Proof of Gauss-Green Theorem

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1 Introduction

In this note, we prove Gauss's theorem.

Gauss's Theorem. Let $\Omega \subset \mathbb{R}^n$ be open and bounded with C^1 boundary. If u is C^1 on a neighborhood O of $\overline{\Omega}$, then for each $i \in \{1, \ldots, n\}$,

$$\int_\Omega u_{x_i}\, dV = \int_{\partial\Omega} u\nu_i\, dS,$$

where ν is the outward pointing unit normal vector to $\partial\Omega$. Equivalently,

$$\int_\Omega \nabla u\, dV = \int_{\partial\Omega} u\nu\, dS.$$

Proof: (1) The first step is to reduce to the case where $u \in C_c^1(\mathbb{R}^n)$. Pick $\phi \in C_c^\infty(O)$ such that $\phi = 1$ on $\overline{\Omega}$. Note that $\phi u \in C_c^1(O) \subset C_c^1(\mathbb{R}^n)$ and $\phi u = u$ on $\overline{\Omega}$. Hence it suffices to prove the theorem for ϕu. Hence we may assume that $u \in C_c^1(\mathbb{R}^n)$.

(2) The second step is to reduce to the case where $\partial\Omega$ is the graph of a C^1 function. Let $x_0 \in \partial\Omega$ be arbitrary. The assumption that $\overline{\Omega}$ has C^1 boundary means that there is a neighborhood U of x_0 in $\mathbb{R}^n$ such that $\partial\Omega \cap U$ is the graph of a C^1 function with $\Omega \cap U$ lying on one side of this graph. More precisely, this means that after a translation and rotation of Ω, there are $r > 0$ and $h > 0$ and a C^1 function $g : \mathbb{R}^{n-1} \to \mathbb{R}$, such that with the notation

$$x' = (x_1, \ldots, x_{n-1}),$$

it holds that

$$U = \{x \in \mathbb{R}^n : |x'| < r \text{ and } |x_n - g(x')| < h\}$$

and for $x \in U$,

$$\begin{aligned} x_n = g(x') &\implies x \in \partial\Omega, \\ -h < x_n - g(x') < 0 &\implies x \in \Omega, \\ 0 < x_n - g(x') < h &\implies x \notin \Omega. \end{aligned}$$

Since $\partial\Omega$ is compact, we can cover $\partial\Omega$ with finitely many neighborhoods $U_1, \ldots, U_N$ of the above form. Note that $\{\Omega, U_1, \ldots, U_N\}$ is an open cover of $\overline{\Omega} = \Omega \cup \partial\Omega$. By using a C^∞ partition of unity subordinate to this cover, it suffices to prove the theorem in the case where either u has compact support in Ω or u has compact support in some U_j. If u has compact support in Ω, then for all $i \in \{1, \ldots, n\}$, $\int_\Omega u_{x_i}\, dV = \int_{\mathbb{R}^n} u_{x_i}\, dV = \int_{\mathbb{R}^{n-1}} \int_{-\infty}^{\infty} u_{x_i}(x)\, dx_i\, dx' = 0$ by the fundamental theorem of calculus, and $\int_{\partial\Omega} u\nu_i\, dS = 0$ since u vanishes on a neighborhood of $\partial\Omega$. Thus the theorem holds for u with compact support in Ω. Thus we have reduced to the case where u has compact support in some U_j.

(3) So assume u has compact support in some U_j. The last step now is to show that the theorem is true by direct computation. Change notation to $U = U_j$, and bring in the notation from (2) used to describe U. This means that we have rotated and translated Ω. This is a valid reduction since the theorem is invariant under rotations and translations of coordinates (this will be proved in detail at the end). Since $u(x) = 0$ for $|x'| \geq r$ and for $|x_n - g(x')| \geq h$, we have for each $i \in \{1, \ldots, n\}$ that

$$\begin{aligned} \int_\Omega u_{x_i}\, dV &= \int_{|x'|<r} \int_{g(x')-h}^{g(x')} u_{x_i}(x', x_n)\, dx_n\, dx' \\ &= \int_{\mathbb{R}^{n-1}} \int_{-\infty}^{g(x')} u_{x_i}(x', x_n)\, dx_n\, dx'. \end{aligned}$$

For $i = n$ we have by the fundamental theorem of calculus that

$$\int_{\mathbb{R}^{n-1}} \int_{-\infty}^{g(x')} u_{x_n}(x', x_n)\, dx_n\, dx' = \int_{\mathbb{R}^{n-1}} u(x', g(x'))\, dx'.$$

Now fix $i \in \{1, \ldots, n-1\}$. Note that

$$\int_{\mathbb{R}^{n-1}} \int_{-\infty}^{g(x')} u_{x_i}(x', x_n)\, dx_n\, dx' = \int_{\mathbb{R}^{n-1}} \int_{-\infty}^{0} u_{x_i}(x', g(x') + s)\, ds\, dx'$$

Define $v : \mathbb{R}^n \to \mathbb{R}$ by $v(x', s) = u(x', g(x') + s)$. By the chain rule,

$$v_{x_i}(x', s) = u_{x_i}(x', g(x') + s) + u_{x_n}(x', g(x') + s) g_{x_i}(x').$$

But since v has compact support, we can integrate out dx_i first to deduce that

$$\int_{\mathbb{R}^{n-1}} \int_{-\infty}^{0} v_{x_i}(x', s)\, ds\, dx' = 0.$$

Thus

$$\begin{aligned}
\int_{\mathbb{R}^{n-1}} \int_{-\infty}^{0} u_{x_i}(x', g(x') + s)\, ds\, dx' &= \int_{\mathbb{R}^{n-1}} \int_{-\infty}^{0} -u_{x_n}(x', g(x') + s) g_{x_i}(x')\, ds\, dx' \\
&= \int_{\mathbb{R}^{n-1}} -u(x', g(x')) g_{x_i}(x')\, dx'.
\end{aligned}$$

In summary, with $\nabla u = (u_{x_1}, \ldots, u_{x_n})$ we have

$$\int_{\Omega} \nabla u\, dV = \int_{\mathbb{R}^{n-1}} \int_{-\infty}^{g(x')} \nabla u\, dV = \int_{\mathbb{R}^{n-1}} u(x', g(x'))(-\nabla g(x'), 1)\, dx'.$$

Recall that the outward unit normal to the graph Γ of g at a point $(x', g(x')) \in \Gamma$ is $\nu(x', g(x')) = \frac{1}{\sqrt{1+|\nabla g(x')|^2}}(-\nabla g(x'), 1)$ and that the surface element dS is given by $dS = \sqrt{1 + |\nabla g(x')|^2}\, dx'$. Thus

$$\int_{\Omega} \nabla u\, dV = \int_{\partial\Omega} u\nu\, dS.$$

This completes the proof.

Poof of invariance under rotation and translation: Fix $u \in C_c^1(\mathbb{R}^n)$. We first show invariance under rotation. Suppose we used an orthogonal matrix R to rotate Ω. We will show that

$$\int_{R\Omega} \nabla(u \circ R^T)(y)\, dy = \int_{\partial R\Omega} (u \circ R^T)(y) \nu_{R\Omega}(y) dS(y) \implies \int_{\Omega} \nabla u(x)\, dx = \int_{\partial\Omega} u(x)\nu(x)\, dS(x).$$

So assume that

$$\int_{R\Omega} \nabla(u \circ R^T)(y)\, dy = \int_{\partial R\Omega} u(R^T y) \nu_{R\Omega}(y)\, dS(y).$$

By the chain rule,

$$\begin{aligned}
\nabla(u \circ R^T)(y) &= D(u \circ R^T)(y)^T \\
&= (Du(R^T y) R^T)^T \\
&= R \nabla u(R^T y).
\end{aligned}$$

Thus
$$\int_{R\Omega} \nabla(u \circ R^T)(y)\, dy = R \int_{R\Omega} \nabla u(R^T y)\, dy = R \int_{\Omega} \nabla u(x)\, dx,$$
where we used the change of variables $x = R^T y$, $dy = |\det R|\, dx = dx$. On the other hand, using the change of variables $x = R^T y$, $y = Rx$, the definition of the surface integral can be used to check that $dS(y) = dS(x)$, so

$$\begin{aligned}
\int_{\partial R\Omega} u(R^T y)\nu_{R\Omega}(y)\, dS(y) &= \int_{R\partial\Omega} u(R^T y)\nu_{R\Omega}(y)\, dS(y) \\
&= \int_{\partial\Omega} u(x)\nu_{R\Omega}(Rx)\, dS(x) \\
&= \int_{\partial\Omega} u(x)R\nu(x)\, dS(x) \\
&= R \int_{\partial\Omega} u(x)\nu(x)\, dS(x).
\end{aligned}$$

Thus
$$\begin{gathered}
R \int_{\Omega} \nabla u(x)\, dx = R \int_{\partial\Omega} u(x)\nu(x)\, dS(x), \\
\int_{\Omega} \nabla u(x)\, dx = \int_{\partial\Omega} u(x)\nu(x)\, dS(x).
\end{gathered}$$
The proof of invariance under translation is similar.