034IN - FONDAMENTI DI AUTOMATICA - FUNDAMENTALS OF AUTOMATIC CONTROL
A.Y. 2023-2024
Part II: Fundamentals of Systems Theory

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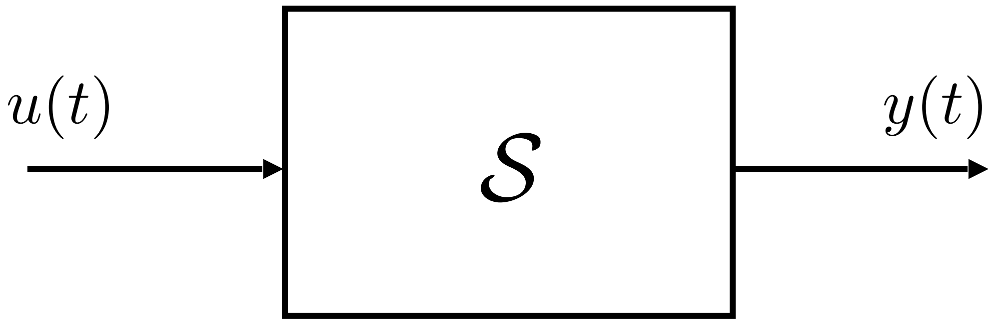
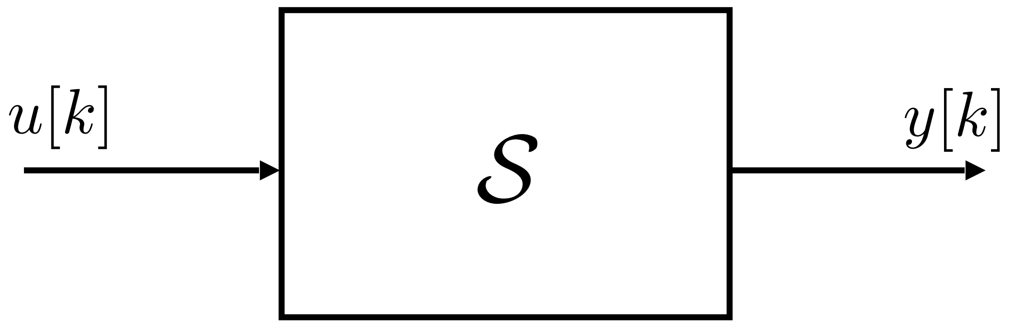
Continuous-time Case



Discrete-time Case



Input

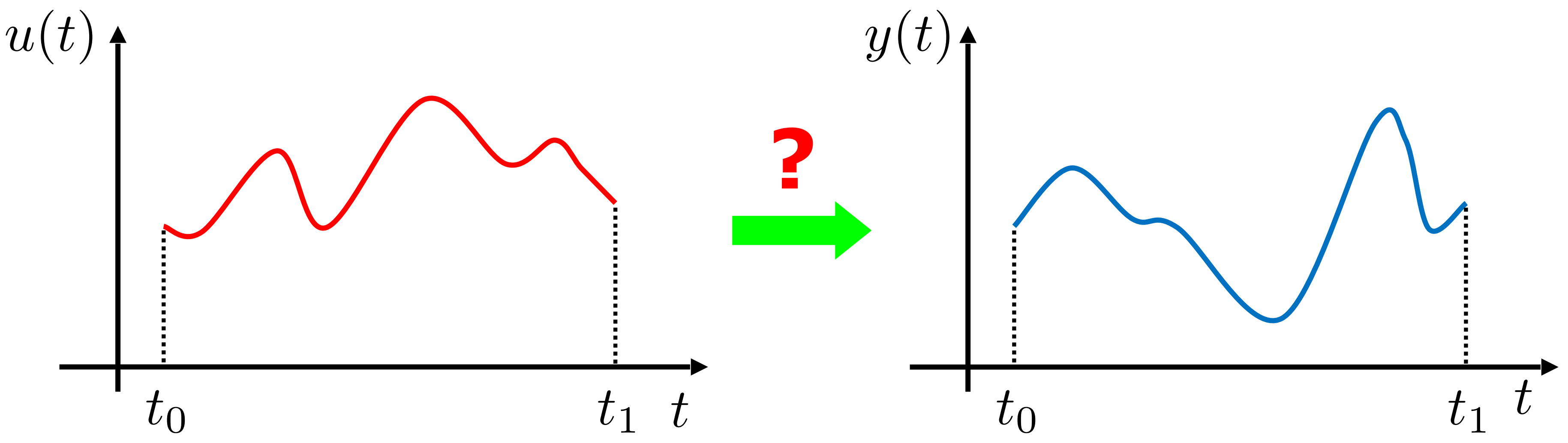
$$u(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ u_m(t) \end{bmatrix} \in \mathbb{R}^m \qquad u[k] = \begin{bmatrix} u_1[k] \\ \vdots \\ u_m[k] \end{bmatrix} \in \mathbb{R}^m$$

Output

$$y(t) = \begin{bmatrix} y_1(t) \\ \vdots \\ y_p(t) \end{bmatrix} \in \mathbb{R}^p \qquad y[k] = \begin{bmatrix} y_1[k] \\ \vdots \\ y_p[k] \end{bmatrix} \in \mathbb{R}^p$$

[**Remark**: the course is focused on **continuous-time** systems except for the **digital** design and implementation of the controller (see Part 10).]

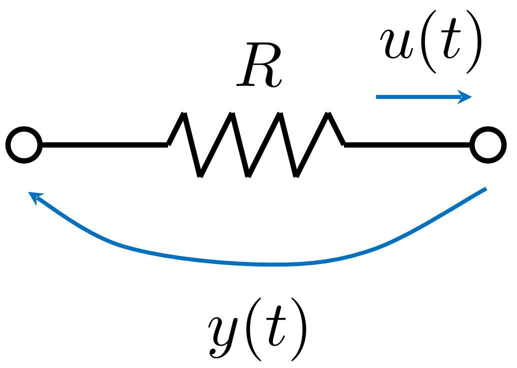
What is the meaning of “Dynamic”?



Can $y(t)$ be determined only through the knowledge of $u(t)$?

If the answer is “no,” then the system is a dynamic one

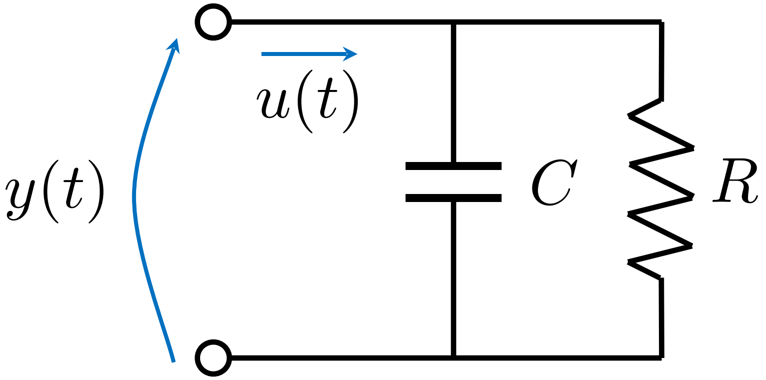
Example 1



$$y(t) = R \cdot u(t)$$

Non dynamic

Example 2

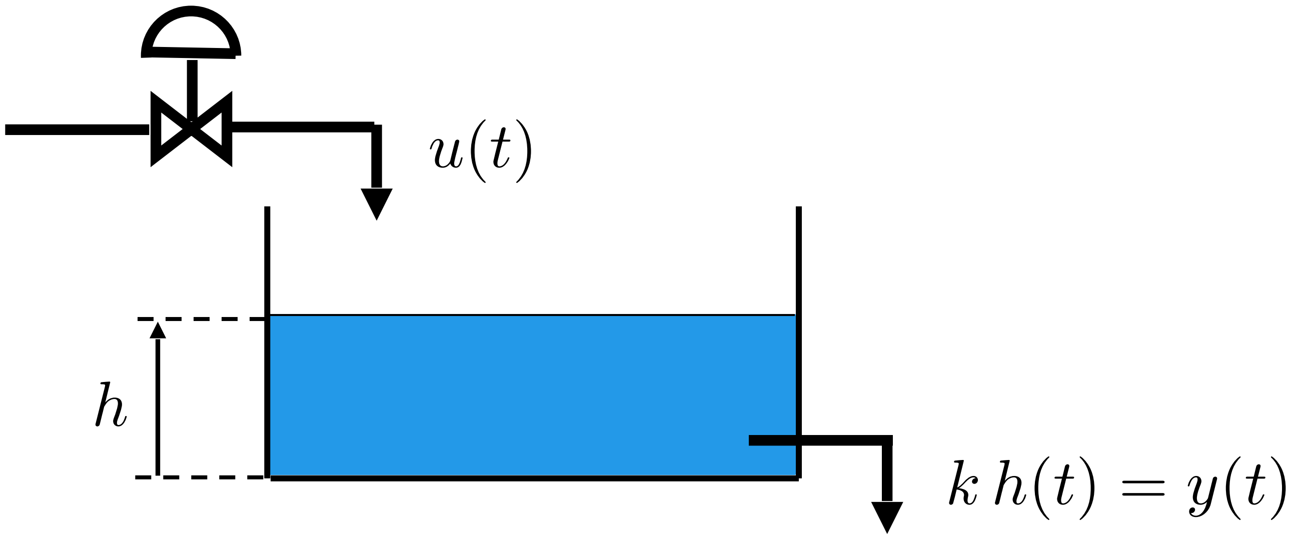


$$\left.\begin{array}{c} u(t), t \in [t_0, t_1] \\ y(t_0) \end{array}\right\}$$

Dynamic

$$y(t), t \in [t_0, t_1]$$

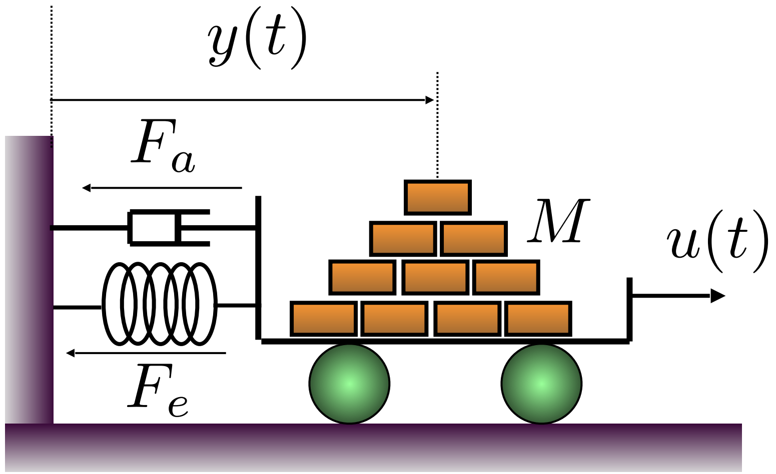
Example 3



$$\left.\begin{array}{c} u(t), t \in [t_0, t_1] \\ h(t_0) \end{array}\right\} \Rightarrow y(t), t \in [t_0, t_1]$$

Dynamic

Example 4



$$\left.\begin{array}{c} u(t), t \in [t_0, t_1] \\ y(t_0) \\ \dot{y}(t_0) \end{array}\right\} \Longrightarrow \quad y(t), t \in [t_0, t_1]$$

Dynamic

Continuous-time Dynamic Systems

Analysis and Properties

State Variables

Variables that must be known at time t_0 to determine $y(t), t \geq t_0$ from $u(t), t \geq t_0$

Order of the system

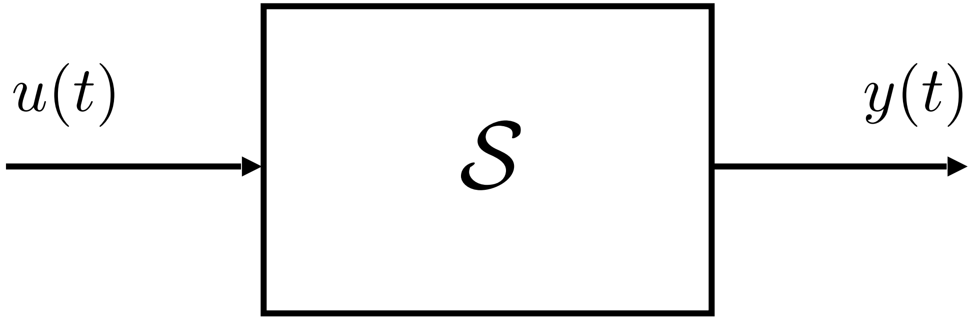
$$x_i(t),\ i = 1, 2, \ldots, n$$

State variables

$$x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} \in \mathbb{R}^n$$

State vector

State-Space Continuous-Time Dynamic Systems



$$u(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ u_m(t) \end{bmatrix} \in \mathbb{R}^m \qquad x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} \in \mathbb{R}^n \qquad y(t) = \begin{bmatrix} y_1(t) \\ \vdots \\ y_p(t) \end{bmatrix} \in \mathbb{R}^p$$

Input — State — Output

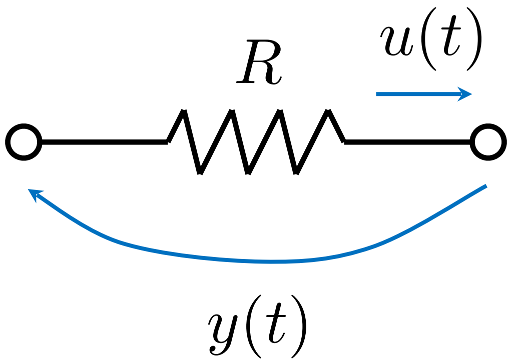
$$\begin{cases} \dot{x}(t) = f[x(t), u(t), t] \\ y(t) = g[x(t), u(t), t] \end{cases} \quad \text{where} \quad f[x(t), u(t), t] := \begin{bmatrix} f_1[x(t), u(t), t] \\ \vdots \\ f_n[x(t), u(t), t] \end{bmatrix}$$

State equations

Basic Definitions

- **Strictly proper** dynamic system:
  - ➡ if function $g(\cdot)$ does not explicitly depend on input
- **Time-invariant** (or **stationary**) dynamic system:
  - ➡ if functions $f(\cdot), g(\cdot)$ do not explicitly depend on time
- **Linear** dynamic system:
  - ➡ if functions $f(\cdot), g(\cdot)$ depend linearly on x, u
- **Single Input Single Output** (**SISO**) dynamic system:
  - ➡ if $m = p = 1$
- **Multi-Input Multi-Output** (**MIMO**) dynamic system:
  - ➡ if $m > 1$ and/or $p > 1$

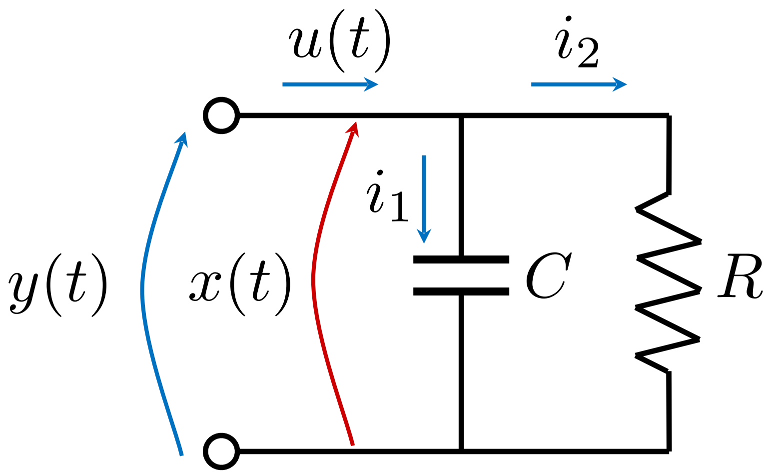
State Equations for Example 1



$$y(t) = R \cdot u(t)$$

Non dynamic $\Longrightarrow$ **No need** to introduce state variables

State Equations for Example 2

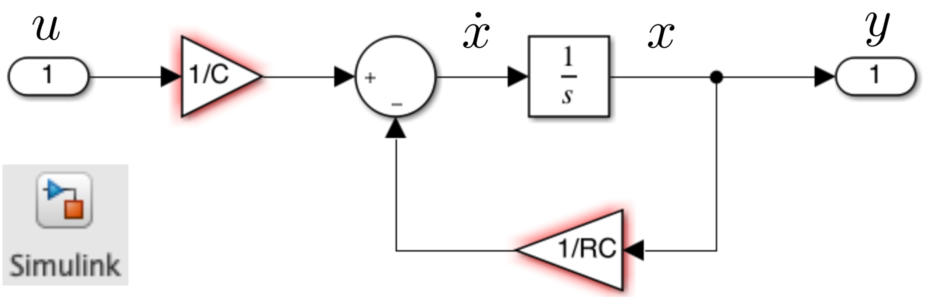


From basic physics and electrical circuit theory:

$$\begin{cases} C\dot{x} = i_1 \\ y = x = R\,i_2 \\ u = i_1 + i_2 \end{cases}$$

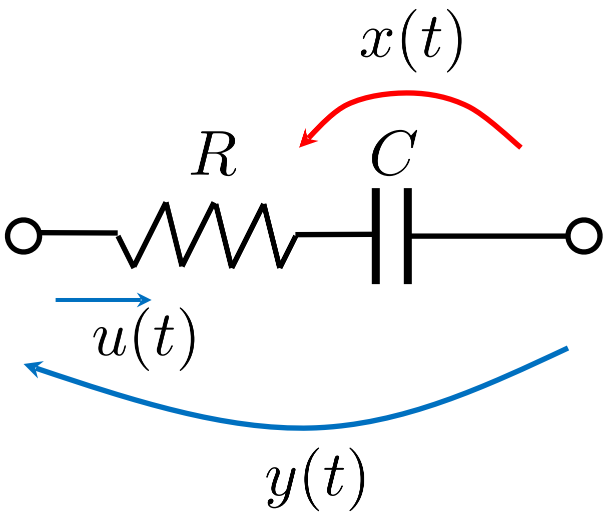
$$x = R\,i_2 = R\,(u - i_1) \Longrightarrow i_1 = u - \frac{1}{R}x \Longrightarrow \dot{x} = \frac{1}{C}i_1 = \frac{1}{C}\left(u - \frac{1}{R}x\right)$$

$$\begin{cases} \dot{x}(t) = -\dfrac{1}{RC}\,x + \dfrac{1}{C}\,u = f(x,u) \\ y = x = g(x) \end{cases}$$



- First order
- Linear
- Time-invariant
- Strictly proper
- SISO

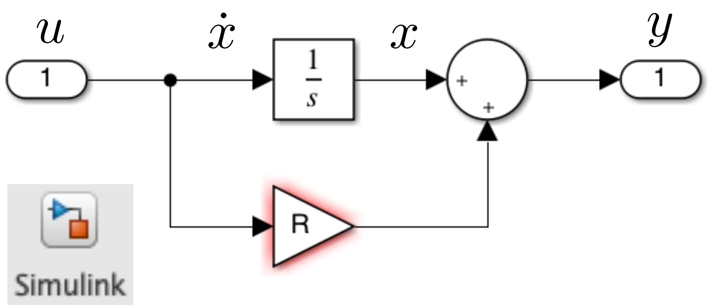
State Equations for Example 2-bis



From basic physics and electrical circuit theory:

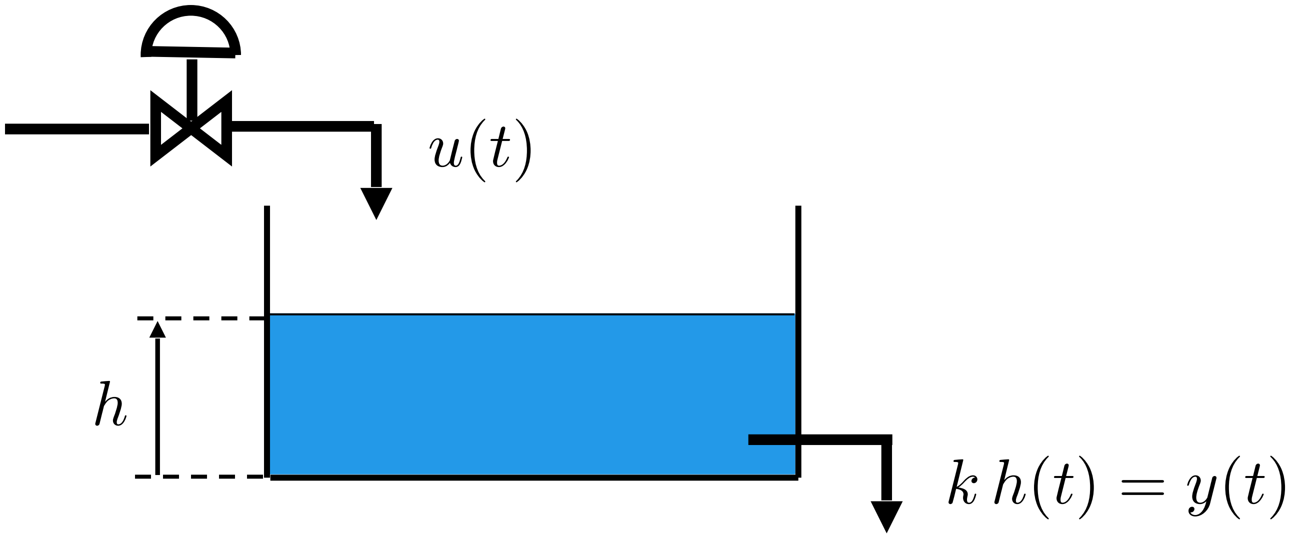
$$\begin{cases} C\dot{x} = u \\ y = x + R\,u \end{cases}$$

$$\begin{cases} \dot{x}(t) = \dfrac{1}{C}\,u = f(x,u) \\ y = x + R\,u = g(x,u) \end{cases}$$



- First order
- Linear
- Time-invariant
- Non strictly proper
- SISO

State Equations for Example 3

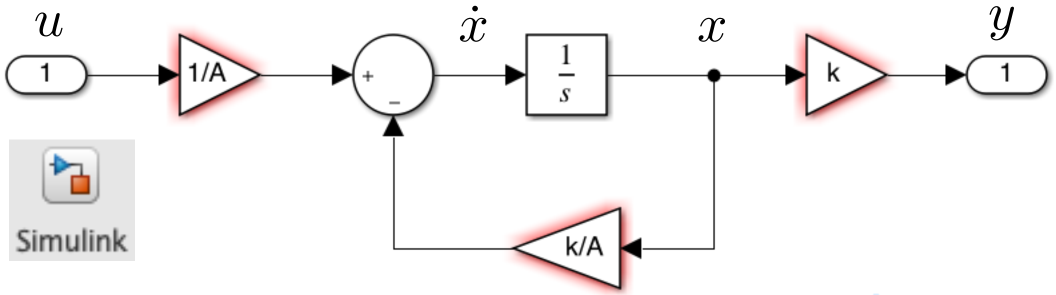


From elementary physics:

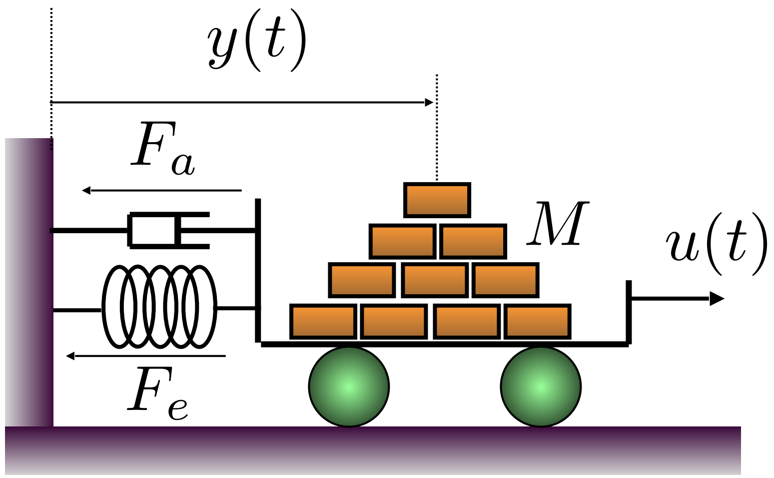
$$\begin{cases} A\,\dot{x} = u - k\,x \\ y = kx \end{cases}$$

$$\begin{cases} \dot{x}(t) = -\dfrac{k}{A}\,x + \dfrac{1}{A}\,u = f(x,u) \\ y = kx = g(x) \end{cases}$$

- First order
- Linear
- Time-invariant
- Strictly proper
- SISO



State Equations for Example 4



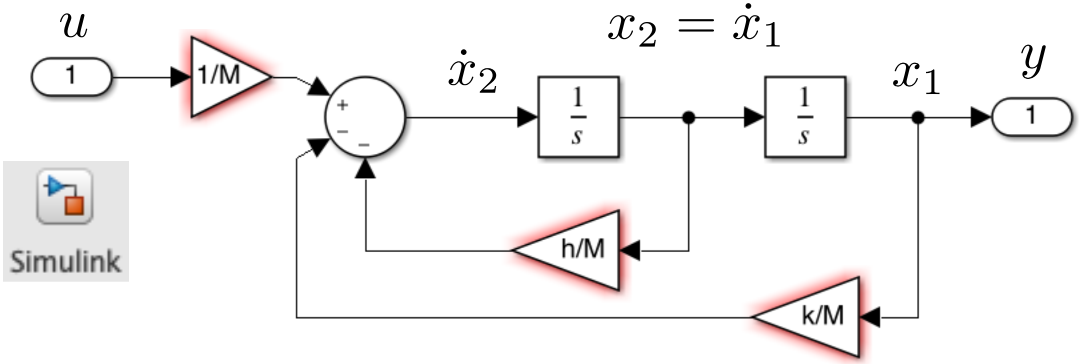
From elementary physics:

$$\begin{cases} F_e = ky \\ F_a = h\dot{y} \\ M\ddot{y} = u - k\,y - h\dot{y} \end{cases}$$

Letting:

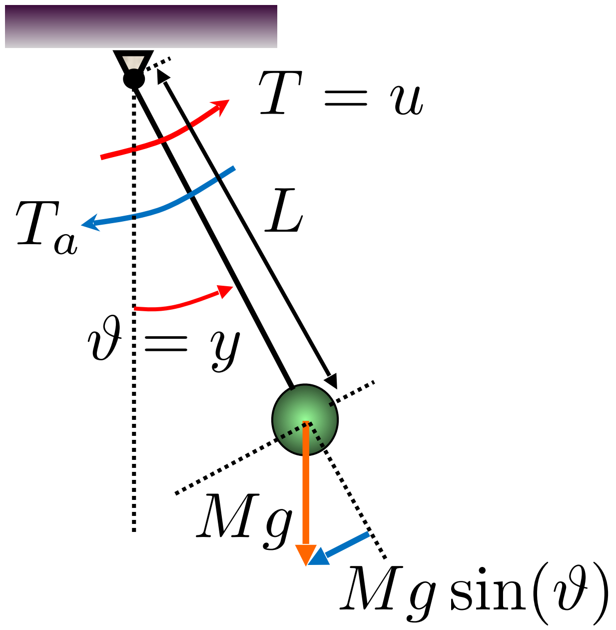
$$\begin{cases} x_1 := y \\ x_2 := \dot{y} \end{cases}, \quad x := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{cases} \dot{x}_1 = \dot{y} = x_2 = f_1(x,u) \\ \dot{x}_2 = \ddot{y} = -\dfrac{k}{M}x_1 - \dfrac{h}{M}x_2 + \dfrac{1}{M}u = f_2(x,u) \\ y = x_1 = g(x) \end{cases}$$



- Second order
- Linear
- Time-invariant
- Strictly proper
- SISO

Example 5



From elementary physics:

$$\begin{cases} T_a = h\dot{\vartheta} \\ J\ddot{\vartheta} = u - h\dot{\vartheta} - MgL\sin(\vartheta) \\ J = ML^2 \end{cases}$$

Letting:

$$\begin{cases} x_1 := \vartheta \\ x_2 := \dot{\vartheta} \end{cases}, \quad x := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{cases} \dot{x}_1 = \dot{\vartheta} = x_2 = f_1(x,u) \\ \dot{x}_2 = \ddot{\vartheta} = -\dfrac{MgL}{J}\sin(x_1) - \dfrac{h}{J}x_2 + \dfrac{1}{J}u = f_2(x,u) \\ y = x_1 = g(x) \end{cases}$$

- Second order
- Nonlinear
- Time-invariant
- Strictly proper
- SISO

Choice of the State Variables - ``Engineering” Criterion

State Variables ⟺ Quantities associated with **storage** of mass, energy, electrical charge, etc.

- **Electrical** systems:
  - ➡ voltage on capacitors, current in inductors
- **Mechanical** systems:
  - ➡ positions, velocities (linear, angular)
- **Thermal** systems:
  - ➡ temperature, enthalpy
- Etc. ...

Choice of the State Variables - ``Mathematical" Criterion

Consider a system modelled via the generic differential equation:

$$\frac{d^n y}{dt^n} = \varphi\left(\frac{d^{n-1}y}{dt^{n-1}}, \ldots, \frac{dy}{dt}, y, u\right)$$

Letting:

$$\begin{cases} x_1 := y \\ x_2 := \frac{dy}{dt} \\ \vdots \\ x_n := \frac{d^{n-1}y}{dt^{n-1}} \end{cases} \quad \text{and} \quad x := \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \Rightarrow \quad \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \vdots \\ \dot{x}_n = \varphi(x, u) \\ y = x_1 \end{cases}$$

Example

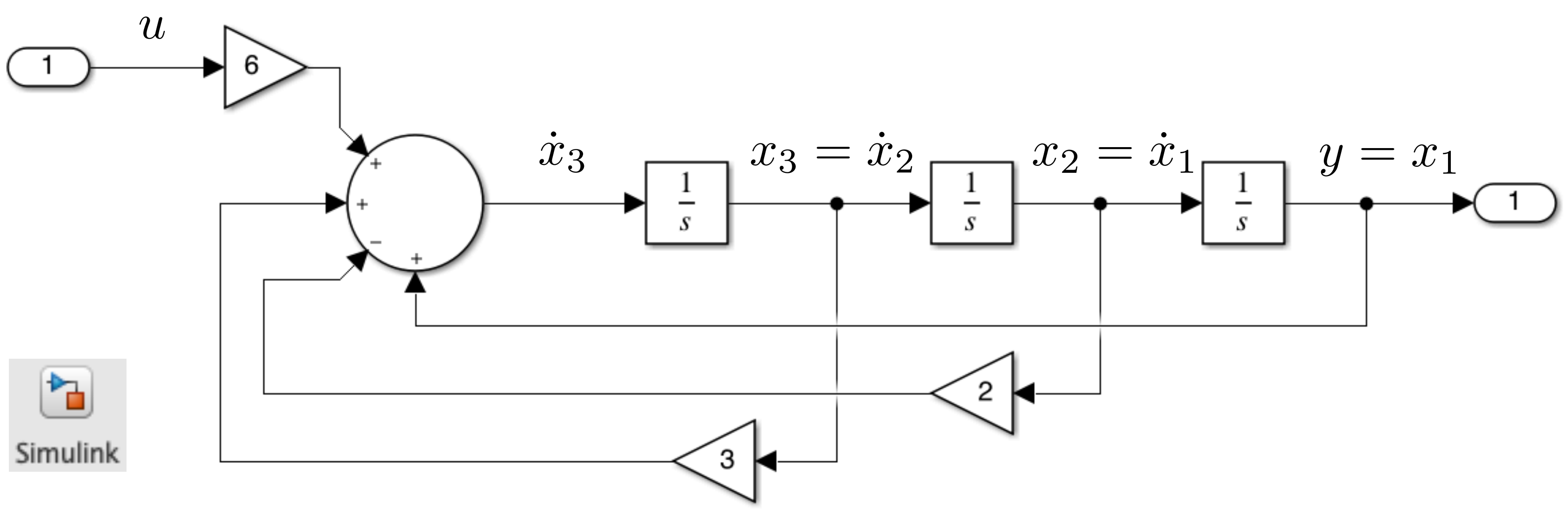
$$\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - y = 6u$$

Livescripts in MS Teams: see Part 2: STATE_SPACE_MODEL_EXAMPLES

Letting:

$$\begin{cases} x_1 := y \\ x_2 := \dfrac{dy}{dt} \\ x_3 := \dfrac{d^2y}{dt^2} \end{cases} \quad \text{and} \quad x := \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\Rightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = 3x_3 - 2x_2 + x_1 + 6u \\ y = x_1 \end{cases}$$



Simulink

Determination of State and Output Trajectories

$$\begin{cases} \dot{x} = f(x,u,t) \\ y = g(x,u,t) \end{cases} \qquad \begin{array}{l} x \in \mathbb{R}^n \\ u \in \mathbb{R}^m \\ y \in \mathbb{R}^p \end{array}$$

$$\left.\begin{array}{l} t_0 \\ x(t_0) \\ u(t),\ t \geq t_0 \end{array}\right\} \Longrightarrow \begin{array}{c} \overbrace{x(t),\ t \geq t_0}^{\text{state trajectory}} \\ \\ \underbrace{y(t),\ t \geq t_0}_{\text{output trajectory}} \end{array}$$

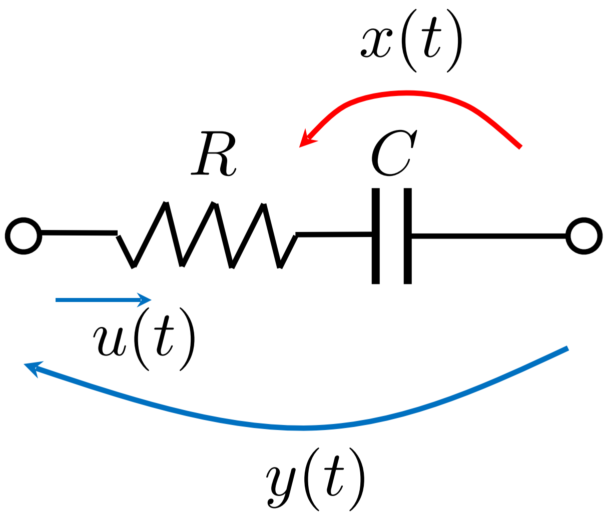
a) integration of the state equation $\Longrightarrow$ $x(t),\ t \geq t_0$

b) substitution of $x(t),\ u(t)$ into the output equation $\Longrightarrow$ $y(t),\ t \geq t_0$

For <u>time-invariant</u> systems we set $t_0 = 0$ without loss of generality

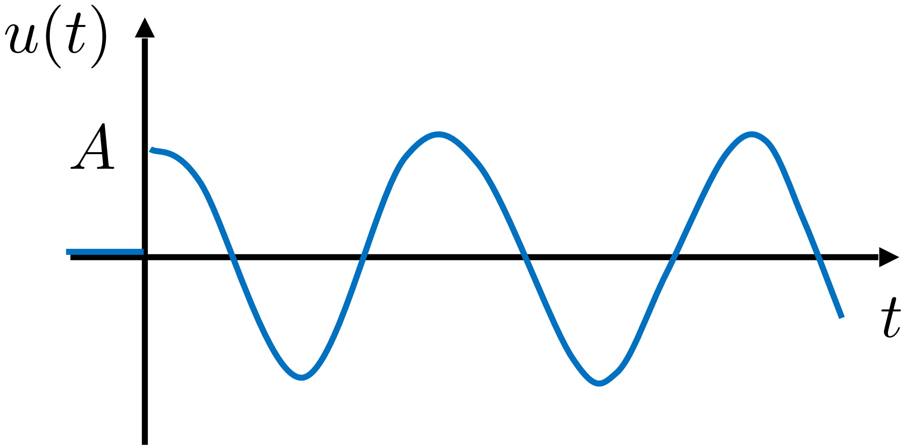
Example 1

Livescripts in MS Teams: see Part 2:
STATE_OUTPUT_TRAJ_CONTROL_TLBX,
STATE_OUTPUT_TRAJ_LAPLACE



$$\begin{cases} \dot{x}(t) = \dfrac{1}{C}\,u \\[2ex] y = x + R\,u \end{cases}$$

$$x(0) = x_0$$
$$u(t) = A\cos(\omega t)\cdot 1(t)$$



Example 1 (contd.)

a) Integration of the state equation:

$$\begin{aligned} x(t) &= x_0 + \frac{A}{C}\int_0^t \cos(\omega\tau)d\tau \\ &= x_0 + \frac{A}{C}\left.\frac{\sin(\omega\tau)}{\omega}\right|_0^t \\ &= x_0 + \frac{A}{C\omega}\sin(\omega t), t \geq 0 \end{aligned}$$

b) Substitution of $x(t)$, $u(t)$ into the output equation:

$$y(t) = x_0 + \frac{A}{C\omega}\sin(\omega t) + RA\cos(\omega t), t \geq 0$$

Example 1 (contd.)

The same result can be obtained using the **Laplace Transform**:

$$\mathcal{L}\{\dot{x}\} = \mathcal{L}\left\{\frac{1}{C}u\right\} \Rightarrow sX(s) - x_0 = \frac{1}{C}U(s)$$

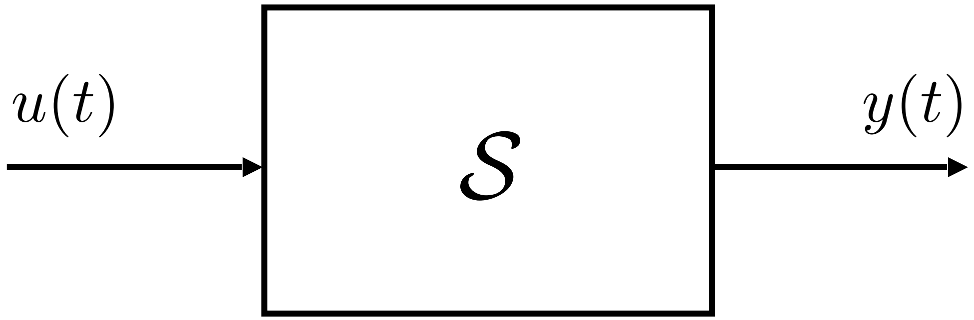
$$X(s) = x_0\frac{1}{s} + \frac{1}{sC}U(s)$$

$$= = x_0\frac{1}{s} + \frac{1}{\cancel{s}C}\frac{A\cancel{s}}{s^2+\omega^2} = x_0\frac{1}{s} + \frac{A}{C}\frac{1}{s^2+\omega^2}$$

$$= x_0\frac{1}{s} + \frac{A}{\omega C}\frac{\omega}{s^2+\omega^2}$$

$$\mathcal{L}^{-1} \quad x(t) = x_0 + \frac{A}{C\omega}\sin(\omega t), t \geq 0$$

Linear Time-Invariant Dynamic Systems



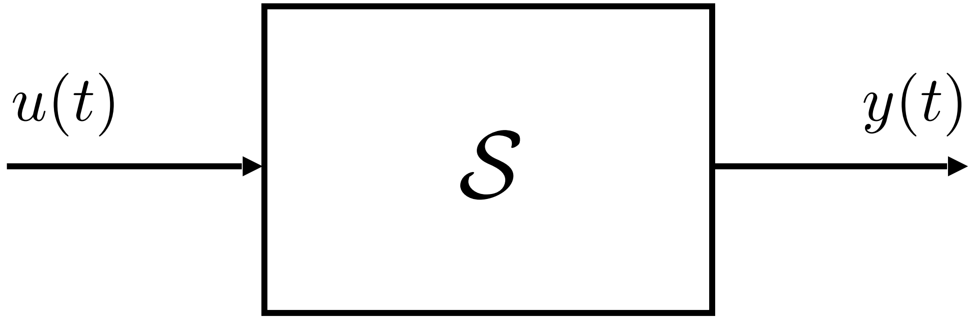
$$u(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ u_m(t) \end{bmatrix} \in \mathbb{R}^m \qquad x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} \in \mathbb{R}^n \qquad y(t) = \begin{bmatrix} y_1(t) \\ \vdots \\ y_p(t) \end{bmatrix} \in \mathbb{R}^p$$

Input — State — Output

$$\begin{cases} \dot{x} = f(x,u) \\ y = g(x,u) \end{cases}$$

System is **linear** if functions $f(\cdot), g(\cdot)$ **depend linearly** on x, u

Linear Time-Invariant SISO Dynamic Systems



$$u(t) \in \mathbb{R}$$

Scalar input

$$x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} \in \mathbb{R}^n$$

State

$$y(t) \in \mathbb{R}$$

Scalar Output

System is **linear** if functions $f(\cdot), g(\cdot)$ **depend linearly** on x, u

$$\begin{cases} \dot{x}_1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + b_1 u \\ \vdots \\ \dot{x}_n = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + b_n u \\ y = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n + du \end{cases}$$

Linear Time-Invariant SISO Dynamic Systems (contd.)

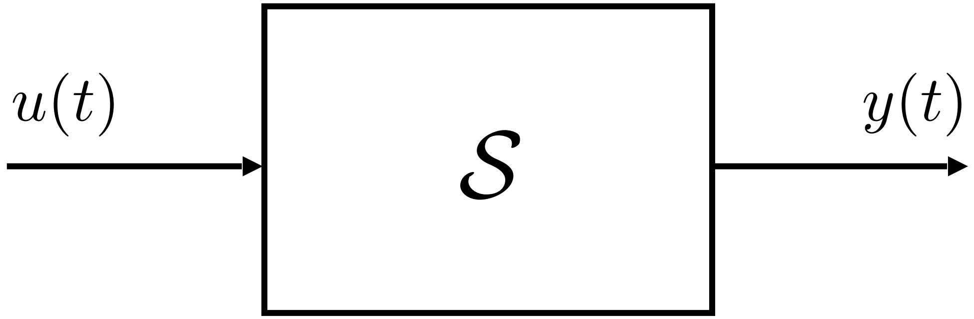
$$A = \underbrace{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}}_{n} \Bigg\} n \qquad B = \underbrace{\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}}_{1} \Bigg\} n$$

$$C = \underbrace{\begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix}}_{n} \Big\} 1 \qquad D = d \in \mathbb{R}$$

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \qquad \begin{matrix} x \in \mathbb{R}^n \\ u \in \mathbb{R} \\ y \in \mathbb{R} \end{matrix} \qquad (A, B, C, D)$$

Compact notation for linear systems

Linear Time-Invariant MIMO Dynamic Systems



$$u(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ u_m(t) \end{bmatrix} \in \mathbb{R}^m$$

Input

$$x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} \in \mathbb{R}^n$$

State

$$y(t) = \begin{bmatrix} y_1(t) \\ \vdots \\ y_p(t) \end{bmatrix} \in \mathbb{R}^p$$

Output

Linear Time-Invariant MIMO Dynamic Systems (contd.)

$$A = \underbrace{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}}_{n} \Big\} n \qquad B = \underbrace{\begin{bmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nm} \end{bmatrix}}_{m} \Big\} n$$

$$C = \underbrace{\begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{p1} & \cdots & c_{pn} \end{bmatrix}}_{n} \Big\} p \qquad D = \underbrace{\begin{bmatrix} d_{11} & \cdots & d_{1m} \\ \vdots & \ddots & \vdots \\ d_{p1} & \cdots & d_{pm} \end{bmatrix}}_{m} \Big\} p$$

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \qquad \begin{array}{l} x \in \mathbb{R}^n \\ u \in \mathbb{R}^m \\ y \in \mathbb{R}^p \end{array} \qquad (A, B, C, D)$$

Calculation of State and Output Trajectories

First, for illustration purposes, we consider the case of **scalar state**:

$$\begin{cases} \dot{x} = ax + bu \\ y = cx + du \end{cases} \qquad \begin{array}{l} x \in \mathbb{R} \\ u \in \mathbb{R} \\ y \in \mathbb{R} \end{array} \qquad \begin{array}{l} x(0) = x_0 \\ u(t),\ t \geq 0 \end{array}$$

By Laplace transformation:

$$\mathcal{L}\{\dot{x}(t)\} = \mathcal{L}\{ax + bu\} \Rightarrow sX(s) - x_0 = aX(s) + bU(s)$$

$$\Rightarrow (s-a)X(s) = x_0 + bU(s) \Rightarrow X(s) = \frac{1}{s-a}x_0 + \frac{b}{s-a}U(s)$$

Since: $\mathcal{L}[f(t) * g(t)] = F(s) \cdot G(s)\,; \quad \mathcal{L}[e^{kt}] = \dfrac{1}{s-k}$

$$\mathcal{L}^{-1} \Rightarrow x(t) = \mathcal{L}^{-1}\{X(s)\} = e^{at}x_0 + \int_0^t e^{a(t-\tau)}bu(\tau)d\tau, t \geq 0$$

Calculation of State and Output Trajectories (contd.)

Now, we consider the general case:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \qquad \begin{array}{l} x \in \mathbb{R}^n \\ u \in \mathbb{R}^m \\ y \in \mathbb{R}^p \end{array} \qquad \begin{array}{l} x(0) = x_0 \\ u(t),\, t \geq 0 \end{array}$$

By Laplace transformation:

$$x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} \Longrightarrow X(s) = \mathcal{L}[x(t)] = \begin{bmatrix} X_1(s) \\ \vdots \\ X_n(s) \end{bmatrix}$$

$$\mathcal{L}[Ax(t)] = A\mathcal{L}[x(t)]$$

$$\mathcal{L}[\dot{x}(t)] = \begin{bmatrix} \mathcal{L}[\dot{x}_1(t)] \\ \vdots \\ \mathcal{L}[\dot{x}_n(t)] \end{bmatrix} = \begin{bmatrix} sX_1(s) - x_1(0) \\ \vdots \\ sX_n(s) - x_n(0) \end{bmatrix} = sX(s) - x(0)$$

Calculation of State and Output Trajectories (contd.)

Hence:

$$\mathcal{L}\{\dot{x}(t)\} = \mathcal{L}\{Ax + Bu\} \quad \Rightarrow \quad sX(s) - x(0) = AX(s) + BU(s)$$

$$\Rightarrow \quad (sI - A)X(s) = x(0) + BU(s)$$

$$\Rightarrow \quad \begin{cases} X(s) & = (sI-A)^{-1}x(0) + (sI-A)^{-1}BU(s) \\ \\ Y(s) & = CX(s) + DU(s) \\ & = C(sI-A)^{-1}x(0) + \left[C(sI-A)^{-1}B + D\right]U(s) \end{cases}$$

Calculation of State and Output Trajectories (contd.)

Therefore:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \qquad \begin{array}{l} x \in \mathbb{R}^n \\ u \in \mathbb{R}^m \\ y \in \mathbb{R}^p \end{array} \qquad \begin{array}{l} x(0) = x_0 \\ u(t),\ t \geq 0 \end{array}$$

$$x(t) = \mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\} x(0) + \mathcal{L}^{-1}\left\{(sI - A)^{-1} BU(s)\right\}$$

$$= e^{At} x_0 + \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau$$

where the **matrix exponential** has been introduced:

$$e^{At} := \sum_{k=0}^{\infty} \frac{(At)^k}{k!} = I + At + \frac{A^2 t^2}{2} + \cdots \quad \text{where} \quad e^{At} = \mathcal{L}^{-1}\{(sI - A)^{-1}\}$$

Calculation of State and Output Trajectories (contd.)

Observe that:

$$\begin{aligned} x(t) &= e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \\ &= x_l(t) + x_f(t) \end{aligned}$$

where:

- $x_l(t)$ denotes the **free state trajectory** that only depends on x_0 **(linearly)**
- $x_f(t)$ denotes the **forced state trajectory** that only depends on $u(t)$ **(linearly)**

$$\begin{aligned} y(t) &= Ce^{At}x_0 + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t) \\ &= y_l(t) + y_f(t) \end{aligned}$$

where:

- $y_l(t)$ denotes the **free output trajectory** that only depends on x_0 **(linearly)**
- $y_f(t)$ denotes the **forced output trajectory** that only depends on $u(t)$ **(linearly)**

Equilibria of Time-Invariant Dynamic Systems

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$$\begin{cases} \dot{x} = f(x,u) \\ y = g(x,u) \end{cases} \qquad \begin{array}{l} x \in \mathbb{R}^n \\ u \in \mathbb{R}^m \\ y \in \mathbb{R}^p \end{array} \qquad u(t) = \bar{u}, \forall\, t$$

- **Equilibrium state**

  Constant trajectory of $x(t)$ for a **constant** input $u(t) = \bar{u}, \forall\, t$

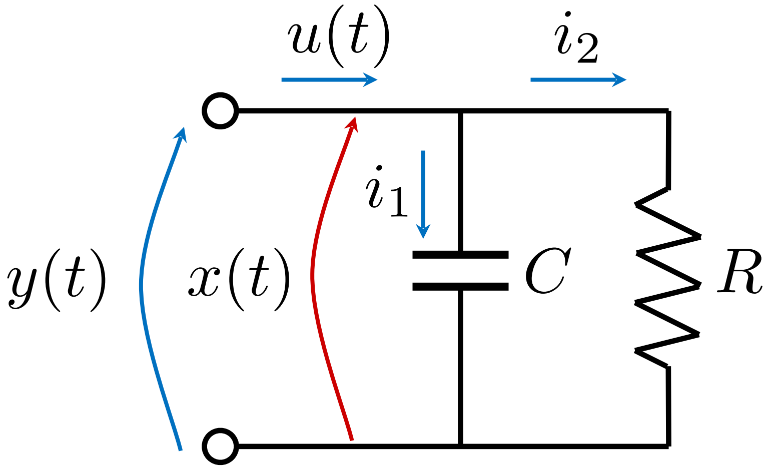
- **All** equilibrium states can be determined selecting inputs $u(t) = \bar{u} \in \mathbb{R}^m$
- **Equilibrium states** $\bar{x}$ are the solutions of the **algebraic equation**

$$0 = f(x, \bar{u}) \Longrightarrow \bar{x}$$

- **Equilibrium outputs** are obtained substituting $\bar{x}, \bar{u}$ into the output equation

$$\bar{y} = g(\bar{x}, \bar{u})$$

Equilibrium: Example 1

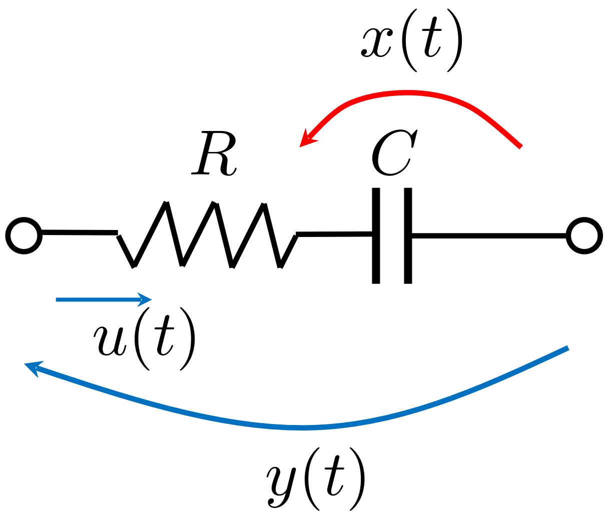


$$\begin{cases} \dot{x}(t) = -\dfrac{1}{RC}\,x + \dfrac{1}{C}\,u \\[2ex] y = x \end{cases}$$

$$u(t) = \bar{u}, \forall\, t \;\Longrightarrow\; 0 = -\frac{1}{RC}\,x + \frac{1}{C}\,\bar{u}$$

$$\Rightarrow\; \bar{x} = R\bar{u} \;\Longrightarrow\; \bar{y} = R\bar{u}$$

Equilibrium: Example 2



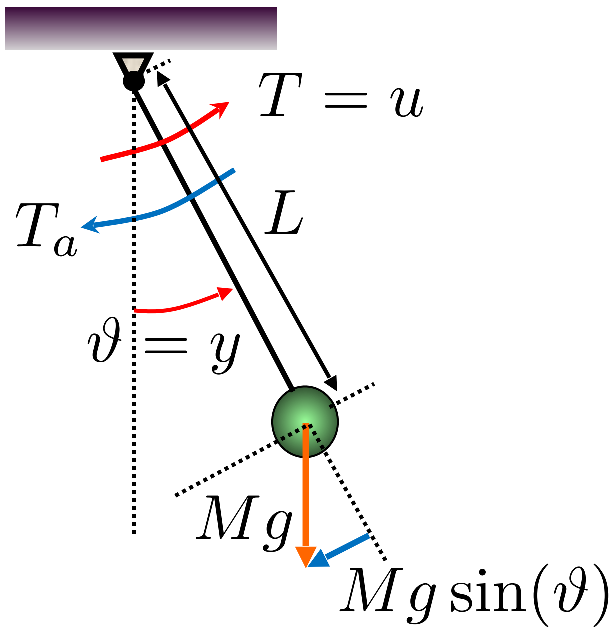
$$\begin{cases} \dot{x}(t) = \dfrac{1}{C}\,u \\ y = x + R\,u \end{cases}$$

$$u(t) = \bar{u}, \forall\, t \quad \Longrightarrow \quad 0 = \frac{1}{C}\,\bar{u}$$

$$\text{If } \bar{u} = 0 \quad \Longrightarrow \quad \exists\ \infty\ \bar{x} \qquad \exists\ \infty\ \bar{y} = \bar{x}$$

$$\text{If } \bar{u} \neq 0 \quad \Longrightarrow \quad \nexists\ \bar{x} \qquad \nexists\ \bar{y}$$

Equilibrium: Example 3



$$\begin{cases} x_1 := \vartheta \\ x_2 := \dot{\vartheta} \end{cases}, \quad x := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{cases} \dot{x}_1 = \dot{\vartheta} = x_2 \\ \dot{x}_2 = -\dfrac{MgL}{J}\sin(x_1) - \dfrac{h}{J}x_2 + \dfrac{1}{J}u \\ y = x_1 \end{cases}$$

$$u(t) = \bar{u}, \forall t \quad \Rightarrow \quad \begin{cases} 0 = x_2 \\ 0 = -\dfrac{MgL}{J}\sin(x_1) - \dfrac{h}{J}x_2 + \dfrac{1}{J}\bar{u} \end{cases}$$

$$\begin{cases} \bar{x}_2 = 0 \\ \sin(\bar{x}_1) = \dfrac{1}{MgL}\bar{u} \end{cases} \xrightarrow{|\bar{u}| \leq MgL} \begin{cases} \bar{x}_2 = 0 \\ \bar{x}_1 = \arcsin\left(\dfrac{\bar{u}}{MgL}\right) \end{cases}$$

$$\exists\, \infty\ \bar{x} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}$$

Equilibria of Linear Time-invariant Systems

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Livescripts in MS Teams: see INTRO_MATLAB_SIMULINK

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \qquad \begin{array}{l} x \in \mathbb{R}^n \\ u \in \mathbb{R}^m \\ y \in \mathbb{R}^p \end{array} \qquad u(t) = \bar{u}, \forall\, t$$

$$0 = Ax + B\bar{u}$$

$$Ax = -B\bar{u}$$

$$\det(A) \neq 0$$

$$\det(A) = 0$$

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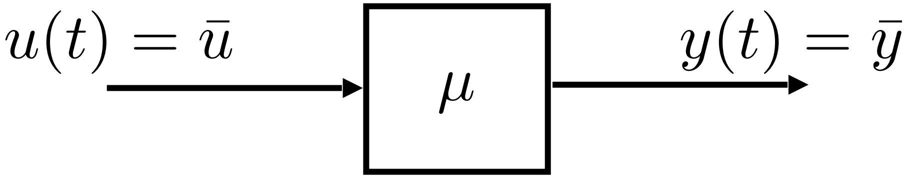
- $\det(A) \neq 0$ $\quad \bar{x} = -A^{-1}B\bar{u}$ One and only one equilibrium state

$$\bar{y} = C\bar{x} + D\bar{u} = -CA^{-1}B\bar{u} + D\bar{u} = \underbrace{\left(-CA^{-1}B + D\right)}_{\textbf{Static Gain}}\bar{u}$$



- $\det(A) = 0$

$\exists\,\infty\,\bar{x},\ \exists\,\infty\,\bar{y}$

$\nexists\,\bar{x},\ \nexists\,\bar{y}$

Equivalent State Equation Representations - Linear case

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \qquad \begin{array}{l} x \in \mathbb{R}^n \\ u \in \mathbb{R}^m \\ y \in \mathbb{R}^p \end{array}$$

Letting:

$$x = T\hat{x}, \quad T \in \mathbb{R}^{n\times n},\ \det(T) \neq 0 \quad \Rightarrow \quad \hat{x} = T^{-1}x$$

$$\begin{cases} \dot{\hat{x}} = T^{-1}(Ax + Bu) = \underbrace{T^{-1}AT}_{\widehat{A}}\hat{x} + \underbrace{T^{-1}B}_{\widehat{B}}u \\ y = \underbrace{CT}_{\widehat{C}}\hat{x} + Du \end{cases}$$

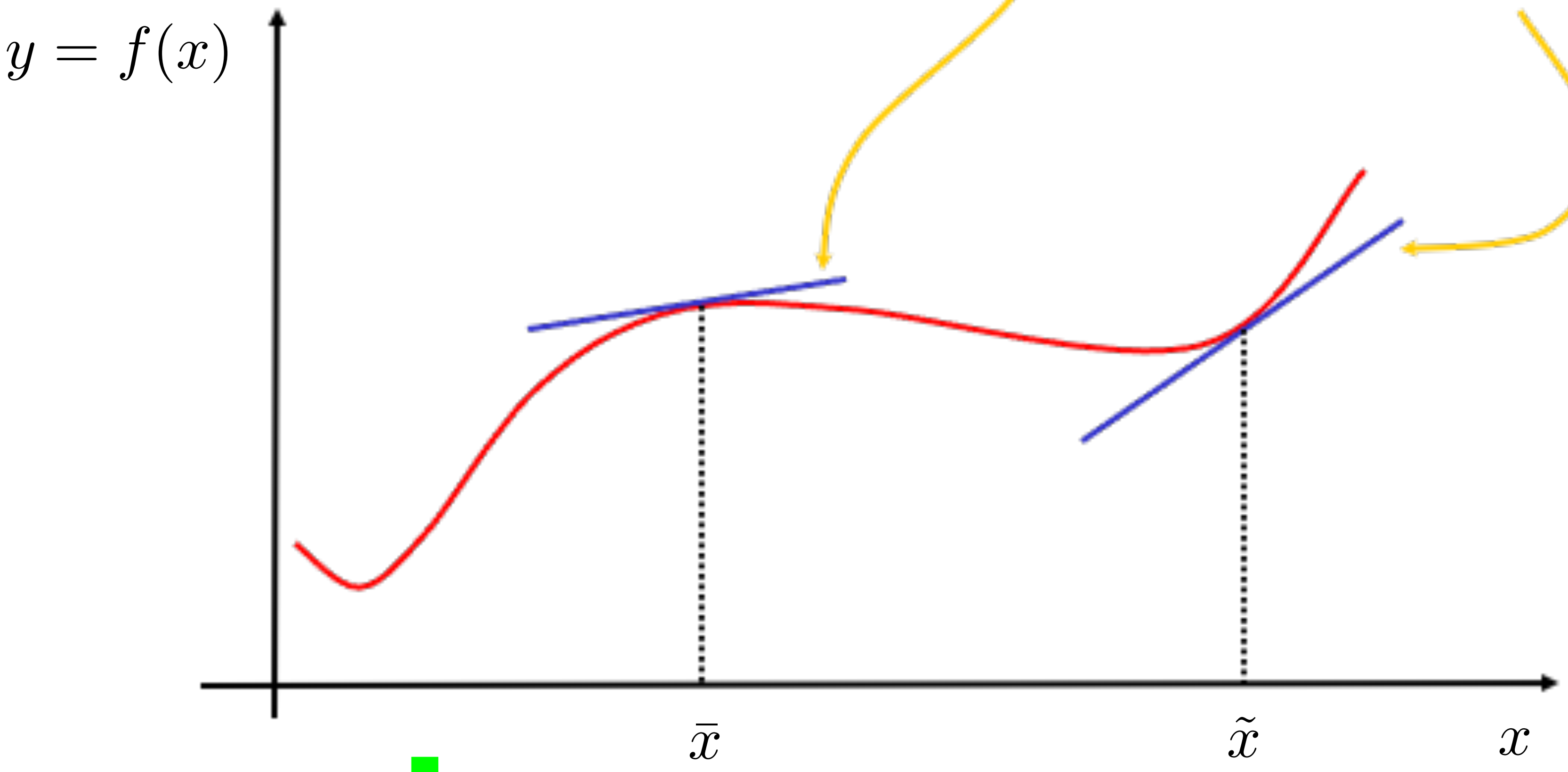
$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \iff \begin{cases} \dot{\hat{x}} = \widehat{A}\hat{x} + \widehat{B}u \\ y = \widehat{C}\hat{x} + Du \end{cases}$$

Linearisation

Basic Idea:

$$\text{slope}: \; f_x(\bar{x}) = \left.\frac{df}{dx}\right|_{x=\bar{x}}$$

$$\text{slope}: \; f_x(\tilde{x}) = \left.\frac{df}{dx}\right|_{x=\tilde{x}}$$



$$y(x) \simeq y(\bar{x}) + f_x(\bar{x})(x - \bar{x})$$

Linearisation (contd.)

Deviations from to the equilibrium:

$$\begin{aligned} \delta u(t) &:= u(t) - \bar{u} \\ \delta x(t) &:= x(t) - \bar{x} \\ \delta y(t) &:= y(t) - \bar{y} \end{aligned} \quad \Longrightarrow \quad \begin{aligned} u(t) &= \delta u(t) + \bar{u} \\ x(t) &= \delta x(t) + \bar{x} \\ y(t) &= \delta y(t) + \bar{y} \end{aligned}$$

- State:

$$\dot{x} = \delta\dot{x} = f(\bar{x} + \delta x, \bar{u} + \delta u) \simeq \underbrace{\cancel{f(\bar{x}, \bar{u})}}_{= 0 \quad \text{(equilibrium)}} + f_x(\bar{x}, \bar{u})\delta x + f_u(\bar{x}, \bar{u})\delta u$$

$$A = f_x(\bar{x}, \bar{u}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{\bar{x}, \bar{u}}$$

$$\delta\dot{x} \simeq \underbrace{f_x(\bar{x}, \bar{u})}_{\substack{n \times n \\ A}}\delta x + \underbrace{f_u(\bar{x}, \bar{u})}_{\substack{n \times m \\ B}}\delta u$$

$$B = f_u(\bar{x}, \bar{u}) = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_m} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_m} \end{bmatrix}_{\bar{x}, \bar{u}}$$

- Output:

$$y(t) = \underbrace{\cancel{\bar{y}}}_{=\text{ at equilibrium}} + \delta y = g(\bar{x} + \delta x, \bar{u} + \delta u) \simeq \underbrace{\cancel{g(\bar{x}, \bar{u})}}_{=\text{ at equilibrium}} + g_x(\bar{x}, \bar{u})\delta x + g_u(\bar{x}, \bar{u})\delta u$$

$$C = g_x(\bar{x}, \bar{u}) = \left.\begin{bmatrix} \dfrac{\partial g_1}{\partial x_1} & \cdots & \dfrac{\partial g_1}{\partial x_n} \\ \vdots & & \vdots \\ \dfrac{\partial g_p}{\partial x_1} & \cdots & \dfrac{\partial g_p}{\partial x_n} \end{bmatrix}\right|_{\bar{x},\bar{u}}$$

$$\delta y \simeq \underbrace{g_x(\bar{x}, \bar{u})}_{\substack{p \times n \\ C}}\delta x + \underbrace{g_u(\bar{x}, \bar{u})}_{\substack{p \times m \\ D}}\delta u$$

$$D = g_u(\bar{x}, \bar{u}) = \left.\begin{bmatrix} \dfrac{\partial g_1}{\partial u_1} & \cdots & \dfrac{\partial g_1}{\partial u_m} \\ \vdots & & \vdots \\ \dfrac{\partial g_p}{\partial u_1} & \cdots & \dfrac{\partial g_p}{\partial u_m} \end{bmatrix}\right|_{\bar{x},\bar{u}}$$

Linearised System

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Summing up:

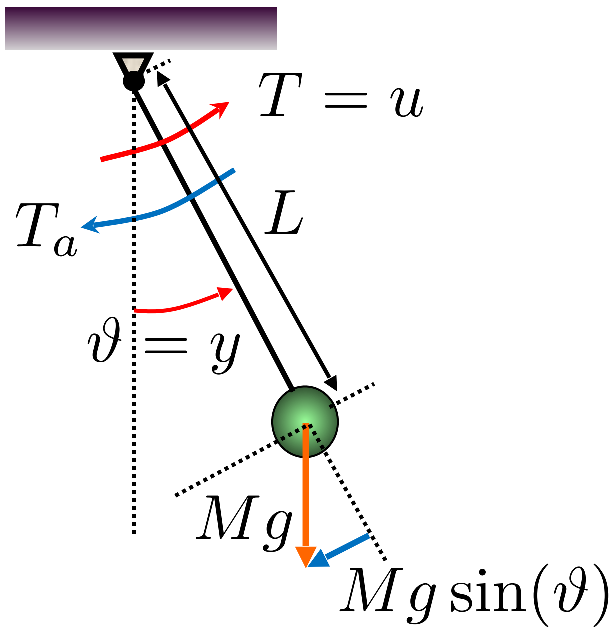
$$\begin{cases} \dot{x} = f(x,u) \\ y = g(x,u) \end{cases} \implies 0 = f(x,\bar{u}) \implies \bar{x} \ \text{(equilibrium state)}$$

$$\delta\dot{x} \simeq \underbrace{f_x(\bar{x},\bar{u})}_{A}\delta x + \underbrace{f_u(\bar{x},\bar{u})}_{B}\delta u$$

$$\delta y \simeq \underbrace{g_x(\bar{x},\bar{u})}_{C}\delta x + \underbrace{g_u(\bar{x},\bar{u})}_{D}\delta u$$

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} ; \quad \begin{array}{l} x \in \mathbb{R}^n \\ u \in \mathbb{R}^m \\ y \in \mathbb{R}^p \end{array}$$

Example



Letting (see slide 27):

$$\begin{cases} x_1 := \vartheta \\ x_2 := \dot{\vartheta} \end{cases}, \quad x := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow \begin{cases} \dot{x}_1 = \dot{\vartheta} = x_2 \\ \dot{x}_2 = -\dfrac{MgL}{J}\sin(x_1) - \dfrac{h}{J}x_2 + \dfrac{1}{J}u \\ y = x_1 \end{cases}$$

$$u(t) = \bar{u} = 0, \forall t \quad \Rightarrow \quad \begin{cases} 0 = x_2 \\ 0 = -\dfrac{MgL}{J}\sin(x_1) - \dfrac{h}{J}x_2 \end{cases}$$

$$\Rightarrow \begin{cases} \bar{x}_2 = 0 \\ \sin(\bar{x}_1) = \dfrac{1}{MgL}\bar{u} \end{cases} \quad \Rightarrow \quad \begin{cases} \bar{x}_2 = 0 \\ \bar{x}_1 = k\pi, \ \forall k \in \mathbb{Z} \end{cases}$$

We pick the two "physical" solutions:

$$\bar{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}; \quad \tilde{x} = \begin{bmatrix} \pi \\ 0 \end{bmatrix}$$

Example (contd.)

We get:

$$f_x(\bar{x},\bar{u}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}_{\bar{x},\bar{u}} = \begin{bmatrix} 0 & 1 \\ -\frac{MgL}{J}\cos(x_1) & -\frac{h}{J} \end{bmatrix}_{\bar{x},\bar{u}} = \begin{bmatrix} 0 & 1 \\ -\frac{MgL}{J} & -\frac{h}{J} \end{bmatrix} = \bar{A}$$

$$f_x(\tilde{x},\bar{u}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}_{\tilde{x},\bar{u}} = \begin{bmatrix} 0 & 1 \\ -\frac{MgL}{J}\cos(x_1) & -\frac{h}{J} \end{bmatrix}_{\tilde{x},\bar{u}} = \begin{bmatrix} 0 & 1 \\ +\frac{MgL}{J} & -\frac{h}{J} \end{bmatrix} = \tilde{A}$$

$$f_u(\bar{x},\bar{u}) = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{bmatrix}_{\bar{x},\bar{u}} = \begin{bmatrix} 0 \\ \frac{1}{J} \end{bmatrix} = \bar{B}\ ; \quad f_u(\tilde{x},\bar{u}) = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{bmatrix}_{\tilde{x},\bar{u}} = \begin{bmatrix} 0 \\ \frac{1}{J} \end{bmatrix} = \tilde{B}$$

$$g_x(\bar{x},\bar{u}) = \begin{bmatrix} \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} \end{bmatrix}_{\bar{x},\bar{u}} = \begin{bmatrix} 1 & 0 \end{bmatrix} = \bar{C}\ ; \ g_x(\tilde{x},\bar{u}) = \begin{bmatrix} \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} \end{bmatrix}_{\tilde{x},\bar{u}} = \begin{bmatrix} 1 & 0 \end{bmatrix} = \tilde{C}$$

$$g_u(\bar{x},\bar{u}) = \left.\frac{\partial g}{\partial u}\right|_{\bar{x},\bar{u}} = 0 = \bar{D}\ ; \quad g_u(\tilde{x},\bar{u}) = \left.\frac{\partial g}{\partial u}\right|_{\tilde{x},\bar{u}} = 0 = \tilde{D}$$