STOR 635 HW 1

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1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a σ-finite measure space, and let $p \in [1, \infty]$. Show that if f, $g \in L^p$, then so is $f + g$ and

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

Show that equality holds for $p \in (1, \infty)$ iff for some nonnegative constants a, b, with at least one being positive, $a f = b g$ a.e.

Proof. First we deal with the case $p = \infty$. Suppose $f, g \in L^\infty$. By definition, we have $f(x) \leq \|f\|_\infty$ for a.e $x \in \Omega$ and $|g(x)| \leq \|g\|_\infty$ for a.e. $x \in \Omega$. Thus for a.e. $x \in \Omega$, we have $|f(x) + g(x)| \leq \|f\|_\infty + \|g\|_\infty$. Thus $\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$.

Now we consider the case $p \in [1, \infty)$. Let $f, g \in L^p$ be arbitrary. The cases where one of $\|f\|_p$ or $\|g\|_p$ are 0 are trivial, so assume $\|f\|_p > 0$ and $\|g\|_p > 0$. We have

$$\begin{aligned} |f + g|^p &\leq (2 \max(|f|, |g|))^p \\ &= 2^p \max(|f|, |g|)^p \\ &\leq 2^p (|f|^p + |g|^p). \end{aligned}$$

Thus $f + g \in L^p$. Let $f_0 = \frac{f}{\|f\|_p}$, $g_0 = \frac{g}{\|g\|_p}$. Note by elementary calculus that the map $t \mapsto t^p$ mapping $[0, \infty)$ into $\mathbb{R}$ is convex, and is strictly convex when $p > 1$. Thus $z \mapsto |z|^p$ is a convex map from $\mathbb{C}$ to $\mathbb{R}$, and is strictly convex when $p > 1$. Thus for any $\lambda \in (0, 1)$,

$$|\lambda f_0 + (1 - \lambda) g_0|^p \leq \lambda |f_0|^p + (1 - \lambda) |g_0|^p. \tag{1}$$

Noting that $\|f_0\|_p = \|g_0\|_p = 1$, integrating both sides gives

$$\|\lambda f_0 + (1 - \lambda) g_0\|_p^p \leq \lambda + 1 - \lambda = 1. \tag{2}$$

Taking pth roots gives

$$\|\lambda f_0 + (1 - \lambda) g_0\|_p \leq 1.$$

Now choosing $\lambda = \frac{\|f\|_p}{\|f\|_p + \|g\|_p} \in (0, 1)$ yields

$$\frac{\|f + g\|_p}{\|f\|_p + \|g\|_p} \leq 1.$$

Hence

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

The conditions for equality are different when $p = 1$ and when $p > 1$, as we now show. Assume $\|f + g\|_p = \|f\|_p + \|g\|_p$. Again, we assume f and g are both nonzero. Assume $p > 1$. Having equality $\|f + g\|_p = \|f\|_p + \|g\|_p$ implies that with $\lambda = \frac{\|f\|_p}{\|f\|_p + \|g\|_p}$, we must have equality in (2) after integrating (1). Hence

$$\int (\lambda |f_0|^p + (1 - \lambda) |g_0|^p - |\lambda f_0 + (1 - \lambda) g_0|^p) = 0.$$

Since the integrand is nonnegative, it follows that it is 0 (a.e.). Hence we have the (a.e.) equality

$$|\lambda f_0 + (1 - \lambda) g_0|^p = \lambda |f_0|^p + (1 - \lambda) |g_0|^p.$$

In this case, since $z \mapsto |z|^p$ is strictly convex and $\lambda \in (0, 1)$, it follows that $f_0 = g_0$ (a.e.). Hence $f = \|f\|_p \frac{g}{\|g\|_p}$. Conversely, if f, g are both nonzero and $f = a g$ for some $a > 0$, then

$$\|f + g\|_p = \|(a + 1) g\|_p = (a + 1)\|g\|_p = \|a g\|_p + \|g\|_p = \|f\|_p + \|g\|_p.$$

Now conside the case $p = 1$. By a similar argument as above, we can conclude the equality

$$|f(x) + g(x)| = |f(x)| + |g(x)| \text{ for a.e. } x \in \Omega.$$

The above is simply equality in the triangle inequality for complex numbers, which, by the Cauchy-Schwarz inequality in $\mathbb{R}^2$, happens if and only if $f(x)$ is a nonnegative multiple of $g(x)$. Thus for a.e. $x \in \Omega$, there exists $\alpha(x) \geq 0$ such that $f(x) = \alpha(x) g(x)$. Thus with

$$a(x) = \begin{cases} \frac{f(x)}{g(x)} & \text{if } g(x) \neq 0 \\ 0 & \text{if } g(x) = 0 \end{cases},$$

we have that a is nonnegative and measurable with $f = a g$ (a.e.). Conversely, if there is a measurable nonnegative function $a : \Omega \to [0, \infty)$ such that $f = a g$, then

$$\int |f + g| = \int (a + 1) |g| = \int a |g| + \int |g| = \int |f| + \int |g|. \qquad \square$$

2. Let $(\Omega, \mathcal{F}, P) = ([0, 1], \mathcal{B}([0, 1]), m)$, where m is the Lebesgue measure on $[0, 1]$. Let $X_n = a_n 1_{[0,1/n]}$. Give necessary and sufficient conditions on the sequence a_n for $\{X_n\}$ to be uniformly integrable.

Proof. Note that $X_n \to 0$ pointwise on $(0, 1]$. Hence $X_n \to 0$ almost everywhere, and therefore in measure, since $m([0, 1]) = 1 < \infty$. By the theorem proved in class (basically the Vitali convergence theorem), it follows that $\{X_n\}$ is uniformly integrable if and only if $X_n \to 0$ in L^1. Since $\|X_n\|_1 = \frac{|a_n|}{n}$, this happens if and only if $\lim_{n \to \infty} \frac{|a_n|}{n} = 0$. $\square$

3. Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space. Suppose that $\mathcal{H}_1$ and $\mathcal{H}_2$ are classes of real measurable functions on Ω such that $\mathcal{H}_2$ is uniformly integrable. Suppose that for every $f \in \mathcal{H}_1$, there is a $g \in \mathcal{H}_2$ such that $|f| \leq |g|$. Show that $\mathcal{H}_1$ is uniformly integrable.

Proof. Let $f \in \mathcal{H}_1$ be arbitrary. Pick $g \in \mathcal{H}_2$ such that $|f| \leq |g|$. Then for any $M > 0$,

$$\begin{aligned} \int_{|f|>M} |f| &\leq \int_{|f|>M} |g| \\ &\leq \int_{|g|>M} |g| \\ &\leq \sup_{h \in \mathcal{H}_2} \int_{|h|>M} |h| \end{aligned}$$

Taking the sup over $f \in \mathcal{H}_1$ yields

$$\sup_{f \in \mathcal{H}_1} \int_{|f|>M} |f| \leq \sup_{h \in \mathcal{H}_2} \int_{|h|>M} |h|.$$

Since, by assumption, the right hand side converges to 0 as $M \to \infty$, the proof is complete. □

4. Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space. Suppose that $\mathcal{H}_1$ and $\mathcal{H}_2$ are uniformly integrable classes of real measurable functions on Ω. Show that $\mathcal{H}_1 \pm \mathcal{H}_2 = \{f_1 \pm f_2 : f_i \in \mathcal{H}_i, i = 1, 2\}$ is a uniformly integrable family as well.

Solution: We first prove a lemma.

Lemma 1. *A family $\mathcal{H} \subset L^1(\Omega, \mu)$ is uniformly integrable if and only if*

$$\lim_{a \to \infty} \sup_{f \in \mathcal{H}} \int (|f| - a)^+ \, d\mu = 0,$$

where $x^+ = \max(x, 0)$.

Proof. $(\Longrightarrow)$ Assume $\mathcal{H}$ is uniformly integrable. Then for any $a \geq 0$, $f \in \mathcal{H}$,

$$\int (|f| - a)^+ \, d\mu = \int_{|f|>a} (|f| - a) \, d\mu \leq \int_{|f|>a} |f| \, d\mu.$$

Hence

$$\sup_{f \in \mathcal{H}} \int (|f| - a)^+ \, d\mu \leq \sup_{f \in \mathcal{H}} \int_{|f|>a} |f| \, d\mu \to 0 \quad \text{as } a \to \infty.$$

$(\Longleftarrow)$ Assume $\lim_{a \to \infty} \sup_{f \in \mathcal{H}} \int (|f| - a)^+ \, d\mu = 0$. Let $\varepsilon > 0$. Pick $a \geq 0$ such that

$$\sup_{f \in \mathcal{H}} \int (|f| - a)^+ \, d\mu \leq \varepsilon.$$

For $f \in \mathcal{H}$, we have

$$\begin{aligned} \int_{|f|>2a} |f| &= \int_{|f|>2a} (|f| - a) + \int_{|f|>2a} a \\ &\leq \int_{|f|>a} (|f| - a)^+ + \int_{|f|>2a} (|f| - a) \\ &\leq \int_{|f|>a} (|f| - a)^+ + \int_{|f|>a} (|f| - a)^+ \\ &\leq 2\,\varepsilon. \end{aligned}$$

Hence $\mathcal{H}$ is uniformly integrable. $\square$

Now we give the proof for problem 4.

Proof. Since the union of two uniformly integrable families of functions is clearly uniformly integrable, it suffices to show that both $\mathcal{H}_1+\mathcal{H}_2$ and $\mathcal{H}_1-\mathcal{H}_2$ are both unfiformly integrable. For $f_1\in\mathcal{H}_1$, $f_2\in\mathcal{H}_2$, $a_1,a_2\geq 0$, we have

$$\begin{aligned}\int (|f_1+f_2|-(a_1+a_2))^+ &\leq \int (|f_1|-a_1+|f_2|-a_2)^+\\ &\leq \int (|f_1|-a_1)^+ + \int (|f_2|-a_2)^+,\end{aligned}$$

where we used the inequality $(x+y)^+\leq x^+ + y^+$, valid for $x,y\in\mathbb{R}$. Pick $a_1,a_2\geq 0$ such that

$$\sup_{f\in\mathcal{H}_i}\int (|f|-a_i)^+\leq\varepsilon \text{ for } i=1,2.$$

Then

$$\sup_{f_1\in\mathcal{H}_1, f_2\in\mathcal{H}_2}\int (|f_1+f_2|-(a_1+a_2))^+\leq 2\,\varepsilon.$$

Thus $\mathcal{H}_1+\mathcal{H}_2$ is uniformly integrable. Since $|-f|=|f|$, it follows that $-\mathcal{H}_2$ is uniformly integrable. Thus $\mathcal{H}_1+(-\mathcal{H}_2)=\mathcal{H}_1-\mathcal{H}_2$ is uniformly integrable. $\square$

5. Prove the generalized dominated convergence theorem: let $\{f_n\}_{n\geq 1}$ be a sequence of measurable functions defined on the measure space $(\Omega,\mathcal{F},\mu)$, and let $\{g_n\}_{n\geq 1}$ be a sequence of non-negative measurable functions on the same space. Suppose the following hold:

- $|f_n|\leq g_n$ for all $n\geq 1$,
- f_n converges pointwise almost everywhere to f and g_n converges pointwise almost everywhere to g,
- $\lim_{n\to\infty}\int g_n\,d\mu=\int g\,d\mu<\infty$.

Then f is in $L^1(\mu)$ and $\lim_{n\to\infty}\int f_n\,d\mu=\int f\,d\mu$. Also $f_n\to f$ in L^1.

Proof. Since $|f_n|\leq|g_n|$, it follows by taking pointwise limits that $|f|\leq g$ a.e. Since $g\in L^1(\mu)$, it follows that $f\in L^1(\mu)$. Since $\lim_{n\to\infty}\int g_n\,d\mu<\infty$, it follows that $g_n\in L^1(\mu)$ for n large. Hence $f_n\in L^1(\mu)$ for n large. Hence $|f_n-f|\in L^1(\mu)$ for n large. Note that $|f_n-f|\leq g_n+g$. By Fatou's lemma applied to $g_n+g-|f_n-f|\geq 0$,

$$\begin{aligned}2\int g &\leq \liminf_{n\to\infty}\int (g_n+g-|f_n-f|)\\ &= \liminf_{n\to\infty}\left(\int g+\int g_n-\int |f_n-f|\right)\\ &= \int g+\int g+\liminf_{n\to\infty}\left(-\int |f_n-f|\right)\\ &= 2\int g-\limsup_{n\to\infty}\int |f_n-f|.\end{aligned}$$

To get the first equality we used linearity of the integral on $L^1(\mu)$, and to get the second equality we used the fact that if (a_n) and (b_n) are sequences of real numbers, then

$$a_n \to a \quad \Longrightarrow \quad \liminf_{n\to\infty} (a_n + b_n) = a + \liminf_{n\to\infty} b_n,$$

which itself follows from the inequality

$$\liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n \le \liminf_{n\to\infty} (a_n + b_n) \le \limsup_{n\to\infty} a_n + \liminf_{n\to\infty} b_n.$$

Thus $\limsup_{n\to\infty} \int |f_n - f| \le 0$, so $f_n \to f$ in $L^1(\mu)$. The fact that $\int f_n \, d\mu \to \int f \, d\mu$ follows from the continuity of $\int$ on $L^1(\mu)$:

$$\left| \int f_n - \int f \right| \le \int |f_n - f| \to 0. \qquad \square$$

6. Let $(\Omega, \mathcal{F}, \mu)$ be a σ-finite measure space and let $p \in [1, \infty)$. Show that if $f_n \to f$ in L^p, then $|f_n|^p \to |f|^p$ in L^1.

Proof. For contradiction, suppose $|f_n|^p$ does not converge to $|f|^p$ in L^1. Then there exists $\varepsilon > 0$ and a subsequence $(|f_{n_k}|^p)_{k=1}^\infty$ such that

$$\| |f_{n_k}|^p - |f|^p \|_1 \ge \varepsilon \text{ for all } k \ge 1. \tag{3}$$

Since $f_{n_k} \to f$ in L^p, it follows by Chebyshev's inequality that $f_{n_k} \to f$ (globally) in measure. Hence there is a subsequence $(f_{n_{k_j}})_{j=1}^\infty$ such that $f_{n_{k_j}} \to f$ a.e. We have $|f_{n_{k_j}}|^p \to |f|^p$ a.e. and

$$\int |f_{n_{k_j}}|^p = \|f_{n_{k_j}}\|_p^p \to \|f\|_p^p = \int |f|^p < \infty \text{ as } j \to \infty.$$

Thus we can apply the result of problem 5 to the sequence $(|f_{n_{k_j}}|^p)$ with itself as the dominating sequence to conclude that $|f_{n_{k_j}}|^p \to |f|^p$ in $L^1(\mu)$. But this contradicts (3). $\square$