Proof of the SVD

by Ameer Qaqish

Abstract

In this article we give a simple proof of the SVD.

Theorem 1. *Let V, W be finite-dimensional inner product spaces over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Suppose X is a subspace of V, Y is a subspace of W, and $T: X \to Y$. Then $T^*: Y^\perp \to X^\perp$.*

Proof. Let $w \in Y^\perp$ be arbitrary. For all $x \in X$, $(T^* w, x) = (w, Tx) = 0$. Thus $T^* w \in X^\perp$. ☐

Theorem 2. *Let V, W be n-dimensional inner product spaces over $\mathbb{F}$ and let $T \in L(V, W)$. Then there exist orthonormal bases $\{v_1, \ldots, v_n\}$ of V, $\{w_1, \ldots, w_n\}$ of W and $\sigma_1, \ldots, \sigma_n \geq 0$ such that $Tv_j = \sigma w_j$ for each $j \in \{1, \ldots, n\}$. Equivalently, there exist isometries $J_1 \in L(\mathbb{F}^n, V)$, $J_2 \in L(\mathbb{F}^n, W)$ and diagonal $D \in M(n, \mathbb{F})$ such that*

$$T = J_2 D J_1^*.$$

Proof. We use $(\cdot, \cdot)$ to denote inner products. Using the definition $(Tv, w) = (v, T^* w)$ and the Cauchy-Schwarz inequality, it is easy to see that

$$\|T\| = \|T^*\| = \sup\{(Tv, w) : v \in V, w \in W, \|v\| = \|w\| = 1\}.$$

By compactness of $S^1 \times S^1$, this supremum is actually a maximum. Thus there exist $v_1 \in V$, $w_1 \in W$ such that $(Tv_1, w_1) = (v_1, T^* w_1) = \sigma_1$, with $\sigma_1 = \|T\|$. By Cauchy-Schwarz, we must have $Tv_1 = \sigma_1 w_1$, $T^* w_1 = \sigma_1 v_1$. Thus $T^*: \operatorname{span}(w_1) \to \operatorname{span}(v_1)$. By theorem (1), $T: \operatorname{span}(v_1)^\perp \to \operatorname{span}(w_1)^\perp$. By induction, there are orthonormal bases $\{v_2, \ldots, v_n\}$ of $\operatorname{span}(v_1)^\perp$, $\{w_2, \ldots, w_n\}$ of $\operatorname{span}(w_1)^\perp$ and $\sigma_2, \ldots \sigma_n \geq 0$ such that $Tv_j = \sigma_j w_j$ for each $j \in \{2, \ldots, n\}$. Thus $\{v_1, \ldots, v_n\}$ is an orthonormal basis of V and $\{w_1, \ldots, w_n\}$ is an orthonormal basis of W and $Tv_j = \sigma_j w_j$ for each $j \in \{1, \ldots, n\}$. ☐