Nonnegative Second Derivative Implies Convexity

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Theorem: If $I \subseteq \mathbb{R}$ is an interval and $f\colon I \to \mathbb{R}$ and for each $x \in I$, $f''(x) \geq 0$, then f is convex on I.

Proof: Let $x_1 < x_2$ and $\lambda \in (0,1)$ be arbitrary. We need to show that

$$(1-\lambda)f(x_1) + \lambda f(x_2) \geq f((1-\lambda)x_1 + \lambda x_2).$$

Let

$$c = (1-\lambda)x_1 + \lambda x_2 = x_1 + \lambda(x_2 - x_1),$$
$$h = x_2 - x_1.$$

Then

$$x_1 = c - \lambda h,$$
$$x_2 = c + (1-\lambda)h.$$

Thus using the Lagrange remainder,

$$\begin{aligned}
f(x_1) &= f(c - \lambda h) && (1)\\
&= f(c) - \lambda h f'(c) + \lambda^2 h^2 \frac{f''(\xi_1)}{2} \qquad \xi_1 \in (x_1, c) && (2)\\
&\geq f(c) - \lambda h f'(c) && (3)
\end{aligned}$$

and

$$\begin{aligned}
f(x_2) &= f(c + (1-\lambda)h) && (4)\\
&= f(c) + (1-\lambda)h f'(c) + (1-\lambda)^2 h^2 \frac{f''(\xi_2)}{2} \qquad \xi_2 \in (c, x_2) && (5)\\
&\geq f(c) + (1-\lambda)h f'(c). && (6)
\end{aligned}$$

Thus

$$\begin{aligned}
(1-\lambda)f(x_1) + \lambda f(x_2) &\geq (1-\lambda)(f(c) - \lambda h f'(c)) + \lambda(f(c) + (1-\lambda)h f'(c)) && (7)\\
&= f(c) && (8)\\
&= f((1-\lambda)x_1 + \lambda x_2). && (9)
\end{aligned}$$