STOR 635 HW 3

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1 Solutions

1. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $\mathcal{G}$ be a sub σ-field of $\mathcal{F}$ and let X be a square integrable random variable in $(\Omega, \mathcal{F}, P)$, that is, $X \in L^2(\Omega, \mathcal{F}, P)$. Show that there exists a unique $Z \in L^2(\Omega, \mathcal{G}, P)$ such that for every $Y \in L^2(\Omega, \mathcal{G}, P)$, $E(XY) = E(ZY)$. Moreover, if $X_1, X_2 \in L^2(\Omega, \mathcal{F}, P)$ such that $X_1 \geq X_2$ almost surely, then $Z_1, Z_2 \in L^2(\Omega, \mathcal{G}, P)$ obtained by the above recipe for X_1, X_2 respectively should satisfy $Z_1 \geq Z_2$ almost surely.

Show that the Z obtained above (for X) satisfies the *projection property*: for any $W \in L^2(\Omega, \mathcal{G}, P)$, $E[(X - Z)^2] \leq E[(X - W)^2]$.

Proof. Note that $L^2(\Omega, \mathcal{G}, P)$ is a linear subspace of $L^2(\Omega, \mathcal{F}, P)$. Since $L^2(\Omega, \mathcal{G}, P)$ is complete, it is closed. By Hilbert space theory, we have the decomposition

$$L^2(\Omega, \mathcal{F}, P) = L^2(\Omega, \mathcal{G}, P) \oplus L^2(\Omega, \mathcal{G}, P)^{\perp}.$$

This means that for every $X \in L^2(\Omega, \mathcal{F}, P)$, there exist unique $Z \in L^2(\Omega, \mathcal{G}, P)$, $V \in L^2(\Omega, \mathcal{G}, P)^{\perp}$ such that

$$X = Z + V.$$

For $X = Z + V$ as above, set $QX = Z$. This defines $Q: L^2(\Omega, \mathcal{F}, P) \to L^2(\Omega, \mathcal{G}, P)$, the orthogonal projection of $L^2(\Omega, \mathcal{F}, P)$ onto $L^2(\Omega, \mathcal{G}, P)$. For future use, we note that Q is linear and that $\|QX\|_{L^2} \leq \|X\|_{L^2}$ since $\|X\|_{L^2}^2 = \|Z\|_{L^2}^2 + \|V\|_{L^2}^2$.

Let $X \in L^2(\Omega, \mathcal{F}, P)$ be arbitrary. Write $X = Z + V$ as above. Note that the inner product on $L^2(\Omega, \mathcal{F}, P)$ is given by $(f, g) = E(fg)$. Thus for $Y \in L^2(\Omega, \mathcal{G}, P)$,

$$\begin{aligned} E(XY) &= (Z + V, Y) \\ &= (Z, Y) + (V, Y) \\ &= (Z, Y) + 0 \\ &= E(ZY). \end{aligned}$$

Now we verify the projection property. Let $W \in L^2(\Omega, \mathcal{G}, P)$ be arbitary. By the Pythagorean theorem,

$$\begin{aligned} E((X - W)^2) &= \|X - W\|_{L^2}^2 \\ &= \|Z - W + V\|_{L^2}^2 \\ &= \|Z - W\|_{L^2}^2 + \|V\|_{L^2}^2. \end{aligned}$$

Thus $E((X - W)^2)$ is minimal at $W = Z$. This proves the projection property.

Suppose $X_1, X_2 \in L^2(\Omega, \mathcal{F}, P)$ such that $X_1 \geq X_2$ almost surely. Let $Z_1 = QX_1$, $Z_2 = QX_2$. To show that $Z_1 \geq Z_2$ a.s., it suffices to show that $Z_1 - Z_2 = Q(X_1 - X_2) \geq 0$ a.s. Thus it suffices to show that if $X \in L^2(\Omega, \mathcal{F}, P)$ and $X \geq 0$ a.s., then $QX \geq 0$ a.s.

So let $X \in L^2(\Omega, \mathcal{F}, P)$ with $X \geq 0$ be arbitrary. Let $Z = QX$. Note that all simple functions are in L^2. Hence for any $A \in \mathcal{G}$,

$$\int_A Z\,dP = \int_A X\,dP \geq 0.$$

Put $A = \{Z < 0\} \in \mathcal{G}$ to get

$$0 \geq \int_{\{Z<0\}} Z\,dP \geq 0.$$

Hence $\int_{\{Z<0\}} Z\,dP = 0$, so $\int_{\{Z<0\}} -Z\,dP = 0$. Since $-Z \geq 0$ on $\{Z < 0\}$, this implies that $-Z = 0$ a.s. on $\{Z < 0\}$. Thus $P(Z < 0) = 0$. Thus $Z \geq 0$ a.s. □

2. Use the above problem to show that conditional expectation exists and is unique, that is, for each $X \in L^1(\Omega, \mathcal{F}, P)$, there exists a unique $Z \in L^1(\Omega, \mathcal{G}, P)$ such that for any $F \in \mathcal{G}$, $\int_F X\,dP = \int_F Z\,dP$.

Proof. Let $X \in L^2(\Omega, \mathcal{F}, P)$ be arbitrary. By the monotonicity proved in problem 1,

$$\begin{aligned} |QX| &= |QX^+ - QX^-| \\ &\leq |QX^+| + |QX^-| \\ &= QX^+ + QX^- \\ &= Q(X^+ + X^-) \\ &= Q\,|X|. \end{aligned}$$

Thus

$$\begin{aligned} \|QX\|_{L^1} &= \int_\Omega |QX|\,dP \\ &\leq \int_\Omega Q\,|X|\,dP \\ &= \int_\Omega |X|\,dP \\ &= \|X\|_{L^1}. \end{aligned}$$

We now use this estimate and the density of $L^2(\Omega, \mathcal{F}, P)$ in $L^1(\Omega, \mathcal{F}, P)$ to extend Q to a linear map on $L^1(\Omega, \mathcal{F}, P)$ satisfying the same estimate. Let $X \in L^1(\Omega, \mathcal{F}, P)$ be arbitrary. Pick $X_n \in L^2(\Omega, \mathcal{F}, P)$ with $X_n \to X$ in L^1; for example, take X_n to be simple functions converging to X pointwise with $|X_n| \leq |X|$. Note that

$$\|QX_n - QX_m\|_{L^1} = \|Q(X_n - X_m)\|_{L^1} \leq \|X_n - X_m\|_{L^1} \to 0 \text{ as } n, m \to \infty.$$

Thus (QX_n) is a Cauchy sequence in $L^1(\Omega, \mathcal{G}, P)$, so there is an L^1 limit

$$QX := \lim_{n\to\infty} QX_n \in L^1(\Omega, \mathcal{G}, P).$$

We need to check that QX is independent of the sequence X_n. If $X_n' \in L^2(\Omega, \mathcal{F}, P)$ also converge to X in L^1, then

$$\left\|QX - \lim_{n\to\infty} QX_n\right\|_{L^1} = \lim_{n\to\infty} \|QX_n - QX_n'\|_{L^1} \leq \lim_{n\to\infty} \|X_n - X_n'\|_{L^1} = \|X - X\|_{L^1} = 0.$$

Thus QX is independent of the sequence X_n. With this in hand, it is easy to check that Q is linear and that $\|QX\|_{L^1} \leq \|X\|_{L^1}$ for all $X \in L^1(\Omega, \mathcal{F}, P)$.

Now let $X \in L^1(\Omega, \mathcal{F}, P)$ be arbitrary. Let $Z = QX \in L^1(\Omega, \mathcal{G}, P)$. Pick $X_n \in L^2(\Omega, \mathcal{F}, P)$ with $X_n \to X$ in L^1. Let $F \in \mathcal{G}$ be arbitrary. For each n, we have by problem 1 that

$$\int_F X_n \, dP = \int_F Q X_n \, dP.$$

By definition, $QX_n \to Z$ in L^1. Thus letting $n \to \infty$ above gives

$$\int_F X \, dP = \int_F Z \, dP.$$

This proves existence. Now suppose $Z' \in L^1(\Omega, \mathcal{G}, P)$ is also a conditional expectation of X given $\mathcal{G}$. Then for every $A \in \mathcal{G}$, $\int_A X \, dP = \int_A Z \, dP = \int_A Z' \, dP$, so

$$\int_A (Z - Z') \, dP = 0.$$

Putting $A = \{Z - Z' > 0\} \in \mathcal{G}$ yields that $Z - Z' = 0$ a.e. on $\{Z - Z' > 0\}$, which implies that $P(Z - Z' > 0) = 0$. A similar argument yields $P(Z - Z' < 0) = 0$. Thus $Z = Z'$ a.e. This proves uniqueness. $\square$

3. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $\mathcal{G}$ be a sub σ-field of $\mathcal{F}$ and let X be an integrable random variable. Let U be a $\mathcal{G}$-measurable random variable such that $E(|UX|) < \infty$. Show that

$$E(UX) = E(U E(X|\mathcal{G})).$$

Proof. Suppose $Y : \Omega \to [0, \infty)$ is $\mathcal{F}$-measurable and integrable (actually, what follows still makes sense when Y is not integrable since $E(Y|\mathcal{G})$ still makes sense). For all $A \in \mathcal{G}$, we have

$$\int 1_A Y dP = \int 1_A E(Y|\mathcal{G}) \, dP.$$

By linearity, it follows that for nonnegative $\mathcal{G}$-measurable simple functions f,

$$\int f Y dP = \int f E(Y|\mathcal{G}) \, dP.$$

By the monotone convergence theorem, the above holds for all $\mathcal{G}$-measurable $f : \Omega \to [0, \infty]$. In particular, if $fY \in L^1$, then $f E(Y|\mathcal{G}) \in L^1$. Note that $|U^{\pm} X^{\pm}| \le |U| |X| \in L^1$. Thus $U^{\pm} E(X^{\pm}|\mathcal{G}) \in L^1$ as well. Thus by linearity of the integral on L^1,

$$\begin{aligned}
\int UX &= \int (U^+ - U^-)(X^+ - X^-) \\
&= \int (U^+ X^+ - U^+ X^- - U^- X^+ + U^- X^-) \\
&= \int (U^+ E(X^+|\mathcal{G}) - U^+ E(X^-|\mathcal{G}) - U^- E(X^+|\mathcal{G}) + U^- E(X^-|\mathcal{G})) \\
&= \int (U^+ E(X|\mathcal{G}) - U^- E(X|\mathcal{G})) \\
&= \int U E(X|\mathcal{G}).
\end{aligned}$$

□

4. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $\mathcal{G}, \mathcal{H}$ be sub σ-fields of $\mathcal{F}$ such that $\mathcal{H} \subset \mathcal{G}$. Let X be an integrable random variable. Show

$$E(E(X|\mathcal{G})|\mathcal{H}) = E(X|\mathcal{H}), \quad E(E(X|\mathcal{H})|\mathcal{G}) = E(X|\mathcal{H}).$$

Proof. Let $A \in \mathcal{H}$. Since $A \in \mathcal{G}$,

$$E(E(X|\mathcal{G})\, 1_A) = E(X\, 1_A) = E(E(X|\mathcal{H})\, 1_A).$$

Thus $E(E(X|\mathcal{G})|\mathcal{H}) = E(X|\mathcal{H})$. Since $E(X|\mathcal{H})$ is $\mathcal{H}$-measurable, it is also $\mathcal{G}$ measurable. Thus trivially, $E(E(X|\mathcal{H})|\mathcal{G}) = E(X|\mathcal{H})$. □

5. Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\mathcal{G}$ be a sub σ-field of $\mathcal{F}$. Show that if $\{X_n\}$ converges to X in L^p $(1 \leq p < \infty)$ then $E(X_n|\mathcal{G}) \to E(X|\mathcal{G})$ in L^p.

Proof. Since conditional expectation is linear, it suffices to show that $\|E(X|\mathcal{G})\|_{L^p} \leq \|X\|_{L^p}$ for all $X \in L^p(\Omega, \mathcal{F}, P)$. So let $X \in L^p(\Omega, \mathcal{F}, P)$ be arbitrary. Note that $\varphi: [0, \infty) \to [0, \infty)$ defined by $\varphi(x) = x^p$ is convex since $\varphi''(x) = p\,(p-1)\,x^{p-2} \geq 0$ for all $x \geq 0$. Thus by monotonicity and Jensen's inequality,

$$|E(X|\mathcal{G})|^p \leq E(|X||\mathcal{G})^p \leq E(|X|^p|\mathcal{G}).$$

Integrating gives

$$\|E(X|\mathcal{G})\|_{L^p}^p \leq E(E(|X|^p|\mathcal{G})) = E(|X|^p) = \|X\|_{L^p}^p.$$ □

6. Let $\{X_i\}_{i\in\mathcal{I}}$ be a uniformly integrable family and $\{\mathcal{G}_j\}_{j\in\mathcal{J}}$ be a collection of sub σ-fields of $\mathcal{F}$. Show that the collection $\mathcal{U} \doteq \{E(X_i|\mathcal{G}_j) : (i,j) \in \mathcal{I} \times \mathcal{J}\}$ is a u.i. family.

Proof. In homework 1, it was proved as a lemma that if μ is a finite measure, a collection $\mathcal{C} \subset L^1(\mu)$ is uniformly integrable if and only if

$$\lim_{a\to\infty} \sup_{f\in\mathcal{C}} \int (|f| - a)^+ \, d\mu = 0.$$

Let $a > 0$ be arbitrary. Note that the map $\varphi: \mathbb{R} \to [0, \infty)$ given by $\varphi(x) = x^+ = \max(x, 0)$ is convex since it is the maximum of two convex functions. Note also that φ is increasing. For $i \in \mathcal{I}$, $j \in \mathcal{J}$, we have by monotonicity and Jensen's inequality that

$$(|E(X_i|\mathcal{G}_j)| - a)^+ \leq (E(|X_i||\mathcal{G}_j) - a)^+ = E(|X_i| - a|\mathcal{G}_j)^+ \leq E((|X_i| - a)^+|\mathcal{G}_j).$$

Taking expectations gives

$$E((|E(X_i|\mathcal{G}_j)| - a)^+) \leq E(E((|X_i| - a)^+|\mathcal{G}_j)) = E((|X_i| - a)^+).$$

Thus

$$\sup_{i\in\mathcal{I}, j\in\mathcal{J}} E((|E(X_i|\mathcal{G}_j)| - a)^+) \leq \sup_{i\in\mathcal{I}} E((|X_i| - a)^+) \to 0 \text{ as } a \to \infty.$$

Thus $\mathcal{U}$ is uniformly integrable. □

7. Let $\boldsymbol{P}$ and $\boldsymbol{Q}$ be two probability measures on $(\Omega, \mathcal{F})$ such that $\boldsymbol{Q} \ll \boldsymbol{P}$. Let $\{\mathcal{G}_j : j \in \mathcal{J}\}$ be a collection of sub σ-fields of $\mathcal{F}$. Let $\boldsymbol{Q}_j \doteq \boldsymbol{Q}|_{\mathcal{G}_j}$ and $\boldsymbol{P}_j = \boldsymbol{P}|_{\mathcal{G}_j}$. Regarding $\boldsymbol{Q}_j, \boldsymbol{P}_j$ as probability measures on $(\Omega, \mathcal{G}_j)$, let X_j be the $\mathcal{G}_j$ measurable random variable such that $X_j = \frac{d\boldsymbol{Q}_j}{d\boldsymbol{P}_j}$. Show $\{X_j : j \in \mathcal{J}\}$ is u.i. on $(\Omega, \mathcal{F}, \boldsymbol{P})$.

Proof. Let $X = \frac{d\boldsymbol{Q}}{d\boldsymbol{P}}$. For all $A \in \mathcal{G}_j$, we have

$$\begin{aligned}\int_A X\, d\boldsymbol{P} &= \boldsymbol{Q}(A)\\ &= \boldsymbol{Q}_j(A)\\ &= \int_A X_j\, d\boldsymbol{P}_j\\ &= \int_A X_j\, d\boldsymbol{P}.\end{aligned}$$

Thus $X_j = E(X|\,\mathcal{G}_j)$. Since $\{X\}$ is uniformly integrable, problem 6 implies $\{E(X|\,\mathcal{G}_j) : j \in \mathcal{J}\} = \{X_j : j \in \mathcal{J}\}$ is uniformly integrable. □

8. Let $X, Y \in L^2$. Suppose $E(X|\,Y) = Y$ and $E(Y|\,X) = X$. Show $X = Y$ a.s.

Proof. We have

$$\begin{aligned}E((X-Y)^2) &= E(E(X^2 + Y^2 - 2\,XY|\,X))\\ &= E(X^2 + E(Y^2|\,X) - 2\,E(XY|\,X)).\end{aligned}$$

Note that in the setting of problem 3, replacing U with $1_A\,U$ for $A \in \mathcal{G}$ yields $E(UX|\,\mathcal{G}) = U\,E(X|\,\mathcal{G})$. Thus since $XY \in L^1$, $E(XY|\,X) = X\,E(Y|\,X) = X^2$. Thus

$$\begin{aligned}E((X-Y)^2) &= E(X^2 + E(Y^2|\,X) - 2\,X^2)\\ &= E(E(Y^2|\,X)) - E(X^2)\\ &= E(Y^2) - E(X^2).\end{aligned}$$

Swapping X and Y gives

$$E((X-Y)^2) = E(X^2) - E(Y^2) = -E((X-Y)^2).$$

Thus $E((X-Y)^2) = 0$, so $X = Y$. □