Midpoint Rule Error Formula

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Consider the integral on a single interval $[-h, h]$. Define the error of the approximation for a function f as

$$E(f) := \int_{-h}^{h} f(x)\,dx - 2f(0).$$

Clearly E is a linear map. $E(1) = E(x) = 0$. For $k \geq 2$,

$$E(x^k) = \begin{cases} \frac{2}{k+1}h^{k+1} & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd} \end{cases}.$$

Thus

$$\begin{aligned} E(f) &= E\left(\sum_{i=0}^{\infty} \frac{f^{(i)}(0)}{i!} x^i\right) && (1) \\ &= \sum_{i=0}^{\infty} \frac{f^{(i)}(0)}{i!} E(x^i) && (2) \\ &= \sum_{i=1}^{\infty} C_{2i} f^{(2i)}(0) h^{2i+1} \qquad C_k := \frac{2}{(k+1)k!} && (3) \end{aligned}$$

Define the composite error as

$$E_c(f) := \int_a^b f(x)\,dx - \sum_{i=1}^{n} 2hf(c_i),$$

where c_i is the midpoint of the ith interval. Clearly E_c is a linear map and the composite error is the sum of the subinterval errors. Thus

$$
\begin{aligned}
E_c(f) &= \sum_{i=1}^{n} \sum_{j=1}^{\infty} C_{2j} f^{(2j)}(c_i) h^{2j+1} && (4)\\
&= \sum_{j=1}^{\infty} \sum_{i=1}^{n} C_{2j} f^{(2j)}(c_i) h^{2j+1} && (5)\\
&= \sum_{j=1}^{\infty} D_{2j} h^{2j} \sum_{i=1}^{n} 2h f^{(2j)}(c_i) \qquad D_k := \frac{C_k}{2} = \frac{1}{(k+1)k!} && (6)\\
&= \sum_{j=1}^{\infty} D_{2j} h^{2j} (I(f^{(2j)}) - E_c(f^{(2j)})) \qquad I(g) := \int_a^b g(x)\,dx && (7)\\
&= \sum_{j=1}^{\infty} D_{2j} I(f^{(2j)}) h^{2j} - \sum_{j=1}^{\infty} D_{2j} h^{2j} E_c(f^{(2j)}) && (8)\\
&= \sum_{j=1}^{\infty} D_{2j} I(f^{(2j)}) h^{2j} - \sum_{j=1}^{\infty} D_{2j} h^{2j} \left(\sum_{k=1}^{\infty} D_{2k} I(f^{(2j+2k)}) h^{2k} - \sum_{k=1}^{\infty} D_{2k} h^{2k} E_c(f^{(2j+2k)}) \right) && (9)\\
&= \sum_{j=1}^{\infty} D_{2j} I(f^{(2j)}) h^{2j} - \sum_{j=1}^{\infty} D_{2j} h^{2j} \sum_{k=1}^{\infty} D_{2k} I(f^{(2j+2k)}) h^{2k} + \sum_{j=1}^{\infty} D_{2j} h^{2j} \sum_{k=1}^{\infty} D_{2k} h^{2k} E_c(f^{(2j+2k)}) && (10)\\
&= \sum_{j=1}^{\infty} D_{2j} I(f^{(2j)}) h^{2j} - \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} D_{2j} D_{2k} I(f^{(2j+2k)}) h^{2j+2k} + \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} D_{2j} D_{2k} h^{2j+2k} E_c(f^{(2j+2k)}) && (11)\\
&= \dots && (12)\\
&= \sum_{i=1}^{\infty} \delta_{2i} h^{2i} \qquad \text{where the } \delta_k \text{ only depend on } f,\ a,\ b && (13)\\
& && (14)
\end{aligned}
$$

Another way to see it:

$$
\begin{aligned}
E_c(f) &= \sum_{i=1}^{n} \sum_{j=1}^{\infty} C_{2j} f^{(2j)}(c_i) h^{2j+1} && (15)\\
&= \sum_{j=1}^{\infty} \sum_{i=1}^{n} C_{2j} f^{(2j)}(c_i) h^{2j+1} && (16)\\
&= \sum_{j=1}^{\infty} D_{2j} h^{2j} \sum_{i=1}^{n} 2h f^{(2j)}(c_i) \qquad D_k := \frac{C_k}{2} = \frac{1}{(k+1)k!} && (17)\\
&= D_2 h^2 \sum_{i=1}^{n} 2h f^{(2)}(c_i) + \sum_{j=2}^{\infty} D_{2j} h^{2j} \sum_{i=1}^{n} 2h f^{(2j)}(c_i) && (18)\\
&= D_2 I(f^{(2)}) h^2 - \sum_{j=1}^{\infty} D_2 D_{2j} h^{2j+2} \sum_{i=1}^{n} 2h f^{(2j+2)}(c_i) + \sum_{j=2}^{\infty} D_{2j} h^{2j} \sum_{i=1}^{n} 2h f^{(2j)}(c_i) && (19)\\
&= D_2 I(f^{(2)}) h^2 - \sum_{j=2}^{\infty} D_2 D_{2j-2} h^{2j} \sum_{i=1}^{n} 2h f^{(2j)}(c_i) + \sum_{j=2}^{\infty} D_{2j} h^{2j} \sum_{i=1}^{n} 2h f^{(2j)}(c_i) && (20)\\
&= \dots && (21)\\
&= \sum_{i=1}^{\infty} \delta_{2i} h^{2i}. && (22)
\end{aligned}
$$

Also we get the asymptotic error estimate

$$
\begin{aligned}
E_c(f) &= D_2(f'(b) - f'(a))h^2 + O(h^4) && (23)\\
&= \frac{f'(b) - f'(a)}{24} H^2 + O(H^4) \qquad H := 2h \text{ is the size of the subinterval} && (24)
\end{aligned}
$$