

DISCRETE

EE23BTECH11006 - Ameen Aazam*

Question : The time-dependent growth of a bacterial population is governed by the equation

$$\frac{dx}{dt} = x \left(1 - \frac{x}{200} \right) \quad (1)$$

where x is the population size at time t . The initial population size is $x_0 = 100$ at $x = 0$. As $t \rightarrow \infty$, the population size of bacteria asymptotically approaches (GATE BM 2023)

Solution: The growth equation is given by,

Parameters	Values	Description
t		Time, Independent variable
$x(t)$		Population size at any time
$x(0)$	100	Initial population
h		Step size
$x(n)$		Discrete-Time approximation of $x(t)$

TABLE 0
PARAMETERS

$$\frac{dx}{dt} = x \left(1 - \frac{x}{200} \right) \quad (2)$$

$$\Rightarrow dx(t)dt = \frac{1}{200} x (200 - x) \quad (3)$$

$$\Rightarrow \frac{1}{200} \left(\int_0^{x(t)} \frac{dx}{200 - x} + \int_0^{x(t)} \frac{dx}{x} \right) = \int_0^t \frac{dt}{200} \quad (4)$$

$$\Rightarrow -\ln(200 - x)|_{100}^{x(t)} + \ln(x)|_{100}^{x(t)} = t \quad (5)$$

$$\Rightarrow \frac{x(t)}{200 - x(t)} = e^t \quad (6)$$

$$\Rightarrow x(t) = \frac{200}{1 + e^{-t}} \quad (7)$$

We can also express the growth equation as,

$$\Rightarrow \int_0^{x(t)} dx = \int_0^t x dt - \frac{1}{200} \int_0^t x^2 dt \quad (8)$$

Now approximating by trapezoidal rule of integration between t_{n-1} to t_n with the step size being h we have,

$$x(t_n) - x(t_{n-1}) = \frac{h}{2} [x(t_n) + x(t_{n-1})] - \frac{h}{400} [x^2(t_n) + x^2(t_{n-1})] \quad (9)$$

Next, replacing $t_n = hn$ we get the difference equation,

$$x(hn) = \frac{-\left(1 - \frac{h}{2}\right) + \sqrt{\left(1 - \frac{h}{2}\right)^2 + \frac{h}{100}p_1}}{h/200} \quad (10)$$

Where,

$$p_1 = \left[\left(1 - \frac{h}{2}\right) x(h(n-1)) - \frac{h}{400} x^2(h(n-1)) \right] \quad (11)$$

And relabeling $hn \longleftrightarrow n$ we will be having the discrete time equation as,

$$x(n) = \frac{-\left(1 - \frac{h}{2}\right) + \sqrt{\left(1 - \frac{h}{2}\right)^2 + \frac{h}{100}p_2}}{h/200} \quad (12)$$

Where,

$$p_2 = \left[\left(1 - \frac{h}{2}\right) x(n-1) - \frac{h}{400} x^2(n-1) \right] \quad (13)$$

Now, plotting both the differential and the difference equations,

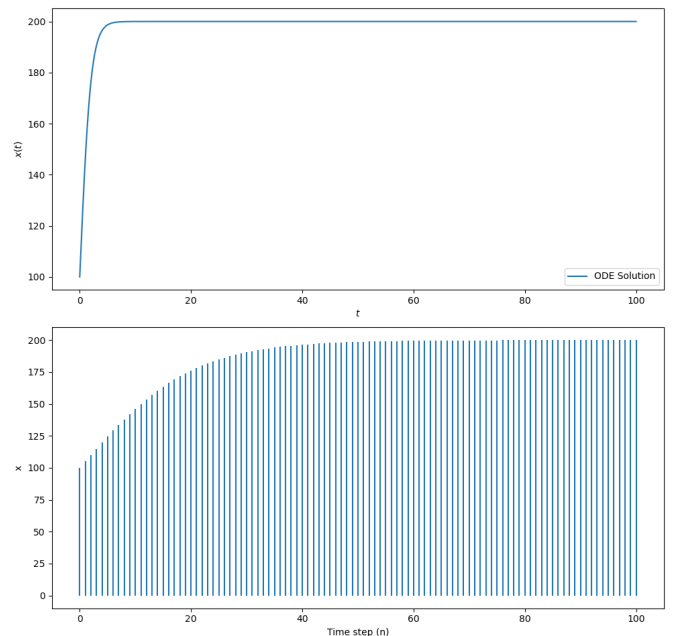


Fig. 0. Approximation in Discrete-Time

As we can see the discrete time plot is following the actual curve, which means (??) is indeed a good approximation of the original continuous-time equation.

And the population size approaches to 200 as $t \rightarrow \infty$.