ECE 581 Homework 2

Due Thursday 5 AM Sept 10, 2015 (2 pages, 30 points total)

Electronic Submission – Please submit via "Assignment" under Sakai

1. (10 points) Given a mixed random variable X with pdf :

$f_X(x) = .0625e^{-x}U(x) + .1875\delta(x+1) + .1875\delta(x) + .5625\delta(x-1)$

where $U(x)$ is the unit step function and $\delta(x)$ is the dirac delta function.

Calculate the following probabilities:
(a) (2 points) $Pr(0 < x \leq 1)$
(b) (2 points) $Pr(0 \leq x < 1)$
(c) (2 points) $Pr(0 < x < 1)$
(d) (2 points) $Pr(0 \leq x \leq 1)$
(e) (2 points) $Pr(2 < x \leq 3)$

2. (10 points total) A random variable, X, has a uniform probability density function defined as follows: $f_X(x) = 1/3$ for $-1 \leq x \leq 2$ and 0 otherwise. This random variable, X, is the input to the transformation $y = g(x)$, defined as follows: $y = 0$ for $x \leq 0$, $2x$ for $0 \leq x \leq 1$, and 2 for $x \geq 1$.

(a) (3 points) Obtain, sketch and accurately and completely label the probability distribution function, $F_Y(y)$, of the output random variable Y.

(b) (3 points) Obtain, sketch and accurately and completely label the probability density function, $f_Y(y)$, of the random variable Y.

(c) (4 points) What is $E[Y]$, the expected value of Y?

3. (10 points total) (Signal detection example) Consider a square image (K x K pixels) of data for two possible hypotheses. Let H_0 denote the hypothesis "noise alone" and H_1 denote the hypothesis signal plus noise. i.e.

$H_0 : \mathbf{X} = \mathbf{N}$
$H_1 : \mathbf{X} = \mathbf{S} + \mathbf{N}$

where $\mathbf{X}, \mathbf{S}$ and $\mathbf{N}$ are each K x K square matrices. $\mathbf{N}$, the noise, is a matrix of statistically independent Gaussian random variables, where each pixel, $n_{i,j}$, has mean zero and variance σ_n^2. In this first example, assume that $\mathbf{S}$ is a completely known signal (image) matrix. $\mathbf{X}$ is the data.

Eventually we would like an optimum (in some sense) algorithm that will process the data $\mathbf{X}$ and give us an output which is a binary decision, either "H_0" noise alone, or "H_1", signal plus noise. The notation is that H_0 or H_1 (without quotes) refers to the truth and "H_0" or "H_1" (with quotes) refers to the respective decisions.

For now, as a first step, let's try processing the data with the following algorithm:

$z = \frac{1}{\sigma_n^2} \sum_{i=1}^{K} \sum_{j=1}^{K} x_{i,j} s_{i,j}$

where $x_{i,j}$ are the elements of the matrix $\mathbf{X}$ and $s_{i,j}$ are the elements of the matrix $\mathbf{S}$.

Problem

(a) (5 points) – Derive an analytical expression for the probability density function of Z conditional to the hypothesis H_0, which we denote by $f(z|H_0)$. This is a transformation of K^2 statistically independent noise alone random variables to one random variable. In your derivation, you may use the fact which we will show later in class, that the sum of statistically independent Gaussian random variables is a Gaussian random variable.

(b) (5 points) – Derive an analytical expression for the probability density function of Z conditional to the hypothesis H_1, which we denote by $f(z|H_1)$.

We will see later in class how we can use the results in this problem to quantitatively evaluate the performance of an optimal signal detection algorithm.