034IN - FONDAMENTI DI AUTOMATICA - FUNDAMENTALS OF AUTOMATIC CONTROL
A.Y. 2023-2024
Part IV: Transfer Function

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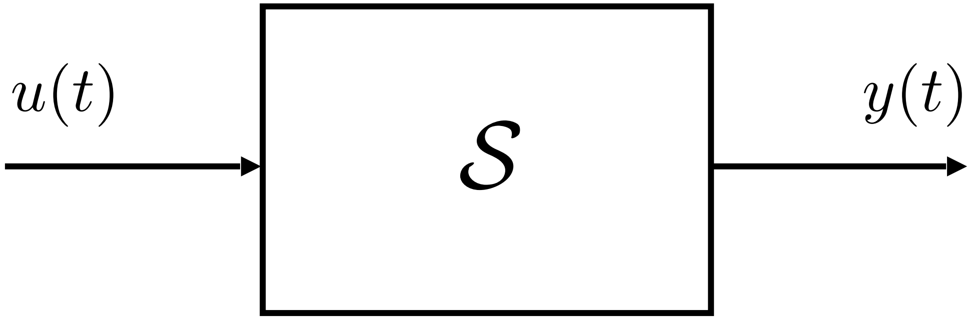
Internal Description of Linear Time-Invariant Systems

Up to now we have considered an **internal dynamic description** based on **state variables**:



$$u(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ u_m(t) \end{bmatrix} \in \mathbb{R}^m \qquad x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} \in \mathbb{R}^n \qquad y(t) = \begin{bmatrix} y_1(t) \\ \vdots \\ y_p(t) \end{bmatrix} \in \mathbb{R}^p$$

Input | State | Output

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \qquad \begin{array}{l} x \in \mathbb{R}^n \\ u \in \mathbb{R}^m \\ y \in \mathbb{R}^p \end{array} \quad (A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{p\times n}, D \in \mathbb{R}^{p\times m})$$

Recall from Part 2:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

$$\mathcal{L}\{\dot{x}(t)\} = \mathcal{L}\{Ax + Bu\} \quad \Longrightarrow \quad sX(s) - x(0) = AX(s) + BU(s)$$

$$(sI - A)X(s) = x(0) + BU(s)$$

$$X(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}BU(s)$$

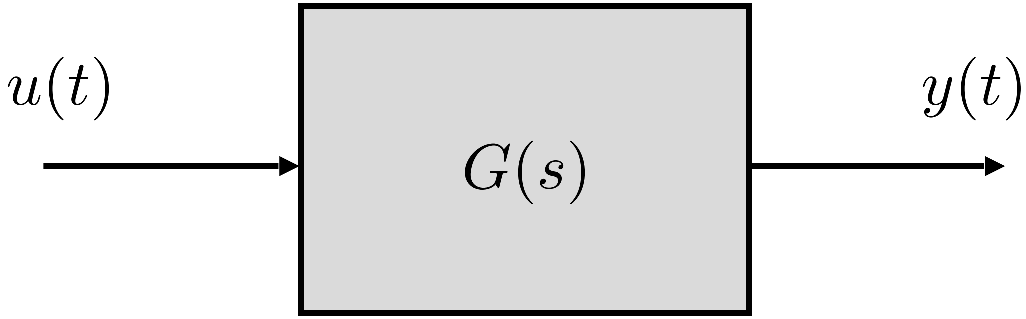
$$\boxed{Y(s) = C(sI - A)^{-1}x(0) + \left[C(sI - A)^{-1}B + D\right]U(s)}$$

Setting $x(0) = 0$:

$$Y(s) = \underbrace{\left[\overbrace{C}^{p \times n}\overbrace{(sI - A)^{-1}}^{n \times n}\overbrace{B}^{n \times m} + \overbrace{D}^{p \times m}\right]}_{p \times m} U(s)$$

$$G(s)$$

Transfer Function



Transfer Function – General Case

$$G(s) = \begin{bmatrix} G_{11}(s) & \cdots & G_{1m}(s) \\ \vdots & & \vdots \\ G_{i1}(s) & \cdots & G_{im}(s) \\ \vdots & & \vdots \\ G_{p1}(s) & \cdots & G_{pm}(s) \end{bmatrix}$$

$$\begin{aligned} Y_i(s) &= \sum_{j=1}^{m} G_{ij}(s) U_j(s) \\ &= G_{i1}(s)U_1(s) + G_{i2}(s)U_2(s) + \cdots \end{aligned}$$

Hence:

$$\begin{cases} x(0) = 0 \\ u_k(t) = 0, \ k \neq j \end{cases} \Rightarrow \quad G_{ij}(s) = \frac{Y_i(s)}{U_j(s)}$$

Transfer Function – SISO Case

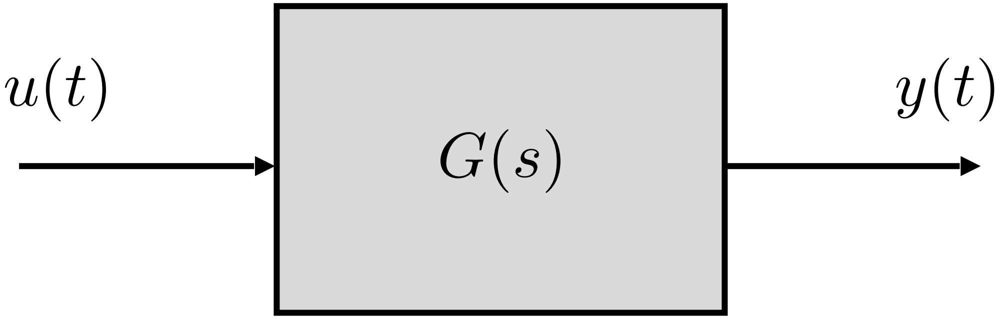
In the Single-Input Single-Output (SISO) case:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \qquad u(t),\, y(t) \in \mathbb{R} \quad x(0) = 0$$

$$Y(s) = \underbrace{\left[\overbrace{C}^{1\times n}\overbrace{(sI - A)^{-1}}^{n\times n}\overbrace{B}^{n\times 1} + \overbrace{D}^{1\times 1}\right]}_{1\times 1} U(s)$$

$G(s)$

Scalar Transfer Function



Transfer Function from Equivalent State Equations

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Recall from Slide 2-40:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \qquad \begin{array}{l} x \in \mathbb{R}^n \\ u \in \mathbb{R}^m \\ y \in \mathbb{R}^p \end{array}$$

Letting:

$$x = T\hat{x}, \quad T \in \mathbb{R}^{n \times n}, \ \det(T) \neq 0 \quad \Rightarrow \quad \hat{x} = T^{-1}x$$

$$\begin{cases} \dot{\hat{x}} = T^{-1}(Ax + Bu) = \underbrace{T^{-1}AT}_{\widehat{A}}\hat{x} + T^{-1}Bu \\ y = \underbrace{CT}_{\widehat{C}}\hat{x} + Du \end{cases}$$

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \Leftrightarrow \begin{cases} \dot{\hat{x}} = \widehat{A}\hat{x} + \widehat{B}u \\ y = \widehat{C}\hat{x} + Du \end{cases}$$

Transfer Function from Equivalent State Equations (contd.)

$$
\begin{aligned}
\widehat{G}(s) &= \widehat{C}(sI - \widehat{A})^{-1}\widehat{B} + \widehat{D} \\
&= C\left[T^{-1}\left(sI - TAT^{-1}\right)^{-1}T\right]B + D \\
&= C\left[T^{-1}\left(sTT^{-1} - TAT^{-1}\right)^{-1}T\right]B + D \\
&= C\left[T^{-1}\left(T(sI - A)T^{-1}\right)^{-1}T\right]B + D \\
&= C\left[T^{-1}T\left(sI - A\right)^{-1}T^{-1}T\right]B + D \\
&= C\left[(sI - A)^{-1}\right]B + D \\
&= G(s)
\end{aligned}
$$

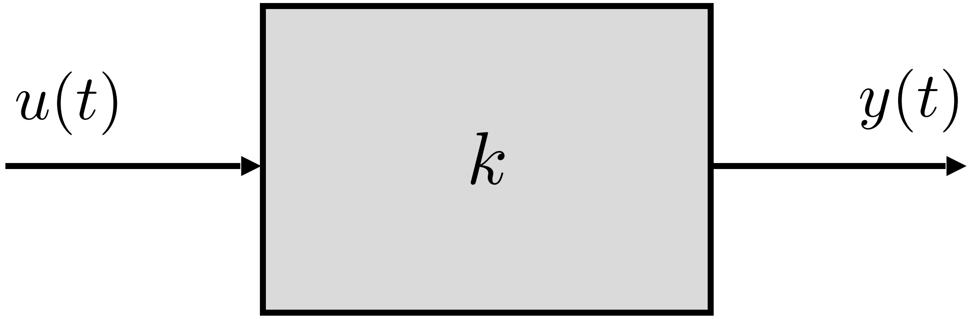
$\Rightarrow$ The transfer function is **unique** and is **not related** to the specific internal state representation

Example 1: Constant Gain

$$y(t) = ku(t)$$

$$Y(s) = kU(s)$$

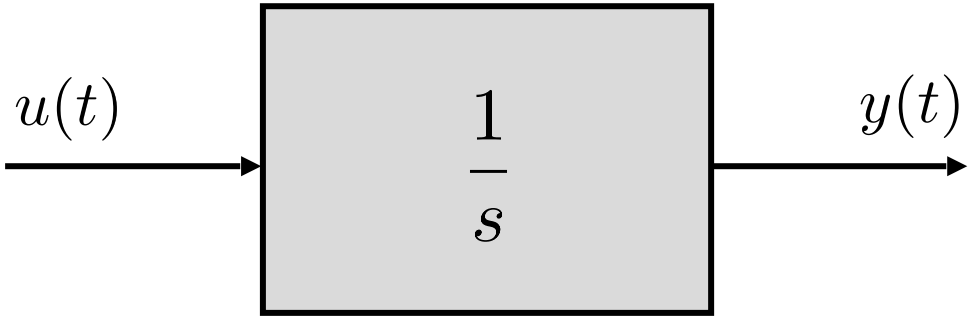
$$G(s) = k$$



Example 2: Integrator

$$\begin{cases} \dot{x} = u \\ y = x \end{cases} \qquad A = 0;\ B = 1;\ C = 1;\ D = 0$$

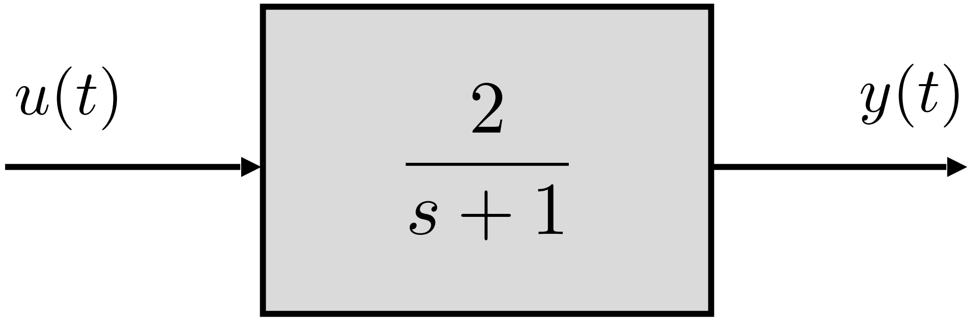
$$G(s) = C\left[(sI - A)^{-1}\right]B + D = \frac{1}{s}$$



Example 3: First-Order System

$$\begin{cases} \dot{x} = -x + u \\ y = 2x \end{cases} \qquad A = -1;\ B = 1;\ C = 2;\ D = 0$$

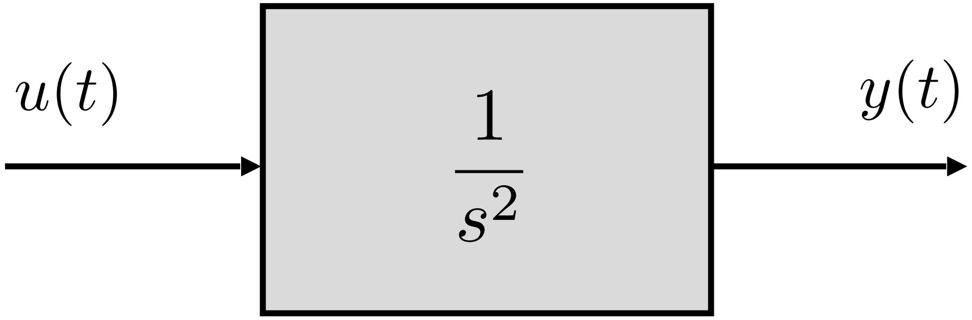
$$G(s) = C\left[(sI - A)^{-1}\right]B + D = \frac{2}{s+1}$$



Example 4: Double Integrator

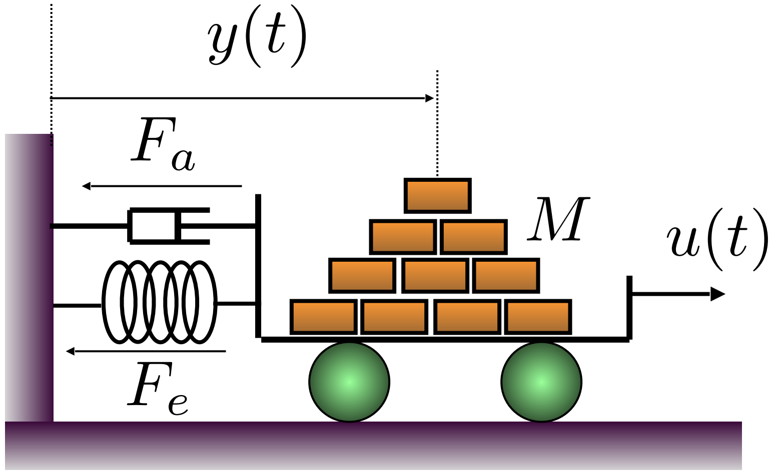
$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \\ y = x_1 \end{cases} \qquad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \; C = \begin{bmatrix} 1 & 0 \end{bmatrix}; \; D = 0$$

$$\begin{aligned} G(s) &= C\left[(sI - A)^{-1}\right] B + D \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ 0 & s \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{1}{s^2} \begin{bmatrix} s & 1 \\ 0 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \frac{1}{s^2} \end{aligned}$$



Example 5: Mechanical System

Livescripts in MS Teams: see Part 4



$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\dfrac{k}{M} x_1 - \dfrac{h}{M} x_2 + \dfrac{1}{M} u \\ y = x_1 \end{cases}$$

$$\begin{cases} \dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{M} & -\frac{h}{M} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} u \\ y = [1 \ \ 0]\, x \end{cases}$$

$$(sI - A)^{-1} = \begin{bmatrix} s & -1 \\ \frac{k}{M} & s + \frac{h}{M} \end{bmatrix}^{-1} = \frac{1}{s^2 + \frac{h}{M} s + \frac{k}{M}} \begin{bmatrix} s + \frac{h}{M} & 1 \\ -\frac{k}{M} & s \end{bmatrix}$$

$$G(s) = C\left[(sI - A)^{-1}\right] B + D$$

$$= \frac{1}{s^2 + \frac{h}{M} s + \frac{k}{M}} [1 \ \ 0] \begin{bmatrix} s + \frac{h}{M} & 1 \\ -\frac{k}{M} & s \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} = \frac{1}{M s^2 + h s + k}$$

Properties of Transfer Functions – SISO Case

From the definition:

$$G(s) = C\left[(sI - A)^{-1}\right]B + D$$

We have:

$$(sI - A)^{-1} = \begin{bmatrix} s - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & s - a_{22} & & \vdots \\ \vdots & & \ddots & \\ -a_{n1} & \cdots & & s - a_{nn} \end{bmatrix}^{-1}$$

$$= \frac{1}{\det(sI - A)} K(s)$$

Properties of Transfer Functions – SISO Case (contd.)

Let's analyse the polynomial $\det(sI - A)$ and the matrix $K(s)$:

- The polynomial $\varphi(s) = \det(sI - A)$ has degree equal to n
- The matrix $K(s)$ is given by:

$$K(s) = \mathrm{adj}(sI - A) = [k_{ij}(s),\, i,j = 1,\ldots,n]$$

where $k_{ij}(s),\, i,j = 1,\ldots,n$ are polynomials of degree $< n$
and $\mathrm{adj}(sI - A)$ is the transpose of the cofactor matrix

Hence:

$$C\,(sI - A)^{-1} B = \frac{1}{\det(sI - A)} \underbrace{CK(s)B}_{M(s)} = \frac{M(s)}{\varphi(s)}$$

where $M(s)$ is a polynomial of degree $< n$

Therefore: $$G(s) = C\,(sI - A)^{-1}\,B + D = \frac{M(s)}{\varphi(s)} + D$$

$$= \frac{M(s) + D\varphi(s)}{\varphi(s)} = \frac{N(s)}{\varphi(s)}$$

where, in the **absence of common factors** among $N(s)$ and $\varphi(s)$:

- if $D = 0$ (**strictly proper system**) the polynomial $N(s)$ has degree $< n$
- otherwise (**non strictly proper system**) the polynomial $N(s)$ has degree n

In the presence of **common factors** among $N(s)$ and $\varphi(s)$ they "cancel out" and the degrees of these polynomials **decrease** accordingly.

Hidden internal dynamics that is not represented by $G(s)$

"Hidden-Dynamics": Example 1

$$\begin{cases} \dot{x} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ \\ y = [1 \;\; 1]\, x \end{cases} \qquad n = 2$$

$$G(s) = [1 \;\; 1] \begin{bmatrix} s-1 & 0 \\ -1 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= [1 \;\; 1] \frac{1}{(s-1)(s+1)} \begin{bmatrix} s+1 & 0 \\ 1 & s-1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{\cancel{(s-1)}}{\cancel{(s-1)}(s+1)} = \frac{1}{s+1}$$

the denominator of $G(s)$ has degree $1 < 2 = n$

"Hidden-Dynamics": Example 1 (contd.)

Moreover:

$$\begin{cases} \dot{x}_1 = x_1 \\ \dot{x}_2 = x_1 - x_2 + u \\ y = x_1 + x_2 \end{cases} \quad \text{with} \quad x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Since $x_1(0) = 0$ from $\dot{x}_1 = x_1$ ⟹ $x_1(t) = 0, \forall\, t \geq 0$

↳

$$\begin{cases} \dot{x}_2 = \cancel{x_1} - x_2 + u = -x_2 + u \\ y = \cancel{x_1} + x_2 = x_2 \end{cases}$$

↳ The dynamics of $x_1(t)$ **does not show up** (it is "**hidden**") and this is consistent with the fact that

$$G(s) = \frac{1}{s+1}$$

"Hidden-Dynamics": Example 2

$$\begin{cases} \dot{x} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u \\ \\ y = [0 \;\; 1]\, x \end{cases} \qquad n = 2$$

$$G(s) = [0 \;\; 1] \begin{bmatrix} s-1 & -1 \\ 0 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= [0 \;\; 1] \frac{1}{(s-1)(s+1)} \begin{bmatrix} s+1 & 1 \\ 0 & s-1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{\cancel{(s-1)}}{\cancel{(s-1)}(s+1)} = \frac{1}{s+1}$$

the denominator of $G(s)$ has degree $1 < 2 = n$

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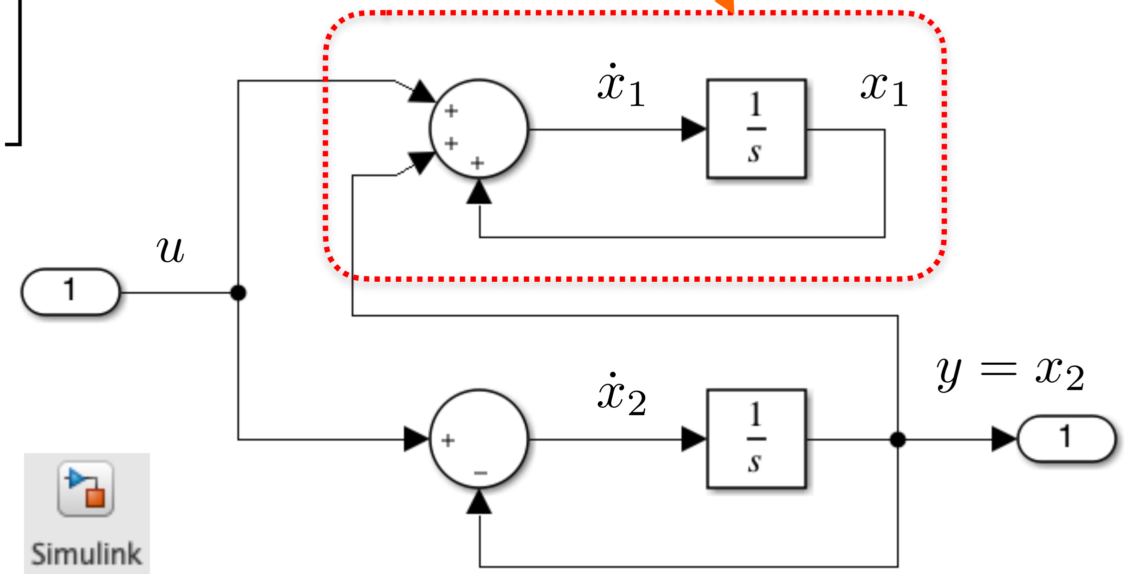
Moreover:

$$\begin{cases} \dot{x}_1 = x_1 + x_2 + u \\ \boxed{\begin{array}{l} \dot{x}_2 = -x_2 + u \\ y = x_2 \end{array}} \end{cases} \quad \text{with} \quad x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$G(s) = \frac{1}{s+1}$$

this dynamics does not influence the input/output relationship



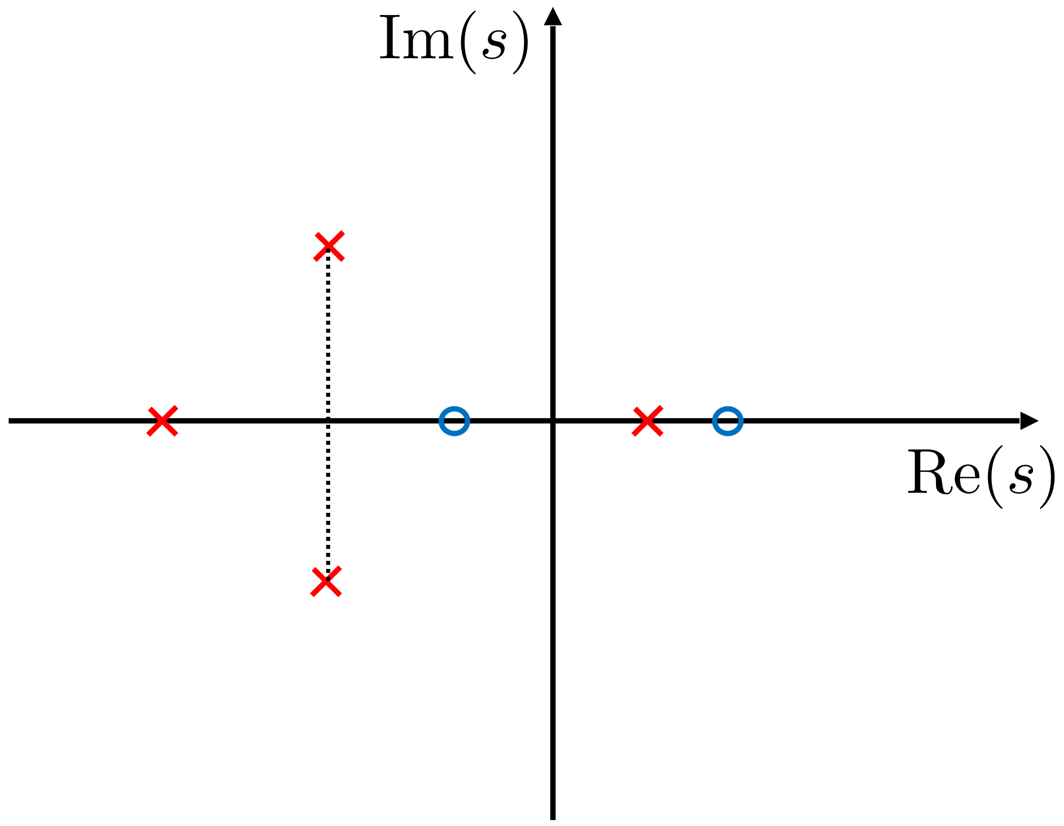
The dynamics of $x_2(t)$ **is not influenced** by the time-evolution of $x_1(t)$ (which is "**hidden**") and this is consistent with the fact that

$$G(s) = \frac{1}{s+1}$$

SISO Transfer Functions – Poles and Zeros

$$G(s) = \frac{N(s)}{\varphi(s)}$$

- Poles: roots of polynomial $\varphi(s)$ $\times$
- Zeros: roots of polynomial $N(s)$ $\circ$



Poles and Zeros – Basic Properties in the SISO case

- Poles of $G(s)$ are also eigenvalues of system's matrix A
- An eigenvalue of system's matrix A might not be a pole of $G(s)$ in case of **common factors** among polynomials $N(s)$ and $\varphi(s)$ (as shown by the examples)
- Stability depends on the poles of $G(s)$:

Asymptotic Stability $\Longleftrightarrow$ $\mathrm{Re}\,(poles) < 0$

in the absence of common factors

- The number of zeros is less or equal to the number of poles

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(A) Example with **real** zeros and poles:

$$G(s) = \frac{4s^2 + 12s}{s^4 + 3s^3 + 2s^2}$$

Parameters:

$$\beta_2 = 4,\ \beta_1 = 12,\ \beta_0 = 0$$
$$\alpha_4 = 1,\ \alpha_3 = 3,\ \alpha_2 = 2,\ \alpha_1 = \alpha_0 = 0$$

$$= 4\,\frac{s(s+3)}{s^2(s+1)(s+2)}$$

Parameters:

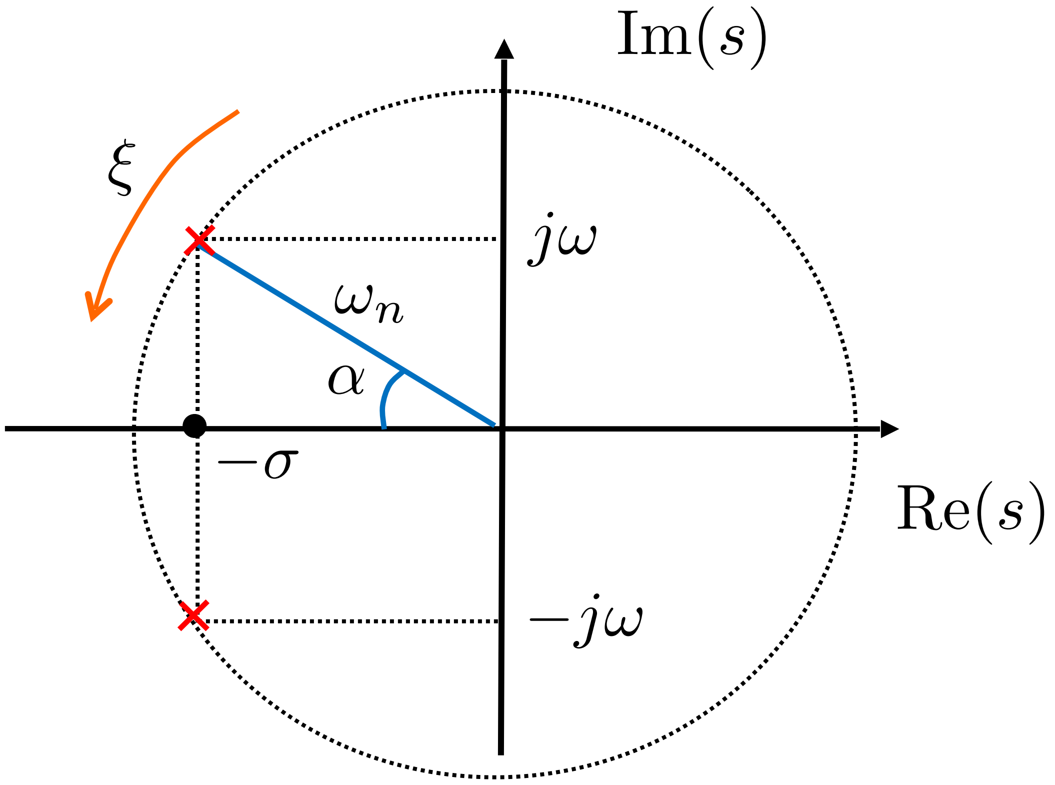
$$\varrho = 4,\ z_1 = -3$$
$$p_1 = 0,\ p_2 = -1,\ p_3 = -2$$

$$= \frac{1}{s}\,\frac{4 \cdot 3}{1 \cdot 2}\,\frac{\left(1 + \frac{s}{3}\right)}{(1+s)\left(1 + \frac{s}{2}\right)}$$

Parameters:

$$\mu = 6,\ T_1 = 1/3$$
$$\tau_1 = 1,\ \tau_2 = 1/2$$

(B) The parametric form in the presence of **complex** zeros/poles is **different**:



$$\omega_n^2 = \sigma^2 + \omega^2$$

$$\omega_n \xi = \sigma$$

$$\omega_n \sqrt{1-\xi^2} = \omega$$

Parameters:

ω_n natural angular frequency:

$\xi = \cos(\alpha)$ damping ratio

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Moreover, we have:

$$G(s) = \frac{\varrho}{(s+\sigma+j\omega)(s+\sigma-j\omega)} = \frac{\varrho}{(s+\sigma)^2+\omega^2}$$

$$= \frac{\varrho}{s^2 + \underbrace{2\sigma}_{2\xi\omega_n} s + \underbrace{\sigma^2+\omega^2}_{\omega_n^2}} = \boxed{\frac{\varrho}{s^2+2\xi\omega_n s+\omega_n^2}}$$

$$\boxed{G(s) = \frac{\varrho/\omega_n^2}{1+\dfrac{2\xi}{\omega_n}s+\dfrac{1}{\omega_n^2}s^2} = \frac{\mu}{1+\dfrac{2\xi}{\omega_n}s+\dfrac{1}{\omega_n^2}s^2}}$$

where: $\mu := \dfrac{\varrho}{\omega_n^2}$

(B) Example with real & complex-conjugate zeros/poles:

$$G(s) = \frac{5s+15}{s^3+6s^2+10s+8}$$

Parameters:

$\beta_1 = 5,\ \beta_0 = 15$
$\alpha_3 = 1,\ \alpha_2 = 6,\ \alpha_1 = 10,\ \alpha_0 = 8$

$$= 5\,\frac{s+3}{(s+4)[(s+1-j)(s+1+j)]}$$

Parameters:

$\varrho = 5,\ z_1 = -3$
$p_1 = -4,\ p_2 = -1+j,\ p_3 = -1-j$

$$= \frac{5 \cdot 3}{4 \cdot 2}\,\frac{\left(1+\frac{s}{3}\right)}{\left(1+\frac{s}{4}\right)\left(1+s+\frac{s^2}{2}\right)}$$

Parameters:

$\mu = \frac{15}{8},\ T_1 = \frac{1}{3}$

$\tau_1 = \frac{1}{4},\ \xi = \frac{1}{\sqrt{2}},\ \omega_n = \sqrt{2}$

In general:

(a) Parameterisation using the coefficients of the polynomials $N(s), \varphi(s)$

$$G(s) = \frac{\beta_m s^m + \beta_{n-1} s^{m-1} + \cdots + \beta_1 s + \beta_0}{\alpha_n s^n + \alpha_{n-1} s^{n-1} + \cdots + \alpha_1 s + \alpha_0}$$

The parameters are the coefficients $\alpha_i,\ i = 0, \ldots, n;\ \beta_j,\ j = 0, \ldots, m$

SISO Transfer Functions – Parametric Forms (contd.)

(b) Parameterisation using poles and zeros of $G(s)$

$$G(s) = \varrho \, \frac{(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

The parameters are the transfer constant ϱ , the zeros (with opposite sign) $z_i,\ i=1,\ldots,m$ and the poles (with opposite sign) $p_j,\ j=1,\ldots,n$

(c) Parameterisation using **time-constants** when

$$z_i \in \mathbb{R},\ i = 1, \ldots, m;\ p_j \in \mathbb{R},\ j = 1, \ldots, n$$

$$G(s) = \varrho \frac{(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

$$= \varrho \frac{1}{s^g} \frac{\prod_{i=1}^{m} z_i \left(\frac{s}{z_i}+1\right)}{\prod_{i=1}^{n} p_i \left(\frac{s}{p_i}+1\right)} = \frac{1}{s^g}\, \varrho \frac{\prod_{i=1}^{m} z_i}{\prod_{i=1}^{n} p_i} \frac{\prod_{i=1}^{m}\left(1+\frac{s}{z_i}\right)}{\prod_{i=1}^{n}\left(1+\frac{s}{p_i}\right)} = \mu \frac{1}{s^g} \frac{\prod_{i=1}^{m}(1+T_i s)}{\prod_{i=1}^{n}(1+\tau_i s)}$$

Hence, in the case $z_i \in \mathbb{R},\ i = 1, \dots, m;\ p_j \in \mathbb{R},\ j = 1, \dots, n$:

$$G(s) = \mu \, \frac{1}{s^g} \, \frac{\prod_{i=1}^{m} (1 + T_i s)}{\prod_{i=1}^{n} (1 + \tau_i s)}$$

The parameters are:

- **Gain:** $\mu := \varrho \, \frac{\prod_{i=1}^{m} z_i}{\prod_{i=1}^{n} p_i}$
- **Time constants:** $\frac{1}{z_i} = T_i \,;\ \frac{1}{p_i} = \tau_i$
- **Type of the system:** $g = (\text{number of poles in } s = 0) - (\text{number of zeros in } s = 0)$

Hence, the parameterisation using **time-constants** when some/all zero(s)/pole(s) are complex, that is, when

$$z_i \in \mathbb{R},\ l = 1, \ldots, m_R;\ z_i \in \mathbb{C},\ h = 1, \ldots, m_C;$$
$$p_j \in \mathbb{R},\ i = 1, \ldots, n_R;\ p_j \in \mathbb{C},\ k = 1, \ldots, n_C$$

takes on the form:

$$G(s) = \frac{1}{s^g}\,\varrho\,\frac{\prod_l z_l}{\prod_i p_i}\,\frac{\prod_h \alpha_{nh}^2}{\prod_k \omega_{nk}^2}\,\frac{\prod_l \left(1+\frac{s}{z_l}\right)}{\prod_i \left(1+\frac{s}{p_i}\right)}\,\frac{\prod_h \left(1+\frac{2\zeta_h}{\alpha_{nh}}s+\frac{1}{\alpha_{nh}^2}s^2\right)}{\prod_k \left(1+\frac{2\xi_k}{\omega_{nk}}s+\frac{1}{\omega_{nk}^2}s^2\right)}$$

$$= \mu\,\frac{1}{s^g}\,\frac{\prod_l \left(1+\frac{s}{z_l}\right)}{\prod_i \left(1+\frac{s}{p_i}\right)}\,\frac{\prod_h \left(1+\frac{2\zeta_h}{\alpha_{nh}}s+\frac{1}{\alpha_{nh}^2}s^2\right)}{\prod_k \left(1+\frac{2\xi_k}{\omega_{nk}}s+\frac{1}{\omega_{nk}^2}s^2\right)}$$

SISO Transfer Functions – Parametric Forms (contd.)

Hence, in the case $z_i \in \mathbb{R},\ l = 1, \ldots, m_R;\ z_i \in \mathbb{C},\ h = 1, \ldots, m_C;$
$p_j \in \mathbb{R},\ i = 1, \ldots, n_R;\ p_j \in \mathbb{C},\ k = 1, \ldots, n_C$

$$G(s) = \mu \, \frac{1}{s^g} \, \frac{\prod_l \left(1 + \frac{s}{z_l}\right)}{\prod_i \left(1 + \frac{s}{p_i}\right)} \, \frac{\prod_h \left(1 + \frac{2\zeta_h}{\alpha_{nh}} s + \frac{1}{\alpha_{nh}^2} s^2\right)}{\prod_k \left(1 + \frac{2\xi_k}{\omega_{nk}} s + \frac{1}{\omega_{nk}^2} s^2\right)}$$

and the parameters are:

- **Gain:** $$\mu := \varrho \, \frac{\prod_l z_l}{\prod_i p_i} \, \frac{\prod_h \alpha_{nh}^2}{\prod_k \omega_{nk}^2}$$
- **Time constants:** $\frac{1}{z_i} = T_i\,;\ \frac{1}{p_i} = \tau_i$
- **Damping ratios:** ζ_h, ξ_k
- **Natural angular frequencies:** $\alpha_{nh}^2, \omega_{nk}^2$
- **Type of the system:** $g = (\text{number of poles in } s = 0) - (\text{number of zeros in } s = 0)$

Static Gain and SISO Transfer Functions

Recall from Part 2:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \qquad \begin{array}{l} x \in \mathbb{R}^n \\ u \in \mathbb{R}^m \\ y \in \mathbb{R}^p \end{array} \qquad u(t) = \bar{u}, t \geq 0$$

$$\det(A) \neq 0 \quad \Rightarrow \quad \bar{x} = -A^{-1}B\bar{u}$$ one and only one equilibrium state

$$\bar{y} = C\bar{x} + D\bar{u} = -CA^{-1}B\bar{u} + D\bar{u} = \underbrace{\left(-CA^{-1}B + D\right)}_{\mu}\bar{u}$$

$$u(t) = \bar{u} \longrightarrow \boxed{\mu} \longrightarrow y(t) = \bar{y}$$

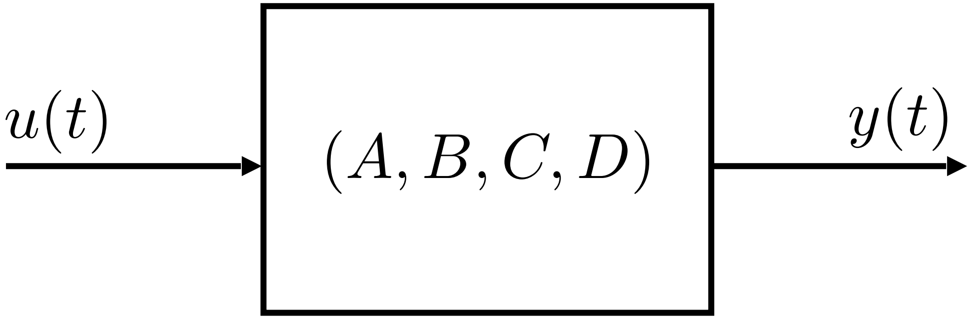
Static Gain

Hence, for **systems of 0-type**, that is when $g = 0$:

$$\boxed{\mu = \left(-CA^{-1}B + D\right)\bar{u} = G(s)|_{s=0} = G(0)}$$

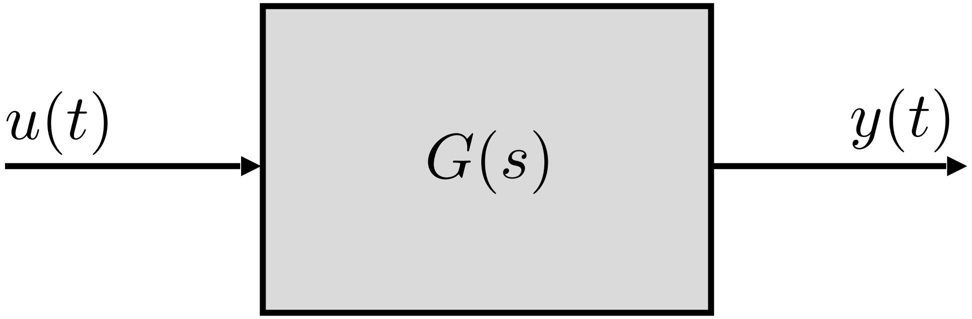
Internal vs. External Representations of Linear Systems

Internal – State Equations



$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

External – Transfer Functions



$$G(s) = \left[C(sI - A)^{-1}B + D\right]$$

$$Y(s) = G(s)U(s) \quad \text{with} \quad x(0) = 0$$

? Realisation Problem

Realisation Problem: from Transfer Function to State Equations

Livescripts in MS Teams: see Part 4

Recall from Slide 2-40:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \qquad \begin{array}{l} x \in \mathbb{R}^n \\ u \in \mathbb{R}^m \\ y \in \mathbb{R}^p \end{array}$$

Letting: $x = T\hat{x}, \quad T \in \mathbb{R}^{n\times n}, \det(T) \neq 0 \Rightarrow \hat{x} = T^{-1}x$

$$\begin{cases} \dot{\hat{x}} = T^{-1}(Ax + Bu) = \underbrace{T^{-1}AT}_{\widehat{A}}\hat{x} + T^{-1}Bu \\ y = \underbrace{CT}_{\widehat{C}}\hat{x} + Du \end{cases}$$

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \Leftrightarrow \begin{cases} \dot{\hat{x}} = \widehat{A}\hat{x} + \widehat{B}u \\ y = \widehat{C}\hat{x} + Du \end{cases} \Rightarrow$$

Infinite number of **equivalent** state equations representing the **same** dynamic system

Realisation Problem for SISO Systems: Control Form

Consider the generic **strictly proper** transfer function

$$G(s) = \frac{\beta_m s^m + \beta_{m-1} s^{m-1} + \cdots + \beta_1 s + \beta_0}{s^n + \alpha_{n-1} s^{n-1} + \cdots + \alpha_1 s + \alpha_0} \quad \text{with} \quad m < n$$

Then:

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 & \cdots & -\alpha_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u \\ \\ y = \begin{bmatrix} \beta_0 & \beta_1 & \cdots & \beta_m & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} \end{cases}$$

Realisation Problem for SISO Systems: Observation Form

Consider the generic **strictly proper** transfer function

$$G(s) = \frac{\beta_m s^m + \beta_{m-1} s^{m-1} + \cdots + \beta_1 s + \beta_0}{s^n + \alpha_{n-1} s^{n-1} + \cdots + \alpha_1 s + \alpha_0} \quad \text{with} \quad m < n$$

Then:

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -\alpha_0 \\ 1 & 0 & \cdots & 0 & -\alpha_1 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & -\alpha_{n-2} \\ 0 & \cdots & 0 & 1 & -\alpha_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_m \\ 0 \\ \vdots \\ 0 \end{bmatrix} u \\ \\ y = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} \end{cases}$$

Realisation Problem for SISO Systems: Remarks

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- Control and Observation Forms are perfectly equivalent in our context
- Many other solutions to the realisation problem are available
- The solution of the realisation problem in the MIMO case is more complicated and is **not dealt with in this introductory course**
- When the transfer function is **not strictly proper,** that is $m = n$, we have:

$$G(s) = \frac{\beta_n s^n + \beta_{n-1} s^{n-1} + \cdots + \beta_1 s + \beta_0}{s^n + \alpha_{n-1} s^{n-1} + \cdots + \alpha_1 s + \alpha_0} = \beta_n + \widetilde{G}(s)$$

where

$$\widetilde{G}(s) = \frac{\gamma_{n-1} s^{n-1} + \gamma_{n-2} s^{n-2} + \cdots + \gamma_1 s + \gamma_0}{s^n + \alpha_{n-1} s^{n-1} + \cdots + \alpha_1 s + \alpha_0}$$

where $\gamma_0, \gamma_1, \ldots, \gamma_{n-2}, \gamma_{n-1}$ are obtained via polynomial division. Then, the scalar D of the state equations is given by

$$D = \beta_n$$

and the realisation of $\widetilde{G}(s)$ can be obtained as in the Control and Observation Forms