STOR 635 HW 9

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1 Solutions

1. Let $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ be a sequence of σ-fields such that $\mathcal{F}_n \supset \mathcal{F}_{n+1}$ for $n\in\mathbb{N}$. Let $\mathcal{F}_\infty = \bigcap_{n\geq 1}\mathcal{F}_n$. Let Y be an integrable random variable. Show that

$$E(Y\,|\,\mathcal{F}_n)\to E(Y\,|\,\mathcal{F}_\infty)$$

a.s. and in L^1.

Proof. Note that $(\mathcal{F}_{-k})_{k\leq -1}$ is a filtration. Also $(E(Y\,|\,\mathcal{F}_{-k}))_{k\leq -1}$ is a Martingale with respect to $(\mathcal{F}_{-k})_{k\leq -1}$ since for any $k\leq -1$, we have $\mathcal{F}_{-(k-1)}\subset\mathcal{F}_{-k}$ and the tower property yields

$$E(E(Y\,|\,\mathcal{F}_{-k})\,|\,\mathcal{F}_{-(k-1)}) = E(Y\,|\,\mathcal{F}_{-k}).$$

By the backwards martingale convergence theorem, $E(Y\,|\,\mathcal{F}_{-k})\to E(Y\,|\,\mathcal{F}_\infty)$ a.s. and in L^1 as $k\to-\infty$. Thus $E(Y\,|\,\mathcal{F}_n)\to E(Y\,|\,\mathcal{F}_\infty)$ a.s. and in L^1 as $n\to\infty$. $\square$

2. Suppose that $\{X_n,\mathcal{F}_n\}_{n=-\infty}^0$ is a martingale. Suppose that for some $p\geq 1$, $E\,|X_0|^p<\infty$. Then X_n converges in L^p to $E(X_0\,|\,\mathcal{F}_{-\infty})$ as $n\downarrow-\infty$.

Proof. By the backwards martingale convergence theorem, $X_n\to E(X_0\,|\,\mathcal{F}_{-\infty})$ a.s. and in L^1. Since we already have convergence in L^1, we may assume that $p>1$. By Doob's martingale L^p inequality, for all $k\leq 0$,

$$\left\|\sup_{k\leq m\leq 0}|X_m|\right\|_{L^p}\leq \frac{p}{p-1}\,\|X_0\|_{L^p}.$$

As $k\to-\infty$, $\sup_{k\leq m\leq 0}|X_m|\nearrow\sup_{m\leq 0}|X_m|$, so by the monotone convergence theorem,

$$\left\|\sup_{m\leq 0}|X_m|\right\|_{L^p}\leq \frac{p}{p-1}\,\|X_0\|_{L^p}.$$

In particular,

$$\begin{aligned}\sup_{m\leq 0}\|X_m\|_{L^p} &\leq \left\|\sup_{m\leq 0}|X_m|\right\|_{L^p}\\ &\leq \frac{p}{p-1}\,\|X_0\|_{L^p}\\ &< \infty.\end{aligned}$$

By the dominated convergence theorem, $X_m\to E(X_0\,|\,\mathcal{F}_{-\infty})$ in L^p as $m\to-\infty$. $\square$

3. (a) Let $\{X_i\}_{i=1}^N$ be an exchangeable collection of real square integrable random variables. Show $\mathrm{Cov}(X_1,X_2)\geq -\frac{1}{N-1}\mathrm{Var}(X_1)$.

(b) Using part (a) show that if $\{X_i\}_{i=1}^{\infty}$ is an exchangeable collection of real square integrable random variables, then $\mathrm{Cov}(X_1, X_2) \geq 0$.

Proof. (a) By exchangeability, each pair (X_i, X_j) with $i \neq j$ has the same distribution. Thus

$$\begin{aligned}
0 &\leq \mathrm{Var}(X_1 + \cdots + X_N) \\
&= \mathrm{Cov}(X_1 + \cdots + X_N, X_1 + \cdots + X_N) \\
&= \sum_{i,j=1}^{N} \mathrm{Cov}(X_i, X_j) \\
&= \sum_{i=1}^{N} \mathrm{Var}(X_i) + \sum_{i \neq j} \mathrm{Cov}(X_i, X_j) \\
&= N\,\mathrm{Var}(X_1) + N\,(N-1)\,\mathrm{Cov}(X_1, X_2).
\end{aligned}$$

Rearranging yields

$$\mathrm{Cov}(X_1, X_2) \geq -\frac{1}{N-1}\,\mathrm{Var}(X_1).$$

(b) Let $(X_i)_{i=1}^{\infty}$ be an exchangeable sequence of real square integrable random variables. For every $N \in \mathbb{N}$, $(X_i)_{i=1}^{N}$ are exchangeable, so by part (a),

$$\mathrm{Cov}(X_1, X_2) \geq -\frac{1}{N-1}\,\mathrm{Var}(X_1).$$

Letting $N \to \infty$ yields

$$\mathrm{Cov}(X_1, X_2) \geq 0.$$

□

4. Let S_n be the simple random walk, that is, $S_0 = 0$ and $S_n = \xi_1 + \cdots + \xi_n$, $n \geq 1$, where $\{\xi_i\}_{i \geq 1}$ are i.i.d. with $P(\xi_1 = 1) = P(\xi_1 = -1) = 1/2$. Using the Hewitt-Savage 0-1 law, show that, almost surely,

$$\limsup_{n \to \infty} \frac{S_n}{\sqrt{n}} = \infty, \quad \liminf_{n \to \infty} \frac{S_n}{\sqrt{n}} = -\infty.$$

Proof. Let $c > 0$ be arbitrary. Let

$$A = \left\{ \frac{S_n}{\sqrt{n}} > c \ \text{ for infinitely many } n \geq 1 \right\}.$$

We claim that A is in the exchangeable σ-algebra $\mathcal{E}$ of $(\xi_i)_{i \in \mathbb{N}}$. Let $k \geq 1$ and $\rho \in S(k)$ be arbitrary. Using notation similar to HW 8, we have

$$\rho^{-1}(A) = \left\{ \frac{X_{\rho(1)} + \cdots + X_{\rho(n)}}{\sqrt{n}} > c \ \text{ for infinitely many } n \geq 1 \right\}.$$

Since $X_{\rho(1)} + \cdots + X_{\rho(n)} = S_n$ when $n \geq k$, it follows that $\rho^{-1}(A) = A$. Thus $A \in \mathcal{E}$. Since $(\xi_i)_{i \in \mathbb{N}}$ are i.i.d, by the Hewitt-Savage 0-1 law, $P(A) \in \{0, 1\}$. Note that $A = \limsup_{n \to \infty} \left\{ \frac{S_n}{\sqrt{n}} > c \right\}$. By the reverse Fatou's lemma, $P(A) \geq \limsup_{n \to \infty} P\left(\frac{S_n}{\sqrt{n}} > c \right)$. By the central limit theorem $\lim_{n \to \infty} P\left(\frac{S_n}{\sqrt{n}} > c \right) = 1 - \Phi(c) > 0$, where Φ is the $N(0,1)$ CDF. Thus $P(A) > 0$, so $P(A) = 1$.

For contradiction, suppose $P\Big(\limsup_{n\to\infty}\frac{S_n}{\sqrt{n}}<\infty\Big)>0$. Then $P\Big(\limsup_{n\to\infty}\frac{S_n}{\sqrt{n}}<c\Big)>0$ for some $c\in\mathbb{Q}$, and hence $P\Big(\frac{S_n}{\sqrt{n}}<c$ for all but finitely many $n\geq 1\Big)>0$. Thus $P\Big(\frac{S_n}{\sqrt{n}}\geq c$ for infinitely many $n\geq 1\Big)<1$. But this contradicts what was proven in the previous paragraph. Thus $P\Big(\limsup_{n\to\infty}\frac{S_n}{\sqrt{n}}<\infty\Big)=0$, so $P\Big(\limsup_{n\to\infty}\frac{S_n}{\sqrt{n}}=\infty\Big)=1$. Applying what was just proved to the simple random walk with increments $-\xi_i$ yields $P\Big(\limsup_{n\to\infty}\frac{-S_n}{\sqrt{n}}=\infty\Big)=1$, which is equivalent to $P\Big(\liminf_{n\to\infty}\frac{S_n}{\sqrt{n}}=-\infty\Big)=1$. □

5. Let X and Y be E_1 and E_2 valued random variables respectively on $(\Omega,\mathcal{F},P)$ where E_1, E_2 are Polish spaces. Let κ be the regular conditional distribution of Y given $\sigma\{X\}$. Let $P_{X,Y}$ be a probability measure on $(E_1\times E_2,\mathcal{B}(E_1)\times\mathcal{B}(E_2))$ defined as

$$P_{X,Y}(C)\doteq\int_{\Omega\times E_2}1_C(X(\omega),y)\,P\otimes\kappa(d\omega,dy).$$

Show that $P_{X,Y}$ is the distribution of (X,Y), namely for $C\in\mathcal{B}(E_1)\otimes\mathcal{B}(E_2)$

$$P((X,Y)\in C)=P_{X,Y}(C).$$

Proof. The formal definition of $P_{X,Y}(C)$ is

$$P_{X,Y}(C)=\int_\Omega P(d\omega)\int_{E_2}\kappa(\omega,dy)\,1_C(X(\omega),y).$$

By the π-λ theorem, to prove that $P_{X,Y}$ is the distribution of (X,Y), it suffices to show that $P((X,Y)\in A_1\times A_2)=P_{X,Y}(A_1\times A_2)$ for all $A_1\in\mathcal{B}(E_1)$, $A_2\in\mathcal{B}(E_2)$. So let $A_1\in\mathcal{B}(E_1)$ and $A_2\in\mathcal{B}(E_2)$ be arbitrary. We have

$$\begin{aligned}
P_{X,Y}(A_1\times A_2) &= \int_\Omega P(d\omega)\int_{E_2}\kappa(\omega,dy)\,1_{A_1}(X(\omega))\,1_{A_2}(y)\\
&= \int_\Omega P(d\omega)\,1_{A_1}(X(\omega))\int_{A_2}\kappa(\omega,dy)\\
&= \int_\Omega P(d\omega)\,1_{A_1}(X(\omega))\,\kappa(\omega,A_2)\\
&= \int_\Omega P(d\omega)\,1_{A_1}(X(\omega))\,P(Y\in A_2\,|\,X)(\omega)\\
&= \int_\Omega P(d\omega)\,1_{A_1}(X(\omega))\,1_{A_2}(Y(\omega))\\
&= \int_{E_1\times E_2}P((X,Y)\in d\,(x,y))\,1_{A_1}(x)\,1_{A_2}(y)\\
&= P((X,Y)\in A_1\times A_2).
\end{aligned}$$

□

6. For a Polish space E and $n\in\mathbb{N}$, let $f_n: E\to\mathbb{R}$ be bounded measurable. Let

$$\varphi_k(x_1,\ldots,x_k)\doteq\prod_{i=1}^{k} f_i(x_i),\qquad k\in\mathbb{N}.$$

Define

$$A_n(\varphi_{k-1})\doteq\frac{1}{n!}\sum_{\rho\in S(n)}\varphi_{k-1}(x_{\rho(1)},\ldots,x_{\rho(k-1)})$$

and

$$A_n(f_k)\doteq\frac{1}{n!}\sum_{\rho\in S(n)} f_k(x_{\rho(1)})=\frac{1}{n}\sum_{i=1}^{n} f_k(x_i).$$

Show that for fixed k,

$$A_n(\varphi_{k-1})\,A_n(f_k)-A_n(\varphi_k)\to 0$$

as $n\to\infty$.

Proof. Since φ_k only depends on the first k coordinates, by simple combinatorics we have for each $k\in\mathbb{N}$ that

$$A_n(\varphi_k)=\frac{1}{(n)_k}\sum_{i_1,\ldots,i_k\in\{1,\ldots,n\}\text{ distinct}}\varphi_k(x_{i_1},\ldots,x_{i_k})=\frac{1}{(n)_k}S_k,$$

where

$$(n)_k=n\,(n-1)\cdots(n-(k-1))$$

and

$$S_k=\sum_{i_1,\ldots,i_k\in\{1,\ldots,n\}\text{ distinct}}\varphi_k(x_{i_1},\ldots,x_{i_k}).$$

From now on, we abbreviate "$i_1,\ldots,i_k\in\{1,\ldots,n\}$ distinct" as "$i_1,\ldots,i_k$".

Let $k\in\mathbb{N}$ be arbitrary. We have

$$\begin{aligned}
A_n(\varphi_{k-1})\,A_n(f_k) &= \frac{1}{(n)_{k-1}}\sum_{i_1,\ldots,i_{k-1}}\varphi_{k-1}(x_{i_1},\ldots,x_{i_{k-1}})\,\frac{1}{n}\sum_{i=1}^{n} f_k(x_i)\\
&= \frac{1}{(n)_{k-1}\,n}\sum_{i_1,\ldots,i_{k-1}}\sum_{i=1}^{n}\varphi_k(x_{i_1},\ldots,x_{i_{k-1}},x_i)\\
&= \frac{1}{(n)_{k-1}\,n}\left(S_k+\sum_{i_1,\ldots,i_{k-1}}\sum_{i\in\{i_1,\ldots,i_{k-1}\}}\varphi_k(x_{i_1},\ldots,x_{i_{k-1}},x_i)\right).
\end{aligned}$$

Thus

$$\begin{aligned}
\left\|A_n(\varphi_{k-1})\,A_n(f_k)-\frac{1}{(n)_{k-1}\,n}S_k\right\|_\infty &\le \frac{1}{(n)_{k-1}\,n}\#\{(i_1,\ldots,i_{k-1},i)\in\{1,\ldots,n\}^k : i_1,\ldots,\\
&\qquad i_{k-1}\text{ are distinct and } i\in\{i_1,\ldots,i_{k-1}\}\}\,\|\varphi_k\|_\infty\\
&= \frac{1}{(n)_{k-1}\,n}(n)_{k-1}\,(k-1)\,\|\varphi_k\|_\infty\\
&= \frac{k-1}{n}\,\|\varphi_k\|_\infty.
\end{aligned}$$

Also

$$\begin{aligned}
\left\| A_n(\varphi_k) - \frac{1}{(n)_{k-1}\, n}\, S_k \right\|_\infty &= \left\| \frac{1}{(n)_k}\, S_k - \frac{1}{(n)_{k-1}\, n}\, S_k \right\|_\infty \\
&= \left(\frac{1}{(n)_k} - \frac{1}{(n)_{k-1}\, n} \right) \|S_k\|_\infty \\
&\le \left(\frac{1}{(n)_k} - \frac{1}{(n)_{k-1}\, n} \right) (n)_k\, \|\varphi_k\|_\infty \\
&= \left(1 - \frac{n-(k-1)}{n} \right) \|\varphi_k\|_\infty \\
&= \frac{k-1}{n}\, \|\varphi_k\|_\infty .
\end{aligned}$$

Hence by the triangle inequality,

$$\|A_n(\varphi_{k-1})\, A_n(f_k) - A_n(\varphi_k)\|_\infty \le \frac{2\,(k-1)}{n}\, \|\varphi_k\|_\infty \to 0 \ \text{ as } n \to \infty. \qquad \square$$