$\boxed{T\,5}$

$$p(x) = \begin{cases} \frac{\theta - 1}{x^{\theta}}, & x \geq 1 \\ 0, & x < 1 \end{cases}, \quad \theta > 1$$

a) $\bar{x}_n$, ОМП

$$L(\theta) = \prod_{i=1}^{n} p(x_i, \theta)$$

$$\ln L(\theta) = \sum_{i=1}^{n} \ln\left(\frac{\theta - 1}{x_i^{\theta}}\right) = n \ln(\theta - 1) - \theta \sum_{i=1}^{n} \ln x_i$$

$$\frac{\partial \ln L(\theta)}{\partial \theta} = \frac{n}{\theta - 1} - \sum_{i=1}^{n} \ln x_i = 0 \Rightarrow \frac{n}{\sum \ln x_i} + 1 = \theta_{max}$$

$$\frac{\partial^2 \ln L(\theta)}{\partial \theta^2} = -\frac{n}{(\theta - 1)^2} < 0 \Rightarrow \theta_{max} \rightarrow max$$

$$\boxed{\tilde{\theta} = \frac{n}{\sum_{i=1}^{n} \ln x_i} + 1}$$

b) медиана: $\int_1^{x_{med}} p(x)dx = \frac{1}{2}$

$$1 - x_{med}^{1-\theta} = \frac{1}{2} \Rightarrow x_{med} = 2^{\frac{1}{\theta - 1}} = t$$

По Т. Фишера: $\frac{\hat{t}(\theta) - t(\theta)}{\sigma}\sqrt{n} \sim N(0,1)$

$$t'(\theta) = 2^{\frac{1}{\theta-1}} \ln 2 \left(-\frac{1}{(\theta-1)^2}\right)$$

$$\cancel{I(\theta) = M\left[\left(\frac{\partial \ln p}{\partial \theta}\right)^2\right] = M\left[\left(\frac{1}{\theta-1} - \theta x^{\theta}\right)^2\right] = \int_{-\infty}^{+\infty}\left(\frac{1}{\theta-1} - \theta x^{\theta}\right)^2 \frac{\theta-1}{x^{\theta}}\,dx =}$$

$$\cancel{= \int_1^{\infty}\left(\frac{1}{(\theta-1)^2} - \frac{2\theta x^{\theta}}{\theta-1} + \theta^2 x^{2\theta}\right)\frac{\theta-1}{x^{\theta}}\,dx = \int_1^{\infty}\frac{1}{(\theta-1)x^{\theta}}\,dx -}$$

$$\cancel{- \int_1^{\infty} 2\theta\,dx + \int_1^{\infty}\theta^2(\theta-1)x^{\theta}\,dx}$$

$$\cancel{I(\theta) = -M\left[\frac{\partial^2 \ln p}{\partial \theta^2}\right] = -M\left[\frac{1}{\theta-1}\right.}$$

$$I(\theta) = M\left[\left(\frac{\partial \ln p}{\partial \theta}\right)^2\right] = \int_{-\infty}^{+\infty}\left(\frac{1}{\theta-1} - \ln x\right)^2 \frac{\theta-1}{x^{\theta}}\,dx = \int_1^{\infty}\left(\frac{1}{(\theta-1)^2} - \frac{2\ln x}{\theta-1} + \ln^2 x\right)\frac{\theta-1}{x^{\theta}}\,dx$$

$$= \int_1^{\infty}\frac{dx}{(\theta-1)x^{\theta}} - \int_1^{\infty}\frac{2\ln x}{x^{\theta}}\,dx + \int_1^{\infty}\frac{\ln^2 x\,(\theta-1)}{x^{\theta}}\,dx =$$

$$= \frac{1}{\theta-1}\,x^{1-\theta}\cdot\frac{1}{1-\theta}\Big|_1^{\infty} + 2\,\frac{x^{-1+\theta}\left[(\theta-1)\cdot\ln x + 1\right]}{(\theta-1)^2}\Big|_1^{\infty} + \frac{2}{(\theta-1)^3} + \frac{2(\theta-1)}{(\theta-1)^3} =$$

$$= \frac{1}{(\theta-1)^2} - \frac{2}{(\theta-1)^2} + \frac{2}{(\theta-1)^2} = \frac{1}{(\theta-1)^2}$$

$$\frac{\tilde{t}(\theta) - t(\theta)}{2^{\frac{1}{\theta-1}}\ln 2\left(-\frac{1}{(\theta-1)^2}\right)}\cdot\frac{\sqrt{n}}{\theta-1} \rightsquigarrow N(0,1)$$

$$-1{,}96 < \frac{X_{med} - 2^{\frac{1}{\theta-1}}}{2^{\frac{1}{\theta-1}}\ln 2}(\theta-1)\sqrt{n} < 1{,}96$$

$$\boxed{2^{\frac{1}{\theta-1}} - \frac{1{,}96\cdot 2^{\frac{1}{\theta-1}}\ln 2}{(\theta-1)\sqrt{n}} < X_{med} < 2^{\frac{1}{\theta-1}} + \frac{1{,}96\, 2^{\frac{1}{\theta-1}}\ln 2}{(\theta-1)\sqrt{n}}}$$

c) $p(y) = \begin{cases} e^{1-y}, & y \geq 1 \\ 0, & y < 1 \end{cases}$

$$p(\theta|\bar{x}_n) = \frac{p(\theta) L(\theta)}{p(\bar{x}_n)} = \begin{cases} \frac{e^{1-\theta} \prod\limits_{i=1}^{n} p(x_i, \theta)}{p(\bar{x}_n)} = B, & \theta \geq 1 \\ 0, & \theta < 1 \end{cases} =>$$

$$p(\theta|\bar{x}_n) = \begin{cases} e^{1-\theta} \prod p(x_i, \theta) B, & \theta \geq 1 \\ 0, & \theta < 1 \end{cases}$$

Нормировка:

$$\int_1^{\infty} e^{1-\theta} \prod p(x_i, \theta) B \, d\theta = 1 \Rightarrow B =$$

$$\int_1^{\infty} e^{1-\theta} \prod_{i=1}^{n} \frac{\theta - 1}{x_i^{\theta}} B \, d\theta = \int_1^{\infty} e^{1-\theta} (\theta - 1)^n \left(\prod_{i=1}^{n} \frac{1}{x_i}\right)^{\theta} B \, d\theta = 1$$

$$\ln p(\theta|\bar{x}_n) = \ln B + 1 - \theta + n \ln(\theta - 1) - \theta \sum \ln x_i$$

$$\frac{\partial \ln}{\partial \theta} = 0 \Rightarrow -1 + \frac{n}{\theta - 1} - \sum \ln x_i = 0 \Rightarrow \boxed{\tilde{\theta} = 1 + \frac{n}{\sum \ln x_i + 1}}$$

$$\begin{cases} \int_1^{f_1} p(\theta|\bar{x}_n) \, d\theta = 0{,}025 \\ \int_{f_2}^{\infty} p(\theta|\bar{x}_n) \, d\theta = 0{,}025 \end{cases} \Rightarrow I = (f_1; f_2)$$

d) $\frac{\hat{\theta}-\theta}{\theta}\sqrt{n I(\theta)} \rightsquigarrow N(0,1)$ $\quad I(\theta) = \frac{1}{(\theta-1)^2}$

$\frac{\hat{\theta}-\theta}{(\hat{\theta}-1)}\sqrt{n} \rightsquigarrow N(0,1)$

$$-1{,}96 < \frac{\tilde{\theta}-\theta}{\hat{\theta}-1}\sqrt{n} < 1{,}96$$

$$\tilde{\theta} - \frac{1{,}96(\hat{\theta}-1)}{\sqrt{n}} < \theta < \tilde{\theta} + \frac{1{,}96(\hat{\theta}-1)}{\sqrt{n}}$$

$$\hat{\theta} = \frac{n}{\sum \ln x_i} + 1 = (\overline{\ln x})^{-1} + 1$$

$$\boxed{(\overline{\ln x})^{-1} + 1 - \frac{1{,}96(\overline{\ln x})^{-1}}{\sqrt{n}} < \theta \; (\overline{\ln x})^{-1} + 1 + \frac{1{,}96(\overline{\ln x})^{-1}}{\sqrt{n}}}$$

e-g) $F(x) = \begin{cases} -x^{1-\theta}+1, & x \geq 1 \\ 0, & x < 1 \end{cases}, \quad \theta > 1$

$\cancel{x = F(x)+1}$ $\quad x = \sqrt[1-\theta]{1-F(x)}$