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Exemple : Hyperboloïde de révolution

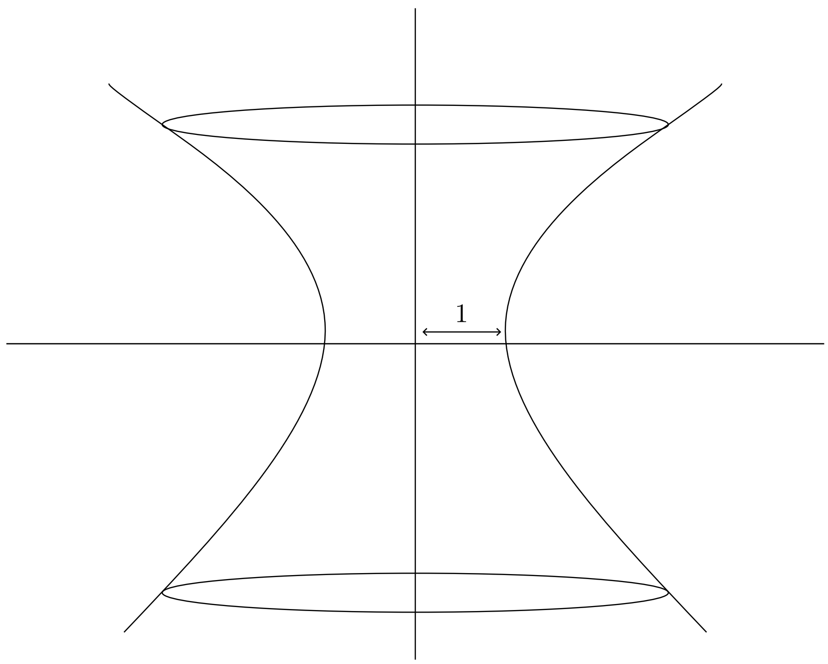

FIGURE 1 – Hyperboloïde de révolution

$$r^2 - z^2 = 1$$

$$2r\mathrm{d}r - 2z - 2z\mathrm{d}z = 0 \ \& \ z = \sqrt{r^2 - 1}$$

$$\mathrm{d}s^2 = \mathrm{d}r^2 + \mathrm{d}^2 + r^2\mathrm{d}\varphi^2 = \left(1 + \frac{r^2}{r^2 - 1}\right)\mathrm{d}r^2 + r^2\mathrm{d}\varphi$$

$$\implies [g_{ij}] = \begin{bmatrix} \frac{2r^2-1}{r^2-1} & 0 \\ 0 & r^2 \end{bmatrix}$$

$$\Gamma^1_{11} = \frac{r}{(r^2 - 1)(2r^2 - 1)}$$
$$\Gamma^1_{22} = \frac{r\left(r^2 - 1\right)}{2r^2 - 1}$$

$$\Gamma^2_{12} = \Gamma^2_{21} = -\frac{1}{r}$$

$$R_{1212} = \frac{-r^2}{(r^2-1)(2r^2-1)}$$
$$R = \frac{-2}{(2r^2-1)^2}$$

$$\dot{u}_i = \frac{1}{2}\partial_i g_{jk} u^i u^k$$

$$i = 2 \implies \quad \dot{u}_\varphi = 0 \implies u_{\varphi=cst} = r^2\dot{\varphi} = h$$

Coordonnées hyperboliques

$$r = \cosh\theta \quad z = \sinh\theta \quad \theta \in [-\infty, \infty]$$

$$r^2 - z^2 = 1$$
$$\mathrm{d}s^2 = \left(\cosh^2\theta + \sinh^2\theta\right)\mathrm{d}\theta^2 + \cosh^2\theta\mathrm{d}\varphi = \cosh 2\theta\mathrm{d}\theta^2 + \cosh^2\theta\mathrm{d}\varphi^2$$

$$\Gamma^1_{11} = -2\Gamma^1_{22} = \tanh 2\theta$$

$$\Gamma^2_{21} = \Gamma^2_{12} = \tanh\theta$$

$$R_{1212} = -\frac{\cosh^2\theta}{\cosh^2 2\theta}$$
$$R = -\frac{2}{\cosh 2\theta}$$

Sphère

$$x^1 = \theta = \mathrm{cst} \quad x^2 = \varphi \in [0, 2\pi]$$

$$\nabla_{lambda} A^i = \frac{\mathrm{d}}{\mathrm{d}\lambda}A^i + \Gamma^i_{jk}A^k i^j$$

$$\nabla_\varphi A^i = \frac{\mathrm{d}A^i}{\mathrm{d}\varphi} + \Gamma^i_{k\varphi}A^k = 0$$

$$\begin{cases} \nabla_\varphi A^\varphi = \frac{\mathrm{d}A^\varphi \varphi}{\mathrm{d}+} \Gamma^2_{12} A^\theta = 0 \\ {}_\varphi A^\theta = \frac{\mathrm{d}A^\theta}{\mathrm{d}\varphi} + \Gamma^1_{22} A^\varphi = 0 \end{cases}$$

$$\begin{cases} \frac{\mathrm{d}A^\varphi}{\mathrm{d}\varphi} + A^\theta \frac{\cos\theta}{\sin\theta} = 0 \\ \frac{\mathrm{d}A^\theta}{\mathrm{d}\varphi} - A^\varphi \sin\theta\cos\theta = 0 \end{cases}$$

$$\begin{cases} \frac{\mathrm{d}^2 A^\varphi}{\mathrm{d}\varphi^2} + \cos^2 A^\varphi = 0 \\ \frac{\mathrm{d}^2 A^\theta}{\mathrm{d}\varphi^2} + \cos\theta A^\theta = 0 \end{cases}$$

$$\begin{cases} A^\varphi(\varphi) = \frac{1}{\sin\theta} \cos(\varphi|\cos\theta|) \\ A^\theta(\varphi) = \mathrm{sign}(\cos\theta)\sin(\varphi|\cos\theta|) \end{cases}$$

$$A^\varphi(\varphi) = \frac{1}{\epsilon}\cos\varphi$$

$$A^\theta(\varphi) = \sin\theta$$

Coordonées polaires planes

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\mathrm{d}\varphi^2$$

$$[g_{ij}] = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix}$$

$$\dot{u}_i = \frac{1}{2}\partial_i g_{kj} i^k u^j$$

$$u^r = u_r = \dot{r}$$
$$i = 1 \implies \dot{u}_r = r\dot{\varphi}^2 = \ddot{r}$$

$$i = 2 \implies \dot{u}_\varphi = 0 \implies u_\varphi = \mathrm{cst}$$
$$U_\varphi = g_{\varphi\varphi} u^\varphi = r^2\dot{\varphi} = h$$

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$$\implies |\mathbf{u}|^2 = 1 = \dot{r}^2 + r^2\dot{\varphi}^2 = \dot{r}^2 + fracr^2h^2r^4$$