DISCRETE

EE23BTECH11006 - Ameen Aazam*

Question : The time-dependent growth of a bacterial population is governed by the equation

$$\frac{dx}{dt} = x\left(1 - \frac{x}{200}\right) \tag{1}$$

where x is the population size at time t. The initial population size is $x_0 = 100$ at $x = 0$. As $t \to \infty$, the population size of bacteria asymptotically approaches .... (GATE BM 2023)

Solution: The growth equation is given by,

Parameters	Values	Description
t		Time, Independent variable
$x(t)$		Population size at any time
$x(0)$	100	Initial population
h		Step size
$x(n)$		Discrete-Time approximation of $x(t)$

TABLE 0
PARAMETERS

$$\frac{dx(t)}{dt} = x(t)\left(1 - \frac{x(t)}{200}\right) \tag{2}$$

$$\implies \int_0^t d[x(t)] = \int_0^t x(t)\,dt - \frac{1}{200}\int_0^t x^2(t)\,dt \tag{3}$$

So the growth curve, we get is as shown in Fig. 0,

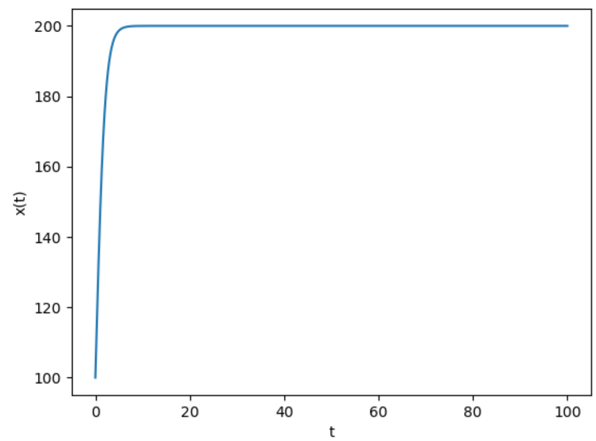

Fig. 0. $x(t)$ vs. t

Now approximating by trapezoidal rule of integration between t_{n-1} to t_n with the step size being h we have,

$$x(t_n) - x(t_{n-1}) = \frac{h}{2}\left[x(t_n) + x(t_{n-1})\right] - \frac{h}{400}\left[x^2(t_n) + x^2(t_{n-1})\right] \tag{4}$$

Next, replacing $t_n = hn$ we get the difference equation,

$$x(hn) = \frac{-\left(1 - \frac{h}{2}\right) + \sqrt{\left(1 - \frac{h}{2}\right)^2 + \frac{h}{100}p_1}}{h/200} \tag{5}$$

Where,

$$p_1 = \left[\left(1 - \frac{h}{2}\right)x(h(n-1)) - \frac{h}{400}x^2(h(n-1))\right] \tag{6}$$

And relabeling $hn \longleftrightarrow n$ we will be having the discrete time equation as,

$$x(n) = \frac{-\left(1 - \frac{h}{2}\right) + \sqrt{\left(1 - \frac{h}{2}\right)^2 + \frac{h}{100}p_2}}{h/200} \tag{7}$$

Where,

$$p_2 = \left[\left(1 - \frac{h}{2}\right)x(n-1) - \frac{h}{400}x^2(n-1)\right] \tag{8}$$

Now, plotting the difference equation,

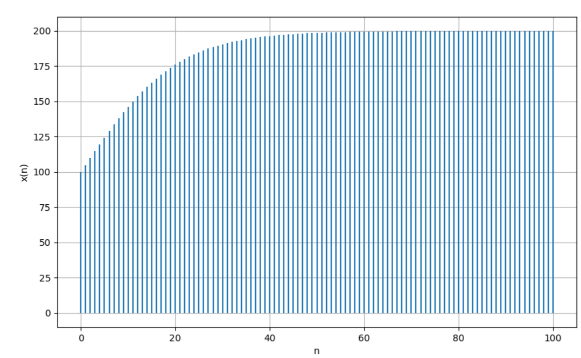

Fig. 0. Approximation in Discrete-Time

As we can see the discrete time plot is following the actual curve, which means (7) is indeed a good approximation of the original continuous-time equation.

And the population size approaches to 200 as $t \to \infty$.