

# Application Assignment: Filter Design

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## 1 Introduction

We are supposed to design the equivalent FIR and IIR filter realizations for filter number 114. This is a bandpass filter whose specifications are available below.

## 2 Filter Specifications

The sampling rate for the filter has been specified as  $F_s = 48$  kHz. If the un-normalized discrete-time (natural) frequency is  $F$ , the corresponding normalized digital filter (angular) frequency is given by  $\omega = 2\pi\left(\frac{F}{F_s}\right)$ .

### 2.1 The Digital Filter

1. *Tolerances:* The passband ( $\delta_1$ ) and stopband ( $\delta_2$ ) tolerances are given to be equal, so we let  $\delta_1 = \delta_2 = \delta = 0.15$ .
2. *Passband:* The passband of filter is  $4 + 0.6j$  kHz to  $4 + 0.6(j + 2)$  kHz, where

$$j = (r - 11000)\% \sigma \quad (1)$$

$$r = \text{roll number (last five digits)} \quad (2)$$

$$\sigma = \text{sum of those digits} \quad (3)$$

In this project,

$$r = 11006 \quad (4)$$

$$\text{so } \sigma = 8 \quad (5)$$

$$\text{and } j = 6\%8 = 6 \quad (6)$$

So the passband range for the bandpass filter is from 7.6 kHz to 8.8 KHz. Hence, the un-normalized discrete time filter passband frequencies are

$$F_{p1} = 8.8\text{kHz} \quad (7)$$

$$F_{p2} = 7.6\text{kHz} \quad (8)$$

The corresponding normalized digital filter passband frequencies are

$$\omega_{p1} = 2\pi \frac{F_{p1}}{F_s} = 0.367\pi \quad (9)$$

$$\omega_{p2} = 2\pi \frac{F_{p2}}{F_s} = 0.317\pi \quad (10)$$

The centre frequency is then given by

$$\omega_c = \frac{\omega_{p1} + \omega_{p2}}{2} \quad (11)$$

$$= 0.342\pi \quad (12)$$

3. *Stopband*: The *transition band* for bandpass filters is  $\Delta F = 0.3$  kHz on either side of the passband. Hence, the un-normalized *stopband* frequencies are

$$F_{s1} = 8.8 + 0.3 = 9.1\text{kHz} \quad (13)$$

$$F_{s2} = 7.6 - 0.3 = 7.3\text{kHz} \quad (14)$$

The corresponding normalized frequencies are

$$\omega_{s1} = 0.379\pi \quad (15)$$

$$\omega_{s2} = 0.304\pi \quad (16)$$

## 2.2 The Analog filter

In the bilinear transform, the analog filter frequency ( $\Omega$ ) is related to the corresponding digital filter frequency ( $\omega$ ) as  $\Omega = \tan \frac{\omega}{2}$ . Using this relation, we obtain the analog passband and stopband frequencies as  $\Omega_{p1} = 0.6502$ ,  $\Omega_{p2} = 0.5436$  and  $\Omega_{s1} = 0.6773$ ,  $\Omega_{s2} = 0.5175$  respectively.

## 3 The IIR Filter Design

*Filter Type*: We are supposed to design filters whose stopband is monotonic and passband equiripple. Hence, we use the *Chebyshev approximation* to design our bandpass IIR filter.

### 3.1 The Analog Filter

1. *Low Pass Filter Specifications*: If  $H_{a,BP}(j\Omega)$  be the desired analog band pass filter, with the specifications provided in Section 2.2, and  $H_{a,LP}(j\Omega_L)$  be the equivalent low pass filter, then

$$\Omega_L = \frac{\Omega^2 - \Omega_0^2}{B\Omega} \quad (17)$$

where

$$\Omega_0 = \sqrt{\Omega_{p1}\Omega_{p2}} = 0.5945 \quad (18)$$

$$B = \Omega_{p1} - \Omega_{p2} = 0.1066 \quad (19)$$

$$(20)$$

The low pass filter has the passband edge at  $\Omega_{Lp} = 1$  and stopband edges at  $\Omega_{Ls1} = 1.4583$  and  $\Omega_{Ls2} = -1.5525$ . We choose the stopband edge of the analog low pass filter as  $\Omega_{Ls} = \min(|\Omega_{Ls1}|, |\Omega_{Ls2}|) = 1.4583$ .

2. *The Low Pass Chebyshev Filter Parameters:* The magnitude squared of the Chebyshev low pass filter is given by

$$|H_{a,LP}(j\Omega_L)|^2 = \frac{1}{1 + \epsilon^2 c_N^2(\Omega_L/\Omega_{Lp})} \quad (21)$$

where  $c_N(x)$  is the Chebyshev's polynomial of order  $N$  and given as,

$$c_N(x) = \begin{cases} \cosh(N \cosh^{-1}(x)) & \text{for } |x| > 1 \\ \cos(N \cos^{-1}(x)) & \text{for } |x| < 1 \end{cases} \quad (22)$$

$$c_0(x) = 1 \quad (23)$$

$$c_1(x) = x \quad (24)$$

$$c_N(x) = 2xc_{N-1}(x) - c_{N-2}(x) \quad (25)$$

$c_N(x)$  and the integer  $N$ , which is the order of the filter, and  $\epsilon$  are design parameters. Since  $\Omega_{Lp} = 1$ , (21) may be rewritten as

$$|H_{a,LP}(j\Omega_L)|^2 = \frac{1}{1 + \epsilon^2 c_N^2(\Omega_L)} \quad (26)$$

Also, the design parameters have the following constraints

$$\begin{aligned} \frac{\sqrt{D_2}}{c_N(\Omega_{Ls})} &\leq \epsilon \leq \sqrt{D_1}, \\ N &\geq \left\lceil \frac{\cosh^{-1} \sqrt{D_2/D_1}}{\cosh^{-1} \Omega_{Ls}} \right\rceil, \end{aligned} \quad (27)$$

where

$$D_1 = \frac{1}{(1 - \delta)^2} - 1 = 0.3841 \quad (28)$$

$$D_2 = \frac{1}{\delta^2} - 1 = 43.4444 \quad (29)$$

After appropriate substitutions, we obtain

$$N \geq 4 \quad (30)$$

$$0.3268 \leq \epsilon \leq 0.6197 \quad (31)$$

In Figure 2, we plot  $|H(j\Omega_L)|$  for a range of values of  $\epsilon$ , for  $N = 4$ . We find that for larger values of  $\epsilon$ ,  $|H(j\Omega_L)|$  decreases in the transition band. We choose  $\epsilon = 0.4$  for our IIR filter design.

3. *The Low Pass Chebyshev Filter:* Thus, we obtain

$$|H_{a,LP}(j\Omega_L)|^2 = \frac{1}{1 + 0.16c_4^2(\Omega_L)} \quad (32)$$

where

$$c_4(x) = 8x^4 - 8x^2 + 1. \quad (33)$$

The poles of the frequency response in (21) lying in the left half plane are in general obtained as

$$p(k) = -\sinh \phi \sin \phi(k) + j \cosh \phi \cos \phi(k) \quad (34)$$

where

$$\phi = \frac{1}{N} \sinh^{-1} \left( \frac{1}{\epsilon} \right) \quad (35)$$

$$\phi(k) = \frac{(2k+1)}{N} \pi \quad k = 0, \dots, N-1 \quad (36)$$

The following code generates the poles for  $N = 4$ , stores it in a .txt file and plots the pole-zero plot in Figure 1,

```
https://github.com/Ameen-etc/Filter-Design/blob/main/codes/pole_zero.py
```

And the poles are stored into the following .txt file,

```
https://github.com/Ameen-etc/Filter-Design/blob/main/codes/poles.txt
```

Thus, for  $N$  even, the low-pass stable Chebyshev filter, with a gain  $G$  has the form (Only the poles on the left side of the  $j\omega$  axis would be considered to ensure stability of the filter)

$$H_{a,LP}(s_L) = \frac{G_{LP}}{(s_L - p(1))(s_L - p(2))(s_L - p(3))(s_L - p(4))} \quad (37)$$

Substituting  $N = 4$ ,  $\epsilon = 0.4$  and  $H_{a,LP}(j) = \frac{1}{\sqrt{1+\epsilon^2}}$ , we obtain

$$H_{a,LP}(s_L) = \frac{0.3125}{s_L^4 + 1.1068s_L^3 + 1.6125s_L^2 + 0.9140s_L + 0.3366} \quad (38)$$

In Figure 2 we plot  $|H(j\Omega)|$  using (32) and (38), thereby verifying that our low-pass Chebyshev filter design meets the specifications.

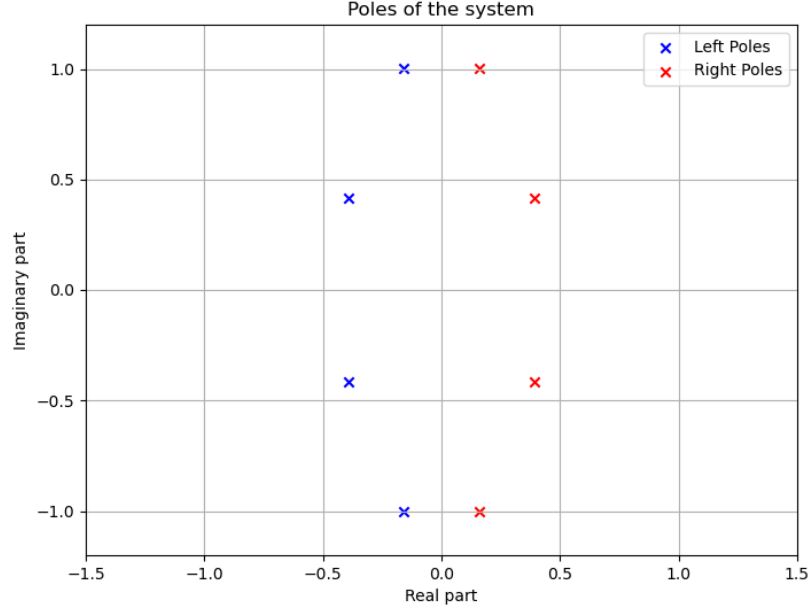


Figure 1: pole-zero plot

4. *The Band Pass Chebyshev Filter:* The analog bandpass filter is obtained from (38) by substituting  $s_L = \frac{s^2 + \Omega_0^2}{Bs}$ . Hence

$$H_{a,BP}(s) = G_{BP} H_{a,LP}(s_L) \Big|_{s_L = \frac{s^2 + \Omega_0^2}{Bs}}, \quad (39)$$

where  $G_{BP}$  is the gain of the bandpass filter. After appropriate substitutions, and evaluating the gain such that  $H_{a,BP}(j\Omega_{p1}) = 1$ , we obtain

$$H_{a,BP}(s) = \frac{4.3489 \times 10^{-5} s^4}{s^8 + 0.1179s^7 + 1.4320s^6 + 0.1262s^5 + 0.7625s^4 + 0.0446s^3 + 0.1789s^2 + 0.0052s + 0.0156} \quad (40)$$

Where,

$$G_{BP} = 1.0788 \quad (41)$$

The above substitution is done by the following code,

```
https://github.com/Ameen-etc/Filter-Design/blob/main/codes/coeff_analog.py
```

And the coefficients are stored into the .txt file,

```
https://github.com/Ameen-etc/Filter-Design/blob/main/codes/coefficients_analog.txt
```

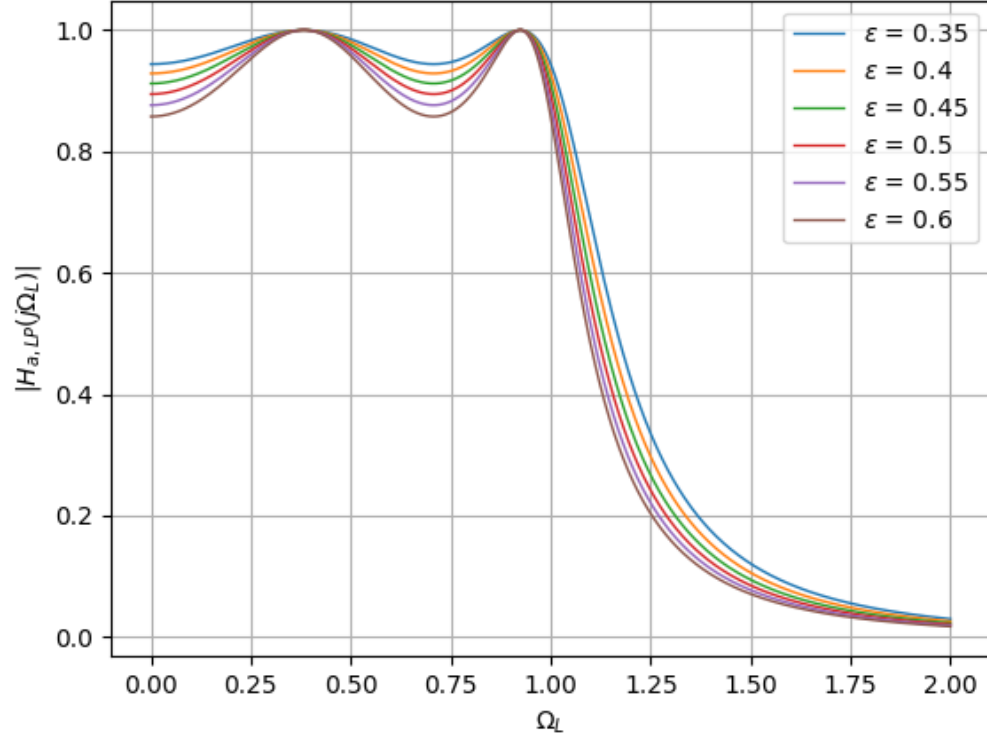


Figure 2: The Analog Low-Pass Frequency Response for  $0.35 \leq \epsilon \leq 0.6$

In Figure 3, we plot  $|H_{a,BP}(j\Omega)|$  as a function of  $\Omega$  for both positive as well as negative frequencies. We find that the passband and stopband frequencies in the figure match well with those obtained analytically through the bilinear transformation.

### 3.2 The Digital Filter

From the bilinear transformation, we obtain the digital bandpass filter from the corresponding analog filter as

$$H_{d,BP}(z) = GH_{a,BP}(s)|_{s=\frac{1-z^{-1}}{1+z^{-1}}} \quad (42)$$

where  $G$  is the gain of the digital filter. From (40) and (42), we obtain

$$H_{d,BP}(z) = G \frac{N(z)}{D(z)} \quad (43)$$

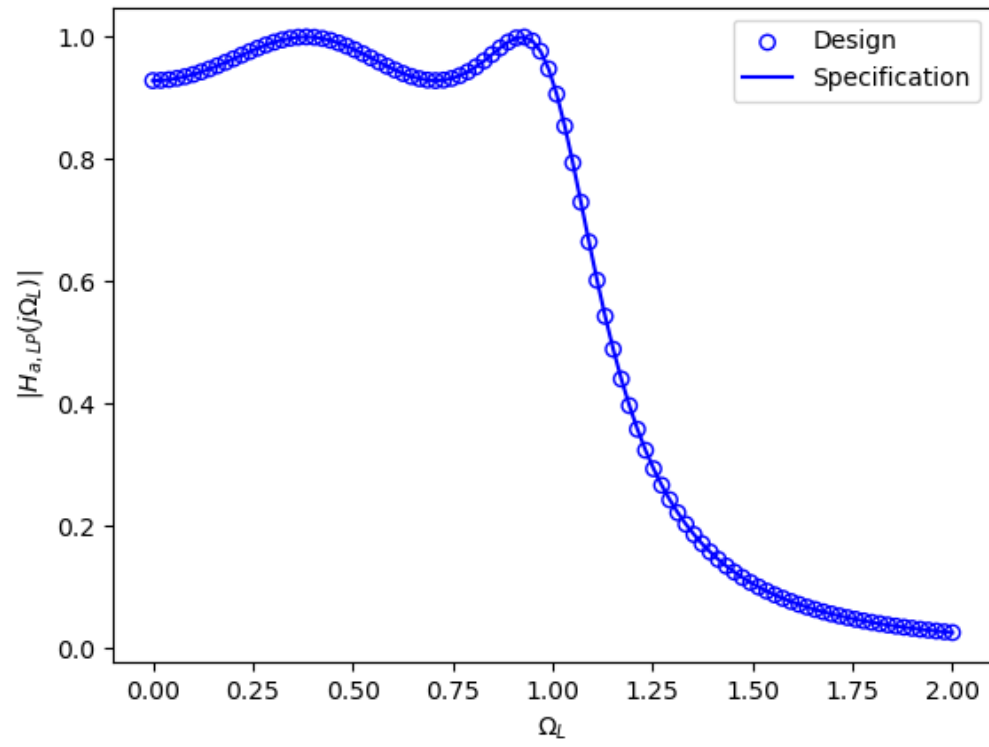


Figure 3: The magnitude response plots from the specifications in Equation 32 and the design in Equation 38

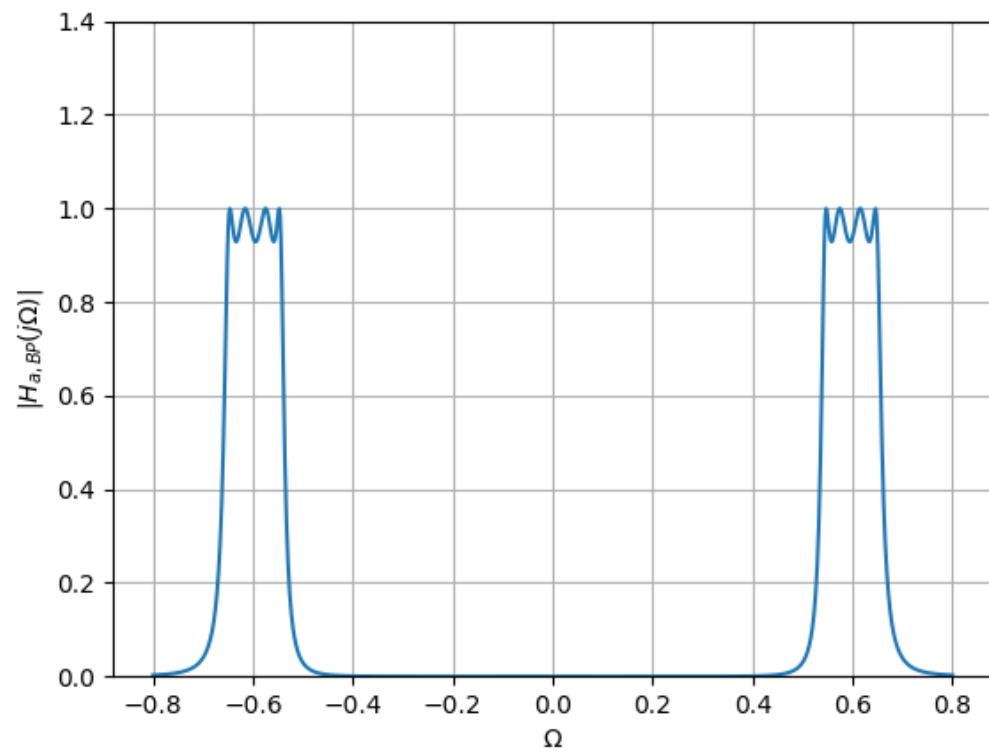


Figure 4: The analog bandpass magnitude response plot from Equation 40



where  $G = 4.3489 \times 10^{-5}$ ,

$$N(z) = 1 - 4z^{-2} + 6z^{-4} - 4z^{-6} + z^{-8} \quad (44)$$

and

$$\begin{aligned} D(z) = & 3.6830 - 13.7277z^{-1} + 33.2138z^{-2} - 51.2028z^{-3} + 59.5578z^{-4} \\ & -49.0243z^{-5} + 30.4476z^{-6} - 12.0480z^{-7} + 3.0950z^{-8} \end{aligned} \quad (45)$$

Again the substitution is done by the code,

[https://github.com/Ameen-etc/Filter-Design/blob/main/codes/coeff\\_digital.py](https://github.com/Ameen-etc/Filter-Design/blob/main/codes/coeff_digital.py)

And the the coefficients are then stored in this .txt file,

[https://github.com/Ameen-etc/Filter-Design/blob/main/codes/coefficients\\_digital.txt](https://github.com/Ameen-etc/Filter-Design/blob/main/codes/coefficients_digital.txt)

The plot of  $|H_{d,BP}(z)|$  with respect to the normalized angular frequency (normalizing factor  $\pi$ ) is available in Figure 5. Again we find that the passband and stopband frequencies meet the specifications well enough.

## 4 The FIR Filter

We design the FIR filter by first obtaining the (non-causal) lowpass equivalent using the Kaiser window and then converting it to a causal bandpass filter.

### 4.1 The Equivalent Lowpass Filter

The lowpass filter has a passband frequency  $\omega_l$  and transition band  $\Delta\omega = 2\pi\frac{\Delta F}{F_s} = 0.0125\pi$ . The stopband tolerance is  $\delta$ .

1. The *passband frequency*  $\omega_l$  is defined as  $\omega_l = \frac{\omega_{p1} - \omega_{p2}}{2}$ . Substituting the values of  $\omega_{p1}$  and  $\omega_{p2}$  from section 2.1, we obtain  $\omega_l = 0.025\pi$ .
2. The *impulse response*  $h_{lp}(n)$  of the desired lowpass filter with cutoff frequency  $\omega_l$  is given by

$$h_l(n) = \frac{\sin(n\omega_l)}{n\pi} w(n), \quad (46)$$

where  $w(n)$  is the Kaiser window obtained from the design specifications.

### 4.2 The Kaiser Window

The Kaiser window is defined as

$$\begin{aligned} w(n) &= \frac{I_0 \left[ \beta N \sqrt{1 - \left( \frac{n}{N} \right)^2} \right]}{I_0(\beta N)}, & -N \leq n \leq N, & \quad \beta > 0 \\ &= 0 & \text{otherwise,} & \end{aligned} \quad (47)$$

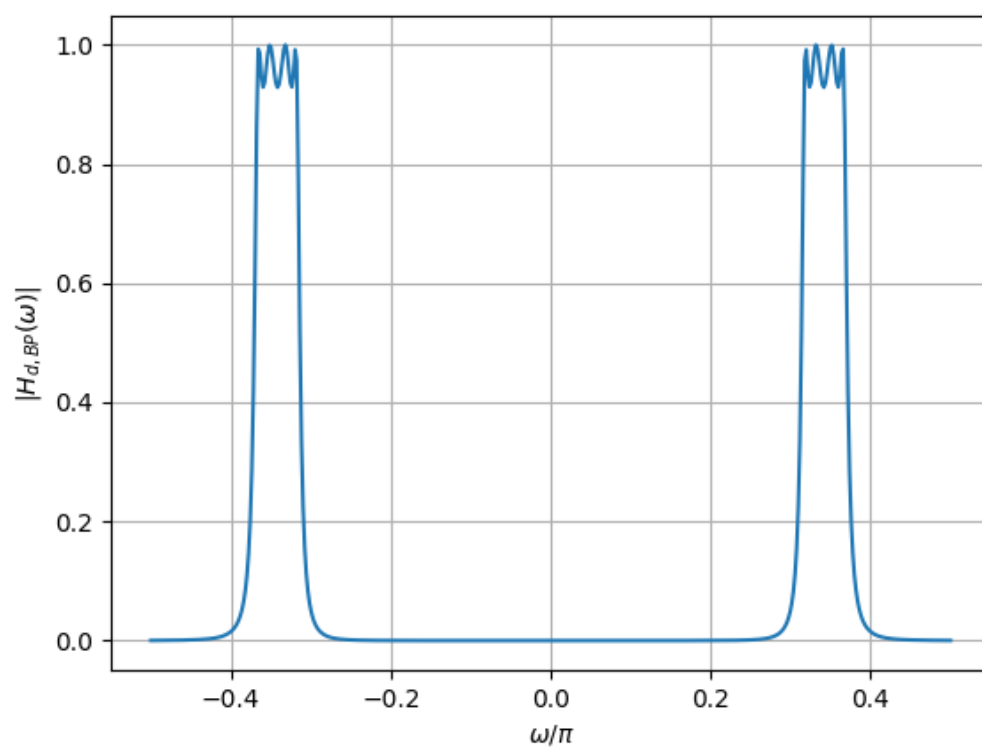


Figure 5: The magnitude response of the bandpass digital filter designed to meet the given specifications

where  $I_0(x)$  is the modified Bessel function of the first kind of order zero in  $x$  and  $\beta$  and  $N$  are the window shaping factors. In the following, we find  $\beta$  and  $N$  using the design parameters in section 2.1.

1.  $N$  is chosen according to

$$N \geq \frac{A - 8}{4.57\Delta\omega}, \quad (48)$$

where  $A = -20 \log_{10} \delta$ . Substituting the appropriate values from the design specifications, we obtain  $A = 16.4782$  and  $N \geq 48$ .

2.  $\beta$  is chosen according to

$$\beta N = \begin{cases} 0.1102(A - 8.7) & A > 50 \\ 0.5849(A - 21)^{0.4} + 0.07886(A - 21) & 21 \leq A \leq 50 \\ 0 & A < 21 \end{cases} \quad (49)$$

In our design, we have  $A = 16.4782 < 21$ . Hence, from (49) we obtain  $\beta = 0$ .

3. We choose  $N = 100$ , to ensure the desired low pass filter response. Substituting in (47) gives us the rectangular window

$$\begin{aligned} w(n) &= 1, \quad -100 \leq n \leq 100 \\ &= 0 \quad \text{otherwise} \end{aligned} \quad (50)$$

From (46) and (50), we obtain the desired lowpass filter impulse response

$$\begin{aligned} h_{lp}(n) &= \frac{\sin(\frac{n\pi}{40})}{n\pi} \quad -100 \leq n \leq 100 \\ &= 0, \quad \text{otherwise} \end{aligned} \quad (51)$$

The response of the filter in (51) is shown in Figure 6.

### 4.3 The FIR Bandpass Filter

The centre of the passband of the desired bandpass filter was found to be  $\omega_c = 0.275\pi$  in Section 2.1. The impulse response of the desired bandpass filter is obtained from the impulse response of the corresponding lowpass filter as

$$h_{bp}(n) = 2h_{lp}(n)\cos(n\omega_c) \quad (52)$$

Thus, from (51), we obtain

$$\begin{aligned} h_{bp}(n) &= \frac{2 \sin(\frac{n\pi}{40}) \cos(0.342n\pi)}{n\pi} \quad -100 \leq n \leq 100 \\ &= 0, \quad \text{otherwise} \end{aligned} \quad (53)$$

The frequency response of the FIR bandpass filter designed to meet the given specifications is plotted in Figure 7.

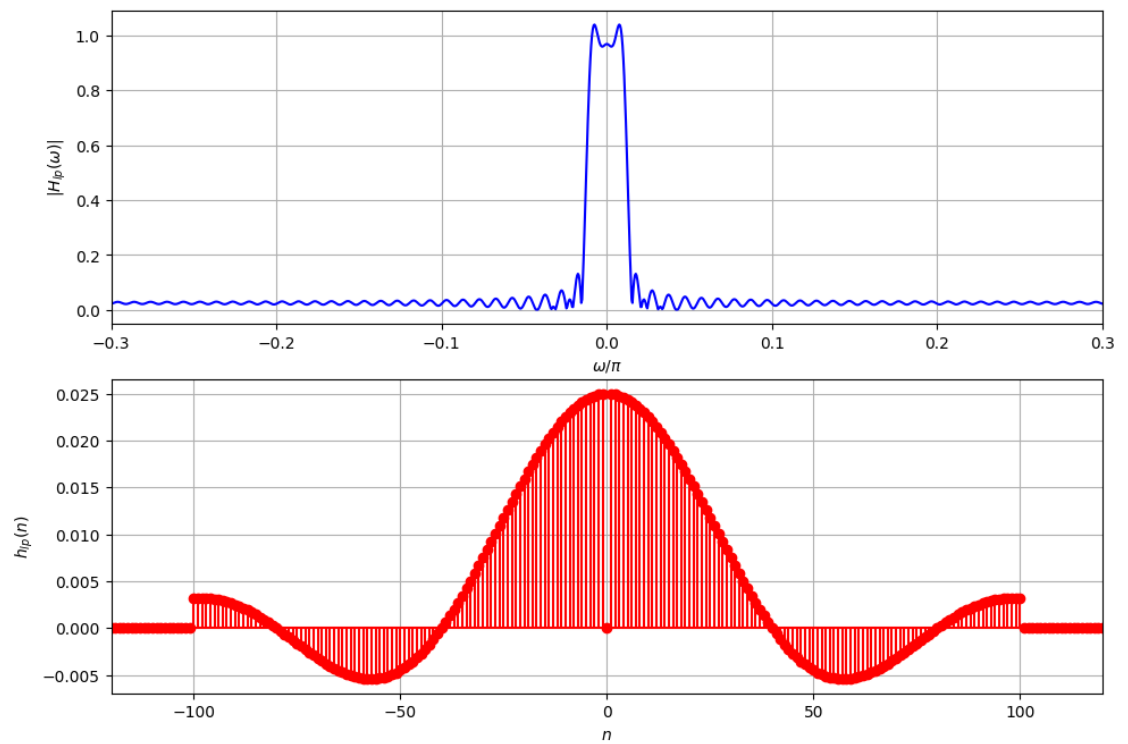


Figure 6: The frequency and the impulse response of the FIR lowpass digital filter designed to meet the given specifications

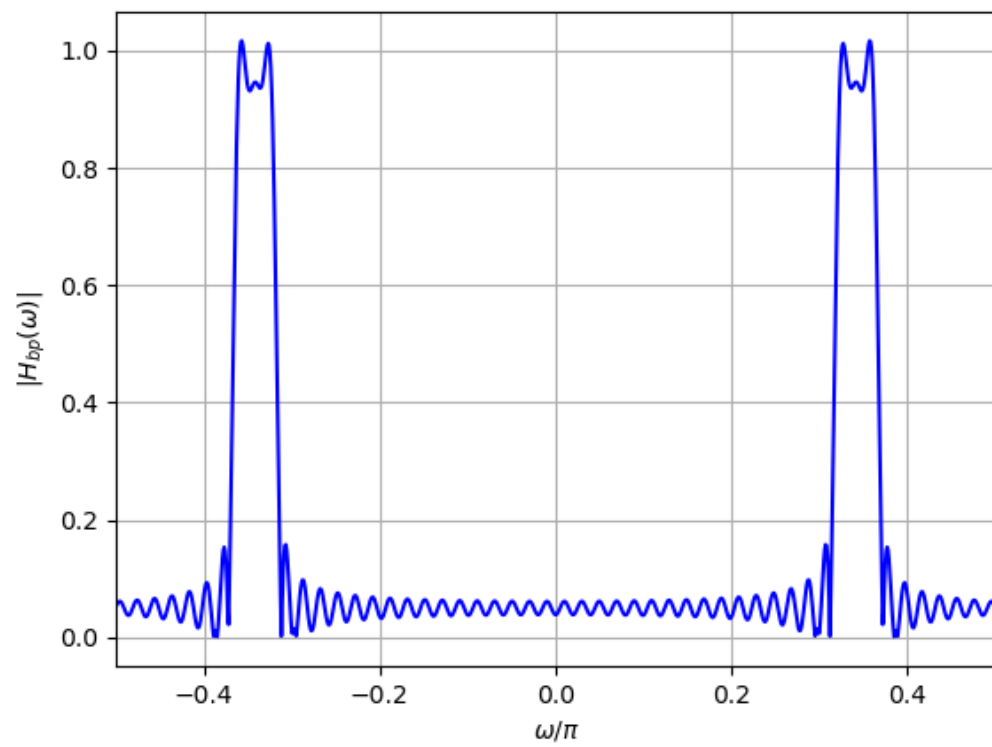


Figure 7: The frequency response of the FIR bandpass digital filter designed to meet the given specifications