UNIVERSITÀ DEGLI STUDI DI TRIESTE

034IN - FONDAMENTI DI AUTOMATICA - FUNDAMENTALS OF AUTOMATIC CONTROL
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Part VI: Stability of Interconnected Systems

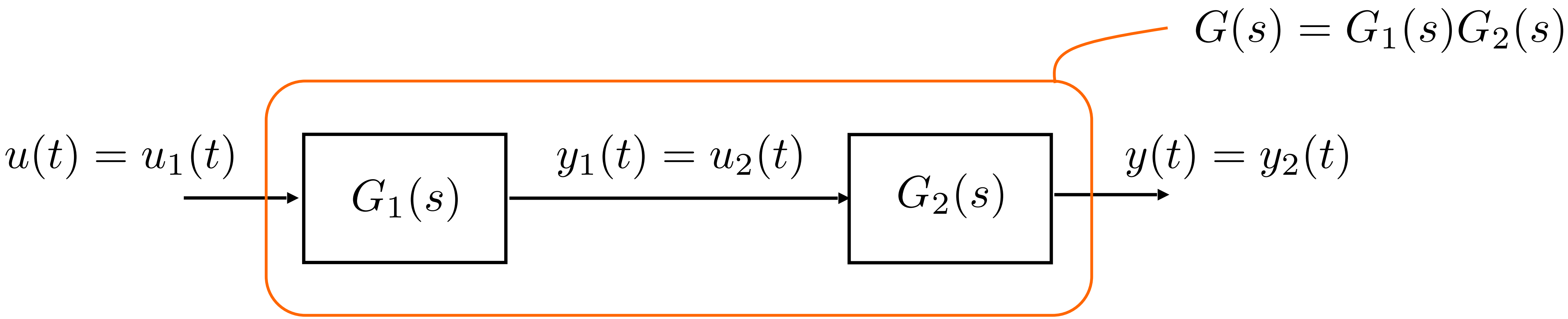
Gianfranco Fenu, Thomas Parisini

Department of Engineering and Architecture

Motivations:

- **Block schemes** may be useful to analyse **stability** of interconnected systems
- The **structure** and **topology** of the block scheme can be exploited under appropriate conditions
- Care should be exercised when **common factors** arise in the context of block-scheme reduction

Stability Analysis - Series Interconnections



$$G_1(s) = \frac{N_1(s)}{\varphi_1(s)}\ ; \quad G_2(s) = \frac{N_2(s)}{\varphi_2(s)}$$

$$G(s) = \frac{N_1(s)N_2(s)}{\varphi_1(s)\varphi_2(s)} = \frac{N(s)}{\varphi(s)}$$

Stability Analysis - Series Interconnections (contd.)

- **Case 1:** no common factors among $N(s)$ and $\varphi(s)$

  ↳ $\text{Poles of } G(s) = \{\text{Poles of } G_1(s)\} \cup \{\text{Poles of } G_2(s)\}$

  ↳ $G(s)$ asymptotically stable $\Longleftrightarrow$ $G_1(s), G_2(s)$ asymptotically stable

- **Case 2:** common factors among $N(s)$ and $\varphi(s)$

  – if $\text{Re poles}[\varphi(s)] \text{ in common} < 0$

  ↳ **hidden internal dynamics** is asymptotically stable

  – if $\text{Re poles}[\varphi(s)] \text{ in common} \geq 0$

  ↳ hidden internal dynamics is not asymptotically stable

  ↳ stability cannot be inferred from $G(s)$

Example 1

$$u(t) \longrightarrow \boxed{\frac{1}{\cancel{s+1}}} \longrightarrow \boxed{\frac{\cancel{s+1}}{s+2}} \longrightarrow y(t)$$

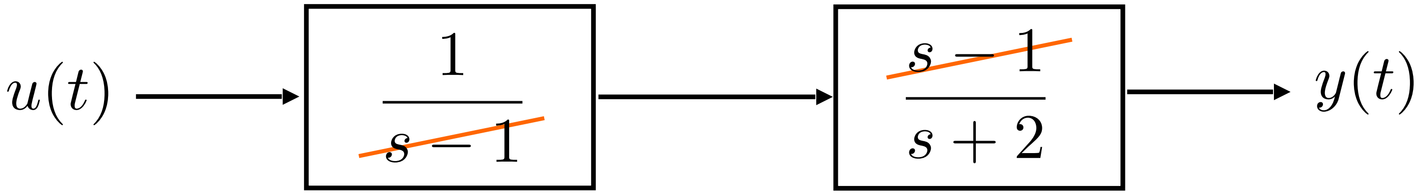
$$G(s) = \frac{1}{s+2}$$ ➡ system asymptotically stable and $G(s)$ shows this

Example 2

$$u(t) \longrightarrow \boxed{\frac{1}{\cancel{s+1}}} \longrightarrow \boxed{\frac{\cancel{s+1}}{s-2}} \longrightarrow y(t)$$

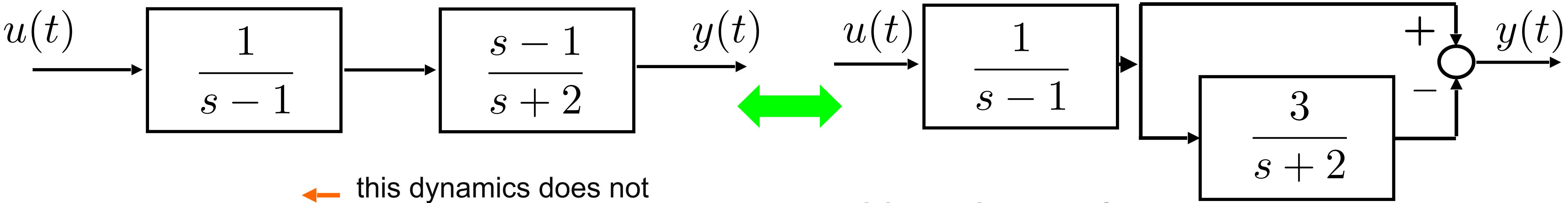
$$G(s) = \frac{1}{s-2}$$ ➡ system unstable and $G(s)$ shows this

Example 3



$$G(s) = \frac{1}{s+2}$$ system unstable but $G(s)$ does **not** show this

In fact:



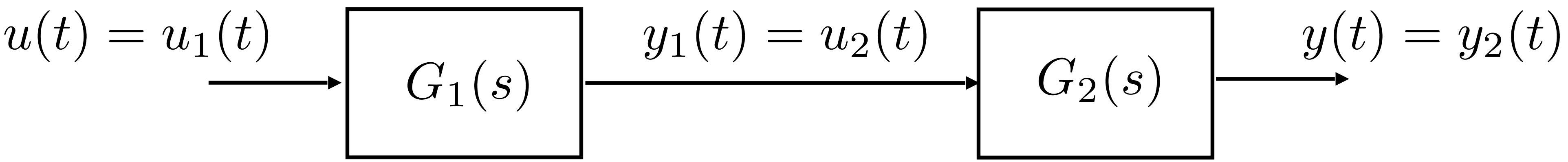
$$\begin{cases} \boxed{\dot{x}_1 = x_1 + u} \\ \dot{x}_2 = -2x_2 + 3x_1 \\ y = x_1 - x_2 \end{cases}$$

this dynamics does not influence the input/output relationship

positive eigenvalue

$$\begin{cases} \dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\ \\ y = [1 \quad -1]\, x \end{cases}$$

Series Interconnections – Time Domain Analysis



$$G_1(s) = c_1(sI - A_1)^{-1}b_1 + d_1\,; \quad G_2(s) = c_2(sI - A_2)^{-1}b_2 + d_2$$

$$\begin{cases} \dot{x}_1 = A_1 x_1 + b_1 u \\ y_1 = c_1 x_1 + d_1 u \end{cases}$$

$$\begin{cases} \dot{x}_2 = A_2 x_2 + b_2 y_1 = A_2 x_2 + b_2 c_1 x_1 + b_2 d_1 u \\ y = c_2 x_2 + d_2 y_1 \end{cases}$$

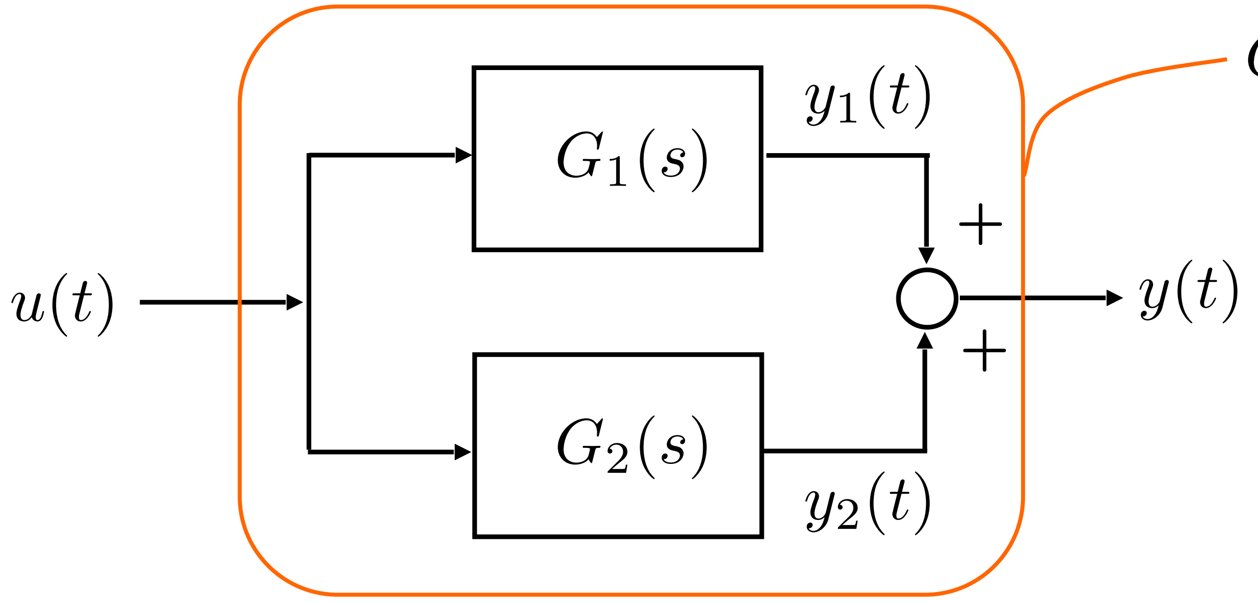
$$\begin{cases} \dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ b_2 c_1 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 d_1 \end{bmatrix} u \\ \\ y = [d_2 c_1 \;\; c_2]\, x + d_1 d_2 u \end{cases}$$

Stability for Series Interconnections: Final Remarks

- Asymptotically Stable Series-Interconnected System $\Longleftrightarrow$ Asymptotically Stable Component Sub-Systems

- Systems' eigenvalues are **not modified** by series-interconnections

Stability Analysis - Parallel Interconnections



$$G(s) = G_1(s) + G_2(s)$$

$$G_1(s) = \frac{N_1(s)}{\varphi_1(s)}\,; \quad G_2(s) = \frac{N_2(s)}{\varphi_2(s)}$$

$$G(s) = \frac{N_1(s)}{\varphi_1(s)} + \frac{N_2(s)}{\varphi_2(s)} = \frac{N_1(s)\varphi_2(s) + N_2(s)\varphi_1(s)}{\varphi_1(s)\varphi_2(s)} = \frac{N(s)}{\varphi(s)}$$

Stability Analysis - Parallel Interconnections (contd.)

- **Case 1:** no common factors among $N(s)$ and $\varphi(s)$

  $\Rightarrow$ $\text{Poles of } G(s) = \{\text{Poles of } G_1(s)\} \cup \{\text{Poles of } G_2(s)\}$

  $\Rightarrow$ $G(s)$ asymptotically stable $\Leftrightarrow$ $G_1(s), G_2(s)$ asymptotically stable

- **Case 2:** common factors among $N(s)$ and $\varphi(s)$

  – if $\text{Re poles}[\varphi(s)] \text{ in common} < 0$

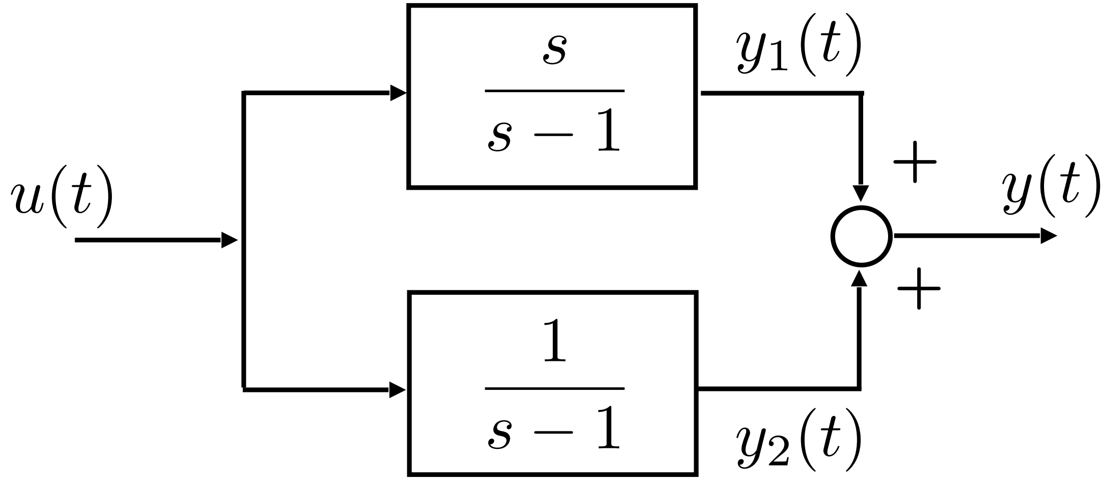
    $\Rightarrow$ **hidden internal dynamics** is asymptotically stable

  – if $\text{Re poles}[\varphi(s)] \text{ in common} \geq 0$

    $\Rightarrow$ **hidden internal dynamics** is not asymptotically stable

    $\Rightarrow$ stability cannot be inferred from $G(s)$

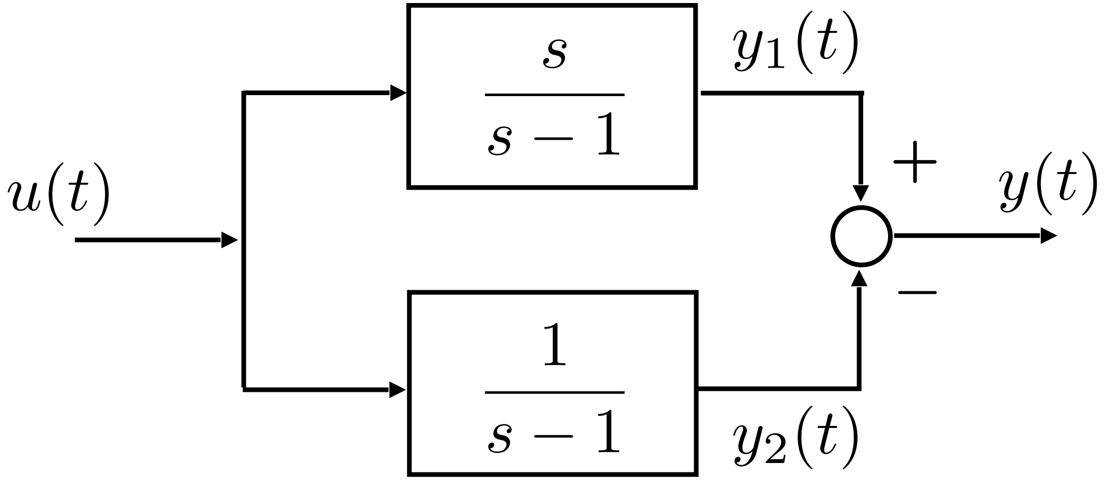
Example 1



$$G(s) = \frac{s+1}{s-1}$$

system unstable and $G(s)$ shows this

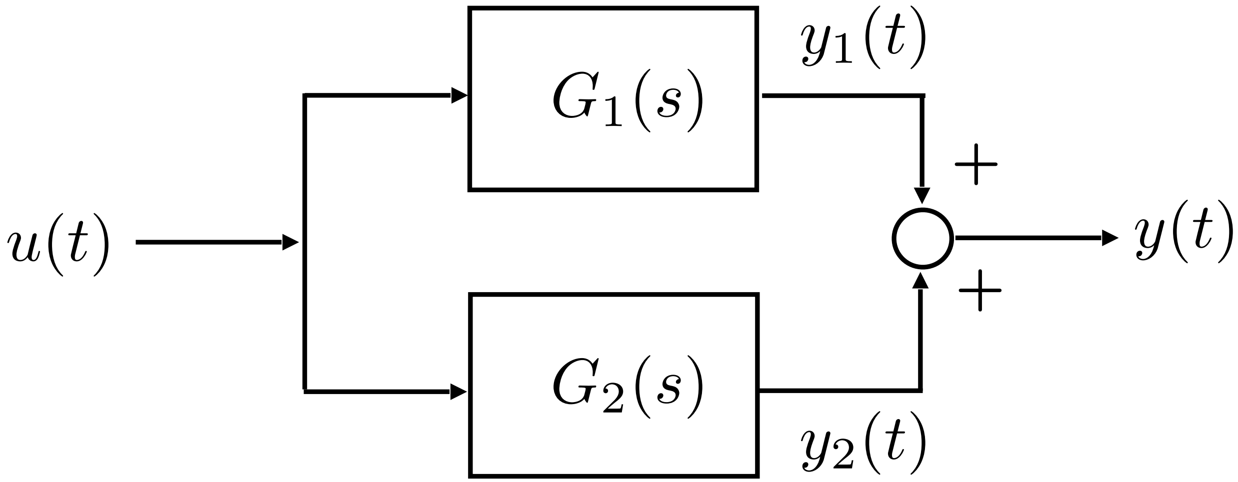
Example 2



$$G(s) = 1$$

system unstable but $G(s)$ does **not** show this

Parallel Interconnections – Time Domain Analysis



$$G_1(s) = c_1(sI - A_1)^{-1}b_1 + d_1\,; \qquad G_2(s) = c_2(sI - A_2)^{-1}b_2 + d_2$$

$$\begin{cases} \dot{x}_1 = A_1 x_1 + b_1 u \\ y_1 = c_1 x_1 + d_1 u \end{cases}$$

$$\begin{cases} \dot{x}_2 = A_2 x_2 + b_2 u \\ y_2 = c_2 x_2 + d_2 u \end{cases}$$

$\Longrightarrow$

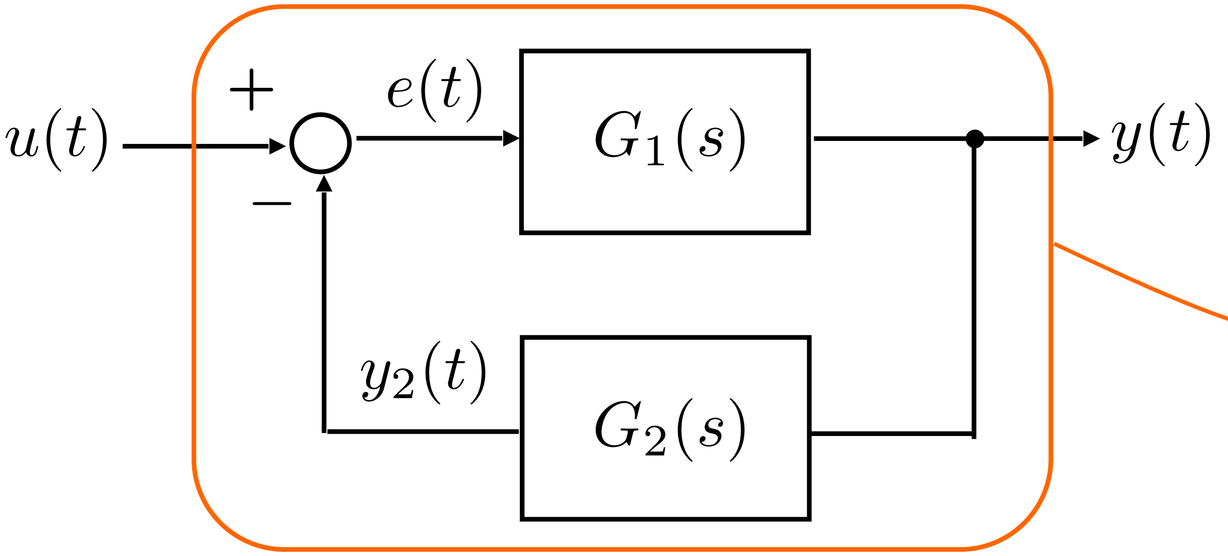
$$\begin{cases} \dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u \\ \\ y = [c_1 \;\; c_2]\, x + (d_1 + d_2)\, u \end{cases}$$

Stability for Parallel Interconnections: Final Remarks

- Asymptotically Stable Parallel-Interconnected System ⟷ Asymptotically Stable Component Sub-Systems

- Systems' eigenvalues are **not modified** by parallel-interconnections

Stability Analysis - Feedback Interconnections



$$G(s) = \frac{G_1(s)}{1 + G_1(s)G_2(s)}$$

$$G_1(s) = \frac{N_1(s)}{\varphi_1(s)}\,; \quad G_2(s) = \frac{N_2(s)}{\varphi_2(s)}$$

$$G(s) = \frac{\dfrac{N_1(s)}{\varphi_1(s)}}{1 + \dfrac{N_1(s)N_2(s)}{\varphi_1(s)\varphi_2(s)}} = \frac{N_1(s)\varphi_2(s)}{\varphi_1(s)\varphi_2(s) + N_1(s)N_2(s)} = \frac{N(s)}{\varphi(s)}$$

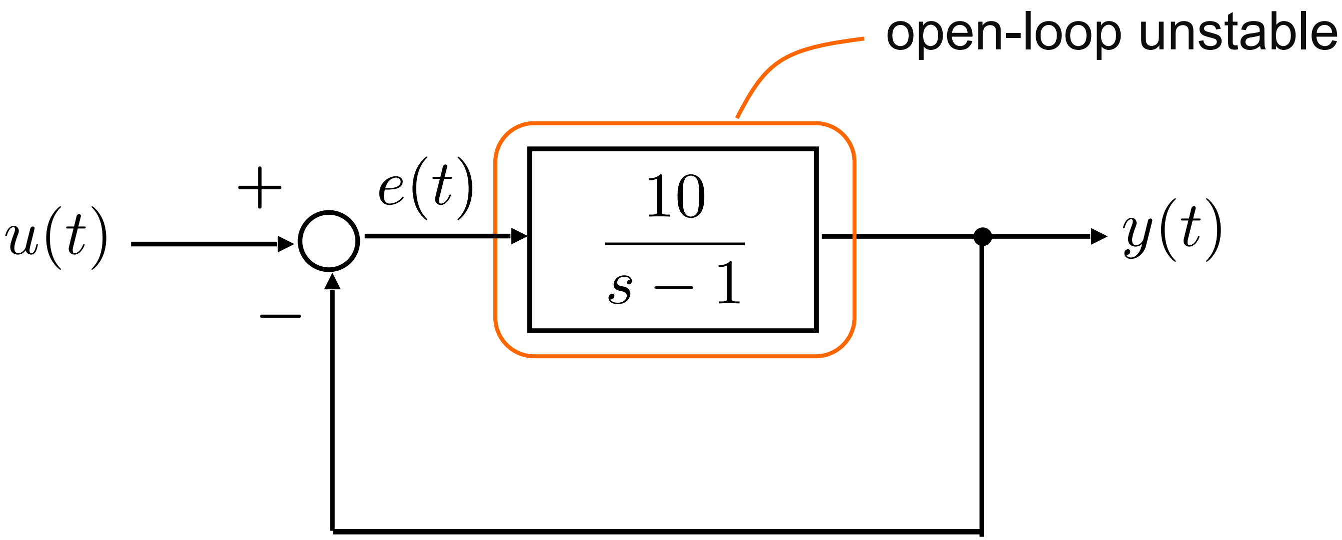
Stability Analysis - Feedback Interconnections

Even in the absence of common factors among $N(s)$ and $\varphi(s)$:

$$\text{Poles of } G(s) = \{\text{roots of } \varphi_1(s)\varphi_2(s) + N_1(s)N_2(s)\}$$

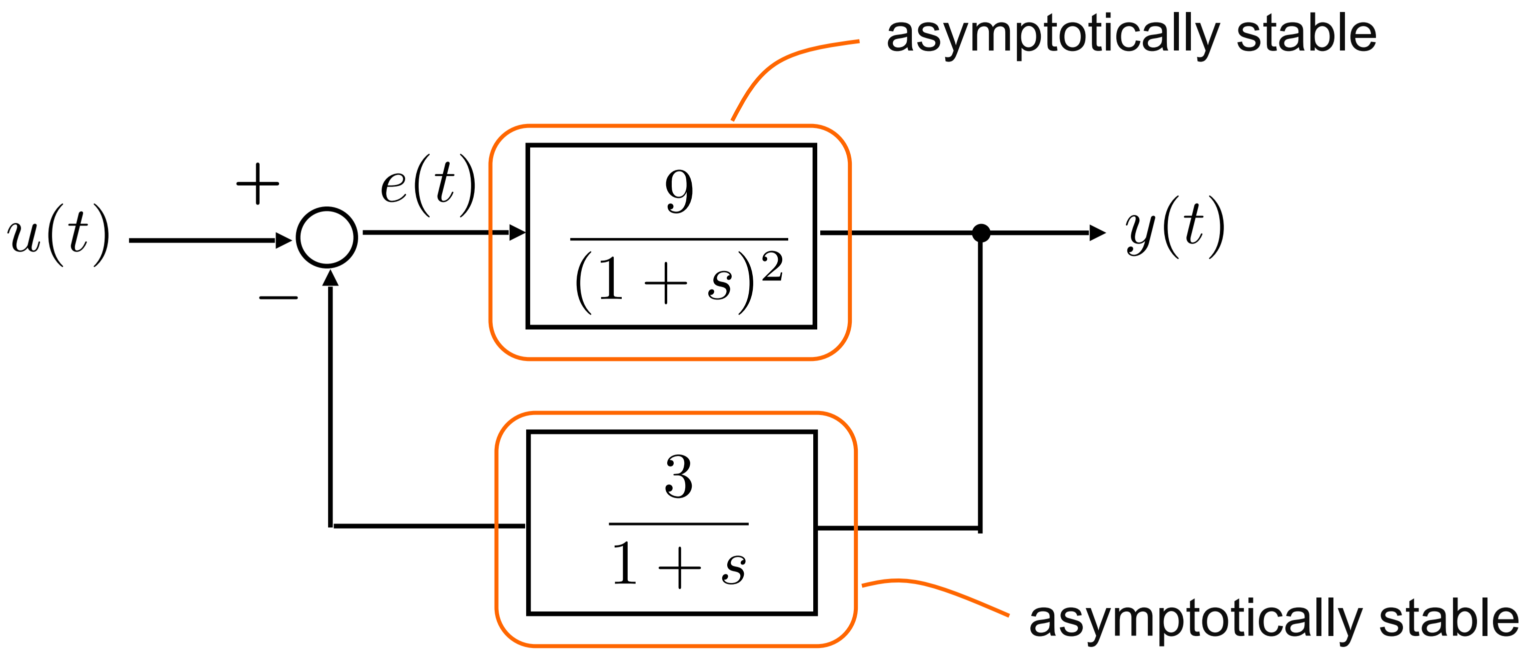
$G(s)$ asymptotically stable $\;\not\Rightarrow\;$ / $\;\not\Leftarrow\;$ $G_1(s), G_2(s)$ asymptotically stable

Example 1



$$G(s) = \frac{\frac{10}{s-1}}{1+\frac{10}{s-1}} = \frac{\frac{10}{s-1}}{\frac{s-1+10}{s-1}} = \frac{10}{s+9}$$ asymptotically stable

Example 2



$$G(s) = \frac{\dfrac{9}{(1+s)^2}}{1 + \dfrac{27}{(1+s)^3}} = \frac{9(1+s)}{(1+s)^3 + 27}$$

Example 2 (contd.)

Hence, the characteristic polynomial is:

$$\varphi(s) = (1+s)^3 + 27 = s^3 + 3s^2 + 3s + 28$$

The Routh table takes on the form:

$$\begin{array}{c|ccc} 1 & \boxed{1} & 3 & 0 \\ 2 & 3 & 28 & 0 \\ 3 & \alpha & 0 & 0 \\ 4 & \beta & 0 & 0 \end{array}$$

r_4

$$\alpha = -\frac{1}{3}\det\begin{bmatrix} 1 & 3 \\ 3 & 28 \end{bmatrix} = -\frac{19}{3}$$

$$\beta = 28$$

Column r_4 of the Routh table shows two sign changes and hence the closed-loop system is unstable

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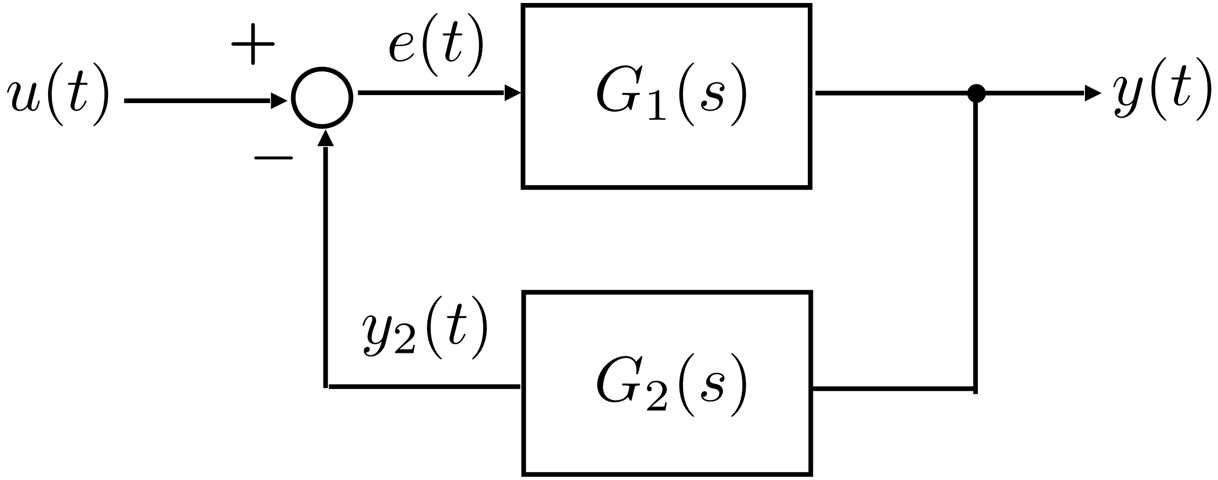
Assume that $G_1(s), G_2(s)$ are strictly proper:



$$G_1(s) = c_1(sI - A_1)^{-1}b_1\,; \quad G_2(s) = c_2(sI - A_2)^{-1}b_2$$

$$\begin{cases} \dot{x}_1 = A_1 x_1 + b_1 e \\ y = c_1 x_1 \end{cases}$$

$$\begin{cases} \dot{x}_2 = A_2 x_2 + b_2 y \\ y_2 = c_2 x_2 \end{cases}$$

$$\Longrightarrow \begin{cases} \dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & -b_1 c_2 \\ b_2 c_1 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ 0 \end{bmatrix} u \\ \\ y = [c_1 \quad 0]\, x \end{cases}$$

Stability for Feedback Interconnections: Final Remarks

- Asymptotically Stable Feedback-Interconnected System $\nLeftrightarrow$ Asymptotically Stable Component Sub-Systems

- Systems' eigenvalues are **modified** by feedback-interconnections

↳ **This is key to enable design of control system that impose the desired behaviour and specifications to the controlled system**