ECE 581 Homework 6.

1.

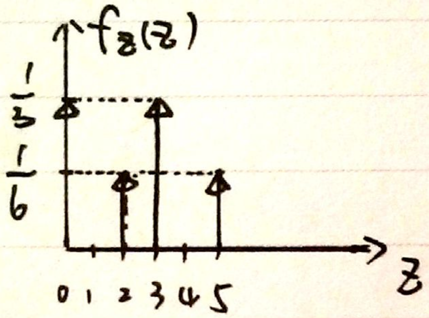
$Y \backslash X$	0	2	
0	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{2}$
3	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{2}$
	$\frac{2}{3}$	$\frac{1}{3}$	

(a) Z: 0 2 3 5

$\Rightarrow P$: $\frac{1}{3}$ $\frac{1}{6}$ $\frac{1}{3}$ $\frac{1}{6}$

$$\therefore f_Z(z) = \frac{1}{3}\delta(z) + \frac{1}{6}\delta(z-2) + \frac{1}{3}\delta(z-3) + \frac{1}{6}\delta(z-5)$$

(b).

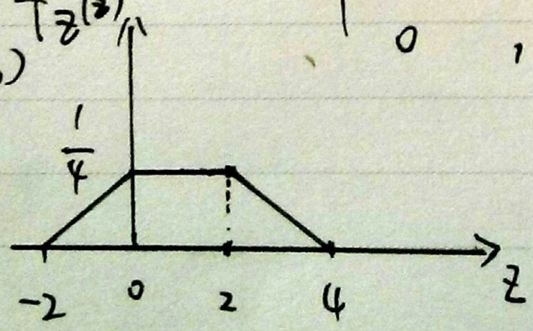


2. $f_X(x) = \begin{cases} \frac{1}{2}, & x \in [-2, 0] \\ 0, & \text{else} \end{cases}$ $\quad f_Y(y) = \begin{cases} \frac{1}{4}, & y \in [0, 4] \\ 0, & \text{else} \end{cases}$

(a). $$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) \cdot f_Y(z-x)\,dx$$

$$= \begin{cases} \frac{1}{8} \cdot 2, & z \in [0, 2] \\ (4-z)\frac{1}{8}, & z \in (2, 4] \\ (z+2)\frac{1}{8}, & z \in [-2, 0) \\ 0, & \text{else.} \end{cases} = \begin{cases} \frac{1}{4}, & z \in [0, 2] \\ (4-z)\frac{1}{8}, & z \in [2, 4] \\ (z+2)\frac{1}{8}, & z \in [-2, 0) \\ 0, & \text{else} \end{cases}$$

(b)



3. (a)

$Y \backslash X$	0	1	2	
0	$\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{1}{2}$
1	$\frac{3}{8}$	0	$\frac{1}{8}$	$\frac{1}{2}$
	$\frac{5}{8}$	$\frac{1}{4}$	$\frac{1}{8}$	

$$f_X(x) = \frac{5}{8}\delta(x) + \frac{1}{4}\delta(x-1) + \frac{1}{8}\delta(x-2)$$

$$f_Y(y) = \frac{1}{2}\delta(y) + \frac{1}{2}\delta(y-1)$$

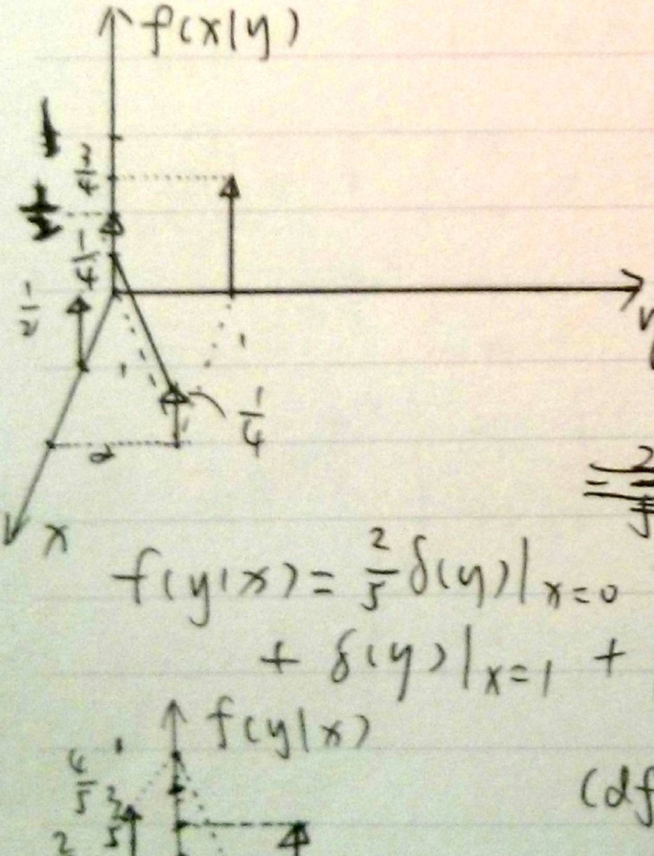
$\because P_r(Y|X=0) = P_r(Y)$ $\therefore$ not statistically independent

(b). $P(X|Y)$

$Y \backslash X$	0	1	2
0	$\frac{1}{2}$	$\frac{1}{2}$	0
1	$\frac{3}{4}$	0	$\frac{1}{4}$

$\therefore f(x|y)$

$$= \frac{1}{2}\delta(x)\Big|_{y=0} + \frac{1}{2}\delta(x-1)\Big|_{y=0} + \frac{3}{4}\delta(x)\Big|_{y=1} + \frac{1}{4}\delta(x-2)\Big|_{y=1}$$



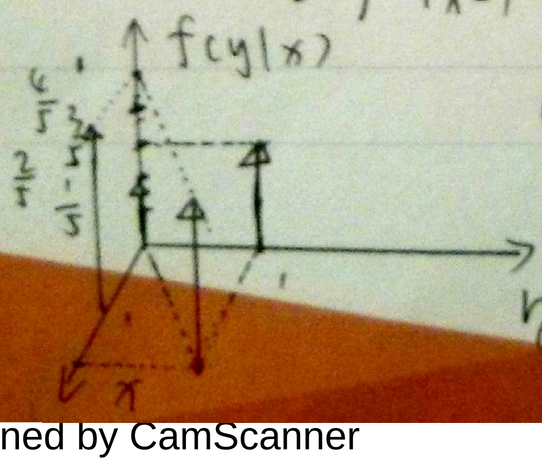
(c) $P(Y|X)$

$Y \backslash X$	0	1	2
0	$\frac{2}{5}$	1	0
1	$\frac{3}{5}$	0	1

$\cancel{f(y|x) = \frac{2}{5}\delta(x,y) + \frac{3}{5}\delta(x,y)}$

$\cancel{= \frac{2}{5}\delta(x)|_{y=0} + \frac{3}{5}\delta(x)}$

$$f(y|x) = \frac{2}{5}\delta(y)\Big|_{x=0} + \frac{3}{5}\delta(y-1)\Big|_{x=0} + \delta(y)\Big|_{x=1} + \delta(y-1)\Big|_{x=2}$$

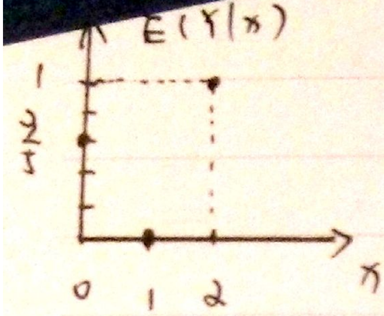


(df) $E(Y|X=x) = \int_{-\infty}^{\infty} y \cdot f(y|x)\, dy$

$\frac{3}{5} = E(Y|0)$

$0 = E(Y|1)$

$1 = E[Y|2]$



(e)

Z:	0	1	2	3
Pr:	$\frac{1}{4}$	$\frac{5}{8}$	0	$\frac{1}{8}$

$$\therefore F_Z(z) = Pr(Z \le z) = \begin{cases} 0, & z \in (-\infty, 0) \\ \frac{1}{4}, & z \in [0, 1) \\ \frac{7}{8}, & z \in [1, 3) \\ 1, & z \in [3, +\infty) \end{cases}$$

4. (a) $\int_{-\infty}^{\infty} f_{X_1,X_2}(x_1, x_2)\,dx_2 = f_{X_1}(x_1)$

$$= \frac{1}{\sqrt{3}\pi} e^{-\frac{x_1^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{2}{3}(x_2 - \frac{1}{2}x_1)^2} dx_2$$

$$= \frac{1}{\sqrt{3}\pi} e^{-\frac{x_1^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{2}{3}u^2} du$$

$$= \frac{1}{\sqrt{3}\pi} e^{-\frac{x_1^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2}} dt \cdot \frac{\sqrt{3}}{2}$$

$$= \frac{1}{\sqrt{2}\pi} e^{-\frac{x_1^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2}} e^{-\frac{t^2}{2}} dt \rightarrow N(0,1) \sim t.$$

$$= \frac{1}{\sqrt{2}\pi} e^{-\frac{x_1^2}{2}} \cdot \sqrt{\pi}$$ (b).

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{x_1^2}{2}}$$

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(b) $f_{x_2|x_1}(x_2|x_1) = \dfrac{f_{x_1,x_2}(x_1,x_2)}{f_{x_2}(x_2)} = \dfrac{\frac{1}{\sqrt{3}\pi} e^{-\frac{2}{3}(x_1^2 - x_1x_2 + x_2^2)}}{\sqrt{2\pi} \cdot e^{-\frac{x_1^2}{2}}}$

$= \sqrt{\dfrac{2}{3\pi}}\, e^{-\frac{2}{3}(\frac{1}{4}x_1^2 - x_1x_2 + x_2^2)}$

$= \dfrac{1}{\sqrt{2\pi}\sqrt{\frac{3}{2}}} e^{-\frac{2}{3}(x_2 - \frac{1}{2}x_1)^2}$ $\therefore \boxed{6 = \frac{\sqrt{3}}{2} \quad \mu = \frac{1}{2}x_1}$

$= \dfrac{1}{\sqrt{2\pi}\frac{\sqrt{3}}{2}} e^{\frac{(x_2 - \frac{1}{2}x_1)^2}{2\frac{3}{4}}} = \dfrac{1}{\sqrt{2\pi}\frac{\sqrt{3}}{2}} e^{\frac{(x_2 - \frac{1}{2}x_1)^2}{2\cdot(\frac{\sqrt{3}}{2})^2}}$