Probability Distributions

1) Random Variable:-

A random variable is defined on a sample space. It quantifies a random phenomenon by associating a real no. with one or more sample points i.e.,

$$X : S \rightarrow R$$

Eg:- X - Number of heads in tossing 3 coins.

$x = \{0, 1, 2, 3\}$

$$p(x) = \frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8}$$

x	0	1	2	3
$p(x)$	1/8	3/8	3/8	1/8

prob. distribution

μ - Mean $= \sum_i x\,p(x)$

variance, $\sigma^2 = \sum_i x^2[p(x)] - \mu^2$

2) Binomial Distribution:-

$B(n, p)$

where n - no. of trials

p - prob. of success.

$$p(x) = {}^nC_x\, p^x q^{n-x} \qquad x = 0, 1, 2, \ldots n$$

$\mu = np$

$\sigma^2 = npq$

3) Random Variables

Discrete	Continous
→ countable values	→ range of values
→ infinitely countable	→ Eg:- Normal,
Eg:-(Poisson's distⁿ)	Exponential
→ $p(x)$ or $f(x)$	→ $f(x)$ - prob.
prob. mass function	density fn
* $p(x) \geq 0$	* $f(x) \geq 0$
$\sum_i p(x) = 1$	$\int f(x)dx = 1$
* $\mu = \sum_i x p(x)$	* $\mu = \int x f(x) dx$
$\sigma^2 = \sum_i x^2 p(x) - \mu^2$	$\sigma^2 = \int x^2 f(x) dx - \mu^2$

4) Poisson Distribution:-

→ Discrete distribution

⇒ Binomial distribution follows Poisson distribution

→ $n \to \infty$ n is very large and p is very small.
$p \to 0$

⇒ $\lambda = np$

$X \sim P(\lambda)$

$p(x) = \frac{e^{-\lambda}\lambda^{x}}{x!}$, $x = 0, 1, 2, \ldots$

→ Applications :- $X \sim B(n,p)$ → Queuing Theory

→ To show that $p(x)$ is a prob. distribution

x	0	1	2	3	
$p(x)$	$e^{-\lambda}$	$\frac{e^{-\lambda}\lambda}{1!}$	$\frac{e^{-\lambda}\lambda^2}{2!}$	$\frac{e^{-\lambda}\lambda^3}{3!}$	$\ldots$

→ To prove $p(x) \geq 0$

$$\Sigma_{i} p(x) = 1$$

Since $\lambda > 0 \Rightarrow e^{-\lambda} \geq 0$

$$\Rightarrow p(x) \geq 0$$

$$\Sigma_{i} p(x) = e^{-\lambda} + \frac{e^{-\lambda}\lambda}{1!} + \frac{e^{-\lambda}\lambda^2}{2!} + \frac{e^{-\lambda}\lambda^3}{3!} + \cdots$$

$$= e^{-\lambda}\left[1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots\right]$$

$$= e^{-\lambda} e^{\lambda}$$

$$= 1.$$

5) ~~To~~ Derive mean and variance of a poisson distribution

$$X \sim P(\lambda)$$

$$p(x) = \frac{e^{-\lambda}\lambda^{x}}{x!}, \quad x = 0, 1, 2, \cdots$$

$$\mu = \sum_{x=0}^{\infty} x p(x)$$

$$= \sum_{x=0}^{\infty} \frac{x e^{-\lambda} \lambda^{x}}{x!} = \sum_{x=1}^{\infty} \frac{e^{-\lambda}\lambda^{(x-1)}}{(x-1)!}$$

$$= \lambda e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^{(x-1)+1}}{(x-1)!}$$

$$= \lambda e^{-\lambda}\left[1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots\right]$$

$$= \lambda e^{-\lambda} e^{\lambda}$$

$$= \lambda$$

$$\Rightarrow \boxed{\mu = \lambda}$$

Variance : $\sigma^2 = \sum x^2 p(x) - \mu^2 \rightarrow (1)$

Consider,

$$\sum x^2 p(x) = \sum (x^2 - x + x) p(x)$$

$$= \sum x(x-1) p(x) + \sum x p(x)$$

$$= \sum x(x-1) p(x) + \mu \rightarrow (2)$$

Consider,

$$\sum_{x=0}^{\infty} x(x-1) p(x)$$

$$\sum_{x=0}^{\infty} x(x-1) \frac{e^{-\lambda} \lambda^x}{x!} \quad , \quad x = 0, 1, 2, \ldots$$

$$= \sum_{x=2}^{\infty} \frac{e^{-\lambda} \lambda^{(x-2)+2}}{(x-2)!}$$

$$= e^{-\lambda} \lambda^2 \sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!}$$

$$= e^{-\lambda} \lambda^2 \left[1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \ldots \right]$$

$$= \cancel{e^{-\lambda}} \lambda^2 \cancel{e^{\lambda}}$$

$$= \lambda^2$$

$$\Rightarrow \boxed{\sum_{x=0}^{\infty} x(x-1) p(x) = \lambda^2} \rightarrow (3)$$

Substitute (3) in (2)

$$\Rightarrow \sum x^2 p(x) = \lambda^2 + \lambda \rightarrow (4)$$

Substitute (4) in (1)

$$\Rightarrow \sigma^2 = \cancel{\lambda^2} + \lambda - \cancel{\lambda^2}$$

$$\Rightarrow \boxed{\sigma^2 = \lambda}$$

Hence,

$$\mu = \sigma^2 = \lambda$$

and $S{:}D = \sigma = +\sqrt{\lambda}$.

Problems :-

1) It is known from past experience that in a certain industrial plant there are on the average 4 industrial accidents per year. Find the prob. that in a given year there will be less than 4 accidents.

Let 'X' be the random variable,

X : Number of accidents in an industrial plant

$X \sim P(\lambda) \quad \lambda = 4$

$$\boxed{p(x) = \frac{e^{-\lambda}\lambda^x}{x!}} \quad x = 0, 1, 2, \ldots$$

To find: $p(x<4) = p(x=0) + p(x=1) + p(x=2) + p(x=3)$

$$p(x<4) = \frac{e^{-\lambda}\lambda^0}{0!} + \frac{e^{-\lambda}\lambda^1}{1!} + \frac{e^{-\lambda}\lambda^2}{2!} + \frac{e^{-\lambda}\lambda^3}{3!}$$

$$= e^{-\lambda}\left[1 + \lambda + \frac{\lambda^2}{2} + \frac{\lambda^3}{6}\right]$$

$$= e^{-4}\left[1 + 4 + \frac{16}{2} + \frac{64}{6}\right] = e^{-4}(23.6667)$$

$$= \underline{0.4335}$$

2) The prob. of a poisson variable taking the values 3 & 4 are equal. Calculate the prob. of the variable taking the value zero.

Given, 'x' is a random variable

$$X \sim P(\lambda)$$

$$P(x) = \frac{e^{-\lambda}\lambda^{x}}{x!} \; ; \; x = 0, 1, 2, \ldots$$

$$P(X=3) = P(X=4)$$

To find :- $P(X=0)$

Soln :- $P(X=3) = P(X=4)$

$$\cancel{\frac{e^{-3}3^{x}}{x!} = \frac{e^{-4}4^{x}}{x!}}$$

$$\cancel{e = \frac{4^{x}}{3^{x}}}$$

$$\frac{e^{-\lambda}\lambda^{3}}{3!} = \frac{e^{-\lambda}\lambda^{4}}{4!}$$

$$\boxed{\lambda = 4}$$

$$P(X=0) = \frac{e^{-4}4^{0}}{0!} = e^{-4}.$$

3) The prob. that a newsreader commits no mistake in reading the news is e^{-3}. Find the prob. that on a particular news broadcast he commits :

a) only 2 mistakes.

b) utmost 3 mistakes.

c) more than 3 mistakes.

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X → Newsreader committing a mistake

$X \sim P(\lambda)$

$P(x) = \frac{e^{-\lambda}\lambda^x}{x!}$; $x = 0,1,2,\ldots$

Given :- $P(X=0) = e^{-3}$.

$\Rightarrow \underline{\lambda = 3}$.

a) $P(X=2) = \frac{e^{-3}3^2}{2!} = \frac{9e^{-3}}{2} = 0.2240$.

b) $P(x \le 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$

$= e^{-3}\left[1 + 3 + \frac{3^2}{2} + \frac{3^3}{6}\right]$

$= e^{-3}(13)$

$= 0.6472$.

c) $P(X>3) = 1 - P(X \le 3)$

$= 1 - 0.6472$

$= 0.3528$.

4) The no. of accidents in a year by a Taxi in a city follows Poisson distⁿ with mean 3. Out of 1000 Taxi drivers find approximately the no. of drivers with:

a) no accidents in a year.

b) more than 3 accidents in a year.

Given :- X - No. of accidents by a taxi driver

$X \sim P(\lambda = 3)$

$P(x) = \frac{e^{-\lambda}\lambda^x}{x!}$, $x = 0,1,2,\ldots$

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a) $P(X=0) = P(\text{No. accidents}) = e^{-3}$.

$\therefore$ No. of taxi drivers with no accidents
$= e^{-3} \times 1000$
~ 49.7
~ 50.

b) $P(X>3) = 1 - P(X \leq 3)$
$= 1 - [P(X=0) + P(X=1) + P(X=2) + P(X=3)]$

$$= 1 - e^{-3}\left[1 + 3 + \frac{3^2}{2!} + \frac{3^3}{3!}\right]$$

$= 0.3528$

No. of taxi drivers with more than 3 accidents in a year $= 1000 \times 0.3528$
≈ 353

5) A communication channel receives independent pulses at the rate of 12 pulses per microsecond. The prob. of transmission error is 0.001 for each μs. Compute the probabilities of:
a) No error during a μs.
b) 1 error per μs.
c) atleast 1 error per μs.
d) 2 errors —"—.
e) utmost 2 errors —"—.

Given:- X – No. of transmission error per μs.

$$X \sim B(n=12,\ p=0.001)$$

Since p is small $\Rightarrow \lambda = np$.

$$\Rightarrow \lambda = 0.012$$

Hence, $X \sim P(\lambda = 0.012)$

$$P(x) = \frac{e^{-\lambda}\lambda^{x}}{x!};\ x = 0,1,2,\ldots$$

a) $P(X=0) = e^{-0.012} = 0.9881.$

b) $P(X=1) = (e^{-0.012})(0.012) = 0.01186.$

c) $P(X \geq 1) = 1 - P(X<1)$

$= 1 - P(X=0)$

$= 1 - 0.9881$

$= 0.0119.$

d) $P(X=2) = \dfrac{(e^{-0.012})(0.012)^2}{2!} = 71.14 \times 10^{-6}.$

e) $P(X \leq 2) = \cancel{1/4}\ \cancel{P(X<2)}\ P(X=0) + P(X=1)$

$= \cancel{1/4}\ [P(X=0) + P(X=1)]$

$= \cancel{1/4}\ [0.9881 + 0.01186]$

$= \cancel{1/4}\ 0.9999.$

$= \cancel{0.0001}.$

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6) In a certain factory, turning out razer blades there is a small chance of 0.002 for any blade to be defected. The blades are supplied in packets of 10. Use Poisson distribution to calculate the approx. no. of packets containing:
i) No defected blade.
ii) 1 defected blade.
iii) 2 — " — respectively in a consignment of 10,000 packets.

X – No. of defected blades in 1 packet

$X \sim B(n=10, p=0.002)$

Since, p is very small,

$\Rightarrow \lambda = np$

$\lambda = 0.02$

Now, $X \sim P(\lambda)$

$P(x) = \frac{e^{-\lambda}\lambda^x}{x!}$; $x = 0,1,2,\ldots$

i) $P(X=0) = e^{-\lambda} = e^{-0.02}$

No. of packets containing no defected blade =

$= 10000 \times e^{-0.02}$

$= 9.8019k \approx 9802$.

ii) $P(X=1) = e^{-\lambda}\lambda = (0.02)e^{-0.02}$

No. of packets containing 1 defected blade =

$= 10000 \times 0.02 \times e^{-0.02}$

$= 196.0397$.

≈ 197.

(iii) $P(X=2) = \dfrac{e^{-\lambda}\lambda^2}{2} = \dfrac{e^{-0.02}(0.02)^2}{2}$

No. of packets containing 2 defected blades =

$$\frac{10000 \times e^{-0.02} \times (0.02)^2}{2}$$

$$\approx 1.96$$

$$\approx 2.$$

7) The no. of accidents per day as recorded in a textile industry over a period of 400 days is given below. Fit a Poisson distⁿ for the given data and calculate the theoretical frequency (expected frequency).

X (no. of acc.)	0	1	2	3	4	5
Frequency	173	168	37	18	3	1

X – no. of accidents per day

$X \sim P(\lambda)$

$$P(x) = \frac{e^{-\lambda}\lambda^x}{x!} \; ; \; x = 0, 1, 2, \ldots$$

Since, λ is not known,

$$\lambda = \mu = \frac{\Sigma fx}{\Sigma f} = \frac{313}{400} = 0.7825$$

x	f	fx	$p(x)$	$400p(x)$	
0	173	0	0.4573	≈182.9	183
1	168	168	0.3578	≈143.12	143
2	37	74	0.1399	≈55.98	56
3	18	54	0.0365	≈ ~~15~~14.6	15
4	3	12	0.0071	≈2.85	3
5	1	5	0.0011	≈0.44	1
	$\Sigma f = 400$	$\Sigma fx = 313$			

$p(x=0) = e^{-\lambda} = e^{-0.7825} = 0.4573$

$p(x=1) = e^{-0.7825}(0.7825) = 0.3578$

$p(x=2) = \dfrac{e^{-0.7825}(0.7825)^2}{2} = 0.1399$

$p(x=3) = \dfrac{e^{-0.7825}(0.7825)^3}{6} = 0.\cancel{0}0365$

$p(x=4) = \dfrac{e^{-0.7825}(0.7825)^4}{24} = 0.0071$

$p(x=5) = \dfrac{e^{-0.7825}(0.7825)^5}{120} = 0.0011$

8) Fit a Poisson distribution for the following data and calculate theoretical frequencies.

x	0	1	2	3	4
f	111	63	22	3	1

Continous Probability distribution:-

X - infinite values / range of values
$f(x)$ - probability function
$f(x) \geq 0$
$\int f(x)dx = 1$

$\mu = \int x f(x) dx$
$\sigma^2 = \int x^2 f(x) dx - \mu^2$

Exponential distribution:-

$$X \sim E(\alpha)$$
$$f(x) = \alpha e^{-\alpha x} \quad ; \alpha, x > 0$$

To show $f(x)$ is a prob. density fn (p.d.f):

Given, $\alpha > 0$, $x > 0$
$$\Rightarrow \alpha e^{-\alpha x} \geq 0$$

$$\int_0^\infty \alpha e^{-\alpha x} dx = \frac{\cancel{\alpha} e^{-\alpha x}}{-\cancel{\alpha}}\Big]_0^\infty$$
$$= -\left[e^{-\infty} - e^0\right] = -[0-1]$$
$$= 1$$

Hence, it is a p.d.f.

Derive an expression for mean and variance of an exponential distribution.

$$X \sim E(\alpha)$$
$$f(x) = \alpha e^{-\alpha x} \quad ; \alpha, x > 0$$

WKT, $\mu = \int_0^\infty x f(x) dx = \int_0^\infty x \alpha e^{-\alpha x} dx$

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$$= \alpha \int_0^{\infty} x e^{-\alpha x} dx$$

$x :\to 1 \to 0$

$e^{\alpha x} : \frac{e^{-\alpha x}}{-\alpha} \to \frac{e^{-\alpha x}}{\alpha^2}$

$$= \alpha \left[-\frac{x e^{-\alpha x}}{\alpha} - \frac{e^{-\alpha x}}{\alpha^2} \right]_0^{\infty}$$

$$= -\left[\left(\frac{\infty}{e^{\infty}} - 0\right) + \frac{1}{\alpha}(0-1) \right]$$

$$\Rightarrow \boxed{\mu = \frac{1}{\alpha}}$$

Variance, $\sigma^2 = \int_0^{\infty} x^2 f(x) dx - \mu^2 \to (1)$

Consider, $\int_0^{\infty} x^2 f(x) dx$

$$= \int_0^{\infty} x^2 (\alpha e^{-\alpha x}) dx$$

$$= \alpha \left[x^2 \left(\frac{e^{-\alpha x}}{-\alpha} \right)_0^{\infty} - (2x)\left(\frac{e^{-\alpha x}}{\alpha^2} \right)_0^{\infty} + (2)\left(\frac{e^{-\alpha x}}{-\alpha^3} \right)_0^{\infty} \right]$$

$$= \alpha \left[-\frac{1}{\alpha}\left(\frac{\infty}{e^{\alpha x}} - \frac{0}{e^0} \right) - \frac{2}{\alpha^2}(0-0) - \frac{2}{\alpha^3}(0-1) \right]$$

$$= \alpha \left[0 - 0 - \frac{2}{\alpha^3}(-1) \right]$$

$$= \frac{2}{\alpha^2} \to (2)$$

Substitute (2) in (1)

$$\Rightarrow \sigma^2 = \frac{2}{\alpha^2} - \frac{1}{\alpha^2}$$

$$\Rightarrow \boxed{\sigma^2 = \frac{1}{\alpha^2}}$$

S.D, $\boxed{\sigma = +\frac{1}{\alpha} = \mu}$

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Let X - Increase in sales per day

$X \sim E(\alpha)$

$f(x) = \alpha e^{-\alpha x} \quad ; \; x, \alpha > 0$

$\mu = \frac{1}{\alpha} = 600$

$\Rightarrow \boxed{\alpha = \frac{1}{600}}$

Let 'A' be the amount

$\rightarrow$ Sales tax is 9%

$\Rightarrow \frac{9}{100} \times A = 81$

$\Rightarrow \boxed{A = 900}$

$P(\text{Sales tax will exceed 81 per day}) = \int_{900}^{\infty} \alpha e^{-\alpha x} \, dx$

$= \left(\frac{1}{600}\right) \int_{900}^{\infty} e^{-\left(\frac{1}{600}\right)x} dx$

$= \left(\frac{1}{\cancel{600}}\right) \left[\frac{e^{-x/600}}{-\cancel{(1/600)}} \right]_{900}^{\infty}$

$= -\left[e^{-\infty} - e^{-\frac{900}{600}} \right]$

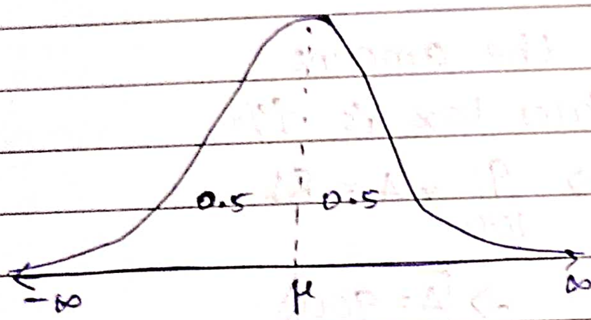
$= e^{-3/2}$

$= \underline{0.2231}$

Normal Distribution :-

* Most widely used
* $X \sim N(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)^2} \quad ; -\infty < x, \mu < \infty$$

* 

→ Asymptotic
→ Mean = Median = Mode (Measures of central tendency)
→ $\mu \pm 3\sigma = 99\%$

* $\int_{-\infty}^{\infty} f(x) = 1$
* $p(a < x < b) = \int_a^b f(x)\,dx$

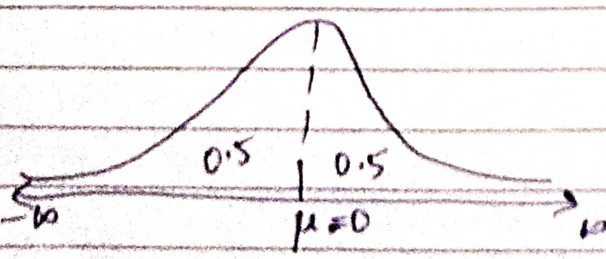
$$= \int_a^b \frac{1}{\sigma\sqrt{2\pi}} e^{-\left(\frac{x-\mu}{\sigma\sqrt{2\pi}}\right)^2} dx$$

Standard Normal Distribution :-

$Z = \frac{x-\mu}{\sigma} \sim N(0,1)$

* $Z \sim N(\mu = 0, \sigma^2 = 1)$

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\left(\frac{x^2}{2\pi}\right)} dz \quad ; -\infty < Z < \infty$$

* 

→ Mean = Median = Mode
→ $\int_{\infty}^{-\infty} f(x)\,dx = 1$
→ $p(a < z < b) = \int_a^b f(z)\,dz$

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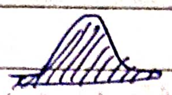
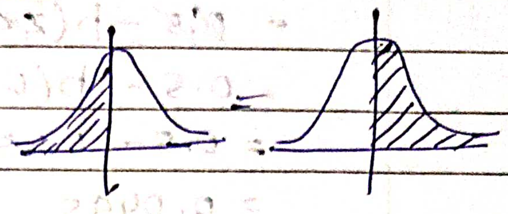
Properties :-

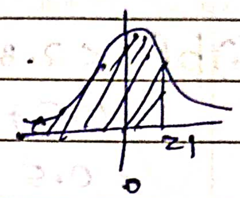
* $\int_{-\infty}^{\infty} f(z)\,dz = P(-\infty<z<\infty) = 1 =$ 

* $P(-\infty<z<0) = P(0<z<\infty) = 0.5$



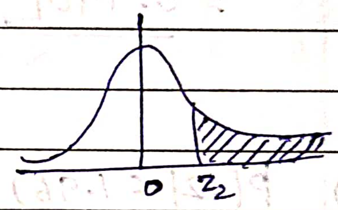
* $P(-\infty<z<z_1) = 0.5 + p(0<z<z_1)$

  $z_1 = +ve$



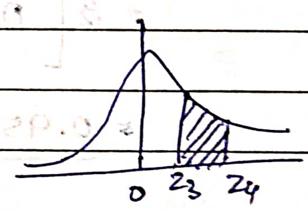
* $P(z>z_2) = 0.5 - p(0<z<z_2)$

  $z_2 = +ve$



* $P(z_3<z<z_4) = p(0<z<z_4) - p(0<z<z_3)$

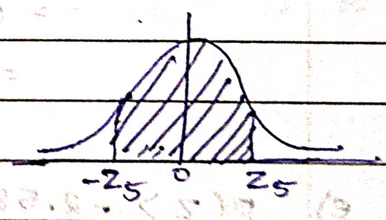
  $z_3, z_4 = +ve$



* $P(|z|<z_5) = P(-z_5<z<z_5)$

  $z_5 = +ve$

  $= 2\,p(0<z<z_5)$



1) Evaluate the following probabilities :-

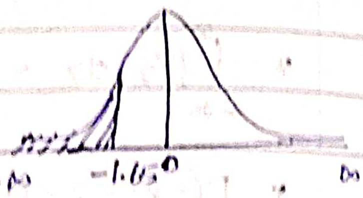
a) $p(z < -1.65)$

$= 0.5 - p(z > 1.65)$

$= 0.5 - p(0 < z < 1.65)$

$= 0.5 - 0.4505$

$= \underline{0.0495}$.



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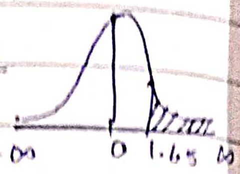
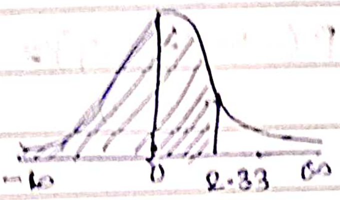


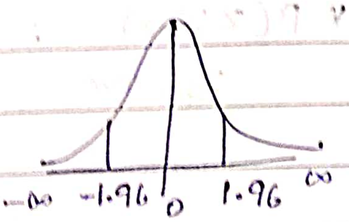
b) $p(z < 2.33)$

$= 0.5 + p(0 < z < 2.33)$

$= 0.5 + 0.4901$

$= \underline{0.9901}$.

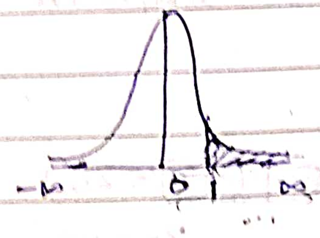




c) $p(|z| < 1.96) = 2 \times p(0 < z < 1.96)$

$= 2[0.4750]$

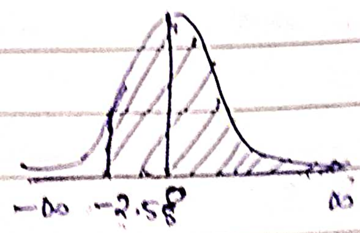
$= \underline{0.9500}$



d) $P(z > 1) = 0.5 - P(0 < z < 1)$

$= 0.5 - 0.3413$

$= \underline{0.1587}$.



e) $p(z > -2.58) = 0.5 + p(0 < z < 2.58)$

$= 0.5 + 0.4951$

$= \underline{0.9951}$.

f) $P(-1.2 < z < -1) =$

$= P(0 < z < 1.2) - P(0 < z < 1)$

$= 0.3849 - 0.3413$

$= \underline{0.0436}$.

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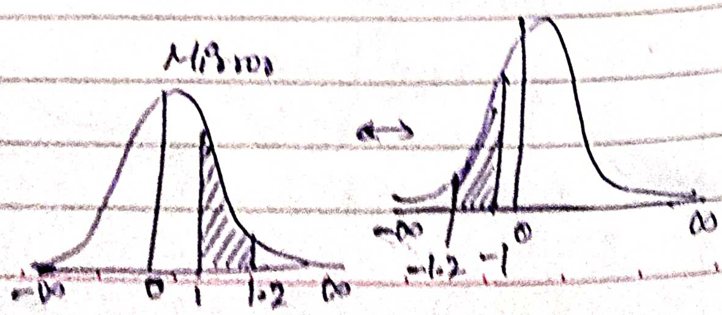
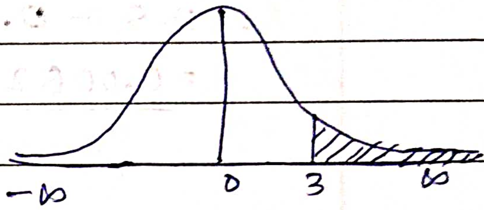


2) ~~Evaluate~~ If x is a normal variable with mean 30, and Standard deviation is 5, evaluate probability of:
a) $P(x>45)$ b) $P(26<x<40)$

$$X \sim N(\mu = 30, \sigma^2 = 5^2)$$

a) $P(x>45) = P\left(\frac{x-\mu}{\sigma} > \frac{45-30}{5}\right) = P(\cancel{x}\, z > 3)$.

$\Rightarrow P(Z>3) = 0.5 - p(0<z<3)$
$= 0.5 - 0.4987$
$= \underline{0.0013}$.

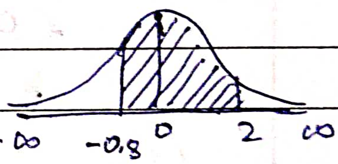


b) $P(26<x<40) = P\left(\frac{26-30}{5} < \frac{x-\mu}{\sigma} < \frac{40-30}{5}\right)$

$= p(-0.8<z<2)$
$= p(0<z<2) + p(0<z<0.8)$

$= 0.4772 + 0.2881$

$= \underline{0.7653}$.



3) If the total cholestrol value for a certain population are approx. normally distributed with a mean of 200 and S.D of 20. Find the prob. that an individual selected at random from this popⁿ will have a cholestral value :
a) Less than 150
b) greater than 225
c) b/w 180 & 200

X - total cholestrol value of an individual in a population.

$$X \sim N(\mu = 200, \sigma^2 = 400)$$

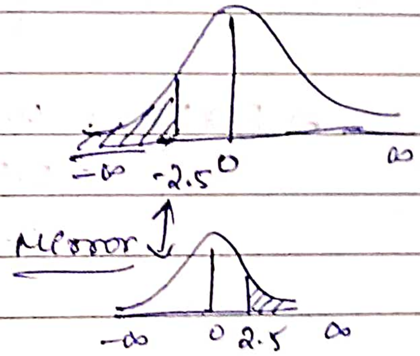
a) $P(x<150) = P\left(\frac{x-\mu}{\sigma} < \frac{150-200}{20}\right)$

$P(z < -2.5)$

$= 0.5 - P(0 < z < 2.5)$

$= 0.5 - 0.4938$

$= \underline{0.0062}$



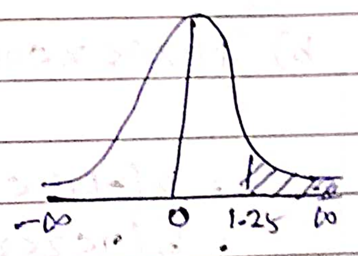
b) $P(x>225) = P\left(\frac{x-\mu}{\sigma} > \frac{225-200}{20}\right)$

$= P(z > 1.25)$

$= 0.5 - P(0 < z < 1.25)$

$= 0.5 - 0.3944$

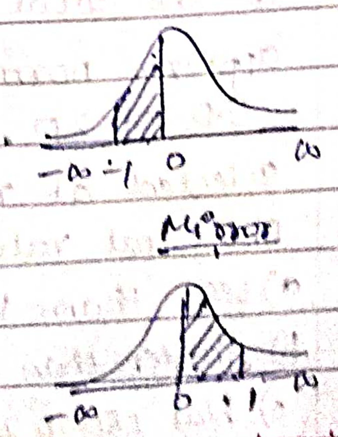
$= \underline{0.1056}$.



c) $P(180 < x < 200) = P\left(\frac{180-200}{20} < \frac{x-\mu}{\sigma} < \frac{200-200}{20}\right)$

$= P(-1 < z < 0)$

$= \cancel{0.8}\ \underline{0.3413}$.



11) The life of a certain electrical lamp is normally distributed with mean 2040 hrs, and SD = 60 hrs. In a consignment of 3000 lamps, find how many would be expected to burn for :

a) more than 2150 hrs

b) less than 1950 hrs

c) b/w 1920 and 2160 hrs.

X - life of the electrical lamp.

$X \sim N(\mu = 2040 \text{ hrs}, \sigma^2 = 3600 \text{ hrs})$

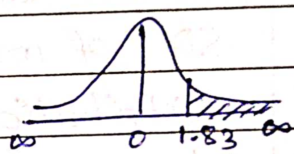
a) $P(x > 2150) = P\left(\frac{x-\mu}{\sigma} > \frac{2150-2040}{60}\right)$

$\Rightarrow P(z > 1.833)$

$= 0.5 - P(0 < z < 1.83)$

$= 0.5 - 0.4664$

$= \underline{0.0336}$



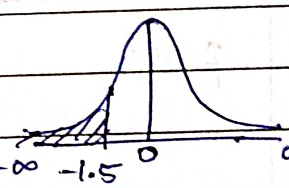
b) $P(x < 1950) = P\left(\frac{x-\mu}{\sigma} < \frac{1950-2040}{60}\right)$

$\Rightarrow P(z < -1.5)$

$= 0.5 - P(0 < z < 1.5)$

$= 0.5 - 0.4332$

$= \underline{0.0668}.$



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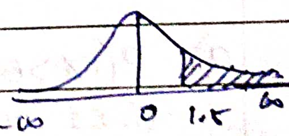


c) $P(1920 < x < 2160)$

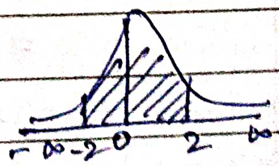
$P\left(\frac{1920-2040}{60} < \frac{x-\mu}{\sigma} < \frac{2160-2040}{60}\right)$ −∞

$\Rightarrow P(-2 < z < 2)$

$= 2P(0 < z < 2)$

$= 2(0.4772)$

$= \underline{0.9544}.$



a) The no. of lamps expected to burn for more than 2150 hours $= 3000 \times (0.0336)$
$= \underline{101}$.

b) The no. of lamps expected to burn for ~~more~~ less than 1950 hours $= 3000 \times (0.0668)$
$= \underline{201}$.

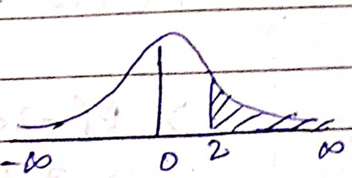
c) The no. of lamps expected to burn ~~for~~ 1920 & 2160 hours $= 3000 \times (0.9544)$
$= \underline{2864}$.

5) In an examination taken by 500 candidates, average & S.D of marks obtained are 40 & 10. Assuming normal distribution, find:
a) how many have scored above 60?
b) how many will pass if 50 is fixed as the minimum for passing?
c) what shd be the min. marks for 350 candidates to pass?

X - marks obtained by a candidate
$X \sim N(\mu = 40, \sigma^2 = 100)$
frequency, $N = 500$

a) $P(x > 60)$ (for 1 candidate) $= P\left(\frac{x-\mu}{\sigma} > \frac{60-40}{10}\right)$

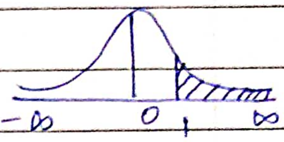
$\Rightarrow P(Z > 2)$
$= 0.5 - P(0 < Z < 2)$
$= 0.5 - 0.4772$
$= 0.0228$



No. of students scoring above 60 marks = 500×0.0228
$\approx$ 12 candidates

b) $P(x > 50) = P\left(\frac{x-\mu}{\sigma} > \frac{50-40}{10}\right)$

$\Rightarrow P(z > 1)$



$= 0.5 - P(0 < z < 1)$
$= 0.5 - 0.3413$
$= 0.1587$.

No. of candidates that will pass the exam = 500×0.1587
$\approx$ 80 candidates

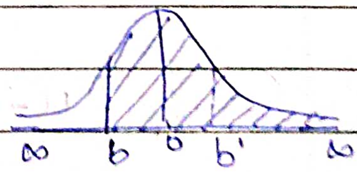
c) Minimum marks required such that 350 candidates will pass the exam is,

$P(x > a) = \frac{350}{500} = 0.7000$.

$P\left(z > \frac{a-40}{10}\right) = 0.7 ~~-0.5~~$
$= ~~0.2000~~$

$\boxed{P(z > b) = 0.7} \longrightarrow (1)$ $\quad b = \frac{a-40}{10}$

Note :-
Check for these conditions,
a) $> \Rightarrow$ right tail
$< \Rightarrow$ left tail
b) > 0.5
< 0.5



$P(z > b) = 0.7$
$0.5 + P(0 < z < b') = 0.7$
$P(0 < z < b') = 0.2$
$\Rightarrow b' = 0.52$

$$b' = -b$$

$$\Rightarrow \boxed{b = -0.52}$$

$$\Rightarrow -0.52 = \frac{a-40}{10}$$

$$\Rightarrow \quad a = 40 - 5.2$$

$$\Rightarrow \boxed{a = 34.7}$$

6) In a normal distribution, 31% of the items are under 45 and 8% are over 64. Find the mean & SD of the distribution.

$$X \sim N(\cancel{\sigma}\mu, \sigma^2)$$

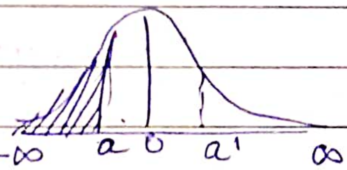
Given,

$$P(x<45) = 0.31$$
$$P(x>64) = 0.08$$

$\rightarrow$ $P(x<45) = 0.31$

$$P(z<a) = 0.31$$

$$a = \frac{45-\mu}{\sigma}$$



$$0.5 - P(0<z<a') = 0.31$$

$$P(0<z<a') = 0.1900$$

$$\Rightarrow \boxed{a' = 0.5}.$$

$$a = -a'$$

$$\Rightarrow \quad a = -0.5$$

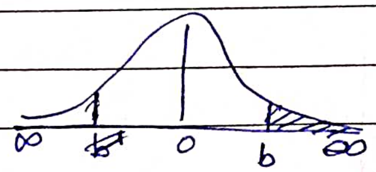
$$-0.5 = \frac{45-\mu}{\sigma}$$

$$\boxed{\mu - 0.5\sigma = 45} \longrightarrow (1)$$

$\rightarrow$ $P(x>64) = 0.08$

$$P(z>b) = 0.08$$

$$b = \frac{64-\mu}{\sigma} = \cancel{0}.$$



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$$P(z>b) = 0.5 - P(0<z<b)$$

$$0.08 = 0.5 - P(0<z<b)$$

$$\Rightarrow P(0<z<b) = 0.4200$$

$$\Rightarrow \underline{b = 1.401}.$$

$$\Rightarrow 1.401 = \frac{64-\mu}{\sigma}$$

$$\boxed{\mu + 1.401\sigma = 64} \rightarrow (2).$$

Solving (1) & (2), we get

~~$\mu = 34.4562$ and $\sigma = -21.09$~~ $\boxed{\mu = 49.99 \sim 50 \text{ and } \sigma = 9.99 \sim 10}$

7) If skulls are classified as A, B, C accordingly to length-breadth index is under 75, b/w 75 & 80 and over 80, find approximately the mean & S.D of a series of skulls in which A are 58%, B is 38% and C is 4%.

X - length-breadth index of a skull

$X \sim N(\mu, \sigma^2)$ (Assuming it is normally distributed)

A - Skulls with index less than 75

B - —ıı— b/w 75 & 80

C - —ıı— Over 80

$$P(A) = P(x<75) = 0.58$$

$$P(B) = P(75<x<80) = 0.38$$

$$\underline{P(C) = P(x>80) = 0.04}$$

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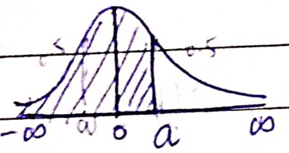
Consider P(A) & P(C),

$P(x<75) = 0.58$

$\Rightarrow P(Z<a) = 0.\cancel{0}58$

$a = \dfrac{75-\mu}{\sigma}$



$P(Z<a) = 0.5 + P(0<Z<a)$

$0.58 = 0.5 + P(0<Z<a)$

$\Rightarrow P(0<Z<a) = 0.08$

$\boxed{a = 0.2}$

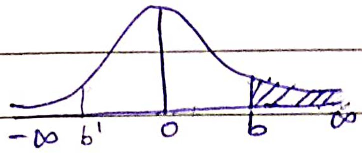
$0.2 = \dfrac{75-\mu}{\sigma}$

$\boxed{\mu + 0.2\sigma = 75} \rightarrow (1)$

$P(x>80) = 0.04$

$\Rightarrow P(Z>b) = 0.04$

$b = \dfrac{80-\mu}{\sigma}$



$P(Z>b) = 0.5 - P(0<Z<b)$

$0.04 = 0.5 - P(0<Z<b)$

$P(0<Z<b) = 0.4600$

$\boxed{b = 1.75}$

$1.75 = \dfrac{80-\mu}{\sigma}$

$\boxed{\mu + 1.75\sigma = 80} \rightarrow (2)$

From (1) and (2)

$\mu = 74.35$ and

$\sigma = 3.2258$

8) In a certain examination, the % of students passing & getting distinctions were 45 and 9 respectively. Calculate the average marks obtained by candidates where min. pass marks & distinction marks are 40 and 75 respectively.

X – marks of a candidate

Assuming, $X \sim N(\mu, \sigma^2)$.

Given,

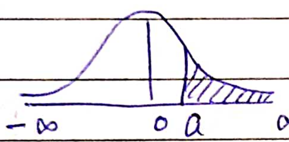
$P(x>40) = 0.45$

$P(x>75) = 0.09$

$P(x>40) = 0.45$

$P(Z>a) = 0.45$

$a = \dfrac{40-\mu}{\sigma}$



$P(Z>a) = 0.5 - P(0<Z<b)$

$P(0<Z<a) = 0.5 - 0.45$

$\Rightarrow P(0<Z<a) = 0.05$

$\Rightarrow a = 0.13$

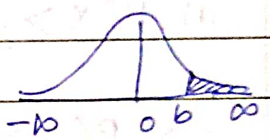
$0.13 = \dfrac{40-\mu}{\sigma}$

$\boxed{\mu + 0.13\sigma = 40} \rightarrow (1)$

$P(x>75) = 0.09$

$P(z>b) = 0.09$

$b = \dfrac{75-\mu}{\sigma}$



$P(Z>b) = 0.5 - P(0<Z<b)$

$P(0<Z<b) = 0.5 - 0.09$

$\Rightarrow P(0<Z<b) = 0.41$

$\Rightarrow b = 1.34$

$1.34 = \dfrac{75-\mu}{\sigma}$

$\boxed{\mu + 1.34\sigma = 75}$
$\rightarrow (2)$

From (1) and (2)

$$\boxed{\begin{array}{ll} \mu = 36.24 & \text{and} \\ \sigma = 28.93 & \end{array}}$$

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9) A manufacturer of a Air-Mail envelopes knows from experience, that weight of the envelope is normally distributed with mean 1.95 , and S.D 0.05 g. About how many envelopes weighing:

a) 2 gm or more.

b) 2.05 g or more

Can be expected in a given packat of 100 envelopes.

X – Weight of the envelope

(Assuming) $X \sim N(\mu = 1.95, \sigma^2 = (0.05)^2)$

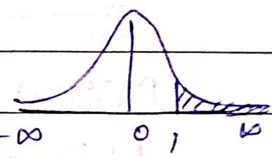
a) $P(X > 2) = P\left(Z > \frac{2 - 1.95}{0.05}\right)$

$\Rightarrow P(Z > 1)$

$= 0.5 - P(0 < Z < 1)$

$= 0.5 - 0.3413$

$= 0.1587.$



No. of envelopes weighing 2 g or more $= 100 \times 0.1587$

$= 15.87$

~ 16.

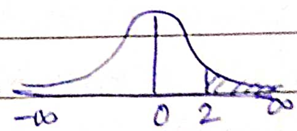
b) $P(X > 2.05) = P\left(Z > \frac{2.05 - 1.95}{0.05}\right)$

$\Rightarrow P(Z > 2)$

$= 0.5 - P(0 < Z < 2)$

$= 0.5 - 0.4772$

$= 0.0228$



No. of envelopes weighing 2.05 g or more $= 100 \times 0.0228$

$= 2.28$

~ 3.

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Joint Probability Distribution :-

X \ Y	y_1	y_2	......	y_m	
x_1	$P(x_1,y_1)$	$P(x_1,y_2)$	....	$P(x_1,y_m)$	$P(x_1)$
x_2	:	:		:	$P(x_2)$
:	:	:		:	:
x_n	$P(x_n,y_1)$	$P(x_n y_2)$	......	$P(x_n y_m)$	$P(x_n)$
	$P(y_1)$	$P(y_2)$	.....	$P(y_m)$	1

$$P(X=x_i, Y=y_j) = p_{ij} = P(x_i, y_i)$$

$$\left.\begin{array}{l} P(x_i\, y_i) \geq 0 \\ \\ \sum_i \sum_i P(x_i\, y_i) = 1 \end{array}\right\}$$

Joint prob. function
$p(x,y) = f(x,y)$

Marginal distribution of X.

x	x_1	x_2	....	x_n
$P(x)$	$P(x_1)$	$P(x_2)$	.....	$P(x_n)$

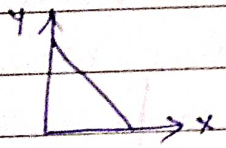
Marginal distribution of Y.

y	y_1	y_2	.....	y_n
$P(y)$	$P(y_1)$	$P(y_2)$	....	$P(y_n)$

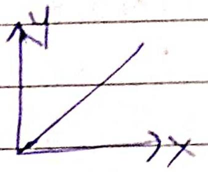
$\rho = r =$ correlation

$-1 \leq \rho \leq 1$

$\rho = -1 \Rightarrow$ perfectly negatively correlated



$\rho = 1$



$\rho = 0 \Rightarrow$ no correlation
$\Rightarrow x$ & y are independent

$$\boxed{\rho(x,y) = \frac{Cov(x,y)}{\sigma_x \sigma_y}}$$

$Cov(x,y)$ – covariance of x and y

Moments
E – Expectation

$$Cov(X,Y) = E(XY) - E(X)E(Y)$$

$$E(X) = \sum_i x p(x) = \mu_x$$
$$E(Y) = \sum_i y p(y) = \mu_y$$

$$\sigma_x^2 = E(x^2) - [E(x)]^2$$
$$E(x^2) = \sum_i x^2 p(x)$$

$$\sigma_y^2 = E(y)^2 - [E(y)]^2$$
$$E(y^2) = \sum_i y^2 p(y)$$

$$E(XY) = \sum_i \sum_i xy\, p(x,y)$$

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1) The JPD of 2 random variables is given below.

X\Y	-3	2	4	
1	0.1	0.2	0.2	0.5
3	0.3	0.1	0.1	0.5
	0.4	0.3	0.3	1

Find:-

a) marginal dist^n of X and Y

b) $\rho(X,Y)$

a) Marginal dist^n of X :-

x	1	3
$p(x)$	0.5	0.5

Marginal dist^n of Y :-

y	-3	2	4
$p(y)$	0.4	0.3	0.3

b) $E(X) = \sum_i x\,p(x)$

$= 1(0.5) + 3(0.5) = 2$

$E(Y) = \sum_i y\,p(y)$

$= -3(0.4) + 2(0.3) + 4(0.3)$

$= 0.6$

$E(X^2) = \sum_i x^2 p(x)$

$= 1^2(0.5) + 3^2(0.5) = 5$

$E(Y^2) = \sum_i y^2 p(xy)$

$= (-3)^2(0.4) + 2^2(0.3) + 4^2(0.3)$

$= 9.6$

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$$E(XY) = \sum_i \sum_j xy\,p(x,y)$$
$$= (1\times -3\times 0.1) + (1\times 2\times 0.2) + (1\times 4\times 0.2) + (3\times -3\times 0.3) + (3\times 2\times 0.1) + (3\times 4\times 0.1)$$
$$= 0.$$

$$\sigma_x^2 = E(x^2) - [E(x)]^2 = 5 - 2^2 = 1$$

$$\sigma_y^2 = E(Y^2) - [E(Y)]^2 = 9.6 - (0.6)^2 = 9.24$$

$$\rho(x,y) = \frac{cov(x,y)}{\sigma_x \sigma_y} = \frac{E(XY) - E(X)E(Y)}{\sigma_x \cdot \sigma_y}$$

$$= \frac{0 - (2)(0.6)}{\sqrt{1} \times \sqrt{9.24}}$$

$$= \frac{-1.2}{\sqrt{9.24}}$$

$$= \cancel{0.1299}$$

$$\boxed{\rho(x,y) = -0.39}$$

2) The joint prob. fn of 2 random variables x and y is given by $P(x,y) = C(2x+y)$ where x and y can assume all integral values such that $0 \le x \le 2$, $0 \le y \le 3$, and '0' otherwise.

$$f(x,y) = \begin{cases} C(2x+y) & ; 0 \le x \le 2,\ 0 \le y \le 3 \\ 0 & ; \text{otherwise} \end{cases}$$

Find:-

a) The value of constant 'c'.

b) Marginal distn of x & y.

c) Prob. of x $P(x \ge 1, y \le 2)$

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$x \backslash y$	0	1	2	3	
0	0	c	2c	3c	6c
1	2c	3c	4c	5c	14c
2	4c	5c	6c	7c	22c
	6c	9c	12c	15c	42c = 1

$f(x=0, y=0)$ $= c(0) = 0$	$f(x=1, y=0)$ $= 2c$	$f(x=2, y=0)$ $= 4c$
$f(x=0, y=1)$ $= c$	$f(x=1, y=1)$ $= 3c$	$f(x=2, y=1)$ $= 5c$
$f(x=0, y=2)$ $= 2c$	$f(x=1, y=2)$ $= 4c$	$f(x=3, y=2)$ $= 6c$
$f(x=0, y=3)$ $= 3c$	$f(x=1, y=3)$ $= 5c$	$f(x=3, y=3)$ $= 7c$

a) wkt, $f(x,y) \geq 0$

$$\sum\sum f(x,y) = 1$$

$$42c = 1$$

$$\Rightarrow \boxed{c = \frac{1}{42}}$$

b) Marginal distr of X :-

X	0	1	2
p(x)	6/42	12/42	22/42

Marginal distr of Y:-

Y	0	1	2	3
P(Y)	6/42	9/42	12/42	15/42

c) $P(X \geq 1, Y \leq 2) = P(X=1, Y=0) + P(X=1, Y=1) + P(X=1, Y=2)$
$+ P(X=2, Y=0) + P(X=2, Y=1) + P(X=2, Y=2)$
$= 9c + 9c + 6c = 24c = \frac{24}{42}$.

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3) A coin is tossed 3 times. Let 'x' denote 0 or 1 accordingly as tail/head occurs on the first toss. Let 'y' denote the total no. of tails that occurs.
a) the jdp of x & y.
b) the marginal distribution of X and Y.
c) E(X+Y)
d) E(XY)

Sample space, S = { HHH, HHT, HTH, HTT, THT, TTH, THH, TTT }

Random Experiment – tossing a coin 3 times

b) $x = \{0, 1\}$

x	0 (tail)	1 (head)
p(x)	4/8	4/8

Marginal distr of X

$y = \{0, 1, 2, 3\}$

y	0	1	2	3
p(y)	1/8	3/8	3/8	1/8

Marginal distr of Y

a) Joint Probability Distribution Table:-

X \ Y	0	1	2	3	
0	0	1/8	2/8	1/8	4/8
1	1/8	2/8	1/8	0	4/8
	1/8	3/8	3/8	1/8	1

$P(x=0, y=0) = 0$ (impossible event)

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c) $E(X+Y) = \sum\sum (x+y)\, p(x,y)$

$= (0+0)0 + (0+1)\frac{1}{8} + (0+2)\frac{2}{8} + (0+3)\frac{1}{8} +$
$(1+0)\frac{1}{8} + (1+1)\frac{2}{8} + (1+2)\frac{1}{8} + (1+3)0$

$= \frac{1}{8} + \frac{4}{8} + \frac{3}{8} + \frac{1}{8} + \frac{4}{8} + \frac{3}{8}$

$= 2\left(\frac{8}{8}\right) = 2$

d) $E(XY) = \sum\sum xy\, p(xy)$

$= (0\times0\times0) + (0\times1)\frac{1}{8} + (0\times2)\frac{2}{8} + (0\times3)\frac{1}{8} +$
$(1\times0\times\frac{1}{8}) + (1\times1\times\frac{2}{8}) + (1\times2\times\frac{1}{8}) + (1\times3\times0)$

$= \frac{2}{8} + \frac{2}{8} = \frac{4}{8}$

$= \frac{1}{2}$

4) 2 cards are selected at random from a box wch contains 5 cards numbered 1, 1, 2, 2 and 3. Find JPD of X & Y where X - denotes the sum and Y - denotes the max. of the 2 no.s drawn. Find correlation co-efficient b/w X & Y.

Sample space, S = { (1,1), (1,2), (1,2), (1,3), (2,1), (2,1), (2,2), (2,3), (1,1) (1,2), (1,2), (1,3), (3,1), (3,1), (3,2), (3,2), (2,1), (2,1), (2,2), (2,3) }

X - Sum of 2 no.s
$X = \{2, 3, 4, 5\}$

Y - max. of 2 no.s
$Y = \{1, 2, 3\}$

Joint prob. distr :-

x \ y	1	2	3	
2	2/20	0	0	2/20
3	0	8/20	0	8/20
4	0	2/20	4/20	6/20
5	0	0	4/20	4/20
	2/20	10/20	8/20	1

Marginal distr of X and Y :-

x	2	3	4	5
p(x)	2/20	8/20	6/20	4/20

y	1	2	3
p(x)	2/20	10/20	8/20

$$\rho(x,y) = \frac{\text{cov}(x,y)}{\sigma_x \sigma_y} \qquad \text{cov}(X,Y) = E(XY) - E(X)E(Y)$$

$$E(X) = \sum x\,p(x) = 2\left(\frac{2}{20}\right) + 3\left(\frac{8}{20}\right) + 4\left(\frac{6}{20}\right) + 5\left(\frac{4}{20}\right)$$

$$= \frac{4+24+24+20}{20} = \frac{72}{20} = 3.6$$

$$E(Y) = \sum y\,p(y) = 1\left(\frac{2}{20}\right) + 2\left(\frac{10}{20}\right) + 3\left(\frac{8}{20}\right)$$

$$= \frac{2+20+24}{20} = \frac{46}{20} = 2.3$$

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$E(x^2) = \sum_i x^2 p(x) = 4\left(\frac{2}{20}\right) + 9\left(\frac{8}{20}\right) + 16\left(\frac{6}{20}\right) + 25\left(\frac{4}{20}\right)$

$= \frac{8 + 72 + 96 + 100}{20} = \frac{276}{20}$

$= 13.8$

$E(y^2) = \sum_i y^2 p(x) = 1\left(\frac{2}{20}\right) + 4\left(\frac{10}{20}\right) + 9\left(\frac{8}{20}\right)$

$= \frac{2 + 40 + 72}{20} = \frac{114}{20}$

$= 5.7$

$E(xy) = \sum_i \sum_i xy\, p(x,y)$

$= (1 \times 2 \times 2/20) + (2 \times 3 \times 8/20) + (2 \times 4 \times 2/20) + (3 \times 4 \times 4/20) + (3 \times 5 \times 4/20)$

$= \frac{4}{20} + \frac{48}{20} + \frac{16}{20} + \frac{48}{20} + \frac{60}{20}$

$= \frac{176}{20}$

$= 8.8$

$\sigma_x^2 = E(x^2) - [E(x)]^2$

$= 13.8 - (3.6)^2$

$= 0.840$

$\sigma_y^2 = E(y^2) - [E(y)]^2$

$= 5.7 - (2.3)^2$

$= 0.410$

$\Rightarrow \sigma_x = 0.916$

$\sigma_y = 0.640$

$Cov(x,y) = E(xy) - E(x)E(y)$

$= 8.8 - (3.6)(2.3)$

$= 0.520$

$\rho = \frac{0.520}{\cancel{0.840 \times 0.640}\ 0.916 \times 0.640} = \underline{0.8870}$

5) 2 fruits are selected at random from a bag containing 2 oranges, 3 apples, 4 mangoes. If X & Y are respectively the no. apples & oranges included among the 2 fruits drawn from the bag. Find prob. associated with all possible pairs of values of X & Y, also find correlation b/w X & Y.

3 apples, 2 oranges, 4 mangoes

X - no. of apples
Y - no. of oranges

$x = \{0, 1, 2\}$
$y = \{0, 1, 2\}$

Joint Prob. Distr Table:-

X \ Y	0	1	2
0	${}^4C_2/{}^9C_2$		
1			
2			

$P(X=1, Y=0) = \frac{{}^3C_1 \times {}^4C_1}{{}^9C_2} = \frac{12}{36}$

$P(X=1, Y=1) = \frac{{}^3C_1 \times {}^2C_1}{{}^9C_2} = \frac{6}{36}$

X \ Y	0	1	2	P
0	${}^4C_2/{}^9C_2 = 6/36$	$\frac{{}^2C_1 \times {}^4C_1}{{}^9C_2} = 8/36$	${}^2C_2/{}^9C_2 = 1/36$	$\frac{15}{36}$
1	$12/36$	$6/36$	0	$\frac{18}{36}$
2	$\frac{{}^3C_2}{{}^9C_2} = \frac{3}{36}$	0	0	$\frac{3}{36}$
	$\frac{21}{36}$	$\frac{14}{36}$	$\frac{1}{36}$	1