$\boxed{№3}$

$$\xi \sim p(x) = \begin{cases} \frac{e^{-\frac{x}{\theta}}}{\theta}, & x \geq 0 \\ 0, & x < 0 \end{cases}, \quad \theta > 0, \quad n = 3$$

$$\tilde{\theta}_1 = \bar{x}, \quad \tilde{\theta}_2 = \frac{x_{min} + x_{max}}{2}, \quad \tilde{\theta}_3 = x_{(2)}$$

а) Несмещённость:

1. $\tilde{\theta}_1 = \bar{x}$

$$M[\tilde{\theta}_1] = M\left[\frac{1}{n}\sum_{i=1}^{n} x_i\right] = \frac{1}{n}\sum_{i=1}^{n} M[x_i] = M_\xi = \theta \Rightarrow \underline{\underline{\tilde{\theta}_1 \text{ - несмещ}}}$$

$$\boxed{\begin{aligned} \bullet M_\xi &= \int_{-\infty}^{\infty} x \frac{e^{-\frac{x}{\theta}}}{\theta} dx = -\int_0^{\infty} x e^{-\frac{x}{\theta}} d\left(-\frac{x}{\theta}\right) = -\int_0^{\infty} x \, d\left(e^{-\frac{x}{\theta}}\right) = \\ &= -x e^{-\frac{x}{\theta}}\Big|_0^{\infty} + \int_0^{\infty} e^{-\frac{x}{\theta}} dx = -x e^{-\frac{x}{\theta}}\Big|_0^{\infty} - \theta \int_0^{\infty} e^{-\frac{x}{\theta}} d\left(-\frac{x}{\theta}\right) = \\ &= \underbrace{-x e^{-\frac{x}{\theta}}\Big|_0^{\infty}}_{=0} - \theta e^{-\frac{x}{\theta}}\Big|_0^{\infty} = \theta \end{aligned}}$$

2. $\tilde{\theta}_2 = \frac{x_{min} + x_{max}}{2}$

$$M[\tilde{\theta}_2] = M\left[\frac{x_{min} + x_{max}}{2}\right] = \frac{1}{2}\left(M[x_{min}] + M[x_{max}]\right) \ominus$$

$$\begin{cases} \varphi_{min}(y) = n(1 - F(y))^{n-1} p(y) \\ \varphi_{max}(y) = n F^{n-1}(y) \, p(y) \end{cases}$$

$\ominus \left\{ F(y) = \int_{-\infty}^{y} p(x)dx = \int_0^y \frac{e^{-\frac{x}{\theta}}}{\theta} dx = 1 - e^{-\frac{y}{\theta}} \right\} \ominus$

$\ominus$ • $M[x_{max}] = \int_{-\infty}^{+\infty} x \psi(x) dx = \int_0^{+\infty} x \left(3(1-e^{-\frac{x}{\theta}})^2 \frac{e^{-\frac{x}{\theta}}}{\theta}\right) dx =$

$= 3\int_0^{+\infty} x(1 - 2e^{-\frac{x}{\theta}} + e^{-\frac{2x}{\theta}}) \frac{e^{-\frac{x}{\theta}}}{\theta} dx = -3\int_0^{+\infty} xe^{-\frac{x}{\theta}} d\left(\frac{x}{-\theta}\right) +$

$+ 3\int_0^{+\infty} xe^{-\frac{2x}{\theta}} d\left(\frac{2x}{-\theta}\right) - \int_0^{+\infty} xe^{-\frac{3x}{\theta}} d\left(\frac{3x}{-\theta}\right) = 3\theta - \frac{3\theta}{2} + \frac{\theta}{3} = \frac{11}{6}\theta$

• $M[x_{min}] = \int_{-\infty}^{+\infty} x \varphi(x) dx = \int_0^{+\infty} x 3 e^{-\frac{3x}{\theta}} \frac{1}{\theta} dx = \int_0^{+\infty} xe^{-\frac{3x}{\theta}} d\left(\frac{3x}{-\theta}\right) = \frac{\theta}{3}$

$\ominus \ \frac{1}{2}\left(\frac{11\theta}{6} + \frac{2\theta}{6}\right) = \frac{13\theta}{12} \Rightarrow \tilde{\theta}_2$ – смещённая

• $\tilde{\theta}_2^* = \frac{6}{13}(x_{min} + x_{max})$

3. $\tilde{\theta}_3 = x_{(2)}$

$x_{(2)} \sim g(x) = 3C_2^1 \, p(x) F(x) (1 - F(x)) = 6(1 - e^{-\frac{x}{\theta}}) e^{-\frac{x}{\theta}} \cdot \frac{e^{-\frac{x}{\theta}}}{\theta} =$

$= \frac{6}{\theta}\left(e^{-\frac{2x}{\theta}} - e^{-\frac{3x}{\theta}}\right)$

$M[\tilde{\theta}_3] = \int_0^{+\infty} \frac{6x}{\theta}\left(e^{-\frac{2x}{\theta}} - e^{-\frac{3x}{\theta}}\right) dx = -3\int_0^{+\infty} xe^{-\frac{2x}{\theta}} d\left(\frac{2x}{-\theta}\right) + 2\int_0^{+\infty} xe^{-\frac{3x}{\theta}} d\left(\frac{3x}{-\theta}\right) =$

$= \frac{3\theta}{2} - \frac{2\theta}{3} = \frac{5}{6}\theta$ – смещённая

• $\tilde{\theta}_3^* = \frac{6}{5} x_{(2)}$

Сравнение по эффективности

• $M[\varphi^2] = \int_0^\infty \frac{x^2 e^{-\frac{x}{\theta}}}{\theta}dx = \overbrace{-x^2 e^{-\frac{x}{\theta}}\Big|_0^\infty}^{=0} + \int_0^\infty 2x e^{-\frac{x}{\theta}} = \underline{2\theta^2}$

• $D[\varphi] = M[\varphi^2] - M^2[\varphi] = 2\theta^2 - \theta^2 = \theta^2$

1) $D[\hat{\theta}_1] = D[\frac{1}{3}\sum_{i=1}^{3} x_i] = \frac{1}{9}\sum_{i=1}^{3} D[\varphi] = \underline{\frac{\theta^2}{3}}$

2) $D[\tilde{\theta}_2^*] = D[\frac{6}{13}(x_{min} + x_{max})] = \frac{36}{169}(D[x_{min}] + D[x_{max}] +$
$+ 2cov(x_{min}, x_{max})) =$

• $D[x_{min}] = M[x_{min}^2] - M^2[x_{min}]$

• $M[x_{min}^2] = \int_0^\infty \frac{3x^2}{\theta}e^{-\frac{3x}{\theta}}dx = -\int_0^\infty x^2 e^{-\frac{3x}{\theta}} d(\frac{3x}{-\theta}) =$

$= \underbrace{-x^2 e^{-\frac{3x}{\theta}}\Big|_0^\infty}_{=0} + \int_0^\infty e^{-\frac{3x}{\theta}} 2x\,dx = -\int_0^\infty \frac{2\theta}{3} e^{-\frac{3x}{\theta}} x\,d(\frac{-3x}{\theta}) = \frac{2\theta^2}{9}$

• $D[x_{min}] = \frac{2\theta^2}{9} - \frac{\theta^2}{9} = \underline{\frac{\theta^2}{9}}$

• $M[x_{max}^2] = -\int_0^\infty 3x^2 e^{-\frac{x}{\theta}} d(\frac{x}{-\theta}) + 3\int_0^{+\infty} x^2 e^{-\frac{2x}{\theta}} d(\frac{2x}{-\theta}) - \int_0^\infty x^2 e^{-\frac{3x}{\theta}} d(\frac{3x}{-\theta}) =$

$= -\int_0^\infty 6\theta e^{-\frac{x}{\theta}} x\,d(\frac{x}{-\theta}) + \int 3\theta e^{-\frac{2x}{\theta}} x\,d(\frac{-2x}{\theta}) + \frac{2\theta^2}{9} =$

$= 6\theta^2 - \frac{3\theta^2}{2} + \frac{2\theta^2}{9} = \frac{85}{18}\theta^2$

• $D[\max] = \frac{85}{18}\theta^2 - \frac{121}{36}\theta^2 = \underline{\frac{49}{36}\theta^2}$

• $cov(X_{max}, X_{min}) = M[X_{max} \cdot X_{min}] - M[X_{max}]M[X_{min}]$

$$k(y,z) = \begin{cases} F^n(z) - (F(z) - F(y))^n, & z \geq y \\ F^n(z), & z < y \end{cases}$$ – совм. закон распр.

$$\varkappa(y,z) = \frac{\partial^2 k}{\partial y \partial z} = n(n-1)(F(z) - F(y))^{n-2} F'(y) F'(z) \cdot [z \geq y) =$$

$$= 6\left(e^{-\frac{y}{\theta}} - e^{-\frac{z}{\theta}}\right) e^{\frac{-z-y}{\theta}} \cdot \frac{1}{\theta^2} \quad (z \geq y \geq 0)$$

$$M[X_{min} \cdot X_{max}] = 6\int_0^{+\infty} z\,dz \int_0^z y\left(e^{-\frac{y}{\theta}} - e^{-\frac{z}{\theta}}\right) e^{\frac{-z-y}{\theta}} \frac{1}{\theta^2}\,dy \ominus$$

• $$\int_0^z y e^{\frac{-z-2y}{\theta}}\,dy = \frac{\theta}{2}\int_0^z y e^{\frac{-z-2y}{\theta}}\,d\left(\frac{-z-2y}{\theta}\right) =$$

$$= -\frac{\theta z}{2} e^{-\frac{3z}{\theta}} + \int_0^z \frac{\theta^2}{4} e^{\frac{-z-2y}{\theta}}\,d\left(\frac{2y+z}{-\theta}\right) =$$

$$= -\frac{\theta z}{2} e^{-\frac{3z}{\theta}} - \frac{\theta^2}{4} e^{-\frac{3z}{\theta}} + \frac{\theta^2}{4} e^{-\frac{z}{\theta}}$$

• $$\int_0^z y e^{\frac{-2z-y}{\theta}}\,dy = -\theta\int_0^z y e^{\frac{-2z-y}{\theta}}\,d\left(\frac{-2z-y}{\theta}\right) = -\theta z e^{-\frac{3z}{\theta}} +$$

$$+\int_0^z -\theta^2 e^{-\frac{2z-y}{\theta}} d\left(-\frac{2z-y}{\theta}\right) = -\theta z e^{-\frac{3z}{\theta}} - \theta^2 e^{-\frac{3z}{\theta}} + \theta^2 e^{-\frac{2z}{\theta}}$$

$$= \frac{6}{\theta^2}\int_0^{+\infty} z\left(e^{-\frac{3z}{\theta}}\left(\frac{\theta z}{2} + \frac{3\theta^2}{4}\right) + \frac{\theta^2}{4}e^{-\frac{z}{\theta}} - \theta^2 e^{-\frac{2z}{\theta}}\right)dz =$$

$$\cancel{-\frac{3\theta}{2}\int_0^{+\infty} z e^{-\frac{z}{\theta}} d\left(\frac{z}{\theta}\right)} = -\frac{3\theta}{2}\int_0^{+\infty} z e^{-\frac{z}{\theta}} d\left(-\frac{z}{\theta}\right) + 3\theta\int_0^{+\infty} z e^{-\frac{2z}{\theta}} d\left(-\frac{2z}{\theta}\right) +$$

$$+ \frac{3}{\theta}\int_0^{+\infty} z^2 e^{-\frac{3z}{\theta}} dz + \frac{3\theta}{2}\int z e^{-\frac{3z}{\theta}} d\left(-\frac{3z}{\theta}\right) =$$

$$= -\frac{3\theta^2}{2} + \frac{3\theta^2}{2} + \frac{\theta^2}{2} - \int_0^{+\infty} z^2 e^{-\frac{3z}{\theta}} d\left(-\frac{3z}{\theta}\right) = \frac{\theta^2}{2} + \frac{2\theta^2}{9} = \frac{13}{18}\theta^2$$

$$\operatorname{cov}(x_{max}, x_{min}) = \frac{13}{18}\theta^2 - \frac{11}{18}\theta^2 = \frac{1}{9}\theta^2$$

- $D[\tilde{\theta}_2^*] = \frac{36}{169}\left(\frac{\theta^2}{9} + \frac{49}{36}\theta^2 + \frac{2}{9}\theta^2\right) = \frac{61}{169}\theta^2$

3) $D[\tilde{\theta}_3^*] = \frac{36}{25} D[\hat{\theta}_3] = \left(M[\tilde{\theta}_3^2] - M^2[\hat{\theta}_3]\right)\frac{36}{25} =$

- $M[\tilde{\theta}_3^2] = \int_0^{+\infty} \frac{6x^2}{\theta}\left(e^{-\frac{2x}{\theta}} - e^{-\frac{3x}{\theta}}\right)dx = -3\int_0^{+\infty} x^2 e^{-\frac{2x}{\theta}} d\left(\frac{2x}{-\theta}\right) + 2\int_0^{+\infty} x^2 e^{-\frac{3x}{\theta}} d\left(\frac{3x}{-\theta}\right) =$

$$= -\frac{4\theta^2}{9} + \frac{3\cdot 2\theta^2}{4} = \frac{3\theta^2}{2} - \frac{4\theta^2}{9} = \frac{19\theta^2}{18}$$

$$= \frac{36}{25}\left(\frac{19}{18}\theta^2 - \frac{25}{36}\theta^2\right) = \frac{13}{25}\theta^2$$

$$\boxed{\frac{\theta^2}{3} < \frac{61}{169}\theta^2 < \frac{13}{25}\theta^2 \Rightarrow D[\tilde{\theta}_1] = \frac{\theta^2}{3} \quad \tilde{\theta}_1 \text{ – эффективнее}}$$

$\cancel{D[\tilde{\theta}_1]\ D[\tilde{\theta}_2^*]\ \dots D_3}$

б) Эф. по К-Р.

$$\cancel{\bullet\ I(\theta) = M_\xi\left[\left(\frac{\partial \ln p(\xi,\theta)}{\partial\theta}\right)^2\right] = \int_{-\infty}^{+\infty}\left(\frac{\partial \ln p}{\partial\theta}\right)^2 p(x,\theta)\,dx =}$$

$$\cancel{\int_0^{+\infty}\frac{\theta}{e^{-\frac{x}{\theta}}}\left(\frac{xe^{-\frac{x}{\theta}}}{\theta^3}-\frac{e^{-\frac{x}{\theta}}}{\theta^2}\right)\frac{e^{-\frac{x}{\theta}}}{\theta}\,dx = -\int_0^{+\infty}\frac{xe^{-\frac{x}{\theta}}}{\theta^2}\,d\left(\frac{x}{-\theta}\right) +}$$

$$\cancel{+\int_0^{+\infty} e^{-\frac{x}{\theta}}\frac{1}{\theta}\,d\left(\frac{x}{-\theta}\right) = \frac{1}{\theta}-\frac{1}{\theta} = 0}$$

$$\bullet\ I(\theta) = M_\xi\left[\left(\frac{\partial \ln p(\xi,\theta)}{\partial\theta}\right)^2\right] = \int_{-\infty}^{\infty}\left(\frac{\partial \ln p}{\partial\theta}\right)^2 p(x,\theta)\,dx =$$

$$= \int_0^{+\infty}\left(\frac{x}{\theta^2}-\frac{1}{\theta}\right)^2\frac{e^{-\frac{x}{\theta}}}{\theta}\,dx = \int_0^{+\infty}\frac{x^2}{\theta^5}e^{-\frac{x}{\theta}}\,dx - \int_0^{+\infty}\frac{2x}{\theta^4}e^{-\frac{x}{\theta}}\,dx +$$

$$+\int_0^{+\infty}\frac{e^{-\frac{x}{\theta}}}{\theta^3}\,dx = -\int_0^{+\infty}\frac{x^2e^{-\frac{x}{\theta}}}{\theta^4}\,d\left(\frac{-x}{\theta}\right) + 2\int_0^{+\infty}\frac{xe^{-\frac{x}{\theta}}}{\theta^3}\,d\left(\frac{x}{-\theta}\right) -$$

$$-\int_0^{\infty}\frac{e^{-\frac{x}{\theta}}}{\theta^2}\,d\left(\frac{x}{-\theta}\right) = \frac{2}{\theta^2} - \frac{2}{\theta^2} - \frac{1}{\theta^2} = \frac{1}{\theta^2}$$

• Регулярность модели

$$p(x,\theta) = \begin{cases}\dfrac{e^{-\frac{x}{\theta}}}{\theta}, & x \geq 0\\ 0, & x < 0\end{cases} \qquad \theta > 0$$

1) $p(x,\theta)$ – дифф по $\theta > 0$

2) $\frac{\partial}{\partial\theta}\int\limits_{-\infty}^{\infty} p(x,\theta)dx = \frac{\partial}{\partial\theta}\int\limits_{0}^{+\infty}\frac{e^{-\frac{x}{\theta}}}{\theta}dx = \frac{\partial}{\partial\theta}\left(-\int\limits_{0}^{+\infty} e^{-\frac{x}{\theta}}d\left(\frac{x}{-\theta}\right)\right) =$

$= \frac{\partial}{\partial\theta}(1) = 0$

$\int\limits_{-\infty}^{\infty}\frac{\partial}{\partial\theta}\left(p(x,\theta)\right)dx = \int\limits_{0}^{\infty}\left(\frac{xe^{-\frac{x}{\theta}}}{\theta^3} - \frac{e^{-\frac{x}{\theta}}}{\theta^2}\right)dx = -\int\limits_{0}^{+\infty}\frac{xe^{-\frac{x}{\theta}}}{\theta^2}d\left(\frac{x}{-\theta}\right) +$

$+\int\limits_{0}^{+\infty}\frac{e^{-\frac{x}{\theta}}}{\theta}d\left(\frac{x}{-\theta}\right) = \frac{1}{\theta} - \frac{1}{\theta} = 0$

$0 = 0$ => модель регулярна

~~Проверка оценок на регулярность~~

1) $\tilde{\theta}_1 = \frac{1}{3}\sum_{i=1}^{3} x_i$ – самая эф среди всех оценок, так же несмещённая и знаем, что регулярная, $\exists I(\theta)$ и модель регулярна =>

$g(\theta) = \theta$

=> $D[\tilde{\theta}_1] \geq \frac{(\theta')^2}{3I(\theta)}$ => $\frac{\theta^2}{3} \geq \frac{\theta^2}{3}$ => $\tilde{\theta}_1$ – эф. по К-Р.

$\tilde{\theta}_2^*$ и $\tilde{\theta}_3^*$ – менее эф. чем $\tilde{\theta}_1$ => они не эффективны по К-Р.