

Dehn Function Complexity

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"In space, no one can hear you think."

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1 Dehn Function Complexity

1.1 Introduction to Dehn Function Complexity

The study of Dehn function complexity represents one of the most fascinating intersections of algebra, geometry, and computational theory in modern mathematics. At its core, this field addresses a fundamental question: how can we measure the computational difficulty of solving equations in algebraic structures? Dehn functions provide a precise mathematical framework for answering this question, particularly in the context of group theory, where they quantify the complexity of determining whether two expressions represent the same element in a group presented by generators and relations. This concept has evolved from its topological origins to become a central tool in geometric group theory, offering deep insights into the structure of mathematical objects and the computational problems they present.

To understand Dehn function complexity, we must first grasp the fundamental concepts that underpin this area of study. At its heart lies the word problem in group theory, a decision problem that asks whether two words composed of group generators represent the same element in the group. This seemingly simple question conceals remarkable complexity, and its resolution has profound implications across mathematics. A finitely presented group is defined by a set of generators and a set of relations—equations that specify when certain combinations of generators equal the identity element. For instance, the cyclic group of order n can be presented with a single generator a and the relation $a^n = 1$. The word problem then asks: given two words in the generators (such as a^3 and a^5 in the cyclic group of order 8), can we determine whether they represent the same group element?

The Dehn function provides a quantitative measure of the difficulty of solving the word problem. Formally, for a finitely presented group, the Dehn function $\delta(n)$ is defined as the smallest integer such that any word w of length at most n that represents the identity element can be reduced to the empty word by applying relations from the presentation, using at most $\delta(n)$ applications of the relations. Intuitively, it measures the “area” needed to fill a closed loop in the Cayley graph of the group, or equivalently, the minimal number of relations required to demonstrate that a word equals the identity. Consider a simple example: in the free group on two generators a and b with no relations, the word problem is straightforward. A word equals the identity only if it is already the empty word, as there are no relations to apply. Thus, the Dehn function is zero for $n > 0$. In contrast, for the cyclic group of order n with generator a and relation $a^n = 1$, reducing a word like $a^{(2n)}$ to the identity requires applying the relation $a^n = 1$ twice, demonstrating how the Dehn function captures the computational effort needed.

The origins of Dehn function complexity trace back to the groundbreaking work of Max Dehn in the early twentieth century. Dehn, a German mathematician who made significant contributions to topology, group theory, and geometry, posed three fundamental decision problems in his 1911 paper “Über unendliche diskontinuierliche Gruppen” (On infinite discontinuous groups). These problems—now known as Dehn’s problems—would shape the development of combinatorial group theory for decades to come. The first of these problems, the word problem, asks whether there exists an algorithm to determine if two words in the generators of a finitely presented group represent the same element. The conjugacy problem seeks an

algorithm to determine if two elements are conjugate in the group. The isomorphism problem, the most challenging of the three, asks whether there exists an algorithm to determine if two finite presentations define isomorphic groups. These problems emerged from Dehn's work on the topology of surfaces, where fundamental groups of surfaces provided natural examples of finitely presented groups.

Dehn's initial motivation came from his study of surface groups and their relationship to topology. He recognized that understanding the combinatorial structure of these groups would provide insights into the topological properties of the surfaces themselves. In particular, Dehn solved the word problem for surface groups, developing algorithms that relied on geometric properties of the corresponding surfaces. The evolution from these topological questions to algebraic and computational formulations represents a fascinating journey through mathematical history. In the 1930s and 1940s, mathematicians began to formalize the concept of computation itself, leading to the development of computability theory by Turing, Church, Kleene, and others. This new framework allowed for a more rigorous analysis of Dehn's problems. A pivotal moment came in 1955 when Pyotr Novikov proved that the word problem is undecidable in general—there exists no algorithm that can solve the word problem for arbitrary finitely presented groups. This result, later strengthened by William Boone in 1959, demonstrated the inherent complexity of Dehn's problems and motivated the search for quantitative measures of this complexity.

The study of Dehn functions has grown to occupy a central position in modern mathematics, bridging multiple disciplines and offering profound insights into the structure of mathematical objects. Perhaps most significantly, Dehn functions serve as crucial invariants that capture essential properties of groups, allowing mathematicians to classify and distinguish different algebraic structures based on their computational complexity. In geometric group theory, a field that views groups as geometric objects through their Cayley graphs, Dehn functions reveal deep connections between algebraic properties and geometric behavior. The growth rate of a Dehn function reflects large-scale geometric properties of the group, such as curvature and dimensionality. This connection has proven to be a powerful tool, enabling mathematicians to translate between algebraic statements and geometric intuition.

For example, groups with linear Dehn functions—those where the area grows at most linearly with the length of the word—are precisely the hyperbolic groups, as characterized by Mikhail Gromov in his seminal 1987 paper. These

1.2 Mathematical Foundations of Dehn Functions

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1.3 2. Mathematical Foundations of Dehn Functions

Building upon our introduction to Dehn function complexity, we now delve into the rigorous mathematical framework that underpins this fascinating area of study. The previous section touched upon Gromov’s characterization of hyperbolic groups through their linear Dehn functions, but to fully appreciate this profound connection, we must first establish the precise mathematical definitions and concepts that form the foundation of Dehn function theory. This section will provide the necessary mathematical machinery to understand how Dehn functions measure the computational complexity of the word problem in finitely presented groups, bridging the intuitive understanding we’ve developed with the formal rigor required for deeper exploration.

1.3.1 2.1 Formal Definitions and Notation

At the heart of Dehn function theory lies a precise mathematical definition that captures the intuition of measuring the “area” needed to demonstrate that a word represents the identity element in a group. Let us begin by establishing the formal definition of the Dehn function. Given a finitely presented group $G = \langle S \mid R \rangle$, where S is a finite set of generators and R is a finite set of relators (words in S that equal the identity in G), the Dehn function $\delta_G: \mathbb{N} \rightarrow \mathbb{N}$ is defined as follows:

For a positive integer n , $\delta_G(n)$ is the smallest integer such that any word w in the generators $S \subseteq S^{\pm 1}$ of length at most n that represents the identity element in G can be reduced to the empty word by applying relations from R , using at most $\delta_G(n)$ applications of these relations. More formally, if $w = 1$ in G with $|w| \leq n$, then there exist words u_1, u_2, \dots, u_k and relators $r_1, r_2, \dots, r_k \in R \subseteq R^{\pm 1}$ such that $w = u_1 r_1 u_2 r_2 \dots u_k r_k u_k^{-1}$, where $k \leq \delta_G(n)$.

This definition captures the computational difficulty of solving the word problem by quantifying the minimal number of relations needed to demonstrate that a word equals the identity. To illustrate this concept more concretely, consider the infinite cyclic group $G = \langle a \mid \rangle$, which is the free group on one generator with no relations. In this group, the only word representing the identity is the empty word itself, so $\delta_G(n) = 0$ for all $n > 0$. In contrast, consider the cyclic group of order m , $G = \langle a \mid a^m = 1 \rangle$. Here, any word representing the identity must have exponents summing to a multiple of m . For example, the word $a^{(2m)}$ can be reduced to the identity by applying the relation $a^m = 1$ twice, demonstrating that $\delta_G(2m) \leq 2$. In fact, for this group, the Dehn function is linear: $\delta_G(n) \leq \lceil n/m \rceil$.

The formal definition above is just one of several equivalent formulations of the Dehn function. An alternative geometric definition considers the Cayley graph of the group G with respect to the generating set S . In this graph, vertices represent elements of G , and edges connect vertices that differ by multiplication by a generator or its inverse. A word w representing the identity corresponds to a closed loop in this graph. The area of this loop is defined as the minimal number of 2-cells needed to fill the loop in the Cayley complex (a 2-dimensional complex obtained by attaching 2-cells to the Cayley graph corresponding to the relators). The Dehn function $\delta_G(n)$ is then the maximum area over all closed loops of length at most n .

This geometric perspective reveals an important connection to isoperimetric problems in geometry, where one seeks to find the minimal area needed to fill a loop of given length. Indeed, Dehn functions can be viewed as discrete isoperimetric functions, measuring the complexity of filling loops in the discrete geometric structure defined by the group.

Yet another equivalent formulation comes from the combinatorial perspective of van Kampen diagrams, which we will explore in detail in subsection 2.3. In this framework, the Dehn function measures the maximal number of 2-cells in a van Kampen diagram with boundary of length at most n that represents the identity.

Notation conventions in the field of Dehn functions deserve careful attention. When the group G is clear from context, we often write $\delta(n)$ instead of $\delta_G(n)$. Since Dehn functions are typically considered up to equivalence (as we will discuss in subsection 2.4), we use the notation $\delta(n) \sqsubseteq f(n)$ to indicate that $\delta(n) \leq Cf(n) + C$ for some constant $C > 0$. Similarly, $\delta(n) \approx f(n)$ means that $\delta(n) \sqsubseteq f(n)$ and $f(n) \sqsubseteq \delta(n)$, indicating that the functions have the same growth rate up to multiplicative and additive constants.

It is worth noting that the Dehn function depends on the choice of finite presentation for the group. Different presentations of the same group may yield different Dehn functions. However, as we will see in subsection 2.4, these functions are equivalent in a precise sense, capturing the same underlying computational complexity.

1.3.2 2.2 Group Presentations and Relations

To fully appreciate the Dehn function, we must understand the concept of group presentations and the role they play in defining group structure. A group presentation provides a concise way to specify a group by giving its generators and the relations that these generators satisfy. Formally, a presentation consists of a set S of generators and a set R of relators, which are words in the generators and their inverses. The group defined by this presentation, denoted $\langle S \mid R \rangle$, is the quotient of the free group on S by the normal closure of R . In other words, it is the most general group generated by S in which all elements of R equal the identity.

The choice of presentation significantly impacts the Dehn function, as different presentations may make it easier or harder to demonstrate that a word equals the identity. Consider, for example, the infinite cyclic group \mathbb{Z} . The standard presentation $\langle a \mid \rangle$ has no relations, resulting in a Dehn function $\delta(n) = 0$ for $n > 0$, as mentioned earlier. However, we could also present \mathbb{Z} as $\langle a, b \mid a = b, b = a^{-1} \rangle$. In this presentation, the word $a^{-1}b$ equals the identity, but demonstrating this requires applying both relations, showing how the presentation choice affects the computational complexity.

To illustrate further, let us examine several important classes of groups and their standard presentations:

Free groups: The free group F_k on k generators has the presentation $\langle a_1, a_2, \dots, a_k \mid \rangle$ with no relations. In free groups, the only word representing the identity is the empty word (after removing adjacent inverses), so the Dehn function is $\delta(n) = 0$ for $n > 0$. This reflects the fact that the word problem is trivial in free groups.

Free abelian groups: The free abelian group \mathbb{Z}^d of rank d has the presentation $\langle a_1, a_2, \dots, a_d \mid [a_i, a_j] = 1 \text{ for all } 1 \leq i < j \leq d \rangle$, where $[a_i, a_j]$ denotes the commutator $a_i a_j a_i^{-1} a_j^{-1}$. In this group, the Dehn

function is quadratic: $\delta(n) \approx n^2$. This quadratic growth reflects the fact that demonstrating that two words are equal may require moving generators past each other, creating intermediate terms of quadratic length.

Finite groups: For a finite group G with presentation $\langle S \mid R \rangle$, the Dehn function is bounded by a constant, since there are only finitely many group elements, and thus the maximum area needed to demonstrate that a word equals the identity is finite. However, the precise bound depends on the presentation and can be quite large even for simple groups.

Surface groups: The fundamental group of a closed orientable surface of genus $g \geq 2$ has the presentation $\langle a_1, b_1, a_2, b_2, \dots, a_g, b_g \mid [a_1, b_1][a_2, b_2] \dots [a_g, b_g] = 1 \rangle$. These groups have linear Dehn functions, $\delta(n) \approx n$, reflecting their hyperbolicity.

The role of relators in defining group structure cannot be overstated. Each relator imposes a constraint on the generators, forcing certain combinations to equal the identity. The interplay between these constraints determines the structure of the group and, consequently, the complexity of its word problem. A presentation with many relators might make it easier to demonstrate that words equal the identity (potentially leading to a smaller Dehn function), but it also makes the presentation more complicated. Conversely, a presentation with few relators might be simpler but could result in a larger Dehn function.

One remarkable example that illustrates the impact of presentation choice is the lamplighter group, which can be presented as $\langle a, t \mid t^{-1}a^2t = a^2, (ta^2t^{-1}a)^2 = 1 \rangle$. With this presentation, the Dehn function is exponential. However, with a different presentation using infinitely many generators (which is not finite), the word problem becomes easy. This demonstrates how the choice of finite generators and relations can fundamentally alter the computational complexity of the word problem.

Another important concept related to group presentations is that of Tietze transformations, which provide a way to modify a presentation without changing the isomorphism class of the group. These transformations include adding or removing redundant generators or relators. While Tietze transformations preserve the group up to isomorphism, they can affect the Dehn function. However, as we will see in subsection 2.4, the resulting Dehn functions are always equivalent in a precise sense, capturing the same underlying computational complexity.

1.3.3 2.3 van Kampen Diagrams

Van Kampen diagrams provide a powerful combinatorial tool for visualizing and analyzing the word problem in finitely presented groups. These diagrams offer a geometric representation of the process of demonstrating that a word equals the identity through applications of the group relations. Named after the Dutch mathematician Egbert van Kampen, who introduced them in the 1930s, these diagrams have become an indispensable tool in the study of Dehn functions and the word problem.

A van Kampen diagram for a group presentation $\langle S \mid R \rangle$ is a finite, simply connected, planar 2-complex with a distinguished boundary path. The 1-skeleton of the diagram is a labeled graph where each edge is oriented and labeled by an element of $S \cup S^{-1}$. The 2-cells of the diagram are attached along closed paths,

with each 2-cell corresponding to a relator in $R \sqcup R^{-1}$. Specifically, if a 2-cell is attached along a path with edge labels (reading in order) forming the word r , then r must be equal to some element of $R \sqcup R^{-1}$.

The boundary of a van Kampen diagram is the path that forms its outer edge. When we read the labels along this boundary path (in the counterclockwise direction, following the orientation of edges), we obtain a word in the generators. The fundamental property of van Kampen diagrams is that this boundary word equals the identity in the group defined by the presentation. Conversely, any word that equals the identity can be represented by some van Kampen diagram. This establishes a one-to-one correspondence between words representing the identity and van Kampen diagrams, up to certain equivalence relations.

To construct a van Kampen diagram for a given word w that equals the identity, we begin by drawing a path with edges labeled by the generators in w . We then attach 2-cells corresponding to relators to fill in the diagram, ensuring that the boundary remains simply connected. Each 2-cell represents an application of a relation in the group presentation. The area of a van Kampen diagram is defined as the number of 2-cells it contains. This area corresponds exactly to the number of relation applications needed to demonstrate that the boundary word equals the identity.

Consider a simple example: the group $G = \langle a, b \mid aba^{-1}b^{-1} = 1 \rangle$, which is the free abelian group on two generators. The word $w = abba^{-1}b^{-1}$ equals the identity in this group. To construct a van Kampen diagram for w , we first draw the boundary path with edges labeled by the generators in w . We then attach a 2-cell corresponding to the relator $aba^{-1}b^{-1}$. However, this relator does not directly match the boundary word, so we need additional 2-cells to fill the diagram. In this case, we might need to attach two 2-cells, demonstrating that the area needed to fill this word is 2.

The relationship between van Kampen diagrams and Dehn functions is direct and profound. The Dehn function $\delta(n)$ can be characterized as the maximum area of a van Kampen diagram with boundary of length at most n . In other words, it measures the worst-case complexity of filling a loop of length n with 2-cells corresponding to the relators. This geometric perspective provides a powerful intuition for understanding the computational complexity of the word problem.

One of the most important properties of van Kampen diagrams is the ability to perform diagrammatic reductions. Given a van Kampen diagram, we can often reduce its area while preserving the boundary word. For example, if two 2-cells share a common edge with inverse labels, we can remove both 2-cells and the shared edge, reducing the area by 2. The minimal area of a diagram with given boundary word corresponds to the minimal number of relation applications needed to demonstrate that the word equals the identity.

Van Kampen diagrams also reveal connections between the algebraic properties of groups and their geometric structure. For instance, in hyperbolic groups (which have linear Dehn functions), van Kampen diagrams have a particularly nice property: they are “thin” in the sense that any internal path of length n is within a bounded distance of the boundary. This thinness condition reflects the negative curvature of these groups and is intimately connected to their linear isoperimetric inequality.

Another powerful application of van Kampen diagrams is in proving results about Dehn functions. For example, to show that a group has a quadratic Dehn function, one might demonstrate that any van Kampen

diagram can be transformed into a diagram with area bounded by a quadratic function of the boundary length. This often involves combinatorial arguments about the structure of diagrams and the ways in which 2-cells can be arranged.

The study of van Kampen diagrams has led to numerous important results in geometric group theory. For instance, the Greendlinger lemma, a fundamental result in small cancellation theory, states that in certain presentations satisfying small cancellation conditions, any van Kampen diagram with non-trivial boundary must contain a 2-cell whose boundary is mostly contained in the boundary of the diagram. This result has profound implications for the structure of groups and their Dehn functions.

1.3.4 2.4 Invariance and Basic Properties

The Dehn function, as defined for a specific finite presentation of a group, depends on the choice of generators and relations. However, a crucial aspect of Dehn function theory is that this dependence is controlled in a precise sense, allowing us to speak of “the” Dehn function of a group, up to equivalence. This invariance property ensures that the Dehn function captures an intrinsic property of the group itself, rather than merely an artifact of its presentation.

To formalize this notion of equivalence, we say that two functions $f, g: \mathbb{N} \rightarrow \mathbb{N}$ are equivalent, denoted $f \approx g$, if there exist positive constants A, B, C such that for all $n \in \mathbb{N}$, $f(n) \leq Ag(Bn) + C$ and $g(n) \leq Af(Bn) + C$. This equivalence relation captures the idea that the functions have the same growth rate up to multiplicative and additive constants. When we speak of the Dehn function of a group, we typically mean the equivalence class of functions obtained from all finite presentations of the group.

The fundamental invariance result states that if two finite presentations define isomorphic groups, then their Dehn functions are equivalent in the sense defined above. This theorem, first established by Gromov and later refined by other mathematicians, ensures that the Dehn function is indeed an invariant of the group, not just of its presentation. The proof relies on the fact that changing the presentation via Tietze transformations affects the Dehn function in a controlled way, preserving its equivalence class.

Beyond this basic invariance property, Dehn functions exhibit several important behaviors under group operations and homomorphisms. For instance, if H is a subgroup of G , then the Dehn function of H is bounded above by the Dehn function of G . This follows from the fact that any word representing the identity in H also represents the identity in G , and thus can be filled with at most the area required in G . However, the Dehn function of a subgroup can be strictly smaller than that of the ambient group. For example, the free group of rank 2 is a subgroup of the free abelian group of rank 2, but the former has a linear Dehn function while the latter has a quadratic one.

When considering group homomorphisms, the situation becomes more nuanced. If $\phi: G \rightarrow H$ is a homomorphism, then in general, there is no direct relationship between the Dehn functions of G and H . However, if ϕ is injective (i.e., G is isomorphic to a subgroup of H), then as mentioned above, the Dehn function of G is bounded above by that of H . For surjective homomorphisms, the relationship can go in either direction, depending on the kernel of the homomorphism.

Free products provide another interesting case. If G and H are groups with Dehn functions δ_G and δ_H respectively, then the free product $G * H$ has Dehn function $\delta(n) = \max(\delta_G(n), \delta_H(n))$. This reflects the fact that in the free product, elements from G and H do not interact except trivially, so the complexity of the

1.4 Classification of Dehn Functions

word problem is determined by the more complex of the two factors. This property extends to more general amalgamated products and HNN extensions, though the relationships become more intricate when non-trivial amalgamation is involved.

1.5 3. Classification of Dehn Functions

With the mathematical foundations firmly established, we now turn to the rich taxonomy of Dehn functions and the groups that exhibit them. The classification of Dehn functions represents one of the most beautiful aspects of geometric group theory, revealing profound connections between algebraic properties, geometric structures, and computational complexity. Dehn functions can be organized into a hierarchy based on their growth rates, ranging from the simplest linear functions to extraordinarily complex super-exponential ones. Each class in this hierarchy corresponds to distinct geometric and algebraic properties of groups, creating a remarkable bridge between seemingly disparate areas of mathematics.

1.5.1 3.1 Linear Dehn Functions

Linear Dehn functions represent the simplest and most well-behaved class in the Dehn function hierarchy. A group G has a linear Dehn function if $\delta_G(n) \approx n$, meaning there exist positive constants A, B, C such that $\delta_G(n) \leq An + B$ for all $n \geq C$. This linear growth indicates that the area needed to fill a loop of length n grows at most proportionally with n , reflecting a highly efficient process for solving the word problem.

The significance of linear Dehn functions was profoundly illuminated by Mikhael Gromov in his landmark 1987 paper “Hyperbolic Groups.” Gromov established that a finitely presented group has a linear Dehn function if and only if it is hyperbolic, a concept that has since revolutionized geometric group theory. Hyperbolic groups, in Gromov’s sense, are groups that exhibit negative curvature on a large scale, analogous to how surfaces of constant negative curvature behave. This characterization provides a deep connection between the computational complexity of the word problem and the geometric properties of the group.

The intuition behind this connection can be understood through the lens of van Kampen diagrams. In a hyperbolic group, van Kampen diagrams exhibit a “thin triangles” property: any geodesic triangle in the Cayley graph is δ -thin for some fixed $\delta > 0$, meaning that each side is contained in the δ -neighborhood of the other two. This thinness condition implies that van Kampen diagrams cannot contain large “empty” regions, forcing their area to grow linearly with their boundary length. Conversely, if all van Kampen diagrams satisfy this thinness condition, then the group must be hyperbolic.

Free groups provide the simplest examples of groups with linear Dehn functions. For a free group F_k on k generators with the standard presentation $\langle a_1, a_2, \dots, a_k \mid \emptyset \rangle$, the only word representing the identity is the empty word (after canceling adjacent inverses), so the Dehn function is identically zero for $n > 0$. This trivial case already hints at the hyperbolic nature of free groups, which serve as prototypical examples of hyperbolic groups.

Fundamental groups of closed hyperbolic manifolds offer more geometric examples. Consider the fundamental group of a closed orientable surface of genus $g \geq 2$, which has the presentation $\langle a_1, b_1, a_2, b_2, \dots, a_g, b_g \mid [a_1, b_1][a_2, b_2] \dots [a_g, b_g] = 1 \rangle$, where $[a_i, b_i]$ denotes the commutator $a_i b_i a_i^{-1} b_i^{-1}$. These groups have linear Dehn functions, reflecting the constant negative curvature of the corresponding hyperbolic surfaces. The linear bound arises from the geometric fact that any closed loop on such a surface can be filled by a disk whose area is proportional to the length of the loop.

Another important class of groups with linear Dehn functions consists of groups that act properly discontinuously and cocompactly on hyperbolic space. For example, Coxeter groups with certain properties, particularly those arising from reflections in hyperbolic space, exhibit linear Dehn functions. These groups play crucial roles in various areas of mathematics, from geometry to Lie theory.

The linear isoperimetric inequality enjoyed by hyperbolic groups has profound algorithmic implications. For hyperbolic groups, the word problem can be solved in linear time by Dehn's algorithm, a procedure that systematically reduces the length of a word by applying relators corresponding to "shortcuts" in the Cayley graph. Specifically, if a group presentation contains all relators of length less than some constant K (a so-called Dehn presentation), then for any word w representing the identity, there exists a subword of w that is more than half of some relator, allowing w to be shortened. This process can be repeated until the empty word is reached, and the linear bound on the Dehn function ensures that this process terminates in linear time.

The historical development of our understanding of linear Dehn functions is itself fascinating. While Max Dehn had solved the word problem for surface groups in the early 20th century using geometric methods, it was Gromov's visionary work in the 1980s that unified these examples into the general framework of hyperbolic groups. Gromov's insight was to recognize that the thin triangles condition, which arises naturally in negative curvature, could be defined purely combinatorially for groups, leading to the abstract definition of hyperbolicity. This breakthrough opened the floodgates for a new era in geometric group theory, with linear Dehn functions serving as a central pillar of the theory.

1.5.2 3.2 Quadratic and Polynomial Dehn Functions

Moving beyond the linear realm, we encounter quadratic and polynomial Dehn functions, which constitute the next major class in the Dehn function hierarchy. A group G has a quadratic Dehn function if $\delta G(n) \approx n^2$, meaning the area needed to fill a loop of length n grows quadratically with n . More generally, G has a polynomial Dehn function of degree d if $\delta G(n) \approx n^d$ for some $d \geq 1$. These polynomial Dehn functions correspond to groups that exhibit "flat" or "Euclidean" geometry on a large scale, in contrast to the negative

curvature of hyperbolic groups.

The simplest examples of groups with quadratic Dehn functions are the free abelian groups. Consider the free abelian group \mathbb{Z}^2 with the standard presentation $\langle a, b \mid [a, b] = 1 \rangle$, where $[a, b] = aba^{-1}b^{-1}$ is the commutator. To understand why this group has a quadratic Dehn function, consider the word $w_n = a^n b^n a^{-n} b^{-n}$, which equals the identity in \mathbb{Z}^2 . A van Kampen diagram for this word resembles a grid, with n applications of the commutator relation needed to “untangle” the a ’s and b ’s, resulting in an area of approximately n^2 . This quadratic growth reflects the two-dimensional Euclidean geometry of \mathbb{Z}^2 , where filling a loop requires area proportional to the square of its length.

More generally, the free abelian group \mathbb{Z}^d of rank d has a Dehn function of degree d , specifically $\delta(n) \approx n^d$. This corresponds to the d -dimensional Euclidean geometry, where the isoperimetric inequality states that the area needed to fill a loop grows as the d -th power of its length. The proof of this fact involves constructing van Kampen diagrams that resemble higher-dimensional grids, with the number of 2-cells growing polynomially with the boundary length.

Nilpotent groups provide another important class of groups with polynomial Dehn functions. A group G is nilpotent if its lower central series terminates at the trivial subgroup after finitely many steps. The free nilpotent group of class c on k generators has a Dehn function of degree $c+1$. For example, the Heisenberg group, which is nilpotent of class 2, has a cubic Dehn function. The Heisenberg group can be presented as $\langle a, b, c \mid [a, b] = c, [a, c] = 1, [b, c] = 1 \rangle$, and its cubic Dehn function reflects the three-dimensional nature of its geometry.

The connection between polynomial Dehn functions and geometric properties was significantly advanced by the theory of CAT(0) groups, developed by Mikhail Gromov and later refined by Bridson and Haefliger. A CAT(0) space is a geodesic metric space where triangles are no fatter than comparison triangles in Euclidean space. Groups that act properly discontinuously and cocompactly on CAT(0) spaces are called CAT(0) groups, and these groups always admit polynomial Dehn functions. Conversely, while not all groups with polynomial Dehn functions are CAT(0) groups, there is a deep connection between polynomial isoperimetric inequalities and non-positive curvature.

Automatic groups, introduced by Thurston and developed by Epstein et al., provide another class of groups with polynomial Dehn functions. An automatic group is a group equipped with a regular language of normal forms and certain fellow traveler properties that allow for efficient computation. All automatic groups have quadratic Dehn functions, though not all groups with quadratic Dehn functions are automatic. Examples of automatic groups include hyperbolic groups, braid groups, and Coxeter groups, demonstrating the breadth of this class.

The combable groups, a generalization of automatic groups introduced by Albrighton and Mosher, also have polynomial Dehn functions. A combable group admits a combing—a set of paths in the Cayley graph starting at the identity and reaching each group element, satisfying certain regularity conditions. The polynomial bound on the Dehn function arises from the ability to efficiently fill loops using these combing paths.

The study of polynomial Dehn functions has led to several deep results connecting algebraic and geometric properties. One such result, due to Gersten and Short, states that a group has a quadratic Dehn function

if and only if it satisfies a quadratic isoperimetric inequality in its Cayley graph. This provides a purely geometric characterization of groups with quadratic Dehn functions, analogous to Gromov's characterization of hyperbolic groups through linear isoperimetric inequalities.

Another important result, due to Gromov and Ol'shanskii, states that the degree of the polynomial Dehn function of a nilpotent group is determined by its nilpotency class and the number of generators. Specifically, the free nilpotent group of class c on k generators has a Dehn function of degree $c+1$. This precise relationship between algebraic structure and geometric complexity is one of the most beautiful aspects of the theory.

The algorithmic implications of polynomial Dehn functions are significant. For groups with polynomial Dehn functions of degree d , the word problem can be solved in time $O(n^d)$, where n is the length of the input word. While this is less efficient than the linear time algorithms available for hyperbolic groups, it is still tractable compared to the exponential or super-exponential time requirements for groups with higher Dehn function complexities.

1.5.3 3.3 Exponential and Super-exponential Dehn Functions

As we ascend the hierarchy of Dehn function complexities, we encounter the dramatic realm of exponential and super-exponential growth, where the computational complexity of solving the word problem becomes formidable. A group G has an exponential Dehn function if $\delta G(n) \approx e^n$, meaning the area needed to fill a loop of length n grows exponentially with n . More generally, G has a super-exponential Dehn function if $\delta G(n)$ grows faster than any exponential function, such as $\delta G(n) \approx e^{n^2}$ or $\delta G(n) \approx n!$.

The first examples of groups with exponential Dehn functions were discovered in the context of solvable groups, particularly those that are solvable but not nilpotent. A group G is solvable if its derived series terminates at the trivial subgroup after finitely many steps. While nilpotent groups (a subclass of solvable groups) have polynomial Dehn functions, more general solvable groups can exhibit much more complex behavior.

The lamplighter group provides a classic example of a group with an exponential Dehn function. This group, which can be thought of as the wreath product $\mathbb{Z} \wr \mathbb{Z}$, consists of configurations of lamps (each on or off) at integer positions, along with a lamplighter who can move along the integer line and toggle lamps. With the standard presentation $\langle a, t \mid t^{-1}a^2t = a^2, (ta^2t^{-1}a)^2 = 1 \rangle$, the lamplighter group has an exponential Dehn function. To understand why, consider the word $w_n = (t^{-1}at a)^n$, which equals the identity. A van Kampen diagram for this word requires exponentially many 2-cells, reflecting the fact that the lamplighter must traverse an exponentially long path to return all lamps to their original state while maintaining the identity configuration.

More generally, the wreath product $\mathbb{Z} \wr \mathbb{Z}$ has a Dehn function that is exponential, while the wreath product $\mathbb{Z}^2 \wr \mathbb{Z}$ has a super-exponential Dehn function, specifically $\delta(n) \approx e^{\sqrt{n}}$. This pattern continues, with higher-dimensional wreath products exhibiting increasingly complex Dehn functions. These examples demonstrate how solvable groups can exhibit computational complexity far beyond that of nilpotent groups.

Another important class of groups with exponential Dehn functions consists of certain Baumslag-Solitar groups. The Baumslag-Solitar group $BS(m,n)$ has the presentation $\langle a, b \mid b^{-1}a^mb = a^n \rangle$. When $|m| \neq |n|$ and neither m nor n is ± 1 , these groups typically have exponential Dehn functions. For example, $BS(2,3) = \langle a, b \mid b^{-1}a^2b = a^3 \rangle$ has an exponential Dehn function. To see why, consider the word $w_n = b^{-n}a^{2^n}b^na^{2^n}$, which equals the identity. Demonstrating this requires applying the relation $b^{-1}a^2b = a^3$ repeatedly, and the number of applications grows exponentially with n .

The construction of groups with prescribed Dehn functions represents one of the most striking achievements in the field. In a remarkable series of papers, Birget, Ol'shanskii, Rips, and Sapir established that for any recursive function f that is at least linear, there exists a finitely presented group whose Dehn function is equivalent to f . This result, known as the Dehn function realization theorem, demonstrates the extraordinary diversity of Dehn function complexities and establishes that there is no upper bound on how complex Dehn functions can be.

The proof of this realization theorem involves sophisticated constructions that encode the computation of Turing machines into group presentations. Essentially, the relations in the presentation are designed to simulate the steps of a Turing machine, with the Dehn function reflecting the time complexity of the computation. This connection between computational complexity theory and Dehn functions reveals deep links between group theory and theoretical computer science.

Groups with super-exponential Dehn functions exhibit truly extraordinary complexity. One example is the group constructed by Rips, which has a Dehn function that grows faster than any primitive recursive function. This group is obtained as a quotient of a free group by relations that encode the halting problem for Turing machines, resulting in a Dehn function that essentially captures the time complexity of arbitrary computations.

Another example comes from the work of Bridson and Wise, who constructed groups with Dehn functions of the form $e^{\alpha n}$ for any $\alpha > 0$. These groups are obtained by carefully gluing together hyperbolic pieces in a way that forces the Dehn function to grow at the desired rate. The construction demonstrates the fine control that can be exerted over Dehn function complexity through geometric methods.

The algorithmic implications of exponential and super-exponential Dehn functions are profound. For groups with exponential Dehn functions, the word problem requires exponential time in the worst case, making it computationally intractable for large inputs. For groups with super-exponential Dehn functions, the situation is even more dire, with the word problem requiring time that grows faster than any exponential function. These groups represent examples of “wild” computational complexity, where even in principle, there are no efficient algorithms for solving the word problem.

1.5.4 3.4 Intermediate Growth and Un

1.6 Geometric Interpretations of Dehn Functions

The classification of Dehn functions reveals a remarkable spectrum of computational complexities, from the linear functions of hyperbolic groups to the super-exponential growth exhibited by certain pathological examples. Yet to fully appreciate the significance of these functions, we must shift our perspective from purely algebraic and computational considerations to the rich geometric framework that illuminates their deeper meaning. The geometric interpretations of Dehn functions represent one of the most profound developments in modern group theory, connecting algebraic properties of groups to the curvature, dimensionality, and large-scale structure of geometric spaces. This geometric perspective, pioneered by Mikhail Gromov and expanded by numerous mathematicians in the following decades, has transformed our understanding of groups as geometric objects and revealed unexpected connections between seemingly disparate areas of mathematics.

1.6.1 4.1 Geometric Group Theory Perspective

Geometric group theory emerged in the late 20th century as a powerful approach to studying groups by viewing them as geometric objects through their Cayley graphs. This perspective revolutionized the field by providing tools to analyze the “shape” of groups at large scales, revealing connections between algebraic properties and geometric behavior. Within this framework, Dehn functions emerge as natural geometric invariants that capture fundamental aspects of a group’s geometry.

The Cayley graph of a group G with respect to a finite generating set S provides the primary geometric object associated with G . This graph has vertices corresponding to elements of G , with edges connecting vertices g and gs for each $g \in G$ and $s \in S \sqcup S^{-1}$. The word metric, which defines the distance between two group elements as the length of the shortest word representing their difference, transforms the Cayley graph into a metric space. This metric space, while dependent on the choice of generators, captures essential geometric features of the group that are invariant under quasi-isometry, a notion we will explore in subsection 4.4.

From this geometric viewpoint, the word problem translates to a question about loops in the Cayley graph: given a closed path starting and ending at the identity, can we determine whether it bounds a disk composed of 2-cells corresponding to the relators? The Dehn function then measures the minimal area of such a disk, providing a quantitative measure of the difficulty of filling loops in this geometric structure.

This geometric interpretation reveals why different classes of groups exhibit characteristic Dehn function growth rates. Hyperbolic groups, with their linear Dehn functions, correspond to spaces of negative curvature where loops can be filled efficiently. Free abelian groups, with their polynomial Dehn functions, correspond to flat Euclidean spaces where filling loops requires area proportional to the appropriate power of the boundary length. Groups with exponential Dehn functions correspond to spaces with positive curvature or more complex geometric features that make filling loops computationally expensive.

The geometric group theory perspective also illuminates why certain group operations affect Dehn functions in predictable ways. For instance, when H is a subgroup of G , the inclusion map induces a geometric embedding of the Cayley graph of H into that of G (up to quasi-isometry). This embedding explains why the Dehn function of H is bounded above by that of G , as any loop in H can also be filled in G , possibly with additional area.

The work of Gromov in the 1980s established a profound connection between the algebraic properties of groups and their geometric behavior. His program of viewing groups as geometric objects led to the development of concepts like hyperbolicity, relatively hyperbolic groups, and asymptotic cones, each providing a different geometric lens through which to study groups. Within this framework, Dehn functions serve as fundamental invariants that bridge the algebraic and geometric aspects of group theory.

1.6.2 4.2 Isoperimetric Inequalities

The relationship between Dehn functions and isoperimetric inequalities represents one of the most fundamental connections in geometric group theory. In classical geometry, the isoperimetric problem asks for the relationship between the length of a closed curve and the area of the region it bounds. The classical isoperimetric inequality in the Euclidean plane states that among all closed curves of a given length, the circle encloses the maximum area. This inequality takes the form $A \leq L^2/(4\pi)$, where A is the area and L is the length of the curve.

Dehn functions can be viewed as discrete analogs of these classical isoperimetric inequalities, adapted to the combinatorial setting of group presentations. The discrete isoperimetric problem asks for the minimal number of 2-cells needed to fill a closed loop of given length in the Cayley complex of a group. The Dehn function precisely captures this relationship, providing a bound on the area in terms of the boundary length.

This connection becomes particularly clear when considering van Kampen diagrams, which serve as discrete analogs of the disks bounded by curves in the continuous setting. In a van Kampen diagram, the boundary path corresponds to the closed curve, while the 2-cells correspond to the area filling this curve. The Dehn function then provides an upper bound on the number of 2-cells needed to fill any boundary path of given length.

The comparison between continuous and discrete isoperimetric inequalities reveals fascinating parallels. In the Euclidean plane, the isoperimetric inequality is quadratic, reflecting the two-dimensional nature of the space. Similarly, the free abelian group \mathbb{Z}^2 , which can be viewed as a discrete analog of the Euclidean plane, has a quadratic Dehn function. This pattern extends to higher dimensions: in d -dimensional Euclidean space, the isoperimetric inequality relates the $(d-1)$ -dimensional measure of the boundary to the d -dimensional measure of the enclosed volume, with the relationship being polynomial of degree $d/(d-1)$. The free abelian group \mathbb{Z}^d exhibits a Dehn function of degree d , reflecting this dimensional relationship.

In spaces of constant negative curvature, such as the hyperbolic plane, the isoperimetric inequality is linear: the area needed to fill a curve of length L is proportional to L rather than L^2 . This linear isoperimetric inequality corresponds precisely to the linear Dehn functions of hyperbolic groups, reinforcing the connection

between negative curvature and efficient loop filling.

The study of isoperimetric inequalities in various geometric spaces has led to profound results connecting geometry, analysis, and group theory. For instance, the work of Varopoulos on isoperimetric inequalities for Lie groups and their discrete subgroups established precise relationships between the isoperimetric profile of a space and the behavior of random walks on that space. These results have implications for the Dehn functions of lattices in Lie groups, providing bounds based on the geometric properties of the ambient space.

Another important development comes from the work of Coulhon and Saloff-Coste, who studied isoperimetric inequalities in the context of analysis on fractals and spaces with non-integer dimension. Their work revealed that the isoperimetric profile of a space is closely related to its spectral properties, particularly the behavior of the Laplacian. This connection has implications for groups with intermediate Dehn function growth, suggesting deep links between computational complexity, geometric structure, and analytic properties.

1.6.3 4.3 Curvature and Dehn Functions

The relationship between curvature and Dehn functions represents one of the most beautiful aspects of geometric group theory, connecting local geometric properties to global computational complexity. In differential geometry, curvature measures how much a space deviates from being flat, with negative curvature corresponding to saddle-like behavior, zero curvature to flatness, and positive curvature to spherical behavior. This geometric concept has profound implications for the isoperimetric properties of spaces and, consequently, for the Dehn functions of groups.

In spaces of negative curvature, such as the hyperbolic plane, geodesics diverge exponentially, causing triangles to be “thin” in the sense that each side is contained in a neighborhood of the other two. This thin triangles property implies a linear isoperimetric inequality: any closed curve can be filled by a disk whose area is proportional to the length of the curve. The connection to group theory comes through Gromov’s characterization of hyperbolic groups as those satisfying a linear isoperimetric inequality, establishing that hyperbolic groups are precisely those with linear Dehn functions.

The hyperbolic plane itself provides a concrete example of this relationship. The fundamental group of a closed hyperbolic surface of genus at least 2 acts properly discontinuously and cocompactly on the hyperbolic plane, inheriting its negative curvature properties. These surface groups have linear Dehn functions, reflecting the linear isoperimetric inequality of the hyperbolic plane. This example illustrates how the curvature of the space on which a group acts determines the Dehn function of the group.

In spaces of zero curvature, such as Euclidean space, geodesics neither converge nor diverge, leading to a quadratic isoperimetric inequality in the plane and higher-degree polynomial inequalities in higher dimensions. Free abelian groups, which act properly discontinuously and cocompactly on Euclidean space, inherit these isoperimetric properties, exhibiting Dehn functions of polynomial growth. The free abelian group \mathbb{Z}^d , for instance, has a Dehn function of degree d , reflecting the d -dimensional Euclidean geometry.

Spaces of positive curvature, such as the sphere, exhibit geodesic convergence, leading to isoperimetric inequalities that are different from those in flat or negatively curved spaces. Finite groups, which can be viewed as acting on spherical buildings or other positively curved spaces, have bounded Dehn functions, reflecting the compactness of these spaces and the finite nature of the groups.

Beyond these classical curvature notions, the concept of Alexandrov curvature provides a more general framework for understanding the relationship between curvature and isoperimetric inequalities. Alexandrov spaces with curvature bounded above by a constant κ generalize the notion of spaces with non-positive curvature (when $\kappa \leq 0$) or non-negative curvature (when $\kappa \geq 0$). Groups acting properly discontinuously and cocompactly on Alexandrov spaces inherit the isoperimetric properties of these spaces, leading to corresponding bounds on their Dehn functions.

The work of Bridson and Haefliger on $CAT(\kappa)$ spaces has been particularly influential in this context. A $CAT(\kappa)$ space is a geodesic metric space where triangles are no fatter than comparison triangles in a space of constant curvature κ . For $\kappa \leq 0$, these spaces generalize the notion of non-positive curvature. Groups acting properly discontinuously and cocompactly on $CAT(0)$ spaces are called $CAT(0)$ groups, and these groups always admit polynomial Dehn functions, reflecting the non-positive curvature of the spaces on which they act.

Gromov's characterization of hyperbolic groups through linear isoperimetric inequalities stands as one of the most profound results connecting curvature and Dehn functions. This theorem states that a finitely presented group is hyperbolic if and only if it satisfies a linear isoperimetric inequality, establishing a precise correspondence between a geometric property (hyperbolicity, which can be defined in terms of thin triangles) and an algebraic-computational property (having a linear Dehn function). This result has been extended and refined in numerous directions, including relatively hyperbolic groups and acylindrically hyperbolic groups, each with their own characteristic isoperimetric behavior.

1.6.4 4.4 Large-scale Geometry and Quasi-isometry

The study of large-scale geometry, also known as coarse geometry, focuses on properties of metric spaces that are preserved under quasi-isometry, a notion of equivalence that ignores local details and captures only the large-scale structure. This perspective has proven particularly fruitful in geometric group theory, as many important properties of groups are invariant under quasi-isometry of their Cayley graphs. Dehn functions, while not strictly quasi-isometry invariants, exhibit a remarkable degree of stability under quasi-isometries, revealing the deep connection between computational complexity and large-scale geometric structure.

A quasi-isometry between two metric spaces (X, d_X) and (Y, d_Y) is a function $f: X \rightarrow Y$ for which there exist constants $A \geq 1$ and $B \geq 0$ such that for all $x, x' \in X$, $(1/A)d_X(x, x') - B \leq d_Y(f(x), f(x')) \leq A d_X(x, x') + B$, and additionally, there exists a constant $C \geq 0$ such that every point of Y is within distance C of the image of f . Quasi-isometries preserve large-scale geometric properties while allowing for bounded distortion at small scales. Two metric spaces are quasi-isometric if there exists a quasi-isometry between them.

The significance of quasi-isometry in group theory stems from the fact that if two groups are quasi-isometric, then they share many important properties. For instance, if G is quasi-isometric to H , and G has a linear Dehn function, then H also has a linear Dehn function. More generally, the “equivalence class” of a Dehn function (up to the equivalence relation defined in section 2.4) is preserved under quasi-isometry, establishing that the growth rate of the Dehn function is a quasi-isometry invariant.

This invariance property has profound implications for the classification of groups based on their Dehn functions. It means that the computational complexity of the word problem, as measured by the Dehn function, is determined by the large-scale geometry of the group rather than by local details of the presentation. This geometric perspective explains why groups that may appear quite different algebraically can have similar Dehn functions: they share the same large-scale geometric structure.

The work of Farb and Mosher on relatively hyperbolic groups illustrates this connection beautifully. A group G is relatively hyperbolic with respect to a collection of subgroups $\{H_1, H_2, \dots, H_k\}$ if G acts on a hyperbolic metric space in a way that generalizes the action of hyperbolic groups on hyperbolic space, with the subgroups H_i playing the role of parabolic subgroups. These relatively hyperbolic groups have Dehn functions that reflect both the hyperbolicity of the ambient space and the complexity of the peripheral subgroups. Specifically, if all the peripheral subgroups have linear Dehn functions, then the relatively hyperbolic group also has a linear Dehn function. If the peripheral subgroups have more complex Dehn functions, the relatively hyperbolic group inherits this complexity in a controlled way.

Another important development comes from the study of mapping class groups of surfaces, which are groups of isotopy classes of homeomorphisms of surfaces. These groups are not hyperbolic but exhibit interesting large-scale geometric properties. The work of Masur and Minsky established that mapping class groups have a hierarchical structure resembling the curve complex of the surface, with Dehn functions that are quadratic. This quadratic growth reflects the geometry of the curve complex and the way in which mapping class groups act on it.

The concept of asymptotic cones, introduced by Gromov and further developed by van den Dries and Wilkie, provides a powerful tool for studying the large-scale geometry of groups. An asymptotic cone of a metric space is a limit object obtained by scaling the metric by a sequence of factors going to infinity and taking a limit in an appropriate sense. Asymptotic cones capture the large-scale geometry of the space, revealing features that are not apparent at finite scales. For groups, the asymptotic cones of their Cayley graphs provide insight into their large-scale geometric structure and, consequently, into their Dehn functions.

For instance, hyperbolic groups have asymptotic cones that are \mathbb{R} -trees, reflecting their tree-like large-scale structure. This tree-like geometry corresponds to the linear Dehn functions of hyperbolic groups. In contrast, free abelian groups have asymptotic cones that are Euclidean spaces, reflecting their flat geometry and corresponding to their polynomial Dehn functions. Groups with exponential Dehn functions often have asymptotic cones with more complex structure, reflecting their more intricate large-scale geometry.

The relationship between large-scale geometry and Dehn functions continues to be an active area of research, with many open questions remaining. For instance, the precise relationship between the dimension of asymptotic cones and the degree of polynomial Dehn functions is not fully understood, particularly for groups with

intermediate growth. Similarly, the large-scale geometric characterization of groups with exponential Dehn functions remains an area of ongoing investigation.

As we have seen, the geometric interpretations of Dehn functions reveal deep connections between algebraic properties of groups, computational complexity, and geometric structure. These connections have transformed our understanding of groups as geometric objects and provided powerful tools for classifying and analyzing them. Yet while the geometric perspective illuminates the “shape” of groups and their Dehn functions, it also raises natural questions about the computational complexity of these functions themselves. How difficult is it to compute or approximate the Dehn function of a given group? What can we say about the algorithmic complexity of problems related to Dehn functions? These questions lead us naturally to the next section, where we explore the computational complexity aspects of Dehn functions in detail.

1.7 Computational Complexity Aspects

The geometric interpretations of Dehn functions reveal deep connections between algebraic properties of groups, computational complexity, and geometric structure. As we’ve seen, these connections transform our understanding of groups as geometric objects and provide powerful tools for classifying and analyzing them. Yet while the geometric perspective illuminates the “shape” of groups and their Dehn functions, it also raises natural questions about the computational complexity of these functions themselves. How difficult is it to compute or approximate the Dehn function of a given group? What can we say about the algorithmic complexity of problems related to Dehn functions? These questions lead us naturally to the computational complexity aspects of Dehn functions, which bridge the abstract mathematical theory with concrete computational problems.

1.7.1 5.1 Relationship to Complexity Classes

The relationship between Dehn functions and computational complexity classes represents one of the most fascinating intersections of group theory and theoretical computer science. Computational complexity theory classifies problems based on the resources (such as time or space) required to solve them, with well-known classes including P (problems solvable in polynomial time), NP (problems whose solutions can be verified in polynomial time), EXP (problems solvable in exponential time), and many others. Dehn functions, which measure the computational difficulty of solving the word problem, naturally connect to these complexity classes through their growth rates.

Perhaps the most direct connection comes from the relationship between Dehn function growth and the time complexity of solving the word problem. For a finitely presented group G with Dehn function $\delta G(n)$, the word problem can be solved in time $O(\delta G(n))$ using a brute-force approach: enumerate all possible sequences of relation applications of length at most $\delta G(n)$ until finding one that reduces the input word to the identity. This establishes that the time complexity of the word problem is bounded above by the Dehn function’s growth rate. Consequently, groups with linear Dehn functions have word problems solvable in linear time, those with quadratic Dehn functions have word problems solvable in quadratic time, and so on.

This connection extends to more refined complexity classes. For instance, groups with linear Dehn functions belong to the class $\text{DTIME}(n)$, the class of problems solvable in linear time. Groups with polynomial Dehn functions of degree d belong to $\text{DTIME}(n^d)$. Groups with exponential Dehn functions belong to $\text{DTIME}(e^n)$, and so forth. This provides a direct mapping from Dehn function growth rates to time complexity classes, establishing a clear link between the geometric properties of groups and computational complexity.

The relationship becomes even more intriguing when considering space complexity. The space complexity of the word problem is bounded by the maximum length of intermediate words in the reduction process. For groups with linear Dehn functions, the space complexity is linear, as the reduction process can be performed without storing intermediate words of excessive length. For groups with polynomial Dehn functions, the space complexity is typically polynomial, though sometimes with a lower degree than the time complexity. This distinction between time and space complexity reveals additional layers of computational structure encoded in the Dehn function.

The connection between Dehn functions and complexity classes extends beyond the word problem to other decision problems in group theory. The conjugacy problem, which asks whether two elements of a group are conjugate (i.e., whether there exists a group element g such that $gxg^{-1} = y$), typically has complexity related to but distinct from that of the word problem. For hyperbolic groups, which have linear Dehn functions, the conjugacy problem is solvable in linear time, similar to the word problem. For automatic groups, which have quadratic Dehn functions, the conjugacy problem is solvable in quadratic time. However, for some groups with exponential Dehn functions, the conjugacy problem can be undecidable, demonstrating how complexity can escalate beyond exponential growth.

Another important connection comes from the study of isomorphism problems. The group isomorphism problem asks whether two finite presentations define isomorphic groups. This problem is known to be undecidable in general, as shown by Adian and Rabin in the 1950s and 1960s. However, for certain classes of groups with restricted Dehn function growth, the isomorphism problem becomes decidable. For instance, the isomorphism problem for hyperbolic groups is decidable, as shown by Sela, though the exact complexity class remains an active area of research.

The relationship between Dehn functions and complexity classes also manifests through the concept of filling functions in computational topology. In computational topology, one studies the complexity of representing various topological constructions algorithmically. The Dehn function of a group corresponds to the complexity of filling loops in the associated classifying space, connecting group-theoretic complexity to topological computation. This connection has led to results in computational topology that leverage the theory of Dehn functions to establish bounds on the complexity of topological algorithms.

Perhaps one of the most fascinating aspects of this relationship is the way it reveals the intrinsic computational nature of mathematical structures. Groups with linear Dehn functions are computationally “tame,” with word problems solvable efficiently. Groups with polynomial Dehn functions are computationally “manageable,” though requiring more resources. Groups with exponential or super-exponential Dehn functions are computationally “wild,” with word problems that may be practically unsolvable for large inputs. This

computational perspective provides a new lens through which to view and classify mathematical structures, complementing traditional algebraic and geometric classifications.

1.7.2 5.2 Algorithms for the Word Problem

The study of algorithms for solving the word problem in finitely presented groups represents a rich intersection of group theory, combinatorics, and computer science. Different classes of groups, characterized by their Dehn function complexities, admit different algorithmic approaches, each with its own strengths and limitations. Understanding these algorithms not only provides practical tools for working with groups but also deepens our insight into the relationship between algebraic structure and computational complexity.

Dehn's algorithm stands as one of the oldest and most elegant algorithms for solving the word problem, particularly effective for groups with linear Dehn functions. Developed by Max Dehn in the early 20th century for surface groups, this algorithm exploits the geometric properties of hyperbolic groups to efficiently determine whether a word represents the identity. The algorithm works as follows: given a word w , repeatedly search for subwords that are more than half of some relator in the presentation, and replace these subwords with the shorter equivalent determined by the relator. If this process terminates with the empty word, then w represents the identity; otherwise, it does not.

For hyperbolic groups, which admit finite presentations where all relators of length less than some constant K are included (so-called Dehn presentations), Dehn's algorithm solves the word problem in linear time. This linear time complexity directly reflects the linear Dehn function of hyperbolic groups. The algorithm's efficiency stems from the geometric property that in hyperbolic groups, any word representing the identity must contain such a "more than half" subword, allowing for systematic reduction. Dehn's algorithm represents one of the earliest examples of a geometric algorithm in group theory, predating the formal development of computational complexity theory by several decades.

The Todd-Coxeter algorithm, developed by J.A. Todd and H.S.M. Coxeter in the 1930s, provides a different approach to solving the word problem, particularly effective for finite groups and certain infinite groups with recursive presentations. This algorithm works by enumerating cosets of a subgroup, building a coset table that records how generators act on these cosets. For the word problem, one considers the trivial subgroup and builds the coset table, which essentially constructs the multiplication table of the group. The word problem is then solved by checking whether the given word acts as the identity on all cosets.

The Todd-Coxeter algorithm is particularly effective for finite groups, where it is guaranteed to terminate, though its time complexity can be exponential in the worst case. For infinite groups, the algorithm may not terminate, but when it does, it provides a certificate that the word represents the identity. The relationship between the Todd-Coxeter algorithm and Dehn functions is less direct than for Dehn's algorithm, but in general, groups with smaller Dehn functions tend to be more amenable to solution by the Todd-Coxeter algorithm.

The Knuth-Bendix algorithm, developed by Donald Knuth and Peter Bendix in 1970, represents a more sophisticated approach based on term rewriting systems. This algorithm attempts to transform a group pre-

sentation into a confluent and terminating rewriting system, which allows for efficient word problem solution by deterministic reduction. The algorithm works by systematically resolving critical pairs between relators, adding new relators until no more critical pairs remain unresolved.

When the Knuth-Bendix algorithm terminates successfully, it produces a confluent rewriting system that solves the word problem in linear time, regardless of the original Dehn function complexity. However, the algorithm may not terminate for presentations that do not admit finite confluent rewriting systems. For groups with linear Dehn functions, the Knuth-Bendix algorithm often terminates quickly, producing an efficient rewriting system. For groups with higher Dehn function complexities, the algorithm may fail to terminate or may produce extremely large rewriting systems.

The small cancellation theory provides another approach to solving the word problem, particularly effective for groups satisfying certain combinatorial conditions on their presentations. Small cancellation conditions, such as $C'(\lambda)$ and $T(\mu)$, restrict how relators can overlap in a presentation, ensuring that van Kampen diagrams have specific local properties. Groups satisfying these small cancellation conditions often have linear or quadratic Dehn functions, and the word problem can be solved efficiently using geometric arguments based on the structure of van Kampen diagrams.

For groups with polynomial Dehn functions, particularly automatic groups, more specialized algorithms exist. Automatic groups, introduced by Thurston and developed by Epstein et al., are equipped with regular languages of normal forms and fellow traveler properties that allow for efficient computation. The word problem in an automatic group can be solved in quadratic time using these automatic structures, reflecting the quadratic Dehn function bound. The algorithm works by simultaneously reducing the input word and keeping track of fellow traveler paths in the Cayley graph, leveraging the regular structure of the automatic language.

For groups with exponential or super-exponential Dehn functions, the word problem becomes significantly more challenging. In these cases, general algorithms may take exponential or even super-exponential time in the worst case. However, for specific classes of groups with exponential Dehn functions, specialized algorithms have been developed. For instance, for certain solvable groups with exponential Dehn functions, algorithms based on the structure of the solvable series can solve the word problem more efficiently than brute-force approaches.

The comparison of these algorithms reveals interesting trade-offs between generality and efficiency. Dehn's algorithm is highly efficient for hyperbolic groups but fails to terminate for many other groups. The Todd-Coxeter algorithm is general but may have exponential time complexity even for simple groups. The Knuth-Bendix algorithm, when it terminates, produces highly efficient rewriting systems but may not terminate for many presentations. This variety of approaches reflects the rich landscape of computational complexity encoded in Dehn functions, with each algorithm tailored to specific classes of groups characterized by their Dehn function growth rates.

1.7.3 5.3 Complexity of Computing Dehn Functions

While the relationship between Dehn functions and the complexity of the word problem is relatively straightforward, a more subtle and profound question concerns the computational complexity of determining the Dehn function itself. Given a finite presentation of a group, how difficult is it to compute or approximate its Dehn function? This question connects the theory of Dehn functions to deep results in computability theory and computational complexity, revealing unexpected layers of undecidability and intractability.

The most striking result in this direction is the undecidability of computing the exact Dehn function for arbitrary finite presentations. This undecidability follows from the celebrated Novikov-Boone theorem, which states that the word problem for finitely presented groups is undecidable in general. If one could compute the Dehn function for an arbitrary finite presentation, then one could solve the word problem by simply checking whether the input word has length n and verifying whether it can be filled with at most $\delta(n)$ 2-cells. Since the word problem is undecidable, computing the Dehn function must also be undecidable.

This undecidability result has profound implications for the theory of Dehn functions. It means that there is no general algorithm that can take an arbitrary finite presentation and output its Dehn function. This undecidability persists even if we relax the requirement of computing the exact Dehn function to merely determining its equivalence class (up to the equivalence relation defined in section 2.4). In other words, there is no algorithm that can determine whether a given finite presentation has a linear, quadratic, exponential, or some other Dehn function growth rate.

The undecidability of computing Dehn functions is closely related to other undecidability results in group theory. For instance, the isomorphism problem for finitely presented groups is undecidable, as shown by Adian and Rabin. The triviality problem—determining whether a given finite presentation defines the trivial group—is also undecidable. These undecidability results form a web of interconnected problems that reveal the inherent computational complexity of group theory.

Despite these general undecidability results, there are important classes of groups for which the Dehn function can be computed or effectively bounded. For hyperbolic groups, the Dehn function is known to be linear, and this can often be verified algorithmically. Specifically, if a finite presentation satisfies a linear isoperimetric inequality, then the group is hyperbolic and has a linear Dehn function. Verifying this inequality can be done algorithmically for specific presentations, though there is no general algorithm that works for all presentations.

For automatic groups, which have quadratic Dehn functions, the situation is more nuanced. While all automatic groups have quadratic Dehn functions, determining whether a given finite presentation defines an automatic group is undecidable in general. However, for specific classes of presentations, particularly those that are “short” or satisfy certain combinatorial conditions, algorithms exist to verify automaticity and thus establish the quadratic Dehn function bound.

The nilpotent groups provide another interesting case. For nilpotent groups, the Dehn function is polynomial, with its degree determined by the nilpotency class and the number of generators. Specifically, the free nilpotent group of class c on k generators has a Dehn function of degree $c+1$. For specific presentations

of nilpotent groups, particularly those in “polycyclic” form, algorithms exist to compute the Dehn function explicitly. These algorithms leverage the structure of the nilpotent series to construct van Kampen diagrams with controlled area, providing precise bounds on the Dehn function.

The problem of approximating Dehn functions, even when exact computation is impossible, has also been studied extensively. One approach is to compute lower and upper bounds on the Dehn function. For lower bounds, one can search for specific words representing the identity that require large area to fill. For upper bounds, one can develop general filling strategies that work for all words of a given length. The gap between these bounds provides an estimate of the Dehn function’s growth rate.

For groups with exponential Dehn functions, particularly those constructed as HNN extensions or amalgamated products, techniques from combinatorial group theory can sometimes provide effective bounds. For instance, the Dehn function of a graph of groups can often be bounded in terms of the Dehn functions of the vertex groups and the edge groups, under certain conditions. These bounds, while not always tight, can establish the exponential growth rate of the Dehn function.

The computational complexity of approximating Dehn functions has connections to other problems in theoretical computer science. For instance, the problem of determining whether a given finite presentation has a polynomial Dehn function is equivalent to determining whether the word problem is solvable in polynomial time. This problem is known to be undecidable, as it would allow one to distinguish between presentations with polynomial and exponential word problem complexity, which is impossible in general.

Another related problem is the complexity of filling functions in computational topology. The Dehn function of a group corresponds to the complexity of filling loops in the associated classifying space. Computing or approximating these filling functions for arbitrary spaces is known to be computationally difficult, with connections to problems in computational geometry and topology.

The study of the complexity of computing Dehn functions reveals a fascinating interplay between algebra, geometry, and computation. While general algorithms are impossible due to undecidability results, important classes of groups admit effective methods for computing or bounding their Dehn functions. This partial computability reflects the rich structure of the theory, with some aspects being algorithmically tractable while others remain fundamentally beyond computation.

1.7.4 5.4 Related Complexity Measures

While the Dehn function serves as a primary measure of the computational complexity of the word problem in finitely presented groups, it is by no means the only such measure. A rich hierarchy of related filling functions and complexity measures has been developed, each capturing different aspects of the computational and geometric complexity of groups. Understanding these related measures provides a more nuanced view of group complexity and reveals deeper connections between algebra, geometry, and computation.

The isodiametric function represents one of the most important related complexity measures. While the Dehn function measures the area (number of 2-cells) needed to fill a loop, the isodiametric function measures the diameter of the filling. Formally, the isodiametric function $\text{Dia}(n)$ is defined as the smallest integer such that

any word w of length at most n that represents the identity can be reduced to the empty word by a sequence of relation applications, with all intermediate words having length at most $\text{Dia}(n)$. This function captures the “width” of the computation needed to solve the word problem, as opposed to the “length” captured by the Dehn function.

The relationship between the Dehn function and the isodiametric function is complex and not fully understood. For hyperbolic groups, both functions are linear, reflecting the efficient computational properties of these groups. For automatic groups, the Dehn function is quadratic, while the isodiametric function is also quadratic. However, there exist groups with quadratic Dehn functions but exponential isodiametric functions, demonstrating that these measures can

1.8 Connections to Topology

differ significantly. This distinction reveals that the computational complexity of groups is multidimensional, with different aspects captured by different filling functions. The isodiametric function has important applications in computational group theory, particularly in the study of parallel algorithms for the word problem.

Another important related complexity measure is the filling length function. This function measures the minimal length of a combing (a homotopy of paths) needed to fill a loop in the Cayley complex. Formally, the filling length function $\text{FillLen}(n)$ is defined as the smallest integer such that any word w of length at most n that represents the identity can be filled by a van Kampen diagram where the distance from any vertex to the boundary is at most $\text{FillLen}(n)$. This function captures the “depth” of the computation needed to solve the word problem, reflecting how far into the interior of the filling one must go.

The filling length function has interesting connections to the geometry of groups. For hyperbolic groups, the filling length function is linear, reflecting the efficient computational properties of these groups. For automatic groups, the filling length function is quadratic, similar to the Dehn function. However, there exist groups with polynomial Dehn functions but exponential filling length functions, again demonstrating the multidimensional nature of group complexity.

The synchronization delay function provides yet another complexity measure, particularly relevant for automatic groups. This function measures the maximal delay between two fellow traveler paths in the Cayley graph of an automatic group. For automatic groups, the synchronization delay is bounded by a constant, reflecting the regular structure of these groups. This bounded synchronization delay is closely related to the quadratic Dehn function bound for automatic groups, though the precise relationship is complex and not fully understood.

These related complexity measures form a hierarchy of filling functions, each capturing different aspects of the computational and geometric complexity of groups. The study of this hierarchy has led to deep results connecting algebra, geometry, and computation. For instance, the work of Bridson and Gersten established precise relationships between different filling functions, showing how bounds on one function imply bounds on another under certain conditions.

The multidimensional nature of group complexity revealed by these related measures has important implications for the classification of groups and the development of algorithms. It shows that the computational complexity of groups cannot be fully captured by a single numerical invariant but requires a more nuanced approach that considers multiple aspects of complexity simultaneously. This perspective has led to the development of more sophisticated algorithms that exploit the structure revealed by these different complexity measures.

1.9 6. Connections to Topology

The rich tapestry of Dehn function complexity we have explored so far reveals deep connections between algebra, geometry, and computation. Yet perhaps the most profound of these connections lies in the relationship between Dehn functions and topology—a relationship that traces back to the very origins of the concept in Max Dehn’s work on surface groups. The topological perspective illuminates Dehn functions from a different angle, revealing them as natural invariants that capture essential properties of topological spaces and the maps between them. This connection between algebraic complexity and topological structure represents one of the most beautiful aspects of the theory, bridging seemingly disparate areas of mathematics in unexpected ways.

1.9.1 6.1 Topological Interpretations and Applications

To understand the topological significance of Dehn functions, we must return to their origins in the study of surfaces and their fundamental groups. When Max Dehn first introduced what we now call Dehn functions, he was motivated by topological questions about surfaces—specifically, how to determine whether a closed curve on a surface bounds a disk. This topological filling problem, which asks for the minimal area of a disk bounded by a given curve, directly corresponds to the algebraic problem of determining the minimal number of relations needed to demonstrate that a word represents the identity in the fundamental group.

This correspondence is not merely coincidental but reflects a deep connection between algebra and topology. The fundamental group of a topological space captures essential information about the loops in that space and how they can be deformed. When a space is path-connected, the fundamental group consists of equivalence classes of loops based at a fixed point, with the group operation given by concatenation of loops. The word problem in this fundamental group asks whether two loops represent the same homotopy class—that is, whether one can be continuously deformed into the other.

The Dehn function of the fundamental group measures the complexity of this homotopy problem. Specifically, it quantifies how efficiently a null-homotopic loop (one that can be continuously deformed to a point) can be filled by a disk. In this topological interpretation, the Dehn function $\delta(n)$ represents the maximal area needed to fill any loop of length at most n that is null-homotopic. This area can be measured in various ways depending on the context—for Riemannian manifolds, it might be the geometric area; for simplicial complexes, it might be the number of 2-simplices.

Covering spaces provide another topological perspective on Dehn functions. The universal cover of a space with fundamental group G is a simply connected space that covers the original space, with the group G acting as deck transformations. The word problem in G can be interpreted as determining whether a loop in the base space lifts to a closed loop in the universal cover. The Dehn function then measures how efficiently one can fill a loop that lifts to a closed loop in the universal cover, reflecting the complexity of the covering space's geometry.

This perspective reveals why hyperbolic groups have linear Dehn functions. The universal cover of a hyperbolic surface is the hyperbolic plane, which has constant negative curvature. In such spaces, the isoperimetric inequality is linear, meaning that the area needed to fill a curve grows linearly with the length of the curve. This linear isoperimetric inequality in the geometric setting translates directly to the linear Dehn function in the algebraic setting, establishing a precise correspondence between geometric curvature and algebraic complexity.

The applications of this topological perspective extend far beyond surface groups. In algebraic topology, Dehn functions play a crucial role in understanding the homotopy types of spaces. For a connected CW-complex X with fundamental group G , the Dehn function of G provides a lower bound on the complexity of the 2-skeleton of X . Specifically, if G has a Dehn function that grows at least as fast as $f(n)$, then any 2-complex with fundamental group G must have 2-cells whose areas grow at least as fast as $f(n)$. This result has important implications for the classification of 2-complexes and their homotopy types.

The topological filling problem also has applications in geometric topology, particularly in the study of minimal surfaces. For a Riemannian manifold, the problem of finding a minimal surface bounded by a given curve is closely related to the isoperimetric problem measured by the Dehn function. While the minimal surface problem is typically more difficult than the combinatorial filling problem captured by the Dehn function, the two are related through the notion of Hausdorff dimension and other geometric invariants. This connection has led to cross-pollination between geometric group theory and differential geometry, with techniques from one field informing the other.

Perhaps one of the most fascinating applications of the topological perspective on Dehn functions comes from the study of aspherical manifolds. A manifold is aspherical if its universal cover is contractible, meaning that all higher homotopy groups vanish. For such manifolds, the homotopy type is completely determined by the fundamental group. The Dehn function of the fundamental group then captures essential information about the geometry of the universal cover. For instance, if the fundamental group has a linear Dehn function, then the universal cover satisfies a linear isoperimetric inequality, reflecting its hyperbolic geometry. This connection between algebraic complexity and geometric structure has been instrumental in the classification of aspherical manifolds and the study of their geometric properties.

1.9.2 6.2 2-Complexes and Presentations

The relationship between group presentations and 2-complexes represents one of the most concrete manifestations of the connection between algebra and topology. Every finite presentation of a group gives rise

to a 2-complex, and conversely, every 2-complex with a single 0-cell determines a presentation of its fundamental group. This correspondence provides a powerful bridge between the algebraic theory of group presentations and the topological theory of 2-complexes, with Dehn functions serving as a key invariant that connects the two perspectives.

Given a finite presentation $\langle S \mid R \rangle$ of a group G , we can construct an associated 2-complex, called the presentation complex, as follows. Start with a single 0-cell (vertex). For each generator in S , attach a 1-cell (edge) to the 0-cell, forming a wedge of circles. For each relator in R , attach a 2-cell along the path determined by the relator. The resulting 2-complex has fundamental group isomorphic to G , and its topology encodes essential information about the presentation.

The Dehn function of the presentation is directly related to the topological properties of this presentation complex. Specifically, the Dehn function $\delta(n)$ measures the maximal area needed to fill any loop of length at most n that is null-homotopic in the complex. This area can be interpreted combinatorially as the number of 2-cells in a minimal filling disk, or geometrically as the area of a minimal surface bounded by the loop. In this way, the Dehn function captures the isoperimetric properties of the presentation complex, quantifying how efficiently null-homotopic loops can be filled.

The homotopy properties of 2-complexes are intimately connected to the Dehn functions of their fundamental groups. A 2-complex is aspherical if it is a $K(G,1)$ space, meaning that all higher homotopy groups vanish. For such complexes, the homotopy type is completely determined by the fundamental group, and the Dehn function captures essential geometric information about the universal cover. A celebrated result of Whitehead states that a presentation complex is aspherical if and only if the presentation satisfies a combinatorial condition known as diagrammatic reducibility, which is closely related to the Dehn function. Specifically, if the Dehn function is linear, then the presentation complex is aspherical if and only if it satisfies a certain local condition on the relators.

The construction of Eilenberg-MacLane spaces provides another important connection between presentations and 2-complexes. An Eilenberg-MacLane space $K(G,1)$ is a path-connected space whose fundamental group is G and whose higher homotopy groups vanish. Every group G admits a $K(G,1)$ space, which is unique up to homotopy equivalence. For a finitely presented group G , one can construct a $K(G,1)$ space by starting with the presentation complex and attaching higher-dimensional cells to kill the higher homotopy groups. The Dehn function of G provides a lower bound on the complexity of the 2-skeleton of any $K(G,1)$ space, reflecting the topological complexity of the group.

The topology and combinatorics of presentation complexes have been extensively studied, leading to deep results connecting algebraic properties of groups to topological properties of their associated complexes. For instance, a presentation complex is simply connected if and only if the group is trivial, reflecting the fact that the trivial group is the only group with a bounded Dehn function. More generally, the homology groups of the presentation complex are related to the homology of the group, with the Dehn function providing information about the efficiency of the presentation in capturing these homological properties.

One particularly striking example of this connection comes from the study of small cancellation theory. A presentation satisfies the small cancellation condition $C'(\lambda)$ if no relator is a product of two fragments of

other relators, each of length less than λ times the length of the relator. For such presentations, the presentation complex has specific local properties that imply global topological consequences. In particular, if $\lambda \leq 1/6$, then the presentation complex is aspherical, and the group has a linear Dehn function. This result establishes a direct connection between combinatorial properties of the presentation, topological properties of the associated complex, and algebraic properties of the group (as captured by the Dehn function).

The study of 2-complexes has also led to important results about the classification of groups with specific Dehn function growth rates. For instance, a group has a linear Dehn function if and only if it acts properly discontinuously and cocompactly on a simply connected 1-dimensional complex (a tree), reflecting its hyperbolicity. A group has a quadratic Dehn function if and only if it acts properly discontinuously and cocompactly on a simply connected 2-dimensional complex with specific geometric properties, reflecting its non-positively curved geometry. These results establish precise connections between algebraic complexity, as measured by Dehn functions, and geometric structure, as captured by group actions on complexes.

The relationship between presentations and 2-complexes also has practical applications in computational topology. The problem of determining whether a given 2-complex is aspherical is closely related to the problem of determining whether the associated group has a linear Dehn function. While both problems are undecidable in general, specific algorithms have been developed for certain classes of presentations and complexes. These algorithms leverage the connection between algebraic and topological properties to solve problems that would be intractable from a purely algebraic or purely topological perspective.

1.9.3 6.3 Higher-dimensional Generalizations

The theory of Dehn functions naturally extends to higher dimensions, giving rise to a hierarchy of isoperimetric functions that capture the complexity of filling problems in higher-dimensional spaces. These higher-dimensional generalizations provide a bridge between the combinatorial group theory we have been studying and the broader landscape of algebraic topology and homotopy theory. By extending the concept of Dehn functions beyond the second dimension, we gain a more comprehensive understanding of the relationship between algebraic complexity and topological structure.

In higher dimensions, we consider n -dimensional filling functions for $n \geq 2$. The classical Dehn function corresponds to the case $n=2$, measuring the complexity of filling 1-dimensional loops with 2-dimensional disks. For $n=3$, we consider the problem of filling 2-dimensional spheres with 3-dimensional balls, and so on for higher dimensions. Formally, the n -dimensional filling function $F_n(k)$ of a space measures the minimal volume needed to fill any n -dimensional cycle of volume at most k with an $(n+1)$ -dimensional chain.

For groups, these higher-dimensional filling functions are defined through the lens of Eilenberg-MacLane spaces. Given a group G , we can consider its $K(G,1)$ space, which has fundamental group G and vanishing higher homotopy groups. The n -dimensional filling function of G is then defined as the n -dimensional filling function of this $K(G,1)$ space. This definition extends the classical Dehn function, which corresponds to the case $n=2$.

The relationship between these higher-dimensional filling functions and the classical Dehn function is com-

plex and not fully understood. For hyperbolic groups, all filling functions are linear, reflecting the efficient computational properties of these groups across all dimensions. For free abelian groups, the n -dimensional filling function is polynomial of degree $n+1$, extending the pattern we observed for the classical Dehn function (which is polynomial of degree 2 for \mathbb{Z}^2 , degree 3 for \mathbb{Z}^3 , and so on). For more general groups, the relationship between different filling functions can be quite intricate, with bounds on one function implying bounds on another under certain conditions.

The study of higher-dimensional filling functions has deep connections to higher-dimensional homotopy theory. In homotopy theory, one studies the classification of spaces up to homotopy equivalence, focusing on invariants like homotopy groups and homology groups. The filling functions provide quantitative measures of the complexity of these homotopy invariants, capturing how efficiently cycles can be filled in the various dimensions. This perspective has led to important results connecting algebraic topology to geometric group theory, with techniques from one field informing the other.

One particularly striking application of higher-dimensional filling functions comes from the study of aspherical manifolds. An aspherical manifold is one whose universal cover is contractible, meaning that all homotopy groups vanish except possibly the fundamental group. For such manifolds, the homotopy type is completely determined by the fundamental group, and the higher-dimensional filling functions capture essential geometric information about the universal cover. For instance, if the fundamental group has linear filling functions in all dimensions, then the universal cover satisfies linear isoperimetric inequalities in all dimensions, reflecting its hyperbolic geometry.

The n -dimensional filling functions for $n > 2$ also have important applications in combinatorial topology. In combinatorial topology, one studies simplicial complexes and their combinatorial properties, particularly those related to homology and hom

1.10 Important Theorems and Results

I need to write section 7 on “Important Theorems and Results” for the Encyclopedia Galactica article on Dehn Function Complexity. This section should cover the significant theorems and results in the theory of Dehn functions, explaining their importance and implications for mathematics.

The previous section (Section 6) was discussing connections to topology, specifically covering topological interpretations and applications, 2-complexes and presentations, and higher-dimensional generalizations. The section was cut off mid-sentence while discussing combinatorial topology and homology groups. I need to create a smooth transition from where that section ended.

The outline for Section 7 includes these subsections: 7.1 Gromov’s Theorem on Hyperbolic Groups 7.2 Bridson’s Theorems on Filling Functions 7.3 Novikov-Boone Theorem and Undecidability 7.4 Gersten’s Results on Automatic Groups 7.5 Recent Breakthroughs

I’ll need to cover each of these subsections with rich detail, examples, and fascinating anecdotes while maintaining the same authoritative yet engaging tone as the previous sections. I should avoid bullet points and instead weave information into flowing paragraphs, using natural transitions.

Let me approach this systematically:

1. I'll start with a transition from the previous section's discussion of combinatorial topology and homology groups.
2. Then I'll cover each subsection in order, explaining the theorems, their significance, and providing examples where appropriate.
3. I'll end with a transition to the next section (Section 8 on Applications in Mathematics and Computer Science).

Now I'll draft the section:

1.11 Section 7: Important Theorems and Results

The study of combinatorial topology and homology groups naturally leads us to the cornerstone theorems that have shaped our understanding of Dehn function complexity. These fundamental results represent the intellectual pillars supporting the entire edifice of geometric group theory, connecting abstract algebraic concepts to concrete geometric and computational properties. The theorems we will explore in this section not only classify groups based on their Dehn function growth rates but also reveal profound connections between seemingly disparate areas of mathematics. Each theorem stands as a monument to mathematical insight, often representing years or even decades of concentrated effort by some of the greatest minds in the field.

1.11.1 7.1 Gromov's Theorem on Hyperbolic Groups

Perhaps no single result has had a more transformative impact on the theory of Dehn functions than Gromov's landmark theorem on hyperbolic groups, published in his seminal 1987 essay "Hyperbolic Groups." This theorem established a breathtaking connection between the linear isoperimetric inequality captured by linear Dehn functions and the geometric notion of negative curvature, revolutionizing geometric group theory and opening entirely new avenues of research. Gromov's insight was to recognize that the thin triangles property, which characterizes spaces of negative curvature, could be abstracted to a purely combinatorial condition for finitely presented groups, leading to a powerful characterization that has shaped the field ever since.

Gromov's theorem states that a finitely presented group is hyperbolic if and only if it satisfies a linear isoperimetric inequality, meaning its Dehn function is linear. This elegant characterization establishes a precise correspondence between a geometric property (hyperbolicity, defined through the thin triangles condition) and an algebraic-computational property (having a linear Dehn function). The theorem's power lies in this bidirectional implication: not only do hyperbolic groups have linear Dehn functions, but any group with a linear Dehn function must be hyperbolic.

To appreciate the significance of this result, we must understand the concept of hyperbolicity in Gromov's sense. A geodesic metric space is δ -hyperbolic if for every geodesic triangle in the space, each side is

contained in the δ -neighborhood of the other two sides. This condition generalizes the property of thin triangles in the hyperbolic plane, where triangles are thinner than their Euclidean counterparts. A finitely generated group is hyperbolic if its Cayley graph (with respect to some finite generating set) is a δ -hyperbolic metric space for some $\delta \geq 0$. Remarkably, this property is independent of the choice of finite generating set, making hyperbolicity an intrinsic property of the group.

Gromov’s theorem connects this geometric notion to the computational complexity of the word problem through the Dehn function. The linear Dehn function of hyperbolic groups means that the word problem can be solved efficiently—specifically, in linear time—using Dehn’s algorithm, as discussed in Section 5.2. This algorithm exploits the geometric structure of hyperbolic groups to systematically reduce words by identifying “shortcuts” in the Cayley graph, reflecting the negative curvature that prevents loops from being “too inefficient.”

The proof of Gromov’s theorem is a masterpiece of mathematical ingenuity, combining geometric intuition with sophisticated combinatorial arguments. One direction of the proof—showing that hyperbolic groups have linear Dehn functions—relies on the geometric properties of hyperbolic space. In a hyperbolic group, van Kampen diagrams exhibit a thinness condition similar to that of geodesic triangles: any internal path in the diagram is close to the boundary. This thinness prevents the diagram from containing large “empty” regions, forcing the area to grow linearly with the boundary length.

The converse direction—showing that groups with linear Dehn functions are hyperbolic—is more subtle and requires a deeper analysis of the geometry of the Cayley graph. Gromov’s approach involved developing a theory of asymptotic cones, limit objects that capture the large-scale geometry of the group. For groups with linear Dehn functions, these asymptotic cones turn out to be R-trees—simply connected metric spaces where all triangles are degenerate—reflecting the tree-like large-scale structure characteristic of hyperbolic groups.

The impact of Gromov’s theorem on mathematics has been profound and far-reaching. Before Gromov’s work, hyperbolic groups were studied primarily as examples of “small cancellation groups” satisfying certain combinatorial conditions on their presentations. Gromov’s theorem revealed that these groups were part of a much broader geometric framework, unifying diverse examples including free groups, fundamental groups of closed hyperbolic manifolds, and many Coxeter groups under a single conceptual umbrella.

One fascinating consequence of Gromov’s theorem is the abundance of hyperbolic groups. In a precise sense, “most” finitely presented groups are hyperbolic, just as “most” manifolds are hyperbolic. This abundance makes hyperbolic groups a natural class to study, and indeed, they have become a central focus of research in geometric group theory.

The theorem also has significant algorithmic implications. For hyperbolic groups, not only is the word problem solvable in linear time, but many other decision problems that are undecidable for general groups become tractable. For instance, the conjugacy problem in hyperbolic groups is solvable in linear time, and the isomorphism problem for hyperbolic groups, while still challenging, is decidable—a result established by Sela in the 1990s.

Gromov’s theorem has inspired numerous generalizations and refinements. For example, relatively hyperbolic groups, introduced by Gromov and developed by Farb and others, generalize hyperbolicity by allowing for the presence of “peripheral” subgroups that may not be hyperbolic themselves. These relatively hyperbolic groups have Dehn functions that reflect both the hyperbolicity of the ambient space and the complexity of the peripheral subgroups, providing a rich generalization of the original theory.

1.11.2 7.2 Bridson’s Theorems on Filling Functions

While Gromov’s theorem provides a complete characterization of groups with linear Dehn functions, the classification of groups with more complex Dehn functions requires a more nuanced approach. This is where the work of Martin Bridson on filling functions has been particularly influential. Bridson’s theorems, developed throughout the 1990s and 2000s, establish precise relationships between different types of filling functions and provide powerful tools for analyzing groups with polynomial and exponential Dehn functions.

One of Bridson’s most significant contributions is his systematic study of the hierarchy of filling functions, which extends beyond the classical Dehn function to include isodiametric functions, filling length functions, and other related measures. These different filling functions capture distinct aspects of the computational complexity of groups, and Bridson’s work has revealed intricate relationships between them.

A key theorem in this direction establishes that for groups satisfying certain geometric conditions, bounds on one filling function imply bounds on another. Specifically, Bridson showed that for groups admitting a proper cocompact action on a $CAT(0)$ space, the classical Dehn function and the isodiametric function have equivalent growth rates. This result is significant because it connects the algebraic complexity of the group (as measured by the Dehn function) to the geometric complexity of the space on which it acts, providing a bridge between algebra and geometry.

Bridson also made substantial contributions to our understanding of groups with quadratic Dehn functions. While Gromov’s theorem completely characterizes groups with linear Dehn functions, the situation for quadratic Dehn functions is more complex. Not all groups with quadratic Dehn functions are $CAT(0)$ groups, and not all $CAT(0)$ groups have quadratic Dehn functions. Bridson’s work clarified this picture by identifying precise geometric conditions that imply quadratic Dehn function bounds.

One particularly elegant result in this direction is Bridson’s characterization of groups with quadratic Dehn functions that act properly discontinuously and cocompactly on $CAT(0)$ cube complexes. These groups, which include many important examples like right-angled Artin groups and certain Coxeter groups, have Dehn functions that are precisely quadratic, reflecting the 2-dimensional nature of the cube complexes on which they act.

Bridson’s theorems also address the relationship between different filling functions in higher dimensions. While the classical Dehn function concerns the filling of 1-dimensional loops with 2-dimensional disks, Bridson extended this framework to higher dimensions, considering the filling of n -dimensional spheres with $(n+1)$ -dimensional balls. His results establish precise relationships between these higher-dimensional

filling functions and the classical Dehn function, providing a more comprehensive understanding of the isoperimetric hierarchy.

Perhaps one of the most striking applications of Bridson's work is in the study of subgroup distortion. The distortion function of a subgroup H of a group G measures how the word metric in H relates to the word metric in G . Bridson's theorems connect these distortion functions to the filling functions of the groups, revealing how the geometric embedding of a subgroup affects its computational properties. For instance, Bridson showed that in a hyperbolic group, the distortion of any finitely generated subgroup is at most exponential, reflecting the efficient computational properties of these groups.

Bridson's collaborative work with Gersten on the isodiametric function of the lamplighter group represents another highlight of this research program. The lamplighter group, as discussed in Section 3.3, has an exponential Dehn function, but Bridson and Gersten showed that its isodiametric function grows even faster—specifically, it is bounded below by a function of the form $e^{\alpha n}$ for some $\alpha > 0$. This result demonstrates that different filling functions can have radically different growth rates, even for the same group, revealing the multidimensional nature of computational complexity in group theory.

The impact of Bridson's theorems extends beyond pure group theory into related areas of mathematics. His work on filling functions has applications in geometric topology, particularly in the study of aspherical manifolds and their fundamental groups. It also has connections to theoretical computer science, especially in the analysis of algorithms for group-theoretic problems. By establishing precise relationships between different measures of complexity, Bridson's work has provided a unified framework for understanding the computational and geometric properties of groups.

1.11.3 7.3 Novikov-Boone Theorem and Undecidability

The theory of Dehn functions exists against a backdrop of fundamental limitations on computation, most dramatically illustrated by the Novikov-Boone theorem on the undecidability of the word problem. This theorem, proved independently by Pyotr Novikov in 1955 and William Boone in 1959, stands as one of the most profound results in mathematical logic and group theory, establishing that there is no general algorithm to solve the word problem for arbitrary finitely presented groups. The implications of this undecidability result for the theory of Dehn functions are far-reaching, revealing inherent limitations on our ability to compute or classify these functions.

The Novikov-Boone theorem states that there exists a finitely presented group with an undecidable word problem. In other words, there is no algorithm that, given an arbitrary finite presentation and an arbitrary word in the generators, can determine whether the word represents the identity element of the group. This result stands in stark contrast to the situation for specific classes of groups like hyperbolic groups, where the word problem is efficiently solvable.

The proof of the Novikov-Boone theorem is a tour de force of mathematical logic and combinatorial group theory. The basic strategy involves encoding the halting problem for Turing machines into the word problem for a specially constructed group. Specifically, given a Turing machine M , one constructs a finite presentation

of a group G_M such that a particular word w_M represents the identity in G_M if and only if M halts on the empty input. Since the halting problem is undecidable (there is no algorithm that can determine whether an arbitrary Turing machine halts on the empty input), it follows that the word problem for G_M is undecidable.

The connection to Dehn functions comes through the relationship between the word problem and the computation of Dehn functions. If one could compute the Dehn function for an arbitrary finite presentation, then one could solve the word problem by simply checking whether the input word has length n and verifying whether it can be filled with at most $\delta(n)$ 2-cells. Since the word problem is undecidable, it follows that computing the Dehn function for arbitrary presentations must also be undecidable.

This undecidability result has profound implications for the theory of Dehn functions. It means that there is no general algorithm that can take an arbitrary finite presentation and output its Dehn function. This undecidability persists even if we relax the requirement of computing the exact Dehn function to merely determining its equivalence class (up to the equivalence relation defined in Section 2.4). In other words, there is no algorithm that can determine whether a given finite presentation has a linear, quadratic, exponential, or some other Dehn function growth rate.

The Novikov-Boone theorem is part of a web of interconnected undecidability results in group theory. The isomorphism problem for finitely presented groups is also undecidable, as shown by Adian and Rabin in the 1950s and 1960s. The triviality problem—determining whether a given finite presentation defines the trivial group—is likewise undecidable. These results collectively reveal the inherent computational complexity of group theory, establishing fundamental limitations on what can be achieved algorithmically.

Despite these general undecidability results, the Novikov-Boone theorem has inspired important positive results about classes of groups with decidable word problems. For instance, while the word problem is undecidable in general, it is decidable for many important classes of groups, including hyperbolic groups, automatic groups, and one-relator groups. For these classes, the Dehn function can often be computed or effectively bounded, revealing a fascinating interplay between decidability and computational complexity.

The historical development of the Novikov-Boone theorem is itself a fascinating story. The question of whether the word problem is decidable was one of the three problems posed by Max Dehn in 1911. Early progress was made by Tarski in the 1940s, who showed that the word problem is decidable for certain classes of groups. However, it was not until the work of Novikov and Boone in the 1950s that the general undecidability was established. Novikov's proof, published in 1955, was the first to demonstrate the undecidability of the word problem, though it was highly complex and difficult to verify. Boone's proof, published in 1959, was more accessible and helped to establish the result firmly in the mathematical literature.

The techniques developed in the proofs of the Novikov-Boone theorem have had a lasting impact on mathematical logic and theoretical computer science. The idea of encoding computational problems into algebraic structures has become a powerful tool in both fields, leading to numerous undecidability results and complexity classifications. In particular, the connection between group theory and computation established by the Novikov-Boone theorem has influenced the development of computability theory and complexity theory, providing concrete examples of undecidable problems that arise naturally in algebra.

1.11.4 7.4 Gersten's Results on Automatic Groups

The theory of automatic groups, developed significantly through the work of Stephen Gersten and his collaborators, represents one of the most successful frameworks for understanding groups with quadratic Dehn functions. Automatic groups, introduced by Jim Cannon and further developed by Epstein, Holt, and others in the 1980s and 1990s, are equipped with formal languages of normal forms and fellow traveler properties that allow for efficient computation. Gersten's contributions to this theory have been particularly influential, establishing precise connections between automaticity and isoperimetric inequalities and providing powerful tools for analyzing the computational complexity of these groups.

An automatic group is a finitely generated group G equipped with a regular language L of words over the generators (called the language of normal forms) such that:

1. Every element of G is represented by at least one word in L .
2. The set of words in L representing the identity is regular.
3. The fellow traveler property holds: there is a constant K such that for any two words u, v in L that represent group elements differing by multiplication by a generator, the paths in the Cayley graph defined by u and v stay within distance K of each other at all times.

These conditions ensure that the group has a highly regular structure that can be exploited algorithmically. Gersten's work established that all automatic groups have quadratic Dehn functions, providing a precise connection between the combinatorial property of automaticity and the geometric property captured by the Dehn function.

The proof of this result is elegant and insightful. Given a word w representing the identity in an automatic group, the fellow traveler property allows one to construct a van Kampen diagram for w by piecing together "strips" corresponding to the fellow traveler paths. The regular structure of the automatic language ensures that these strips can be arranged in a controlled way, leading to a quadratic bound on the area of the diagram. This quadratic bound reflects the 2-dimensional nature of the geometry of automatic groups, which often act on non-positively curved spaces.

Gersten

1.12 Applications in Mathematics and Computer Science

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1.13 Section 8: Applications in Mathematics and Computer Science

Gersten’s work on automatic groups, with its elegant connections between combinatorial properties and isoperimetric inequalities, exemplifies how the theory of Dehn functions transcends pure mathematics to find applications across diverse disciplines. The rich framework developed for understanding Dehn function complexity has proven to be a powerful tool not only within group theory itself but also in topology, geometry, computer science, and even cryptography. This section explores these multifaceted applications, demonstrating how a concept originating from Max Dehn’s early 20th-century topological investigations has evolved into a cornerstone of modern mathematical science with far-reaching practical implications.

1.13.1 8.1 Applications in Group Theory

Within group theory itself, Dehn functions serve as powerful invariants that enable mathematicians to classify, distinguish, and understand the structure of groups in profound ways. The growth rate of a group’s Dehn function provides immediate insight into its algebraic and geometric properties, allowing group theorists to categorize groups into meaningful classes and develop structure theorems that reveal deeper connections.

One of the most fundamental applications of Dehn functions in group theory is in the classification and distinction of groups. Since the Dehn function is invariant under quasi-isometry (as discussed in Section 4.4), it captures an intrinsic property of the group rather than merely a feature of its presentation. This invariance makes Dehn functions particularly valuable for distinguishing between groups that might appear similar from other perspectives. For instance, while both the free abelian group \mathbb{Z}^2 and the Heisenberg group are nilpotent, their Dehn functions differ significantly—quadratic for \mathbb{Z}^2 and cubic for the Heisenberg group—immediately revealing their distinct geometric structures.

The study of subgroup distortion represents another important application area. The distortion function of a subgroup H of a group G measures how the word metric in H relates to the word metric in G . Dehn functions play a crucial role in understanding these distortion properties. A theorem by Bridson and Gersten establishes that in a hyperbolic group, the distortion of any finitely generated subgroup is at most exponential, reflecting the efficient computational properties of these groups. This connection between Dehn functions and subgroup distortion has led to important results about the structure of subgroups in groups with various Dehn function complexities.

Dehn functions also find applications in the study of group extensions, amalgamations, and HNN extensions—fundamental constructions in combinatorial group theory. The Dehn function of a group constructed through these operations can often be bounded in terms of the Dehn functions of the constituent groups and the complexity of the gluing data. For instance, for an amalgamated free product $G = A *_C B$, where C is a common subgroup of A and B , the Dehn function of G can be bounded in terms of the Dehn functions of A and B and the distortion of C in A and B . These bounds provide quantitative control over how the complexity of

the word problem behaves under group constructions, enabling mathematicians to build new groups with prescribed computational properties.

The theory of relatively hyperbolic groups, developed by Gromov and refined by Farb, Bowditch, and others, represents another area where Dehn functions have been instrumental. A group G is relatively hyperbolic with respect to a collection of subgroups $\{H_1, H_2, \dots, H_k\}$ if G acts on a hyperbolic metric space in a way that generalizes the action of hyperbolic groups on hyperbolic space. The Dehn function of G in this case reflects both the hyperbolicity of the ambient space and the complexity of the peripheral subgroups. Specifically, if all the peripheral subgroups have linear Dehn functions, then the relatively hyperbolic group also has a linear Dehn function. If the peripheral subgroups have more complex Dehn functions, the relatively hyperbolic group inherits this complexity in a controlled way. This relationship has been used to establish precise results about the structure and properties of relatively hyperbolic groups.

In the realm of solvable groups, Dehn functions have provided crucial insights into the relationship between algebraic structure and computational complexity. While nilpotent groups (a subclass of solvable groups) have polynomial Dehn functions, more general solvable groups can exhibit exponential or even super-exponential Dehn functions. This distinction has led to important results about the structure of solvable groups and their classification based on Dehn function complexity. For instance, the work of Ol'shanskii and Sapir has shown that certain solvable groups of exponential Dehn function complexity cannot be embedded into finitely presented groups with polynomial Dehn functions, revealing deep connections between solvability, embeddability, and computational complexity.

The applications of Dehn functions in group theory extend to the study of group actions on metric spaces. When a group acts properly discontinuously and cocompactly on a metric space, the Dehn function of the group is closely related to the isoperimetric properties of the space. This relationship has been used to establish results about the existence and properties of such actions. For example, a group with a linear Dehn function can act properly discontinuously and cocompactly on a hyperbolic metric space, while a group with a quadratic Dehn function can act on a non-positively curved space. These connections have led to a rich interplay between geometric group theory and the theory of group actions, with Dehn functions serving as a bridge between algebraic properties of groups and geometric properties of spaces.

1.13.2 8.2 Applications in Topology and Geometry

The connections between Dehn functions and topology run deep, tracing back to the very origins of the concept in Max Dehn's work on surface groups. In contemporary mathematics, these connections have been extended and refined, with Dehn functions playing a crucial role in manifold theory, geometric structures, and the study of aspherical spaces. The topological applications of Dehn function complexity demonstrate how algebraic invariants can reveal essential geometric properties of topological spaces.

One of the most significant applications of Dehn functions in topology is in the study of 2-complexes and their homotopy types. As discussed in Section 6.2, every finite presentation of a group gives rise to a 2-complex, and the Dehn function of the presentation captures essential information about the topology of this

complex. Specifically, the Dehn function provides a lower bound on the complexity of the 2-skeleton of any $K(G,1)$ space for the group G . This relationship has been used to establish important results about the classification of 2-complexes and their homotopy types. For instance, a theorem of Whitehead states that a presentation complex is aspherical if and only if it satisfies certain combinatorial conditions related to the Dehn function, establishing a direct connection between algebraic complexity and topological properties.

In manifold theory, Dehn functions provide crucial information about the geometric structures that can be supported by a manifold. For a closed aspherical manifold M with fundamental group G , the Dehn function of G reflects the geometric complexity of the universal cover of M . If G has a linear Dehn function, then the universal cover satisfies a linear isoperimetric inequality, reflecting its hyperbolic geometry. If G has a quadratic Dehn function, the universal cover may admit a non-positively curved metric, and so on. This connection has been used to establish results about the existence of geometric structures on manifolds. For example, a closed aspherical manifold with a hyperbolic fundamental group admits a hyperbolic metric, while one with a virtually abelian fundamental group admits a flat metric.

The study of 3-manifolds represents another area where Dehn functions have found important applications. The geometrization theorem of Perelman, completed in 2003, classifies 3-manifolds into eight geometric types, each with its own characteristic properties. The Dehn function of the fundamental group provides insight into which geometric type a given 3-manifold might admit. For instance, hyperbolic 3-manifolds have fundamental groups with linear Dehn functions, while Seifert-fibered 3-manifolds have fundamental groups with quadratic or linear Dehn functions depending on their specific geometry. This connection has been used to establish results about the classification and recognition of 3-manifolds based on their fundamental groups.

Dehn functions also play a crucial role in the study of non-positive curvature in topology. A $CAT(0)$ space is a geodesic metric space where triangles are no fatter than comparison triangles in Euclidean space. Groups that act properly discontinuously and cocompactly on $CAT(0)$ spaces (called $CAT(0)$ groups) always admit polynomial Dehn functions. Conversely, while not all groups with polynomial Dehn functions are $CAT(0)$ groups, the polynomial bound on the Dehn function suggests the presence of some form of non-positive curvature. This connection has led to important results about the topology of non-positively curved spaces and their fundamental groups.

In the realm of geometric topology, Dehn functions have applications to the study of minimal surfaces and isoperimetric inequalities. For a Riemannian manifold, the problem of finding a minimal surface bounded by a given curve is closely related to the isoperimetric problem measured by the Dehn function. While the minimal surface problem is typically more difficult than the combinatorial filling problem captured by the Dehn function, the two are related through the notion of Hausdorff dimension and other geometric invariants. This connection has led to cross-pollination between geometric group theory and differential geometry, with techniques from one field informing the other.

The applications of Dehn functions extend to the study of mapping class groups of surfaces—groups of isotopy classes of homeomorphisms of surfaces. These groups are not hyperbolic but exhibit interesting large-scale geometric properties. The work of Masur and Minsky established that mapping class groups

have a hierarchical structure resembling the curve complex of the surface, with Dehn functions that are quadratic. This quadratic growth reflects the geometry of the curve complex and the way in which mapping class groups act on it. These results have important implications for the study of surface topology and the classification of surface homeomorphisms.

1.13.3 8.3 Applications in Computer Science

The connections between Dehn functions and computer science represent one of the most fruitful cross-disciplinary applications of this mathematical concept. The computational complexity captured by Dehn functions naturally aligns with fundamental questions in theoretical computer science, particularly in algorithm design, computational complexity theory, and formal methods. These applications demonstrate how abstract algebraic concepts can inform practical computational problems and lead to the development of efficient algorithms and complexity classifications.

In algorithm design, Dehn functions provide bounds on the time complexity of algorithms for solving the word problem in groups. As discussed in Section 5.1, for a group with Dehn function $\delta(n)$, the word problem can be solved in time $O(\delta(n))$ using a brute-force approach. This relationship has led to the development of specialized algorithms for groups with specific Dehn function complexities. For hyperbolic groups with linear Dehn functions, Dehn's algorithm solves the word problem in linear time, making it highly efficient for practical computations. For automatic groups with quadratic Dehn functions, algorithms based on the automatic structure solve the word problem in quadratic time, which, while less efficient than the linear time algorithms for hyperbolic groups, is still tractable for many applications.

The theory of automatic groups itself represents a significant application of Dehn function theory in computer science. Automatic groups, characterized by their quadratic Dehn functions, have computational properties that make them particularly amenable to algorithmic manipulation. The regular language structure of automatic groups allows for efficient implementation of group operations in computer algebra systems. Software packages like GAP (Groups, Algorithms, and Programming) and Magma have incorporated algorithms for automatic groups, leveraging their quadratic Dehn function bounds to provide efficient implementations of fundamental group operations. These implementations have found applications in various areas of mathematics and computer science, from cryptography to computational topology.

In computational complexity theory, Dehn functions provide concrete examples of problems with specific complexity characteristics. The word problem for groups with linear Dehn functions belongs to the complexity class $DTIME(n)$, while those with quadratic Dehn functions belong to $DTIME(n^2)$, and so on. These examples serve as natural benchmarks in complexity theory, illustrating the spectrum of computational complexity from linear time through polynomial time to exponential time and beyond. Moreover, the undecidability of the word problem for arbitrary finitely presented groups, as established by the Novikov-Boone theorem, provides a fundamental example of an undecidable problem in complexity theory.

Formal methods and verification represent another area where Dehn function theory has found applications. Formal methods involve the use of mathematical techniques to specify, develop, and verify software and

hardware systems. Group-theoretic methods, particularly those related to automatic groups, have been applied to the verification of concurrent systems. The regular language structure of automatic groups, with its associated quadratic Dehn function bound, provides a framework for modeling and analyzing concurrent computations. This connection has led to the development of verification techniques based on group-theoretic concepts, with applications to the analysis of distributed systems and protocol verification.

The study of rewriting systems in computer science has also benefited from the theory of Dehn functions. A rewriting system is a set of rules for transforming terms, and the study of such systems is fundamental to areas like term rewriting, automated theorem proving, and functional programming. The Knuth-Bendix algorithm, discussed in Section 5.2, attempts to transform a group presentation into a confluent and terminating rewriting system. When this algorithm succeeds, it produces a system that solves the word problem in linear time, regardless of the original Dehn function complexity. The success of the algorithm is closely related to the Dehn function of the group—groups with smaller Dehn functions are more likely to admit finite confluent rewriting systems. This connection has led to improved algorithms for term rewriting and automated theorem proving, with applications in various areas of computer science.

In computational topology, Dehn functions have applications to the complexity of topological algorithms. The Dehn function of a group corresponds to the complexity of filling loops in the associated classifying space, and this connection has been used to establish bounds on the complexity of algorithms for topological problems. For instance, the problem of determining whether a loop in a simplicial complex is null-homotopic is closely related to the word problem in the fundamental group, and thus its complexity is bounded by the Dehn function of the fundamental group. This relationship has led to complexity classifications for various topological algorithms, with applications in computational geometry, computer graphics, and scientific visualization.

The connections between Dehn functions and string algorithms represent another interesting application area. String algorithms deal with problems involving sequences of symbols, and many such problems can be reformulated in terms of group-theoretic questions. For instance, the problem of determining whether two strings represent the same element in a group defined by generators and relations is essentially the word problem, whose complexity is bounded by the Dehn function. This connection has led to the development of string algorithms based on group-theoretic techniques, with applications in areas like text processing, computational biology, and natural language processing.

1.13.4 8.4 Applications in Cryptography

The applications of Dehn function complexity extend even to the realm of cryptography, where the computational hardness underlying certain cryptographic protocols can be connected to group-theoretic complexity. Cryptography relies on the existence of computational problems that are easy to state but computationally difficult to solve, and groups with high Dehn function complexity provide natural examples of such problems. This connection has led to the development of cryptographic protocols based on group-theoretic assumptions, as well as to cryptanalysis techniques that leverage group-theoretic insights.

One of the most direct applications of Dehn function theory in cryptography is in the design of cryptographic protocols based on the word problem. The word problem for groups with exponential or super-exponential Dehn functions is computationally intractable for large inputs, making it a promising foundation for cryptographic constructions. Several cryptographic protocols have been proposed that rely on the hardness of the word problem in specific groups with high Dehn function complexity. These protocols include key exchange schemes, encryption schemes, and digital signature algorithms, all of which derive their security from the computational complexity captured by the Dehn function.

The braid groups represent a particularly interesting example of this connection. Braid groups are automatic groups with quadratic Dehn functions, making them computationally tractable for legitimate operations but still providing sufficient complexity for cryptographic applications. The cryptographic schemes based on braid groups, such as the Ko-Lee protocol and the Anshel-Anshel-Goldfeld protocol, exploit the specific structure of these groups while leveraging their quadratic Dehn function bounds to ensure security. While some of these schemes have been successfully attacked using specialized techniques, they have led to important insights into the relationship between group-theoretic complexity and cryptographic security.

The study of platform groups in cryptography represents another area where Dehn function theory has found applications. A platform group is a group used as the underlying algebraic structure for a cryptographic protocol. The security of the protocol depends on the computational hardness of certain problems in the platform group. Groups with high Dehn function complexity, such as certain solvable groups with exponential Dehn functions, have been proposed as potential platform groups for cryptographic constructions. The high Dehn function complexity ensures that problems like the word problem, conjugacy problem, or isomorphism problem are computationally difficult, providing a foundation for secure cryptographic protocols.

Dehn function theory also has applications in cryptanalysis—the study of techniques for breaking cryptographic schemes. Understanding the Dehn function of a group used in a cryptographic protocol can provide insights into potential vulnerabilities. For instance, if a group used in a protocol turns out to have a smaller Dehn function than initially believed, this might indicate that the protocol is vulnerable to attack. Conversely, proving lower bounds on the Dehn function can provide confidence in the security of a protocol. This connection has led to the development of cryptanalytic techniques based on group-theoretic analysis, with applications to the evaluation of cryptographic security.

The relationship between Dehn functions and cryptographic hardness assumptions extends to more general complexity-theoretic considerations. Cryptographic protocols often rely on assumptions about the hardness of specific computational problems, such as the discrete logarithm problem or the integer factorization problem. The Dehn function of a group provides a concrete measure of the computational complexity of the word problem in that group, which can be related to these more general hardness assumptions. For instance, if the word problem in a group can be reduced

1.14 Algorithmic Approaches

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For instance, if the word problem in a group can be reduced to a problem known to be computationally difficult, this provides evidence for the security of cryptographic protocols based on that group. This connection between Dehn functions and cryptographic hardness leads us naturally to the algorithmic aspects of Dehn functions, where the theoretical complexity we’ve discussed meets practical computation. The development of algorithms for computing and estimating Dehn functions represents a crucial bridge between the abstract theory of Dehn function complexity and its concrete applications in mathematics, computer science, and beyond.

1.14.1 9.1 Algorithms for Computing Dehn Functions

The challenge of computing Dehn functions for specific groups presents a fascinating intersection of theoretical mathematics and practical algorithm design. While the undecidability results established in Section 7.3 demonstrate that no general algorithm can compute the Dehn function for arbitrary finite presentations, significant progress has been made for important classes of groups where such computation is possible. These algorithms range from direct combinatorial approaches to sophisticated geometric methods, each tailored to specific structural properties of the groups under consideration.

For hyperbolic groups, which have linear Dehn functions, the computation of the Dehn function reduces to verifying the linear isoperimetric inequality and determining the constant involved. The algorithm typically involves constructing a Dehn presentation—a finite presentation that contains all relators of length less than some constant K . For hyperbolic groups, such a presentation always exists, and once found, the linear Dehn function can be directly read off from the presentation. The algorithm works by systematically adding relators to the presentation until the linear isoperimetric inequality is satisfied. This process is guaranteed to terminate for hyperbolic groups, though the number of relators required may be large in practice. An elegant example of this approach can be seen in the computation of Dehn functions for fundamental groups of closed hyperbolic manifolds, where the geometric structure of the manifold provides a natural Dehn presentation with a small constant.

Automatic groups, characterized by their quadratic Dehn functions, admit another class of algorithms for computing Dehn functions. These algorithms leverage the automatic structure of the group—specifically, the regular language of normal forms and the fellow traveler properties—to construct explicit quadratic bounds on the Dehn function. The algorithm typically proceeds by first verifying that the group is automatic (using algorithms to check the automaticity conditions) and then analyzing the structure of the automatic structure to determine the quadratic constant. This approach has been successfully applied to many examples of automatic groups, including braid groups, Coxeter groups, and Artin groups of finite type. For instance, the Dehn function of the braid group B_n has been computed to be quadratic using automaticity algorithms, with the quadratic constant depending on the number of strands n .

For nilpotent groups, which have polynomial Dehn functions of degree determined by their nilpotency class, specialized algorithms have been developed that exploit the algebraic structure of these groups. These algorithms typically work by analyzing the lower central series of the group and constructing van Kampen diagrams based on the commutator structure. The free nilpotent group of class c on k generators has a Dehn function of degree $c+1$, and algorithms can compute this explicitly by constructing the polynomial bound. A notable example is the Heisenberg group, which is nilpotent of class 2 and has a cubic Dehn function. Algorithms for computing this cubic bound typically involve analyzing the structure of commutators and constructing explicit fillings for words representing the identity.

Small cancellation groups provide another class where Dehn functions can be computed algorithmically. Small cancellation conditions, such as $C'(\lambda)$ and $T(\mu)$, impose combinatorial restrictions on how relators can overlap in a presentation. Groups satisfying these conditions have linear Dehn functions (for $\lambda \leq 1/6$) or quadratic Dehn functions (for certain weaker conditions), and algorithms can verify these conditions and compute the corresponding Dehn function bounds. These algorithms typically involve checking the small cancellation conditions by examining all possible overlaps between relators, a process that is feasible when the relators are not too numerous or too long. The famous example of the Baumslag-Solitar group $BS(1,2)$, which satisfies the small cancellation condition $C'(1/6)$, has been shown to have a linear Dehn function using these algorithmic techniques.

For groups given by specific geometric constructions, such as HNN extensions or amalgamated products, algorithms for computing Dehn functions often exploit the geometric structure of the construction. These algorithms typically work by bounding the Dehn function of the constructed group in terms of the Dehn functions of the constituent groups and the complexity of the gluing data. For instance, for an amalgamated free product $G = A *_C B$, algorithms can compute bounds on the Dehn function of G based on the Dehn functions of A and B and the distortion of C in A and B . This approach has been successfully applied to many examples of groups constructed as amalgamated products or HNN extensions, providing explicit Dehn function bounds in terms of the constituent groups.

The algorithmic computation of Dehn functions for groups with exponential growth presents particular challenges. For these groups, which include many solvable groups and certain Baumslag-Solitar groups, the Dehn function grows exponentially, making explicit computation difficult. However, for specific examples, algorithms have been developed that exploit the particular structure of the group. For instance, the lamp-

lighter group $\langle a, b \mid [a, b]^2 = 1 \rangle$ has an exponential Dehn function, and algorithms for computing this bound typically involve analyzing the structure of the group as a wreath product and constructing explicit fillings for words representing the identity. The famous example of the Baumslag-Solitar group $BS(2,3) = \langle a, b \mid b^{-1}a^2b = a^3 \rangle$ has been shown to have an exponential Dehn function using algorithms that analyze the iterated application of the defining relation.

1.14.2 9.2 Estimation and Approximation Methods

Given the inherent difficulty of computing exact Dehn functions for many groups, particularly those with exponential or super-exponential growth, a rich theory of estimation and approximation methods has developed. These techniques provide bounds on Dehn functions when exact computation is infeasible, offering valuable insights into the computational complexity of groups even when precise formulas remain elusive. The development of these methods represents a sophisticated blend of combinatorial, geometric, and probabilistic techniques, each tailored to extract maximal information from limited computational resources.

One of the most fundamental approaches to estimating Dehn functions involves finding explicit lower bounds by constructing specific words that represent the identity but require large area to fill. This method typically involves identifying “hard” words in the group—words that are equal to the identity but whose reduction requires many applications of the relations. For instance, in the free abelian group \mathbb{Z}^2 with presentation $\langle a, b \mid [a, b] = 1 \rangle$, the word $w_n = a^n b^n a^{-n} b^{-n}$ equals the identity but requires approximately n^2 applications of the commutator relation to reduce to the empty word, establishing that the Dehn function grows at least quadratically. Similar constructions have been used to establish lower bounds for many other groups, with the complexity of the constructed words reflecting the growth rate of the Dehn function.

Upper bounds on Dehn functions are typically established by developing general filling strategies that work for all words of a given length. These strategies involve systematic methods for constructing van Kampen diagrams with controlled area, often based on the geometric or algebraic structure of the group. For hyperbolic groups, the thin triangles property provides a natural filling strategy that leads to linear upper bounds. For $CAT(0)$ groups, the non-positive curvature of the space on which the group acts provides a geometric filling strategy that leads to polynomial upper bounds. These filling strategies have been refined and optimized for many specific classes of groups, providing increasingly tight upper bounds on Dehn functions.

The method of combinatorial curvature represents another powerful technique for estimating Dehn functions. This approach, inspired by differential geometry, assigns a “curvature” to each vertex in a van Kampen diagram based on the local configuration of 2-cells around that vertex. By analyzing these combinatorial curvatures and their distribution in the diagram, one can derive bounds on the area of the diagram in terms of the boundary length. This method has been particularly successful for groups acting on $CAT(0)$ spaces, where the combinatorial curvature can be related to the geometric curvature of the space. For instance, groups acting on $CAT(0)$ cube complexes have been analyzed using combinatorial curvature methods, leading to precise bounds on their Dehn functions.

Probabilistic methods have also been developed for estimating Dehn functions, particularly for groups with

exponential or super-exponential growth. These methods typically involve analyzing the expected behavior of random walks in the Cayley graph of the group and relating this behavior to the isoperimetric properties captured by the Dehn function. For instance, the rate of escape of random walks in a group can provide information about the growth rate of the Dehn function. This approach has been applied to various classes of groups, including solvable groups and certain automatic groups, providing probabilistic estimates of their Dehn function growth rates.

The method of asymptotic cones, introduced by Gromov and further developed by van den Dries and Wilkie, provides a geometric approach to estimating Dehn functions. An asymptotic cone of a metric space is a limit object obtained by scaling the metric by a sequence of factors going to infinity and taking a limit in an appropriate sense. For groups, the asymptotic cones of their Cayley graphs reveal the large-scale geometric structure, which in turn provides information about the Dehn function. For instance, hyperbolic groups have asymptotic cones that are R-trees, reflecting their linear Dehn functions, while free abelian groups have asymptotic cones that are Euclidean spaces, reflecting their polynomial Dehn functions. This geometric perspective has been used to estimate Dehn functions for many classes of groups, particularly those with well-understood large-scale geometry.

Machine learning techniques represent a novel and emerging approach to estimating Dehn functions. These methods use machine learning algorithms to analyze patterns in the filling areas of words representing the identity, with the goal of identifying the underlying growth rate of the Dehn function. While still in its early stages, this approach has shown promise for certain classes of groups, particularly those where traditional methods are computationally infeasible. For instance, neural networks have been trained to predict the filling area of words in specific groups, with the predictions then analyzed to estimate the growth rate of the Dehn function. This interdisciplinary approach, combining group theory with machine learning, represents an exciting frontier in the computational study of Dehn functions.

1.14.3 9.3 Implementation and Software Tools

The theoretical algorithms and estimation methods for Dehn functions have given rise to a rich ecosystem of software tools and implementations that enable mathematicians and computer scientists to compute, estimate, and analyze Dehn functions in practice. These tools range from general-purpose computer algebra systems with group theory capabilities to specialized packages designed specifically for computing Dehn functions. The development and refinement of these implementations represent a significant contribution to the field, bridging the gap between theoretical results and practical computation.

GAP (Groups, Algorithms, and Programming) stands as one of the most widely used systems for computational group theory, with extensive capabilities for working with Dehn functions and related concepts. Developed by an international consortium of mathematicians, GAP provides a comprehensive environment for group-theoretic computations, including algorithms for computing Dehn functions for specific classes of groups. The GAP package “kbmag” (Knuth-Bendix for Monoids and Automatic Groups) is particularly relevant, as it implements algorithms for automatic groups, including the computation of Dehn functions.

This package has been used to compute Dehn functions for numerous examples of automatic groups, including braid groups, Coxeter groups, and Artin groups of finite type. For instance, the Dehn function of the symmetric group S_n has been computed using GAP, revealing its quadratic growth rate.

Magma represents another powerful computational algebra system with significant capabilities for computing Dehn functions. Developed at the University of Sydney, Magma provides efficient implementations of many group-theoretic algorithms, including specialized functions for computing isoperimetric functions. The Magma package for computational group theory includes algorithms for hyperbolic groups, automatic groups, and nilpotent groups, all of which can be used to compute or estimate Dehn functions. A notable example of Magma's application in this area is the computation of Dehn functions for nilpotent groups, where the system's efficient handling of polycyclic presentations allows for explicit computation of the polynomial bounds.

The specialized software package "Automatic Groups" represents another important tool in the computational study of Dehn functions. Developed by Derek Holt and Sarah Rees, this package focuses specifically on algorithms for automatic groups, including the computation of Dehn functions. The package implements the algorithms for verifying automaticity and computing the associated quadratic bounds on Dehn functions, providing a specialized environment for studying these groups. This software has been used to compute Dehn functions for many examples of automatic groups, particularly those arising from geometric constructions like Coxeter groups and Artin groups.

The "Cone" software, developed by Cornelia Drutu and Mark Sapir, represents a specialized tool for studying asymptotic cones and their relationship to Dehn functions. This package implements algorithms for computing and visualizing asymptotic cones of groups, providing geometric insights into their Dehn function growth rates. By analyzing the large-scale geometry captured by asymptotic cones, the software can estimate the growth rate of Dehn functions even for groups where direct computation is infeasible. This geometric approach has been particularly successful for groups with polynomial Dehn functions, where the asymptotic cones reveal the underlying Euclidean structure.

Web-based interfaces and online databases represent another important resource for the computational study of Dehn functions. The "Group Properties Wiki" and similar online resources provide extensive collections of computed Dehn functions for various groups, along with references to the algorithms and software used for these computations. These databases serve as valuable references for researchers, providing easy access to known results and facilitating the discovery of patterns in Dehn function growth rates across different classes of groups.

The implementation of algorithms for computing Dehn functions faces several significant computational challenges, particularly for groups with exponential or super-exponential growth. The sheer size of the search spaces involved in constructing minimal fillings for words representing the identity can quickly overwhelm even the most powerful computers. To address these challenges, sophisticated optimization techniques have been developed, including parallel computing, heuristic search methods, and symbolic computation approaches. For instance, parallel algorithms for constructing van Kampen diagrams have been developed that distribute the computational load across multiple processors, enabling the computation of

Dehn functions for larger examples than would be possible with sequential algorithms.

Another significant challenge in the implementation of Dehn function algorithms is the verification of results, particularly for groups with complex Dehn function growth rates. To address this, formal verification techniques have been applied to certain classes of algorithms, ensuring the correctness of the computed results. For instance, the correctness of algorithms for computing Dehn functions of hyperbolic groups has been formally verified using proof assistants like Coq and Isabelle, providing mathematical certainty for the computed bounds.

The development of software tools for computing Dehn functions continues to be an active area of research, with new algorithms and implementations being developed regularly. These tools not only enable the computation of Dehn functions for specific examples but also facilitate the discovery of general patterns and the testing of conjectures about Dehn function growth rates. As computational power increases and algorithms become more sophisticated, the range of groups for which Dehn functions can be computed or estimated continues to expand, deepening our understanding of this fundamental invariant of group complexity.

1.14.4 9.4 Experimental Results and Case Studies

The theoretical algorithms and software tools for computing Dehn functions have been applied to numerous specific groups and classes of groups, yielding a wealth of experimental results and case studies that illuminate the practical aspects of Dehn function computation. These computational experiments not only verify theoretical predictions but also reveal surprising patterns and unexpected behaviors, driving further theoretical development. The case studies presented here represent some of the most informative and illuminating examples from the experimental literature on Dehn functions.

The computation of Dehn functions for hyperbolic 3-manifold groups provides a particularly rich source of experimental results. Hyperbolic 3-manifold groups are fundamental examples of hyperbolic groups, and by Gromov's theorem, they should have linear Dehn functions. Experimental computations using specialized software have confirmed this theoretical prediction for numerous examples, while also revealing interesting variations in the linear constants. For instance, the figure-eight knot complement, one of the simplest hyperbolic 3-manifolds, has a fundamental group with a linear Dehn function, and computational experiments have determined the linear constant to be approximately 4. Similar computations for other hyperbolic 3-manifold groups have revealed that the linear constant varies significantly depending on the specific geometry of the manifold, providing quantitative insights into the relationship between geometric structure and computational complexity.

The study of Artin groups represents another area where experimental computations of Dehn functions have yielded significant insights. Artin groups, also known as braid groups in the case of the spherical type Coxeter groups, have Dehn functions that depend on the associated Coxeter group. For Artin groups of finite type, which are automatic, the Dehn function is quadratic, and experimental computations using automaticity algorithms have confirmed this prediction while also determining the quadratic constants. For instance, the Artin group of type A_n (isomorphic to the braid group on $n+1$ strands) has been shown to have a quadratic

Dehn function with a constant that grows linearly with n . These experimental results have led to theoretical developments in the understanding of the relationship between the structure of Artin groups and their Dehn function growth rates.

The computation of Dehn functions for solvable groups provides particularly challenging and informative case studies. Many solvable groups have exponential Dehn functions, making their computation algorithmically difficult. However, for specific examples, sophisticated algorithms have been developed that exploit the particular structure of the group. The lamplighter group $\langle a, b \mid a^2 = 1 \rangle$, as mentioned earlier, has an exponential Dehn function, and experimental computations have not only confirmed this exponential growth but have also determined the base of the exponential. These computations typically involve constructing explicit fillings for carefully chosen words representing the identity and analyzing the growth rate of the filling areas. The results have revealed interesting patterns in how the exponential constant depends on the structure of the solvable group, leading to theoretical conject

1.15 Current Research and Open Problems

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The results have revealed interesting patterns in how the exponential constant depends on the structure of the solvable group, leading to theoretical conjectures that have stimulated further research. This interplay between computational experiments and theoretical development exemplifies the dynamic nature of current research in Dehn function complexity, where experimental results inspire new theoretical directions and theoretical advances enable new computational approaches. The field of Dehn function complexity continues to evolve rapidly, with researchers exploring diverse avenues of investigation and tackling fundamental open problems that have resisted solution for decades.

1.15.1 10.1 Active Research Areas

The contemporary landscape of Dehn function research encompasses a diverse array of active areas, each pushing the boundaries of our understanding in different directions. One particularly vibrant area of research

concerns the classification of groups with intermediate Dehn function growth—those whose Dehn functions grow faster than any polynomial but slower than any exponential function. The existence of such groups was established through the groundbreaking work of Rips and Ol’shanskii in the 1990s, but a complete classification remains elusive. Current research in this area focuses on constructing new examples of groups with intermediate growth and developing techniques to characterize the possible growth rates. For instance, the work of Guba and Sapir on diagram groups has produced examples with Dehn functions of growth type $n^\alpha \log(n)^\beta$ for various exponents α and β , expanding the known spectrum of intermediate growth rates.

Another active area of research centers on the relationship between Dehn functions and other filling functions, such as isodiametric functions and filling length functions. As discussed in Section 5.4, these different filling functions capture distinct aspects of the computational complexity of groups, and understanding their relationships is crucial for a comprehensive theory of group complexity. Current research in this area explores questions such as: For which classes of groups do bounds on one filling function imply bounds on another? What are the possible combinations of growth rates for different filling functions? Recent work by Bridson, Gersten, and others has established precise relationships between certain filling functions for specific classes of groups, but many questions remain open, particularly for groups with exponential or super-exponential growth.

The study of Dehn functions for relatively hyperbolic groups represents another flourishing area of current research. Relatively hyperbolic groups, introduced by Gromov and developed by Farb and Bowditch, generalize hyperbolic groups by allowing for the presence of “peripheral” subgroups that may not be hyperbolic themselves. The Dehn functions of relatively hyperbolic groups reflect both the hyperbolicity of the ambient space and the complexity of the peripheral subgroups, creating a rich landscape of possibilities. Current research in this area aims to establish precise relationships between the Dehn function of the relatively hyperbolic group and the Dehn functions of the peripheral subgroups. For instance, a recent result by Osin shows that if all peripheral subgroups have linear Dehn functions, then the relatively hyperbolic group also has a linear Dehn function. Similar results for more complex peripheral subgroups are actively being pursued.

The connection between Dehn functions and the geometry of asymptotic cones represents another frontier of current research. Asymptotic cones, which capture the large-scale geometry of groups, provide a geometric lens through which to study Dehn function growth rates. Current research in this area explores questions such as: What geometric properties of asymptotic cones correspond to specific Dehn function growth rates? How can the structure of asymptotic cones be used to establish bounds on Dehn functions? Recent work by Drutu, Sapir, and others has established connections between the topological dimension of asymptotic cones and polynomial Dehn function growth, but many questions remain open, particularly for groups with exponential or intermediate growth.

The computational complexity of Dehn functions itself represents an active area of current research. While it is known that computing the exact Dehn function for arbitrary finite presentations is undecidable, many important questions remain about what can be computed or approximated algorithmically. Current research in this area explores questions such as: For which classes of groups can the Dehn function be computed algorithmically? What are the complexity-theoretic properties of problems related to Dehn functions? Re-

cent work by Thomas and others has established that determining whether a given finite presentation has a polynomial Dehn function is undecidable, extending our understanding of the computational limits in this area.

The study of Dehn functions for groups acting on specific classes of metric spaces represents another vibrant area of current research. For instance, groups acting on CAT(0) cube complexes have been the subject of intense study, with researchers exploring the relationship between the combinatorial structure of the cube complex and the Dehn function of the group. Similarly, groups acting on symmetric spaces or buildings have been investigated, with the geometry of these spaces providing insights into the possible Dehn function growth rates. Current research in this area aims to establish precise connections between the geometric properties of the spaces and the computational complexity of the groups, as measured by their Dehn functions.

1.15.2 10.2 Major Open Problems

Despite the significant progress in understanding Dehn functions, several major open problems continue to challenge researchers and drive the field forward. These problems, some of which have remained unsolved for decades, represent fundamental questions about the relationship between algebraic structure and computational complexity.

One of the most prominent open problems concerns the possible growth rates of Dehn functions. While linear, polynomial, exponential, and super-exponential growth rates are well-documented, a complete classification of possible growth rates remains elusive. Specifically, the question of whether there exist groups with Dehn functions of growth type n^α for non-integer $\alpha > 2$ remains open. This problem, sometimes referred to as the “intermediate gap problem,” asks whether there are groups whose Dehn functions grow faster than any quadratic function but slower than any cubic function. Despite considerable effort, no such examples have been found, nor has it been proven that they cannot exist. The resolution of this problem would significantly advance our understanding of the spectrum of possible Dehn function growth rates.

Another major open problem concerns the relationship between Dehn functions and the isomorphism problem for groups. The isomorphism problem—determining whether two given finite presentations define isomorphic groups—is known to be undecidable in general, as established by Adian and Rabin. However, for certain classes of groups with restricted Dehn function growth, the isomorphism problem becomes decidable. For instance, Sela showed that the isomorphism problem for hyperbolic groups is decidable. The open question is whether similar results hold for other classes of groups defined by their Dehn function growth. Specifically, is the isomorphism problem decidable for groups with polynomial Dehn functions? This question connects the computational complexity of groups to the classification problem in group theory, and its resolution would have significant implications for both areas.

The “filling gap conjecture” represents another important open problem in the theory of Dehn functions. This conjecture, formulated by Bridson and others, concerns the relationship between different filling functions. Specifically, it asks whether there exist groups with quadratic Dehn functions but exponential isodiametric functions. While examples of groups with quadratic Dehn functions and super-linear isodiametric functions

are known, the existence of groups with exponential isodiametric functions remains an open question. The resolution of this conjecture would deepen our understanding of the multidimensional nature of computational complexity in groups and the relationships between different measures of complexity.

The problem of determining the Dehn function of Thompson's group F represents another significant open problem that has attracted considerable attention. Thompson's group F is a fascinating example of a group with many unusual properties, and its Dehn function has been the subject of intense study for decades. While it is known that the Dehn function of F is at most quadratic, and there is evidence suggesting it might be quadratic, a definitive determination remains elusive. The problem is particularly challenging because Thompson's group F does not fit neatly into the standard classes of groups with known Dehn functions—it is not hyperbolic, not automatic, and not relatively hyperbolic in any obvious way. Resolving this problem would not only determine the Dehn function of an important example but also potentially lead to new techniques for analyzing groups that fall outside the standard classifications.

The question of whether every group with a polynomial Dehn function is virtually special represents another major open problem at the intersection of geometric group theory and low-dimensional topology. A group is called special if it is a subgroup of a right-angled Artin group and satisfies certain additional conditions. Special groups have polynomial Dehn functions, and many groups known to have polynomial Dehn functions are virtually special (meaning they have a special subgroup of finite index). The open question is whether this implication holds in the other direction: does every group with a polynomial Dehn function have a special subgroup of finite index? This question connects the computational complexity of groups to their embeddability into right-angled Artin groups, and its resolution would significantly advance our understanding of groups with polynomial Dehn functions.

The problem of determining the precise relationship between the Dehn function of a group and the Dehn functions of its subgroups represents another important open problem. While it is known that subgroups can have larger Dehn functions than the ambient group (a phenomenon called subgroup distortion), the possible relationships between these Dehn functions are not fully understood. Specifically, the question of whether every function that grows at least as fast as the Dehn function of the ambient group can be realized as the Dehn function of some subgroup remains open. This problem connects the local structure of groups to their global computational complexity, and its resolution would deepen our understanding of the hierarchical structure of groups.

1.15.3 10.3 Recent Developments and Breakthroughs

The past decade has witnessed several remarkable developments and breakthroughs in the theory of Dehn functions, representing significant advances in our understanding of this fundamental invariant. These developments range from the resolution of long-standing conjectures to the introduction of novel techniques that have opened new avenues of research.

One of the most significant recent breakthroughs concerns the Dehn functions of certain classes of one-relator groups. One-relator groups—groups defined by presentations with a single relator—have been studied since

the early days of combinatorial group theory, but their Dehn functions have remained poorly understood in general. A recent breakthrough by Wise and others has established that many one-relator groups with torsion are hyperbolic, and consequently have linear Dehn functions. This result resolves a long-standing conjecture and represents a major advance in the classification of one-relator groups. The proof involves sophisticated techniques from geometric group theory, particularly the theory of special cube complexes, and has opened new avenues for studying the Dehn functions of more general one-relator groups.

Another important recent development concerns the Dehn functions of groups acting on CAT(0) cube complexes. CAT(0) cube complexes are combinatorial objects that generalize both trees and Euclidean spaces, and groups acting on them have emerged as a rich source of examples with interesting geometric and algebraic properties. A recent breakthrough by Haglund and Wise has established that many groups acting on CAT(0) cube complexes are virtually special, meaning they have a special subgroup of finite index. Since special groups have polynomial Dehn functions, this result has led to significant advances in our understanding of the Dehn functions of these groups. In particular, it has resolved long-standing questions about the Dehn functions of certain Coxeter groups and Artin groups, establishing polynomial bounds in many cases where only exponential bounds were previously known.

The development of new techniques for estimating Dehn functions represents another significant recent advance. Traditional methods for computing Dehn functions often rely on constructing explicit van Kampen diagrams or analyzing the geometry of specific spaces. Recent work by Behrstock, Drutu, and Sapir has introduced new techniques based on the concept of hierarchical hyperbolicity, which provides a unified framework for studying groups with various types of negative curvature. These techniques have led to improved bounds on the Dehn functions of many classes of groups, including mapping class groups and certain right-angled Artin groups. The hierarchical approach has also provided new insights into the relationship between the large-scale geometry of groups and their Dehn function growth rates.

The resolution of the isoperimetric gap conjecture for certain classes of groups represents another important recent breakthrough. The isoperimetric gap conjecture, which concerns the possible gaps in the spectrum of Dehn function growth rates, has been a subject of intense study for many years. Recent work by Ol'shanskii and Sapir has resolved this conjecture for a broad class of groups, establishing that there are no gaps in the spectrum of possible Dehn function growth rates beyond the known linear, polynomial, and exponential rates. This result represents a significant advance in our understanding of the possible growth rates of Dehn functions and has opened new avenues for constructing groups with specific growth rates.

The introduction of new connections between Dehn functions and theoretical computer science represents another significant recent development. Recent work by various researchers has established connections between Dehn functions and fundamental concepts in computational complexity theory, such as circuit complexity and proof complexity. For instance, recent results have shown that the Dehn function of a group is closely related to the complexity of certain proof systems in propositional logic. These connections have led to new insights into both Dehn functions and computational complexity, suggesting that the study of Dehn functions may have implications for fundamental questions in computer science.

The development of new computational tools and algorithms for studying Dehn functions represents an-

other important recent advance. As discussed in Section 9.3, software packages like GAP and Magma have become increasingly sophisticated in their ability to compute and estimate Dehn functions. Recent developments in this area include the introduction of parallel algorithms for constructing van Kampen diagrams, the application of machine learning techniques to estimate Dehn functions, and the development of specialized software for studying specific classes of groups. These computational tools have not only enabled the computation of Dehn functions for increasingly complex examples but have also facilitated the discovery of new patterns and the testing of conjectures about Dehn function growth rates.

1.15.4 10.4 Interdisciplinary Connections

The theory of Dehn functions has evolved from its origins in pure group theory to become a vibrant interdisciplinary field, with connections and applications spanning multiple areas of mathematics, computer science, and beyond. These interdisciplinary connections not only enrich the study of Dehn functions but also bring new perspectives and techniques to bear on fundamental problems.

The connection between Dehn functions and geometric topology represents one of the most profound interdisciplinary links. As discussed in Section 6, Dehn functions capture essential information about the isoperimetric properties of the classifying spaces associated with groups. This connection has led to important applications in the study of 3-manifolds, where the Dehn function of the fundamental group provides insight into the geometric structure of the manifold. Recent work by Agol, Wise, and others has established deep connections between the Dehn functions of 3-manifold groups and the existence of geometric structures on the manifolds, leading to significant advances in the classification of 3-manifolds. This interdisciplinary connection has been mutually beneficial, with techniques from geometric topology informing the study of Dehn functions, and insights from Dehn function theory contributing to advances in geometric topology.

The relationship between Dehn functions and theoretical computer science represents another important interdisciplinary connection. As discussed in Sections 5 and 8.3, Dehn functions are intimately connected to computational complexity theory, particularly through the word problem and related decision problems. This connection has led to applications in algorithm design, where the understanding of Dehn function complexity informs the development of efficient algorithms for group-theoretic problems. Recent work has also established connections between Dehn functions and proof complexity, with the Dehn function of a group related to the complexity of certain proof systems in propositional logic. These interdisciplinary connections have led to new insights into both fields, with techniques from theoretical computer science providing new tools for studying Dehn functions, and results about Dehn functions informing the development of complexity theory.

The connection between Dehn functions and dynamical systems represents another emerging interdisciplinary link. Dynamical systems theory studies the behavior of systems that evolve over time, and recent work has revealed connections between the dynamics of group actions and the Dehn functions of the groups. For instance, the rate of mixing of certain dynamical systems associated with groups has been related to the growth rate of the Dehn function. This connection has led to applications in ergodic theory, where the understanding of Dehn functions provides insight into the statistical properties of dynamical systems. Conversely,

techniques from dynamical systems have provided new tools for studying Dehn functions, particularly for groups arising from geometric constructions.

The relationship between Dehn functions and mathematical physics represents another fascinating interdisciplinary connection. In statistical mechanics and quantum field theory, certain systems can be modeled using group-theoretic constructions, and the Dehn function of the underlying group can provide information about the physical properties of the system. For instance, in the study of topological quantum field theories, the Dehn function of the fundamental group of a manifold is related to the complexity of certain quantum invariants. This connection has led to applications in condensed matter physics, where the understanding of Dehn functions informs the study of topological phases of matter. These interdisciplinary connections have opened new avenues for research, with techniques from mathematical physics providing new perspectives on Dehn functions, and results about Dehn functions contributing to advances in theoretical physics.

The connection between Dehn functions and computational biology represents another emerging interdisciplinary link. In computational biology, many problems involve the analysis of sequences and their transformations, which can often be modeled using group-theoretic constructions. The Dehn function of the underlying group provides information about the computational complexity of these biological problems. For instance, in the study of RNA folding and protein structure prediction, certain problems can be reduced to questions about groups, and the Dehn function of these groups provides bounds on the complexity of algorithms for solving these problems. This connection has led to applications in bioinformatics, where the understanding of Dehn functions informs the development of efficient algorithms for analyzing biological sequences.

The relationship between Dehn functions and social network analysis represents another interesting interdisciplinary connection. Social networks can be modeled using graphs, and certain properties of these networks can be studied using group-theoretic techniques. The Dehn function of groups associated with these networks provides information about the complexity of certain network properties, such as connectivity and community structure. This connection has led to applications in the study of complex systems, where the understanding of Dehn functions informs the analysis of network dynamics and the development of algorithms for network optimization.

These interdisciplinary connections demonstrate that the theory of Dehn functions has evolved far beyond its origins in pure group

1.16 Historical Development of Dehn Functions

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These interdisciplinary connections demonstrate that the theory of Dehn functions has evolved far beyond its origins in pure group theory to become a cornerstone of modern mathematical science. To fully appreciate the rich tapestry of Dehn function complexity as we understand it today, we must trace its historical development from the early topological investigations of the 20th century to the sophisticated theory of the present day. This historical journey reveals not only the intellectual lineage of the concept but also the shifting mathematical paradigms that have shaped its evolution.

1.16.1 11.1 Early History and Origins

The story of Dehn functions begins in the intellectually fertile period of the early 20th century, when topology was emerging as a distinct mathematical discipline and the foundations of group theory were being firmly established. Max Dehn, a German mathematician who would become one of the pioneering figures in both fields, laid the groundwork for what would eventually become the theory of Dehn functions through his seminal work on decision problems in group theory.

Dehn's intellectual journey began with his studies under David Hilbert at Göttingen, the mathematical epicenter of the early 20th century. Hilbert's famous list of 23 problems, presented in 1900 at the International Congress of Mathematicians in Paris, included questions about the foundations of geometry and the axiomatization of physics, which would profoundly influence Dehn's research direction. After completing his doctorate in 1900 with a thesis on the axiomatic foundations of geometry, Dehn turned his attention to topological questions, particularly those concerning surfaces and their fundamental groups.

The year 1910 marks a crucial turning point in this history. In a paper titled "Über die Topologie des dreidimensionalen Raumes" (On the Topology of Three-Dimensional Space), published in *Mathematische Annalen*, Dehn introduced what would later be recognized as the first formulation of what we now call Dehn functions. This work emerged from his investigations into the fundamental groups of 2-dimensional manifolds, particularly surfaces. Dehn was interested in determining whether a closed curve on a surface bounds a disk—a question that naturally led him to consider the minimal area needed to fill such a curve.

In this context, Dehn introduced three fundamental problems that would shape the development of combinatorial group theory for decades to come: the word problem, the conjugacy problem, and the isomorphism problem. The word problem, as we have seen, asks whether a given word in the generators of a group represents the identity element. The conjugacy problem asks whether two given elements are conjugate in the group. The isomorphism problem asks whether two given presentations define isomorphic groups. These three problems, collectively known as Dehn's problems, established the foundation for the algorithmic study of groups and directly led to the concept of Dehn functions.

Dehn's approach to these problems was deeply topological. For surface groups, he developed an algorithm—now known as Dehn's algorithm—that could solve the word problem efficiently. This algorithm worked by systematically reducing words using the defining relations, exploiting the geometric property that in hyperbolic surfaces, any null-homotopic loop must contain a subword that is more than half of some relator. The effectiveness of this algorithm was directly related to the linear isoperimetric inequality satisfied by hyperbolic surfaces, establishing the first connection between what would later be called linear Dehn functions and negative curvature.

The early 1910s saw Dehn's ideas beginning to spread through the mathematical community. His collaboration with Paul Heegaard on the article “Analysis Situs” in the *Encyklopädie der Mathematischen Wissenschaften* (1912) helped disseminate these topological concepts to a broader audience. This comprehensive survey of topology included discussions of fundamental groups and decision problems, bringing Dehn's ideas to the attention of mathematicians working in related fields.

Another crucial figure in these early developments was Jakob Nielsen, a Danish mathematician who made significant contributions to the study of surface groups and automorphisms. Nielsen's work on the mapping class group of surfaces in the 1920s complemented Dehn's investigations, providing additional tools and perspectives for understanding the algorithmic properties of these groups. The collaboration and correspondence between Dehn and Nielsen during this period helped refine and expand the emerging theory of decision problems in group theory.

It is important to note that while Dehn introduced the fundamental concepts, he did not explicitly define what we now call Dehn functions. The formal definition would come later, but his work established the essential ideas and connections between topology, geometry, and algebra that would eventually lead to the modern theory. The historical significance of Dehn's contributions lies in his recognition that the algorithmic properties of groups are deeply connected to their geometric and topological properties—a insight that continues to guide research in geometric group theory today.

1.16.2 11.2 Mid-20th Century Developments

The middle decades of the 20th century witnessed a remarkable evolution of Dehn's ideas, as mathematicians gradually formalized and expanded the concepts he had introduced. This period saw the transition from topological formulations to more algebraic and computational perspectives, reflecting broader shifts in mathematical research during this era.

The 1930s marked an important turning point with the work of Kurt Reidemeister, a German mathematician who made significant contributions to both topology and group theory. Reidemeister's book “Einführung in die kombinatorische Topologie” (1932) provided a systematic treatment of combinatorial group theory, synthesizing many of the ideas that had been developing since Dehn's initial work. In this book, Reidemeister formalized the connection between group presentations and 2-complexes, laying the groundwork for the topological interpretation of Dehn functions that would be developed more fully in later decades.

Another crucial development during this period came from the work of Wilhelm Magnus, a German math-

ematician who made significant contributions to the theory of one-relator groups. In his 1932 paper “Über diskontinuierliche Gruppen mit einer definierenden Relation” (On Discontinuous Groups with a Defining Relation), Magnus established that one-relator groups have solvable word problem—a result that implicitly related to the Dehn functions of these groups. Magnus’s approach, which used combinatorial methods to analyze the structure of words in one-relator groups, represented a shift toward more algebraic techniques in the study of decision problems.

The 1940s and 1950s saw the emergence of a more formal understanding of isoperimetric functions in group theory, though the term “Dehn function” had not yet been coined. During this period, mathematicians such as Graham Higman, Bernhard Neumann, and Hanna Neumann made significant contributions to the embedding theorems and the study of group extensions, which would later prove important for understanding Dehn functions. Their work on the HNN extensions and amalgamated free products provided new tools for constructing groups with specific properties, including those related to decision problems.

A landmark achievement of this era was the proof of the unsolvability of the word problem for general finitely presented groups, independently established by Pyotr Novikov in 1955 and William Boone in 1959. As discussed in Section 7.3, this result demonstrated that no general algorithm could solve the word problem for arbitrary finite presentations, establishing fundamental limitations on what could be achieved algorithmically. The proofs of Novikov and Boone were technically complex and represented major achievements in mathematical logic and combinatorial group theory. They also implicitly established that computing Dehn functions for arbitrary presentations must be undecidable—a result that would be explicitly recognized later.

The 1960s witnessed further developments in the understanding of isoperimetric inequalities in combinatorial group theory. The work of Gilbert Baumslag and Donald Solitar on one-relator groups led to the introduction of the Baumslag-Solitar groups, which would later become important examples of groups with exponential Dehn functions. Their 1962 paper “Some two-generator one-relator non-Hopfian groups” introduced these groups and studied their properties, though the full significance of their Dehn function complexity would not be recognized until later.

Another significant development during this period came from the work of James Stallings, an American mathematician who made important contributions to geometric group theory. His 1968 paper “On torsion-free groups with infinitely many ends” introduced new geometric techniques for studying groups, particularly those acting on trees. While not explicitly concerned with Dehn functions, Stallings’s work helped establish the geometric perspective that would later prove crucial for understanding the connection between group theory and geometry.

The late 1960s and early 1970s saw the gradual emergence of the formal concept of isoperimetric functions in group theory. Mathematicians such as Roger Lyndon and Paul Schupp, in their influential 1977 book “Combinatorial Group Theory,” systematized many of the ideas that had been developing since Dehn’s time. Their book provided a comprehensive treatment of decision problems in group theory and included discussions of isoperimetric inequalities, though the term “Dehn function” was still not in widespread use.

It is worth noting that during this mid-20th century period, the concept of Dehn functions was not yet explicitly defined or studied as a distinct topic. Instead, the ideas were developing within the broader context

of combinatorial group theory and decision problems. The formal definition and systematic study of Dehn functions as distinct invariants would come later, but the foundations were being laid through the work of these mathematicians and others who built upon Dehn’s initial insights.

1.16.3 11.3 The Modern Era of Dehn Functions

The modern era of Dehn functions dawned in the 1980s, marked by a revolutionary shift in perspective that transformed the study of group theory from primarily algebraic to deeply geometric. This transformation was largely catalyzed by the groundbreaking work of Mikhail Gromov, whose visionary ideas would reshape the landscape of geometric group theory and establish Dehn functions as central objects of study.

Gromov’s seminal 1987 essay “Hyperbolic Groups,” published in the collection “Essays in Group Theory,” represented a watershed moment in the history of Dehn functions. In this work, Gromov introduced the concept of hyperbolic groups—groups whose Cayley graphs satisfy a combinatorial version of the thin triangles property characteristic of hyperbolic geometry. What made this work revolutionary for the theory of Dehn functions was Gromov’s theorem establishing that a finitely presented group is hyperbolic if and only if it satisfies a linear isoperimetric inequality—in other words, if and only if its Dehn function is linear. This elegant characterization established a precise correspondence between a geometric property (hyperbolicity) and an algebraic-computational property (having a linear Dehn function), unifying previously disparate concepts under a single conceptual framework.

Gromov’s work did more than just establish this correspondence; it introduced a entirely new way of thinking about groups as geometric objects. The large-scale geometry of groups, captured through concepts like quasi-isometry, became a central focus of study, and Dehn functions emerged as fundamental invariants that reflect this geometric structure. The impact of Gromov’s ideas was immediate and profound, inspiring a generation of mathematicians to explore the geometric aspects of group theory and leading to rapid advances in the understanding of Dehn functions.

The late 1980s and early 1990s saw the emergence of automatic groups as another important class of groups with well-understood Dehn functions. The theory of automatic groups, developed by Jim Cannon, David Epstein, Derek Holt, and others, provided a systematic framework for studying groups with quadratic Dehn functions. Automatic groups are characterized by the existence of regular languages of normal forms with certain fellow traveler properties, and the 1986 book “Word Processing in Groups” by Epstein et al. established that all automatic groups have quadratic Dehn functions. This work provided a rich source of examples of groups with quadratic Dehn functions and introduced powerful algorithmic techniques for studying them.

The 1990s witnessed significant advances in the understanding of groups with polynomial Dehn functions. The work of Martin Bridson and Stephen Gersten was particularly influential in this period. Their systematic study of filling functions—including the classical Dehn function, isodiametric functions, and filling length functions—established precise relationships between these different measures of complexity and provided a more nuanced understanding of the multidimensional nature of computational complexity in groups. Bridson’s 1999 paper “The geometry of the word problem” became a standard reference in the field, synthesizing

many of the developments in the geometric study of Dehn functions up to that point.

Another significant development during this period was the study of groups with intermediate Dehn function growth—those whose Dehn functions grow faster than any polynomial but slower than any exponential function. The existence of such groups was established through the work of Rips and Ol’shanskii in the 1990s, expanding the known spectrum of possible Dehn function growth rates and challenging previous assumptions about the classification of groups based on their Dehn functions. These examples demonstrated that the landscape of Dehn function complexity is richer and more varied than had been previously suspected.

The early 2000s saw further refinements in the understanding of Dehn functions, particularly through the development of new geometric techniques. The work of Cornelia Drutu and Mark Sapir on asymptotic cones provided new insights into the large-scale geometry of groups and its relationship to Dehn function growth rates. Their 2005 paper “Tree-graded spaces and asymptotic cones of groups” established important connections between the topological dimension of asymptotic cones and polynomial Dehn function growth, providing a geometric framework for understanding polynomial Dehn functions.

The concept of relatively hyperbolic groups, introduced by Gromov and developed by Farb, Bowditch, and others, also became increasingly important during this period. Relatively hyperbolic groups generalize hyperbolic groups by allowing for the presence of “peripheral” subgroups that may not be hyperbolic themselves. The Dehn functions of relatively hyperbolic groups reflect both the hyperbolicity of the ambient space and the complexity of the peripheral subgroups, creating a rich landscape of possibilities. The work of Denis Osin in 2006 on relatively hyperbolic groups established precise relationships between the Dehn function of the relatively hyperbolic group and the Dehn functions of the peripheral subgroups, further expanding our understanding of Dehn function complexity.

The late 2000s and 2010s witnessed the emergence of new connections between Dehn functions and other areas of mathematics, including low-dimensional topology, geometric analysis, and theoretical computer science. The solution of the virtual Haken conjecture by Ian Agol in 2012, building on the work of Daniel Wise and others, established deep connections between the Dehn functions of 3-manifold groups and the existence of geometric structures on the manifolds. This work represented a major breakthrough in geometric topology and demonstrated the continuing importance of Dehn functions in contemporary mathematical research.

Throughout this modern era, the term “Dehn function” became firmly established in the mathematical literature, and the systematic study of these functions as distinct invariants became a central focus of geometric group theory. The development of sophisticated computational tools and algorithms, as discussed in Section 9, enabled the computation and estimation of Dehn functions for increasingly complex examples, further advancing our understanding of this fundamental invariant.

1.16.4 11.4 Key Historical Milestones

The historical development of Dehn functions can be traced through a series of key milestones that mark significant advances in our understanding of this concept. These milestones represent both theoretical break-

throughs and technical developments that collectively shaped the modern theory of Dehn function complexity.

The year 1910 stands as the foundational milestone, when Max Dehn introduced what would later be recognized as the first formulation of Dehn functions in his paper “Über die Topologie des dreidimensionalen Raumes.” This work established three fundamental problems—the word problem, conjugacy problem, and isomorphism problem—that would shape the development of combinatorial group theory for decades to come. Dehn’s algorithm for solving the word problem in surface groups implicitly relied on the linear isoperimetric inequality satisfied by these groups, establishing the first connection between what would later be called linear Dehn functions and negative curvature.

Another crucial milestone came in 1932 with the publication of Wilhelm Magnus’s paper on one-relator groups. Magnus established that one-relator groups have solvable word problem, implicitly relating to the Dehn functions of these groups. This work represented an important step toward understanding the algorithmic properties of specific classes of groups and laid the groundwork for later developments in the study of Dehn functions for one-relator groups.

The years 1955 and 1959 mark a watershed moment in the history of Dehn functions with the independent proofs by Pyotr Novikov and William Boone of the unsolvability of the word problem for general finitely presented groups. This result established fundamental limitations on what could be achieved algorithmically and implicitly demonstrated that computing Dehn functions for arbitrary presentations must be undecidable. The technical complexity of these proofs represented major achievements in mathematical logic and combinatorial group theory.

The publication of Gromov’s “Hyperbolic Groups” in 1987 represents perhaps the most significant milestone in the modern theory of Dehn functions. Gromov’s theorem establishing that a finitely presented group is hyperbolic if and only if it has a linear Dehn function transformed the study of group theory from primarily algebraic to deeply geometric. This work introduced an entirely new way of thinking about groups as geometric objects and established Dehn functions as central invariants that reflect this geometric structure.

The 1991 publication of “Word Processing in Groups” by Epstein et al. marked another important milestone with the systematic development of the theory of automatic groups. This work established that all automatic groups have quadratic Dehn functions and provided a rich source of examples of groups with quadratic Dehn functions. The introduction of automatic structures also provided powerful algorithmic techniques for studying these groups, further advancing our understanding of Dehn function complexity.

The late 1990s and early 2000s witnessed several important milestones in the understanding of groups with intermediate and exponential Dehn function growth. The work of Rips and Ol’shanskii in the 1990s established the existence of groups with intermediate Dehn function growth, expanding the known spectrum of possible growth rates. Similarly, the detailed analysis of the Baumslag-Solitar group $BS(2,3)$ by Gersten

1.17 Conclusion and Future Directions

Similarly, the detailed analysis of the Baumslag-Solitar group $BS(2,3)$ by Gersten established its exponential Dehn function, providing a concrete example of this growth rate and stimulating further research into groups with exponential complexity. This historical journey brings us to the present state of Dehn function theory, where we can now reflect on the rich tapestry of concepts, results, and applications that have been developed, and contemplate the future directions this field might take.

1.17.1 12.1 Synthesis of Key Concepts

The theory of Dehn function complexity that has emerged over the past century represents a remarkable synthesis of algebraic, geometric, and computational perspectives on group theory. At its core, the Dehn function captures a fundamental aspect of group complexity—the difficulty of solving the word problem—but it does so in a way that reveals deep connections between seemingly disparate areas of mathematics. Through this exploration, we have seen how Dehn functions serve as a bridge between the combinatorial structure of groups, the geometry of spaces on which they act, and the computational complexity of algorithmic problems.

The fundamental concept of the Dehn function as a measure of isoperimetric complexity has proven to be extraordinarily versatile. For hyperbolic groups, the linear Dehn function reflects the negative curvature of the spaces on which these groups act, capturing the essence of Gromov’s revolutionary insight that geometric properties can be characterized algebraically. For automatic groups, the quadratic Dehn function reflects the regular structure of the automatic languages associated with these groups, connecting algebraic properties to formal language theory. For nilpotent groups, the polynomial Dehn function of degree determined by the nilpotency class reflects the hierarchical structure of these groups, connecting algebraic structure to computational complexity.

The relationship between Dehn functions and other filling functions has emerged as a crucial aspect of the theory. The isodiametric function, filling length function, and synchronization delay function each capture different aspects of the computational complexity of groups, and the study of their interrelationships has revealed the multidimensional nature of group complexity. This multidimensional perspective has been particularly important for understanding groups where different filling functions exhibit different growth rates, such as the lamplighter group with its exponential Dehn function but even faster isodiametric function.

The topological interpretation of Dehn functions has proven to be equally rich. Through the connection between group presentations and 2-complexes, Dehn functions capture essential information about the isoperimetric properties of topological spaces. This perspective has been particularly fruitful in the study of aspherical manifolds, where the Dehn function of the fundamental group reflects the geometric complexity of the universal cover. The higher-dimensional generalizations of Dehn functions further extend this topological connection, providing a framework for understanding filling problems in higher dimensions that connects to homotopy theory and algebraic topology.

The computational aspects of Dehn functions have been equally important. The relationship between Dehn function growth rates and standard complexity classes has provided concrete examples of problems with specific complexity characteristics. The algorithms for solving the word problem based on Dehn functions, from Dehn's algorithm for hyperbolic groups to the automatic structures for automatic groups, demonstrate how the theoretical understanding of Dehn function complexity can be translated into practical computational methods. The undecidability results for Dehn functions, stemming from the Novikov-Boone theorem, establish fundamental limitations on what can be achieved algorithmically, connecting group theory to mathematical logic and computability theory.

1.17.2 12.2 Broader Implications

The theory of Dehn function complexity extends far beyond its origins in pure group theory, with implications that resonate throughout mathematics and computer science. These broader implications reflect the fundamental nature of the concepts involved and the deep connections between different areas of mathematical science.

In mathematics, the study of Dehn functions has transformed our understanding of groups as geometric objects. The geometric group theory perspective, catalyzed by Gromov's work, has revealed that groups can be studied as geometric spaces in their own right, with their large-scale geometry captured through concepts like quasi-is