n,m})x$ and vanishes by exactness of the bottom row.

(iii) $\Rightarrow$ (i): Let A, (J_n) and J be as in Definition 3.8. Using the five-lemma and the split exact sequence $0 \to KK(D,A) \to KK(D,A^+) \to KK(D,\mathbb{C}) \to 0$, we reduce the proof to the case when A is unital. Let $x \in KK(D,A/J)$. Since the map $KK(D^+,A/J) \to KK(D,A/J)$ is surjective, x lifts to some element $x^+ \in KK(D^+,A/J)$. By [31, Thm. 8.3.3], since D^+ is nuclear, there is a $*$-homomorphism $\Phi : D^+ \to A/J \otimes \mathcal{O}_\infty \otimes \mathcal{K}$ such that $KK(\Phi) = x^+$ and hence if set $\varphi = \Phi|_D$, then $KK(\varphi) = x$. Since D is weakly semiprojective, there are n and a $*$-homomorphism $\psi : D \to A/J_n \otimes \mathcal{O}_\infty \otimes \mathcal{K}$ such that $\|\pi_n\psi(d) - \varphi(d)\| < \delta$ for all $d \in \mathcal{G}$, where $\mathcal{G}$ and δ are as in the definition of KK-stability. Therefore $KK(\pi_n\psi) = KK(\varphi)$ and hence $KK(\psi)$ is a lifting of x to $KK(D,A/J_n)$.

(i) $\Rightarrow$ (iii): D is KK-stable by Proposition 3.9. It remains to show that D is weakly semiprojective. Since any nonunital Kirchberg algebra is isomorphic to the stabilization of a unital one (see [31, Prop. 4.1.3]) and since by Proposition 3.10 D is KK-semiprojective if and only if $D \otimes \mathcal{K}$ is KK-semiprojective, we may assume that D is unital. Let A, (J_n), $\pi_{m,n} : A/J_m \to A/J_n$ $(m \le n)$ and $\pi_n : A/J_n \to A/J$ be as in the definition of weak semiprojectivity. By [2, Cor. 2.15], we may assume that A and the $*$-homomorphism $\varphi : D \to A$ (for which we want to construct an approximative lifting) are unital. In particular φ is injective since D is simple. Set $B = \varphi(D) \subset A/J$ and $B_n = \pi_n^{-1}(B) \subset A/J_n$. The corresponding maps $\pi_{m,n} : B_m \to B_n$ $(m \le n)$ and $\pi_n : B_n \to B$ are surjective and they induce an isomorphism $\varinjlim (B_n, \pi_{n,n+1}) \cong B$.

Given $\varepsilon > 0$ and $\mathcal{F} \subset D$ (a finite set) we are going to produce an approximate lifting $\varphi_n : D \to B_n$ for φ. Since 1_B is a properly infinite projection, it follows by [2, Props. 2.18 and 2.23] that the unit 1_n of B_n is a properly infinite projection, for all sufficiently large n. Since D is KK-semiprojective, there exist m and an element $h \in KK(D,B_m)$ which lifts $KK(\varphi)$ such that $K_0(h)[1_D] = [1_m]$. By [31, Thm. 8.3.3], there is a full $*$-homomorphism $\eta : D \to B_m \otimes \mathcal{K}$ such that $KK(\eta) = h$. By [31, Prop. 4.1.4], since both $\eta(1_D)$ and 1_m are full and properly infinite projections in $B_m \otimes \mathcal{K}$, there is a partial isometry $w \in B_m \otimes \mathcal{K}$ such that $w^*w = \eta(1_D)$ and $ww^* = 1_m$. Replacing η by $w\,\eta(-)\,w^*$, we may assume that $\eta : D \to B_m$ is unital. Then $KK(\pi_m\eta) = KK(\pi_m)h = KK(\varphi)$. By Theorem 3.1, $\pi_m\eta \approx_{uh} \varphi$. Thus there is a unitary $u \in B$ such that $\|u\pi_m\eta(d)u^* - \varphi(d)\| < \varepsilon$ for all $d \in \mathcal{F}$. Since $C(\mathbb{T})$ is semiprojective, there is $n \ge m$ such that u lifts to a unitary unitary $u_n \in B_n$. Then $\varphi_n = u_n\,\pi_{m,n}\eta(-)\,u_n^*$ is a $*$-homomorphism from D to B_n such that $\|\pi_n\varphi_n(d) - \varphi(d)\| < \varepsilon$ for all $d \in \mathcal{F}$. □

Corollary 3.12. *Any separable nuclear semiprojective C*-algebra is KK-semiprojective.*

Proof. This is very similar to the proof of the implication (iii) $\Rightarrow$ (i) of Theorem 3.11. Alternately, the statement follows from Corollary 3.6 and Theorem 3.11. □

Blackadar has shown that a semiprojective Kirchberg algebra satisfying the UCT has finitely generated K-theory groups [31, Prop. 8.4.15]. A similar argument gives the following:

Proposition 3.13. *Let D be a separable C*-algebra satisfying the UCT. Then D is KK-semiprojective if and only $K_*(D)$ is finitely generated.*

Proof. If $K_*(D)$ is finitely generated, then D is KK-semiprojective by [33]. Conversely, assume that D is KK-semiprojective. Since D satisfies the UCT, we infer that if $G = K_i(D)$ $(i = 0, 1)$, then G is semiprojective in the category of countable abelian groups, in the sense that if $H_1 \to H_2 \to \cdots$ is an inductive system of countable abelian groups with inductive limit H, then the natural map $\varinjlim \mathrm{Hom}(G, H_n) \to \mathrm{Hom}(G, H)$ is surjective. This implies that G is finitely generated. Indeed, taking $H = G$, we see that id_G lifts to $\mathrm{Hom}(G, H_n)$ for some finitely generated subgroup H_n of G and hence G is a quotient of H_n. □

4. Approximation of $C(X)$-algebras

In this section we use weak semiprojectivity to approximate a $C(X)$-algebra A by $C(X)$-subalgebras given by pullbacks of n-fibered monomorphisms into A.

Lemma 4.1. *Let D be a finite direct sum of simple C*-algebras and let $\varphi, \psi : D \to A$ be $*$-homomorphisms. Suppose that $\mathcal{H} \subset D$ contains a full element from each simple direct summand of D. If $\|\psi(d) - \varphi(d)\| \leq \|d\|/2$ for all $d \in \mathcal{H}$, then φ is injective if and only if ψ is injective.*

Proof. Let us note that φ is injective if and only if $\|\varphi(d)\| = \|d\|$ for all $d \in \mathcal{H}$. Therefore if φ is injective, then $\|\psi(d)\| \geq \|\varphi(d)\| - \|\psi(d) - \varphi(d)\| \geq \|d\|/2$ for all $d \in \mathcal{H}$ and hence ψ is injective. □

A sequence (A_n) of subalgebras of A is called *exhaustive* if for any finite subset $\mathcal{F}$ of A and any $\varepsilon > 0$ there is n such that $\mathcal{F} \subset_\varepsilon A_n$.

Lemma 4.2. *Let $\mathcal{C}$ be a class consisting of finite direct sums of separable simple weakly semiprojective C*-algebras. Let X be a compact metrizable space and let A be a $C(X)$-algebra. Let $\mathcal{F} \subset A$ be a finite set, let $\varepsilon > 0$ and suppose that $A(x)$ admits an exhaustive sequence of C*-algebras isomorphic to C*-algebras in $\mathcal{C}$ for some $x \in X$. Then there exist a compact neighborhood U of x and a $*$-homomorphism $\varphi : D \to A(U)$ for some $D \in \mathcal{C}$ such that $\pi_U(\mathcal{F}) \subset_\varepsilon \varphi(D)$. If A is a continuous $C(X)$-algebra, then we may arrange that φ_z is injective for all $z \in U$.*

Proof. Let $\mathcal{F} = \{a_1, \dots, a_r\}$ and ε be given. By hypothesis there exist $D \in \mathcal{C}$, $\{c_1, \dots, c_r\} \subset D$ and a $*$-monomorphism $\iota : D \to A(x)$ such that $\|\pi_x(a_i) - \iota(c_i)\| < \varepsilon/2$, for all $i = 1, ..., r$. Set $U_n = \{y \in X : d(x, y) \leq 1/n\}$. Choose a full element d_j in each direct summand of D. Since D is weakly semiprojective, there is a $*$-homomorphism $\varphi : D \to A(U_n)$ (for some n) such that $\|\pi_x\varphi(c_i) - \iota(c_i)\| < \varepsilon/2$ for all $i = 1, ..., r$, and $\|\pi_x\varphi(d_j) - \iota(d_j)\| \leq \|d_j\|/2$ for all d_j. Therefore

$$\|\pi_x\varphi(c_i) - \pi_x(a_i)\| \leq \|\pi_x\varphi(c_i) - \iota(c_i)\| + \|\pi_x(a_i) - \iota(c_i)\| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

and φ_x is injective by Lemma 4.1. By Lemma 2.1(i), after increasing n and setting $U = U_n$ and $\varphi = \pi_U\varphi$, we have

$$\|\varphi(c_i) - \pi_U(a_i)\| = \|\pi_U\big(\varphi(c_i) - a_i\big)\| < \varepsilon,$$

for all $i = 1, ..., r$. This shows that $\pi_U(\mathcal{F}) \subset_\varepsilon \varphi(D)$. If A is continuous, then after shrinking U we may arrange that $\|\varphi_z(d_j)\| \geq \|\varphi_x(d_j)\|/2 = \|d_j\|/2$ for all d_j and all $z \in U$. This implies that φ_z in injective for all $z \in U$. □

Lemma 4.3. *Let X be a compact metrizable space and let A be a separable continuous $C(X)$-algebra the fibers of which are stable Kirchberg algebras. Let $\mathcal{F} \subset A$ be a finite set and let $\varepsilon > 0$. Suppose that there exist a KK-semiprojective stable Kirchberg algebra D and $\sigma \in KK(D, A)$ such that $\sigma_x \in KK(D, A(x))^{-1}$ for some $x \in X$. Then there exist a closed neighborhood U of x and a full $*$-homomorphism $\psi : D \to A(U)$ such that $KK(\psi) = \sigma_U$ and $\pi_U(\mathcal{F}) \subset_\varepsilon \psi(D)$.*

Proof. By [31, Thm. 8.4.1] there is an isomorphism $\psi_0 : D \to A(x)$ such that $KK(\psi_0) = \sigma_x$. Let $\mathcal{H} \subset D$ be such that $\psi_0(\mathcal{H}) = \pi_x(\mathcal{F})$. Set $U_n = \{y \in X : d(x, y) \leq 1/n\}$. By Theorem 3.11 D is KK-stable and weakly semiprojective. By Proposition 3.7 there exists a $*$-homomorphism $\psi_n : D \to A(U_n)$ (for some n) such that $\|\pi_x\psi_n(d) - \psi_0(d)\| < \varepsilon$ for all $d \in \mathcal{H}$ and $KK(\pi_x\psi_n) = KK(\psi_0) = \sigma_x$. Since $\varinjlim_m KK(D, A(U_m)) = KK(D, A(x))$, we deduce that there is $m \geq n$ such that $KK(\pi_{U_m}\psi_n) = \sigma_{U_m}$. By increasing m we may arrange that $\pi_{U_m}(\mathcal{F}) \subset_\varepsilon \pi_{U_m}\psi_n(D)$ since we have seen that $\pi_x(\mathcal{F}) = \psi_0(\mathcal{H}) \subset_\varepsilon \pi_x\psi_n(D)$. We can arrange that ψ_z is injective for all $z \in U$ by reasoning as in the proof of Lemma 4.2. We conclude by setting $U = U_m$ and $\psi = \pi_{U_m}\psi_n$. □

The following lemma is useful for constructing fibered morphisms.

Lemma 4.4. *Let $(D_j)_{j \in J}$ be a finite family consisting of finite direct sums of weakly semiprojective simple C*-algebras. Let $\varepsilon > 0$ and for each $j \in J$ let $\mathcal{H}_j \subset D_j$ be a finite set such that for each direct summand of D_j there is an element of $\mathcal{H}_j$ of norm $\geq \varepsilon$ which is contained and is full in that summand. Let $\mathcal{G}_j \subset D_j$ and $\delta_j > 0$ be given by Proposition 3.7 applied to D_j, $\mathcal{H}_j$ and $\varepsilon/2$. Let X be a compact metrizable space, let*

$(Z_j)_{j\in J}$ be disjoint nonempty closed subsets of X and let Y be a closed nonempty subset of X such that $X = Y \cup \big(\cup_j Z_j\big)$. Let A be a continuous $C(X)$-algebra and let $\mathcal{F}$ be a finite subset of A. Let $\eta : B(Y) \to A(Y)$ be a $*$-monomorphism of $C(Y)$-algebras and let $\varphi_j : D_j \to A(Z_j)$ be $*$-homomorphisms such that $(\varphi_j)_x$ is injective for all $x \in Z_j$ and $j \in J$, and which satisfy the following conditions:

(i) $\pi_{Z_j}(\mathcal{F}) \subset_{\varepsilon/2} \varphi_j(\mathcal{H}_j)$, *for all* $j \in J$,
(ii) $\pi_Y(\mathcal{F}) \subset_\varepsilon \eta(B)$,
(iii) $\pi^{Z_j}_{Y\cap Z_j}\varphi_j(\mathcal{G}_j) \subset_{\delta_j} \pi^Y_{Y\cap Z_j}\eta(B)$, *for all* $j \in J$.

Then, there are $C(Z_j)$-linear $*$-monomorphisms $\psi_j : C(Z_j)\otimes D_j \to A(Z_j)$, satisfying

$$\|\varphi_j(c) - \psi_j(c)\| < \varepsilon/2, \text{ for all } c \in \mathcal{H}_j, \text{ and } j \in J, \tag{6}$$

and such that if we set $E = \bigoplus_j C(Z_j)\otimes D_j$, $Z = \cup_j Z_j$, and $\psi : E \to A(Z) = \bigoplus_j A(Z_j)$, $\psi = \oplus_j \psi_j$, then $\pi^Z_{Y\cap Z}(\psi(E)) \subseteq \pi^Y_{Y\cap Z}(\eta(B))$, $\pi_Z(\mathcal{F}) \subset_\varepsilon \psi(E)$ and hence

$$\mathcal{F} \subset_\varepsilon \eta(B) \oplus_{Y\cap Z} \psi(E) = \chi(B \oplus_{\pi\eta,\pi\psi} E),$$

where χ is the isomorphism induced by the pair (η,ψ). If we assume that each D_j is KK-stable, then we also have $KK(\varphi_j) = KK(\psi_j|_{D_j})$ for all $j \in J$.

Proof. Let $\mathcal{F} = \{a_1,\dots,a_r\} \subset A$ be as in the statement. By (i), for each $j \in J$ we find $\{c^{(j)}_1,\dots,c^{(j)}_r\} \subseteq \mathcal{H}_j$ such that $\|\varphi_j(c^{(j)}_i) - \pi_{Z_j}(a_i)\| < \varepsilon/2$ for all i. Consider the $C(X)$-algebra $A \oplus_Y \eta(B) \subset A$. From (iii), Lemma 2.1(iv) and Lemma 2.5 we obtain

$$\varphi_j(\mathcal{G}_j) \subset_{\delta_j} \pi_{Z_j}(A \oplus_Y \eta(B)).$$

Applying Proposition 3.7 we perturb φ_j to a $*$-homomorphism $\psi_j : D_j \to \pi_{Z_j}(A\oplus_Y \eta(B))$ satisfying (6), and hence such that $\|\varphi_j(c^{(j)}_i) - \psi_j(c^{(j)}_i)\| < \varepsilon/2$, for all i,j. Therefore

$$\|\psi_j(c^{(j)}_i) - \pi_{Z_j}(a_i)\| \le \|\psi_j(c^{(j)}_i) - \varphi_j(c^{(j)}_i)\| + \|\varphi_j(c^{(j)}_i) - \pi_{Z_j}(a_i)\| < \varepsilon.$$

This shows that $\pi_{Z_j}(\mathcal{F}) \subset_\varepsilon \psi_j(D_j)$. From (6) and Lemma 4.1 we obtain that each $(\psi_j)_x$ is injective. We extend ψ_j to a $C(Z_j)$-linear $*$-monomorphism $\psi_j : C(Z_j)\otimes D_j \to \pi_{Z_j}(A\oplus_Y \eta(B))$ and then we define E, ψ and Z as in the statement. In this way we obtain that $\psi : E \to (A\oplus_Y \eta(B))(Z) \subset A(Z)$ satisfies

$$\pi_Z(\mathcal{F}) \subset_\varepsilon \psi(E). \tag{7}$$

The property $\psi(E) \subset (A\oplus_Y \eta(B))(Z)$ is equivalent to $\pi^Z_{Y\cap Z}(\psi(E)) \subseteq \pi^Y_{Y\cap Z}(\eta(B))$ by Lemma 2.6(b). Finally, from (ii), (7) and Lemma 2.6 (c) we get $\mathcal{F} \subset_\varepsilon \eta(B)\oplus_{Y\cap Z}\psi(E)$. □

Let $\mathcal{C}$ be as in Lemma 4.2. Let A be a $C(X)$-algebra, let $\mathcal{F} \subset A$ be a finite set and let $\varepsilon > 0$. An $(\mathcal{F},\varepsilon,\mathcal{C})$-*approximation* of A

$$\alpha = \{\mathcal{F}, \varepsilon, \{U_i, \varphi_i : D_i \to A(U_i), \mathcal{H}_i, \mathcal{G}_i, \delta_i\}_{i\in I}\}, \tag{8}$$

is a collection with the following properties: $(U_i)_{i\in I}$ is a finite family of closed subsets of X, whose interiors cover X and $(D_i)_{i\in I}$ are C*-algebras in $\mathcal{C}$; for each $i \in I$, $\varphi_i : D_i \to A(U_i)$ is a $*$-homomorphism such that $(\varphi_i)_x$ is injective for all $x \in U_i$; $\mathcal{H}_i \subset D_i$ is a finite set such that $\pi_{U_i}(\mathcal{F}) \subset_{\varepsilon/2} \varphi_i(\mathcal{H}_i)$ and such that for each direct summand of D_i there is an element of $\mathcal{H}_i$ of norm $\geq \varepsilon$ which is contained and is full in that summand; the finite set $\mathcal{G}_i \subset D_i$ and $\delta_i > 0$ are given by Proposition 3.7 applied to the weakly semiprojective C*-algebra D_i for the input data $\mathcal{H}_i$ and $\varepsilon/2$; if D_i is KK-stable, then $\mathcal{G}_i$ and δ_i are chosen such that the second part of Proposition 3.7 also applies.

Lemma 4.5. *Let A and $\mathcal{C}$ be as in Lemma 4.2. Suppose that each fiber of A admits an exhaustive sequence of C*-algebras isomorphic to C*-algebras in $\mathcal{C}$. Then for any finite subset $\mathcal{F}$ of A and any $\varepsilon > 0$ there is an $(\mathcal{F}, \varepsilon, \mathcal{C})$-approximation of A. Moreover, if A, D and σ are as in Lemma 4.3 and $\sigma_x \in KK(D, A(x))^{-1}$ for all $x \in X$, then there is an $(\mathcal{F}, \varepsilon, \mathcal{C})$-approximation of A such that $\mathcal{C} = \{D\}$ and $KK(\varphi_i) = \sigma_{U_i}$ for all $i \in I$.*

Proof. Since X is compact, this is an immediate consequence of Lemmas 4.2, 4.3 and Proposition 3.7. □

It is useful to consider the following operation of restriction. Suppose that Y is a closed subspace of X and let $(V_j)_{j\in J}$ be a finite family of closed subsets of Y which refines the family $(Y \cap U_i)_{i\in I}$ and such that the interiors of the V_j's form a cover of Y. Let $\iota : J \to I$ be a map such that $V_j \subseteq Y \cap U_{\iota(j)}$. Define

$$\iota^*(\alpha) = \{\pi_Y(\mathcal{F}), \varepsilon, \{V_j, \pi_{V_j}\varphi_{\iota(j)} : D_{\iota(j)} \to A(V_j), \mathcal{H}_{\iota(j)}, \mathcal{G}_{\iota(j)}, \delta_{\iota(j)}\}_{j\in J}\}.$$

It is obvious that $\iota^*(\alpha)$ is a $(\pi_Y(\mathcal{F}), \varepsilon, \mathcal{C})$-approximation of $A(Y)$. The operation $\alpha \mapsto \iota^*(\alpha)$ is useful even in the case $X = Y$. Indeed, by applying this procedure we can refine the cover of X that appears in a given $(\mathcal{F}, \varepsilon, \mathcal{C})$-approximation of A.

An $(\mathcal{F}, \varepsilon, \mathcal{C})$-approximation α (as in (8)) is subordinated to an $(\mathcal{F}', \varepsilon', \mathcal{C})$-approximation, $\alpha' = \{\mathcal{F}', \varepsilon', \{U_{i'}, \varphi_{i'} : D_{i'} \to A(U_{i'}), \mathcal{H}_{i'}, \mathcal{G}_{i'}, \delta_{i'}\}_{i'\in I'}\}$, written $\alpha \prec \alpha'$, if

(i) $\mathcal{F} \subseteq \mathcal{F}'$,
(ii) $\varphi_i(\mathcal{G}_i) \subseteq \pi_{U_i}(\mathcal{F}')$ for all $i \in I$, and
(iii) $\varepsilon' < \min\big(\{\varepsilon\} \cup \{\delta_i, \, i \in I\}\big)$.

Let us note that, with notation as above, we have $\iota^*(\alpha) \prec \iota^*(\alpha')$ whenever $\alpha \prec \alpha'$.

The following theorem is the crucial technical result of our paper. It provides an approximation of continuous $C(X)$-algebras by subalgebras of category $\leq \dim(X)$.

Theorem 4.6. *Let $\mathcal{C}$ be a class consisting of finite direct sums of weakly semiprojective simple C*-algebras. Let X be a finite dimensional compact metrizable space and let A be a separable continuous $C(X)$-algebra the fibers of which admit exhaustive sequences of C*-algebras isomorphic to C*-algebras in $\mathcal{C}$. For any finite set $\mathcal{F} \subset A$ and any $\varepsilon > 0$ there* *exist* $n \leq \dim(X)$ *and an* n*-fibered* $\mathcal{C}$*-monomorphism* $(\psi_0, \dots, \psi_n)$ *into* A *which induces a* $*$*-monomorphism* $\eta : A(\psi_0, ..., \psi_n) \to A$ *such that* $\mathcal{F} \subset_\varepsilon \eta(A(\psi_0, ..., \psi_n))$.

Proof. By Lemma 4.5, for any finite set $\mathcal{F} \subset A$ and any $\varepsilon > 0$ there is an $(\mathcal{F}, \varepsilon, \mathcal{C})$-approximation of A. Moreover, for any finite set $\mathcal{F} \subset A$, any $\varepsilon > 0$ and any n, there is a sequence $\{\alpha_k : 0 \leq k \leq n\}$ of $(\mathcal{F}_k, \varepsilon_k, \mathcal{C})$-approximations of A such that $(\mathcal{F}_0, \varepsilon_0) = (\mathcal{F}, \varepsilon)$ and α_k is subordinated to α_{k+1}:

$$\alpha_0 \prec \alpha_1 \prec \cdots \prec \alpha_n.$$

Indeed, assume that α_k was constructed. Let us choose a finite set $\mathcal{F}_{k+1}$ which contains $\mathcal{F}_k$ and liftings to A of all the elements in $\bigcup_{i_k \in I_k} \varphi_{i_k}(\mathcal{G}_{i_k})$. This choice takes care of the above conditions (i) and (ii). Next we choose ε_{k+1} sufficiently small such that (iii) is satisfied. Let α_{k+1} be an $(\mathcal{F}_{k+1}, \varepsilon_{k+1}, \mathcal{C})$-approximation of A given by Lemma 4.5. Then obviously $\alpha_k \prec \alpha_{k+1}$. Fix a tower of approximations of A as above where $n = \dim(X)$.

By [6, Lemma 3.2], for every open cover $\mathcal{V}$ of X there is a finite open cover $\mathcal{U}$ which refines $\mathcal{V}$ and such that the set $\mathcal{U}$ can be partitioned into $n+1$ nonempty subsets consisting of elements with pairwise disjoint closures. Since we can refine simultaneously the covers that appear in a finite family $\{\alpha_k : 0 \leq k \leq n\}$ of approximations while preserving subordination, we may arrange not only that all α_k share the same cover $(U_i)_{\in I}$, but moreover, that the cover $(U_i)_{i \in I}$ can be partitioned into $n+1$ subsets $\mathcal{U}_0, \dots, \mathcal{U}_n$ consisting of mutually disjoint elements. For definiteness, let us write $\mathcal{U}_k = \{U_{i_k} : i_k \in I_k\}$. Now for each k we consider the closed subset of X

$$Y_k = \bigcup_{i_k \in I_k} U_{i_k},$$

the map $\iota_k : I_k \to I$ and the $(\pi_{Y_k}(\mathcal{F}_k), \varepsilon_k, \mathcal{C})$-approximation of $A(Y_k)$, induced by α_k, which is of the form

$$\iota_k^*(\alpha_k) = \{\pi_{Y_k}(\mathcal{F}_k), \varepsilon, \{U_{i_k}, \varphi_{i_k} : D_{i_k} \to A(U_{i_k}), \mathcal{H}_{i_k}, \mathcal{G}_{i_k}, \delta_{i_k}\}_{i_k \in I_k}\},$$

where each U_{i_k} is nonempty. We have

$$\pi_{U_{i_k}}(\mathcal{F}_k) \subset_{\varepsilon_k/2} \varphi_{i_k}(\mathcal{H}_{i_k}), \tag{9}$$

by construction. Since $\alpha_k \prec \alpha_{k+1}$ we obtain

$$\mathcal{F}_k \subseteq \mathcal{F}_{k+1}, \tag{10}$$

$$\varphi_{i_k}(\mathcal{G}_{i_k}) \subseteq \pi_{U_{i_k}}(\mathcal{F}_{k+1}), \text{ for all } i_k \in I_k, \tag{11}$$

$$\varepsilon_{k+1} < \min\left(\{\varepsilon_k\} \cup \{\delta_{i_k},\, i_k \in I_k\}.\right) \tag{12}$$

Set $X_k = Y_k \cup \cdots \cup Y_n$ and $E_k = \oplus_{i_k} C(U_{i_k}) \otimes D_{i_k}$ for $0 \le k \le n$. We shall construct a sequence of $C(Y_k)$-linear $*$-monomorphisms, $\psi_k : E_k \to A(Y_k)$, $k = n, ..., 0$, such that $(\psi_k, \dots, \psi_n)$ is an $(n-k)$-fibered monomorphism into $A(X_k)$. Each map

$$\psi_k = \oplus_{i_k} \psi_{i_k} : E_k \to A(Y_k) = \oplus_{i_k} A(U_{i_k})$$

will have components $\psi_{i_k} : C(U_{i_k}) \otimes D_{i_k} \to A(U_{i_k})$ whose restrictions to D_{i_k} will be perturbations of $\varphi_{i_k} : D_{i_k} \to A(U_{i_k})$, $i_k \in I_k$. We shall construct the maps ψ_k by induction on decreasing k such that if $B_k = A(X_k)(\psi_k, \dots, \psi_n)$ and $\eta_k : B_k \to A(X_k)$ is the map induced by the $(n-k)$-fibered monomorphism $(\psi_k, \dots, \psi_n)$, then

$$\pi_{X_{k+1} \cap U_{i_k}}\big(\psi_{i_k}(D_{i_k})\big) \subset \pi_{X_{k+1} \cap U_{i_k}}\big(\eta_{k+1}(B_{k+1})\big), \forall\, i_k \in I_k, \tag{13}$$

and

$$\pi_{X_k}(\mathcal{F}_k) \subset_{\varepsilon_k} \eta_k(B_k). \tag{14}$$

Note that (13) is equivalent to

$$\pi_{X_{k+1} \cap Y_k}\big(\psi_k(E_k)\big) \subset \pi_{X_{k+1} \cap Y_k}\big(\eta_{k+1}(B_{k+1})\big). \tag{15}$$

For the first step of induction, $k = n$, we choose $\psi_n = \oplus_{i_n} \widetilde{\varphi}_{i_n}$ where $\widetilde{\varphi}_{i_n} : C(U_{i_n}) \otimes D_{i_n} \to A(U_{i_n})$ are $C(U_{i_n})$-linear extensions of the original φ_{i_n}. Then $B_n = E_n$ and $\eta_n = \psi_n$. Assume that $\psi_n, \dots, \psi_{k+1}$ were constructed and that they have the desired properties. We shall construct now ψ_k. Condition (14) formulated for $k+1$ becomes

$$\pi_{X_{k+1}}(\mathcal{F}_{k+1}) \subset_{\varepsilon_{k+1}} \eta_{k+1}(B_{k+1}). \tag{16}$$

Since $\varepsilon_{k+1} < \delta_{i_k}$, by using (11) and (16) we obtain

$$\pi_{X_{k+1} \cap U_{i_k}}\big(\varphi_{i_k}(\mathcal{G}_{i_k})\big) \subset_{\delta_{i_k}} \pi_{X_{k+1} \cap U_{i_k}}\big(\eta_{k+1}(B_{k+1})\big), \text{ for all } i_k \in I_k. \tag{17}$$

Since $\mathcal{F}_k \subseteq \mathcal{F}_{k+1}$ and $\varepsilon_{k+1} < \varepsilon_k$, condition (16) gives

$$\pi_{X_{k+1}}(\mathcal{F}_k) \subset_{\varepsilon_k} \eta_{k+1}(B_{k+1}). \tag{18}$$

Conditions (9), (17) and (18) enable us to apply Lemma 4.4 and perturb $\widetilde{\varphi}_{i_k}$ to a $*$-monomorphism $\psi_{i_k} : C(U_{i_k}) \otimes D_{i_k} \to A(U_{i_k})$ satisfying (13) and (14) and such that

$$KK(\psi_{i_k}|_{D_{i_k}}) = KK(\varphi_{i_k}) \tag{19}$$

if the algebras in $\mathcal{C}$ are assumed to be KK-stable. We set $\psi_k = \oplus_{i_k} \psi_{i_k}$ and this completes the construction of $(\psi_0, \dots, \psi_n)$. Condition (14) for $k = 0$ gives $\mathcal{F} \subset_\varepsilon \eta_0(B_0) = \eta(A(\psi_0, \dots, \psi_n))$. Thus $(\psi_0, \dots, \psi_n)$ satisfies the conclusion of the theorem. Finally let us note that it can happen that $X_k = X$ for some $k > 0$. In this case $\mathcal{F} \subset_\varepsilon A(\psi_k, ..., \psi_n)$ and for this reason we write $n \le \dim(X)$ in the statement of the theorem. □

Proposition 4.7. *Let X be a finite dimensional compact metrizable space and let A be a separable continuous $C(X)$-algebra the fibers of which are stable Kirchberg algebras. Let D be a KK-semiprojective stable Kirchberg algebra and suppose that there exists $\sigma \in KK(D, A)$ such that $\sigma_x \in KK(D, A(x))^{-1}$ for all $x \in X$. For any finite subset $\mathcal{F}$ of A and any $\varepsilon > 0$ there is an n-fibered $\mathcal{C}$-monomorphism $(\psi_0, \dots, \psi_n)$ into A such that $n \leq \dim(X)$, $\mathcal{C} = \{D\}$, and each component $\psi_i : C(Y_i) \otimes D \to A(Y_i)$ satisfies $KK(\psi_i) = \sigma_{Y_i}$, $i = 0, \dots, n$. Moreover, if $\eta : A(\psi_0, \dots, \psi_n) \to A$ is the induced $*$-monomorphism, then $\mathcal{F} \subset_\varepsilon \eta(A(\psi_0, \dots, \psi_n))$ and $KK(\eta_x)$ is a KK-equivalence for each $x \in X$.*

Proof. We repeat the proof of Theorem 4.6 while using only $(\mathcal{F}_i, \varepsilon_i, \{D\})$-approximations of A provided by the second part of Lemma 4.5. The outcome will be an n-fibered $\{D\}$-monomorphism $(\psi_0, \dots, \psi_n)$ into A such that $\mathcal{F} \subset_\varepsilon \eta(A(\psi_0, \dots, \psi_n))$. Moreover we can arrange that $KK(\psi_i) = \sigma_{Y_i}$ for all $i = 0, \dots, n$, by (19), since $KK(\varphi_{i_k}) = \sigma_{U_{i_k}}$ by Lemma 4.5. If $x \in X$, and $i = \min\{k : x \in Y_k\}$, then $\eta_x \equiv (\psi_i)_x$, and hence $KK(\eta_x)$ is a KK-equivalence. □

Remark 4.8. Let us point out that we can strengthen the conclusion of Theorem 4.6 and Proposition 4.7 as follows. Fix a metric d for the topology of X. Then we may arrange that there is a closed cover $\{Y_0', ..., Y_n'\}$ of X and a number $\ell > 0$ such that $\{x : d(x, Y_i') \leq \ell\} \subset Y_i$ for $i = 0, ..., n$. Indeed, when we choose the finite closed cover $\mathcal{U} = (U_i)_{i \in I}$ of X in the proof of Theorem 4.6 which can be partitioned into $n+1$ subsets $\mathcal{U}_0, \dots, \mathcal{U}_n$ consisting of mutually disjoints elements, as given by [6, Lemma 3.2], and which refines all the covers $\mathcal{U}(\alpha_0), ..., \mathcal{U}(\alpha_n)$ corresponding to $\alpha_0, ..., \alpha_n$, we may assume that $\mathcal{U}$ also refines the covers given by the interiors of the elements of $\mathcal{U}(\alpha_0), ..., \mathcal{U}(\alpha_n)$. Since each U_i is compact and I is finite, there is $\ell > 0$ such that if $V_i = \{x : d(x, U_i) \leq \ell\}$, then the cover $\mathcal{V} = (V_i)_{i \in I}$ still refines all of $\mathcal{U}(\alpha_0), ..., \mathcal{U}(\alpha_n)$ and for each $k = 0, ..., n$, the elements of $\mathcal{V}_k = \{V_i : U_i \in \mathcal{U}_k\}$, are still mutually disjoint. We shall use the cover $\mathcal{V}$ rather than $\mathcal{U}$ in the proof of the two theorems and observe that $Y_k' \stackrel{def}{=} \bigcup_{i_k \in I_k} U_{i_k} \subset \bigcup_{i_k \in I_k} V_{i_k} = Y_k$ has the desired property. Finally let us note that if we define $\psi_i' : E(Y_i') \to A(Y_i')$ by $\psi_i' = \pi_{Y_i'} \psi_i$, then $(\psi_0', \dots, \psi_n')$ is an n-fibered $\mathcal{C}$-monomorphism into A which satisfies the conclusion of Theorem 4.6 and Proposition 4.7 since $\pi_{Y_i'}(\mathcal{F}) \subset_\varepsilon \psi_i'(E_i)$ for all $i = 0, \dots, n$ and $X = \bigcup_{i=1}^n Y_i'$.

5. Representing $C(X)$-algebras as inductive limits

We have seen that Theorem 4.6 yields exhaustive sequences for certain $C(X)$-algebras. In this section we show how to pass from an exhaustive sequence to a nested exhaustive sequence using semiprojectivity. The remainder of the paper does not depend on this section.

Proposition 5.1. *Let X, A and $\mathcal{C}$ be as in Theorem 4.6. Let $(\psi_0, \dots, \psi_n)$ be an n-fibered $\mathcal{C}$-monomorphism into A with components $\psi_i : E_i \to A(Y_i)$. Let $\mathcal{F}_i \subset E_i$, $\mathcal{F} \subset A(\psi_0, ..., \psi_n)$ be finite sets and let $\varepsilon > 0$. Then there are finite sets $\mathcal{G}_i \subset E_i$ and $\delta_i > 0$, $i = 0, ..., n$, such that for any $C(X)$-subalgebra $A' \subset A$ which satisfies $\psi_i(\mathcal{G}_i) \subset_{\delta_i} A'(Y_i)$, $i = 0, ..., n$, there is an n-fibered $\mathcal{C}$-monomorphism $(\psi'_0, \dots, \psi'_n)$ into A', with $\psi'_i : E_i \to A'(Y_i)$ and such that (i) $\|\psi_i(a) - \psi'_i(a)\| < \varepsilon$ for all $a \in \mathcal{F}_i$ and all $i \in \{0, ..., n\}$, (ii) $(\psi_j)_x^{-1}(\psi_i)_x = (\psi'_j)_x^{-1}(\psi'_i)_x$ for all $x \in Y_i \cap Y_j$ and $0 \leq i \leq j \leq n$. Moreover $A(\psi_0, \dots, \psi_n) = A'(\psi'_0, \dots, \psi'_n)$ and the maps $\eta : A(\psi_0, ..., \psi_n) \to A$ and $\eta' : A'(\psi'_0, \dots, \psi'_n) \to A'$ induced by $(\psi_0, \dots, \psi_n)$ and $(\psi'_0, \dots, \psi'_n)$ satisfy (iii) $\|\eta(a) - \eta'(a)\| < \varepsilon$ for all $a \in \mathcal{F}$.*

Proof. We start by making three simplifications. First we may assume that $p_i(\mathcal{F}) \subset \mathcal{F}_i$ (by enlarging $\mathcal{F}_i$) and so (iii) will follow from (ii) ($p_i : A(\psi_0, ..., \psi_n) \to E_i$ is the natural projection map). Second we may assume that $E_0 = C(Y_0) \otimes D_0$ with $D_0 \in \mathcal{C}$ since the perturbations corresponding to disjoint closed sets can be done independently of each other. Third we may assume that $\mathcal{F}_0 \subset D_0$ since we are working with morphisms on E_0 which are $C(Y_0)$-linear. We enlarge $\mathcal{F}_0$ so that for each direct summand of D_0, $\mathcal{F}_0$ contains an element of norm $\geq 2\varepsilon$ which is full in that direct summand.

The proof is by induction on n. If $n = 0$ the statement follows from Proposition 3.7 and Lemma 4.1. Assume now that the statement is true for $n-1$. Let E_i, ψ_i, A, A', $\mathcal{F}_i$, $\mathcal{F}$ and ε be as in the statement. Let $\mathcal{G}$ and δ be given by Proposition 3.3 applied to the C*-algebra D_0 for the input data $\mathcal{F}_0$ and $\varepsilon/2$. We may assume that $\mathcal{F}_0 \subset \mathcal{G}$ and $\delta < \varepsilon$. Let $\mathcal{G}_0$ and δ_0 be given by Proposition 3.7 applied to D_0 for the input data $\mathcal{G}$ and $\delta/2$. With notation as in Definition 2.8 we have an induced map $\eta_1 : B_1 = A(X_1)(\psi_1, \dots, \psi_n) \to A(X_1)$. Let us choose finite sets $\mathcal{H} \subset B_1$ and $\mathcal{H}_i \subset E_i$, $i = 1, \dots, n$ such that

$$\mathcal{F}_i \subset \mathcal{H}_i, \quad \text{and} \quad (\eta_1)^{-1}_{X_1 \cap Y_0} \pi^{Y_0}_{X_1 \cap Y_0} \psi_0(\mathcal{G}) \subset \pi^{X_1}_{X_1 \cap Y_0}(\mathcal{H}). \tag{20}$$

Note that the existence of $\mathcal{H}$ follows from (3). Let $\mathcal{G}_1, ...\mathcal{G}_n$ and $\delta_1, ..., \delta_n$ be given by the inductive assumption for $n-1$ applied to $A(X_1)$, $(\psi_1, \dots, \psi_n)$, $\mathcal{H}_i$, $\mathcal{H}$ and $\delta/2$. We need to show that $\mathcal{G}_0, \mathcal{G}_1, ...\mathcal{G}_n$ and $\delta_0, \delta_1, ..., \delta_n$ satisfy the statement. By assumption there exists an $(n-1)$-fibered $\mathcal{C}$-monomorphism $(\psi'_1, \dots, \psi'_n)$ into $A'(X_1)$ with components $\psi'_i : E_i \to A'(Y_i)$ such that

(a) $\|\psi_i(a) - \psi'_i(a)\| < \delta/2 < \varepsilon$ for all $a \in \mathcal{H}_i$ and all $i \in \{1, \dots, n\}$,
(b) $(\psi_j)_x^{-1}(\psi_i)_x = (\psi'_j)_x^{-1}(\psi'_i)_x$ for all $x \in Y_i \cap Y_j$ and $1 \leq i \leq j \leq n$,
(c) $\|\eta_1(h) - \eta'_1(h)\| < \delta/2$ for all $h \in \mathcal{H}$.

It remains to construct $\psi'_0 : E_0 \to A'(Y_0)$ such that $\|\psi_0(a) - \psi'_0(a)\| < \varepsilon$ for all $a \in \mathcal{F}_0$ and such that (ii) holds for $i = 0$ and $j \in \{1, \dots, n\}$. The latter condition is readily seen to be equivalent to the equation $(\eta'_1)_{X_1 \cap Y_0} \Gamma = \pi^{Y_0}_{X_1 \cap Y_0} \psi'_0$, where $\Gamma = (\eta_1)^{-1}_{X_1 \cap Y_0} \pi^{Y_0}_{X_1 \cap Y_0} \psi_0$.

The setup is illustrated by the following diagram:

$$\begin{tikzcd} B_1 \arrow[r, "\pi^{Y_1}_{X_1\cap Y_0}"] \arrow[d, "\eta_1'"] & B_1(X_1\cap Y_0) \arrow[d, "(\eta_1')_{X_1\cap Y_0}"] & D_0 \arrow[l, "\Gamma"'] \arrow[d, dotted, "\psi_0'"] \\ A'(X_1) \arrow[r, "\pi^{Y_1}_{X_1\cap Y_0}"'] & A'(X_1\cap Y_0) & A'(Y_0) \arrow[l, "\pi^{Y_0}_{X_1\cap Y_0}"] \end{tikzcd}$$

By assumption, $\psi_0(\mathcal{G}_0) \subset_{\delta_0} A'(Y_0)$. By Proposition 3.7 there is a $*$-homomorphism $\gamma_0 : D_0 \to A'(Y_0)$ such that

$$\|\gamma_0(d) - \psi_0(d)\| < \delta/2 \tag{21}$$

for all $d \in \mathcal{G}$. Let us verify that γ_0 is an approximate lifting of $(\eta_1')_{X_1\cap Y_0}\Gamma|_{D_0}$. If $d \in \mathcal{G}$, then $\Gamma(d) = \pi^{X_1}_{X_1\cap Y_0}(h)$ for some $h \in \mathcal{H}$ by (20) and hence $(\eta_1)_{X_1\cap Y_0}\pi^{X_1}_{X_1\cap Y_0}(h) = \pi^{Y_0}_{X_1\cap Y_0}\psi_0(d)$. From (c) and (21) we have

$$\begin{aligned} \|(\eta_1')_{X_1\cap Y_0}\Gamma(d) - \pi^{Y_0}_{X_1\cap Y_0}\gamma_0(d)\| \;&\le\; \|(\eta_1')_{X_1\cap Y_0}\pi^{X_1}_{X_1\cap Y_0}(h) - (\eta_1)_{X_1\cap Y_0}\pi^{X_1}_{X_1\cap Y_0}(h)\| \\ &+\; \|\pi^{Y_0}_{X_1\cap Y_0}\psi_0(d) - \pi^{Y_0}_{X_1\cap Y_0}\gamma_0(d)\| < \delta/2 + \delta/2 = \delta \end{aligned}$$

for all $d \in \mathcal{G}$. By Proposition 3.3 there is a $*$-homomorphism $\psi_0' : D_0 \to A'(Y_0)$ such that $(\eta_1')_{X_1\cap Y_0}\Gamma = \pi^{Y_0}_{X_1\cap Y_0}\psi_0'$ and $\|\psi_0'(c) - \gamma_0(c)\| < \varepsilon/2$ for all $c \in \mathcal{F}_0$. Since $\mathcal{F}_0 \subset \mathcal{G}$ and $\delta < \varepsilon$, by combining the last estimate with (21) we obtain that $\|\psi_0(a) - \psi_0'(a)\| < \varepsilon$ for all $a \in \mathcal{F}_0$. By Lemma 4.1 this also shows that each $(\psi_0')_x$ is injective since each $(\psi_0)_x$ is injective. Therefore the $C(Y_0)$-linear extension of ψ_0 to E_0 is a $*$-monomorphism satisfying the desired properties. $\square$

The following result gives an inductive limit representation for $C(X)$-algebras whose fibers are inductive limits of finite direct sums of simple semiprojective C*-algebras. For example the fibers can be arbitrary AF-algebras or Kirchberg algebras which satisfy the UCT and whose K_1-groups are torsion free. Indeed, by [31, Prop. 8.4.13], these algebras are isomorphic to inductive limits of sequences of Kirchberg algebras (D_n) with finitely generated K-theory groups and torsion free K_1-groups. The algebras D_n are semiprojective by [35].

Theorem 5.2. *Let $\mathcal{C}$ be a class consisting of finite direct sums of semiprojective simple C*-algebras. Let X be a finite dimensional compact metrizable space and let A be a separable continuous $C(X)$-algebra such that all its fibers admit exhaustive sequences consisting of C*-algebras isomorphic to C*-algebras in $\mathcal{C}$. Then A is isomorphic to the inductive limit of a sequence of continuous $C(X)$-algebras A_k such that $\mathrm{cat}_{\mathcal{C}}(A_k) \le \dim(X)$.*

Proof. By Theorem 4.6 and Proposition 5.1 we find a sequence $(\psi_0^{(k)}, ..., \psi_n^{(k)})$ of n-fibered $\mathcal{C}$-monomorphisms into A which induces $*$-monomorphisms $\eta^{(k)} : A_k = A(\psi_0^{(k)}, ..., \psi_n^{(k)}) \to A$ with the following properties. There is a sequence of finite sets $\mathcal{F}_k \subset A_k$ and a sequence of $C(X)$-linear $*$-monomorphisms $\mu_k : A_k \to A_{k+1}$ such that

(i) $\|\eta^{(k+1)}\mu_k(a) - \eta^{(k)}(a)\| < 2^{-k}$ for all $a \in \mathcal{F}_k$ and all $k \geq 1$,
(ii) $\mu_k(\mathcal{F}_k) \subset \mathcal{F}_{k+1}$ for all $k \geq 1$,
(iii) $\bigcup_{j=k+1}^{\infty} \big(\mu_{j-1} \circ \cdots \circ \mu_k\big)^{-1}(\mathcal{F}_j)$ is dense in A_k and $\bigcup_{j=k}^{\infty} \eta^{(j)}(\mathcal{F}_j)$ is dense in A for all $k \geq 1$.

Arguing as in the proof of [31, Prop. 2.3.2], one verifies that

$$\varphi_k(a) = \lim_{j\to\infty} \eta^{(j)} \circ \big(\mu_{j-1} \circ \cdots \circ \mu_k\big)(a)$$

defines a sequence of $*$-monomorphisms $\varphi_k : A_k \to A$ such that $\varphi_{k+1}\mu_k = \varphi_k$ and the induced map $\varphi : \varinjlim_k (A_k, \mu_k) \to A$ is an isomorphism of $C(X)$-algebras. ☐

Remark 5.3. By similar arguments one proves a unital version of Theorem 5.2.

6. When is a fibered product (locally) trivial

For C*-algebras A, B we endow the space $\mathrm{Hom}(A, B)$ of $*$-homomorphisms with the point-norm topology. If X is a compact Hausdorff space, then $\mathrm{Hom}(A, C(X) \otimes B)$ is homeomorphic to the space of continuous maps from X to $\mathrm{Hom}(A, B)$ endowed with the compact-open topology. We shall identify a $*$-homomorphism $\varphi \in \mathrm{Hom}(A, C(X) \otimes B)$ with the corresponding continuous map $X \to \mathrm{Hom}(A, B)$, $x \mapsto \varphi_x$, $\varphi_x(a) = \varphi(a)(x)$ for all $x \in X$ and $a \in A$. Let D be a C*-algebra and let A be a $C(X)$-algebra. If $\alpha : D \to A$ is a $*$-homomorphism, let us denote by $\widetilde{\alpha} : C(X) \otimes D \to A$ its (unique) $C(X)$-linear extension and write $\widetilde{\alpha} \in \mathrm{Hom}_{C(X)}(C(X) \otimes D, A)$. For C*-algebras D, B we shall make without further comment the following identifications

$$\mathrm{Hom}_{C(X)}(C(X) \otimes D, C(X) \otimes B) \equiv \mathrm{Hom}(D, C(X) \otimes B) \equiv C(X, \mathrm{Hom}(D, B)).$$

For a C*-algebra D we denote by $\mathrm{End}(D)$ the set of full (*and unital if D is unital*) $*$-endomorphisms of D and by $\mathrm{End}(D)^0$ the path component of id_D in $\mathrm{End}(D)$. Let us consider

$$\mathrm{End}(D)^* = \{\gamma \in \mathrm{End}(D) : KK(\gamma) \in KK(D, D)^{-1}\}.$$

Proposition 6.1. *Let X be a compact metrizable space and let D be a KK-semiprojective Kirchberg algebra. Let $\alpha : D \to C(X) \otimes D$ be a full (and unital, if D is unital) $*$-homomorphism such that $KK(\alpha_x) \in KK(D, D)^{-1}$ for all $x \in X$. Then there is a full $*$-homomorphism $\Phi : D \to C(X \times [0, 1]) \otimes D$ such that $\Phi_{(x,0)} = \alpha_x$ and $\Phi_{(x,t)} \in \mathrm{Aut}(D)$ for all $x \in X$ and $t \in (0, 1]$. Moreover, if $\Phi_1 : D \to C(X) \otimes D$ is defined by $\Phi_1(d)(x) = \Phi_{(x,1)}(d)$, for all $d \in D$ and $x \in X$, then $\alpha \approx_{uh} \Phi_1$.*

Proof. Since X is a metrizable compact space, X is homeomorphic to the projective limit of a sequence of finite simplicial complexes (X_i) by [16, Thm. 10.1, p.284]. Since D is KK-semiprojective, $KK(D, \varinjlim C(X_i) \otimes D) = KK(D, C(X) \otimes D)$ by Theorem 3.11. By Theorem 3.1, there is i and a full (and unital if D is unital) $*$-homomorphism $\varphi : D \to C(X_i) \otimes D$ whose KK-class maps to $KK(\alpha) \in KK(D, C(X) \otimes D)$. To summarize, we have found a finite simplicial complex Y, a continuous map $h : X \to Y$ and a continuous map $y \mapsto \varphi_y \in \mathrm{End}(D)$, defined on Y, such that the full (and unital if D is unital) $*$-homomorphism $h^*\varphi : D \to C(X) \otimes D$ corresponding to the continuous map $x \mapsto \varphi_{h(x)}$ satisfies $KK(h^*\varphi) = KK(\alpha)$. We may arrange that $h(X)$ intersects all the path components of Y by dropping the path components which are not intersected. Since $\alpha_x \in \mathrm{End}(D)^*$ by hypothesis, and since $KK(\alpha_x) = KK(\varphi_{h(x)})$, we infer that $\varphi_y \in \mathrm{End}(D)^*$ for all $y \in Y$. We shall find a continuous map $y \mapsto \psi_y \in \mathrm{End}(D)^*$ defined on Y, such that the maps $y \mapsto \psi_y\varphi_y$ and $y \mapsto \varphi_y\psi_y$ are homotopic to the constant map ι that takes Y to id_D. It is clear that it suffices to deal separately with each path component of Y, so that for this part of the proof we may assume that Y is connected. Fix a point $z \in Y$. By [31, Thm. 8.4.1] there is $\nu \in \mathrm{Aut}(D)$ such that $KK(\nu^{-1}) = KK(\varphi_z)$ and hence $KK(\nu\varphi_z) = KK(\mathrm{id}_D)$. By Theorem 3.1, there is a unitary $u \in M(D)$ such that $u\nu\varphi_z(-)u^*$ is homotopic to id_D. Let us set $\theta = u\nu(-)u^* \in \mathrm{Aut}(D)$ and observe that $\theta\varphi_z \in \mathrm{End}(D)^0$. Since Y is path connected, it follows that the entire image of the map $y \mapsto \theta\varphi_y$ is contained in $\mathrm{End}(D)^0$. Since $\mathrm{End}(D)^0$ is a path connected H-space with unit element, it follows by [39, Thm. 2.4, p462] that the homotopy classes $[Y, \mathrm{End}(D)^0]$ (with no condition on basepoints, since the action of the fundamental group $\pi_1(\mathrm{End}(D)^0, \mathrm{id}_D)$ is trivial by [39, 3.6, p166]) form a group under the natural multiplication. Therefore we find $y \mapsto \psi'_y \in \mathrm{End}(D)^0$ such that $y \mapsto \psi'_y\theta\varphi_y$ and $y \mapsto \theta\varphi_y\psi'_y$ are homotopic to ι. It follows that $y \mapsto \psi_y \overset{def}{=} \psi'_y\theta$ is the homotopic inverse of $y \mapsto \varphi_y$ in $[Y, \mathrm{End}(D)^*]$. Composing with h we obtain that the maps $x \mapsto \varphi_{h(x)}\psi_{h(x)}$ and $x \mapsto \psi_{h(x)}\varphi_{h(x)}$ are homotopic to the constant map that takes X to id_D. By the homotopy invariance of KK-theory we obtain that

$$KK(\widetilde{h^*\varphi}\, h^*\psi) = KK(\widetilde{h^*\psi}\, h^*\varphi) = KK(\iota_D),$$

where $\widetilde{h^*\varphi}$ and $\widetilde{h^*\psi}$ denote the $C(X)$-linear extensions of the corresponding maps and $\iota_D : D \to C(X) \otimes D$ is defined by $\iota_D(d) = 1_{C(X)} \otimes d$ for all $d \in D$. Let us recall that $KK(h^*\varphi) = KK(\alpha)$ and hence $KK(\widetilde{h^*\varphi}) = KK(\widetilde{\alpha})$. If we set $\Psi = h^*\psi$, then

$$KK(\widetilde{\alpha}\, \Psi) = KK(\widetilde{\Psi}\, \alpha) = KK(\iota_D).$$

By Theorem 3.1 $\widetilde{\alpha}\,\Psi \approx_u \iota_D$ and $\widetilde{\Psi}\,\alpha \approx_u \iota_D$, and hence $\widetilde{\alpha}\,\widetilde{\Psi} \approx_u \mathrm{id}_{C(X)\otimes D}$ and $\widetilde{\Psi}\,\widetilde{\alpha} \approx_u \mathrm{id}_{C(X)\otimes D}$. By [31, Cor. 2.3.4], there is an isomorphism $\Gamma : C(X) \otimes D \to C(X) \otimes D$ such that $\Gamma \approx_u \widetilde{\alpha}$. In particular Γ is $C(X)$-linear and $\Gamma_x \in \mathrm{Aut}(D)$ for all $x \in X$. Replacing Γ by $u\Gamma(\cdot)u^*$ for some unitary $u \in M(C(X) \otimes D)$ we can arrange that $\Gamma|_D$ is arbitrarily close to α. Therefore $KK(\Gamma|_D) = KK(\alpha)$ since D is KK-stable. By Theorem 3.1 there is a continuous map $(0,1] \to U(M(C(X) \otimes D))$, $t \mapsto u_t$, with the property that

$$\lim_{t\to 0} \|u_t \Gamma(a) u_t^* - \alpha(a)\| = 0, \text{ for all } \quad a \in D.$$

Therefore the equation

$$\Phi_{(x,t)} = \begin{cases} \alpha_x, & \text{if } t = 0, \\ u_t(x)\Gamma_x u_t(x)^*, & \text{if } t \in (0,1], \end{cases}$$

defines a continuous map $\Phi : X \times [0,1] \to \mathrm{End}(D)^*$ which extends α and such that $\Phi(X \times (0,1]) \subset \mathrm{Aut}(D)$. Since α is homotopic to Φ_1, we have that $\alpha \approx_{uh} \Phi_1$ by Theorem 3.1. □

Proposition 6.2. *Let X be a compact metrizable space and let D be a KK-semiprojective Kirchberg algebra. Let Y be a closed subset of X. Assume that a map $\gamma : Y \to \mathrm{End}(D)^*$ extends to a continuous map $\alpha : X \to \mathrm{End}(D)^*$. Then there is a continuous extension $\eta : X \to \mathrm{End}(D)^*$ of γ, such that $\eta(X \setminus Y) \subset \mathrm{Aut}(D)$.*

Proof. Since the map $x \to \alpha_x$ takes values in $\mathrm{End}(D)^*$, by Proposition 6.1 there exists a continuous map $\Phi : X \times [0,1] \to \mathrm{End}(D)^*$ which extends α and such that $\Phi(X \times (0,1]) \subset \mathrm{Aut}(D)$. Let d be a metric for the topology of X such that $\mathrm{diam}(X) \leq 1$. The equation $\eta(x) = \Phi(x, d(x,Y))$ defines a map on X that satisfies the conclusion of the proposition. □

Lemma 6.3. *Let X be a compact metrizable space and let D be a KK-semiprojective Kirchberg algebra. Let Y be a closed subset of X. Let $\alpha : Y \times [0,1] \cup X \times \{0\} \to \mathrm{End}(D)$ be a continuous map such that $\alpha_{(x,0)} \in \mathrm{End}(D)^*$ for all $x \in X$. Suppose that there is an open set V in X which contains Y and such that α extends to a continuous map $\alpha_V : V \times [0,1] \cup X \times \{0\} \to \mathrm{End}(D)$. Then there is $\eta : X \times [0,1] \to \mathrm{End}(D)^*$ such that η extends α and $\eta_{(x,t)} \in \mathrm{Aut}(D)$ for all $x \in X \setminus Y$ and $t \in (0,1]$.*

Proof. By Proposition 6.2 it suffices to find a continuous map $\widehat{\alpha} : X \times [0,1] \to \mathrm{End}(D)^*$ which extends α. Fix a metric d for the topology of X and define $\lambda : X \to [0,1]$ by $\lambda(x) = d(x, X \setminus V)\big(d(x, X \setminus V) + d(x,Y)\big)^{-1}$. Let us define $\widehat{\alpha} : X \times [0,1] \to \mathrm{End}(D)$ by $\widehat{\alpha}_{(x,t)} = \alpha_V(x, \lambda(x)t)$ and observe that $\widehat{\alpha}$ extends α. Finally, since $\widehat{\alpha}_{(x,t)}$ is homotopic to $\widehat{\alpha}_{(x,0)} = \alpha_{(x,0)}$, we conclude that the image of $\widehat{\alpha}$ in contained in $\mathrm{End}(D)^*$. □

Proposition 6.4. *Let X be a compact metrizable space and let D be a KK-semiprojective stable Kirchberg algebra. Let A be a separable $C(X)$-algebra which is locally isomorphic to $C(X) \otimes D$. Suppose that there is $\sigma \in KK(D,A)$ such that $\sigma_x \in KK(D, A(x))^{-1}$ for all $x \in X$. Then there is an isomorphism of $C(X)$-algebras $\psi : C(X) \otimes D \to A$ such that $KK(\psi|_D) = \sigma$.*

Proof. Since X is compact and A is locally trivial it follows that $\mathrm{cat}_{\{D\}}(A) < \infty$. By Lemma 2.9, $A \cong pAp \otimes \mathcal{O}_\infty \otimes \mathcal{K}$ for some projection $p \in A$. By Theorem 3.1, there is a full $*$-homomorphism $\varphi : D \to A$ such that $KK(\varphi) = \sigma$. We shall construct an isomorphism of $C(X)$-algebras $\psi : C(X) \otimes D \to A$ such that ψ is homotopic to $\widetilde{\varphi}$, the $C(X)$-linear extension of φ. Let d be a metric for the topology of X. We denote by $B(x, r)$ the closed ball of radius r centered at x. Using the compactness of X and the local triviality of A, we find points $x_1, \dots, x_m \in X$ and numbers $r_1, \dots, r_m > 0$ such that $X = \bigcup_{i=1}^m B(x_i, r_i)$ and if we set $V_i = B(x_i, 2r_i)$, then there are $C(V_i)$-linear isomorphisms $\nu_i : A(V_i) \to C(V_i) \otimes D$, $i = 1, \dots, m$. The morphism $\varphi_i = \nu_i \pi_{V_i} \varphi : D \to C(V_i) \otimes D$ corresponds to a map $\varphi_i : V_i \to \mathrm{End}(D)^*$. For $i, k \in \{1, \dots, m\}$ let us consider the sets: $V_i^{(k)} = B(x_i, 2r_i - r_i(k-1)/m)$, $S_i = V_1^{(i+1)} \cup V_2^{(i)} \cup \dots \cup V_i^{(2)}$, $T_i = V_1^{(i)} \cup V_2^{(i-1)} \cup \dots \cup V_i^{(1)}$. Let us observe that $V_i^{(m)} \subset V_i^{(m-1)} \subset \dots \subset V_i^{(1)} = V_i$, $S_i \subset T_i$, $S_i \cup V_{i+1} = T_{i+1}$, that T_i is a neighborhood of S_i and that $T_m = X$. This array of sets is needed in order to ensure the existence of the local extension required by Lemma 6.3. We shall construct inductively a sequence of homotopies $H_i : D \to A(T_i) \otimes C[0, 1]$, with components $(H_i)_{(x,t)} : D \to A(x)$, such that the restriction of $H_{i+1} : D \to A(T_{i+1}) \otimes C[0, 1]$ to S_i is equal to the restriction of H_i to S_i, i.e. $(H_{i+1})_{(x,t)} = (H_i)_{(x,t)}$ for all $x \in S_i$ and $t \in [0, 1]$. Moreover, we shall also have that $(H_i)_{(x,0)} = \varphi_x$ and that $(H_i)_{(x,t)}$ is an isomorphism for each $x \in T_i$, $t > 0$, $i = 1, \dots, m$. We start with $i = 1$ and regard $\varphi_1 = \nu_1 \pi_{V_1} \varphi : D \to C(V_1) \otimes D$ as a continuous map $x \to (\varphi_1)_x = (\nu_1)_x \varphi_x \in \mathrm{End}(D)^*$ defined on V_1. By Proposition 6.2 applied for $Y = V_1$, $X = V_1 \times [0, 1]$, $\gamma = \varphi_1$ and $\alpha_{(x,t)} = (\varphi_1)_x$ there is a homotopy $\eta_1 : V_1 \times [0, 1] \to End(D)^*$ such that $(\eta_1)_{(x,0)} = (\varphi_1)_x$ and $(\eta_1)_{(x,t)} \in \mathrm{Aut}(D)$ for $t > 0$. We set $(H_1)_{(x,t)} = (\nu_1)_x^{-1} \circ (\eta_1)_{(x,t)}$. Suppose now that $H_1, \dots, H_i$ were constructed. We shall construct H_{i+1} by restricting H_i to $S_i \times [0, 1]$, and then by extending this restriction to $T_{i+1} \times [0, 1] = (S_i \cup V_{i+1}) \times [0, 1]$. Clearly it suffices to extend H_i from $(S_i \cap V_{i+1}) \times [0, 1]$ to $V_{i+1} \times [0, 1]$ and then set $(H_{i+1})_{(x,t)} = (H_i)_{(x,t)}$ for $x \in S_i \setminus V_{i+1}$ and $t \in [0, 1]$. To this purpose we define a continuous map $\alpha : (T_i \cap V_{i+1}) \times [0, 1] \cup V_{i+1} \times \{0\} \to \mathrm{End}(D)$ by $\alpha_{(x,t)} = (\nu_{i+1})_x \circ (H_i)_{(x,t)}$ for $x \in T_i \cap V_{i+1}$ and $t \in [0, 1]$ and $\alpha_{(x,0)} = (\varphi_{i+1})_x$ for $x \in V_{i+1}$. Since $T_i \cap V_{i+1}$ is a neighborhood of $S_i \cap V_{i+1}$ in V_{i+1} and since $(\varphi_{i+1})_x \in \mathrm{End}(D)^*$ for all $x \in V_{i+1}$, we can apply Lemma 6.3 to obtain a continuous map $\eta_{i+1} : V_{i+1} \times [0, 1] \to \mathrm{End}(D)^*$ such that η_{i+1} extends the restriction of α to $(S_i \cap V_{i+1}) \times [0, 1] \cup V_{i+1} \times \{0\}$ and $(\eta_{i+1})_{(x,t)} \in \mathrm{Aut}(D)$ for all $x \in V_{i+1}$ and $t > 0$. We conclude the construction of H_{i+1} by defining $(H_{i+1})_{(x,t)} = (\nu_{i+1})_x^{-1} \circ (\eta_{i+1})_{(x,t)}$ for $x \in V_{i+1}$ and $t \in [0, 1]$. Finally we observe that since $T_m = X$, $H_m : D \to C[0, 1] \otimes A$ is a homotopy from φ to some full $*$-homomorphism ψ such that $\psi_x \in \mathrm{Aut}(D)$ for all $x \in X$. Therefore $\widetilde{\psi} : C(X) \otimes D \to A$ is an isomorphism of $C(X)$-algebras homotopic to $\widetilde{\varphi}$. $\square$

Lemma 6.5. *Let X be a compact metrizable space and let D be a KK-semiprojective stable Kirchberg algebra. Let Y, Z be closed subsets of X such that $X = Y \cup Z$. Let $\gamma : D \to C(Y \cap Z) \otimes D$ be a full $*$-homomorphism. Assume that there is a full $*$-homomorphism $\alpha : D \to C(Y) \otimes D$ such that $\alpha_x \in KK(D,D)^{-1}$ for all $x \in Y$ and such that $\alpha_x = \gamma_x$ for all $x \in Y \cap Z$. Then the pullback $C(Y) \otimes D \oplus_{\pi_{Y\cap Z}, \widetilde{\gamma}\pi_{Y\cap Z}} C(Z) \otimes D$ is isomorphic to $C(X) \otimes D$.*

Proof. By Prop. 6.2 there is a $*$-homomorphism $\eta : D \to C(Y) \otimes D$ such that $\eta_x = \gamma_x$ for $x \in Y \cap Z$ and such that $\eta_x \in \mathrm{Aut}(D)$ for $x \in Y \setminus Z$. One checks immediately that the pair $(\widetilde{\eta}, \mathrm{id}_{C(Z)\otimes D})$ defines a $C(X)$-linear isomorphism:
$\theta : C(X) \otimes D = C(Y) \otimes D \oplus_{\pi_{Y\cap Z}, \pi_{Y\cap Z}} C(Z) \otimes D \to C(Y) \otimes D \oplus_{\pi_{Y\cap Z}, \widetilde{\gamma}\pi_{Y\cap Z}} C(Z) \otimes D.$ □

Lemma 6.6. *Let D be a KK-semiprojective stable Kirchberg algebra. Let Y, Z and Z' be closed subsets of a compact metrizable space X such that Z' is a neighborhood of Z and $X = Y \cup Z$. Let B be a $C(Y)$-algebra locally isomorphic to $C(Y) \otimes D$ and let E be a $C(Z')$-algebra locally isomorphic to $C(Z') \otimes D$. Let $\alpha : E(Y \cap Z') \to B(Y \cap Z')$ be a $*$-monomorphism of $C(Y \cap Z')$-algebras such that $KK(\alpha_x) \in KK(E(x), B(x))^{-1}$ for all $x \in Y \cap Z'$. If $\gamma = \alpha_{Y\cap Z}$, then $B(Y) \oplus_{\pi_{Y\cap Z}, \gamma\, \pi_{Y\cap Z}} E(Z)$ is locally isomorphic to $C(X) \otimes D$.*

Proof. Since we are dealing with a local property, we may assume that $B = C(Y) \otimes D$ and $E = C(Z') \otimes D$. To simplify notation we let π stand for both $\pi^Y_{Y\cap Z}$ and $\pi^Z_{Y\cap Z}$ in the sequel. Let us denote by H the $C(X)$-algebra $C(Y) \otimes D \oplus_{\pi, \gamma\pi} C(Z) \otimes D$. We must show that H is locally trivial. Let $x \in X$. If $x \notin Z$, then there is a closed neighborhood U of x which does not intersect Z, and hence the restriction of H to U is isomorphic to $C(U) \otimes D$, as it follows immediately from the definition of H. It remains to consider the case when $x \in Z$. Now Z' is a closed neighborhood of x in X and the restriction of H to Z' is isomorphic to $C(Y \cap Z') \otimes D \oplus_{\pi, \gamma\pi} C(Z) \otimes D$. Since $\gamma : Y \cap Z \to \mathrm{End}(D)^*$ admits a continuous extension $\alpha : Y \cap Z' \to \mathrm{End}(D)^*$, it follows that $H(Z')$ is isomorphic to $C(Z') \otimes D$ by Lemma 6.5. □

Proposition 6.7. *Let X, A, D and σ be as in Proposition 4.7. For any finite subset $\mathcal{F}$ of A and any $\varepsilon > 0$ there is a $C(X)$-algebra B which is locally isomorphic to $C(X) \otimes D$ and there exists a $C(X)$-linear $*$-monomorphism $\eta : B \to A$ such that $\mathcal{F} \subset_\varepsilon \eta(B)$ and $KK(\eta_x) \in KK(B(x), A(x))^{-1}$ for all $x \in X$.*

Proof. Let $\psi_k : E_k = C(Y_k) \otimes D \to A(Y_k)$, $k = 0, ..., n$ be as in the conclusion of Proposition 4.7, strengthen as in Remark 4.8. Therefore we may assume that there is another n-fibered $\{D\}$-monomorphism $(\psi'_0, ..., \psi'_n)$ into A such that $\psi'_k : C(Y'_k) \otimes D \to A(Y'_k)$, Y'_k is a closed neighborhood of Y_i, and $\pi_{Y_k}\psi'_k = \psi_k$, $k = 0, ..., n$. Let X_k, B_k, η_k and γ_k be as in Definition 2.8. B_0 and η_0 satisfy the conclusion of the proposition, except that we need to prove that B_0 is locally isomorphic to $C(X)\otimes D$. We prove by induction on decreasing k that the $C(X_k)$-algebras B_k are locally trivial. Indeed $B_n = C(X_n)\otimes D$ and assuming that B_k is locally trivial, it follows by Lemma 6.6 that B_{k-1} is locally trivial, since by (5)

$$B_{k-1} \cong B_k \oplus_{\pi\eta_k,\pi\psi_{k-1}} E_{k-1} \cong B_k \oplus_{\pi,\gamma_k\pi} E_{k-1}, \quad (\ \pi = \pi_{X_k\cap Y_{k-1}})$$

and $\gamma_k : E_{k-1}(X_k \cap Y_{k-1}) \to B_k(X_k \cap Y_{k-1})$, $(\gamma_k)_x = (\eta_k)_x^{-1}(\psi_{k-1})_x$, extends to a $*$-monomorphism $\alpha : E_{k-1}(X_k \cap Y'_{k-1}) \to B_k(X_k \cap Y'_{k-1})$, $\alpha_x = (\eta_k)_x^{-1}(\psi'_{k-1})_x$ and $KK(\alpha_x)$ is a KK-equivalence since both $KK((\eta_k)_x)$ and $KK((\psi_{k-1})_x)$ are KK-equivalences. $\square$

7. When is a $C(X)$-algebra (locally) trivial

In this section we prove Theorems 1.1 – 1.3 and of some of their consequences.

Proof of Theorem 1.2.

Proof. Let X denote the primitive spectrum of A. Then A is a continuous $C(X)$-algebra and its fibers are stable Kirchberg algebras (see [7, 2.2.2]). Since A is separable, X is metrizable by Lemma 2.2. By Proposition 6.7 there is a sequence of $C(X)$-algebras $(A_k)_{k=1}^{\infty}$ locally isomorphic to $C(X)\otimes D$ and a sequence of $C(X)$-linear $*$-monomorphisms $(\eta_k : A_k \to A)_{k=1}^{\infty}$, such that $KK(\eta_k)_x$ is a KK-equivalence for each $x\in X$ and $(\eta_k(A_k))_{k=1}^{\infty}$ is an exhaustive sequence of $C(X)$-subalgebras of A. Since D is weakly semiprojective and KK-stable, after passing to a subsequence of (A_k) if necessary, we find a sequence $(\sigma_k)_{k=1}^{\infty}$, $\sigma_k \in KK(D, A_k)$ such that $KK(\eta_k)\sigma_k = \sigma$ for all $k\geq 1$. Since both $KK(\eta_k)_x$ and σ_x are KK-equivalences, we deduce that $(\sigma_k)_x \in KK(D, A_k(x))^{-1}$ for all $x \in X$. By Proposition 6.4, for each $k \geq 1$ there is an isomorphism of $C(X)$-algebras $\varphi_k : C(X)\otimes D \to A_k$ such that $KK(\varphi_k) = \sigma_k$. Therefore if we set $\theta_k = \eta_k\varphi_k$, then θ_k is a $C(X)$-linear $*$-monomorphism from $B \overset{def}{=} C(X)\otimes D$ to A such that $KK(\theta_k) = \sigma$ and $(\theta_k(B))_{k=1}^{\infty}$ is an exhaustive sequence of $C(X)$-subalgebras of A. Using again the weak semiprojectivity and the KK-stability of D, and Lemma 4.1, after passing to a subsequence of $(\theta_k)_{k=1}^{\infty}$, we construct a sequence of finite sets $\mathcal{F}_k \subset B$ and a sequence of $C(X)$-linear $*$-monomorphisms $\mu_k : B \to B$ such that

(i) $KK(\theta_{k+1}\mu_k) = KK(\theta_k)$ for all $k \geq 1$,
(ii) $\|\theta_{k+1}\mu_k(a) - \theta_k(a)\| < 2^{-k}$ for all $a \in \mathcal{F}_k$ and all $k\geq 1$,
(iii) $\mu_k(\mathcal{F}_k) \subset \mathcal{F}_{k+1}$ for all $k \geq 1$,
(iv) $\bigcup_{j=k+1}^{\infty} \big(\mu_{j-1}\circ\cdots\circ\mu_k\big)^{-1}(\mathcal{F}_j)$ is dense in B and $\bigcup_{j=k}^{\infty}\theta_j(\mathcal{F}_j)$ is dense in A for all $k \geq 1$.

Arguing as in the proof of [31, Prop. 2.3.2], one verifies that

$$\Delta_k(a) = \lim_{j\to\infty} \theta_j \circ \big(\mu_{j-1}\circ\cdots\circ\mu_k\big)(a)$$

defines a sequence of $*$-monomorphisms $\Delta_k : A_k \to A$ such that $\Delta_{k+1}\mu_k = \Delta_k$ and the induced map $\Delta : \varinjlim_k(B, \mu_k) \to A$ is an isomorphism of $C(X)$-algebras. Let us show that $\varinjlim_k(B, \mu_k)$ is isomorphic to B. To this purpose, in view of Elliott's intertwining argument, it suffices to show that each map μ_k is approximately unitarily equivalent to a $C(X)$-linear automorphism of B. Since $KK(\theta_k) = \sigma$, we deduce from (i) that $KK((\mu_k)_x) = KK(\mathrm{id}_D)$ for all $x \in X$. By Proposition 6.1, this property implies that each map μ_k is approximately unitarily equivalent to a $C(X)$-linear automorphism of B. Therefore there is an isomorphism of $C(X)$-algebras $\Delta : B \to A$. Let us show that we can arrange that $KK(\Delta|_D) = \sigma$. By Theorem 3.1, there is a full $*$-homomorphism $\alpha : D \to B$ such that $KK(\alpha) = KK(\Delta^{-1})\sigma$. Since $KK(\Delta_x^{-1})\sigma_x \in KK(D, D)^{-1}$, by Proposition 6.1 there is $\Phi_1 : D \to C(X) \otimes D$ such that $\widetilde{\Phi}_1 \in \mathrm{Aut}_{C(X)}(B)$ and $KK(\Phi_1) = KK(\Delta^{-1})\sigma$. Then $\Phi = \Delta\widetilde{\Phi}_1 : B \to A$ is an isomorphism such that $KK(\Phi|_D) = KK(\Delta\Phi_1) = \sigma$. □

Proof of Theorem 1.3.

Proof. For the first part we apply Theorem 1.2 for $D = \mathcal{O}_2 \otimes \mathcal{K}$ and $\sigma = 0$. For the second part we assert that if D is a Kirchberg such that all continuous $C[0, 1]$-algebras with fibers isomorphic to D are locally trivial then D is stable and $KK(D, D) = 0$. This implies that D is KK-equivalent to $\mathcal{O}_2$ and hence that $D \cong \mathcal{O}_2 \otimes \mathcal{K}$ by [31, Thm. 8.4.1]. The Kirchberg algebra D is either unital or stable [31, Prop. 4.1.3]. Let $\psi : D \to D$ be a $*$-monomorphism such that $KK(\psi) = 0$ and such that $\psi(1_D) < 1_D$ if D is unital. Let us consider the continuous $C[0, 1]$-algebra $E = \{(f, d) \in C[0, 1] \otimes D : f(1) \in \psi(D)\}$. By assumption E is trivial on some neighborhood of 1. Therefore there is $s \in [0, 1)$ such that $C[s, 1] \otimes D \cong E[s, 1]$. We deduce that there are $\beta \in \mathrm{Aut}(D)$ and $\theta \in \mathrm{Hom}(D, C[s, 1] \otimes D)$ such that $\theta_t \in \mathrm{Aut}(D)$ for $s \le t < 1$ and $\theta_1 = \psi\beta$. Therefore D must be nonunital (and hence stable), since otherwise 1_D would be homotopic to its proper subprojection $\psi(1_D)$. Moreover $KK(\theta_s) = KK(\psi\beta) = 0$ and hence $KK(D, D) = 0$ since θ_s is an automorphism. □

Dixmier and Douady [15] proved that a continuous field with fibers $\mathcal{K}$ over a finite dimensional locally compact Hausdorff space is locally trivial if and only it verifies Fell's condition, i.e. for each $x_0 \in X$ there is a continuous section a of the field such that $a(x)$ is a rank one projection for each x in a neighborhood of x_0. We have a analogous result:

Corollary 7.1. *Let A be a separable C^*-algebra whose primitive spectrum X is Hausdorff and of finite dimension. Suppose that for each $x \in X$, $A(x)$ is KK-semiprojective, nuclear, purely infinite and stable. Then A is locally trivial if and only if for each $x \in X$ there exist a closed neighborhood V of x, a Kirchberg algebra D and $\sigma \in KK(D, A(V))$ such that $\sigma_v \in KK(D, A(v))^{-1}$ for each $v \in V$.*

Proof. One applies Theorem 1.2 for $D \otimes \mathcal{K}$ and $A(V)$. □

We turn now to unital $C(X)$-algebras.

Theorem 7.2. *Let A be a separable unital $C(X)$-algebra over a finite dimensional compact Hausdorff space X. Suppose that each fiber $A(x)$ is nuclear simple and purely infinite. Then A is isomorphic to $C(X) \otimes D$, for some KK-semiprojective unital Kirchberg algebra D, if and only if there is $\sigma \in KK(D, A)$ such that $K_0(\sigma)[1_D] = [1_A]$ and $\sigma_x \in KK(D, A(x))^{-1}$ for all $x \in X$. For any such σ there is an isomorphism of $C(X)$-algebras $\Phi : C(X) \otimes D \to A$ such that $KK(\Phi|_D) = \sigma$.*

Proof. We verify the nontrivial implication. X is metrizable by Lemma 2.2. A is a continuous $C(X)$-algebra by Lemma 2.3. By Theorem 1.2, there is an isomorphism $\Phi : C(X) \otimes D \otimes \mathcal{K} \to A \otimes \mathcal{K}$ such that $KK(\Phi) = \sigma$. Since $K_0(\sigma)[1_D] = [1_A]$, and since $A \otimes \mathcal{K}$ contains a full properly infinite projection, we may arrange that $\Phi(1_{C(X)\otimes D} \otimes e_{11}) = 1_A \otimes e_{11}$ after conjugating Φ by some unitary $u \in M(A \otimes \mathcal{K})$. Then $\varphi = \Phi|_{C(X)\otimes D \otimes e_{11}}$ satisfies the conclusion of the theorem. □

Proof of Theorem 1.4.

Proof. Let D be a KK-semiprojective unital Kirchberg algebra D such that every unital $*$-endomorphism of D is a KK-equivalence. Suppose that A is a separable unital $C(X)$-algebra over a finite dimensional compact Hausdorff space the fibers of which are isomorphic to D and that there is $\sigma \in KK(D, A)$ such that $K_0(\sigma)[1_D] = [1_A]$. We shall prove that $A \cong C(X) \otimes D$. By Theorem 7.2, it suffices to show that $\sigma_x \in KK(D, A(x))^{-1}$ for all $x \in X$. Fix $x \in X$. By assumption, there is an isomorphism $\varphi : A(x) \to D$. The K_0-morphism induced by $KK(\varphi)\sigma_x$ maps $[1_D]$ to itself. By Theorem 3.1 there is a unital $*$-homomorphism $\psi : D \to D$ such that $KK(\psi) = KK(\varphi)\sigma_x$. By assumption we must have $KK(\psi) \in KK(D, D)^{-1}$ and hence $\sigma_x \in KK(D, A(x))^{-1}$ since φ is an isomorphism.

Now let us show that if A and D are as above, then A is locally trivial since the local existence of σ is automatic. Let $x \in X$ and set $V_k = \{y \in X : d(y, x) \leq 1/k\}$. Then $A(x) \cong \varinjlim_k A(V_k)$. By assumption, there is an isomorphism $\eta : D \to A(x)$. Since D is KK-semiprojective, $KK(\eta)$ lifts to some KK-element $\sigma \in KK(D, A(V_k))$ such that $K_0(\sigma)[1_D] = [1_{A(V_k)}]$. Therefore $A(V_k) \cong C(V_k) \otimes D$ by the first part of the proof.

Conversely, let us assume that all separable unital continuous $C[0,1]$-algebras with fibers isomorphic to D are locally trivial. Let ψ be a given unital $*$-endomorphism of D. Consider the continuous $C[0,1]$-algebra $E = \{(f, d) \in C[0,1] \otimes D : f(1) \in \psi(D)\}$. By assumption, E is trivial on some neighborhood of 1 and hence there is $s \in [0,1)$ such that $C[s,1] \otimes D \cong E[s,1]$. Therefore there are $\beta \in \mathrm{Aut}(D)$ and $\theta \in \mathrm{Hom}(D, C[s,1] \otimes D)$ such that $\theta_t \in \mathrm{Aut}(D)$ for $s \leq t < 1$ and $\theta_1 = \psi\beta$. It follows that $KK(\theta_s) = KK(\psi\beta)$ and hence $KK(\psi)$ is invertible since θ_s and β are automorphisms of D. □

Proof of Theorem 1.1

Proof. Let A be as in Theorem 1.1 and let $n \in \{2, 3, ...\} \cup \{\infty\}$. It is known that $\mathcal{O}_n$ satisfies the UCT. Moreover $K_0(\mathcal{O}_n)$ is generated by $[1_{\mathcal{O}_n}]$ and $K_1(\mathcal{O}_n) = 0$. Therefore any unital $*$-endomorphism of $\mathcal{O}_n$ is a KK-equivalence. It follows that A is locally trivial by Theorem 1.4. Suppose now that $n = 2$. Since $KK(\mathcal{O}_2, \mathcal{O}_2) = KK(\mathcal{O}_2, A) = 0$, we may apply Theorem 1.4 with $\sigma = 0$ and obtain that $A \cong C(X) \otimes \mathcal{O}_2$. Suppose now that $n = \infty$. Let us define $\theta : K_0(\mathcal{O}_\infty) \to K_0(A)$ by $\theta(k[1_{\mathcal{O}_\infty}]) = k[1_A]$, $k \in \mathbb{Z}$. Since $\mathcal{O}_\infty$ satisfies the UCT, θ lifts to some element $\sigma \in KK(\mathcal{O}_\infty, A)$. By Theorem 1.4 it follows that $A \cong C(X) \otimes \mathcal{O}_\infty$. Finally let us consider the case $n \in \{3, 4, ...\}$. Then $K_0(\mathcal{O}_n) = \mathbb{Z}/(n-1)$. Since $\mathcal{O}_n$ satisfies the UCT, the existence of an element $\sigma \in KK(\mathcal{O}_n, A)$ such that $K_0(\sigma)[1_{\mathcal{O}_n}] = [1_A]$ is equivalent to the existence of a morphism of groups $\theta : \mathbb{Z}/(n-1) \to K_0(A)$ such that $\theta(\bar{1}) = [1_A]$. This is equivalent to requiring that $(n-1)[1_A] = 0$. □

As a corollary of Theorem 1.1 we have that $[X, \mathrm{Aut}(\mathcal{O}_\infty)]$ reduces to a point. The homotopy groups of the endomorphisms of the stable Cuntz-Krieger algebras were computed in [9]. Let $v_1, \dots, v_n$ be the canonical generators of $\mathcal{O}_n$, $2 \leq n < \infty$.

Theorem 7.3. *For any compact metrizable space X there is a bijection $[X, \mathrm{Aut}(\mathcal{O}_n)] \to K_1(C(X) \otimes \mathcal{O}_n)$. The k^{th}-homotopy group $\pi_k(\mathrm{Aut}(\mathcal{O}_n))$ is isomorphic to $\mathbb{Z}/(n-1)$ if k is odd and it vanishes if k is even. In particular $\pi_1(\mathrm{Aut}(\mathcal{O}_n))$ is generated by the class of the canonical action of $\mathbb{T}$ on $\mathcal{O}_n$, $\lambda_z(v_i) = zv_i$.*

Proof. Since $\mathcal{O}_n$ satisfies the UCT, we deduce that $\mathrm{End}(\mathcal{O}_n)^* = \mathrm{End}(\mathcal{O}_n)$. An immediate application of Proposition 6.1 shows that the natural map $\mathrm{Aut}(\mathcal{O}_n) \hookrightarrow \mathrm{End}(\mathcal{O}_n)$ induces an isomorphism of groups $[X, \mathrm{Aut}(\mathcal{O}_n)] \cong [X, \mathrm{End}(\mathcal{O}_n)]$. Let $\iota : \mathcal{O}_n \to C(X) \otimes \mathcal{O}_n$ be defined by $\iota(v_i) = 1_{C(X)} \otimes v_i$, $i = 1, ..., n$. The map $\psi \mapsto u(\psi) = \psi(v_1)\iota(v_1)^* + \cdots + \psi(v_n)\iota(v_n)^*$ is known to be a homeomorphism from $\mathrm{Hom}(\mathcal{O}_n, C(X) \otimes \mathcal{O}_n)$ to the unitary group of $C(X) \otimes \mathcal{O}_n$. Its inverse maps a unitary w to the $*$-homomorphism ψ uniquely defined by $\psi(v_i) = w\iota(v_i)$, $i = 1, ..., n$. Therefore

$$[X, \mathrm{Aut}(\mathcal{O}_n)] \cong [X, \mathrm{End}(\mathcal{O}_n)] \cong \pi_0(U(C(X) \otimes \mathcal{O}_n)) \cong K_1(C(X) \otimes \mathcal{O}_n).$$

The last isomorphism holds since $\pi_0(U(B)) \cong K_1(B)$ if $B \cong B \otimes \mathcal{O}_\infty$, by [30, Lemma 2.1.7]. One verifies easily that if $\varphi \in \mathrm{Hom}(\mathcal{O}_n, C(X) \otimes \mathcal{O}_n)$, then $u(\widetilde{\psi}\varphi) = \widetilde{\psi}(u(\varphi))u(\psi)$. Therefore the bijection $\chi : [X, \mathrm{End}(D)] \to K_1(C(X) \otimes \mathcal{O}_n)$ is an isomorphism of groups whenever $K_1(\widetilde{\psi}) = \mathrm{id}$ for all $\psi \in \mathrm{Hom}(\mathcal{O}_n, C(X) \otimes \mathcal{O}_n)$. Using the $C(X)$-linearity of $\widetilde{\psi}$ one observes that this holds if the $n-1$ torsion of $K_0(C(X))$ reduces to $\{0\}$, since in that case the map $K_1(C(X)) \to K_1(C(X) \otimes \mathcal{O}_n)$ is surjective by the Künneth formula. $\square$

Corollary 7.4. *Let X be a finite dimensional compact metrizable space. The isomorphism classes of unital separable $C(SX)$-algebras with all fibers isomorphic to $\mathcal{O}_n$ are parameterized by $K_1(C(X) \otimes \mathcal{O}_n)$.*

Proof. This follows from Theorems 1.1 and 7.3, since the locally trivial principal H-bundles over $SX = X \times [0,1]/X \times \{0,1\}$ are parameterized by $\pi_{m-1}(H)$ if H is a path connected group [19, Cor. 8.4] and since $\dim(SX) \leq \dim(X) + 1$. Here we take $H = \mathrm{Aut}(\mathcal{O}_n)$. $\square$

We need some preparation for the proof of Theorem 1.5. Let G be a group, let $g \in G$ and set $\mathrm{End}(G,g) = \{\alpha \in \mathrm{End}(G) : \alpha(g) = g\}$. The pair (G,g) is called *weakly rigid* if $\mathrm{End}(G,g) \subset \mathrm{Aut}(G)$ and *rigid* if $\mathrm{End}(G,g) = \{\mathrm{id}_G\}$.

Proposition 7.5. *If G is a finitely generated abelian group, then (G,g) is* weakly rigid *if and only if $(G,g) \cong (s\mathbb{Z} \oplus \mathbb{Z}/m, k \oplus \bar{q})$ where (s,k,m,q) is a quadruple of integers as in Theorem 1.5.*

Proof. We need to show that either $G = \{0\}$ or (G,g) is isomorphic to one of the following (mutually nonisomorphic) pointed groups: (i) $(\mathbb{Z}, k)$, where $k \geq 1$, (ii) $(\mathbb{Z}/m, \bar{q})$, where if $m = p_1^{r_1} \cdots p_n^{r_n}$ is decomposition of $m \geq 2$ into distinct primes, then $q = p_1^{s_1} \cdots p_n^{s_n}$ for some $0 \leq s_i < r_i$, $i = 1, ..., n$, (iii) $(\mathbb{Z} \oplus \mathbb{Z}/m, k \oplus \bar{q})$, where m and q are as in (ii) and $k \in qp_1p_2...p_n\,\mathbb{Z}$, $k \geq 1$. The proof is split in a number of steps.

(1) Let us observe that if $(G \oplus H, g \oplus h)$ is weakly rigid, then so are (G,g) and (H,h). If (G,g) is weakly rigid and $G \neq 0$ then $g \neq 0$.

(2) Let us observe that $(\mathbb{Z}^2, g)$ is not weakly rigid for any g. Indeed, if $g = (a,b) \neq 0$, then the matrix $\begin{pmatrix} 1+b^2 & -ab \\ -ab & 1+a^2 \end{pmatrix}$ defines an endomorphism α of $\mathbb{Z}^2$ such that $\alpha(g) = g$, but α is not invertible since $\det(\alpha) = 1 + a^2 + b^2 > 1$.

(3) Let p be a prime and let $r, s \geq 1$. Then $(\mathbb{Z}/p^r \oplus \mathbb{Z}/p^s, g)$ is not weakly rigid for any g. Indeed, if $g = \bar{a} \oplus \bar{b}$ where $a, b \in \mathbb{Z}$, then there are $k, \ell \in \mathbb{Z}$ such that $a - kb$ is divisible by p^r or $b - \ell a$ is divisible by p^s. Define an endomorphism α of $\mathbb{Z}/p^r \oplus \mathbb{Z}/p^s$ by $\alpha(\bar{1}, \bar{0}) = (\bar{0}, \bar{0})$, $\alpha(\bar{0}, \bar{1}) = (\bar{k}, \bar{1})$ in the first case and $\alpha(\bar{1}, \bar{0}) = (\bar{1}, \bar{\ell})$, $\alpha(\bar{0}, \bar{1}) = (\bar{0}, \bar{0})$ in the second case. Then $\alpha(g) = g$ but α is not injective.

(4) Let us observe that as a consequence of (1), (2) and (3), if (G,g) is weakly rigid for some g, then G is isomorphic to $s\mathbb{Z} \oplus \mathbb{Z}/p_1^{r_1} \oplus \cdots \oplus \mathbb{Z}/p_n^{r_n}$ where $s \in \{0,1\}$ and p_i are distinct primes (if G has torsion). Thus $G \cong \mathbb{Z}$ or $G \cong \mathbb{Z}/m$ or $G \cong \mathbb{Z} \oplus \mathbb{Z}/m$, where $m = p_1^{r_1} p_2^{r_2} ... p_n^{r_n}$ if $m \geq 2$.

(5) It is obvious that $(\mathbb{Z}, g)$ is weakly rigid if and only if $g \neq 0$.

(6) Let us show that if $q \in \mathbb{Z}$ and $m = p_1^{r_1} p_2^{r_2} ... p_n^{r_n} \geq 2$, then $(\mathbb{Z}/m, \bar{q})$ is weakly rigid if and only if q is not divisible by $p_i^{r_i}$ for any $i = 1, ..., n$. In this case $\bar{q} = q_0 \bar{u}$ for some invertible element $\bar{u} \in \mathbb{Z}/m$ and $q_0 = p_1^{s_1} \cdots p_n^{s_n}$ where s_i are as in (ii). Moreover $(\mathbb{Z}/m, \bar{q}) \cong (\mathbb{Z}/m, \bar{q}_0)$.

Let $\alpha \in \mathrm{End}(\mathbb{Z}/m, \bar{q})$. Then $\alpha(\bar{1}) = \bar{b}$ for some $b \in \mathbb{Z}$. Assume first that q is not divisible by $p_i^{r_i}$ for any $i = 1, ..., n$. From $\alpha(\bar{q}) = \bar{q}$ we obtain $(b-1)\bar{q} = \bar{0}$ and hence $(b-1)q$ is divisible by m. This can happen only if $b - 1$ is divisible by each p_i and hence b is not divisible by any p_i. Therefore $\bar{b}$ is invertible in $\mathbb{Z}/m$ and hence α is an automorphism. Conversely, assume that $q = \ell p_i^{r_i}$ for some $i \in \{1, ..., n\}$ and $\ell \in \mathbb{Z}$. Let $m_i = m/p_i^{r_i}$ and choose $m_i^* \in \mathbb{Z}$ such that $m_i^* m_i - 1$ is divisible by $p_i^{r_i}$. Then $b = 1 - m_i^* m_i$ is such that $\alpha(\bar{q}) = b\bar{q} = \bar{q}$ but α is not an automorphism since $\bar{b}$ is not invertible in $\mathbb{Z}/m$ as its image vanishes in the quotient $\mathbb{Z}/p_i^{r_i}$ of $\mathbb{Z}/m$.

(7) Let us observe that $(\mathbb{Z} \oplus \mathbb{Z}/m, k \oplus \bar{q})$ is weakly rigid if and only if $k \neq 0$ and whenever $ka + (b-1)q$ is divisible by m for some integers a and b, $\bar{b}$ is invertible in $\mathbb{Z}/m$. Indeed, if $\alpha \in \mathrm{End}(\mathbb{Z} \oplus \mathbb{Z}/m, k \oplus \bar{q})$ and $k \neq 0$, then $\alpha(1, \bar{0}) = (1, \bar{a})$ and $\alpha(0, \bar{1}) = (0, \bar{b})$ for some integers a and b satisfying the equation $k\bar{a} + (b-1)\bar{q} = \bar{0}$ in $\mathbb{Z}/m$, and α is bijective if and only if $\bar{b}$ is invertible in $\mathbb{Z}/m$. If $\bar{b}^*$ is the inverse of $\bar{b}$, then $\beta(1, \bar{0}) = (1, -\overline{ab^*})$ and $\beta(0, \bar{1}) = (0, \overline{b^*})$ defines an inverse for α.

(8) Let us show now that if $k \geq 0$, $0 \leq q \leq m - 1$ and $m \geq 2$ are integers, then $(\mathbb{Z} \oplus \mathbb{Z}/m, k \oplus \bar{q})$ is weakly rigid if and only if k, q and m are as in (iii). Assume first that $(\mathbb{Z} \oplus \mathbb{Z}/m, k \oplus \bar{q})$ is weakly rigid. Then $(\mathbb{Z}, k)$ and $(\mathbb{Z}/m, \bar{q})$ are weakly rigid. In view of (5) and (6) it remains to prove that $k \in qp_1 p_2 ... p_n \mathbb{Z}$. First we show that $k \in qp_i \mathbb{Z} + p_i^{r_i} \mathbb{Z}$ for each $i = 1, ..., n$. Seeking a contradiction, let us suppose that $\bar{k}$ is not divisible by $\bar{q}\bar{p}_i$ in $\mathbb{Z}/p_i^{r_i}$ for some i. Then we have factorizations $k = p_i^t k_1$ and $q = p_i^{s_i} q_1$ where $s_i \geq t \geq 0$ and neither k_1 or q_1 are divisible by p_i. Choose $k_1^* \in \mathbb{Z}$ such that $k_1^* k_1 - 1$ is divisible by $p_i^{r_i}$. If we set $a = k_1^* q_1 p_i^{s_i - t}$, then $ak - q$ is divisible by $p_i^{r_i}$. Let m_i and m_i^* be as in (6). Then $m_i m_i^* ak - m_i m_i^* q$ is divisible by m. Since $(\mathbb{Z} \oplus \mathbb{Z}/m, k \oplus \bar{q})$ is weakly rigid, the image of $1 - m_i m_i^*$ in $\mathbb{Z}/m$ must be invertible by (7). This is a contradiction since $1 - m_i m_i^*$ is divisible by $p_i^{r_i}$. Therefore there are integers $\ell_1, ..., \ell_n$ such that $k - \ell_i q p_i$ is divisible by $p_i^{r_i}$ for each i. Let us set $k_0 = \sum_{i=1}^n \ell_i q p_i m_i m_i^*$ and observe that k_0 is divisible by $qp_1 \cdots p_n$. We have $k - k_0 \in m\mathbb{Z}$ by the Chinese Remainder Theorem, and hence $k \in qp_1 \cdots p_n \mathbb{Z}$.

Conversely, assume that k, m, q are as in (iii). Therefore $k = qp_1 \cdots p_n c$ and $m = qp_1 \cdots p_n d$ for some $c, d \in \mathbb{Z}$. By (7), in order to prove that the pair $(\mathbb{Z} \oplus \mathbb{Z}/m, k \oplus \bar{q})$ is weakly rigid, it suffices to show that if $ka + (b-1)q$ is divisible by m for some integers a, b, then $\bar{b}$ is invertible in $\mathbb{Z}/m$. Assume that $ka + (b-1)q = \ell m$ for some $\ell \in \mathbb{Z}$. Then $qp_1 \cdots p_n ca + (b-1)q = qp_1 \cdots p_n \ell d$. This implies that $b - 1$ is divisible by $p_1 \cdots p_n$ and hence b and m are relatively prime. $\square$

Corollary 7.6. *If G is a finitely generated abelian group, then the pair (G, g) is* rigid *if and only if $G = \{0\}$ or $(G, g) \cong (\mathbb{Z}, k)$ where $k \geq 1$, or $(G, g) \cong (\mathbb{Z}/m, \bar{1})$ where $m \geq 2$.*

Proof. The statement follows from Proposition 7.5 and the following two observations. If k, m, q are as in Proposition 7.5(iii), then $(\mathbb{Z} \oplus \mathbb{Z}/m, k \oplus \bar{q})$ is not rigid since its endomorphism α defined by $\alpha(1, \bar{0}) = (1, \bar{a})$, $\alpha(0, \bar{1}) = (0, \bar{1})$, where $a = m/qp_1 \cdots p_n$, fixes $k \oplus \bar{q}$ but $\alpha \neq \mathrm{id}$. If m, q are as in Proposition 7.5(ii), then $(\mathbb{Z}/m, \bar{q})$ is not rigid unless $\bar{q}$ is invertible in $\mathbb{Z}/m$. Indeed, if $\bar{q}_0$ is a nonzero element of $\mathbb{Z}/m$ such that $\bar{q}_0\bar{q} = \bar{0}$, then $\alpha(x) = x + \bar{q}_0 x$ is an endomorphism of $(\mathbb{Z}/m, \bar{q})$ but $\alpha \neq \mathrm{id}$. □

Proof of Theorem 1.5

Proof. Let D be a unital Kirchberg algebra such that D satisfies the UCT and $K_*(D)$ is finitely generated. Then D is KK-semiprojective by Proposition 3.13 and $KK(D, D)^{-1} = \{\alpha \in KK(D, D) : K_*(\alpha) \text{ is bijective}\}$. By Theorem 3.1, all unital endomorphisms of D are KK-equivalences if and only if the pair $(K_0(D) \oplus K_1(D), [1_D] \oplus 0)$ is weakly rigid. Equivalently, $K_1(D) = 0$ and the pair $(K_0(D), [1_D])$ is weakly rigid. By Proposition 7.5 the pair $(K_0(D), [1_D])$ is weakly rigid if and only if $(K_0(D), [1_D]) \cong (s\mathbb{Z} \oplus \mathbb{Z}/m, k \oplus \bar{q})$ for some quadruple (s, k, m, q) as in Theorem 1.5. Equivalently, $D \cong D(s, k, m, q)$ by the classification theorem of Kirchberg and Phillips. We conclude the proof of the part of Theorem 1.5 that concerns automatic local triviality, by applying Theorem 1.4. Let us note that by Theorem 1.4, a separable unital $C(X)$-algebra A over a finite dimensional compact Hausdorff space X with all fibers isomorphic to $D(s, k, m, q)$ is trivial if and only there is a morphism of pointed groups $\theta : (s\mathbb{Z} \oplus \mathbb{Z}/m, k \oplus \bar{q}) \to (K_0(A), [1_A])$. It is readily seen that such a morphism exists if and only if $[1_A] \in k\, K_0(A) + q\, \mathrm{Tor}(K_0(A), \mathbb{Z}/m)$.

Next let us deal with the issue of automatic triviality. Let D be a unital Kirchberg algebra such that D satisfies the UCT and $K_*(D)$ is finitely generated. Suppose that all separable unital locally trivial $C(X)$-algebras with fibers isomorphic to D are trivial whenever X is a finite CW-complex of dimension at most three. We shall prove that D is isomorphic to either $\mathcal{O}_2$ or $\mathcal{O}_\infty$. First we show that if $\alpha \in KK(D, D)$ and $K_0(\alpha)[1_D] = [1_D]$ then $\alpha = KK(\mathrm{id}_D)$. By Theorem 3.1 for each α as above, there is $\varphi \in \mathrm{Aut}(D)$ such that $KK(\varphi) = \alpha$. Consider the $C(\mathbb{T})$-algebra $A = \{f \in C[0, 1] \otimes D : f(1) = \varphi(f(0))\}$. By assumption A is isomorphic to $C(\mathbb{T}) \otimes D$, and hence the principal $\mathrm{Aut}(D)$-bundle associated to A is trivial. By [19, Thm. 8.2 p85] φ is homotopic to id_D and hence $\alpha = KK(\varphi) = KK(\mathrm{id}_D)$. Since D satisfies the UCT, this implies that $K_1(D) = 0$ and that the pair $(K_0(D), [1_D])$ is rigid. We deduce from Corollary 7.6 that $K_0(D) = 0$ in which case $D \cong \mathcal{O}_2$ or that $(K_0(D), [1_D]) \cong (\mathbb{Z}/m, \bar{1})$ where $m \geq 2$, in which case $D \cong \mathcal{O}_{m+1}$, or that $(K_0(D), [1_D]) \cong (\mathbb{Z}, k)$, $k \geq 1$, in which case $D \cong M_k(\mathcal{O}_\infty)$ by the classification theorem of Kirchberg and Phillips. By Corollary 7.4, we can eliminate the case $D \cong \mathcal{O}_{m+1}$, $m \geq 2$.

To conclude the proof, in view of Theorem 1.1, it suffices to show that for each $k \geq 2$ there is a nontrivial $\mathrm{Aut}(M_k(\mathcal{O}_\infty))$-principal bundle over some CW complex of dimension 3. Let Y be the two-dimensional space obtained by attaching a disk to a circle by a degree-k map. Then $K_0(C(Y) \otimes \mathcal{O}_\infty) \cong \mathbb{Z} \oplus \mathbb{Z}/k$. By Theorem 3.1 there is a unital $*$-homomorphism $\varphi : M_k(\mathcal{O}_\infty) \to M_k(C(Y) \otimes \mathcal{O}_\infty)$ such that $K_0(\varphi)(1) = (1, \bar{1})$. By Proposition 6.1, there is a $*$-homomorphism $\Phi : M_k(\mathcal{O}_\infty) \to M_k(C(Y) \otimes \mathcal{O}_\infty)$ such that $\Phi_y \in \mathrm{Aut}(M_k(\mathcal{O}_\infty))$ for all $y \in Y$ and $KK(\Phi) = KK(\varphi)$. In particular the map $y \mapsto \Phi_y$ is not homotopic to a constant map $Y \to \mathrm{Aut}(M_k(\mathcal{O}_\infty))$. We deduce from [19, Thm. 8.2 p85] that the $\mathrm{Aut}(M_k(\mathcal{O}_\infty))$-principal bundle constructed over the suspension of Y with characteristic map $y \mapsto \Phi_y$ is not trivial. □

8. $C(X)$-algebras and the Universal Coefficient Theorem

Kirchberg has shown that any nuclear separable C*-algebra is equivalent in KK-theory to a Kirchberg algebra [31, Prop. 8.4.5]. Here we extend this result in the context of continuous $C(X)$-algebras and $KK_{C(X)}$-theory (see Theorem 8.4). In combination with Theorem 4.6, this leads to a new permanence property for the class of nuclear C*-algebras satisfying the UCT (see Theorem 1.6). Except for the proof of Theorem 1.6, we assume that X is a compact metrizable space throughout this section.

Lemma 8.1 ([4, Prop. 3.2])**.** *If A is a continuous $C(X)$-algebra, then there is a split short exact sequence of $C(X)$-algebras*

$$0 \longrightarrow A \longrightarrow A^+ \underset{\alpha}{\rightleftarrows} C(X) \longrightarrow 0 \tag{22}$$

where A^+ is unital, α is $C(X)$-linear and $\alpha(1) = 1$.

Consider the category of separable $C(X)$-algebras where the morphisms from A to B are the elements of $KK_{C(X)}(A, B)$ with composition given by the Kasparov product. The isomorphisms in this category are the KK-invertible elements denoted by $KK_{C(X)}(A, B)^{-1}$. Two $C(X)$-algebras are $KK_{C(X)}$-equivalent if they are isomorphic objects in this category. In the sequel we shall use twice the following elementary observation (valid in any category). If composition with $\gamma \in KK_{C(X)}(A, B)$ induces a bijection $KK_{C(X)}(B, C) \to KK_{C(X)}(A, C)$ for $C = A$ and $C = B$, then $\gamma \in KK_{C(X)}(A, B)^{-1}$.

Lemma 8.2. *Let A be a separable nuclear continuous $C(X)$-algebra. Then there exist a separable nuclear unital continuous $C(X)$-algebra $A^\flat$ and $C(X)$-linear monomorphisms $\alpha : C(X) \otimes \mathcal{O}_2 \to A^\flat$ and $\jmath : A \to A^\flat$ such that α is unital and $KK_{C(X)}(\jmath) \in KK_{C(X)}(A, A^\flat)^{-1}$.*

Proof. Let $p \in \mathcal{O}_\infty$ be a non-zero projection with $[p] = 0$ in $K_0(\mathcal{O}_\infty)$. Then there is a unital $*$-homomorphism $\mathcal{O}_2 \to p\mathcal{O}_\infty p$ which induces a $C(X)$-linear unital $*$-monomorphism

$\mu : C(X) \otimes \mathcal{O}_2 \to C(X) \otimes p\mathcal{O}_\infty p$. We tensor the exact sequence (22) by $p\mathcal{O}_\infty p$ and then take the pullback by μ. This gives a split exact sequence of unital $C(X)$-algebras:

$$\begin{tikzcd} 0 \arrow[r] & A \otimes p\mathcal{O}_\infty p \arrow[r] & A^+ \otimes p\mathcal{O}_\infty p \arrow[r, shift left] & C(X) \otimes p\mathcal{O}_\infty p \arrow[l, shift left, "\alpha"] \arrow[r] & 0 \\ 0 \arrow[r] & A \otimes p\mathcal{O}_\infty p \arrow[r, "j"] \arrow[u, equal] & A^\flat \arrow[r, shift left] \arrow[u] & C(X) \otimes \mathcal{O}_2 \arrow[l, shift left, "\alpha"] \arrow[r] \arrow[u, "\mu"'] & 0 \end{tikzcd}$$

The map $A^\flat \to A^+ \otimes p\mathcal{O}_\infty p$ is a unital $C(X)$-linear $*$-monomorphism, so that $A^\flat$ is a continuous $C(X)$-algebra. It is nuclear being an extension of nuclear C*-algebras. By [1, Thm. 5.4] for any separable $C(X)$-algebra B there is an exact sequence of groups

$$0 \to KK_{C(X)}(C(X) \otimes \mathcal{O}_2, B) \longrightarrow KK_{C(X)}(A^\flat, B) \xrightarrow{j^*} KK_{C(X)}(A \otimes p\mathcal{O}_\infty p, B) \to 0.$$

$KK_{C(X)}(C(X)\otimes\mathcal{O}_2, B) = 0$ since the class of the identity map of $C(X)\otimes\mathcal{O}_2$ vanishes in $KK_{C(X)}$. Therefore j^* is bijective and so $KK_{C(X)}(\jmath) \in KK_{C(X)}(A, A^\flat)^{-1}$. We conclude the proof by observing that map $A \to A\otimes p\mathcal{O}_\infty p$, $a \mapsto a\otimes p$, induces a $KK_{C(X)}$-equivalence. □

Proposition 8.3. *Let (A_i, φ_i) be an inductive system of separable nuclear $C(X)$-algebras with injective connecting maps. If $\varphi_i \in KK_{C(X)}(A_i, A_{i+1})^{-1}$ for all i, and $\Phi : A_1 \to \varinjlim(A_i, \varphi_i) = A_\infty$ is the induced map, then $\Phi \in KK_{C(X)}(A_1, A_\infty)^{-1}$.*

Proof. We use Milnor's $\varprojlim^1$-sequence for $KK_{C(X)}$-theory. Its proof is essentially identical to the proof of the corresponding sequence for KK-theory (argue as in [33] using [1]).

$$0 \to \varprojlim{}^1 KK_{C(X)}(A_i, B) \longrightarrow KK_{C(X)}(A_\infty, B) \longrightarrow \varprojlim KK_{C(X)}(A_i, B) \to 0$$

Since $\varprojlim^1 \big(G_i, \lambda_i\big) = 0$ and $G_1 \cong \varprojlim \big(G_i, \lambda_i\big)$ for any sequence of abelian groups $(G_i)_{i=1}^\infty$ and group isomorphisms $\lambda_i : G_i \to G_{i+1}$, the $\varprojlim^1$-sequence shows that for any separable $C(X)$ algebra B, the map $KK_{C(X)}(A_\infty, B) \to KK_{C(X)}(A, B)$ induced by Φ is bijective. Therefore $KK_{C(X)}(\Phi) \in KK_{C(X)}(A, A_\infty)^{-1}$. □

We need the following $C(X)$-equivariant construction which parallels a construction of Kirchberg as presented in [31]. A similar deformation technique has appeared in [11].

Theorem 8.4. *Let A be a separable nuclear continuous $C(X)$-algebra. Then there exist a separable nuclear continuous unital $C(X)$-algebra $A^\sharp$ whose fibers are Kirchberg C*-algebras and a $C(X)$-linear $*$-monomorphism $\Phi : A \to A^\sharp$ such that Φ is a $KK_{C(X)}$-equivalence. For any $x \in X$ the map $\Phi_x : A(x) \to A^\sharp(x)$ is a KK-equivalence.*

Proof. By Proposition 8.2 we may assume that A is unital and that there is a unital $C(X)$-linear $*$-monomorphism $\alpha : C(X) \otimes \mathcal{O}_2 \to A$. By [5, Thm. 2.5] there is a unital

$C(X)$-linear $*$-monomorphism $\beta : A \to C(X) \otimes \mathcal{O}_2$. Let s_1, s_2 be the images in A of the canonical generators v_1, v_2 of $\mathcal{O}_2 \subset C(X) \otimes \mathcal{O}_2$ under the map α. Set $\theta = \alpha \circ \beta : A \to A$ and define $\varphi : A \to A$ by $\varphi(a) = s_1\, a\, s_1^* + s_2\, \theta(a)\, s_2^*$. The unital $*$-homomorphism $\varphi_x : A(x) \to A(x)$ induced by φ satisfies $\varphi_x \pi_x = \pi_x \varphi$ and $\varphi_x(b) = s_1(x)\, b\, s_1(x)^* + s_2(x)\, \theta_x(b)\, s_2(x)^*$. Moreover θ_x factors through $\mathcal{O}_2$ since $\theta_x = \alpha_x \circ \beta_x$. Let $A^\sharp$ be the continuous $C(X)$-algebra obtained as the limit of the inductive system

$$A \xrightarrow{\varphi} A \xrightarrow{\varphi} A \xrightarrow{\varphi} \cdots$$

and let $\Phi : A \to A^\sharp$ be the induced map. The commutative diagram

$$\begin{array}{ccccccccc} A & \xrightarrow{\varphi} & A & \xrightarrow{\varphi} & A & \xrightarrow{\varphi} & \cdots & \longrightarrow & A^\sharp \\ \downarrow{\scriptstyle \pi_x} & & \downarrow{\scriptstyle \pi_x} & & \downarrow{\scriptstyle \pi_x} & & & & \downarrow{\scriptstyle \pi_x} \\ A(x) & \xrightarrow{\varphi_x} & A(x) & \xrightarrow{\varphi_x} & A(x) & \xrightarrow{\varphi_x} & \cdots & \longrightarrow & A^\sharp(x) \end{array}$$

shows that the fiber $A^\sharp(x)$ of A is isomorphic to $\varinjlim(A(x), \varphi_x)$. By the proof of [31, Prop. 8.4.5] $A^\sharp(x)$ is a unital Kirchberg algebra. It remains to prove that the map $\Phi : A \to A^\sharp$ induces a $KK_{C(X)}$-equivalence. By Proposition 8.3 it suffices to verify that $KK_{C(X)}(\varphi) = KK_{C(X)}(\mathrm{id}_A)$. This follows from the equation $\varphi(a) = s_1\, a\, s_1^* + s_2\, \theta(a)\, s_2^*$, since θ factors through $C(X) \otimes \mathcal{O}_2$ and hence $KK_{C(X)}(\theta) = 0$. $\square$

Lemma 8.5. *Let A be a $C(X)$-algebra such that $\mathrm{cat}_{\mathcal{C}}(A) = n < \infty$ where $\mathcal{C}$ is the class of all Kirchberg algebras satisfying the UCT. Then A satisfies the UCT.*

Proof. We shall prove by induction on n that if $\mathrm{cat}_{\mathcal{C}}(A) \leq n$, then A satisfy the UCT. If $n = 0$, then $A \cong \oplus_i C(Z_i) \otimes D_i$ and all its ideals satisfy the UCT since each D_i is simple and satisfies the UCT. By a result of [33], if two out of three separable nuclear C*-algebras in a short exact sequence satisfy the UCT, then all three of them satisfy the UCT. For the inductive step we use the exact sequence (1), with B elementary and $\mathrm{cat}_{\mathcal{C}}(D) \leq n-1$. $\square$

Theorem 8.6 ([12])**.** *Let A be a nuclear separable C*-algebra. Assume that for any finite set $\mathcal{F} \subset A$ and any $\varepsilon > 0$ there is a C*-subalgebra B of A satisfying the UCT and such that $\mathcal{F} \subset_\varepsilon B$. Then A satisfies the UCT.*

Proof. For the convenience of the reader we sketch an alternate proof in the case when B is nuclear. It is just this case that is needed in the sequel. By assumption, A admits an exhaustive sequence (A_n) consisting of nuclear separable C*-subalgebras which satisfy the UCT. We may assume that A is unital and its unit is contained in each A_n. Let us replace the pair $A_n \subseteq A$ by $A_n \otimes p\mathcal{O}_\infty p \subseteq A \otimes p\mathcal{O}_\infty p$ with p as in Lemma 8.2. If we use the map $\theta : A \otimes p\mathcal{O}_\infty p \hookrightarrow \mathcal{O}_2 \subset 1_A \otimes p\mathcal{O}_\infty p$ and $s_1, s_2 \in 1_A \otimes p\mathcal{O}_\infty p$ to construct $\varphi : A \otimes p\mathcal{O}_\infty p \to A \otimes p\mathcal{O}_\infty p$ as in Theorem 8.4, then $\varphi(B \otimes p\mathcal{O}_\infty p) \subset B \otimes p\mathcal{O}_\infty p$ for any subalgebra B of A. Therefore we can make the construction $A \mapsto A^\sharp$ functorial with respect to subalgebras. This shows that $A^\sharp$ admits an exhaustive sequence $(A_n^\sharp)$ consisting of nuclear separable C*-subalgebras which satisfy the UCT since each $A_n^\sharp$ is KK-equivalent to A_n. We can write each $A_n^\sharp$ as an inductive limit of a sequence of Kirchberg algebras satisfying the UCT and having finitely generated K-theory groups. Those algebras are weakly semiprojective (Examples 3.2). Thus $A^\sharp$ admits an exhaustive sequence (B_n) consisting of weakly semiprojective C*-algebras which satisfy the UCT. Arguing as in the proof of Theorem 5.2, we see that $A^\sharp$ is isomorphic to the inductive limit of a subsequence (B_{i_n}) of (B_n) and hence $A^\sharp$ satisfies the UCT [33]. Therefore A satisfies the UCT since it is KK-equivalent to $A^\sharp$. □

Proof of Theorem 1.6. Let A be as in the statement and consider the open set $Y = \{x \in X : A(x) \neq 0\}$. By replacing X by Y and viewing $A \cong C_0(Y)A$ as a $C(Y)$-algebra we may assume that all the fibers of A are nonzero. Let X^+ be the one-point compactification of X. Then $C(X^+)$ is separable by Lemma 2.2. By [4, Prop. 3.2], there is a unital $C(X^+)$-algebra A^+ which contains A as an ideal and such that $A^+/A \cong C(X^+)$. Thus we have reduced the proof to the case when X is compact and metrizable and A is unital. By Theorem 8.4 we may assume that the fibers of A are Kirchberg C*-algebras satisfying the UCT. By Theorem 4.6, A admits an exhaustive sequence (A_k) such that each A_k verifies the assumptions of Lemma 8.5 and hence A_k satisfies the UCT. We conclude the proof by applying Theorem 8.6. Let us note that the above proof only requires a weaker version of Theorem 8.4 which states that $\Phi : A \to A^\sharp$ and each Φ_x are KK-equivalences. Its proof requires only the usual $\varprojlim^1$-sequence for KK-theory.

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