Systems with Intermittent Chaos

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Chaotic Systems

Definition

A dynamical system is **chaotic** if it has a sensitive dependence to initial conditions, i.e., an arbitrarily small change may result in a large change in the future.

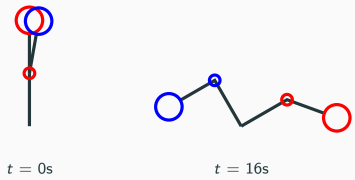
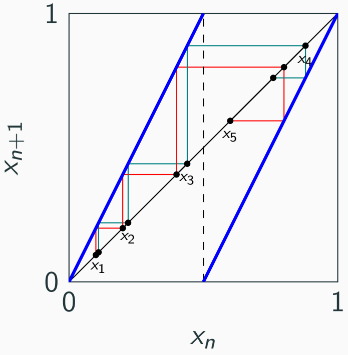
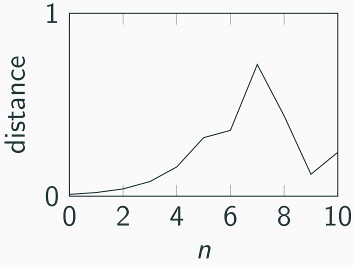


Figure 1: Double pendulum with slightly different starting positions.

Simple One-Dimensional Model of a Chaotic System



(a) Cobweb Plot after 4 iterations



(b) Distance over time

Figure 2: $x_{n+1} = 2x_n \bmod 1$

Predicting Behavior

Question

Given an initial condition, what is the distribution of orbits?

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We start by finding the cumulative distribution function (CDF).

$$\begin{aligned}\mathbb{P}(x_n \in [0, y)) &= \mathbb{P}\left(x_{n-1} \in (2x \bmod 1)^{-1}([0, y))\right)\\ &= \mathbb{P}\left(x_{n-1} \in [0, \frac{y}{2}) \text{ or } x_{n-1} \in [\frac{1}{2}, \frac{y+1}{2})\right)\\ &= \mathbb{P}\left(x_{n-1} \in [0, \frac{y}{2})\right) + \mathbb{P}\left(x_{n-1} \in [\frac{1}{2}, \frac{y+1}{2})\right).\end{aligned}$$

Predicting Behavior: PDF

So

$$\int_0^y \rho_n(x)dx = \int_0^{\frac{y}{2}} \rho_{n-1}(x)dx + \int_{\frac{1}{2}}^{\frac{y+1}{2}} \rho_{n-1}(x)dx.$$

Differentiating to determine the probability density function (PDF),

$$\rho_n(y) = \frac{1}{2}\rho_{n-1}(\frac{y}{2}) + \frac{1}{2}\rho_{n-1}(\frac{y+1}{2}).$$

This forms a Markov operator $\mathcal{P}$ acting on the space of PDFs.

Density Over Time

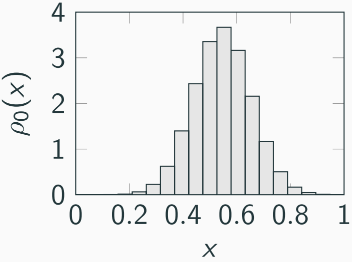
Similar to Markov chains, under suitable conditions $\rho_\infty = \lim_{n\to\infty} \mathcal{P}^n \rho_0$, for any non-zero PDF ρ_0, is an eigenvector of $\mathcal{P}$. In such a case, $\mathcal{P}\rho_\infty = \rho_\infty$.

Let ρ^* be such that $\mathcal{P}\rho^* = k\rho^*$. Since $\mathcal{P}\rho^*$ is a PDF over $[0, 1]$,

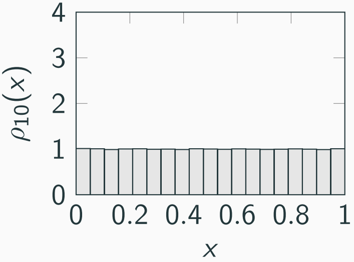
$$1 = \int_0^1 k\rho^*(x)dx = k\int_0^1 \rho^*(x)dx = k.$$

So 1 is the only eigenvalue of $\mathcal{P}$. Therefore $\mathcal{P}\rho_\infty = \rho_\infty$.

Stationary Distribution



(a) Initial Distribution



(b) Distribution after 10 applications

Figure 3: Change in distribution over time.

Thus for our $2x$ mod 1 map we have that $\rho_\infty(x) = 1$.

Definition

In a dynamical system, **intermittency** is the alternation between phases of nearly periodic and chaotic dynamics.

Figure 4: Spatial Intermittency in airflow[1]

[1] Aerographers Mate, http://meteorologytraining.tpub.com/14312/css/14312_84.htm

New Model: Adding Intermittency

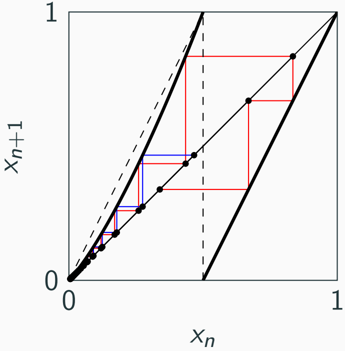
Sensitivity Controlled by α

$$x_{n+1} = \begin{cases} \color{blue}{x_n + 2^\alpha x_n^{1+\alpha}} & \text{if } 0 \leq x_n < \frac{1}{2} \\ \color{red}{2x_n - 1} & \text{if } \frac{1}{2} \leq x_n \leq 1 \end{cases} \qquad 0 < \alpha < 1$$

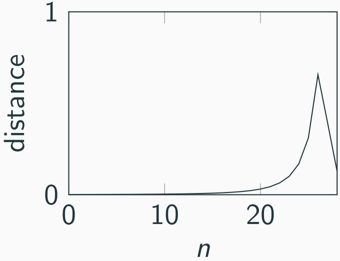
Very Sensitive ($2x$ mod 1)

Parameter

New Model: Orbit



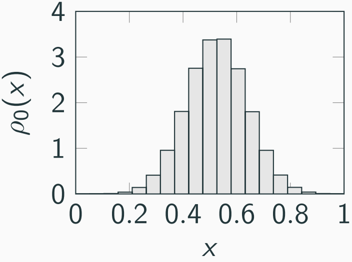
(a) Cobweb Plot after 28 iterations



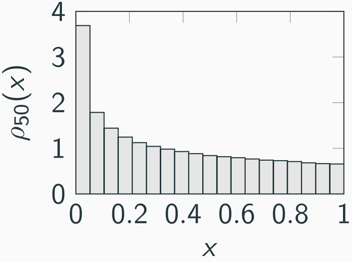
(b) Distance over time

Figure 5: Orbits starting at $x_0 = 0.005$ and $x_0 = 0.006$ with $\alpha = 0.6$.

New Model: Stationary Distribution



(a) Initial Distribution



(b) Distribution after 50 applications

Figure 6: Change in distribution over time with $\alpha = 0.6$.

The intermittency of the system introduces a singularity at zero.
Suggesting $\rho_\infty(x) \approx x^{-\beta}$ for some $\beta > 0$.

Future Questions

- What exactly is ρ_∞ for our model? (Near 0, $\rho_\infty(x) \approx x^{-\alpha}$)
- How does noise affect our model?
- How well does the model reflect real-world systems?

Questions?