Minkowski's Integral Inequality

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Abstract

In this note, we use Jensen's inequality to deduce Minkowski's integral inequality.

Lemma 1. *Let* $(X, \mathcal{F}, \mu)$ *and* $(Y, \mathcal{G}, \nu)$ *be* σ*-finite measure spaces and let* $p \in [1, \infty)$*. Suppose* $f : X \times Y \to [0, \infty]$ *is* $\mathcal{F} \otimes \mathcal{G} - \mathcal{B}([0, \infty])$ *measurable. Assume*

(i) $\|f^y\|_p \in (0, \infty)$ *for a.e.* $y \in Y$*,*

(ii) $\nu(Y) > 0$*.*

Then

$$\left\| \int_Y f^y(\cdot)\, dy \right\|_p \le \int_Y \|f^y\|_p\, dy.$$

Proof. We use the notation $f^y(\cdot) = f(\cdot, y) : X \to [0, \infty]$. For $y \in Y$, define $f_0^y = \frac{f^y}{\|f^y\|_p}$. Let $\theta : Y \to [0, \infty]$ be an arbitrary measurable function with $\int_Y \theta(y)\, dy = 1$. By Jensen's inequality, for every $x \in X$,

$$\left(\int_Y f_0^y(x)\, \theta(y)\, dy \right)^p \le \int_Y f_0^y(x)^p\, \theta(y)\, dy.$$

Thus

$$\begin{aligned} \int_X \left(\int_Y f_0^y(x)\, \theta(y)\, dy \right)^p dx &\le \int_X \int_Y f_0^y(x)^p\, \theta(y)\, dy\, dx \\ &= \int_Y \theta(y) \int_X f_0^y(x)^p\, dx\, dy \\ &= \int_Y \theta(y)\, dy \\ &= 1. \end{aligned}$$

Now letting $\theta(y) = \frac{\|f^y\|_p}{\int_Y \|f^y\|_p\, dy}$ gives

$$\int_X \left(\int_Y f^y(x)\, dy \right)^p dx \le \left(\int_Y \|f^y\|_p\, dy \right)^p.$$

Taking p-th roots yields

$$\left\| \int_Y f^y(\cdot)\, dy \right\|_p \le \int_Y \|f^y\|_p\, dy. \qquad \square$$

Theorem 2. *Let everything be as in Lemma 1, except that we no longer assume (i) and (ii). Then we still have*

$$\left\| \int_Y f^y(\cdot)\, dy \right\|_p \le \int_Y \|f^y\|_p\, dy.$$

Proof. If $\nu(Y)=0$, then the result holds trivially. Thus we may assume $\nu(Y)>0$.

If $\{y\in Y:\|f^y\|_p=\infty\}$ has positive measure, then $\int_Y\|f^y\|_p\,dy=\infty$, so the result holds. Thus we may assume that $\|f^y\|_p<\infty$ for a.e. $y\in Y$.

Let

$$E=\{y\in Y:\|f^y\|_p>0\}=\{y\in Y: f^y \text{ is not a.e. equal to } 0\}.$$

Applying Lemma 1 to $X\times Y$ gives

$$\left\|\int_E f^y(\cdot)\,dy\right\|_p\leq\int_E\|f^y\|_p\,dy.$$

Clearly $\int_E\|f^y\|_p\,dy=\int_Y\|f^y\|_p\,dy$. We claim that $\|\int_E f^y(\cdot)\,dy\|_p=\|\int_Y f^y(\cdot)\,dy\|_p$. To show this, it suffices to show that $\int_E f^y(x)\,dy=\int_Y f^y(x)\,dy$ for a.e. $x\in X$. For this, it suffices to show that $\int_{Y\setminus E}f(x,y)\,dy=0$ for a.e. $x\in X$. We have

$$\int_X\int_{Y\setminus E}f(x,y)\,dy\,dx=\int_{Y\setminus E}\int_X f(x,y)\,dx\,dy=0.$$

Thus $\int_{Y\setminus E}f(x,y)\,dy=0$ for a.e. $x\in X$ as desired. Thus

$$\left\|\int_Y f^y(\cdot)\,dy\right\|_p=\left\|\int_E f^y(\cdot)\,dy\right\|_p\leq\int_E\|f^y\|_p\,dy=\int_Y\|f^y\|_p\,dy. \qquad\square$$