034IN - FONDAMENTI DI AUTOMATICA - FUNDAMENTALS OF AUTOMATIC CONTROL
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Part VII: Step-Response Analysis

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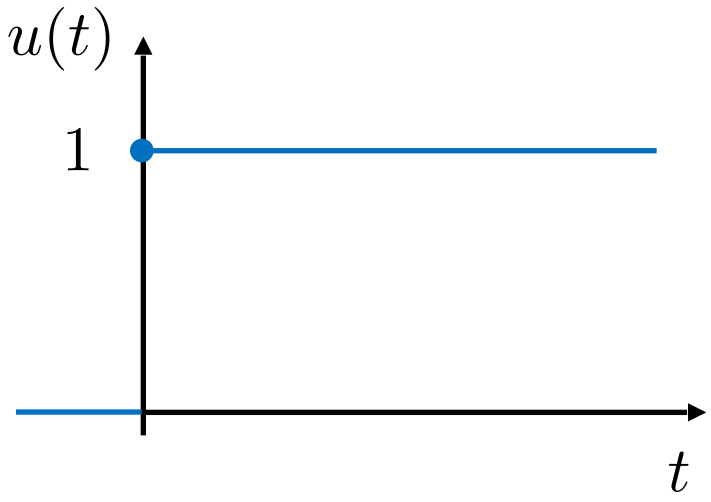
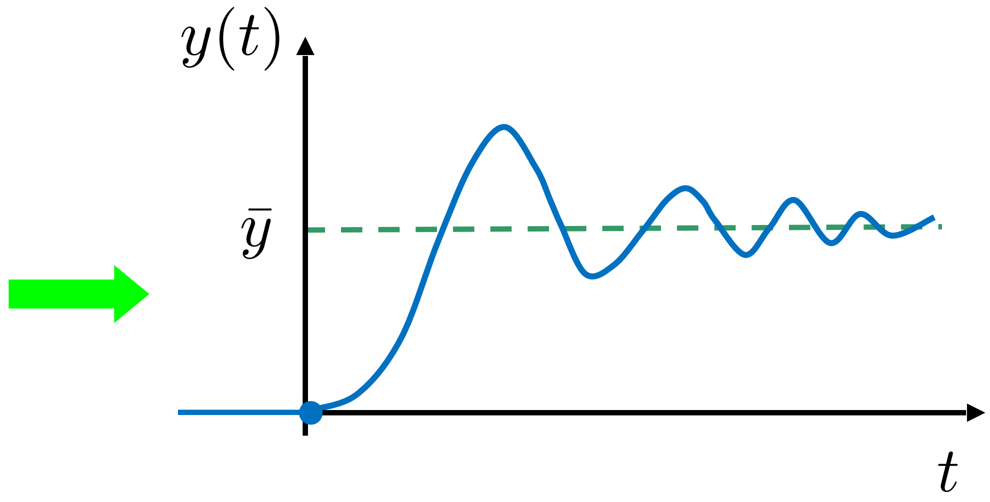
Step Response

$$x(0) = 0\,; \quad u(t) = 1(t)$$





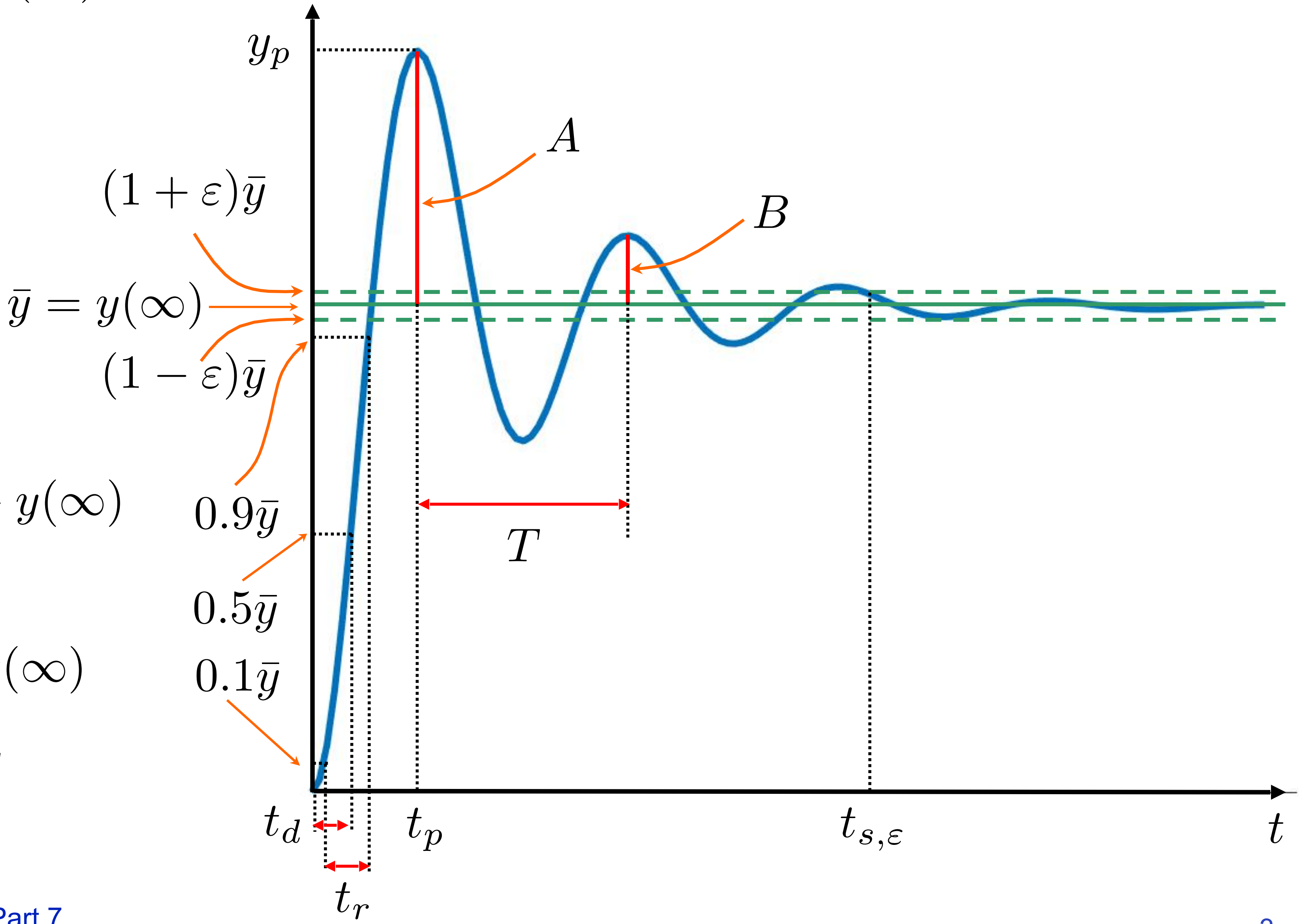
- For asymptotically stable systems, the step response describes the way the systems ”moves” from an equilibrium to another
- The characteristics of the step response are a key element in the **requirements** for a control systems

Characteristic Parameters of the Step Response

- Steady-state value: $\bar{y} = y(\infty)$
- Settling time: $t_{s,\varepsilon}$
- Rise time: t_r
- Delay time: t_d
- Peak time: t_p
- Peak value: y_p
- Max. overshoot: $A = y_p - y(\infty)$
- Max. % overshoot:

$$\Delta\% = 100 \cdot A/y(\infty)$$

- "Period" of oscillations: T
- Damping factor: B/A



Step Response: First Order Systems

- **Case A)**

$$G(s) = \frac{\mu}{1+s\tau}\,; \quad \mu > 0;\ \tau > 0$$

strictly proper first-order system

↳ asymptotic stability

- **Case B)**

$$G(s) = \frac{\mu(1+sT)}{1+s\tau}\,; \quad \mu > 0;\ \tau > 0$$

non strictly proper first-order system

↳ asymptotic stability

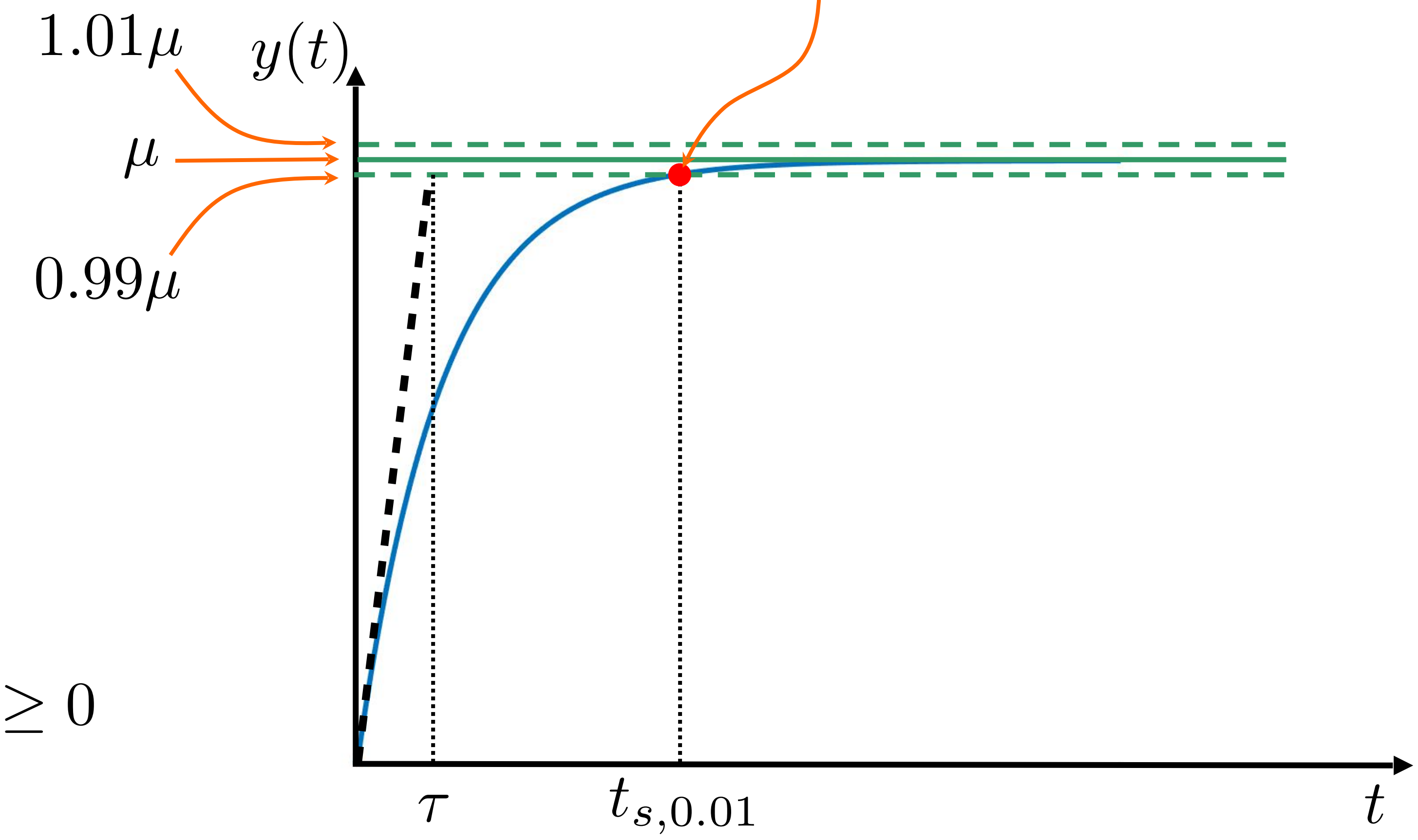
Step Response: First Order Systems (contd.)

- **Case A)**

$$
\begin{aligned}
y(t) &= \mathcal{L}^{-1}\left[G(s)\frac{1}{s}\right] \\
&= \mathcal{L}^{-1}\left[\frac{\mu}{s(1+s\tau)}\right] \\
&= \mathcal{L}^{-1}\left[\frac{\mu}{s} - \frac{\mu\tau}{1+s\tau}\right] \\
&= \mu\left(1-e^{-t/\tau}\right), \quad t \geq 0
\end{aligned}
$$

$\varepsilon = 0.01$

$\cancel{\mu}\left(1-e^{-t/\tau}\right) = 0.99\cancel{\mu}$



Settling-Time Calculation

For example, the settling time for $\varepsilon = 0.01$ can be characterised as follows:

$$1 - e^{-t/\tau} = 0.99 \quad \Rightarrow \quad e^{-t/\tau} = 0.01 \quad \Rightarrow \quad e^{t/\tau} = 100$$

$$\boxed{t_{s,0.01} = \tau \ln 100 \simeq 4.6\tau}$$

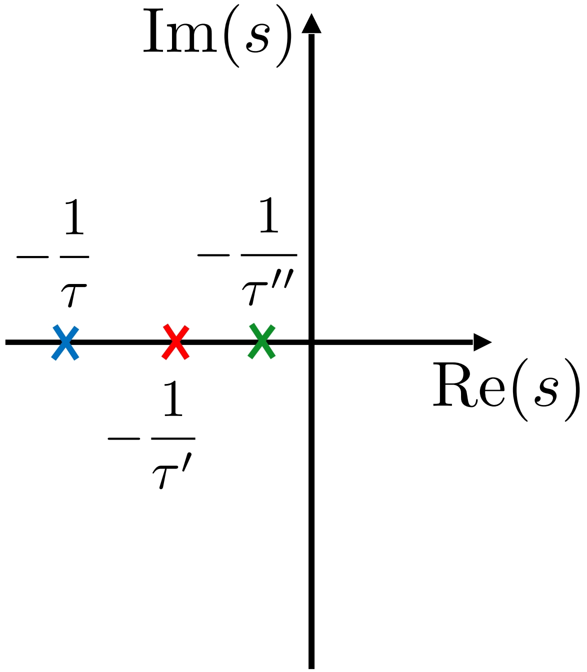
The calculation of the rising time t_r and the delay time t_d follows similar lines.

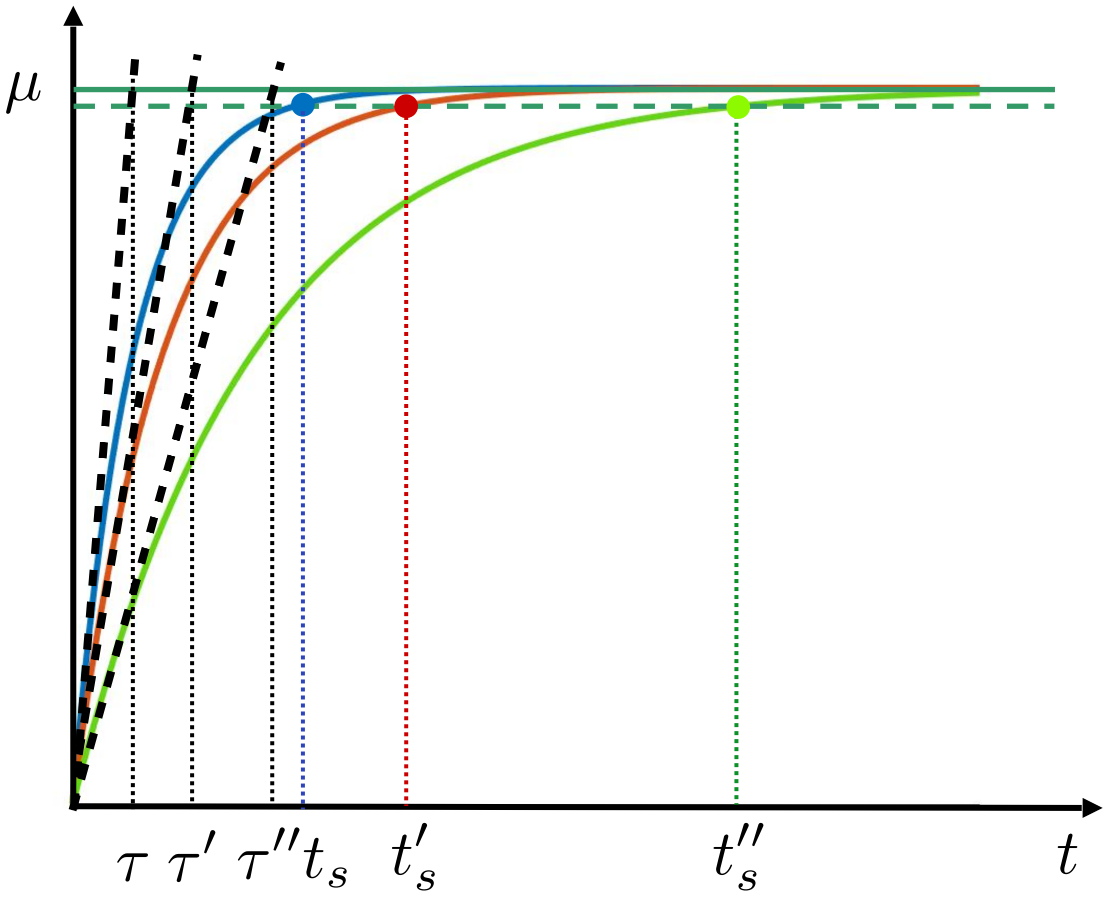
The following approximations are useful:

$$\boxed{t_r \simeq 2.2\tau \qquad t_d \simeq 0.7\tau \qquad t_{s,0.05} \simeq 3\tau \qquad t_{s,0.01} \simeq 4.6\tau}$$

Remark: without loss of generality, from now on we shall use t_s as a shorthand for $t_{s,0.01}$

Qualitative Analysis of the Step Response





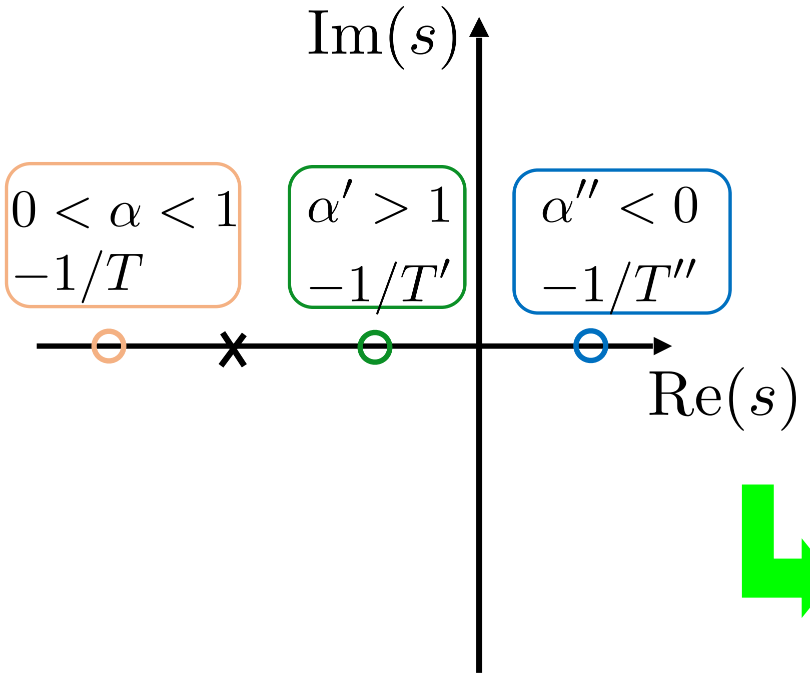
Step Response: First Order Systems (contd.)

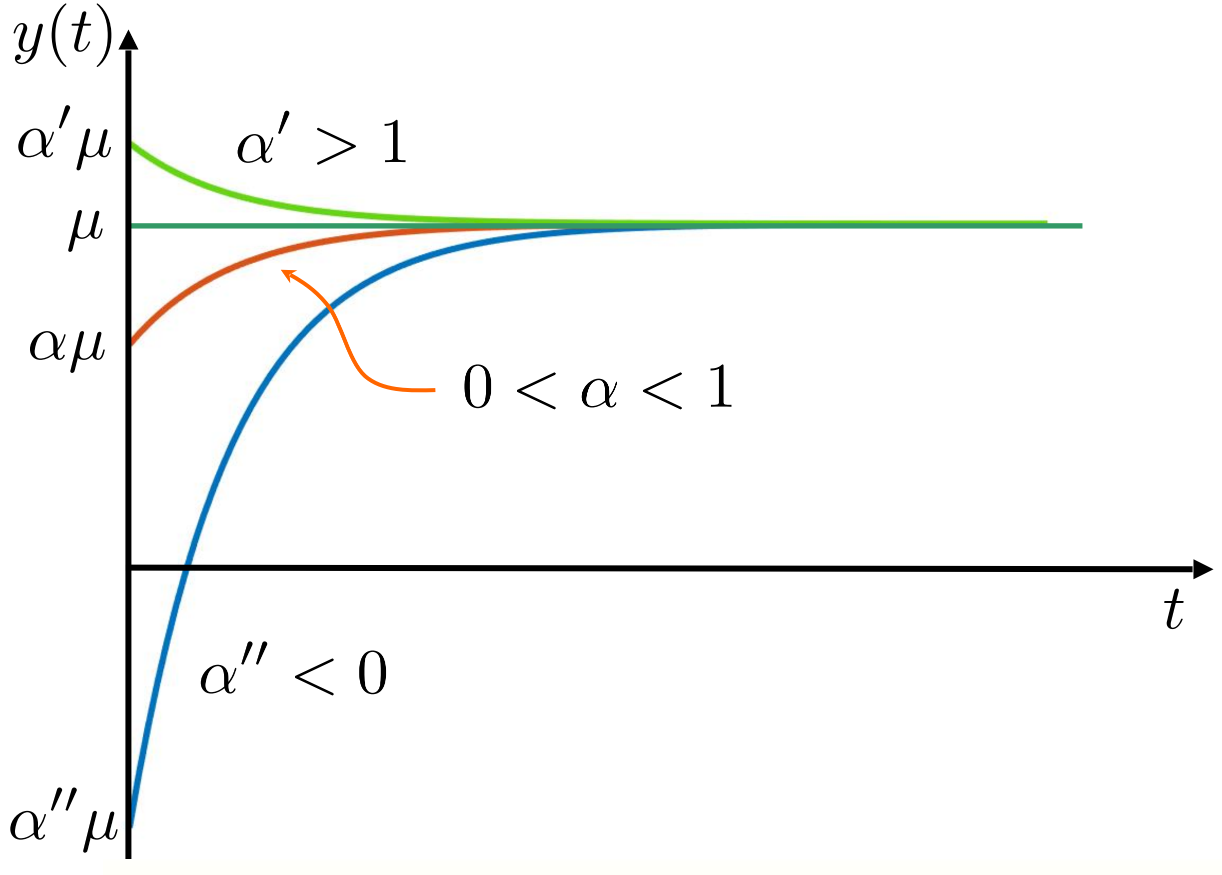
- **Case B)**

$$
\begin{aligned}
y(t) &= \mathcal{L}^{-1}\left[G(s)\frac{1}{s}\right] \\
&= \mathcal{L}^{-1}\left[\frac{\mu(1+sT)}{s(1+s\tau)}\right] \\
&= \mathcal{L}^{-1}\left[\frac{\mu}{s}+\frac{\mu(T-\tau)}{1+s\tau}\right] \\
&= \mu\left(1+(\alpha-1)e^{-t/\tau}\right), \quad t \geq 0 \ \text{ with } \ T=\alpha\tau
\end{aligned}
$$

Note that (the system is not strictly proper): $\lim_{t\to 0^+} y(t) = \mu\frac{T}{\tau} \neq 0$

Qualitative Analysis of the Step Response





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- **Case A)**

$$G(s) = \frac{\mu}{(1+s\tau_1)(1+s\tau_2)}$$ real poles, no zeros

- **Case B)**

$$G(s) = \frac{\mu(1+sT)}{(1+s\tau_1)(1+s\tau_2)}$$ real poles, one zero

- **Case C)**

$$G(s) = \frac{\varrho}{(s+\sigma+j\omega)(s+\sigma-j\omega)}$$ complex poles, no zeros

- **Case D)**

$$G(s) = \frac{\varrho(1+sT)}{(s+\sigma+j\omega)(s+\sigma-j\omega)}$$ complex poles, one zero

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- **Case A)**

$$G(s) = \frac{\mu}{(1+s\tau_1)(1+s\tau_2)}\ ; \quad \mu > 0\ ; \quad \tau_1 \neq \tau_2$$

$$\left.\begin{aligned} \tau_1 &> 0 \\ \tau_2 &> 0 \end{aligned}\right\} \text{ asymptotic stability}$$

Without loss of generality, assume $\tau_1 > \tau_2$

Step Response: Second Order Systems (contd.)

$$y(t) = \mathcal{L}^{-1}\left[G(s)\frac{1}{s}\right] = \mathcal{L}^{-1}\left[\frac{\mu}{s(1+s\tau_1)(1+s\tau_2)}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{A}{s} + \frac{B}{1+s\tau_1} + \frac{C}{1+s\tau_2}\right]$$

where

$$A = \left.\frac{\mu}{(1+s\tau_1)(1+s\tau_2)}\right|_{s=0} = \mu$$

$$B = \left.\frac{\mu}{s(1+s\tau_2)}\right|_{s=-1/\tau_1} = \frac{\mu}{-\frac{1}{\tau_1}(1-\frac{\tau_2}{\tau_1})} = \frac{\mu\tau_1^2}{\tau_2-\tau_1}$$

$$C = \left.\frac{\mu}{s(1+s\tau_1)}\right|_{s=-1/\tau_2} = \frac{\mu}{-\frac{1}{\tau_2}(1-\frac{\tau_1}{\tau_2})} = \frac{\mu\tau_2^2}{\tau_1-\tau_2}$$

Step Response: Second Order Systems (contd.)

Hence:

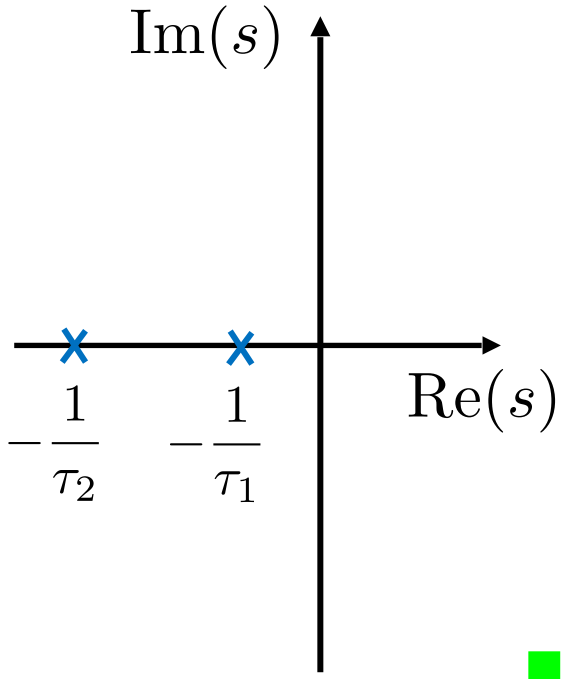
$$y(t) = \mathcal{L}^{-1}\left[\frac{\mu}{s} + \frac{\frac{\mu\tau_1^2}{\tau_2-\tau_1}}{1+s\tau_1} + \frac{\frac{\mu\tau_2^2}{\tau_1-\tau_2}}{1+s\tau_2}\right]$$

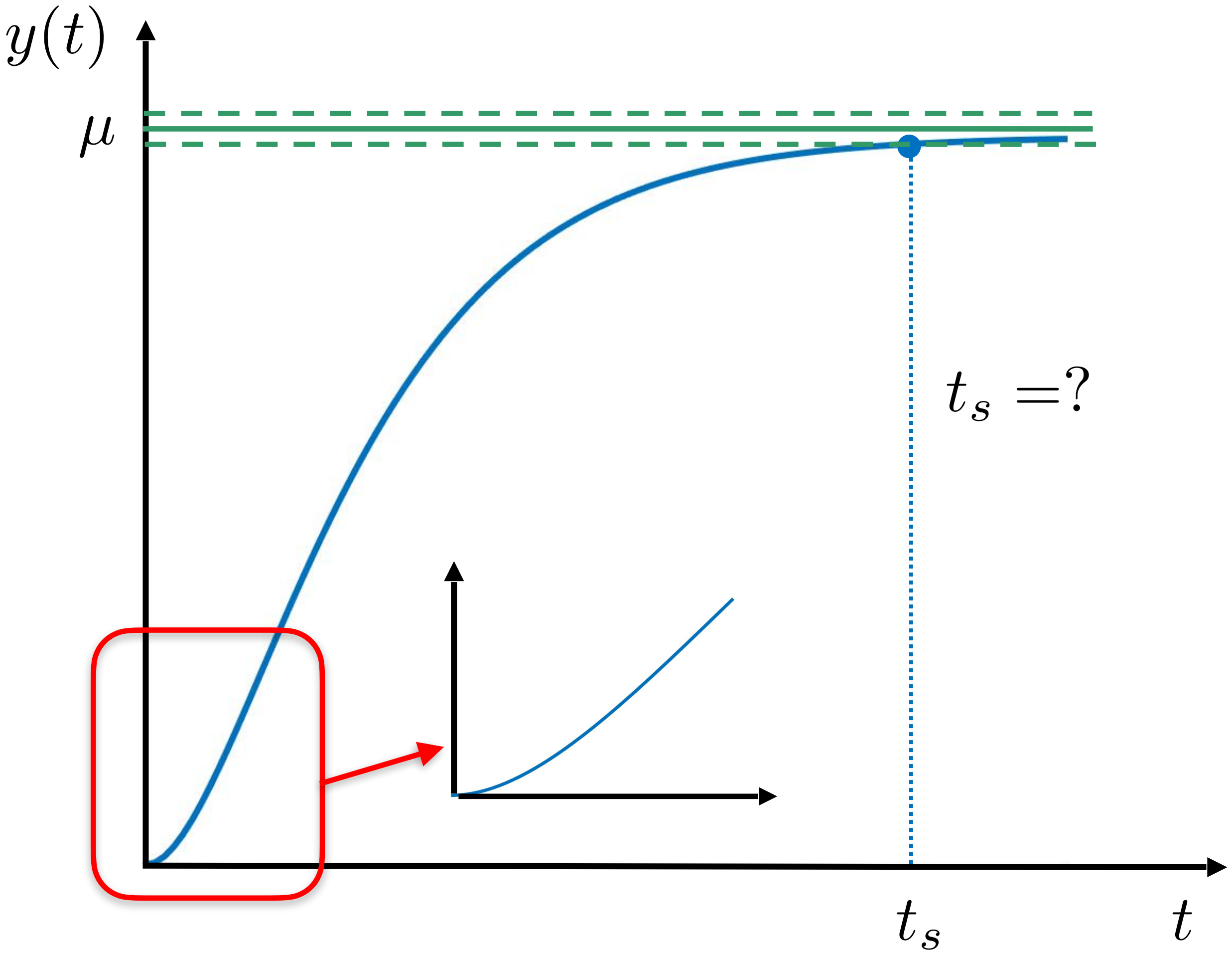
$$= \mu\left(1 - \frac{\tau_1}{\tau_1-\tau_2}e^{-t/\tau_1} + \frac{\tau_2}{\tau_1-\tau_2}e^{-t/\tau_2}\right), \quad t \geq 0$$

Characteristics:

- $y(\infty) = \mu > 0$
- $y(0) = 0$
- $\dot{y}(0) = 0$
- $\ddot{y}(0) = \dfrac{\mu}{\tau_1\tau_2} > 0$

Qualitative Analysis of the Step Response





Approximate Calculation of the Settling Time

If $\tau_1 \gg \tau_2$:

$$y(t) = \mu \left(1 - \frac{\tau_1}{\tau_1 - \cancel{\tau_2}} e^{-t/\tau_1} + \cancel{\frac{\tau_2}{\tau_1 - \tau_2}} e^{-t/\tau_2} \right), \quad t \geq 0$$

$$\simeq \mu \left(1 - e^{-t/\tau_1} \right), \quad t \geq 0$$

$$t_s \simeq 4.6\tau_1$$

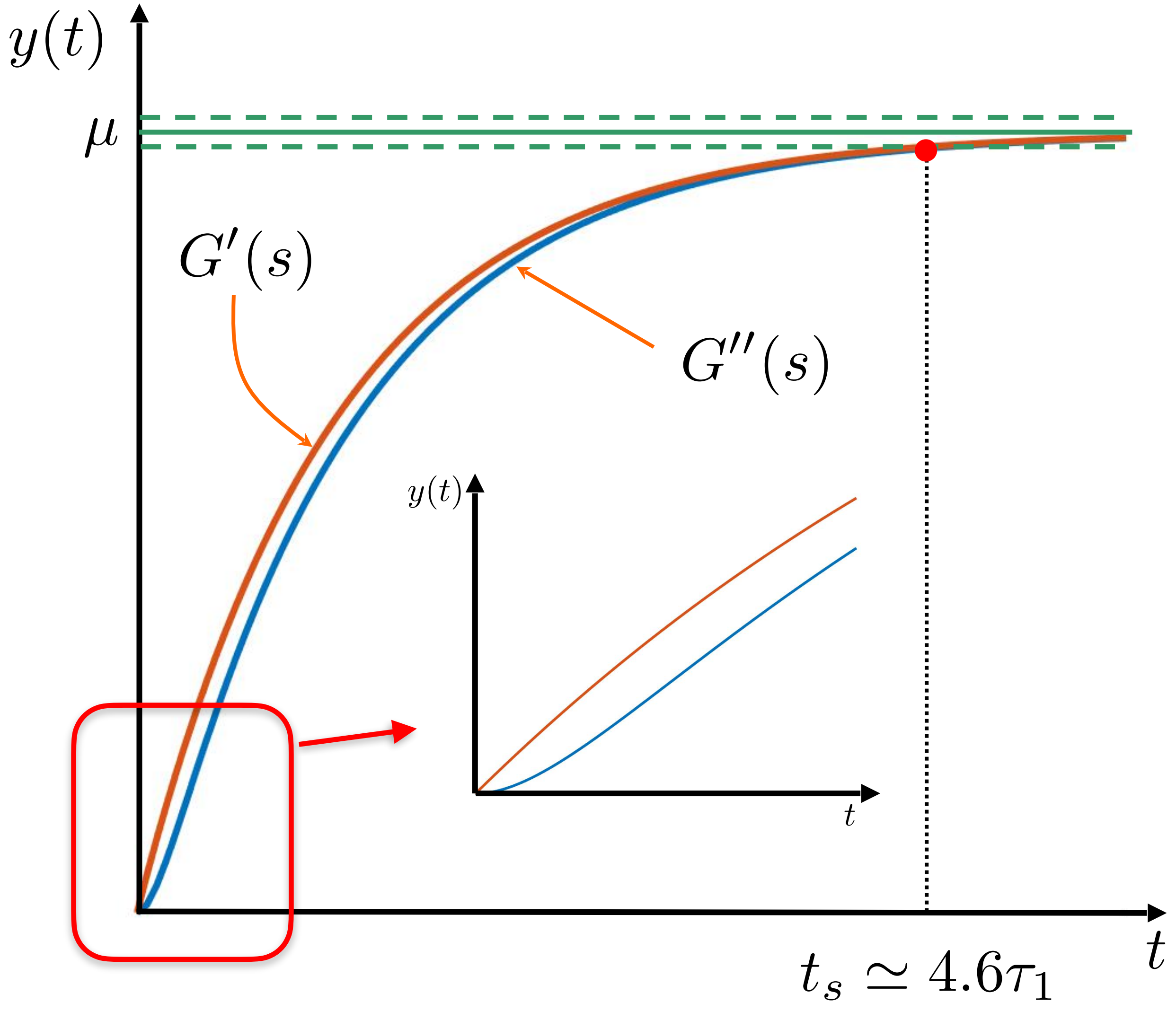
In general, in the absence of zeros, the most influential poles on the qualitative behaviour of the step response are the ones **closer to the imaginary axis**.

Qualitative Analysis: Comparison Between First and Second Order

$$G'(s) = \frac{\mu}{1 + s\tau_1}$$

$$G''(s) = \frac{\mu}{(1 + s\tau_1)(1 + s\tau_2)}; \; \tau_1 \gg \tau_2$$

- The main difference lies in the initial transient behaviour
- For a given settling time, the step-response in the second-order case without zeros has a "slower" dynamics



- **Case B)**

$$G(s) = \frac{\mu(1+sT)}{(1+s\tau_1)(1+s\tau_2)}\,; \quad \mu > 0\,; \quad \tau_1 \neq \tau_2$$

$$\left.\begin{array}{l}\tau_1 > 0 \\ \tau_2 > 0\end{array}\right\} \Longrightarrow \text{asymptotic stability}$$

Without loss of generality, assume $\tau_1 > \tau_2$

Step Response: Second Order Systems (contd.)

$$
\begin{aligned}
y(t) &= \mathcal{L}^{-1}\left[G(s)\frac{1}{s}\right] = \mathcal{L}^{-1}\left[\frac{\mu}{s(1+s\tau_1)(1+s\tau_2)}\right] \\
&= \mathcal{L}^{-1}\left[\frac{A}{s} + \frac{B}{1+s\tau_1} + \frac{C}{1+s\tau_2}\right]
\end{aligned}
$$

where

$$
\begin{aligned}
A &= \left.\frac{\mu(1+sT)}{(1+s\tau_1)(1+s\tau_2)}\right|_{s=0} = \mu \\
B &= \left.\frac{\mu(1+sT)}{s(1+s\tau_2)}\right|_{s=-1/\tau_1} = \frac{\mu(1-T/\tau_1)}{-\frac{1}{\tau_1}(1-\frac{\tau_2}{\tau_1})} = \frac{\mu\tau_1(\tau_1-T)}{\tau_2-\tau_1} \\
C &= \left.\frac{\mu(1+sT)}{s(1+s\tau_1)}\right|_{s=-1/\tau_2} = \frac{\mu(1-T/\tau_2)}{-\frac{1}{\tau_2}(1-\frac{\tau_1}{\tau_2})} = \frac{\mu\tau_2(\tau_2-T)}{\tau_1-\tau_2}
\end{aligned}
$$

Step Response: Second Order Systems (contd.)

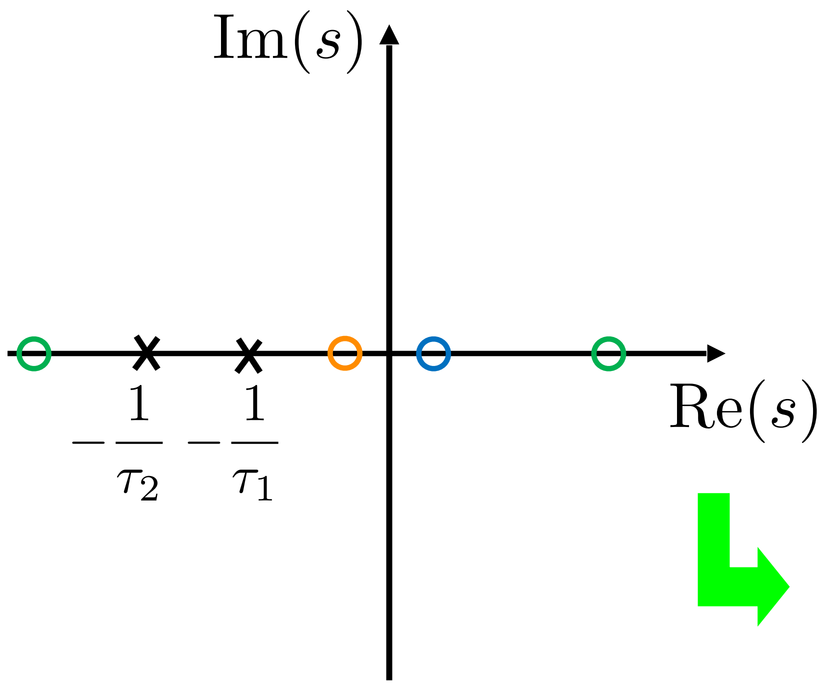
Hence:

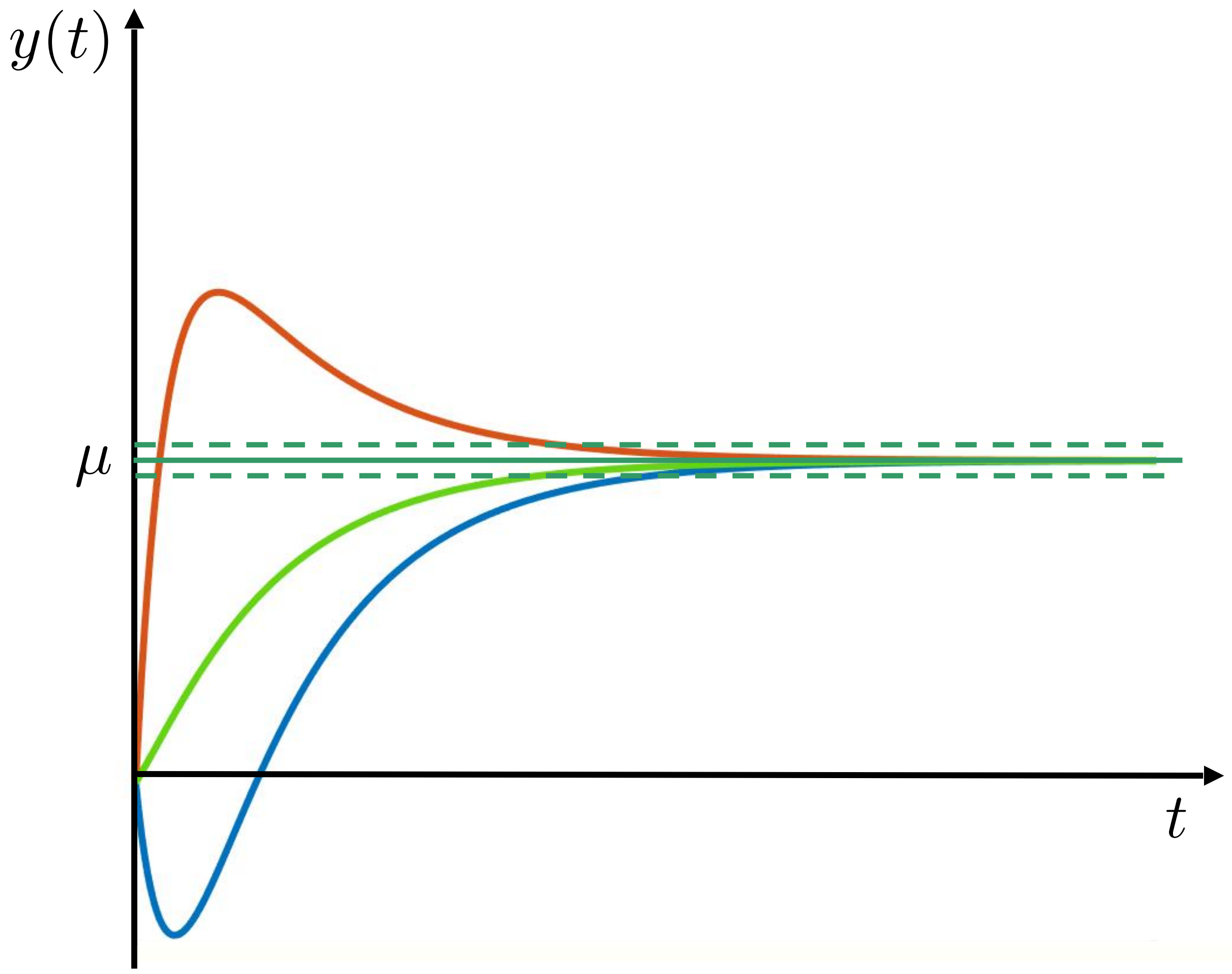
$$y(t) = \mu \left(1 - \frac{\tau_1 - T}{\tau_1 - \tau_2} e^{-t/\tau_1} + \frac{\tau_2 - T}{\tau_1 - \tau_2} e^{-t/\tau_2} \right), \quad t \geq 0$$

Characteristics:

- $y(\infty) = \mu > 0$
- $y(0) = 0$
- $\dot{y}(0) = \dfrac{\mu T}{\tau_1 \tau_2} \begin{cases} > 0, & \text{if } T > 0 \\ < 0, & \text{if } T < 0 \end{cases}$

Qualitative Analysis of the Step Response





- zero with little influence
- overshoot
- undershoot

Step Response: Second Order Systems (contd.)

- **Case C)**

$$G(s) = \frac{\varrho}{(s+\sigma+j\omega)(s+\sigma-j\omega)}$$

$$\mu = G(0) = \frac{\varrho}{\sigma^2+\omega^2}$$

poles: $-\sigma \pm j\omega$

$\sigma > 0$ $\Rightarrow$ asymptotic stability

$\omega > 0$

$\varrho > 0$

Step Response: Second Order Systems (contd.)

$$Y(s) = \frac{G(s)}{s} = \frac{A}{s} + \frac{Bs + C}{s^2 + 2\sigma s + \sigma^2 + \omega^2}$$

$$As^2 + 2A\sigma s + A\sigma^2 + A\omega^2 + Bs^2 + Cs = \varrho$$

$$\begin{cases} A + B = 0 \\ 2A\sigma + C = 0 \\ A(\sigma^2 + \omega^2) = \varrho \end{cases} \longrightarrow \begin{cases} A = \dfrac{\varrho}{\sigma^2 + \omega^2} = \mu \\ B = -\mu \\ C = -2\sigma\mu \end{cases}$$

$$Y(s) = \mu\left[\frac{1}{s} - \frac{s + 2\sigma}{s^2 + 2\sigma s + \sigma^2 + \omega^2}\right] = \mu\left[\frac{1}{s} - \frac{s + \sigma + \sigma}{(s+\sigma)^2 + \omega^2}\right]$$

$$= \mu\left[\frac{1}{s} - \frac{s + \sigma}{(s+\sigma)^2 + \omega^2} - \frac{\sigma}{\omega}\frac{\omega}{(s+\sigma)^2 + \omega^2}\right]$$

Step Response: Second Order Systems (contd.)

Hence: $$y(t) = \mu \left[1 - e^{-\sigma t}\cos(\omega t) - \frac{\sigma}{\omega} e^{-\sigma t}\sin(\omega t) \right], \quad t \geq 0$$

$$= \mu \left[1 - e^{-\sigma t}\left(\cos(\omega t) + \frac{\sigma}{\omega}\sin(\omega t) \right) \right], \quad t \geq 0$$

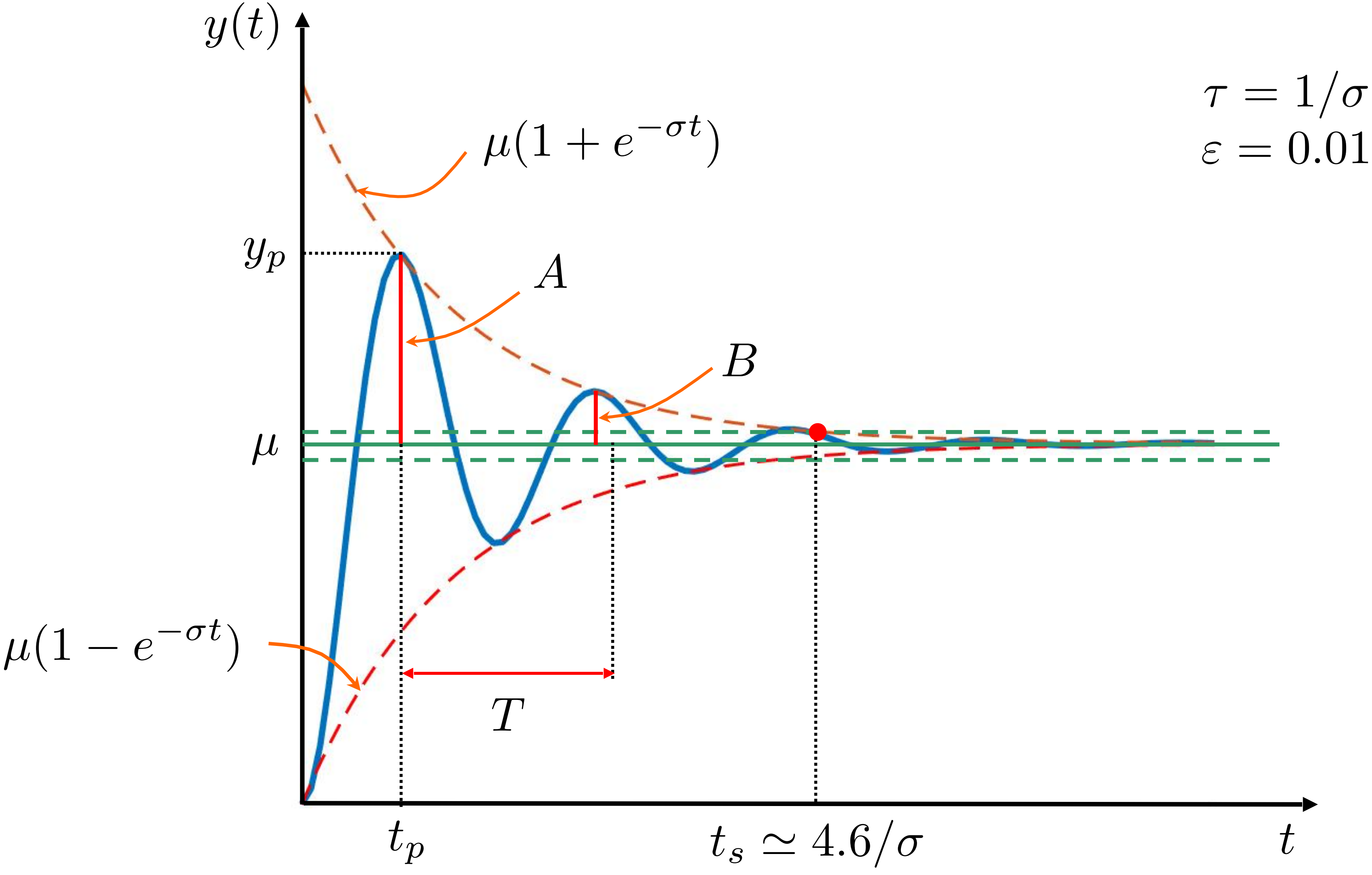
$$= \mu \underbrace{\left[1 - \frac{\sqrt{\sigma^2 + \omega^2}}{\omega} e^{-\sigma t}\sin(\omega t + \varphi) \right]}_{\text{damped oscillations}}, \quad t \geq 0$$

$$\text{where } \varphi = \arccos\left(\frac{\sigma}{\sqrt{\sigma^2 + \omega^2}} \right)$$

Characteristics:

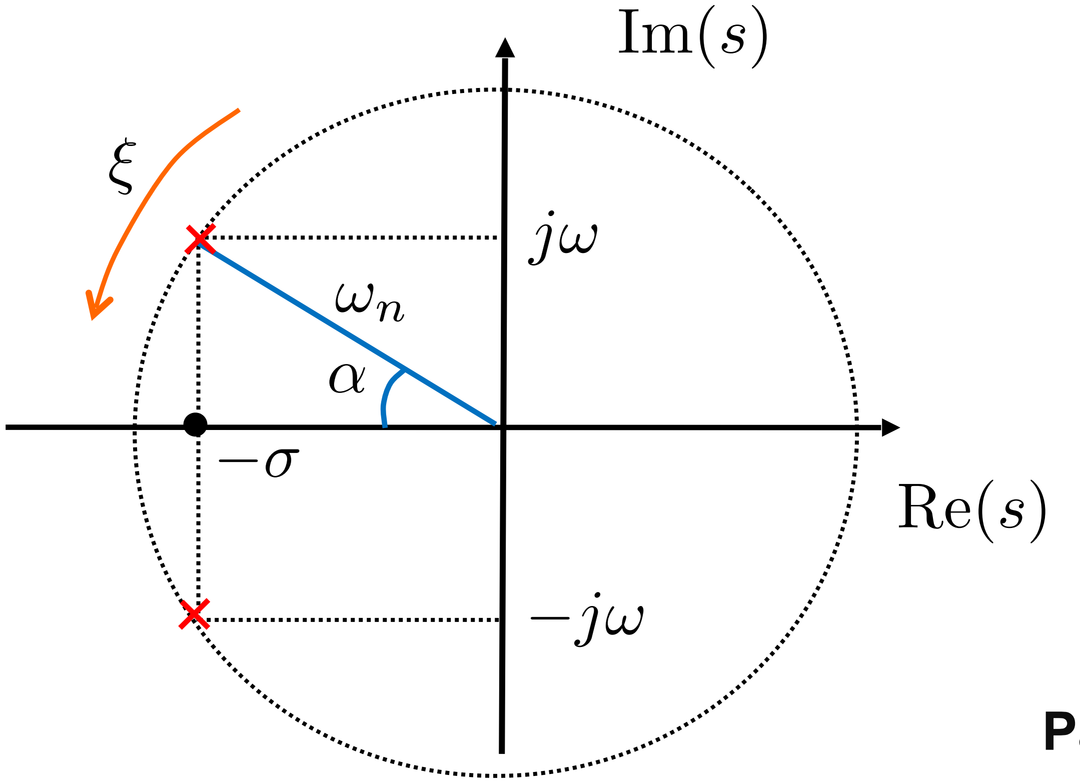
- $y(\infty) = \mu > 0$
- $y(0) = 0$
- $\dot{y}(0) = 0$
- $\ddot{y}(0) = \varrho > 0$

Qualitative Analysis of the Step Response



Characteristic Parameters of the Step Response

Recall from Part 4:



$$\omega_n^2 = \sigma^2 + \omega^2$$

$$\omega_n \xi = \sigma$$

$$\omega_n \sqrt{1-\xi^2} = \omega$$

$$G(s) = \frac{\varrho}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

Parameters:

ω_n natural angular frequency:

$\xi = \cos(\alpha)$ damping ratio

Characteristic Parameters of the Step Response (contd.)

and:

$$
\begin{aligned}
G(s) &= \frac{\varrho}{(s+\sigma+j\omega)(s+\sigma-j\omega)} = \frac{\varrho}{(s+\sigma)^2+\omega^2} \\
&= \frac{\varrho}{s^2 + \underbrace{2\sigma s}_{2\xi\omega_n} + \underbrace{\sigma^2+\omega^2}_{\omega_n^2}} = \frac{\varrho}{s^2+2\xi\omega_n s+\omega_n^2}
\end{aligned}
$$

$$
G(s) = \frac{\varrho/\omega_n^2}{1+\dfrac{2\xi}{\omega_n}s+\dfrac{1}{\omega_n^2}s^2} = \frac{\mu}{1+\dfrac{2\xi}{\omega_n}s+\dfrac{1}{\omega_n^2}s^2}
$$

where: $\mu := \dfrac{\varrho}{\omega_n^2}$

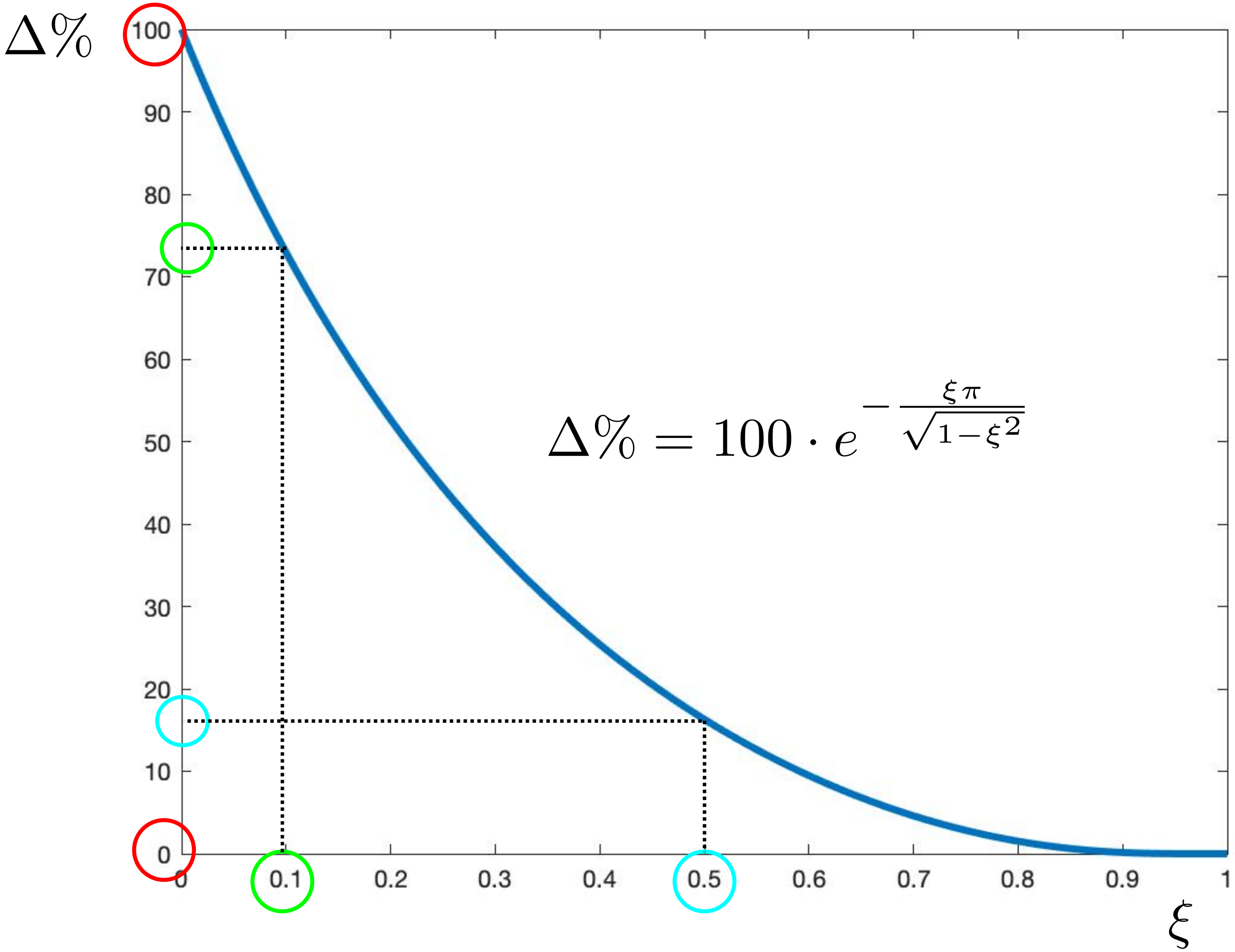
Characteristic Parameters of the Step Response (contd.)

Hence:

- Settling time: $t_s \simeq \dfrac{4.6}{\sigma} = \dfrac{4.6}{\xi\omega_n}$
- Peak time: $t_p = \dfrac{\pi}{\omega} = \dfrac{\pi}{\omega_n\sqrt{1-\xi^2}}$
- Peak value: $y_p = \mu\left[1+e^{-\frac{\sigma\pi}{\omega}}\right] = \mu\left[1+e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}}\right]$
- Maximum percentage overshoot: $\Delta\% = 100 \cdot \dfrac{A}{\mu} = e^{-\sigma\pi/\omega} = 100 \cdot e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}}$
- "Period" of oscillations: $T = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{\omega_n\sqrt{1-\xi^2}}$
- Damping factor: $\dfrac{B}{A} = \cdots = \Delta^2 = e^{-2\sigma\pi/\omega} = e^{-\frac{2\xi\pi}{\sqrt{1-\xi^2}}}$

only depend on ξ
but **not** on ω_n

Maximum Percentage Overshoot



Limit Cases

- **No damping:** $\xi = 0$

$$G(s) = \frac{\varrho}{s^2 + \omega_n^2} \qquad \text{poles: } \pm j\omega_n$$

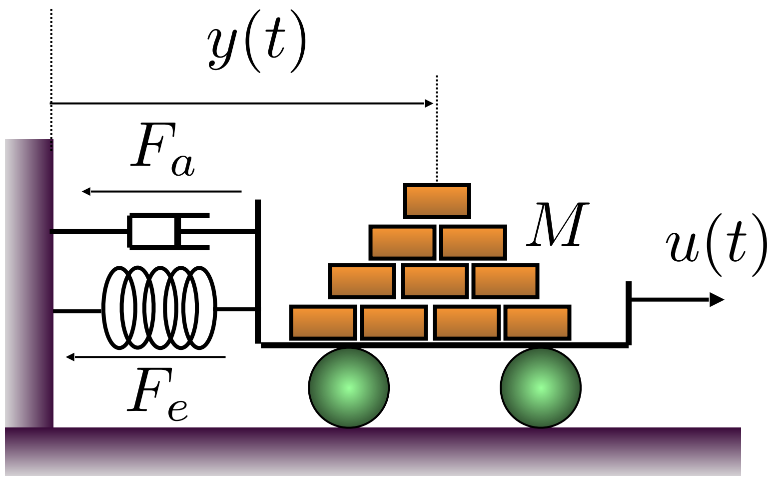
Undamped oscillations

- **Full damping:** $\xi = 1$

$$G(s) = \frac{\varrho}{(s + \omega_n)^2} \qquad \text{poles: } -\omega_n; -\omega_n$$

No oscillations at all

Example 1



$$G(s) = \frac{1}{Ms^2 + hs + k} = \frac{1/M}{s^2 + \frac{h}{M}s + \frac{k}{M}}$$

$$s^2 + 2\xi\omega_n s + \omega_n^2 = s^2 + \frac{h}{M}s + \frac{k}{M}$$

Hence:

$$\mu = G(0) = \frac{1}{k}$$

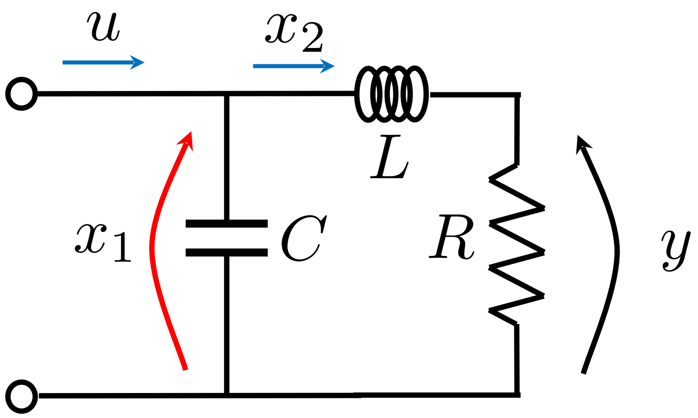
$$2\xi\omega_n = \frac{h}{M}$$

$$\omega_n^2 = \frac{k}{M}$$

$$\omega_n = \sqrt{\frac{k}{M}}$$

$$\xi = \frac{h}{2\sqrt{kM}}$$

Example 2



$$\begin{cases} C\dot{x}_1 = u - x_2 \\ L\dot{x}_2 = x_1 - Rx_2 \\ y = R\,x_2 \end{cases}$$

$$A = \begin{bmatrix} 0 & -1/C \\ 1/L & -R/L \end{bmatrix} \qquad B = \begin{bmatrix} 1/C \\ 0 \end{bmatrix} \qquad C = [0 \;\; R]$$

$$G(s) = [0 \;\; R] \begin{bmatrix} s & 1/C \\ -1/L & s + R/L \end{bmatrix}^{-1} \begin{bmatrix} 1/C \\ 0 \end{bmatrix} = \cdots = \frac{R/(LC)}{s^2 + \dfrac{R}{L}s + \dfrac{1}{LC}}$$

$$\omega_n = \frac{1}{\sqrt{LC}}\,; \quad \xi = \frac{R}{2}\sqrt{\frac{C}{L}}\,; \quad \mu = R$$

Step Response: Second Order Systems (contd.)

- **Case D)**

$$G(s) = \frac{\mu(1+sT)}{1+\frac{2\xi}{\omega_n}s+\frac{1}{\omega_n^2}s^2}\,; \quad 0<\xi<1\,;\ \omega_n>0\,;\ \mu>0$$

Characteristics of the step response:

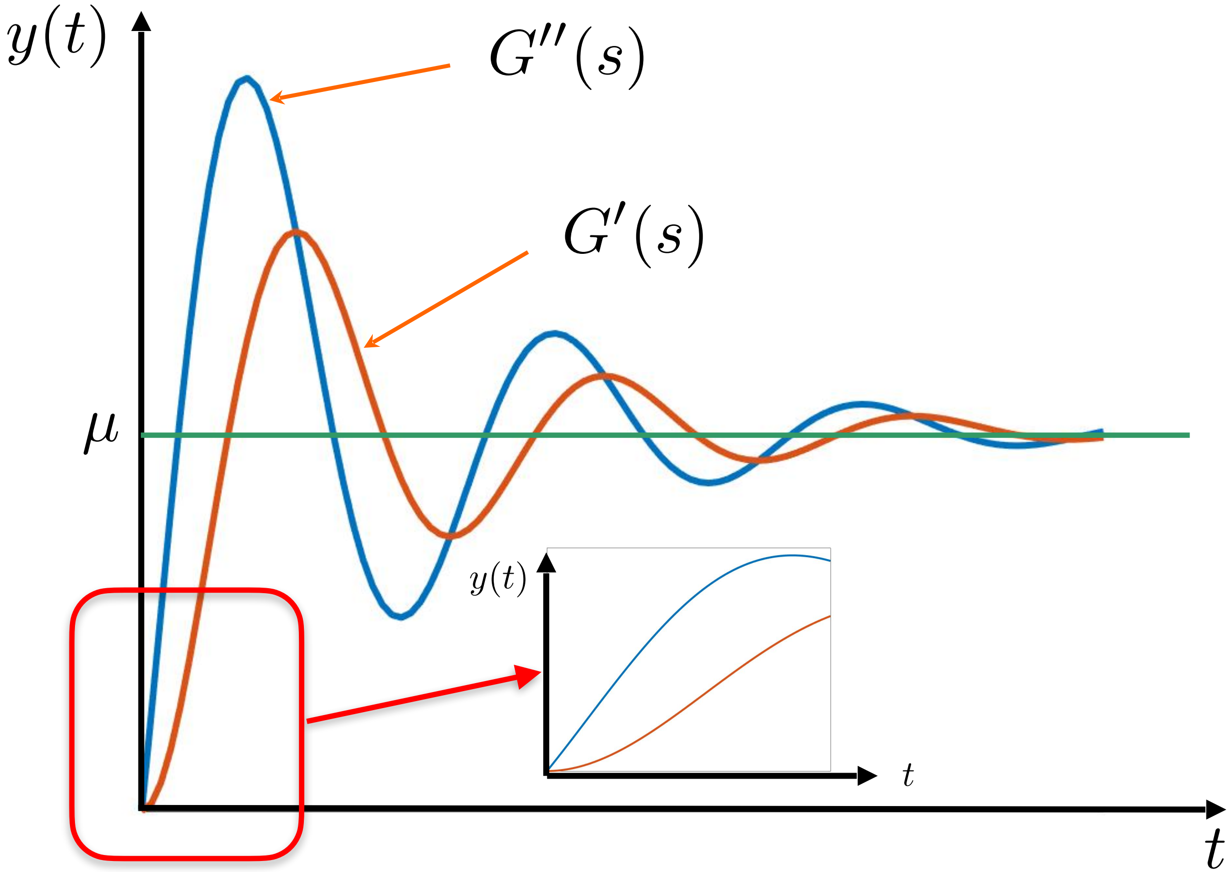
- $y(\infty) = \mu > 0$
- $y(0) = 0$
- $\dot{y}(0) = \mu T \omega_n^2 \begin{cases} >0, & \text{if } T>0 \\ <0, & \text{if } T<0 \end{cases}$

Qualitative Analysis:
Comparison between Case C) (no zeros) and Case D) (one zero)

$$G'(s) = \frac{\mu}{1 + \frac{2\xi}{\omega_n}s + \frac{1}{\omega_n^2}s^2}$$

$$G''(s) = \frac{\mu(1+sT)}{1 + \frac{2\xi}{\omega_n}s + \frac{1}{\omega_n^2}s^2}$$

- Again, the main difference lies in the initial transient behaviour
- For a given settling time, the step-response in Case C) without zeros has a "slower" dynamics



Step Response for Systems of Order > 2

For simplicity, consider the case of real poles only:

$$G(s) = \frac{\mu}{s^g} \frac{\prod_{i=1}^{m}(1+sT_i)}{\prod_{i=1}^{n}(1+s\tau_i)}$$

Recall (in the absence of common factors in $G(s)$):

Asymptotic Stability $\Longleftrightarrow$ $\begin{array}{c}\mathrm{Re(poles)} < 0 \\ g \leq 0\end{array}$

$$y(t) = \mathcal{L}^{-1}\left[G(s)\frac{1}{s}\right]$$

Step Response for Systems of Order > 2 (contd.)

- **Initial Value Theorem**

$$\lim_{t \to 0^+} y(t) = \lim_{s \to \infty} \cancel{s} \, \frac{1}{\cancel{s}} \, G(s) \begin{cases} = 0\,, & \text{if } m < n \\ \neq 0\,, & \text{if } m = n \end{cases}$$

- **Final Value Theorem**

$$\lim_{t \to \infty} y(t) = \lim_{s \to 0} \cancel{s} \, \frac{1}{\cancel{s}} \, G(s) \begin{cases} = \mu\,, & \text{if } g = 0 \\ = 0\,, & \text{if } g < 0 \end{cases}$$

Dominant Poles Approximation

Again, for simplicity, consider the case of real poles:

$$Y(s) = G(s)\frac{1}{s} = \frac{\alpha_0}{s} + \frac{\alpha_1}{1+s\tau_1} + \cdots + \frac{\alpha_n}{1+s\tau_n}$$

$$y(t) = \mathcal{L}^{-1}\left[G(s)\frac{1}{s}\right]$$

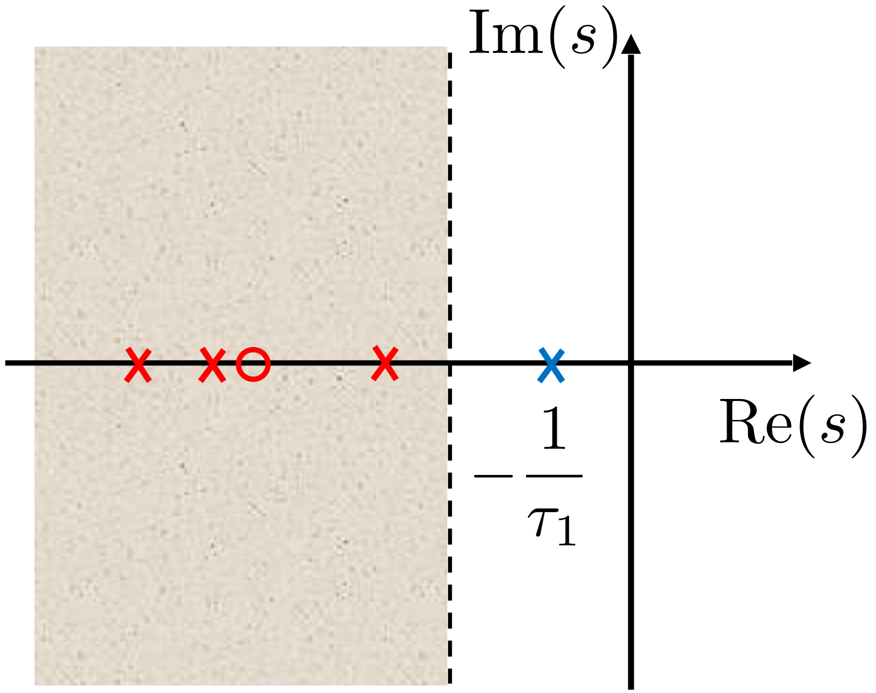
$$= \alpha_0 + \frac{\alpha_1}{\tau_1}e^{-t/\tau_1} + \cdots + \frac{\alpha_n}{\tau_n}e^{-t/\tau_n}$$

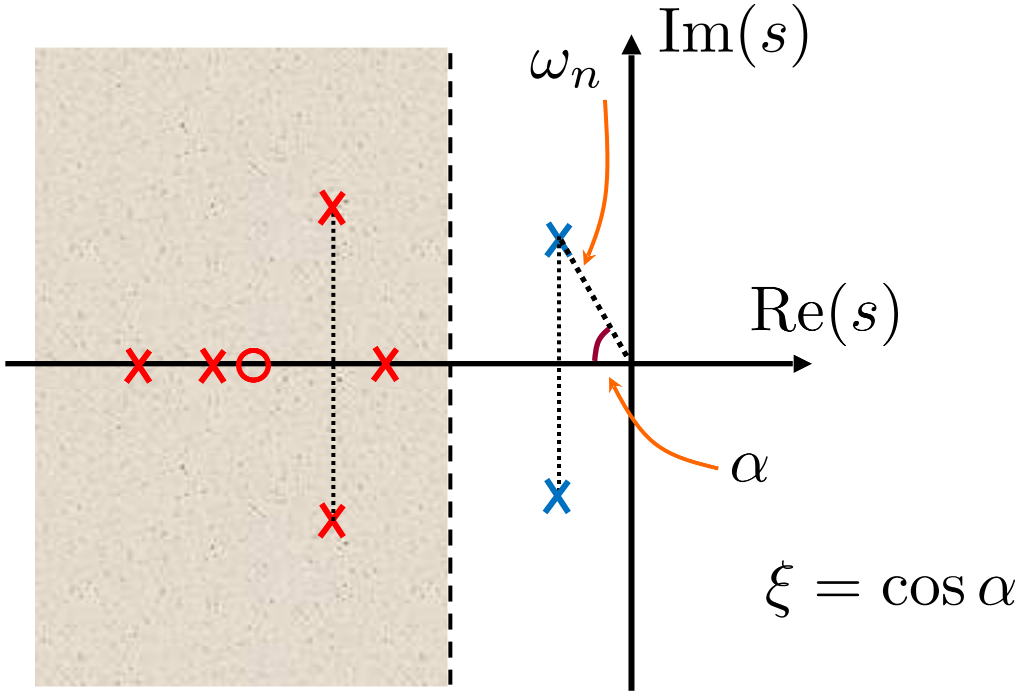
Assuming: $\tau_1 > \tau_2 > \cdots > \tau_n$

$$y(t) = \alpha_0 + \frac{\alpha_1}{\tau_1}e^{-t/\tau_1} + \cdots + \frac{\alpha_n}{\tau_n}e^{-t/\tau_n}$$

$$\simeq \alpha_0 + \frac{\alpha_1}{\tau_1}e^{-t/\tau_1}$$ dominant component, hence: $t_s \simeq 4.6\tau_1$

Dominant Poles Approximation: Real Poles





- When using the **dominant poles** approximation:
  - It is important to “preserve” the gain
  - Zeros located close to the imaginary axis have to be properly taken into account
- This approximation is useful in **qualitative** analysis and the for initial and rough controller’s design steps

Example

$$G(s) = \frac{400(1+s)}{(1+0.2s)(1+0.1s)(s^2+2s+4)}$$

$\omega_n = 2$

$\xi = 1/2$

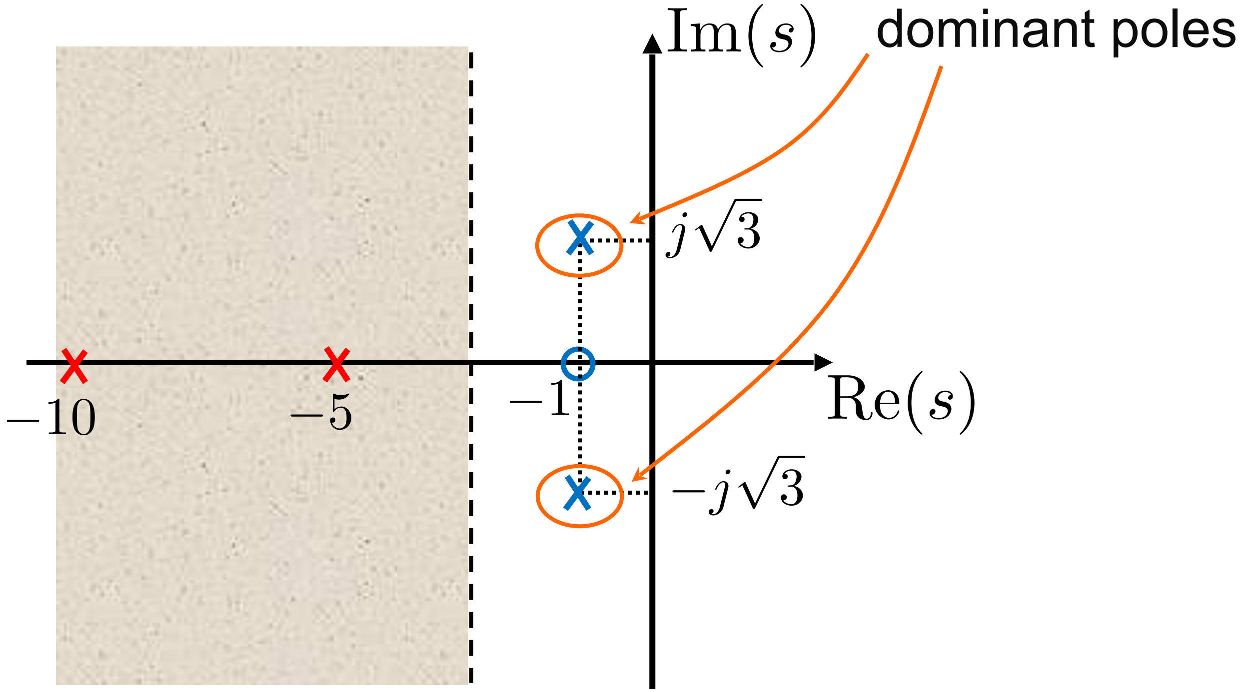
$\mu = G(0) = 100$

poles: -5

-10

$-1 \pm j\sqrt{3}$

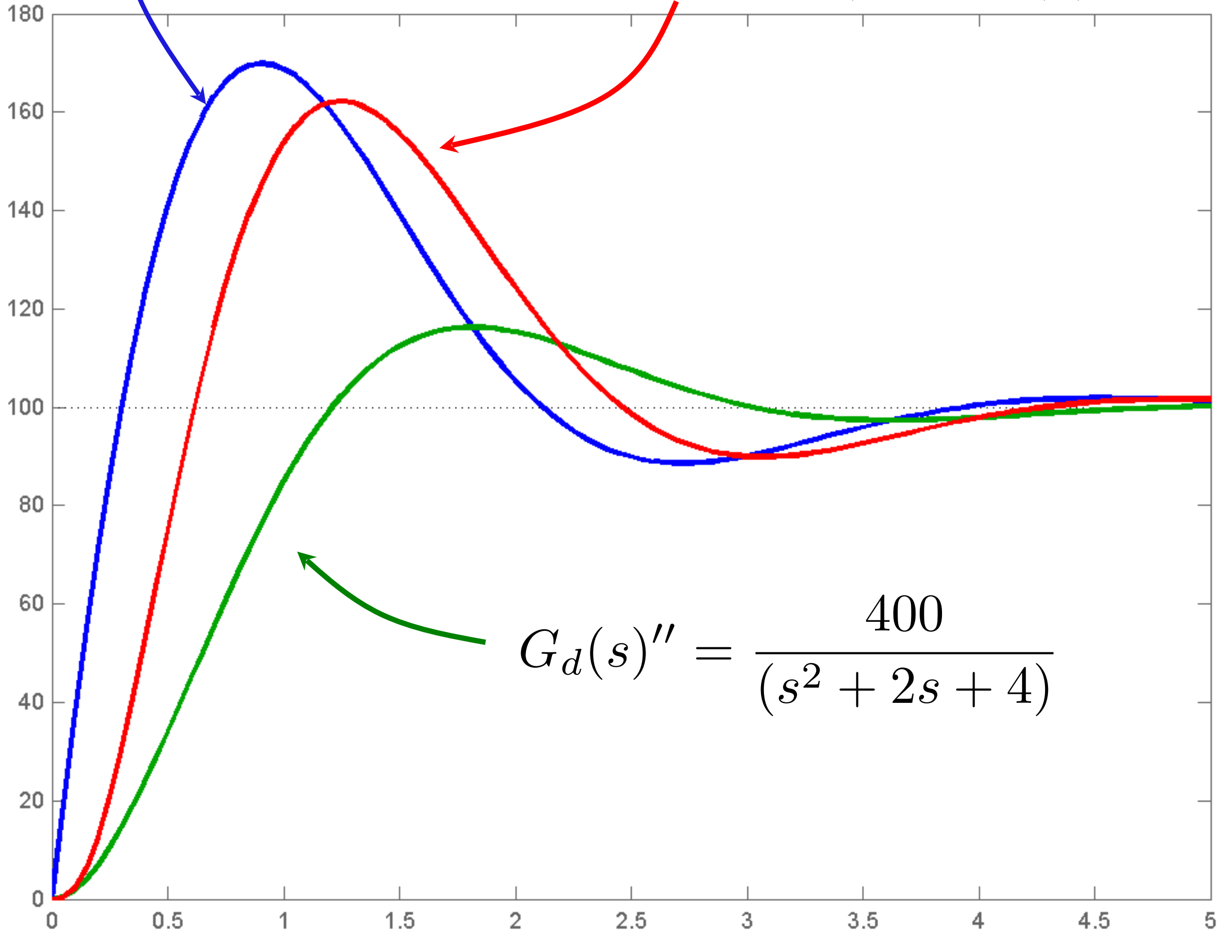
zero: -1



Example (contd.)

$$G_d(s)' = \frac{400(1+s)}{(s^2+2s+4)}$$

$$G(s) = \frac{400(1+s)}{(1+0.2s)(1+0.1s)(s^2+2s+4)}$$



$$G_d(s)'' = \frac{400}{(s^2+2s+4)}$$