$$\boxed{\text{T 5}}$$

$$p(x) = \begin{cases} \dfrac{\theta - 1}{x^{\theta}}, & x \geq 1 \\ 0, & x < 1 \end{cases}, \quad \theta > 1$$

a) $\bar{x}_n$, ОМП

$$L(\theta) = \prod_{i=1}^{n} p(x_i, \theta)$$

$$\ln L(\theta) = \sum_{i=1}^{n} \ln\left(\frac{\theta - 1}{x_i^{\theta}}\right) = n \ln(\theta - 1) - \theta \sum_{i=1}^{n} \ln x_i$$

$$\frac{d \ln L(\theta)}{d\theta} = \frac{n}{\theta - 1} - \sum_{i=1}^{n} \ln x_i = 0 \Rightarrow \frac{n}{\sum \ln x_i} + 1 = \theta_{max}$$

$$\frac{d^2 \ln L(\theta)}{d\theta^2} = -\frac{n}{(\theta - 1)^2} < 0 \Rightarrow \theta_{max} \rightarrow max$$

$$\boxed{\widehat{\theta} = \frac{n}{\sum\limits_{i=1}^{n} \ln x_i} + 1}$$

b) медиана: $\displaystyle\int_{1}^{x_{med}} p(x)\,dx = \frac{1}{2}$

$$1 - x_{med}^{1-\theta} = \frac{1}{2} \Rightarrow x_{med} = 2^{\frac{1}{\theta - 1}} = t$$

По Т. Фишера: $\dfrac{\widehat{t}(\theta) - t(\theta)}{\sigma} \sqrt{n} \sim N(0,1)$

$$t'(\theta) = 2^{\frac{1}{\theta - 1}} \ln 2 \left(-\frac{1}{(\theta - 1)^2}\right)$$

$$\overline{I}(\theta) = M\left[\left(\frac{\partial^2 \ln p}{\partial \theta^2}\right)^2\right] = M\left[\left(\frac{1}{\theta-1} - \theta x^\theta\right)^2\right] = \int_{-\infty}^{+\infty}\left(\frac{1}{\theta-1} - \theta x^\theta\right)^2 \frac{\theta-1}{x^\theta}\,dx =$$

$$= \int_1^\infty\left(\frac{1}{(\theta-1)^2} - \frac{2\theta x^\theta}{\theta-1} + \theta^2 x^{2\theta}\right)\frac{\theta-1}{x^\theta}\,dx = \int_1^\infty \frac{1}{(\theta-1)x^\theta}\,dx -$$

$$- \int_1^\infty 2\theta\,dx + \int_1^\infty \theta^2(\theta-1)x^\theta\,dx$$

$$\overline{I}(\theta) = -M\left[\frac{\partial^2 \ln p}{\partial \theta^2}\right] = -M\left[\frac{1}{\theta-1}\right]$$

$$I(\theta) = M\left[\left(\frac{\partial \ln p}{\partial \theta}\right)^2\right] = \int_{-\infty}^{+\infty}\left(\frac{1}{\theta-1} - \ln x\right)^2 \frac{\theta-1}{x^\theta}\,dx = \int_1^\infty\left(\frac{1}{(\theta-1)^2} - \frac{2\ln x}{\theta-1} + \ln^2 x\right)\frac{\theta-1}{x^\theta}\,dx$$

$$= \int_1^\infty \frac{dx}{(\theta-1)x^\theta} - \int_1^\infty \frac{2\ln x}{x^\theta}\,dx + \int_1^\infty \frac{\ln^2 x\,(\theta-1)}{x^\theta}\,dx =$$

$$= \frac{1}{\theta-1}\,x^{1-\theta}\cdot\frac{1}{1-\theta}\Big|_1^\infty + 2\frac{x^{-1+\theta}\left[(\theta-1)\cdot\ln x + 1\right]}{(\theta-1)^2}\Big|_1^\infty + \frac{2}{(\theta-1)^3} + \frac{2(\theta-1)}{(2\theta-1)^3} =$$

$$= \frac{1}{(\theta-1)^2} - \frac{2}{(\theta-1)^2} + \frac{2}{(\theta-1)^2} = \frac{1}{(\theta-1)^2}$$

$$\frac{\widetilde{t}(\theta) - t(\theta)}{2^{\frac{1}{\theta-1}}\ln 2\left(-\frac{1}{(\theta-1)^2}\right)}\cdot\frac{\sqrt{n}}{\theta-1} \rightsquigarrow N(0,1)$$

$$-1{,}96 < \frac{x_{med} - 2^{\frac{1}{\theta-1}}}{2^{\frac{1}{\theta-1}}\ln 2}(\theta-1)\sqrt{n} < 1{,}96$$

$$\boxed{2^{\frac{1}{\theta-1}} - \frac{1{,}96\cdot 2^{\frac{1}{\theta-1}}\ln 2}{(\theta-1)\sqrt{n}} < x_{med} < 2^{\frac{1}{\theta-1}} + \frac{1{,}96\,2^{\frac{1}{\theta-1}}\ln 2}{(\theta-1)\sqrt{n}}}$$

c) $p(y) = \begin{cases} e^{1-y} & , y \geq 1 \\ 0 & , y < 1 \end{cases}$

$p(\theta | \bar{x}_n) = \dfrac{p(\theta) L(\theta)}{p(\bar{x}_n)} = \begin{cases} \dfrac{e^{1-\theta} \prod\limits_{i=1}^{n} p(x_i, \theta)}{p(\bar{x}_n)} = B, & \theta \geq 1 \\ 0, & \theta < 1 \end{cases} \Rightarrow$

$p(\theta | \bar{x}_n) = \begin{cases} e^{1-\theta} \prod p(x_i, \theta) B, & \theta \geq 1 \\ 0, & \theta < 1 \end{cases}$

Нормировка:

$\int\limits_1^{\infty} e^{1-\theta} \prod p(x_i, \theta) B \, d\theta = 1 \Rightarrow B =$

$\int\limits_1^{\infty} e^{1-\theta} \prod\limits_{i=1}^{n} \dfrac{\theta-1}{x_i^{\theta}} B \, d\theta = \int\limits_1^{\infty} e^{1-\theta} (\theta-1)^n \left(\prod\limits_{i=1}^{n} \dfrac{1}{x_i} \right)^{\theta} B \, d\theta =$

$\ln p(\theta | \bar{x}_n) = \ln B + 1 - \theta + n \ln(\theta-1) - \theta \sum \ln x_i$

$\dfrac{\partial \ln}{\partial \theta} = 0 \Rightarrow -1 + \dfrac{n}{\theta-1} - \sum \ln x_i = 0 \Rightarrow \boxed{\tilde{\theta} = 1 + \dfrac{n}{\sum \ln x_i + 1}}$

$\begin{cases} \int\limits_1^{f_1} p(\theta | \bar{x}_n) \, d\theta = 0.025 \\ \int\limits_{f_2}^{\infty} p(\theta | \bar{x}_n) \, d\theta = 0.025 \end{cases} \Rightarrow I = (f_1; f_2)$

d) $\dfrac{\hat{Q} - Q}{\hat{Q}'} \sqrt{n\, I_{(\theta)}} \rightsquigarrow N(0,1)$ $I_{(\theta)} = \dfrac{1}{(\theta-1)^2}$

$\dfrac{\tilde{Q} - Q}{(\hat{Q}-1)} \sqrt{n} \rightsquigarrow N(0,1)$

$$-1,96 < \dfrac{\tilde{Q} - Q}{\hat{Q} - 1} \sqrt{n} < 1,96$$

$$\tilde{Q} - \dfrac{1,96\,(\hat{Q}-1)}{\sqrt{n}} < Q < \tilde{Q} + \dfrac{1,96\,(\hat{Q}-1)}{\sqrt{n}}$$

$$\hat{Q} = \dfrac{n}{\Sigma\, l_2 x_i} + 1 = (\overline{l_n x})^{-1} + 1$$

$$\boxed{(\overline{l_n x})^{-1} + 1 - \dfrac{1,96\,(\overline{l_n x})^{-1}}{\sqrt{n}} < \theta\ (\overline{l_n x})^{-1} + 1 + \dfrac{1,96\,(\overline{l_n x})^{-1}}{\sqrt{n}}}$$

e-q) $F(x) = \begin{cases} -x^{1-\theta} + 1, & x \geq 1 \\ 0, & x < 1 \end{cases}$, $\theta > 1$

~~$x \leq F(x) + 1$~~ $x = \sqrt[1-\theta]{1 - F(x)}$