STOR 635 HW 4

BY AMEER QAQISH

February 7, 2022

1 Solutions

1. Does convergence in probability imply conditional convergence in probability? That is, if $X_1, X_2, \ldots, X$ are random variables defined on the same probability space $(\Omega, \mathcal{F}, P)$ such that $X_n \to X$ in probability and $\mathcal{G}$ is any sub-σ-field of $\mathcal{F}$, does one necessarily have for every $\varepsilon > 0$, $P(|X_n - X| > \varepsilon \,|\, \mathcal{G}) \to 0$ almost surely? What about L^p-convergence for $p \in (0, \infty)$ (i.e. if $X_n \to X$ in L^p, does this imply $E(|X_n - X|^p \,|\, \mathcal{G}) \to 0$ almost surely)? Prove or disprove with a counter-example.

Answer. Convergence in probability does not imply conditional convergence in probability. Define a sequence of random variables $X_1, X_2, \ldots$ on $([0,1], \mathcal{B}([0,1]), P = \text{Lebesgue measure})$ by $X_1 = 1_{\left[0, \frac{1}{2}\right]}$, $X_2 = 1_{\left[\frac{1}{2}, 1\right]}$, $X_3 = 1_{\left[0, \frac{1}{4}\right]}$, $X_4 = 1_{\left[\frac{1}{4}, \frac{2}{4}\right]}$, $X_5 = 1_{\left[\frac{2}{4}, \frac{3}{4}\right]}$, $X_6 = 1_{\left[\frac{3}{4}, 1\right]}$, $X_7 = 1_{\left[0, \frac{1}{8}\right]}$, $\ldots$. Since $P(X_n = 0) \to 1$ as $n \to \infty$, $P(X_n > \varepsilon) = P(X_n = 1) = 1 - P(X_n = 0) \to 0$ for any $\varepsilon < 1$. Thus $X_n \to 0$ in probability. We have

$$P(X_n > \varepsilon \,|\, \mathcal{B}([0,1])) = 1_{\{X_n > \varepsilon\}} = 1_{(\varepsilon, \infty)}(X_n).$$

For any $\omega \in [0,1]$, $X_n(\omega) = 1$ for infinitely many n, so $1_{(\varepsilon,\infty)}(X_n(\omega))$ does not converge to 0. Thus for almost every $\omega \in [0,1]$, $P(X_n > \varepsilon \,|\, \mathcal{B}([0,1]))(\omega)$ does not converge to 0.

L^p convergence does not imply that $E(|X_n - X|^p \,|\, \mathcal{G}) \to 0$ almost surely. Let X_1, $X_2, \ldots$ be as above, and let $p \in (0, \infty)$. We have $E(X_n^p) = E(X_n) \to 0$ as $n \to \infty$. Thus $X_n \to 0$ in L^p. We have

$$E(X_n^p \,|\, \mathcal{B}([0,1])) = X_n^p = X_n.$$

For every $\omega \in [0,1]$, $X_n(\omega) = 1$ for infinitley many n, so $X_n(\omega)$ does not converge to 0. Thus for almost every $\omega \in [0,1]$, $E(X_n^p \,|\, \mathcal{B}([0,1]))(\omega)$ does not converge to 0.

2. Let $\{X_i\}_{i \geq 1}$ be i.i.d. integrable random variables with $E(X_i) = \mu$. Let $S_n = X_1 + \cdots + X_n$. Show that $E(X_1 \,|\, S_n)$ converges a.s. to $E(X_1)$.

Proof. Let $n \in \mathbb{N}$ be arbitrary. First we show that $E(X_i \,|\, S_n) = E(X_1 \,|\, S_n)$ for all $i \leq n$. Let $i \leq n$. Every set in $\sigma(S_n)$ is of the form $S_n^{-1}(A)$ with $A \in \mathcal{B}(\mathbb{R})$. For such a set,

$$\begin{aligned}
\int X_i \, 1_{S_n^{-1}(A)} \, dP &= \int X_i \, 1_A(S_n) \, dP \\
&= \int X_i \, 1_A(X_1 + \cdots + X_n) \, dP \\
&= \int x_i \, 1_A(x_1 + \cdots + x_n) \, dP_{(X_1, \ldots, X_n)}.
\end{aligned}$$

Since the X_is are independent, $P_{(X_1,\ldots,X_n)} = P_{X_1} \times \ldots \times P_{X_n}$. Since the X_is are identically distributed, $P_{X_i} = P_{X_1}$ for all i. Hence

$$\int x_i\, 1_A(x_1 + \cdots + x_n)\, dP_{(X_1,\ldots,X_n)} =$$

$$\int x_i\, 1_A(x_1 + \cdots + x_n)\, d\,P_{X_1}(x_1) \cdots dP_{X_1}(x_i) \cdots d\,P_{X_1}(x_n) =$$

$$\int x_i\, 1_A(x_1 + \cdots + x_i + \cdots + x_n)\, d\,P_{X_1}(x_i) \cdots dP_{X_1}(x_1) \cdots d\,P_{X_1}(x_n) =$$

$$\int x_1\, 1_A(x_i + \cdots + x_1 + \cdots + x_n)\, d\,P_{X_1}(x_1) \cdots dP_{X_1}(x_i) \cdots d\,P_{X_1}(x_n) =$$

$$\int x_1\, 1_A(x_1 + \cdots + x_n)\, d\,P_{X_1}(x_1) \cdots d\,P_{X_1}(x_n) = \int X_1\, 1_A(S_n)\, dP.$$

Thus $\int X_i\, 1_A(S_n)\, dP = \int X_1\, 1_A(S_n)\, dP$ for all $A \in \mathcal{B}(\mathbb{R})$. Thus $E(X_i \mid S_n) = E(X_1 \mid S_n)$.

Thus

$$n\, E(X_1 \mid S_n) = E(X_1 \mid S_n) + \cdots + E(X_n \mid S_n) = E(S_n \mid S_n) = S_n.$$

Thus

$$E(X_1 \mid S_n) = \frac{S_n}{n}.$$

By the strong law of large numbers, $\frac{S_n}{n} \to E(X_1)$ a.s. □

3. Let for a square integrable random variable X and a sub σ-field $\mathcal{G}$ of $\mathcal{F}$, $\mathrm{var}(X \mid \mathcal{G}) \doteq E(X^2 \mid \mathcal{G}) - E(X \mid \mathcal{G})^2$. Show that

$$\mathrm{var}(X) = E(\mathrm{var}(X \mid \mathcal{G})) + \mathrm{var}(E(X \mid \mathcal{G})).$$

Proof. We have

$$E(\mathrm{var}(X \mid \mathcal{G})) = E(X^2) - E(E(X \mid \mathcal{G})^2),$$

$$\mathrm{var}(E(X \mid \mathcal{G})) = E(E(X \mid \mathcal{G})^2) - E(E(X \mid \mathcal{G}))^2 = E(E(X \mid \mathcal{G})^2) - E(X)^2.$$

Thus

$$E(\mathrm{var}(X \mid \mathcal{G})) + \mathrm{var}(E(X \mid \mathcal{G})) = E(X^2) - E(X)^2 = \mathrm{var}(X).$$ □

4. Consider the Polya urn model: Start with R_0 red balls and B_0 black balls in an urn. At each step, draw a ball uniformly at random and put back two balls of the color drawn back into the urn. Let $\mathcal{F}_n = \sigma\{R_0, \ldots, R_n\}$, $n \geq 0$. Show that $X_n := R_n / (R_n + B_n)$ is a martingale with respect to $\mathcal{F}_n$.

Proof. We assume that R_0 and B_0 are fixed numbers. For each $n \geq 0$, let $N_n = R_0 + B_0 + n$ be the number of balls in the urn after the nth draw. Note that N_n is not random. Since $B_n = N_n - R_n$, B_n is $\mathcal{F}_n$-measurable. Thus X_n is $\mathcal{F}_n$-measurable. We have $0 \leq X_n \leq 1$, so $X_n \in L^\infty$ and is therefore integrable. Let $D_n = 1$ if the nth draw is a red ball, and let $D_n = 0$ if it is a blue ball. We have

$$X_{n+1} = 1_{\{D_n = 1\}} \frac{X_n\, N_n + 1}{N_n + 1} + 1_{\{D_n = 0\}} \frac{X_n\, N_n}{N_n + 1}.$$

Taking the expectation given X_n and using $P(D_n=1\,|\,X_n=r)=r$ gives

$$\begin{aligned}E(X_{n+1}\,|\,X_n=r) &= r\,\frac{r\,N_n+1}{N_n+1}+(1-r)\,\frac{r\,N_n}{N_n+1}\\ &= \frac{r^2\,N_n+r+r\,N_n-r^2\,N_n}{N_n+1}\\ &= r.\end{aligned}$$

Thus $E(X_{n+1}\,|\,X_n)=X_n$. Thus $(X_n)_{n\geq 0}$ is a martingale with respect to $(\mathcal{F}_n)_{n\geq 0}$. □

5. Let $\xi_1, \xi_2, \ldots$ be independent with $E(\xi_i)=0$ and $\mathrm{Var}(\xi_m)=\sigma_m^2<\infty$. Let $s_n^2=\sum_{m=1}^n \sigma_m^2$ and $S_n=\sum_{m=1}^n \xi_m$. Show that $S_n^2-s_n^2$ is a martingale.

Proof. $S_n^2-s_n^2$ is adapted with respect to the natural filtration $\mathcal{F}_n=\sigma(S_1,\ldots,S_n)=\sigma(\xi_1,\ldots,\xi_n)$. $S_n^2-s_n^2$ is integrable since $E(S_n^2)=s_n^2<\infty$. We have

$$\begin{aligned}E(S_{n+1}^2\,|\,\mathcal{F}_n) &= E((S_n+\xi_{n+1})^2\,|\,\mathcal{F}_n)\\ &= E(S_n^2+\xi_{n+1}^2+2\,S_n\,\xi_{n+1}\,|\,\mathcal{F}_n)\\ &= S_n^2+E(\xi_{n+1}^2\,|\,\mathcal{F}_n)+2\,S_n\,E(\xi_{n+1}\,|\,\mathcal{F}_n)\\ &= S_n^2+E(\xi_{n+1}^2)+2\,S_n\,E(\xi_{n+1})\\ &= S_n^2+\sigma_{n+1}^2.\end{aligned}$$

Thus

$$E(S_{n+1}^2-s_{n+1}^2\,|\,\mathcal{F}_n)=E(S_{n+1}^2\,|\,\mathcal{F}_n)-s_{n+1}^2=S_n^2-s_n^2.$$

Thus $S_n^2-s_n^2$ is a martingale. □

6. Let $\{X_n\}$, $\{Y_n\}$ be square integrable martingales with respect to the filtration $\{\mathcal{F}_n\}$. Let $X_0=Y_0=0$. Show that

$$E(X_n\,Y_n)=\sum_{k=1}^n E((X_k-X_{k-1})\,(Y_k-Y_{k-1})), \quad n\geq 1.$$

Proof. For each n,

$$X_n=\sum_{j=1}^n (X_j-X_{j-1}), \quad Y_n=\sum_{k=1}^n (Y_k-Y_{k-1}).$$

Thus

$$E(X_n\,Y_n)=\sum_{j,k=1}^n E((X_j-X_{j-1})\,(Y_k-Y_{k-1})).$$

Let $j<k$ be arbitrary. Since (Y_n) is a martingale, $E(Y_k\,|\,\mathcal{F}_j)=Y_j$ and $E(Y_{k-1}\,|\,\mathcal{F}_j)=Y_j$. Thus

$$\begin{aligned}E((X_j-X_{j-1})\,(Y_k-Y_{k-1})) &= E(E((X_j-X_{j-1})\,(Y_k-Y_{k-1})\,|\,\mathcal{F}_j))\\ &= E((X_j-X_{j-1})\,E(Y_k-Y_{k-1}\,|\,\mathcal{F}_j))\\ &= E((X_j-X_{j-1})\,0)\\ &= 0.\end{aligned}$$

Thus

$$E(X_n Y_n) = \sum_{k=1}^{n} E((X_k - X_{k-1})(Y_k - Y_{k-1})). \qquad \square$$

7. Let $\{Y_n\}$ be a sequence of $\{\mathcal{F}_n\}$-adapted integrable random variables. Let

$$X_n = \sum_{j=1}^{n} (Y_j - E(Y_j \mid \mathcal{F}_{j-1})), \quad n \geq 1$$

and $X_0 = 0$. Show that $\{X_n\}$ is a $\{\mathcal{F}_n\}$-martingale.

Proof. For each j, since Y_j is integrable, $E(Y_j \mid \mathcal{F}_{j-1})$ is integrable. Therefore X_n is integrable. Clearly X_n is $\mathcal{F}_n$-measurable. We have

$$\begin{aligned} E(X_{n+1} \mid \mathcal{F}_n) &= \sum_{j=1}^{n+1} (E(Y_j \mid \mathcal{F}_n) - E(E(Y_j \mid \mathcal{F}_{j-1}) \mid \mathcal{F}_n)) \\ &= \sum_{j=1}^{n} (Y_j - E(Y_j \mid \mathcal{F}_{j-1})) + E(Y_{n+1} \mid \mathcal{F}_n) - E(Y_{n+1} \mid \mathcal{F}_n) \\ &= \sum_{j=1}^{n} (Y_j - E(Y_j \mid \mathcal{F}_{j-1})) \\ &= X_n. \end{aligned}$$

Thus $\{X_n\}$ is an $\mathcal{F}_n$-martingale. $\square$