STATISTICS AND DISCRETE MATHEMATICS

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Date: 05/10/2024

Unit-1

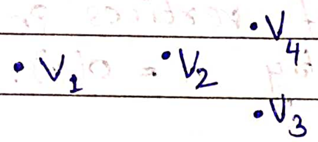
GRAPH THEORY

1) **Graph** :- An ordered pair of vertex and edges set.

$G = (V, E)$ V = non-empty set
E = empty / non-empty set

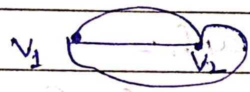
Ex:- a) 

$G = \{(V_1, V_2), V_1V_2\}$

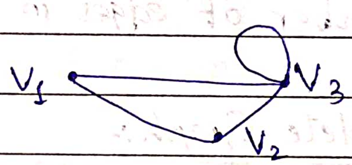
b) Null graph :- 

Edges :-
1) Simple :- 

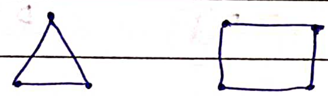
2) Multiple :- 

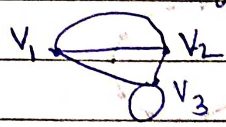
c) Multigraph :- 

Multiple edges

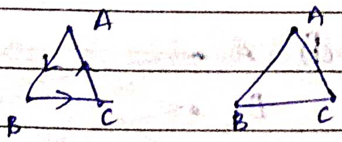
d) Self loop :- 

e) Simple graph :- No self-loops / multiple edges.

 (All polygons)

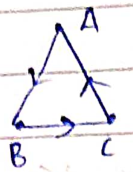
f) General graph :- It contains self-loops as well as multiple edges. 

g) Directed and un-directed graphs :-



Directed Undirected

1) Labelled and Unlabelled Graph:-



Labelled

Unlabelled

2) Order and Degree of a Graph:-

a) Order of a Graph :-

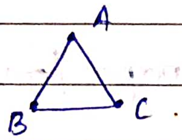
Number of vertices in a graph = $|V|$
Denoted by $n = O(V)$.

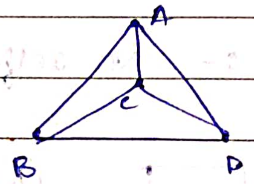
b) Size of a graph:-

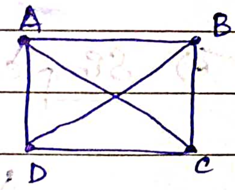
$m = |E|$.
Number of edges in G.

c) Complete Graph:-

$O(V) \geq 2$

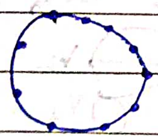






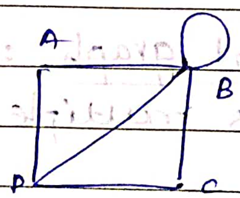
Exercises:-

a)



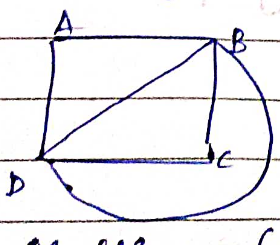
Simple graph

b)



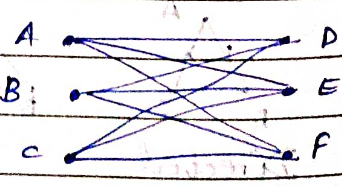
General graph

c)



Multi-graph

d)



d) Complete - Bi-partite Graph:-

Bi-partite Graph

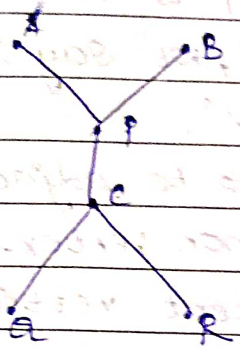
$G = (V_1, V_2)$

$V = V_1$ disjoint V_2

$V = \{A, B, C, D, E, F\}$

$V_1 = \{A B C\}$

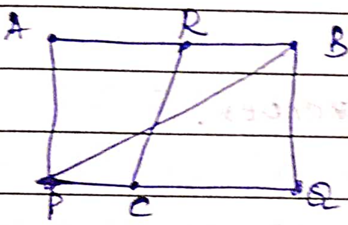
$V_2 = \{D, E, F\}$

e) 

$V_1 = \{A, B, C\}$

$V_2 = \{P, Q, R\}$

→ Bi-partite Graph

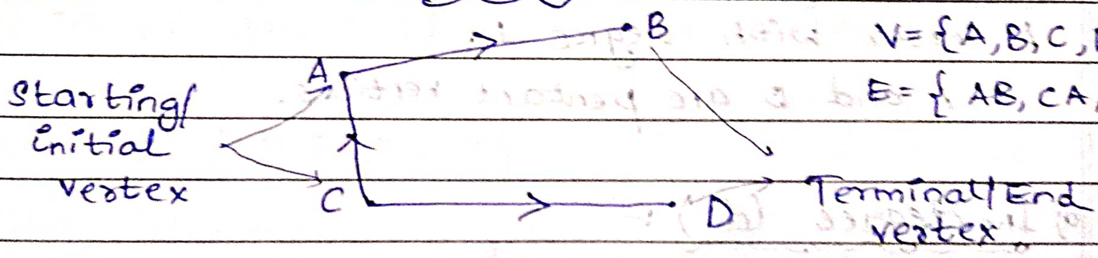
f) 

$V_1 = \{A, B, C\}$

$V_2 = \{P, Q, R\}$

Bi-partite

3) Directed Graph :- (Di-Graph):-

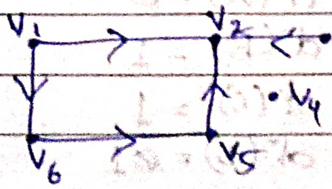


$V = \{A, B, C, D\}$

$E = \{AB, CA, CD\}$

→ Isolated vertex - No starting/end vertex.

→ Non-Isolated vertex → source ($d^-(v) = 0$) / sink

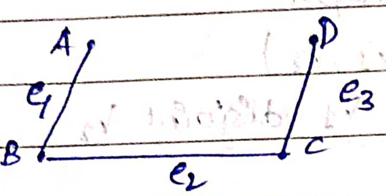


V_1 - source
V_2 - sink
V_3 - source
V_4 - isolated
V_5 - Neither source nor sink
V_6 -

~~Pentant vert~~

4) Degree Sequence of the graph :-

→ Incidence :-



- When a vertex 'V' of a graph 'G' is an end vertex of an edge 'e' of the graph 'G', we say that the edge 'e' is incident on vertex 'V'.
- 2 non-parallel edges are set to be adjacent edges if they are incident on a common vertex.
- 2 vertices are said to be adjacent vertices if there is an edge joining them.

→ Degree of a Vertex :-

No. of edges incident on a vertex.

$d(A) = 1$ $\quad d(C) = 2$

$d(B) = 2$ $\quad d(D) = 1$

5) Pentant Vertex :-

A vertex with degree 1.

∴ A and D are pentant vertices.

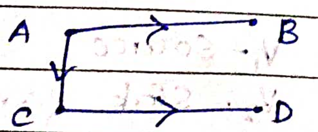
6) a) In-degree (d^-) :-

No. of edges coming into the vertex.

b) Out-degree (d^+) :-

No. of edges coming out of the vertex.

Eg :-



$d^-(A) = 0$ $\quad d^+(A) = 2$

$d^-(B) = 1$ $\quad d^+(B) = 0$

$d^-(C) = 1$ $\quad d^+(C) = 1$

$d^-(D) = \cancel{0}1$ $\quad d^+(D) = 0$

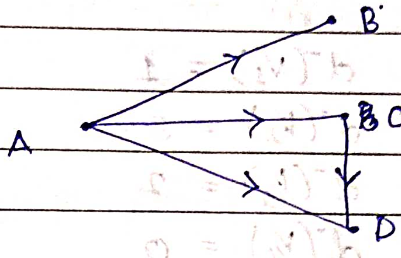
4) First Property of Di-Graph :-

$$\sum_{1} deg(-V) = \sum_{1} deg(+V)$$

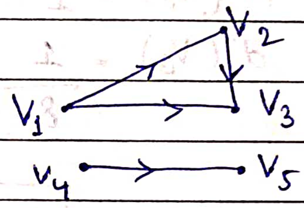
Problems :-

Draw a graph G in the following cases:-

1) $V = \{A, B, C, D\}$
$E = \{AB, AC, AD, CD\}$



2) $V = \{V_1, V_2, V_3, V_4, V_5\}$
$E = \{V_1V_2, V_1V_3, V_2V_3, V_4V_5\}$

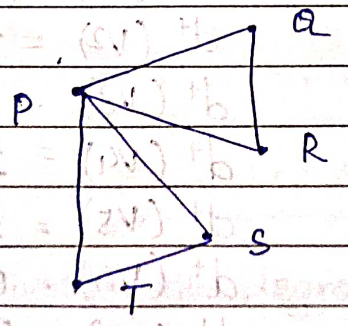


3) Let P, Q, R, S, T represent 5 cricket teams, Suppose that the teams P, Q, R have played one game with each other and the teams P, S, T have played one game with each other. Represent the situation in a graph and hence determine:

a) The teams that have not played with each other.

b) No. of games played by each team.

$V = \{P, Q, R, S, T\}$

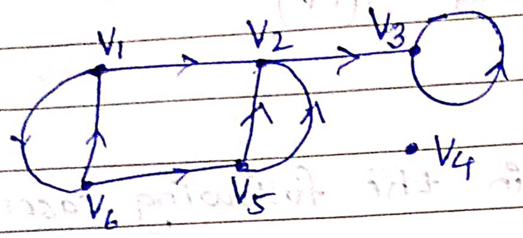


a) Q and S, Q and T
R and S, R and T

b) $d(P) - 4$ $d(S) - 2$
$d(Q) - 2$ $d(T) - 2$
$d(R) - 2$

4) Verify the first theorem of Di-Graph for the following graphs :-

a)



$d^-(V_1) = 1$ $\quad$ $d^+(V_1) = 2$

$d^-(V_2) = 3$ $\quad$ $d^+(V_2) = 1$

$d^-(V_3) = 2$ $\quad$ $d^+(V_3) = 1$

$d^-(V_4) = 0$ $\quad$ $d^+(V_4) = 0$

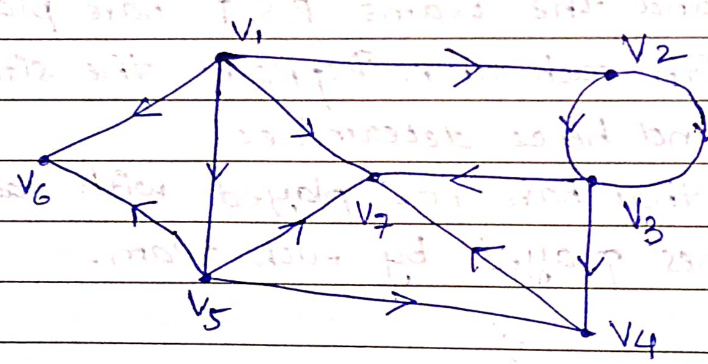
$d^-(V_5) = 1$ $\quad$ $d^+(V_5) = 2$

$d^-(V_6) = 1$ $\quad$ $d^+(V_6) = 2$

$\underline{8}$ $\quad$ $\underline{8}$

$$\Rightarrow \boxed{\sum_i d^-(V) = \sum_i d^+(V)}$$

b)



(Source) $d^-(V1) = 0$ $\quad$ $d^+(V1) = 4$

$d^-(V2) = 1$ $\quad$ $d^+(V2) = 2$

$d^-(V3) = 2$ $\quad$ $d^+(V3) = 2$

$d^-(V4) = 2$ $\quad$ $d^+(V4) = 1$

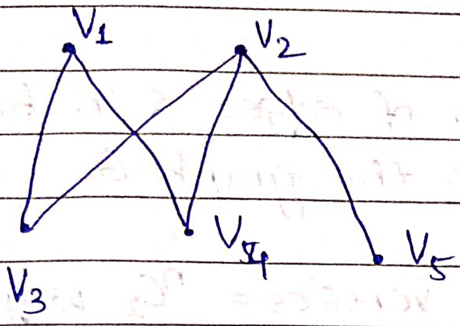
$d^-(V5) = 1$ $\quad$ $d^+(V5) = 3$

$d^-(V6) = 2$ $\quad$ $d^+(V6) = 0$

(Sink) $d^-(V7) = 4$ $\quad$ $d^+(V7) = 0$

$\underline{12} = \underline{12}$

Bipartite Graph :-



$V = \{V_1, V_2, V_3, V_4, V_5\}$

$V_1 \cup V_2 = V$

$V_1 \cap V_2 = \phi$

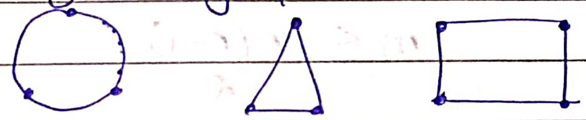
$V_1 = \{V_1, V_2\}$

$V_2 = \{V_3, V_4, V_5\}$

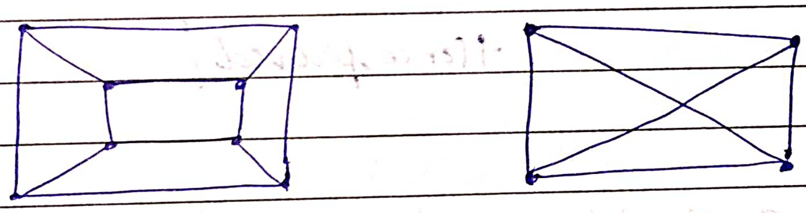
Regular Graph :- (K-regular graph)

where k - degree of each vertex.

$k=2$ regular graphs



$k=3$ regular graphs



Hypercube

Handshaking Property :-

$$\boxed{\sum_i deg(V) = 2m}$$

The sum of the degrees of all the vertices in a graph is an even number and this no. is equal to twice the number of edges in the ~~graph~~ graph.

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If $G=(V,E)$ is a simple graph, PT $2|E| \leq |V|^2-|V|$

Proof:-

Let 'm' be the no. of edges & 'n' be the no. of vertices in the graph G.

No. of pairs of vertices $= {}^nC_2$ ways

$$= \frac{n!}{2!(n-2)!}$$

$$= \frac{n(n-1)\cancel{(n-2)!}}{2!\cancel{(n-2)!}}$$

$$= \frac{n(n-1)}{2}$$

If 'm' is size of the graph

$$m \leq \frac{n(n-1)}{2}$$

$$2m \leq n^2-n$$

$$\boxed{2|E| \leq |V|^2-|V|}$$

Hence proved!

S.T., a complete graph with 'n' vertices has ~~half~~ $\frac{1}{2}(n)(n-1)$ edges.

In a complete graph, there exists b/w every pair of vertices as such the no. of edges in a complete graph is equal to no. of pair of vertices.

If no. of vertices is 'n' then the no. of pair of vertices is ${}^nC_2 = \frac{1}{2}(n)(n-1)$.

Thus, the no. edges in a complete graph with 'n' vertices is $\frac{n(n-1)}{2}$.

Bi-partite Graph :-

Suppose a simple graph 'G' is such that its vertex set V is the union of 2 of its mutually disjoint non-empty subsets V_1 & V_2 which are such that each edge in G joins a vertex in V_1 and a vertex in V_2, then G is called as Bi-partite graph.

Complete Bi-partitite Graph :-

A bi-partite graph $G = (V_1, V_2 ; E)$ is called a complete bi-partite graph if there is an edge between every vertex in V_1 and every vertex in V_2.

complete bi-partite graph is denoted by

$K_{r,s}$ where $r \leq s$ denotes the

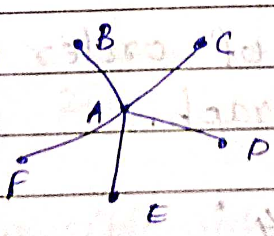
$r = |V_1|$

$s = |V_2|$

$|V| = r+s$ NO. of vertices /Order.

$|E| = rs$ Size of the graph.

Ex:- a)



$V_1 = \{A\}$

$V_2 = \{B, C, D, E, F\}$

Complete - Bi-partite graph

$\boxed{K_{1,5}}$

b)

$K_{2,3}$

How many vertices & edges in a complete Bi-partite Graph $K_{4,7}$ and $K_{7,11}$.

for, $K_{4,7}$ $|V| = 11$

$|E| = 28$

for $K_{7,11}$ $|V| = 18$

$|E| = 77$

If the graph $K_{r,12}$ has 72 edges, what is the size of the graph.

$r = \frac{72}{12} = 6.$

$|V| = r = 6.$

Note :-
Let 'G' = (V, E) be a simple graph of order 'n' and size 'm'. If G is a bi-partite graph then,

$$\boxed{4m \leq n^2}$$

Problems :-

1) Can a simple graph of order 4 and size 5 be a bi-partite graph.

$m = 5, \; n = 4$

$\therefore 20 \nleq 16$

Hence, it can't be a CBP graph.

2) Check whether following graphs exists or not.

a) Simple graph of order 3 and size 2

$2|E| \leq |V|^2 - |V|$

$V = 3$
$E = 2$

$4 \leq 9 - 3$

$4 \leq 6$

∴ The following graph exists.

Note :-

1) Simple
$m \leq \frac{n(n-1)}{2}$

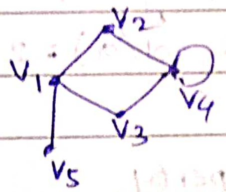
2) Complete
$m = \frac{n(n-1)}{2}$

3) Bi-partite
$4m \leq n^2$

Degree Sequence of a graph :-

Arranging the degrees in ascending order.

Eg :-



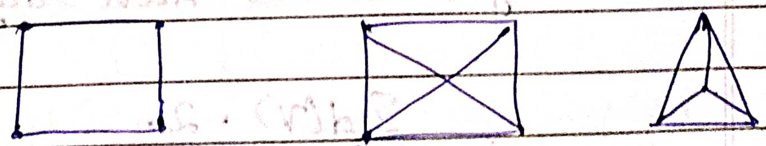
$d(V_1) = 3$
$d(V_2) = 2$
$d(V_3) = 2$
$d(V_4) = 4$
$d(V_5) = 1$

1, 2, 2, 3, 4

degree of graph – min. of all degrees.

$d(G) = 1$.

k-Regular Graph :- $\delta(G) \geq 2$

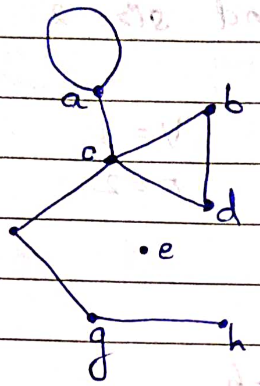


2-regular graph 3-regular graph

Note :-

For any possible integer, a loop-free k-regular graph with 2^k vertices is called a k-dimensional hypercube and is denoted Q_k.

1) For the given graph, indicate degrees of each vertex and verify handshaking property.



$d(a) = 3$ $d(e) = 0$
$d(b) = 2$ $d(f) = 2$
$d(c) = 4$ $d(g) = 2$
$d(d) = 2$ $d(h) = 1$

$\Sigma d(V) = 16$

No. of edges :- $m = 8$

$\boxed{2m = \Sigma d(V)}$

Hence Verified

2) Can there be a graph consisting of vertices a, b, c, d with $d(a) = 2$, $d(b) = 3$, $d(c) = 2$, $d(d) = 2$.

Using Hand-shaking property,
$\Sigma d(V) = 2m$

$2m = 9$

$m = 9/2$ Not possible

3) Can there be a graph of 12 vertices such that two of the vertices have degree 3 each & remaining 10 vertices have degree 4 each.

$\Sigma d(V) = 2m$
$3(2) + 4(10) = 2m$
$6 + 40 = 2m$
$46 = 2m$
$\boxed{m = 23}$

Possible

4) For a graph $G = (V,E)$, what is the largest possible value for $|V|$, if $|E| = 19$ and degree of a vertex is ≥ 4 ($d(v) \geq 4 \ \forall v \in V$).

Given $G = (V, E)$ $\quad$ $\deg(V) \geq 4$

$|E| = 19 = m$ $\quad$ summation on b.s

Let $|V| = n$

$\sum d(V) = 2m$

$\sum \deg(V) \geq 4n$

From Hand-shaking property.

$2m \geq 4n$

$n \leq \frac{2m}{4}$

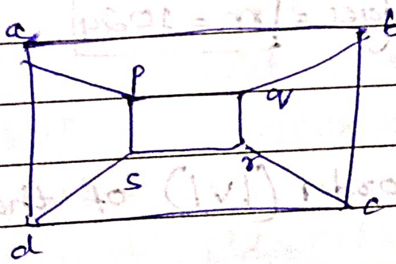
$n \leq \frac{2(19)}{4}$

$\boxed{n = 9} \rightarrow$ max. value of n.

5) S.T the hypercube Q_3 is a bi-partite graph which is not a complete bi-partite graph.

No. of ~~de~~ vertices $= 2^k = 2^3 = 8$ vertices

degree of each vertex $= 3$



$V = \{a, b, c, d, p, q, r, s\}$

$V = V_1$ disjoint V_2.

$V_1 = \{a, c, q, \cancel{\ }, s\}$ $\Rightarrow$ $V_1 \cup V_2 = V$

$V_2 = \{b, d, p, r\}$ $\quad$ $V_1 \cap V_2 = \phi$.

vertex 'a' from set V_1 is not connected to vertex 'r' ~~fro~~ of set V_2.
Hence, it is not a complete bi-partite graph.

6) P.T k-dimensional hypercube Q_k has $k2^{k-1}$ edges. Hence, determine the no. of edges in Q_8.

For a k-dimensional hypercube,
No. of vertices $= 2^k$

From Hand-shaking property,
$$\sum \deg(v) = 2m$$
$$k2^k = 2m$$

$$\Rightarrow \boxed{m = k2^{k-1}}$$
No. of edges.

In Q_8,
No. of vertices is $2^8 = 256$

No. of edges, $m = k2^{k-1}$ $(k=8)$
$$m = 8(2^{8-1})$$
$$m = 8.2^7 = 2^{10}$$
No. of edges $= \boxed{m = 1024}$

7) Determine the order $(|V|)$ of the graph in the following cases:

a) G is a cubic graph with 9 edges.
b) G has 10 edges with 2 vertices of degree 4 and all other vertices of degree 3.

c) G is a regular graph with 15 edges.

Note :-
A cubic graph is a graph with each ~~te~~ vertex of degree 3. (Q_3).

a) From handshaking property,

$$\sum d(v) = 2m$$
$$3(|V|) = 2(9)$$
$$|V| = \frac{18}{3}$$
$$\therefore \boxed{|V| = 6 \text{ vertices}}.$$

b)
$$\sum d(v) = 2m$$
$$\cancel{2(4) + 8(3)}$$
$$2(4) + 3(|V| - 2) = 2\times 10$$
$$3|V| = 20 - 8 = 12 + 6$$
$$|V| = \frac{18}{3}$$
$$\boxed{|V| = 6}$$

c)
$$m = k2^{k-1}$$
$$50 = k2^{k-1}$$
$$100 = k2^{k}$$

G is a regular graph, Let 'k' be the degree of each vertex & 'n' be no. of vertices.

$$\sum \deg(v) = 2m$$
$$kn = 2m = 2\times 15$$
$$n = \frac{30}{k} \quad k = \frac{30}{n}$$

n is a divisor of 30.

$\therefore$ 1, 2, 3, 5, 6, 10, 15, 30.

8) Let G be a graph of order 9 such that each vertex has a degree 5 or 6. Prove that atleast 5 vertices have degree 6 or atleast 6 vertices have degree 5.

$n = |V| = 9$

Let 'p' be the no. of vertices of degree 5
and 'q' ~~be~~ $= 9-p$ — " — of degree 6.

$0 \leq p \leq 9$ $\quad 0 \leq q \leq 9$

By H.S.P

$\Sigma \deg(v) = 2m$

$5p + 6q = 2m$

$5p + 6(9-p) = 2m$

$5p + 54 - 6p = 2m$

$\boxed{54 - p = 2m}$

$p = 0, 2, 4, 6, 8$

If $p = 0 \Rightarrow q = 9$
$p = 2 \Rightarrow q = 7$
$p = 4 \Rightarrow q = 5$
$p = 6 \Rightarrow q = 3$
$p = 8 \rightarrow q = 1$

We observe that in all the above cases,
$q \geq 5$ or $p \geq 6$
This means atleast 5 vertices have degree 6 or atleast 6 vertices have degree 5.

9) S.T., there is no graph with 28 edges and 12 vertices in the following cases:-

i) Degree of vertex is either 3 or 4
ii) —"— 3 or 6

Given, $m = 28$ and $n = 12$.

From H.S.P,

$$\sum (\deg(V)) = 2m$$
$$k \cdot n = 2 \times 28$$
$$\text{or } k = \frac{2 \times 28}{n} = \frac{2 \times 28}{\cancel{12}\,6} = \frac{14}{3}$$

$$k = \frac{14}{3}$$

$$\cancel{k \leq 4} \quad \boxed{\cancel{k = 4}}$$

X

i) Let p be no. of vertices of degree 3
& $q = 12 - p$ —"— 4.

$$\sum (\deg(V)) = 2m$$
$$3p + 4q = 2 \times 28 = 56$$
$$3p + 4(12 - p) = 56$$
$$3p + 48 - 4p = 56$$
$$48 - p = 56$$
$$p = 48 - 56$$
$$\boxed{p = -8}$$

p can't be negative
Hence, the graph doesn't exist.

ii) Let 'p' → degree 3 & 'q' → degree 6.

$$3p + (12 - p)6 = 56$$
$$3p + 72 - 6p = 56$$
$$-3p = 56 - 72$$

$$-3p = -16$$
$$p = 16/3$$

10) If a graph with n vertices & m edges is k-regular S.T., i) $m = \frac{kn}{2}$

ii) Does there exist a cubic graph with 11 vertices

iii) Does there exist a 4-regular graph with a) 15 edges & b) 10 edges.

Given, In a k-regular graph,

i) No. of vertices = n & k

No. of edges = m.

From H.S.P,

$nk = 2m$

$\Rightarrow \boxed{m = \frac{kn}{2}}$.

ii) Cubic Graph $\Rightarrow k = 3$,

$m = \frac{3 \times 11}{2} = \frac{33}{2}$

m can't be fraction.

$\Rightarrow$ Graph can't exist.

iii) $k = 4$,

a) $m = 15$ edges

$m = \frac{kn}{2} = \frac{\cancel{4}^{2} n}{\cancel{2}_{1}}$

$n = 15/2$

(Graph Can't exist)

b) $m = 10$ edges

$m = \frac{kn}{2} = 2n$

$n = \frac{10}{2} = 5$

(Graph exist)

11) S.T., every simple graph of order ≥ 2 must have atleast 2 vertices of the same degree.

Let 'G' be a simple graph and 'n' be the no. of vertices,

Using contradiction method,

Let each ~~degree~~ vertex be of different degree,

Hence, each vertex will be of degree of 0, 1, 2, 3, ..., n-1 (Simple Graph)

Since every vertex must have a different degree and all such degrees must be between 0 and n-1.

- Let 'A' be a vertex whose degree is '0', and 'p' be the vertex of degree is 'n-1'.

Then, (n-1) edges are incident on 'p', this 'p' is joined to all other vertices by an edge and in particular to vertex 'A' also.

Hence, the degree of 'A' is not '0'.

This is a contradiction to our assumption, Hence all vertices of G can not have different edges, atleast 2 of them must have same degree.

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12) Is there a simple graph with 1, 1, 3, 3, 3, 4, 6, 7 as the degrees of their vertices.

~~Let us consider the graph with 8.~~
Let us assume there is a graph such that the vertices are '8' and are 'a, b, c, d, e, f, g, h' arranged in the order of degree as given.

Since there are 8 vertices and the vertex 'h' is of degree 7, h should have an edge on all other vertices in particular h should have an edge on vertices 'a' & 'b' whose degree is '1'.

Then, a and b are not joined to any other vertex and in particular to vertex 'g' which is of degree 6.
Since, the graph is simple, there cannot be an edge joining the vertex 'g' itself.
$\therefore$ 'g' can be joined only to 5 vertices c, d, e, f and h,
then vertex 'g' cannot have degree '6'.

This is a contradiction and hence there is no simple graph stated in the question.

13) For a graph with n vertices and m edges if δ is the minimum and Δ is maximum of degrees of vertices. S.T.,

$$\delta \leq \frac{2m}{n} \leq \Delta.$$

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Let $d_1, d_2, d_3, \ldots, d_n$ be the degrees of vertices then by ~~HBS~~ HSP,

$$\sum \deg(v) = 2m$$

$$d_1 + d_2 + d_3 + \ldots + d_n = 2m \longrightarrow (1)$$

Let δ = minimum of the degrees.

$$\delta = \min(d_1, d_2, \ldots, d_n)$$

$$d_1 \geq \delta, \ d_2 \geq \delta, \ldots\ldots, d_n \geq \delta) n$$

$$\Rightarrow d_1 + d_2 + d_3 + \ldots + d_n \geq n\delta$$

From (1)

$$\boxed{2m \geq n\delta} \longrightarrow (2)$$

Let $\Delta = \max(d_1, d_2, \ldots, d_n)$.

$$d_1 \leq \Delta, \ d_2 \leq \Delta, \ldots\ldots, d_n \leq \Delta$$

$$\Rightarrow d_1 + d_2 + d_3 + \ldots + d_n \leq n\Delta$$

From (1)

$$\boxed{2m \leq n\Delta} \longrightarrow (3)$$

From (2) and (3),

$$n\delta \leq 2m \leq n\Delta$$

$$\Rightarrow \boxed{\delta \leq \frac{2m}{n} < \Delta}$$

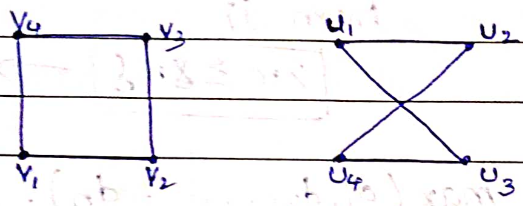
Hence proved.

Isomorphism:-

Consider 2 graphs G and G'.
Suppose there exists a function $f: V \longrightarrow V'$
such that
- f is a one to one correspondence
- For all vertices a, b of G i.e., $\{a, b\}$ is an edge of G if and only if $\{f(a), f(b)\}$ is an edge of G',

then f is called an isomorphism between G and G' and we say that G and G' are isomorphic graphs.

Eg:-



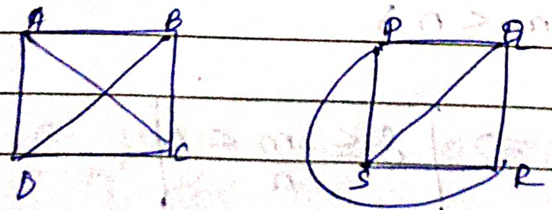
$$f: V \longrightarrow V'$$
$$G \simeq G'$$

$V_4 \longleftrightarrow u_1$ $\qquad$ $V_4V_1 \longleftrightarrow u_1u_2$
$V_3 \longleftrightarrow$
$V_2 \longleftrightarrow$
$V_1 \longleftrightarrow$

Problems:-

1) Verify given graphs are isomorphic or not :-

a)

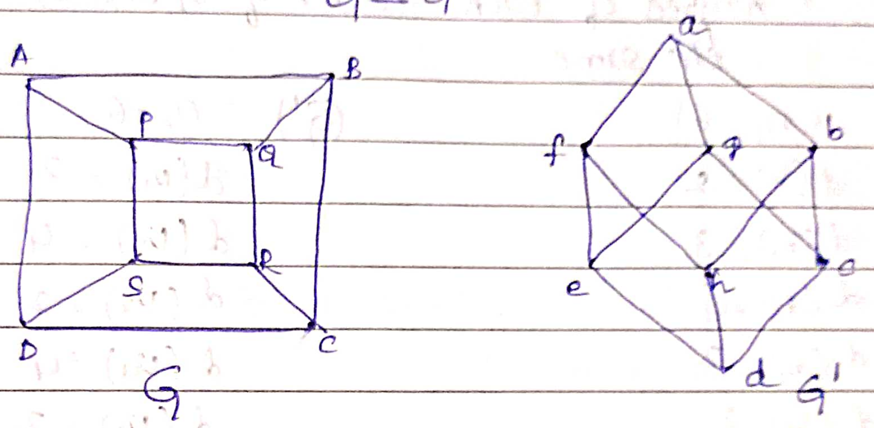


No. of vertices in G and G' are 4.
No. of degree of each vertex is 3.

$A \longleftrightarrow P$	$AB \longleftrightarrow PQ$
$B \longleftrightarrow Q$	$AC \longleftrightarrow PR$
$D \longleftrightarrow S$	$AD \longleftrightarrow PS$
$C \longleftrightarrow R$	$BC \longleftrightarrow QR$
	$BD \longleftrightarrow QS$
	$CD \longleftrightarrow RS$

$G \cong G'$

b)



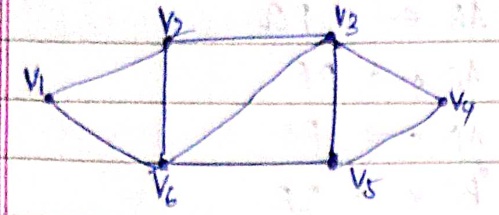
No. of vertices in G and G' are 8.

No. of degree of each vertex is 3.

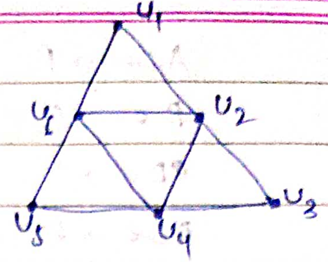
$A \longleftrightarrow a$	$AB \longleftrightarrow ab$
$B \longleftrightarrow b$	$AP \longleftrightarrow af$
$D \longleftrightarrow g$	$AD \longleftrightarrow ag$
$P \longleftrightarrow f$	$BC \longleftrightarrow bc$
$Q \longleftrightarrow h$	$BQ \longleftrightarrow bh$
$C \longleftrightarrow c$	$CD \longleftrightarrow cg$
$S \longleftrightarrow e$	$CR \longleftrightarrow cd$
$R \longleftrightarrow d$	$PQ \longleftrightarrow fh$
	$PS \longleftrightarrow fe$
	$QR \longleftrightarrow hd$
	$RS \longleftrightarrow de$
	$DS \longleftrightarrow ge$

Hence $G \cong G'$

c)



G



G'

No. of vertices in G and G' are 6

Degrees of each vertex of G and G' are not the same

(G) $n_1 = 6$

$d(V_1) = 2$

$d(V_2) = 3$

$d(V_3) = 4$

$d(V_4) = 2$

$d(V_5) = 3$

$d(V_6) = 4$

(G') $n_2 = 6$

$d(U_1) = 2$

$d(U_2) = 4$

$d(U_3) = 2$

$d(U_4) = 4$

$d(U_5) = 2$

$d(U_6) = 4$

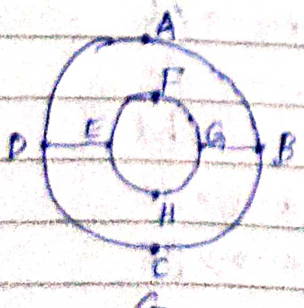
From G and G' no. of ~~deg~~ vertices with degree 4 are not equal

Hence, there cannot be edge adjacency and edge correspondence.

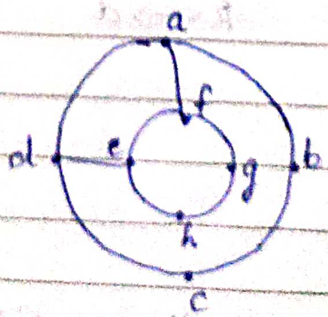
∴ G and G' are not isomorphic.

$G \not\cong G'$

d)



G



G'

No. of vertices in G and G' are 8

Degree of each vertex in G and G'

G	G'
d(A) = 2	d(a) = 3
d(B) = 3	d(b) = 2
d(C) = 2	d(c) = 2
d(D) = 3	d(d) = 3
d(E) = 3	d(e) = 3
d(F) = 2	d(f) = 3
d(G) = 3	d(g) = 2
d(H) = 2	d(h) = 2

No. of vertices with degree 2 = 4

No. of vertices with degree 3 = 4

$A \longleftrightarrow a$

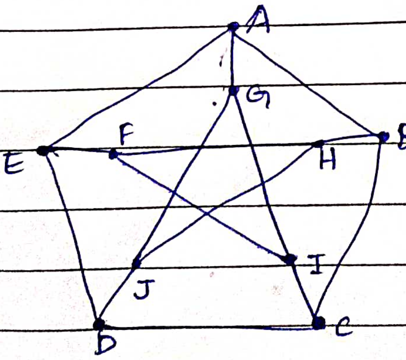
$B \longleftrightarrow b$

$D \longleftrightarrow d$

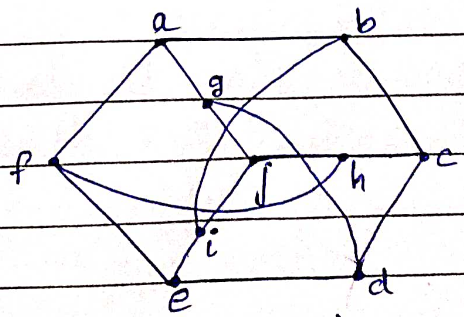
Vertex 'A' has 2 vertex adjacencies whereas the corresponding vertex 'a' has 3 vertex adjacencies.

Thus $G \ncong G'$

e)



G

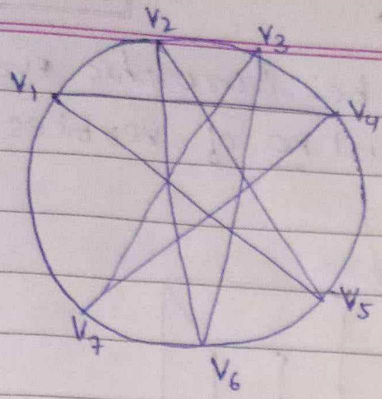


G'

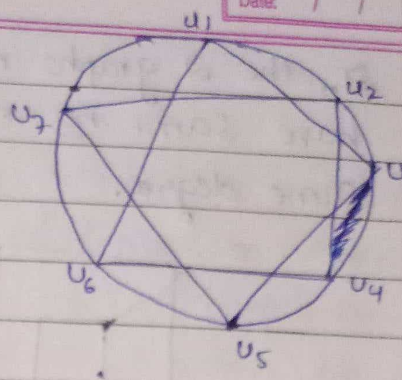
No. of vertices of G and G' are 10.
Degree of each vertex in G and G' is 3.

$A \longleftrightarrow a$	$AB \longleftrightarrow ab$
$B \longleftrightarrow b$	$AG \longleftrightarrow ag$
$G \longleftrightarrow g$	$AE \longleftrightarrow af$
$E \longleftrightarrow f$	$BH \longleftrightarrow bi$
$H \longleftrightarrow i$	$BC \longleftrightarrow bc$
$C \longleftrightarrow c$	$GJ \longleftrightarrow gh$
$J \longleftrightarrow h$	$GI \longleftrightarrow gd$
$I \longleftrightarrow d$	$EF \longleftrightarrow ef$
$F \longleftrightarrow \ell$	$ED \longleftrightarrow$
$D \longleftrightarrow j$	

f)



G



G'

$V_1 \longleftrightarrow u_1$

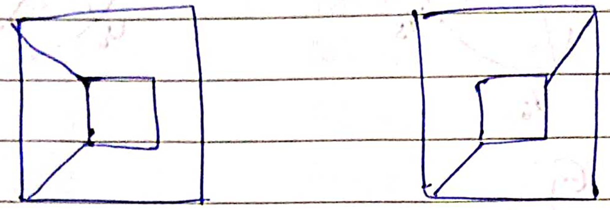
$V_4 \longleftrightarrow u_3$

$V_5 \longleftrightarrow u_6$

$V_2 \longleftrightarrow u_2$

$V_7 \longleftrightarrow u_7$

2) S.T., the 2 graphs need not be isomorphic even they have same no. edges, equal no. of vertices with same degree.



G

$n=8$

$m=10$

G'

$n'=8$

$m'=10$

degree of 4 vertices = 2

degree of 4 vertices = 3

Sub-graphs :-

Given 2 graphs G and G', we say that G' is a subgraph of G if the following conditions hold:

i) all the vertices & edges of G' are in G

ii) each edge of G' has the same end vertices in G as in G'

Spanning Subgraph :-

Given a graph $G=(V,E)$ if there is a subgraph $G'=(V',E')$ such that $V'=V$, then G' is called a spanning subgraph of G.

Induced Subgraph :-

Given a graph $G = (V, E)$,
Suppose there is a subgraph $G' = (V', E')$ of G such that every edge $\{A, B\}$ of G where $A, B \in V'$ is an edge of G' also, then G' is called an induced subgraph of G and is denoted by $\langle V' \rangle$.

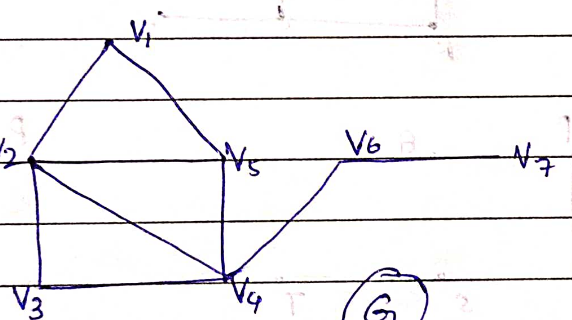
Edge disjoint and Vertex disjoint Subgraphs :-

Let G be a graph and G_1 and G_2 ~~be are~~ two subgraphs of G, then :

i) G_1 and G_2 are said to be edge disjoint, if they do not have any edge in common.

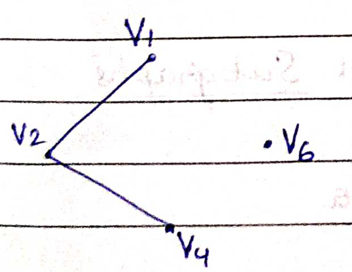
ii) G_1 and G_2 are said to be vertex disjoint, if they do not have any vertex and any edge in common.

Eg:-



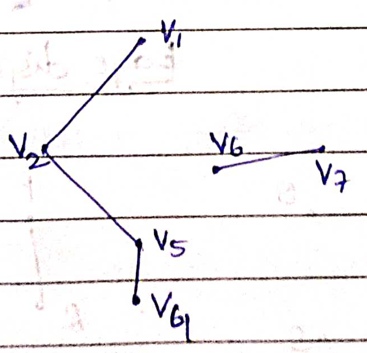
$G = (V, E)$

G



a) Subgraph

$G' = (V', E')$
$V' \subseteq V$; $V' \neq \phi$
$E' \subseteq E$.

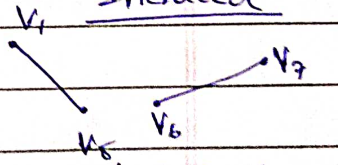


b) Spanning

$G' = (V', E')$
$V' = V$

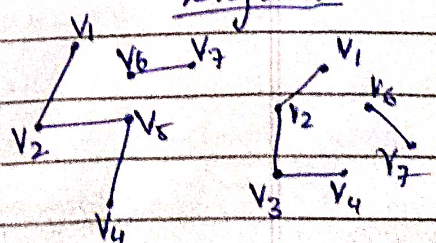
$V' = \{V_1, V_5, V_6, V_7\}$

c) Induced



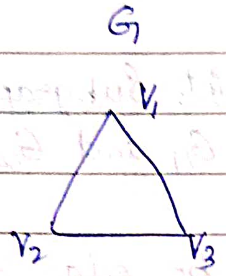
$G' = (V', E')$
$V' \subseteq V$
$\langle V' \rangle$

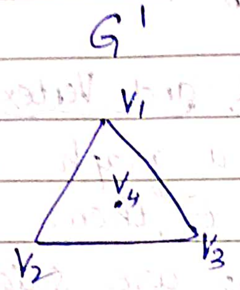
d) Edge & Vertex Disjoint



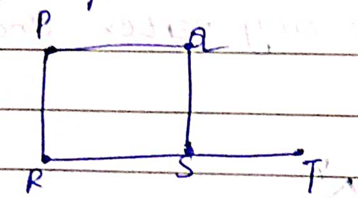
Problems :-

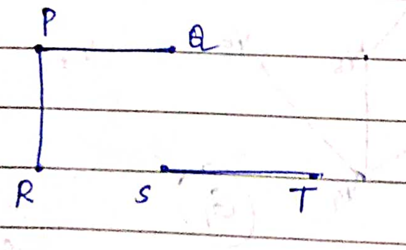
1) Given a graph G, can there exist a graph G' such that G is a subgraph of G' but not a spanning subgraph of G' and yet G and G' have the same size.

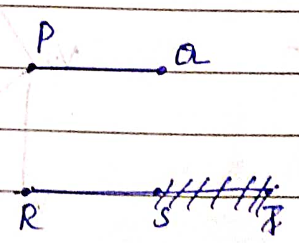




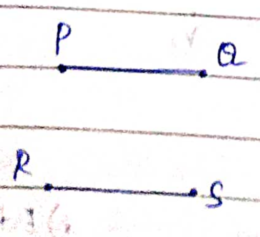
2) Find 2 edge disjoint & 2 vertex disjoint subgraphs.

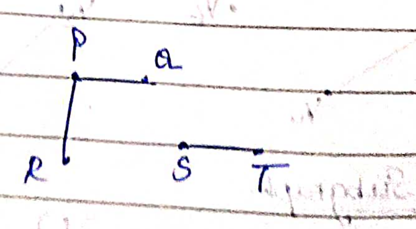




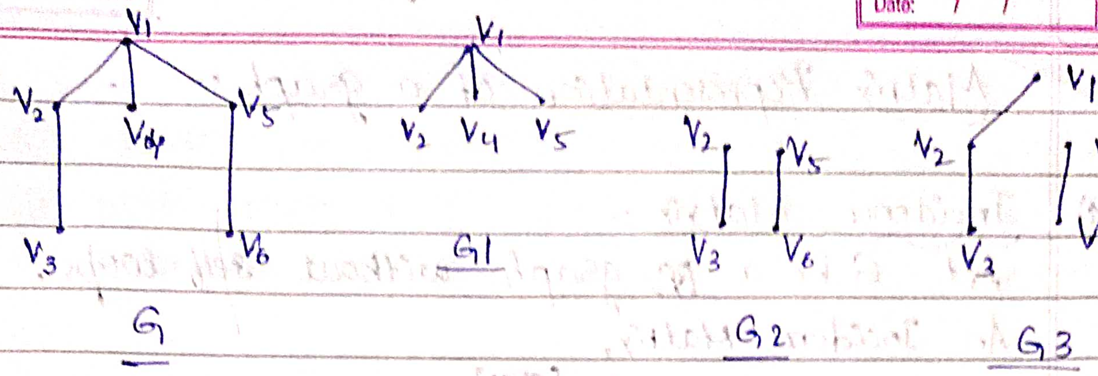


Edge disjoint Subgraphs



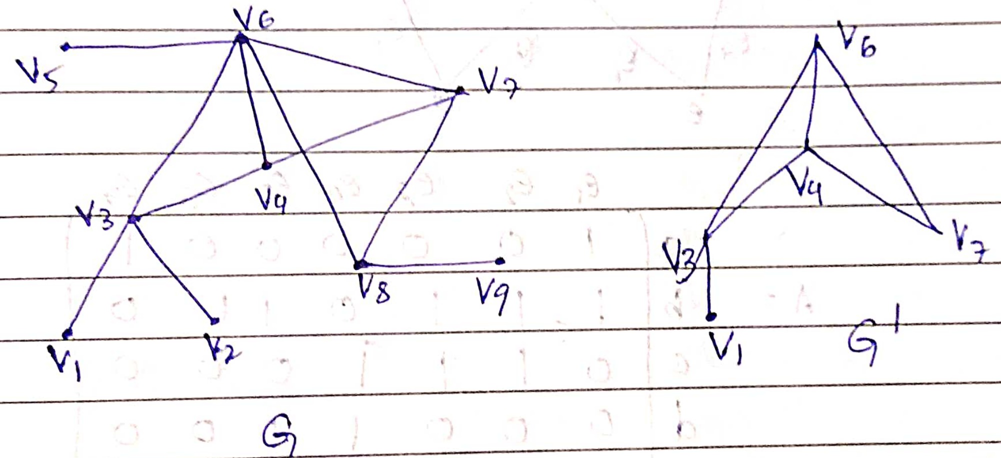


3)



G1 – induced subgraph
G2 – induced subgraph
G3 – Subgraph

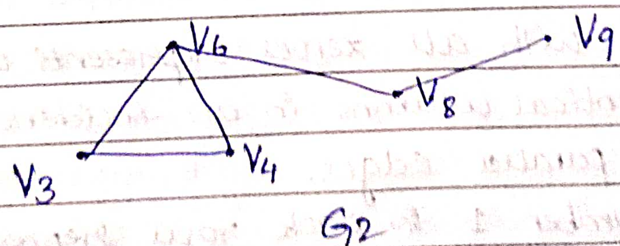
4)



a) Is G' an induced or a spanning subgraph?
b) Draw the subgraphs G2 of G induced by $\langle V \rangle =$
$\langle V \rangle = \langle V_3, V_4, V_6, V_8, V_9 \rangle$

a) Induced subgraph.

b)



Matrix Representation of a graph :-

1) Incidence Matrix :-

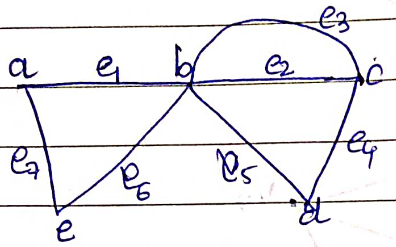
Let G be a graph without self-loops.

An Incidence Matrix,

$$A = [a_{ij}]_{p \times q}$$

where, $a_{ij} = \begin{cases} 1 ; & \text{if } e_j \text{ is incident on vertex } v_i \text{ in } G \\ 0 ; & \text{otherwise} \end{cases}$

Eg :-



$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{array} \\ \left[\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array} \right]_{5 \times 7} \end{array}$$

Observations :-

- We have each edge containing exactly 2 end vertices therefore each column contains 1's in exactly 2 places.
- A row with all zeroes represents an isolated vertex.
- Two identical columns in an incidence matrix corresponds to the parallel edges.
- The number 1 in each row represents the edge incident from the vertex corresponding to the row therefore the sum of 1's in each row represents the degree of a vertex corresponding to the row.

- If a row contains 1's at exactly one place, then the vertex corresponding to the row is called a pentant vertex and the edge corresponding to the column at which 1 occur is a pentant edge.

2) Adjacency Matrix :-

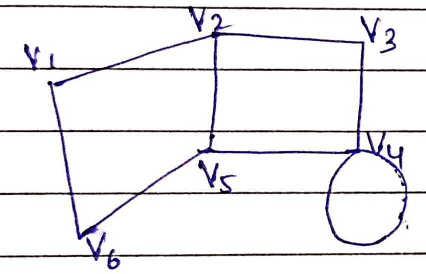
Let $G = (P, q)$ be a graph without parallel edges, then the adjacency matrix is given as,

$$X = [x_{ij}]_{p \times p}$$

where,

$$x_{ij} = \begin{cases} 1 & ; \text{ if } v_i v_j \in E(G) \\ 0 & ; \text{ otherwise} \end{cases}$$

Eg:-



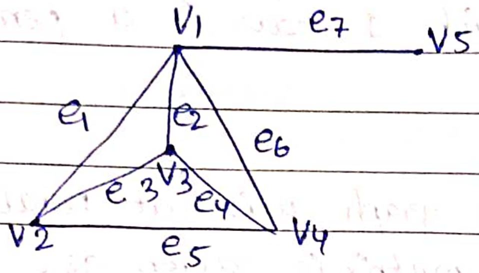
$$X = \begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \end{array} \begin{array}{c} \begin{array}{cccccc} V_1 & V_2 & V_3 & V_4 & V_5 & V_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}_{p \times p} \end{array}$$

Observations :-

- Adjacency matrix of a graph is a symmetric binary matrix.
- Diagonal elements of an adjacency matrix are all zero if and only if G does not have a loop.
- We get no information about parallel edges thus ~~they~~ we avoid parallel edges in definition of adjacency.
- No. of vertices having self-loop is equal to the no. of non-zero diagonal entries.
- If two rows are interchanged, then the corresponding columns are interchanged.
- No. of 1's in a row/column gives the degree of the vertex corresponding to row/column, counting the diagonal elements twice.

Problems:-

1) For the given matrix, write adjacency and incidence matrix:



Incidence Matrix,

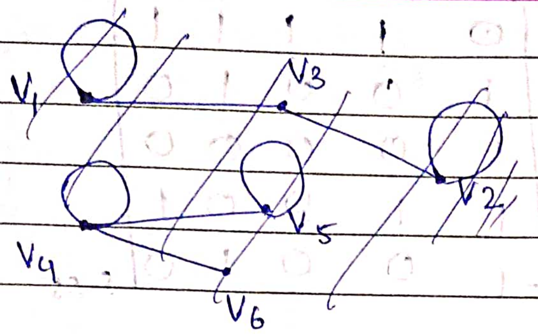
$$A = \begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{array} \begin{array}{c} \begin{array}{ccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{5\times 6} \end{array}$$

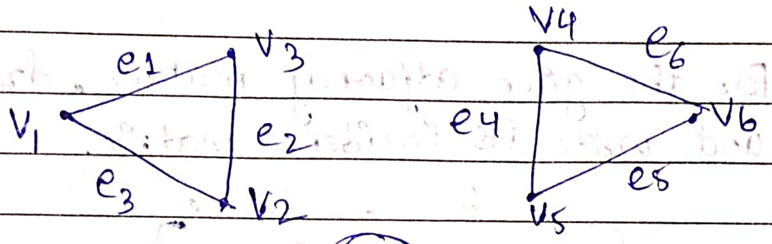
Adjacency Matrix,

$$X = \begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{array} \begin{array}{c} \begin{array}{ccccc} V_1 & V_2 & V_3 & V_4 & V_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}_{5\times 5} \end{array}$$

2) For the given Incidence Matrix, draw the graph G, and hence write adjacency matrix.

a)
$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}_{6\times 6}$$



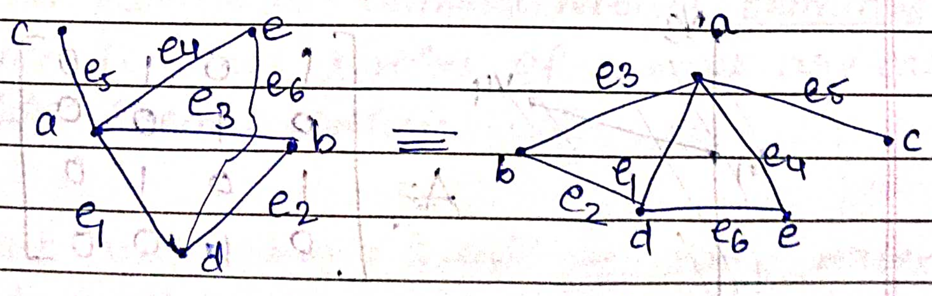


G

$$
X = \begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \end{array}
\begin{array}{c}
\begin{array}{cccccc} V_1 & V_2 & V_3 & V_4 & V_5 & V_6 \end{array} \\
\begin{bmatrix}
0 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 0
\end{bmatrix}_{6 \times 6}
\end{array}
$$

b)

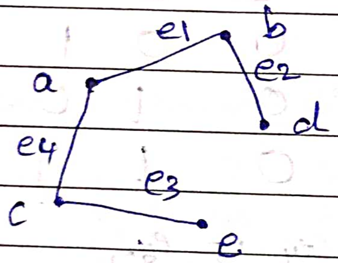
$$
A = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array}
\begin{array}{c}
\begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\
\begin{bmatrix}
1 & 0 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1
\end{bmatrix}_{5 \times 6}
\end{array}
$$



$$X = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{matrix} a & b & c & d & e \end{matrix} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}_{5\times5} \end{array}$$

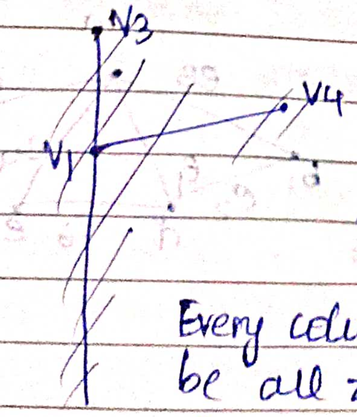
3) For the given adjacency matrix, draw the graph and write its incidence matrix.

$$X = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{matrix} a & b & c & d & e \end{matrix} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}_{5\times5} \end{array}$$



$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{matrix} e_1 & e_2 & e_3 & e_4 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}_{5\times4} \end{array}$$

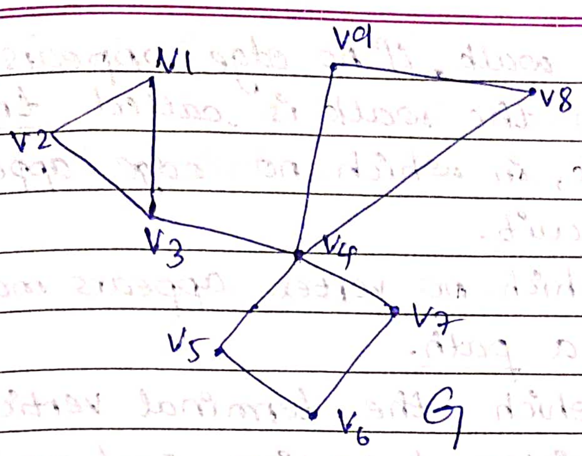
4) Does there exist a graph G corresponding to the incidence matrix.



$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}_{4\times4}$$

Every column shd contain two 1's and cannot be all zero.

Hence such a graph does not exist.



- Let G be a graph having atleast one edge, consider a finite alternating sequence of vertices & edges which begins and ends with vertices and which is such that each edge in the sequence is ~~co~~ incident on vertices preceding & following it in the sequence.
  Such a sequence is called a walk in the graph G.

- In a walk, a vertex ~~and~~ or an edge or both can appear more than once.
- The no. of edges present in a walk is called its length.
- The vertex with which a walk begins is called initial vertex of walk and —"— ends is called a final vertex of walk.
- The initial and final vertices of a walk are together called as terminal vertices/terminus.
- The non-terminal vertices of a walk are called as ~~init~~ internal vertices.

- A walk which begins & ends at same vertex is called a closed walk.
- A walk which is not closed i.e., that begins and ends at diff. vertices is called an open walk.

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- If in an open walk, if no edge appears more than once, then the walk is called trail.
- A closed walk, in which no edge appears more than once — circuit.
- A trail in which no vertex appears more than once is called a path.
- A circuit in which the terminal vertices does not appear as internal vertices and no internal vertex is repeated is called a cycle.

Eg:-

- Walk:— $V_1 e_1 V_2 e_3 V_3 e_4 V_4$ (Length:-3)
  → open walk:— $V_1 e_1 V_2 e_3 V_3 e_4 V_4$
  → closed walk:— $V_1 e_1 V_2 e_3 V_3 e_2 V_1$

- Trail :- $V_9 e_{11} V_4 e_4 V_3$
  Circuit :- $V_9 e_{11} V_4 e_5 V_5 e_6 V_6 e_7 V_7 e_8 V_4 e_9 V_8 e_{10} V_9$
  Path :- $V_9 e_{11} V_4 e_4 V_3$
  Cycle :- $V_9 e_{11} V_4 e_9 V_8 e_{10} V_9$

Connected Graphs:-

A graph G is said to be connected if there is atleast path from one vertex to another vertex.

(OR)

A graph G is said to be connected if every pair of distinct vertices in G are connected.

Disconnected Graphs:-

A graph G is said to be disconnected if there is no path between distinct vertices in G.

A Graph G consists one or more connected graph each such connected graph is a subgraph of G and is called a component of G.

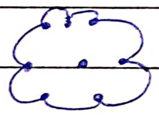
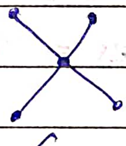
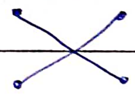
The number of components of G is denoted by K(G).

Note :-
- In a connected graph, $K(G) = 1$.
- For a disconnected graph, $K(G) \geq 2$.

- If a graph has exactly 2 vertices of odd degree, then there must be a path connecting these vertices.
- A connected graph with 'n' vertices has atleast $(n-1)$ edges.

Problems :-

1) Identify whether graphs are connected / disconnected

a)  b)  c) 

Disconnected   Connected   Disconnected

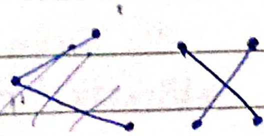
2) Let G be a graph with 'n' vertices, where n is even and $n > 2$. If the degree of each vertex in G is $\frac{1}{2}(n-2)$, disprove that G is connected.

Given, $n > 2$ and n is even.

Degree $= \frac{1}{2}(n-2)$

Let $n = 4$

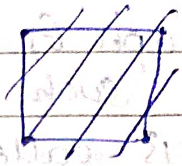
$\Rightarrow$ Degree $= \frac{1}{2}(4-2) = 1$

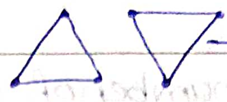


Hence disproved!

Let $n = 6$

Then degree, $d = \frac{1}{2}(6-2) = 2$





→ Disconnected

3) Let G be a disconnected graph with even order with 2 components with each of which is complete. P.T., G has a minimum of $\frac{n(n-2)}{4}$ edges.

Let G be a disconnected graph with 2 components. Let 'x' be the no. of vertices in first component and 'y' be —ll— in second component of G.

Maxima & Minima

$y = f(x)$

i) $y' = 0$ (critical points)

ii) $y'' \to > 0$ (min)

$\quad\;\; \to < 0$ (max)

$y'' = 0$ (in

For a complete graph,

$$\text{no. of edges} = {}^nC_2 = \frac{n(n-1)}{2}$$

For 1st component,

$$\text{No. of edges} = \frac{x(x-1)}{2}$$

For 2nd component,

$$\text{No. of edges} = \frac{(n-x)(n-x-1)}{2}$$

$$\text{Total edges in } G = \frac{x(x-1)}{2} + \frac{(n-x)(n-x-1)}{2}$$

$$\boxed{m = \frac{x(x-1)}{2} + \frac{(n-x)(n-x-1)}{2}}$$

$$m = \frac{1}{2}\left[2x^2 - 2nx - n + n^2\right]$$

Diff. 'm' wrt 'x'

$$m' = \frac{1}{2}\left[2x - 1 + (-n - n + 2x + 1)\right]$$

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$$m' = \frac{2x - \cancel{1} - 2n + 2x + \cancel{1}}{2} = 2x - n$$

$$m' = 2x - n = 0$$

$$\boxed{x = \frac{n}{2}} \rightarrow \text{critical point.}$$

Diff. 'm'' wrt 'x'

$$\frac{d^2m}{dx^2} = m'' = \frac{4}{2} = 2 > 0 \Rightarrow \underline{\text{minima}}$$

$$m_{(x=\frac{n}{2})} = \cancel{\frac{n}{2}\left(\frac{n}{2}-1\right)} \frac{1}{2}\left[2\left(\frac{n^2}{4}\right) - 2n\left(\frac{n}{2}\right) - n + n^2\right]$$

$$m = \frac{1}{2}\left[\frac{n^2}{2} - \cancel{n^2} - n + \cancel{n^2}\right] = \frac{n}{4}(n-2).$$

9)

4) If G is a simple graph with n vertices where degree of every vertex is atleast $\left(\frac{n-1}{2}\right)$. P.T G is connected.

→ Take any 2 vertices 'u' and 'v' of G, then they are either adjacent or not-adjacent.

→ If they are adjacent, then G is connected, otherwise ~~G~~ each has atleast $\frac{(n-1)}{2}$ neighbours because degree of each vertex is $\left(\frac{n-1}{2}\right)$.

→ Therefore u and v taken together have atleast (n-1) neighbours.

→ But since, G has a total of 'n' vertices, the total no. of neighbours which u & v can have together is (n-2).

→ ∴ Atleast one vertex 'x' is a neighbour of both 'u' and 'v'.

→ Hence, there is an edge b/w 'u & x' and also between 'x & v'.

→ Thus, there is a path b/w u and v.
As such G must be connected.

5) P.T., a connected graph G remains connected after removing an edge E from G if and only if E is a part of some cycle in G.

→ Suppose 'e' is part of some cycle in G, then the end vertices ~~say~~ of E (say A & B) are joined by atleast 2 paths,
one is 'e' and another is 'C-e'.

→ Hence, removal of e from G does not affect the connectivity of G bcoz even after removal of e, the end vertices of ~~E~~ e are connected through the path 'C-e'.

→ Conversely, suppose 'e' is not a part of any cycle 'C', then the end vertices of 'e' are connected by utmost 1 path.
→ Hence, the removal of 'e' from G disconnects these end points, this means that 'G-e' is a disconnected graph.
→ Thus, if 'e' is not a part of any cycle in G, then 'G-e' is connected.

→ Using contrapositive statement, it is equivalent to say that if 'G-e' is connected, then 'e' belongs to some part of cycle 'c' in G.

Consider a connected graph G, if there is a circuit in G wch contains all the edges of G, then the circuit is called Euler circuit / Eulerian line / Euler tour.

A connected graph which contains an Euler circuit, it is called Euler / Eulerian graph.

If there is a trail in G wch contains all the edges of G, then that trail is called an Euler trail / Unicursal line.

A connected graph that contains an Euler trail is called a Semi-Euler graph / Unicursal graph.

Let G be a connected graph, if there is a cycle in G that contains all the vertices of G, then that cycle is called a Hamilton cycle in G.

A Graph that contains a Hamilton cycle is called a Hamilton graph / Hamiltonian graph.

A path in a connected graph which includes every vertex in a graph is called a Hamiltonian path in a graph.

Note/Remarks:-

1) A connected graph G has an Euler circuit if and only if all vertices of G are of even degree.

2) A connected graph G has an Euler circuit iff G can be decomposed into 'n' disjoint cycles.

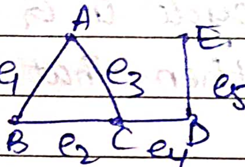
3) If in a simple connected graph with 'n' vertices ($n \geq 3$), the sum of the degrees of every pair of non-adjacent vertices $\geq n$, then the graph is Hamiltonian.

4) If in a simple connected graph with 'n' vertices ($n \geq 3$), the degree of every vertex is $\geq (n/2)$, then the graph is Hamiltonian.

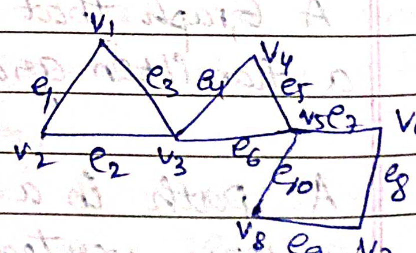
5) In the complete graph with 'n' vertices, where 'n' is odd number and ($n \geq 3$), there are $\left(\frac{n-1}{2}\right)$ 'n' disjoint Hamiltonian cycles.

Examples:-

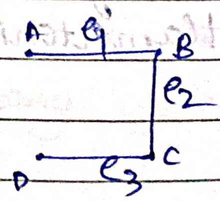
a) Euler trail:-
$C e_3 A e_1 B e_2 C e_4 D e_5 E$



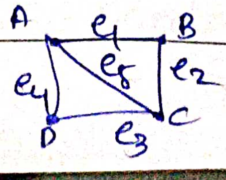
b) Euler circuit:-
$V_1 e_1 V_2 e_2 V_3 e_6 V_5 e_{10} V_8 e_9 V_7 e_8 V_6 e_7$
$V_5 e_5 V_4 e_4 V_3 e_3 V_1$



c) Hamiltonian path:-
$A e_1 B e_2 C e_3 D$

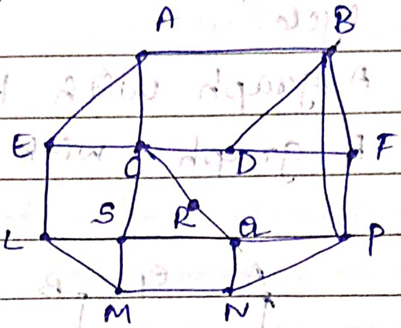


d) Hamiltonian cycle:-
$A e_1 B e_2 C e_3 D e_4 A$



Problems:-

1) ST, it is a hamiltonian graph.

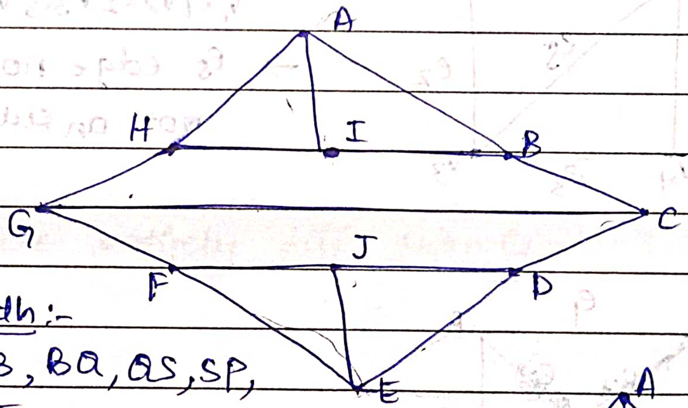


→ AB, BD, DF, FP, PN, NM, ML, LS, SQ, QR, RC, CE.
→ closed walk,
→ No edge repetition.
→ No vertex repetition.
→ All vertices included.

2) ST, the Peterson graph does not have a Hamilton cycle in it but Has Hamilton path in it.

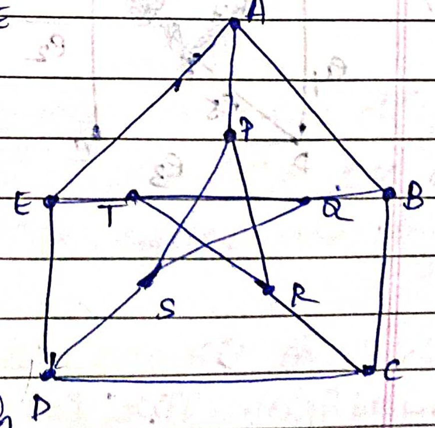
Note:-

Petersen Graph:-
→ It is a regular graph with $n=10$ and $\deg(V)=3$.



Hamilton path:-
AE, ED, DC, CB, BQ, QS, SP, PR, RT, TE

(or)



Consider, non-adjacent vertices A and C,
$\deg(A)=3$ and $\deg(C)=3$
$\deg(A)+\deg(B)=6 \ngeq 10$
∴ It is not a Hamilton graph bcoz no Hamilton cycle.

3) Draw the graph

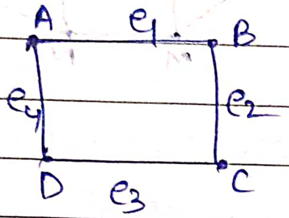
a) Graph with Hamiltonian cycle & Euler circuit

b) A graph which has Euler circuit but no hamilton cycle

c) A graph with Hamilton cycle but no Euler circuit.

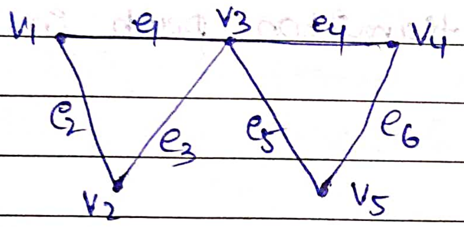
d) A graph with neither of the two.

a)



$A e_1 B e_2 C e_3 D e_4 A$

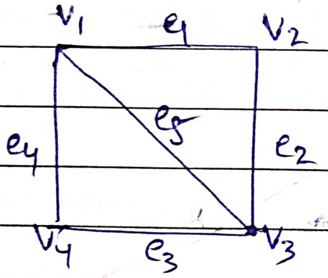
b)



$V_1 e_2 V_2 e_3 \underline{V_3} e_5 V_5 e_6 V_4 e_4 \underline{V_3} e_1 V_1$

$\rightarrow V_3$ (vertex repeated), Hence no Hamilton cycle.

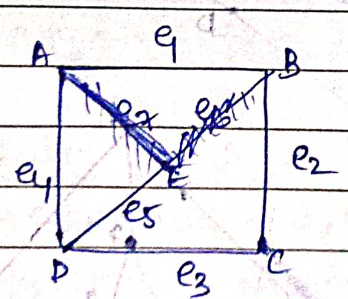
c)



$V_1 e_1 V_2 e_2 V_3 e_3 V_4 e_4 V_1$

$\rightarrow e_5$ edge not included, hence not an euler circuit.

d)



4) 7 members of a team meet everyday at lunch. They decide to sit such that every member has diff. neighbours everyday at each lunch. How many days can this arrangement last (using Graph theory).

→ The problem can be represented with a graph G with 7 vertices such each vertex represents a member and an edge joining 2 vertices represents that they are sitting next to each other.

→ Let x & y be any 2 arbitrary vertices such that they are adjacent to each other. Since every member is allowed to sit next to every other member, G is a complete graph of 7 vertices.

→ Every seating arrangement around the table is clearly a Hamilton cycle.

→ The first day of their meeting, they can sit in any order and it will be a Hamilton cycle.

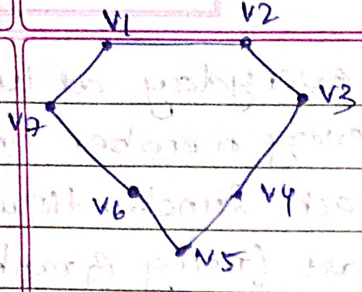
→ In 2nd day, they have to sit such that every member has diff. neighbours, hence C_2 is another Hamilton cycle entirely diff. than C_1.

→ Clearly C_1 and C_2 are edge disjoint cycles.

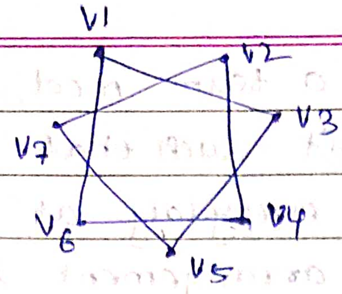
→ |||ly all possible Hamilton cycles are edge disjoint. Since, G is a complete graph with odd no. of vertices, the possible Hamilton cycles are $\left(\frac{n-1}{2}\right)$

i.e., $\frac{7-1}{2} = 3$.

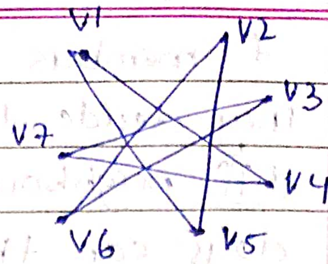
→ Hence, 7 members of the team can sit in 3 diff. ways so that every member has diff. neighbours each day.



C1



C2



C3

5) Let G be a simple graph with 'n' vertices and 'm' edges where m is atleast 3. If $m \geq \frac{1}{2}(n-1)(n-2)+2$, PT G is a Hamilton graph.

Let 'u' and 'v' be 2 non-adjacent vertices in 'G', Let 'x' and 'y' be their respective degrees.

If we delete 'u' and 'v' from G we get a subgraph with (n-2) vertices, and 'q' edges.

$$q \leq \frac{(n-2)(n-3)}{2} \longrightarrow (1)$$

Also,

$$m = q + x + y$$

$$\Rightarrow x + y = m - q \longrightarrow (2)$$

$$-q \geq -\frac{(n-2)(n-3)}{2}$$

$$x+y \geq \frac{(n-1)(n-2)}{2} + 2 - \frac{(n-2)(n-3)}{2}$$

$$x+y \geq \frac{1}{2}\left[n^2-3n+2+4-(n^2-5n+6)\right]$$

$$x+y \geq \frac{\cancel{n^2}-3n+\cancel{6}-\cancel{n^2}+5n-\cancel{6}}{2} \geq \frac{2n}{2}$$

$$\boxed{x+y \geq n}$$

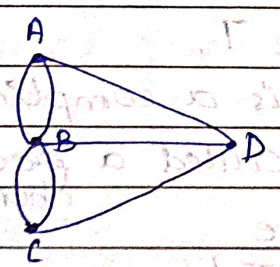
Therefore, in the theorem, if in a simple connected graph with 'n' vertices $(n \geq 3)$, the sum of the degrees of every pair of non-adjacent vertices is $\geq n$ when the graph is Hamiltonian.

6) ST a connected graph with exactly 2 vertices of odd degree has an Euler trail.

Let a & b be the only 2 vertices with odd degree in a connected graph.
Join these vertices by an edge 'e' even ~~after~~ if there is an edge already.
Then a & b become vertices of even degree.
Since, all other vertices in graph G are of even degree, the graph $G1 = G+e$ is connected and has vertices of even degree.

Therefore, G1 contains an Euler circuit wch must include 'e'. The trail got by deleting the edge 'e' in Euler circuit is an Euler trail in G.

7) Kongsberg Bridge Problem:-



→ To solve this problem, there must be atleast one Euler trail.

→ To be an Euler trail, all vertices must have even degree.

→ deg(A) = 3
deg(B) = 5
deg(C) = 3
deg(D) = 3

→ But the condition is not satisfied, Hence no solution exists.

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• A connected graph is a tree

• A connected graph is said to be minimally connected if the removal of any one edge from it disconnects the graph.

A Subgraph P of a connected graph G is called a Spanning tree of G if T is a tree and T contains all vertices of G.

Since a spanning tree of Graph G is a subgraph of G that contains all vertices of G, T is a maximal subgraph of G, for this reason Spanning tree is also called maximal tree.

A Spanning tree's ~~subgra~~ edges is called its branches.

If T is a spanning tree of graph G, then the edges of G wch are not in T are called the chords ~~pods~~ of G with respective T.

The set of all chords ~~pods~~ of G is a compliment of T in G and this set is called a chord ~~cord~~ set or ~~core~~ co tree of T ($\bar{T}$), therefore

$$\boxed{G = T \cup \bar{T}}$$

Let 'G' be a graph and suppose there is a positive real no. associated with each edge of G, then G is called a weighted graph and the 've' real no. associated with an edge e is called the

RAJDHANI

weight of graph G.

Suppose we consider all spanning trees of a connected weighted graph and find the ~~ways~~ weights of each of these spanning trees, a spanning tree whose weight is atleast is called the minimal spanning tree of a graph. This tree is not unique.

Remarks / Note :-

1) In a tree, there is one & only one path b/w every pair of vertices.

2) In a graph G, if there is ~~only~~ one and only one path b/w every pair of vertices, then Graph G is a tree.

3) A tree with 'n' vertices has 'n-1' edges.

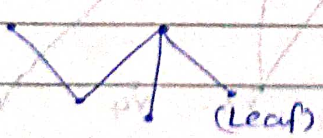
4) Any connected graph with 'n' vertices & 'n-1' edges is a tree.

5) A connected graph G is a tree iff adding an edge b/w any 2 vertices in G creates exactly one cycle in G.

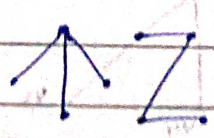
6) A graph is connected iff it has a spanning tree.

7) With respect to any of its spanning trees, a connected graph of 'n' vertices and 'm' edges has (n-1) branches and 'm-n+1' chords.

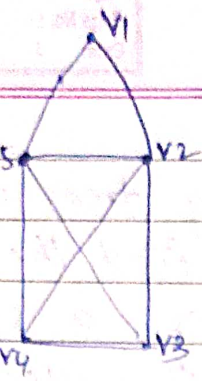
Examples :-



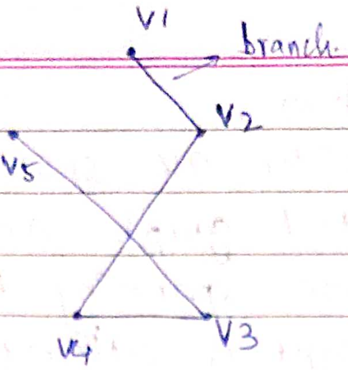
Tree



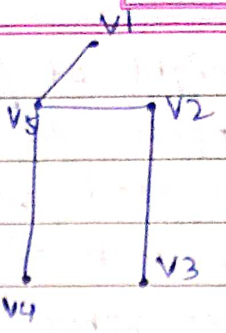
Forest



G



(T)
Spanning Tree



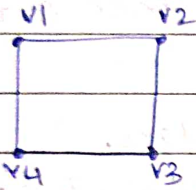
$(\overline{T})$
Co-Tree

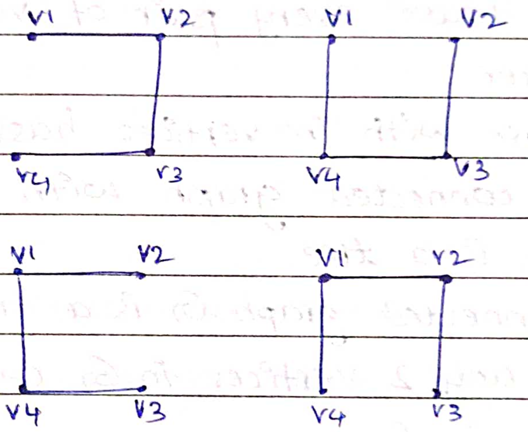
$$\boxed{G = T \cup \overline{T}}$$

Problems :-

1) Find all the spanning trees of graph G.

a)

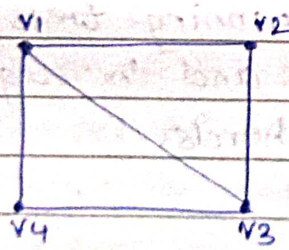


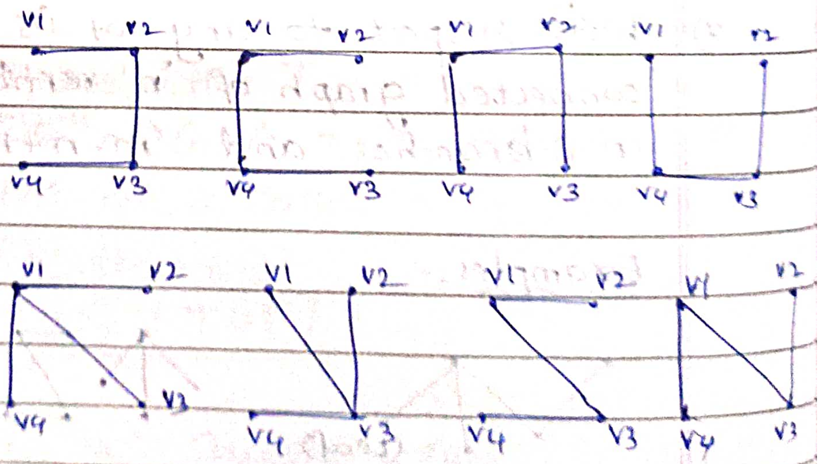


'n' = 4

'm' = edges = n-1 = 3

b)

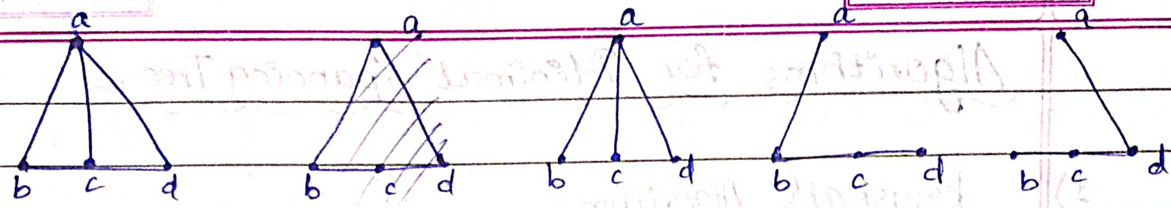




n = 4

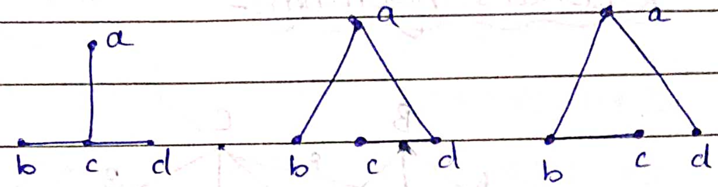
m = 3

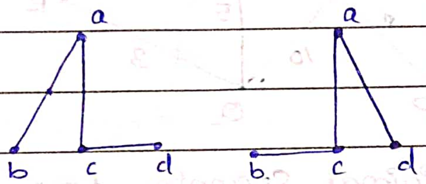
c)



$n=4$

$m=3$





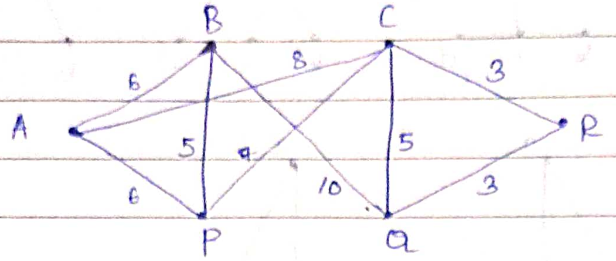
2) ST, a hamilton path is a spanning tree.

- A Hamilton path if such exists in a connected graph G, is a path wch contains every vertex of G and such that if G has 'n' vertices, then the path P has (n-1) edges.
- Thus, P is a connected subgraph of G with 'n' vertices and 'n-1' edges.
- Therefore, P is a tree.

Since, P contains all vertices of G, it is a spanning tree of G.

Algorithms for Minimal Spanning Tree :-

1) **Kruskal's Algorithm :-**



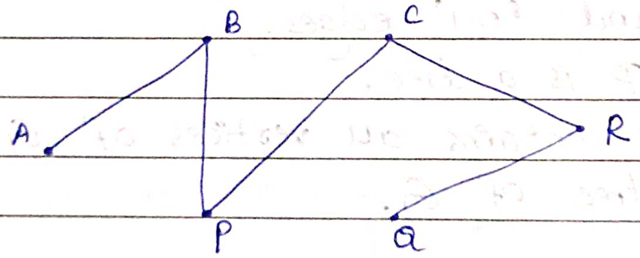
(weighted)

i) In minimal spanning tree,
'n' vertices and 'n-1' edges
$\Rightarrow$ $n=6$ and $m=5$

ii) Place the edges in ascending order.

Edges	CR	QR	CQ	BP	AB	AP	PC	AC	BQ
weight	3	3	5	5	6	6	7	8	10

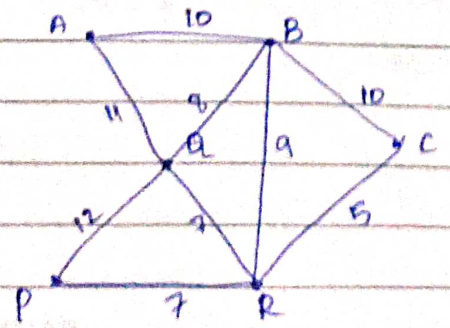
iii) Select the edges such that no cycle is formed.



$wt(T) = 24$

Problems :-

1)

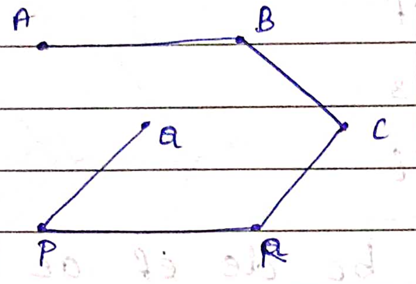


i) $n=6$
$m=5$

ii)

Edges	RC	QR	PR	QB	BR	BC	AB	AQ	PQ
weight	5	7	7	8	9	10	10	11	12

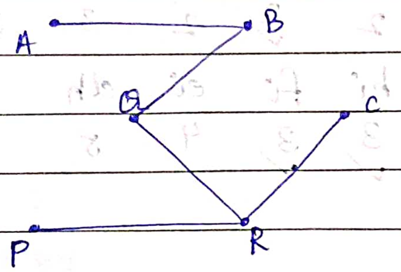
iii)



$10+10+5+7+12$

$Wt(T) = 44$

(X)

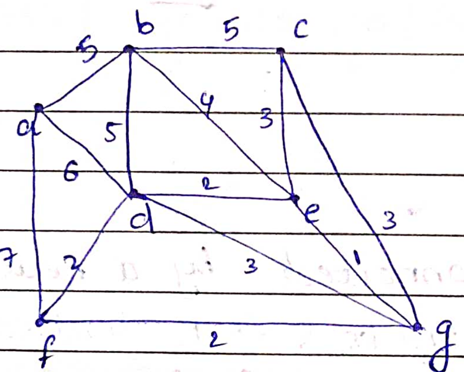


$10+8+7+5+7$

$Wt(T) = 37$

(✓)

3)



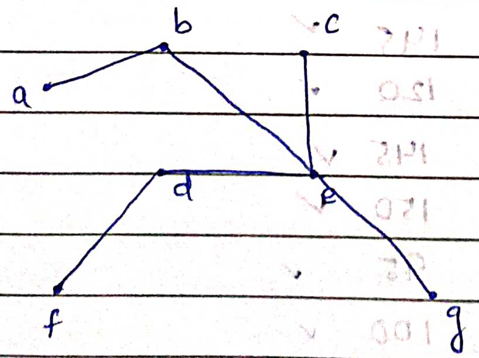
i) $n = 7$

$m = 6$

ii)

Edges	eg	de	df	fg	ce	dg	cg	be	bd	ab	bc	ad	af
weight	1	2	2	2	3	3	3	4	5	5	5	6	7

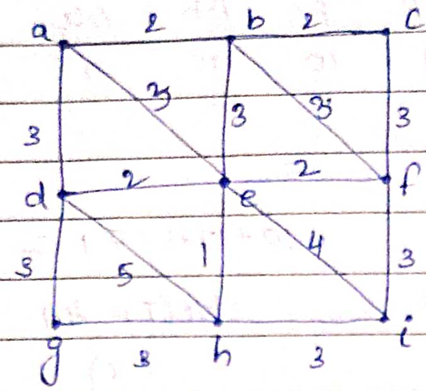
iii)



$1+2+3+4+5+2$

$Wt(T) = 17$

3)



i) $n = 9$

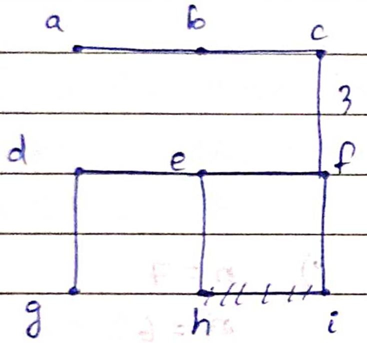
$m = 8$

ii)

Edges	eh	ab	bc	de	ef	ae	be	bf	cf	ad
weights	1 ✓	2 ✓	2	2 ✓	2	3	3	3	3	3

dg	gh	hi	fi	ei	dh
3 ✓	3	3 ✓	3 ✓	4	5

iii)



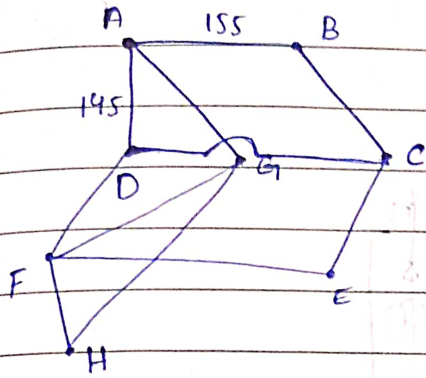
$2+2+3+3+3+1+2+3$

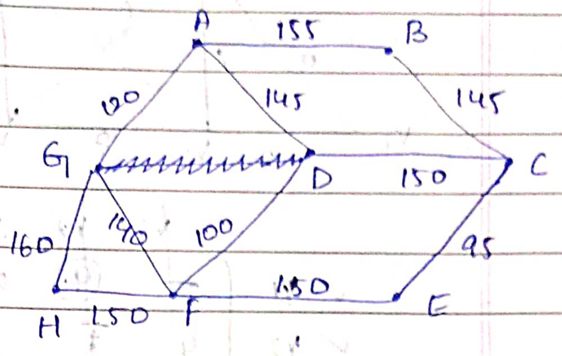
$wt(T) = \underline{19}$

2) 8 cities are to be connected by a new railway network. The possible tracks & costs involved to laiden (in cr), are summarised in following table.

Road track	Costs (in cr.)
AB	155 ✓
AD	145 ✓
AG	120 ✓
BC	145 ✓
CD	150 ✓
CE	95 ✓
DF	100 ✓
EF	150 ✓
FG	140 ✓
FH	150 ✓
GH	160 ✓

Determine a railway network of minimal cost that connects all the cities.





(G)

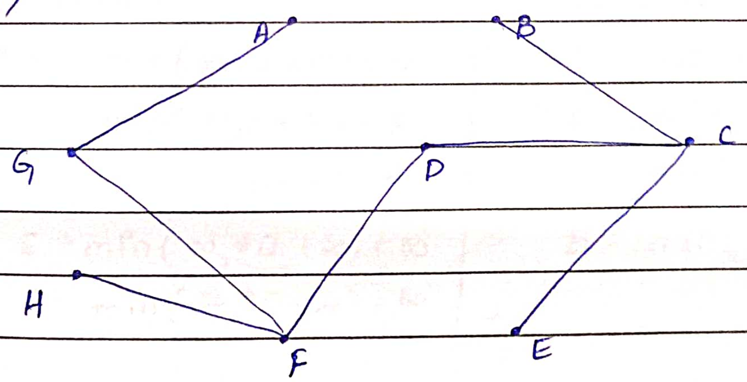
i) $n = 8$

$m = \cancel{8}7$

ii)

Edges	CE	DF	AG	FG	AD	BC	CD	EF	FH	AB	GH
weights	95	100	120	140	145	145	150	150	150	155	160
	✓	✓	✓	✓		✓	✓		✓		

iii)



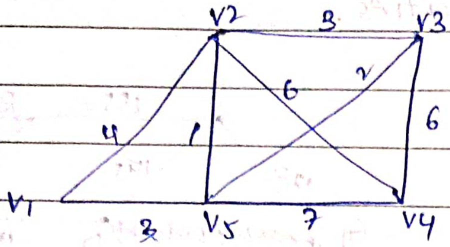
$= 95+100+120+140+145+150+150$

$= 90+450+360$

$= 900$

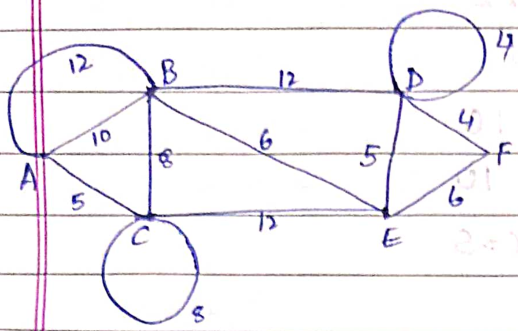
$\boxed{wt(T) = 900}$.

2) Prims Algorithm :-

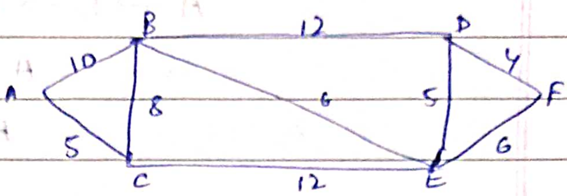


$$
\begin{array}{c|ccccc}
 & V1 & V2 & V3 & V4 & V5 \\
\hline
\checkmark\ V1 & - & 4 & \infty & \infty & 5 \\
\checkmark\ V2 & 4 & - & 3 & 6 & 1 \\
\checkmark\ V3 & \infty & 3 & - & 6 & 2 \\
\checkmark\ V4 & \infty & 6 & 6 & - & 7 \\
\checkmark\ V5 & 5 & 1 & 2 & 7 & -
\end{array}
$$

1) Using Djkstra's algorithm, find shortest distance b/w vertex A to F.



i) Delete selfloops & parallel edges



ii)

	A	B	C	D	E	F
A	$\boxed{0}$ ✓	∞	∞	∞	∞	∞
C	$\boxed{0}$	10	$\boxed{5}$ ✓	∞	∞	∞
B	$\boxed{0}$	$\boxed{10}$ ✓	$\boxed{5}$	∞	$\boxed{17}$	∞
E	$\boxed{0}$	$\boxed{10}$	$\boxed{5}$	22	$\boxed{16}$ ✓	∞
D	$\boxed{0}$	$\boxed{10}$	$\boxed{5}$	$\boxed{21}$ ✓	$\boxed{17}$	22
F	$\boxed{0}$	$\boxed{10}$	$\boxed{5}$	$\boxed{21}$	$\boxed{17}$	$\boxed{22}$ ✓

Vertex A :-

✓ B = min(∞, 0 + 10) = 10
C = min(∞, 0 + 5) = 5
D = min(∞, 0 + ∞) = ∞
E = min(∞, 0 + ∞) = ∞
F = min(∞, 0 + ∞) = ∞

Vertex C :-

D = min(∞, 5 + ∞) = ∞
E = min(∞, 5 + 12) = 17
F = min(∞, 5 + ∞) = ∞
B = min(10, 5 + 8) = 10

Vertex B :-

D = min(∞, 10 + 12) = 22
E = min(17, 10 + 6) = 16
F = min(∞, 10 + ∞) = ∞

Vertex E :-

D = min(22, 16 + 5) = 21
F = min(∞, 16 + 6) = 22

Vertex D :-

F = min(22, 21 + 4) = 22

iii) Backtracing :-

A → B → E → F = 22

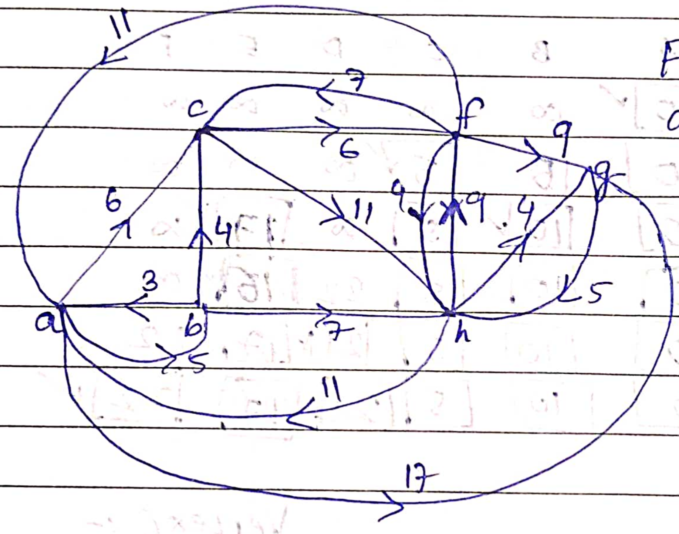
A → B → E → D = 21

A → B → E = 16

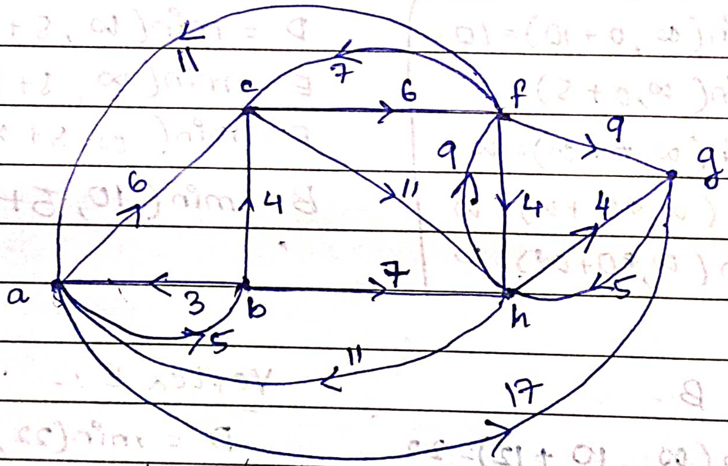
A → B = 10

A → C = 5

2)



From vertex C to all other vertices



a	b	c	d	f	h	g
$\boxed{0}$	∞	∞	∞	∞		
$\boxed{0}$						
$\boxed{0}$						
$\boxed{0}$						
$\boxed{0}$						
$\boxed{0}$						
$\boxed{0}$						

Vertex c :-

b = min(∞, 0+∞) = ∞

a = min(∞, 0+∞) = ∞

f = min(∞, 0+6) = 6

h = min(∞, 0+11) = 11

g = min(∞, 0+∞) = ∞

	c	a	b	f	h	g
c	$\boxed{0}$ ✓	∞	∞	∞	∞	∞
f	$\boxed{0}$	∞	∞	$\boxed{6}$ ✓	11	∞
h	$\boxed{0}$	17	∞	$\boxed{6}$	$\boxed{10}$ ✓	15
g	$\boxed{0}$	17	∞	$\boxed{6}$	$\boxed{10}$	$\boxed{14}$ ✓
a	$\boxed{0}$	$\boxed{17}$ ✓	∞	$\boxed{6}$	$\boxed{10}$	$\boxed{14}$
b	$\boxed{0}$	$\boxed{17}$	$\boxed{22}$ ✓	$\boxed{6}$	$\boxed{10}$	$\boxed{14}$

Vertex f :-

a = min(∞, 6+11) = 17

b = min(∞, 6+∞) = ∞

h = min(11, 6+4) = 10

g = min(∞, 6+9) = 15

Vertex h :-

a = min(17, 10+11) = 17

b = min(∞, 10+∞) = ∞

g = min(15, 10+4) = 14

Vertex g :-

a = min(17, 14+∞) = 17

b = min(∞, 14+∞) = ∞

Vertex a :-

b = min(∞, 17+5) = 22

$\boxed{c \to f \to a \to b \Rightarrow 22}$

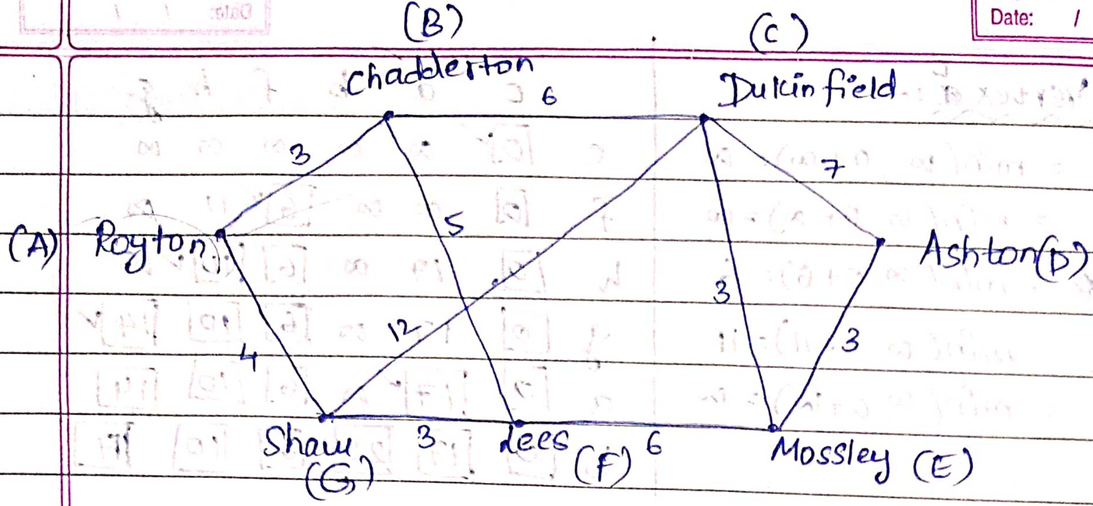
c → f → a = 17

c → f → h → g = 14

c → f → h = 10

c → f = 6

3) The diagram below shows roads connecting villages near to the city Royton. The nos. on each arc represent the distance in miles along each road. Leon lives in Royton and works in Ashton ~~an~~ Use Dijkshetra's algorithm to find the minimum distance for Leon's jouoney to work.



~~A~~ ~~C~~ ~~F~~

	A	B	C	D	E	F	G
A	$\boxed{0}$ ✓	∞	∞	∞	∞	∞	∞
B	$\boxed{0}$	$\boxed{3}$ ✓	∞	∞	∞	∞	4
F	$\boxed{0}$	$\boxed{3}$	9	∞	∞	$\boxed{8}$ ✓	∞
C	$\boxed{0}$	$\boxed{3}$	$\boxed{9}$ ✓	∞	14	$\boxed{8}$	11
G	$\boxed{0}$	$\boxed{3}$	$\boxed{9}$	16	12	$\boxed{8}$	$\boxed{11}$ ✓
E	$\boxed{0}$	$\boxed{3}$	$\boxed{9}$	16	$\boxed{12}$ ✓	$\boxed{8}$	$\boxed{11}$
D	$\boxed{0}$	$\boxed{3}$	$\boxed{9}$	$\boxed{15}$ ✓	$\boxed{12}$	$\boxed{8}$	$\boxed{11}$

Vertex A :-

$B = \min(\infty, 0+3) = 3$

$C = \min(\infty, 0+\infty) = \infty$

$D = \min(\infty, 0+\infty) = \infty$

$E = \min(\infty, 0+\infty) = \infty$

$F = \min(\infty, 0+\infty) = \infty$

$G = \min(\infty, 0+4) = 4$

Vertex B :-

$C = \min(\infty, 3+6) = 9$

$D = \min(\infty, 3+\infty) = \infty$

$E = \min(\infty, 3+\infty) = \infty$

$F = \min(\infty, 3+5) = 8$

$G = \min(\infty, 3+\infty) = \infty$

Vertex F :-

$C = \min(9, 8+\infty) = 9$

$D = \min(\infty, 8+\infty) = \infty$

$E = \min(\infty, 8+6) = 14$

$G = \min(\infty, 8+3) = 11$

Vertex C :-

$D = \min(\infty, 9+7) = 16$

$E = \min(14, 9+3) = 12$

$G = \min(11, 9+12) = 11$

Vertex G:-

$D = \min(16, 11+\infty) = 16$

$E = \min(12, 11+\infty) = 12$

Vertex E :-

$D = \min(16, 12+3) = 15$

$A \rightarrow B \rightarrow C \rightarrow E \rightarrow D = 15$

$A \rightarrow B \rightarrow C \rightarrow E = 12$

$A \rightarrow B \rightarrow F \rightarrow G = 11$

$A \rightarrow B \rightarrow C = 9$

$A \rightarrow B \rightarrow F = 8$

$A \rightarrow B = 3$

Royton → Chadderton → Dukinfield → Mossley → Ashton = 15