DISCRETE

EE23BTECH11006 - Ameen Aazam*

Question : The time-dependent growth of a bacterial population is governed by the equation

$$\frac{dx}{dt} = x\left(1 - \frac{x}{200}\right) \tag{1}$$

where x is the population size at time t. The initial population size is $x_0 = 100$ at $x = 0$. As $t \to \infty$, the population size of bacteria asymptotically approaches .... (GATE BM 2023)

Solution: The growth equation is given by,

Parameters	Values	Description
t		Time, Independent variable
$x(t)$		Population size at any time
$x(0)$	100	Initial population
h		Step size
$x(n)$		Discrete-Time approximation of $x(t)$

TABLE 0
PARAMETERS

$$\frac{dx}{dt} = x\left(1 - \frac{x}{200}\right) \tag{2}$$

$$\implies dx(t)dt = \frac{1}{200}x(200 - x) \tag{3}$$

$$\implies \frac{1}{200}\left(\int_0^{x(t)} \frac{dx}{200 - x} + \int_0^{x(t)} \frac{dx}{x}\right) = \int_0^t \frac{dt}{200} \tag{4}$$

$$\implies -\ln(200 - x)\Big|_{100}^{x(t)} + \ln(x)\Big|_{100}^{x(t)} = t \tag{5}$$

$$\implies \frac{x(t)}{200 - x(t)} = e^t \tag{6}$$

$$\implies x(t) = \frac{200}{1 + e^{-t}} \tag{7}$$

We can also express the growth equation as,

$$\implies \int_0^{x(t)} dx = \int_0^t x dt - \frac{1}{200}\int_0^t x^2 dt \tag{8}$$

Now approximating by trapezoidal rule of integration between t_{n-1} to t_n with the step size being h we have,

$$\begin{aligned} x(t_n) - x(t_{n-1}) =& \frac{h}{2}\left[x(t_n) + x(t_{n-1})\right] - \\ & \frac{h}{400}\left[x^2(t_n) + x^2(t_{n-1})\right] \end{aligned} \tag{9}$$

Next, replacing $t_n = hn$ we get the difference equation,

$$x(hn) = \frac{-\left(1 - \frac{h}{2}\right) + \sqrt{\left(1 - \frac{h}{2}\right)^2 + \frac{h}{100}p_1}}{h/200} \tag{10}$$

Where,

$$p_1 = \left[\left(1 - \frac{h}{2}\right)x(h(n-1)) - \frac{h}{400}x^2(h(n-1))\right] \tag{11}$$

And relabeling $hn \longleftrightarrow n$ we will be having the discrete time equation as,

$$x(n) = \frac{-\left(1 - \frac{h}{2}\right) + \sqrt{\left(1 - \frac{h}{2}\right)^2 + \frac{h}{100}p_2}}{h/200} \tag{12}$$

Where,

$$p_2 = \left[\left(1 - \frac{h}{2}\right)x(n-1) - \frac{h}{400}x^2(n-1)\right] \tag{13}$$

Now, plotting both the differential and the difference equations,

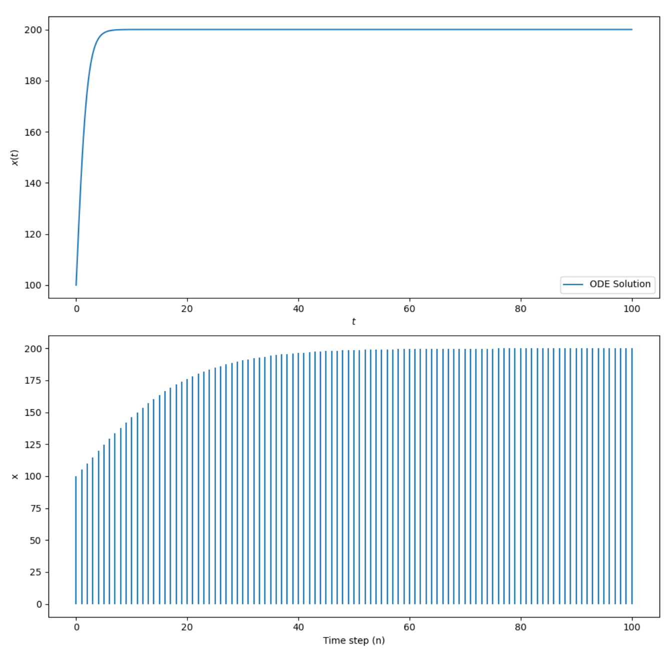

Fig. 0. Approximation in Discrete-Time

As we can see the discrete time plot is following the actual curve, which means (12) is indeed a good approximation of the original continuous-time equation.
And the population size approaches to 200 as $t \to \infty$.