Hausdorff Measure

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Abstract

We introduce Hausdorff's r-dimensional outer measures, defined on a metric space (X, d). We show that Hausdorff's n-dimensional measure coincides with Lebesgue measure on $\mathbb{R}^n$. We define Hausdorff dimension and compute the dimension and Hausdorff measure of the middle-thirds Cantor set.

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1 Hausdorff's r-Dimensional Outer Measures

Most of the arguments appearing here are adapted from chapter 12 of [1].

1.1 Introduction

Huasdorff measure is a way to assign meaningful measures to "lower dimensional" subsets of $\mathbb{R}^n$ such as curves and surfaces which have Lebesgue measure 0. It is worthy of note that for "smooth surfaces", there is a way of doing this simply as follows. Suppose $m \leq n$ and $O \subset \mathbb{R}^m$ is open. Suppose $\phi : O \to \mathbb{R}^n$ is a C^1 map which is a homeomorphism onto it's image and has injective derivative $D\phi(x)$ at each $x \in O$. Then for Borel $E \subset \phi(O)$ we define

$$V_m(E) = \int_{\phi^{-1}(E)} J_\phi(x)\, dx, \quad J_\phi(x) = \sqrt{\det(D\phi(x)^T D\phi(x))}$$

V_m is a Borel measure on $\phi(O)$ that gives the "m-dimensional" surface area of subsets of $\phi(O)$.

Let (X, d) be a metric space. We proceed to define Hausdorff's r-dimensional outer measure $\mathcal{H}^r$ on X for real $r \geq 0$. For $S \subset X$, $\delta > 0$, set

$$h^*_{r,\delta}(S) = \inf \left\{ \sum_{j=1}^{\infty} \operatorname{diam}(B_j)^r : S \subset \bigcup_{j=1}^{\infty} B_j, \operatorname{diam}(B_j) \leq \delta \right\}.$$

Here the sets B_j appearing in the infimum can be any subsets of X. However, we get the same result if we require them to be closed since $\operatorname{diam}(\overline{B_j}) = \operatorname{diam}(B_j)$. If $X = \mathbb{R}$, we can require the B_j to be closed intervals since any $B_j \subset \mathbb{R}$ is contained in the interval $I_j = [\inf(B_j), \sup(B_j)]$ and $\operatorname{diam}(I_j) = \operatorname{diam}(B_j)$.

Note that $\operatorname{diam}(\emptyset) = 0$. Thus $h^*_{r,\delta}$ is an outer measure (recall the construction of outer measures from a set function $\rho: \mathcal{E} \to [0, \infty]$ that satisfies $\rho(\emptyset) = 0$). Note that as δ decreases, $h^*_{r,\delta}$ increases. Thus we can set

$$h^*_r(S) = \lim_{\delta \searrow 0} h^*_{r,\delta}(S).$$

That h^*_r is an outer measure is a consequence of the following result:

Proposition 1. *Suppose μ^*_j is an increasing sequence of outer measures on a set X. For $S \subset X$, define $\mu^*(S) = \lim_{j\to\infty} \mu^*_j(S)$. μ^* is an outer measure.*

Proof. We have

$$\mu^*(\emptyset) = \lim_{j\to\infty} \mu^*_j(\emptyset) = \lim_{j\to\infty} 0 = 0.$$

If $A \subset B$, then

$$\mu^*(A) = \lim_{j\to\infty} \mu^*_j(A) \le \lim_{j\to\infty} \mu^*_j(B) = \mu^*(B).$$

If $A_1, A_2, \ldots \subset X$, then we have

$$\mu^*_j\left(\bigcup_{n=1}^{\infty} A_n\right) \le \sum_{n=1}^{\infty} \mu^*_j(A_n) \le \sum_{n=1}^{\infty} \mu^*(A_n).$$

Taking $j \to \infty$ yields $\mu^*(\bigcup_{n=1}^{\infty} A_n) \le \sum_{n=1}^{\infty} \mu^*(A_n)$. $\square$

We want to show that every Borel set is h^*_r-measurable. To do so, we use the following result, due to Caratheodory:

Definition 2. *An outer measure μ^* on a metric space (X, d) is called a metric outer measure if*

$$d(A, B) = \inf\{d(x, y) : x \in A, y \in B\} > 0 \quad \Longrightarrow \quad \mu^*(A \cup B) = \mu^*(A) + \mu^*(B).$$

Proposition 3. *If μ^* is a metric outer measure on a metric space (X, d), then every Borel subset of X is μ^*-measurable.*

Proof. It suffices to show that every closed subset of X is μ^*-measurable. Let $F \subset X$ be an arbitrary closed subset of X. We need to show that if $Y \subset X$ and $\mu^*(Y) < \infty$, then

$$\mu^*(Y) \ge \mu^*(Y \cap F) + \mu^*(Y \setminus F).$$

For $n \in \mathbb{N}$, set

$$B_n = \left\{x \in Y \setminus F : d(x, F) \ge \frac{1}{n}\right\}.$$

Note that $d(B_n, Y \cap F) \ge \frac{1}{n}$. Since F is closed, $x \in F \iff d(x, F) = 0$. Thus

$$B_n \nearrow \{x \in Y \setminus F : d(x, F) > 0\} = Y \setminus F.$$

We have

$$\mu^*(Y) \ge \mu^*((Y \cap F) \cup B_n) = \mu^*(Y \cap F) + \mu^*(B_n).$$

Thus we will be done if we can show that $\lim_{n\to\infty} \mu^*(B_n) = \mu^*(Y \setminus F)$. We have $B_n \subset Y \setminus F$, so $\limsup_{n\to\infty} \mu^*(B_n) \le \mu^*(Y \setminus F)$. It remains to show that $\liminf_{n\to\infty} \mu^*(B_n) \ge \mu^*(Y \setminus F)$.

Set

$$C_n = B_{n+1} \setminus B_n = \left\{ x \in Y \setminus F : \frac{1}{n+1} \le d(x, F) < \frac{1}{n} \right\}.$$

By countable subadditivity, we have for all n that

$$\mu^*(Y \setminus F) \le \mu^*(B_n) + \sum_{j=n}^{\infty} \mu^*(C_j).$$

Thus to get $\liminf_{n\to\infty} \mu^*(B_n) \ge \mu^*(Y \setminus F)$, it suffices to show that the sum $\sum_{j=1}^{\infty} \mu^*(C_k)$ is finite. Note that for $j \in \mathbb{N}$,

$$x \in C_n, y \in C_{n+j} \quad \Longrightarrow \quad d(x, y) \ge d(x, F) - d(y, F) \ge \frac{1}{n+1} - \frac{1}{n+j}.$$

Thus for $j \ge 2$, $d(C_n, C_{n+j}) > 0$. Applying the metric outer measure hypothesis inductively gives for all N that

$$\mu^*\left(\bigcup_{j=1}^{N} C_{2j} \right) = \sum_{j=1}^{N} \mu^*(C_{2j}),$$

$$\mu^*\left(\bigcup_{j=1}^{N} C_{2j-1} \right) = \sum_{j=1}^{N} \mu^*(C_{2j-1}).$$

Both of these quantities are $\le \mu^*(Y)$ by monotonicity. Thus by taking $N \to \infty$ and then adding we get

$$\sum_{j=1}^{\infty} \mu^*(C_j) \le 2\,\mu^*(Y) < \infty. \qquad \square$$

Remark. The converse of proposition (3) is true and is easy to prove.

In order to apply proposition (3), we show that h_r^* is a metric outer measure.

Lemma 4. *h_r^* is a metric outer measure.*

Proof. Suppose $A, B \subset X$ with $d(A, B) > 0$. Let $\delta < d(A, B)$ be arbitrary. Let B_1, $B_2, \ldots$ be sets with $\operatorname{diam}(B_j) \le \delta$ that cover $A \cup B$. Since $\operatorname{diam}(B_j) \le \delta < d(A, B)$, it follows that each B_j intersects at most one of A or B. Thus

$$\sum_{j=1}^{\infty} \mu^*(B_j) = \sum_{j: B_j \cap A \ne \emptyset} \mu^*(B_j) + \sum_{j: B_j \cap B \ne \emptyset} \mu^*(B_j) \ge h_{r,\delta}^*(A) + h_{r,\delta}^*(B).$$

Taking the infimum over all such covers $\{B_j\}$ gives

$$h_{r,\delta}^*(A \cup B) \ge h_{r,\delta}^*(A) + h_{r,\delta}^*(B).$$

Subadditivity gives $h_{r,\delta}^*(A \cup B) = h_{r,\delta}^*(A) + h_{r,\delta}^*(B)$, and then taking $\delta \to 0$ gives $h_r^*(A \cup B) = h_r^*(A) + h_r^*(B)$. $\square$

By proposition (3), all Borel subsets of X are h_r^*-measurable. For any $A \subset X$, we set

$$\mathcal{H}^r(A) = \gamma_r\, h_r^*(A), \quad \gamma_r = \frac{2^{-r}\,\pi^{\frac{r}{2}}}{\Gamma\left(\frac{r}{2}+1\right)}.$$

For $n \in \mathbb{N}$, it can be shown via the Gaussian integral and polar coordinates that γ_n is the Lebesgue measure of the ball of diameter 1 in $\mathbb{R}^n$, i.e. $\gamma_n = \mathcal{L}^n\left(B\left(0, \frac{1}{2}\right)\right)$, where $\mathcal{L}^n$ denotes Lebesgue measure on $\mathbb{R}^n$.

1.2 $\mathcal{H}^n = \mathcal{L}^n$

The reason for γ_r in the definition of $\mathcal{H}^r$ is so that $\mathcal{H}^n(S) = \mathcal{L}^n(S)$ for all Lebesgue measurable $S \subset \mathbb{R}^n$, a nontrivial fact which we proceed to prove. We now focus on $\mathcal{H}^n$ with $X = \mathbb{R}^n$. To start, we show that Lebesgue-measurable S are $\mathcal{H}^n$-measurable. Since we already know Borel sets are $\mathcal{H}^n$-measurable, it suffices to show that

$$\mathcal{L}^n(S) = 0 \quad \Longrightarrow \quad \mathcal{H}^n(S) = 0.$$

So suppose $\mathcal{L}^n(S) = 0$. We have

$$\mathcal{L}^n(S) = \inf\left\{\sum_{j=1}^{\infty} \mathcal{L}^n(Q_j) : Q_j \text{ are cubes, } S \subset \bigcup_{j=1}^{\infty} Q_j\right\}.$$

Let $\varepsilon > 0$. There are cubes Q_j such that $S \subset \bigcup_{j=1}^{\infty} Q_j$ and $\sum_{j=1}^{\infty} \mathcal{L}^n(Q_j) < \varepsilon$. By subadditivity,

$$\mathcal{H}^n(S) \le \sum_{j=1}^{\infty} \mathcal{H}^n(Q_j).$$

Note that if a cube Q has side length s, then $\operatorname{diam}(Q) = \sqrt{n}\, s$, so Q is contained in a ball B of diameter $\sqrt{n}\, s$ and hence

$$\mathcal{H}^n(Q) \le \gamma_n\, (\sqrt{n}\, s)^n = \gamma_n\, n^{n/2} \mathcal{L}^n(Q).$$

Thus

$$\mathcal{H}^n(S) \le \sum_{j=1}^{\infty} \mathcal{H}^n(Q_j) \le \gamma_n\, n^{n/2} \sum_{j=1}^{\infty} \mathcal{L}^n(Q_j) \le \gamma_n\, n^{n/2}\, \varepsilon.$$

Letting $\varepsilon \to 0$ gives $\mathcal{H}^n(S) = 0$.

Now we focus on proving $\mathcal{L}^n(S) \le \mathcal{H}^n(S)$ for Lebesgue-measurable $S \subset \mathbb{R}^n$. Let $S \subset \mathbb{R}^n$ be an arbitrary Lebesgue-measurable set. Let $\delta > 0$ be arbitrary. Suppose $C_1, C_2, \ldots$ are closed sets with $S \subset \bigcup_{j=1}^{\infty} C_j$. We have

$$\mathcal{L}^n(S) \le \sum_{j=1}^{\infty} \mathcal{L}^n(C_j).$$

If we can show that $\mathcal{L}^n(C_j) \le \gamma_n \operatorname{diam}(C_j)^n$, then we will have

$$\mathcal{L}^n(S) \le \sum_{j=1}^{\infty} \gamma_n \operatorname{diam}(C_j)^n,$$

and taking the infimum over all covers $\{C_j\}$ will give $\mathcal{L}^n(S) \le \gamma_n h_{n,\delta}(S)$, and then letting $\delta \to 0$ will give $\mathcal{L}^n(S) \le \mathcal{H}^n(S)$. The inequality $\mathcal{L}^n(C_j) \le \gamma_n \operatorname{diam}(C_j)^n$ is the isodiametric inequality, which we now prove.

Definition 5. *Let $\omega \in \mathbb{R}^n$ with $|\omega| = 1$. Set*

$$P_\omega = \{x \in \mathbb{R}^n : (x, \omega) = 0\}.$$

P_ω is the hyperplane with normal vector ω. For $y \in \mathbb{R}^n$, set

$$L_y^\omega = \{y + t\,\omega : t \in \mathbb{R}\}.$$

Define

$$\Sigma_\omega(S) = \left\{y + t\,\omega : y \in P_\omega, L_y^\omega \cap S \neq \emptyset, |t| \le \frac{1}{2}\int_{-\infty}^{\infty} \chi_S(y + t\,\omega)\,dt\right\}.$$

$\Sigma_\omega(S)$ is called the Steiner symmetrization of S about the hyperplane P_ω.

Proposition 6. *Suppose $S \subset \mathbb{R}^n$ is Lebesgue-measurable and $\omega \in \mathbb{R}^n$ with $|\omega| = 1$. Then*

(i) $\Sigma_\omega(S)$ is measurable,
(ii) $\mathcal{L}^n(\Sigma_\omega(S)) = \mathcal{L}^n(S)$,
(iii) $\operatorname{diam}(\Sigma_\omega(S)) \le \operatorname{diam}(S)$.

Proof. If $R : \mathbb{R}^n \to \mathbb{R}^n$ is a unitary linear map, then

$$\begin{aligned} R\Sigma_\omega(S) &= \left\{Ry + t\,R\omega : y \in P_\omega, L_y^\omega \cap S \neq \emptyset, |t| \le \frac{1}{2}\int_{-\infty}^{\infty} \chi_S(y + t\,\omega)\,dt\right\} \\ &= \left\{z + t\,R\omega : z \in RP_\omega, RL_y^\omega \cap RS \neq \emptyset, |t| \le \frac{1}{2}\int_{-\infty}^{\infty} \chi_{RS}(z + t\,R\omega)\,dt\right\}. \end{aligned}$$

We have $RP_\omega = P_{R\omega}$ and $RL_y^\omega = L_{Ry}^{R\omega}$. Thus

$$R\Sigma_\omega(S) = \Sigma_{R\omega}(RS). \tag{1}$$

By picking R such that $R\omega = e_n$ and by invariance of diam and $\mathcal{L}^n$ under unitary maps, we may assume $\omega = e_n$. Define $f : \mathbb{R}^{n-1} \to [0, \infty]$ by

$$f(y) = \int_{-\infty}^{\infty} \chi_S(y, t)\,dt.$$

By Tonelli's theorem, f is measurable. We have

$$\Sigma_{e_n}(S) = \left\{(y, t) : y \in \mathbb{R}^{n-1}, |t| \le \frac{1}{2} f(y)\right\} \setminus \{(y, 0) : L_y^{e_n} \cap S = \emptyset\}.$$

The first set on the right hand side is measurable since $t \mapsto |t|$ and f are both measurable, and the second set is measurable since it is a subset of a null set. Thus $\Sigma_{e_n}(S)$ is measurable. By Tonelli's theorem,

$$\mathcal{L}^n(\Sigma_{e_n}(S)) = \int_{\mathbb{R}^{n-1}} \int_{-\frac{1}{2}f(y)}^{\frac{1}{2}f(y)} 1\,dt\,dy = \int_{\mathbb{R}^{n-1}} f(y)\,dy = \mathcal{L}^n(S).$$

It remains to prove (iii). We may assume that $\operatorname{diam}(S)<\infty$. By replacing S with $\overline{S}$, we may assume that S is closed. Let $x, y \in \Sigma_{e_n}(S)$ be arbitrary. Write $x=(x', x_n)$, $y=(y', y_n)$. Set

$$\begin{aligned} a_x &= \inf\{t\in\mathbb{R} : (x', t) \in S\} \\ b_x &= \sup\{t\in\mathbb{R} : (x', t) \in S\} \\ a_y &= \inf\{t\in\mathbb{R} : (y', t) \in S\} \\ b_y &= \sup\{t\in\mathbb{R} : (y', t) \in S\} \end{aligned}$$

Since S is closed, $(x', a_x), (x', b_x), (y', a_y), (y', b_y) \in S$. Without loss of generality, assume $b_x - a_y \geq b_y - a_x$. We claim that $|x-y| \leq |(x', b_x) - (y', a_y)|$. We have

$$\begin{aligned} |x_n - y_n| &\leq |x_n| + |y_n| \\ &\leq \frac{1}{2} f(x') + \frac{1}{2} f(y') \\ &\leq \frac{1}{2}(b_x - a_x) + \frac{1}{2}(b_y - a_y) \\ &= \frac{1}{2}(b_x - a_y) + \frac{1}{2}(b_y - a_x) \\ &\leq b_x - a_y \end{aligned}$$

Thus $|x-y| \leq |(x', b_x) - (y', a_y)| \leq \operatorname{diam}(S)$. Taking the supremum over all $x, y \in \Sigma_{e_n}(S)$ gives $\operatorname{diam}(\Sigma_{e_n}(S)) \leq \operatorname{diam}(S)$. ☐

Proposition 7. *Let $u \in \mathbb{R}^n$ with $|u|=1$. Let R_u be the reflection across P_u, i.e.*

$$R_u x = x - 2(x, u) u.$$

Suppose $S \subset \mathbb{R}^n$ is measurable with $R_u S = S$. If $\omega \in \mathbb{R}^n$ with $|\omega|=1$ and $(\omega, u)=0$, then $R_u \Sigma_\omega(S) = \Sigma_\omega(S)$.

Proof. By equation (1),

$$R_u \Sigma_\omega(S) = \Sigma_{R_u \omega}(R_u S) = \Sigma_\omega(S).$$ ☐

Proposition 8. *Let $\omega \in \mathbb{R}^n$ with $|\omega|=1$. Then $R_\omega \Sigma_\omega(S) = \Sigma_\omega(S)$.*

Proof. This follows immediately from the definition of $\Sigma_\omega(S)$. ☐

Proposition 9. *For Lebesgue-measurable $S \subset \mathbb{R}^n$, $\mathcal{L}^n(S) \leq \gamma_n \operatorname{diam}(S)^n$. In other words, $\mathcal{L}^n(S) \leq \mathcal{L}^n(B)$, where B is a ball with the same diameter as S.*

Proof. Let $S_1 = \Sigma_{e_1}(S), S_2 = \Sigma_{e_2}(S_1), \ldots, S_n = \Sigma_{e_n}(S_{n-1})$. We have $R_{e_j} S_j = S_j$ for each $j \in \{1, \ldots, n\}$ by proposition (8). Thus by proposition (7), $R_{e_j} S_n = S_n$ for each $j \in \{1, \ldots, n\}$. Since $-x = R_{e_1} \ldots R_{e_n} x$, we have $-S_n = S_n$. Thus

$$x \in S_n \Longrightarrow \operatorname{diam}(S_n) \geq |x - (-x)| = 2|x| \Longrightarrow |x| \leq \frac{\operatorname{diam}(S_n)}{2}.$$

Thus $S_n \subset B\left(0, \frac{\operatorname{diam}(S_n)}{2}\right)$, so

$$\mathcal{L}^n(S) = \mathcal{L}^n(S_n) \leq \mathcal{L}^n\left(B\left(0, \frac{\operatorname{diam}(S_n)}{2}\right)\right) \leq \mathcal{L}^n\left(B\left(0, \frac{\operatorname{diam}(S)}{2}\right)\right).$$ ☐

Now we focus on establishing the reverse inequality $\mathcal{H}^n(S) \leq \mathcal{L}^n(S)$. By outer regularity of $\mathcal{L}^n$, we only need to establish this when $S = U$ is open.

Proposition 10. *Let $U \subset \mathbb{R}^n$ be open with $\mathcal{L}^n(U) < \infty$. For any $\delta > 0$, there exist disjoint balls $B_j \subset U$ with $\operatorname{diam}(B_j) \leq \delta$ such that $\mathcal{L}^n(U \setminus \bigcup_{j=1}^{\infty} B_j) = 0$.*

Proof. For each $x \in U$, pick a ball B_x with $B_x \subset U$ and $\operatorname{diam}(B_x) \leq \delta$. Since $\frac{3^n}{4^n} \mathcal{L}^n(U) < \mathcal{L}^n(U)$, by Weiner's covering lemma, there exist disjoint balls $B_{1,j} = B_{x_j}, 1 \leq j \leq N_1$ such that $\sum_{j=1}^{N_1} \mathcal{L}^n(B_{1,j}) > 4^{-n} \mathcal{L}^n(U)$. We have

$$\mathcal{L}^n(U) - \sum_{j=1}^{N_1} \mathcal{L}^n(B_{1,j}) \leq (1 - 4^{-n}) \mathcal{L}^n(U).$$

Repeating the same argument for $U_1 = U \setminus \bigcup_{j=1}^{N_1} \overline{B_{1,j}}$, we get disjoint balls $B_{2,j} \subset U_1$, $1 \leq j \leq N_2$ of diameter $\leq \delta$ such that

$$\begin{aligned} \mathcal{L}^n(U) - \sum_{j=1}^{N_1} \mathcal{L}^n(B_{1,j}) - \sum_{j=1}^{N_2} \mathcal{L}^n(B_{2,j}) &\leq (1 - 4^{-n}) \left(\mathcal{L}^n(U) - \sum_{j=1}^{N_1} \mathcal{L}^n(B_{1,j}) \right) \\ &\leq (1 - 4^{-n})^2 \mathcal{L}^n(U). \end{aligned}$$

Continue inductively for each $k \in \mathbb{N}$ to produce $B_{k,j}, 1 \leq j \leq N_k$ with

$$\mathcal{L}^n(U) - \sum_{j=1}^{N_1} \mathcal{L}^n(B_{1,j}) - \cdots - \sum_{j=1}^{N_k} \mathcal{L}^n(B_{k,j}) \leq (1 - 4^{-n})^k \mathcal{L}^n(U).$$

Set $V = \bigcup_{k=1}^{\infty} \bigcup_{j=1}^{N_k} B_{k,j}$. V is a disjoint union of balls of diameter $\leq \delta$, and for each $k \in \mathbb{N}$,

$$\mathcal{L}^n(U) - \mathcal{L}^n(V) \leq (1 - 4^{-n})^k \mathcal{L}^n(U).$$

Letting $k \to \infty$ gives $\mathcal{L}^n(V) = \mathcal{L}^n(U)$. □

Proposition 11. *If $U \subset \mathbb{R}^n$ is open, then $\mathcal{H}^n(U) \leq \mathcal{L}^n(U)$.*

Proof. We may assume that $\mathcal{L}^n(U) < \infty$. By proposition (10), for each $k \in \mathbb{N}$, there exist disjoint balls $B_{k,j} \subset U$, $j \in \mathbb{N}$ of diameter $\leq \delta_k = 2^{-k}$ such that

$$\mathcal{L}^n(U \setminus V_k) = 0, \quad V_k = \bigcup_{j=1}^{\infty} B_{k,j}.$$

Let $V = \cap_{k=1}^{\infty} V_k$. For each $k \in \mathbb{N}$,

$$\gamma_n h^*_{n,\delta_k}(V) \leq \gamma_n h^*_{n,\delta_k}(V_k) \leq \sum_{j=1}^{\infty} \gamma_n h^*_{n,\delta_k}(B_{k,j}) \leq \sum_{j=1}^{\infty} \mathcal{L}^n(B_{k,j}) = \mathcal{L}^n(V_k) = \mathcal{L}^n(U).$$

Taking $k \to \infty$ gives

$$\mathcal{H}^n(V) \leq \mathcal{L}^n(U).$$

Thus using $\mathcal{H}^n \ll \mathcal{L}^n$,

$$\mathcal{H}^n(U) = \mathcal{H}^n(V) + \mathcal{H}^n(U \setminus V) = \mathcal{H}^n(V) \leq \mathcal{L}^n(U).$$ □

Remark 12. With $\mathcal{H}^n = \mathcal{L}^n$ in hand, it can be shown that on an m-dimensional manifold $M \subset \mathbb{R}^N$, $\mathcal{H}^m$ agrees with the usual surface measure on M obtained by integration of $\sqrt{g}$.

1.3 Hausdorff Dimension

Now we introduce Hausdorff dimension and give some examples. Let X be a metric space. Let $S \subset X$. We define the Hausdorff dimension of S as

$$\mathrm{Hdim}(S) = \sup\{r \in [0,\infty) : \mathcal{H}^r(S) > 0\} \in [0,\infty].$$

Proposition 13. *Let $S \subset X$, $r \in [0,\infty)$. If $r < \mathrm{Hdim}(S)$, then $\mathcal{H}^r(S) = \infty$. If $r > \mathrm{Hdim}(S)$, then $\mathcal{H}^r(S) = 0$.*

Proof. Suppose $r < \mathrm{Hdim}(S)$. Then there exists $s > r$ such that $\mathcal{H}^s(S) > 0$. For any $B \subset X$ with $\mathrm{diam}(B) \le \delta$ we have

$$\mathrm{diam}(B)^s = \mathrm{diam}(B)^r \,\mathrm{diam}(B)^{s-r} \le \delta^{s-r} \,\mathrm{diam}(B)^r.$$

Consequently,

$$h^*_{s,\delta}(S) \le \delta^{s-r}\, h^*_{r,\delta}(S). \tag{2}$$

Thus $h^*_{r,\delta}(S) \ge \delta^{-(s-r)}\, h^*_{s,\delta}(S)$. Taking $\delta \to 0$ gives $\mathcal{H}^r(S) = \infty$. If $r > \mathrm{Hdim}(S)$, then $\mathcal{H}^r(S) = 0$ by definition of sup. ☐

Proposition 14. *$\mathcal{H}^0$ is the counting measure on X.*

Proof. Since singletons are closed, they are measurable. Thus it suffices to show that $\mathcal{H}^0(\{x\}) = 1$ for all $x \in X$. So let $x \in X$ be arbitrary. Let $\delta > 0$ be arbitrary. Note that $\mathrm{diam}(B)^0 = \left\{\begin{smallmatrix} 1 & \text{if } B \neq \emptyset \\ 0 & \text{if } B = \emptyset \end{smallmatrix}\right.$ (we need $0^0 = 0$ here). Thus

$$h^*_{0,\delta}(\{x\}) = \inf\left\{N : \{x\} \subset \bigcup_{j=1}^{N} B_j, \mathrm{diam}(B_j) \le \delta\right\}$$

Since $\{x\} \subset B\big(x, \frac{\delta}{2}\big)$, $h^*_{0,\delta}(\{x\}) \le 1$. On the other hand, any cover of $\{x\}$ must have at least 1 set, so $h^*_{0,\delta}(\{x\}) \ge 1$. Taking $\delta \to \infty$ and noting that $\gamma_0 = 1$ gives $\mathcal{H}^0(\{x\}) = 1$. ☐

Now we compute the Hausdorff dimension and measure of the middle thirds Cantor set K. It is defined as follows. Set $K_0 = [0,1]$. For $\nu \ge 1$, let K_ν be the result of removing the open middle third of each interval in $K_{\nu-1}$. Thus K_ν is a disjoint union of 2^ν closed intervals of length $3^{-\nu}$. Set $K = \cap_{\nu=1}^{\infty} K_\nu$. For any $r \ge 0$,

$$h^*_{r,3^{-\nu}}(K_\nu) \le 2^\nu\, 3^{-\nu r} = \left(\frac{2}{3^r}\right)^{\nu}.$$

The right hand side is independent of ν when $\frac{2}{3^r} = 1$, i.e $r = \frac{\log(2)}{\log(3)}$. For this value of r, $h^*_{r,3^{-\nu}}(K) \le h^*_{r,3^{-\nu}}(K_\nu) \le 1$, so $\mathcal{H}^r(K) \le \gamma_r$. Establishing the reverse inequality is more difficult:

Proposition 15. *Let K be the middle thirds Cantor set. Then for $r = \frac{\log(2)}{\log(3)} \approx 0.6309$,*

$$\mathcal{H}^r(K) = \gamma_r \approx 1.0351.$$

Proof. We need to prove that $h_r^*(K) \geq 1$. It suffices to prove that $h_{r,\delta}^*(K) \geq 1$ for all $\delta > 0$. For this it suffices to prove that if $K \subset \bigcup_{k=1}^{\infty} I_k$ with $I_k \subset \mathbb{R}$ bounded closed intervals, then

$$\sum_{k=1}^{\infty} \ell(I_k)^r \geq 1.$$

By enlarging the length of each interval by a factor of $(1+\varepsilon)$, we can assume that each I_k is an open interval. By compactness of K, we can assume that the cover $\{I_k\}$ is finite.

Set $O = \bigcup_k I_k$. We claim that $O \supset K_\nu$ for some ν. If not, then for each ν, $K_\nu \setminus O \neq \emptyset$, and compactness of $K_\nu \setminus O$ implies $\bigcap_{\nu=1}^{\infty} (K_\nu \setminus O) = K \setminus O \neq \emptyset$, a contradiction. Thus there exists ν such that $K_\nu \subset O$. By increasing ν if necessary, we can assume that each interval in K_ν is contained in some I_k (choose ν such that $3^{-\nu} < \delta$, where δ is a Lebesgue number for the cover $\{I_k\}$).

By shrinking each I_k if necessary, we can arrange that each interval in K_ν intersects only one I_k. By further shrinking each I_k and then taking closures, we can arrange that the endpoints of each I_k are also endpoints of intervals in K_ν.

Now for each I_k we have that either

(a) I_k is equal to an interval in K_ν, or

(b) $I_k \setminus K_\nu$ contains an open interval L_k with $\ell(L_k) \geq \frac{\ell(I_k)}{3}$.

In case (b), we can partition I_k into 3 intervals J_k, L_k, J_k', where J_k and J_k' are closed intervals. Using $3^r = 2$ and concavity of $\varphi(t) = t^r$ for $r \in (0,1)$ gives

$$\begin{aligned}
\ell(I_k)^r &= (\ell(J_k) + \ell(L_k) + \ell(J_k'))^r \\
&\geq \left(\ell(J_k) + \frac{\ell(J_k) + \ell(J_k')}{2} + \ell(J_k')\right)^r \\
&= \left(\frac{3}{2}(\ell(J_k) + \ell(J_k'))\right)^r \\
&= 2\left(\frac{1}{2}(\ell(J_k) + \ell(J_k'))\right)^r \\
&\geq 2\,\frac{1}{2}(\ell(J_k)^r + \ell(J_k')^r) \\
&= \ell(J_k)^r + \ell(J_k')^r.
\end{aligned}$$

Thus replacing I_k with J_k and J_k' does not increase the sum $\sum_k \ell(I_k)^r$. Iterating this finitley many times, we can replace intervals until the I_k are exactly the intervals making up K_ν, without increasing the sum. Now we have $\sum_k \ell(I_k) = \left(\frac{2}{3^r}\right)^\nu = 1$, so we are done. ☐

Bibliography

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