Finding the Optimal 2 Stage RK Method

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Assume a method of the form

$$\begin{aligned} k_1 &= f(t_n, y_n) \\ k_2 &= f(t_n + ah, y_n + bhk_1) \\ y_{n+1} &= y_n + h(ck_1 + dk_2). \end{aligned}$$

To analyze truncation error we assume $y_n = y(t_n)$. Then

$$\begin{aligned} k_2 &= \left(f + ahf_t + bhk_1 f_y + \frac{1}{2}a^2h^2 f_{tt} + abh^2 k_1 f_{ty} + \frac{1}{2}b^2h^2k_1^2 f_{yy}\right)(t_n, y_n) + O(h^3) \\ &= \left(f + h(af_t + bff_y) + h^2\left(\frac{1}{2}a^2 f_{tt} + abff_{ty} + \frac{1}{2}b^2 f^2 f_{yy}\right)\right)(t_n, y_n) + O(h^3). \end{aligned}$$

Thus

$$y_{n+1} = y_n + \left(h(c+d)f + h^2(adf_t + bdff_y) + h^3 d\left(\frac{1}{2}a^2 f_{tt} + abff_{ty} + \frac{1}{2}b^2 f^2 f_{yy}\right)\right)(t_n, y_n) + O(h^4).$$

We have

$$\begin{aligned} f' &= f_t + f_y f, \\ f'' &= f_{tt} + f_{ty} f + (f_{yt} + f_{yy} f)f + f_y(f_t + f_y f) \\ &= f_{tt} + 2f_{ty} f + f_{yy} f^2 + f_y f_t + f_y^2 f. \end{aligned}$$

So

$$y(t_n + h) = y_n + \left(hf + \frac{h^2}{2}(f_t + f_y f) + h^3 \frac{f''}{6}\right)(t_n, y_n) + O(h^4).$$

Equating coefficients gives

$$\begin{aligned} c + d &= 1, \\ ad &= \frac{1}{2}, \\ bd &= \frac{1}{2}. \end{aligned}$$

To minimize the LTE we need to minimize

$$\begin{aligned} \frac{y_{n+1} - y(t_n + h)}{h^3} &= d\left(\frac{1}{2}a^2 f_{tt} + abf f_{ty} + \frac{1}{2}b^2 f^2 f_{yy}\right) - \frac{1}{6}(f_{tt} + 2f_{ty}f + f_{yy}f^2 + f_y f_t + f_y^2 f) + O(h) \\ &= da^2\left(\frac{1}{2}f_{tt} + f f_{ty} + \frac{1}{2}f^2 f_{yy}\right) - \frac{1}{6}(f_{tt} + 2f_{ty}f + f_{yy}f^2 + f_y f_t + f_y^2 f) + O(h). \end{aligned}$$

so to make as many terms 0 as possible we need

$$\begin{aligned} \frac{da^2}{2} &= \frac{1}{6}, \\ da^2 &= \frac{1}{3} \\ \frac{da^2}{2} &= \frac{1}{6}, \end{aligned}$$

So

$$da^2 = \frac{1}{3}.$$

This uniquely determines the coefficients

$$\begin{aligned} a &= \frac{2}{3}, \\ b &= \frac{2}{3}, \\ c &= \frac{1}{4}, \\ d &= \frac{3}{4}. \end{aligned}$$

The LTE is

$$y_{n+1} - y(t_n + h) = -\frac{1}{6}\left(f_y f_t + f_y^2 f\right)(t_n, y_n)h^3 + O(h^4).$$