Ross Theoretical Excercise 5.2

Ameer Qaqish

5.2 Let Y be a continuous random variable. Show that

$$E(Y) = \int_{y=0}^{\infty} P(Y > y)\,dy - \int_{y=0}^{\infty} P(Y < -y)\,dy.$$

Proof: We have

$$\begin{aligned}
\int_{y=0}^{\infty} P(Y > y)\,dy &= \int_{y=0}^{\infty}\int_{x=y}^{\infty} f_Y(x)\,dx\,dy && (1)\\
&= \int_{x=0}^{\infty}\int_{y=0}^{x} f_Y(x)\,dy\,dx && (2)\\
&= \int_{x=0}^{\infty} x f_Y(x)\,dx. && (3)
\end{aligned}$$

Also

$$\begin{aligned}
\int_{y=0}^{\infty} P(Y < -y)\,dy &= \int_{y=0}^{\infty}\int_{x=-\infty}^{-y} f_Y(x)\,dx\,dy && (4)\\
&= \int_{x=-\infty}^{0}\int_{y=0}^{-x} f_Y(x)\,dy\,dx && (5)\\
&= \int_{x=-\infty}^{0} -x f_Y(x)\,dx && (6)\\
&= -\int_{x=-\infty}^{0} x f_Y(x)\,dx. && (7)
\end{aligned}$$

Thus

$$\int_{y=0}^{\infty} P(Y > y)\, dy - \int_{y=0}^{\infty} P(Y < -y)\, dy = \int_{x=0}^{\infty} x f_Y(x)\, dx + \int_{x=-\infty}^{0} x f_Y(x)\, dx \tag{8}$$

$$= \int_{x=-\infty}^{\infty} x f_Y(x)\, dx \tag{9}$$

$$= E(Y). \tag{10}$$

5.3 Let X be a continuous random variable. Show that

$$E(g(X)) = \int_{x=-\infty}^{\infty} g(x) f_X(x)\, dx.$$

Proof: From 5.2, We have

$$E(g(X)) = \int_{y=0}^{\infty} P(g(X) > y)\, dy - \int_{y=0}^{\infty} P(g(X) < -y)\, dy \tag{11}$$

$$= \int_{y=0}^{\infty} P(g(X) \geq y)\, dy - \int_{y=0}^{\infty} P(g(X) < -y)\, dy \tag{12}$$

We have

$$\int_{y=0}^{\infty} P(g(X) \geq y)\, dy = \int_{y=0}^{\infty} P(X \in \{x : g(x) \geq y\})\, dy \tag{13}$$

$$= \int_{y=0}^{\infty} \int_{x:g(x)\geq y} f_X(x)\, dx\, dy \tag{14}$$

$$= \int_{x:g(x)\geq 0} \int_{y=0}^{g(x)} f_X(x)\, dy\, dx \tag{15}$$

$$= \int_{x:g(x)\geq 0} g(x) f_X(x)\, dx. \tag{16}$$

Also

$$\int_{y=0}^{\infty} P(g(X) < -y)\, dy = \int_{y=0}^{\infty} \int_{x:g(x)<-y} f_X(x)\, dx\, dy \tag{17}$$

$$= \int_{x:g(x)<0} \int_{y=0}^{-g(x)} f_X(x)\, dy\, dx \tag{18}$$

$$= \int_{x:g(x)<0} -g(x) f_X(x)\, dx \tag{19}$$

$$= -\int_{x:g(x)<0} g(x) f_X(x)\, dx. \tag{20}$$

Thus

$$E(g(X)) = \int_{x:g(x)\geq 0} g(x) f_X(x)\, dx + \int_{x:g(x)<0} g(x) f_X(x)\, dx \tag{21}$$

$$= \int_{x\in\mathbb{R}} g(x) f_X(x)\, dx \tag{22}$$

$$= \int_{x=-\infty}^{\infty} g(x) f_X(x)\, dx. \tag{23}$$