Removable Singularities of Harmonic Functions

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Theorem 1. *Let B be the unit ball in $\mathbb{R}^n$. Suppose $u \in C^2(B \setminus 0) \cap C(\overline{B} \setminus 0)$ is harmonic in $B \setminus 0$ and bounded. Then u can be extended to a harmonic function on B.*

Proof. Consider u as a distribution on B. Since u is harmonic on $B \setminus 0$, we have $\operatorname{supp}(\Delta u) \subset \{0\}$. By a famous result, this implies that there is $N \in \mathbb{N}_0$ such that

$$\Delta u = \sum_{|\alpha| \leq N} c_\alpha \, \partial^\alpha \, \delta,$$

where $c_\alpha \in \mathbb{C}$. Assume $n \geq 3$ (the $n = 2$ case can be handled similary). Let $\Gamma(x) = C_n \, |x|^{2-n}$ be the fundamental solution for the Laplace operator, satisfying $\Delta \Gamma = \delta$. Then for $|\alpha| \leq N$, $\Delta \partial^\alpha \, \Gamma = \partial^\alpha \, \Delta \Gamma = \partial^\alpha \, \delta$. Let

$$v = \sum_{|\alpha| \leq N} c_\alpha \, \partial^\alpha \, \Gamma.$$

We have $u = w + v$, with $\Delta w = 0$. By Weyl's theorem, $w \in C^\infty(B)$. Thus w is bounded near 0. Thus v is bounded near 0. Now Lemma 2 below finishes the proof. □

Lemma 2. *Assume $n \geq 3$. Suppose $v = \sum_{|\alpha| \leq N} c_\alpha \, \partial^\alpha \, \Gamma$ with $c_\alpha \in \mathbb{C}$. If v is bounded near 0, then $v = 0$.*

Proof. Suppose v is bounded near 0. Let $s = 2 - n \leq -1$. Note that for $x \neq 0$, $r > 0$, $f(r\,x) = r^s \, f(x)$. Thus if $|\alpha| = k$, then $r^k \, \partial^\alpha \, f(r\,x) = r^s \, \partial^\alpha \, f(x)$, so $\partial^\alpha \, f(r\,x) = r^{s-k} \, \partial^\alpha \, f(x)$. Thus

$$v = \sum_{k=0}^{N} \sum_{|\alpha| = k} c_\alpha \, \partial^\alpha \, \Gamma =: \sum_{k=0}^{N} g_k,$$

Where each g_k satisfies $g_k(r\,x) = r^{s-k} \, g(x)$ for $x \neq 0$, $r > 0$. Thus for $x \neq 0$,

$$\begin{aligned} v(x) &= \sum_{k=0}^{N} |x|^{s-k} \, g_k\!\left(\frac{x}{|x|}\right) \\ &= |x|^{s-N} \sum_{k=0}^{N} |x|^{N-k} \, g_k\!\left(\frac{x}{|x|}\right) \\ &= |x|^{s-N} \left(g_N\!\left(\frac{x}{|x|}\right) + O(|x|) \right). \end{aligned}$$

If $g_N \neq 0$, then by continuity of g_N, there is ω with $|\omega| = 1$ such that $|g_N(\omega)| = \sup_{|x|=1} |g_N(x)| > 0$ and

$$|v(r\,\omega)| = r^{s-N} (|g_N(\omega)| + O(r)) \to \infty \text{ as } r \searrow 0,$$

contradicting that v is bounded near 0. Thus $g_N = 0$. Thus $v = \sum_{k=0}^{N-1} g_k$. Repeating the same argument shows that $g_{N-1} = 0$, and then $g_{N-2} = 0$, and so forth, showing that $g_k = 0$ for each k. $\square$

Remark 3. The assumption that u be bounded is stronger than necessary. The proof of lemma 2 shows that if $v \neq 0$, then $v = \Omega(|x|^{2-n})$ as $x \to 0$. Thus the theorem remains true if we only assume that $u = o(|x|^{2-n})$.