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Depth–energy and depth–force relationships in open channel flows: Analytical findings

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Abstract

In the present work the depth–specific energy relationship and the depth–total force relationship in open channel flows of wide rectangular cross-section are analytically inverted.

The nondimensional expressions of the specific energy and of the total force, as functions of the nondimensional water depth, are considered. The inversion of such functions consists of finding the roots of third degree algebraic equations; simple analytical solutions are obtained. More specifically, for a given specific discharge and for each meaningful value of the specific energy, a subcritical and a supercritical depth are found analytically. Similarly, for a given specific discharge and for each meaningful value of the total force, a subcritical and a supercritical depth are found analytically. For both functions, it is also shown that the third root corresponds to a negative depth, which can be discarded on the basis of physics.

Examples from classical open channel hydraulics and a numerical application show the consistency of these analytical solutions.

Keywords: Open channel flows; Total force; Specific energy; Analytical results

1. Introduction

A short look to any handbook of hydraulic engineering or to classical textbooks on free surface flows [1–3] is sufficient to guess the importance of the concepts of specific energy and total force in the crucial aspects of open channel flows. For example, the complex and rich physics of fluid motion around obstacles, at abrupt contractions or expansions of the channel width, near localized steps or sudden changes in bed elevation, in flowing out from different hydraulic structures, at the occurrence of a sudden change of the flow status (supercritical–subcritical), giving rise to stationary and moving hydraulic jump, requires the mastery of the concepts of specific energy and total force [1–5].

Often, to reproduce the main features of a rapidly varied flow, it is sufficient to understand if a problem is dominated by energy conservation or by total force conservation. In the former case, the meticulous evaluation of the small sources of dissipation changes the details of the problem, but not the substantial aspects, whilst, in the latter case, the accurate estimate of the forces supported by the boundaries gives final refinements, leaving more or less unchanged the main aspects of the flow field.

Two immediate examples of that are the flow under a sluice gate and the stationary hydraulic jump: in the former case, the specific energy is conserved and a portion of the total force of the flow is supported by the gate; in the latter case, neglecting the small force supported by the wetted boundary, the total force of the stream is conserved and the specific energy is dissipated in the inviscid shock. In the former case, abrupt variations of the flow field are determined by the geometry of the obstacle; in the latter case, from the internal nonlinearity of the flow field, which is governed by hyperbolic equations, admitting discontinuous solutions [6].

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The same concepts can be useful in the different context of shallow flow computational dynamics. While the numerical treatment of the homogeneous form of shallow water equations can be considered a solved problem [7–11], at least from the engineering point of view, the source terms treatment in the momentum balance equation is still an open question [12–15]. This is particularly true when the balancing of the scheme is put into evidence for practical purpose calculations [16–18]. The development of novel models in this field often requires accurate procedures, involving the energy or the total force conservation.

In this context, the present work is conceived and carried out. A simple, practical, correct, conservative (not propagating errors) framework to manipulate, in an "exact" form, the specific energy *vs* depth relationship and the total force *vs* depth relationship is found. Such a framework allows to pass easily from one of these physical quantities to the others, using analytical expressions.

A rectangular cross-section of an open wide channel, neglecting bank effects, is considered. Expressions of specific energy (depth plus kinetic energy) and of total force (hydrostatic plus inertial force) as functions of water depth are the starting point of the present work.

The specific energy and the total force are made nondimensional, writing them as functions of the nondimensional water depth. Inversion of such functions involves algebraic equations of third degree.

A new, analytical, solution of the problem is found here. A certain value of the flow discharge per unit width is considered as known, which gives an unique value for the critical depth, critical specific energy and critical total force. For each value of the specific energy (greater than the critical value), a subcritical and a supercritical value of the corresponding depth are found analytically. Similarly, for each value of the total force (greater than the critical value), a subcritical and a supercritical value of the corresponding depth are found analytically. The further, real, root is shown to be negative both for energy and force, so that it can be rejected, having no physical meaning. Similarly, for physical reasons, analysis of inversion for energy values (force values) smaller than critical value is not performed.

The proposed solution is simple, both in the case of specific energy and in the case of total force. Such solutions have the important advantage that the nature of the three algebraic roots is known *a priori*: for the proper range of parameters, the first solution is always negative (so it must be discarded), the second solution is always subcritical and the third solution is always supercritical.

Some simple examples from classical open channel hydraulics show how such analytical results can be used in engineering practice.

A remarkable added value of the herein presented method is to be recognized in its utilization inside numerical applications, concerning shallow water equations integration. In such applications, particularly at singularities which require the control over the specific energy and the total force near the critical stage, the inversion of the depth–specific energy relationship or of the depth–total force relationship can be faced a huge number of times, like it happens in nonstationary simulations which can require millions or ten of millions time steps. In such cases, analytical expressions allows: (a) to obtain precise estimates up to error machine, knowing that a specific part of the procedure does not propagate errors in computations; (b) to completely avoid iterative methods, which represent potential sources of computational inefficiency; (c) to avoid ambiguities or departure from convergence when working in a narrow range around the critical condition.

All these aspects stress the potentials of the found analytical solutions as an efficient tool in open channel hydraulics.

2. Depth, energy, force

The elementary theory of open channel flows is given as stated. Chow [1] and Henderson [2] can be used as the reference textbooks.

A rectangular cross-section channel, in which the depth is much more smaller than the width, is considered. The bank influence is also neglected and therefore, without loose of generality, the following results can be referred to the unit channel width.

Let be Q the liquid discharge, q the unit width (specific) discharge so that $q = Q/b$, being b the channel width.

Let be: Y, Y_c the current and critical depth; E, E_c the current and critical specific energy; F, F_c the current and critical total force.

The classical expressions of the specific energy and of the total force as functions of the depth are summarized as follows (g is gravity and ρ fluid density):

$$E = Y + \frac{q^2}{2gY^2}; \tag{1}$$

$$F = \frac{1}{2}\rho g b Y^2 + \frac{\rho b q^2}{Y}. \tag{2}$$

The critical values of the specific energy and of the total force can be written as functions of the critical depth as follows:

$$Y_c = \sqrt[3]{\frac{q^2}{g}}; \quad E_c = \frac{3}{2}Y_c; \quad F_c = \frac{3}{2}\rho g b Y_c^2. \tag{3}$$

Making nondimensional the depth, the energy and the force using their critical values:

$$\eta = \frac{Y}{Y_c}; \quad \Gamma = \frac{E}{E_c}; \quad \Phi = \frac{F}{F_c}, \tag{4}$$

the following equations can be obtained:

$$\Gamma = \frac{2}{3}\eta + \frac{1}{3}\left(\frac{1}{\eta}\right)^2; \tag{5}$$

$$\Phi = \frac{1}{3}\eta^2 + \frac{2}{3}\left(\frac{1}{\eta}\right). \tag{6}$$

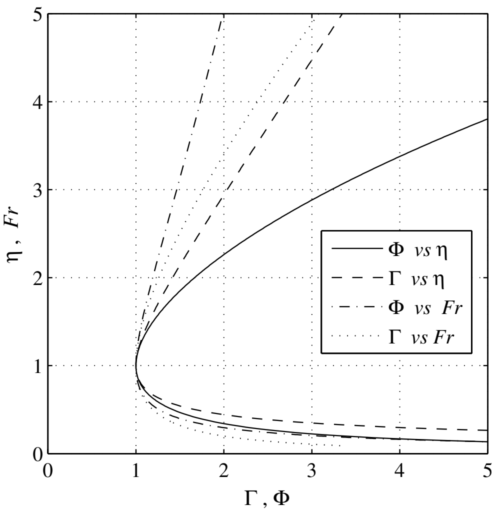


Fig. 1. Specific energy and total force *vs* depth and *Fr*, respectively.

These relationships are plotted in Fig. 1. The physically meaningful domains for the variables are:

$$0 < \eta < \infty; \quad 1 \leqslant \Gamma < \infty; \quad 1 \leqslant \Phi < \infty. \tag{7}$$

These results can be expressed also using the Froude number, $Fr = U/\sqrt{gY}$ (U is the mean velocity of the flow, such that $UY = q$) as independent variable (Fig. 1). In fact, it is easy to show that $Fr = \eta^{-3/2}$.

If the nondimensional depth is given, the nondimensional energy and the nondimensional force can be easily calculated. Inversely, the following results are presented in the next sections:

- Given a certain, physically meaningful, value of nondimensional specific energy, $\Gamma_0 \geqslant 1$, the algebraic equation $\Gamma(\eta) = \Gamma_0$ has three real roots, corresponding to a negative depth (to discard), a positive, subcritical depth $\eta_{sb} \geqslant 1$, a positive, supercritical depth $0 < \eta_{sp} \leqslant 1$. Such three roots can be analytically determined.
- Given a certain, physically meaningful, value of nondimensional total force, $\Phi_0 \geqslant 1$, the algebraic equation $\Phi(\eta) = \Phi_0$ has three real roots, corresponding to a negative depth (to discard), a positive, subcritical depth $\eta_{sb} \geqslant 1$, a positive, supercritical depth $0 < \eta_{sp} \leqslant 1$. Such three roots can be analytically determined.

3. Depth–energy relationship inversion

In this section, the three algebraic solutions $[\eta_1, \eta_2, \eta_3]$ of the equation:

$$\Gamma_0 - \frac{2}{3}\eta - \frac{1}{3}\left(\frac{1}{\eta}\right)^2 = 0, \tag{8}$$

are found.

This can be performed by a symbolic solver software, but further manipulations based on complex analysis and physical meaning of some terms make the final findings much more friendly to use. Moreover, considering specific energy values greater than the critical value, the quantity $\sqrt{\Gamma_0^3 - 1}$ is certainly real, so that $\sqrt{1-\Gamma_0^3} = \mathrm{i}\sqrt{\Gamma_0^3 - 1}$, being $i = \sqrt{-1}$. In these calculations, the rationalization:

$$\left(2 - \Gamma_0^3 + 2\sqrt{1-\Gamma_0^3}\right)^{-1/3} = \left(1 - \mathrm{i}\sqrt{\Gamma_0^3 - 1}\right)^{2/3}\Gamma_0^{-2} \tag{9}$$

is also used. Finally, the following definition:

$$\alpha = \arctan\left(\sqrt{\Gamma_0^3 - 1}\right) \tag{10}$$

is adopted. Being $1 \leqslant \Gamma_0 < \infty$, it is: $0 \leqslant \alpha < \pi/2$.

3.1. First root

$$\begin{aligned}\eta_1 &= \frac{\Gamma_0}{2} - \frac{\Gamma_0^2}{2\left(2 - \Gamma_0^3 + 2\sqrt{1-\Gamma_0^3}\right)^{1/3}} - \frac{1}{2}\left(2 - \Gamma_0^3 + 2\sqrt{1-\Gamma_0^3}\right)^{1/3}\\ &= \frac{\Gamma_0}{2} - \frac{1}{2}\left(\Gamma_0^{3/2}\mathrm{e}^{-\mathrm{i}\alpha}\right)^{2/3} - \frac{1}{2}\left(\Gamma_0^{3/2}\mathrm{e}^{\mathrm{i}\alpha}\right)^{2/3}\\ \Rightarrow \eta_1 &= \frac{\Gamma_0}{2}\left[1 - 2\cos\left(\frac{2}{3}\alpha\right)\right].\end{aligned} \tag{11}$$

Being $\eta_1 < 0$ for $\cos\left(\frac{2}{3}\alpha\right) < \frac{1}{2}$, that is for $\alpha < \pi/2$, in the domain of interest η_1 is real and negative.

3.2. Second root

$$\begin{aligned}\eta_2 &= \frac{\Gamma_0}{2} + \frac{(1+\mathrm{i}\sqrt{3})\Gamma_0^2}{4\left(2 - \Gamma_0^3 + 2\sqrt{1-\Gamma_0^3}\right)^{1/3}}\\ &\quad + \frac{(1-\mathrm{i}\sqrt{3})}{4}\left(2 - \Gamma_0^3 + 2\sqrt{1-\Gamma_0^3}\right)^{1/3}\\ &= \frac{\Gamma_0}{2} + \frac{1}{2}\mathrm{e}^{\mathrm{i}\pi/3}\left(\Gamma_0^{3/2}\mathrm{e}^{-\mathrm{i}\alpha}\right)^{2/3} + \frac{1}{2}\mathrm{e}^{-\mathrm{i}\pi/3}\left(\Gamma_0^{3/2}\mathrm{e}^{\mathrm{i}\alpha}\right)^{2/3}\\ \Rightarrow \eta_2 &= \frac{\Gamma_0}{2}\left[1 + 2\cos\left(\frac{\pi - 2\alpha}{3}\right)\right].\end{aligned} \tag{12}$$

Being $0 \leqslant \alpha < \pi/2$, $\cos\left(\frac{\pi-2\alpha}{3}\right) \geqslant \frac{1}{2}$ for any α in the prescribed interval, in the domain of interest η_2 is real and greater than unity, that is $\eta_2 \geqslant 1$ is the subcritical solution: $\eta_2 = \eta_{sb}$.

3.3. Third root

$$\begin{aligned}\eta_3 &= \frac{\Gamma_0}{2} + \frac{(1-\mathrm{i}\sqrt{3})\Gamma_0^2}{4\left(2 - \Gamma_0^3 + 2\sqrt{1-\Gamma_0^3}\right)^{1/3}}\\ &\quad + \frac{(1+\mathrm{i}\sqrt{3})}{4}\left(2 - \Gamma_0^3 + 2\sqrt{1-\Gamma_0^3}\right)^{1/3}\\ &= \frac{\Gamma_0}{2} + \mathrm{e}^{-\mathrm{i}\pi/3}\left(\Gamma_0^{3/2}\mathrm{e}^{-\mathrm{i}\alpha}\right)^{2/3} + \mathrm{e}^{\mathrm{i}\pi/3}\left(\Gamma_0^{3/2}\mathrm{e}^{\mathrm{i}\alpha}\right)^{2/3}\\ \Rightarrow \eta_3 &= \frac{\Gamma_0}{2}\left[1 + 2\cos\left(\frac{\pi + 2\alpha}{3}\right)\right].\end{aligned} \tag{13}$$

Being $0 \leqslant \alpha < \pi/2$, $-\frac{1}{2} < \cos\left(\frac{\pi+2\alpha}{3}\right) \leqslant \frac{1}{2}$ for any α in the prescribed interval, in the domain of interest η_3 is real and less than unity, that is $\eta_3 \leqslant 1$ is the supercritical solution: $\eta_3 = \eta_{\mathrm{sp}}$.

4. Depth–total force relationship inversion

In this section, the three algebraic solutions $[\eta_1, \eta_2, \eta_3]$ of the equation:

$$\Phi_0 - \frac{1}{3}\eta^2 - \frac{2}{3}\left(\frac{1}{\eta}\right) = 0, \tag{14}$$

are found.

This can be performed again by a symbolic solver, but further manipulations of some terms make the final findings friendly to use. Considering total force values greater than the critical value, the quantity $\sqrt{\Phi_0^3 - 1}$ is certainly real, so that $\sqrt{1-\Phi_0^3} = \mathrm{i}\sqrt{\Phi_0^3 - 1}$. In these calculations, the rationalization:

$$\left(1+\sqrt{1-\Phi_0^3}\right)^{-1/3} = \left(1-\mathrm{i}\sqrt{\Phi_0^3-1}\right)^{1/3}\Phi_0^{-1} \tag{15}$$

is used. Finally, the following definition:

$$\theta = \arctan\left(\sqrt{\Phi_0^3-1}\right) \tag{16}$$

is adopted. Being $1 \leqslant \Phi_0 < \infty$, it is: $0 \leqslant \theta < \pi/2$.

4.1. First root

$$\begin{aligned}\eta_1 &= -\frac{\Phi_0}{\left(1+\sqrt{1-\Phi_0^3}\right)^{1/3}} - \left(1+\sqrt{1-\Phi_0^3}\right)^{1/3} \\ &= -\left(\Phi_0^{3/2}\mathrm{e}^{-\mathrm{i}\theta}\right)^{1/3} - \left(\Phi_0^{3/2}\mathrm{e}^{\mathrm{i}\theta}\right)^{1/3} \\ &\Rightarrow \eta_1 = -2\sqrt{\Phi_0}\cos\left(\frac{\theta}{3}\right).\end{aligned} \tag{17}$$

Being $\cos\left(\frac{\theta}{3}\right) \geqslant 0$, for any admissible value of θ, in the domain of interest η_1 is real and negative, that is $\eta_1 < 0$.

4.2. Second root

$$\begin{aligned}\eta_2 &= \frac{(1+\mathrm{i}\sqrt{3})}{2}\frac{\Phi_0}{\left(1+\sqrt{1-\Phi_0^3}\right)^{1/3}} \\ &\quad + \frac{(1-\mathrm{i}\sqrt{3})}{2}\left(1+\sqrt{1-\Phi_0^3}\right)^{1/3} \\ &= \mathrm{e}^{\mathrm{i}\pi/3}\left(\Phi_0^{3/2}\mathrm{e}^{-\mathrm{i}\theta}\right)^{1/3} + \mathrm{e}^{-\mathrm{i}\pi/3}\left(\Phi_0^{3/2}\mathrm{e}^{\mathrm{i}\theta}\right)^{1/3} \\ &\Rightarrow \eta_2 = 2\sqrt{\Phi_0}\cos\left(\frac{\pi-\theta}{3}\right).\end{aligned} \tag{18}$$

Being $0 \leqslant \theta < \pi/2$, $\cos\left(\frac{\pi-\theta}{3}\right) \geqslant \frac{1}{2}$ for any θ in the prescribed interval, in the domain of interest η_2 is real and greater than unity, that is $\eta_2 \geqslant 1$ is the subcritical solution: $\eta_2 = \eta_{\mathrm{sb}}$.

4.3. Third root

$$\begin{aligned}\eta_3 &= \frac{(1-\mathrm{i}\sqrt{3})}{2}\frac{\Phi_0}{\left(1+\sqrt{1-\Phi_0^3}\right)^{1/3}} \\ &\quad + \frac{(1+\mathrm{i}\sqrt{3})}{2}\left(1+\sqrt{1-\Phi_0^3}\right)^{1/3} \\ &= \mathrm{e}^{-\mathrm{i}\pi/3}\left(\Phi_0^{3/2}\mathrm{e}^{-\mathrm{i}\theta}\right)^{1/3} + \mathrm{e}^{\mathrm{i}\pi/3}\left(\Phi_0^{3/2}\mathrm{e}^{\mathrm{i}\theta}\right)^{1/3} \\ &\Rightarrow \eta_3 = 2\sqrt{\Phi_0}\cos\left(\frac{\pi+\theta}{3}\right).\end{aligned} \tag{19}$$

Being $0 \leqslant \theta < \pi/2$, $0 < \cos\left(\frac{\pi+\theta}{3}\right) \leqslant \frac{1}{2}$ for any θ in the prescribed interval, in the domain of interest η_3 is real and less than unity, that is $\eta_3 \leqslant 1$ is the supercritical solution: $\eta_3 = \eta_{\mathrm{sp}}$.

5. Summary and further considerations

All results having physical meaning are summarized in Table 1.

With some simple algebraic manipulations, it is also obviously possible re-obtaining the conjugate depths ratio, satisfying the equal total force condition $\Phi(\eta_{\mathrm{sp}}) = \Phi(\eta_{\mathrm{sb}})$, as:

Table 1
Physically meaningful solutions

Equation	Subcritical solution	Supercritical solution
$\Gamma_0 - \frac{2}{3}\eta - \frac{1}{3}\left(\frac{1}{\eta}\right)^2 = 0$	$\eta_{\mathrm{sb}} = \frac{\Gamma_0}{2}\left[1+2\cos\left(\frac{\pi-2\alpha}{3}\right)\right]$	$\eta_{\mathrm{sp}} = \frac{\Gamma_0}{2}\left[1+2\cos\left(\frac{\pi+2\alpha}{3}\right)\right]$
$\Phi_0 - \frac{1}{3}\eta^2 - \frac{2}{3}\left(\frac{1}{\eta}\right) = 0$	$\eta_{\mathrm{sb}} = 2\sqrt{\Phi_0}\cos\left(\frac{\pi-\theta}{3}\right)$	$\eta_{\mathrm{sp}} = 2\sqrt{\Phi_0}\cos\left(\frac{\pi+\theta}{3}\right)$

$$\frac{\eta_{sb}}{\eta_{sp}} = \frac{1}{2}\left(\sqrt{1+\frac{8}{\eta_{sp}^3}}-1\right) = \frac{1}{2}\left(\sqrt{1+8Fr_{sp}^2}-1\right); \quad (20)$$

$$\frac{\eta_{sp}}{\eta_{sb}} = \frac{1}{2}\left(\sqrt{1+\frac{8}{\eta_{sb}^3}}-1\right) = \frac{1}{2}\left(\sqrt{1+8Fr_{sb}^2}-1\right). \quad (21)$$

With closely similar computations, expressions for the iso-energy depths ratio, satisfying the equal specific energy condition $\Gamma(\eta_{sp}) = \Gamma(\eta_{sb})$, can be obtained:

$$\frac{\eta_{sb}}{\eta_{sp}} = \frac{\sqrt{1+8\eta_{sp}^3}+1}{4\eta_{sp}^3} = \frac{Fr_{sp}^2}{4}\left(\sqrt{1+\frac{8}{Fr_{sp}^2}}+1\right); \quad (22)$$

$$\frac{\eta_{sp}}{\eta_{sb}} = \frac{\sqrt{1+8\eta_{sb}^3}+1}{4\eta_{sb}^3} = \frac{Fr_{sb}^2}{4}\left(\sqrt{1+\frac{8}{Fr_{sb}^2}}+1\right). \quad (23)$$

6. Practical examples

Different examples are here presented, to show the extreme simplicity, efficiency and utility of the proposed method. The following computations, except for the last example Section 6.3, are carried out analytically. For readability reasons only three significant digits are retained in shown results, even if a double precision computation of each variable is performed to avoid significant effects of propagation errors. It is important to note that arbitrary precision can be easily achieved using this method. For all the examples the values $g = 9.81$ m s^{-2} and $\rho = 1000$ kg m^{-3} are used for gravity and fluid density, respectively.

6.1. Flow under a sluice gate

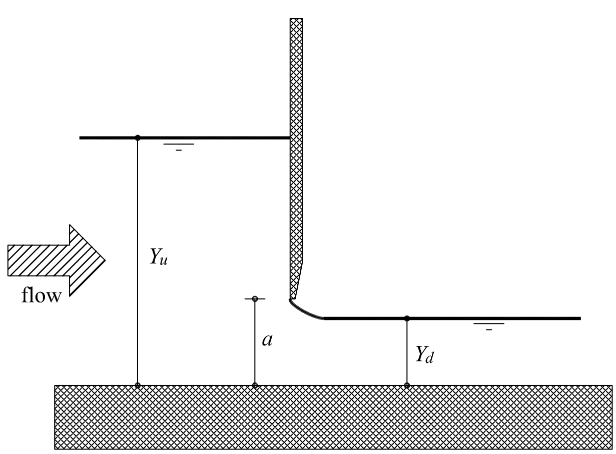
A wide rectangular channel is characterized by a subcritical normal flow with the following parameters: Discharge: $Q = 200$ m^{3}s^{-1}; width: $b = 100$ m; longitudinal bed (mild) slope: $S_0 = 2\times10^{-4}$; Froude number in normal flow: $Fr_0 = U_0/\sqrt{gY_0} = 0.40$.

Such data imply a unit discharge $q = Q/b = 2$ m^2 s^{-1}; a nondimensional Chézy coefficient in normal flow $C_0 = Fr_0/\sqrt{S_0} \simeq 28.3$; a mean water depth in normal flow $Y_0 = [q/(Fr_0\sqrt{g})]^{2/3} \simeq 1.37$ m; a mean velocity in normal flow $U_0 = q/Y_0 \simeq 1.46$ m s^{-1}. The specific energy in normal flow is $E_0 = Y_0 + U_0^2/(2g) \simeq 1.48$ m. For the given unit discharge, the critical depth is: $Y_c = \sqrt[3]{q^2/g} \simeq 0.742$ m; the corresponding critical energy is $E_c = \frac{3}{2}Y_c \simeq 1.11$ m.

A sluice gate inserted into the flow, has an assigned opening: $a = 0.60$ m, is considered. A simple sketch is depicted in Fig. 2. The depth downstream the obstacle can be immediately computed using the theoretical value [4] for the contraction coefficient: $C_c = \pi/(\pi+2) \simeq 0.611$, so that the downstream depth is $Y_d = C_c\, a \simeq 0.367$ m, corresponding to a nondimensional downstream depth of $\eta_d = Y_d/Y_c \simeq 0.494$. The depth upstream the sluice gate is obtained by imposing the specific energy conservation: $E_u = E_d$. Using prescribed data, it is: $\Gamma_u = E_u/E_c = \Gamma_d = E_d/E_c \simeq 1.69$.

To calculate the depth upstream the gate, α value is computed using Eq. (10), thus $\alpha = \arctan(\sqrt{\Gamma_d^3 - 1}) \simeq 1.10$ rad $\simeq 63°$; the nondimensional depth upstream of the gate is computed by Eq. (12), obtaining: $\eta_u \simeq 2.46$, corresponding to a dimensional depth: $Y_u = \eta_u Y_c \simeq 1.82$ m (Fig. 2). Upstream along the channel, the classical profile M1 [1] takes place.

The nondimensional total forces upstream and downstream the sluice gate are $\Phi_u = \frac{1}{3}\eta_u^2 + 2/(3\eta_u) \simeq 2.28$ and $\Phi_d = \frac{1}{3}\eta_d^2 + 2/(3\eta_d) \simeq 1.43$; the critical total force is $F_c = \frac{3}{2}g\rho bY_c^2 \simeq 809$ kN, so that the dimensional upstream and downstream total forces are $F_u = \Phi_u F_c \simeq 1848$ kN and $F_d = \Phi_d F_c \simeq 1157$ kN, respectively. Neglecting the bottom friction and the gravity component in the flow direction, as usual, the force on the gate is easily obtained from the integral momentum balance, which results to be: $F_{gate} = F_u - F_d \simeq 691$ kN, obviously exerted in the flow direction.



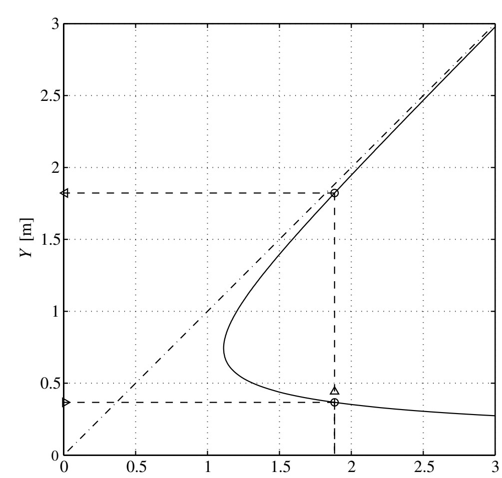


Fig. 2. Flow under a sluice gate: sketch and depth–energy relationship.

It is worth noting that the sluice gate can be seen has a counterpart of the hydraulic jump: in the former, conservation of energy leads to a loss of total force (released on the gate); in the latter, conservation of the total force leads to a loss of specific energy (dissipated in the inviscid shock).

6.2. Flow in a narrowing due to bridge piers, with hydraulic jump

A wide rectangular channel is considered. It is characterized by a subcritical normal flow with the same parameters assumed in the previous example Section 6.1: Discharge: $Q = 200\ \mathrm{m^3\ s^{-1}}$; width: $b = 100$ m; longitudinal bed (mild) slope: $S_0 = 2 \times 10^{-4}$; Froude number in normal flow: $Fr_0 = U_0/\sqrt{gY_0} = 0.40$.

As before, such data imply a unit discharge $q = 2\ \mathrm{m^2\ s^{-1}}$; a nondimensional Chézy coefficient in normal flow $C_0 \simeq 28.3$; a mean water depth in normal flow $Y_0 \simeq 1.37$ m; a mean velocity in normal flow $U_0 \simeq 1.46\ \mathrm{m\ s^{-1}}$. The specific energy in normal flow is $E_0 \simeq 1.48$ m. For the given unit discharge, the critical depth is: $Y_c \simeq 0.742$ m; the corresponding critical energy is $E_c \simeq 1.11$ m.

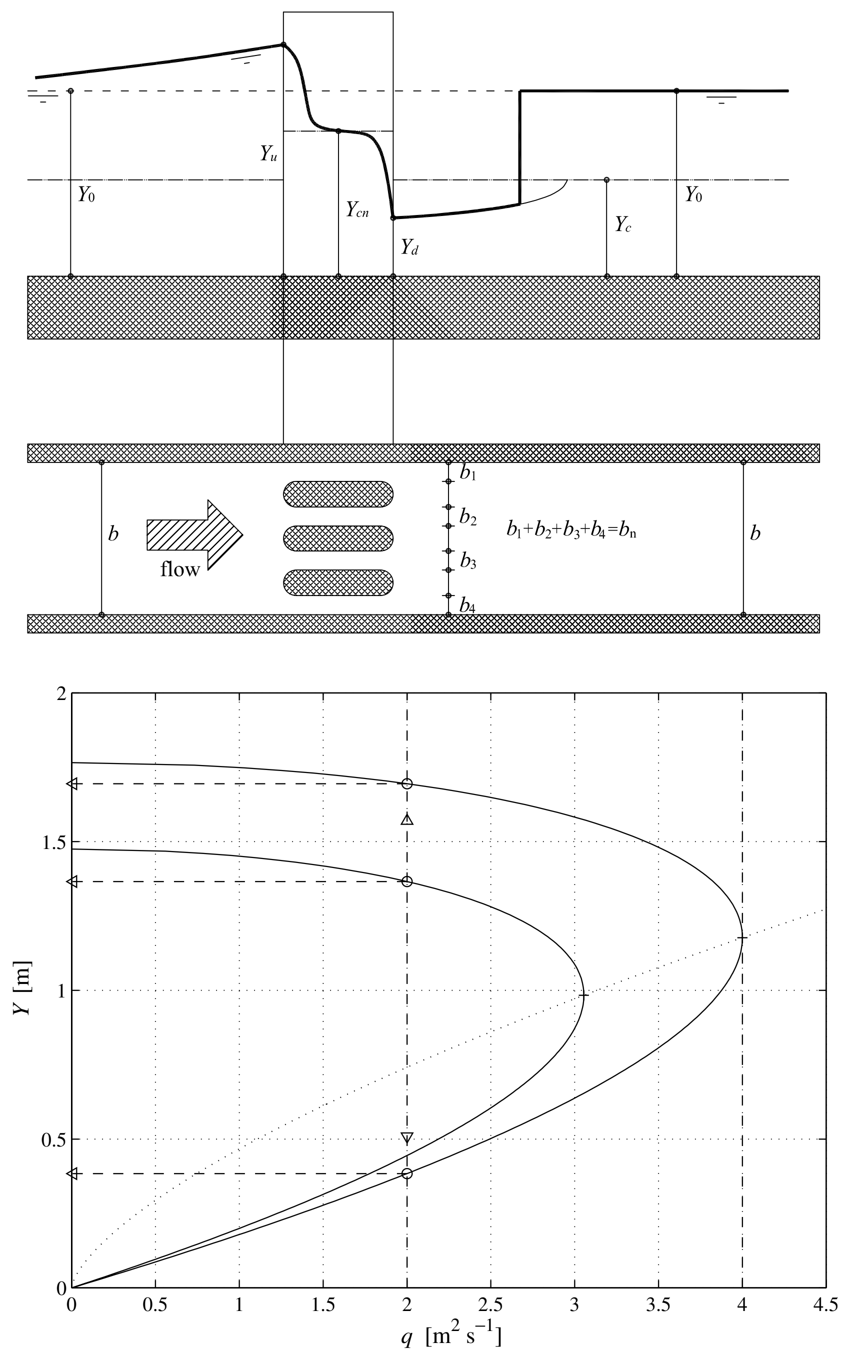


Fig. 3. Sketch of the flow through bridge piers and depth–specific discharge relationship.

A localized narrowing of the cross-section due to the presence of bridge piers is considered, as depicted in Fig. 3.

The narrowing ratio due to the piers is: $r_n = b_n/b = 0.5$, due to a reduced width in the narrowing equal to $b_n = 50$ m. As a consequence, the unit discharge in the narrowing is increased to: $q_n = Q/b_n = 4\ \mathrm{m^2\,s^{-1}}$. The critical depth in the narrowing is increased to $Y_{cn} = \sqrt[3]{q_n^2/g} \simeq 1.18$ m, and consequently the critical specific energy in the narrowing is: $E_{cn} = E_{\min} = \frac{3}{2} Y_{cn} \simeq 1.77$ m.

This is the minimum value of the specific energy which is necessary and sufficient to pass through the bridge piers. Because the normal energy of the flow is less than this value ($E_0 < E_{\min}$), the flow slow down before the bridge and a hydraulic jump after the obstacle occurs. For such a value of energy, a ratio $\Gamma_{\min} = E_{\min}/E_c \simeq 1.59$ is obtained; the corresponding subcritical and supercritical depths are located immediately upstream and downstream the narrowing.

The following results are obtained: $\alpha = \arctan\left(\sqrt{\Gamma_{\min}^3 - 1}\right) \simeq 1.05\ \mathrm{rad} \simeq 60^\circ$ (Eq. (10)); subcritical nondimensional depth occurring upstream of the narrowing: $\eta_u \simeq 2.29$ (Eq. (12); supercritical nondimensional depth occurring downstream of the narrowing: $\eta_d \simeq 0.518$ (Eq. (13); the corresponding dimensional depths are: $Y_u \simeq 1.69$ m and $Y_d \simeq 0.384$ m (Fig. 3). In the upstream trench of the channel, a M1 profile occurs; in the downstream trench, the flow is supercritical and a M3 profile occurs, up to the hydraulic jump. Downstream the jump, the normal flow is established. A simple integral momentum balance between the upstream and downstream the narrowing gives the total force supported by the whole ensemble of the bridge piers. The nondimensional total forces upstream and downstream the bridge are respectively: $\Phi_u \simeq 2.03$ and $\Phi_d \simeq 1.38$. Being the critical total force $F_c \simeq 809$ kN, the dimensional upstream and downstream total forces are $F_u \simeq 1645$ kN and $F_d \simeq 1114$ kN, respectively. Neglecting friction just in correspondence of the bridge and the gravity component in the flow direction, as usual, the force on the bridge piers is obtained: $F_{\mathrm{bridge}} = F_u - F_d \simeq 531$ kN.

6.3. Numerical solution of a dam-break flow over a rectangular bump

To show the usefulness of the analytical inversion of the depth–energy relationship in the context of unsteady flow numerical models, few exemplifying numerical results are presented.

The attention is here focused on a specific test case, originally proposed in [14] for the validation of numerical models based on shallow water equations [10,19]. It consists of the dam-break flow simulation over a discontinuous bottom, described by the following equation:

$$z(x) = \begin{cases} 8\ \mathrm{m} & \text{if } |x - 750| \leqslant 1500/8\ \mathrm{m}, \\ 0 & \text{otherwise}, \end{cases} \tag{24}$$

with $0 \leqslant x \leqslant 1500$ m. The initial conditions are:

$$q(x,0) = 0; \quad Y(x) + z(x) = \begin{cases} 20\ \mathrm{m} & \text{if } x \leqslant 750\ \mathrm{m}, \\ 15\ \mathrm{m} & \text{otherwise}. \end{cases} \tag{25}$$

The total duration of the simulation is 60 s.

The numerical results shown in Fig. 4 are obtained by a new Finite Volume WENO scheme, fourth-order accurate in space and time. The details of the model are here omitted because are outside the purposes of the present work, but an exhaustive description of the scheme is given in [20]. Here the attention is focused on the novel method introduced for managing the bed discontinuities. This is developed introducing a correction of the numerical flux based on the total energy local conservation, requiring the inversion of the depth–energy relationship [20]. This evaluation must be performed a huge amount of times (two times for each cell interface, for each computational time step) and, therefore, the availability of an analytical method to accomplish this task is essential to achieve an overall computational efficiency. Moreover, the use of an analytical direct approach, instead of an iterative procedure, drastically reduces the numerical errors. In Fig. 4, the numerical

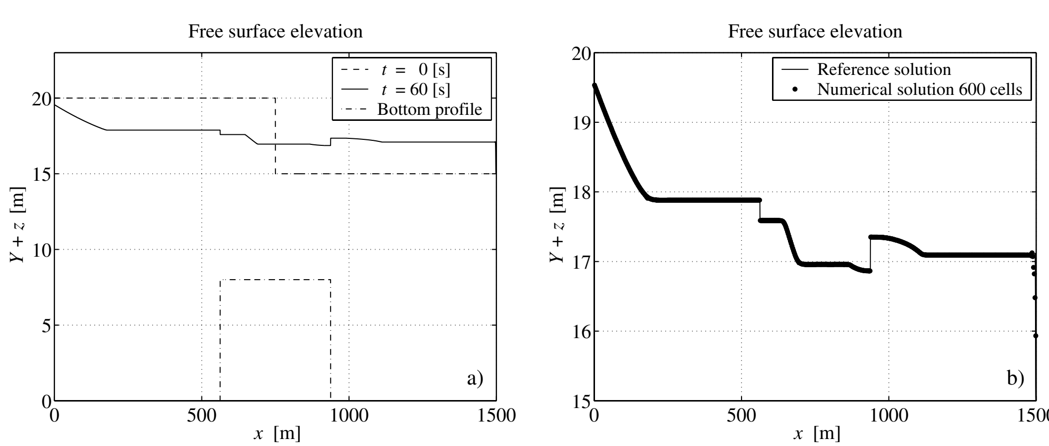


Fig. 4. Dam-break over a rectangular bump.

solution, computed using 600 cells, is compared with a reference solution computed using 6000 cells. The agreement between the numerical results and the reference results is very good. In particular, the reproduction of the steady discontinuities induced by the bottom steps is nearly exact. A comparison with results obtained on the same test case using different numerical schemes [14,10,19] put into evidence a significant improvement which is essentially due to the new flux correction.

7. Conclusions

This work is mainly conceived in the context of free surface flows in wide channels of rectangular cross-section channels. A simple, original, analytical method is presented, which allows to obtain the two depths (one subcritical, one supercritical) corresponding to a given value (greater than the critical value) of the specific energy. Likewise, a strictly similar method allows to obtain the two depths (one subcritical, one supercritical) corresponding to a given value (greater than the critical value) of the total force of the flow. These simple relationships are considered to be useful engineering tools to solve common problems of classical hydraulics, such as localized obstacles (bridge piers, localized bumps) or abrupt changes in cross-section width or bed elevation. These tools can be used in quite different contexts: to obtain exact solutions of real problems for artificial, prismatic, wide channels; to obtain approximate solutions for real rivers which can be represented, at a large scale overview, as wide rectangular channels; to develop numerical schemes requiring the inversion of the depth–energy relationship or the depth–total force relationship a huge number of times. The same results can be also applied to obtain exact solutions for two dimensional shallow flows, where our reasoning is referred to each single vertical.

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