On the realizability of the critical points of a realizable list

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The NIEP

The *Nonnegative Inverse Eigenvalue Problem* (NIEP) seeks to characterize the spectra of nonnegative matrices. More specifically,

- What are the properties of the spectra of nonnegative matrices?
- What are the sufficient conditions for a list to be the spectra of a nonnegative matrix?

We say a spectrum Λ is *realizable* if there is a nonnegative matrix A such that $\sigma(A) = \Lambda$. Furthermore, we refer to A as the *realizing matrix*.

Necessary Conditions

Conjugacy	$\Lambda = \overline{\Lambda}$
Perron	$\rho(\Lambda) := \max\{\lvert\lambda\rvert : \lambda \in \Lambda\} \in \Lambda$
Moment	$s_k(\Lambda) := \sum_{i=1}^{n} \lambda_i^k \geq 0 \quad \forall k \in \mathbb{N}$
J-LL	$s_k^m(\Lambda) \leq n^{m-1} s_{km}(\Lambda) \quad \forall k, m \in \mathbb{N}$

Monov's Conjecture

Conjecture. ([6]) *Let Λ be the spectrum of a nonnegative matrix A, and let Λ' be the critical points of $p(\lambda) = \det(\lambda I - A)$. The moments $s_k(\Lambda')$ are nonnegative for all $k \in \mathbb{N}$.*

Johnson [4] posed a more general question.

Johnson's Conjecture

Conjecture. *Let Λ be the spectrum of a nonnegative matrix A, and let Λ' be the critical points of $p(\lambda) = \det(\lambda I - A)$. Then Λ' is realizable.*

Suleĭmanova Spectra [8]

If $s_1(\Lambda) \geq 0$ and Λ contains exactly one nonnegative eigenvalue, then

$$C(p_\Lambda) = \begin{bmatrix} 0 & -a_0 \\ I_{n-1} & -a \end{bmatrix}, \quad a := [a_1 \dots a_{n-1}]^\top$$

is nonnegative as the coefficients of p_Λ are nonpositive [3, 7].

Ciarlet Spectra [1]

- Ciarlet spectra: $n\lambda_n + \lambda_1 \geq 0,\ \Lambda \subset \mathbb{R}$.
- All Ciarlet spectra are realizable [1].
- All Ciarlet spectra's critical points are realizable by a matrix of the form:

$$a_{ij} = \frac{1}{n} \begin{cases} (n-1)\lambda_{i+1} + \lambda_1 & i = j \\ \lambda_1 - \lambda_{i+1} & i \neq j. \end{cases}$$

Verified Necessary Conditions

The following necessary conditions have been verified:

$$\Lambda' = \overline{\Lambda'}$$
$$s_1(\Lambda') = \frac{n-1}{n} s_1(\Lambda) \geq 0$$
$$s_2(\Lambda') = \frac{n-2}{n} s_2(\Lambda) + \frac{1}{n^2} s_1^2(\Lambda) \geq 0$$
$$s_1^2(\Lambda') \leq (n-1) s_2(\Lambda')$$

Johnson's conjecture has also been established for $n \leq 4$ ($n \leq 6$ when $s_1(\Lambda) = 0$) [2].

Main Result

Theorem 5.12. Let Λ be a list and U be a unitary matrix such that $\exists i \in \{1, \dots, n\}$ where $\lvert u_{ij}\rvert = \frac{1}{\sqrt{n}}$, $\forall j = 1, \dots, n$. If $A = UDU^*$, then $\sigma(A_{(n)}) = \Lambda'$.

Corollary

A matrix H is called a *complex Hadamard matrix* if $HH^* = nI$ and $\lvert h_{ij}\rvert = 1$.

- If $UDU^* \geq 0$, where $U = \frac{1}{\sqrt{n}} H$, then Theorem 5.12 applies.
- The spectracone

$$\mathcal{C}(H) := \{x \in \mathbb{C}^n \mid HDH^{-1} \geq 0\}$$

encompasses a large class of realizable spectra [5].

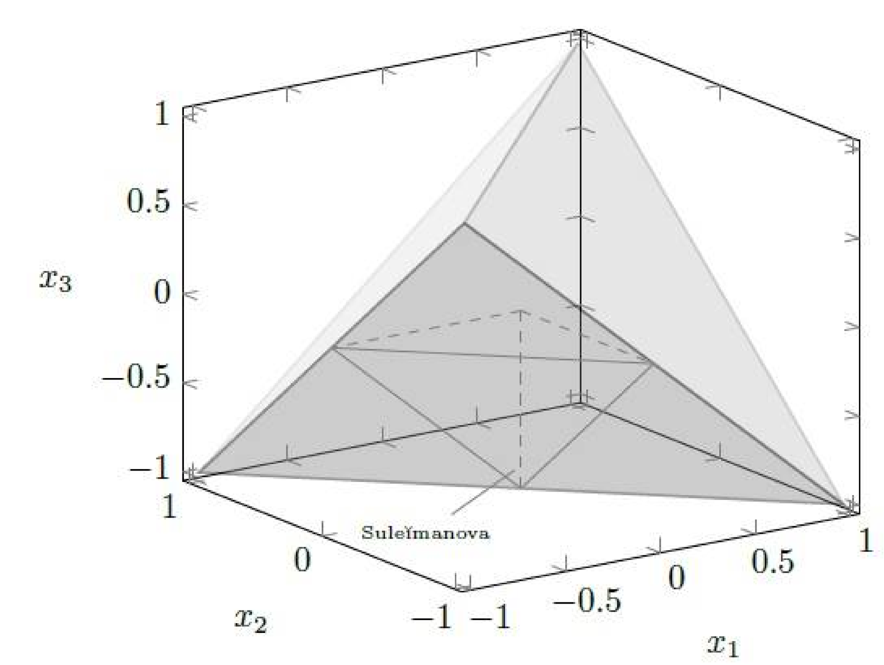

Figure 1: $\mathcal{C}(H)$ [5]

Example: Hadamard Matrices

Consider the following spectra:

$$\Lambda = \{6, -1, -2, -3\}$$

Define the matrix U as follows:

$$U = \frac{1}{\sqrt{n}} H = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

Notice then that

$$A = U \operatorname{diag}(\Lambda) U^* = \frac{1}{2} \begin{bmatrix} 0 & 4 & 5 & 3 \\ 4 & 0 & 3 & 5 \\ 5 & 3 & 0 & 4 \\ 3 & 5 & 4 & 0 \end{bmatrix}.$$

Since $\lvert u_{ij}\rvert = \frac{1}{\sqrt{n}}$,

$$\sigma(A_{(n)}) = \Lambda' = \{4.0279, -1.4378, -2.5901\}.$$

Application: Gauss-Lucas

Theorem 5.12 provides a new proof to a classic result:

Gauss-Lucas: Given a polynomial with real roots $\lambda_1, \dots, \lambda_n$ and critical points $\mu_1, \dots, \mu_{n-1}$, then

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n.$$

Open Questions

- Johnson's conjecture is still unsolved in a general case for spectra of size $n \geq 5$.
- Is there an analog to Johnson's conjecture in relation to the anti-derivative of the characteristic polynomial?

Final remarks

Preprint: https://arxiv.org/abs/1712.05454.

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