LINMA2361 - Nonlinear Dynamical Systems

Analysis of the Saltzman-Lorenz equations

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1 Introduction

Saltzman-Lorenz system is a well know example in Nonlinear Dynamics. In this project I will study this system through the numerous literature that concerns it and better understand why it is of such interest. This dynamical system first describe in the famous article by Lorenz[6] is a very simplified version of atmospheric convection that surprisingly displays the behavior of a strange attractor.

In this paper I will introduce Saltzman-Lorenz and first study some general salient properties of it that will be helpful to set the framework for the analysis of this complex system. Hence we will talk about dissipation, local instability and general stability. Then I will go into the classical methodology of study in Dynamical system by computing the fixed points and analyze their stability. After that we will have the knowledge needed to propose a first analyze of the phase portrait. Lastly with regards to the previous considerations we will understand the chaotic behavior and why we can call the Saltzman-Lorenz equations to be a strange attractor.

Generally I will try to give some insights into the physical intuition we can have beyond the many technical mathematical considerations but more analysis on this subject can be found in Thompson and al.[10] or Alligood and al.[1] Because the literature over the Saltzman-Lorenz model is already very rich I take the choice to draw a portrait of the main properties of this complex system. Lastly I will produced some figures from numerical simulations[1] that will help to insist on particular characteristics.

2 The Saltzman-Lorenz System

The Oberbeck and Bussinesq equations describe convection for a fluid with a two-dimensional layer heated from below[5]. From those equations Saltzman started to generate a set of 52 ordinary differential equations of quadratic form in order to study the development of turbulence in the atmosphere.[3] To perform qualitative analysis Lorenz proposed a further idealized version with three equations only that we will call the *Saltzman-Lorenz model*[2](2.3),

$$\dot{X} = -\sigma X + \sigma Y, \tag{2.1}$$

$$\dot{Y} = -XZ + \rho X - Y, \tag{2.2}$$

$$\dot{Z} = XY - bZ. \tag{2.3}$$

In this highly idealized model of a fluid, the warm fluid below rises and the cool fluid above sinks, setting up a clockwise or counterclockwise current.[1]. In this form, X is proportional to the intensity of convective motion. A positive value $X > 0$ implies a clockwise circulation while for $X < 0$ the motion is counterclockwise. In addition Y is proportional to the temperature difference between the ascending and descending currents and Z to the distortion of the temperature profile with respect to the linearity. Three parameters are involved here, σ is the Prandtl number defined as the ratio of momentum diffusivity to thermal diffusivity, b is related to the buoyancy and ρ is related to the Rayleigh number i.e the higher ρ, the more we expect turbulence in the flow.[3] Moreover, we can suppose that $\rho, \sigma, b > 0$.

2.1 A paradox ?

As Lorenz[6] does in his paper, by standard techniques he was able before going into numerical simulations of the phase portrait to sketch some of the main features of the Saltzman-Lorenz

[1]All the simulations that are either reproduct or product can be found on the Github for this project at the following address : `https://github.com/AmauryLaridon/LINMA2361-Project-Analysis-of-the-Saltzman-Lorenz-equations`

[2]Historical elements proves that the contribution of Saltzman is far from being negligible hence the model should be called the Saltzman-Lorenz model. Further informations can be found in Maasch and al[7]

model but once he has taken into account all these preliminary derivations, he was confronted with what seemed like a paradox.
One by one he had eliminated all the know possibilities for the long-term behavior of his system : he showed that in a certain range of parameters, there could be no stable fixed points and no stable limit cycles, yet he also proved that all trajectories remain confined to a bounded region and are eventually attracted to a set of volume 0. As we will see that set is the strange attractor, and the motion on it is chaotic.[9]

2.2 Preliminary properties of the system

Before going into the more classical analysis of the dynamical system with equilibrium points, bifurcations and phase portrait analysis we will go trough some properties of the Saltzman-Lorenz system which will be helpful in order to understand the complexity of this dynamical system.

2.2.1 Nonlinearity

Saltzman-Lorenz system is formed by three deterministic and autonomous differential equations. Despite its very complex and chaotic solutions those do not appear from a stochastic forcing but from the nonlinearity of the system itself.

2.2.2 Symmetry with z-axis

The system is symmetrical with regard to the z-axis because $\dot{\mathbf{X}}(-X(t), -Y(t), Z(t)) = \dot{\mathbf{X}}(X(t), Y(t), Z(t))$. In other words, if $(x(t), y(t), z(t))$ is a solution so is $(-x(t), -y(t), z(t))$. This property will be useful in the analysis of the system and will lead to simplifications.

2.2.3 Dissipation of the system

We may wonder how a specific volume in the phase space evolve under the flow. To do so we want to compute the evolution of an infinitesimal volume V with respect to time. An intuitive proof in terms of vector calculus in Strogatz[9] leads to the well know properties that,

$$\dot{V}(t) = \int_V \nabla \cdot \mathbf{f} \tag{2.4}$$

with $\mathbf{f}$ being obviously in this case the vector function defined by the set (2.3). Note that the right hand side of the last equation is exactly the trace of the Jacobian matrix. Hence,

$$\nabla \cdot \mathbf{f} = -(\sigma + 1 + \beta). \tag{2.5}$$

Since the divergence is constant in (2.4) it reduces to $\dot{V} = -(\sigma + b + 1)V$ which has the solution $V(t) = V(0)e^{-(\sigma+b+1)t}$. While $\sigma, \beta > 0$ we have $\dot{V}(t) < 0$ and in phase space, volumes shrink exponentially fast ![9] In other words, any small volume in phase space will shrink for any combination of positive parameters. In his paper, Lorenz[6] shows with respect to that property of volume contraction that a closed, simply connected region $D \subset \mathbb{R}^3$ containing the origin can be found such that the vector field is directed everywhere inwards on the boundary i.e D contains an attracting set $A = \cap_{t \leq 0} \phi_t(D)$.[5]
It is important to notice that having any phase space volume converging towards zero doesn't mean that it shrinks to a point! Mathematically the volume shrinks to a volume of *zero measure* which may be a surface or something more complex as we will see.

The volume contraction property imposes strong constraints on the possible solutions of the Lorenz equations that we can derived with two examples coming from Strogatz[9]

- **They are no quasi-periodic solutions of the Saltzman-Lorenz equations**. Following Lorenz[6], a trajectory $P(t)$ will be called *quasi-periodic* if for some arbitrarily large time interval $\tau, P(t+\tau)$ ultimately remains arbitrarily close to $P(t)$. Indeed, if there were a quasi-periodic solution, it would have to lie on the surface of a torus (cfr. Section 8.6 Strogatz[9]) and this torus will be invariant under the flow i.e, the volume inside the torus would be constant in time. But this is a contradiction with the dissipative property just derived.

- **It is impossible for the Lorenz system to have either repelling fixed points or repelling closed orbits**. Again there is a contradiction with the volume contraction property. Indeed repellers are *sources* of volume because if we suppose to encase a repeller with a closed surface of initial conditions nearby in phase space, a short time later, the surface will have expanded as the corresponding trajectories are driven away. In other words the volume inside the surface would increase which is clearly a contradiction.

Therefore, we can already conclude by elimination that all fixed points must be sinks or saddles, and closed orbits (if they exist) must be stable or saddle-like.[9] Moreover, with this dissipative property we expect at least one attractor but this is not enough to guarantee that trajectories will not fly to infinity ![3]

2.2.4 Stability in the large

Lorenz[6] showed that all solutions are attracting in a defined region and so there is a global stability property which means that orbits do not diverge to infinity but stay trapped in some region around the origin. He did this by finding a function $U(x, y, z)$ that is decreasing along any trajectory that meets the boundary of the bounded region defined by $U \leq C$ for some constant C. Indeed if we consider for $\rho > 1$,

$$U = \rho X^2 + \sigma Y^2 + \sigma (Z - 2\rho)^2, \tag{2.6}$$

$$\dot{U} = -2\sigma(\rho X^2 + Y^2 + (Z - \rho)^2 - b\rho^2) \tag{2.7}$$

Therefore, $\dot{U} < 0$ for $(\rho X^2 + Y^2 + (Z-\rho)^2 - b\rho^2) > 0$ which is any point outside the ellipsoid defined by $\rho X^2 + Y^2 + b(Z-\rho)^2 = b\rho^2$.[3] This means it can never leave the U region, since U must increase to get out, moreover any point outside U is attract into U. A proper proof of the existence of the trapping region can be found in Alligood[1].

2.2.5 Instability in the small

Without proving it mathematically (because we will do later) it is already know that Saltzman-Lorenz equations displays chaotic trajectories. There is indeed the regime in which orbits do not settle down to stationary periodic or quasi-periodic motion due to a sort of local instability.

3 Equilibrium points and stability analysis

3.1 Equilibrium points

To start our qualitative study in a completely canonical way, we first look for the equilibrium points or fixed points of the system. By definition [9] an equilibrium point is a noted value $\mathbf{x}^*$ in this three dimensional case $x^* \in \mathbb{R}^3$ if and only if $f(x^*) = (0, 0, 0)$ where $f : \mathbb{R}^3 \to \mathbb{R}^3$ is a smooth function encoding the dynamics of the system being studied. Therefore we find,

$$0 = -\sigma X + \sigma Y, \tag{3.1}$$
$$0 = -XZ + \rho X - Y, \tag{3.2}$$
$$0 = XY - bZ. \tag{3.3}$$

which yields to,

$$X = Y, \tag{3.4}$$
$$X(1 + Z - \rho) = 0, \tag{3.5}$$
$$X^2 = bZ. \tag{3.6}$$

Finally if $\rho \leq 1$ the only fixed point is the trivial one,

$$\boxed{\mathbf{x}^* = (0, 0, 0)} \tag{3.7}$$

which represent the steady and motionless state. If $\rho > 1$ two new fixed points appears[3] ($\mathbf{x}^*$ still exists),

$$\boxed{\mathbf{C}^{\pm} = (\pm\sqrt{b(\rho-1)}, \pm\sqrt{b(\rho-1)}, \rho - 1).} \tag{3.8}$$

We therefore have two more fixed points which are symmetric on both side of the z-axis and they take real value only if $\rho > 1$.

3.2 Stability Analysis

The linear stability analysis[4] is performed with the Jacobian matrix,

$$df(X, Y, Z) = \begin{pmatrix} -\sigma & \sigma & 0 \\ (\rho - Z) & -1 & -X \\ Y & X & -b \end{pmatrix}. \tag{3.9}$$

3.2.1 Subcritical flow $\rho < 1$

For reason that will be explained soon we already divide our stability analysis in what we called the *Subcritical flow* when $\rho < 1$ and the *Supercritical flow* for $\rho > 1$.
In the subcritical flow regime, only the trivial fixed point exists. Hence for linear stability analysis we compute,

$$df(0, 0, 0) = \begin{pmatrix} -\sigma & \sigma & 0 \\ \rho & -1 & 0 \\ 0 & 0 & -b \end{pmatrix} \tag{3.10}$$

and after some algebra we find the following eigenvalues for $df(0, 0, 0)$,

$$\boxed{\lambda_0 = -b \quad ; \quad \lambda_{\pm} = -\frac{\sigma+1}{2} \pm \sqrt{\left(\frac{\sigma+1}{2}\right)^2 + \sigma(\rho - 1)}}. \tag{3.11}$$

While $\rho < 1$ we have that all eigenvalues are negative hence, **the trivial fixed point is local attractor**

[3] Following Lorenz[6] notations we will noted them C^+ and C^-.
[4] The analysis methodology of fixed point is inspired by Crucifix[3] because I found it particularly readable and orderly.

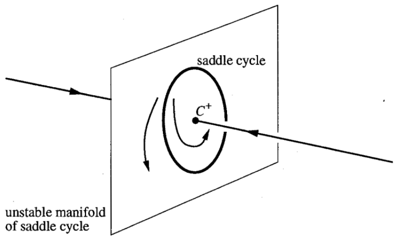

Figure 1: *Phase portrait near C^+ for $\rho < \rho_H$. The fixed point is stable and encircled by a saddle cycle, a new type of unstable limit cycle only possible in phases spaces of at least three dimensions. Figure from Strogatz.[9]*

A nicer and stronger way to prove this can be done by considering the Lyapunov function $V = X^2 + \sigma Y^2 + \sigma Z^2$. Since we have the conditions that $V : \mathbb{R}^3 \to \mathbb{R}$, $V(\mathbf{x}^*) = 0$ and $V(x, y, z) > 0$ if $\mathbf{x} \neq \mathbf{x}^*$ and that we can find a neighborhood U around $\mathbf{x}^*$ such that $\dot{V}(\mathbf{x}) \leq 0$ in $U_{\{\mathbf{x}^*\}}$ then by the Lyapunov function theorem $\mathbf{x}^*$ **is globally stable for** $\rho < 1$**.** Lastly we realize that the eigenvalues $\lambda_\pm = 0$ at the first critical value $\rho_c = 1$. Physically, when $\rho = 1$ the pure conductive solution having zero velocity and linear temperature gradient becomes unstable to a solution containing steady convective rolls or cells[5] as we will see in the next subsection.

3.2.2 Supercritical flow $\rho > 1$

If we still consider the trivial fixed point it is worth mentioning that with $\rho > 1$, λ_+ becomes positive. The fixed point $\mathbf{x}^*$ has become a saddle-point at $b = 0$ with the particularity that $\lambda_0, \lambda_- < 0$. In other words, **the saddle node bifurcation** leads to a local stable manifold of dimension two and an unstable local manifold of dimension one associated to the eigenvectors $\mathbf{v}_{\lambda_+}$. Physically for low values of ρ which corresponds to low thermal driving or high viscosity and thermal conductivity, the rest state $\mathbf{x}^*$ can be able to be a stable equilibrium but as ρ increases the equilibrium lose stability as the influx of heat produces a convective roll. The buoyancy of warm fluid overcomes damping forces and the convection motion start.[10]

As previously computed, while $\rho > 1$ two new fixed points exists $\mathbf{C}^\pm$. **Hence at** $\rho = 1$ **a pitchfork bifurcation occurs.** Because of the symmetry with the z-axis we can already consider that they will behave similarly. What is the stability of those fixed point ? We should perform the same eigenvalues analysis of $df(\mathbf{C}^\pm)$. Since the calculations are much longer (without being really interesting) I do not write them here but they can be found for example in Crucifix[3]. The conclusion is that **for** $1 < \rho < \rho_H = \sigma\frac{\sigma+b+3}{\sigma-b-1}$**,** $\mathbf{C}^\pm$ **are attractive. For** $\rho < \rho_1 < \rho_H$ **they are sinks and sinking spirals for** $\rho_1 \leq \rho < \rho_H$ where the saddle cycle has a two dimensional unstable manifold represent by the sheet (cfr Figure(1)) and a two dimensional stable manifold not shown.[9] The value ρ_1 is found during calculation and represent the value at which eigenvalues becomes complex (cfr Crucifix[3] for more informations).

As $\rho \to \rho_H$ from below, the cycle shrinks down around the fixed point. **When** $\rho = \rho_H$ **at the nontrivial points it is a Hopf bifurcation that occurs.** At this Hopf bifurcation, the fixed point absorbs the saddle cycle and change intro a saddle point.[9]
What happens now for $\rho > \rho_h$? For this domain, all three fixed points are unstable but as written previously an attracting set $A = \cap_{n\leq 0}\phi_t(D)$ still exists.[5] Is it possible that the Hopf bifurcation give rise to stable periodic orbits ? It has been shown by Marsden and McCracken[8] that the bifurcation is *subcritical* so that the unstable orbits shrink down upon the sinks as ρ increases towards ρ_h and no closed orbits exist near these fixed points for $\rho > \rho_h$. Physically the steady convection rolls represented by the symmetric pair of nontrivial solutions become unstable and are therefore replaced by some other large amplitude motion.[5]

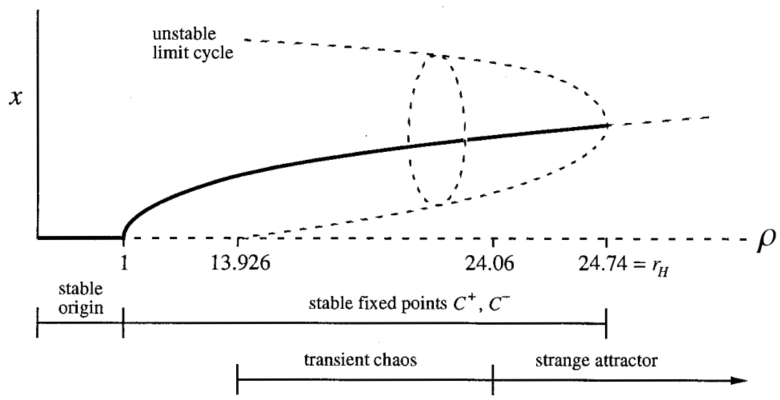

Figure 2: *Bifurcation diagram for small values of ρ while $\sigma = 10$ and $b = \frac{8}{3}$ Figure from Strogatz.[9]*

We can sketch a approximate bifurcation diagram Fig.(2) for varying values of ρ. Taking all this conclusion together it seems that trajectories must have a bizarre kind of long-term behavior because they are repelled from one unstable object after another yet at the same time they are confined to a bounded set of zero volume and they manage to move on this set forever without intersecting themselves or others ![9] This very complex movements will make more sens in the next sections about Strange attractor and Chaos analysis.

4 Phase space and fractals of Saltzman-Lorenz equations

Historically Lorenz make numerical simulations with the computer available at that time with the following values $\sigma = 10, b = \frac{8}{3}$ and $\rho = 28$. For the rest of this paper we will call the set of those values the *Default Values of Parameters* (DVP). The value of ρ is such that $\rho > \rho_c$ so that we just passed the Hopf bifurcation value $\rho_H \approx 24,74$ so he knew that something strange had to occur. With an initial condition $(0, 1, 0)$ near the saddle point at the origin the solution settles into an irregular oscillation (after an initial transient) that persists as $t \to \infty$ but never repeat exactly. The motion is said top be **aperiodic**.[9] Once the solution is viualized as a trajectory in phase space[5] a very complex structure emerges Fig.(3)
Obviously the trajectory never cross itself otherwise the uniqueness of solution would be violated so this impression from Fig.(3) comes from the projection of the 3-dimension space phase into a plan.
What can we say about this complex movement of the trajectory ? It starts near the origin then swings to the right and dives into the center of a spiral on the left. Without stopping the solution spiral around $\mathbf{C}^+$ and then after $\mathbf{C}^+$. The number of circuits made on either seems to be unpredictable from one cycle to the next.[9]
When the trajectory is viewed in three-dimension Fig.(4) it appears to settle onto a thin set that looks like a pair of butterfly wings.
Figure (4) suggests that it is a pair of surfaces that merge into the other and Lorenz[6] gives an explanation that these two surfaces only appear to merge. The illusion comes from the strong volume contraction and a poor numerical resolution. Moreover Lorenz develop a deep idea in his paper,

> *It would seem, then, that the two surfaces merely appears to merge, and remain distinct surface. Following these surfaces along a path parallel to a trajectory and circling C^+ and C^-, we see that each surface is really a pair of surfaces, so that,*

[5]The phase space analysis in this section is essentially based on Strogatz[9]

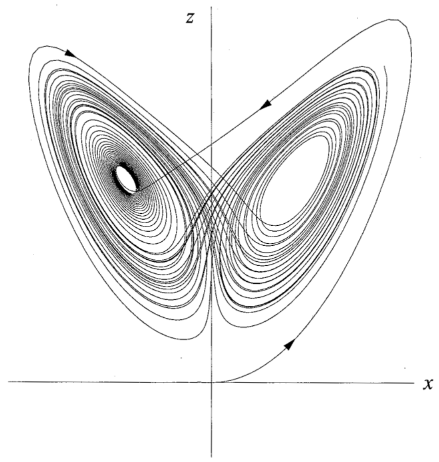

Figure 3: *Plot of a solution in the form* $X(t)$ *against* $Z(t)$ *with value* $\sigma = 10, b = \frac{8}{3}$ *and* $\rho = 28$. *The initial condition is* $(0, 1, 0)$. *Figure from Strogatz.[9]*

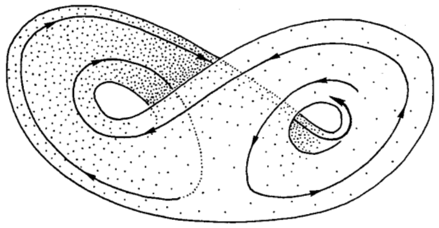

Figure 4: *Trajectory of the Saltzman-Lorenz system in the strange-attractor regime. Figure from Strogatz.[9]*

> *where they appear to merge, there are really four surfaces. Continuing this process for another circuit, we see that there are really eight surfaces, etc and so we finally conclude that there is an infinite complex of surfaces.*
> LORENZ - DETERMINISTIC NON PERIODIC FLOW[6]

Nowadays we call this *infinite complex of surfaces* a fractal as a set of points with zero volume but infinite surface area ![9] In the next section we will be able to compute the fractal dimension of what we will call the Lorenz Attractor.

5 Strange attractor and chaos

In this section we want to prove or at least give insights why the Saltzman-Lorenz model is chaotic. Despite the different nuances that exists in the definition of *chaos* I consider here the one propose by Strogatz[9]

> *Chaos is aperiodic long-term behavior in a deterministic system that exhibits sensitive dependence on initial conditions.*
> STROGATZ - NONLINEAR DYNAMICS AND CHAOS[9]

The deterministic feature of the dynamical system is rather straightforward by definition of the Saltzman-Lorenz system and has previously mentioned they are no random or noisy inputs in the system. For the two others properties in the definition more work has to be done.

5.1 Aperiodic long term behavior and Lorenz map

The aperiodic long term behavior means that there are trajectories which do not settle down to fixed points, periodic orbits or quasi-periodic orbits as $t \to \infty$. Since for $\rho > \rho_H$ they are no longer stable fixed points it is clear that trajectories can not settle down to them. For the possibility of quasi-periodic orbits this has been forbidden due to the property of dissipation but what about periodic orbits ? To prove this we will use the brilliant analyze of Lorenz with what will be called *Lorenz map*. Since : "*the trajectory apparently leaves one spiral after exceeding some critical distance from the center [...] it seems that some single feature of a given circuit (z_n the nth local maximum of $Z(t)$) should predict the same feature of the following circuit." Lorenz[6]*
Therefore, the idea of Lorenz is to analyze an iterated map instead of the complex set of equations in order to extract some order from this chaotic system.[9] We will compute also numerically those values using a Julia programm written by M.Crucifix in his Annexe[3]. The code is available in the Github attach to this project.
The idea of Lorenz was then to measured the local maxima of $Z(t)$ and finally plot Z_{n+1} against Z_n. The plot is shown in Figure(5)
The function $Z_{n+1} = f(Z_n)$ is what we call the **Lorenz map**[6] Now we have the tool needed to answer the following hard question : how do we know that there is no stable limit cycle or that trajectories will settle down into a periodic behavior ? By using his map, Lorenz was able to give a plausible counterargument that stable limit cycles do not occur the parameter values studied.[9] Indeed, if we look at Figure(5) we see that the function satisfies everywhere,

$$|f'(z)| > 1. \tag{5.1}$$

In fact, this property implies that if any limit cycles exist, it is necessarily unstable. Indeed, any deviation from one point grows with each iteration so that the original closed orbit is unstable. A generalisation proof which show thal *all* closed orbits are unstable can be found in Strogatz.[9]

[6] The graph in Figure(5) is not actually a curve, it does have some thickness so strictly speaking $f(z)$ is not a well defined function.

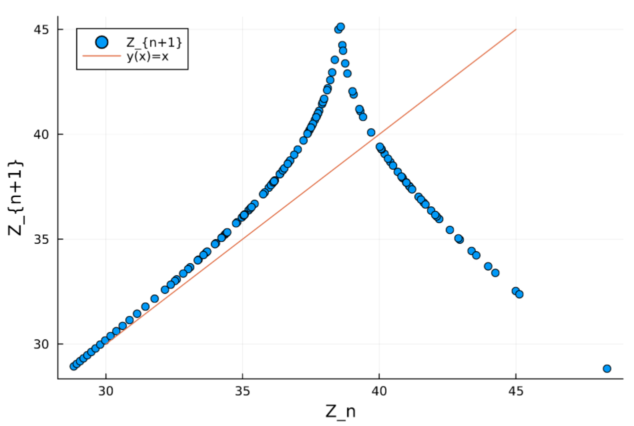

Figure 5: *Plot of Z_{n+1} as a function of Z_n. Figure from the simulation available on the Github.*

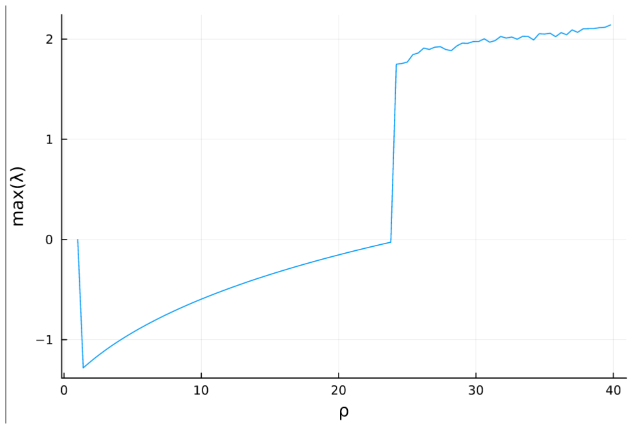

Figure 6: *Largest Lyapunov coefficient depending on ρ. Figure from the simulation available on the Github.*

5.2 Sensitive dependence on initial conditions and Lyapunov coefficients

Let's prove that two trajectories starting very close together will rapidly diverge from each other, and thereafter have totally different futures. We suppose that $\mathbf{x(t)}$ is a point on the attractor at time t and we consider a nearby point $\mathbf{x(t)} + \delta\mathbf{(t)}$ where δ is a tiny separation of initial length $||\delta_0|| = 10^{-15}$.In numerical studies we can find that[9] ,

$$||\delta(t)|| \approx ||\delta_0||e^{\lambda t}. \tag{5.2}$$

Which means that neighboring trajectories separate exponentially fast. By definition we call λ the largest positive Lyapunov exponent[7] We compute numerically the value of the largest Lyapunov exponent for different values of ρ with the Benettin[2] algorithm.
We can clearly see the apparition of chaos due to the divergence of nearby trajectories at the value $\rho_H \approx 24, 74$. For DVP we find that $\lambda = 1, 995$ which confirms our exponential

[7]We must not forget that for a n-dimensional system there are n Lyapunov exponent which correspond to the scaling coefficient that affects the n different axes of an elementary volume under the flow. Moreover, λ depends slightly on the trajectory so we have to average over many different points on the same trajectory to get the true value of λ[9].

divergence of nearby trajectories[8].
Lastly from the Lyapunov spectrum composed by the the set of the n Lyapunov exponents it is possible to obtain the Lyapunov dimension. This number is an estimation of the Hausdorff dimension d which is a measure of the fractal dimension of an object. For Saltzman-Lorenz model with DVP Grassberger and Procaccia[4] compute that $d \approx 2,06 \pm 0,01$.

With this last demonstration we show that Saltzman-Lorenz equations are chaotic in the DVP. The importance of exponential divergence of nearby trajectories should not be underestimated. Indeed, the consequence that long-term prediction becomes impossible because any small uncertainties becomes amplified enormously fast was the major theoretical breakthrough of Lorenz[6].

5.3 Saltzman-Lorenz system, a strange attractor

We recall the definition of an attractor[9] to be a closed set A with the following properties,

- *A is an invariant set :* any trajectory that starts in A stays in A for all time.
- *A attracts an open set of initial conditions :* A attracts all trajectories that start sufficiently close to it.
- *A is minimal*

It is clear with the previous developments that the Saltzman-Lorenz model has an attractor for DVP. It is worth mentioning that it is generally accepted that the bounded set with zero volume is minimal even if no one has managed to prove in computer experiments that Lorenz attractor is truly an attractor in this technical sense.[9]
Finally since Lorenz attractor has sensitive dependance on initial conditions we call it by definition a **strange attractor**. It is impressive to know that even if the existence of a strange attractor has been conjectured by Lorenz in 1963 it was only in 2002 that Tucker[11] proposed a rigorous demonstration.

5.4 Numerical simulations of Saltzman-Lorenz equations

Several numerical simulations of the Lorenz attractor already exist. An animation of the simulation with the DVP and the initial condition $\mathbf{x_0} = (1, 1, 1)$ can be found on the Github of the project. A snapshot of this simulation can be found in Figure(7)
The chaotic and complex behavior of the solutions is quite visible once we plot other initial conditions. Indeed, after a few iterations, the trajectories of two points initially very close can end up anywhere in the whole attractor. Moreover, the initial instability of the state of rest can be seen easily. What happens is that as the sinking cold fluid is replaced by even colder fluid from above the three variables grow rapidly because the cold water is replaced by even colder fluid from above and the rising warm fluid by warmer fluid from below. It can be noted that the strength of the convection far exceeds that of steady convection.[6]

6 Analysis of the Lorenz Attractor for large values of ρ

7 Conclusion

This project as a global overview of Saltzman-Lorenz system trough the numerous literature has allowed to show the deep complexity that a system, which may seems simple, exhibits. For a certain range of parameters it has been shown that the two nonlinearities in the Saltzman-Lorenz equations are enough to produce chaos. In this case, because the equations contains

[8]Due to the volume contraction under the flow we have that $\sum_i^3 \lambda_i < 0$ which doesn't contradict the fact that $\lambda \equiv max(\lambda_i) > 0$.

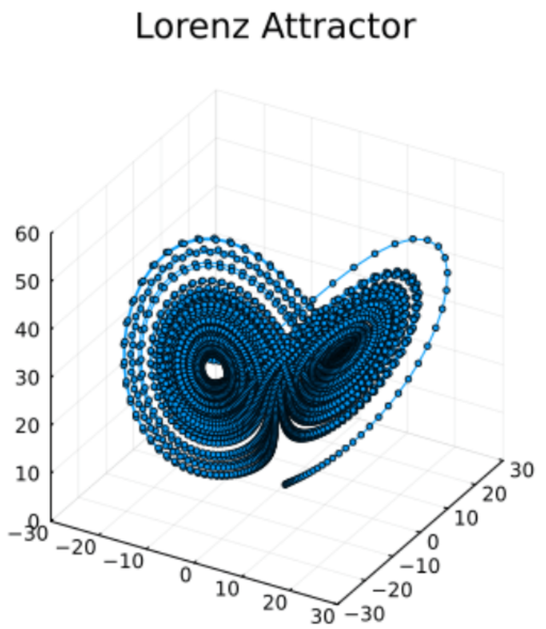

Figure 7: *Lorenz attractor for DVP and* $\mathbf{x_0} = (1, 1, 1)$*. Figure from the simulation available on the Github.*

terms representing advection which is a nonlinear phenomena we are in a system where a constant forcing can lead to a variable response.[6]
The Lorenz attractor was the first mathematical system who proves that the weather system is chaotic and hence destroyed the possibility of having one day meteorological predictions with a very high precision on durations longer than a few days. Indeed there will always be an error, however small it may be, between the measured value of a variable in the empiric world and the exact value it has. Therefore due to divergence of nearby trajectories the prediction of a model will display a trajectory with a non negligible distance from the trajectory of the exact initial state of the system.

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