Quaternary Science Reviews 29 (2010) 1968–1982

Contents lists available at ScienceDirect

Quaternary Science Reviews

journal homepage: www.elsevier.com/locate/quascirev

Total irradiation during any time interval of the year using elliptic integrals

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ARTICLE INFO

Article history:
Received 15 November 2009
Received in revised form
30 April 2010
Accepted 7 May 2010

ABSTRACT

The calculation of the total solar energy received during a given interval of time over the year requires more attention than the calculation of the daily incoming solar radiation (daily insolation). It depends indeed upon whether the time interval is defined in terms of the true longitude or of the calendar date. The details of such a calculation based on elliptic integrals and on the sum of the daily irradiations are given in this paper. Numerical examples show the very high accuracy of both methods. The analytical expression and the spectral analysis of the long-term variations of such irradiation received during any time interval over the year show that they depend almost exclusively upon obliquity with a very small contribution of eccentricity, not at all upon precession. The eccentricity signal comes from the variation of the so-called solar constant through the variation of the mean distance from the Earth to the Sun. The precession signal is eliminated through the second law by Kepler. The correlation between total irradiation and obliquity reverses its sign at a specific latitude of the summer hemisphere which is a function of the selected time interval. At this latitude the obliquity signal disappears and only a pure eccentricity signal is left in the total irradiation, but with a very weak amplitude. Amplitude of the continuous wavelet transforrm shows that the amplitude variation of both the daily irradiance (mostly precession) and of the total irradiation (obliquity) vanishes around the Mid-Pleistocene Transition.

1. Introduction

Since the 1970s, there is an impressive interest in the astronomical theory of paleoclimates which started with the revival of the Milankovitch theory (Berger, 1988; Berger et al., 1984; Milankovitch, 1941). On one hand, geological data provide quantitative support to the relationship between the long-term variations of climate and of the astronomical parameters characterizing the orbit of the Earth around the Sun and its axis of rotation (Hays et al., 1976; Lisiecki and Raymo, 2005; Shackleton, 2000). The processes involved in such a relationship rely on the related incoming solar radiation (the so-called insolation, Berger, 1978). On the other hand, these insolation values are also used to force any model which attempts to reproduce and explain past climate changes at time scales going from decades to millions of years. Sensitivity experiments have been made over the last millennium using high frequency variations of the astronomically-forced insolation (Bertrand et al., 2002a,b). Centennial scale events have been studied in relationship with the astronomical forcing, like the 8.2 ka BP Holocene cooling (Renssen et al., 2007), the Heinrich Event 1 (Crucifix and Berger, 2002), and the Dansgaard–Oeschger events (Claussen et al., 2003; Sanchez-Goñi et al., 2008). At the thousand years time scale, sensitivity analyses of the climate system to the astronomical forcing were made for the Holocene (Crucifix et al., 2002; Ganopolski et al., 1998; Joussaume et al., 1999; Goosse et al., 2007; Renssen et al., 2005) and the last 20 ka (Timm and Timmermann, 2007). The astronomical theory was also tested over the glacial-interglacial cycles using models of different complexities (Berger and Loutre, 2002; Berger et al., 1998a,b; Vettoretti and Peltier, 2004; Yin and Berger, 2010).

Besides the influence of precession and eccentricity on climatic variations, the importance of obliquity is more and more acknowledged since the discovery that the Matuyama period is characterized by a strong 41 ka periodicity (Ruddiman et al., 1986), a periodicity which could be reproduced using a model of intermediate complexity in response to the insolation forcing (Berger et al., 1999). More recent studies show that the obliquity signal is actually also present during pre-Quaternary times (Berger, 1989; Lourens et al., 2010) and over the whole Pleistocene (Huybers, 2007; Jouzel et al., 2007). The imprint of its changes in the EPICA Dome C record has been used to stress its role in shaping the exceptionally long MIS-11 interglacial (Masson-Delmotte et al., 2006) and to a lesser degree the difference beween interglacials before and after the Mid-Bruhnes Event (Yin and Berger, 2010).

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doi:10.1016/j.quascirev.2010.05.007

Since obliquity is exclusively present in the total irradiation received during any interval of time of the year (Berger and Pestiaux, 1984; Milankovitch, 1941), the direct impact of the annual insolation has been simulated (Loutre et al., 2004) and the integrated summer insolation used to explain the early Pleistocene glacial cycles (Huybers, 2006). This direct impact of obliquity, in addition to the more traditional forcing due to the latitudinal and seasonal distribution of the daily irradiance (Berger, 1979), is at the origin of this paper. Different ways to compute the irradiance (in Wm-2) and the irradiation (in Jm-2) received by the Earth from the Sun are presented. The calculation of the total irradiation during any time interval of the year using elliptical integrals, and the properties and symmetries of the relevant equations are discussed thoroughly. In parallel, the opportunity is taken to remind the definition of some fundamental variables used in the astronomical theory, stressing, for example, the differences between the true and the mean longitudes of the Sun and between the sidereal and tropical years. The hypotheses used in deducting the equations are underlined. It is our hope that this paper will help to better understand the theoretical concepts underlying the calculation of insolation, although some of them will probably remain with us for some more years, in particular the choice of a calendar in climate models (Joussaume and Braconnot, 1997).

The first section reminds the classical formula of the instantaneous insolation and of the daily irradiance on which the calculation of the total irradiation is based. The next section explains how to define a total irradiation over any interval of time. The following sections generalize the formula used by Milankovitch for the total irradiation and introduce the elliptic integrals to compute it. Some examples for time intervals defined in terms of the true or the mean longitudes are finally given with their spectral properties and their latitudinal behavior.

2. Incoming solar radiation

The long-term variations of the energy available on the Earth surface at a given latitude on the assumption of a perfectly transparent atmosphere (the incoming solar radiation or insolation) are a single valued function of the energy, S_a, received by the Earth from the Sun per unit of time per unit of area on a theoretical surface perpendicular to the Sun's rays at the distance a (Appendix A), the semi-major axis of the Earth's orbit around the Sun (the ecliptic), its eccentricity, e, its obliquity, ε, and the longitude of the perihelion, $\tilde{\omega}$, measured from the moving vernal point (Appendix B).

The present-day value of these parameters are:

$S_a = 1366\ \mathrm{W\,m^{-2}}$
$a \sim 150 \times 10^6$ km
$e = 0.016724$
$\varepsilon = 23.446^\circ$
$\tilde{\omega} = 102.04^\circ$.

The long-term variations of e, ε and $\tilde{\omega}$ for one to a few million years before and after the present are given in Berger (1978) and Berger and Loutre (1991). For more distant times, Laskar et al. (2004) should be used.

3. Insolation at a given instant of time and latitude

A given instant during the course of the year corresponds to the declination δ of the Sun and to the distance from the Earth to the Sun, r. The irradiance (in $\mathrm{W\,m^{-2}}$), W, received on a horizontal surface located at a latitude ϕ is given by (more details in Berger et al., 1993):

$$W = S_a\left(\frac{a}{r}\right)^2 \cos z \tag{1}$$

where z, the zenith distance of the Sun is given by:

$$\cos z = \sin\phi\sin\delta + \cos\phi\cos\delta\cos H \tag{2}$$

- The declination, δ, is related to the true longitude of the Sun, λ, by:

$$\sin\delta = \sin\lambda\sin\varepsilon \tag{3}$$

Over one year, δ varies between two extreme values, $-\varepsilon$ and $+\varepsilon$, whereas λ varies from 0 to 360°. But because of (3) and the long-term variations of ε, the insolation for a given δ does not correspond always to the same position of the Earth on its orbit given by λ.

- H is the hour angle of the Sun which defines the time of the day. It is expressed either in units of angle or of time. Accordingly it varies from 0 to 360° or 0–24 h. In time units, it is related to the legal time, TL, through

$$H = TL + FH - WT - 12 - LG + ET \tag{4}$$

where FH is the number of the time zone (0 for Greenwich and positive to the West), WT allows to define the daylight savings time (e.g., in Europe $WT = 1$ for summer time and 0 for winter time), LG is the geographical longitude expressed in time units and measured positively to the West and ET the equation of time (Appendix C).

- r, the Earth–Sun distance, is given by the ellipse equation (Fig. 1):

$$r = \frac{a(1-e^2)}{1+e\ \cos v}\ \text{and}\ \rho = \frac{r}{a} \tag{5}$$

- v being the true anomaly is related to the true longitude, λ, by

$$\lambda = v + \omega \tag{6}$$

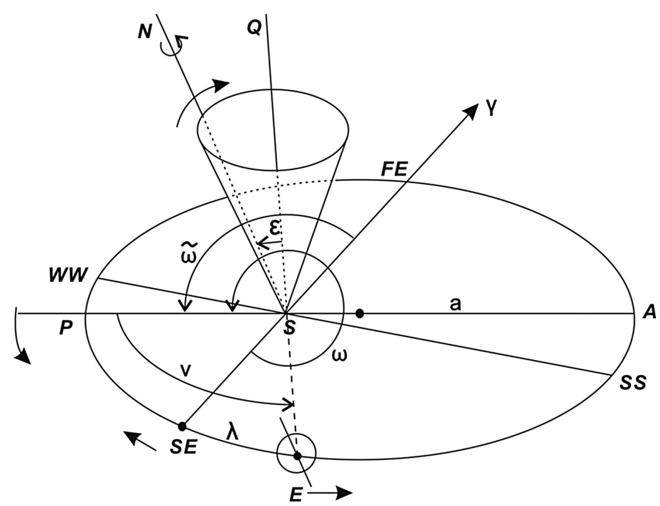

Fig. 1. Heliocentric coordinates of the Earth around the Sun, S, on its elliptical orbit (the ecliptic). P is the perihelion, A, the aphelion, SE, the spring equinox, SS, the summer solstice, FE, the fall equinox, and WW, the winter solstice. γ is the vernal point, $\tilde{\omega}$, the longitude of the perihelion, ω, the longitude of the perigee, λ, the true longitude and v, the true anomaly. SQ is perpendicular to the ecliptic, SN is the axis of the Earth rotation and ε is the obliquity.

As in this formula λ and ω, the longitude of the perigee, are measured from the Spring equinox, 180° has to be added to the numerical value of $\tilde{\omega}$ calculated by Berger (1978) or Berger and Loutre (1991) ($\omega = \tilde{\omega} + 180°$).

With (2) and (5), (1) becomes:

$$W = S_a \frac{(1 + e \cos\nu)^2}{(1 - e^2)^2} (\sin\phi \sin\delta + \cos\phi \cos\delta \cos H) \quad (7)$$

In paleoclimate studies, over a given year, ε, e and ω are assumed to remain constant ; over a given day, r, λ and δ are assumed to be constant (equal to their value at noon) and H varies from 0 at solar noon to 360° (or from 0 to 24 h in time units).

The long-term variations for a given latitude, day and hour thus clearly depend upon e and ω which appear in (7) through (5) and (6) and upon ε which is implicitly contained in δ according to (3). Moreover, the behaviour of each factor in (7) is governed by a different orbital parameter: ε drives the long-term variations of $\cos z$; ω the long-term variations of $(1 + e \cos(\lambda - \omega))^2$ and e the long-term variations of $(1 - e^2)^2$. This last factor is particularly interesting and it must be recognized that the daily insolation is depending upon $(1 - e^2)^{-2}$ and not upon $(1 - e^2)^{-1/2}$ which drives the total energy received by the Earth over one year (Appendix A).

The daily cycle of insolation given by (7) is fundamentally different whether the latitude belongs to the polar zone or not. In the non-polar zone, the Sun rises and sets everyday. Beyond the polar circles (defined by $|\phi| = 90° - \varepsilon$) and depending on the time in the year, the Sun either rises and sets over the day, or does not rise (polar night) or does not set (polar day).

Sunset or sunrise corresponds to $z = 90°$, which means from (2):

$$\sin\phi \ \sin\delta + \cos\phi \cos\delta \cos H = 0$$

This gives the value of the hour angle, H_0, at sunrise ($H = -H_0$) and sunset ($H = H_0$) with:

$$\cos H_0 = -\mathrm{tg}\,\phi \ \mathrm{tg}\,\delta \quad (8)$$

H_0 does exist only for: $-1 \le \mathrm{tg}\,\phi \mathrm{tg}\,\delta \le +1$ which means that the latitudes for which there is a daily sunrise and sunset are given by

$$-(90° - |\delta|) \le \phi \le (90° - |\delta|) \quad (9)$$

Therefore, the latitudes where there is no sunset ($H_0 = 12h$) are defined by:

$$|\phi| > 90° - |\delta| \quad \text{with} \quad \begin{cases} \phi > 0 \ \text{if} \ \delta > 0 \\ \phi < 0 \ \text{if} \ \delta < 0 \end{cases}$$

and the latitudes where there is no sunrise ($H_0 = 0$)

$$|\phi| > 90° - |\delta| \quad \text{with} \quad \begin{cases} \phi > 0 \ \text{if} \ \delta < 0 \\ \phi < 0 \ \text{if} \ \delta > 0 \end{cases}$$

For the polar latitudes ($|\phi| \ge 90° - \varepsilon$), the relations (8) and (9) can be expressed in terms of the true longitude of the Sun, λ. In such a case, the polar night ($H_0 = 0$) and the polar day ($H_0 = 180°$ or $12h$) are defined by:

$$\mathrm{tg}\,H_0 = \frac{\sqrt{\cos^2\phi - \sin^2\delta}}{-\sin\phi \sin\delta} = 0$$

$$\text{or} \ \cos^2\phi = \sin^2\varepsilon \sin^2\lambda \quad (10)$$

This equation has four roots (λ_i, $i = 1,4$) which,

$$\text{with} \ |\sin\lambda_i| = \frac{\cos\phi}{\sin\varepsilon} \quad (11)$$

are such that

$$0 \le \lambda_1 \le 90° \quad \lambda_2 = 180° - \lambda_1 \quad \lambda_3 = 180° + \lambda_1 \quad \lambda_4 = 360° - \lambda_1 \quad (12)$$

This means that the year is subdivided into 4 parts for the polar latitudes:

$\lambda_4 \le \lambda < \lambda_1$	the Sun rises and sets every day
$\lambda_1 \le \lambda < \lambda_2$	the polar day (night) in the Northern (Southern) Hemisphere
$\lambda_2 \le \lambda < \lambda_3$	the Sun rises and sets every day
$\lambda_3 \le \lambda < \lambda_4$	the polar night (day) in the Northern (Southern) Hemisphere

For example, at the present-day, at 70°N the polar day lasts from $\lambda \sim 60°$ to 120° (21 May to 24 July) and the polar night from $\lambda \sim 240°$ to 300° (22 November to 2 January).

4. Total daily insolation

The total daily insolation W_d is obtained by integrating (1) over 24 h of true solar time, t_s centered at solar noon ($H = 0$).

$$W_d = S_a \left[\sin\phi \int_{t_1}^{t_2} \frac{\sin\delta}{\rho^2} dt + \cos\phi \int_{t_1}^{t_2} \frac{\cos\delta \cos H}{\rho^2} dt \right] \quad (13)$$

where t_1 and t_2 are the time of sunrise and sunset respectively. These sunrise and sunset are actually not symmetrical relatively to the true solar noon because the declination δ varies over the day, as well as ρ. This is solved by keeping δ and ρ constant equal to their noon value, the error committed being negligible (of the order of 1‰ at the maximum) as λ varies by about 1° per day only.

To resolve (13), we must finally represent the diurnal course of the irradiance as a function of the mean solar time, t, which, by definition, is regular, the true solar time being not regular because of the elliptical shape of the orbit and the second law of Kepler. The relation between the two is given by the equation of time, ET (Appendix C), provided in the Astronomical Ephemeris for each day.

$$t_s = t + ET \quad (14)$$

Therefore

$$\frac{dt_s}{dt} = 1 + \frac{dET}{dt}$$

and neglecting the small variations of ET compared to 1, we have $dt \sim dt_s$. As the solar hour-angle, H, in time unit is called t_s, we have

$$H = \frac{2\pi}{\tau} t_s$$

where τ is the interval of 24 h expressed in units coherent with the units of S_a. Therefore,

$$dH = \frac{2\pi}{\tau} dt_s \simeq \frac{2\pi}{\tau} dt$$

and assuming r, δ and λ are constant over the day, the integration of (1) or (13) gives:

$$W_d = \int_{24h} W \ dt = \frac{S_a \tau}{2\pi} \left(\frac{a}{r}\right)^2 \int_{-H_0}^{H_0} \cos z \ dH$$

which means:

- for the latitudes where there is a daily sunrise and sunset:

$$|\phi| < 90° - |\delta|$$

$$W_d = \frac{S_a \tau}{\pi}\left(\frac{a}{r}\right)^2 \quad (H_0 \sin\phi \sin\delta + \cos\phi \cos\delta \sin H_0) \tag{15}$$

length of the day in hours of mean solar time
$= (24h/\pi) \cos^{-1}(-\mathrm{tg}\,\phi\ \mathrm{tg}\,\delta)$

- for the latitudes $|\phi| \geq 90° - |\delta|$

1. the polar night is defined by: $\phi \cdot \delta \leq 0$

$$\begin{aligned} &\text{and we have}: \quad W_d = 0 \\ &\text{length of the day} = 0 \end{aligned} \tag{16}$$

2. the polar day is defined by: $\phi \cdot \delta > 0$

$$\begin{aligned} &\text{and we have}: \quad W_d = S_a \tau \left(\frac{a}{r}\right)^2 \sin\phi \sin\delta \\ &\text{length of the day} = 24\ \text{h} \end{aligned} \tag{17}$$

The poles belong always either to the polar day or to the polar night, the equinoxes defining the transition between these two alternatives.

A program for calculating the long-term variations of the daily insolation for any calendar day (directly related to the mean longitude) or any date corresponding to a given value of the true longitude is available at ftp://ftp.astr.ucl.ac.be/pub/berger/berger78. These two different ways to compute daily insolation originate from Berger (1978). They were used later by Laskar et al. (1993, 2004) in their insolation programs to compute the monthly averages (http://www.imcce.fr/Equipes/ASD/insola/eart/online/).

5. Total irradiation during any interval of time

The irradiance at each point on the Earth has a discontinuous course through the year because as soon as the Sun sets below the horizon ($z \geq 90°$ in (1)), it ceases completely and, as it cannot become negative, is set to zero. Here we assume that the Sun is a point source, not a disk, therefore neglecting the problem of twilight, of the rising and setting of the upper solar limb and the influence of atmospheric refraction. If we consider the irradiation of a given unit surface during an interval of time which is longer than the stay of the Sun above the horizon (the day), this discontinuity must be considered.

This is done by defining the 24-h average irradiance $\overline{W}_d$:

$$\begin{aligned} \overline{W}_d &= \frac{W_d}{\int_{-\pi}^{\pi} \frac{dt}{dH} dH} = \frac{W_d}{\tau} \\ &= \frac{S_a}{\pi}\left(\frac{a}{r}\right)^2 (H_0 \sin\phi \sin\delta + \cos\phi \cos\delta \cos H_0) \\ &\cos H_0 = -\mathrm{tg}\,\phi\,\mathrm{tg}\,\delta \end{aligned} \tag{18}$$

r and δ being kept constant over the day and assigned their value at solar noon, $\overline{W}_d$ is also the mean irradiance of the parallel ϕ at true solar noon (Milankovitch, 1941) with H_0 equal to the geographical longitude of the limited circle of insolation counted from the meridian which the Sun is just crossing.

If H_0 is set to 0 or 180° when $|\cos H_0|$ becomes greater than 1 for a given latitude ϕ and declination δ, this formula remains valid for any ϕ and δ.

From (18), the total energy received during an interval of time $[t_0, t_0 + \tau]$ of one day of length τ can be calculated

$$W_{[t_0, t_0+\tau]} = \int_{t_0}^{t_0+\tau} W\,dt = \int_{t_0}^{t_0+\tau} \overline{W_{d,t_0}}\,dt = \tau \overline{W_{d,t_0}}$$

with t_0 being the time at 0 h of a day N during the year and τ the 24-h interval. As already said, the error due to the assumption that δ and ρ are kept constant over 24 h is very small, less than 0.01 W m^{-2}. Consequently, we may also write:

$$W_{[t_0, t_0+\tau]} = \int_{t_0}^{t_0+\tau} \overline{W_{d,\,t}}\,dt$$

where now λ, δ, ρ and $\overline{W_{d,t}}$ are continuous functions of time.

For any interval of time $[t_0, t_0 + n\tau]$ of an integer number of days, n, we will have:

$$W_{[t_0, t_0+n\tau]} = \int_{t_0}^{t_0+n\tau} \overline{W_d}\,dt = \tau \sum_{i=1}^{n} \overline{W_{d,i}} \tag{19}$$

$\overline{W_d}$ varies from one day to the other because r and δ vary, but the other variables e, ε and ω are assumed to remain constant over the whole year. This formula is valid for an interval of time made of a whole number, n, of days. If this is not the case, the energy, W_1 and W_2 received during part of the first day, Δ_1, and part of the last day, Δ_2, must be added. For any interval $[t_1, t_2]$ of $(\Delta_1 + n + \Delta_2)$ days with t_1 corresponding to day $N - 1$ of the year and t_2 to day $N + n$, we have therefore (Loutre, 1993):

$$\begin{aligned} &[t_1, t_2] = [N - \Delta_1, N + n + \Delta_2] \\ &W_{[t_1,\,t_2]} = W_1 + \sum_{i=N+1}^{N+n} W_{d,i} + W_2 \end{aligned} \tag{20}$$

where $W_{d,i}$ is the irradiation received during a full day, $N + i$, of the interval. To be coherent with the calculation of $\overline{W_d}$, W_1 and W_2 are assumed to be proportional to the fractions Δ_1 and Δ_2 of 24 h which receive insolation during the first and the last days, respectively and to the irradiation W_d of these days:

$$\begin{aligned} W_1 &= \Delta_1 W_{d,N} \\ W_2 &= \Delta_2 W_{d,N+n+1} \end{aligned}$$

For computing the total irradiation over an interval of time $[t_1, t_2]$, we therefore sum up the daily irradiation, W_d, for each day of the interval or have to calculate

$$W_{[t_1, t_2]} = \int_{t_1}^{t_2} W\,dt = \int_{t_1}^{t_2} \overline{W_d}\,dt \tag{21}$$

where the integral of the discontinuous function W is replaced by the integral of $\overline{W_d}$ which is a continuous function of time.

This means that we have two ways for calculating $W_{[t_1, t_2]}$. Either we use (20) or (21). Equation (21) involves elliptic integrals, the subject of the next section. Equation (20) uses explicitly the daily irradiation for each day of the interval. Both, however, request to make a transformation of the time variable t. Indeed, as in equations (18) and (21), $\overline{W_d}$ is an implicit function of t through δ and therefore λ, t must be replaced by λ in the integral. This is done through the second law of Kepler:

$$r^2 \frac{d\nu}{dt} = \frac{2\pi ab}{T} = \frac{2\pi}{T} a^2 \sqrt{1 - e^2}$$

with the true anomaly v being equal to $\lambda - \omega$.

T is the period of revolution of the Earth around the Sun, the sideral year of 365.25636 mean solar days (Appendix D). The sideral year is a constant because through the third law of Kepler it is related to the semi-major axis of the Earth's orbit, a, which is almost invariant in the planetary system (Appendix A).

Assuming ω is constant over a year, we write:

$$d\nu = d\lambda$$

The second law of Kepler leads to:

$$dt = \frac{T\rho^2}{2\pi\sqrt{1-e^2}}d\lambda$$

$$\text{and} \quad W = \int_{t_1}^{t_2} \overline{W_d}\, dt = \frac{S_a}{\pi}\int_{t_1}^{t_2}\frac{1}{\rho^2}(H_0 \sin\phi\sin\delta + \cos\phi\cos\delta\sin H_0)\,dt$$

becomes

$$W = \frac{S_a T}{2\pi^2\sqrt{1-e^2}}\int_{\lambda'(t_1)}^{\lambda''(t_2)}(H_0\sin\phi\sin\delta + \cos\phi\cos\delta\sin H_0)\,d\lambda \tag{22}$$

where if S_a is expressed in W m^{-2}, T must be in seconds.

These equations show that when the integral is calculated between two calendar days (t_1,t_2), the total energy is a function of both precession and obliquity. In this case, the length of the interval in days is constant. However, if the bounds of the integral are given in true longitude, equation (22) clearly shows that the total energy received during such an interval $[\lambda', \lambda'']$ does not depend anymore of the precession parameter and is primarily a function of obliquity with a very weak signal of eccentricity related to $(1-e^2)^{-1/2}$. In such a case, the length in days of the interval $[\lambda', \lambda'']$ is however varying in time as a function of precession only, leading to an average being again function of precession and obliquity.

In (22), the bounds of the integral must now be re-written in terms of λ. Actually there is a set of transformations of variables which allows to do so in astronomy:

λ is related to the true anomaly, v, by $v = \lambda - \omega$
v is related to the eccentric anomaly, E, by

$$tg(v/2) = \sqrt{(1+e)/(1-e)}\,tg(E/2)$$

E is related to time t and the mean anomaly M by the Kepler equation: $E - e\sin E = n(t - t^*) = M$
where t^* is the time of the perihelion passage
n is the mean motion of the Earth around the Sun

The true anomaly is the position angle of the Earth on the ecliptic measured counterclockwise from the perihelion. The mean anomaly is the anomaly of the Earth if it would move around the Sun at a constant speed. It means that both M and v are equal to zero at the perihelion. There are also direct relationships between M and v and therefore between $\lambda_m = M + \omega$ and $\lambda = v + \omega$ (Brouwer and Clemence, 1961):

$$\lambda_m = \lambda - 2\left[e\sin(\lambda-\omega) - \frac{3}{8}e^2\sin 2(\lambda-\omega) + \frac{e^3}{6}\sin 3(\lambda-\omega)\ldots\right] \tag{23}$$

If the daily insolation is computed for a specific calendar date or if we would like to have the total energy over a given calendar interval of time during the year $[\lambda_{m_1}, \lambda_{m_2}]$, the following strategy has to be used:

1) We let the origin of time be 21.0 March, the time of the Spring equinox ($\lambda = 0$).
2) We determine the mean longitude of the Sun at this date, λ_{m0}, through the application of (23)
3) For each value of λ_m obtained through an increment $\Delta\lambda_m$, i.e., $\lambda_m = \lambda_{m0} + \Delta\lambda_m$, ($\Delta\lambda_m$ for one day equals 360°/T, T being the number of day in one year), we determine λ from:

$$\lambda = \lambda_m + \left(2e - \frac{1}{4}e^3\right)\sin(\lambda_m - \omega) + \left(\frac{5}{4}\right)e^2\sin 2(\lambda_m-\omega) + \left(\frac{13}{12}\right)e^2\sin 3(\lambda_m - \omega)\ldots \tag{24}$$

4) The daily insolation for such a calendar date and the total insolation over a given time interval are then obtained through the formulas (18)–(20).

Equations (23) and (24) allow therefore to relate the uniform time (defined through the mean anomaly or the mean longitude) to the angular position of the Earth (given by the true anomaly or the true longitude) which speed varies according to the second Law by Kepler.

The technique involving equation (20) is the traditional way to compute numerically the energy received on the Earth over a given time interval during the year (Berger, 1978; Laskar et al., 1993). There is however a more direct way which is to make analytically the integral of the 24-hr mean irradiance over a given period of time $[t_1, t_2]$. This is done in the next sections 6 and 7.

6. Milankovitch seasonal insolation

In order to solve the integral in (22), Milankovitch (1941) decided to expand $\overline{W_d}$ in trigonometrical series of λ. For any latitude where the Sun rises and sets every day, the two terms of the daily irradiation

$$H_0 \sin\phi\sin\delta$$

$$\text{and} \quad \cos\phi\cos\delta\sin H_0 = \cos\phi\sqrt{1-\sin^2\varepsilon(1+tg^2\phi)\sin^2\lambda}$$

$$\text{with} \quad H_0 = \frac{\pi}{2} + sin^{-1}(tg\,\phi\;tg\,\delta)$$

can be expanded in series of λ. This series is convergent provided $\sin^2\varepsilon(1+tg^2\phi) \le 1$ and $tg^2\phi\; tg^2\delta < 1$, both conditions being satisfied for the non-polar zones. A generalisation of the Milankovitch formula was made by Berger (1973, 1975) leading to

$$\overline{W_d} = \frac{S_a}{\pi\rho^2}\left(b_0 + \frac{\pi}{2}\sin\lambda\sin\phi\sin\varepsilon + \sum_{l=1}^{\infty}(-1)^l b_l \cos 2l\lambda\right) \tag{25}$$

$$\text{where} \quad b_0 = \cos\phi\left(1 + \sum_{i=1}^{\infty} C_{2i}^{i}\frac{a_i}{2^{2i}}\right)$$

$$\underset{l\neq 0}{b_l} = \cos\phi\sum_{i=l}^{\infty}C_{2i}^{i-l}\frac{a_i}{2^{2i-1}}$$

$$a_i = (-1)^i(\sin\varepsilon)^{2i}C_{1/2}^{i}\left(1+\sum_{j=1}^{i}n^{2j}\left(C_i^j - \frac{2^j i!}{(2j-1)(i-j)!(2j-2)!!}\right)\right)$$

$$\text{with} \quad C_i^j = \frac{i!}{j!(i-j)!} \quad i! = i(i-1)\ldots 1 \quad 0! = 1$$

$$i!! = i(i-2)\ldots 1 \text{ if } i \text{ is odd} \quad i!! = i(i-2)\ldots 2 \text{ if } i \text{ is even}$$

$$n = tg\phi$$

For any interval $[\lambda', \lambda'']$ during the year, the integral of (25) in terms of λ leads to:

$$W_{[\lambda',\lambda'']} = \frac{S_a T}{2\pi^2\sqrt{1-e^2}}\Big\{b_0(\lambda''-\lambda') - \frac{\pi}{2}\sin\varepsilon\sin\phi\,(\cos\lambda'' - \cos\lambda') - \frac{b_1}{2}(\sin 2\lambda'' - \sin 2\lambda') + \frac{b_2}{4}(\sin 4\lambda'' - \sin 4\lambda') - \frac{b_3}{6}(\sin 6\lambda'' - \sin 6\lambda') + \frac{b_4}{8}(\sin 8\lambda'' - \sin 8\lambda')\ldots\Big\} \tag{26}$$

Or if $\lambda' = 0$, $\lambda'' = \lambda$:

$$W_{[0,\lambda]} = \frac{S_a T}{2\pi^2\sqrt{1-e^2}}\left\{b_0\lambda + (1-\cos\lambda)\frac{\pi}{2}\sin\varepsilon\sin\phi + \sum_{l=1}^{\infty}\frac{(-1)^l}{2l}b_l\sin 2l\lambda\right\} \tag{27}$$

Providing the Sun rises and sets every day of the interval, this formula is valid for both the non-polar and the polar regions. For the polar latitudes, the boundaries must however be taken according to (12). The validity of (27) for the polar zones comes from the fact that the intervals with daily sunrise and sunset correspond to

$$0 \le \lambda < \lambda_1 \quad \lambda_2 \le \lambda < \lambda_3 \quad \lambda_4 \le \lambda \le 2\pi$$

which are characterized by

$$-1 < \mathrm{tg}\,\phi\ \ \mathrm{tg}\,\delta < 1$$

$$\text{or}\quad \sin^2\varepsilon\left(1+\mathrm{tg}^2\phi\right)\sin^2\lambda < 1$$

the two conditions for the series in (26) to be convergent.

For the polar latitudes, during any interval $[\lambda', \lambda'']$ of the polar day, the integration is more straightforward because $H_0 = \pi$ leads to an easy expression of $\overline{W_d} = (S_a/\rho^2)\sin\phi\sin\delta$ and

$$W = \frac{S_a\,T}{2\pi\sqrt{1-e^2}}\int_{\lambda'}^{\lambda''}\sin\phi\,\sin\varepsilon\,\sin\lambda\,\mathrm{d}\lambda = \frac{S_a\,T}{2\pi\sqrt{1-e^2}}\sin\phi\,\sin\varepsilon(\cos\lambda' - \cos\lambda'') \tag{28}$$

For the whole polar day

$$W = \frac{S_a T}{\pi\sqrt{1-e^2}}\sin|\phi|\sqrt{\sin^2\varepsilon - \cos^2\phi} \tag{29}$$

which holds for both the northern and the southern hemispheres.

For the polar night, we have evidently $W = 0$.

For the days when the Sun rises and sets, Milankovitch assumed that all $a_i (i > 7)$ could be neglected because the highest degree of $\sin\varepsilon$ in a_i is $2i$. Accordingly he neglected all terms after b_7 in the series expansion. For the latitudes below 70° and for the present day obliquity, the technique used by Milankovitch to compute a_i and b_i shows that the series converge rapidly and the truncation is acceptable. For the latitudes above 70° however, this is not the case anymore: for example, a_7 for $\phi = 70°$ is equal to about 8×10^{-3}, it means 100 times larger than a_7 for $\phi = 60°$ which is equal to about 8×10^{-5}. As it cannot be neglected, b_7 must also be computed which would have requested Milankovitch to use a heavy numerical procedure to compute the energy received in these high polar latitudes when the Sun sets and rises everyday. This problem can be avoided by using the elliptic integrals as it was introduced by Meech in 1856, applied by Wiener in 1876 and by Fempl in 1957 and further developed by Berger in 1973, but surprisingly not used by Milankovitch.

7. Elliptic integrals

7.1. Non polar latitudes: $-(\pi/2-\varepsilon) \le \phi \le (\pi/2-\varepsilon)$

For the latitudes where the Sun rises and sets every day

$$W_{[t_1,t_2]} = c\int_{\lambda'(t_1)}^{\lambda''(t_2)}(H_0\sin\phi\sin\delta + \cos\phi\cos\delta\sin H_0)\mathrm{d}\lambda$$

where

$$c = \frac{S_a\,T}{2\pi^2}\frac{1}{\sqrt{1-e^2}}$$

can be expressed in terms of elliptic integrals (Berger, 1975). Its integration by parts leads indeed to

$$W(\phi,\lambda',\lambda'') = c\Big\{\sin\phi\sin\varepsilon[-H_0\cos\lambda]_{\lambda'}^{\lambda''} + \sin\phi\,\mathrm{tg}\phi\int_{\lambda'}^{\lambda''}\frac{1}{\left(1-k^2\sin^2\lambda\right)^{1/2}}\mathrm{d}\lambda - \sin\phi\,\mathrm{tg}\,\phi\cos^2\varepsilon\int_{\lambda'}^{\lambda''}\frac{1}{\left(1-k^2\sin^2\lambda\right)^{1/2}\left(1-\sin^2\varepsilon\sin^2\lambda\right)}\mathrm{d}\lambda + \cos\phi\int_{\lambda'}^{\lambda''}\left(1-k^2\sin^2\lambda\right)^{1/2}\mathrm{d}\lambda\Big\} \tag{30}$$

with $k^2 = \sin^2\varepsilon/\cos^2\phi$ which for the non-polar latitudes is ≤ 1 and with the hour angle at sunset given by:

$$H_0 = \cos^{-1}(-\ \mathrm{tg}\,\phi\,\mathrm{tg}\,\delta) = \cos^{-1}\left\{-\frac{\mathrm{tg}\phi\,\sin\varepsilon\,\sin\lambda}{\sqrt{1-\sin^2\varepsilon\,\sin^2\lambda}}\right\}$$

which leads to:

$$\mathrm{d}H_0 = \frac{\mathrm{tg}\,\phi\,\sin\varepsilon\,\cos\lambda}{\sqrt{\left(1-k^2\sin^2\lambda\right)\left(1-\sin^2\varepsilon\,\sin^2\lambda\right)}}\mathrm{d}\lambda$$

This formula explains how the length of the day varies through the year. For a given Hemisphere, the sign of $\mathrm{d}H_0$ depends only upon the sign of $\cos\lambda$. For the Northern Hemisphere ($\phi > 0$), the length of the illuminated day $2H_0$ increases from the spring equinox (when it is equal to 12h for any latitude) to the summer solstice (when it reaches its maximum). It decreases to the fall equinox and further to the winter solstice (when it reaches its minimum). From there, it starts to increase up to the spring equinox.

Moreover (30) shows that: $W(\phi,\lambda',\lambda'') = W(-\phi,\pi+\lambda',\pi+\lambda'')$ which means that the irradiation received by the latitude ϕ when the Sun goes from λ' to λ'' is equal to the irradiation received by the latitude $-\phi$ when the Sun goes from $\pi+\lambda'$ to $\pi+\lambda''$. This is not true for the average irradiance because the times for covering these ecliptical arcs are different.

If the time interval starts at the Spring equinox ($\lambda' = 0$) and ends for $\lambda'' = \lambda$, (30) becomes

$$W(\phi,0,\lambda) = c\Big\{\sin\phi\sin\varepsilon[-H_0\cos\lambda]_0^\lambda + \cos\phi E(\lambda,k) + \sin\phi\,\mathrm{tg}\,\phi F(\lambda,k) - \sin\phi\,\mathrm{tg}\,\phi\cos^2\varepsilon\Pi\left(\lambda,\sin^2\varepsilon,k\right)\Big\} \tag{31}$$

These three integrals, $F(\lambda,k)$, $E(\lambda,k)$ and $\Pi(\lambda,\sin^2\varepsilon,k)$ are the Legendre canonical elliptic integrals of the first, second and third kinds of argument λ, modulus k and parameter $\sin^2\varepsilon$ which according to Byrd and Friedman (1971) are given by:

$$F(\lambda,k) = \int_0^\lambda \frac{\mathrm{d}\lambda}{\sqrt{1-k^2\sin^2\lambda}} \tag{32}$$

$$E(\lambda,k) = \int_0^\lambda \sqrt{1-k^2\sin^2\lambda}\,\mathrm{d}\lambda$$

$$\Pi\left(\lambda,\sin^2\varepsilon,k\right) = \int_0^\lambda \frac{\mathrm{d}\lambda}{\left(1-\sin^2\varepsilon\sin^2\lambda\right)\sqrt{1-k^2\sin^2\lambda}}$$

The reader must be aware that the Legendre's normal forms are not the only standard forms possible. Moreover, in the definition of the third integral, some authors write $(1+n\sin^2\lambda)$ in the integrand while, here, it is more convenient to follow the notation of Byrd and Friedman (1971) as well as of Abramowitz and Stegun (1965) and to use $(1-m\sin^2\lambda)$ with $m = \sin^2\epsilon$. These remarks might be useful when using available computer code where elliptic integrals are used as in Analyseries Software (http://www.lsce.ipsl.fr/logiciels/index.php).

Because $0 < \sin^2\varepsilon < k^2$, the third elliptic integral can also be written in this hyperbolic case in term of the Jacobi Zeta function, Z, and the Ω_3 function.

$$\Pi\left(\lambda,\sin^2\varepsilon,k\right) = F(\lambda,k) + \frac{\cos\phi}{\cos\varepsilon\sin\phi}[F(\lambda,k)Z(\beta,k) - \Omega_3]$$

$$\text{with } \beta = \sin^{-1}\left(\sin^2\varepsilon/k\right) = \frac{\pi}{2} - \phi$$

$$Z(\beta,k) = E(\beta,k) - \frac{E}{K}F(\beta,k)$$

$$E = E\left(\frac{\pi}{2},k\right) \qquad K = F\left(\frac{\pi}{2},k\right) \tag{33}$$

$$\Omega_3 = \sum_{j=1}^{\infty} \frac{\sin\frac{j\pi F(\beta,k)}{K}\sin\frac{j\pi F(\lambda,k)}{K}}{j\sinh\left(j\pi\frac{K'}{K}\right)}$$

$$K' = F\left(\tfrac{\pi}{2},k'\right) \qquad k'^2 = 1-k^2$$

With this (31) becomes:

$$W = \frac{S_\mathrm{a}T}{2\pi^2\sqrt{1-e^2}}\Big\{\sin\phi\sin\varepsilon[-\cos\lambda H_0]_0^\lambda + \cos\phi E(\lambda,k) + F(\lambda,k)\left[\frac{\sin^2\phi\sin^2\varepsilon}{\cos\phi} - \sin\phi\cos\varepsilon\left(E(\beta,k) - \frac{E}{K}F(\beta,k)\right)\right] + \sin\phi\cos\varepsilon\,\Omega_3\Big\} \tag{34}$$

(34) is particularly interesting for the irradiation received during the astronomical seasons because for $\lambda = \pi/2$, $F(\lambda,k) = K(k)$, $\sin(j\pi) = 0$, $\Omega_3 = 0$ and therefore, the expression of W does not contain anymore the elliptic integral of the third kind. Finally, as the integrands of the three elliptic integrals (32) do not change for the two transformations,

$$\lambda \to \pi + \lambda$$

$$\lambda \to -\lambda$$

some properties of W follow as used in section 8 and the difference between the astronomical seasons arises always from the term in H_0.

As a most simple illustration of (31), at the equator

$$\phi = 0 \quad H_0 = \frac{\pi}{2} \quad \text{for any } \lambda$$

$$k = \sin\varepsilon$$

$$\text{and} \quad W(0,0,\lambda) = cE(\lambda,k) \tag{35}$$

with no seasonal contrast.

At the polar circles, (31) leads to

$$W\left(\frac{\pi}{2}-\varepsilon,0,\lambda\right) = c\left\{\sin\varepsilon\cos\varepsilon[-H_0\cos\lambda]_0^\lambda + \frac{\cos^2\varepsilon}{2}\left[\ln\frac{1+\sin\varepsilon\sin\lambda}{1-\sin\varepsilon\sin\lambda}\right]_0^\lambda + \sin\varepsilon\sin\lambda\right\} \tag{36}$$

because in this case, $k = 1$ and the elliptic integrals can be resolved analytically:

$$F(\lambda,1) = \ln(tg\,\lambda + \sec\lambda)$$

$$E(\lambda,1) = \sin\lambda$$

$$\Pi\left(\lambda,\sin^2\varepsilon,1\right) = \sec^2\varepsilon\left[\ln(tg\lambda + \sec\lambda) - \sin\varepsilon\ln\sqrt{\frac{1+\sin\varepsilon\sin\lambda}{1-\sin\varepsilon\sin\lambda}}\right]$$

7.2. Polar latitudes

For the polar latitudes $|\phi| > \pi/2 - \varepsilon$, three cases have to be discussed.

1. For the polar night, obviously

$$W_{[t_1,t_2]} = 0 \quad H_0 = 0$$

with $\lambda_3 \le \lambda < \lambda_4$ for the Northern Hemisphere ($\lambda_1 \le \lambda < \lambda_2$ for the Southern Hemisphere)

2. For the long day $\lambda_1 \le \lambda < \lambda_2$ in the Northern Hemisphere ($\lambda_3 \le \lambda < \lambda_4$ for the Southern Hemisphere)

$$H_0 = \pi$$

$$W = \frac{TS_\mathrm{a}}{\pi\sqrt{1-e^2}}\sin|\phi|\sqrt{\sin^2\varepsilon - \cos^2\phi} \tag{37}$$

For part of this long day (λ', λ''), we have:

$$W = \pi c \sin\phi \sin\varepsilon(\cos\lambda' - \cos\lambda'') \tag{38}$$

3. For the time interval during which the Sun rises and sets every day

$\lambda_2 \le \lambda < \lambda_3$ and $\lambda_4 \le \lambda < \lambda_1$

For these polar latitudes, the elliptic integrals will still be used provided their modulus is less than 1. In this case, a transformation of coordinates ($\lambda \to \psi$) must be done with:

$$\sin\psi = \frac{\sin\varepsilon}{\cos\phi}\sin\lambda = k\sin\lambda \tag{39}$$

For all polar latitudes where there is a sunrise and a sunset, λ is such that:

$$\sin^2\lambda < \frac{\cos^2\phi}{\sin^2\varepsilon} \quad \text{and therefore} \quad k^2\sin^2\lambda < 1$$

For these latitudes, $|\phi| > \pi/2 - \varepsilon$, the elliptic integrals can still be used provided that, in (31), they are replaced by their equivalents obtained from the reciprocal modulus transformation:

$$E(\lambda,k) = \frac{\sin\varepsilon}{\cos\phi}E\left(\psi,k^{-1}\right) + \frac{\cos^2\varepsilon\phi - \sin^2\varepsilon}{\sin\varepsilon\cos\phi}F\left(\psi,k^{-1}\right)$$

$$F(\lambda,k) = k^{-1}F\left(\psi,k^{-1}\right)$$

$$\Pi\left(\lambda,\sin^2\varepsilon,k\right) = \frac{1}{k}\Pi\left(\psi,\frac{\sin^2\varepsilon}{k^2},k^{-1}\right) = \frac{1}{k}\Pi\left(\psi,\cos^2\phi,k^{-1}\right)$$

After substitution in (31), we have:

$$\begin{aligned} W = &\frac{S_a T}{2\pi^2\sqrt{1-e^2}}\left\{\sin\phi\sin\varepsilon\left(-\cos\lambda\arccos\left(\frac{-tg\phi\sin\varepsilon\sin\lambda}{\sqrt{1-\sin^2\varepsilon\sin^2\lambda}}\right)\right)_0^\lambda\right. \\ &+\sin\varepsilon E\left(\psi,k^{-1}\right)+\frac{\cos^2\varepsilon}{\sin\varepsilon}F\left(\psi,k^{-1}\right) \\ &\left.-\frac{\sin^2\phi\cos^2\varepsilon}{\sin\varepsilon}\Pi\left(\psi,\cos^2\phi,k^{-1}\right)\right\} \end{aligned} \tag{40}$$

If we use the Zeta and Ω_3 functions, we obtain:

$$\begin{aligned} W = &\frac{S_0 T}{2\pi^2\sqrt{1-e^2}}\left\{\sin\phi\sin\varepsilon(-H_0\cos\lambda)_0^\lambda + \sin\varepsilon\, E\left(\psi,k^{-1}\right)\right. \\ &\left.+AF\left(\psi,k^{-1}\right)+\sin\phi\cos\varepsilon\,\Omega_3^*\right\} \end{aligned}$$

$$\text{where} \quad A = \frac{\cos^2\varepsilon\cos^2\phi}{\sin\varepsilon} - \sin\phi\cos\varepsilon\left(E\left(\varepsilon,k^{-1}\right) - \frac{E}{K}F\left(\varepsilon,k^{-1}\right)\right) \tag{41}$$

$$\Omega_3^* = \sum_{j=1}^{\infty}\frac{\sin\left(\frac{j\pi F\left(\varepsilon,k^{-1}\right)}{K}\right)\sin\left(\frac{j\pi F\left(\psi,k^{-1}\right)}{K}\right)}{j\sin h\left(j\pi\frac{K'}{K}\right)}$$

$$k'^* = 1 - \frac{1}{k^2} = \frac{\sin^2\varepsilon - \cos^2\phi}{\sin^2\varepsilon}$$

$$K' = K\left(k'^*\right)$$

Table 1a
Irradiation in 10^6 J m^{-2} for an interval defined in true longitude (0° to 70°) calculated from 1. Elliptic integrals and 2. Sum of daily irradiations.

(0° to 70°) Latitude	Method	
	1.	2.
85	1808.689	1808.690
75	1889.350	1889.349
65	2078.156	2078.156
55	2321.582	2321.582
45	2527.724	2527.724
35	2672.679	2672.679
25	2745.971	2745.971
15	2742.701	2742.701
5	2661.698	2661.698
−5	2504.900	2504.900
−15	2277.071	2277.071
−25	1985.658	1985.658
−35	1640.784	1640.784
−45	1255.600	1255.600
−55	847.883	847.883
−65	447.659	447.659
−75	151.597	15.1596
−85	16.481	16.481

8. Numerical examples

8.1. Properties of the elliptic integrals

The computation of the elliptic integrals are done for

$$0 \le \lambda \le \frac{\pi}{2}$$

This is not restrictive and the integrals over any interval can be made. This is based on the properties of the integrands of the elliptic integrals and on the properties of $E(\lambda,k)$, $F(\lambda,k)$ and $\Pi(\lambda,\sin^2\varepsilon,k)$ themselves:

$$E(-\lambda,k) = -E(\lambda,k) \quad F(-\lambda,k) = -F(\lambda,k)$$

$$E(m\pi\pm\lambda,k) = 2mE \pm E(\lambda,k)$$

$$F(m\pi\pm\lambda,k) = 2mK \pm F(\lambda,k)$$

$$\Pi\left(m\pi\pm\lambda,\sin^2\varepsilon,k\right) = 2m\Pi\left(\sin^2\varepsilon,k\right) \pm \Pi\left(\lambda,\sin^2\varepsilon,k\right)$$

with the following complete elliptic integrals defined for $\lambda = \pi/2$:

$$F\left(\frac{\pi}{2},k\right) = K(k)$$

$$E\left(\frac{\pi}{2},k\right) = E(k)$$

$$\Pi\left(\frac{\pi}{2},\sin^2\varepsilon,k\right) = \Pi\left(\sin^2\varepsilon,k\right)$$

Table 1b
Irradiation in 10^6 J m^{-2} for an interval defined in true longitude (30° to 75°) calculated from 1. Elliptic integrals and 2. Sum of daily irradiations.

(30° to 75°) Latitude	Method	
	1.	2.
85	1653.912	1653.912
55	1722.220	1722.220
0	1631.727	1631.727
−55	362.240	362.240
−85	0.0	0.0

Table 1c
Irradiation in 10^6 J m^{-2} for an interval defined in true longitude (270° to 360°) calculated from 1. Elliptic integrals and 2. Sum of daily irradiations.

(270° to 360°) Latitude	Method	
	1.	2.
85	16.481	16.484
55	944.453	944.453
0	3295.649	3295.649
−55	3184.186	3184.186
−85	2740.286	2740.289

The classical way to calculate numerically the elliptic integrals is given by Bulirsch (1965, 1969a). The basic principle of the iterative procedure is common to the three elliptic integrals (Chapeau, 1976; de Maere d'Aertrycke, 1986). Through a transformation of variables, the elliptic integrals are changed into elliptic integrals of the same kind but with a smaller modulus k, up to $k = 0$ for which their numerical values are straightforward. The algorithm to compute the elliptic integrals of the first kind is based on the descending Landen transformation. The integrals of the second kind are estimated from the transformation by Bartky (1938) after decomposition into integrals of the first kind and complete integrals of the second and third kinds. The integrals of the third kind are calculated from the Theta and Jacobi functions from an extension of the Bartky transformation. According to Bulirsch (1969b), the error relative to all the algorithms used in computing the irradiation is less than 10^{-9}.

Table 2a
Irradiation in 10^6 J m^{-2} for an interval defined in calendar days (21 March to 30 May) calculated from 1. Elliptic integrals and 2. Sum of daily irradiations.

21/3 to 30/5 Latitude	Method	
	1.	2.
85	1709.303	1709.299
75	1796.287	1796.288
65	1992.492	1992.492
55	2236.806	2236.807
45	2443.530	2443.530
35	2590.353	2590.353
25	2667.251	2667.252
15	2669.435	2669.435
5	2595.670	2595.670
−5	2447.716	2447.716
−15	2230.069	2230.069
−25	1949.824	1949.824
−35	1616.662	1616.662
−45	1243.161	1243.161
−55	846.232	846.232
−65	453.963	453.963
−75	156.553	156.553
−85	18.184	18.180

8.2. Time interval given in true longitude of the Sun

For computing the total irradiation received during a time interval defined by the true longitude λ: $\lambda' \leq \lambda < \lambda''$, the formula (31) and (40) using the elliptic integrals can be directly used as their bounds are given in λ. But if we want to sum up the total daily

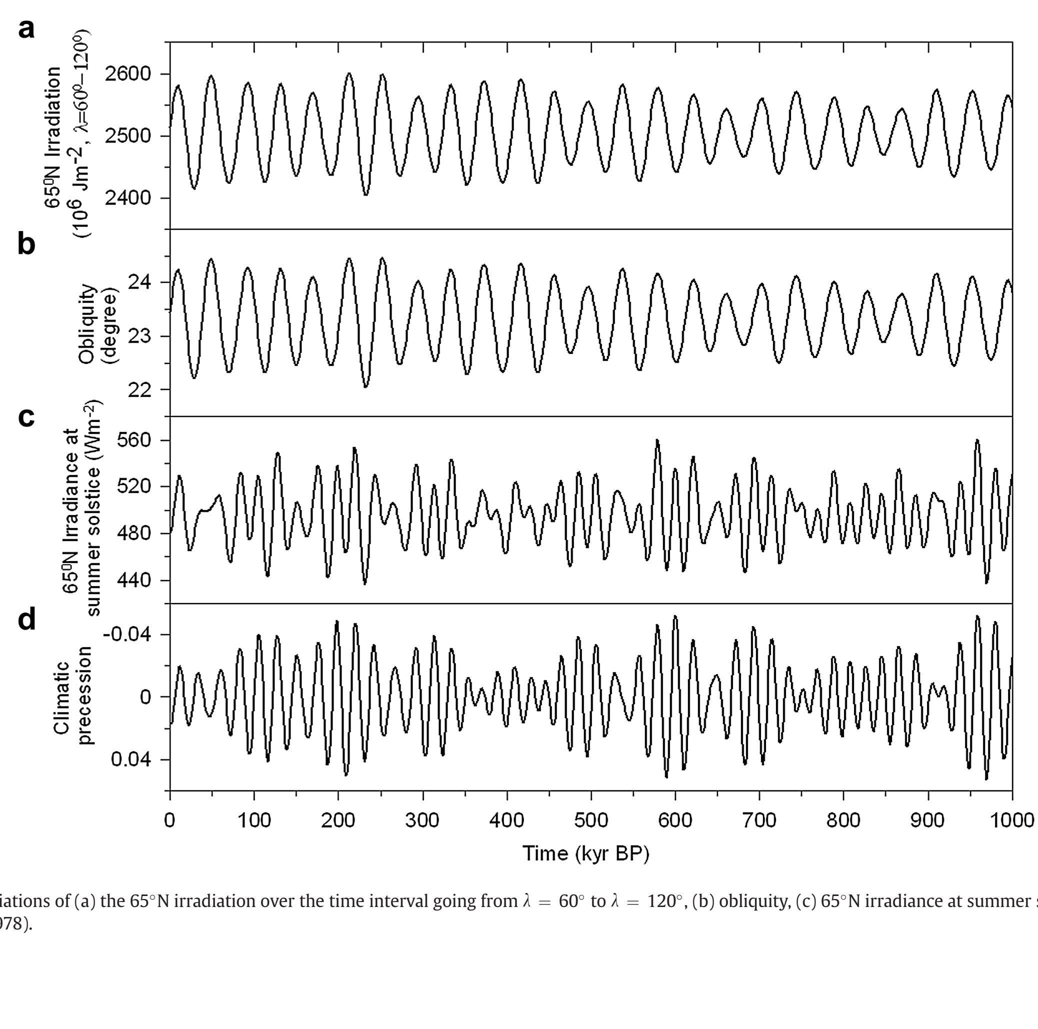

Fig. 2. Long-term variations of (a) the 65°N irradiation over the time interval going from $\lambda = 60°$ to $\lambda = 120°$, (b) obliquity, (c) 65°N irradiance at summer solstice and (d) climatic precession (Berger, 1978).

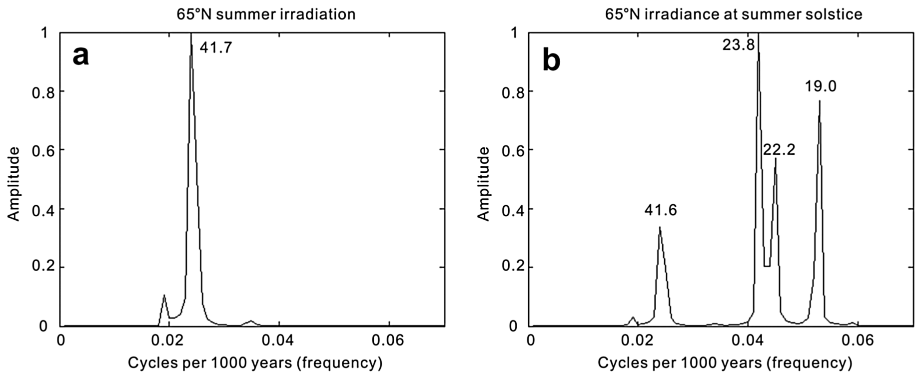

Fig. 3. Fourier spectra of the long-term variations of (a) the 65°N irradiation over the time interval going from $\lambda = 60°$ to $\lambda = 120°$ and (b) the 65°N irradiance at the summer solstice. Periods in kyr are indicated.

irradiation for each day of the interval, (19) has to be used and the bounds in λ must be transformed to be written in terms of time:

1) The bounds in λ, the true longitude, must be converted into the mean longitude,λ_m, using (23).
2) The time in mean solar days elapsed between λ' and λ'' is then given by $L = (\lambda''_m - \lambda'_m)/n$ where $n = 360°/T$ is the mean motion of the Earth or the daily increment of λ_m if T is expressed in mean solar days.
3) Depending upon the latitude, the first and last days might not be fully illuminated. To take this into account, reference must be made to 21 March at 0h ($H = -12h$) by adding half-a-day to the number of days elapsed since 21 March ($H = 0$). For these partly illuminated days, the irradiation is calculated according to (20).
4) For each day i, $(1 \le i \le l+1)$ of the interval, the mean longitude is given by

$$\lambda_{mi} = \lambda'_m + (i-1)n$$

and the corresponding λ_i is calculated through (24) for finally computing $W_{d,i}$ through (18).

5) The total irradiation over the interval $\lambda' < \lambda < \lambda''$ is then given by

$$W(\phi, \lambda', \lambda'') = \sum_{i=1}^{l+1} W_i$$

Table 1a shows a very good agreement between the two techniques used (elliptic integrals or sum of daily irradiations) for the interval $0 \le \lambda \le 70°$ for the present-day with

$$\lambda' = 0, \ \lambda_{mo} = -1°.87$$
$$\lambda'' = 70, \ \lambda_{m,l+1} = 68°.98$$

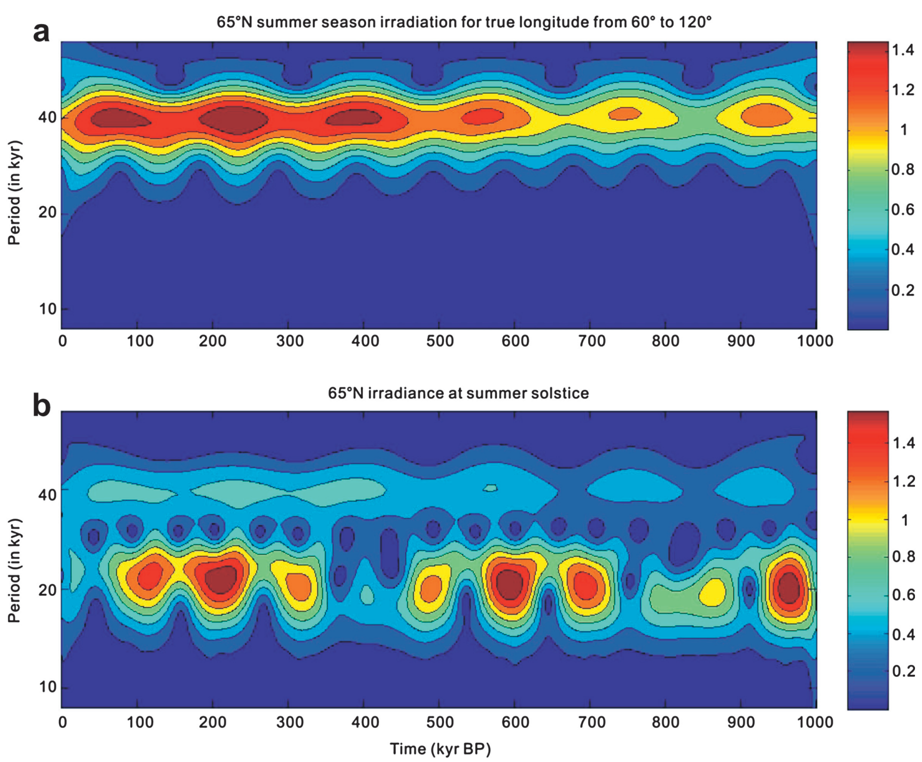

Fig. 4. Amplitude of the continuous wavelet transform of the long-term variations of (a) the 65°N irradiation over the time interval going from $\lambda = 60°$ to $\lambda = 120°$ and (b) the 65°N irradiance at the summer solstice.

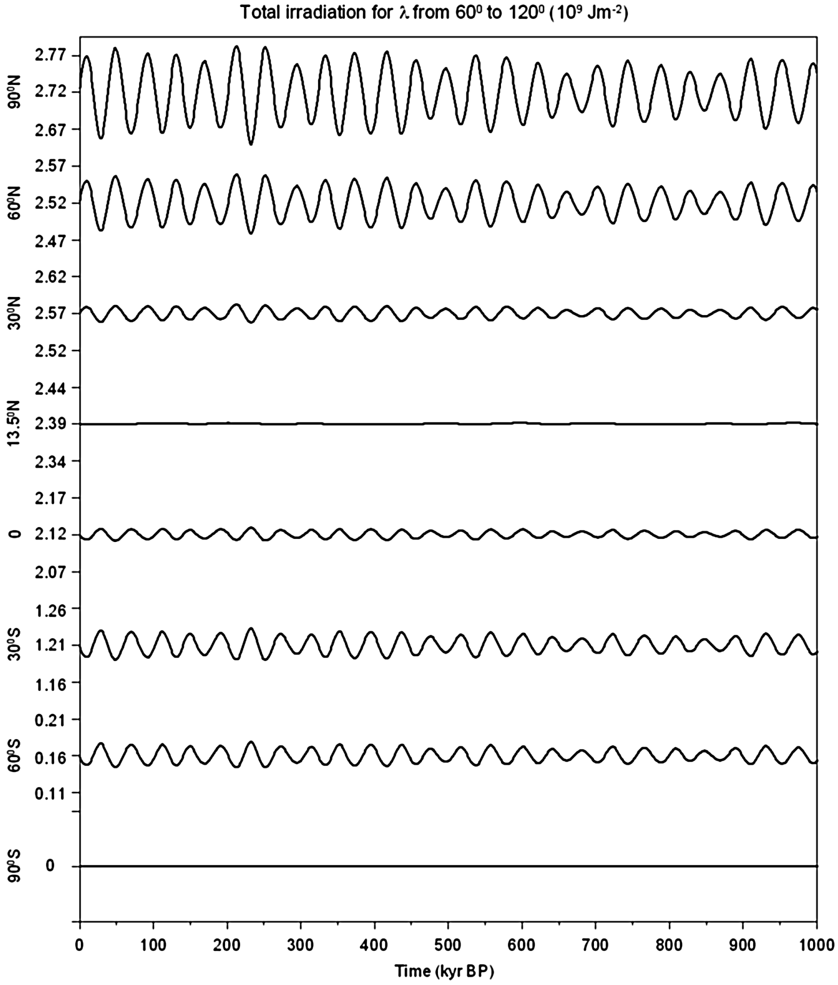

Fig. 5. Long-term variations of the total irradiation (in 10^9 J m^{-2}) over the time interval going from $\lambda = 60°–120°$ for latitudes between the North and South Poles.

$\Delta\lambda_m = 70°.85$ or $L = 71.89$ mean solar days elapsed since 21 March noon ($H = 0$) or 72.39 days since 21 March 0*h*.

The interval starts March 21 and ends June 1 which insolation is given by:

$$W_{l+1} = 0.39{*}W(\lambda_{l+1})$$

Tables 1b and 1c provide other examples, for (30° to 75°) and (270° to 360°), confirming the excellent agreement between the two techniques.

8.3. *Time interval given in calendar days* $[t_1 \le t < t_{l+1}]$

An interval is defined here such as it starts at day t_1 0h ($H = -12h$) and ends at day t_{l+1} 0h. This means that day t_{l+1} is excluded and the interval contains therefore l full days. Remember that the origin of the calendar is prescribed at 21 March for a true longitude equal to zero at noon ($H = 0$) and with the mean longitude,λ_{m0}, calculated for $\lambda = 0$.

If the daily insolations are calculated and summed up, we have to determine for each day of the interval, the number of days, i,

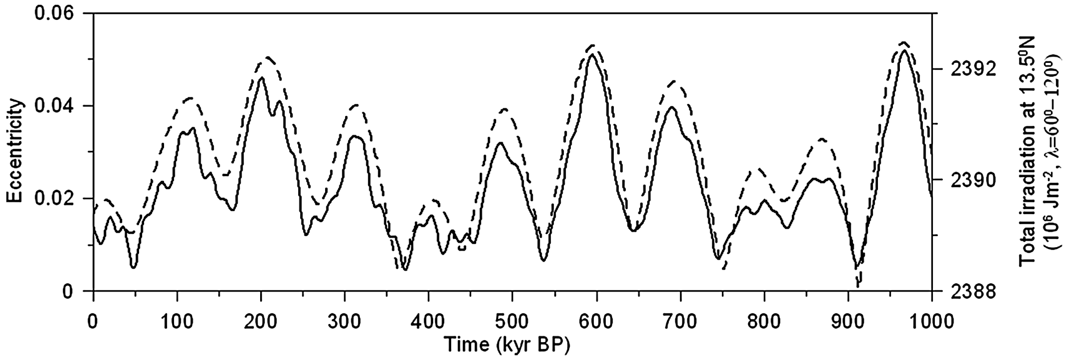

Fig. 6. Long-term variations of eccentricity (dashed line) and of the total irradiation at 13.5°N(full line) over the time interval going from $\lambda = 60°$ to $\lambda = 120°$.

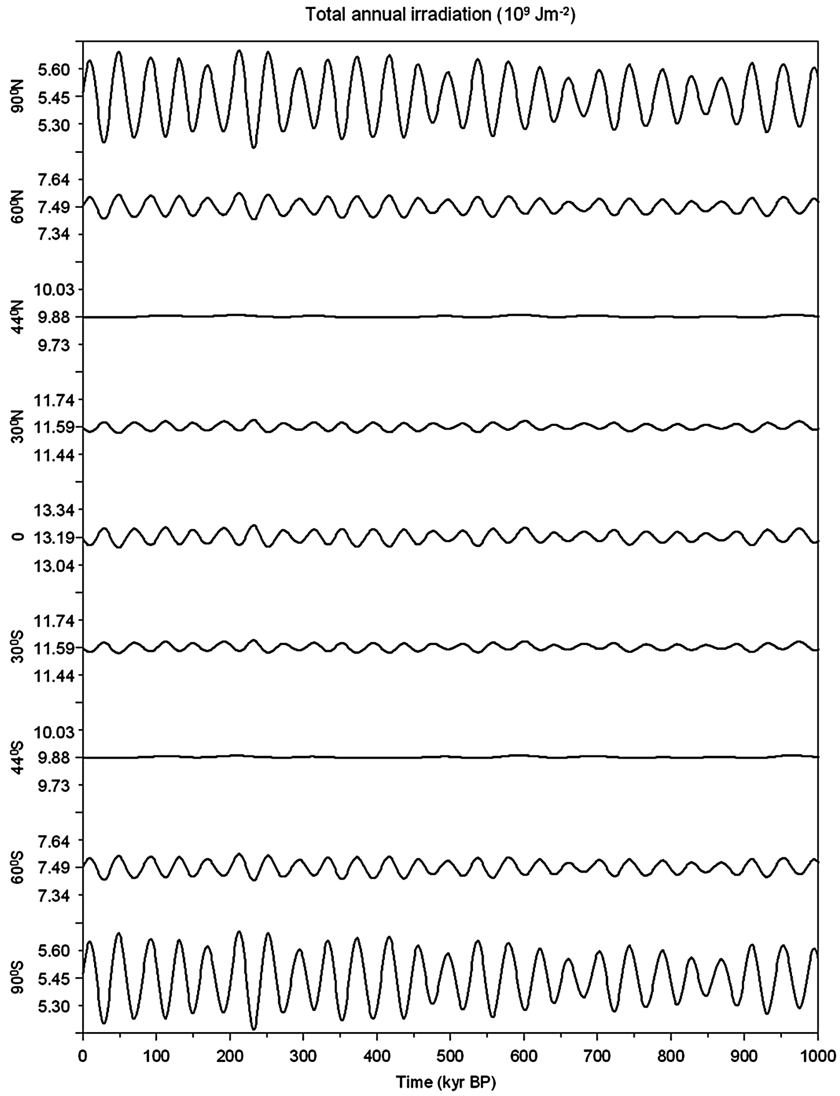

Fig. 7. Long-term variations of the total annual irradiation (in 10^9 J m^{-2}) for latitudes between the North and South Poles.

elapsed since 21 March. For calculating their mean longitude, λ_{mi}, from λ_{m0}, the number of days elapsed since 21 March at noon, $i-0.5$, must be used and we have:

$$\lambda_{mi} = \lambda_{mo} + (i-0.5) \times n$$

Then the corresponding true longitude,λ_i, is calculated through (24) and the irradiation W_i is given by (18) with the total energy $W = \sum_{i=1}^{l} W_i$. Since the sidereal year is equal to 365.2564 days, a fraction, 0.2564, of 24 h is added between 20 and 21 March. The irradiation over this period of time is proportional to the daily irradiation for the day starting 20 March midnight ($H = +12h$).

A more direct way will be to use the elliptic integrals (31). In this case, each bound of the interval expressed in time must be transformed into the true longitude. The mean longitude corresponding to t_1 and t_{l+1} are first calculated from the equation by Kepler, being given the number of days, i and $i+l$ respectively elapsed since the Spring equinox. But, as this equation provides the value of the mean longitude at noon ($H = 0$), and as the daily insolation is calculated

Table 2b
Irradiation in 10^6 J m^{-2} for an interval defined in calendar days (24 April to 30 July) calculated from 1. Elliptic integrals and 2. Sum of daily irradiations.

24/4 to 30/7 Latitude	Method	
	1.	2.
85	3885.474	3885.474
55	3789.089	3789.089
0	3322.148	3322.148
−55	594.138	594.138
−85	0.0	0.0

Table 2c
Irradiation in 10^6 J m^{-2} for an interval defined in calendar days (22 December to 21 March) calculated from 1. Elliptic integrals and 2. Sum of daily irradiations.

22/12 to 21/3 Latitude	Method	
	1.	2.
85	16.879	14.880
55	937.041	937.041
0	3303.076	3303.076
−55	3206.134	3206.134
−85	2774.390	2774.391

from $0h$ ($H = -12h$) to $24h$ ($H = 12h$), one must subtract an angle swept over half-a-day and the boundaries of the integrals in mean solar longitude are:

$$\lambda_{m1} = \lambda_{m0} + i \times n - n/2$$

$$\lambda_{ml+1} = \lambda_{m0} + (i + l) \times n - n/2$$

The corresponding true longitudes are then calculated from (24) and the total irradiance from (31). Table 2a shows the excellent agreement between the two techniques used for the present-day and a time interval (21 March to 30 May). This is confirmed by examples given for two other intervals: (24 April to 30 July) and (22 December to 21 March), spanning different times during the year.

8.4. Spectral properties

In order to show clearly the difference between the classical daily insolation and the irradiation over a given time interval during the year, Fig. 2 gives, for the last 1 million years, (panel a) the 65°N irradiation calculated over a time interval defined in true longitude, centered on the summer solstice ($\lambda = 90°$) and going from $\lambda = 60°$ to $\lambda = 120°$ and (panel c) the 65°N irradiance at the summer solstice.

Obliquity is also represented (Fig. 2, panel b) to show its perfect correlation with the total irradiation. This includes the amplitude modulation (seen on the wavelet spectrum of Fig. 4a) with a decrease in the amplitude of the obliquity variation back in time up to 850 ka BP (this decrease is related to a period of 1.3 million years shown in Berger et al., 1998a,b). This positive correlation between the total irradiation and obliquity (i.e. a large obliquity leading to a large irradiation) is however a characteristic of the latitudes north of 13.5°N for this particular time interval (60°–120°) (Fig. 5). At around 13.5°N, the obliquity signal indeed disappears in the total irradiation and only a pure eccentricity signal is left (but very weak, less than 0.2%, which means that at such latitude and around, the total irradiation is almost constant with time, Fig. 6). South of this latitude, the irradiation is negatively correlated with obliquity, a large obliquity leading to a low irradiation. For other time intervals, including the full year, there is also such specific latitude from where the correlation between total irradiation and obliquity reverses its sign. For the total annual irradiation, however, because of the symmetry between the hemispheres, the obliquity signal vanishes around both 44°N and 44°S. North of 44°N and south of 44°S (Fig. 7), the irradiation is positively correlated with obliquity, but between these latitudes, the correlation reverses its sign. In such a case, the latitudinal gradient between low and high latitudes (e.g. 0° and 60°) is therefore enhanced when the obliquity is weak. In trying to associate long-term variations of proxy climate record to obliquity, the choice of an assumed-to-be sensitive latitude is therefore critical.

In parallel, the spectrum (Fig. 3b) of the 65°N daily irradiance at the summer solstice shows a mix of obliquity and precession, with precession dominating, contrary to the spectrum of the total irradiation which displays a pure obliquity signal (Fig. 3a). The wavelet spectrum (Fig. 4b) of the daily irradiance shows also clearly a precession amplitude modulated by eccentricity with minima at 400 ka BP and 800 ka BP, as well as a weaker obliquity signal fading away progressively towards the past.

These spectra show that the amplitude of the variation of the total irradiation and of the daily irradiance (and therefore of obliquity and precession) are fading away progressively back in time, almost vanishing completely together around 850 ka BP. This is also the time of the Mid-Pleistocene Transition when the spectrum of climate proxy records changes (e.g. Berger et al., 1999), a coincidence which deserves more investigations.

9. Conclusions

The total irradiation available during intervals of time over the year (and therefore obliquity) becomes more and more popular in addition to the daily insolation which is used to force the general circulation models (Drysdale et al., 2009; Paillard, 2010). Although the calculation of the daily irradiance and of its long-term variations is well known (Milankovitch, 1941 and improvements in the astronomical elements in Berger, 1978; Berger and Loutre, 1991; Laskar et al., 2004), the calculation of total irradiation recommended in these papers needed to be confirmed by a more accurate and straightforward technique. The definite model for this, based on elliptic integrals, is developed in this paper and its accuracy demonstrated. In both examples dealing with bounds of selected time intervals defined either in calendar days or in true longitude, the difference between the two methods used (the sum of the daily irradiations or the elliptic integrals, Tables 1 and 2) is very small, amounting to less than 0.01 W m^{-2}. In the method using the sum of the daily irradiations, ρ and δ are assumed to remain constant over the whole day. In the method using the elliptic integrals, they are assumed to vary continuously but the daily irradiation is spread all over 24 h (not only from sunrise to sunset, but also over the night).

Spectral analyses of the daily insolation and of the total irradiation confirm earlier findings (Berger and Pestiaux, 1984; Milankovitch, 1941) of their total different behaviour, dominated by precession for the daily insolation and almost exclusively by obliquity for the total irradiation. In addition, wavelet spectra show that their amplitude variation almost vanishes together at around the Mid-Pleistocene Transition, a key turn towards the climate of the Late Pleistocene.

The advantage of the elliptic integrals is to provide an analytical expression of the total irradiation. This allows to show directly the properties of symmetry or to discuss the relative influence of the different parameters involved.

A computer programme for calculating the total irradiation over any period of time using the elliptic integrals written in Fortran is made available on ftp://ftp.astr.ucl.ac.be/pub/berger/ellipticintegrals.

Acknowledgements

This work is supported by the European Research Council Advanced Grant EMIS (N°227348 of the Programme “Ideas”). Q.Z. Yin is postdoctoral follow of Belgian National Fund for Scientific Research (F.R.S-FNRS). Spectrum programs are kindly provided by J.L. Mélice.

Appendix A

The mean distance r_m, from the Earth to the Sun – which defines the so-called solar constant S_0 – is not a constant. r_m is indeed defined in such a way that the area (πr_m^2) of the circle with radius r_m is equal to the area of the ellipse (πab). Therefore:

$$r_m^2 = a^2\sqrt{1 - e^2}$$

and consequently r_m depends upon e.

To avoid this problem, we will actually consider the energy received by the Earth from the Sun, S_a, as measured at the distance of the semi-major axis of the ecliptic, a, because, according to J.L. Lagrange (1736–1813), P.S. Laplace (1749–1827) and S.D. Poisson (1781–1840), the variation of a, up to the second order in the mass, is only periodic with a constant amplitude (see also Laskar et al., 2004: Fig. 11). From this, the total energy received by the Earth over one year of length T is equal to $W_E^T = S_0 \pi R^2 T = (S_a/\sqrt{1-e^2}) \pi R^2 T$.

Appendix B

The vernal point, γ, is the direction in which the Sun is seen in the sky from the Earth at the Spring (March) equinox. It is the origin from which the ecliptical longitude is measured in celestial mechanics. In a heliocentric system, γ coincides with the position of the Earth at the fall (September) equinox. According to this, the longitude of the perihelion, $\tilde{\omega}$, calculated in Berger (1978), has a present-day value of 102°.04. In a geocentric system, γ coincides with the Spring equinox and the longitude of the perigee, ω, is equal to $\tilde{\omega} + 180°$. This ω value is used in the insolation calculation because the longitudes are measured in this case from the Spring equinox.

Appendix C

The equation of time, *ET*, expresses the difference between the true solar time and the mean solar time. It is independent of the latitude and is only a function of the date. Presently its minimum value is −14 min reached on 11 February and its maximum is reached on 3 November with +16 min. The mean solar time is the hour angle of the mean Sun. The mean Sun is a virtual Sun which revolves around the Earth at a constant speed on an orbit which is assumed to be circular and lying in the equatorial plane. It coincides with the true Sun at the Perihelion (presently around 3 January). The mean motion n is equal to $2\pi/T$, T being the sidereal year (Appendix D).

Appendix D

The sidereal year is the time elapsed between two successive passages of the Sun through the same point in the sky. It is equal to 365.25636 mean solar days. The tropical year is the time elapsed between two successive passages of the Sun through the vernal point. It is presently 365.24219876 mean solar days. Because the vernal point precesses (it was advancing westwards in 1900 by 50″1564 in a tropical year), the tropical year is correspondingly shorter than the sideral year. In everyday life, we use the civil year which is equal to $365.2425 = 365 + (1/4) - (3/400)$ mean solar days corresponding to the intercalary prescription of the Gregorian calendar introduced in 1582 by Pope Gregory XIII. After 3 years of 365 days, follows a leap-year (divisible by 4) of 366 days except for the years of the centuries not divisible by 400.

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