# LPHYS2163 Project #2
# Discovering barotropic Rossby waves in the free atmosphere: a numerical approach

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## 1 Scope of the project and practical aspects

- The project is done in groups of two or three students. The members of the group can work on separate questions, but everyone should be able to understand the whole project and answer questions on any part of the project.
- Each group should deliver, by Wednesday 21 December 2022 16:00:
  - The code developed to solve the problem, written in the group's preferred programming language. The code should be directly working and testable to produce realistic results. The code should follow standard practices (comments, indentation) to make it easy to be understood by someone external to the code.
  - A 5-page document (excluding title and references but including figures) summarizing the key findings of the project;
  - Any other supplementary material (videos, animations, ...)

  to `francois.massonnet@uclouvain.be`. Individual meetings can be organized to track your progress and solve possible issues.

## 2 Project guidelines

### 2.1 Deriving the equations to solve

The governing equation that you will solve in this project is the equation of conservation of absolute vorticity in the $\beta$-plane for a barotropic fluid of constant depth that is assumed to be divergence-free. As seen in the course, this equation reads

$$\frac{\mathrm{D}_h(\zeta + f)}{\mathrm{D}t} = 0 \tag{1}$$

where $\zeta$ is the vertical component of the relative vorticity of the fluid $\zeta := \mathbf{k} \cdot (\nabla \times \mathbf{u})$ and $f$ is the planetary (or ambient) vorticity.

**Question 1.** Show that Eq. 1, when expanded in Cartesian coordinates, can be re-written in "flux form" as

$$\frac{\partial \zeta}{\partial t} = -F := -\left(\frac{\partial(\zeta u)}{\partial x} + \frac{\partial(\zeta v)}{\partial y} + \beta v\right)$$

While absolute vorticity is conserved with the flow (Eq. 1), the above equation highlights that the local tendency of relative vorticity is governed by the flux of relative vorticity, but that a small correction has to be applied to take into account the latitudinal dependence of ambient vorticity. The $\beta$ term is the one encountered in the course when discussing approximations to the variations of the Coriolis parameter $f$ with latitude.

As seen in the lectures, a two-dimensional divergence-free flow can be expressed in terms of a streamfunction:

$$\begin{aligned} u &= -\frac{\partial \psi}{\partial y} \\ v &= \frac{\partial \psi}{\partial x} \end{aligned}$$

with $\psi(t, x, y)$ a scalar function. This approach is convenient, because by construction, lines of constant value of $\psi$ follow the flow. The streamfunction approach is therefore useful to study graphically the motion within a fluid.

Since the vertical component of relative vorticity is defined by $\zeta := \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$, it comes that

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \nabla_h^2 \psi = \zeta \tag{2}$$

This is called a Poisson equation.

In summary, we have four unknowns ($u$, $v$, $\zeta$, $\psi$) and four equations to solve: one prognostic equation for the relative vorticity, two diagnostic equations for the components of velocity given by the streamfunction, and one diagnostic equation relating the streamfunction and the relative vorticity. The prognostic equation is readily nonlinear, calling for a numerical approach to solve it.

## 2.2 Experimental setup

The default spatial domain for solving the equations has the following characteristics:

- The domain is rectangular with width $L_x = 12\ 000$ km (west-east direction) by $L_y = 6\ 000$ km (south-north).
- The domain is centered at a reference latitude of 45°N
- In both $x$- and $y$-directions, the domain is discretized with a spatial resolution of $\Delta s = 200$ km. This implies that the domain contains $M \times N = 60 \times 30$ sample points.

As initial condition, we suppose the following streamfunction:

$$\psi^0(x,y) = \frac{g}{f_0}\left(100.\sin(kx).\cos(jy)\right) \tag{3}$$

Here, $g$ is the acceleration of gravity and $f_0$ is the Coriolis parameter taken at the reference latitude mentioned above. The parameters $k = 2\pi/W_x$ with $W_x = 6000$ km and $j = 2\pi/W_y$ with $W_y = 3000$ km are the wavenumbers corresponding to the wavelengths $W_x$ and $W_y$ of the initial field in the $x$ and $y$ directions, respectively. These wavelengths are chosen to satisfy periodicity at the boundaries and represent large-scale disturbances in the flow that are typically encountered at mid-latitudes.

Note also that the streamfunction is such that the corresponding geopotential height variations reach up to 100 m around a reference, which is what is typically seen in large-scale flows.

Given the initial streamfunction $\psi^0$, the corresponding initial relative vorticity field $\zeta^0$ can be calculated by making use of Eq. 2. The Laplacian operator $\nabla^2$ can be discretized by using a centered-difference scheme, such that at any point of the discretized domain $(x,y)$, we have:

$$\begin{aligned}\nabla_h^2\psi(x,y) &= \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} \\ &\approx \frac{1}{\Delta s^2}\left[-4\psi(x,y) + \psi(x+\Delta s,y) + \psi(x-\Delta s,y) + \ldots\right. \\ &\quad \left.\psi(x,y+\Delta s) + \psi(x,y-\Delta s)\right]\end{aligned} \tag{4}$$

For points at the boundary of the domain, we need to make assumptions by either using prescribed values for the streamfunction, or by assuming periodicity in the $x$- and $y$- directions. We will follow the latter approach: the point "to the left" of a point on the western boundary is the point with the same $y$ coordinate but on the eastern boundary; the point "to the top" of a point on the northern boundary is the point with the same $x$ coordinate but on the southern boundary; and so on.

Similarly, from the initial streamfunction, we can compute the initial velocity field at $(x,y)$ by discretizing the derivatives as:

$$u(x,y) = -\frac{\partial\psi}{\partial y} \approx -\frac{\psi(x,y+\Delta s) - \psi(x,y-\Delta s)}{2\Delta s}$$

$$v(x,y) = \frac{\partial\psi}{\partial x} \approx \frac{\psi(x+\Delta s,y) - \psi(x-\Delta s),y}{2\Delta s}$$

Again, periodicity in both $x$ and $y$ directions is assumed.

**Question 2.** Plot the two-dimensional field $\psi^0$, the initial relative vorticity field $\zeta^0$, and the initial velocity field (with arrows since this is a vector field)

We are now ready to integrate numerically the barotropic vorticity equation in Cartesian coordinates. We need to choose ourselves a time step. For linear equations, a convenient criterion to avoid the development of numerical instabilities is to choose the time step $\Delta t$ such that in one time step, the flow makes a distance less than the nominal spatial resolution. For nonlinear equations, this condition is not sufficient. In our case, a time step of $\Delta t = 1$ hour should prevent the development of such instabilities. The total duration of the simulation can be set to 4 days but can be shortened for initial tests.

## 2.3 Numerical integration

Now that we have initial conditions, we can proceed with solving the set of four equations from one time step to the next. Let us suppose that we are at time step $t$, called the "current" time step. We wish to find the values of the variables $\zeta$, $\psi$, $u$ and $v$ at the "next" time step $t + \Delta t$. For this, we will need information about the variables at the current time step and at the "previous" time step $t - \Delta t$. We know the current fields either because we have calculated them at the previous time step, or because we are at the first time step and we know the initial conditions.

At each time step, we proceed with four operations:

1. The current velocity components $u$ and $v$ at each point are diagnosed from the current streamfunction (that was calculated in 4. at the previous time step);
2. The current flux term is calculated using the current velocity components (calculated in 1.) and the current relative vorticity (calculated in 3. at the previous time step):

   $$F = \frac{\partial(\zeta u)}{\partial x} + \frac{\partial(\zeta v)}{\partial y} + \beta v$$

   Note that as for above, periodicity in the boundary conditions is assumed.
3. The relative vorticity at the next time step is calculated by discretizing the conservation equation with a centered difference scheme, using the relative vorticity at the previous time step and the flux term at the current time step (calculated in 2.):

   $$\begin{aligned} \frac{\partial \zeta}{\partial t} = -F \quad &\Leftrightarrow \quad \frac{\zeta(t+\Delta t) - \zeta(t-\Delta t)}{2\Delta t} \approx -F \\ &\Rightarrow \quad \zeta(t+\Delta t) = -F.2\Delta t + \zeta(t-\Delta t) \end{aligned}$$

   Since for the first time step ($t = 0$), the "previous time step" does not exist, a simple forward scheme can be used instead, for that particular case:

   $$\zeta(t+\Delta t) = -F.\Delta t + \zeta(t)$$

4. Given the newly available relative vorticity at the next time step (just calculated in 3.), the streamfunction at the next time step is calculated by solving the Poisson equation

$$\nabla^2\psi = \zeta$$

Note however that, compared to the previous section where the initial relative vorticity was diagnosed from the initial streamfunction, the roles are now reversed: we have to find the field $\psi$ that satisfies the Poisson equation, given the field $\zeta$. For each of the $M \times N$ points of the domain, we can establish one linear equation involving five unknowns: the streamfunction at the point itself plus the streamfunction at the four neighbouring points (left, right, up and down) – see Eq. 4. Again, for points on the boundaries of the domain, periodicity is assumed. The Poisson equation can thus be discretized as a set of $M \times N$ linear equations involving $M \times N$ unknowns (the value of the streamfunction at each point) and equated to the value of the relative vorticity at the next time step. The streamfunction can be solved by matrix inversion (or using any sort of pre-built algorithm in your coding software).

> **Question 3.** Integrate the barotropic vorticity equation following the sequence above. Check the fields at each time step to ensure that the results are physically consistent. In the case that numerical instabilities develop, you may want to decrease the time step or to decrease the spatial resolution.

When the default simulation is producing realistic results, it is time to turn to more physical considerations.

## 2.4 Interpretation of the results and sensitivity experiments

For the sake of visualization, it is recommended to produce, for each time step, one figure with the 2-D fields of relative vorticity, streamfunction, and velocity. By doing so, the dynamic evolution of the fields can be investigated qualitatively and quantitatively, if needed.

> **Question 4.** Comment on the general behavior of the numerical solution. What do you observe? What are the characteristics (speed, direction, rotational character) of the flow?

Now, make a couple of sensitivity experiments by changing for example:

– The wavelength of the initial streamfunction: go from $W_x = W_y = 6\,000$ km (default values) to $W_x = 12\,000$ km (keeping $W_y$ unchanged). This produces disturbances with larger initial amplitude

- The reference latitude from 45° N to 80° N or 20° N, or even negative values (for the Southern Hemisphere)
- A value of $0$ for $\beta$, i.e., making the $f$-plane approximation instead of the $\beta$-plane approximation
- Add a mean background flow. We note that such a flow ($u = \bar{U} =$ constant) is a perfectly valid solution to the equations of motion, since this flow has no relative vorticity and is divergence free. This flow is often called the "mean flow" or "background flow" and can be added to our solution, as indeed the depth-averaged value of the zonal flow is non-zero (see, e.g. Marshall and Plumb 2008, Fig. 5.20). You can assume a mean flow of $\overline{U} = 20$ m.s$^{-1}$ but explore how the numerical solution depends on the chosen value.

**Question 5.** Comment on the sensitivity tests performed with your model.

## 2.5 Advices, tips and tricks

- To avoid any mistakes, convert all physical variables in the International System of Units.
- In your code, make maximal use of symbolic variables and avoid "hard-coded" variables.
- In your code, use functions as much as possible (e.g. to compute spatial derivatives, Laplacian operators).
- All parameters given in the guidelines (domain size, time step, etc.) are suggestions. You are encouraged to develop your model following those parameters to make sure your code works well. Of course, you are free (and encouraged) to explore alternative sets of parameters and hypotheses!
- This document and the slides of the lectures are self-sufficient to reach the goals of the work, but reading of other material (see below) is encouraged to root your understanding in deeper grounds.

## 2.6 Further reading

The goal of this project is to introduce you to the concept of (barotropic free-atmosphere) Rossby waves through the lens of numerical modeling. You are encouraged to make your own research on this topic to interpret your results. The following articles and references might be useful:

- The seminal paper of Charney and co-authors: Charney, J. G., Fjörtoft, R., Neumann, J. V. (1950). Numerical Integration of the Barotropic Vorticity Equation. Tellus, 2(4), 237–254. https://doi.org/10.1111/j.2153-3490.1950.tb00336.x

- Chapter 7 (in particular Section 7.7.1 "Free Barotropic Rossby Waves") and Section 13.4 "The barotropic vorticity equation in finite differences" of Holton, J. R., Hakim, G. J. (2013). An introduction to dynamic meteorology (Fifth edition). Academic Press.
- Chapter 9 (in particular Section 9.4 "Planetary Waves (Rossby Waves)" of Cushman-Roisin, B., Beckers, J.-M. (2011). Introduction to geophysical fluid dynamics: Physical and numerical aspects (2nd ed). Academic Press.
- Chapter 5 (in particular Section 5.7.1 "Waves in a single layer" of Vallis, G. K. (2006). Atmospheric and oceanic fluid dynamics: Fundamentals and large-scale circulation. Cambridge University Press.