Radon-Nikodym Theorem Proof

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This proof is based on von Neumann's proof.

Radon-Nikodym Theorem. Let ν, μ be finite measures on a measure space (X, F). Then there exists $h \in L^1(X, \mu)$ and a measure ρ on (X, F) such that

$$d\nu = h\, d\mu + d\rho,$$

$$\rho \perp \mu.$$

Proof. Define linear functionals $\phi_1, \phi_2 \colon L^2(X, \nu + \mu) \to \mathbb{C}$ by

$$\phi_1(f) = \int_X f\, d\nu,$$

$$\phi_2(f) = \int_X f\, d\mu.$$

Note that for $f \in L^2(\nu + \mu)$ we have by Holder's inequality and finiteness of ν that

$$\begin{aligned} |\phi_1(f)| &\leq \int_X |f|\ d\nu \\ &\leq \|1\|_{L^2(X,\nu)} \|f\|_{L^2(X,\nu)} \\ &\leq \sqrt{\nu(X)}\, \|f\|_{L^2(X,\nu+\mu)} \\ &< \infty. \end{aligned}$$

Thus ϕ_1 is well defined and continuous. Similarly, ϕ_2 is well defined and continuous. Thus by the Hilbert space representation theorem, there exist $h_1, h_2 \in L^2(X, \nu + \mu)$ such that for every $f \in L^2(X, \nu + \mu)$,

$$\int_X f\, d\nu = \int_X f h_1\, d(\nu + \mu) \tag{1}$$

and

$$\int_X f\,d\mu = \int_X f h_2\,d(\nu+\mu). \tag{2}$$

In particular, (1) and (2) hold for every bounded measurable f, and therefore for any simple function. Putting $f = \chi_{\{h_1 \le -\frac{1}{n}\}}$ in (1) gives $\nu(\{h_1 \le -\frac{1}{n}\}) \le -\frac{1}{n}(\nu(\{h_1 \le -\frac{1}{n}\}) + \mu(\{h_1 \le -\frac{1}{n}\}))$, which implies $\nu(\{h_1 \le -\frac{1}{n}\}) = \mu(\{h_1 \le -\frac{1}{n}\}) = 0$ for all $n \in \mathbb{N}$. Thus $h_1 \ge 0$ a.e. $\nu+\mu$. Similarly, $h_2 \ge 0$ a.e. $\nu+\mu$. Thus by changing representatives of the equivalence classes of h_1 and h_2 in $L^2(\nu+\mu)$, we may assume that $h_1(x) \ge 0$ and $h_2(x) \ge 0$ for all $x \in X$. Since (1) and (2) hold for simple functions, by the monotone convergence theorem, (1) and (2) hold for all measurable $f\colon X \to [0,\infty]$. Putting $f = \chi_{\{h_2=0\}}$ in (2) gives $\mu(\{h_2=0\}) = 0$. Let $Z = \{h_2 \ne 0\}$. For any measurable set A,

$$\begin{aligned}
\nu(A) &= \nu(A\cap Z) + \nu(A\cap Z^c)\\
&= \int_{A\cap Z} h_1\,d(\nu+\mu) + \nu(A\cap Z^c)\\
&= \int_{A\cap Z} \frac{h_1}{h_2} h_2\,d(\nu+\mu) + \nu(A\cap Z^c)\\
&= \int_{A\cap Z} \frac{h_1}{h_2}\,d\mu + \nu(A\cap Z^c).
\end{aligned}$$

Note that $\int_{A\cap Z} \frac{h_1}{h_2}\,d\mu = \int_A \frac{h_1}{h_2}\,d\mu$ since $\mu(Z^c) = 0$. Thus $h = \frac{h_1}{h_2}$ and the measure ρ defined by

$$\rho(A) = \nu(A\cap Z^c)$$

have the desired properties. □