On the Integrality Gap of the Steiner Tree Problem with Three Terminals

Master's Thesis Proposal, Department of Mathematics

Supervisor: Prof. Stefano Gualandi
Co-supervisor: Ambrogio Maria Bernardelli

1 Introduction

Given an undirected, edge-weighted, connected graph $G = (V, E)$ with n nodes and positive costs c_{ij} on each edge $\{i, j\} \in E,\ i, j \in V$, and a subset of nodes $T \subset V$ of cardinality $t \geq 2$, the Steiner Tree Problem (STP) involves finding the minimum-cost tree that spans the set of *terminals* T. The STP is generally modeled and solved via integer linear programming, with many diverse publications released over the years. Among them, the *Bidirected cut formulation* has attracted much attention, thanks to exceptional empirical performances. The core of the formulation consists in replacing each undirected edge $\{i, j\}$ with two arcs (i, j) and (j, i) and introducing a decision variable x_{ij} for each arc. For a given root node $r \in T$, the formulation is presented below:

$$\min_{\mathbf{x}\in\{0,1\}^{2\times|E|}} \quad \sum_{\{i,j\}\in E} c_e(x_{ij} + x_{ji}) \tag{1a}$$

$$\text{s.t.} \quad x_{ij} + x_{ji} \leq 1, \qquad e = \{i, j\} \in E, \tag{1b}$$

$$x\left(\delta^-(W)\right) \geq 1, \qquad W \subset V \setminus \{r\},\ W \cap T \neq \emptyset, \tag{1c}$$

$$x_{ij} \in \{0, 1\}, \tag{1d}$$

where $\delta^-(W) := \{(i, j) \mid i \notin W,\ j \in W\}$. This model can be relaxed by replacing constrain (1d) with

$$0 \leq x_{ij} \leq 1. \tag{2}$$

We will abbreviate the ILP with DCUT and the relaxed version RDCUT.

2 Goals of the thesis

Given an STP instance, i.e., a graph G and a subset of node T, we will denote by $\mathrm{DCUT}(G, T)$ and by $\mathrm{RDCUT}(G, T)$ the optimal value of the ILP and the optimal value of the relaxed version, respectively. We define the *integrality gap* of the instance (G, T) by

$$\mathrm{IG}(G, T) := \frac{\mathrm{DCUT}(G, T)}{\mathrm{RDCUT}(G, T)}.$$

The problem of finding intsances with maximum integrality gap is linked to the search of fractional vertices of the polytope $P_{\mathrm{DCUT}}(n, t)$ defined by the constraint matrix.

Two well-known cases of the STP are the *shortest path* ($|T| = t = 2$) and the *minimum spanning tree* ($t = n$), for which $P_{\mathrm{DCUT}}(n, t)$ is integral, i.e, every vertex is integer, and so no instance with an integrality gap greater than 1 can be found. Note that there exist instances with a non-trivial integrality gap, for example for the case $n = 7$, $t = 4$. For the case with three terminals, i.e., $t = 3$, after several numerical tests on complete metric graphs, we were not able to find neither instances with an integrality gap greater than 1 nor non-integer optimal solutions. This led us to formulate two different conjectures.

Conjecture 1. *Any vertex of $P_{DCUT}(n, 3)$ optimum for a metric cost is integer.*

Conjecture 2. *Given a metric graph, there exists an optimal solution which is integer.*

Note in particular that Conjecture 1 implies Conjecture 2. Note also that the conjectures cannot be proven using total unimodularity of the constraint matrix because, for example, the matrix is not totally unimodular for the cases $(n, t) = (4, 3), (5, 3)$. The polytope linked to the case $(n, t) = (5, 3)$ is not even integral.

The main goal of the thesis is to prove these conjectures. Additionally, noteworthy accomplishments include demonstrating them in weaker forms or special cases.

3 Starting points

Suggested lectures:

- an introduction to the problem and its formulations [4, 3];
- the characterization of integer solutions for the case with three terminals [2];
- a study of the relation between integrality gap and fractional vertices [1].

The first steps could consist in studying the conjectures

- in the simple case of the cost being in the set $\{1, 2\}$;
- for the small cases $n = 4, 5$;
- for a stronger formulation proposed in [2];

both from a theoretical and computational point of view.

References

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