ECE 581 Homework 8

Due Thursday 5 AM October 22, 2015, (10 hmwk points total) Show work.
Electronic Submission – Please submit via "Assignment" under Sakai

Problem 8-1 (5 points Total) Given the following random process $X(t) = A + \cos(\omega t + \theta)$ where θ is a uniform random variable in the range $-\pi$ to π and A is a constant. Determine analytical expressions for the following:
(a) (2 point) The mean of $X(t)$. Sketch and completely label it.
(b) (3 points) The autocorrelation function of $X(t)$. Sketch and completely label it.
(c) (1 points The autocovariance function of $X(t)$. Sketch and completely label it.

Problem 8-2 (5 points Total) Consider the random process $X(t) = A\cos(\omega t + \theta)$ where A and ω are known and the random variable θ has the pdf $f(\theta) = \frac{1}{2\pi}$ for $0 \le \theta \le 2\pi$ and 0 otherwise.

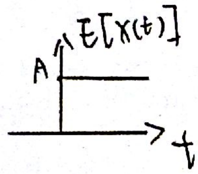
(a) (2 points) Derive an analytical expression for the first-order probability density function, $f_X(x;t)$ of this random process and sketch and completely label it.
(b) (2 points) Derive the expected value, $E[X(t)]$ of this random process? Sketch and completely label it..
(c) (1 point) Derive the variance, Var$[X(t)]$, of this random process. Sketch and completely label it.

8-1: (a) $E[X(t)] = E[A+\cos(wt+\theta)]$

$= A + E[\cos(wt+\theta)]$

$= A + \int_{-\pi}^{\pi} \frac{1}{2\pi}\cos(wt+\theta)\,d\theta$

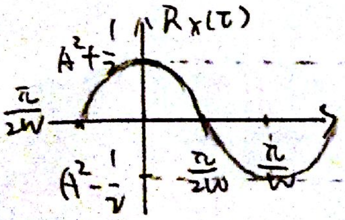
$= A + \frac{1}{2\pi}\sin(\theta+wt)\Big|_{-\pi}^{\pi} = A$



(b) $E[X(t_1)X(t_2)] = E\{[A+\cos(wt_1+\theta)][A+\cos(wt_2+\theta)]\}$

$= A^2 + E[\frac{1}{2}\cos w\tau + \frac{1}{2}\cos[w(t_1+t_2)+2\theta]]$

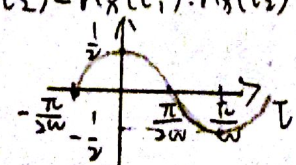
$= A^2 + \frac{1}{2}\cos w\tau$, $\tau = t_1 - t_2$



(c) Autocovariance

$= R_X(t_1,t_2) - \mu_X(t_1)\cdot\mu_X(t_2)$

$= \frac{1}{2}\cos w\tau$



8-2: (a) $X(t) = A\cos(wt+\theta)$

$wt = -\varphi + 2k\pi \Rightarrow X(t) = A\cos(\theta-\varphi)$

$\varphi \in [0, 2\pi]$

$\therefore P(X \le x) = P[A\cos(\theta-\varphi) \le x]$

$\because f(\theta) = \frac{1}{2\pi}$, $\theta \in [0, 2\pi] \Rightarrow$

$\boxed{\text{when } x \in [A\cos\varphi, A]}$

$P(X\le x) = P[0 \le \theta \le \varphi - \arccos(\frac{x}{A})] + P[\varphi + \arccos(\frac{x}{A}) \le \theta \le 2\pi]$

$\Rightarrow f_X(x;t) = \frac{dP(X\le x)}{dx} = \frac{1}{\pi}\frac{1}{\sqrt{A^2-x^2}}$

$\boxed{\text{when } x \in [-A, A\cos\varphi]}$

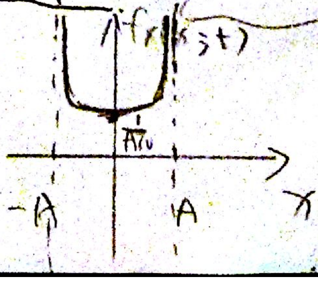
$P(X\le x) = P[\varphi + \arccos(\frac{x}{A}) \le \theta \le \varphi - \arccos(\frac{x}{A}) + 2\pi]$

$\Rightarrow f_X(x;t) = \frac{dP(X\le x)}{dx} = \frac{1}{\pi}\frac{1}{\sqrt{A^2-x^2}}$

$\Rightarrow f_X(x;t) = \begin{cases} \frac{1}{\pi}\frac{1}{\sqrt{A^2-x^2}}, & x \in [-A, A] \\ 0, & \text{else.} \end{cases}$

basically, $f_X(x;t) = \boxed{\text{the number of joint point on } X(t)} * f_\theta[h(x)]/|h'(x)|$

$\boxed{h(x) = \arccos(\frac{x}{A}) - wt}$.



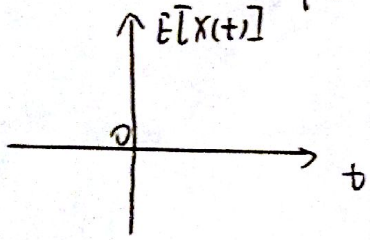
(b). $X(t) = A\cos(wt+\theta)$

$$f_\theta(\theta) = \begin{cases} \frac{1}{2\pi}, & \theta\in[0,2\pi] \\ 0, & else \end{cases}$$

$$E[X(t)] = E[A\cos(wt+\theta)]$$

$$= \int_0^{2\pi} A\cdot\cos(wt+\theta)\cdot\frac{1}{2\pi}\,d\theta$$

$$= \frac{A}{2\pi}\sin(wt+\theta)\Big|_0^{2\pi} = 0$$



(c). Autocovariance

$$= R_X(t_1,t_2) - m_X(t_1)\,m_X(t_2)$$

$$= R_X(t_1,t_2)$$

$$= E[A\cos(wt_1+\theta)\,A\cos(wt_2+\theta)]$$

$$= A^2E[\cos(wt_1+\theta)\cos(wt_2+\theta)]$$

$$= A^2\frac{1}{2}\cdot E[\cos w\tau + \cos[w(t_1+t_2)+2\theta]]$$

$$= \frac{A^2}{2}\cos w\tau\ ,\quad \tau \triangleq t_1-t_2.$$