Stone-Weierstrass Theorem Proof

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This proof is the proof given in the book Measure Theory and Integration by Michael Taylor, but with the details filled in.

Weierstrass Approximation Theorem. If $I = [a, b]$ is an interval in $\mathbb{R}$, then the space $P(I)$ of polynomials on I is dense in $C(I)$.

Proof. Omitted.

Stone-Weierstrass Theorem. Let X be a compact topological space, and A a subalgebra of $C(X)$. Suppose $1 \in A$ and that A separates points of X, that is, for distinct $p, q \in X$, there exists $f_{pq} \in A$ such that $f_{pq}(p) \neq f_{pq}(q)$. Then A is dense in $C(X)$.

Proof. We prove the theorem in several steps. Note that $\overline{A}$ is an algebra.

Step 1. If $f \in \overline{A}$ and $\phi \in C(\mathbb{R})$, then $\phi \circ f \in \overline{A}$.

Proof. Since X is compact, $f(X)$ is compact, so there is a compact interval J in $\mathbb{R}$ such that $f(X) \subset J$. Take $a_n \in A$ converging to f, $p_n \in P(J)$ converging to $\phi|_J$. Enlarging J if necessary, we may assume that $a_n(X) \subset J$ for all $n \in \mathbb{N}$. Then $p_n \circ a_n \in A$, and for all $x \in X$,

$$\begin{aligned}|p_n \circ a_n(x) - \phi \circ f(x)| &\leq |p_n \circ a_n(x) - \phi \circ a_n(x)| + |\phi \circ a_n(x) - \phi \circ f(x)| \\ &\leq \|p_n - \phi|_J\| + \omega_{\phi|_J}(\|a_n - f\|),\end{aligned}$$

where $\omega_{\phi|_J}$ is a modulus of continuity for $\phi|_J$. Thus $p_n \circ a_n$ converges to $\phi \circ f$, so $\phi \circ f \in \overline{A}$.

Step 2. If $f_1, f_2 \in \overline{A}$, then $\max(f_1, f_2), \min(f_1, f_2) \in \overline{A}$.

Proof. This follows from the identites

$$\max(f_1, f_2) = \frac{f_1 + f_2}{2} + \frac{|f_1 - f_2|}{2},$$

$$\min(f_1, f_2) = \frac{f_1 + f_2}{2} - \frac{|f_1 - f_2|}{2}.$$

Step 3. For distinct $p, q \in X$, there exists $f_{pq} \in \overline{A}$ such that $f_{pq}(p) = 1$ and $f_{pq}(q) = 0$.

Proof. Since X separates points, there exists $f_{pq} \in \overline{A}$ such that $f_{pq}(p) \neq f_{pq}(q)$. Let $g_{pq}(x) = \frac{f_{pq}(x) - f_{pq}(q)}{f_{pq}(p) - f_{pq}(q)}$. Then $g_{pq} \in \overline{A}$, $g_{pq}(p) = 1$, and $g_{pq}(q) = 0$.

Step 4. For distinct $p, q \in X$, there exists $f_{pq} \in \overline{A}$ such that $0 \leq f_{pq} \leq 1$ on X, $f_{pq} = 1$ on a neighborhood of p, and $f_{pq} = 0$ on a neighborhood of q.

Proof. By step 3, there exists $f_{pq} \in \overline{A}$ such that $f_{pq}(p) = 1$ and $f_{pq}(q) = 0$. Define $\phi \colon \mathbb{R} \to \mathbb{R}$ by

$$\phi(y) = \begin{cases} 0 & \text{if } y \leq \frac{1}{3} \\ 3(y - \frac{1}{3}) & \text{if } \frac{1}{3} \leq y \leq \frac{2}{3} \\ 1 & \text{if } y \geq \frac{2}{3} \end{cases}.$$

Then $\phi \in C(\mathbb{R})$ and $\phi \circ f_{pq} \in \overline{A}$ has the desired properties. Note that this implies X is Hausdorff.

Step 5. If $p \in X$ and U is a neighborhood of p, then there exists $f_{pU} \in \overline{A}$ such that $0 \leq f_{pU} \leq 1$ on X, $f_{pU} = 1$ on a neighborhood of p, and $f_{pU} = 0$ off U.

Proof. By step 4, for each $q \in U^c$, there exists $f_{pq} \in \overline{A}$ such that $0 \leq f_{pq} \leq 1$ on X, $f_{pq} = 1$ on a neighborhood U_q of p, and $f_{pq} = 0$ on a neighborhood V_q of q. Since $\{V_q \mid q \in U^c\}$ is an open cover of U^c, there is a finite subcover $\{V_{q_1}, \ldots, V_{q_N}\}$. Let $f_{pU} = \min(f_{pq_1}, \ldots, f_{pq_n}) \in \overline{A}$. Then $0 \leq f_{pU} \leq 1$ on X, $f_{pU} = 1$ on $\cap_{i=1}^N U_{q_i}$, and $f_{pU} = 0$ on $\cup_{i=1}^N V_{q_i} \supset U^c$.

Step 6. For each compact $K \subset X$, open $U \supset K$, there exists $f_{KU} \in \overline{A}$ such that $0 \leq f_{KU} \leq 1$ on X, $f_{KU} = 1$ on K, and $f_{KU} = 0$ off U.

Proof. By step 5, for each $p \in K$, there exists $f_{pU} \in \overline{A}$ such that $0 \leq f_{pU} \leq 1$ on X, $f_{pU} = 1$ on a neighborhood U_p of p, and $f_{pU} = 0$ off U. Since $\{U_p \mid p \in$

$K\}$ is an open cover of K, there is a finite subcover $\{U_{p_1}, \ldots, U_{p_N}\}$. Let $f_{KU} = \max(f_{p_1 U}, \ldots, f_{p_N U}) \in \overline{A}$. Then $0 \leq f_{KU} \leq 1$ on X, $f_{KU} = 1$ on $\cup_{i=1}^N U_{p_i} \supset K$, and $f_{KU} = 0$ off U.

Step 7. Now we prove the theorem. Let $g \in C(X)$ be arbitrary. We may assume that $0 \leq g \leq 1$ on X, since if $g(X) \subset [-M, M]$, we could consider $\frac{g+M}{2M}$ instead. Let $N \in \mathbb{N}$ be arbitrary. For $i = 0, 1, \ldots, N$, let $K_i = g^{-1}([\frac{i}{N}, \frac{i+1}{N}])$, $U_i = g^{-1}((\frac{i-1}{N}, \frac{i+2}{N}))$. By step 7, for each i, there exists $f_i \in \overline{A}$ such that $0 \leq f_i \leq 1$ on X, $f_i = 1$ on K_i, and $f_i = 0$ off U_i. Let $f = \max_{0 \leq i \leq N} \frac{i}{N} f_i \in \overline{A}$. Now let $x \in X$ be arbitrary. Let $j \in \{0, 1, \ldots, N\}$ be such that $\frac{j}{N} \leq g(x) < \frac{j+1}{N}$. Note that $f_i(x) = 0$ for all $i \leq j - 2$ and $i \geq j + 2$. Thus $f(x) = \max(\frac{j-1}{N} f_{j-1}(x), \frac{j}{N} f_j(x), \frac{j+1}{N} f_{j+1}(x))$. Thus $f(x) \geq \frac{j}{N}$ and $f(x) \leq \frac{j+1}{N}$. Thus $|g(x) - f(x)| \leq \frac{1}{N}$. Thus $\|f - g\| \leq \frac{1}{N}$. Since N was arbitrary, this proves the theorem. $\square$