Unit-5

Markov Chain and Queuing Theory

→ Markov chain :-

- Stochastic process
- Prediction
- Present / Current state depends on previous state.
- $P(X_n = x_i / X_{r-1} = x_j)$ Conditional probability.

→ Terminologies :-

- Stochastic matrix – square matrix + elements are probabilities.

$P = (p_{ij})_{m \times m}$

Ex:- $P = \begin{bmatrix} 1/2 & 1/2 \\ 0 & 1 \end{bmatrix} \begin{matrix} \rightarrow 1/2 + 1/2 = 1 \\ \rightarrow 0 + 1 = 1 \end{matrix}$

→ Each row is called probability vector

$V = (V_1 \; V_2 \; \dots \; V_n)_{1 \times n}$

$V_1, V_2, \dots V_n \geq 0$

$V_1 + V_2 + V_3 + \dots + V_n = 1$

Eg:- $(1, 0)$, $(1/3 \;\; 1/3 \;\; 1/3)$

$(1/6 \;\; 2/6 \;\; 3/6)$

- Unique fixed Probability Vector [V]

$\boxed{VP = V}$

V- prob. vector

P- stochastic matrix

- Regular stochastic matrix [P]

A stochastic matrix is called a regular stochastic matrix P^n such that all entries are strictly positive.

Problems :-

1) Verify that the matrix $A = \begin{pmatrix} 0 & 1 \\ 0.3 & 0.7 \end{pmatrix}$ is a regular stochastic matrix.

a) $A = \begin{bmatrix} 0 & 1 \\ 0.3 & 0.7 \end{bmatrix}$

$A^2 = \begin{bmatrix} 0.3 & 0.7 \\ 0.21 & 0.79 \end{bmatrix}$

Since, every entry in A^2 is strictly positive (>0)

$\Rightarrow$ A is a regular stochastic matrix.

b) $A = \begin{bmatrix} 0 & 0 & 1 \\ 0.5 & 0.25 & 0.25 \\ 0 & 1 & 0 \end{bmatrix}$

$A^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0.125 & 0.3125 & 0.5625 \\ 0.5 & 0.25 & 0.25 \end{bmatrix}$

Since, not every entry in A^2 is strictly positive,

~~hence A is not a regular stochastic matrix.~~

Hence we calculate A^3.

$A^3 = \begin{bmatrix} 0.5 & 0.25 & 0.25 \\ 0.1562 & 0.6406 & 0.2031 \\ 0.125 & 0.8152 & 0.5625 \end{bmatrix}$

Since, each entry in A^3 is strictly positive,

hence A is ~~not~~ a regular stochastic matrix.

2) Find the unique fixed probability vector for the regular stochastic matrix $A = \begin{pmatrix} 0.75 & 0.25 \\ 0.5 & 0.5 \end{pmatrix}$.

$$A = \begin{bmatrix} 0.75 & 0.25 \\ 0.5 & 0.5 \end{bmatrix}$$

$$\boxed{VA = V}$$

$V \rightarrow$ always a row matrix,

$V = (V_1 \ V_2)_{1\times 2}$

$V_1 + V_2 = 1$ and $V_1 V_2 \geq 0$

$$(V_1 \ V_2)\begin{pmatrix} 0.75 & 0.25 \\ 0.5 & 0.5 \end{pmatrix} = (V_1 \ V_2)$$

$$\left.\begin{array}{l} 0.75V_1 + 0.5V_2 = V_1 \\ 0.25V_1 + 0.5V_2 = V_2 \end{array}\right\} \text{Homogeneous System of Equations}$$

$0.5V_2 = 0.25V_1$

$\Rightarrow \boxed{V_1 = 2V_2}$.

$\cancel{0.75(2V_2) + 0.5V_2 = 2V_2}$

$\cancel{\Rightarrow \boxed{V_2 = 1}}$

Making non-homogeneous system of equations, i.e., Simultaneous Equations,

$0.5V_2 - 0.25V_1 = 0$

and $V_2 + V_1 = 1$

$\Rightarrow \boxed{V_2 = 0.33 \text{ and } V_1 = 0.67}$.

3) Find the U.F.P.V for the transition matrix $P = \begin{bmatrix} 0.1 & 0.3 & 0.6 \\ 0. & 0.4 & 0.6 \\ 0 & 0.2 & 0.8 \end{bmatrix}$

$$\boxed{VP = V}.$$

$$V = [V_1 \ V_2 \ V_3]_{1\times 3}$$

$V_1 + V_2 + V_3 = 1$ and

$V_1, V_2, V_3 \geq 0$

$$[V_1 \ V_2 \ V_3]_{1\times 3}\begin{bmatrix} 0.1 & 0.3 & 0.6 \\ 0 & 0.4 & 0.6 \\ 0 & 0.2 & 0.8 \end{bmatrix}_{3\times 3} = [V_1 \ V_2 \ V_3]_{1\times 3}$$

$0.1V_1 + 0V_2 + 0V_3 = V_1 \quad \Rightarrow V_1 = 0$

$0.3V_1 + 0.4V_2 + 0.2V_3 = V_2$

$0.6V_1 + 0.6V_2 + 0.8V_3 = V_3$

$[0.1V_1 \quad 0.3V_1 + 0.4V_2 + 0.2V_3 \quad 0.6V_1 + 0.6V_2 + 0.8V_3] = [V_1 \ V_2 \ V_3]$

$-0.6V_2 + 0.2V_3 = 0$

$V_2 + V_3 = 1$

$\Rightarrow \boxed{V_1 = 0, V_2 = 0.25, 0.75}$

4) a) Find UFPV. $A = \begin{bmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{bmatrix}$

$\boxed{VP = V}$ where $V_1 + V_2 + V_3 = 1$

$V = [V_1 \ V_2 \ V_3]_{1\times 3}$ and $V_1, V_2, V_3 \geq 0$

Page No.
Date

$$[V_1 \quad V_2 \quad V_3] \begin{bmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{bmatrix} = [V_1 \quad V_2 \quad V_3]$$

$$= \begin{bmatrix} \frac{V_2}{6} & V_1 + \frac{V_2}{2} + \frac{2V_3}{3} & \frac{V_2}{3} + \frac{V_3}{3} \end{bmatrix} = [V_1 \quad V_2 \quad V_3]$$

$$\frac{V_2}{6} = V_1 \implies V_2 = 6V_1$$

$$1 + 3 - 6 = -2$$

$$V_1 + \frac{V_2}{2} + \frac{2V_3}{3} = V_2 \implies \frac{2V_3}{3} - 2V_1 = 0$$

$$\frac{V_2}{3} + \frac{V_3}{3} = V_3 \implies 2V_1 - \frac{2V_3}{3} = 0$$

Equations,

$$2V_1 + 0V_2 - \frac{2V_3}{3} = 0$$

$$V_1 + V_2 + V_3 = 1$$

$$\implies V_1 = 1/10, \quad V_2 = 3/5, \quad V_3 = 3/10$$

b) $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/4 & 1/4 \end{bmatrix} = \begin{pmatrix} \frac{2}{9} & \frac{1}{3} & \frac{4}{9} \end{pmatrix}$

Markov Chain :-

5) A habitual gambler is a member of 2 clubs A & B. He visits either of the clubs everyday for playing cards. He never visits club A on 2 consecutive days. But, if he visits club B on a particular day, then the next day he is as likely to visit club B or club A.

i) Find the transition matrix of this Markov chain.

ii) Show that the matrix is a regular stochastic matrix.

iii) Find UFPV. (In long run, how often does he visits the clubs).

iv) If the person had visited club B on Monday, find the probability that he visits club A on Thursday.

State 1 :- Visiting club A

State 2 :- Visiting club B

Stochastic matrix $\Longleftrightarrow$ transition matrix

i)
$$P = \begin{array}{c|cc} & \text{State 1} & \text{State 2} \\ \hline \text{State 1} & 0 & 1 \\ \text{State 2} & 1/2 & 1/2 \end{array}$$

ii)
$$P^2 = \begin{bmatrix} 0.5 & 0.5 \\ 0.25 & 0.75 \end{bmatrix}$$

Regular Stochastic Matrix $\Leftrightarrow$ Irreducible Markov Chain

$\Rightarrow$ Markov chain is irreducible.

iii)
$$VP = V$$

UFPV $\Leftrightarrow$ In the long run or steady state $(n \to \infty)$

$$V = (V_1 \ V_2)_{1\times 2}$$

$$(V_1 \ V_2)\begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix} = (V_1 \ V_2)$$

$$\begin{pmatrix} \frac{V_2}{2} & V_1 + \frac{V_2}{2} \end{pmatrix} = (V_1 \ V_2)$$

$$\Rightarrow \frac{V_2}{2} = V_1 \text{ and } V_1 + \frac{V_2}{2} = V_2$$

$$\boxed{V_1 - \frac{V_2}{2} = 0} \text{ and } \boxed{V_1 + V_2 = 1}$$

$$\Rightarrow \boxed{V_1 = 0.33 \text{ and } V_2 = 0.66}$$

$$V = (0.33 \ \ 0.67)_{1\times 2}$$

In the long run, the gambler visits club A with prob. 1/3 and club B with prob. 2/3.

iv) Higher transition matrix : $P^n = P^0 P^n$

$$P^0 = \begin{array}{cc} A & B \\ (0 & 1) \end{array}$$

0 → Monday
1 → Tuesday
2 → Wednesday
3 → Thursday

$$\Rightarrow P^3 = P^0 P^3$$

$$P^3 = \begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix}^3 = \begin{pmatrix} 0.25 & 0.75 \\ 0.375 & 0.625 \end{pmatrix}$$

$$P^3 = P^0 P^3 = (0 \ 1)\begin{pmatrix} 0.25 & 0.75 \\ 0.375 & 0.625 \end{pmatrix}$$

$$\boxed{P^3 = (0.375 \ \ 0.625)}$$

Gambler visiting club A on Thursday is 0.375 and club B is 0.625.

6) A computer device can be either in a busy mode (state 1) processing a task, or in an idle mode (state 2), when there are no tasks to process. Being in a busy mode, it can finish a task and enter an idle mode any minute with the probability 0.2. Thus, with the prob. 0.8 it stays another minute in a busy mode. Being in an idle mode, it receives a new task any minute with the prob. 0.1 and enters a busy mode. Thus, it stays another minute in an idle mode with the prob. 0.9. The initial state is idle. Let X_i be the state of the device after n minutes.

a) Find the distribution of X_2

b) Find the steady-state distribution of X

State 1 :- Busy mode

State 2 :- Idle mode

$$X_i \quad P = \begin{array}{c c} & \begin{array}{cc} B & I \end{array} \\ \begin{array}{c} B \\ I \end{array} & \begin{bmatrix} 0.8 & 0.2 \\ 0.1 & 0.9 \end{bmatrix}_{2\times 2} \end{array}$$

Initial state :- $p^0 = \begin{array}{cc} B & I \\ (0 & 1) \end{array}$

$$X_2 = p^{(2)} = p^0 p^2 = (0\ 1)\begin{bmatrix} 0.66 & 0.34 \\ 0.17 & 0.83 \end{bmatrix} = (0.17\ 0.83)$$

Steady state $\Rightarrow VP = V$

$V = (v_1\ v_2)$

$v_1, v_2 \geq 0$ and $v_1 + v_2 = 1$

$$(v_1\ v_2)\begin{pmatrix} 0.8 & 0.2 \\ 0.1 & 0.9 \end{pmatrix} = (v_1\ v_2)$$

$0.8v_1 + 0.1v_2 = v_1$

$0.2v_1 + 0.9v_2 = v_2$ $\Rightarrow$ $0.2v_1 - 0.1v_2 = 0$ and $v_1 + v_2 = 1$

$\Rightarrow (v_1\ v_2) = (1/3\ \ 2/3)$

7) A gambler's luck follows a pattern. If he wins a game, the prob. of winning the next game is 0.6. However, if he loses a game, the prob. of losing the next game is 0.7. There is an even chance of gambler winning the first game. If so

a) What is prob. of he winning the second game?

b) What is prob. of he winning the third game?

c) In the long run, how often he will win?

$$P = \begin{array}{c c} & \begin{array}{cc} W & L \end{array} \\ \begin{array}{c} W \\ L \end{array} & \begin{pmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{pmatrix} \end{array}$$

$$p^0 = \begin{array}{cc} W & L \\ (1/2 & 1/2) \end{array}$$

1st game – 0

2nd – 1

3rd – 2

a) $\longrightarrow p^{(1)} = p^0 p^1 = (1/2\ \ 1/2)\begin{pmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{pmatrix}$

$$\boxed{p^{(1)} = (0.45\quad 0.55)}$$

Prob. of winning the second game.

b) $p^{(2)} = p^0 p^2 = (1/2\ \ 1/2)\begin{pmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{pmatrix}^2$

$$\boxed{p^{(2)} = (0.435\quad 0.565)}$$

Prob. of winning the third game.

c) $VP = V$

$V = (v_1\ v_2)$

where $v_1, v_2 \geq 0$ and $v_1 + v_2 = 1$

$$(v_1\ v_2)\begin{bmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{bmatrix} = (v_1\ v_2)$$

$$0.6v_1 + 0.3v_2 = v_1$$
$$0.4v_1 + 0.7v_2 = v_2 \Rightarrow \boxed{\begin{aligned} 0.4v_1 - 0.3v_2 &= 0 \\ v_1 + v_2 &= 1 \end{aligned}}$$

$$\Rightarrow \boxed{(v_1\ v_2) = (0.428\ \ 0.572)}$$

8) The pattern of sunny & rainy ~~d~~ days — on the planet Rainbow is a homogeneous markov chain with 2 states. Every sunny day is followed by another sunny day with prob. 0.8. Every rainy day is followed by another rainy day with prob. 0.6.

a) Today is sunny is planet rainbow, what is the chance of rain the day after tomorrow?

b) Compute the prob. that April 1, next year is rainy on planet rainbow.

→ A - sunny
B - rainy

$$\rightarrow P = \begin{array}{c} \\ A \\ B \end{array}\begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{pmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{pmatrix} \end{array}$$

0 - today
1 - tmmr
2 - day after tmmr

$$\rightarrow P^0 = \begin{array}{c} \begin{array}{cc} A & B \end{array} \\ (1\ \ 0) \end{array}$$

a) $P^{(2)} = P^0 P^2$

$$= (1\ 0)\begin{pmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{pmatrix}^2$$

$$= [0.72\ \ 0.28]$$

Prob. that rain occurs day after tommorrow is 0.28.

b) Steady state (long run) ($\because n \to \infty$)

$VP = V$

$V = (v_1\ v_2)$

where $v_1, v_2 \geq 0$ and $v_1 + v_2 = 1$

$$(v_1\ v_2)\begin{pmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{pmatrix} = (v_1\ v_2)$$

$$0.8v_1 + 0.4v_2 = v_1$$
$$0.2v_1 + 0.6v_2 = v_2$$

$$0.2v_1 - 0.4v_2 = 0 \text{ and}$$
$$v_1 + v_2 = 1$$

$$\Rightarrow \boxed{v_1 = 2/3 \text{ and } v_2 = 1/3}$$

9) A housewife buys 3 brands of soap A, B and C. She never buys the same brand on successive weeks. If she buys brand A in the week, she buys brand B ~~on~~ in next week. If she buys brand other than A in a week, then the next week she is 3 times as likely to buy the brand A as the ~~over~~ other brand. Supposing that she has bought brand B in the first week, find prob. of her buying brand B in the fourth week. In the long run, how often that she buys brand C soap?

→ States :- A — buying brand A soap
B — " — B "
C — " — C "

$$\rightarrow \quad P = \begin{array}{c|ccc} & A & B & C \\ \hline A & 0 & 1 & 0 \\ B & \frac{3}{4} & 0 & \frac{1}{4} \\ C & \frac{3}{4} & \frac{1}{4} & 0 \end{array}_{3\times3}$$

$3x + x = 1 \Rightarrow x = 1/4$

a) ⇒ $P^0 = \begin{array}{ccc} A & B & C \\ (0 & 1 & 0) \end{array}$

0 – 1st
1 – 2nd
2 – 3rd
3 – 4th week

$$P^{(3)} = P^0 P^3 = (0\ 1\ 0)\begin{pmatrix} 0 & 1 & 0 \\ 0.75 & 0 & 0.25 \\ 0.75 & 0.25 & 0 \end{pmatrix}^3$$

$$\Rightarrow \boxed{P^{(3)} = (0.6093 \quad 0.1875 \quad 0.2031)}$$

Probability that she ~~is~~ will buy brand B on the fourth week is 0.1875

b) long run ⇒ Steady State

$VP = V$

$V = (v_1\ v_2\ v_3)$ where $v_1, v_2, v_3 \geq 0$ and $v_1 + v_2 + v_3 = 1$

$$(v_1\ v_2\ v_3)\begin{pmatrix} 0 & 1 & 0 \\ 0.75 & 0 & 0.25 \\ 0.75 & 0.25 & 0 \end{pmatrix} = (v_1\ v_2\ v_3)$$

$0.75v_2 + 0.75v_3 = v_1$ and $v_1 + 0.25v_3 = v_2$

$v_1 + v_2 + v_3 = 1$

~~$v_1 = 3/7$, $v_2 =$, $v_3 = 0.1143$~~

$v_1 = 0.4286$, $v_2 = 0.4571$, $v_3 = 0.1143$.

The prob. that she buys brand C soap in the long run is 0.1143.

10) An auto-insurance company classifies its customers in 3 categories : poor, satisfactory and preferred. No one ~~knows~~ moves from poor to preferred or from preferred to poor in one year. 40% of customers in the poor category becomes satisfactory, 30% of those in the satisfactory category moves to preferred while 10% becomes poor, 20% of those in the preferred category are downgraded to satisfactory category.

a) Write the transition matrix for the model.

b) Is the markov chain irreducible?

→ States :- Poor (A), Satisfactory (B), Preferred (C)

a) → Transition matrix :-

$$P = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 0.6 & 0.4 & 0 \\ 0.1 & 0.6 & 0.3 \\ 0 & 0.2 & 0.8 \end{pmatrix}_{3\times 3} \end{array}$$

b) Irreducible matrix ⇒ Regular - stochastic matrix

$$P^2 = \begin{pmatrix} 0.4 & 0.48 & 0.12 \\ 0.12 & 0.46 & 0.42 \\ 0.02 & 0.28 & 0.7 \end{pmatrix}$$

Hence, the transition matrix is irreducible.

11) A man's profession can be classified as professional, skilled labourer or un-skilled labourer. Assume that of the sons of prof. men 80% are prof., 10% are skilled & 10% are unskilled. In the case of ~~son~~ of skilled lab., 60% are skilled, 20% - prof. & 20% unskilled. Finally in the case of unskilled labourers, 80% of sons are unskilled 25% - other 2 categories. Assume that every man has atleast one son & forms a markov chain by following a profession of a randomly chosen son of a given family through several generations.

a) Set up the matrix of transition prob.

b) find prob. that a randomly chosen grandson of an unskilled labourer is a prof. man.

c) In the long run, what is the prob. that the (~~great grandson of a labourer~~ skilled) is a professional man?

→ States :- Professional (p), Skilled Labourers (S) and Unskilled labourers (u).

a)

$$P = \begin{array}{c} \\ p \\ s \\ u \end{array} \begin{array}{c} \begin{array}{ccc} p & s & u \end{array} \\ \begin{pmatrix} 0.8 & 0.1 & 0.1 \\ 0.2 & 0.6 & 0.2 \\ 0.25 & 0.25 & 0.5 \end{pmatrix}_{3\times 3} \end{array}$$

(present generation) → rows; (Next Generation) → columns

b) $P^0 = (0 \ 0 \ 1)_{1\times 3}$

professional man, skilled, unskilled

0 - Father
1 - Son
2 - grandson
3 - g.g.son

$$p^{(2)} = P^0 P^2 = (1 0 0) \begin{pmatrix} 0.8 & 0.1 & 0.1 \\ 0.2 & 0.6 & 0.2 \\ 0.25 & 0.25 & 0.5 \end{pmatrix}^2$$

$$p^{(2)} = (0.375 \quad 0.3 \quad 0.325)$$

⇒ Prob. that a randomly chosen grandson of an unskilled labourer is a prof. man is 0.375.

c) long run ⇒ Steady state

$$VP = V$$

Let $V = (V_1 \ V_2 \ V_3)$

where $V_1, V_2, V_3 > 0$

$V_1 + V_2 + V_3 = 0$.

Assume that a man's profession

$$(V_1\ V_2\ V_3)_{1\times3}\begin{pmatrix}0.8 & 0.1 & 0.1\\0.2 & 0.6 & 0.2\\0.25 & 0.25 & 0.5\end{pmatrix}_{3\times3} = (V_1\ V_2\ V_3)_{1\times3}$$

$$0.8V_1 + 0.2V_2 + 0.25V_3 = V_1$$
$$0.1V_1 + 0.6V_2 + 0.25V_3 = V_2$$
$$0.1V_1 + 0.2V_2 + 0.5V_3 = V_3$$

$$\Rightarrow -0.1V_1 - 0.4V_2 + 0.25V_3 = 0$$
$$0.1V_1 + 0.2V_2 - 0.5V_3 = 0$$
and $V_1 + V_2 + V_3 = 1$

$$\Rightarrow (V_1\ V_2\ V_3) = (0.5263\ \ 0.2631\ \ 0.2105)$$

Hence, prob. that son is a professional man in the long run is 0.5263.

12) Consider a game of ladder climbing, there are 5 levels in the game, level 1 is the lowest (bottom) and level 5 is highest (top). A player starts at the bottom, each time a fair coin is tossed. If it turns up heads, the player moves up one rung (step), if tails, the player moves down to the very bottom. One at the top level, the player moves to the very bottom if a tails turns up and stays at the top if head turns up.

a) find transition prob matrix.

b) find the steady state distn of the Markov chain.

→ States :- level 1, level 2, level 3, level 4, level 5
(L1) (L2) (L3) (L4) (L5)

a)
$$P = \begin{array}{c|ccccc} & L1 & L2 & L3 & L4 & L5\\ \hline L1 & 0.5 & 0.5 & 0 & 0 & 0\\ L2 & 0.5 & 0 & 0.5 & 0 & 0\\ L3 & 0.5 & 0 & 0 & 0.5 & 0\\ L4 & 0.5 & 0 & 0 & 0 & 0.5\\ L5 & 0.5 & 0 & 0 & 0 & 0.5\end{array}_{5\times5}$$

b) Steady state ⇒ UFPV.

$$V = (V_1\ V_2\ V_3\ V_4\ V_5)_{1\times5}$$

where, $V_1, V_2, V_3, V_4, V_5 \geq 0$ and $V_1+V_2+V_3+V_4+V_5 = 1$

$$(V_1\ V_2\ V_3\ V_4\ V_5)_{1\times5}\begin{pmatrix}0.5 & 0.5 & 0 & 0 & 0\\0.5 & 0 & 0.5 & 0 & 0\\0.5 & 0 & 0 & 0.5 & \cancel{0.5}0\\0.5 & 0 & 0 & 0 & 0.5\\0.5 & 0 & 0 & 0 & 0.5\end{pmatrix}_{5\times5}$$

$$= (V_1\ V_2\ V_3\ V_4\ V_5)$$

$0.5(V_1+V_2+V_3+V_4+V_5) = V_1$ ⇒ $\boxed{V_1 = 0.5}$

$\cancel{-0.5V_1 + 0.5V_2 = 0}$

$\cancel{0.5V_1 - V_2 + 0V_3 = 0}$ $\quad 0.5V_1 = V_2$

$\cancel{0.5V_1 + 0V_2 - V_3 + 0.5V_4 = 0}$ $\quad 0.5V_2 = V_3$

$\cancel{0.5V_1 - V_4 + 0.5V_5 = 0}$ $\quad 0.5V_3 = V_4$

$\cancel{0.5V_1 + 0.5V_5 = 0}$ $\quad 0.5V_4 + 0.5V_5 = V_5$

$V_1 + V_2 + V_3 + V_4 + V_5 = 1$

$$\cancel{v_2 - 0.5v_3 = 0.25} \qquad \cancel{v_2 = 0.}$$
$$\cancel{v_3 - 0.5v_4 = 0.25} \qquad \cancel{v_3 = 0.4375}$$
$$\cancel{v_4 - 0.5v_5 = 0.25} \qquad \cancel{v_4 = 0.378}$$
$$\cancel{= 0.25}$$
$$\cancel{v_1 + v_2 + v_3 + v_4 + v_5 = 1} \qquad \boxed{\cancel{v_5 = v_1 = 0.25}}$$

$$\Rightarrow \boxed{v_1 = 0.5,\ v_2 = 0.25,\ v_3 = 0.125,\ v_4 = 0.0625,\ v_5 = 0.0625}$$

13) 2 boys B1 and B2, 2 girls G1 and G2 are throwing ball from one to the other. Each boy throws the ball to the other boy with prob. (1/2) and to each girl with prob ($\frac{1}{4}$), on the other hand, each girl throws the ball to each boy with prob (1/2) and never to the other girl. In the long run, how often does each receive the ball?

→ States :- B1, B2, G1 and G2.

$$P = \begin{array}{c|cccc} & B1 & B2 & G1 & G2 \\ \hline B1 & 0 & 0.5 & 0.25 & 0.25 \\ B2 & 0.25 & 0 & 0.25 & 0.25 \\ G1 & 0.5 & 0.5 & 0 & 0 \\ G2 & 0.5 & 0.5 & 0 & 0 \end{array}_{4\times4}$$

Long run ⇒ Steady State ⇒ UFPV

$$V = (v_1\ v_2\ v_3\ v_4)_{1\times4}$$
where, $v_1, v_2, v_3, v_4 \geq 0$ and
$v_1 + v_2 + v_3 + v_4 = 1$

$$(v_1\ v_2\ v_3\ v_4)_{1\times4} \begin{pmatrix} 0 & 0.5 & 0.25 & 0.25 \\ 0.25 & 0 & 0.25 & 0.25 \\ 0.5 & 0.5 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 \\ & & 0 & 0 \end{pmatrix}_{4\times4} = (v_1\ v_2\ v_3\ v_4)_{1\times4}$$

$$-v_1 + 0.5v_2 + 0.5v_3 + 0.5v_4 = 0$$
$$0.5v_1 - v_2 + 0.5v_3 + 0.5v_4 = 0$$
$$0.25v_1 + 0.25v_2 - v_3 + 0v_4 = 0$$
$$v_1 + v_2 + v_3 + v_4 = 1$$

$$\Rightarrow \boxed{v_1 = \frac{1}{3},\ v_2 = \frac{1}{3},\ v_3 = \frac{1}{6},\ v_4 = \frac{1}{6}}$$

14) In a certain city, the weather on a day is reported as sunny, cloudy or rainy. If a day is sunny, the prob. that the next day is sunny is 70%, cloudy is 20% and rainy is 10%. If a day is cloudy, the prob. that the next day is sunny is 30%, cloudy 20%, and rainy 50%. If a day is rainy, the prob. of next day being sunny is 30%, cloudy 30% and rainy is 40%. If a Sunday is sunny, find the prob. that the wednesday is rainy.

→ States :- Sunny (S), Cloudy (C) and Rainy (R)

$$P = \begin{array}{c|ccc} & S & C & R \\ \hline S & 0.7 & 0.2 & 0.1 \\ C & 0.3 & 0.2 & 0.5 \\ R & 0.3 & 0.3 & 0.4 \end{array}_{3\times3}$$

0 – Sunday
1 – Monday
2 – Tuesday
3 – Wednesday

Initial vector, p^0 = $\begin{array}{ccc} S & C & R \\ (1 & 0 & 0) \end{array}_{1\times3}$

$$p^{(3)} = p^0 P^3 = (1\ 0\ 0) \begin{pmatrix} 0.7 & 0.2 & 0.1 \\ 0.2 & 0.2 & 0.5 \\ 0.3 & 0.3 & 0.4 \end{pmatrix}^3$$

$$\Rightarrow \boxed{p^{(3)} = (0.532 \quad 0.221 \quad 0.247)}$$

$\Rightarrow$ Prob. that wednesday is rainy is 0.247.

15) Check whether the following Markov Chain is irreducible:-

a)

$$A = \begin{pmatrix} 1/2 & 1/3 & 1/5 \\ 1/3 & 1/3 & 1/3 \\ 0 & 1/9 & 8/9 \end{pmatrix}_{3\times3}$$

$$A^2 = \begin{pmatrix} 0.3611 & 0.2962 & 0.3425 \\ 0.2777 & 0.2592 & 0.4629 \\ 0.0370 & 0.1358 & 0.8271 \end{pmatrix}$$

Hence, A is an irreducible Markov Chain.

16) When Lily is sad which isn't very usual, she either goes for a run, gobbles down ice-cream or takes a nap. From historic data, if she spends nap on a sad day, the next day, it is 60% likely she will go for a run, 20% she will stay in bed the next day and 20% chance she will take ice-cream. When she is sad and goes for a run, there is 60% chances she will go for a run the next day, 30% she gobbles ice-cream and 10% nap. Finally, when she indulges in ice-cream, she continues to have ice-cream 10% the next day, 70% for run and 20% for nap the next day.

a) Construct the transition prob. matrix.

b) In the long run, find the prob. that lily will be gobbling the ice-cream.

c) Supposing Lily is sad on Monday and gobbles an ice cream, find the prob. that she is sleeping on Thursday.

$\rightarrow$ States $\rightarrow$ Run (R), Ice-cream (I), Nap (N).

a)

$$P = \begin{array}{c|ccc} & R & I & N \\ \hline R & 0.6 & 0.3 & 0.1 \\ I & 0.7 & 0.1 & 0.2 \\ N & 0.6 & 0.2 & 0.2 \end{array}_{3\times3}$$

b) Long run $\Rightarrow$ UFPV

$$V = (v_1 \quad v_2 \quad v_3)_{1\times3}$$

where, $v_1, v_2, v_3 \geq 0$ and $v_1+v_2+v_3 = 1$

$$(v_1\ v_2\ v_3)_{1\times3} \begin{pmatrix} 0.6 & 0.3 & 0.1 \\ 0.7 & 0.1 & 0.2 \\ 0.6 & 0.2 & 0.2 \end{pmatrix}_{3\times3} = (v_1\ v_2\ v_3)$$

$$-0.4v_1 + 0.7v_2 + 0.6v_3 = 0$$

$$0.3v_1 - 0.9v_2 + 0.2v_3 = 0$$

$$v_1 + v_2 + v_3 = 1$$

$$\Rightarrow \boxed{(V_1 \; V_2 \; V_3) = (0.6238 \quad 0.2385 \quad 0.1376)}$$

Hence, the prob. that Lily will be gobbling the ice-cream is 0.2385.

c) Initial vector, $p^0 = \underset{\text{ice-cream}}{(0 \quad 1 \quad 0)}_{1\times 3}$

0 - Monday
1 - Tuesday
2 - Wednesday
3 - Thursday

$$p(3) = p^0 p^3 = (0 \; 1 \; 0)\begin{pmatrix} 0.6 & 0.3 & 0.1 \\ 0.7 & 0.1 & 0.2 \\ 0.6 & 0.2 & 0.2 \end{pmatrix}^3$$

$$\Rightarrow \boxed{p(3) = (0.626 \quad 0.235 \quad 0.139)}$$

Hence, prob. that Lily will be sleeping on Thursday when she is sad is 0.139.

Queuing Theory

→ Danish Mathematician - Erlang.

→ Characteristics of a queing process.

* Arrival process - random
  Poission / Exponential distribution.
* Customer's behaviour
  - → Balking
  - → Reneging
  - → Jockeying
  - → Priorities
* Service Pattern : Poisson / Exponential distr.
* Service Discipline : FIFO, LILO, FILO, LILO
* Waiting Room : Capacity of a queue

→ Queuing Model :- Kendall's Notation

A | S | C | K | N | D

where, A - Arrival pattern [Poisson / Expo → M]

S - Mathematical distr of servicetime [Poisson/Expo → M]

K - Capacity of Queue [omitted if unlimited]

C - No. of services

N - No. of possible customers [omitted if unlimited]

D - Queue discipline [FIFO - omitted]

→ M/M/1 Model :-

1) At any instant of time the probability that there are 'n' arrivals (customers) in the queue (waiting line) including those being serviced is given by

Page No.
Date

a) $P_n = (1-\rho)\rho^n \quad ; n \geq 0$

where $\rho = \frac{\lambda}{\mu}$ = utilization factor

b) $P_0 = (1-\rho) \Rightarrow$ System is idle

c) Average (Expected no. of units in the system)

$$L_s = \frac{\rho}{1-\rho} = \frac{\lambda}{\mu-\lambda}$$

d) Average or Expected Queue Length

$$L_q = L_s - \rho$$
$$= \frac{\lambda^2}{\mu(\mu-\lambda)}$$

e) Mean or Expected waiting time of customers in the system (Including service time)

$$W_s = \frac{1}{\mu-\lambda}$$

f) Expected waiting time of customers in the queue

$$W_q = W_s - \frac{1}{\mu} = \frac{\lambda}{\mu(\mu-\lambda)}$$

g) Average length of non-empty queue (i.e., expected no. of customers in the queue when a queue is already existing)

$$P(L/L>0)$$

Page No.
Date

$$L_n = \frac{1}{1-\rho} = \frac{\mu}{\mu-\lambda}$$

h) Average waiting time in a non-empty queue,

$$W_n = \frac{1}{\mu-\lambda}$$

i) Probability that, $P(\text{Queue Length} \geq n) = \rho^n$

j) Little's law :- $L_s = \lambda W_s$

$$\boxed{L_q = \lambda W_q}$$

2) Note:- a) λ - arrival rate

μ - service rate

b) $\frac{1}{\lambda}$ = average arrival time

$\frac{1}{\mu}$ = average service time

Problems:-

1) A barber takes 25 minutes to complete one hair-cut on an average. If the customers arrive at an average interval of 40 min, how long on an avg. the customer must have waited for the service?

Data:- Service time :- $\frac{1}{\mu} = 25$ min

Arrival time :- $\frac{1}{\lambda} = 40$ min

To find :- W_q (waiting time before service)

Soln :- $W_q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{1/40}{1/25\left(\frac{1}{25}-\frac{1}{40}\right)}$

$\boxed{W_q = 41.67 \text{ min}}$

2) The rate of arrivals of planes at an international airport is 20 per hour & the airport can land 30 planes per hour on an average. When there is a ~~confect~~ congestion the planes are forced to fly over the field in the stack awaiting the landing of other planes that arrived earlier.

Find :-

a) How many planes would be flying over the field on an average?

b) How long a plane would be in the stack and in the process of landing?

Data :- $\lambda = 20/\text{hr}$.
$\mu = 30/\text{hr}$.

To find :-
i) L_q
ii) W_s

Soln :-

i) $L_q = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{(20)^2}{30(30-20)} = 1.33\text{hr} \approx \underline{80 \text{ minutes}}$.

ii) $W_s = \frac{1}{(\mu-\lambda)} = \frac{1}{(30-20)} = 0.1\text{hr}$ or 6 minutes

3) In a railway marshalling yard, goods train arrives at rate of 30 trains per day, assuming that the inter-arrival time follows an expo. distr. & the service time (time taken to hump a train) & distr. is also exponential with an average of 36 minutes. Calculate :-

a) Average no. of trains in the yard.

b) The prob. that the queue size exceeds 9.

c) Expected waiting time in the queue.

d) Average no. of trains in the queue.

Data :- $\lambda = 30$ trains/day $= 30/(24\times60) = 1/48$ minutes

$\frac{1}{\mu} = 36$ minutes

To find :- a) L_s b) $P(Q \geq 10) = \rho^{10}$
c) W_q d) L_q

Soln :-

a) $L_s = \frac{\lambda}{\mu-\lambda} = \frac{1/48}{1/36 - 1/48} = 3$ trains

b) $P(Q \geq 10) = \rho^{10} = \left(\frac{\lambda}{\mu}\right)^{10} = \left(\frac{30\times36}{48}\right)^{10} = 0.0563$

c) $W_q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{1/48}{1/36(1/36 - 1/48)} = \underline{108 \text{ minutes.}}$

d) $L_q = L_s - \rho = 3 - \left(\frac{36}{48}\right) = 2.25$

4) In a 24 hour service

5) The arrivals at telephone booth are considered to be following Poisson Law of distribution with an average time of 10 minutes b/w one arrival & the next. length of the telephone call is assumed to be distributed exponentially with a mean of 3 minutes.

a) what is the prob. that a person arriving at the booth will have to wait?

b) what is the average length of Queue that forms from time to time?

c) The telephone department will install the 2nd booth when convince that an arrival would expect to wait atleast 5 min for the phone. By how much must the flow of arrival be increased in order to justify the second booth?

Data:- $1/\lambda = 10$ min To find:- a)

$1/\mu = 3$ min

Sol^n:-

a) $P(Q \geq 1) = \rho' = \left(\frac{\lambda}{\mu}\right) = \frac{3}{10} = 0.3$

b) time to time ⇒ existing Queue ⇒ L_n.

$L_n = \frac{\mu}{\mu-\lambda} = \frac{1/3}{1/3 - 1/10} = 1.4 \sim 2.$

c) $W_q \geq 3$

Let '$\cancel{\lambda}$' 'λ'' be the new arrival rate,

$\Rightarrow \frac{\lambda'}{\mu(\mu-\lambda')} \geq 3$

$$\lambda' \geq 3\mu(\mu-\lambda')$$
$$\lambda' \geq 3\mu^2 - \frac{3}{3}\lambda'\mu$$
$$\lambda'(1+\tfrac{3}{3}) \geq 3\mu^2$$
$$\lambda' \geq \frac{3\mu^2}{2}$$
$$\lambda' \geq \frac{3(1/3)^2}{2} \geq \frac{1}{6}$$
$$\Rightarrow \boxed{\lambda' \geq \frac{1}{6}}$$

$$\lambda' - \lambda = \frac{1}{6} - \frac{1}{10} = 0.067$$

⇒ If the arrival rate increases by 0.0667, then the statement is justified.

6) In a departmental store one cashier is there to serve the customers & the customers pick up their needs ~~by~~ by themselves. The arrival rate is 9 customers for every 10 minutes and the cashier can serve 10 customers in 59 minutes. Assuming Poisson arrival rate & expo. distr. for service, find:-
a) average no. of customers in the system.
b) —11— in the queue.
c) average time the customer spends in the system.
d) average time a customer waits before serviced.

Data:- $\lambda = 9/10$ per minute
$\mu = 10/5$ per minute

a) $$Ls = \frac{\lambda}{\mu-\lambda} = \frac{9/10}{2-9/10} = 0.82 \sim 1.$$

b) $$Lq = \frac{\lambda^2}{\mu(\mu-\lambda)} = 0.3682 \sim 0.1$$

c) $$Ws = \frac{1}{\mu-\lambda} = 0.91 \sim 1$$

d) $$Wq = \frac{\lambda}{\mu(\mu-\lambda)} = 0.4091$$

7) The capacity of a communication line is 2000 bits per sec. The line is used to transmit 8 bits of characters. It is required to transmit a total of 12000 ~~bi~~ characters per minute. Find the average response time & expected no. of characters waited to be transmitted.

Data:-
$\mu = \frac{2000}{8}$ characters per second. (250)

$\lambda = \frac{12000}{60}$ characters per second. (200)

To find:- Ws, Lq

Soln:- $$Ws = \frac{1}{\mu-\lambda} = \frac{1}{250-200} = 0.02$$

$$Lq = \frac{\lambda^2}{\mu(\mu-\lambda)} = 3.2$$

Page No.
Date

8) Customers arrive at a first class ticket counter of railway at rate of 12 per hour. There is a clerk servicing the customers at rate of 30 per hour.
a) What is the prob. that there are no customers in the counter?
b) —"— more than 2 customers in the queue.
c) —"— a customer is being served and no one is waiting.

Data:- $\lambda = 12/hr$
$\mu = 30/hr$

To find:- a) P_0
b) $P(Q>2)$
c) P_1

Soln:- a) $P_0 = 1-\rho$
$= 1-\left(\frac{\lambda}{\mu}\right) = 1-\left(\frac{12}{30}\right)$
$= 0.6$

b) $P(Q>2) = \cancel{1-P(Q} \rho^3$
$= (0.4)^3$
$= 0.064$

c) $P_1 = P_0\rho' = (0.6)(0.4)$
$= 0.24$

* In a 24 hr service station vehicles arrive at the rate of 30 per day on an average. The average servicing time for a vehicle is 36 mins. Find (i) The mean number of vehicles waiting in the system.

(ii) The probability that queue size exceeds 9.

=>

$\lambda = \frac{30}{24 \times 60} = 0.2083 \qquad \frac{1}{\mu} = 36$

(i) $L_s = \frac{\lambda}{\mu - \lambda} = \frac{0.2083}{0.0277 - 0.2083} = \frac{0.2083}{-0.1806}$

$\simeq 1.15$

(ii) $P(Q \geq 10) = \rho^{10} = \left(\frac{\lambda}{\mu}\right)^{10} = \left(0.2083\right) = 0.056$

*