DISCRETE

EE23BTECH11006 - Ameen Aazam*

Question : The time-dependent growth of a bacterial population is governed by the equation

$$\frac{dx}{dt} = x\left(1 - \frac{x}{200}\right) \tag{1}$$

where x is the population size at time t. The initial population size is $x_0 = 100$ at $x = 0$. As $t \to \infty$, the population size of bacteria asymptotically approaches .... (GATE BM 2023)

Solution: The growth equation is given by,

Parameters	Values	Description
t		Time, Independent variable
$x(t)$		Population size at any time
$x(0)$	100	Initial population
h		Step size
$x(n)$		Discrete-Time approximation of $x(t)$

TABLE 0
PARAMETERS

$$\frac{dx}{dt} = x\left(1 - \frac{x}{200}\right) \tag{2}$$

$$dx(t)dt = \frac{1}{200}x(200 - x) \tag{3}$$

$$\frac{1}{200}\left(\int_0^{x(t)} \frac{dx}{200-x} + \int_0^{x(t)} \frac{dx}{x}\right) = \int_0^t \frac{dt}{200} \tag{4}$$

$$-\ln(200-x)\big|_{100}^{x(t)} + \ln(x)\big|_{100}^{x(t)} = t \tag{5}$$

$$\frac{x(t)}{200 - x(t)} = e^t \tag{6}$$

$$\implies x(t) = \frac{200}{1 + e^{-t}} \tag{7}$$

We can also express the growth equation as,

$$\int_0^{x(t)} dx = \int_0^t x dt - \frac{1}{200}\int_0^t x^2 dt \tag{8}$$

Now approximating by trapezoidal rule of integration between t_{n-1} to t_n with the step size being h we have,

$$x(t_n) - x(t_{n-1}) = \frac{h}{2}\left[x(t_n) + x(t_{n-1})\right] - \frac{h}{400}\left[x^2(t_n) + x^2(t_{n-1})\right] \tag{9}$$

Next, replacing $t_n = hn$ we get the difference equation,

$$x(hn) = \frac{-\left(1 - \frac{h}{2}\right) + \sqrt{\left(1 - \frac{h}{2}\right)^2 + \frac{h}{100}p_1}}{h/200} \tag{10}$$

Where,

$$p_1 = \left[\left(1 - \frac{h}{2}\right)x(h(n-1)) - \frac{h}{400}x^2(h(n-1))\right] \tag{11}$$

And relabeling $hn \longleftrightarrow n$ we will be having the discrete time equation as,

$$x(n) = \frac{-\left(1 - \frac{h}{2}\right) + \sqrt{\left(1 - \frac{h}{2}\right)^2 + \frac{h}{100}p_2}}{h/200} \tag{12}$$

Where,

$$p_2 = \left[\left(1 - \frac{h}{2}\right)x(n-1) - \frac{h}{400}x^2(n-1)\right] \tag{13}$$

Now, plotting both the differential and the difference equations,

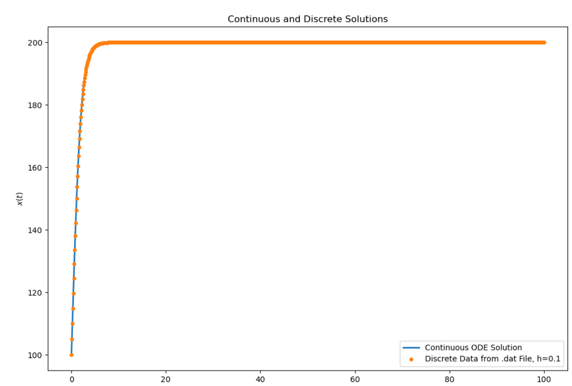

Fig. 0. Plot in Continuous and Discrete Time

As we can see the discrete time plot is following the actual curve, which means (12) is indeed a good approximation of the original continuous-time equation.
And the population size approaches to 200 as $t \to \infty$.