Advanced Financial Economics

Series 2

Andy Tran

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Exercise 1

We know that $\beta = 0.5$ and $H = 2$, $(e_1^1, e_2^1) = (1, 0)$ and $(e_1^2, e_2^2) = (0, 1)$.

a) CARA Utilities

From the lecture we know that we have to solve

$$\log(1 + r) + \log(\beta) = \frac{1}{H}\sum_{i=1}^{H}(e_2^h - e_1^h)$$

Using the given values we get

$$\log(1 + r) + \log(0.5) = \frac{1}{2}((0 - 1) + (1 - 0)) = 0 \Rightarrow \log(1 + r) = -\log(0.5) = \log(2)$$

This yields $r = 1$.

b) CRRA Utilities

From the lecture we know that we have to solve

$$\beta(1 + r) = \left(\frac{\sum_h e_2^h}{\sum_h e_1^h}\right)^{\sigma}$$

Using the given values we get

$$0.5(1 + r) = \left(\frac{1}{1}\right)^2 = 1 \Rightarrow 1 + r = \frac{1}{0.5} = 2 \Rightarrow r = 1$$

c) Quadratic Utilities

As both utility functions are the same, the marginal utility function will have to satisfy

$$\beta(1 + r) = \frac{1 - \frac{1}{5}c_1}{1 - \frac{1}{5}c_2}$$

We can use that $\sum_h s^h = \sum_h(e_1^h - c_1^h) = 0$, or equivalently $\sum_h e_1^h = \sum_h c_1^h$ and note that $\sum_h c_2^h = \sum_h e_2^h + (1 + r)\sum_h s^h \Rightarrow \sum_h c_2^h = \sum_h e_2^h$, due to market clearing.

$$(1 - \frac{1}{5}c_1) = \beta(1 + r)(1 - \frac{1}{5}c_2)$$

Summing over each household gives us:

$$
\begin{aligned}
&\sum_h (1 - \frac{1}{5}c_1^h) = \beta(1+r)\sum_h (1 - \frac{1}{5}c_2^h) \\
&\Rightarrow 2\left(1 - \frac{1}{5}\sum_h c_1^h\right) = \beta(1+r)\cdot 2\left(1 - \frac{1}{5}\sum_h c_2^h\right) \\
&\Rightarrow 2\left(1 - \frac{1}{5}\sum_h e_1^h\right) = \beta(1+r)\cdot 2\left(1 - \frac{1}{5}\sum_h e_2^h\right)
\end{aligned}
$$

Plugging in the given values yields:

$$
\begin{aligned}
&2\left(1 - \frac{1}{5}\right) = 0.5(1+r)\cdot 2\left(1 - \frac{1}{5}\right) \\
&\Rightarrow \frac{8}{5} = 0.5(1+r)\cdot\frac{8}{5} \\
&\Rightarrow 1 = 0.5(1+r) \\
&\Rightarrow 1 + r = 2
\end{aligned}
$$

which gives us $r = 1$.

d) Some other utilities

We have the first marginal utility

$$
\begin{aligned}
\frac{\frac{1}{c_1+0.1}}{\frac{1}{c_2+0.1}} = \frac{c_2+0.1}{c_1+0.1} = 0.5(1+r) \\
c_2^R + 0.1 = 0.5(1+r)(c_1^R + 0.1)
\end{aligned}
$$

For the other agent we get the marginal utility:

$$
\frac{c_2^M}{c_1^M} = \beta(1+r)
$$

which is equivalent to:

$$
c_2^M = 0.5(1+r)c_1^M
$$

Adding them both up we get

$$
c_2^R + 0.1 + c_2^M + = 0.5(1+r)(c_1^R + 0.1 + c_1^M)
$$

we can use that $c_2^R + c_2^M = 1, c_1^R + c_1^M = 1,$

$$
\begin{aligned}
&1 + 0.1 = 0.5(1+r)(1+0.1) \\
&\Rightarrow 1.1 = 0.5(1+r)(1.1) \\
&\Rightarrow r = 1
\end{aligned}
$$

e) Another utility

The first marginal utility gives us:

$$
\begin{aligned}
&c_2 = (1+r) \\
&\Rightarrow s^R(1+r) + e_2 = (1+r) \\
&\Rightarrow s^R(1+r) + 0 = (1+r)
\end{aligned}
$$

Assuming that $(1+r) \neq 0$, we retrieve $s^R = 1$

The second marginal utility gives us:

$$
\begin{aligned}
&c_2 = 0.5(1+r) \\
&\Rightarrow s^M(1+r) + e_2 = 0.5(1+r)
\end{aligned}
$$

and using that $s^M = -s^R = -1$ we get:

$$
\begin{aligned}
&-1(1+r) + 1 = 0.5(1+r) \\
&\Rightarrow -(1+r) + 1 = 0.5(1+r) \\
&\Rightarrow -1 - r + 1 = 0.5 + 0.5r \\
&\Rightarrow -r = 0.5 + 0.5r \\
&\Rightarrow -1.5r = 0.5 \\
&\Rightarrow r = -\frac{1}{3}
\end{aligned}
$$

f) Last utility

The first margin utility gives us:

$$
\begin{aligned}
&\frac{c_2^R}{c_1^R} = 1 + r \\
&\Rightarrow \frac{s^R(1+r) + e_2^R}{e_1^R - s^R} = 1 + r \\
&\Rightarrow \frac{s^R(1+r) + 0}{1 - s^R} = 1 + r \\
&\Rightarrow \frac{s^R(1+r)}{1 - s^R} = 1 + r \\
&\Rightarrow s^R(1+r) = (1+r)(1 - s^R) \\
&\Rightarrow s^R(1+r) = (1+r) - s^R(1+r) \\
&\Rightarrow 2s^R(1+r) = (1+r) \\
&\Rightarrow 2s^R = 1 \\
&\Rightarrow s^R = 0.5
\end{aligned}
$$

The second marginal utility gives us:

$$\begin{aligned}
&\frac{c_2^M}{c_1^M} = 0.5(1+r) \\
&\Rightarrow \frac{s^M(1+r)+e_2^M}{e_1^M - s^M} = 0.5(1+r) \\
&\Rightarrow \frac{-s^R(1+r)+e_2^M}{e_1^M + s^R} = 0.5(1+r) \\
&\Rightarrow \frac{-0.5(1+r)+1}{0+0.5} = 0.5(1+r) \\
&\Rightarrow \frac{-0.5(1+r)+1}{0.5} = 0.5(1+r) \\
&\Rightarrow -1(1+r)+2 = 0.5(1+r) \\
&\Rightarrow -1-r+2 = 0.5+0.5r \\
&\Rightarrow 1-r = 0.5+0.5r \\
&\Rightarrow 0.5 = 1.5r \\
&\Rightarrow r = \frac{1}{3}
\end{aligned}$$

Exercise 2: T-periods, H-agents

Since the discount factor is the same, from the lecture we know that the Euler Equation is:

$$\begin{aligned}
&\frac{e^{-c_{t-1}}}{e^{-c_t}} = \beta(1+r_t),\ \forall t \in [1,T] \\
&\Rightarrow \frac{e^{-c_{t-1}}}{e^{-c_t}} = \beta(1+r_t) \\
&\Rightarrow e^{c_t - c_{t-1}} = \beta(1+r_t) \\
&\Rightarrow c_t - c_{t-1} = \log(\beta(1+r_t)) \\
&(e_t^h + s_{t-1}^h(1+r_t)) - (e_{t-1}^h + s_{t-2}^h(1+r_{t-1}) - s_{T-1}) = \log(\beta(1+r))
\end{aligned}$$

If we sum up over all agents, and using the condition that the markets clear, we easily see that:

$$\begin{aligned}
&\sum_h e_t^h + 0 - \sum_h e_{t-1}^h + 0 - 0 = H\log(\beta(1+r_t)) \\
&\Rightarrow \sum_h (e_t^h - e_{t-1}^h) = H\log(\beta(1+r_t)) \\
&\Rightarrow \sum_h 1 = H\log(\beta(1+r_t)) \\
&\Rightarrow H = H\log(\beta(1+r_t)) \\
&\Rightarrow 1 = \log(\beta(1+r_t)) \\
&\Rightarrow 1 - \log(\beta) = \log(1+r_t) \\
&\Rightarrow e^{1-\log(\beta)} = 1 + r_t \\
&\Rightarrow \frac{e^1}{e^{\log(\beta)}} = 1 + r_t \\
&\Rightarrow \frac{e}{\beta} = 1 + r_t \\
&\Rightarrow r_t = \frac{e}{\beta} - 1
\end{aligned}$$

Thus the interest rate is always the same for each timestep we are at.