GRAPH THEORY :

- Graph is mathematical object consisting a set of;

  V = nodes (vertices, pts) non empty set.

  E = edges (lines, curves) may be empty.

  edge e represent $\rightarrow \{u, v\}$ or uv.

  $G = (V, E)$

no. of vertices = ORDER = $\lvert V \rvert = n$
no. of edges = SIZE = $m = \lvert E \rvert = m$

Null Graph:

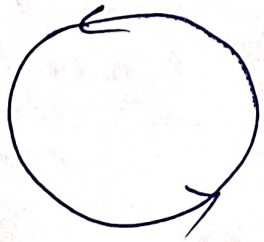
Graph containing no edges.

Trivial G:

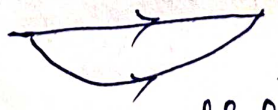
null graph with only one vertex.

Directed graph:

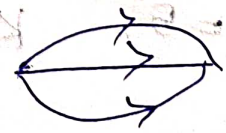
Graph is digraph if V is nonempty & E is set of ordered pairs of elements taken from V set. otherwise it is undirected G.



directed loop



parallel direc edges

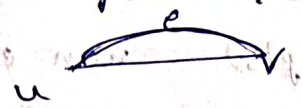


multiple d.e.

undirected

Self loop

Parallel edges



$e = \{u, v\} \in E(G)$

adjacent edges:

have common pts with no parallel edges.

Simple graph: no multiple edges (ME) & no SLoop

- Free graph: no self loop
- multigraph: ME ✓ no loop

General graph: ME ✓, SL ✓

Complete Graph (full graph):

Simple graph of order ≥ 2 in which there is an edge b/w every pair of vertices.

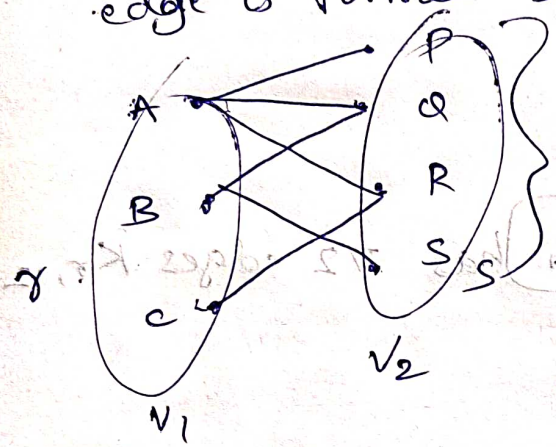
with $(n \geq 2)$ represented by (K_n)

$n = 5$, KURATOWSKI'S FIRST GRAPH

Bipartites :

$G = G(V, E)$ $\quad V = V_1 \cup V_2$

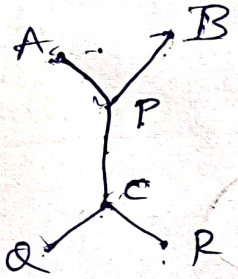
edge is formed b/w vertex of V_1 & V_2.

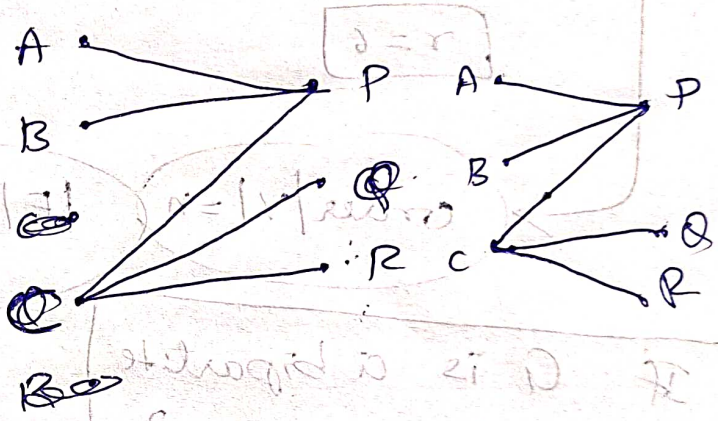


$G = G(V_1, V_2 ; E)$

$n = 7$

$m = 6$





Complete bipartite Graph:

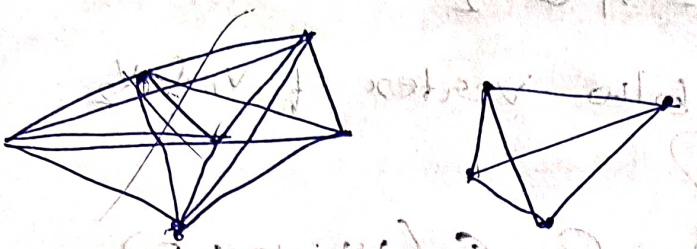
if there is an edge between vertex in V_1 & vertex in V_2.

V_1 (r elements) $\quad V_2$ (s elements)

$r \leq s$, denoted by $K_{r,s}$

$n = r + s$, $\quad m = r \times s$

No. of edges in a complete graph



(M) Complete bipartite graph has 72 edges $K_{r,12}$

$r = ?$

$12r = 72$ (6)

$\boxed{r = 6}$

order $|V| = n$ $|E| = m$ size.

If G is a bipartite graph $4m \leq n^2$

K-Dimensional Hypercube Q_k :

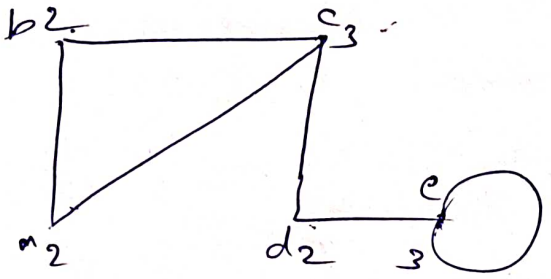
$\Downarrow$

2^k vertices (k-cube) $\Longrightarrow Q_k$

Hand shaking property [1st theorem of GT]

The sum of degrees of all the vertices in the graph is even no. & this no. = $2 \times$ no. edges.

if several ppl shake hands total no. of hand shakes must be equal just two hands are involved.



$\Sigma d(v) = 12$

$= \underline{\underline{6 \times 2}}$.

$\therefore$ no. of edges $= 6$.

Prob: In every graph, no. of vertices of odd degree ~~vertices~~ is even.

Let G be a graph with n vertices out of which k vertices are of odd degree & remaining $n-k$ are of even degree.

Let $v_1, v_2, \ldots, v_k$ be odd degree vertices,

$v_{k+1}, v_{k+2}, \ldots v_n$ even " ".

By hand shaking property, (HSP)

$$\sum_{i=1}^{n} d(v) = 2m$$

$$\sum_{i=1}^{k} d(v_i) + \sum_{i=k+1}^{n} d(v_i) = 2m$$

$$\sum_{i=1}^{k} d(v_i) = 2m - \overbrace{\sum_{i=k+1}^{n} d(v_i)}^{\text{this is always even,}} \quad \text{——— (1)}$$

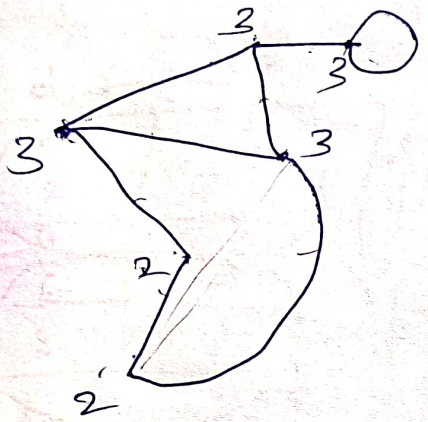
RHS $\sum_{i=k+1}^{n} d(v_i)$ is an even number ($\because$ sum of even no. is even).

$\sum_{i=1}^{k} d(v_i)$ is also an even no. ($\because$ 2m - even is even)

odd no.s should be added even times so that its sum will be even.

$\therefore \quad d(v_1) + d(v_2) + \dots + d(v_k) = \text{even}$

$\therefore$ k is even.



$\underline{k = 4}$, $\underline{n-k} = 2$

$\underline{n = 6}$,

$\Sigma d(v) = 16 = \underbrace{8}_{\text{no. of edges}} \times 2$

---

(2) Can there be a graph with $n=12$ i,e: 2 of the vertices have degree $= \underline{3}$ & remain 10 vertices have $d = \underline{4}$ each:

By HSP,

$(3 \times 2) + (4 \times 10) = 46 = \underbrace{23}_{\text{no. of edges}} \times 2$

since sum of degree is even no. we can construct graph with given data.

Size = no. of edges = m Cardinality = n

degree = edges from the pt.

n = order = no. of vertices.

odd degree vertices means $d(v_i)$ = is odd

even " " " $d(v_i)$ = is even.

③ Order of graph in following cases

① G is a cubic graph of 9 edges

② G has 10 edges with 2 vertices of $d=4$ & other

③ G is regular graph with 15 edges.

Sol ① Let order of graph G be n.

By HSP,

$$\sum_{i=1}^{n} d(v_i) = 2m$$

$$\underbrace{3+3+\cdots}_{n \text{ times}} = 2(9)$$

$$\boxed{n=6}$$

② $(2\times4)+(3\times(n-2)) = 20 = (10)\times 2$ (m)

$$8+3n-6 = 20$$

$$\boxed{n = 18/3 = 6}$$

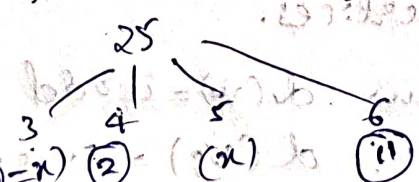
③ $\underbrace{k+\cdots}_{} = 30 = 15\times 2$ (m)

$$kn = 30$$

$$n = \frac{30}{k}$$

n can be, 30, 15, 6, 5, 3, 2, 1

④ Degree of every vertice of graph G of order $n=25$ & size (no. of edges) $= 62$ is either 3, 4, 5 or 6 there 2 vertices of $d=4$ & 11 vertice of $d=6$. How many vertices of $d=5$ are there?



$$66+8+3(19-x)+5x = 124$$

$$\overset{74}{36}+2x$$

$$\boxed{x=12}$$

(5) Let G be a graph of order 9 ($n=9$) each vertex has $d=5$ or 6 each. PT atleast 5 vertex have $d=6$. (or) atleast 6 vertices have $d=5$.

By HSP,

$$5x + 6(9-x) = 2m$$

$$5x - 6x + 54 = 2m$$

$$54 - x = 2m$$

$$x = 54 - 2m$$

$$\begin{array}{ccc} & 9 & \\ x & & 9-x \\ (5) & & (6) \end{array}$$

x is even no. $\in (0, 2, 4, 6, 8)$

$9-x \in (9, 7, 5, 3, 1)$

$d=5$	$d=6$
0	9
2	7
4	5
6	3
8	1

Hence, either $d=6$ vertices atleast 5 or $d=5$ vertices atleast 6.

$\therefore$ $x \geq 6$ or $y = 9-x \geq 5$

---

(6) Let G be a bipartite graph of order $\boxed{n=22}$ with partition sets V_1 & V_2, where cardinality of $|V_1| = 12$ (no. of vertices) suppose every vertex in V_1 has $d=3$ while $|V_2| \Rightarrow$ has $d = 2$ or 4. How many vertices have $d=2$ & $d=4$.

$n = 22$

V_1: 12, $d=3$

V_2: 10, $d=2$ or 4, x, $10-x$

$$(12 \times 3) + 2x + 40 - 4x = 2m$$

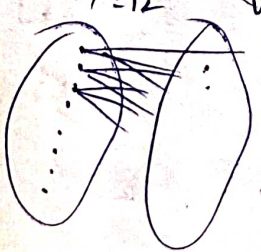
$$2x = 76 - 2m$$

$$\boxed{x = 38 - m}$$

$$x = 2$$

$\cancel{{}^{22}C_2 = \frac{22 \times 21}{2}}$

$V_1 = 12$, $V_2 = 10$



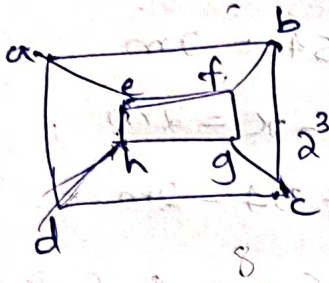
$$= 12 \times 3 = 36 = m$$

(7) ST the hypercube Q_3 is a <u>bipartite</u> graph but not a complete bipartite graph.

Q_2 $2^2 = 4$

$\left\{ V_1 \cup V_2 = V \;,\; V_1 \cap V_2 = \text{null} \right.$
$\left. \text{no vertices shld be adjacent} \right\}$

Q_3



$2^3 = 8$

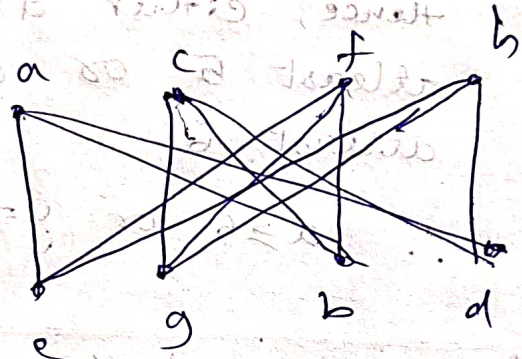
$Q_4 \rightarrow 2^4 = 16$, $Q_n = 2^n$

$V_1 = \{a, c, f, h\}$

$V_2 = \{e, g, b, d\}$

$V_1 \cup V_2 = V$ & every edge of Q_3 joints vertices V_1 to V_2 $\therefore Q_3$ is bipartite.
$V_1 \cap V_2 = \text{null}$

G:



$4 \times \frac{k}{2}$

$3 \times 4 = 12$ edges

For complete bipartite graph every V_1 should be connected with every V_2

$\therefore$ this is bipartite graph.

---

(8) PT k dimensional hypercube Q_k In an HC Q_k d of every vertice is k & no. of vertices is 2^k

$$\sum_{i=1}^{2^k} d(V_i) = 2m$$

$$k + k + \cdots = 2m$$

$$\boxed{m = k \cdot 2^{k-1}} \rightarrow \text{proved.}$$

(9) What is dimension of hypercube with 524288 edges.

$$K \cdot 2^{K-1} = 524288 = 2^{19-1} \qquad 2 \qquad 2^4 \cdot 2^{15}$$

$$K \cdot 2^{K-1} = 16 \cdot 2^{16-1}$$

$$\boxed{K = 16}$$

---

(10) How many vertices in hypercube with 4980736

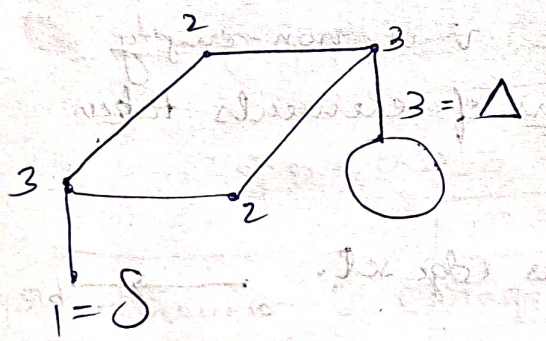
$$K \cdot 2^{K-1} = 49{,}80{,}736 \qquad \text{near to } 2^{22} \; 2^{19}$$

$$2^{16} \times 2^{8} \times 19$$

$$K \cdot 2^{K-1} = 2^{19-1} \times 19$$

$$\boxed{K = 19}$$

$\therefore$ no. of vertices $= 2^{19} = 524288$.



$\delta = 1$, $\Delta = 3$

$$1 \le \left(2 \times \frac{7}{6}\right) \le 3.$$

<u>Theorem</u>: For a Graph with n vertices, m edges if δ is min Δ is max of degrees of vertices then

$$\delta \le \frac{2m}{n} \le \Delta$$

<u>Proof</u>:

Let G be a graph n vertice & m edges.

Let $d_1, d_2, \ldots d_n$ degrees of vertices $v_1 v_2 \ldots v_n$.

$$\delta = \min\{d_1, d_2, \ldots d_n\}$$

$\Delta = \max\{d_1, d_2, \ldots, d_n\}$

$\delta \le d_1, \delta \le d_2, \ldots, \delta \le d_n$

$\sum_{i=1}^{n} d_i = 2m$ (HSP)

$n\delta \le 2m$

$\delta \le \frac{2m}{n}$ — (1)

$\Delta \ge d_1, \Delta \ge d_2, \ldots$

$\Delta n \ge 2m$

$\Delta \ge \frac{2m}{n}$ — (2)

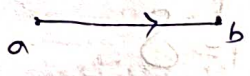
from (1) & (2),

$$\boxed{\delta \le \frac{2m}{n} \le \Delta}$$

Directed Graph (Digraph):

A graph $D = (V, E)$ where V is non-empty set E is set of order pairs of elements taken from set V.

V is vertex set, E is edge set.



Every directed edge of digraph is determined by 2 vertices a vertex from which it begins (initial V) & vertex at which it ends (terminal V)

For a directed edge ini & ter vertex need not be diff. A DE which begins & ends at same

vertex is called self loop.

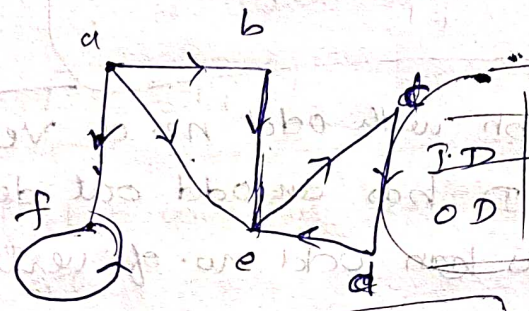
* 2 or more directed edges having same initial & terminal vertex are called parallel or multiple edges.

In degree & Out Degree of a vertex:-

If v is a vertice of diagraph D no. of edges for which v is initial vertice is called out degree of v denoted by ~~d(v)~~ or $d^+(v)$ or $od(v)$

& no. of edges for which v is terminal vertex is called In Degree of v. → $d^-(v)$ or $id(v)$

D:



	a	b	c	d	e	f	
ID	0	1	1	1	3	2	8
OD	3	1	1	1	1	1	8

$ID = OD$ for every Digraph

$\sum ID + \sum OD = 2m$

$\boxed{m = 8}$

1st theorem :- (Diagraph)

If in every Diagraph D sum of OD of all vertices = ID of all vertices = no. of edges in D.

Proof: Let D be digraph with n vertices & m edges

Let r_1 be no. of edges going out of v_1,

r_2 " " v_2 .....

$d^+(v_1) = r_1$, $d^+(v_2) = r_2$ ..... $d^+(v_n) = r_n$

Since every edge terminate at some vertex, Since there are m edges,

$$r_1 + r_2 + \dots = m$$

$$\boxed{\sum_{i=1}^{n} d^+(v_i) = m}$$

Similarly s_1 no. edges coming into v_1

$d^-(v_1) = s_1, d^-(v_2) = s_2$ ...

$$\boxed{\sum_{i=1}^{n} d^-(v_i) = m}$$

$$\therefore \sum_{i=1}^{n} d^+(v_i) = \sum_{i=1}^{n} d^-(v_i) = m$$

(Pr 1) Let D be a digraph with odd no. of vertices if each vertex of D has an odd out degree prove that D has an odd no. of vertices with odd in degree.

Soln

Let D be a digraph with odd no. of vertices n = odd. Since out degree of every vertex is odd $\sum_{i=1}^{n} d^+(v_i) = m$ is odd

$\therefore$ m is an odd no. ($\because$ odd × odd)

now, $\boxed{\sum_{i=1}^{n} d^-(v_i) = m}$ ($\because$ by FT of D)

n & m & $\sum_{i=1}^{n} d^-(v_i)$ is odd

Out of n vertices let k vertices $(v_1 \dots v_k)$ have odd ID & $n-k$ " $(v_{k+1} \dots v_n)$ even ID

$$\sum_{i=1}^{k} d^-(v_i) + \underbrace{\sum_{i=k+1}^{n} d^-(v_i)}_{\text{even always}} = \text{odd}$$

$\sum_{i=1}^{k} d(v_i) = \text{odd} - \text{even} = \text{odd}$

$\therefore$ k is an odd no. ($\because$ in order to make sum to be odd, odd no.s shld be added odd no. of times).

Isomorphism:

(Imp)

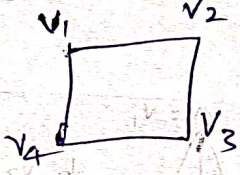
Consider 2 graphs $G(V, E)$ & $G'(V', E')$ suppose there exist a funcⁿ $f: V \to V'$ such that, f is $1 \to 1$ correspondance ($1 \to 1$ Corr. - one to one & onto funcⁿ). & for all vertices a, b in G ab is an edge in G. iff $f(a)f(b)$ is an edge in G'

Then G is said to be isomorphic to G' written as $\boxed{G \cong G'}$.

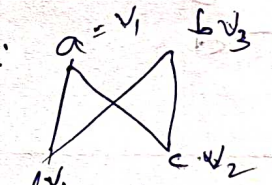
---

(1) Check whether following graphs are isomorphic or not.

(i) G1:



G2:



Soln

$n = 4, m = 4$

$\sum d(v) = 8$

Degree sequence

2 2 2 2

$n = 4, m = 4$

$\sum d(v) = 8$

2 2 2 2

Vertex correspondance:

$v_1 \leftrightarrow a$

$v_2 \leftrightarrow c$

$v_3 \leftrightarrow b$

$v_4 \leftrightarrow d$

Edge correspondance

$v_1 v_2 \leftrightarrow ac$

$v_2 v_3 \leftrightarrow bc$

$v_3 v_4 \leftrightarrow bd$

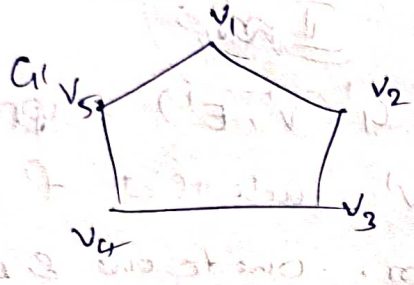
$v_4 v_1 \leftrightarrow ad$

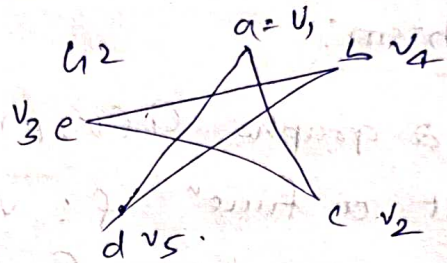
These represents one-one correspondance b/w edges of 2 Graphs under which adjacent vertices in 1st G $\longleftrightarrow$ adjacent vertices in 2nd & vice versa.

$\therefore \; G_1 \cong G_2$

---

(2)





$n=5, \; m=5$

$\Sigma d = 10$

2 2 2 2 2

vertex corr

$v_1 \leftrightarrow a$
$v_2 \leftrightarrow c$
$v_3 \leftrightarrow e$
$v_4 \leftrightarrow b$
$v_5 \leftrightarrow d$

$n=5 \; m=5$

$\Sigma d = 10$

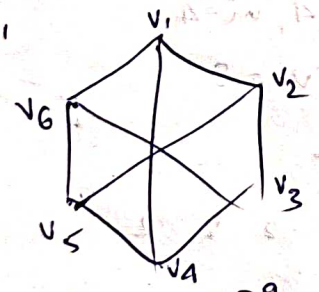
2 2 2 2 2

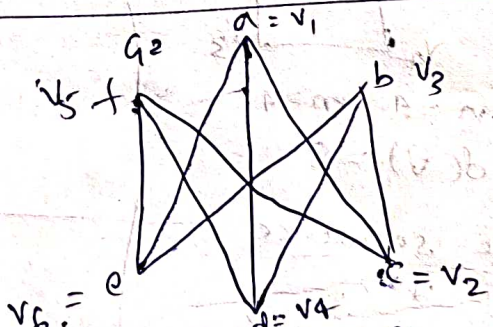
edge corr

$v_1 v_2 \leftrightarrow ac$
$v_2 v_3 \leftrightarrow ce$
$v_3 v_4 \leftrightarrow be$
$v_4 v_5 \leftrightarrow bd$
$v_5 v_1 \leftrightarrow ad$

---

(3)





$n=6, \; m=9$

$\Sigma d = 18$

3 3 3 3 3 3

Vertex correspondance:

$v_1 \leftrightarrow a$
$v_2 \leftrightarrow c$
$v_3 \leftrightarrow b$
$v_4 \leftrightarrow d$
$v_5 \leftrightarrow f$
$v_6 \leftrightarrow e$

$n=6, \; m=9$

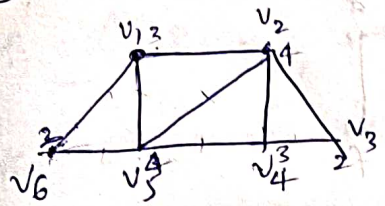
$\Sigma d = 18$

3 3 3 3 3 3

edge corr.

$v_1 v_2 \leftrightarrow ac$
$v_2 v_3 \leftrightarrow cb$
$v_3 v_4 \leftrightarrow bd$
$v_4 v_5 \leftrightarrow df$
$v_5 v_6 \leftrightarrow fe$
$v_6 v_1 \leftrightarrow ea$
$v_6 v_3 \leftrightarrow eb$
$v_5 v_2 \leftrightarrow fc$
$v_1 v_4 \leftrightarrow ad$

(4) G1

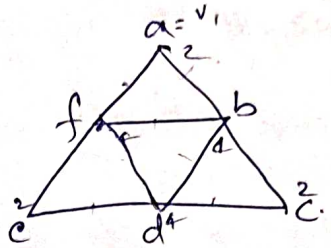


$n=6$, $m=9$

$\Sigma d = 18$

2 2 3 3 4 4

G2



$n=6$, $m=9$

$\Sigma d = 18$

2 2 2 4 4 4

vertex corr:

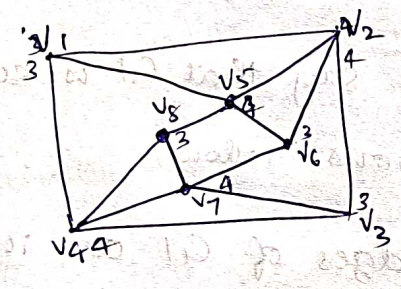
edge corr:

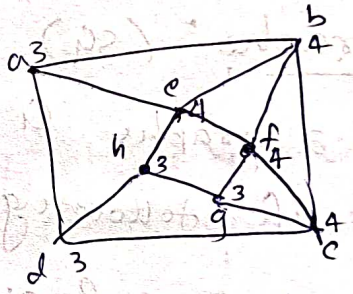
As degree seq of G1 & G2 are not same

$G1 \not\cong G2$ as there can't be 1-1 correspondance.

---

(5)





$n=8$, $m=14$

$\Sigma d = 28$

3 3 3 3 4 4 4 4

$12 + 16$

$n=8$, $m=14$

$\Sigma d = 28$

3 3 3 3 4 4 4 4

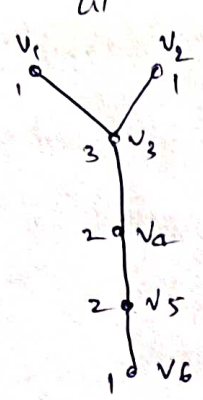
vertex corr

edge corr:

A degree 3 vertex in G1 is adjacent to 3 vertices of with degree 4 whereas in G2 such a vertex doesn't exist. Hence there can't be 1-1 correspondance b/w vertices of G1 & G2.

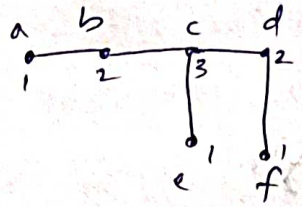
$\therefore G1 \not\cong G2$.

(6)



1 1 1 2 2 3

$n = 6, m = 5$



1 1 1 2 2 3

$n = 6, m = 5$

A degree 3 vertice in G1 is adjacent to a degree of 1 whereas in G2 $d = 3$ is adj to 2 deg of 2.

$\therefore$ $G1 \not\cong G2$.

Subgraphs: (SG)

Given 2 graphs G & G1 we say that G1 is subgraph of G. if following conditions follow:

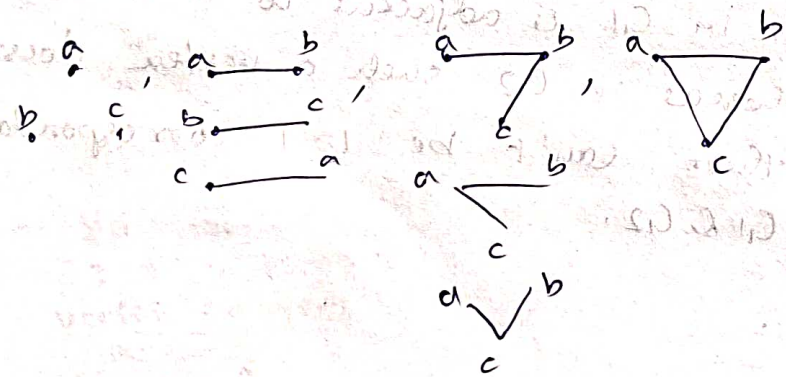
all the vertices & all the edges of G1 are in G

each edge of G1 as same in vertices in G as in G1

Ex: G & G1

$G1 \subseteq G$

possible SG



Note: ★ A subgraph of a graph is also a graph

★ Every graph is a subgraph of itself

★ Every simple graph of n vertices is a

subgraph of complete graph taken.

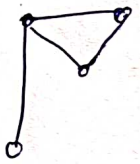
* A simple vertex in graph G is subgraph of G

* A single edge in graph together with its end vertices is subgraph of G.

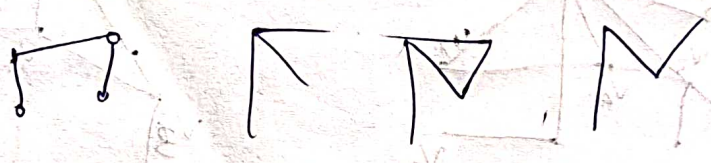
---

(1) Spanning subgraph :- (all vertex shld be included).

G1



spanning SG.



SG which aren't spanning

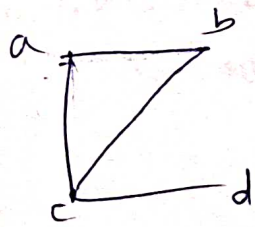
Given a graph $G(V,E)$ if there is a SG $G1(V1,E1)$ of G i,e: $V=V1$ then G1 is spanning SG of G.

---

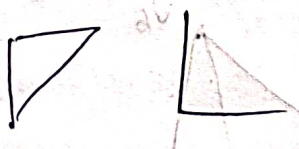
(2) Induced subgraph :-

Given a graph $G(V,E)$ suppose there is a SG $G1(V_1, E_1)$ of G i,e: every edge ab of G where $a,b \in V_1$ is an edge of G1 also. then G1 is called Induced SG of G. denoted by $\langle V_1 \rangle$ (induced by V_1).

G



G1



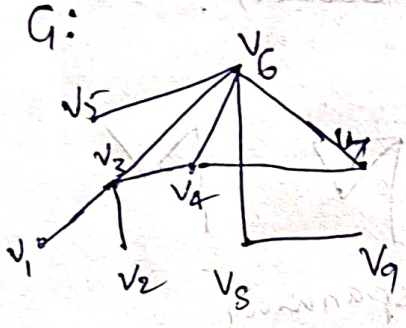
Problems:

① Verify that G1 is an induced SG of G. Is this a spanning subgraph of G.

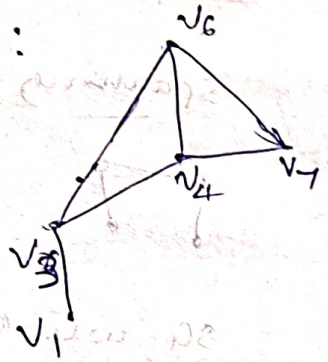
② Draw the SG G2 of G induced by the set

$V_2 = \{V_3, V_4, V_6, V_8, V_9\}$

G:



G1:



Sol^n $V_1 = \{V_1, V_3, V_4, V_6, V_7\}$ $\subseteq V$

① and edges in G1 are in G also.

Each edge in G1 has same end vertices as in G.

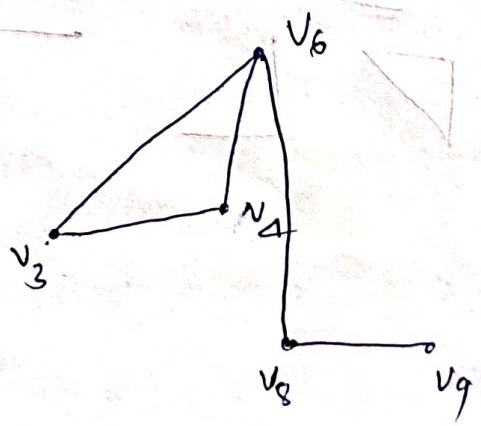
$\boxed{\therefore \text{ G1 is subgraph of G}}$ $[G1 \subseteq G]$

→ 2 vertices are adjacent in G1 whenever they are adjacent in G.

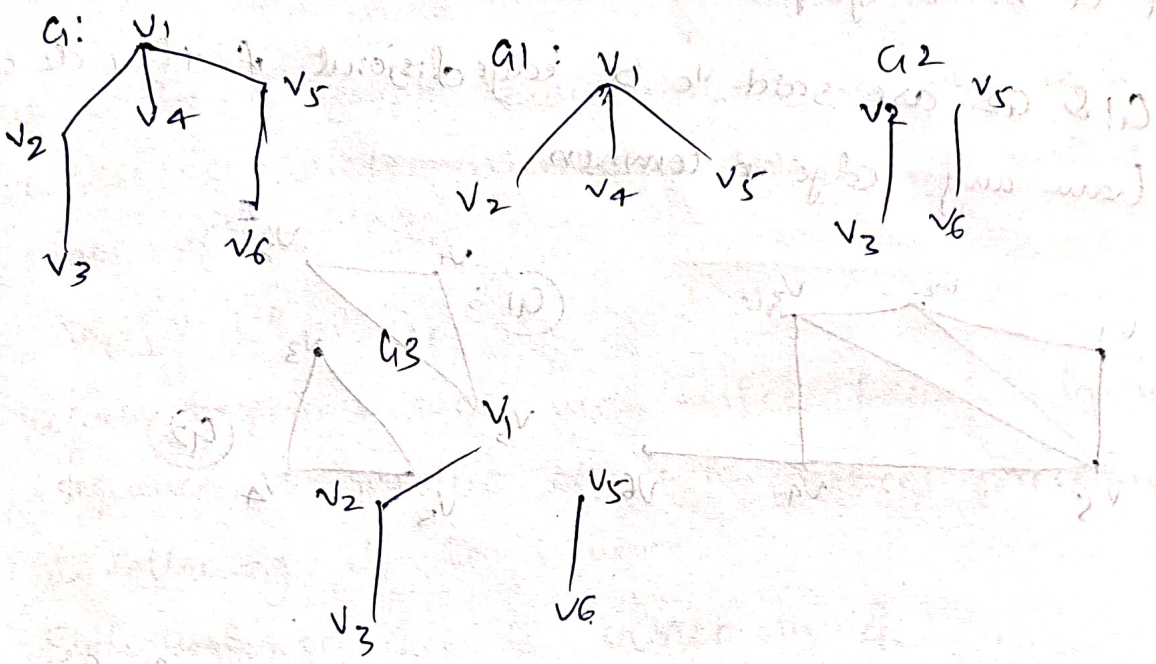
$\therefore$ G1 is induced SG of G.

$V_1 \neq V$ ($\therefore$ G1 is not spanning SG of G)

② G2



③ Consider graph G verify that graph G1 & G2 shown below are ISG of G whereas graph G3 is n'T ISG of G.



Sol^n $V_1 = \{v_1, v_2, v_4, v_5\} \subseteq V$.

& edges in G1 are in G too.

each edge in G1 has same end vertices as in G.

∴ G1 is subgraph of G

∴ $G2 \subseteq G$, $G1 \subseteq G$, $G3 \subseteq G$.

In G1 & G2 ⇒ 2 vertices are adjacent in G1 & G2 whenever they are adjacent in G.

∴ G1 & G2 are ISG.

In G3 ⇒ b/w v_1 & v_5 there is an edge in G but in G3 no edge is there. hence it is not an ISG.

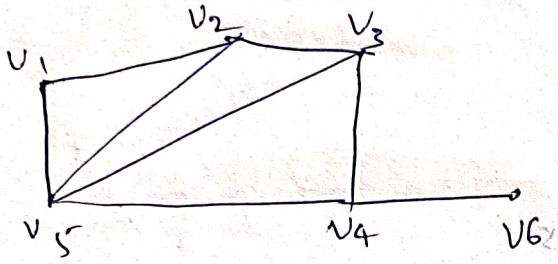
∴ G3 not an ISG.

Edge and Vertex Disjoint :

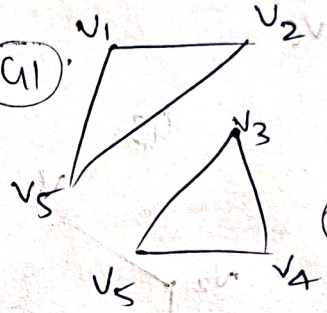
Let G be a graph and G1, G2 be 2 SG of G.

G1 & G2 are said to be edge disjoint if they do not have any edge in common.

G:



G1

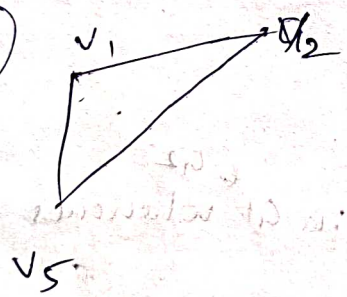


G2

G1 & G2 are said to be vertex disjoint if they do not have any common edge or vertex.

G1 & G2 are EDGE DISJOINT.

G3



G4

G3 & G4 are VERTEX DISJOINT

Note: ① Edge disjoint SG may have common vertices

② SG that have no vertex in common can't possibly have an edge in common.

∴ i,e vertex DSJ SG ~~may~~ must be edge disjoint but converse need not be true

Walk:

Alternative sequence of vertices & edges (start from vertex)

Walk.

Let G be a graph having atleast one edge in G consider a finite alternating sequence of vertices & edges of the form.

$v_i \, e_j \, v_{i+1} \, e_{j+1} \, v_{i+2} \, e_{j+2} \ldots v_m$

which begins & ends with vertices & each edge in the sequence is incident ~~of~~ on the vertices preceding & following it in sequence.

Such ~~vertex~~ sequence is WALK in G.

Note: (1) In a walk vertex or edge or both can appear more than once.

(2) The no. of edges present in a walk is called its LENGTH.

(3) Vertex with which walk begins is called initial vertex & ends is called terminal vertex.

(4) Vertices which are not initial or terminal are called internal vertices.

(5) If initial vertex = u
terminal " = v

such walk is denoted by uv walk.

(6) A walk which begins & ends at same vertex is called a closed walk. walk which is not closed is OPEN walk.

Length of walk = no. of edges.

CLASSIFICATIONS OF A WALK:

WALK

~~open~~

① trail → no edge repeats

② Path → no vertex, no edge repeats
↳ is trail where no ~~edge~~ vertex repeats

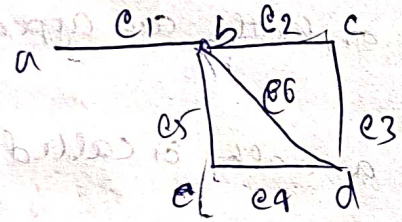
closed

① circuit → no edge repeats

② Cycle → no edge, no vertex repeats
↓
a circuit in which terminal vertex does not appear as internal vertex
no internal vertex repeats

~~Fix a walk~~

G:



trail:

w_1: $a e_1 b$

w_2: $a e_1 b e_2 c$

w_3: $a e_1 b e_2 c e_3 d e_4 c$

} paths too

w_4: $a e_1 b e_2 c e_3 d e_6 b$ → not path

Ckt:

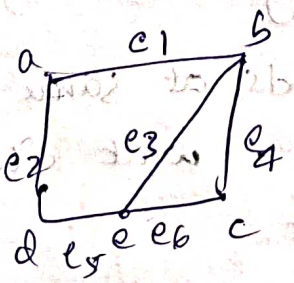
w_4: $b e_2 c e_3 d e_4 e e_5 b$

w_5: $b e_2 c e_3 d e_6 b$

w_7: $\cancel{b e_3 d}$

Problem: ① Consider the graph given below.

Find all paths from a to c



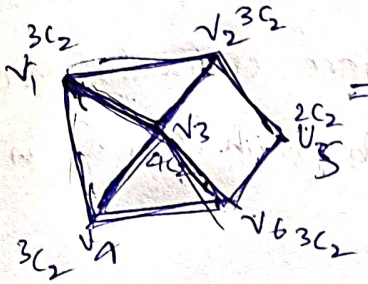
$a e_1 b e_3 e e_6 c$ → 3

$a e_1 b e_4 c$ → 2

$a e_2 d e_5 e e_6 c$ → 3

$a e_2 d e_5 e e_3 b e_4 c$ → 4

② Determine no. of diff paths of length 2 in graph



$\Rightarrow$ ${}^6P_3 = \frac{6\times5\times4}{6} = 120$

${}^5C_2 = \frac{5\times4}{2} = 10$

${}^5C_2 = \frac{5\times4}{2} = 10$

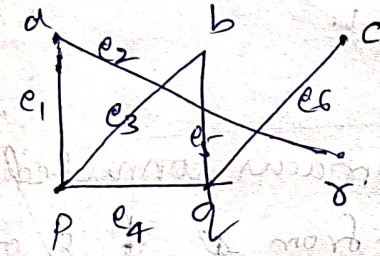
from vertex v_1 no. of pair of edges incident of $v_1 = 3$.

3 edges are incident on v_1, no. of paths ${}^3C_2 = 3$.

$3\times4 + 6 + 1 = 19$

---

③ Find all ~~graphs~~ cycles present in graph.



q e_5 b e_3 p e_4 q

Connected Graph :

Two vertices in G are said to be connected if there is atleast one path from one vertex to other.

Graph G is a connected graph if every pair of distinct vertices in G are connected.

Otherwise G is a disconnected graph.

Note: A digraph D is said to be connected or disconnected according to its underlying graph.

(A Graph obtained from digraph by neglecting directns) is connected or disconnected.

⇒ In Graph G all walks are connected sub graphs of G.

⇒ Every graph G consist of one or more connected graphs each such connected graph is sub graph of G & is called component of G.

⇒ A connected graph has only one component while disconnected graph has 2 or more.

Note: ① A path with n vertices has n-1 edges.

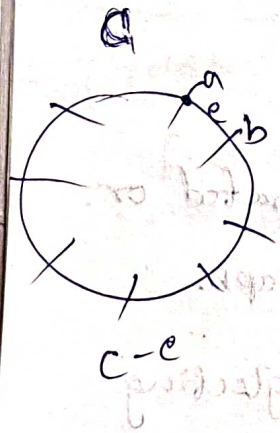
② A cycle with n vertices has n edges.

③ In a cycle degree of vertex = 2

Prove that connected graph G remains connected after removing an edge e from G if & only if e is part of some cycle in G.

proof:

Suppose e is part of some cycle C of G then the end vertices of G say 'a' & 'b'.



Joined by atleast 2 paths

Removal of e from G wn't affect connectivity of G. bcz even after removal of e end vertices of e are still connected.

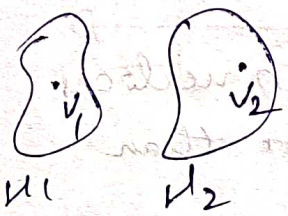
Conversely ⇒ (C-e) removal → a b are still connected

Suppose e is not a part of cycle in G then end vertices of e are connected atmost one path. Hence removal e from G disconnects these end vertices implies $G_1 - e$ is disconnected graph.

Qn. If a Graph has exactly 2 vertices of odd degree then there must be a path connecting these vertices.

Sol^n Let v_1 & v_2 be 2 vertices of odd degree.

Assume there's no path connecting them. the graph is disconnected so, v_1 & v_2 belongs to 2 diff compone-nts of G say H_1, H_2



Consequently each of H_1 & H_2 contains only one vertex of odd degree. This is not possible as no. of odd degree vertices in H_1 & H_2 shld be even.

∴ Therefore there must be a path connecting 2 odd degree vertices.

If G is a simple graph with n vertices in which the degree of every vertex is atleast $\frac{(n-1)}{2}$, PT G is connected graph.

Sol^n

Take any two vertices u & v of G. They are either adjacent or not adjacent.

If u & v are adjacent then G is connected.

Suppose u & v are not adjacent:

$$d(v) \geq \left(\frac{n-1}{2}\right) \qquad v \in V$$

Since degree of vertex atleast $\left(\frac{n-1}{2}\right)$ u & v together will have atleast $(n-1)$ neighbours.

Since G has n vertices. Total no. of neighbours which u & v can have is only $(n-2)$.

$\therefore$ Atleast one vertex say x is neighbour of u & v. Hence there is an edge b/w (u & x) & (x & v). Therefore there is path b/w u & v.

Prove that a simple Graph with n vertices must be connected if it has more than $\frac{(n-1)*(n-2)}{2}$ edges.

Consider a SG with n vertices - choose $(n-1)$ vertices, say $v_1, v_2, \ldots v_{n-1}$ of G.

$\therefore$ total max edges we can have ${}^{(n-1)}C_2$

$$= \frac{(n-1)(n-2)}{2}.$$

Then if we have more than $\frac{(n-1)(n-2)}{2}$ edges atleast one edge shld be drawn b/w v_n & to some vertex v_i.

Hence G must be connected.

3) ST a connected graph with exactly 2 vertices of odd deg has Euler trail!

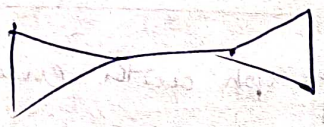
Let a, b be only 2 vertices of odd deg in a connected graph G join these vertices by an edge e (even if there is an edge b/w them).

Then a, b become vertices of even deg. Since all vertices in G are of even deg. Graph $G1 = G \cup e$ is connected whose vertices are of even degree

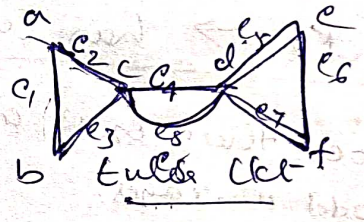
$\therefore$ G1 has euler ckt which must include e.

The trail got by deleting e from this euler ckt is Euler trail in G.

G



G1



Euler ckt : $c e_2 a e_1 b e_3 c e_4 d e_5 e e_6 f e_7 d e_8 c$.

Euler trail : $c e_2 a e_1 b e_3 c e_4 d e_5 e e_6 f e_7 d$.

Eulerian ckt:

① A connected Graph G has Euler ckt if and only if all vertices of G are of even degree.

Find all +ve integers $n \geq 2$ for which the complete graph K_n contains Euler ckt. For what value of n does K_n have Euler trail but not a Euler ckt.

For $n \geq 3$ K_n contains Euler ckt if & only if n is odd number. ($\because$ deg v in complete graph $K_n = n-1$)

$\therefore$ odd − 1 = even

If G has Euler ckt then n−1 must be even

Only for K_2 → has Euler trail. ~~not~~ not for any other.

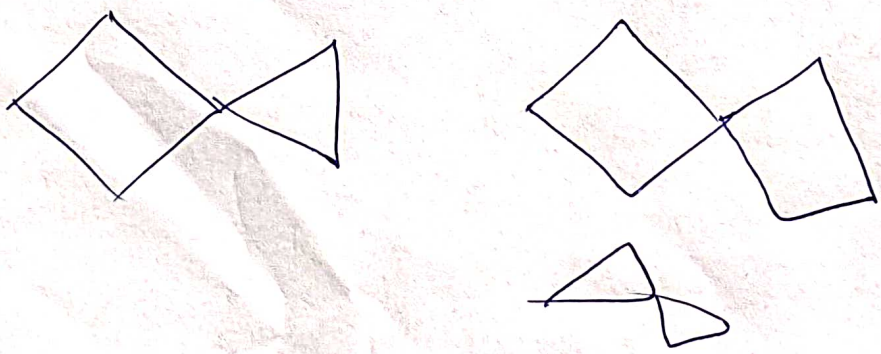
---

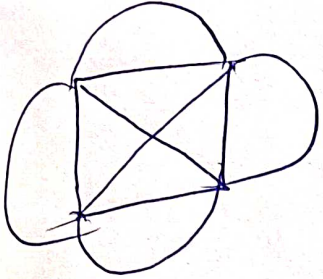
② (a) Does there exist a Euler graph with even no. vertices & odd no. edges.

(b) Does there exist a Euler graph with odd no. of vertices & even no. edges.

(a) Yes, Suppose C is a ckt with even no. of vertices & V be any vertex on C & C' with odd no. vertices passing through V C & C' have no edge in common. circuit C consist edges of C & C' is Euler graph of desired type.

(b) Yes, In a suppose C & C' ckt with odd no. of vertices & even no. of vertices then C is Euler graph of desired type.





Eulerian Circuit & Eulerian Trail.

Consider a connected Graph G. if there is a circuit in G that contains all edge in G then that circuit is called Eulerian circuit in G.

If there is trail in G contain all edge in G then that trail is called Eulerian trail in G.

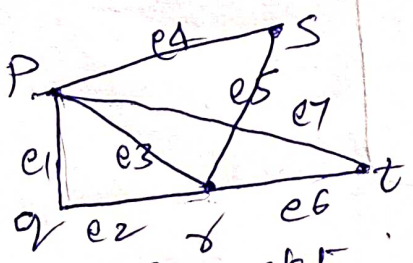
Note:
In trail & ckt no edge can appear more than once but a vertex can appear more than once.
This property is carried to Euler trail & Euler ckt also.

Since Euler ckt & Euler trail includes all edges. They automatically include all vertices.

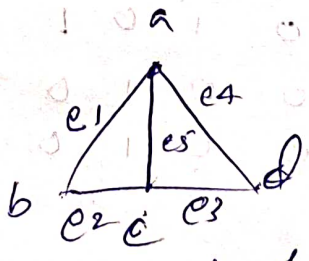
Note: A connected graph that contains Euler ckt is called Eulerian graph.

A connected graph that contains Euler trail is called semi euler graph.

Ex:



Euler ckt



Euler trail.

Graphs in Coding :

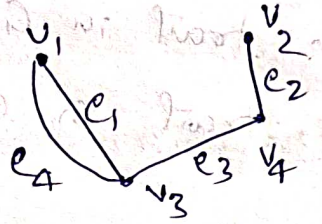
↳ through matrix representation

Incidence matrix:

Can't have self loops Pendent vertex = 1

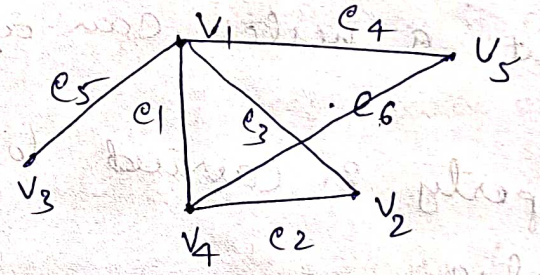
(1)

$$\begin{array}{c|cccc} & e_1 & e_2 & e_3 & e_4 \\ \hline v_1 & 1 & 0 & 0 & 1 \\ v_2 & 0 & 1 & 0 & 0 \\ v_3 & 1 & 0 & 1 & 1 \\ v_4 & 0 & 1 & 1 & 0 \end{array}$$



(2)

$$\begin{array}{c|cccccc} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ \hline v_1 & 1 & 0 & 1 & 1 & 1 & 0 \\ v_2 & 0 & 1 & 1 & 0 & 0 & 0 \\ v_3 & 0 & 0 & 0 & 0 & 1 & 0 \\ v_4 & 1 & 1 & 0 & 0 & 0 & 1 \\ v_5 & 0 & 0 & 0 & 1 & 0 & 1 \end{array}$$



Adjacency matrix

Can't have parallel edges.

(3) without actually drawing the graph, ST there exist no connected graph whose incidence matrix is.

$$\begin{array}{c|cccccc} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ \hline v_1 & 1 & 0 & 0 & 0 & 0 & 1 \\ v_2 & 1 & 1 & 0 & 1 & 0 & 1 \\ v_3 & 0 & 1 & 0 & 1 & 0 & 0 \\ v_4 & 0 & 0 & 1 & 0 & 1 & 0 \\ v_5 & 0 & 0 & 1 & 0 & 1 & 0 \end{array}$$

$$\rightarrow \begin{bmatrix} \overbrace{\begin{matrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}}^{A(G1)} & \overbrace{\begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \\ 1 & 1 \end{matrix}}^{0} \\ \begin{matrix} 0 & 0 & 0 & 0 & 0 \end{matrix} & \underbrace{\begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix}}_{A(G2)} \end{bmatrix} \qquad \begin{bmatrix} A(G1) & 0 \\ 0 & A(G2) \end{bmatrix} \rightarrow \text{disconnected.}$$

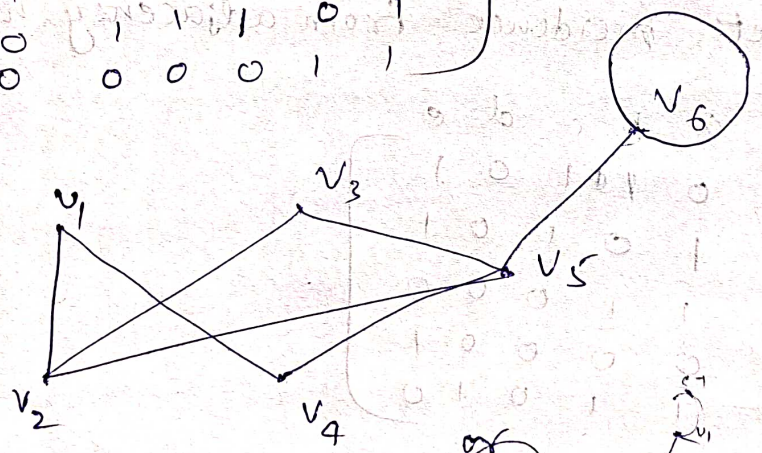
Reduced to this ↑

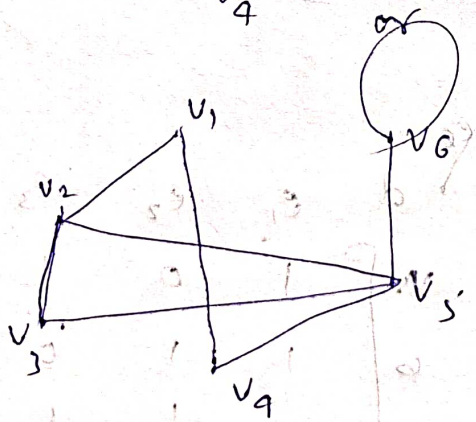
& WKT such matrix represents disconnected Graph.

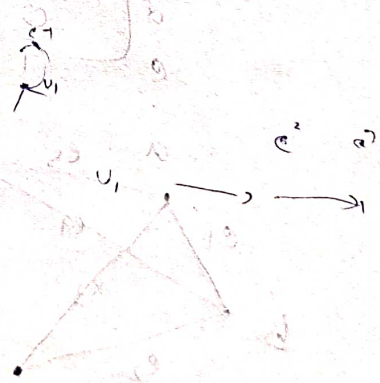
---

④ Draw adjacency matrix

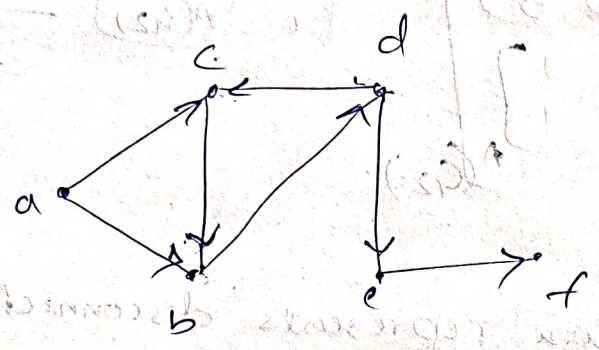
$$\begin{array}{c|cccccc} & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \\ \hline v_1 & 0 & 1 & 0 & 1 & 0 & 0 \\ v_2 & 1 & 0 & 1 & 0 & 1 & 0 \\ v_3 & 0 & 1 & 0 & 0 & 1 & 0 \\ v_4 & 1 & 0 & 0 & 0 & 1 & 0 \\ v_5 & 0 & 1 & 1 & 1 & 0 & 1 \\ v_6 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}$$







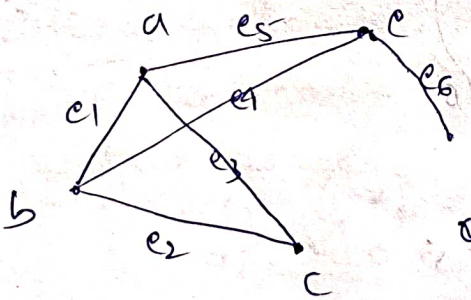
(5) Construct matrix



$$
\begin{array}{c|cccccc}
 & a & b & c & d & e & f \\
\hline
a & 0 & 0 & 1 & 0 & 0 & 0 \\
b & 0 & 0 & 0 & 1 & 0 & 0 \\
c & 0 & 0 & 0 & 0 & 0 & 0 \\
d & 0 & 0 & 1 & 0 & 1 & 0 \\
e & 0 & 0 & 0 & 0 & 0 & 1 \\
f & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
$$

(6) Construct incidence from adjacency matrix.

$$
x(G) = \begin{array}{c|ccccc}
 & a & b & c & d & e \\
\hline
a & 0 & 1 & 1 & 0 & 1 \\
b & 1 & 0 & 1 & 0 & 1 \\
c & 1 & 1 & 0 & 0 & 0 \\
d & 0 & 0 & 0 & 0 & 1 \\
e & 1 & 1 & 0 & 1 & 0
\end{array}
$$

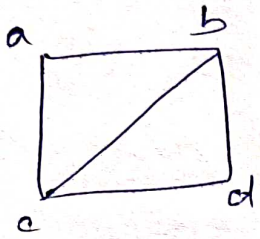


$$
\begin{array}{c|cccccc}
 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\
\hline
a & 1 & 0 & 1 & 0 & 1 & 0 \\
b & 1 & 1 & 0 & 1 & 0 & 0 \\
c & 0 & 1 & 1 & 0 & 0 & 0 \\
d & 0 & 0 & 0 & 0 & 0 & 1 \\
e & 0 & 0 & 0 & 1 & 1 & 1
\end{array}
$$

Hamiltonian Graph:

Let G be connected graph if there is cycle in G that contains all vertices in G that cycle is Hamiltoni - an cycle in G.

- ① A Hamilton cycle G with n vertices consist exactly n edges.
- ② Hamiltonian cycle includes all vertices of G does not imply it includes edges of G.



abdc → cycle.

Let's consider a graph G in which a member 'x' is represented by a vertex and possibility of his sitting next to another member 'y' is represented by edge b/w x & y. Since every member is allowed to sit next to any other member G is a comp-te graph. Since there are 'n' members there are 'n' vertices. Every seating arrangement around table is hamilton cycle. On first daly, they can sit in any order, this will be hamilton cycle say C_1.

On second day, If they have sit in way where every member has diff neighbours, we must have hamilton cycle C_2 which is edge disjt with C_1. If same arrangement has to be thru on subsequent days then for each day we have to find hamilton cycle which is edge disjt with hamilton cycles formed earlier.

From above, no. such cycles $= \left(\frac{n-1}{2}\right)$

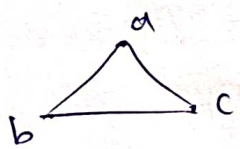
$\frac{2K+1-1}{2} = K$.

Hence seating arrangement of desired type can last for K days.

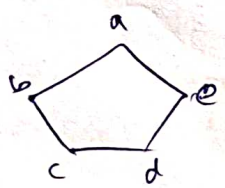
Ex: when $n=3$ $\Rightarrow$ $\frac{3-1}{2} = 1$

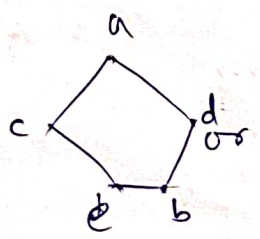
$= 2(1)+1$, $1 \to K$

Let a, b, c, be members.

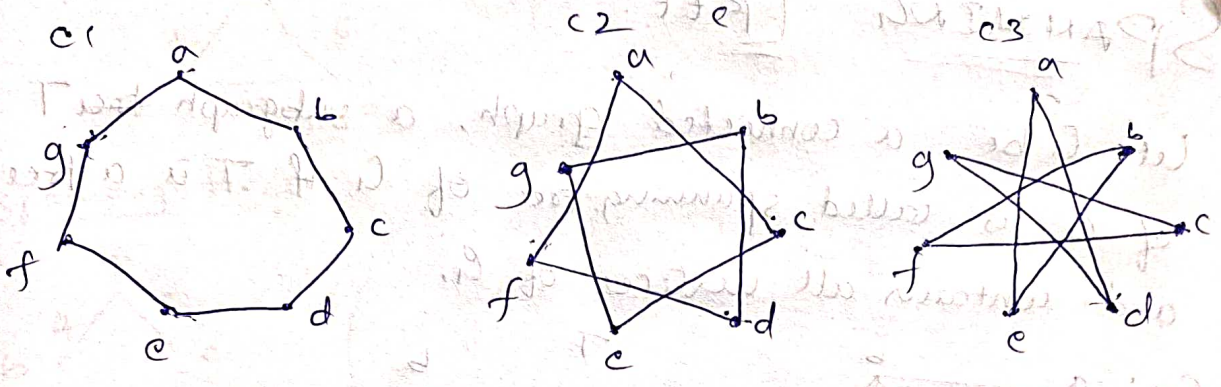


$n = 5 = 2(2)+1$, $2 \to K$.

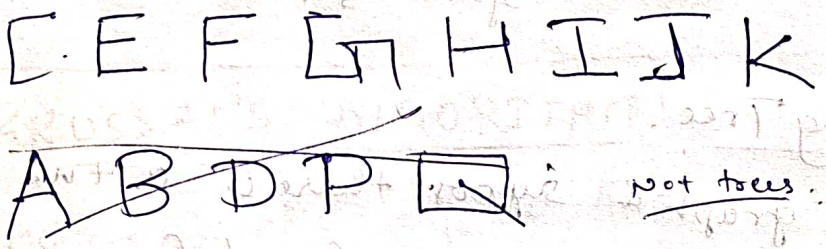




$n = 7 = 2(3) + 1 \rightarrow k = 3$



TREES :-



Trees

Not trees.

Tree is a connected acyclic graph denoted by T.

Note: ① Tree has to be simple graph bcz loops & parallel edges forms cycles.

② A tree has atleast 2 pendent vertices (leaves).

③ A disconnected graph where each component is a tree is called forest.

④ In a tree there is only one path b/w every pair of vertices.

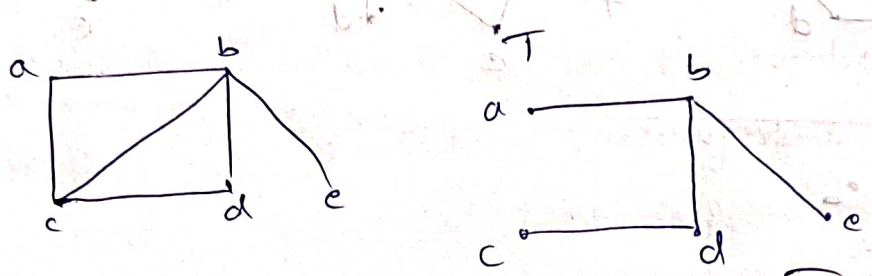
⑤ A tree with n vertices will have n-1 edges.

⑥ A connected graph G is tree if adding an edge b/w any two vertices creates exactly one cycle in G.

SPANNING TREE:

Let G be a connected graph, a subgraph ~~tree~~ T of G is called spanning tree of G if T is a tree and contains all vertices of G.

Ex:



spanning Tree

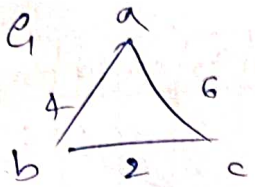
Minimal Spanning Tree:

Let G be a graph & suppose there is a +ve real no. associated with each edge of G. Then G is called a weighted graph & the real no. associated with edge 'e' is called weight of edge 'e'. Let G be a weighted graph & T be its spanning tree.

Every branch of T is an edge of G which has some wt. The sum of wts of all branches in T is wt of T.

If G is connected weighted graph then weight of an edge 'e' is denoted by 'w(e)' & weight of spanning tree T of G is denoted w(T). Suppose if we consider all spanning trees of connected weighted graph & find weight of every span T. Then a ST with least wt is minimal spanning Tree & this tree is not unique.

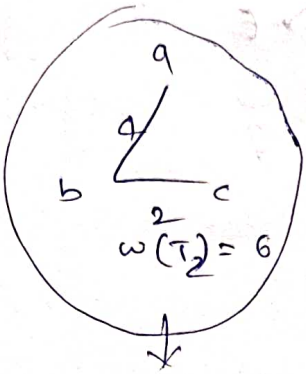
Ex: G_1



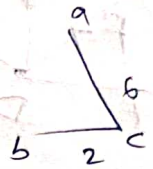
$w(G) = 12$

spanning trees

$w(T_1) = 10$



$w(T_2) = 6$

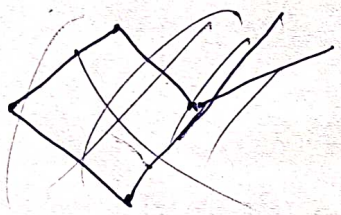


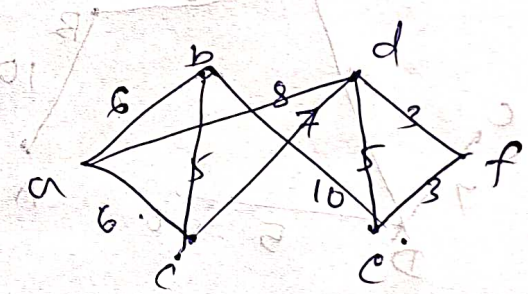
$w(T_3) = 8$

minimal Spanning Tree

KRUSKAL'S ALGORITHM:

⇒ Use Kruskal's algo to find minimal spanning Tree of weighted graph.

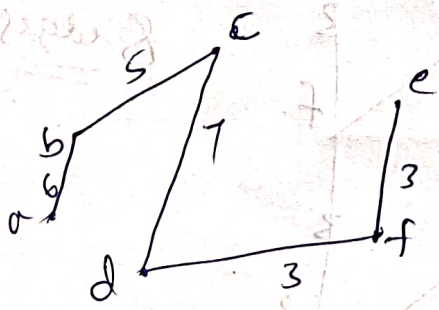




Graph G has 6 vertices MST will have 5 edges.

edges	df	ef	de	bc	ab	ac	cd	bd	be
Weight	3	3	5	5	6	6	7	8	10
Selection	✓	✓	✗	✓	✓	✗	✓	✗	

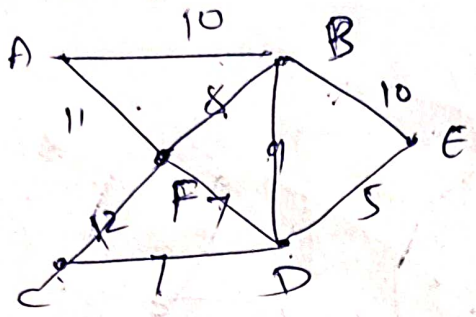
T:



$3+3+5+6+7$

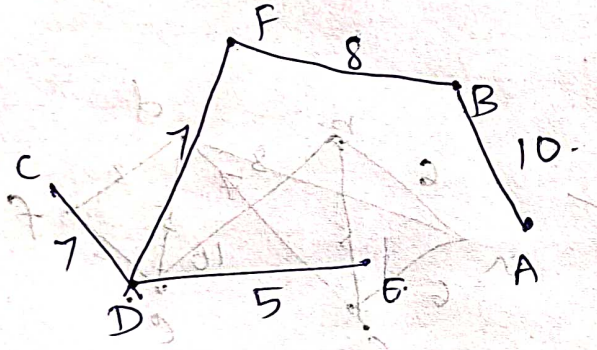
$w(T) = 24$

(2)



In MST → 5 edges

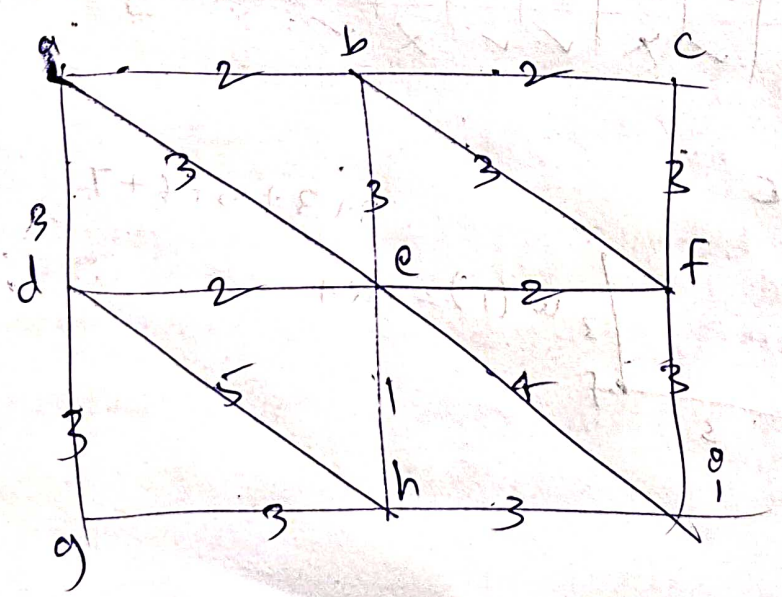
edges	DE	CD	FD	BF	BD	BE	AB	AF	CF
wt.	5	7	7	8	9	10	10	11	12
selectn	✓	✓	✓	✓	✗	✗	✓	✗	✗



$w(T) = 5 + 7 + 7 + 8 + 10$

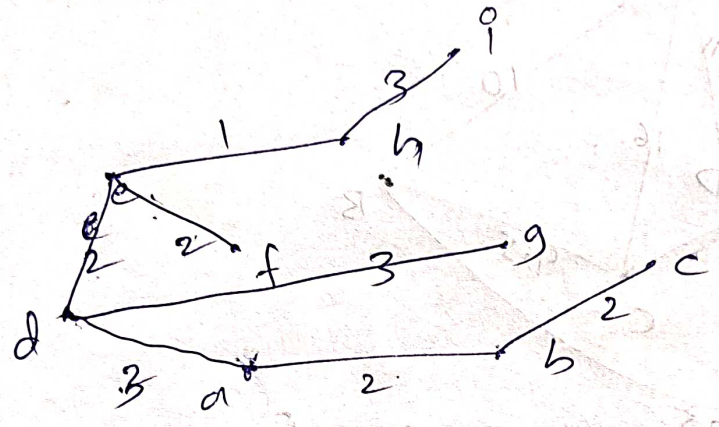
$= 37$

(3)



8 edges

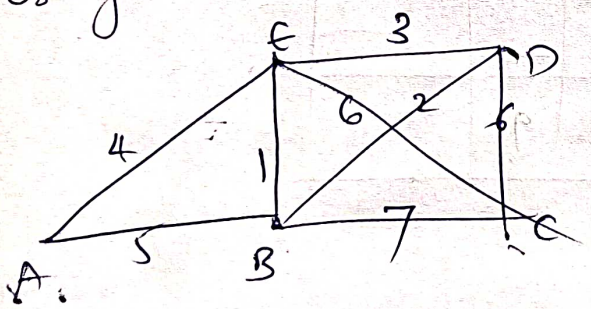
edge	gh	ab	bc	de	ef	ad	ae	be	bf	cf	dg	gh	hi	if	ei	dh
wt	1	2	2	2	2	3	3	3	3	3	3	3	3	3	4	5
sele	✓	✓	✓	✓	✓	✓	×	×	×	×	✓	×	✓	×	×	×



$$w(T) = 2\times5 + 1 + 2\times3$$
$$= 16 + 1 = 18$$

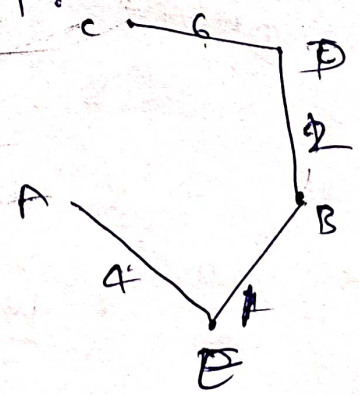
PRIM'S ALGORITHM: (PA)

Using PA find MST



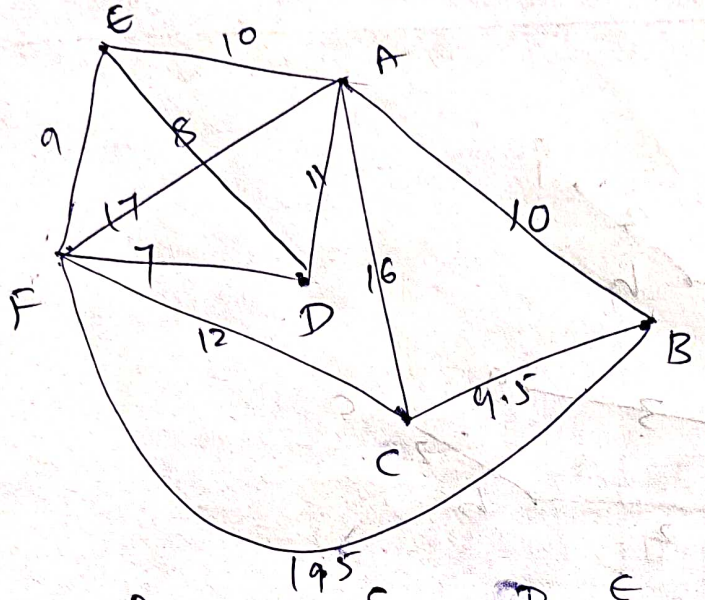
	A	B	C	D	E
A	~~∞~~	5	∞	∞	4
B	5	~~∞~~	7	2	1
C	∞	7	~~∞~~	6	6
D	∞	2	6	~~∞~~	3
E	4	1	6	3	~~∞~~

T:

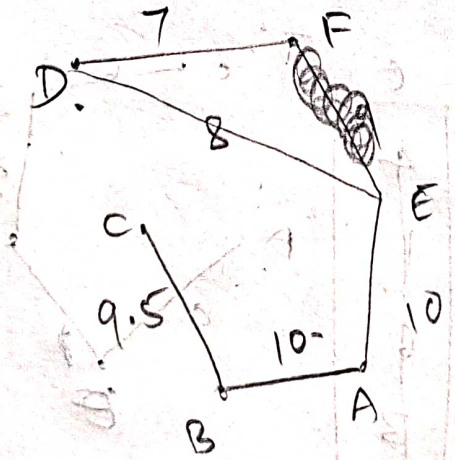


$= w(T) = 13$

② PA Find MST wt.?



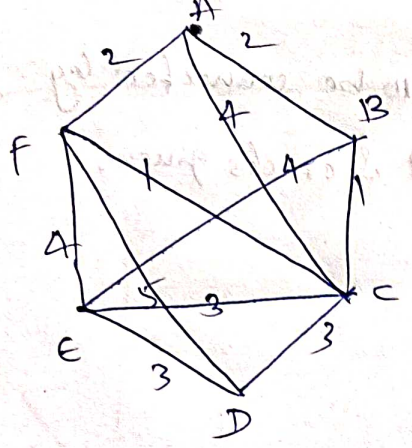
	A	B	C	D	E	F
A	—	10	16	11	10	17
B	10	—	9.5	∞	∞	19.5
C	16	9.5	—	∞	∞	12
D	11	∞	∞	—	8	7
E	10	∞	∞		—	
F	17	19.5	12	7	9	—



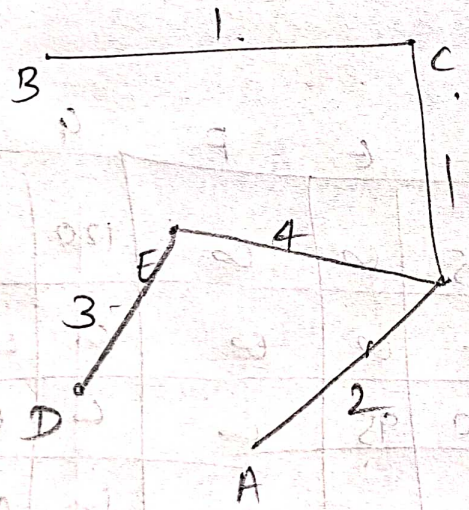
$w(T) = 20 + 7 + 18.5$

$= \underline{45.5}$

③

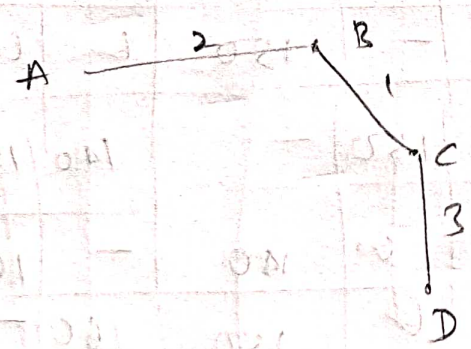


	A	B	C	D	E	F
A	–	2	4	∞	∞	2
B	2	–	1	∞	4	∞
C	4	1	–	3	3	1
D	∞	∞	3	–	3	5
E	∞	4	3	3	–	4
F	2	∞	1	5	4	–



$w(T) = 1+1+2+3+4$

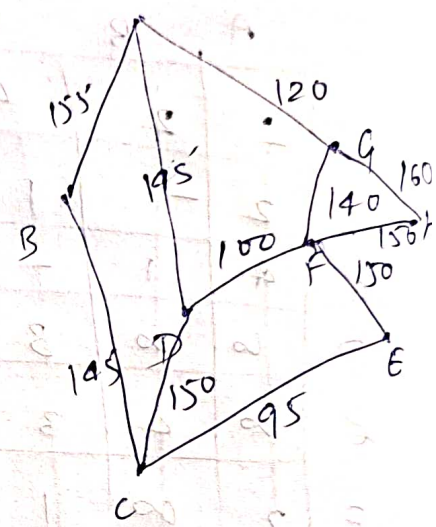
$= 11$



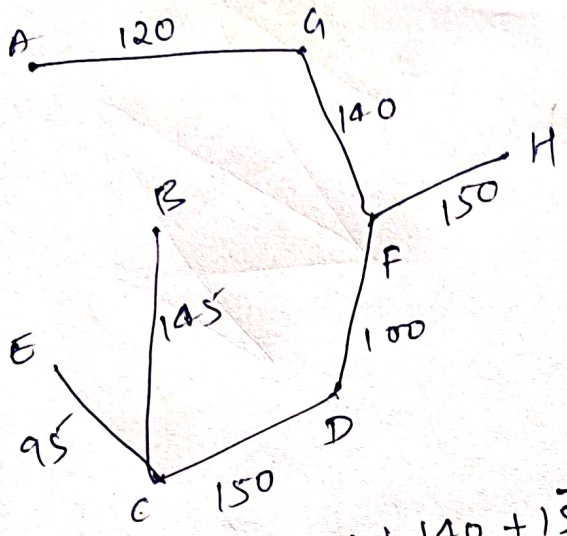
$= 10.$

(4) 8 cities A B C D E F G H to be connected by railway track. possible cost & tracks given.

	Crores
A & B	155
A & D	145
A & G	120
B & C	145
C & D	150
C & E	95
D & F	100
E & F	150
F & G	140
F & H	150
G & H	160



	A	B	C	D	E	F	G	H
A	—	155	∞	145	∞	∞	120	∞
B	155	—	145	∞	∞	∞	∞	∞
C	∞	145	—	150	95	∞	∞	∞
D	145	∞	150	—	∞	100	∞	∞
E	∞	∞	95	∞	—	150	∞	∞
F	∞	∞	∞	100	150	—	140	150
G	120	∞	∞	∞	∞	140	—	160
H	∞	∞	∞	∞	∞	150	160	—



$= 120 + 140 + 150 + 100 + 150 + 95 + 145$

400

260

240

M.Cost = 900 Crores.