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INSOLATION VALUES FOR THE CLIMATE OF THE LAST 10 MILLION YEARS

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New values for the astronomical parameters of the Earth's orbit and rotation (eccentricity, obliquity and precession) are proposed for paleoclimatic research related to the Late Miocene, the Pliocene and the Quaternary They have been obtained from a numerical solution of the Lagrangian system of the planetary point masses and from an analytical solution of the Poisson equations of the Earth–Moon system The analytical expansion developed in this paper allows the direct determination of the main frequencies with their phase and amplitude Numerical and analytical comparisons with the former astronomical solution BER78 are performed so that the accuracy and the interval of time over which the new solution is valid can be estimated The corresponding insolation values have also been computed and compared to the former ones This analysis leads to the conclusion that the new values are expected to be reliable over the last 5 Ma in the time domain and at least over the last 10 Ma in the frequency domain

INTRODUCTION

Research in the astronomical theory of paleoclimates involves four main steps (Berger, 1988):

1 The theoretical computation of the long-term variations of the Earth's orbital parameters and related geometrical insolations
2 The design of climatic models to transfer the insolation into climate
3 The collection of geological data and their interpretation in terms of climate
4 The comparison of these proxy data to the simulated climatic variables

This paper will focus only on the first point

The energy available at any given latitude ϕ on the Earth, on the assumption of a perfectly transparent atmosphere and of a constant solar output, is a single-valued function of the semi-major axis, a, of the Earth's orbit (the ecliptic), its eccentricity, e, its obliquity (the tilt of the equator on the ecliptic), ε and of the longitude of the perihelion measured from the moving vernal equinox, $\tilde{\omega}$ (Berger, 1978a,b). The eccentricity is a measure of the shape of the Earth's orbit around the Sun It changes the mean distance from the Earth to the Sun and therefore the total amount of energy received by the Earth The geographical and seasonal pattern of this insolation depends on ε and on the climatic precessional parameter, $e \sin \tilde{\omega}$, that describes how the precession of the equinoxes affects the seasonal configuration of the Earth–Sun distance

The first calculations of these parameters date back to the 19th century (Le Verrier, 1855, see Berger, 1988 for a review) Milankovitch (1941) was however the first to complete a full astronomical theory of the Pleistocene ice ages, computing the orbital elements and the subsequent changes in the insolation and climate (Imbrie and Imbrie, 1979, Berger and Andjelic, 1988) In the late 1960's judicious use of radioactive dating and other techniques gradually clarified the details of the Quaternary time scale, better instrumental methods came on the scene using oxygen isotope as ice age relics, ecological methods of core interpretation were perfected, global climates of the past were reconstructed and climate models became available (see for example Berger, 1990, for a review of the significant steps made to improve the astronomical theory of paleoclimates over the last 20 years) With these improvements in dating and in interpreting the geological data in terms of paleoclimates, it became necessary to investigate more critically the computation of the astronomical elements (Berger, 1984) and of appropriate insolation parameters (Berger and Pestiaux, 1984) This has allowed us to test first, the astronomical theory in the frequency (Hays *et al* , 1976, Berger, 1977b, Imbrie *et al* , 1989, Berger, 1989b) and in the time domain (Berger *et al* , 1990) and, further, to calibrate the Quaternary (Imbrie *et al* , 1984, Martinson *et al* , 1987) and Pliocene (Shackleton *et al* , 1990) time scales

A first improvement to the Milankovitch solution came in the 1950's from Brouwer and van Woerkom (1950) and later from Sharaf and Boudnikova (1967) and Anolik *et al* (1969) But a serious step forward was made with the analytical solution by Bretagnon (1974) for the planetary point masses and the calculation of e, ε and $e \sin \tilde{\omega}$ by Berger (1976, 1977a) which lead to his 1978 solution (Berger, 1978a,b), referred to here as BER78 This solution was assumed to provide valuable information over the last 1 5 Ma in the time domain and over a much longer period in the frequency domain (Berger, 1984) The next significant improvement was related to the numerical integration made by Laskar a few years ago (Laskar, 1986, 1988) This calculation was at the origin of a new astronomical solution calculated by Berger *et al.* (1988) and used for extending the validity of the paleoclimatic parameters and insolation over the last 5 to 10 million years (Berger and Loutre, 1988) It is the purpose of this paper to present the final and most accurate version of this solution

In order to appreciate the improvement of the accuracy in the computation of the astronomical parameters of the Earth's orbit and rotation, it is necessary to introduce some elementary notions of celestial mechanics Two systems will have to be considered One will deal with the motion of nine planetary point masses around the Sun, the other will consider the rotation of the Earth as a result of the luni–solar attraction

PLANETARY SYSTEM

Galilean Frame of Reference

Every two particles in the universe attract each other with a force that is directly proportional to the product of their masses, and inversely proportional to the square of the distance between them, such is the Newtonian law of gravitational attraction Applying this law to the case of N celestial bodies of the solar system (the Sun and the planets), we are able to express the equations of motion

$$m_j\vec{r}_j = \sum_{i \neq j}^{N} \frac{G m_i m_j \vec{r}_{ji}}{r_{ji}^3} \qquad j = 1, \quad N \quad (1)$$

where $\vec{r}_j$ denotes the radius vector of the particle P_j, with regard to the fixed origin, $\vec{r}_{ji}$ is the distance vector between P_j and P_i and is equal to $\vec{r}_i - \vec{r}_j$, the distance r_{ji} is equal to r_{ij}, m_i is the mass of P_i, G is the Gaussian gravitational constant derived from Kepler's third law

The conservation of linear momentum, which can be obtained by adding all the N equations (1) together, tells us that the centre of mass of the N particles moves uniformly in a straight line Indeed, we have

$$\sum_{j=1}^{N} m_j r_j = 0$$

and if we define the position, $\vec{r}_c$, of this centre of mass by

$$\vec{r}_c = \frac{\left(\sum_{j=1}^{N} m_j\vec{r}_j\right)}{\left(\sum_{j=1}^{N} m_j\right)}$$

we obtain $\vec{r}_c = 0$

Therefore, assuming that the centre of mass of the system is taken as the origin it does not change the equations of motion (1) Defining a force function U_j for each particle P_j

$$U_j = \sum_{\substack{i=1 \\ i \neq j}}^{N} \frac{G m_i}{r_{ij}} \qquad (2)$$

the equations of motion can also be written as

$$m_j r_j = m_j (\nabla U_j) \qquad (3)$$

where ∇U_j denotes the vector

$$\left(\frac{\partial U_j}{\partial x_j}, \frac{\partial U_j}{\partial y_j}, \frac{\partial U_j}{\partial z_j}\right)$$

Heliocentric Coordinate System

Instead of referring the position of the N particles in a Galilean frame of reference, they will be referred to with regard to one of them (the Sun) Consequently, the radius vector of the planet P_j is given by $\vec{\varrho}_j = \vec{r}_{Sj} = \vec{r}_j - \vec{r}_S$ (Fig 1) From equation (1) we have

$$\vec{r}_j = \sum_{\substack{i \neq S \\ i \neq j}}^{N} \frac{G m_i \vec{r}_{ji}}{r_{ji}^3} + \frac{G m_S \vec{r}_{jS}}{r_{jS}^3}$$

$$\vec{r}_S = \sum_{\substack{i \neq S \\ i \neq j}}^{N} \frac{G m_i \vec{r}_{Si}}{r_{Si}^3} + \frac{G m_j \vec{r}_{Sj}}{r_{Sj}^3}$$

Subtracting them, we obtain

$$\vec{\varrho}_j = -\frac{G(m_j + m_S)\vec{\varrho}_j}{\varrho_j^3} + \sum_{\substack{i \neq S \\ i \neq j}}^{N} \frac{G m_i \vec{r}_{ji}}{r_{ji}^3} - \sum_{\substack{i \neq S \\ i \neq j}}^{N} \frac{G m_i \vec{\varrho}_i}{\varrho_i^3} \qquad (4)$$

In equation (4), the first term of the right hand side represents the action of the Sun on P_j, the second represents the action of the other planets on P_j and the third one can be considered as a perturbation due to the choice of the reference frame, as it represents the action of the planets (except P_j) on the Sun (i e the new origin of the coordinate system)

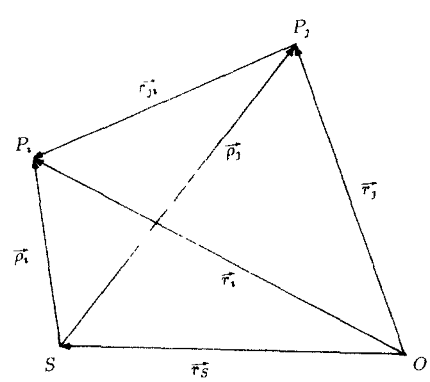

FIG 1 Radius vector of the Sun and of the planets with respect to the centre of mass (O) of the planetary system

As in the Galilean frame of reference, a force function can be defined:

$$U_j = \frac{G\,(m_S + m_j)}{\varrho_j} + R_j \tag{5}$$

with R_j, the disturbing function given by

$$R_j = \sum_{i \neq j} G m_i \left(- \frac{\vec{r_j}\;\vec{r_i}}{r^3_{ji}} + \frac{1}{\varrho_i} \right) \tag{6}$$

Accordingly, the equations of motion take the same form as (3)

$$m_j \ddot{\vec{\varrho_j}} = m_j \frac{\partial U_j}{\partial \vec{\varrho_j}}. \tag{7}$$

This equation may also be written in the x-coordinate.

$$\frac{d^2 x_j}{dt^2} = \frac{\partial U_j}{\partial x_j} \quad 1 \leqslant j \leqslant 9 \text{ (8 if Pluto is excluded)} \tag{8}$$

with similar equations in y and z

Keplerian Elements

A transformation from the coordinates (x_j, y_j, z_j) and velocity components ($\dot{x}_j$, $\dot{y}_j$, $\dot{z}_j$) into to the 6 *osculating* elements (semi-major axis, a; eccentricity, e; inclination i of the orbit on the reference plane; longitude of the ascending node, Ω; longitude of the perihelion, π, and mean longitude, λ, reckoned from the origin of time, Fig. 2) gives rise to the Lagrange equations (6 times the number of planets) relating all the orbital elements of the planets together and describing their motion around the Sun (Brouwer and Clemence, 1961)

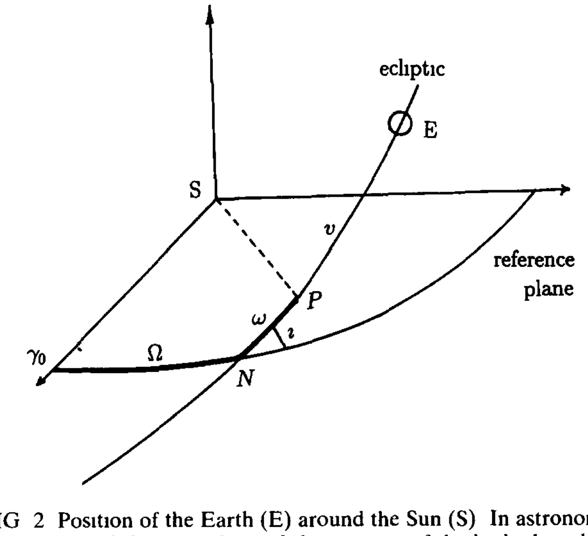

FIG 2 Position of the Earth (E) around the Sun (S) In astronomy, it is usual to define an orbit and the position of the body describing that orbit by six quantities called the elements Three of them define the orientation of the orbit with respect to a set of axes, two of them define the size and the shape of the orbit (a and e respectively), and the sixth (with time) defines the position of the body within the orbit at that time In the case of a planet moving in an elliptic orbit about the Sun, it is convenient to take a set of rectangular axes in and perpendicular to the plane of reference, with the origin at the centre of the Sun The x-axis may be taken towards the ascending node N, the y-axis being in the plane of reference and 90° from x, while the z-axis is taken to be perpendicular to this reference plane so that the three axes form a right-handed coordinate system γ_o the reference point from where the angles are measured As the reference plane is usually chosen to be the ecliptic at a particular fixed date of reference (named epoch of reference in celestial mechanics, Woolard and Clemence, 1966), γ_o is, in such a case, the vernal equinox at that fixed date (the vernal equinox is also referred to as the First Point of Aries indicating the position of the Sun when it crosses the celestial equator from the austral to the boreal hemisphere) P is the perihelion, Ω the longitude of the ascending node, ω, the argument of the perihelion, $\pi = \Omega + \omega$ the longitude of the perihelion, i the inclination, v the true anomaly, $\lambda = \pi + v$ the longitude of the Earth in its orbit

$$\frac{da}{dt} = \frac{2}{na} \frac{\partial R}{\partial \lambda}$$

$$\frac{de}{dt} = - \frac{(1-e^2)^{1/2}}{ena^2} \frac{\partial R}{\partial \pi} - \frac{(1-e^2)^{1/2}}{ena^2} \left[1 - (1-e^2)^{1/2}\right] \frac{\partial R}{\partial \lambda}$$

$$\frac{di}{dt} = \frac{-1}{na^2 (1-e^2)^{1/2} \sin i} \frac{\partial R}{\partial \Omega} - \frac{\tan \frac{i}{2}}{na^2 (1-e^2)^{1/2}} \left(\frac{\partial R}{\partial \pi} + \frac{\partial R}{\partial \lambda} \right) \tag{9}$$

$$\frac{d\Omega}{dt} = \frac{1}{na^2 (1-e^2)^{1/2} \sin i} \frac{\partial R}{\partial i}$$

$$\frac{d\pi}{dt} = \frac{(1-e^2)^{1/2}}{ena^2} \frac{\partial R}{\partial e} + \frac{\tan \frac{i}{2}}{na^2 (1-e^2)^{1/2}} \frac{\partial R}{\partial i}$$

$$\frac{d\lambda}{dt} = n - \frac{2}{na} \frac{\partial R}{\partial a} + \frac{(1-e^2)^{1/2}}{ena^2} \left[1 - (1-e^2)^{1/2}\right] \frac{\partial R}{\partial e} + \frac{\tan \frac{i}{2}}{na^2 (1-e^2)^{1/2}} \frac{\partial R}{\partial i}$$

However, these equations possess some inconvenient features for orbits with small eccentricities and/or small inclinations· the appearance of the eccentricity and of $\sin i$ in the denominator of the expressions for $d\pi/dt$ and $d\Omega/dt$ leads to serious problems related to small denominators when e and i approach zero. As all planetary orbits lie almost exactly in the same plane and differ only slightly from circles, it is thus desirable to use a modified form of these equations by setting

$$\begin{aligned} h &= e \sin \pi \\ k &= e \cos \pi \\ p &= \sin i \sin \Omega \\ q &= \sin i \cos \Omega \end{aligned} \tag{10}$$

So that the equations (9) become

$$\frac{da}{dt} = \frac{2}{na}\frac{\partial R}{\partial \lambda}$$

$$\frac{dh}{dt} = \frac{(1-e^2)^{1/2}}{na^2}\frac{\partial R}{\partial k} - \frac{h\,(1-e^2)^{1/2}}{na^2\,[1+(1-e^2)^{1/2}]}\frac{\partial R}{\partial \lambda} + \frac{kp}{2na^2\,(1-e^2)^{1/2}}\frac{\partial R}{\partial p} + \frac{kq}{2na^2\,(1-e^2)^{1/2}}\frac{\partial R}{\partial q}$$

$$\frac{dk}{dt} = -\frac{(1-e^2)^{1/2}}{na^2}\frac{\partial R}{\partial h} - \frac{k\,(1-e^2)^{1/2}}{na^2\,[1+(1-e^2)^{1/2}]}\frac{\partial R}{\partial \lambda} - \frac{hp}{2na^2\,(1-e^2)^{1/2}}\frac{\partial R}{\partial p} - \frac{hq}{2na^2\,(1-e^2)^{1/2}}\frac{\partial R}{\partial q}$$

$$\frac{dp}{dt} = \frac{1}{4na^2\,(1-e^2)^{1/2}}\frac{\partial R}{\partial q} - \frac{p}{2na^2\,(1-e^2)^{1/2}}\frac{\partial R}{\partial \lambda} - \frac{pk}{2na^2\,(1-e^2)^{1/2}}\frac{\partial R}{\partial h} + \frac{ph}{2na^2\,(1-e^2)^{1/2}}\frac{\partial R}{\partial k} \tag{11}$$

$$\frac{dq}{dt} = -\frac{1}{4na^2\,(1-e^2)^{1/2}}\frac{\partial R}{\partial p} - \frac{q}{2na^2\,(1-e^2)^{1/2}}\frac{\partial R}{\partial \lambda} - \frac{qk}{2na^2\,(1-e^2)^{1/2}}\frac{\partial R}{\partial h} + \frac{qh}{2na^2\,(1-e^2)^{1/2}}\frac{\partial R}{\partial k}$$

$$\frac{d\lambda}{dt} = -\frac{2}{na}\frac{\partial R}{\partial a} + \frac{(1-e^2)^{1/2}}{na^2\,[1+(1-e^2)^{1/2}]}\left(h\frac{\partial R}{\partial h} + k\frac{\partial R}{\partial k}\right) + \frac{1}{2na^2\,(1-e^2)^{1/2}}\left(p\frac{\partial R}{\partial p} + q\frac{\partial R}{\partial q}\right)$$

In order to obtain the long periodic terms, only the long period part of the disturbing function R is retained (Bretagnon, 1974, 1984, Berger, 1984) and expanded in powers of the two following small parameters

1 The ratio of the masses of the planets to that of the Sun
2 Their eccentricities (e) and inclinations (i) on the reference plane

This system (11) might be solved numerically or analytically But in any case, due to the complexity of R, this disturbing function will have to be truncated For example, following Milankovitch (1941), Brouwer and van Woerkom (1950), Dziobek (1963) and Bretagnon (1974), R can be expanded to the second degree of the variables h, k, p, q In that case for a planet P_j disturbed by the seven others (Pluto is excluded), R takes the following form

$$R_j = \sum_{i<j} \mu \frac{\mathbf{m}_i}{a_j} R_{ij} + \sum_{i>j} \mu \frac{\mathbf{m}_i}{a_i} R_{ij} \tag{12}$$

with μ given by the third law of Kepler

$$\mu = Gm_S = \frac{n_j^2\, a_j^3}{1+\mathbf{m}_j} = \frac{n_i^2\, a_i^3}{1+\mathbf{m}_i}$$

where n_j represents the mean motion of P_j ($2\pi/T_j$, T_j being the planet's period of revolution), $\mathbf{m}_j$ is the mass of P_j relative to the mass of the Sun ($\mathbf{m}_j = \frac{m_j}{m_S}$) and

$$\begin{aligned} R_{ij} = \frac{a_i}{r_{ij}} &= C_{ij} + A_{ij}\,(h_j^2 + k_j^2 + h_i^2 + k_i^2) \\ &- 4A_{ij}\,(p_j^2 + q_j^2 + p_i^2 + q_i^2) + \\ &B_{ij}\,(k_jk_i + h_jh_i) + 8A_{ij}\,(q_jq_i + p_jp_i) \end{aligned} \tag{13}$$

where A_{ij}, B_{ij} and C_{ij} are functions of $\alpha_{ij} = \frac{a_i}{a_j}$

In such a case an analytical solution for (11) can be found With (13), (11) indeed becomes

$$\frac{dh_j}{dt} = + \sum_{i \neq j} [j, i]\,(2A_{ji}k_j + B_{ji}k_i)$$

$$\frac{dk_j}{dt} = - \sum_{i \neq j} [j, i]\,(2A_{ji}h_j + B_{ji}h_i)$$

$$\frac{dp_j}{dt} = - \sum_{i \neq j} [j, i]\,(2A_{ji}q_j - A_{ji}q_i)$$

$$\frac{dh_j}{dt} = + \sum_{i \neq j} [j, i]\,(2A_{ji}p_j - A_{ji}p_i)$$

where

$$[j, i] = \frac{n_j \alpha_{ji} \mathbf{m}_i}{1 + \mathbf{m}_j} \text{ for } i > j$$

$$[j, i] = \frac{n_j \mathbf{m}_i}{1 + \mathbf{m}_j} \text{ for } i < j.$$

In the Lagrange method, the second degree terms of the quantities h, k, p, q of the so-called long period part of the disturbing function are retained. The resolution of the system of the differential equations thus obtained gives the Lagrange solution:

$$h = \sum_{k=1}^{m} M_k \sin(g_k t + \beta_k)$$

$$k = \sum_{k=1}^{m} M_k \cos(g_k t + \beta_k) \tag{14}$$

$$p = \sum_{k=1}^{n} N_k \sin(s_k t + \delta_k)$$

$$q = \sum_{k=1}^{n} N_k \cos(s_k t + \delta_k) \tag{15}$$

where $m = n = 8$

Long period terms of higher degree can then be introduced in the Lagrange equations Their solution takes the same form as (14) and (15), with more terms ($m > 8$ and $n > 8$). In that way, solutions are accurate to the first or to the second power with respect to their masses and to the first or to the third degree with respect to the planetary e's and i's, respectively if terms of power equal to or higher than 2 or 3 of their masses and terms of degree equal to or higher than 3 or 5 in e's and i's are neglected in the disturbing function (12 and 13) used to write the Lagrange equations

EARTH–MOON SYSTEM

After having computed the motion of the planetary point masses around the Sun, the method by Sharaf and Boudnikova (1967) for the Earth–Moon system can be used to obtain an analytical expansion for the long-term variations of the other two variables involved in the astronomical theory of paleoclimates (Fig. 3): the precession in longitude (ψ) involved in the calculation of $\tilde{\omega}$ and the obliquity of the ecliptic or tilt (ε), i.e. the inclination of the ecliptic of a particular date on the equator of that date. The Poisson equations for the Earth–Moon system provide the long-term variations of the luni-solar precession in longitude ψ_f and of the inclination ε_f of the equator on the mean ecliptic of epoch (Woolard and Clemence, 1966, Lieske *et al.*, 1977). These equations expanded to the second degree in eccentricity and inclination can be written as

$$\frac{d\varepsilon_f}{dt} = \tilde{P} \cos \varepsilon_f \sum_i N_i \sin(s_i t + \delta_i + \psi_f) - \frac{1}{2} \tilde{P} \sin \varepsilon_f \sum_i N_i^2 \sin 2(s_i t + \delta_i + \psi_f) - \tilde{P} \sin \varepsilon_f \sum_i \sum_{j > i} N_i N_j \sin[(s_i + s_j)t + \delta_i + \delta_j + 2\psi_f] \tag{16}$$

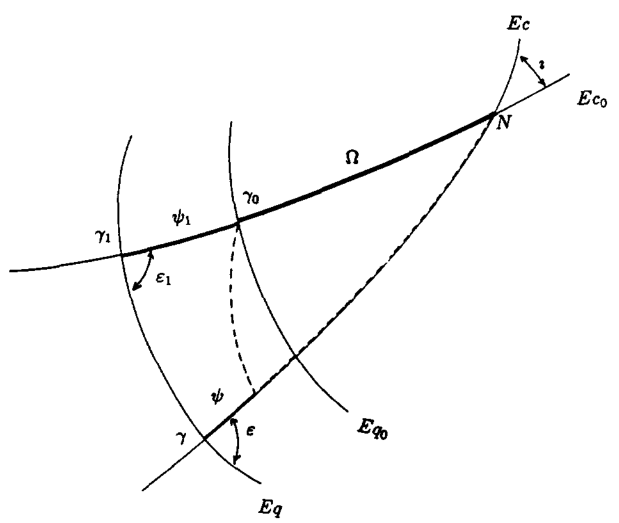

FIG 3 Precession and obliquity γ the vernal equinox of date, γ_0 the vernal equinox of reference, $\gamma_0\gamma_1 = \psi_f$ the luni-solar precession in longitude, ψ the general precession in longitude which provides the longitude of the moving perihelion $\tilde{\omega}$ through $\tilde{\omega} = \pi + \psi$, ε_f the inclination of the equator of date on the ecliptic of reference and ε the obliquity (Berger, 1984), as we are interested by the long-term variations of the astronomical elements, their short-term variations are removed and γ and γ_0 are more adequately referred to as mean vernal equinox

$$\frac{d\psi_f}{dt} = \bar{P}\cos\varepsilon_f \left[1 - \frac{3}{2}\sum_i N_i^2 - 3\frac{P_o}{\bar{P}}\sum_i \sum_{j>i} M_i M_j \cos(\beta_i - \beta_j)\right]$$

$$+3\bar{P}\cos\varepsilon_f \frac{P_o}{\bar{P}}\sum_i \sum_{j>i} M_i M_j \cos[(g_i - g_j)t + \beta_i - \beta_j]$$

$$+ \bar{P}\cos\varepsilon_f\,(\cot\varepsilon_f - \tan\varepsilon_f)\sum_i N_i \cos(s_i t + \psi_f)$$

$$- \frac{1}{2}\bar{P}\cos\varepsilon_f \sum_i N_i^2 \cos 2(s_i t + \delta_i + \psi_f)$$

$$- 3\bar{P}\cos\varepsilon_f \sum_i \sum_{j>i} N_i N_j \cos[(s_i - s_j)t + \delta_i - \delta_j]$$

$$- \bar{P}\cos\varepsilon_f \sum_i \sum_{j>i} N_i N_j \cos[(s_i + s_j)t + \delta_i + \delta_j + 2\psi_f] \quad (17)$$

where $\bar{P}$ is the so-called precessional constant of Newcomb ($\bar{P}$ = 54 ″9066) and P_o = 17 ″3919

Clearly (16) is dependent upon the solution of the system (15) in p and q, whereas (17) depends upon both (14) and (15) and therefore involves the elements of the expansion of (h, k) and (p, q) These equations (16) and (17) are solved order by order with respect to the small parameters (eccentricity and inclination) which leads to

$$\varepsilon_f = \mathbf{h} - \sum_i a_i N_i \cos[(s_i + \mathbf{k})t + \delta_i + \alpha]$$

$$- \sum_i a_{ii} N_i^2 \cos 2[(s_i + \mathbf{k})t + \delta_i + \alpha]$$

$$- \sum_i \sum_{j>i} a_{ij} N_i N_j \cos[(s_i + s_j + 2\mathbf{k})t + \delta_i + \delta_j + 2\alpha]$$

$$- \sum_i \sum_{j>i} a'_{ij} N_i N_j \cos[(s_i - s_j)t + \delta_i - \delta_j] \quad (18)$$

$$\psi_f = \tilde{\mathbf{k}}t + \alpha + \sum_i b_i N_i \sin[(s_i + \mathbf{k})t + \delta_i + \alpha]$$

$$+ \sum_i b_{ii} N_i^2 \sin 2[(s_i + \mathbf{k})t + \delta_i + \alpha]$$

$$+ \sum_{i,j} b_{ij} N_i N_j \sin[(s_i + s_j + \mathbf{k})t + \delta_i\ \delta_j + 2\alpha]$$

$$+ \sum_{i,j} b'_{ij} N_i N_j \sin[(s_i - s_j)t + \delta_i - \delta_j]$$

$$+ \sum_{i,j} b''_{ij} M_i M_j \sin[(g_i - g_j)t + \beta_i - \beta_j] \quad (19)$$

where $\mathbf{h}$ and α are constants of integration, and $\mathbf{k}$ and $\tilde{\mathbf{k}}$ are given by

$$\mathbf{k} = P\cos\mathbf{h} \quad (20)$$

$$\tilde{\mathbf{k}} = \bar{P}\cos\mathbf{h}\left[1 - \sum_i N_i^2\left(\frac{3}{2} + \frac{3}{4}a_i^2 - \frac{5}{2}a_i\quad \frac{1}{2}a_i^2(a_i - 1)\tan^2 h\right) - 3\frac{P_o}{\bar{P}}\sum_{i,j} M_i M_j \cos(\beta_i - \beta_j)\right] \quad (21)$$

The a, a', b and b''s are complicated functions of $\mathbf{h}$, $\mathbf{k}$, s, g, P_o and $\bar{P}$

The value of ψ and ε are related to ψ_f and ε_f through the spherical triangle Ω, γ_1, γ of Fig 3 where Ω is the ascending node of the ecliptic of date on the ecliptic of epoch, γ_1 is the direction of the intersection of the equator of date with the ecliptic of epoch, and γ the mean equinox of date (Berger, 1978a) So, ψ and ε can be determined by

$$\varepsilon = \varepsilon - \sum_i C_i N_i \cos[(s_i + \mathbf{k})t + \delta_i + \alpha]$$

$$- \sum_i C_{ii} N_i^2 \cos 2[(s_i + \mathbf{k})t + \delta_i + \alpha]$$

$$- \sum_i \sum_{j>i} C_{ij} N_i N_j \cos[(s_i + s_j + 2\mathbf{k})t + \delta_i + \delta_j + 2\alpha]$$

$$- \sum_i \sum_{j\ i} C'_{ij} N_i N_j \cos[(s_i - s_j)t + \delta_i - \delta_j] \quad (22)$$

$$\psi = \tilde{\mathbf{k}}t + \alpha$$

$$+ \sum_i G_i N_i \sin[(s_i + \mathbf{k})t + \delta_i + \alpha]$$

$$+ \sum_i G_{ii} N_i^2 \sin 2[(s_i + \mathbf{k})t + \delta_i + \alpha]$$

$$+ \sum_i \sum_{j>i} G_{ij} N_i N_j \sin[(s_i + s_j + 2\mathbf{k})t + \delta_i + \delta_j + 2\alpha]$$

$$+ \sum_i \sum_{j>i} G'_{ij} N_i N_j \sin[(s_i - s_j) + \delta_i - \delta_j]$$

$$+ \sum_i \sum_{j>i} G''_{ij} M_i M_j \sin[(g_i - g_j)t + \beta_i - \beta_j] \quad (23)$$

where

$$\varepsilon^* = \mathbf{h} - \sum_i N_i^2 \left[\frac{1}{2} a_i \ (a_i - 1) \tan \mathbf{h} + \frac{1}{4} (2a_i - 1) \cot \mathbf{h}\right] \tag{24}$$

The C, C', G and G''s are complicated functions of the a, a', b, b''s and $\mathbf{h}$

The general precession in longitude and the obliquity can, therefore, be written in the same expansion form as the orbital elements given by (14) and (15)

$$\varepsilon = \varepsilon^* + \sum_i A_i \cos(\gamma_i t + \zeta_i) \tag{25}$$

$$\psi = \tilde{\mathbf{k}}t + \alpha + \sum_i S_i \sin(\xi_i t + o_i) \tag{26}$$

A_i, γ_i, ζ_i, S_i, ξ, o_i are given by identification of (25) to (22) and (26) to (23). The constants $\mathbf{h}$, $\mathbf{k}$, $\tilde{\mathbf{k}}$, ε^* and α are computed by solving the set of equations (22) and (23) corresponding to the initial conditions and by using the relations between them: (20), (21) and (24) (see the next section for illustration)

LONG-TERM VARIATIONS OF THE EARTH'S ORBITAL ELEMENTS, PRECESSION AND OBLIQUITY

The 1978 Solution

The method stated above has been used by Berger (1977b, 1978a,b) to give the analytical expressions of and calculate the astro-insolation parameters currently used for paleoclimate reconstruction In these papers, the orbital elements h, k, p, q have been computed from Bretagnon's (1974) analytical solution which takes into account all the long period terms of the disturbing function up to the fourth degree in e's and i's and the short period terms (secular part) of the disturbing function that give, to the second power of the masses, important long period terms in the solution For this contribution of the short period terms, all the terms to the third degree in e's and i's which lead to a significant modification of the frequencies have been kept This leaves Bretagnon with solution (14) for (h, k) which has 24 terms and (15) for (p, q) which contains 17 terms All these terms have been ordered and regrouped by Berger so that the final expressions (14) and (15) contain only 19 and 15 terms respectively The amplitude, frequency and phase of all these terms are given in Table 1 of Berger (1976) The development of ε and ψ have been obtained by using the method developed by Sharaf and Boudnikova (1967) which allows the terms to the second degree in the eccentricity of the Earth's orbit to be included To reach a sufficient accuracy, Berger (1978a) has kept 240 and 411 terms respectively in (25) and (26) The constants of integration $\mathbf{k}$, $\mathbf{h}$ and α were deduced from the initial conditions for 1950 0, the reference plane being of 1850.0

$$\left\{\begin{aligned} \varepsilon_0 &= 23°4458 \\ \psi_0 &= 1°3960 \\ \left.\frac{d\psi}{dt}\right|_0 &= 50\,''2686 \end{aligned}\right. \tag{27}$$

In addition, for this solution, $\mathbf{k}$ in (18) and (19) was given by the same expression as for $\tilde{\mathbf{k}}$, it means that $\mathbf{k}$ was not given by (20) but rather by (21), according to the expansion by Sharaf and Boudnikova (1967) who used a mixed procedure of integration step by step (see Berger *et al.*, 1988 for more details)

This computation leads to the following values

$$\begin{aligned} \varepsilon^* &= 23°320\,556 \\ \mathbf{h} &= 23°394\,901 \\ \alpha &= 3°392\,506 \\ \mathbf{k}\,(=\tilde{\mathbf{k}}) &= 50\,''439\,273 \end{aligned} \tag{28}$$

and the amplitudes, phases, and frequencies of (14), (15), (26) and (25) were also published in Berger (1978b) with a computer programme available in Berger (1978a).

This solution, labelled here as BER78, was compared with many previous ones (Berger, 1977a). Moreover, a sensitivity analysis to the number of terms kept in the disturbing function and to the order of masses lead to the conclusion that this solution would be quite accurate over the last 1 5 Ma in the time domain and the frequencies acceptable for a much longer period (Berger, 1984). The insolations have the same accuracy (Berger and Pestiaux, 1984) because the formula used to compute them was without approximation (Berger, 1978a,b), they were used to validate the astronomical theory (e g. Kutzbach, 1985, Kutzbach and Guetter, 1986; Prell and Kutzbach, 1987, Saltzman *et al.*, 1984; Berger *et al.*, 1990) and/or calibrate the geological data (Imbrie *et al*, 1984, 1989, Martinson *et al.*, 1987)

The 1990 Solution

This former astro-insolation solution (BER78) will be compared in this paper to a new solution (BER90) built up from the Laskar's numerical solution for (e, π, i, Ω) (Laskar, 1986 and 1988) and according to the procedure developed in a previous section of this paper for ε and $e \sin \tilde{\omega}$

The main characteristics of BER90 can be summarized as follows. the Lagrange equations giving the secular evolution of the planets are expanded up to the second order of the masses and the fifth degree in e's and i's, including lunar and relativistic contributions This secular system is integrated over 30 million years (-10 to $+20$ million years) A modified Fourier analysis is then performed to fit a quasi-periodic function to the numerical values obtained The elements of the Earth's orbit are referred to the mean ecliptic and the mean equinox of 2000 0 To allow a comparison with the solution BER78, the time origin adopted has been moved to 1950 0 The constants of integration as well as the values of the disturbing planetary masses (Table 1) used in the Laskar's

TABLE 1 Values of the planetary masses used by Bretagnon in 1974 and in 1982 (this is the ratio of the solar mass to the mass of each planet)

Planets	Bretagnon (1974)	Bretagnon (1982)
Mercury	6,000,000	6 023 600
Venus	408,500	408 523 5
Earth + Moon	328,900	328,900 5
Mars	3 099,000	3 098,710
Jupiter	1,047 355	1 047 355
Saturn	3 498	3 498 500
Uranus	22,869	22 869
Neptune	19 314	19 314

computation are those used by Bretagnon for his VSOP82 solution (Bretagnon, 1982) The VSOP82 solution, representing the secular variations of the planetary orbits, is made of the perturbations developed up to the third power of the masses for all the planets and up to the sixth power for the four outer planets It also contains the perturbations of the Moon onto the Earth–Moon system

According to the sensitivity analysis made by Berger (1977a), this improvement in the planetary masses relative to those used in the solution BER78 is not expected to significantly change the numerical values of the solution It is the accuracy with which the perturbing function is known which mostly influences the accuracy of the solution of the planetary point masses system

The method by Sharaf and Boudnikova (1967) (see previous section) has then been used to obtain an expansion for the obliquity (22) and the precession (23) corresponding to the orbital elements given by Laskar Moreover, according to (20) and (21), the constants **k** and $\tilde{\mathbf{k}}$ are now calculated through different expressions leading to expansions (22) and (23) which are strictly to the second degree with respect to the Earth's eccentricity In such a case, **h** and α can be considered as the only two constants of integration, **k** and $\tilde{\mathbf{k}}$ being computed by (20) and (21), while the initial condition for $\frac{d\psi}{dt}$ is used to test the accuracy of the computation

Using the initial conditions for 1950 0 referred to the reference plane of 2000 0

$$\begin{aligned} \varepsilon_0 &= 23°4458 \\ \psi_0 &= -0°69824 \end{aligned} \qquad (29)$$

we deduced the following constants of integration

$$\begin{aligned} \mathbf{h} &= 23\ 399\ 935 \\ \alpha &= 1\ 600\ 753 \end{aligned} \qquad (30)$$

From them, we computed the value of **k**, $\tilde{\mathbf{k}}$ and ε

$$\begin{aligned} \mathbf{k} &= 50''390\ 811 \\ \tilde{\mathbf{k}} &= 50''417\ 262 \\ \varepsilon &= 23\ 333\ 410 \end{aligned} \qquad (31)$$

The initial value of $\frac{d\psi}{dt}$, i e the derivative of (26) computed at t = 0

$$\left.\frac{d\psi}{dt}\right|_{t=0} = 50''273\ 147$$

can be compared to the initial value given in (27)

$$\frac{d\psi}{dt} = 50''2686$$

giving the accuracy of the computation a complete out of phase of the climatic precession will occur only in a time span of more than 140 Ma

PALEOCLIMATIC SOLUTION

Knowing the expansion of the astronomical elements e, i, π, Ω (14 and 15), ε (25) and ψ (26) (Tables 3 4, 5, 7), it is possible to calculate the expansion of all the astro-insolation parameters As the eccentricity can be computed through $e^2 = h^2 + k^2$ the expansion for e^2 is given by

$$e^2 = \sum_i M_i^2 + \sum_i \sum_j 2M_iM_j \cos(\chi_i - \chi_j)$$

with $\chi_i = g_it + \beta_i$

TABLE 2 Characteristics of the 2 different solutions for the orbital elements the obliquity and the precession

Author	Label	Accuracy			Reference	
		degree (e, i)	order masses	ε ψ[d]	Epoch	Ecliptic
Bretagnon–Berger	BER78	3	2	2[b]	1950	1850
Laskar–Berger	BER90	5[a]	2	2	1950	2000

[a] Order of the expansion of the equations (the order of the solution might be higher)
[b] Berger has computed ε ψ from the (e, π) and (i Ω) system of Bretagnon
[c] Berger has computed ε ψ from the (e π) and (i, Ω) system of Laskar
[d] Degree of the expansion of (ε, ψ) with respect to the Earth's eccentricity according to the formulae by Sharaf and Boudnikova (1967)

TABLE 3 Amplitudes, mean rates, phases and periods of the 5 largest amplitude in the trigonometrical expansion of the (e, π) system (equations 14)

	Amplitudes		Mean Rate ("/year)		Phase (°)		Period (years)	
	BER78	BER90	BER78	BER90	BER78	BER90	BER78	BER90
1	0.018608	0.018970	4.20721	4.24898	28.6	30.6	308043	305014
2	0.016275	0.016318	7.34609	7.45549	193.8	199.7	176420	173831
3	0.013007	0.012989	17.85726	17.92249	128.3	151.7	72576	72311
4	0.009888	0.008136	17.22055	17.37774	320.2	309.8	75259	74578
5	0.003367	0.003870	16.84673	5.56847	99.3	77.0	76929	232738

The labels BER78 and BER90 correspond respectively to Bretagnon (1974) and Laskar (1988) from which the developments of e, π, i, Ω originate. However, the numbers given here are not those published by these authors because both solutions have been assigned to the same standard astronomical epoch of reference (origin of time is 1950.0). The sign of the amplitude of terms 3 and 5 in column BER78 has been changed relatively to the values given in Berger (1978a, b) in agreement with the change of the phase by 180°. This has been done to allow an easier comparison between the solutions.

TABLE 4 Amplitudes, mean rates, phases and periods of the 5 largest amplitude terms in the trigonometrical expansion of the (i, Ω) system (equations 15)

	Amplitudes		Mean Rate ("/year)		Phase (°)		Period (years)	
	BER78	BER90	BER78	BER90	BER78	BER90	BER78	BER90
1	0.027672	0.027538	0.0	0.0	106.2	107.6	—	—
2	0.020040	0.015973	−18.82930	−18.85013	248.5	245.5	68829	68752
3	0.012076	0.010306	−5.61094	−5.60436	12.0	17.2	230977	231248
4	0.007609	0.008047	−17.81877	−17.76134	277.4	287.2	72732	72967
5	0.005083	0.005695	−6.77103	−7.05274	305.0	143.8	191404	183758

The labels BER78 and BER90 correspond respectively to Bretagnon (1974) and Laskar (1988) from which the developments of e, π, i, Ω originate. However, the numbers given here are not those published by these authors because both solutions have been assigned to the same standard astronomical epoch of reference (origin of time is 1950.0).

TABLE 5 Amplitudes, mean rates, phases and periods of the 5 largest amplitude terms in the trigonometrical expansion of the general precession (26)

	Amplitudes (")		Mean Rate ("/year)		Phase (°)		Period (years)	
	BER78	BER90	BER78	BER90	BER78	BER90	BER78	BER90
1	7391.02	5911.4	31.60997	31.54068	251.9	247.2	41000	41090
2	2555.15	3597.2	32.62050	0.04374	280.8	230.4	39730	29630307
3	2022.76	2865.5	24.17220	0.04782	128.3	45.3	53615	27101064
4	1973.65	2691.7	0.63672	32.62947	168.1	288.8	2035441	39719
5	1240.23	2217.7	31.98378	0.09238	292.7	352.0	40521	14029011

The sign of the amplitude of term 4 in column BER78 has been changed relatively to the values given in Berger (1978a, b) in agreement with the change of the phase by 180°. This has been done to allow an easier comparison between the solutions.

The expansion for e can then be obtained, defining m^2 and a_i through:

$$m^2 = \sum_i M_i^2, \quad a_i = \frac{M_i}{m}$$

and

$$\sum_i \sum_{j>i} a_i a_j \cos(\chi_i - \chi_j) = \sum_k b_k \cos\gamma_k$$

we finally obtain (Berger, 1978a):

$$e = m \Big[1 - 0.25 \sum_k b_k^2 + \sum_k e_k \cos\gamma_k - 0.25 \sum_k b_k^2 \cos 2\gamma_k - 0.5 \sum_k \sum_{l>k} b_k b_l \cos(\gamma_k + \gamma_l)$$

$$
\begin{aligned}
&-0.5 \sum_k \sum_{l>k} b_k b_l \cos(\gamma_k - \gamma_l) \\
&+0.125 \sum_k b_k^3 \cos 3\gamma_k \\
&+0.375 \sum_k \sum_{l>k} b_k^2 b_l \cos(2\gamma_k + \gamma_l) \\
&+0.375 \sum_k \sum_{l>k} b_k b_l^2 \cos(2\gamma_l + \gamma_k) \\
&+0.375 \sum_k \sum_{l>k} b_k^2 b_l \cos(2\gamma_k - \gamma_l) \\
&+0.375 \sum_k \sum_{l>k} b_k b_l^2 \cos(2\gamma_l - \gamma_k) \\
&+0.75 \sum_k \sum_{l>k} \sum_{m>l} \\
&\quad b_k b_l b_m \,[\cos(\gamma_k + \gamma_l + \gamma_m) \\
&\quad + \cos(\gamma_k + \gamma_l - \gamma_m) + \cos(\gamma_k - \\
&\quad \gamma_l + \gamma_m) + \cos(\gamma_k - \gamma_l - \gamma_m)] \qquad (32)
\end{aligned}
$$

with

$$e_k = b_k + 0.375 b_k^3 + 0.75\, b_k \sum_{l \neq k} b_l^2$$

On the other hand, the longitude of the perihelion measured from the equinox of date is given by $\tilde{\omega} = \pi + \psi$. The precession ($\psi$), given by (23), can also be written in short as $\psi = \tilde{\mathbf{k}}t + \alpha + \delta\psi$ where $\delta\psi$ represents the periodic part of ψ. The climatic precession then becomes

$$
\begin{aligned}
e \sin \tilde{\omega} &= e \sin(\pi + (\tilde{\mathbf{k}}t + \alpha) + \delta\psi) \\
&= e \sin(\pi + \tilde{\mathbf{k}}t + \alpha) \cos \delta\psi \\
&\quad + e \cos(\pi + \tilde{\mathbf{k}}t + \alpha) \sin \delta\psi
\end{aligned}
$$

With equations (14) and (23), and limiting the expansion to the second order in M_i and N_i the climatic precession can be written as

$$
\begin{aligned}
e \sin \tilde{\omega} = &\sum_i M_i \sin[(g_i + \tilde{\mathbf{k}})t + \beta_i + \alpha] \\
&+ \sum_i \sum_j \frac{1}{2} G_i N_i M_j \sin \\
&\quad [(s_i + g_j + \mathbf{k} + \tilde{\mathbf{k}})t + \delta_i + \beta_j + 2\alpha] \\
&+ \sum_i \sum_j \frac{1}{2} G_i N_i M_j \sin \\
&\quad [(s_i - g_j + \mathbf{k} - \tilde{\mathbf{k}})t + \delta_i - \beta_j] \qquad (33)
\end{aligned}
$$

The classical astro-insolation elements (e, $e \sin \omega$ and ε — already given in (25)) can therefore be written in the same expansion form as for h, k, p and q given by (14) and (15)

$$
\begin{aligned}
e &= e^* + \sum_i E_i \cos(\gamma_i t + \phi_i) \\
e \sin \tilde{\omega} &= \sum_i P_i \sin(\alpha_i t + \eta_i) \qquad (34) \\
\varepsilon &= \varepsilon^* + \sum_i A_i \cos(\gamma_i t + \zeta_i)
\end{aligned}
$$

e^*, E_i, λ_i, ϕ_i, P_i, α_i, η_i are given by identification of (34) to (32) and (33)

The amplitudes, frequencies, phases of (14), (15) and (34) are given in Berger (1978a, b) for BER78, as for BER90, the most important terms of (34) are given in Tables 6, 7 and 8 of this paper. The main advantages of such developments, in addition to providing the numerical values of the elements, are that they allow an intercomparison with the previous solution(s) (see next section) and also give directly the most important frequencies of these fundamental parameters.

It is important to stress that the limited number of terms given in Tables 3 to 8 does not allow an accurate computation of the respective elements, many more terms have been used for the computation of (34). Let us remember that the values related to e, π, i, Ω are associated with an analytical expression used by Laskar to fit the numerical values he obtained from a numerical integration of the Lagrange equations. To obtain a good fit, 80 terms (the first of which are given in Tables 3 and 4) had to be kept in the trigonometrical expressions (14) and (15) which normally lead to a large number of terms in the analytical expressions (34) for respectively the eccentricity, the climatic precession parameter and the obliquity (Berger and Loutre 1990).

Re-assembling all these terms in such a way that all the frequencies would be different from term to term, ordering them to immediately have the most important ones and analysing the accuracy of the numerical values obtained from (34) according to the number of terms kept in each of the expansions lead to the following conclusions

For the obliquity, from the 6480 terms of (22), 6320 have different arguments and 704 have an amplitude larger than 0''.1, leading to deviations generally less than 0°.0005. 89 with an amplitude larger than 5'' already lead to deviations generally less than 0°.01

For the climatic precession, among the 12880 terms, 1522 have an amplitude larger than 10^{-6}, leading to deviations less than 1°.5 for $\tilde{\omega}$, less than 0.0005 for e and less than 7×10^{-4} for the climatic precession. With the 92 terms having an amplitude larger than 10^{-4} the precision is not very much lower, it reaches 10^{-3} for the climatic precession and 2° for $\tilde{\omega}$

TABLE 6 Amplitudes, mean rates, phases and periods of the 5 largest amplitude terms in the trigonometrical expansion of climatic precession (second equation in 34)

	Amplitudes		Mean Rate ("/year)		Phase (°)		Period (years)	
	BER78	BER90	BER78	BER90	BER78	BER90	BER78	BER90
1	0 018608	0 018970	54 64648	54 66624	32 0	32 2	23716	23708
2	0 016275	0 016318	57 78537	57 87275	197 2	201 3	22428	22394
3	0 013007	0 012989	68 29654	68 33975	131 7	153 4	18976	18964
4	0 009888	0 008136	67 65982	67 79501	323 6	311 4	19155	19116
5	0 003367	0 003870	67 28601	55 98574	102 8	78 6	19261	23149

The sign of the amplitude of terms 3 and 5 in column BER78 has been changed relatively to the values given in Berger (1978a, b) in agreement with the change of the phase by 180° This has been done to allow an easier comparison between the solutions

TABLE 7 Amplitudes, mean rates, phases and periods of the 5 largest amplitude terms in the trigonometrical expansion of obliquity (equation 25)

	Amplitudes (")		Mean Rate ("/year)		Phase (°)		Period (years)	
	BER78	BER90	BER78	BER90	BER78	BER90	BER78	BER90
1	−2462 22	−1969 00	31 60997	31 54068	251 9	247 14	41000	41090
2	−857 32	−903 50	32 62050	32 62947	280 8	288 79	39730	39719
3	−629 32	−631 67	24 17220	32 08588	128 3	265 33	53615	40392
4	−414 28	−602 81	31 98378	24 06077	292 7	129 70	40521	53864
5	−311 76	−352 88	44 82834	30 99683	15 4	43 20	28910	41811

TABLE 8 Amplitudes, mean rates, phases and periods of the 5 largest amplitude terms in the trigonometrical expansion of eccentricity (first equation of 34)

	Amplitudes		Mean Rate ("/year)		Phase (°)		Period (years)	
	BER78	BER90	BER78	BER90	BER78	BER90	BER78	BER90
1	0 011029	0 011268	3 13889	3 20651	165 2	169 2	412885	404178
2	0 008733	0 008819	13 65006	13 67352	99 7	121 2	94945	94782
3	0 007493	0 007419	10 51117	10 46700	294 5	312 0	123297	123818
4	0 006724	0 005600	13 01334	13 12877	291 6	279 2	99590	98715
5	0 005812	0 004759	9 87446	9 92226	126 4	110 1	131248	130615

The sign of the amplitude of terms 2 and 3 in column BER78 has been changed relatively to the values given in Berger (1978a, b) in agreement with the change of the phase by 180° This has been done to allow an easier comparison between the solutions

For the eccentricity as the series expansion is slowly convergent and the number of terms is huge, the accuracy of the numerical values for e can be very poor if uncontrolled truncations are made. Keeping all the terms (11479) for which the amplitude is larger than 4×10^{-6} leads to a deviation of about 2×10^{-6}

For the general precession in longitude the 8492 terms with an amplitude larger than 0 "01 give rise to a deviation less than 8×10^{-5} degree With 2394 terms, the deviation is generally less than 6×10^{-3} degree With 262 terms whose amplitudes are larger than 50", it is of the order of 0 °2

Therefore, in order to avoid any loss of accuracy by having to limit the expansions in order to provide the expressions (34) with an acceptable, managable number of terms, only the numerical values of e, ε, $e \sin \tilde{\omega}$

TABLE 9 Value of the different constants in the development of the astro-climatic elements for BER78 and BER90 (t_o = 1950 0)

	BER78	BER90
ε^* (°)	23 320 556	23 333 410
k ("/year)	50 439 273	50 390 811
$\bar{\mathbf{k}}$ ("/year)	50 439 273	50 417 262
α (°)	3 392 506	1 600 753

and insolation for the last 10 million years will be available upon request from the first author

INTERCOMPARISON OF THE ASTRO-PALEOCLIMATIC PARAMETERS

First, let us point out that even if the planes of reference for the solution BER78 and BER90 are not the same (1850 0 and 2000 0) the comparison between the two solutions for the obliquity and climatic precession is entirely valid as the values calculated for paleoclimatic research are instantaneous (i e they refer to the reference planes of the date and not of the epoch)

The accuracy of the solution depends essentially upon the accuracy and the number of terms kept in the perturbation function (Berger, 1976, 1984) In the case of BER90, it also depends on the numerical process used to obtain the analytical development of the elements the Fourier analysis for the Earth's orbital elements (h, k, p and q) was limited to 80 terms giving rise to an accuracy of about 0 1% for the eccentricity and 1 5% for the inclination, but this accuracy does not depend on time and affects only the values at times close to the present-day (Laskar, 1988)

Analytical Comparison

Tables 3 to 8 provide, for BER78 and BER90, the characteristics of the 5 largest terms in the orbital systems (e, π) and (i, Ω) as well as in ε, ψ, $e \sin \tilde{\omega}$ and e The frequencies will provide automatically the spectra of the astro-insolation parameters, as needed in the validation process of the astronomical theory (e g Imbrie *et al* , 1984, Berger, 1989a, b)

For the obliquity the number of terms for the solution BER90 is much larger than for BER78 for BER90 there are 149 terms for which the amplitude is larger than 1″ whereas there are only 47 for BER78 The 4 first terms of BER90 have to be compared respectively with term numbers 1, 2, 4, 3 of BER78, the 5th term does not have any corresponding term in BER78 Comparison of the two solutions shows only weak differences in the frequencies for the most important terms, the differences in the frequencies generate differences in the periods of the order of a few tens to hundreds of years, they amount to 200 years for the 53,864 yr-period but are generally less important for the other terms On the contrary, the amplitudes in BER90 are significantly different from those of BER78 −1969″ for the first term of BER90 against − 2462″ for the corresponding term in BER78 This difference of 20% is slightly compensated for by the appearance of a new 41,000-yr term (number 5 in BER90), the 29,000-yr term of BER78 ranking only 6 in BER90 with an amplitude of −266″ against −312″ Differences can even reach 50% for the 4th term of BER90 but are only a few percent for the 3rd one As for the phase, the differences are less than 30° for the important terms

The comparison of ψ between BER78 and BER90 is more complicated Very large periodicities (about 30 Ma) appear in BER90 with important amplitudes These periods are characteristics of the existence of almost commensurable characteristic frequencies If they are omitted, the second and third terms of BER78 can be compared to terms 4 and 7 respectively in BER90, the frequencies and phases being in good agreement and their respective amplitudes quite comparable

As was also the case for the obliquity, the number of terms in the expansion of the climatic precession is larger for BER90 than for BER78 in BER90 there are 110 terms for which the amplitude is larger than 5×10^{-5}, whereas there are only 46 in BER78 As there are many frequencies close to each other, it is not very easy to find the terms which have to be compared Nevertheless, the 5 first terms of BER90 can be compared respectively with term numbers 1, 2, 3, 4 6 of BER78 The 5th term of BER78 corresponds to the 9th term of BER90, which gives more weight to the 23,000-yr period in BER90 This does not preclude a good agreement between the 2 solutions the periods generally differ only by a few tens to a few hundreds of years, the amplitudes are quite close to each other (the absolute difference is only $3\ 7 \times 10^{-4}$ for the first term, i e a relative difference of about 2%) but for some terms the difference is more important (the amplitude of the 9th term of BER90 corresponding to a 19,000-yr component is about half the corresponding amplitude of the 5th term of BER78), the difference between the phases is generally less than 20°

For the time series for climatic precession, as well as for obliquity, the difference in amplitudes are compensated for by new terms having more or less similar frequencies and phases

The analysis of the eccentricity is much more complicated the number of terms is greater for BER90 than for BER78 (90 against 42 for all terms with an amplitude larger than 4×10^{-4}), the difference in the periods increases with their length (the difference is 8,700 years for the first term but only 163 years for the second) which makes the comparison term by term more delicate Nevertheless, assuming that the first terms effectively correspond to each other, the change in the phases between BER90 and BER78 are of the same order of magnitude than for the other elements (generally about 20° for the most important terms), the agreement between the amplitudes remains very good with a difference of the order of 10^{-4} (but it becomes more important — of the order of 10^{-3} — for the other terms)

This comparison leads to the conclusion that the solutions BER78 and BER90 can only become different before 1 5 million years ago (a few times the period of 400,000 years corresponds to the largest amplitude for the eccentricity)

Numerical Comparison

From the analytical expressions (34), time series can be generated and compared together, for example over the previous 5 million years and the next million years centred at 1950 0 The variations in time of the eccentricity the obliquity and the climatic precession corresponding to BER90 are presented in Figs 4 to 9 for a time interval going from 0 to 6 Ma BP They are compared to the solution BER78 for the time interval 0 to 3 Ma BP in Figs 4, 5 and 6

In BER90, during the last 5 million years (next million years), the eccentricity is seen to vary between 0.000267 (0.001694) and 0.057133 (0.052614) with an average quasi-period of 96,805 (93,100) years. Simultaneously, the obliquity of the Earth's orbit has varied between 22 °08 (22 °28) and 24 °54 (24 °32) with an average quasi-period of 41,074 (41,174) years. While the climatic precession oscillates between −0.05625 (−0.05193) and 0.05623 (0.05201) with an average quasi-period of 21,000 (21,378). A characteristic feature of the time evolution of the eccentricity is the almost complete disappearance of the 100,000-year cycle between 2.4 Ma BP and 2.8 Ma BP (Fig. 6a), as well as between 4.4 Ma BP and 4.8 Ma BP (Fig. 8a), leaving only the 400,000-year cycle. The obliquity is characterised by very small changes in amplitude between 3 Ma BP and 3.5 Ma BP (Fig. 7b), and between 4 Ma BP and 4.5 Ma BP (Fig. 8b).

A visual check to Figs 4, 5 and 6 shows that the solution BER90 is in good agreement with BER78 over the last 1.5 × 10^6 years, for the eccentricity, as well as for the obliquity and climatic precession. They become divergent only before 1.5 × 10^6 BP. The eccentricity curves look totally different at the 100 ka time scale starting 1.5 Ma BP. The amplitude of the 100,000 year-cycle disappears in the two solutions leaving only the 400,000-year envelope but for different time intervals.

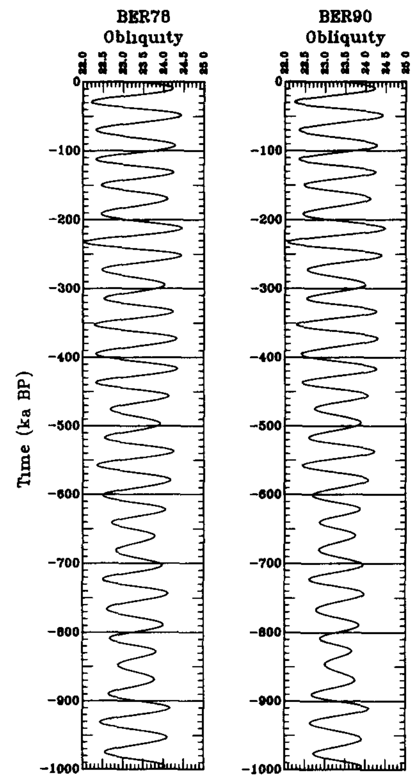

FIG. 4b. Comparison between BER78 and BER90 of the long-term variations of the obliquity from 1 Ma BP to the present (1950.0 A.D.).

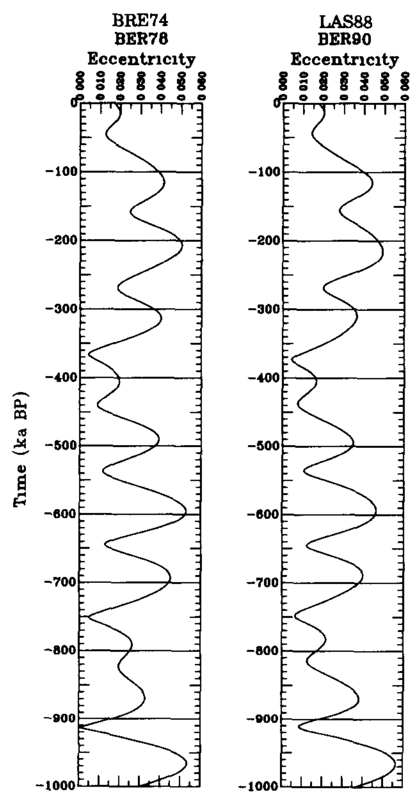

FIG. 4a. Comparison between BER78 and BER90 of the long-term variations of the eccentricity from 1 Ma BP to the present (1950.0 A.D.). BRE 74 refers to Bretagnon (1974) and LAS 88 to Laskar (1988).

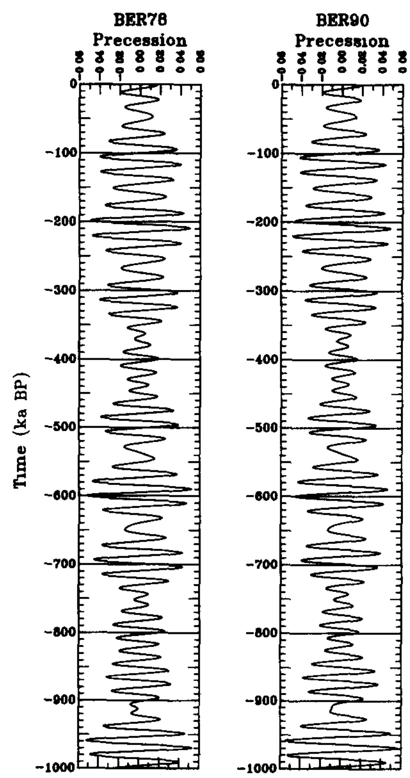

FIG. 4c. Comparison between BER78 and BER90 of the long-term variations of the climatic precession from 1 Ma BP to the present (1950.0 A.D.).

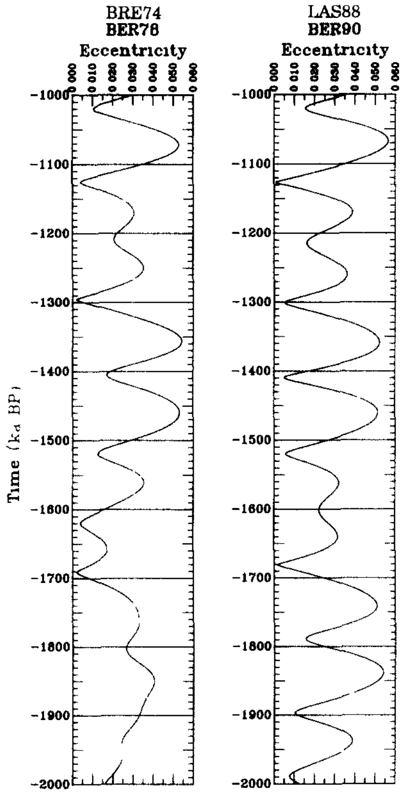

FIG 5a Comparison between BER78 and BER90 of the long-term variations of the eccentricity from 2 Ma BP to 1 Ma BP

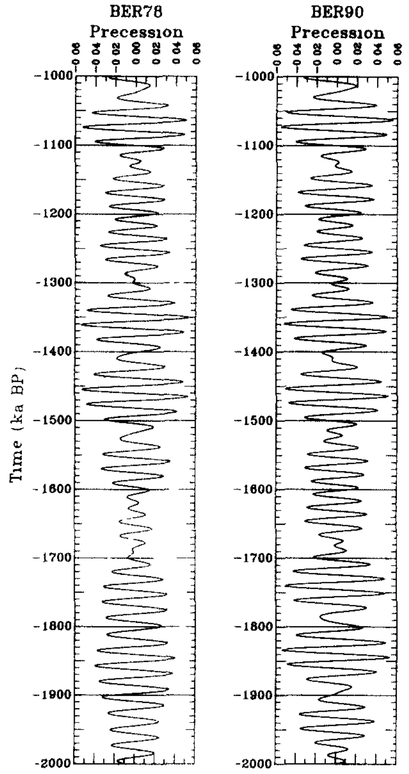

FIG 5c Comparison between BER78 and BER90 of the long-term variations of the climatic precession from 2 Ma BP to 1 Ma BP

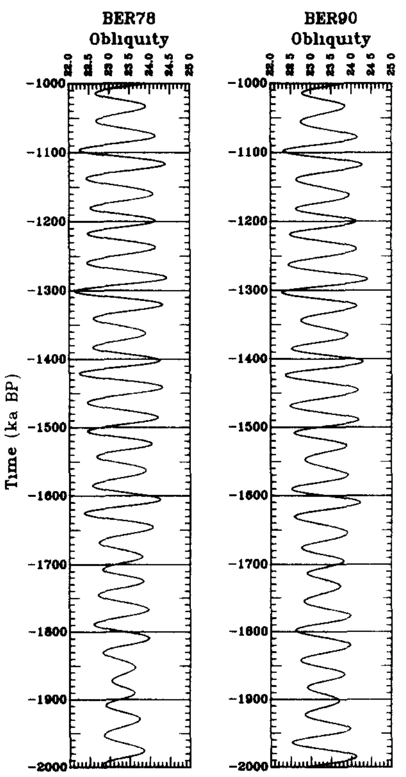

FIG 5b Comparison between BER78 and BER90 of the long-term variations of the obliquity from 2 Ma BP to 1 Ma BP

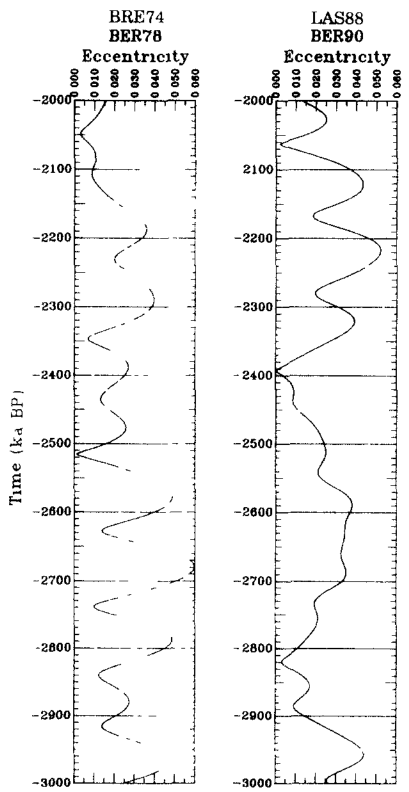

FIG 6a Comparison between BER78 and BER90 of the long-term variations of the eccentricity from 3 Ma BP to 2 Ma BP

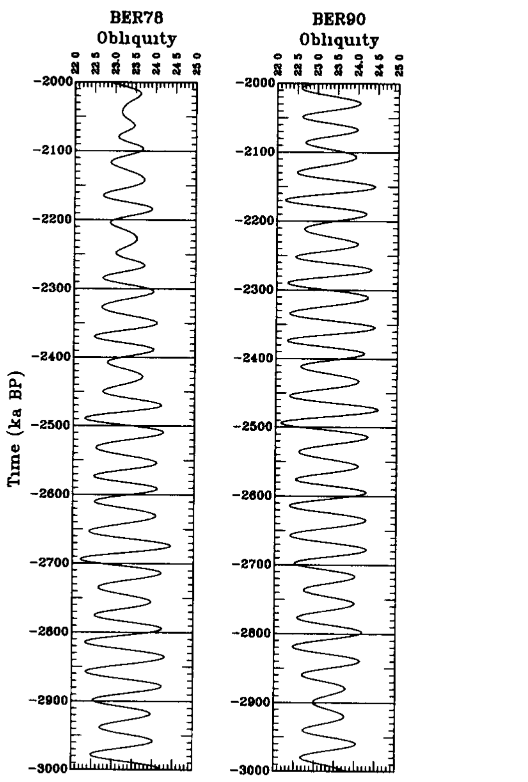

FIG 6b Comparison between BER78 and BER90 of the long-term variations of the obliquity from 3 Ma BP to 2 Ma BP

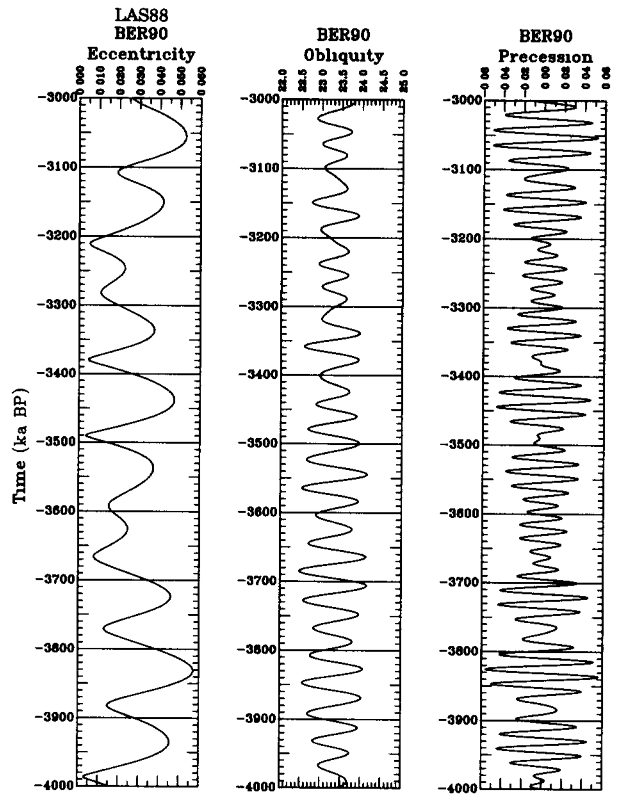

FIG 7 Long-term variations of the eccentricity (a), the obliquity (b) and the climatic precession (c) from 4 Ma BP to 3 Ma BP for BER90

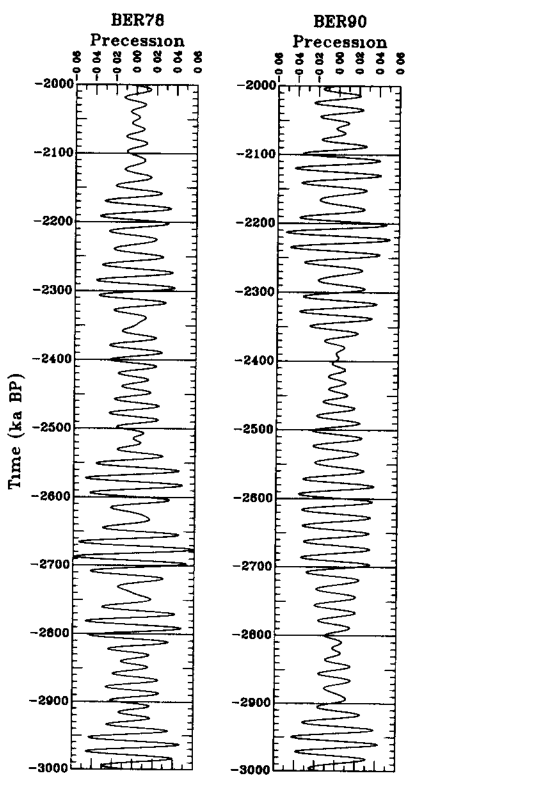

FIG 6c Comparison between BER78 and BER90 of the long-term variations of the climatic precession from 3 Ma BP to 2 Ma BP

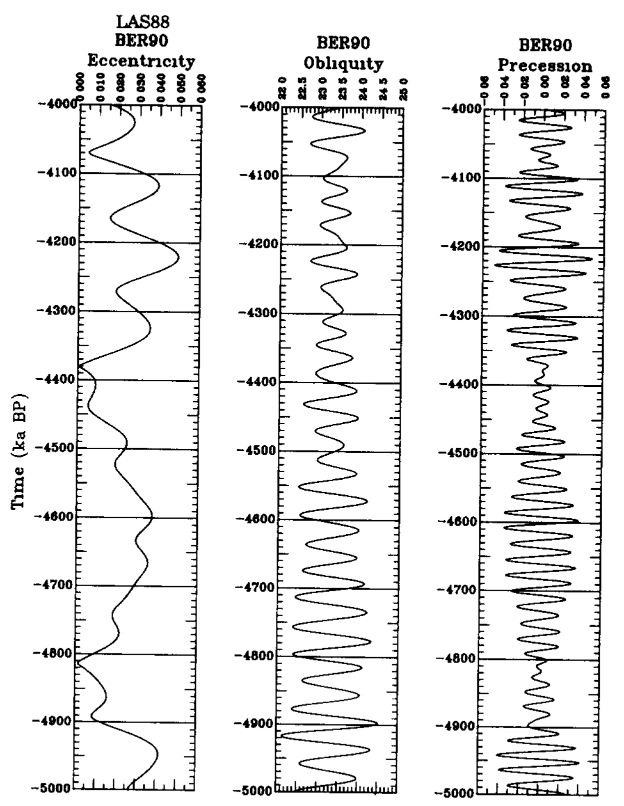

FIG 8 Long-term variations of the eccentricity (a), the obliquity (b) and the climatic precession (c) from 5 Ma BP to 4 Ma BP for BER90

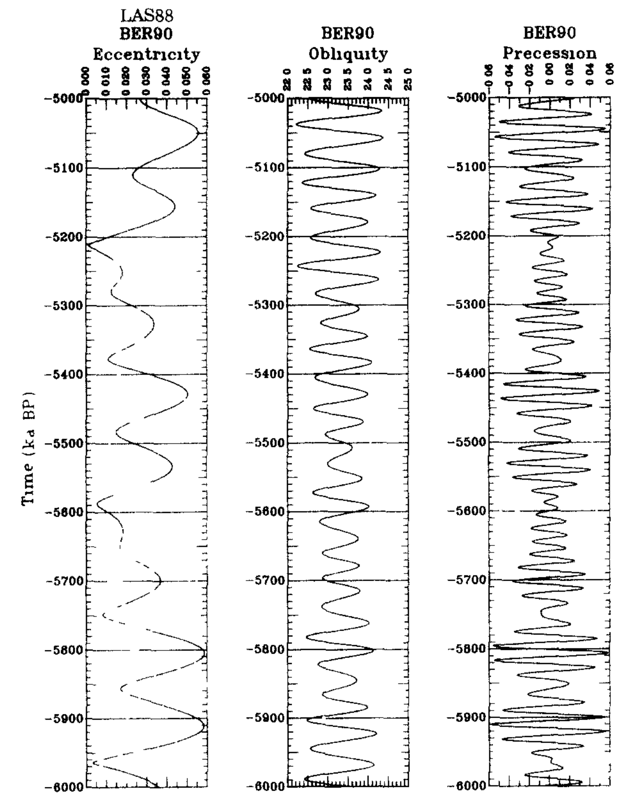

FIG. 9. Long-term variations of the eccentricity (a), the obliquity (b) and the climatic precession (c) from 6 Ma BP to 5 Ma BP for BER90.

this characteristic shape, reflecting the 400,000-year component in BER78 between 1.7 Ma BP and 2.1 Ma BP, can be found in BER90 between 2.4 Ma BP and 2.8 Ma BP. The obliquity is out of phase by one fourth of a cycle (10,000 years) at 1.9 Ma BP with BER90 leading BER78. The first discrepancy between the two solutions for climatic precession appears at 1.3 Ma BP but the first significant difference occurs between 1.6 and 1.7 Ma BP. This confirms the conclusion made in 1984: comparing the solution BER78 with other solutions and estimating the influence of the perturbations not taken into account (relativistic and lunar), Berger (1984) concluded that the time series for the precession, eccentricity and obliquity can be considered reliable for the last 1.5×10^6 years only.

In the case of the eccentricity, the difference in phase between BER78 and BER90 is much less than a half period over the last 5 Ma. For the obliquity, the main difference arises between 3 and 3.5 Ma BP during which the new solution is far less regular than BER78 (i.e. less sinusoidal). The two solutions for the climatic precession are out of phase by half a mean cycle (10,000 years) at 2 Ma BP, but they recover very rapidly: at 2.2 Ma BP the phase difference is again small.

THE INSOLATION VALUES

Simulation of the past climate requires the calculation of the daily or monthly insolation instead of, or in addition to, the Milankovitch caloric season's insolation (Berger, 1978c). The daily mid-month or monthly mean insolation can be derived from a simple but accurate set of formulae (Berger, 1978a, b). For the sake of comparison, the mid-month daily insolation, defined from a constant increment of the true longitude of the Sun, starting at the spring equinox, are computed for each 10 degrees of latitude. Let us recall that these values represent the insolation at around the 20th of each month. In the same way, monthly mean insolation values, averaged over 10 degree latitudinal zones will also be displayed for the intercomparison between BER78 and BER90.

Analysis of the insolation values obtained from BER90 brings some general conclusions: insolation is dominated by precession mainly in the equatorial regions but the obliquity signal is reinforced at the solstices and at high latitudes. The role of eccentricity in modulating the precessional component in the variation of insolation is very visible through the 400,000 year cycle (see for example the monthly mean insolation for March 20–30°N (Figs 12, 13 and 14)).

For the last 1.5 million years the BER78 and BER90 insolation values are very similar (Figs 10 and 11). This confirms the limit of validity already given for the orbital parameters. The same characteristics hold therefore for the two solutions over that time span. For example for the last 200,000 years, the most significant deviations of the 65°N July mid-month insolation from the 1950.0 AD value (427 W/m^2) are found to be located around 185 ka BP (−28 W/m^2), 160 ka BP (−9 W/m^2), 137 ka BP (−11 W/m^2), 114 ka BP (−35 W/m^2), 93 ka BP (−6 W/m^2), 70 ka BP (−19 W/m^2), 41 ka BP (−10 W/m^2) and 22 ka BP (−9 W/m^2) as far as the negative deviations are concerned, and around 197 ka BP (46 W/m^2), 173 ka BP (54 W/m^2), 148 ka BP (28 W/m^2), 126 ka BP (60 W/m^2), 104 ka BP (48 W/m^2), 82 ka BP (40 W/m^2), 56 ka BP (34 W/m^2), 33 ka BP (15 W/m^2) and 10 ka BP (43 W/m^2) for the positive deviations.

In addition to the analysis of this 65°N July insolation, it might also be significant to compare the monthly mean insolation values given by the 2 solutions for June over the latitudinal band 80–90°N and for December 70–80°S, for March 20–30°N and for September 10–20°S. These latitudes and months are indeed among those which were retained in the insolation climate index (Berger *et al.*, 1981) as showing a statistically significant correlation with δO^{18} records.

The behaviour of the curves for March 20–30°N and September 10–20°S are very similar and their intercomparison will be restricted to the insolation curve for March 20–30°N. Globally, the insolation curves for March 20°–30°N (Figs 12–14) are very similar for the two solutions until around 1.4×10^6 BP. One of the characteristics of the BER78 curve is a well-marked beat between 1.7 and 2.1 million years for which the 400,000 year 'envelope' corresponds to the disappearance of the 100,000 year eccentricity cycle, and to a slight damping of the amplitude of obliquity, and to a less extent of precession. This same feature appears later (2.4 to 2.8 Ma BP) in BER90. In a more detailed analysis, we can see that until 200 ka BP, there exists

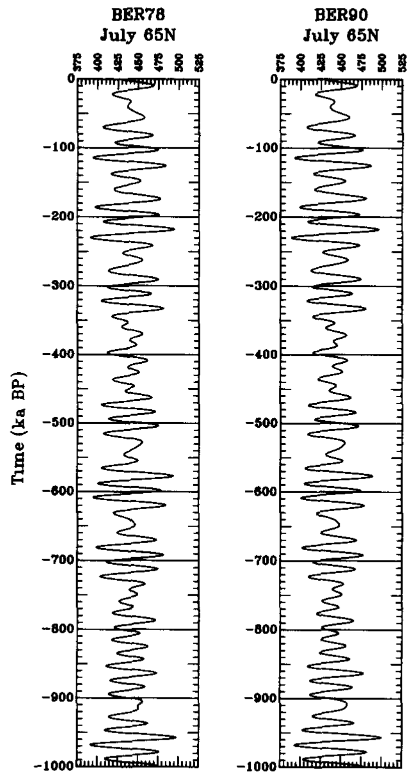

FIG 10 Comparison between BER78 and BER90 of the long-term variations of the 65N mid-month July insolation from 1 Ma BP to the present

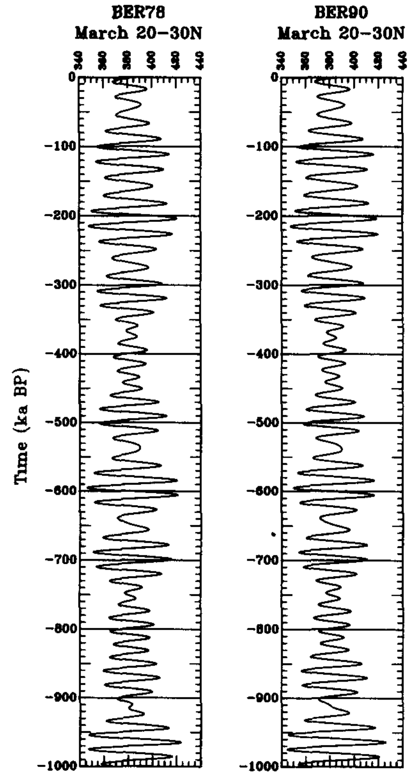

FIG 12 Comparison between BER78 and BER90 of the long-term variations of the monthly mean insolation for March 20N-30N from 1 Ma BP to the present

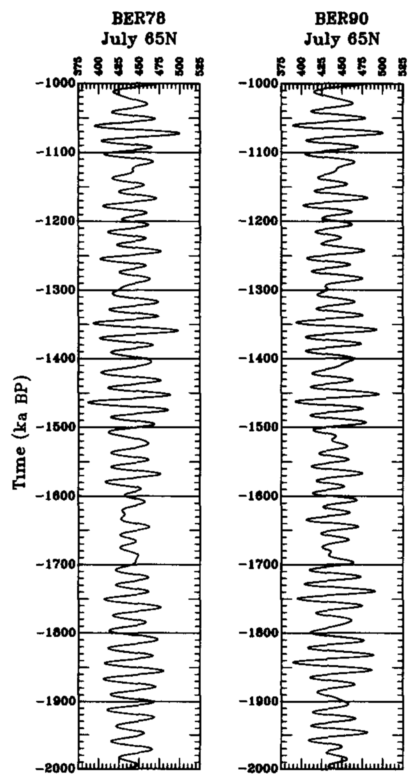

FIG 11 Comparison between BER78 and BER90 of the long-term variations of the 65N mid-month July insolation from 2 Ma BP to 1 Ma BP

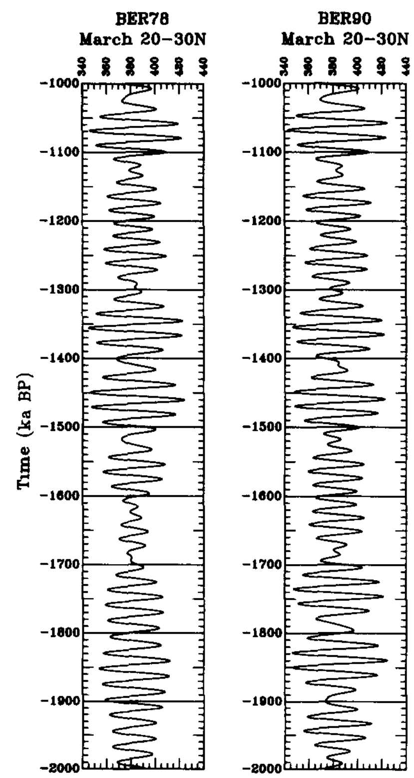

FIG 13 Comparison between BER78 and BER90 of the long-term variations of the monthly mean insolation for March 20N-30N from 2 Ma BP to 1 Ma BP

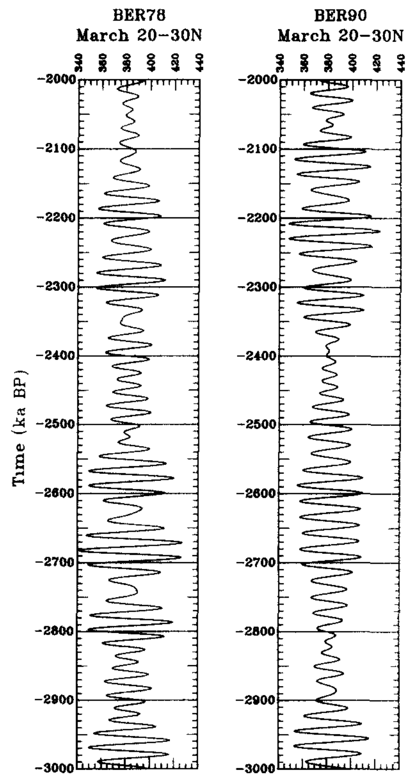

FIG 14 Comparison between BER78 and BER90 of the long-term variations of the monthly mean insolation for March 20N-30N from 3 Ma BP to 2 Ma BP

some tiny differences in amplitudes between the two solutions (less than 5 W/m^2) Before 400 ka BP the amplitude exhibits increasing differences reaching 10 W/m^2, with the amplitude of BER78 sometimes being larger than in BER90 (400 ka to 830 ka), sometimes smaller (830 ka to 1100 ka) Before 1 5 Ma BP, the differences become more significant, in particular between 1 9 and 2 7 Ma BP At 1 9 Ma BP, BER90 leads BER78 and its amplitude is smaller, but before 1 95 Ma, BER90 starts to lag behind BER78, at around 2 Ma, the two solutions are completely out of phase and they remain out of phase over more or less 100,000 years the amplitude of BER78 being smaller than BER90 For the next 300,000 years, there are periods during which the two solutions are in phase (2 08–2 10 Ma BP, 2 17–2 22 Ma BP, 2 28–2 33 Ma BP) alternating with periods where BER90 lags behind BER78 Between 2 4 and 2 7 Ma BP the mean quasi-period of BER90 (22,150 years) becomes larger than in BER78 (20,570 years) giving rise to periods during which the two solutions are in phase (around 2 6 Ma BP) or totally out of phase (around 2 5 Ma BP) Finally, at 2 7 Ma BP the two solutions are again in phase

The comparison of the solutions over other latitudinal zones and months shows similar behaviour beats can be seen in the June monthly mean insolation averaged over the latitudinal band between 80 and 90°N, but do not occur at the same time for the two solutions The differences between the amplitudes of the insolation curves increase with time back in the past From a few W/m^2 during the last 200,000 years to 10 W/m^2 around 700 ka, it reaches 20 W/m^2 at 1 3 Ma BP Before 1 3 Ma BP, the two solutions become different A characteristic feature of the insolation around 4 4 Ma is the small variations in amplitude related to the small value of the eccentricity at that time (e is almost 0 at 4 38 Ma BP) and to the small changes in precession and obliquity The same feature has a striking appearance in the 65°N mid-month insolation values for July (Fig 15)

In the case of the monthly mean values for December 70–80S some tiny differences can already be seen around 1 Ma BP (few W/m^2 in amplitude), but they are much more perceptible before for example, differences in amplitude like around 1 4 Ma BP and appearance of new relative maxima at 1 29 and 1 41 Ma BP

For the winter latitudes at December 60–70N and June 50–60S, the amplitude of the variations are very small and consequently the comparison more difficult Broadly for December 60–70N the two solutions differ very few from each other (less than 1 W/m^2) over the last 900,000 years Between 900 ka and 1 5 Ma, the differences in amplitude increase and phase lags appear at different times Before 1 5 Ma, the two solutions can hardly be compared sometimes they look very similar (2 4 Ma to 2 7 Ma) while at other times they are totally different (1 9 Ma to 2 3 Ma)

CONCLUSION

The BER90 solution includes for (e, π) and (i, Ω), terms depending upon the second power as to the disturbing masses and on the fifth degree with respect to the planetary eccentricities and inclinations

The conclusions drawn from former comparisons of different astronomical solutions for the astro-insolation parameters (Berger, 1984) are confirmed

- The accuracy of the solution depends upon the accuracy of the constants, the initial conditions and the expansions themselves
- The values of obliquity and precession are strongly dependent upon the accuracy of the system (i, Ω), more than upon the accuracy with which the Poisson equations can be solved
- Back to 1 Ma, the two solutions BER78 and BER90 are very similar Between 1 Ma and 1 5 Ma some differences arise, which are not very important, but, earlier than that, the two solutions become very different, so that for periods previous to 1 5 Ma the solution BER90 must be used when insolation values are needed to force climate models A preliminary comparison of BER90 with two numerical integrations of a set of equations for the dynamics of both the planetary point masses and the Earth–Moon system (Laskar, personal communication in Berger *et al* , 1988, Quinn *et al* , 1991) leads us to conclude that these values remain reli-

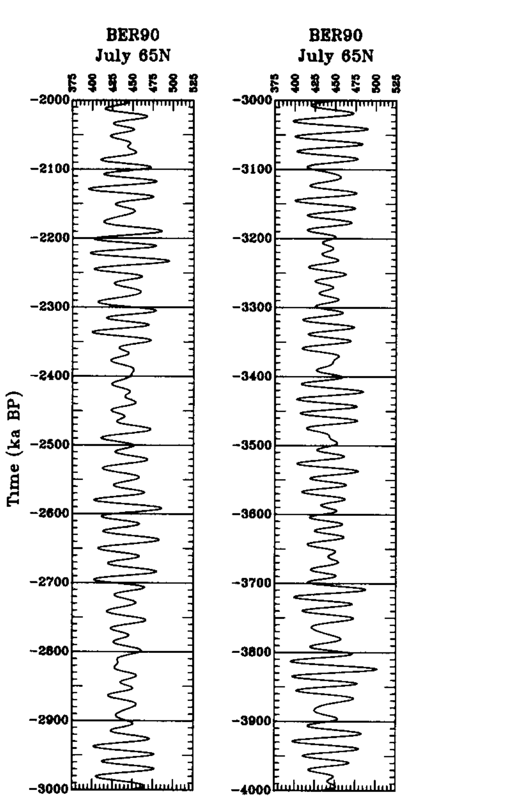

FIG 15a-b Long-term variations of the 65N mid-month July insolation from 3 Ma BP to 2 Ma BP (a) and from 4 Ma BP to 3 Ma BP (b)

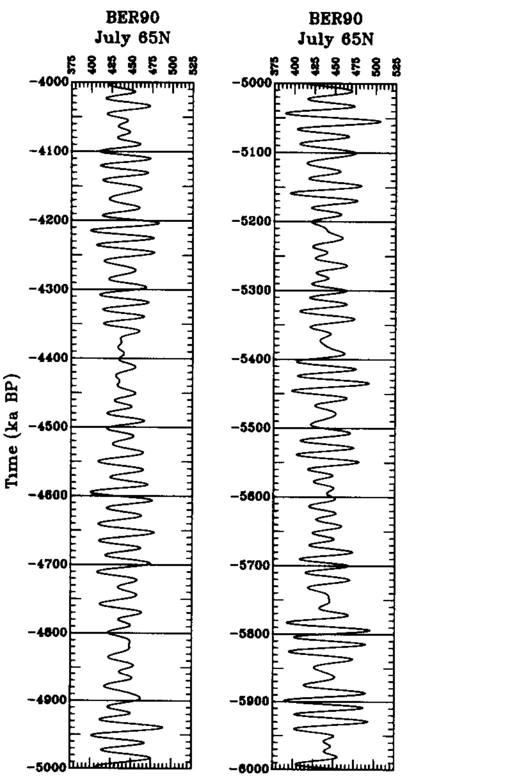

FIG 15c-d Long-term variations of the 65N mid-month July insolation from 5 Ma BP to 4 Ma BP (c) and from 6 Ma BP to 5 Ma BP (d)

able until 5 to 10 Ma ago, but most probably not for periods earlier than 10 Ma, as this seems to be the limit of validity of the astronomical solution Indeed, before 10 Ma, the orbits of the inner planets look chaotic any two orbits with nearby initial conditions diverge (Laskar, 1989, 1990)

- The intercomparison between BER78 and BER90 shows also that BER78 might continue to be used without any problem up to 1 Ma BP However, it is preferable to BER90 for the last glacial–interglacial cycles because of its better accuracy close to present-day times, the reproduction of the present-day conditions from BER90 indeed suffers from the fit carried-out by Laskar (1988) to represent its numerical values for *h*, *k*, *p*, *q* by trigonometrical series Fortunately, this numerical procedure does not affect the solution outside the time origin
- Finally, it is highly significant for the reliability of the solution that three solutions obtained independently for e, ε, $e \sin \tilde{\omega}$ all agree over the last 3 Ma at least Moreover, use of these new values has already shown a much better and more natural fit with the geological records over the last 5 Ma (Shackleton *et al* , 1990; Hilgen, 1991)

The slight advantage of the analytical procedure used here for BER90, and earlier for BER78, allows a straightforward way to obtain all the frequencies and related amplitudes and phases which characterize the astronomical parameters without having to rely on spectral techniques at all Improvements made not only in the straightforward numerical integration of the planetary point masses and Earth–Moon systems, but also in the analytical procedures to only deal directly with the long term variations (Bretagnon, 1990, Bretagnon and Simon, 1990), are therefore very encouraging

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ADDITIONAL DATA

The disk at the end of this issue contains the orbital and insolation data referred to in this paper The disk is a 3 5 inch double density — double sides — 720 KB containing four IBM® format ASCII files

File _ 90 TOP contains the introductory information for the three data files

File 1 _ 90 DAT contains 0–5 Myr BP
- first column time in ka (negative for the past, origin (0) is 1950 A.D)
- second column eccentricity, ECC
- third column: longitude of perihelion from moving vernal equinox in degree and decimals, OMEGA
- fourth column obliquity in degree and decimals, OBL
- fifth column climatic precession, ECC SIN(OMEGA)
- sixth column: mid-month insolation 65N for July in W/m^2
- seventh column mid-month insolation 65S for January in W/m^2
- eighth column mid-month insolation 15N for July in W/m^2
- ninth column mid-month insolation 15S for January in W/m^2

File 2 _ 90. DAT contains: 0–1 Ma BP
- first column time in ka (negative for the past, origin (0) is 1950 A D.)
- second to eighth column: mid-month insolation 90N, 60N, 30N, 0, 30S, 60S, 90S for December in W/m^2

File 3 _ 90 DAT contains 0–1 Ma BP
- first column time in ka (negative for the past, origin (0) is 1950 A D)
- second to eighth column: mid-month insolation 90N, 60N, 30N, 0, 30S, 60S, 90S for June in W/m^2