10
1 36
2 25
~~3 29~~

100 Total

ECE 581 September 17, 2015 Exam 1 **(75 minutes) 100 points total** Show work to get credit.
Closed book, closed notes, no computers, calculators, phones or electronics.
3 problems on 2 pages

(10 points) I accept the Duke University Community Standard (Reference http://www.integrity.duke.edu/new.html) I will not communicate with anyone in either section of this course or anyone else relevant to this exam, using any communication media during the total time spanned by the exams for both sections.

....Shengxin Qian..............................(Sign your name)

1. (36 points total Given a mixed (discrete and continuous) random variable X with probability density function
$f_X(x) = x[U(x) - U(x-1)] + \frac{1}{4}\delta(x+2) + \frac{1}{8}\delta(x-1) + \frac{1}{8}\delta(x)$ where $U(x)$ is the unit step function, and $\delta(x)$ is the dirac delta function.

(a) (1 point) Sketch accurately and completely label the probability density function $f_X(x)$ vs. x

(b) (3 points) Obtain and sketch accurately and completely label the probability distribution function $F_X(x)$ vs. x

Double check the expression given above for the probability density function and the expression you got for the probability distribution function of the random variable X before proceeding.
Calculate numerical values for each of the following:
(b) (4 points) $E[X]$
(c) (4 points) $E[X^2]$
(d) (4 points) $Pr(0 < X \leq 1)$
(e) (4 points) $Pr(0 \leq X < 1)$
(f) (4 points) $Pr(0 < X < 1)$
(g) (4 points) $Pr(0 \leq X \leq 1)$
(i) (4 points) $Pr(X = \frac{1}{2})$
(j) (4 points) $F_X(0)$

2. (25 points total) You are given a function (transformation) $y = g(x)$ that is defined as the absolute value of the trigonometric sine function: $y = |\sin(x)|$ for $0 \leq x \leq 2\pi$ and zero otherwise. Now consider an input random variable, X, whose probability density function is:

$$f_X(x) = \frac{3}{16}\delta(x) + \overset{2/16}{\frac{1}{8}}\delta(x - \frac{3\pi}{2}) + \overset{4/16}{\frac{1}{4}}\delta(x - \frac{\pi}{2}) + \overset{2/16}{\frac{1}{8}}\delta(x - \pi) + \frac{5}{16}\delta(x - 2\pi)$$

(a) (5 points total) (i) (1 point) Sketch and accurately and completely label the function $y = g(x)$.
(ii) (4 points)Also, sketch and accurately and completely label the probability density function, $f_X(x)$, of the random variable X.

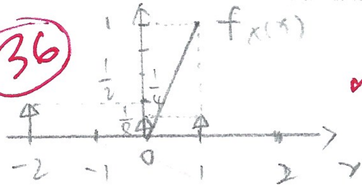
(b) (10 points total) Determine the probability *distribution* function, $F_Y(y)$ of the output random variable Y. $\boxed{\text{Sketch}}$ and accurately and completely label $F_Y(y)$.

(c) (10 points total) Instead of the above probability density function $f_X(x)$, let the input random variable X have a probability *distribution* function $F_X(x) = U(x - \frac{\pi}{2})$, where $U(x)$ is the unit step function. What is the probability *distribution* function $F_Y(y)$ of the random variable Y? Sketch and completely label the probability *distribution* function, $F_Y(y)$.

3. (29 points total) The joint probability density function of the two random variables X and Y is given by $f_{XY}(x,y) = \frac{1}{4}\delta(x-0, y-0) + \frac{3}{8}\delta(x-0, y-1) + \frac{1}{4}\delta(x-1, y-0) + \frac{1}{8}\delta(x-2, y-1)$. The notation $\frac{1}{8}\delta(x-a, y-b)$ represents a probability in the x y plane of $\frac{1}{8}$ at the point $(x = a, y = b)$

(a) (5 points) Sketch and completely label accurately this joint probability density function. Double check this before proceeding.
(b) (5 points) What is the marginal pdf, $f_X(x)$? Sketch and completely label it.
(c) (5 points) What is the marginal pdf, $f_Y(y)$? Sketch and completely label it.
(d) (5 points) What is the correlation of X and Y? (final numerical answer)
(e) (5 points) Are X and Y uncorrelated random variables? (No credit without proof)
(f) (4 points) Are X and Y statistically independent random variables? (No credit without proof)

1 (a) 36



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(b). $F_X(x) = \begin{cases} 0, & x<-2 \\ \frac{1}{4}, & x\in[-2,0) \\ \frac{3}{8}, & x=0 \\ \frac{3}{8}+\frac{1}{2}x^2, & x\in(0,1) \\ 1, & x\in[1,+\infty) \end{cases}$

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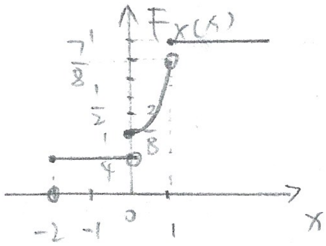

(b) $E[X] = \int_{-\infty}^{+\infty} x\cdot f_X(x)dx$

4 $= \int_{-\infty}^{+\infty} \frac{1}{4}\delta(x+2)\cdot x dx$

$+\int_{-\infty}^{\infty}\frac{1}{8}\delta(x-1)dx\cdot x + \int_{-\infty}^{+\infty}\frac{1}{8}\delta(x)\cdot x dx$

$+ \int_0^1 x\cdot x dx$

$= \frac{1}{4}(-2) + \frac{1}{8}\cdot + 0 + \frac{1}{3}x^3\Big|_0^1$

$= -\frac{1}{2}+\frac{1}{8}+\frac{1}{3} = -\frac{1}{24}$

(c) $E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)dx$

4 $= \frac{1}{4}\cdot(-2)^2 + \frac{1}{8} + \int_0^1 x^3 dx$

$= \frac{4}{4}+\frac{1}{8}+\frac{1}{4} = \frac{11}{8}$

(d). $Pr(0<X\le 1) = \frac{1}{8} + \int_0^1 x\cdot dx$

4 $= \frac{1}{8}+\frac{1}{2} = \frac{5}{8}$

(e) $Pr(0\le x<1) = \frac{1}{8}+\int_0^1 x\cdot dx$

4 $= \frac{1}{8}+\frac{1}{2} = \frac{5}{8}$

4 (f) $Pr(0<X<1) = \int_0^1 x dx = \frac{1}{2}$

4 (g) $Pr(0\le x\le 1) = \frac{1}{2}+\frac{1}{8}+\frac{1}{8} = \frac{3}{4}$

4 (i) $Pr(x=\frac{1}{2})$ $\cancel{F_X(\frac{1}{2}) = \frac{3}{8}+\frac{1}{2}(\frac{1}{2})^2}$

$= \lim_{\Delta x\to 0}\int_{\frac{1}{2}}^{\frac{1}{2}+\Delta x} x\cdot dx$

$= \frac{1}{2}x^2\Big|_{\frac{1}{2}}^{\frac{1}{2}+\Delta x}\lim_{\Delta x\to 0} = 0.$

4 (j) $F_X(0) = \frac{3}{8}.$

2. $g(x) = \begin{cases} |\sin(x)|, & x\in[0,2\pi] \\ 0, & \text{oth} \end{cases}$

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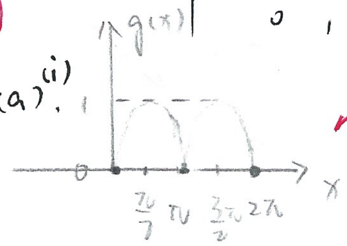
(a)(i)



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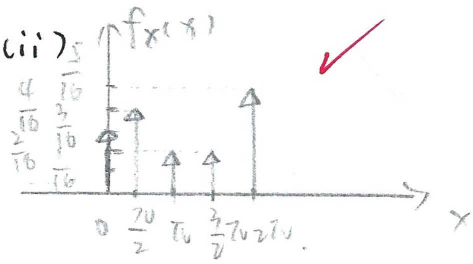
(ii)



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(b) $F_Y(y) = Pr(Y\le y) \to$ the ~~next pa~~ back si

$= Pr(-y\le \sin x\le y)$

$\cancel{= Pr[\arcsin(y) \le x \le \ldots]}$

$= Pr(0\le x\le \arcsin(y))$

$+ Pr[(\pi+\arcsin(y))\le x\le(\pi-\arcsin(y))]$

$+ Pr[(2\pi-\arcsin(y))\le x\le 2\pi]$

2.(b).

~~$F_Y(y) = Pr[$~~

~~$Pr[0 \le x \le \arcsin(y)] + Pr[\pi + \arcsin(y) \le x \le \pi - \arcsin(y)] + Pr[2\pi - \arcsin(y) \le x \le 2\pi] =$~~

$Pr(x, y)$

$y \backslash x$	0	$\frac{\pi}{2}$	π	$\frac{3}{2}\pi$	2π
0	$\frac{3}{16}$	0	$\frac{1}{8}$	0	$\frac{5}{16}$
1	0	$\frac{1}{4}$	0	$\frac{1}{8}$	0

~~$\therefore F_Y(y) = \frac{5}{8},\ y = 0;\ \frac{3}{8},\ y = 1$~~

$$F_Y(y) = \begin{cases} 0, & y < 0 \\ \frac{5}{8}, & y = 0 \\ \frac{5}{8}, & y \in (0, 1) \\ 1, & y \in [1, +\infty) \end{cases}$$

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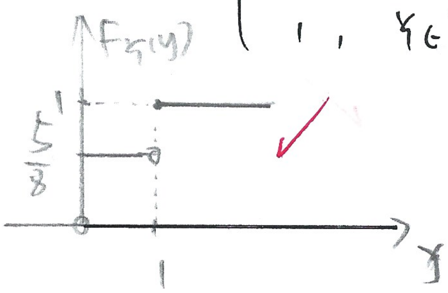

(c). $F_X(x) = \begin{cases} 1, & x \in [\frac{\pi}{2}, +\infty) \\ 0, & \text{oth} \end{cases}$

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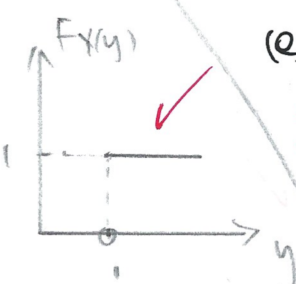
$\therefore$ Pr

$y \backslash x$	$\frac{\pi}{2}$
1	1

$\therefore Pr(Y = 1) = 1$

$\therefore F_Y(y) = \begin{cases} 0, & \text{other} \\ 1, & y \in [1, +\infty) \end{cases}$



3.(a).

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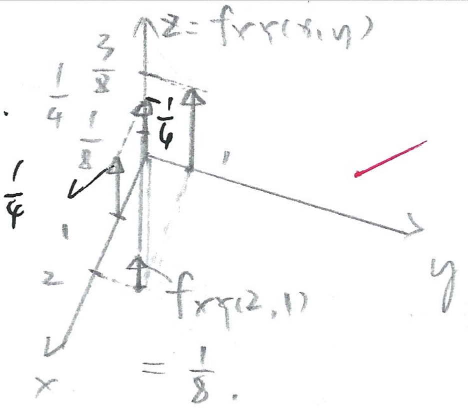

$f_{XY}(2,1) = \frac{1}{8}$.

(b). ~~$f_{XY}(x,y)$~~ $Pr: XY$

$y \backslash x$	0	1	2
0	$\frac{1}{4}$	$\frac{1}{4}$	0
1	$\frac{3}{8}$	0	$\frac{1}{8}$

$\therefore f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy$

$= \frac{5}{8}\delta(x) + \frac{1}{4}\delta(x-1) + \frac{1}{8}\delta(x-$

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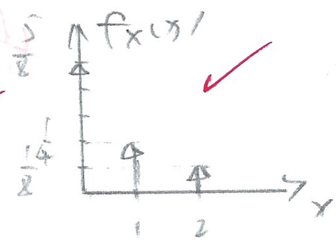

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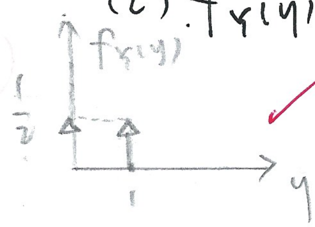
(c) $f_Y(y) = \frac{1}{2}\delta(y) + \frac{1}{2}\delta(y-1)$



(d) $E(XY) = 0 \cdot \frac{1}{4} + 0 \cdot \frac{1}{4} + 2 \cdot \frac{1}{8} + 0 \cdot \frac{3}{8} = \frac{1}{4}$

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(e) $E(X) = 0 \cdot \frac{5}{8} + 1 \cdot \frac{1}{4} + 2 \cdot \frac{1}{8} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

5

$E[Y] = \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 1 = \frac{1}{2}$

$\therefore E(X) \cdot E(Y) = E(XY)$

uncorrelated

(Shanpxin Qian)

3. (f) independent $\Leftrightarrow f_{XY}(x,y) = f_X(x) \cdot f_Y(y)$

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$\because \; f_{XY}(x,y) = \frac{1}{4}\delta(x-0, y-0) + \frac{3}{8}\delta(x-0, y-1)$
$+ \frac{1}{4}\delta(x-1, y-1) + \frac{1}{8}\delta(x-2, y-1)$

$f_X(x) \cdot f_Y(y) = \left[\frac{1}{2}\delta(y) + \frac{1}{2}\delta(y-1)\right] \cdot \left[\frac{5}{8}\delta(x) + \frac{1}{4}\delta(x-1) + \frac{1}{8}\delta(x-2)\right]$

$= \frac{5}{16}\delta(x-0, y-0) + \frac{1}{8}\delta(x-1, y-0)$
$+ \frac{1}{16}\delta(x-2, y-0)$
$+ \frac{5}{16}\delta(x-0, y-1) + \frac{1}{8}\delta(x-1, y-1)$
$+ \frac{1}{16}\delta(x-2, y-1)$

$\therefore \; f_X(x) \cdot f_Y(y) \neq f_{XY}(x,y)$

$\therefore$ not statistically independent.

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statistic independent

$\Pr(Y=0 \mid X=0) \quad \Pr(Y=0)$
$= \frac{1}{4} \qquad = \frac{1}{2}$

$\because \Pr(Y=0 \mid X=0) \neq \Pr(Y=0)$

not