$\boxed{№ 4}$

$\xi \sim R[\theta, 2\theta]$

$$p(x) = \begin{cases} \frac{1}{\theta}, & x \in [\theta, 2\theta] \\ 0, & x \notin [\theta, 2\theta] \end{cases}$$

а) ОММ:

$$\alpha_1(\theta) = M\xi = \int_{\theta}^{2\theta} \frac{x}{\theta} dx = \frac{x^2}{2\theta}\Big|_{\theta}^{2\theta} = 2\theta - \frac{\theta}{2} = \frac{3}{2}\theta$$

$$\tilde{\alpha}_1 = \frac{1}{n}\sum x_i = \overline{x}$$

$$\alpha_1(\theta) = \tilde{\alpha}_1 \Rightarrow \frac{3}{2}\theta = \overline{x} \to \boxed{\tilde{\theta}_1 = \frac{2}{3}\overline{x}}$$

ОМП:

$$L(\theta) = \frac{1}{\theta^n}(\theta \le x_i \le 2\theta) = \frac{1}{\theta^n}(\max x_i < 2\theta) \to \max$$

$$\boxed{\tilde{\theta}_2 = \frac{x_{max}}{2}}$$

$$\boxed{\tilde{\theta}_3 = \frac{1}{5}(x_{min} + 2x_{max})}$$

б) 1. $\tilde{\theta}_1 = \frac{2}{3}\overline{x}$

- Несмещаемость:

$$M[\tilde{\theta}_1] = \frac{2}{3} M\left[\frac{1}{n}\sum x_i\right] = \frac{2}{3} M\xi = \frac{2}{3} \cdot \frac{3}{2}\theta = \theta$$ несмещ

- Состоятельность:

$$D[\hat{\Theta}_1] = \frac{4}{9} D\left[\frac{1}{n}\sum x_i\right] = \frac{4}{9n} D_\xi \xrightarrow[n\to\infty]{} 0 \Rightarrow \underline{\text{состоят}}$$

② $\tilde{\Theta}_2 = \frac{x_{max}}{2}$

~~$M[\Theta_2]$~~ • Несмещённость:

$$M[\tilde{\Theta}_2] = \frac{1}{2} M[x_{max}] = \frac{1}{2}\int_\Theta^{2\Theta} x n F^{n-1}(x)\, p(x)\, dx = \boxed{\cancel{\frac{1}{2}\int_\Theta^{2\Theta} x n \left(\frac{x}{\Theta}\right)^{n-1}\frac{1}{\Theta}dx}} =$$

$$\boxed{\cancel{= \frac{n\Theta}{2}\int_\Theta^{2\Theta}\left(\frac{x}{\Theta}\right)^n d\left(\frac{x}{\Theta}\right) = \frac{n\Theta}{2}\left(\frac{x}{\Theta}\right)^{n+1}\frac{1}{n+1}\Bigg|_\Theta^{2\Theta} = \frac{2^n n\Theta}{n+1} - \frac{n\Theta}{2}}}$$

$$= \frac{n}{2}\int_\Theta^{2\Theta} x\left(\frac{x-\Theta}{\Theta}\right)^{n-1}\frac{1}{\Theta}dx = \frac{1}{2}\int_\Theta^{2\Theta} x\, d\left(\frac{x-\Theta}{\Theta}\right)^n = \frac{1}{2} x\left(\frac{x-\Theta}{\Theta}\right)^n\Bigg|_\Theta^{2\Theta} -$$

$$- \frac{1}{2}\int_\Theta^{2\Theta}\left(\frac{x-\Theta}{\Theta}\right)^n dx = \Theta - \frac{\Theta}{2}\int_\Theta^{2\Theta}\left(\frac{x-\Theta}{\Theta}\right)^n d\left(\frac{x-\Theta}{\Theta}\right) =$$

$$= \Theta - \frac{\Theta}{2}\left(\frac{x-\Theta}{\Theta}\right)^{n+1}\cdot\frac{1}{n+1}\Bigg|_\Theta^{2\Theta} = \Theta - \frac{\Theta}{2(n+1)} = \Theta\,\frac{2n+1}{2(n+1)} \ - \underline{\text{смещ}}$$

$$\boxed{\tilde{\Theta}_2' = \frac{2n+2}{2n+1}\,\frac{x_{max}}{2} = \frac{n+1}{2n+1}x_{max}} - \underline{\text{несмещ}}$$

• Состоят:

$$D[\tilde{\Theta}_2'] = \left(\frac{n+1}{2n+1}\right)^2 D[x_{max}]$$

$$D[x_{max}] = M[x_{max}^2] - M^2[x_{max}]$$

$$M[x_{max}^2] = \int_\Theta^{2\Theta} x^2 n\left(\frac{x-\Theta}{\Theta}\right)^{n-1}\frac{1}{\Theta}dx = \int_\Theta^{2\Theta} x^2 d\left(\frac{x-\Theta}{\Theta}\right)^n = x^2\left(\frac{x-\Theta}{\Theta}\right)^n\Bigg|_\Theta^{2\Theta} -$$

$$- \int_\Theta^{2\Theta}\left(\frac{x-\Theta}{\Theta}\right)^n 2x\,dx = 4\Theta^2 - \frac{2\Theta}{n+1}\int_\Theta^{2\Theta} x\, d\left(\frac{x-\Theta}{\Theta}\right)^{n+1} =$$

$$= 4\theta^2 - \frac{2\theta}{n+1}\left(2\theta - \int_\theta^{2\theta}\left(\frac{x-\theta}{\theta}\right)^{n+1}dx\right) = 4\theta^2\frac{n}{n+1} + \frac{2\theta^3}{(n+1)(n+2)} =$$

$$= \frac{2\theta^2(2n^2+4n+1)}{(n+1)(n+2)}$$

$$D[X_{max}] = \frac{2\theta^2(2n^2+4n+1)}{(n+1)(n+2)} - \frac{\theta^2(2n+1)^2}{(n+1)^2} = \frac{n\theta^2}{(n+1)^2(n+2)}$$

$D[X_{max}] \underset{n\to\infty}{\to} 0 \Rightarrow \tilde{\theta}_2' -$ состоят

$$D[\tilde{\theta}_2'] = \frac{n\theta^2}{(n+2)(2n+1)} \underset{n\to\infty}{\to} 0$$

③ $\tilde{\theta}_3 = \frac{1}{5}(X_{min} + 2X_{max})$

• Несмещённость:

$$M[\tilde{\theta}_3] = \frac{1}{5}M[X_{min}] + \frac{2}{5}M[X_{max}]$$

$$M[X_{min}] = \int_\theta^{2\theta} x\, n\left(1-\frac{x-\theta}{\theta}\right)^{n-1}\frac{1}{\theta}dx = \frac{\theta(n+2)}{(n+1)}$$

$$M[\tilde{\theta}_3] = \frac{1}{5}\frac{\theta(n+2)}{(n+1)} + \frac{2}{5}\frac{\theta(2n+1)}{(n+1)} =$$

$$= \frac{\theta(5n+4)}{5(n+1)} = \theta\frac{5n+4}{5n+5}$$ – смещ.

$$\tilde{\theta}_3' = \frac{5n+5}{5n+4}\tilde{\theta}_3 \Rightarrow \boxed{\tilde{\theta}_3' = \frac{n+1}{5n+4}(X_{min}+2X_{max})}$$ – несмещ.

• Состоят:

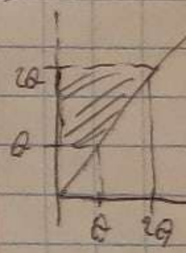
$$D[\tilde{\theta}_3'] = \frac{(\lambda+1)^2}{(5\lambda+4)^2}\left(D[x_{min}] + 4D[x_{max}] + 4\,cov(x_{min}, x_{max})\right)$$

$$M[x_{min}^2] = \int_{\theta}^{2\theta} x^2 \lambda \left(1 - \frac{x-\theta}{\theta}\right)^{\lambda-1} \frac{1}{\theta}\, dx = \frac{(\lambda(\lambda+5)+8)\theta^2}{(\lambda+1)(\lambda+2)}$$

$$D[x_{min}] = \frac{\lambda^2+5\lambda+8}{(\lambda+1)(\lambda+2)}\theta^2 - \frac{\theta^2(\lambda+2)^2}{(\lambda+1)^2} = \frac{\lambda}{(\lambda+1)^2(\lambda+2)}\theta^2$$

$$cov(x_{min}, x_{max}) = M[x_{min} \cdot x_{max}] - M[x_{min}]M[x_{max}]$$

$$K(y, z) = \begin{cases} F^{\lambda}(z) - (F(z) - F(y))^{\lambda}, & z \geq y \\ F^{\lambda}(z), & z < y \end{cases}$$ – совм. зак. распр.



$$\varkappa(y, z) = \frac{\partial^2 K}{\partial y \partial z} = \lambda(\lambda-1)(F(z) - F(y))^{\lambda-2} F'(y) F'(z) \quad z \geq y =$$

$$= \lambda(\lambda-1)\left(\frac{z-y}{\theta}\right)^{\lambda-2} \frac{1}{\theta^2}$$

$$M[x_{min} \cdot x_{max}] = \iint_{-\infty}^{+\infty} zy\, \varkappa(y,z)\, dy dz = \int_{\theta}^{2\theta} dy \int_{\theta}^{\theta} zy\, \lambda(\lambda-1)\left(\frac{z-y}{\theta}\right)^{\lambda-2} \frac{1}{\theta^2} =$$

$$= \frac{\theta^2(2\lambda+5)}{\lambda+2}$$

$$cov(x_{min}, x_{max}) = \frac{\theta^2(2\lambda+5)}{\lambda+2} - \frac{\theta(\lambda+2)}{\lambda+1} \cdot \frac{\theta(2\lambda+1)}{(\lambda+1)} = \frac{\theta^2}{(\lambda+1)^2(\lambda+2)}$$

$$D[\tilde{\theta}_3'] = \frac{\cancel{(\lambda+1)}^2}{(5\lambda+4)^2}\left(\frac{5\lambda\theta^2}{\cancel{(\lambda+1)}^2(\lambda+2)} + \frac{4\theta^2}{\cancel{(\lambda+1)}^2(\lambda+2)}\right) = \frac{\cancel{\lambda}\theta^2}{(5\lambda+4)(\lambda+2)} \xrightarrow[\lambda\to\infty]{} 0$$ <u>состоят</u>

c) $D[\widehat{\Theta}_1] = \frac{4}{9n} D_\varphi$

$M_{\varphi^2} = \int_\Theta^{2\Theta} x^2 \frac{1}{\Theta} dx = \frac{7\Theta^2}{3} \Rightarrow D_\varphi = \frac{1}{3}\Theta^2$

$D[\widehat{\Theta}_1] = \frac{16}{27n}\Theta^2$

$D[\widehat{\Theta}_2'] = \frac{n\Theta^2}{(n+2)(2n+1)}$

$D[\Theta_3'] = \frac{\Theta^2}{(5n+4)(n+2)}$

$\widetilde{\Theta}_3'$ – самая эф. оц.

самая асимпт. эф. оц.

d)

$$F(x) = \begin{cases} 0, & x < \Theta \\ \frac{x - \Theta}{\Theta}, & x \in [\Theta, 2\Theta] \\ 1, & x > 2\Theta \end{cases}$$

$G(\overline{X}_n, \Theta) = \frac{\max(\overline{X}_n)}{\Theta}$

$F_G(y) = P(G \le y) = P(\max(\overline{X}_n) \le \Theta y) = (P(X_1 \le \Theta y))^n = (F(\Theta y))^n$

$$F_G(y) = \begin{cases} 0, & y < 1 \\ (y-1)^n, & 1 \le y \le 2 \\ 1, & y > 2 \end{cases}$$

$$p_G(y) = \begin{cases} 0, & y \notin [1, 2] \\ n(y-1)^{n-1}, & y \in [1, 2] \end{cases}$$

$$\int_{\cancel{f_1}}^{f_1} n(y-1)^{n-1}\,dy = \cancel{1-0{,}95}\,/2 = \frac{1-0{,}95}{2}$$

$\cancel{y=0}$ $(f_1-1)^n = 0{,}025 \Rightarrow f_1 = \sqrt[n]{0{,}025}+1$

$$\int_{f_2}^{2} n(y-1)^{n-1}\,dy = \frac{1-0{,}95}{2}$$

$\cancel{1-(f_2-1)^n = 0{,}975 \Rightarrow f_2 = \sqrt[n]{0{,}025}+1}$

$1-(f_2-1)^n = 0{,}025 \Rightarrow f_2 = \sqrt[n]{0{,}975}+1$

Доверит. интервал. (точный):

$$\boxed{\left(\sqrt[n]{0{,}025}+1;\ \sqrt[n]{0{,}975}+1\right)} \Rightarrow \boxed{\frac{\max x_i}{1+\sqrt[n]{0{,}975}} < \theta < \frac{\max x_i}{1+\sqrt[n]{0{,}025}}}$$

e) ОММ

$\tilde{\theta} = \frac{2}{3}\bar{x} = \frac{2}{3}\tilde{\alpha}_1$

$f(\alpha) = \frac{2}{3}\alpha_1 \Rightarrow \nabla f(\alpha) = \frac{2}{3}$

$k = (k_{11}) = \alpha_2 - \alpha_1^2 = \cancel{D\tilde{\xi}}$ $\Rightarrow \sigma = \sqrt{\frac{4}{9}(\alpha_2-\alpha_1^2)}$

$$\frac{\tilde{\theta}-\theta}{\sqrt{\frac{4}{9}(\tilde{\alpha}_2-\tilde{\alpha}_1)^2}}\sqrt{n} \rightsquigarrow N(0,1) \sim p(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$$

$\cancel{f_1: \int_{-\infty}^{f_1}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx = 0{,}025}$ $\quad -1{,}96 < \dfrac{\tilde{\theta}-\theta}{\frac{2}{3}\sqrt{\tilde{\alpha}_2-\tilde{\alpha}_1^2}}\sqrt{n} < 1{,}96$

$$\boxed{\frac{-1{,}96\cdot\frac{2}{3}\sqrt{\tilde{\alpha}_2-\tilde{\alpha}_1^2}}{\sqrt{n}}+\tilde{\theta} < \theta < \frac{1{,}96\cdot\frac{2}{3}\sqrt{\tilde{\alpha}_2-\tilde{\alpha}_1^2}}{\sqrt{2}}+\tilde{\theta}}$$ — дов. интервал асимпт.