A MILP approach to a structured ensemble Binarized Neural Network

Matematica per l'Intelligenza Artificiale e il Machine Learning - Giovani ricercatori

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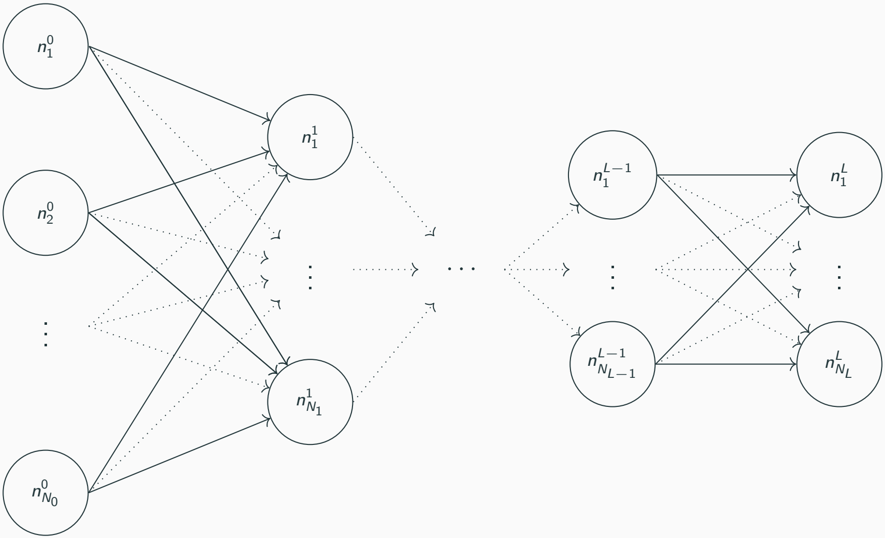
2022-11-24

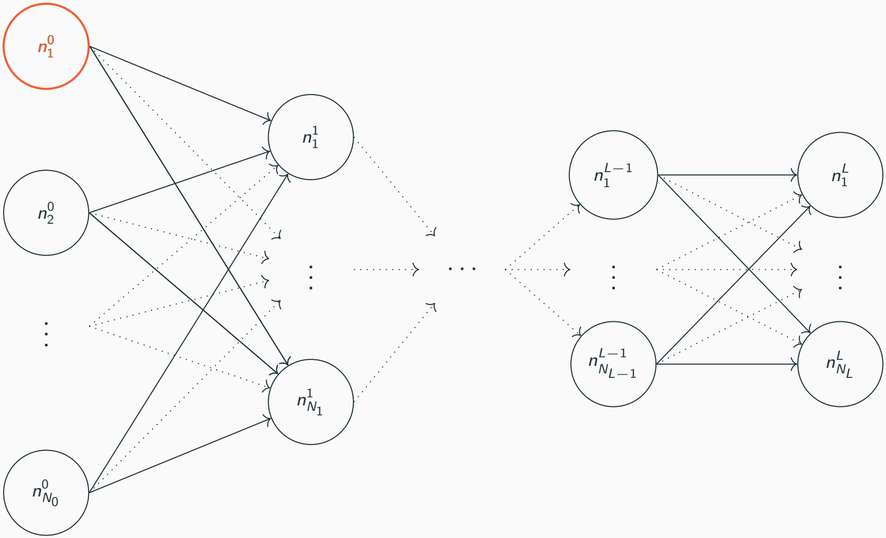
Overview

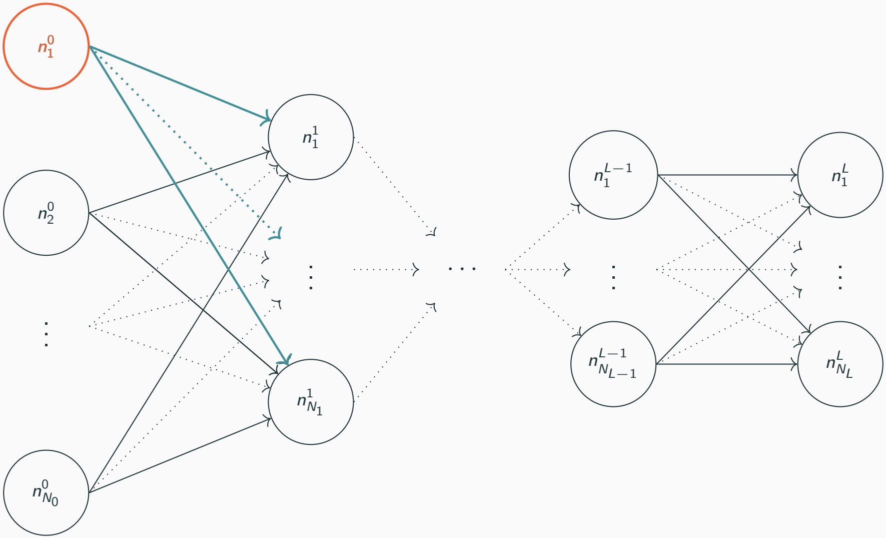
Introduction

Neural Networks - NNs

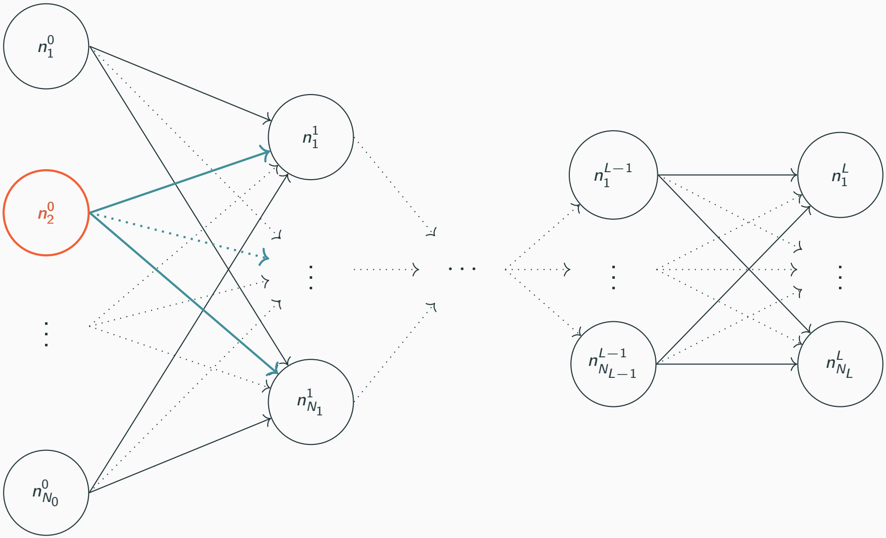


Neural Networks - NNs

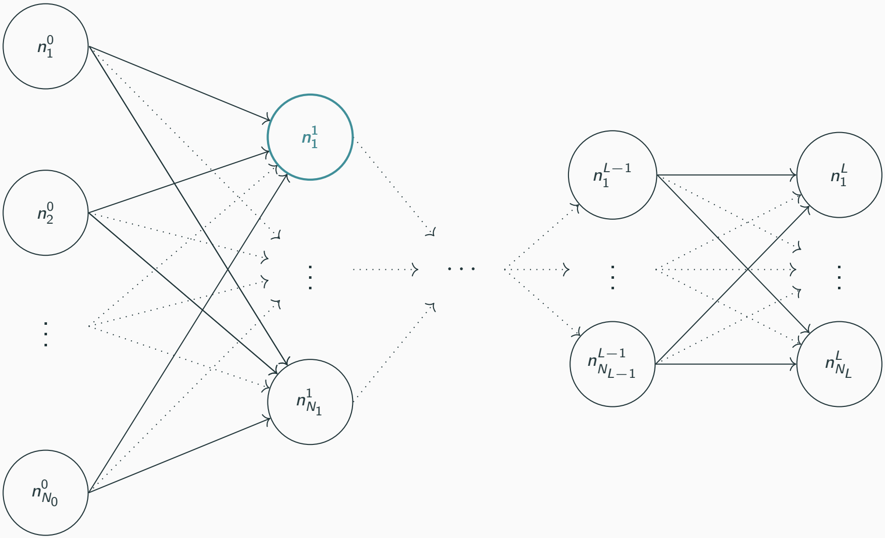



n_1^0
n_2^0
$n_{N_0}^0$
n_1^1
$n_{N_1}^1$
n_1^{L-1}
$n_{N_{L-1}}^{L-1}$
n_1^L
$n_{N_L}^L$

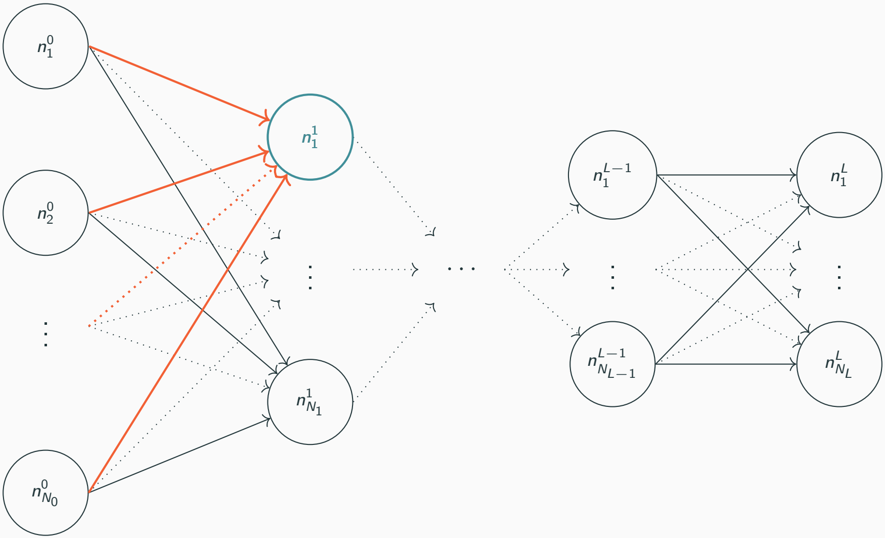
Neural Networks - NNs

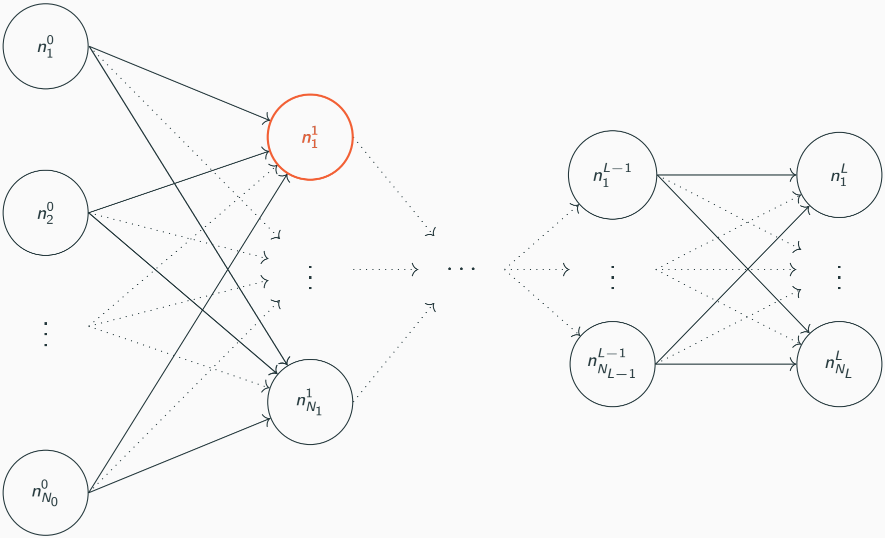


Neural Networks - NNs



Neural Networks - NNs





n_1^0
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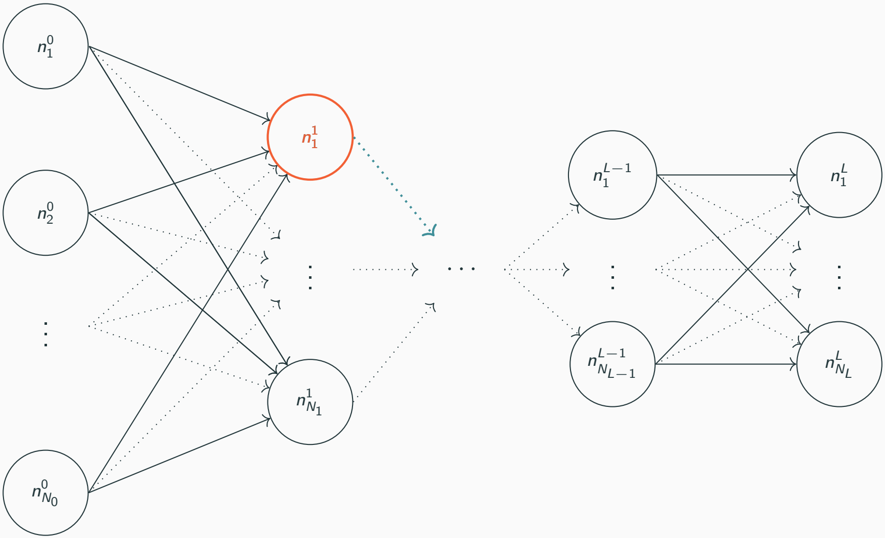
Neural Networks - NNs



Binarized Neural Networks - BNNs

BNNs are getting increasing attention thanks to their compactness and versatility.
In this kind of NN, every neuron $j \in N_l$ is connected to every neuron $i \in N_{l-1}$ by a weight $w_{ilj} \in \{-1, 0, 1\}$. Given a value x for input neurons, the preactivation $a_{lj}(x)$ of neuron $j \in N_l$ and the activation $p_j(x)$ are, respectively,

$$a_{lj}(x) = \sum_{i \in N_{l-1}} w_{ilj} \cdot p_{(l-1)i}(x) \qquad \text{and} \qquad p_{lj}(x) = \begin{cases} x_j & \text{if } l = 0, \\ +1 & \text{if } l > 0, a_{lj}(x) \geq 0, \\ -1 & \text{otherwise.} \end{cases} \tag{1}$$

Binarized Neural Networks - BNNs

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Recent works[1] show that this kind of networks are hard to train with GD-based algorithm in a context of few-shot learning. Instead, MILP approaches are being researched.

[1] Toro Icarte, R., Illanes, L., Castro, M.P., Cire, A.A., McIlraith, S.A. and Beck, J.C.: Training binarized neural networks using MIP and CP. In: Proceedings of CP'19. vol 11802, pp. 401–417. Springer (2019).

Mathematical Models

Models

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(M-M) Max-Margin: a way of finding robust BNNs by maximizing the margins of their neurons. Intuitively, neurons with larger margins requires bigger changes on their inputs and weights to change their activation values;[1]

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(M-W) Min-Weight: a way of finding simple BNNs by minimizing the number of connections;[1]

(S-M) Sat-Margin: a way of finding BNNs by maximizing the number of correct predictions. At the same time each correctly predicted sample is confidently predicted.[2]

[1] Toro Icarte, R., Illanes, L., Castro, M.P., Cire, A.A., McIlraith, S.A. and Beck, J.C.: Training binarized neural networks using MIP and CP. In: Proceedings of CP'19. vol 11802, pp. 401–417. Springer (2019).

[2] Thorbjarnarson, T., Yorke-Smith, N.: On Training Neural Networks with Mixed Integer Programming. arXiv preprint arXiv:2009.03825 (2020).

Max-Margin

$$\max_{\mathbf{c},\mathbf{u},\mathbf{w},\mathbf{m}} \quad \sum_{l \in \mathcal{L}} \sum_{j \in N_l} m_{lj} \tag{2}$$

Max-Margin

$$
\begin{aligned}
\max_{\mathbf{c},\mathbf{u},\mathbf{w},\mathbf{m}} \quad & \sum_{l\in\mathcal{L}} \sum_{j\in N_l} m_{lj} && && (2)\\
\text{s.t.} \quad & \sum_{i\in N_{L-1}} c_{iLj}^k \geq m_{Lj} && \forall j \in N_L, k \in T : y_j^k = 1, && (3)\\
& \sum_{i\in N_{L-1}} c_{iLj}^k \leq -\epsilon - m_{Lj} && \forall j \in N_L, k \in T : y_j^k = -1, && (4)
\end{aligned}
$$

Max-Margin

$$\max_{\mathbf{c},\mathbf{u},\mathbf{w},\mathbf{m}} \quad \sum_{l\in\mathcal{L}}\sum_{j\in N_l} m_{lj} \tag{2}$$

$$\text{s.t.} \quad \sum_{i\in N_{L-1}} c^k_{iLj} \geq m_{Lj} \qquad \forall j \in N_L, k \in T : y^k_j = 1, \tag{3}$$

$$\sum_{i\in N_{L-1}} c^k_{iLj} \leq -\epsilon - m_{Lj} \qquad \forall j \in N_L, k \in T : y^k_j = -1, \tag{4}$$

$$u^k_{lj} = 1 \implies \sum_{i\in N_{l-1}} c^k_{ilj} \geq m_{lj} \qquad \forall l \in \mathcal{L}^{L-1}, j \in N_l, k \in T, \tag{5}$$

$$u^k_{lj} = 0 \implies \sum_{i\in N_{l-1}} c^k_{ilj} \leq -\epsilon - m_{lj} \qquad \forall l \in \mathcal{L}^{L-1}, j \in N_l, k \in T, \tag{6}$$

Max-Margin

$$c^k_{i1j} = x^k_i \cdot w_{i1j} \qquad \forall i \in N_0, j \in N_1, k \in T, \tag{7}$$

$$c^k_{ilj} - w_{ilj} + 2u^k_{(l-1)i} \leq 2 \qquad \forall l \in \mathcal{L}_2, i \in N_{l-1}, j \in N_l, k \in T, \tag{8}$$

$$c^k_{ilj} + w_{ilj} - 2u^k_{(l-1)i} \leq 0 \qquad \forall l \in \mathcal{L}_2, i \in N_{l-1}, j \in N_l, k \in T, \tag{9}$$

$$c^k_{ilj} - w_{ilj} - 2u^k_{(l-1)i} \geq -2 \qquad \forall l \in \mathcal{L}_2, i \in N_{l-1}, j \in N_l, k \in T, \tag{10}$$

$$c^k_{ilj} + w_{ilj} + 2u^k_{(l-1)i} \geq 0 \qquad \forall l \in \mathcal{L}_2, i \in N_{l-1}, j \in N_l, k \in T, \tag{11}$$

Max-Margin

$$
\begin{aligned}
& c_{i1j}^{k} = x_i^k \cdot w_{i1j} && \forall i \in N_0, j \in N_1, k \in T, && (7)\\
& c_{ilj}^{k} - w_{ilj} + 2u_{(l-1)i}^{k} \le 2 && \forall l \in \mathcal{L}_2, i \in N_{l-1}, j \in N_l, k \in T, && (8)\\
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& c_{ilj}^{k} + w_{ilj} + 2u_{(l-1)i}^{k} \ge 0 && \forall l \in \mathcal{L}_2, i \in N_{l-1}, j \in N_l, k \in T, && (11)\\
& w_{ilj} \in \{-1, 0, 1\} && \forall l \in \mathcal{L}, i \in N_{l-1}, j \in N_l, && (12)\\
& u_{lj}^{k} \in \{0, 1\} && \forall l \in \mathcal{L}^{L-1}, j \in N_l, k \in T, && (13)\\
& c_{i1j}^{k} \in [-\mathfrak{b}, \mathfrak{b}] && \forall i \in N_0, j \in N_1, k \in T, && (14)\\
& c_{ilj}^{k} \in \{-1, 0, 1\} && \forall l \in \mathcal{L}_2, i \in N_{l-1}, j \in N_l, k \in T, && (15)\\
& m_{lj} \in \mathbb{R}_{\ge 0} && \forall l \in \mathcal{L}, j \in N_l. && (16)
\end{aligned}
$$

Max-Margin

$$
\begin{aligned}
& c_{i1j}^{k} = x_i^k \cdot w_{i1j} && \forall i \in N_0, j \in N_1, k \in T, && (7)\\
& c_{ilj}^{k} - w_{ilj} + 2u_{(l-1)i}^{k} \leq 2 && \forall l \in \mathcal{L}_2, i \in N_{l-1}, j \in N_l, k \in T, && (8)\\
& c_{ilj}^{k} + w_{ilj} - 2u_{(l-1)i}^{k} \leq 0 && \forall l \in \mathcal{L}_2, i \in N_{l-1}, j \in N_l, k \in T, && (9)\\
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Min-Weight (0-margin)

$$\min_{\mathbf{c},\mathbf{u},\mathbf{w},\mathbf{v}} \quad \sum_{l \in \mathcal{L}} \sum_{i \in N_{l-1}} \sum_{j \in N_l} v_{ilj} \tag{17}$$

Min-Weight (0-margin)

$$
\begin{aligned}
\min_{\mathbf{c},\mathbf{u},\mathbf{w},\mathbf{v}} \quad & \sum_{l\in\mathcal{L}} \sum_{i\in N_{l-1}} \sum_{j\in N_l} v_{ilj} && && (17)\\
\text{s.t.} \quad & \sum_{i\in N_{L-1}} c_{iLj}^k \geq 0 && \forall j \in N_L, k \in T : y_j^k = 1, && (18)\\
& \sum_{i\in N_{L-1}} c_{iLj}^k \leq -0 - \epsilon && \forall j \in N_L, k \in T : y_j^k = -1, && (19)
\end{aligned}
$$

Min-Weight (0-margin)

$$
\begin{aligned}
\min_{\mathbf{c},\mathbf{u},\mathbf{w},\mathbf{v}} \quad & \sum_{l\in\mathcal{L}} \sum_{i\in N_{l-1}} \sum_{j\in N_l} v_{ilj} && && (17)\\
\text{s.t.} \quad & \sum_{i\in N_{L-1}} c^k_{iLj} \geq 0 && \forall j \in N_L, k \in T : y^k_j = 1, && (18)\\
& \sum_{i\in N_{L-1}} c^k_{iLj} \leq -0 - \epsilon && \forall j \in N_L, k \in T : y^k_j = -1, && (19)\\
& u^k_{lj} = 1 \implies \sum_{i\in N_{l-1}} c^k_{ilj} \geq 0 && \forall l \in \mathcal{L}^{L-1}, j \in N_l, k \in T, && (20)\\
& u^k_{lj} = 0 \implies \sum_{i\in N_{l-1}} c^k_{ilj} \leq -0 - \epsilon && \forall l \in \mathcal{L}^{L-1}, j \in N_l, k \in T, && (21)
\end{aligned}
$$

Min-Weight (0-margin)

$$c_{i1j}^k = x_i^k \cdot w_{i1j} \qquad \forall i \in N_0, j \in N_1, k \in T, \tag{22}$$

$$c_{ilj}^k - w_{ilj} + 2u_{(l-1)i}^k \leq 2 \qquad \forall l \in \mathcal{L}_2, i \in N_{l-1}, j \in N_l, k \in T, \tag{23}$$

$$c_{ilj}^k + w_{ilj} - 2u_{(l-1)i}^k \leq 0 \qquad \forall l \in \mathcal{L}_2, i \in N_{l-1}, j \in N_l, k \in T, \tag{24}$$

$$c_{ilj}^k - w_{ilj} - 2u_{(l-1)i}^k \geq -2 \qquad \forall l \in \mathcal{L}_2, i \in N_{l-1}, j \in N_l, k \in T, \tag{25}$$

$$c_{ilj}^k + w_{ilj} + 2u_{(l-1)i}^k \geq 0 \qquad \forall l \in \mathcal{L}_2, i \in N_{l-1}, j \in N_l, k \in T, \tag{26}$$

Min-Weight (0-margin)

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& c_{i1j}^{k} = x_i^k \cdot w_{i1j} && \forall i \in N_0, j \in N_1, k \in T, && (22)\\
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& c_{ilj}^{k} + w_{ilj} + 2u_{(l-1)i}^{k} \geq 0 && \forall l \in \mathcal{L}_2, i \in N_{l-1}, j \in N_l, k \in T, && (26)\\
& -v_{ilj} \leq w_{ilj} \leq v_{ilj} && \forall l \in \mathcal{L}, i \in N_{l-1}, j \in N_l, && (27)
\end{aligned}
$$

Min-Weight (0-margin)

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& c^k_{i1j} = x^k_i \cdot w_{i1j} && \forall i \in N_0, j \in N_1, k \in T, && (22)\\
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& -v_{ilj} \leq w_{ilj} \leq v_{ilj} && \forall l \in \mathcal{L}, i \in N_{l-1}, j \in N_l, && (27)\\
& w_{ilj} \in \{-1, 0, 1\} && \forall l \in \mathcal{L}, i \in N_{l-1}, j \in N_l, && (28)\\
& u^k_{lj} \in \{0, 1\} && \forall l \in \mathcal{L}^{L-1}, j \in N_l, k \in T, && (29)\\
& c^k_{i1j} \in [-\mathfrak{b}, \mathfrak{b}] && \forall i \in N_0, j \in N_1, k \in T, && (30)\\
& c^k_{ilj} \in \{-1, 0, 1\} && \forall l \in \mathcal{L}_2, i \in N_{l-1}, j \in N_l, k \in T, && (31)\\
& v_{ilj} \in \{0, 1\} && \forall l \in \mathcal{L}, i \in N_{l-1}, j \in N_l. && (32)
\end{aligned}
$$

Min-Weight (0-margin)

$$
\begin{aligned}
& c_{i1j}^k = x_i^k \cdot w_{i1j} && \forall i \in N_0, j \in N_1, k \in T, && (22)\\
& c_{ilj}^k - w_{ilj} + 2u_{(l-1)i}^k \leq 2 && \forall l \in \mathcal{L}_2, i \in N_{l-1}, j \in N_l, k \in T, && (23)\\
& c_{ilj}^k + w_{ilj} - 2u_{(l-1)i}^k \leq 0 && \forall l \in \mathcal{L}_2, i \in N_{l-1}, j \in N_l, k \in T, && (24)\\
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& -v_{ilj} \leq w_{ilj} \leq v_{ilj} && \forall l \in \mathcal{L}, i \in N_{l-1}, j \in N_l, && (27)\\
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Sat-Margin

$$\max_{\mathbf{c},\mathbf{u},\mathbf{w},\hat{\mathbf{y}},\mathbf{q}} \quad \sum_{k\in T}\sum_{j\in N_L} q_j^k \tag{33}$$

Sat-Margin

$$
\begin{aligned}
\max_{\mathbf{c},\mathbf{u},\mathbf{w},\hat{\mathbf{y}},\mathbf{q}} \quad & \sum_{k\in T}\sum_{j\in N_L} q_j^k && && (33)\\
& q_j^k = 1 \implies \hat{y}_j^k \cdot y_j^k \geq 1/2 && \forall j \in N_L, k \in T, && (34)\\
& q_j^k = 0 \implies \hat{y}_j^k \cdot y_j^k \leq 1/2 - \epsilon && \forall j \in N_L, k \in T, && (35)\\
& q_j^k \in \{0,1\} && \forall j \in N_L, k \in T, && (36)\\
& \hat{y}_j^k = \frac{2}{N_{L-1}+1}\sum_{i\in N_{L-1}} c_{iLj}^k && \forall j \in N_L, k \in T, && (37)
\end{aligned}
$$

Sat-Margin

$$
\begin{aligned}
&\max_{\mathbf{c},\mathbf{u},\mathbf{w},\hat{\mathbf{y}},\mathbf{q}} && \sum_{k\in T}\sum_{j\in N_L} q_j^k && && (33)\\
& && q_j^k = 1 \implies \hat{y}_j^k \cdot y_j^k \geq 1/2 && \forall j \in N_L, k \in T, && (34)\\
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& && \hat{y}_j^k = \frac{2}{N_{L-1}+1}\sum_{i\in N_{L-1}} c_{iLj}^k && \forall j \in N_L, k \in T, && (37)\\
& && u_{lj}^k = 1 \implies \sum_{i\in N_{l-1}} c_{ilj}^k \geq 0 && \forall l \in \mathcal{L}^{L-1}, j \in N_l, k \in T, && (38)\\
& && u_{lj}^k = 0 \implies \sum_{i\in N_{l-1}} c_{ilj}^k \leq -\epsilon && \forall l \in \mathcal{L}^{L-1}, j \in N_l, k \in T, && (39)
\end{aligned}
$$

Sat-Margin

$$c^k_{i1j} = x^k_i \cdot w_{i1j} \qquad \forall i \in N_0, j \in N_1, k \in T, \tag{40}$$

$$c^k_{ilj} - w_{ilj} + 2u^k_{(l-1)i} \leq 2 \qquad \forall l \in \mathcal{L}_2, i \in N_{l-1}, j \in N_l, k \in T, \tag{41}$$

$$c^k_{ilj} + w_{ilj} - 2u^k_{(l-1)i} \leq 0 \qquad \forall l \in \mathcal{L}_2, i \in N_{l-1}, j \in N_l, k \in T, \tag{42}$$

$$c^k_{ilj} - w_{ilj} - 2u^k_{(l-1)i} \geq -2 \qquad \forall l \in \mathcal{L}_2, i \in N_{l-1}, j \in N_l, k \in T, \tag{43}$$

$$c^k_{ilj} + w_{ilj} + 2u^k_{(l-1)i} \geq 0 \qquad \forall l \in \mathcal{L}_2, i \in N_{l-1}, j \in N_l, k \in T, \tag{44}$$

Sat-Margin

$$
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& c^k_{ilj} + w_{ilj} + 2u^k_{(l-1)i} \geq 0 && \forall l \in \mathcal{L}_2, i \in N_{l-1}, j \in N_l, k \in T, && (44)\\
& w_{ilj} \in \{-1, 0, 1\} && \forall l \in \mathcal{L}, i \in N_{l-1}, j \in N_l, && (45)\\
& u^k_{lj} \in \{0, 1\} && \forall l \in \mathcal{L}^{L-1}, j \in N_l, k \in T, && (46)\\
& c^k_{i1j} \in [-\mathfrak{b}, \mathfrak{b}] && \forall i \in N_0, j \in N_1, k \in T, && (47)\\
& c^k_{ilj} \in \{-1, 0, 1\} && \forall l \in \mathcal{L}_2, i \in N_{l-1}, j \in N_l, k \in T. && (48)
\end{aligned}
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$$

Thorbjarnarson, T., Yorke-Smith, N.: On Training Neural Networks with Mixed Integer Programming. arXiv preprint arXiv:2009.03825 (2020).

Methodology

Our model: Ctrl + C, Ctrl + V

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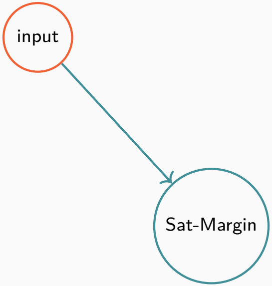

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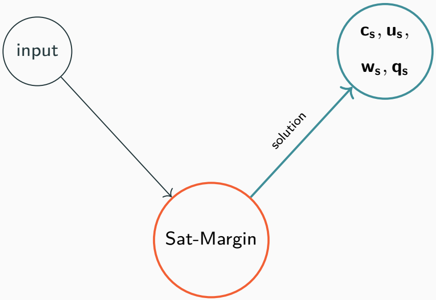

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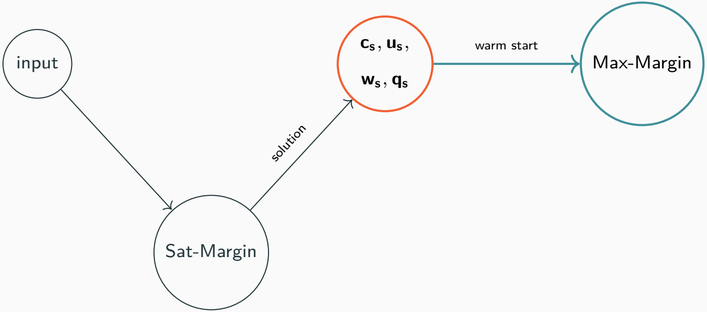

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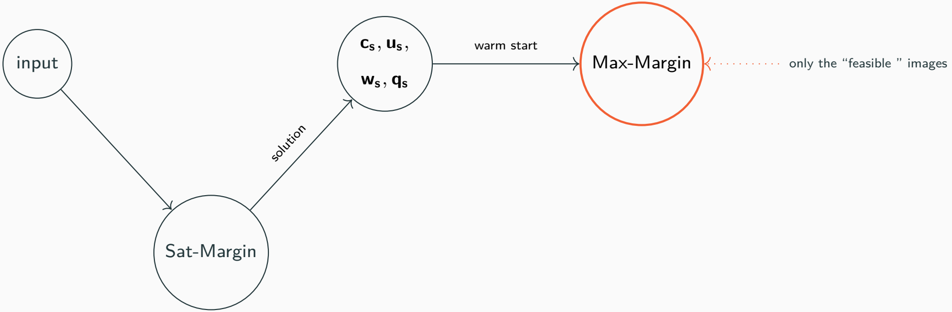

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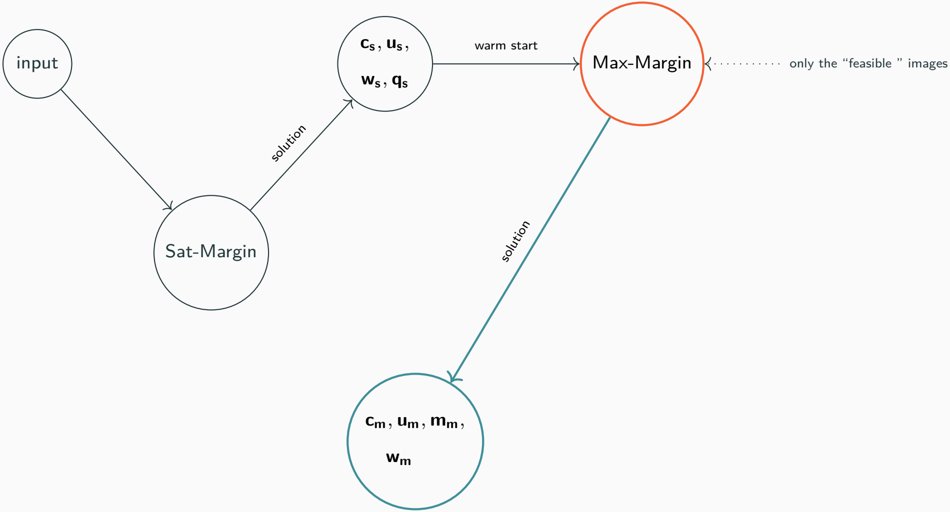

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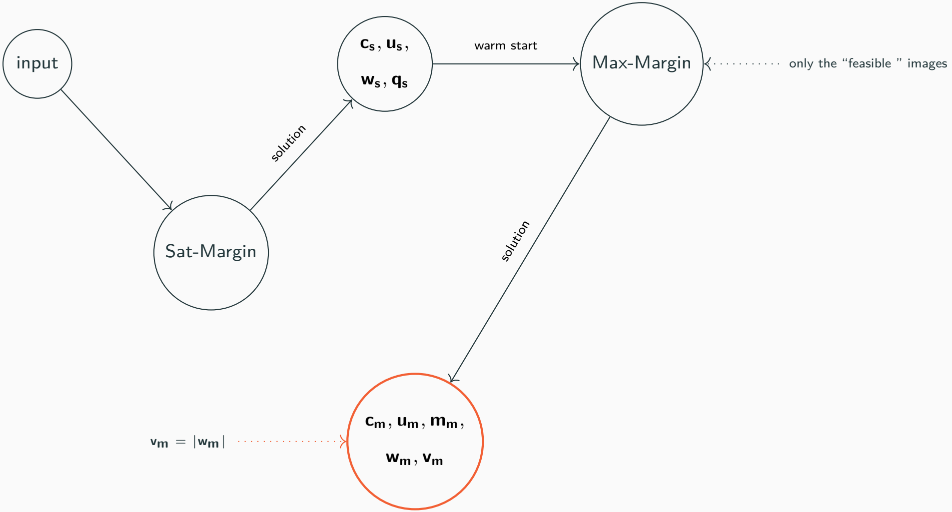

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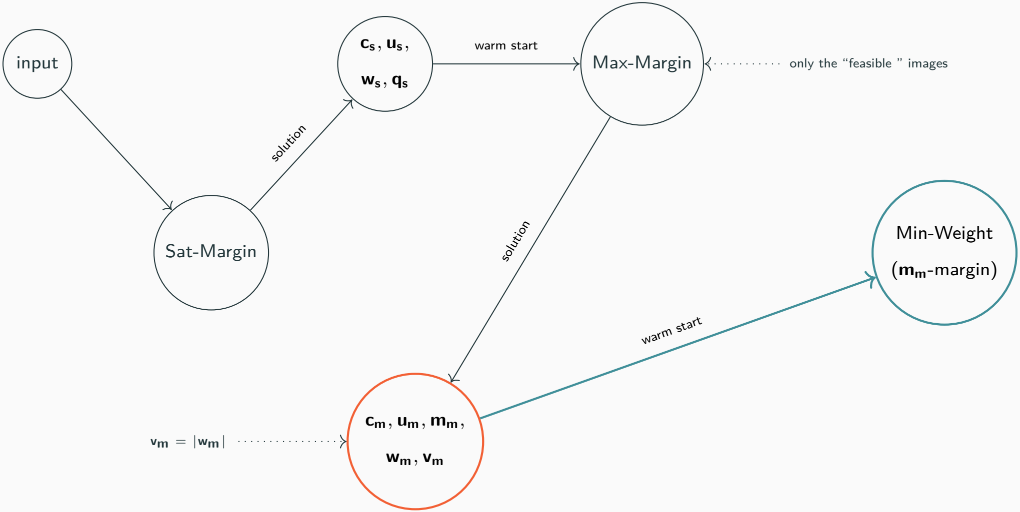

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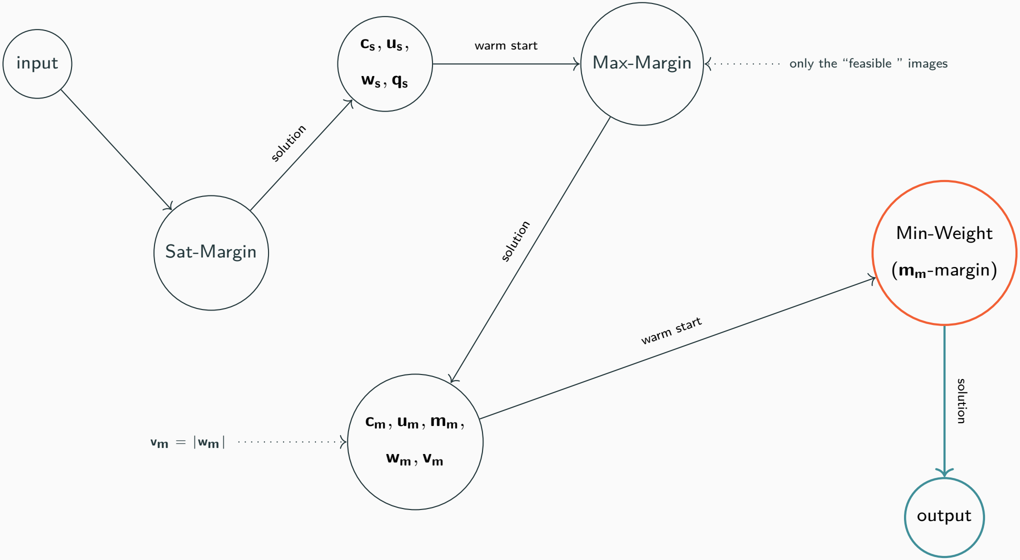

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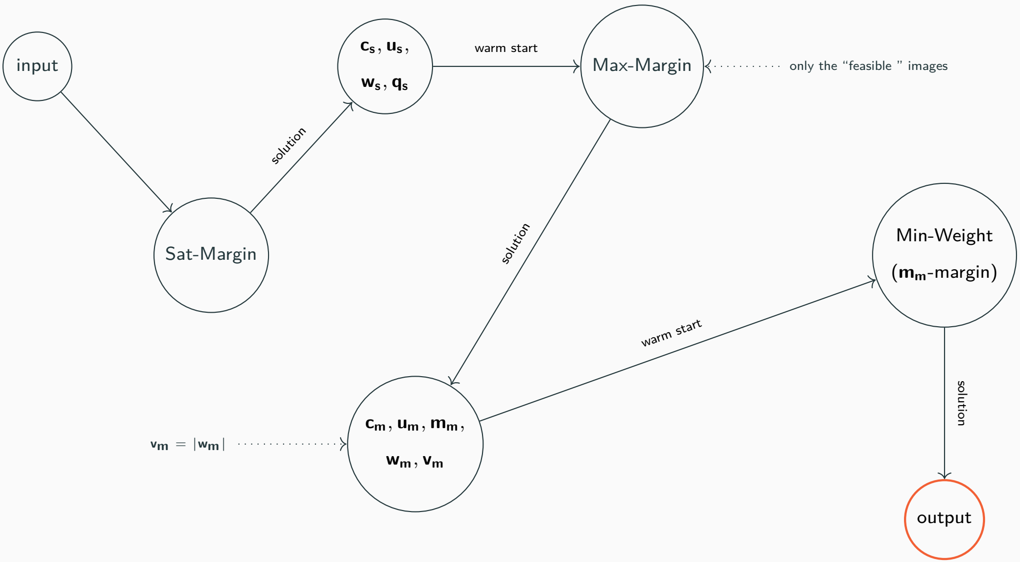

Structured ensemble

Suppose we have a labelling problem and that our set of labels is $\mathcal{I}$.

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We set a parameter $1 < k \leq n = |\mathcal{I}|$.

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- We set a parameter $1 < k \leq n = |\mathcal{I}|$.
- We train one network $\mathcal{N}_{\mathcal{J}}$ for every $\mathcal{J} \in \mathcal{P}(\mathcal{I})_k$.
- When testing an input, we feed it to our list of networks $(\mathcal{N}_{\mathcal{J}})_{\mathcal{J} \in \mathcal{P}(\mathcal{I})_k}$ and we obtain a list of labels $(\mathfrak{e}_{\mathcal{J}})_{\mathcal{J} \in \mathcal{P}(\mathcal{I})_k}$.

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- We then apply a majority voting system.

For the sake of simplicty, suppose $\mathcal{I} = \{0, 1, \ldots, 9\}$ and $k = 2$. So every $\mathcal{J}$ is a set of type $\{i, j\}$, $i, j \in \{0, 1, \ldots, 9\}$, $i \neq j$. We denote with $\mathfrak{e}_{\{i,j\}}$ the output of the network $\mathcal{N}_{\{i,j\}}$ and with $\hat{\mathfrak{e}}_{\{i,j\}}$ the only element of the set $\{i, j\} \setminus \mathfrak{e}_{\{i,j\}}$.

Majority Voting - Example 1

input

Majority Voting - Example 1

$\mathcal{N}_{\{0,1\}}$

$\mathcal{N}_{\{0,2\}}$

$\vdots$

input

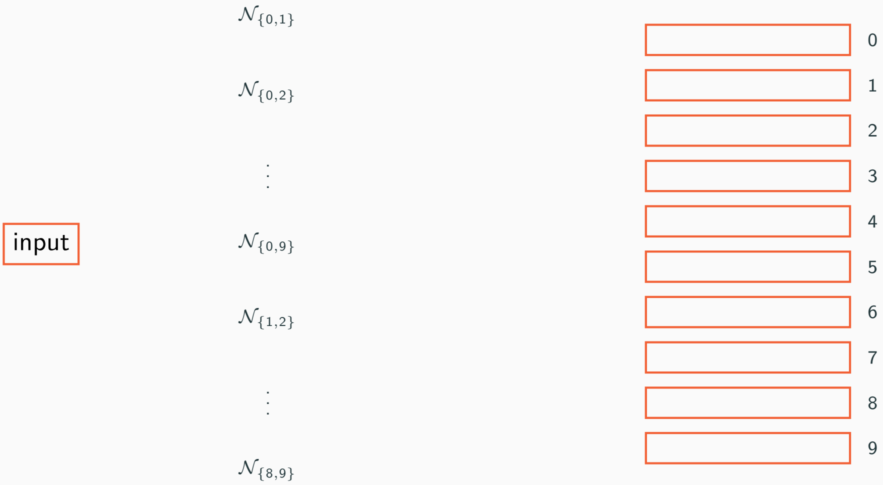
$\mathcal{N}_{\{0,9\}}$

$\mathcal{N}_{\{1,2\}}$

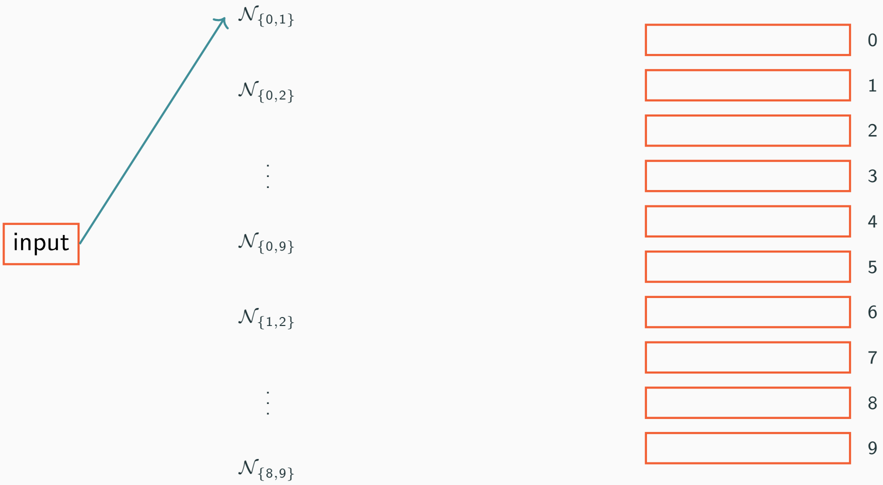
$\vdots$

$\mathcal{N}_{\{8,9\}}$

Majority Voting - Example 1



Majority Voting - Example 1



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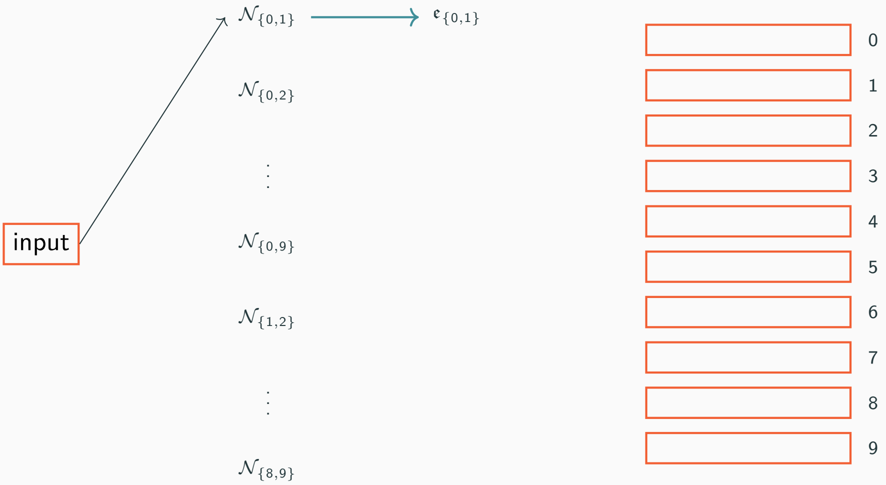

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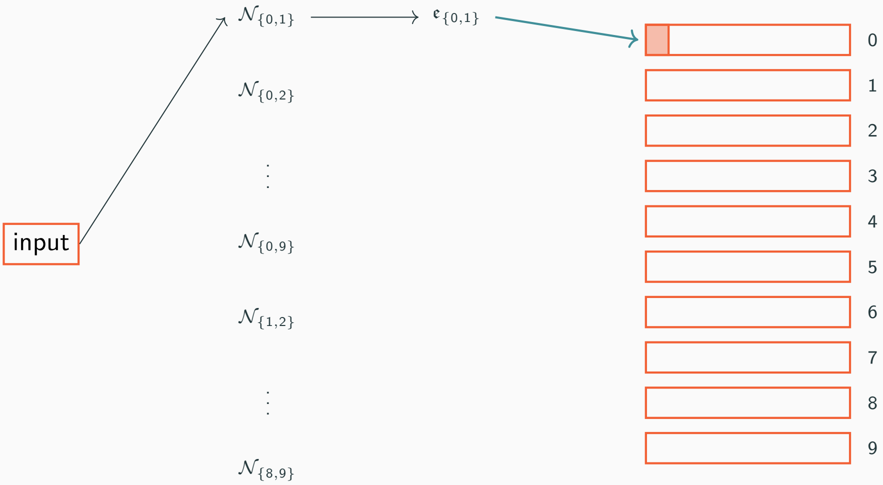

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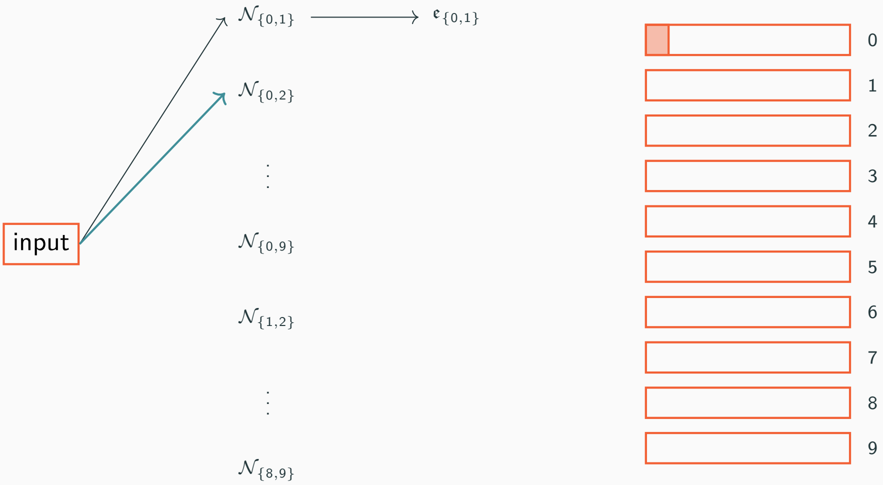

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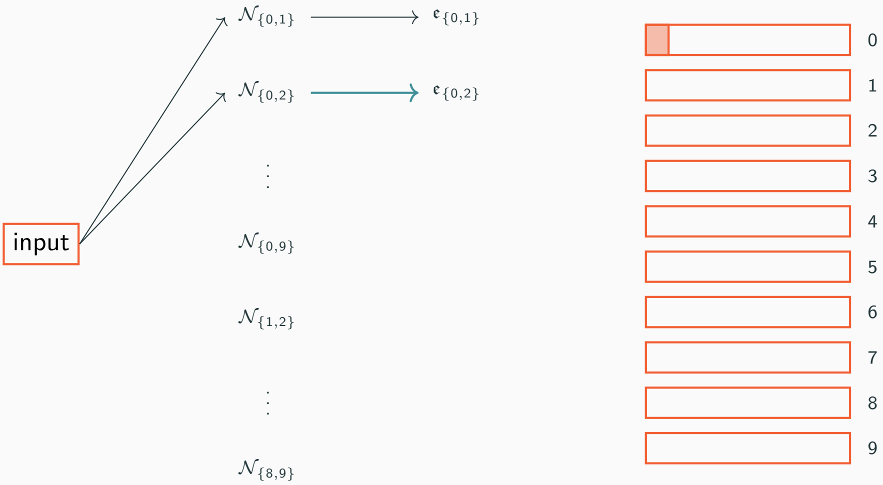

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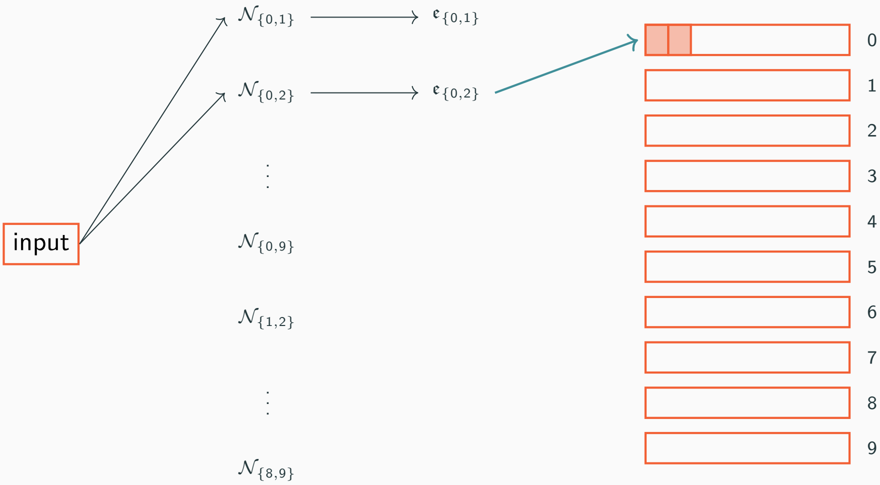

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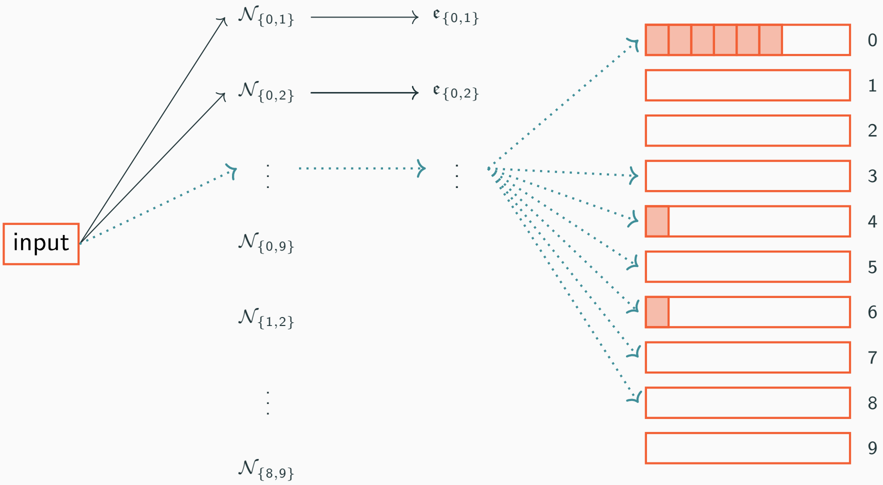

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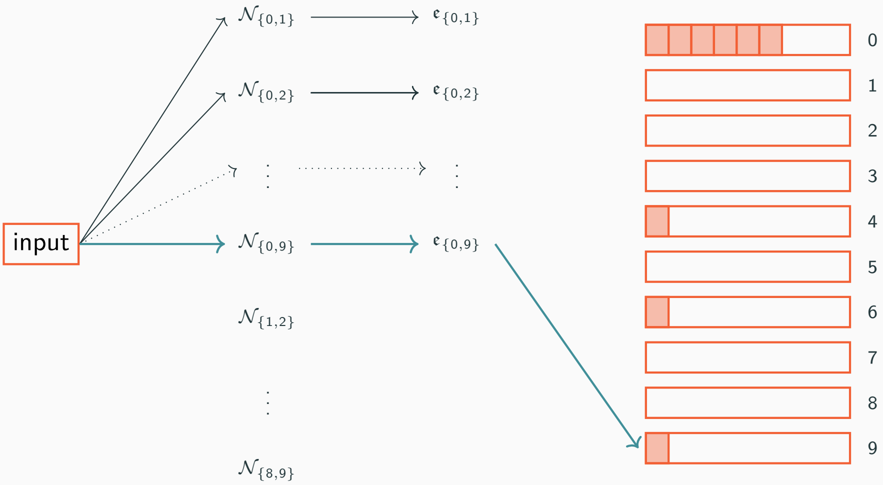

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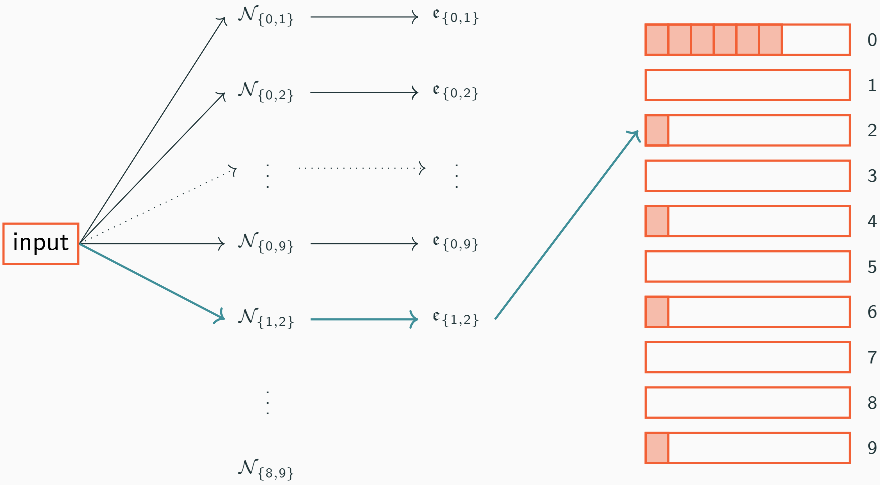

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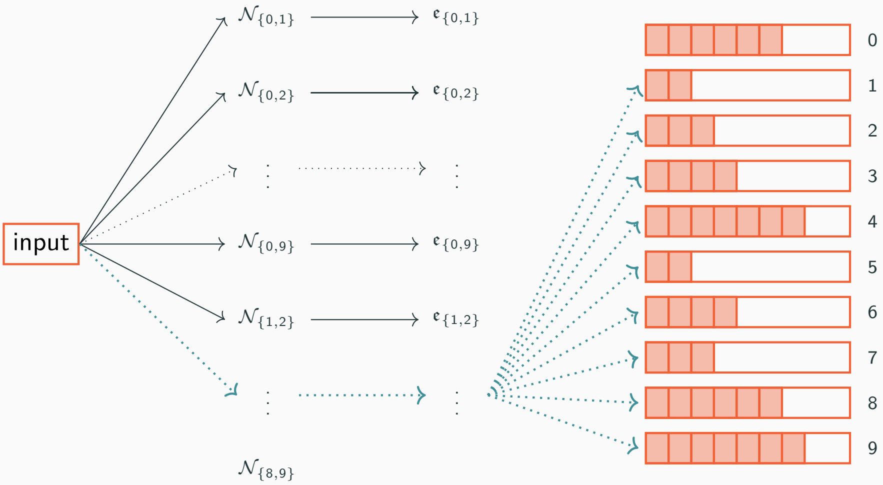

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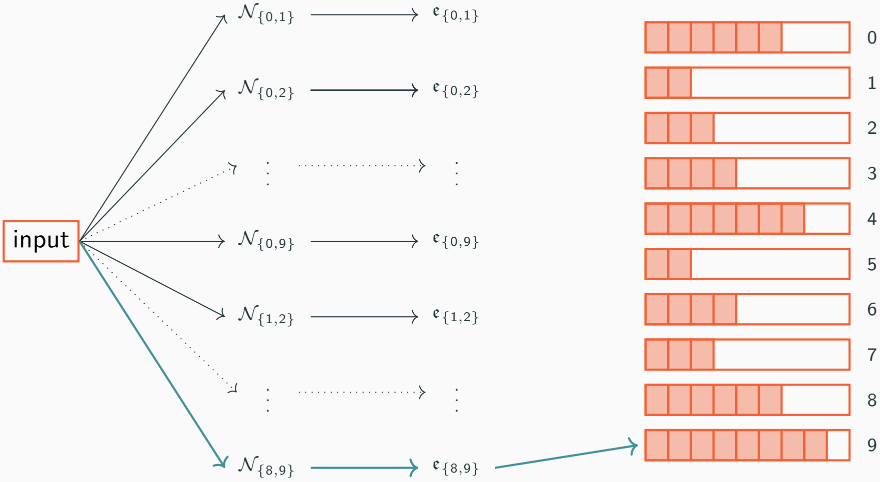

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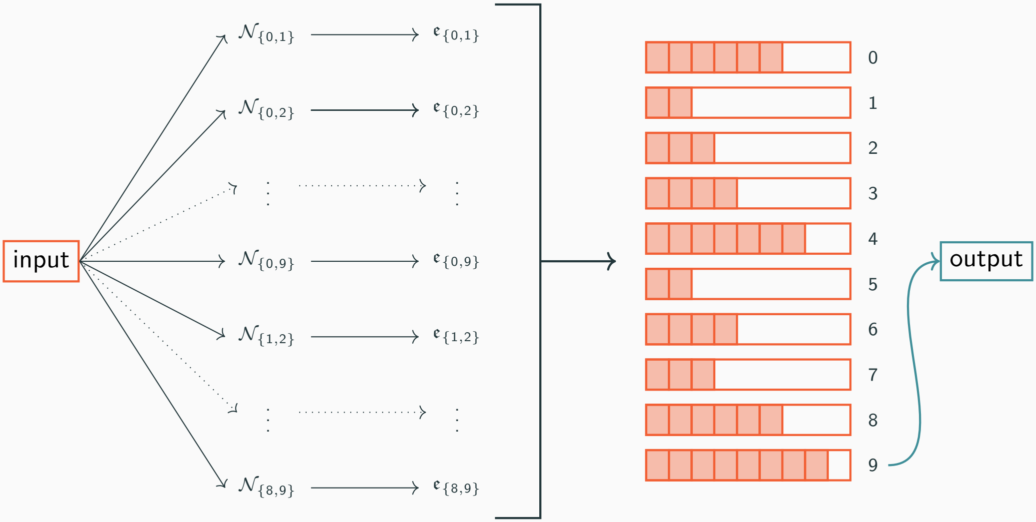

What about ex-aequo?

For every $k \in \{0, 1, \ldots, 9\}$ we define

$$C_k = \{\{i, j\} \in \mathcal{P}(\{0, 1, \ldots, 9\})_2 \mid \mathfrak{e}_{\{i,j\}} = k\}$$

and we say that a label k is a dominant label if $|C_k| \geq |C_l|$ for every $l \in \{0, 1, \ldots, 9\}$.

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(a) there exists one $k \in \{0, 1, \ldots, 9\}$ such that $|C_k| > |C_l|$ for every $l \in \{0, 1, \ldots, 9\} \setminus \{k\}$ (there exists exactly one dominant label) $\implies$ our input is labelled as k;

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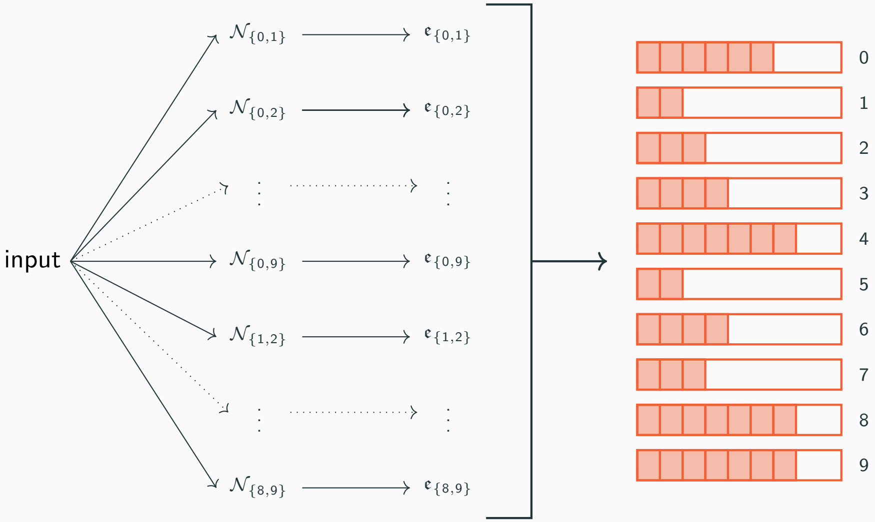
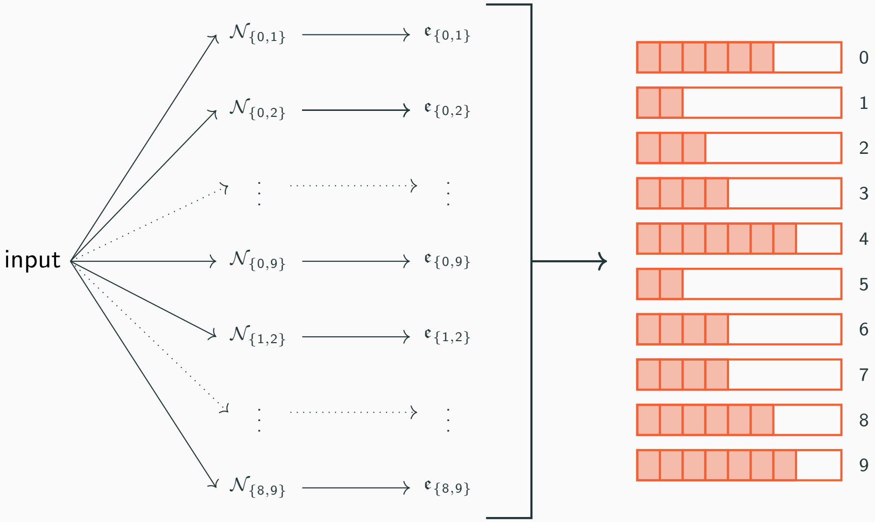
(c) there exist three or more dominant labels $\implies$ our input is labelled as -1.

Majority voting - Example 2



Majority voting - Example 3



Consequences

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(4) there exist three or more dominant labels but none of them is the correct one;

(5) there exist exactly two dominant labels k_1, k_2 but neither $\mathfrak{e}_{\{k_1,k_2\}}$ nor $\hat{\mathfrak{e}}_{\{k_1,k_2\}}$ is the correct one;

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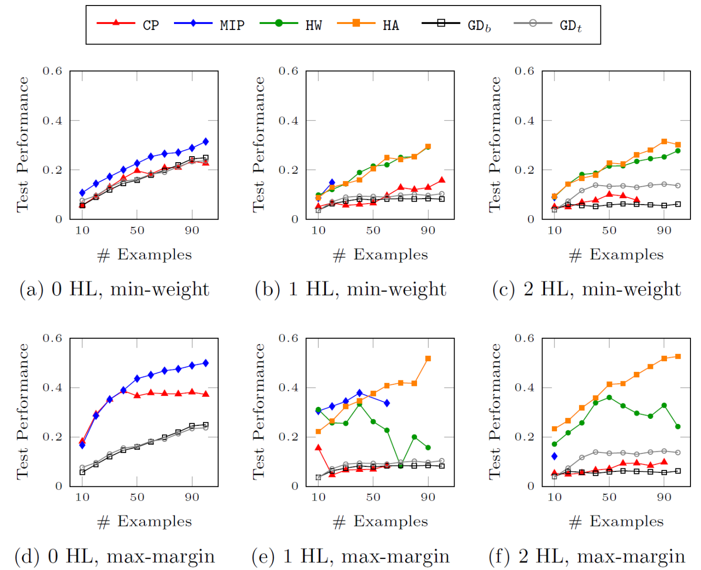
(6) there exists exactly one dominant label but it is not the correct one.

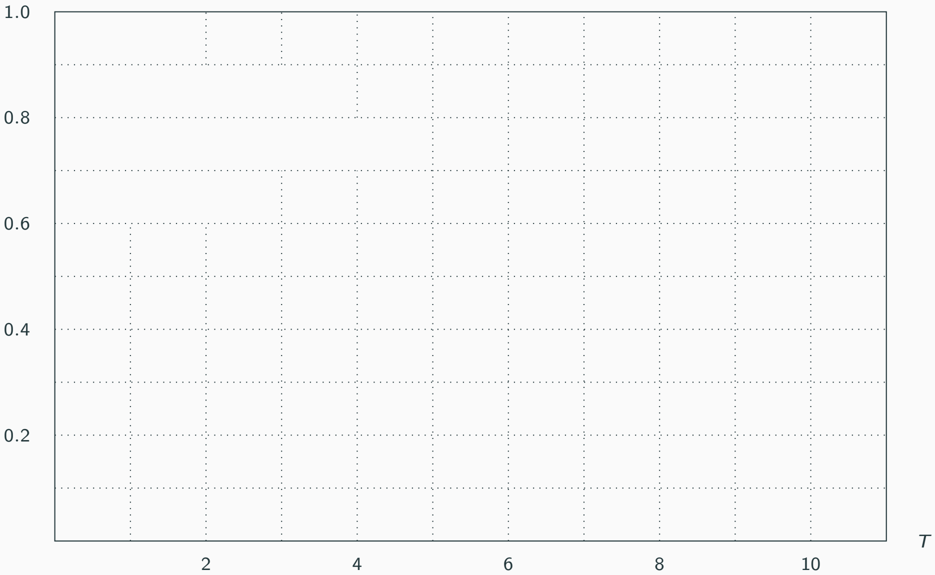
Computational Analysis

Some previous results



Toro Icarte, R., Illanes, L., Castro, M.P., Cire, A.A., McIlraith, S.A. and Beck, J.C.: Training binarized neural networks using MIP and CP. In: Proceedings of CP'19. vol 11802, pp. 401–417. Springer (2019).

Comparison - Test Accuracy



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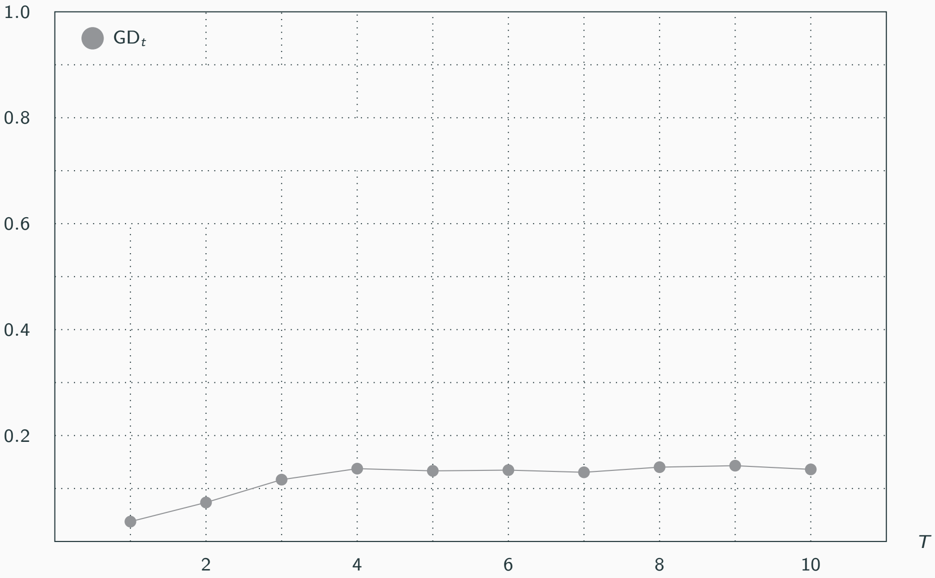

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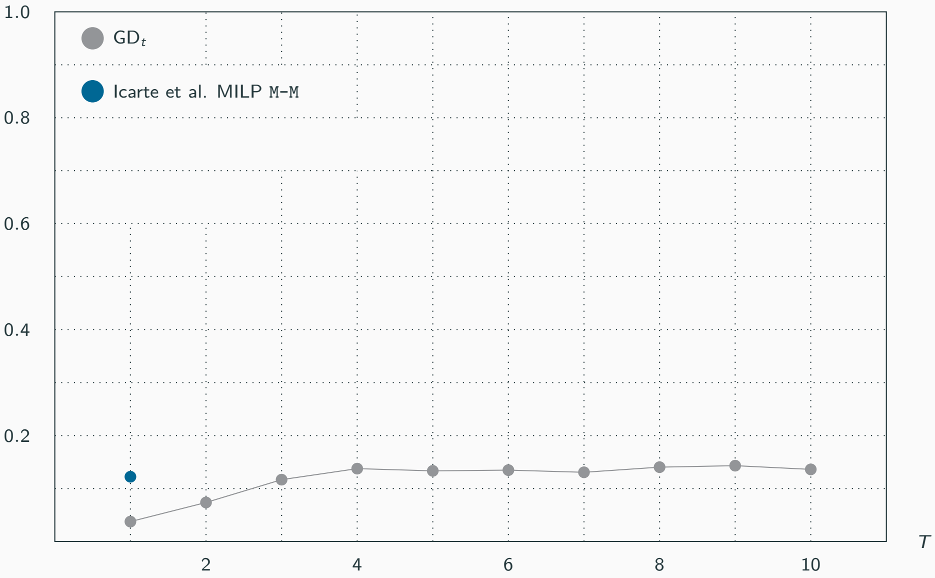

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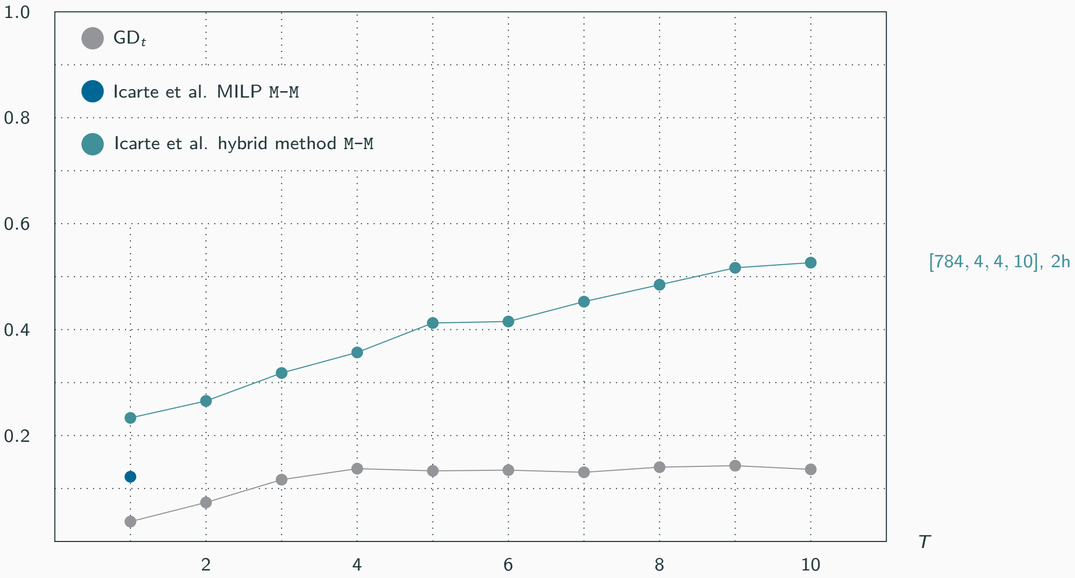

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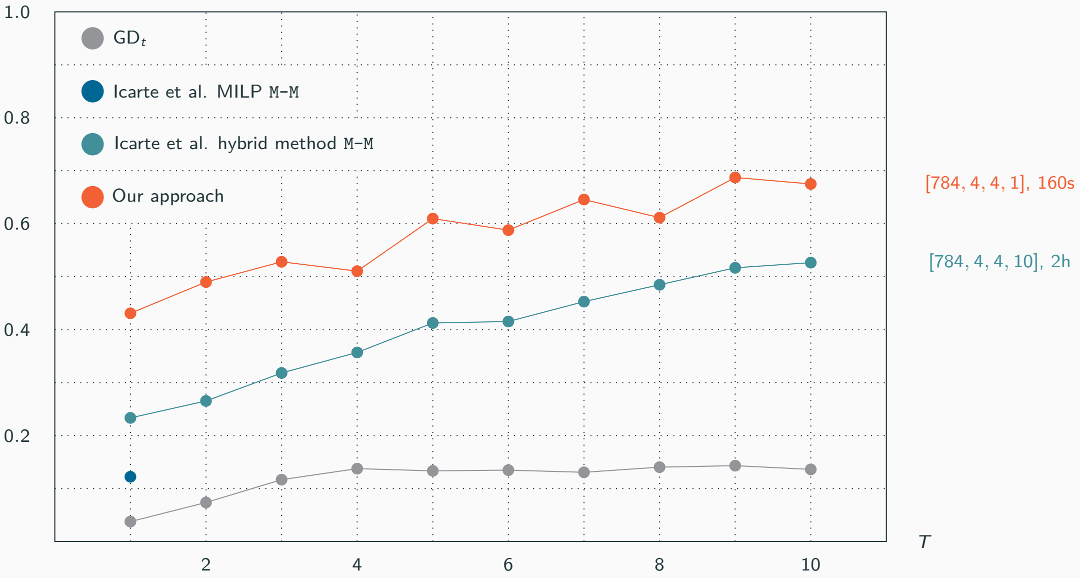


Bigger tests

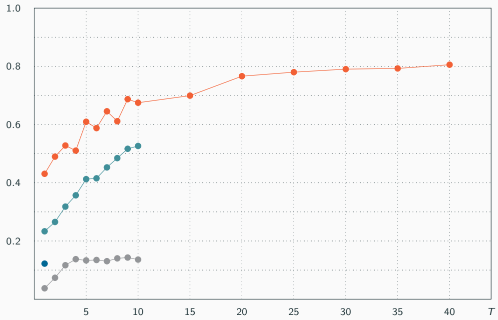
Structure:

networks architecture: [784, 10, 3, 1];

time limit for each network: 290s + 290s + 20s;

tested images: 5000.

T	%Correct	%Incorrect	%−1	Feas. S-M	%$\lvert w \rvert$ M-M	%$\lvert w \rvert$ M-W
10	67.56	29.94	2.50	20	51.59	25.46
15	69.94	26.96	3.10	30	53.25	26.32
20	76.62	21.18	2.20	40	52.73	21.95
25	78.00	20.20	1.80	50	53.86	22.92
30	79.04	19.06	1.90	60	57.58	23.51
35	79.28	19.02	1.70	70	57.36	23.92
40	80.56	17.38	2.06	80	58.97	25.65



T	%Status 0	%Status 1	%Status 2	%Status 3	%Status 4	%Status 5	%Status 6
10	65.82	1.74	1.52	2.20	0.30	2.16	26.26
15	68.46	1.48	1.18	2.58	0.52	2.12	23.66
20	74.76	1.86	1.12	1.72	0.48	1.56	18.50
25	76.64	1.36	1.34	1.46	0.34	1.78	17.08
30	77.68	1.36	1.40	1.56	0.34	1.54	16.12
35	77.66	1.62	1.36	1.54	0.16	1.26	16.40
40	78.60	1.96	1.66	1.90	0.16	1.80	13.92

Confusion matrix

Actual \ Predicted	0	1	2	3	4	5	6	7	8	9	−1
0	448	0	0	2	1	17	6	0	5	1	20
1	0	487	2	4	0	1	2	0	2	1	1
2	12	58	335	19	2	8	17	2	25	2	16
3	0	9	11	406	0	19	2	12	24	5	12
4	0	1	14	0	382	1	19	6	4	66	7
5	3	5	1	35	2	400	7	3	25	8	11
6	17	3	15	1	9	23	418	2	3	0	9
7	1	20	26	14	7	1	1	383	6	35	6
8	8	13	9	31	7	32	4	5	367	12	12
9	7	5	5	6	43	0	0	15	8	401	9

Networks architecture: [784, 10, 3, 1];
time limit for each network: 290s + 290s + 20s;
training images per digit: 40;
tested images: 5000.

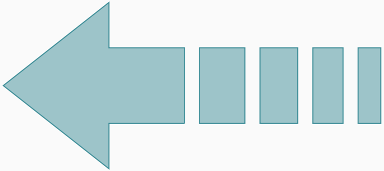
Conclusions

Final Remarks & Future Perspectives

Different approaches from the literature and how to model them with MILPs;

a way of combining these approaches to preserve feasibility, robustness, and simplicity;

a structured ensemble method with its majority voting system.



Final Remarks & Future Perspectives

Different approaches from the literature and how to model them with MILPs;

a way of combining these approaches to preserve feasibility, robustness, and simplicity;

a structured ensemble method with its majority voting system.

Apply a dimensionality reduction method to use more data or data with bigger dimensions;

alternative model formulations to improve the solver performances;

study the theoretical framework.

That's all Folks!

Any Questions?

You can also send me an e-mail at

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