Permutative Matrices and the Real Nonnegative Inverse Eigenvalue Problem

Amber Thrall

Under the guidance of Dr. Pietro Paparella

The Problem

Given a finite set of real numbers σ, is there a real entry-wise nonnegative matrix whose spectrum (set of eigenvalues) is σ? Furthermore, what are the conditions on σ for this to be true?

Inverse Eigenvalue Problem

The *inverse eigenvalue problem* (IEP) is a long-standing problem regarding the set of eigenvalues for matrices.

- Given a set of numbers σ, is there a matrix A with that set of eigenvalues (or spectra)?
- If there is such a matrix, σ is said to be *realizable.*
- The problem is further restricted by restricting A to a specific class (see Figure 1).
- Our primary concern: *real nonnegative IEP* (RNIEP). A's entries must be real and nonnegative.

A well-established condition within the RNIEP is that if a given spectra σ is realizable, then

$$s_k(\sigma) := \sum_{i=1}^{n} \lambda_i^k \geq 0, \ \forall k \in \mathbb{N} \tag{1}$$

$$\rho(\sigma) := \max_{i \leq i \leq n} |\lambda_i| \in \sigma. \tag{2}$$

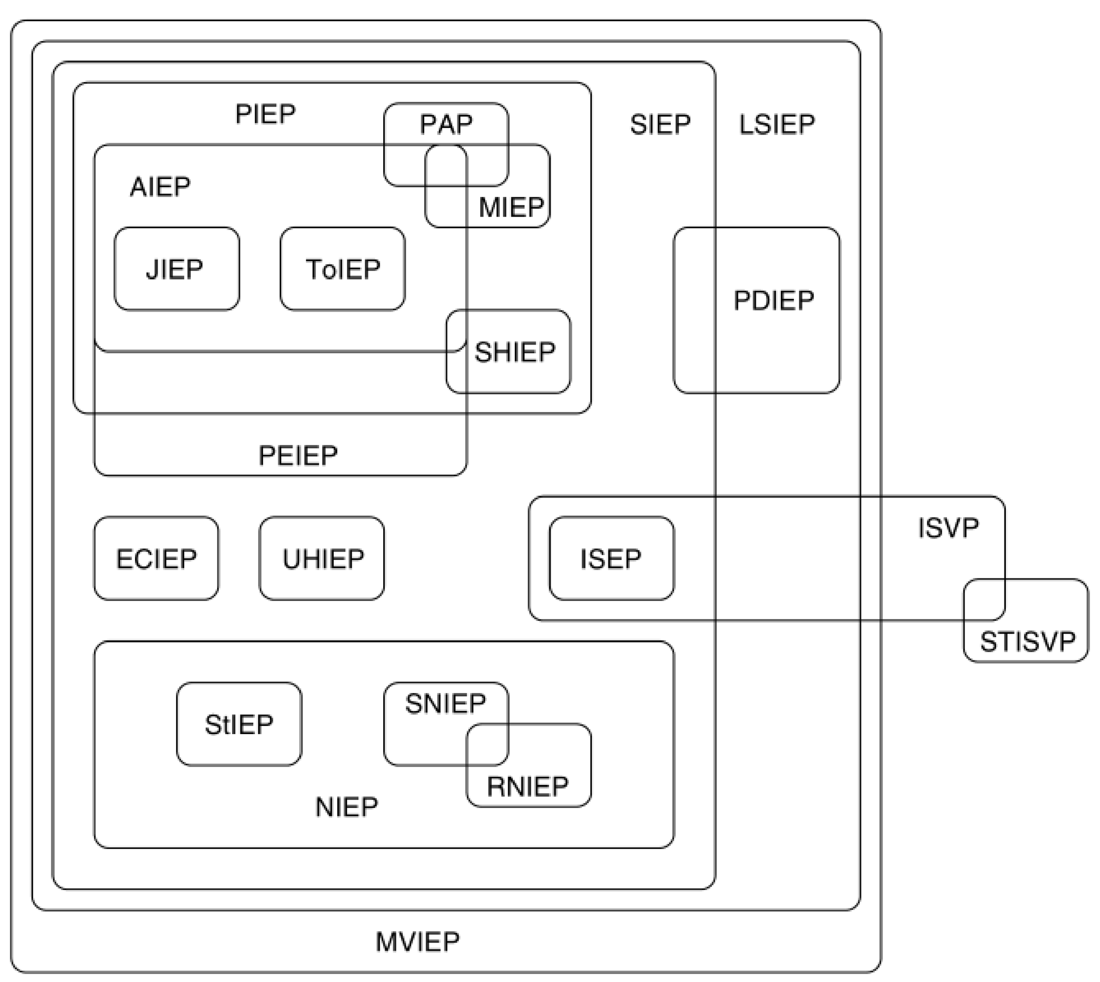

Figure 1: Various classifications of Inverse Eigenvalue Problem [1]

Suleĭmanova Spectra

A *Suleĭmanova spectrum* σ, must satisfy the following:

- $s_1(\sigma) \geq 0$.
- σ contains exactly one positive element.

There are several important results for Suleĭmanova spectra:

- All Suleĭmanova spectra are realizable by a real nonnegative matrix [2].
- All Suleĭmanova spectra are also realizable by a symmetric nonnegative matrix (SNIEP) [3].

Permutative Matrices

Permutative matrices are matrices whose rows are permutations of the first row.

$$P = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix}$$

Figure 2: Permutative matrix for $\sigma = \{6, -1, -2\}$.

- Every Suleĭmanoa spectrum is realizable by a permutative matrix [4].
- Every permutative matrix is diagonalizable.
- Permutative matrices' eigenvectors are known for each eigenvalue.

Important Result

Every Suleĭmaonva spectrum is realizable. [2]

Diagonalizable

Given a permutative matrix P and it's eigenvectors $v_1, \ldots, v_n$, the following equation is proven to be true:

$$P = S\Lambda S^{-1} \tag{3}$$

where $\Lambda = diag(\lambda_1, \ldots, \lambda_n)$ and $S = [v_1| \ldots |v_n]$.

- Equation (3) only holds if S is invertible.
- Since we have proven that $\det(S) \neq 0$ is always true, S is always invertible.

This result is significant for symmetry.

- Every Suleĭmanova spectrum is symmetrically realizable [3].
- Not all spectra can be realized by a diagonalizable nonnegative matrix.
- Permutative matrices show that symmetry is not necessary for realizability.

Hadamard Matrices

Hadamard matrices are a special type of square matrix, whose size is a power of 2, defined as

$$H_{2^k} := \begin{bmatrix} H_{2^{k-1}} & H_{2^{k-1}} \\ H_{2^{k-1}} & -H_{2^{k-1}} \end{bmatrix}, \tag{4}$$

where $H_1 = [1]$.

- Due to their recursive nature, all Hadamard matrices are symmetric.
- All Hadamard matrices are invertible, where $HH^T = nI$.
- According to Johnson and Paparella [5], $H\Lambda H^{-1}$ is nonnegative, where $\Lambda = diag(\lambda_1, \ldots, \lambda_n)$.

Current Results

In our study we have already provided the following:

- A Suleĭmanoa spectrum of length $h+1$, can be realized using Hadamard matrices of size h.
- All permutative matrices can be diagonalized.

Additional Questions

We are currently investigating the following related questions:

- Can Hadamard matrices prove beneficial for realizing Suleĭmanova spectra of length $h-1$?
- How may permutative matrices be of use in other areas?

References

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[3] M. Fielder, "Eigenvalues of nonnegative symmetric matrices," *Linear Algebra Appl*, vol. 9, pp. 119–142, 1974.

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Contact Information

- A. Thrall: arthrall@uw.edu
- P. Paparella: pietrop@uw.edu

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