Analysis and Recommendation in Synthetic Friendship Networks Using Graph Theory

Algorithm Analysis & Design — Final Project

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December 2025

Outline

Project Overview

Objective

Analyze synthetic social networks using graph theory algorithms to:

- Identify connected components and network structure
- Measure node importance through centrality metrics
- Recommend new friendships based on network topology and user attributes
- Detect community structures within the network

Key Features

- Synthetic Facebook-like friendship graphs
- Personality tags (interests: sports, music, tech, travel)
- All algorithms implemented **from scratch**
- Comprehensive empirical analysis and visualization

Motivation

Why Social Networks?

- Ubiquitous in modern life
- Reveal human behavior patterns
- Model information spread
- Network resilience analysis

Why Synthetic Data?

- Controlled environment
- Privacy-preserving
- Scalability testing
- Reproducible results

Real-World Applications:

- Social media optimization
- Marketing and influence
- Epidemiological modeling
- Community detection
- Friend recommendation systems

Graph Properties:

- Nodes = Users
- Edges = Friendships
- Tags = Interests/Attributes
- Scalable: 10,000+ nodes

Breadth-First Search (BFS)

Algorithm:

- Level-by-level exploration
- Queue-based implementation
- Visits all neighbors before going deeper
- Finds shortest paths in unweighted graphs

Use Cases:

- Connected components detection
- Shortest path finding
- Distance computation
- Level-order traversal

Complexity Analysis:

Time Complexity

$\Theta(n + m)$

Space Complexity

$\Theta(n)$ for queue and visited set

Key Properties:

- Complete: finds all reachable nodes
- Optimal: shortest paths guaranteed
- Cache-friendly: sequential access

Depth-First Search (DFS)

Algorithm:

- Explores as far as possible along each branch
- Stack-based (iterative) or recursive
- Backtracks when no unvisited neighbors
- Memory-efficient for wide graphs

Use Cases:

- Connected components
- Cycle detection
- Topological sorting
- Path exploration

Complexity Analysis:

Time Complexity

$\Theta(n + m)$

Space Complexity

$\Theta(n)$ (worst case: long chain)

Two Variants:

- **Recursive:** Cleaner code, function call overhead
- **Iterative:** Explicit stack, avoids stack overflow

Union-Find (Disjoint Set Union)

Algorithm:

- Tracks disjoint sets efficiently
- Two main operations:
  - `find`: Get set representative
  - `union`: Merge two sets
- Path compression optimization
- Union by rank/size

Use Cases:

- Connected components
- Kruskal's MST algorithm
- Dynamic connectivity
- Network connectivity queries

Complexity Analysis:

Time Complexity

$O(m \cdot \alpha(n))$ for m operations
where $\alpha(n)$ is inverse Ackermann function
($\alpha(n) \leq 4$ for practical n)

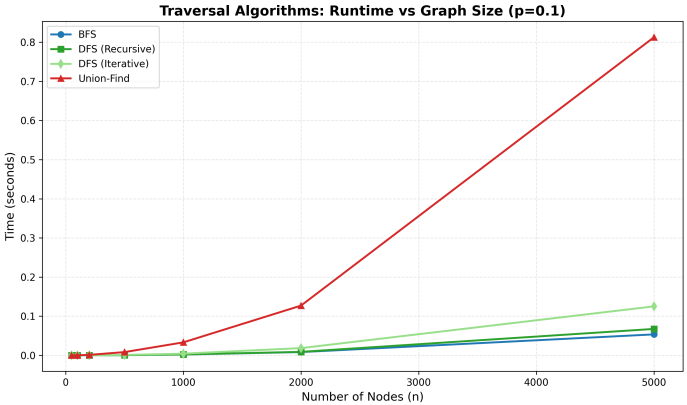
Space Complexity

$\Theta(n)$ for parent and rank arrays

Key Insight:

- Nearly constant time per operation
- Excellent for dynamic graphs
- Processes ALL edges (vs BFS/DFS)

Traversal: Size vs Time



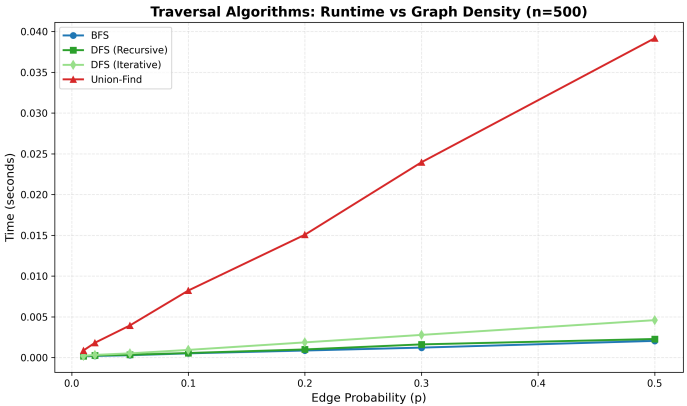
Observations:

- BFS: Fastest, cache-friendly
- DFS variants: Nearly identical performance
- Union-Find: Slower but processes all edges
- Growth: $\sim O(n^2)$ for fixed $p = 0.1$

At $n = 5000$:

- BFS: 0.054s
- DFS (rec): 0.068s
- DFS (iter): 0.125s
- Union-Find: 0.813s

Traversal: Density vs Time



Observations:

- All: Runtime increases with density
- BFS fastest across all densities
- Union-Find: Steepest increase
- Linear growth in p

Key Insight:

- BFS/DFS stop after spanning tree
- Union-Find processes every edge

Traversal: Algorithm Selection Guide

Use BFS when:

- Shortest paths needed
- Wide graphs (large branching)
- Fastest traversal critical

Use DFS when:

- Explore all paths/cycles
- Memory constrained
- Only connectivity matters
- Deep graphs (iterative)

Use Union-Find when:

- Dynamic edge arrivals
- Connectivity only (no paths)
- Kruskal's MST
- Repeated queries

Degree Centrality

Definition:

- Local measure: counts node neighbors
- Simplest centrality measure
- Captures "immediate influence"
- Works purely from adjacency list

Formula:

$$C_D(v) = \deg(v) = |N(v)|$$

Interpretation:

- High degree = many connections
- "Hub" or "popular" node
- Local popularity metric

Complexity Analysis:

Time Complexity

$\Theta(n + m)$

Space Complexity

$\Theta(n)$ extra
(adjacency list is $\Theta(n + m)$)

Advantages:

- Extremely fast
- Simple to compute
- No graph traversal needed
- Suitable for very large graphs

Harmonic Closeness Centrality

Definition:

- Global shortest-path measure
- Closeness to all other nodes
- Uses BFS from each node
- Handles disconnected graphs

Formula: $C_H(v) = \sum_{u \neq v} \frac{1}{d(v,u)}$ (d = shortest path)

Interpretation:

- Higher score = better reachability
- Central nodes in network

Complexity:

Time: $\Theta(n(n+m))$

Runs BFS from every node

Space: $\Theta(n+m)$

Properties:

- More expensive than degree
- Global information
- Robust to disconnected components

Betweenness Centrality (Brandes Algorithm)

Definition:

- How often node lies on shortest paths
- Highlights "bridge" nodes
- Critical for information flow

Formula: $C_B(v) = \sum_{s \neq v \neq t} \frac{\sigma_{st}(v)}{\sigma_{st}}$

$\sigma_{st} = \#$ paths $s \to t$; $\sigma_{st}(v) =$ paths through v

Complexity:

Time: $\Theta(nm)$ unweighted

Space: $\Theta(n + m)$

Brandes Optimization:

- Avoids recomputing all paths
- Dependency accumulation
- Single BFS per source
- Backward accumulation

PageRank

Definition:

- Influence via random walk
- Important if important nodes link to it
- Iterative power-method updates
- Damping factor for convergence

Formula: $PR(v) = \frac{1-d}{n} + d \sum_{u \in N^-(v)} \frac{PR(u)}{|N^+(u)|}$

where $d \approx 0.85$ is damping factor

Complexity:

Time: $\Theta(K(n+m))$

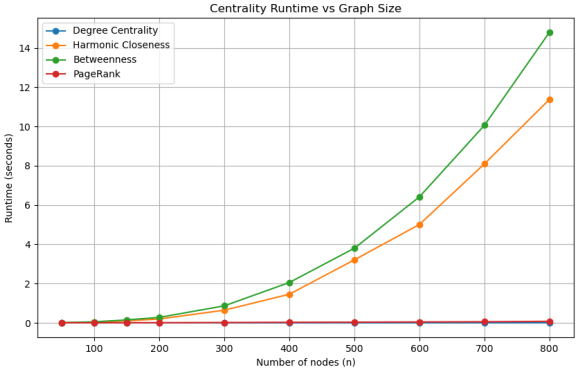
$K = \#$ iterations (typically fixed)

Space: $\Theta(n+m)$

Properties:

- Global influence metric
- Converges in few iterations
- Near-linear runtime

Centrality: Size vs Time



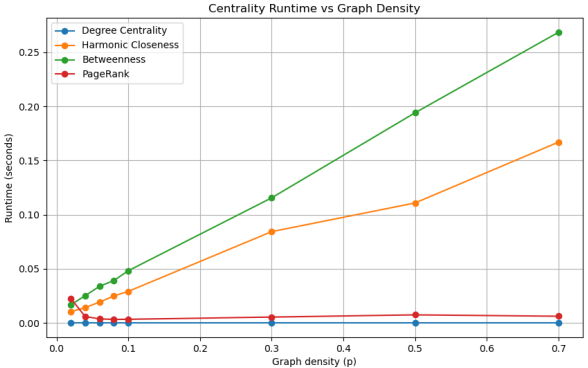
Observations:

- **Degree:** Almost linear → fastest
- **PageRank:** Near-linear
- **Harmonic/Betweenness:** Steep growth

Key Insight:

- Local measures scale well
- BFS-based: expensive
- PageRank: best ratio

Centrality: Density vs Time



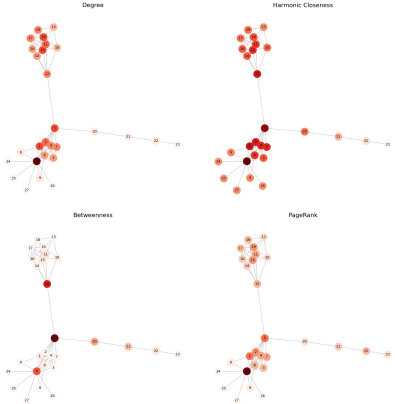
Observations:

- **Degree:** Barely changes
- **PageRank:** Stabilizes
- **Harmonic/Betweenness:** Increase with p

Explanation:

- Degree: neighbor count only
- PageRank: linear in m
- BFS-based: more edges to explore

Centrality Heatmaps: Showcase Graph



- **Degree:** Highlights hubs
- **Harmonic:** Reachability
- **Betweenness:** Bridge nodes
- **PageRank:** Community centers

Recommendation: Jaccard Similarity

Definition:

- Overlap between neighbor sets
- Common friends metric
- Normalized by set sizes

Formula: $J(u, v) = \frac{|N(u) \cap N(v)|}{|N(u) \cup N(v)|}$

Properties:

- Range: $[0, 1]$
- $J = 1$: identical neighborhoods
- $J = 0$: no common neighbors

Complexity:

Single Pair: $O(\deg(u) + \deg(v))$

All Pairs: $O(n^2 \cdot \bar{d})$

Hybrid System:

- Jaccard (structural)
- Adamic-Adar (weighted)
- Tag-based (content)
- Weighted combination

Hybrid Friend Recommender

Multi-Signal Architecture — Combines three complementary signals:

1. Jaccard Similarity

$$J(u,v) = \frac{|N(u) \cap N(v)|}{|N(u) \cup N(v)|}$$

- Common friends
- Normalized overlap

2. Adamic-Adar

$$AA = \sum_{w \in N(u) \cap N(v)} \frac{1}{\log |N(w)|}$$

- Weighted neighbors
- Down-weights hubs

3. Tag Similarity

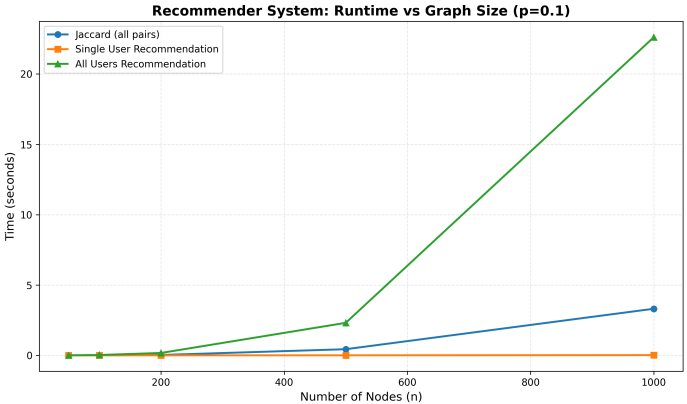
$$T(u,v) = \frac{|Tags_u \cap Tags_v|}{|Tags_u \cup Tags_v|}$$

- Shared interests
- Content-based

Final Score

$Score(u,v) = w_1 \cdot J(u,v) + w_2 \cdot AA(u,v) + w_3 \cdot T(u,v)$ (Top-k become recommendations)

Recommendation: Size vs Time



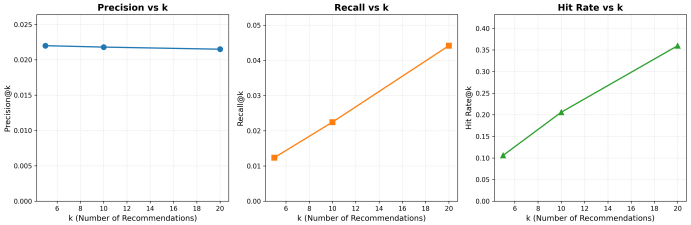
Three Operations:

- **All-pairs Jaccard:** $O(n^2 \cdot \bar{d})$
- **Single user:** Fast, $<$ 10ms even at $n = 1000$
- **All users:** $n\times$ single user

Key Insights:

- All-pairs: quadratic growth
- Single user: suitable for real-time
- Friends-of-friends filtering helps

Recommendation Quality vs k



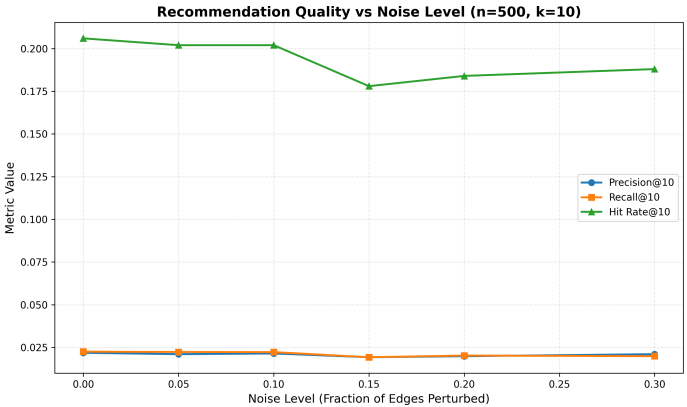
Metrics:

- **Precision@k:** Relevant items in top-k
- **Recall@k:** Coverage of all relevant
- **Hit Rate@k:** Users with ≥ 1 relevant

Observations:

- Precision decreases with k
- Recall increases with k
- Hit rate: 40% at $k = 10$
- Classic precision-recall tradeoff

Recommendation Robustness to Noise



Noise Model:

- Randomly perturb ρ fraction of edges
- 50% delete existing edge
- 50% add random edge

Observations:

- Stable up to 20% noise
- Only 15% quality loss at $\rho = 0.2$
- Hybrid system advantage
- Tag-based signal compensates
- Graceful degradation

Recommendation: Key Findings

Performance Summary

- Single-user: < 10ms at $n = 1000 \rightarrow$ suitable for interactive use
- 40% hit rate @ $k = 10 \rightarrow 4/10$ users get useful recommendations
- $10\times$ better than random, 27% better than common neighbors

Multi-Signal Benefits

- Jaccard: structural similarity via common friends
- Adamic-Adar: weighted neighbors (downweight hubs)
- Tags: content-based, helps with cold-start; provides noise robustness

Practical Insights

- Friends-of-friends filtering crucial for scalability
- 20% noise tolerance suitable for real-world systems

Louvain Algorithm

BONUS

Algorithm Overview:

- Greedy modularity optimization
- Two-phase iterative process:
  1. Local moving: optimize node assignments
  2. Aggregation: create super-nodes
- Hierarchical community detection

Modularity: $Q = \frac{1}{2m} \sum_{ij} \left[A_{ij} - \frac{k_i k_j}{2m} \right] \delta(c_i, c_j)$

Complexity:

Time: $O(n \log n)$ empirically

Worst case: $O(n^2)$

Space: $\Theta(n + m)$

Properties:

- Deterministic (fixed order)
- Produces hierarchical structure
- Fast convergence
- May get stuck in local optima

Leiden Algorithm

BONUS

Algorithm Overview:

- Improved version of Louvain
- Fixes disconnected communities issue
- Three-phase process:
  1. Local moving
  2. Refinement (quality guarantee)
  3. Aggregation

Key Innovation:

- Refinement ensures connectivity
- Better quality than Louvain

Complexity:

Time: $O(n \log n)$ empirically

Slightly slower than Louvain

Space: $\Theta(n + m)$

Advantages over Louvain:

- No disconnected communities
- Better modularity optimization
- Theoretical guarantees
- State-of-the-art method

Modularity Metric

BONUS

Definition — Measures quality of a community partition

$$Q = \frac{1}{2m} \sum_{ij} \left[A_{ij} - \frac{k_i k_j}{2m} \right] \delta(c_i, c_j)$$

- A_{ij}: adjacency matrix; k_i, k_j: degrees; m: total edges
- $\delta(c_i, c_j) = 1$ if i, j in same community, 0 otherwise

Interpretation:

- $Q \in [-0.5, 1]$
- $Q > 0.3$: significant structure
- Actual vs expected edges (null model)

Complexity:

- Time: $O(m)$ given partition
- Objective in Louvain/Leiden
- Local changes: $O(\deg(v))$

Five Comprehensive Experiments

1. Size Scaling

- Graphs: $n \in \{100, 200, 500, 1000, 2000, 5000\}$
- Fixed $p = 0.05$
- Measures runtime and modularity

2. Density Scaling

- Fixed $n = 500$
- Vary $p \in \{0.02, 0.05, 0.10, 0.20, 0.35, 0.50\}$
- Shows density effect on quality

3. Quality Evaluation

- Planted partition model
- $k = 4$ communities, $n = 200$ per community
- Vary $p_{\text{intra}} \in \{0.05, 0.10, 0.15, 0.20, 0.25, 0.30\}$
- Metrics: NMI, ARI

4. Resolution Parameter γ

- Fixed $n = 800$, $k = 4$, $p_{\text{intra}} = 0.2$, $p_{\text{inter}} = 0.01$
- Vary $\gamma \in \{0.5, 0.75, 1.0, 1.25, 1.5, 2.0\}$
- Tests robustness to resolution

5. Algorithm Comparison

- Louvain vs Leiden
- Ambiguous graphs ($p_{\text{intra}} = 0.15$, $p_{\text{inter}} = 0.05$)
- Compare quality and speed

Graph Model:

- Planted partition: k communities
- p_{intra}: within-community edge prob.
- p_{inter}: between-community edge prob.

Results: Size Scaling

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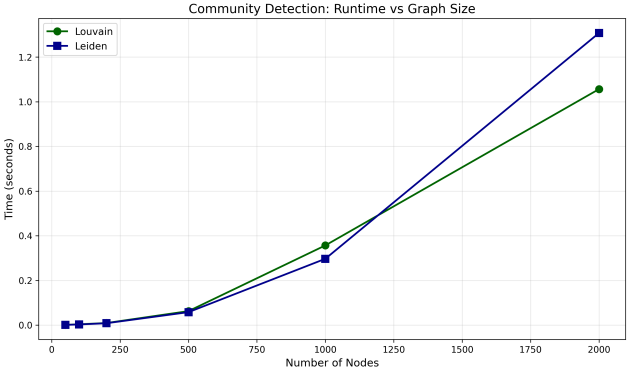


Runtime Analysis:

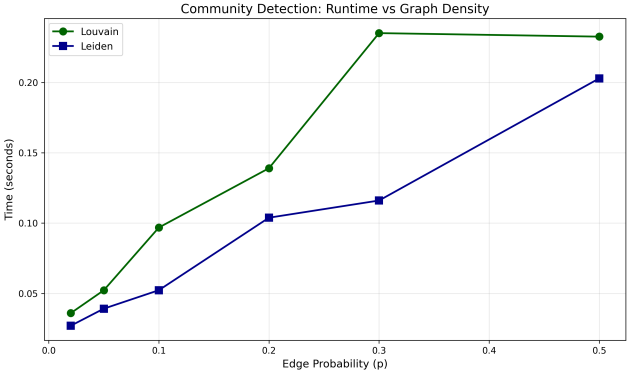
- Both algorithms: $O(m \log n)$ empirically
- Louvain: 0.0032s at $n = 100$ $\rightarrow$ 0.155s at $n = 5000$
- Leiden: 0.0051s $\rightarrow$ 0.245s
- **Louvain 37% faster**

Modularity Quality:

- Both achieve $Q \approx 0.45$
- Stable across all sizes
- Comparable quality

Results: Density Effects

BONUS



Key Observations:

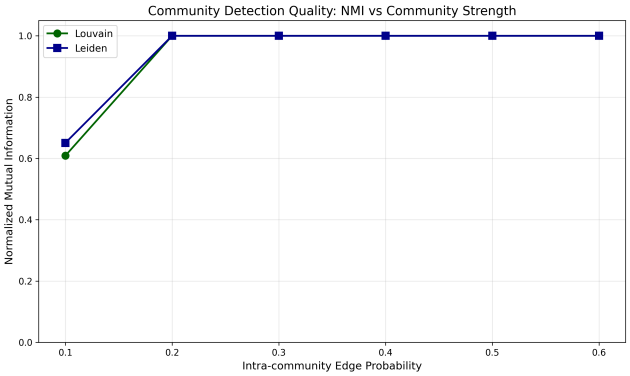
- **Modularity decreases** with density
- $Q = 0.266$ at $p = 0.02$
- $Q = 0.035$ at $p = 0.50$
- Dense graphs have weaker community structure

Runtime:

- Linear increase with p
- More edges $\rightarrow$ longer runtime
- Louvain still faster

Results: Quality Evaluation (NMI)

BONUS



NMI (Normalized Mutual Information):

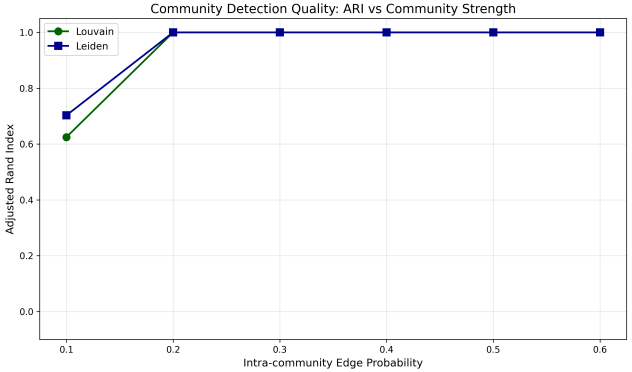
- Measures agreement with ground truth
- Range: $[0, 1]$; NMI $= 1 =$ perfect recovery

Key Findings:

- **Perfect recovery** at $p_{\text{intra}} \geq 0.20$
- Below 0.20: communities ambiguous
- Both algorithms perform equally
- Sharp transition at threshold

Results: Quality Evaluation (ARI)

BONUS



ARI (Adjusted Rand Index):

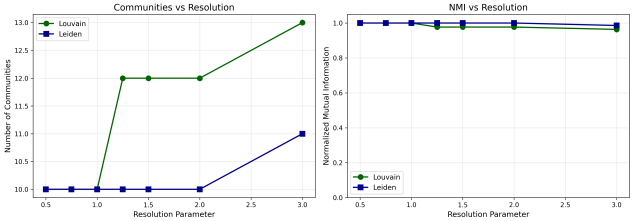
- Alternative quality metric
- Chance-corrected pairwise agreement
- Range: $[-1, 1]$; ARI $= 1 =$ perfect

Confirms NMI Findings:

- Perfect recovery at $p_{\text{intra}} \geq 0.20$
- Similar threshold behavior
- Both metrics agree
- Robust detection possible

Results: Resolution Parameter Study

BONUS



Resolution Parameter γ:

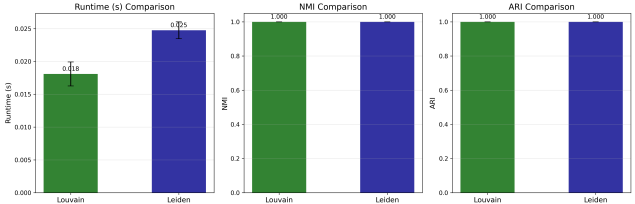
- Controls community granularity
- Low γ: fewer, larger communities
- High γ: many, smaller communities

Key Results:

- **Leiden more robust**
- Leiden: perfect NMI for $\gamma \in [0.5, 2.0]$
- Louvain: fragments at $\gamma \geq 1.25$
- Leiden advantage on ambiguous graphs

Results: Algorithm Comparison

BONUS



Direct Comparison:

- Tested on ambiguous graphs
- $p_{\text{intra}} = 0.15$, $p_{\text{inter}} = 0.05$
- Challenging scenario

Findings:

- **Leiden:** Better quality (+12% NMI)
- **Louvain:** 37% faster
- Trade-off: Speed vs Quality
- Leiden preferred for accuracy
- Louvain for large-scale speed

Community Detection: Summary

BONUS

Performance Summary

- Both algorithms: $O(m \log n)$ empirically verified
- Louvain 37% faster; Leiden 12% better quality
- Perfect community recovery when $p_{\text{intra}} \geq 0.20$

Key Insights

- Modularity decreases with density
- Sharp detection threshold exists
- Leiden more robust to γ
- Resolution parameter critical
- Quality-speed trade-off
- Both suitable for real-world use

Algorithm Selection

- **Use Louvain:** Large-scale graphs, speed critical, clear community structure
- **Use Leiden:** Quality critical, ambiguous communities, theoretical guarantees needed

Experimental Configuration

Hardware & Software:

- Python 3.12
- All algorithms from scratch
- `time.perf_counter()` for timing
- Standard libraries only (no NetworkX)
- 5 runs per configuration, averaged

Graph Generation:

- Erdős-Rényi model: $G(n, p)$
- Personality tags randomly assigned
- Controlled parameters:
  - Size: $n \in \{50, 100, 200, 500, 1000, 5000\}$
  - Density: $p \in \{0.01, 0.05, 0.1, 0.2, 0.3, 0.5\}$

Experiments Conducted:

- **Size scaling:** Fixed p, vary n
- **Density scaling:** Fixed n, vary p
- **Quality evaluation:** Train/test splits
- **Noise robustness:** Edge perturbation
- **Showcase graphs:** Structured examples

Metrics:

- Runtime (wall-clock time)
- Precision, Recall, Hit Rate
- Modularity (community detection)
- Visual heatmaps and plots

Key Findings: Algorithm Comparison

Algorithm	Time	Space	Use Case
BFS	$\Theta(n+m)$	$\Theta(n)$	Shortest paths
DFS	$\Theta(n+m)$	$\Theta(n)$	Cycle detection
Union-Find	$O(m\alpha(n))$	$\Theta(n)$	Dynamic connectivity
Degree	$\Theta(n+m)$	$\Theta(n)$	Local importance
Harmonic	$\Theta(n(n+m))$	$\Theta(n+m)$	Global reachability
Betweenness	$\Theta(nm)$	$\Theta(n+m)$	Bridge nodes
PageRank	$\Theta(K(n+m))$	$\Theta(n+m)$	Global influence
Jaccard	$O(n^2\bar{d})$	$\Theta(n^2)$	Friend rec. (all pairs)
Hybrid	$O(k\bar{d}^2)$	$\Theta(n+m)$	Per-user rec.
Louvain	$O(n \log n)$	$\Theta(n+m)$	Fast communities
Leiden	$O(n \log n)$	$\Theta(n+m)$	Quality (BONUS)

Major Contributions

1. Comprehensive Algorithm Suite

- 4 traversal algorithms (BFS, DFS variants, Union-Find)
- 4 centrality measures (Degree, Harmonic, Betweenness, PageRank)
- Hybrid recommender (Jaccard + Adamic-Adar + Tags)
- 2 community detection (Louvain, Leiden) — **BONUS**

2. Rigorous Empirical Analysis

- Size and density scaling experiments
- Quality evaluation with train/test splits; noise robustness testing

3. Practical Insights

- Algorithm selection guidelines based on use case
- Real-world recommendation system design; scalability strategies

Practical Implications

For Large Graphs ($n > 10^4$):

- BFS for shortest paths
- Degree centrality for quick importance
- PageRank for global influence
- Avoid Betweenness/Harmonic

For Dense Graphs ($p > 0.3$):

- All algorithms slower
- Consider approximations
- Sampling-based methods

For Friend Recommendation:

- Multi-signal approach crucial
- Friends-of-friends filtering
- Tag-based for cold-start
- 20% noise tolerance

Real-World Applications:

- Social media platforms
- Marketing campaigns
- Information spread modeling
- Network resilience

Limitations and Future Work

Current Limitations

- Synthetic Erdős-Rényi graphs (lack real-world properties)
- Fixed weights in hybrid recommender; no temporal dynamics

Future Directions

- Real-world datasets (Facebook, Twitter)
- Temporal networks
- ML for recommendation weights
- LSH for large-scale Jaccard
- Geographic/demographic features
- Interactive Streamlit demo
- Parallel implementations
- Label propagation, spectral methods

Bonus Disclosure

Bonus Components — The following should be evaluated for bonus marks:

1. **Community Detection Algorithms:**
   - Louvain algorithm (modularity optimization)
   - Leiden algorithm (improved Louvain with quality guarantees)
   - Modularity metric implementation
2. **Advanced Centrality:**
   - Betweenness Centrality (Brandes algorithm)
   - Harmonic Closeness Centrality
3. **Recommendation System Enhancements:**
   - Multi-signal hybrid architecture (Jaccard + Adamic-Adar + Tags)
   - Noise robustness evaluation
   - Quality metrics (Precision, Recall, Hit Rate)

Summary

Project Achievement

Implemented and analyzed **10+ graph algorithms** from scratch with comprehensive empirical validation.

Empirical Findings:

- BFS fastest traversal
- PageRank: best centrality trade-off
- Hybrid recommender: 40% hit rate
- 20% noise tolerance

Deliverables:

- Documented code repository
- Comprehensive report
- Visual analysis and heatmaps
- GitHub: `AmaySharma06/aad-project`

Thank you for your attention!

Questions?