STOR 635 HW 6

BY AMEER QAQISH

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1 Solutions

1. Let $X_n = \sum_{i=1}^n \xi_i$ for $n \geq 1$ and $X_0 = 0$, where $\{\xi_i\}$ are iid random variables with $P(\xi_i = 1) = P(\xi_i = -1) = 1/2$. The sequence $\{X_n\}$ is called a symmetric random walk. Let $|X_n| = M_n + A_n$ be the Doob decomposition of $\{|X_n|\}$. Show that $A_n = \#\{i \leq n-1 : |X_i| = 0\}$. $\{A_n\}$ is referred to as the local time of $\{X_n\}$ at 0.

Proof. For $n \in \mathbb{N}_0$, let $\mathcal{F}_n = \sigma(X_1, \ldots, X_n) = \sigma(\xi_1, \ldots, \xi_n)$. We have

$$A_n = \sum_{i=1}^n (E(|X_i| \,|\, \mathcal{F}_{i-1}) - |X_{i-1}|).$$

Note that

$$|X_i| = \begin{cases} |X_{i-1}| + \xi_i & \text{if } X_{i-1} > 0 \\ |X_{i-1}| - \xi_i & \text{if } X_{i-1} < 0 \\ 1 & \text{if } X_{i-1} = 0 \end{cases}.$$

Thus

$$\begin{aligned} E(|X_i| \,|\, \mathcal{F}_{i-1}) &= E((|X_{i-1}| + \xi_i)\, 1_{\{X_{i-1} > 0\}} + (|X_{i-1}| - \xi_i)\, 1_{\{X_{i-1} < 0\}} + 1_{\{X_{i-1} = 0\}} \,|\, \mathcal{F}_{i-1}) \\ &= 1_{\{X_{i-1} > 0\}}\, (|X_{i-1}| + E(\xi_i)) + 1_{\{X_{i-1} < 0\}}\, (|X_{i-1}| - E(\xi_i)) + 1_{\{X_{i-1} = 0\}} \\ &= \begin{cases} |X_{i-1}| & \text{if } X_{i-1} \neq 0 \\ 1 & \text{if } X_{i-1} = 0 \end{cases}. \end{aligned}$$

Thus

$$A_n = \sum_{i=1}^n 1_{\{X_{i-1} = 0\}} = \#\{i \in \{0, \ldots, n-1\} : X_i = 0\}. \qquad \square$$

2. Suppose X_n^1 and X_n^2 are supermartingales with respect to $\mathcal{F}_n$ and N is a finite stopping time such that $X_N^1 \geq X_N^2$ a.s. Then

(i) $Y_n = X_n^1\, 1_{\{N > n\}} + X_n^2\, 1_{\{N \leq n\}}$ is a supermartingale.

(ii) $Z_n = X_n^1\, 1_{\{N \geq n\}} + X_n^2\, 1_{\{N < n\}}$ is a supermartingale.

Proof. (i) Let $n \in \mathbb{N}$ be arbitrary. Clearly Y_n is $\mathcal{F}_n$-measurable and since $|Y_n| \leq |X_n^1| + |X_n^2|$, Y_n is integrable. We have

$$Y_n - Y_{n-1} = \begin{cases} X_n^1 - X_{n-1}^1 & \text{if } n \leq N-1 \\ X_n^2 - X_{n-1}^1 & \text{if } n = N \\ X_n^2 - X_{n-1}^2 & \text{if } n \geq N+1 \end{cases} = \begin{cases} X_n^1 - X_{n-1}^1 & \text{if } N \geq n+1 \\ X_n^2 - X_{n-1}^1 & \text{if } N = n \\ X_n^2 - X_{n-1}^2 & \text{if } N \leq n-1 \end{cases}.$$

Adding and subtracting $(X_n^1 - X_{n-1}^1)\,1_{\{N=n\}}$ to the above equation yields

$$Y_n - Y_{n-1} = (X_n^1 - X_{n-1}^1)\,1_{\{N\geq n\}} + (X_n^2 - X_n^1)\,1_{\{N=n\}} + (X_n^2 - X_{n-1}^2)\,1_{\{N\leq n-1\}}.$$

Since $\{N\geq n\}\in\mathcal{F}_{n-1}$ and X^1 is a supermartingale,

$$E((X_n^1 - X_{n-1}^1)\,1_{\{N\geq n\}}\,|\,\mathcal{F}_{n-1}) = E(X_n^1 - X_{n-1}^1\,|\,\mathcal{F}_{n-1})\,1_{\{N\geq n\}} \leq 0.$$

Similarly, since $\{N\leq n-1\}\in\mathcal{F}_{n-1}$,

$$E((X_n^2 - X_{n-1}^2)\,1_{\{N\leq n-1\}}) \leq 0.$$

The assumption $X_N^1 \geq X_N^2$ yields $(X_n^2 - X_n^1)\,1_{\{N=n\}} = (X_N^2 - X_N^1)\,1_{\{N=n\}} \leq 0$. Thus $E((X_n^2 - X_n^1)\,1_{\{N=n\}}\,|\,\mathcal{F}_{n-1}) \leq 0$. Thus $E(Y_n - Y_{n-1}\,|\,\mathcal{F}_{n-1}) \leq 0+0+0=0$. Thus Y is a supermartingale.

(ii) Let $n\in\mathbb{N}$ be arbitrary. Clearly Z_n is $\mathcal{F}_n$-measurable and integrable. We have

$$Z_n - Z_{n-1} = \begin{cases} X_n^1 - X_{n-1}^1 & \text{if } n\leq N \\ X_n^2 - X_{n-1}^1 & \text{if } n = N+1 \\ X_n^2 - X_{n-1}^2 & \text{if } n\geq N+2 \end{cases} = \begin{cases} X_n^1 - X_{n-1}^1 & \text{if } N\geq n \\ X_n^2 - X_{n-1}^1 & \text{if } N = n-1 \\ X_n^2 - X_{n-1}^2 & \text{if } N\leq n-2 \end{cases}.$$

Since $\{N\geq n\}\in\mathcal{F}_{n-1}$ and X^1 is a supermartingale, $E\left((X_n^1 - X_{n-1}^1)\,1_{\{N\geq n\}}\,|\,\mathcal{F}_{n-1}\right) \leq 0$. Since $\{N\leq n-2\}\in\mathcal{F}_{n-1}$ and X^2 is a supermartingale, $E((X_n^2 - X_{n-1}^2)\,1_{\{N\leq n-2\}}\,|\,\mathcal{F}_{n-1}) \leq 0$. We have

$$X_n^2 - X_{n-1}^1 = X_n^2 - X_{n-1}^2 + X_{n-1}^2 - X_{n-1}^1.$$

Thus

$E((X_n^2 - X_{n-1}^1)\,1_{\{N=n-1\}}\,|\,\mathcal{F}_{n-1}) = E(X_n^2 - X_{n-1}^2\,|\,\mathcal{F}_{n-1})\,1_{\{N=n-1\}} + E((X_N^2 - X_N^1)\,1_{\{N=n-1\}}\,|\,\mathcal{F}_{n-1}) \leq 0+0.$

Thus $E(Z_n - Z_{n-1}\,|\,\mathcal{F}_{n-1}) \leq 0+0+0=0$. Thus Z is a supermartingale. □

3. Let $X_n = \sum_{i=1}^n \xi_i$ for $n\geq 1$ and $X_0 = 0$, where $\{\xi_i\}$ are iid random variables with $P(\xi_i = 1) = P(\xi_i = -1) = 1/2$. The sequence $\{X_n\}$ is called a symmetric random walk. Fix $a<0$, $b>0$ and define stopping times

$$\tau_a \doteq \inf\{n : X_n = a\}, \quad \tau_b \doteq \inf\{n : X_n = b\}, \quad \tau_{a,b} = \tau_a\wedge\tau_b.$$

Compute $E(\tau_{a,b})$ by completing the following steps.

(a) Show that $\tau_a<\infty$ a.s. and $\tau_b<\infty$ a.s.

(b) Show $X_{n\wedge\tau_{a,b}} \to X_{\tau_{a,b}}$ in L^1.

(c) Using optional sampling theorem show that $P(\tau_{a,b} = \tau_a) = \frac{b}{b-a}$.

(d) Show that $Y_n \doteq X_n^2 - n$ is a martingale.

(e) Use the last two parts and optional sampling theorem to show $E(\tau_{a,b}) = |a|\,b$.

Proof. (a) For $k \in \mathbb{Z}$, let $f(k) = P(\tau_k < \infty)$. It suffices to show that $f(k) = 1$ for all $k \in \mathbb{Z}$. Since $\tau_0 = 0$, $f(0) = 1$. Let $k \geq 1$ be arbitrary. We have

$$\begin{aligned} f(k) &= P(\tau_k < \infty) \\ &= P(\xi_1 = 1)\, P(\tau_k < \infty \mid \xi_1 = 1) + P(\xi_1 = -1)\, P(\tau_k < \infty \mid \xi_1 = -1). \end{aligned}$$

Note that

$$\begin{aligned} P(\tau_k < \infty \mid \xi_1 = 1) &= P(\xi_1 + \cdots + \xi_n = k \;\text{ for some } n \in \mathbb{N}_0 \mid \xi_1 = 1) \\ &= P(\xi_1 + \cdots + \xi_n = k \;\text{ for some } n \in \mathbb{N} \mid \xi_1 = 1) \\ &= P(\xi_2 + \cdots + \xi_n = k - 1 \;\text{ for some } n \in \mathbb{N} \mid \xi_1 = 1) \\ &= P(\xi_2 + \cdots + \xi_n = k - 1 \;\text{ for some } n \in \mathbb{N}) \\ &= P(\xi_1 + \cdots + \xi_{n-1} = k - 1 \;\text{ for some } n \in \mathbb{N}) \\ &= P(\xi_1 + \cdots + \xi_n = k - 1 \;\text{ for some } n \in \mathbb{N}_0) \\ &= P(\tau_{k-1} < \infty) \\ &= f(k-1). \end{aligned}$$

To get the 5th equality, we used the fact that $(\xi_1, \xi_2, \dots)$ has the same distribution as $(\xi_2, \xi_3, \ldots)$. A similar computation shows that $P(\tau_k < \infty \mid \xi_1 = -1) = f(k+1)$. Hence

$$f(k) = \frac{1}{2} f(k-1) + \frac{1}{2} f(k+1).$$

This is a linear recurrence relation with characteristic polynomial $\frac{1}{2} r^2 - r + \frac{1}{2} = \frac{1}{2}(r-1)^2$. By the theorem for solutions of linear recurrences, there exist $C_1, C_2 \in \mathbb{R}$ such that for all $k \in \mathbb{N}_0$, $f(k) = C_1 k + C_2$. Since $f(k) \in [0,1]$ for all $k \in \mathbb{N}_0$, it follows that $C_1 = 0$. Since $f(0) = 1$, $C_2 = 1$. Thus $f(k) = 1$ for all $k \in \mathbb{N}_0$. By symmetry, $f(k) = 1$ for all $k \in -\mathbb{N}_0$. Hence $f(k) = 1$ for all $k \in \mathbb{Z}$.

(b) For every $\omega \in \Omega$ for which $\tau_{a,b}(\omega) < \infty$, we have $\tau_{a,b}(\omega) \wedge n = \tau_{a,b}(\omega)$ for n large, and hence $X_{\tau_{a,b} \wedge n}(\omega) = X_{\tau_{a,b}}(\omega)$ for n large. Thus $X_{\tau_{a,b} \wedge n} \to X_{\tau_{a,b}}$ a.s. Since $|X_{\tau_{a,b} \wedge n}| \leq \max(|a|, b) < \infty$ for all n, it follows by the dominated convergence theorem that $X_{\tau_{a,b} \wedge n} \to X_{\tau_{a,b}}$ in L^1.

(c) By the optional sampling theorem, $(X_{\tau_{a,b} \wedge n})_{n \in \mathbb{N}_0}$ is an $\mathcal{F}_n$-martingale, where $\mathcal{F}_n = \sigma(\xi_1, \dots, \xi_n)$. Thus for each $n \in \mathbb{N}_0$,

$$E(X_{\tau_{a,b} \wedge n}) = E(X_{\tau_{a,b} \wedge 0}) = E(X_0) = 0.$$

Since $X_{\tau_{a,b} \wedge n} \to X_{\tau_{a,b}}$ in L^1, $E(X_{\tau_{a,b}}) = \lim_{n \to \infty} E(X_{\tau_{a,b} \wedge n}) = 0$. Thus

$$E(X_{\tau_{a,b}}) = a\, P(\tau_{a,b} = \tau_a) + b\, P(\tau_{a,b} = \tau_b) = 0.$$

Using $P(\tau_{a,b} = \tau_b) = 1 - P(\tau_{a,b} = \tau_a)$ yields $P(\tau_{a,b} = \tau_a) = \frac{b}{b-a}$.

(d) Note that $|X_n| \le n$, so $X_n \in L^\infty$. Thus $X_n \in L^2$. Let $X_n^2 = M_n + A_n$ be the Doob decomposition of (X_n^2). Using $E(X_i \,|\, \mathcal{F}_{i-1}) = X_{i-1}$, we have

$$\begin{aligned} A_n &= \sum_{i=1}^n (E(X_i^2 \,|\, \mathcal{F}_{i-1}) - X_{i-1}^2) \\ &= \sum_{i=1}^n \mathrm{Var}(X_i \,|\, \mathcal{F}_{i-1}) \\ &= \sum_{i=1}^n E((X_i - X_{i-1})^2 \,|\, \mathcal{F}_{i-1}) \\ &= \sum_{i=1}^n E(\xi_i^2 \,|\, \mathcal{F}_{i-1}) \\ &= \sum_{i=1}^n 1 \\ &= n. \end{aligned}$$

Thus $M_n = X_n^2 - n$ is a martingale.

(e) By part (d) and the optional stopping theorem, $(X_{\tau_{a,b}\wedge n}^2 - (\tau_{a,b} \wedge n))_{n \in \mathbb{N}_0}$ is a martingale. Thus $E(X_{\tau_{a,b}\wedge n}^2 - (\tau_{a,b} \wedge n)) = E(X_0^2) = 0$ for all $n \in \mathbb{N}_0$. Thus $E(\tau_{a,b} \wedge n) = E(X_{\tau_{a,b}\wedge n}^2)$ for all $n \in \mathbb{N}_0$. By the monotone convergence theorem, $E(\tau_{a,b} \wedge n) \nearrow E(\tau_{a,b})$. Since $|X_{\tau_{a,b}\wedge n}| \le \max(|a|, b) < \infty$ for all n, it follows by the dominated convergence theorem that $X_{\tau_{a,b}\wedge n} \to X_{\tau_{a,b}}$ in L^2. Thus $E(X_{\tau_{a,b}\wedge n}^2) \to E(X_{\tau_{a,b}}^2)$. Thus taking $n \to \infty$ in the equation $E(\tau_{a,b} \wedge n) = E(X_{\tau_{a,b}\wedge n}^2)$ yields $E(\tau_{a,b}) = E(X_{\tau_{a,b}}^2)$. Thus

$$\begin{aligned} E(\tau_{a,b}) &= E(X_{\tau_{a,b}}^2) \\ &= a^2 P(\tau_{a,b} = \tau_a) + b^2 P(\tau_{a,b} = \tau_b) \\ &= a^2 \frac{b}{b-a} + b^2 \frac{-a}{b-a} \\ &= -a\, b \\ &= |a|\, b. \end{aligned}$$

□

4. Let $\{X_n\}_{n \ge 0}$ be a submartingale with $\sup X_n < \infty$ a.s. Let $\xi_n = X_n - X_{n-1}$ and suppose $E(\sup_n \xi_n^+) < \infty$. Show that X_n converges a.s.

Proof. For each $k \ge 0$, let $N_k = \inf\{n \in \mathbb{N}_0 : X_n > k\}$. Let $k \ge 0$ be arbitrary. Note that N_k is a stopping time. By the optional stopping theorem, $(X_{N_k \wedge n})_{n \in \mathbb{N}_0}$ is a submartingale with respect to the same filtration as $(X_n)_{n \in \mathbb{N}_0}$. Let $n \in \mathbb{N}_0$ be arbitrary. On the set $\{n < N_k\}$, we have

$$X_{N_k \wedge n}^+ = X_n^+ \le k^+ = k.$$

On the set $\{n \ge N_k\}$, we have

$$X_{N_k \wedge n} = X_{N_k} = X_{N_k - 1} + \xi_{N_k} \le k + \xi_{N_k}$$

and hence

$$X_{N_k\wedge n}^+ \le k + \xi_{N_k}^+ \le k + \sup_m \xi_m^+.$$

Thus $X_{N_k\wedge n} \le k + \sup_m \xi_m^+$ a.s. Thus $E(X_{N_k\wedge n}^+) \le k + E(\sup_m \xi_m^+)$. Thus $\sup_n E(X_{N_k\wedge n}^+) \le k + E(\sup_m \xi_m^+) < \infty$. By the martingale convergence theorem, there is a random variable $X_\infty^{N_k}$ such that $X_{N_k\wedge n} \to X_\infty^{N_k}$ a.s.

For every $k \ge 0$, note that on the set $\{N_k = \infty\}$, $X_n = X_{N_k\wedge n} \to X_\infty^{N_k}$ a.s. Thus $X_n(\omega)$ converges for almost every $\omega \in \bigcup_{k=1}^\infty \{N_k = \infty\}$. Note that $\bigcup_{k=1}^\infty \{N_k = \infty\} = \bigcup_{k=1}^\infty \{X_n \le k \text{ for all } n \in \mathbb{N}_0\} = \{\sup_n X_n < \infty\}$. By assumption, $P(\sup_n X_n < \infty) = 1$. Thus X_n converges a.s. $\square$