

## Project 1: Rogue waves and the nonlinear Shroedinger equation

The **nonlinear Shroedinger equation** is a nonlinear partial differential equation for a complex wavefunction  $\psi(x, t)$  that emerges in different applications in non-linear optics, plasma physics and oceanography, among others. The equation in dimensionless form reads

$$i \frac{\partial \psi}{\partial t} = -\frac{1}{2} \frac{\partial^2 \psi}{\partial x^2} + K |\psi|^2 \psi \quad (1.1)$$

where  $K$  is a parameter. In the context of water waves formation, the equation describes the evolution of a carrier wave with amplitude  $A(x, t) = |\psi(x, t)|$  and phase  $\theta(x, t) = \arg(\psi(x, t))$ . The solutions of the equation are characterised by the formation of so called **rogue waves**: sudden

amplifications of the amplitude  $|\psi(x, t)|$  that develop from nothing, and that disappear without leaving traces.

The figure on the left shows the evolution of the wave amplitude  $|\psi(x, t)|$  on a periodic domain  $x \in [-20, 20]$  with  $K = -1$  starting from the initial condition

$$\psi(x, 0) = 0.5 + 0.01 \cos(2\pi x/40)$$

The initial condition is a smooth function with a nearly flat wave amplitude. After an initial evolution in which nothing seems to happen, the solution suddenly develops an isolated rogue wave, followed by a train of similar waves separated by flat intermediate states.

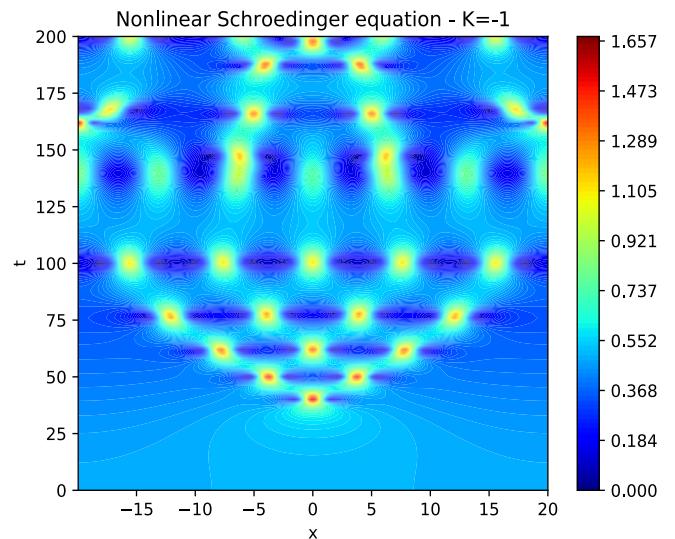


Figure 1

### NUMERICAL METHOD

Equation 1.1 on a periodic domain of size  $L$  can be simulated with a **split-operator** or **split-step** Fourier pseudo-spectral method. The idea is to observe that the time evolution

$$\frac{\partial \psi}{\partial t} = \frac{i}{2} \frac{\partial^2 \psi}{\partial x^2} - iK |\psi|^2 \psi$$

is given by the application of two evolution operators, one linear and one nonlinear

$$\frac{\partial \psi}{\partial t} = \mathcal{L}\psi + \mathcal{N}\psi, \quad \text{with} \quad \mathcal{L}\psi = \frac{i}{2} \frac{\partial^2 \psi}{\partial x^2} \quad \text{and} \quad \mathcal{N}\psi = -iK |\psi|^2 \psi$$

The full equation has formal solution  $\psi(x, t) = e^{t(\mathcal{L}+\mathcal{N})}\psi(x, 0)$ . If we consider advancements in time of small size  $\Delta t$  we have an approximation due to the Baker-Hausdorff formula as

$$\psi(x, t + \Delta t) = e^{\Delta t(\mathcal{L}+\mathcal{N})}\psi(x, t) \approx e^{\Delta t\mathcal{L}}e^{\Delta t\mathcal{N}}\psi(x, t)$$

where the error on the approximation is  $O(\Delta t^2)$ . One can therefore think to create an approximated time stepping scheme by applying first the nonlinear evolution operator, and then on the output of that apply the linear evolution operator.

In the specific case of the nonlinear Shroedinger equation, this strategy is particularly effective because both the linear and non-linear evolutions can be solved analytically if considered separately: the nonlinear part can be solved in physical space as

$$\frac{\partial \psi}{\partial t} = \mathcal{N}\psi = -iK|\psi|^2\psi \quad \rightarrow \quad \psi(x, t + \Delta t) = e^{-iK|\psi(x, t)|^2\Delta t}\psi(x, t)$$

and the linear part can be solved in spectral space as

$$\frac{\partial \psi}{\partial t} = \mathcal{L}\psi = \frac{i}{2}\frac{\partial^2 \psi}{\partial x^2} \quad \rightarrow \quad \hat{\psi}_k(t + \Delta t) = e^{-\frac{i}{2}\left(\frac{2\pi}{L}k\right)^2\Delta t}\hat{\psi}_k(t)$$

Indicating with  $\mathcal{F}$  and  $\mathcal{F}^{-1}$  the discrete Fourier transform and its inverse, the update operation from time  $t$  to time  $t + \Delta t$  is written as

$$\psi(x, t + \Delta t) = \mathcal{F}^{-1}[e^{-\frac{i}{2}\left(\frac{2\pi}{L}k\right)^2\Delta t}\mathcal{F}[e^{-iK|\psi(x, t)|^2\Delta t}\psi(x, t)]]$$

Algorithmically, the chain of operations for the scheme is the following:

1) given  $\psi(x, t)$ , advance the nonlinear part by  $\Delta t$  computing the partial update

$$g(x, t; \Delta t) = e^{-iK|\psi|^2\Delta t}\psi(x, t)$$

2) compute the Fourier transform of the partial update

$$\hat{g}_k(t; \Delta t) = \mathcal{F}[g(x, t; \Delta t)]$$

3) advance the linear part by  $\Delta t$  computing the fully updated solution in spectral space

$$\hat{\psi}_k(t + \Delta t) = e^{-\frac{i}{2}\left(\frac{2\pi}{L}k\right)^2\Delta t}\hat{g}_k(t; \Delta t)$$

4) compute the inverse Fourier transform obtaining the updated solution in physical space

$$\psi(x, t + \Delta t) = \mathcal{F}^{-1}[\hat{\psi}_k(t + \Delta t)]$$

Then go back to point 1 to repeat the cycle for the next timestep.

## PROJECT DESCRIPTION

- 1) write a code that solves the nonlinear Shroedinger equation using the split-step method described above, and run it to replicate Figure 1. Take a domain  $x \in [-20,20]$ ,  $K=-1$ , and initial condition

$$\psi(x,0) = 0.5 + 0.01 \cos(2\pi x/40)$$

In terms of numerical setup, you can discretize the domain in  $N=1024$  grid points and take a timestep  $\Delta t = 0.01$ . Plot also a snapshot at an instant of time and a time series at a location of your choice to show the amplification of the rogue waves. Then repeat the experiment for  $K=1$ , describing what happens. In general the cases  $K<0$  and  $K>0$  are called respectively the focussing and defocussing cases, can you guess from the results why?

- 2) an analytical solution of the nonlinear Shroedinger equation on an unbounded domain  $x \in (-\infty, +\infty)$  with  $K=-1$  is the so called Peregrine breather

$$\psi(x,t) = \left[ 1 - \frac{4(1+2it)}{1+4x^2+4t^2} \right] e^{it},$$

that is often taken as a “fundamental” model of rogue waves. Set up a numerical experiment to reproduce a solution of this type on a bounded periodic domain of size  $L$ . Note that in the formula the maximum amplitude is for  $t = 0$  and  $x = 0$ : choose the starting time and the spatial domain accordingly in order to show both the amplification and decay phase of the wave. The results will be similar but not identical to the formula, because the Peregrine breather solution is valid on an infinite domain. Show that increasing the size of the domain  $L$  you obtain better and better approximations of the Peregrine breather solution (start from say  $L=6$ ).

## Project 2: Solitons and the Korteweg-de Vries equation

The **Korteweg-de Vries equation** is a nonlinear partial differential equation that models the evolution of the height  $u(x, t)$  of a fluid in shallow water conditions. You find different versions of the equation in the literature. The “canonical” form is

$$\frac{\partial u}{\partial t} + 6u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 \quad (2.1)$$

Note that this equation is different from the one introduced in the finite differences projects. You can find more information the equation and its history in the paper referenced in the solitons projects of the finite differences part. Independently from the specific form, the equation is famous for admitting so called **soliton** solutions, that is positive travelling wave solutions decaying at infinity, that therefore behave as solitary travelling wave packets. The shape of the solitons remain unaffected during the evolution due to a delicate balance between dispersion and nonlinearity.

Equation 2.1 admits an analytical solution for a single soliton wave

$$u(x, t) = \frac{c}{2} \cosh^{-2} \left[ \frac{\sqrt{c}}{2} (x - ct - a) \right]$$

travelling at constant speed  $c$ , starting from  $x = a$  at  $t = 0$ . Figure 2 shows the evolution of two soliton waves (for which we do not have an analytical solution) travelling at different speed on a bounded domain. When they meet there is a phase shift due to nonlinear interactions, but afterwards each continues its trajectory maintaining its original shape and speed.

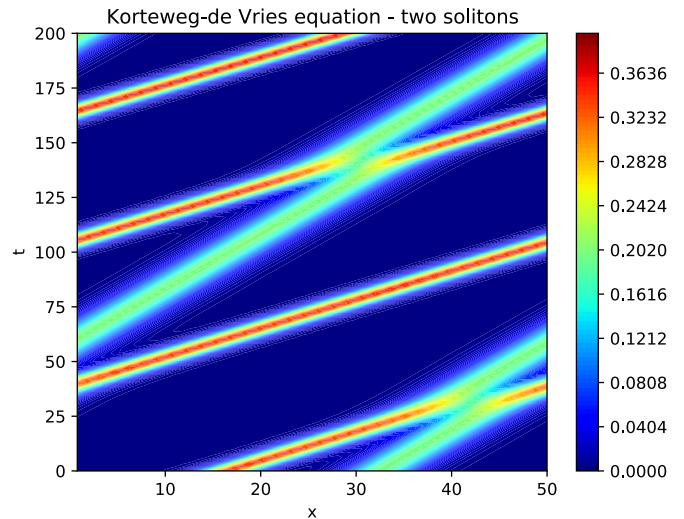


Figure 2

### NUMERICAL METHOD

Equation 2.1 on a periodic domain of size  $L$  can be simulated with a **split-operator** or **split-step** Fourier pseudo-spectral method. Like in the case of the nonlinear Shroedinger equation, the evolution equation is given by the application of two evolution operators, one linear and one nonlinear

$$\frac{\partial u}{\partial t} = \mathcal{L}u + \mathcal{N}u, \quad \text{with} \quad \mathcal{L}u = -\frac{\partial^3 u}{\partial x^3} \quad \text{and} \quad \mathcal{N}u = -6u \frac{\partial u}{\partial x} = -3 \frac{\partial u^2}{\partial x}$$

One can think to create an approximated time stepping scheme by applying first one evolution operator, and then on the output of that apply the other evolution operator.

Contrary to the case of the nonlinear Shroedinger equation, here only the linear dynamics has analytical solution: in spectral space we have

$$\frac{\partial u}{\partial t} = \mathcal{L}u = -\frac{\partial^3 u}{\partial x^3} \quad \rightarrow \quad \hat{u}_k(t + \Delta t) = e^{i\left(\frac{2\pi}{L}k\right)^3 \Delta t} \hat{u}_k(t)$$

However we can still use the split-step idea. In this case we first perform an update of the linear part in spectral space, and then advance the nonlinear part with a time-stepping scheme of choice. Indicating with  $\mathcal{F}$  and  $\mathcal{F}^{-1}$  the discrete Fourier transform and its inverse, the update operation from time  $t$  to time  $t + \Delta t$  is the following:

- 1) given  $u(x, t)$ , compute the Fourier transform

$$\hat{u}_k(t) = \mathcal{F}[u(x, t)]$$

- 2) advance the linear part by  $\Delta t$  computing the partial update in spectral space

$$\hat{g}_k(t; \Delta t) = e^{i\left(\frac{2\pi}{L}k\right)^3 \Delta t} \hat{u}_k(t)$$

- 3) apply the inverse Fourier transform obtaining the partial update in physical space, compute its square, and compute the spatial derivative of that with the spectral method

$$g(x, t; \Delta t) = \mathcal{F}^{-1}[\hat{g}_k(t; \Delta t)]$$

$$\frac{\partial g^2(x, t; \Delta t)}{\partial x} = \mathcal{F}^{-1}[i \frac{2\pi}{L} k \mathcal{F}[g^2(x, t; \Delta t)]]$$

- 4) apply a time stepping scheme of choice to advance the nonlinear part in physical space and obtain the fully updated solution. For example, with 1st order Euler forward

$$u(x, t + \Delta t) = g(x, t; \Delta t) - 3 \frac{\partial g^2(x, t; \Delta t)}{\partial x} \Delta t$$

Then go back to point 1 to repeat the cycle for the next timestep.

## PROJECT DESCRIPTION

- 1) write a code for the Korteweg-de Vries equation using the split-step method in periodic boundary conditions, and run it to reproduce Figure 2. Take initial conditions

$$u(x, 0) = \frac{c_1}{2} \cosh^{-2} \left[ \frac{\sqrt{c_1}}{2} (x - a_1 L) \right] + \frac{c_2}{2} \cosh^{-2} \left[ \frac{\sqrt{c_2}}{2} (x - a_2 L) \right]$$

on  $x \in [0, L]$ , with  $L = 50$ , ( $c_1 = 0.75$ ,  $a_1 = 0.33$ ) and ( $c_2 = 0.4$ ,  $a_2 = 0.65$ ). In terms of numerical setup, you can discretize the domain in  $N=256$  grid points and

take a timestep  $\Delta t = 0.0004$ . This initial condition corresponds to the linear combination of the analytical solutions for two separate soliton waves at  $t = 0$ . Plot a few snapshots (or maybe make a video if you have time) to discuss what happens when the fast soliton collides with the slow soliton

- 2) since the equation is nonlinear, this initial condition does not evolve like the linear combination of the analytical solutions for the two solitons for  $t > 0$ . To visualise this, plot a space-time plot for the the linear combination of the analytical solutions for the two solitons. Discuss the comparison between the two figures. Is there a range when you can approximate the numerical solution with the linear combination of the analytical solutions for the two waves? Can you make a quantitative comparison between numerical and analytical solution limited to this range? Then take an initial condition corresponding to only one of the two solitons, and make a full quantitative comparison between numerical and analytical solution (that now is well defined).

## Project 3: Pattern formation in the Swift-Hohenberg equation

The **Swift-Hohenberg equation** is a nonlinear partial differential equation introduced to study thermal fluctuations in a fluid near the Rayleigh-Benard convective instability. The equation in 1D for the temperature field  $u(x, t)$  reads

$$\frac{\partial u}{\partial t} = (r - 1)u - 2\frac{\partial^2 u}{\partial x^2} - \frac{\partial^4 u}{\partial x^4} - u^3 \quad (3.1)$$

The parameter  $r$  measures how far the heating is above the minimum value required for convection. For  $r < 0$  the heating is too small to cause convection, while for  $r > 0$  convection occurs, and temperature organises in a pattern of convective cells.

In 2D the Swift-Hohenberg equation features a wide phenomenology of **patterns**, including stripes, spots and spirals. In 1D any initial condition ends up in a stationary pattern of convective rolls like the one shown in Figure 3. The amplitude of the pattern increases with  $r$ , and is zero (no convection) for  $r < 0$ , so that the equation shows a **bifurcation** behaviour at the critical point  $r = 0$ .

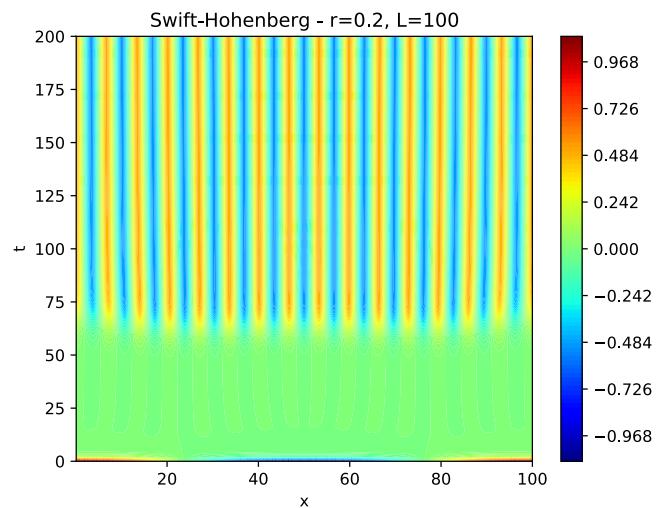


Figure 3

### NUMERICAL METHOD

For this equation we use a mixed **Crank-Nicholson/Adam Bashforth** scheme. Again the idea is to treat the linear and nonlinear operators separately

$$\frac{\partial u}{\partial t} = \mathcal{L}u + \mathcal{N}u, \quad \text{with} \quad \mathcal{L}u = (r - 1)u - 2\frac{\partial^2 u}{\partial x^2} - \frac{\partial^4 u}{\partial x^4} \quad \text{and} \quad \mathcal{N}u = -u^3$$

The linear part is updated using an implicit method, the Crank-Nicholson scheme, and the nonlinear part is updated using an explicit method, the Adam Bashforth scheme (that you have seen in the finite differences part of the course). The Crank-Nicholson scheme is

$$\frac{u(x, t + \Delta t) - u(x, t)}{\Delta t} = \frac{1}{2} (\mathcal{L}u(x, t + \Delta t) + \mathcal{L}u(x, t))$$

and the Adam Bashforth scheme is

$$\frac{u(x, t + \Delta t) - u(x, t)}{\Delta t} = \frac{3}{2} \mathcal{N}u(x, t) - \frac{1}{2} \mathcal{N}u(x, t - \Delta t)$$

The total update operation is therefore

$$\frac{u(x, t + \Delta t) - u(x, t)}{\Delta t} = \frac{1}{2} (\mathcal{L}u(x, t + \Delta t) + \mathcal{L}u(x, t)) + \left( \frac{3}{2} \mathcal{N}u(x, t) - \frac{1}{2} \mathcal{N}u(x, t - \Delta t) \right)$$

Since the Crank-Nicholson scheme is implicit,  $u(x, t + \Delta t)$  appears on both sides of the formula. Rearranging to solve for  $u(x, t + \Delta t)$  we obtain

$$\left( 1 - \frac{\Delta t}{2} \mathcal{L} \right) u(x, t + \Delta t) = \left( 1 + \frac{\Delta t}{2} \mathcal{L} \right) u(x, t) + \left( \frac{3}{2} \mathcal{N}u(x, t) - \frac{1}{2} \mathcal{N}u(x, t - \Delta t) \right) \Delta t$$

We exploit the fact that in Fourier space the application of the linear operator becomes a simple multiplication  $\mathcal{F}[\mathcal{L}u(x, t)] = f_{\mathcal{L}}(k)\hat{u}_k(t)$ , where  $f_{\mathcal{L}}(k)$  is a function of  $k$  that depends on the specific form of  $\mathcal{L}$  (we have seen extensively the cases when  $\mathcal{L}$  is the first or second derivative in  $x$ ). Taking the Fourier transform of both sides we have

$$\hat{u}_k(t + \Delta t) = \frac{1 + \frac{\Delta t}{2} f_{\mathcal{L}}(k)}{1 - \frac{\Delta t}{2} f_{\mathcal{L}}(k)} \hat{u}_k(t) + \frac{\frac{3}{2} \mathcal{F}[\mathcal{N}u(x, t)] - \frac{1}{2} \mathcal{F}[\mathcal{N}u(x, t - \Delta t)]}{1 - \frac{\Delta t}{2} f_{\mathcal{L}}(k)} \Delta t$$

Applying the specific forms of  $\mathcal{L}$  and  $\mathcal{N}$  for the Swift-Hohenberg equation we finally have

$$\hat{u}_k(t + \Delta t) = \frac{1 + \frac{\Delta t}{2} f_{\mathcal{L}}(k)}{1 - \frac{\Delta t}{2} f_{\mathcal{L}}(k)} \hat{u}_k(t) - \frac{\frac{3}{2} \mathcal{F}[u^3(x, t)] - \frac{1}{2} \mathcal{F}[u^3(x, t - \Delta t)]}{1 - \frac{\Delta t}{2} f_{\mathcal{L}}(k)} \Delta t \quad (3.2)$$

$$f_{\mathcal{L}}(k) = r - 1 + 2 \left( \frac{2\pi}{L} k \right)^2 - \left( \frac{2\pi}{L} k \right)^4 \quad (3.3)$$

The algorithm is then

- 1) given  $u(x, t - \Delta t)$  and  $u(x, t)$ , use the Fourier transform to compute  $\hat{u}_k(t)$ ,  $\mathcal{F}[u^3(x, t)]$  and  $\mathcal{F}[u^3(x, t - \Delta t)]$
- 2) apply equations 3.2 and 3.3 to update the solution in spectral space to  $\hat{u}_k(t + \Delta t)$
- 3) apply the inverse Fourier transform to obtain the updated solution in physical space to  $u(x, t + \Delta t)$

Note that the scheme requires the knowledge of the solution at the two previous time steps to update the next one. This creates a problem when computing the first timestep after the initial condition at time  $t = 0$ . A way to circumvent the problem is to approximate for the evolution of the first step  $u^3(x, -\Delta t) = u^3(x, 0)$ , and from the second timestep then proceed with the normal loop.

## PROJECT DESCRIPTION

- 1) try to replicate Figure 3. Take a periodic domain  $x \in [0,100]$ , and initial condition

$$u(x,0) = \cos(2\pi x/100) + 0.1 \cos(4\pi x/100)$$

Discretize the domain in  $N = 1024$  grid points and take a timestep  $\Delta t = 0.05$ . Once you obtain Figure 3, play with the values of  $r$  and  $L$  (and try some other initial condition). What determines the wavelength of the resulting pattern once reached a stationary solution?

- 2) it is possible to characterise the bifurcation taking  $r$  as control parameter and the norm of the stationary pattern as order parameter. For different values of  $r$  compute

$$A^2 = \lim_{t \rightarrow +\infty} \frac{1}{L} \int_0^L u^2(x,t) dx$$

where  $t \rightarrow +\infty$  means taking the solution for a value  $t$  sufficiently large that the amplitude of the pattern is stable (for small values of  $r$  you may need longer simulations). Plot  $A$  as a function of  $r$ , you should obtain a transition phase-like diagram, with critical point at  $r = 0$ , that is  $A = 0$  for  $r < 0$  and  $A > 0$  for  $r > 0$ . Can you estimate empirically a scaling law  $A \sim r^p$  for  $r \rightarrow 0^+$ ?

## Project 4: Patterns and chaos in the Kuramoto-Sivashinsky equation

The **Kuramoto-Sivashinsky equation** is a nonlinear partial differential equation introduced to study fluid velocity fluctuations at the front contour of a flame, and that has been studied extensively beyond that, thanks to a surprisingly rich variety of behaviours. The equation in 1D reads

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + \nu \frac{\partial^4 u}{\partial x^4} = 0 \quad (1)$$

where  $\nu$  is a parameter taking the role of a viscosity and  $x \in [0, L]$  with periodic boundary conditions. For a given value of  $\nu$ , the solutions of the equation show a rich variety of dynamics changing the size of the domain  $L$ . Like in the case of the Swift-Hohenberg equation, the characteristics of the solution at stationary state do not depend on the initial condition.

For  $L$  smaller than a critical value  $L_\nu$  (which depends on the value of  $\nu$ ) the system evolves to the trivial solution  $u(x, t) = 0$ , while for large values of  $L$  the system evolves to a **chaotic** regime like the one shown in Figure 4. For intermediate values of  $L$  between the critical length  $L_\nu$  and the onset of chaos, the system ends up in a variety of dynamical stationary states which includes regular patterns like we have seen in the Swift-Hohenberg equation and travelling waves like in the KdV equation, among others.

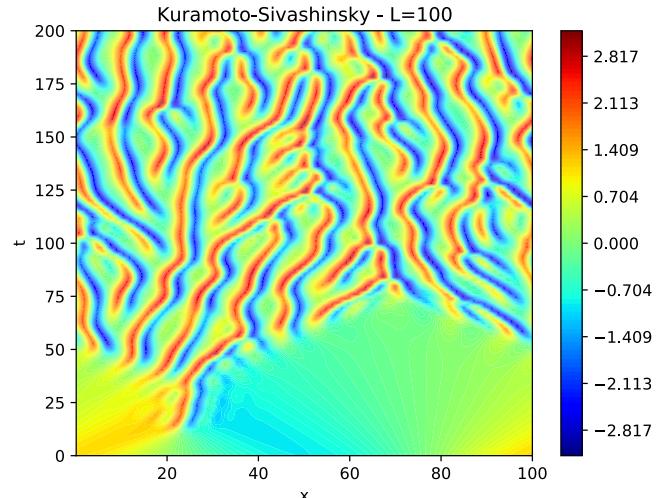


Figure 4

### NUMERICAL METHOD

We use again the mixed **Crank-Nicholson/Adam Bashforth** scheme, here repeated in full for ease of reference. The idea is to treat the linear and nonlinear operators separately

$$\frac{\partial u}{\partial t} = \mathcal{L}u + \mathcal{N}u, \text{ with } \mathcal{L}u = -\frac{\partial^2 u}{\partial x^2} - \nu \frac{\partial^4 u}{\partial x^4} \text{ and } \mathcal{N}u = -u \frac{\partial u}{\partial x} = -\frac{1}{2} \frac{\partial u^2}{\partial x}$$

The linear part is updated using an implicit method, the Crank-Nicholson scheme, and the nonlinear part is updated using an explicit method, the Adam Bashforth scheme (that you have seen in the finite differences part of the course). The Crank-Nicholson scheme is

$$\frac{u(x, t + \Delta t) - u(x, t)}{\Delta t} = \frac{1}{2} (\mathcal{L}u(x, t + \Delta t) + \mathcal{L}u(x, t))$$

and the Adam Bashforth scheme is

$$\frac{u(x, t + \Delta t) - u(x, t)}{\Delta t} = \frac{3}{2} \mathcal{N}u(x, t) - \frac{1}{2} \mathcal{N}u(x, t - \Delta t)$$

The total update operation is therefore

$$\frac{u(x, t + \Delta t) - u(x, t)}{\Delta t} = \frac{1}{2} (\mathcal{L}u(x, t + \Delta t) + \mathcal{L}u(x, t)) + \left( \frac{3}{2} \mathcal{N}u(x, t) - \frac{1}{2} \mathcal{N}u(x, t - \Delta t) \right)$$

Rearranging to solve for  $u(x, t + \Delta t)$  we obtain

$$\left( 1 - \frac{\Delta t}{2} \mathcal{L} \right) u(x, t + \Delta t) = \left( 1 + \frac{\Delta t}{2} \mathcal{L} \right) u(x, t) + \left( \frac{3}{2} \mathcal{N}u(x, t) - \frac{1}{2} \mathcal{N}u(x, t - \Delta t) \right) \Delta t$$

We exploit the fact that in Fourier space the application of the linear operator becomes a simple multiplication  $\mathcal{F}[\mathcal{L}u(x, t)] = f_{\mathcal{L}}(k)\hat{u}_k(t)$ , where  $f_{\mathcal{L}}(k)$  is a function of  $k$  that depends on the specific form of  $\mathcal{L}$  (we have seen extensively the cases when  $\mathcal{L}$  is the first or second derivative in  $x$ ). Taking the Fourier transform of both sides we have

$$\hat{u}_k(t + \Delta t) = \frac{1 + \frac{\Delta t}{2} f_{\mathcal{L}}(k)}{1 - \frac{\Delta t}{2} f_{\mathcal{L}}(k)} \hat{u}_k(t) + \frac{\frac{3}{2} \mathcal{F}[\mathcal{N}u(x, t)] - \frac{1}{2} \mathcal{F}[\mathcal{N}u(x, t - \Delta t)]}{1 - \frac{\Delta t}{2} f_{\mathcal{L}}(k)} \Delta t$$

Applying the specific forms of  $\mathcal{L}$  and  $\mathcal{N}$  for the Kuramoto-Sivashinsky equation we have

$$\begin{aligned} \hat{u}_k(t + \Delta t) &= \frac{1 + \frac{\Delta t}{2} f_{\mathcal{L}}(k)}{1 - \frac{\Delta t}{2} f_{\mathcal{L}}(k)} \hat{u}_k(t) + \frac{\frac{3}{2} \mathcal{F}\left[-\frac{1}{2} \frac{\partial}{\partial x} u^2(x, t)\right] - \frac{1}{2} \mathcal{F}\left[-\frac{1}{2} \frac{\partial}{\partial x} u^2(x, t - \Delta t)\right]}{1 - \frac{\Delta t}{2} f_{\mathcal{L}}(k)} \Delta t \\ f_{\mathcal{L}}(k) &= \left(\frac{2\pi}{L} k\right)^2 - \nu \left(\frac{2\pi}{L} k\right)^4 \end{aligned}$$

The spatial derivatives in the nonlinear terms are computed with the spectral method, obtaining eventually

$$\hat{u}_k(t + \Delta t) = \frac{1 + \frac{\Delta t}{2} f_{\mathcal{L}}(k)}{1 - \frac{\Delta t}{2} f_{\mathcal{L}}(k)} \hat{u}_k(t) - \frac{i}{2} \frac{2\pi}{L} k \frac{\frac{3}{2} \mathcal{F}[u^2(x, t)] - \frac{1}{2} \mathcal{F}[u^2(x, t - \Delta t)]}{1 - \frac{\Delta t}{2} f_{\mathcal{L}}(k)} \Delta t \quad (4.2)$$

$$f_{\mathcal{L}}(k) = \left(\frac{2\pi}{L} k\right)^2 - \nu \left(\frac{2\pi}{L} k\right)^4 \quad (4.3)$$

The algorithm is then

- 1) given  $u(x, t - \Delta t)$  and  $u(x, t)$ , use the Fourier transform to compute  $\hat{u}_k(t)$ ,  $\mathcal{F}[u^2(x, t)]$  and  $\mathcal{F}[u^2(x, t - \Delta t)]$

- 2) apply equations 4.2 and 4.3 to update the solution in spectral space to  $\hat{u}_k(t + \Delta t)$
- 3) apply the inverse Fourier transform to obtain the updated solution in physical space  $u(x, t + \Delta t)$

Note that the scheme requires the knowledge of the solution at the two previous time steps to update the next one. This creates a problem when computing the first timestep after the initial condition at time  $t = 0$ . A way to circumvent the problem is to approximate for the evolution of the first step  $u^2(x, -\Delta t) = u^2(x, 0)$ , and from the second timestep then proceed with the normal loop.

## PROJECT DESCRIPTION

- 1) replicate Figure 4. Take  $\nu = 1$ , a domain  $x \in [0, L]$  with  $L = 100$  and initial condition

$$u(x, 0) = \cos(2\pi x/L) + 0.1 \cos(4\pi x/L)$$

Discretize the domain in  $N = 1024$  grid points and take a timestep  $\Delta t = 0.05$ . Once you obtain Figure 4, make more experiments reducing the domain size  $L$ . You will see many different behaviours. Present some plots to showcase the variety of solutions you find.

- 2) perform an analysis to find the critical length  $L_\nu$  for  $\nu = 1$ . In order to do so, analyse for different values of  $L$  the norm of the solution at stationary state computing

$$A^2 = \lim_{t \rightarrow +\infty} \frac{1}{L} \int_0^L u^2(x, t) dx$$

where  $t \rightarrow +\infty$  means taking the solution for a value  $t$  sufficiently large that the amplitude of the pattern is stable. Plot  $A$  as a function of  $L$ , you will obtain a transition phase-like diagram, that is there will be a critical point  $L = L_\nu$  for which  $A = 0$  for  $L < L_\nu$  and  $A > 0$  for  $L > L_\nu$ . What is the value of  $L_\nu$ ? If you have time, repeat the analysis for different values of  $\nu$ , and for each of them identify  $L_\nu$  (depending on the values of  $\nu$  you use you may need to take longer simulations to reach a stationary state). Can you estimate a scaling law  $L_\nu \sim \nu^p$ ?