# On the Realizability of the Critical Points of a Realizable Multiset

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# Nonnegative Inverse Eigenvalue Problem

What is the NIEP?

- Broadly: characterize the spectra of entrywise nonnegative matrices.
- Specifically: given a multiset (herein *list*) $\Lambda = \{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{C}$, find necessary and sufficient conditions on $\Lambda$ such that there is a nonnegative $A$ with spectrum $\Lambda$.
- If such a matrix exists, $\Lambda$ is called **realizable**, and $A$ is called a **realizing matrix** for $\Lambda$.

# Example Case

**Example ([Pap16])**

Are any of the following lists realizable?

A. $\{15, -1, -2, -3, -4\}$

B. $\{17, -3, -4, -5, -6\}$

C. A and B

D. None of the above.

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D. None of the above.

**Answer:** A,

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 3 & 4 & 5 \\ 3 & 2 & 1 & 4 & 5 \\ 4 & 2 & 3 & 1 & 5 \\ 5 & 2 & 3 & 4 & 1 \end{bmatrix}.$$

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There are many known sufficient conditions, but some can be quite complicated and hence don't provide much insight.

# Suleĭmanova Spectra

Suleĭmanova presented a specific class of spectra which are always realizable [Pap16]. A **Suleĭmanova spectrum** $\Lambda$ has the following characteristics:

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- $\Lambda$ has exactly one positive element.

It has been shown that for any Suleĭmanova spectrum, only the leading coeffiecent of the characteristic polynomial $p$ is positive. Furthermore, the critical points of $p$ form a Suleĭmanova spectrum.

# *Critical* Conjectures

**Conjecture (Monov [Mon08])**

*Let $\Lambda = \{\lambda_1, \dots, \lambda_n\}$ be realizable, and $p(t) = \prod_{i=1}^{n}(t - \lambda_i)$. Let $M = \{\mu_1, ..., \mu_{n-1}\}$ be the critical points of $p$. Then $s_k(M) \geq 0$ for all $k \in \mathbb{N}$.*

**Conjecture (C. Johnson)**

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This conjecture has been proven when $n \leq 4$ (or $n \leq 6$ when $s_1(\Lambda) = 0$) by Cronin and Laffey [CL14].

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## Conjecture

*Let $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ be realizable, where $\lambda_1 \geq \cdots \geq |\lambda_n|$. There exists a realizable multiset $M = \{\mu_1, \ldots, \mu_{n-1}, 0\}$ where $|\mu_1| \geq \cdots \geq |\mu_{n-1}|$ such that*

$$\sum_{i=1}^{k} |\mu_i| \leq \sum_{i=1}^{k} |\lambda_i|,$$

*for all $k \in \langle n \rangle$. In other words, $\Lambda$* **weakly majorizes** *$M$.*

# Verified Necessary Conditions

- It can be shown that

$$s_1(M) = \frac{n-1}{n} s_1(\Lambda).$$

  where $\Lambda$ is a list and $M$ are its critical points. Thus if $\Lambda$ is realizable, $s_1(M) \geq 0$.
- $M$ also satisfies the self-conjugacy condition, since the coefficients of $p'_\Lambda$ are real.
- We've also shown that $s_2(M) = \frac{n-2}{n} s_2(\Lambda) + \frac{1}{n^2} s_1^2(\Lambda) \geq 0$.
- Higher moments can be found with a formula provided by Monov [Mon08].

# D-Companion Matrix

**Definition ([CN06])**

Let $\{\lambda_1, ..., \lambda_n\}$ be the roots of some polynomial $p$ counting multiplicity. Then the matrix

$$A = D\left(I - \frac{1}{n}J\right) + \frac{\lambda_1}{n}J \quad \in M_{n-1}(\mathbb{C})$$

is called the **d-companion matrix**, where $D = \text{diag}(\lambda_2, ..., \lambda_n)$, and $I$ and $J$ are the identity and all-ones matrix respectively. The eigenvalues of A are the critical points of $p$.

# D-Companion Matrix

Straightforward computation gives the following formula:

$$a_{ij} = \frac{1}{n} \begin{cases} (n-1)\lambda_{i+1} + \lambda_1 & i = j \\ \lambda_1 - \lambda_{i+1} & i \neq j. \end{cases}$$

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If $\Lambda \subset \mathbb{R}$ and $\lambda_1 = \rho(A)$, the matrix $A$ is a **Metzler** matrix, i.e., there is a nonnegative matrix $B$ such that $A = B - sI$.

# D-Companion Matrix

**Theorem (Ciarlet[Cia68])**

*Let* $\Lambda = \{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{R}$ *be a list in descending order. If* $n\lambda_n + \lambda_1 \geq 0$, *then* $\Lambda$ *is realizable.*

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**Theorem**

*Let $\Lambda = \{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{R}$ be a realizable list in descending order, and let $M$ be its critical points. If $(n-1)\lambda_n + \lambda_1 \geq 0$, then $M$ is realizable.*

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Note that a Ciarlet spectrum also satisfies the above critical points theorem.

# Normal and Unitary Matrices

**Definition**

A square matrix A is called **normal** if $AA^* = A^*A$, where $A^* := \overline{A^T}$, the conjugate transpose of $A$.

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**Definition**

A square matrix $U$ is called **unitary** if $U^{-1} = U^*$.

Note: All unitary matrices are normal.

**Remark**

*A matrix is normal if and only if it is unitarily diagonalizable.*

# Hadamard Matrices

## Definition

A square matrix H is called **Hadamard** if $H_{ij} \in \pm 1$ and $HH^T = nI_n$.

Note that the rows of $H$ are mutually orthogonal. Further note that $\frac{1}{\sqrt{n}}H$ is unitary.

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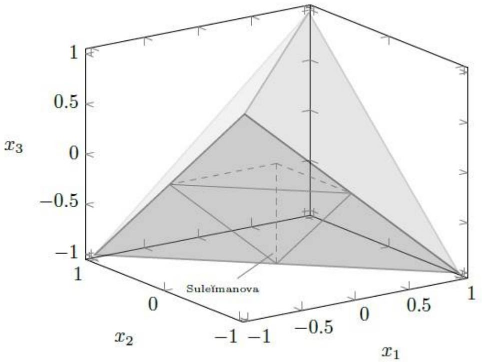
## Example

$$H_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} H_1 & H_1 \\ H_1 & -H_1 \end{bmatrix}$$

# The Great Pyramid of Hadamard

## Visualization of $n = 4$ Spectrum [JP16]




For each point $(x_1, x_2, x_3) \in \mathbb{R}^3$ in the shaded region, the list $\Lambda = \{1, x_1, x_2, x_3\}$ is realizable by $H\,\mathrm{diag}(\Lambda)H^{-1}$.

# Discrete Fourier Transform Matrix

## Definition (DFT)

The **discrete Fourier transformation (DFT)** matrix is given by:

$$F := \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \dots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \dots & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \dots & \omega^{(n-1)^2} \end{bmatrix},$$

where $\omega = e^{-2\pi i/n}$.

Note that the DFT matrix is a complex Hadamard matrix, i.e. $FF^* = nI$.

# Differentiators and Trace Vectors [Per03]

**Definition (Differentiator)**

Let $A \in M_n(\mathbb{C})$ and let $P \in M_n(\mathbb{C})$ be a orthogonal projection from $\mathbb{C}^n$ onto $\mathbb{C}^{n-1}$. Set $B = PAP|_{\mathcal{R}(P)}$. Then $P$ is a differentiator of $A$ if

$$p_B(x) = \frac{1}{n} p_A'(x)$$

where $p_A$ and $p_B$ are the characteristic polynomials of $A$ and $B$ respectively.

# Differentiators and Trace Vectors [Per03]

**Definition (Trace vector)**

Let $A \in M_n(\mathbb{C})$ and $z \in \mathbb{C}$. Then $z$ is a trace vector if $z^* A^k z = \frac{1}{n} \operatorname{tr}(A^k)$ for all $k \in \mathbb{N}$.

**Theorem (Pereira)**

*Every matrix in $M_n\mathbb{C}$ has a trace vector.*

**Theorem (Pereira)**

*Let $A \in M_n(\mathbb{C})$. Then the projection $P$ is a differentiator of $A$ if and only if $P$ is a projection onto $z^\perp$ for some trace vector $z$ of $A$, i.e., $P = I - zz^*$.*

# Differentiators and Trace Vectors

## Example

Let

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

and $z = e_3$ so that

$$B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Then $p_A(t) = t^3 - 1$ and $p_B(t) = t^2$, thus

$$p_B = \frac{1}{n} p_A'$$

# Critical Points and Submatrices

**Theorem**

*Let $\Lambda \subset \mathbb{C}$, and $M$ be its critical points, and let $U \in M_n(\mathbb{C})$ be a unitary matrix such that for some $i \in \langle n \rangle$, $|u_{ij}| = \frac{1}{\sqrt{n}}$ for all $j \in \langle n \rangle$. If $A = UDU^*$, where $D = \mathrm{diag}(\Lambda)$, then $\sigma(A_i) = M$, where $A_i$ denotes the $i$ $^{th}$ principal submatrix*

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**Corollary**

*Let $\Lambda$ be realizable by a matrix of the form $A = U\,\mathrm{diag}(\Lambda)U^*$ with $U$ as in the above theorem, and let $M$ be its critical points. Then $M$ is realizable and its realizing matrix is $A_{(i)}$.*

# Result on Hadamard/DFT similarities

## Theorem

*Let $\Lambda$ be realizable by $A = H\,\mathrm{diag}(\Lambda)H^{-1}$ or by $A = F\,\mathrm{diag}(\Lambda)F^{-1}$ and let $M$ be the critical points of $\Lambda$. Then $M$ is realizable. Furthermore, $M$ is realized by any principal submatrix of $A$.*

This result lends itself to a new proof of a classical result on polynomials.

## Theorem

*Let $p$ be a polynomial with real roots $\Lambda = \{\lambda_1, \dots, \lambda_n\}$ and critical points $M = \{\mu_1, \dots, \mu_{n-1}\}$, each listed in descending order. Then $\Lambda$ and $M$ interlace, that is*

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \cdots \geq \mu_{n-1} \geq \lambda_n.$$

## Open Questions

- For $A \geq 0$, does there exist $0 \leq B \in M_{n-1}(\mathbb{R})$ such that

$$p_B = \frac{1}{n} p_A'?$$

- Is $\rho(M) = \max\{|\mu| : \mu \in M\} \in M$?
- Does $M$ satisfy the J-LL condition?
- What can be said about the zeros of the anti-derivative?
- Where is the best coffee in Seattle?

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- Does $M$ satisfy the J-LL condition?
- What can be said about the zeros of the anti-derivative?
- Where is the best coffee in Seattle? QED Coffee.

# References


P. G. Ciarlet.
Some results in the theory of nonnegative matrices.
*Linear Algebra and Appl.*, 1(1):139–152, 1968.

Anthony G. Cronin and Thomas J. Laffey.
Perturbations on constructible lists preserving realizability in the NIEP and questions of Monov.
*Linear Algebra Appl.*, 445:206–222, 2014.

Wai Shun Cheung and Tuen Wai Ng.
A companion matrix approach to the study of zeros and critical points of a polynomial.
*J. Math. Anal. Appl.*, 319(2):690–707, 2006.

Olga Holtz.
M-matrices satisfy newton's inequalities.
*Proc. Amer. Math. Soc.*, 133(3):711–717, 2004.

Charles R. Johnson and Pietro Paparella.
Perron spectratopes and the real nonnegative inverse eigenvalue problem.
*Linear Algebra Appl.*, 493:281–300, 2016.

Vladimir V. Monov.
A family of symmetric polynomials of the eigenvalues of a matrix.
*Linear Algebra Appl.*, 429(8-9):2199–2208, 2008.

Pietro Paparella.
Realizing Suleimanova spectra via permutative matrices.
*Electron. J. Linear Algebra*, 31:306–312, 2016.

Rajesh Pereira.
Differentiators and the geometry of polynomials.
*J. Math. Anal. Appl.*, 285(1):336–348, 2003.