UNIT 2 : Combinatorics.

- Branch of Discrete Mathematics which deals with counting techniques. There are 2 basic rules of counting called SUM rule & PRODUCT rule.

- Suppose T1 & T2 are 2 tasks to be performed. If T_1 can be performed in m different ways & T_2 in n different ways. If these 2 tasks cannot be performed simultaneously then 1 of the 2 tasks can be performed in m+n ways.

$p \to q$

Inverse $\sim p \to \sim q$

Converse $q \to p$

Contra-positive $\sim q \to \sim p$

- Suppose that 2 tasks T1 & T2 are to be performed 1 after the other. If T_1 can be performed in m diff. ways & T_2 in n diff. ways ∴ Both the tasks can be performed in $m \times n$ ways.

Q) 4 different MATH books, 5 different CS books & 2 different C.T. books are to be arranged in a shelf. How many different arrangements are possible if
i) Books of particular subject must be together.
ii) MATH books are together.

Ans) i) MATH books can be arranged in 4! ways, CS books in 5! ways & C.T. books in 2! ways.
3 groups of books can be arranged in 3! ways.
Total no. of ways $= 3! \times 4! \times 5! \times 2!$.

ii) Consider 4 MATH books as one book, no. of books to be arranged is 8 (1+5+2) which can be arranged in 8! ways.

- Math books can be arranged in 4! ways.
- Total no. of ways $= 8! \times 4!$.

Q) How many arrangements are there for letters in word SOCIOLOGICAL? In how many of these arrangements
a) 'a' & 'g' are adjacent.
b) All vowels are adjacent.

A) $\dfrac{12!}{3! \times 2! \times 2! \times 2!}$

i) $\dfrac{11!}{3! \times 2! \times 2! \times 2!} \times 2!$

- a & g are adjacent so treat them as one letter.
- No. of permutation, a & g themselves can be arranged in 2! ways

ii) $\dfrac{7!}{2! \times 2!} \times \dfrac{6!}{3! \times 2!}$ (6 vowels considered as 1 alphabet)
$[12 - 1 + 6 = 7]$.

Q) A QP contains 10 questions of which 7 are to be answered. In how many ways can a student select 7 questions:
i) If he can choose any 7.
ii) If he should select 3 from first 5 & 4 from last 5.
iii) If he should select atleast 3 from first 5.

Ans) i) $^{10}C_7 = \dfrac{10!}{7!\,3!} = 120$.

ii) $^5C_3 \times {^5C_4} = \dfrac{5 \times 4}{2} \times 5 = 50$.

iii) $^5C_3 \cdot {^5C_4} + {^5C_4}\,{^5C_3} + {^5C_5}\,{^5C_2}$
$= 50 + 50 + 10 = 110$.

Q) How many 4 digit numbers can be formed with 10 digits $(0, \dots 9)$ if

i) Repetitions are allowed.

ii) Repetitions are not allowed

iii) Last digits must be zero repetitions are not allowed

Ans) i) $9 \times 10 \times 10 \times 10 = 9000$ ii) $9 \times 9 \times 8 \times 7 = 81 \times 56 = 4536$

$$\begin{array}{r} 81 \\ \times 56 \\ \hline 486 \\ 4050 \\ \hline 4536 \end{array}$$

iii) $9 \times 8 \times 7 \times 1 = 504$

Q) In how many ways can 3 men & 3 women be seated around a table if

i) No restriction

ii) 2 particular women must not sit together

iii) Each woman must be seated b/w 2 men.

Circular permutation,

Ans) i) $(n-1)! = 5! = 120$

ii) $5! - 2 \times 4! = 120 - 48 = 72$

iii) $2! \times 2! = 4$

Multinomial Theorem

We know that, $(x+y)^n = \sum_{r=0}^{n} {}^nC_r\, x^{n-r} y^r$

$= \sum_{r=0}^{n} \frac{n!}{(n-r)!\, r!} \cdot x^{n-r} \cdot y^r$ | let, $(n-r) = n_1$

$= \sum_{r=0}^{n} \frac{n!}{n_1!\, n_2!} \cdot x^{n-n_2} \cdot y^{n_2}$ | $r = n_2$

$$\sum \frac{n!}{n_1!\, n_2!} \cdot x^{n_1} \cdot y^{n_2} = \sum \binom{n}{n_1 \;\; n_2} x^{n_1} y^{n_2}$$

- Generalization of Binomial theorem is Multinomial theorem.

Multinomial Theorem

$$(x_1 + x_2 + x_3 + \dots + x_k)^n = \sum \binom{n}{n_1\, n_2 \dots n_k} x_1^{n_1} x_2^{n_2} \dots x_k^{n_k}$$

$$\binom{n}{n_1\, n_2 \dots n_k} = \frac{n!}{n_1!\, n_2! \dots n_k!}$$

For any +ve integer n & k. The co-efficient of $x_1^{n_1}, x_2^{n_2}$ upto $x_k^{n_k}$ in the expansion of $(x_1 + x_2 + \dots + x_k)^n$ is $\frac{n!}{n_1!\, n_2! \dots n_k!}$. Each n_i is positive integer and $n_1 + n_2 + \dots + n_k = n$.

Q1) Prove the following positive integer n. → Binomial

① $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots \binom{n}{n} = 2^n$

② $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \dots + \binom{n}{n}(-1)^n = 0$

Soln) W.K.T., $(x+y)^n = \sum_{r=0}^{n} {}^nC_r\, x^{n-r} y^r$

Let, $x = y = 1$,

$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots \binom{n}{n}$

Let, $x = 1$, $y = -1$

$$\Rightarrow 0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \dots + (-1)^n \binom{n}{n}$$

Q2) Find sum of all co-efficients in the expansion of $(x+y)^n$ & $(x_1 + x_2 + \dots + x_k)^n$. Hence find sum of all co-efficients in expansion of:

① $(x+y)^{10}$ ② $(x+y+z+w)^5$ ③ $(x+2y-3z+4u-5v)^6$

A) i) $(x+y)^n$, Ans) 2^n (write proof in exam).

$$= \sum_{r=0}^{n} {}^nC_r x^{n-r} y^r \Rightarrow 2^n = {}^nC_0 + {}^nC_1 + \dots + {}^nC_n \quad (x = y = 1)$$

ii) $(x_1 + x_2 + \dots + x_k)^n$ Let, $x_1 + x_2 + x_3 \dots = 1$

$$= \sum \binom{n}{n_1, n_2 \dots n_k} x_1^{n_1} x_2^{n_2} \dots x_k^{n_k}$$

$$k^n = \sum \binom{n}{n_1, n_2 \dots n_k}$$

① $(x+y)^{10}$ Sum of co-efficients $= 2^{10}$

② $(x+y+z+w)^5 = 4^5$

③ $(x+2y-3z+4u-5v)^6 = \sum \binom{6}{n_1, n_2 \dots n_5} x^{n_1} (2y)^{n_2} (-3z)^{n_3} (4u)^{n_4} (-5v)^{n_5}$

$$= \sum \binom{6}{n_1, n_2 \dots n_5} x^{n_1} y^{n_2} z^{n_3} u^{n_4} v^{n_5}\, 2^{n_2} (-3)^{n_3} (4)^{n_4} (-5)^{n_5}$$

Let, $x = y = z = u = v = 1$

$$(-1)^6 = \sum \binom{6}{n_1, n_2 \dots n_5} 2^{n_2} (-3)^{n_3} (4)^{n_4} (-5)^{n_5}$$

Sum of co-efficients $= 1$

Q3) Find co-efficient of $x^9 y^3$ in expansion of $(2x-3y)^{12}$.

A) $${}^{12}C_3 (2x)^9 (-3y)^3 = \frac{12 \times 11 \times 10}{6} \times 2^9 \times (-27) . x^9 . y^3$$

Co-efficient $: -2^{10} . 3^3 . 110$.

Q4) Find co-efficient of x^{12} in expansion of $x^3 (1-2x)^{10}$.

A) $$(1-2x)^{10} = \sum_{r=0}^{10} \binom{10}{r} (1)^{10-r} (-2x)^r$$

$r = 9$.

${}^{10}C_9 (-2x)^9$ $\qquad x^9 . x^3 \to x^{12}$.

Ans) $-10 . 2^9$.

Q5) Find co-efficient of x^k in expansion of $(1+x+x^2)(1+x)^n$ where n is a +ve integer. k lies b/w 0 & $n+2$.

A) $$(1+x)^n = \sum_{r=0}^{n} {}^nC_r (1)^{n-r} (x)^r.$$

$$(1+x+x^2)(1+x)^n = \sum_{r=0}^{n} {}^nC_r (1)^{n-r} (x)^r + \sum_{r=0}^{n} {}^nC_r (1) x^{n-r} x^{r+1} + \sum_{r=0}^{n} {}^nC_r (1) x^{n-r} x^{r+2}$$

Substitute k, k-1 & k-2 in the 3 terms respectively.

Co-efficient of $x^k = {}^nC_k + {}^nC_{k-1} + {}^nC_{k-2}$.

Q6) Find Co-efficient of xyz^2 in $(2x-y-z)^4$.

A) $(2x-y-z)^4 = \sum \binom{4}{n_1\ n_2\ n_3} (2x)^{n_1} (-y)^{n_2} (-z)^{n_3}$.

$n_1 = 1 \quad n_2 = 1 \quad z = 2$.

$\binom{4}{1\ 1\ 2} (2x)(-y)(-z)^2$.

Co-efficient $= \dfrac{4!}{2!} \times 2 \times (-1)(1)$

$= -4! = -24$.

Q7) Find Co-efficient of $x^{11}y^4$ in expansion of $(2x^3 - 3xy^2 + z^2)^6$.

A) $= \sum \binom{6}{n_1\ n_2\ n_3} (2x^3)^{n_1} \cdot (-3xy^2)^{n_2} \cdot (z^2)^{n_3}$

$x \to 3n_1 + n_2 = 11$, $n_3 = 0$

$y \to 2n_2 = 4$, $n_2 = 2$

$n_1 = 3$.

$\dfrac{6!}{3!\,2!\,0!} \cdot (2)^3 \cdot (-3)^2 = \dfrac{6 \times 5 \times 4}{2} \times 2^{\cancel{3}\,4} \times 9$

$= 30 \times 16 \times 9 \qquad \left(\binom{\cancel{3}}{3} = 0\right)$

$= 480 \times 9$

$= 4320$

Q8) Find Co-efficient of $a^2b^3c^2d^5$ in expansion of $(a + 2b - 3c + 2d + 5e)^{16}$.

$= \sum \binom{16}{n_1\ n_2\ n_3\ n_4\ n_5} a^{n_1} (2b)^{n_2} (-3c)^{n_3} (2d)^{n_4} 5^{n_5}$

$\boxed{2+3+2+5+4 = 16}$

$n_1 = 2,\ n_2 = 3,\ n_3 = 2,\ n_4 = 5,\ n_5 = 4$

$\binom{16}{2\ 3\ 2\ 5\ n_5} a^2 (2)^3 b^3 (-3)^2 c^2 (2)^5 d^5\ 5^4$.

$= \dfrac{16!}{2!\,2!\,3!\,2!\,4!\,8} \times 5^4 \times 2^8 \times (9) \times 2^5$ (x)

$= 2 \times 16!$.

Ans) $\dfrac{16!}{2!\,2!\,3!\,4!\,5!} \times (8)(9)(2^5)(5^4)$.

Q) Find Co-efficient of x^9y^3 in expansion of $(x+2y)^{12}$.

$$(x+2y)^{12} = \sum_{r=0}^{12} \binom{12}{r} (x)^{12-r} (2y)^r$$

$\boxed{r=3}$

$$^{12}C_3 (x)^9 (2y)^3 = \frac{\cancel{12}^{4} \times 11 \times 10}{\cancel{3} \times \cancel{2}} \times \cancel{8}^{4} \times x^9 y^3$$

$$= 110 \times 16 \times (x^9y^3)$$

$\rightarrow 1760$ is Co-efficient.

Q) Find Co-efficient of x^5y^2 in $(2x-3y)^7$ expansion.

A) $\binom{7}{r} (2x)^{7-r} (-3y)^r \qquad r=2.$

$$\frac{7\times6}{2\times1} \times (2)^5 \times (-3)^2$$

$\Rightarrow \; 7\times6 \times 2^4 \times 9 = 378 \times 16$

$= 6048$

$$\begin{array}{r} 378 \\ \times 16 \\ \hline 2268 \\ 3780 \\ \hline 6048 \end{array}$$

Q) Co-efficient of xyz^{-2} in $(x-2y+3z^{-1})^4$.

$$= \sum \binom{4}{n_1\, n_2\, n_3} (x)^{n_1} (-2y)^{n_2} (3z^{-1})^{n_3}$$

$n_1=1 \quad n_2=1 \quad n_3=2$

$$= \frac{4!}{1!\,1!\,2!} \cdot x \cdot (-2y) \cdot 9z^{-2}$$

$= 12 \times (-2) \times 9$

$= -216$

Q) Co-efficient of $w^3x^2yz^2$ in expansion of $(2w-x+3y-2z)^8$.

$$\binom{8}{n_1\, n_2\, n_3\, n_4} \cdot (2w)^{n_1} (-x)^{n_2} (3y)^{n_3} (-2z)^{n_4}$$

$$\frac{8!}{\cancel{3!}\,\cancel{2!}\,1!\,\cancel{2!}} \times \cancel{8} \times 1 \times \cancel{3} \times \cancel{4}$$

$= 4 \times 81$

$= 161280$

$$\begin{array}{r} 720 \\ \times 56 \\ \hline 4320 \\ 36000 \\ \hline 40320 \end{array}$$

Q) P.T. if n is non negative integer, $\frac{1}{2}\left[(1+x)^n + (1-x)^n\right]$

$= \binom{n}{0} + \binom{n}{2}x^2 + \dots + \binom{n}{k}x^k$ where $k = \begin{cases} n & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd} \end{cases}$

W.K.T. $(1+x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r \quad -①$ | $(1-x)^n = \sum_{r=0}^{n} \binom{n}{r} (x)^r (-1)^r \quad -②$

$① + ② \rightarrow 2\left[\binom{n}{0} + \binom{n}{2}x^2 + \dots + \binom{n}{k}x^k\right]$

$$\frac{① + ②}{2} = \binom{n}{0} + \binom{n}{2}x^2 + \dots + \binom{n}{k}x^k$$

Q) Find the sum of all co-efficients in expansion of $(x+y+z)^{12}$.

$$(x+y+z)^{12} \rightarrow \sum \binom{12}{n_1\, n_2\, n_3} (x)^{n_1} (y)^{n_2} (z)^{n_3}$$

$x, y, z = 1$

$$\sum \binom{12}{n_1\, n_2\, n_3} = 3^{12}$$

Q) $(2s - 3t + 5u + 6v - 11w + 3x + 2y)^{10}$

All 1,

$(2-3+5+6-11+3+2)^{10} = 4^{10} = \sum \binom{10}{n_1\, n_2\, n_3\, n_4\, n_5\, n_6\, n_7}$ Answer

$\sum \binom{10}{n_1 n_2 \dots n_7} = \frac{4^{10}}{(2^{n_1} \cdot 2^{n_1} \cdot (-3)^{n_2} \cdot (5^{n_3}) \cdot (6)^{n_4} \cdot (-11)^{n_5} \cdot (3)^{n_6}} \Bigg| \cdot 2^{n_1} \cdot (3)^{n_2} \cdot (5)^{n_3} \cdot (6)^{n_4} \cdot (-11)^{n_5} \cdot (3)^{n_6} \cdot (2)^{n_7}$

Principle of inclusion & exclusion

- If S is a finite set. Order of S denoted by $|S|$ is number of elements in S.
- If A & B are subsets of S then,

$$|A \cup B| = |A| + |B| - |A \cap B| \quad \rightarrow ①$$
$$|\overline{A \cup B}| = |S| - |A \cup B| = |\bar{A} \cap \bar{B}|$$

- No. of elements of $A \cup B$ includes all elements of A & B, excludes elements common to A & B.

$$|\bar{A} \cap \bar{B}| = |S| - |A| - |B| + |A \cap B| \quad \rightarrow ②$$

- ① & ② are reffered as principle of inclusion & exclusion of 2 sets.

- P.I.E. for 'n' sets.

Let S be a finite set. $A_1, A_2, \dots A_n$ be subsets of S. Principles of inclusion & exclusion of $A_1, A_2, \dots A_n$ states that,

$|A_1 \cup A_2 \cup A_3 \dots A_n| = [|A_1| + |A_2| + \dots + |A_n|] - [|A_1 \cap A_2| + |A_2 \cap A_3| + \dots |A_i \cap A_j|] + \sum [|A_i| + |A_j| + |A_k|] - + \dots (-1)^{n-1} [|A_1| \cap |A_2| \cap \dots |A_n|] \rightarrow ⓐ$

$|\bar{A_1} \cap \bar{A_2} \dots \cap \bar{A_n}| = S - |A_1 \cup A_2 \dots \cup A_n|$ $\rightarrow ⓑ$

Q) Suppose A_1 represents set of all those elements of S which satisfy a certain condition C_1, A_2 represents set of all those elements of S satisfying condition C_2 & so on. Let, $N = |S|$, $N(C_i) = |A_i|$

$N(C_i C_j) = |A_i \cap A_j|$ $\quad N(C_i C_j C_k) = |A_i \cap A_j \cap A_k|$

$N(C_i \text{ or } C_j) = |A_i \cup A_j|$ $\quad N(C_i \text{ or } C_j \text{ or } C_k)$ or C_n

$= S_1 - S_2 + S_3 - + \dots (-1)^{n-1} S_n$

$\bar{N} = S_0 - (S_1 - S_2 + S_3 + \dots + (-1)^{n-1} S_n)$

S_0 → cardinality of S.

Generalization, No. of elements in S satisfying exactly m out of n conditions is given by,

- $E_m = S_m - \binom{m+1}{1} S_{m+1} + \binom{m+2}{2} S_{m+2} - + \dots + (-1)^{n-m} \binom{n}{n-m} S_n$

- No. of elements in S that satisfy atleast m of n conditions.

$L_m = S_m - \binom{m}{m-1} S_{m+1} + \binom{m+1}{m-1} S_{m+2} \dots + (-1)^{n-m} \binom{n-1}{m-1} S_n$

Q) Among hostel students, 12 study Math, 20 study Phy, 20 chem. & 8 study Bio.
There are 5 students for M & P, 7 for M & C, 4 for M & Bi,
16 for P & C, 4 for P & Bi, 3 for C & Bi. 3 for M, P & C, 2 for M, P & Bi, 2 for P, C & Bi, 3 for M, C, & Bi, finally 2 for M, C, P, Bi.
71 students who don't study any of these subjects. Find total no. of hostel students.

A) $|M \cup P \cup C \cup B| = 12+20+20+8-(5+7+4+16+4+3 \cancel{+3} \cancel{+2}) + (3+3+2+1+3) - 2$

$= 60-(39)+(13)-2-3$

$= 73-41 = 32-3 = 29$

$|S| = |\bar{A} \cap \bar{B}| + |\bar{M} \cap \bar{P} \cap \bar{C} \cap \bar{B}| + |M \cup P \cup C \cup B|$

$= 32+71 = 103-3 = 100$

Q) How many integers between 1 & 300 are divisible by atleast one of ① (5, 6, 8 ?) & none of ② (5, 6, 8 ?)

A) Let A, B & C be set of dividends of 5, 6, 8.

$|A| = \frac{300}{5} = 60$ $|B| = \frac{300}{6} = 50$ $|C| = \left\lfloor\frac{300}{8}\right\rfloor = 37$

flooring 37.5 → 37

$|A \cap B| = \frac{300}{30} = 10$, $|A \cap C| = \frac{300}{40} = 7$

$|B \cap C| = \frac{300}{\cancel{6 \times 8}_{4}} = \frac{6}{12}$ $= \boxed{8}\ 12$ $|A \cap B \cap C| = \frac{\cancel{300}^{10}}{\cancel{5 \times 6 \times 8}_{24}} = 1$

$|A \cup B \cup C| = |A| + |B| + |C| - (|A \cap B| + |A \cap C| + |B \cap C|) + |A \cap B \cap C|$

$= 147 - 17 - 6 + 1$ (x).

$= 147 - 10 - 7 - 12 + 2$

$= 120$

② Not divisible by 5, 6, 8 $= |\bar{A} \cap \bar{B} \cap \bar{C}|$

$= S - |A \cup B \cup C|$

$= 300 - 120 = 180$

Q) Determine no. of integers b/w 1 & 300 which are:

a) Divisible by exactly 2 of 5, 6, & 8.

b) Divisible by atleast 2 of 5, 6 & 8.

$E_2 = S_2 - \binom{3}{1} S_3 = (10+7+12) - (3)(2)$

$= 29 - 6 = 23$

$L_2 = S_2 - \binom{2}{1} S_3 = 29 - 4 = 25$

$|A \cap B| + |A \cap C| + |B \cap C| - 3|A \cap B \cap C|$

$|A \cap B| + |A \cap C| + |B \cap C| - 2|A \cap B \cap C|$.

Q) Find no. of permutations of English letters which contain,

1) Exactly 2 2) Atleast 2 3) Exactly 3 4) Atleast 3 of the patterns {CAR, DOG, PUN, BYTE} (numbered 1, 2, 3, 4) $\boxed{n=4}$

- 26 letters total
- No repetition
- CAR no one change in pattern.

$\boxed{C|A|R}$ 23 $24! = |A_{CAR}|$

$|A_{DOG}| = |A_{PUN}| = 24!$

$|A_{BYTE}| = 23!$ $|S| = 26!$

$$E_2 = S_2 - \binom{3}{1} S_3 + \binom{4}{2} S_4$$

$$L_2 = S_2 - \binom{2}{1} S_3 + \binom{3}{1} S_4$$

$$|A_{CAR} \cap A_{DOG}| = 2 \times (26-6+1)!$$

DOGCAR: CARDOG

$|A_{CAR} \cap A_{PUN}| = |A_{DOG} \cap A_{PUN}| = 2 \times 21!$

$|A_{CAR} \cap A_{BYTE}| = |A_{PUN} \cap A_{BYTE}|$

$= |A_{DOG} \cap A_{BYTE}| = 2 \times 20!$

Ans) i) $|A_1| = 24!$ $|A_2| = |A_3| = 24!$ $|A_4| = 23!$

$|A_1 \cap A_2| = |A_1 \cap A_3| = |A_2 \cap A_3| = \cancel{|A_2 \cap A_4|} = 22!$

$|A_1 \cap A_4| = 21!$ $|A_3 \cap A_4| = 21!$

$|A_2 \cap A_4| = 21!$

$|A_1 \cap A_2 \cap A_4| = |A_2 \cap A_3 \cap A_4| = |A_1 \cap A_3 \cap A_4| = 19!$

$|A_1 \cap A_2 \cap A_3| = 20!$ $|A_1 \cap A_2 \cap A_3 \cap A_4| = 17!$

i) Exactly 2: $E_2 = S_2 - \binom{3}{1} S_3 + \binom{4}{2} S_4$

$= 3 \times 22! + 3 \times 21! - 3[1 \times 20! + 3 \times 19!] + 6 \times 17!$

ii) Atleast 2: $L_2 = S_2 - \binom{2}{1} S_3 + \binom{3}{1} S_4$

$= 3 \times 22! + 3 \times 21! - 2[1 \times 20! + 3 \times 19!] + 3 \times [17!]$.

iii) Exactly 3: $E_3 = S_3 - \binom{4}{1} S_4$

$= 20! + 3 \times 19! - 4 \times 17!$

iv) Atleast 3:

$L_3 = S_3 - \binom{3}{2} S_4$

$= 20! + 3 \times 19! - 3 \times 17!$

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Q) In how many ways can the integers 1, 2, 3 ... upto 10 be arranged in a line, so that no even integer is in its natural space/pos.

1 2 3 4 5 6 7 8 9 10

$|A_1| = |A_2| \dots = |A_5| = 9!$ $\quad {}^5C_1 \times 9!$

$A_1 \rightarrow$ 2 is in its position

$|A_1 \cap A_2| = 8! = |A_2 \cap A_3| = |A_3 \cap A_4| = |A_4 \cap A_5|$

$\therefore = |A_1 \cap A_3| = \dots$

$S_2 = {}^5C_2 \times 8!$ $\qquad S_0 = 10!$

$S_3 = {}^5C_3 \times 7!$ $\qquad S_4 = {}^5C_4 \times 6!$ $\qquad S_5 = {}^5C_5 \times 5!$

$|\bar{A_1} \cap \bar{A_2} \cap \bar{A_3} \cap \bar{A_4} \cap \bar{A_5}|$

$|S| -$

$= 5 \times 9! - {}^5C_2 \times 8! + {}^5C_3 \times 7! - {}^5C_4 \times 6! + {}^5C_5 \times 5!$

$|S| - |A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5| = \bar{A_1} \cap \bar{A_2} \cap \bar{A_3} \cap \bar{A_4} \cap \bar{A_5}$

$= 10! - (5 \times 9! - {}^5C_2 \times 8! + {}^5C_3 \times 7! - {}^5C_4 \times 6! + {}^5C_5 \times 5!)$

$= 2170680.$

Q) In how many ways can 26 letters of English alphabets be permuted so that none of the patterns CAR, DOG, PUN, BYTE occurs.

A) $|\bar{A_1} \cap \bar{A_2} \cap \bar{A_3} \cap \bar{A_4}| = |S| - |A_1 \cup A_2 \cup A_3 \cup A_4|$

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$= 26! - [3 \times 24! + 23! - \{3 \times 22! + 3 \times 21!\} + \{3 \times 19! + 20!\} - 17!]$.

Q) In how many ways 5 nos. of a's, 4 nos. of b's & 3 nos. of c's can be arranged so that all the identical letters are not in a single block.

A) $|S| = \dfrac{12!}{5!\,4!\,3!}$ $\qquad A \rightarrow$ Set of permutations where a's are together.

B, C.

$|A| = \dfrac{8!}{4!\,3!}$ $\qquad |B| = \dfrac{9!}{5!\,3!}$ $\qquad |C| = \dfrac{10!}{5! \times 4!}$

$|A \cap B| = \dfrac{5!}{3!}$ $\qquad |B \cap C| = \dfrac{7!}{5!}$ $\qquad |A \cap C| = \dfrac{6!}{4!}$

$|A \cap B \cap C| = 3!$

$|\bar{A} \cap \bar{B} \cap \bar{C}| = |S| - |A \cup B \cup C| = \dfrac{12!}{5!\,4!\,3!} - \left[\dfrac{8!}{4!\,3!} + \dfrac{9!}{5!\,3!} + \dfrac{10!}{5!\,4!}\right]$

$+ \dfrac{5!}{3!} + \dfrac{7!}{5!} + \dfrac{6!}{4!} - 3!$.

Catalan Number

- The integers $b_0, b_1, b_2 \dots$ $\quad b_0 = 1$, $b_n = \dfrac{C(2n, n)}{n+1}$

$b_n = \dfrac{{}^{2n}C_n}{n+1}$ $\Bigg|$ $b_n = \dfrac{(2n)!}{(n!)^2\,(n+1)}$

$b_1 = 1, b_2 = 2, b_3 = 5, b_4 = 14, b_5 = 42, \dots$

☆ $b_n = \binom{2n}{n} - \binom{2n}{n-1}$

- X-Y plane with 'O' as origin & 'P' as a point with n, n as its co-ordinate.
- n is +ve integer. Suppose we wish to reach point 'P' from 'O' by making moves R & U. $R: (x, y) \to (x+1, y)$
  $U: (x, y) \to (x, y+1)$.
- Now there is a restriction that we may touch the line $y = x$, but never rise above it.
- A path in which we move from O to P under stated restrictions is called a good path from O to P.

→ No. of k such paths is the Catalan Number

No. of good paths = Total no. of paths − (No. of paths which cross $y = x$)

NOTE) For $n \geq 1$, Catalan no. b_n represents no. of good paths from origin 'O' to point P (n, n) (n is a +ve integer) $b_n = \frac{(2n)!}{(n!)^2 (n+1)}$

Q) Using moves $R(x, y) \to (x+1, y)$ $U(x, y) \to (x, y+1)$

Find no. of ways one can go from

i) $(0,0)$ to $(3,3)$ & not rise above the line $y = x$.

ii) $(0,0)$ to $(6,6)$ | $y = x$

iii) $(2,1)$ to $(7,6)$ | $y = x - 1$

iv) $(3,8)$ to $(10,15)$ $y = x + 5$.

A) i) $b_3 = \frac{(2 \times 3)!}{(3!)^2 (4)} = 5$.

ii) $b_6 = \frac{(2 \times 6)!}{(6!)^2 \times 7} = 132$

iii) Shift origin $(0,0)$ to $(2,1)$. $(7,6) \to (5,5)$

$X = x - 2$, $Y = y - 1$.

No. of good paths from $(2,1)$ to $(7,6)$ = No. of good paths from $(0,0)$ to $(5,5)$

Line ($y = x$)

$b_5 = \frac{10!}{5! \times 5! \times 6} = \frac{10 \times 9 \times 8 \times 7}{5 \times 4 \times 3 \times 2 \times 1} = 42$.

iv) Shift O $(0,0)$ to $(3,8)$ $X = x - 3$, $Y = y - 8$

⇒ $(10,15) \longrightarrow (7,7)$ line ($y = x$)

No. of good paths $= b_7 = \frac{14!}{7! \times 8!}$

$= \frac{14 \times 13 \times 12 \times 11 \times 10 \times 9}{7 \times 6 \times 5 \times 4 \times 3 \times 2} = 33 \times 13 = 429$.

<u>Derangements</u>

Permutations of n distinct objects in which none of the objects is in its natural place is called derangements

Eg) series : 1, 2, 3. Permutations where 1 is not in 1st place and 2 is not in 2nd place & 3 is not in 3rd place.

For n numbers, derangements $= D_n$.

$(n = 1)$, $D_n = 0$. $(n = 2)$, $D_n = 1$.

$(n = 3)$, $D_n = 2$.

$$\Rightarrow d_n = n! \left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + \frac{(-1)^n}{n!} \right]$$

for $n \geq 1$,

$$d_n = n! \sum_{k=0}^{n} \frac{(-1)^n}{k!}$$

NOTE) $e^{-1} = \sum_{k=0}^{n} \frac{(-1)^n}{k!}$ | $d_n = n!(0.36768)$ $\downarrow$ Only to verify.

Q) Derangements for 1, 2, 3, 4.

$d_4 = 4!\left[\frac{12-4+1}{24}\right] = 9$.

Q) There are n pairs of children's gloves in a box. Each pair is of a different colour. Suppose the right-handed gloves are distributed at random to n children & there after left gloves are also distributed at random. Find probability that:

i) No child gets a matching pair.
ii) Every child gets a matching pair.
iii) Exactly one child gets a matching pair.
iv) Atleast 2 children.

A)

i) No. of derangements $d_n = n!e^{-1}$

Probability $= \frac{d_n}{n!} = e^{-1}$

ii) $(n=1)$ $\cancel{d_1} = \cancel{1}$. Only 1 way. (d_1) | Probability $= \frac{1}{n!}$

iii) Exactly one child $\rightarrow d_{(n-1)}$ | Prob $= \frac{d_{(n-1)}}{n!} \rightarrow \frac{(n-1)!\,e^{-1}}{n(n-1)!} = \frac{e^{-1}}{n}$

iv) Atleast 2 children

$\hookrightarrow$ (No child, 1 child $\rightarrow x$)

$\sum_{k=0}^{n-2} d_k$. $1 - e^{-1} - \frac{e^{-1}}{n}$ | $\left[1-(d_n+d_{n-1})\right]$

UNIT-3 : Probability

- **Poisson Distribution** : Regarded as limiting form of binomial distribution. $n \to \infty$, $p \to 0$ (Probability of success) $np \to \lambda$ (λ = const.).

$$p(x) = \frac{\lambda^x e^{-\lambda}}{x!}$$

- $p(x) \geq 0$ (+ve terms in R.H.S.) & $\sum p(x) = e^{-\lambda}\left(1 + \lambda + \frac{\lambda^2}{2!} \dots\right)$

$= e^{-\lambda} \cdot e^{\lambda}$ [(MacLaurin Series)]

- It is a probability distribution.

→ Mean & Variance of Poisson Distribution.

$$\text{Mean} = \frac{\sum [p(x) \cdot (x)]}{\sum p(x)} \qquad \sum p(x) = 1$$

$$= \sum_{x=0}^{\infty} \frac{x e^{-\lambda} \lambda^x}{x!} = \lambda e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}$$

$$= \lambda e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}$$

(Approx., ignore $x=0$)

OR
$x - 1 = y$
y : 0 to ∞
$\frac{\lambda^y}{(y+1)!}$

$$= \lambda e^{-\lambda}\left[1 + \frac{\lambda}{1} + \frac{\lambda^2}{2!} + \dots\right]$$

$$= \lambda\left(e^{-\lambda} e^{\lambda}\right) = \lambda.$$

$$\text{Variance} = \sigma^2 = \frac{\sum f_i (x - \mu)^2}{\sum f_i}$$

$$\sigma^2 = \frac{\sum [p_i (x_i - \lambda)^2]}{\underbrace{\sum p_i}_{\to 1}} = \sum p_i [x_i^2 + \lambda^2 - 2x_i\lambda]$$

$$= \sum p_i x_i^2 + \underbrace{\sum p_i}_{1} \lambda^2 - 2\lambda \underbrace{\sum p_i x_i}_{\lambda}$$

$$= \sum p_i x_i^2 + \lambda^2 - 2\lambda^2 = \sum p_i x_i^2 - \lambda^2.$$

$$\sigma^2 = \sum_{x=0}^{\infty} \left[\frac{x^2 \cdot e^{-\lambda} \lambda^x}{x!}\right] - \lambda^2 = e^{-\lambda} \sum_{x=0}^{\infty} \left[\frac{x^2 \lambda^x}{x!}\right] - \lambda^2.$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \left[\frac{[x(x-1) + x]\lambda^x}{x!}\right] - \lambda^2.$$

$$= e^{-\lambda}\left(\sum_{x=0}^{\infty} \left[\frac{(x)(x-1)\lambda^x}{x!}\right] + \sum_{x=0}^{\infty} \left[\frac{x\lambda^x}{x!}\right]\right) - \lambda^2.$$

$$= e^{-\lambda}\left[\sum_{x=0}^{\infty} \left(\frac{\cancel{x}\cancel{(x-1)}\lambda^x}{\cancel{x}\cancel{(x-1)}(x-2)!}\right) + \sum_{x=0}^{\infty} \left(\frac{\lambda^x}{(x-1)!}\right)\right] - \lambda^2.$$

$$= e^{-\lambda}\left[\lambda^2\left(\sum_{x=0}^{\infty}\left(\frac{\lambda^{x-2}}{(x-2)!}\right)\right) + \lambda\left(\sum_{x=0}^{\infty}\left(\frac{\lambda^{x-1}}{(x-1)!}\right)\right)\right] - \lambda^2$$

↳ Expansion. (MacLaurin Series)

$$= e^{-\lambda}\left[\lambda^2 e^{\lambda} + \lambda e^{\lambda}\right] - \lambda^2$$

$$= \cancel{\lambda^2} + \lambda - \cancel{\lambda^2} = \lambda.$$

$S.D. = \sigma = \sqrt{\lambda}$ $(\sigma^2 = \lambda)$

Q) No. of accidents in a year to a taxi in a city follows Poisson distribution, with mean = 3. Out of 1000 taxi drivers find approximately no. of drivers with no accident in a year | more than 3 accidents in a year.

A) $\mu = \lambda = 3$ $n = 1000$, $p = \frac{3}{1000}$

$X \to$ No. of accidents in 1 year.

$p(x) = \frac{e^{-\lambda} \lambda^x}{x!}$ $p(x) = \frac{3^x e^{-3}}{x!}$

i) $P(X=0)$,

$p(0) = \frac{3^0 e^{-3}}{0!} = 0.049$.

No. of taxi's with no accidents
$= (1000) \times (0.049) = 49$.

ii) $P(x > 3) = 1 - P(x \le 3)$

$= 1 - [P(x=0) + P(x=1) + P(x=2) + P(x=3)]$

$= 1 - \left[e^{-3} + e^{-3}.3 + \frac{e^{-3}.3^2}{2} + \frac{e^{-3}.3^3}{6}\right]$

$= 1 - e^{-3}[13]$

$= \cancel{1-e^{-3}[}\ 0.3527$

No. of taxi's with ~~3~~ 4 or more accidents $= 0.3527 \times 1000$
$= 353$,

Q) 2 % of fuses manufactured by a firm are found to be defective. Find probability that a box containing 200 fuses.

(i) No defective fuses   (ii) 3 or more defective fuses

$x =$ No. of defective fuses.

$n = 200$, $p = 0.02$ $\lambda = 4$.

i) $P(x=0) = \frac{4^0 . e^{-4}}{0!} = 0.01831$ //

No. of fuse

ii) $P(X \ge 3) = 1 - [P(X=0) + P(X=1) + P(X=2)]$.

$= 1 - \left[\frac{4^0 . e^{-4}}{0!} + \frac{4 . e^{-4}}{1!} + \frac{\overset{8}{\cancel{16}} . e^{-4}}{\cancel{2!}}\right]$.

$= 1 - 13[e^{-4}] = 0.7618$ //

Q) A distributor of bean seeds determines from extensive test that 5 % of large batch of seeds will not germinate. He sells the seeds in packets of 200. He guarantees 98 % germination, determine probability that a particular packet will violate the guarantee.

$x \to$ No. of seeds which do not germinate per packet.

A) $\lambda = \frac{5}{100} \times 200 = 10$.

98% germination (4/3/2/1/0 seeds don't germinate),

$P(x>4)$

$= 1 - [P(x=0) + P(x=1) + P(x=2) + P(x=3) + P(x=4)]$

$= 1 - \left[\frac{e^{-10}\cdot 10^0}{0!} + \frac{e^{-10}\cdot 10^1}{1!} + \frac{e^{-10}\cdot 10^2}{2!} + \frac{e^{-10}\cdot 10^3}{3!} + \frac{e^{-10}\cdot 10^4}{4!}\right]$

$= 1 - e^{-10}\left[1 + 10 + 50 + \frac{1000}{6} + \frac{10000}{24}\right]$

$= 1 - e^{-10}\left[61 + \frac{14000}{24}\right]$

$= 1 - e^{-10}\left[644.33\right]$

$= 97.07\% \cdot 0.9707$

97% of the time, guarantee is violated.

Q) In a certain factory turning out razor blades, there is a 0.002 chance that blade is defective. Blades are supplied in packs of 10. Use Poisson distribution to calculate approximate number of packets containing:

1) No defective blades 2) ≥1 defective 3) 2 defective

in consignment of 10,000 packets.

A) $X \rightarrow$ No. of defective blades for packet.

$p = 0.002 \quad n = 10 \quad \lambda = 10 \times \frac{2}{1000} = 0.02$

i) $p(x=0) = \frac{e^{-0.02}\cdot(0.02)^0}{0!} = 0.98019863$

No. of ~~bls~~ packets $=$
$0.98019 \times 10000 = 9802$ //

ii) $p(x=1) = \frac{e^{-0.02}\cdot(0.02)^1}{1!} = 0.019603$

No. of packets $= 196$ //

iii) $p(x=2) = \frac{e^{-0.02}\cdot(0.02)^2}{2!} = 1.9603 \times 10^{-4}$

No. of packets $= 1.9603 \simeq 2$.

Q) A car-hiring firm has 2 cars which it hires out day by day. The number of demands for a car on each day is distributed as Poisson distribution with mean = 1.5. Calculate the probability that on a certain day

1) There is no demand 2) Demands are refused.

$X \rightarrow$ No. of demands for car.

$\lambda = 1.5$

1) $p(x=0) = \frac{e^{-1.5}\cdot(1.5)^0}{0!} = 0.2231$ //

2) $p(x=1) = \frac{e^{-1.5}\cdot(1.5)^1}{1!} = 0.33469$ // $\simeq 0.3347$

3) $1 - [p(x=0) + p(x=1) + p(x=2)] = 1 - [0.2231 + 0.3347 + 0.2510]$

$= 0.1912$ //

classmate

Date ______ Page ______

Q) A communication channel receives independent pulses at the rate of 12 pulses per micro-second. The probability of transmission error is 0.001 for each micro second. Calculate probabilities of i) no error during μs ii) 1 ~~no~~ error per micro second iii) Atleast 1 error per μs iv) 2 errors per μs v) Atmost 2 errors per μs.

A) $x \to$ No. of errors per micro-second.

$p = 0.001 \quad \lambda = 0.001 \times 12 = 0.012$.

$$p(x) = \frac{e^{-0.012}(0.012)^x}{x!}$$

i) $p(0) = e^{-0.012} = 0.98807$

ii) $p(1) = e^{-0.012}(0.012) = 0.011856$

iii) $p(0) + p(1) = 0.999928$ (Atmost 1.)

iv) $p(2) = \frac{e^{-0.012}(0.012)^2}{2} = 7.1141 \times 10^{-5}$.

v) $p(0) + (p(1)) + p(2) = 0.999999141$

vi) Atleast 1 error $= 1 - P(x=0) = 0.012$.

Q) The frequency of accidents per shift in a factory is as shown in following table.

(APS)	0	1	2	3	4
Freq. ~~Shifts~~.	180	92	24	3	1

- Calculate mean number of accidents per shift & the corresponding Poisson distribution, compare with actual observation.

A) $x \to$ No. of APS. $\quad \text{Mean} = \frac{\sum f_i x_i}{\sum f_i}$

$= \frac{144}{324} = 0.444 \quad 0.51 = \mu$.

$$p(x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-0.51} \cdot (0.51)^x}{x!}$$

x	$f(x)$	$p(x)$	Expected frequency.
0	180	0.6004	180.12 ≃ 180
1	92	0.3062	91.86 ≃ 92
2	24	0.0780	23.4 ≃ 23
3	3	0.01327	3.96 ≃ 4
4	1	1.6926×10^{-3}	0.50778 ≃ 1

No. of Shifts $= \sum f(x)$ $= 300$.

(Add 1 or 2 if any problem) 300

Q) Fit a Poisson dist. for data. Calculate theoretical frequencies.

x	0	1	2	3	4
f	111	63	22	3	1

$\sum f_i = 200$.

$$\mu = \frac{63 + 44 + 9 + 4}{200} = \frac{120}{200} = 0.6 = \lambda.$$

$$p(x) = \frac{e^{-0.6}(0.6)^x}{x!}$$

x	$f(x)$	$p(x)$	Theor. freq.
0	111	0.5488	109.76 → 110
1	63	0.32928	65.856 → 66
2	22	0.098786	19.757 → 20
3	3	0.014757	3.951 → 4
4	1	2.9635×10^{-3}	0.5937 → 1
			201
			200

Continous Probability Distribution

- Probability function → $f(x) \geq 0$. $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

→ If a random variable takes non countable infinite number of values, is called Continous R.V.

→ If a variant can take any value in an interval, it will give rise to continous distribution.

→ If for every $x \in$ Range of C.R.V. 'X' we assign a real number $f(x)$ satisfying above conditions.

→ $f(x)$: C.P.F. & X is C.R.V. $(-\infty < x < \infty)$

NOTE: If (a,b) is a sub-interval of range space of 'X', then probability that x lies in (a,b) is probability

$\longrightarrow P(a<x<b) = \int_a^b f(x)\,dx$.

$\int_{-\infty}^{+\infty} f(x)\,dx = 1$ geometrically means that area bounded by the curve $f(x)$ & x-axis is 1.

- Mean & Variance.

$$\mu = \int_{-\infty}^{\infty} f(x)\,dx \cdot x$$

$$\text{Variance} = \sigma^2 = \int_{-\infty}^{\infty} f(x)(x-\mu)^2\,dx.$$

i) Exponential distribution: The C.P.D. having C.P.F.

$$f(x) = \begin{cases} \alpha e^{-\alpha x} & x>0 \\ 0 & \text{Otherwise.} \end{cases} \quad \alpha > 0.$$

- Since α & x are +ve $\Rightarrow f(x) \geq 0$.

$$\therefore \int_{-\infty}^{\infty} f(x)\,dx = \underbrace{\int_{-\infty}^{0} f(x)\,dx}_{0} + \int_0^{\infty} f(x)\,dx.$$

$$= \int_0^{\infty} \alpha e^{-\alpha x}\,dx = \left[\cancel{\alpha}\,\frac{e^{-\alpha x}}{-\cancel{\alpha}}\right]_0^{\infty}$$

$$= -\left[e^{-\infty} - e^{0}\right] = -[-1] = 1.$$

→ Mean $= \int_{-\infty}^{\infty} f(x)\,x\,dx = \int_{-\infty}^{0} x f(x)\,dx + \int_0^{\infty} x f(x)\,dx.$

$$= \int_0^{\infty} -\alpha x\, e^{-\alpha x}\,dx \qquad \alpha x = y,\ \alpha\,dx = dy,\ dx = \frac{dy}{\alpha}.$$

$$= \alpha\left[\frac{x \cdot e^{-\alpha x}}{\alpha} - \frac{e^{-\alpha x}}{\alpha^2}\right]_0^{\infty} = -\alpha\left[\frac{-1}{\alpha^2}\right] = \frac{1}{\alpha}.$$

Variance, $\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$.

$-\infty$ to 0 : $f(x)=0$.

$= \int_0^{\infty} (x-\mu)^2 \alpha e^{-\alpha x} dx$.

$= \alpha \int_0^{\infty} \overset{u}{(x-\mu)^2} \overset{v}{e^{-\alpha x}} dx$

$= \alpha \left[(x-\mu)^2 \frac{e^{-\alpha x}}{-\alpha} - 2(x-\mu)\frac{e^{-\alpha x}}{\alpha^2} + 2\frac{e^{-\alpha x}}{-\alpha^3} \right]_0^{\infty}$

$= -\alpha \left[\mu^2 \left(\frac{-1}{\alpha}\right) + \frac{2\mu}{\alpha^2} - \frac{2}{\alpha^3} \right]$

$= +\alpha \times \frac{+1}{\alpha^3} = \frac{1}{\alpha^2}$

$\sigma^2 = \frac{1}{\alpha^2}$ $\quad \sigma = \frac{1}{\alpha} = S.D$

Q) The duration of telephone conversation has been found to have an exponential distribution with mean = 3 minutes. Find probability that the conversation may last for more than a minute, less than 3 minutes.

i) ~~$1 - [P(0<x<1)]$~~ $P(x>1)$ =

Mean $= 3 = \frac{1}{\alpha}$ $\quad \alpha = \frac{1}{3}$.

$f(x) = \begin{cases} e^{-x/3}/3 & x>0 \\ 0 & \text{otherwise.} \end{cases}$

$P(x>1) = \int_1^{\infty} \frac{e^{-x/3}}{3} dx = \frac{1}{3} \times (-3) \left[0 - e^{-1/3}\right]$

$= e^{-1/3} = 0.7165$

$P(0<x<3) = \frac{1}{3} \int_0^{3} e^{-x/3} dx$.

$= \frac{1}{3} \times [-3] \left(-e^{-1}\right) = 1 - \frac{1}{e} = 0.632$

Q) Duration of shower E.D., $\mu = 5$ $\quad \alpha = \frac{1}{5}$

Prob. that shower will last for:

i) 10 mins or more $\quad$ ii) less than 10 min $\quad$ iii) B/w 10 & 12.

i) $P(x>10) = \int_{10}^{\infty} \frac{e^{-x/5}}{5} dx = \frac{1}{5} \times (-5) \left[0 - e^{-2}\right]$

$= 0.1353$

ii) $P(x<10) = \int_0^{10} \frac{e^{-x/5}}{5} dx = \frac{1}{5} \times (-5) \left[e^{-2} - 1\right]$.

$= 1 - e^{-2} = 0.8646$

iii) $P(10<x<12) = \int_{10}^{12} \frac{e^{-x/5}}{5} = \frac{1}{5} \times (-5) \left[e^{-12/5} - e^{-2}\right]$

$= [0.1353 - 0.0907] = 0.04459$.

Q) After the appointment of new Sales manager, the Sales in 2-wheelers Showroom is exponentially distributed $(\mu = 4,\ \alpha = \frac{1}{4})$. 2 days taken at random. What is the probability that:

i) On both the days, Sale is over 5 units

ii) Sale is over 5 units on atleast one of the two days.

A) $f(x) = \frac{1}{4} e^{-x/4}$

$P(x>5) = \int_5^{\infty} \frac{1}{4} e^{-x/4}\,dx$

$= e^{-5/4} = 0.2865.$

i) $\underset{\text{2 days}}{P(x>5)} = (0.2865)^2 = 0.08208$

$P(x \le 5) = 0.7135.$

$\underset{\text{Atleast one}}{P(x \le 5)} = 2 \times (0.7135)(0.2865) + (0.2865)^2.$

$= 0.4909$

$= 0.4909$

Q) The life of a compressor manufactured by a company is known to be 200 months on an average, E.D. followed. Find probability that life of a compressor of that company is:

i) Less than 200 ii) b/w 100 & 25x12 = 300

$\alpha = \frac{1}{200}$

i) $\int_0^{200} \frac{e^{-x/200}}{200}\,dx = \left[1 - e^{-1}\right]$

$= 0.6321$

ii) $\int_{100}^{300} \frac{e^{-x/300}}{300}\,dx = \left[e^{-1/3} - e^{-2/3}\right]$

$= 0.2031$

o Normal Distribution.

$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ $\quad -\infty < x < \infty$

$-\infty < \mu < \infty$

$\sigma > 0$

$f(x) \ge 0$ & $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx$

$\frac{x-\mu}{\sqrt{2}\,\sigma} = t \Rightarrow x = t\sqrt{2}\,\sigma + \mu$

$dx = \sqrt{2}\,\sigma\,dt$

$\Rightarrow \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-t^2} \sqrt{2}\,\sigma\,dt = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2}\,dt.$

$= \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-t^2} t^2\,dt = \frac{2}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2} = 1.$

(Derivation not needed.

$$\text{Mean} = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{-\infty}^{\infty} \frac{x\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}}\,dx$$

Put, $\frac{x-\mu}{\sigma\sqrt{2}} = t \Rightarrow dx = \sqrt{2}\,\sigma\,dt$

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} (\mu + \sqrt{2}\sigma t)\, e^{-t^2}\,dt = \mu.$$

S.D. of normal derivation

$$\text{Variance} = \int_{-\infty}^{\infty} f(x)\,(x-\mu)^2\,dx.$$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx$$

$$= \sigma^2.$$

$SD = \sigma^2.$

Q) In a test on 2000 electric bulbs, it was found that life of a particular make was normally distributed with an average life of 2040 hrs & S.D. of 60 Hrs. Estimate no. of bulbs likely to burn for
i) More than 2150 hrs.
ii) Less than 1950 hrs.
iii) More than 1920 hrs & less than 2160 hrs.

A)

$\mu = 2040$ hrs $\quad$ S.D. $= \sigma = 60$ hrs.

$x \rightarrow$ life of an electric bulb. $\quad \phi(z)$

$Z \Rightarrow \frac{x-\mu}{\sigma} = \frac{x-2040}{60} \quad = \phi(-x)$

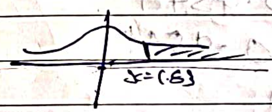
i) Prob. of $x > 2150$.

$x - 2040 > 2150 - 2040$

$\frac{x-2040}{60} > \frac{110}{60}$

$\frac{x-2040}{60} > 1.83$

$P(x > 2150) = P(Z > 1.83)$.



$P(x > 1.83) = 0.5 - \phi(1.83)$

$= 0.5 - (0.4664)$

$= 0.0336.$

No. of bulbs which will last for more than 2150 hrs $= 0.0336 \times 2000 = 67.2 \simeq 67.$

2) Prob. of $x < 1950$

$\frac{x-2040}{60} < \frac{1950-2040}{60}$

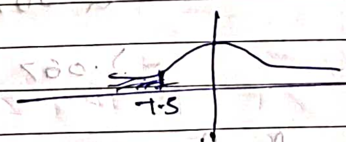
$Z = \frac{x-2040}{60} \quad || \quad P(x < -1.5). \quad Z < -1.5.$

$P(x < 1950)$

$= 0.5 - \phi(1.5)$

$= 0.5 - 0.4332$

$= 0.0668$

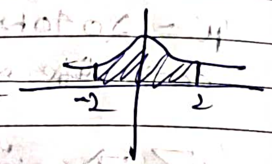


No. of bulbs $= (0.0668) \times (2000)$

$= 133.6 \simeq 134.$

3) Prob. of $1920 < x < 2160$

$\frac{-120}{60} < \frac{x - 5040}{60} < \frac{120}{60}$



$-2 < z < 2$.

$P(-2<z<2) = \phi(2) + \phi(-2)$

$= 2(0.4772)$

$= 0.9544$.

No. of bulbs $= 2000 \times 0.9544$

$\simeq 1909$

Q) Mean height of 500 Students = 151 cm
SD = 15 cm (Normal distribution), Find no. of Students whose height lie b/w 120 & 155 cm.

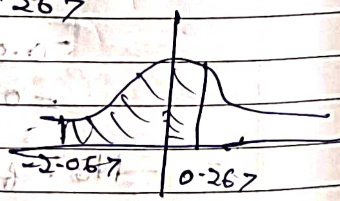
$z = \frac{x - \mu}{\sigma}$ (Student height) $\frac{x - 151}{15}$

$\frac{120 - 151}{15}$ $\qquad$ $\frac{155 - 151}{15}$

-2.067 $\qquad$ 0.267 $\cancel{0.266}$

$-2.067 < z < 0.267$

$\phi(2.067) + \phi(0.267)$



$= 0.4808 + 0.1064$

$= \cancel{0.6251}$. 0.5872

No. of Students $= 500 \times (\cancel{0.6251})$ 0.5872

$= 312.55$.

$\simeq 313$.

$\simeq 294$.

Q) Manufacturer of AIRMAIL envelopes knows from past experience that weight of envelope is normally distributed with mean = 1.95 g. S.D. = 0.05 g. About how many envelopes weigh
i) 2 g or more ii) 2.05 g or more
iii) can be expected? (Pack of 100 envelopes)

A) $z = \frac{x - 1.95}{0.05} = 1$ $\qquad$ $\frac{2 - 1.95}{0.05} = 1$

$\frac{2.05 - 1.95}{0.05}$

i) $z \geq 1$ $\qquad$ $= \frac{0.1}{0.05} = 2$.

$= 0.5 - \phi(1)$

$= 0.5 - 0.3413 = 0.1587$

$\simeq \frac{158.7}{10} \simeq \frac{159}{10} \simeq 16$.

ii) $z > 2$

$= 0.5 - \phi(2) = 0.5 - 0.4772$

$= 0.0228$.

No. of mail $= 2.28$.

$\simeq 2$

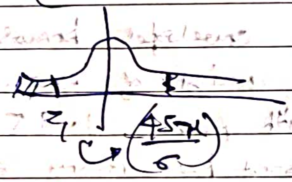
Q) In a normal distribution 31% of items are under 45 & 8% are over 64. Find Mean & Variance.

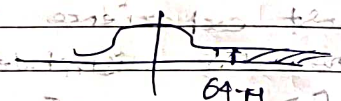
$\frac{x-\mu}{\sigma} = z$ $\quad \frac{45-\mu}{\sigma}$ $\quad P(x<45) = 0.31$

$P(x>64) = 0.08$

$P\left(z < \frac{45-\mu}{\sigma}\right) = 0.31$

$P\left(z > \frac{64-\mu}{\sigma}\right) = 0.08$





$0.5 - 0.31 = \phi\left(\frac{45-\mu}{\sigma}\right)$ $\quad 0.08 = 0.5 - \phi\left(\frac{64-\mu}{\sigma}\right)$

$\phi\left(\frac{45-\mu}{\sigma}\right) = 0.19$ $\quad \phi\left(\frac{64-\mu}{\sigma}\right) = 0.42$

$= \phi(1.41)$

$= \phi(0.5)$

$64 - \mu = 1.41\sigma$

$45 - \mu = 0.5\sigma$

$64 - 1.41\sigma = \mu$

$45 + 0.5\sigma = \mu$

$45 + 0.5\sigma = 64 - 1.41\sigma$

$\mu = 50$

$\sigma = 10.$ $\quad \cancel{0}1.91\sigma = \cancel{19}\ 19$ $\quad \sigma = 10$

$\sigma = 20.829$

$\sigma = \frac{\mu - 45}{0.5}$ $\quad \sigma = \frac{64-\mu}{1.41}$

$1.41\mu - 63.45 = 32 - 0.5\mu$

$\mu = 49.973$ $\quad 1.91\mu = 95.45$

LOGIC DESIGN

UNIT-1: Basic gates & combinational logic circuits.

Logic gate: is the basic digital electronic circuit that has one or more inputs & single output.

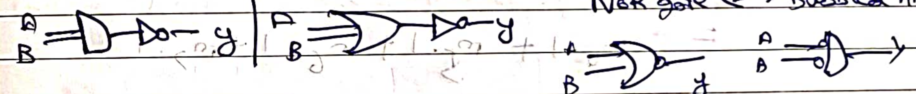
Low voltage 'logic' → 0, High voltage 'logic' → 1.

- Basic, universal & Special gates.
- Basic: AND, OR & NOT gate.
- Universal: NAND, NOR $\quad$ Special: EX-OR, EX-NOR.

→ Inverter, AND & OR gates.

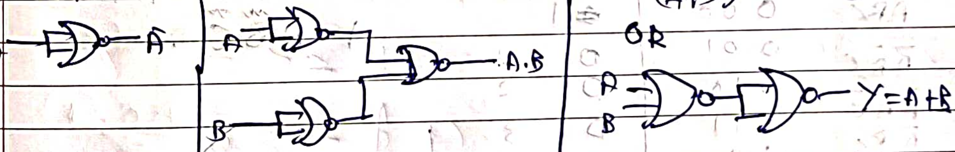
→ +ve & -ve logic $\quad$ LOGIC 0 - OFF $\quad$ LOGIC 1 - ON.

NOR gate ⟺ Bubbled AND.



$\overline{(A+B)} = \bar{A}\cdot\bar{B}$.

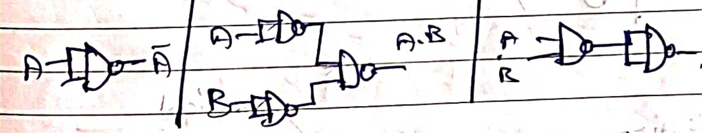
Universal NOR 7402



NAND gate ⟺ Bubbled OR

$\bar{A}+\bar{B} \Leftrightarrow \overline{A.B}$

Universal NAND 7400



+ve	-ve
OR	AND
AND	OR
NOR	NAND
NAND	NOR

Combinational Logic Circuit.

- SOP / POS
  AND-OR   OR-AND
  NAND-NAND   NOR-NOR.

Minterms

- SOP.   A , B $A'.B'$, $A'.B$, $A.B'$, $A.B$.
  m_0, m_1, m_2, m_3.

$Y = m_0.0 + m_1.1 + m_2.1 + m_3.1$.

$Y = \sum m(1,2,3)$.

$\bar{A}B + A\bar{B} + AB$.

A B	Y
0 0	0
0 1	1
1 0	1

$Y = f(A,B,C) = \sum m(0,5,6,7)$.

$= m_0.1 + m_5.1 + m_6.1 + m_7.1$

	A B C	Y	
$\bar{A}\bar{B}\bar{C}$	0 0 0	1	0
$\bar{A}\bar{B}C$	0 0 1	0	1
$\bar{A}B\bar{C}$	0 1 0	0	2
$\bar{A}BC$	0 1 1	0	3
$A\bar{B}\bar{C}$	1 0 0	1	4
$A\bar{B}C$	1 0 1	1	5
$AB\bar{C}$	1 1 0	1	6
ABC	1 1 1	0	7

$Y = \bar{A}\bar{B}\bar{C} + A\bar{B}C + AB\bar{C}$
$+ AB\bar{C}$.

Simplification.

$Y = \bar{A}B + A\bar{B} + AB = \bar{A}B + A\bar{B} + AB + AB$ ← Canonical

$Y = B(\bar{A}+A) + A(B+\bar{B})$

$= B.1 + A.1$

$= A+B \rightarrow$ Non-Canonical.

eg)

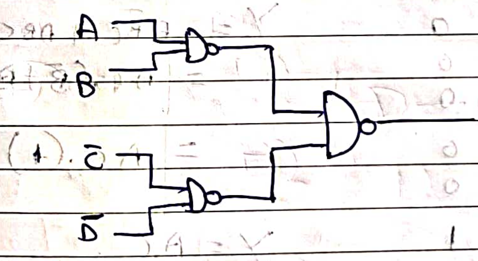
A B C	Y
0 0 0	0
0 0 1	0
0 1 0	0
0 1 1	1
1 0 0	0
1 0 1	1
1 1 0	1
1 1 1	1

$Y = \bar{A}BC + A\bar{B}C + AB\bar{C} + ABC$
$= BC(A+\bar{A}) + A(\bar{B}C + B\bar{C})$

ASSIGNMENT - 1.

1) Implement $AB + \bar{C}\bar{D}$ with only 3 NAND gates. Draw logic diagram also. Assume inverted input is available.

A) $AB + \bar{C}\bar{D} = \overline{\overline{AB} . \overline{\bar{C}\bar{D}}}$



$Y = AB + \bar{C}\bar{D} = \overline{\overline{AB}.\overline{\bar{C}\bar{D}}}$

2) Implement the following function using universal gates only : $((A+B).C)'.D$

A) $\overline{((A+B).C)}.D = (\overline{A+B} + \bar{C}).D$

$= \overline{\overline{(\overline{A+B} + \bar{C})} + \bar{D}}$

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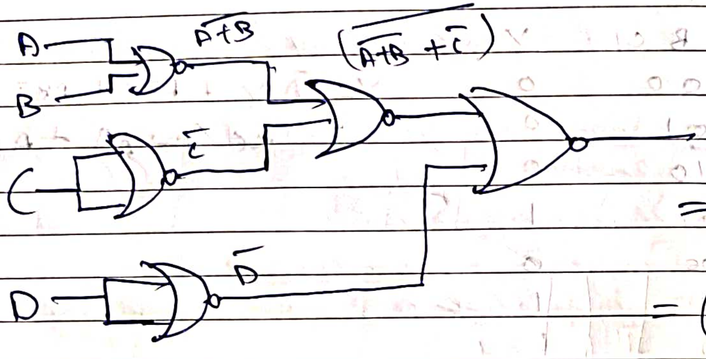


$$= \overline{((\overline{A+B}) \cdot C)' \cdot D}$$

$$= (\overline{A+B} \cdot C) \cdot D$$

$$= (\overline{A+B} + \bar{C}) \cdot D$$

$$= (\overline{\overline{A+B}+\bar{C}}) + \bar{D}$$

3) Write the truth table of the logic circuit having 3 inputs A, B & C and the output expressed as $Y = A\bar{B}C + ABC$. Also simplify the expression using Boolean Algebra and implement the logic circuit using NAND gates.

TRUTH TABLE

A	B	C	Y
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

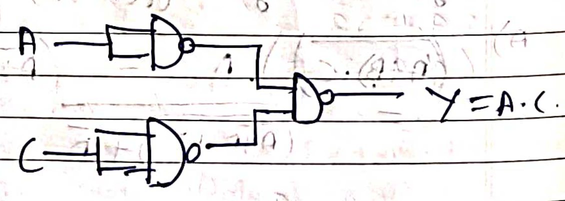
$$Y = A\bar{B}C + ABC$$

$$= AC.(\bar{B}+B)$$

$$= AC.(1)$$

$$Y = AC$$

Logic Circuit :



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4) i) Simplify the equation $AB + (\overline{AC}) + A\bar{B}C(AB + C)$ to obtain '1' after applying basic laws.

ii) Prove that $A(\bar{A}+C)(\bar{A}B+C)(\bar{A}BC+\bar{C}) = 0$.

A) i) $AB + (\overline{AC}) + \overset{\to 0}{A \cdot A}(\overset{\to 0}{B \cdot \bar{B}}) \cdot C + C \cdot \bar{C} \cdot A \cdot \bar{B}$

$$= AB + \overline{AC} + 0.AC + A\bar{B}C$$

$$= AB + \overline{AC} + A\bar{B}C$$

$$= \overline{AB \cdot AC} \quad A(B + \bar{B}C) + \overline{AC}$$

$$= A(B+C) + \overline{AC} = AB + (AC + \overline{AC}) +$$

$$= AB + 1 = 1$$

$$= AB + (1)$$

ii) $A(\bar{A}+C)(\bar{A}B+C)(\bar{A}BC+\bar{C})$

$$= (\underbrace{A\bar{A}}_{\to 0} + A.C)(\bar{A}B+C)(\bar{A}BC+\bar{C})$$

$$= (A.C)(\bar{A}B+C)(\bar{A}BC+\bar{C})$$

$$= (\underbrace{A.\bar{A}.B.C}_{\to 0} + A.C)(\bar{A}BC+\bar{C})$$

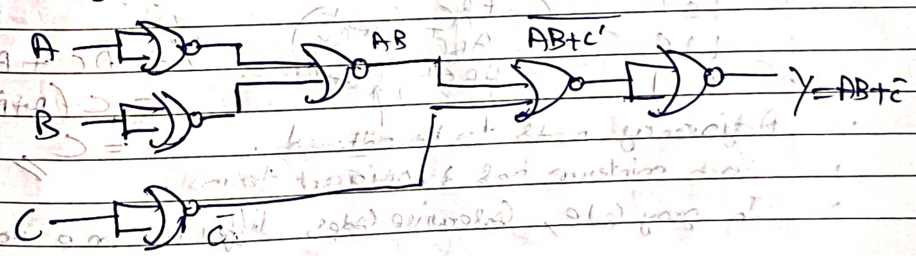
$$= (A.C)(\bar{A}BC + \bar{C})$$

$$= (A.\bar{A}.B.C) + (A.C.\bar{C})$$

$$= 0 + 0$$

$$= 0$$

5) Realize $Y = AB + C'$ using only NOR gate.



Truth table to Karnaugh Map. (K-map)

$\Longleftrightarrow$

$\bar{A}\bar{B}$	$\bar{A}B$
$A\bar{B}$	AB

$\bar{A}\bar{B}$	$\bar{A}B$
$A\bar{B}$	AB

	B'	B
A'	$A'B'$	$A'B$
A	AB'	AB

$\longrightarrow$

	B' 0	B 1
0 A'	00 (0)	01 (1)
1 A	10 (2)	11 (3)

A	B	Y
0	0	0
0	1	1
1	0	1
1	1	1

Implicants (Output's 1)

	B'	B
A'	0	1
A	1	1

Literal. eg: $AC'D \rightarrow 3$ Literals. $A'BC'D \rightarrow 4$ Literals.

· Implicants can be group of minterms.

eg)

A	B	C	Minterms
0	0	0	$\bar{A}\bar{B}\bar{C}$
0	0	1	$\bar{A}\bar{B}C$
0	1	0	$\bar{A}B\bar{C}$
0	1	1	$\bar{A}BC$
1	0	0	$A\bar{B}\bar{C}$
1	0	1	$A\bar{B}C$
1	1	0	$AB\bar{C}$
1	1	1	ABC

$Y = f(A, B, C)$.

	$\bar{B}\bar{C}$	$\bar{B}C$	BC	$B\bar{C}$
$\bar{A}$	0 (0)	1 (1)	1 (3)	0 (2)
A	0 (4)	1 (5)	1 (7)	0 (6)

Only 1 variable change

$Y = \bar{A}C + AC$
$= C(A + \bar{A})$
$= C$

· Adjacency needs to be adhered.
· Each minterm has 3 adjacent terms.
· In gray code, consecutive codes differ in one variable only.

Q) $Y = f(A, B, C) = \sum m(2, 3, 4, 6)$.
$= A'BC' + A'BC + AB'C' + ABC'$.

	00 $B'C'$	01 $B'C$	11 BC	10 BC'
0 A'	0 (0)	0 (1)	1 (3)	1 (2)
1 A	1 (4)	0 (5)	0 (7)	1 (6)

$\rightarrow P_1$

P_2

$= A'B + AC'$

Prime Implicants

$A'B$, B, AC'

Essential.

Non Essential

Q) $Y = f(A, B, C) = \sum m(0, 2, 4, 6)$

	$B'C'$	$B'C$	BC	BC'
A'	1 (0)	0 (1)	0 (3)	1 (2)
A	1 (4)	0 (5)	0 (7)	1 (6)

$Y = C'$.

Fundamental product = 1

4-Bit Gray Code.

G_3	G_2	G_1	G_0
0	0	0	0
0	0	0	1
0	0	1	1
0	0	1	0
0	1	1	0
0	1	1	1
0	1	1	0
0	1	0	0
1	1	0	0
1	1	0	1
1	1	1	1
1	0	1	1
1	0	1	0
1	0	0	0
1	0	0	1

1110 ←

G_3	G_2	G_1	G_0
0	0	0	0
0	0	0	1
0	0	1	1
0	0	1	0
0	1	1	0
0	1	1	1
0	1	0	1
0	1	0	0
1	1	0	0
1	1	0	1
1	1	1	1
1	1	1	0
1	0	1	0
1	0	1	1
1	0	0	1
1	0	0	0

4 - Variable K-Map.

Q1)

AB \ CD	C'D' 00	C'D 01	CD 11	CD' 10
A'B' 00	0 (0)	1 (1)	1 (3)	1 (2)
A'B 01	0 (4)	0 (5)	0 (7)	1 (6)
AB 11	1 (12)	1 (13)	0 (15)	1 (14)
AB' 10	1 (8)	1 (9)	0 (11)	1 (10)

$Y = f(A, B, C, D)$

$= \sum m(1, 2, 3, 6, 8, 9, 10, 12, 13, 14)$.

$= AC' + CD' + DA'B'$

$= \cancel{A + T + D}$.

Q2) Simplify $Y = \sum m(0, 1, 2, 4, 5, 6, 8, 9, 10, 12, 13)$.

	C'D'	C'D	CD	CD'
A'B'	1 (0)	1 (1)	0 (3)	1 (2)
A'B	1 (4)	1 (5)	0 (7)	1 (6)
AB	1 (12)	1 (13)	0 (15)	0 (14)
AB'	1 (8)	1 (9)	0 (11)	1 (10)

Octet, Quad, single.

$C' + A'D' + B'D'$.

$C' + D'(A' + B')$.

Q3) $Y = \sum m(3, 4, 5, 7, 9, 13, 14, 15)$.

	C'D'	C'D	CD	CD'
A'B'	0 (0)	0 (1)	1 (3)	0 (2)
A'B	1 (4)	1 (5)	1 (7)	0 (6)
AB	0 (12)	1 (13)	1 (15)	1 (14)
AB'	0 (8)	1 (9)	0 (11)	0 (10)

$BD + BD' + B'D$.

$\cancel{Y = A'BC' + A'CD + AC'D + ABC}$.

Don't Care Condition (Output = X) $\to$ 0 or 1

eg) $Y = \sum m(1, 3) + \sum d(5, 6, 7)$

- In K-Map, all ones have to be covered. But, all don't cares need not be covered.

A \ BC				
	0	1	1	0
	0	x	x	x

$Y = C$

Product of Sum method

- OR-AND or NOR-NOR network.

- Sum term: Disjunction of literals, each literal is either boolean variable / complement.

eg) $A + B$ $A' + B + C'$ A.

- For a function of n variables, a sum term in which each of n variables appears once is called a maxterm.

$M_0 = A + B$ | 00   Uncomplemented $\to$ 0.

$M_1 = A + B'$ | 01   1 $\to$ Comp.

$M_2 = A' + B$ | 10

$M_3 = A' + B'$ | 11.   Fundamental Sum gives 0.

AND gate. $Y = (A + B) \cdot (A + B') \cdot (A' + B)$ (OR-AND network)

00 0 ✓ $= M_0 \cdot M_1 \cdot M_2$

01 0 ✓ $Y = f(A, B) = \pi M(0, 1, 2)$.

10 0 ✓

11 1

	B'	B
A'	0	0
A	0	1

$Y = AB$

Q) $f(A, B, C, D) = \pi M(0, 2, 4, 10, 11, 14, 15)$.

AB \ CD	C+D	C+D'	C'+D'	C'+D
A+B	0 (0)	1 (1)	1 (3)	0 (2)
A+B'	0 (4)	1 (5)	1 (7)	1 (6)
A'+B'	1 (12)	1 (13)	0 (15)	0 (14)
A'+B	1 (8)	1 (9)	0 (11)	0 (10)

Quad $A' + C'$

Corners $A + D + B$

~~Quad~~ (0, 4) 2-cell: $A + C + D$

Statistics and Discrete Mathematics

- Joint Probability Distribution ($x, y \rightarrow$ Discrete random variable)

Expectation : $E(X) = \mu_x = \sum_{i=1}^{n} x_i f(x_i)$

Variance : $V(X) = \sum_{i=1}^{n} (x_i - \mu)^2 f(x_i) = E(x-\mu)^2$

$\mu_y = \sum_{j=1}^{n} y_j g(y_j)$ $E(XY) = \sum x_i y_j J_{ij}$

Covariance :

$$cov(X,Y) = E(XY) - \mu_x \mu_y .$$

Correlation of X & Y is :

$$\rho(X,Y) = \frac{cov(X,Y)}{\sigma_x \sigma_y}$$

$$\sigma_x^2 = E(X^2) - \mu_x^2 \qquad \sigma_y^2 = E(Y^2) - \mu_y^2 .$$

Q) A fair coin is tossed thrice, the random variables X & Y are defined as follows. X = 0 or 1 according head / tail occurs on 1st toss. Y = No. of heads. i) Determine distribution of X & Y ii) Joint distribution of X & Y iii) $E(X)$, $E(Y)$, $E(XY)$, $cov(X,Y)$, $\rho(X,Y)$.

Outcomes : HHH, HHT, HTT, TTT, TTH, THH, THT, HTH.

$X = \{0, 1\}$ $Y = \{0, 1, 2, 3\}$.

(2)

X \ Y	0	1	2	3	f(x)
0	0	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	$\frac{4}{8}$
1	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	0	$\frac{1}{8}$
g(y)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	1

$P(0,0) = 0 \quad P(0,1) = \frac{1}{8} \quad P(0,2) = \frac{2}{8}$

$P(0,3) = \frac{1}{8} \quad P(1,0) = \frac{1}{8} \quad P(1,1) = \frac{2}{8}$

$P(1,2) = \frac{1}{8} \quad P(1,3) = 0$

(1) · Distribution of X

X	0	1
f(x)	$\frac{4}{8}$	$\frac{4}{8}$

· Distribution of Y

Y	0	1	2	3
f(x)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

(3) $E(X)$

$\sum_{i=1}^{m} x_i f(x_i) = 0\left(\frac{4}{8}\right) + 1\left(\frac{4}{8}\right) = \frac{4}{8}$.

$E(Y) = 0\left(\frac{1}{8}\right) + 1\left(\frac{3}{8}\right) + 2\left(\frac{3}{8}\right) + 3\left(\frac{1}{8}\right)$

$= \frac{12}{8} = \frac{3}{2}$

$E(XY) = (0)(0)(0) + (0)(1)\left(\frac{1}{8}\right) + (0)(2)\left(\frac{2}{8}\right) + (0)(3)\left(\frac{1}{8}\right)$

$+ (1)(0)\left(\frac{1}{8}\right) + 1(1)\left(\frac{2}{8}\right) + 1(2)\left(\frac{1}{8}\right) + (1)(0)(3)$

$= \frac{4}{8} = \frac{1}{2}$ //

$\text{Cov}(X, Y) = E(XY) - \mu_x \mu_y = \frac{1}{2} - \left(\frac{3}{2}\right)\left(\frac{4}{8}\right)$

$= \frac{-1}{4}.$

$\rho(X, Y) = \frac{\text{cov}(X, Y)}{\sigma_x \sigma_y}.$ $\quad \sigma_x^2 = E(X^2) - \mu_x^2$

$E(X^2) = \sum x^2 p(x_i) = \frac{4}{8}.$

$\sigma_x^2 = \frac{4}{8} - \frac{1}{4} = \frac{1}{4}$ $\quad \sigma_x = \frac{1}{2}.$

$\sigma_y^2 = E(Y^2) - \mu_y^2$

$E(Y^2) = \sum y^2 p(y_i) = \frac{3}{8} + \frac{4 \times 3}{8} + \frac{9}{8}.$

$= \frac{24}{8} = 3.$

$\sigma_y^2 = E(Y^2) - \mu_y^2 = 3 - \frac{9}{4} = \frac{3}{4}.$

$\sigma_y = \frac{\sqrt{3}}{2}.$