STOR 635 HW 2

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1. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $0 < p < \infty$. Show that the simple functions in L^p are dense in L^p, i.e., for every $f \in L^p$, there exists a sequence of simple functions $s_n \in L^p$ such that $\|s_n - f\|_p \to 0$ as $n \to \infty$.

Answer. First we prove some auxiliary theorems.

Theorem 1. *Suppose $f:\Omega \to [0,\infty]$ is measurable. Then there exists a sequence of nonnegative simple functions $s_n:\Omega \to [0,\infty)$ such that $s_n \nearrow f$ pointwise, that is, for each $x \in \Omega$, $s_n(x) \nearrow f(x)$.*

Proof. For each $n \in \mathbb{N}$, define $s_n:\Omega \to [0,\infty)$ by

$$s_n(x) = \begin{cases} k\,2^{-n} & \text{if } k \in \left\{0,1,\ldots,\frac{n}{2^{-n}}-1\right\} \text{ and } f(x) \in [k\,2^{-n},(k+1)\,2^{-n}) \\ n & \text{if } f(x) \geq n \end{cases}.$$

We have

$$s_n = \sum_{k=0}^{n2^n-1} k\,2^{-n}\,1_{f^{-1}([k2^{-n},(k+1)2^{-n}))} + n\,1_{f^{-1}([n,\infty])},$$

so each s_n is a nonnegative simple function. It is clear that $s_n \leq f$. Noting that the dyadic partition gets more refined as n increases, it is easy to check that $s_n(x) \leq s_{n+1}(x)$ for all $x \in \Omega$. Let $x \in \Omega$. If $f(x) = \infty$, then $s_n(x) = n \nearrow f(x)$. If $f(x) < \infty$, then $|s_n(x) - f(x)| \leq 2^{-n}$ for $n > f(x)$, so $s_n(x) \nearrow f(x)$. Thus $s_n(x) \nearrow f(x)$ for all $x \in \Omega$. □

Theorem 2. *Let $f:\Omega \to \mathbb{C}$ be measurable. Then there exist simple functions $s_n:\Omega \to \mathbb{C}$ such that $|s_n| \leq |f|$ and $s_n \to f$ pointwise.*

Proof. First assume f is real valued. By Theorem 1, there exist nonnegative simple functions φ_n and ψ_n such that $\varphi_n \nearrow f^+$ and $\psi_n \nearrow f^-$ pointwise. For each $n \in \mathbb{N}$, $\varphi_n - \psi_n$ is a real valued simple function and $|\varphi_n - \psi_n| \leq \varphi_n + \psi_n \leq f^+ + f^- = |f|$. We have $\varphi_n - \psi_n \to f$ pointwise, so the proof is finished for real valued f. To prove the theorem for complex valued f, apply the real valued case to $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$. □

Theorem 3. *Suppose $f_n:\Omega \to \mathbb{C}$ are measurable functions with $f_n \to f$ pointwise. Suppose there exists $g \in L^p$ such that $|f_n| \leq g$ for all n. Then $f_n \to f$ in L^p.*

Proof. We have

$$|f_n - f|^p \leq (2\max(|f_n|,|f|))^p \leq 2^p(|f_n|^p + |f|^p) \leq 2^p(g^p + g^p) \in L^1.$$

By the dominated convergence theorem,

$$\int |f_n - f|^p\,d\mu \to 0.$$

□

Now we prove problem 1.

Proof of Problem 1. Let $f \in L^p$. By Theorem 2, there exists a sequence of simple functions $(s_n)_{n=1}^{\infty}$ such that $|s_n| \leq |f|$ and $s_n \to f$ pointwise. By Theorem 3, $s_n \to f$ in L^p. ☐

2. Consider the measure space $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), \mu)$, where μ is a Lebesgue-Stieltjes measure (that is, μ assigns finite measure to all bounded intervals in $\mathbb{R}^n$). Fix any $0 < p < \infty$. Use question 1 to show the following:
(i) Continuous functions are dense in L^p.
(ii) L^p is separable, that is, it has a countable dense subset.

Answer. (i) We state without proof a theorem alluded to in the hint.

Theorem 4. *Let X be a second-countable locally compact Hausdorff space, e.g. $X = \mathbb{R}^n$. Then every measure ν on $\mathcal{B}(X)$ that is finite on compact sets is regular, that is, for all $E \in \mathcal{B}(X)$,*

$$\begin{aligned} \nu(E) &= \inf\{\nu(U) : U \text{ is open and } U \supset E\} \\ &= \sup\{\nu(K) : K \text{ is compact and } K \subset E\} \end{aligned}$$

Proof. This theorem is implied by Theorem 7.8 on page 217 of [1]. ☐

Now we prove a special case of Urysohn's lemma.

Theorem 5. *Let X be a metric space. Let $C \subset X$ be closed and $U \subset X$ be open with $C \subset U$. Then there is a continuous function $f : X \to [0,1]$ with $f = 1$ on C and $f = 0$ off U.*

Proof. Set

$$f(x) = \frac{d(x, U^c)}{d(x, C) + d(x, U^c)},$$

where for $A \subset X$, $d(x, A) = \inf\{d(x, y) : y \in A\}$. Using the triangle inequality, it can be shown that $x \mapsto d(x, A)$ is continuous. Note that $d(x, A) = 0$ if and only if $x \in \overline{A}$. Thus if A is closed, then $d(x, A) = 0$ if and only if $x \in A$. Since $C \cap U^c = \emptyset$, this implies that the denominator of $f(x)$ is never 0, so f is well defined. The claimed properties of f follow easily. ☐

Theorem 6. *Let X be a metric space. If $K \subset X$ is compact, $C \subset X$ is closed, and $K \cap C = \emptyset$, then $d(K, C) := \inf\{d(x, y) : x \in K, y \in C\} > 0$.*

Proof. If $C = \emptyset$, then $d(K, C) = \inf(\emptyset) = \infty$. So suppose $C \neq \emptyset$. The function $f : K \to [0, \infty)$ defined by $f(x) = d(x, C)$ is continuous and $d(K, C) = \inf\{f(x) : x \in K\}$. Since K is compact, this inf is actually a minimum, so there exists $x_0 \in K$ such that $d(K, C) = d(x_0, C)$. Since C is closed and $x_0 \notin C$, we have $d(x_0, C) > 0$. ☐

Now we give the proof of (i).

Proof of (i). We will show that $C_c(\mathbb{R}^n)$, the space of continuous functions $f : \mathbb{R}^n \to \mathbb{C}$ with compact support, is dense in L^p. Since $C_c(\mathbb{R}^n)$ is a linear subspace of L^p, it's closure $\overline{C_c(\mathbb{R}^n)}$ in L^p is a linear subspace of L^p. We need to show that $\overline{C_c(\mathbb{R}^n)} = L^p$. By problem 1, simple functions are dense in L^p. Thus it suffices to show that if $\varphi \in L^p$ is a simple function, then $\varphi \in \overline{C_c(\mathbb{R}^n)}$.

So let φ be an arbitrary simple function in L^p. Write $\varphi(\Omega)\setminus\{0\}=\{a_1,a_2,\ldots,a_N\}$ (this set is finite by definition of simple function). We have $\varphi=\sum_{i=1}^N a_i 1_{E_i}$, where $E_i=\varphi^{-1}(\{a_i\})\in\mathcal{B}(\mathbb{R}^n)$. Thus $|\varphi|^p=\sum_{i=1}^N |a_i|^p 1_{E_i}$. Since $\int|\varphi|^p\,d\mu=\sum_{i=1}^N |a_i|^p\mu(E_i)<\infty$, it follows that $\mu(E_i)<\infty$ for each i. Since $\overline{C_c(\mathbb{R}^n)}$ is a linear subspace of L^p, it suffices to show that if $E\in\mathcal{B}(\mathbb{R}^n)$ and $\mu(E)<\infty$, then $1_E\in\overline{C_c(\mathbb{R}^n)}$.

So let $E\in\mathcal{B}(\mathbb{R}^n)$ with $\mu(E)<\infty$ be arbitrary. Let $\varepsilon>0$ be arbitrary. Since any compact $K\subset\mathbb{R}^n$ is contained in a bounded interval, the hypothesis that μ assigns finite measure to all bounded intervals in $\mathbb{R}^n$ implies that $\mu(K)$ is finite for all compact $K\subset\mathbb{R}^n$. By Theorem 4, there exist compact $K\subset E$ and open $U\supset E$ such that $\mu(E)<\mu(K)+\frac{\varepsilon}{2}$ and $\mu(U)-\frac{\varepsilon}{2}<\mu(E)$. Since $\mu(E)<\infty$, it follows that $\mu(U)<\infty$, so

$$\mu(U\setminus K)=\mu(U)-\mu(K)<\varepsilon.$$

By Theorem 6, $d(K,U^c)>0$. Pick $\delta\in(0,d(K,U^c))$. Let $V=\{x\in\mathbb{R}^n: d(x,K)<\delta\}$. By Theorem 5, there is a continuous function $f:\mathbb{R}^n\to[0,1]$ such that $f=1$ on K and $f=0$ off V. Since $f=0$ off V, it follows that $\mathrm{supp}(f)\subset\overline{V}$. Since $\overline{V}\subset A:=\{x\in\mathbb{R}^n: d(x,K)\le\delta\}$ and A is bounded (since K is), it follows that $\overline{V}$ is closed and bounded, hence compact. Thus $f\in C_c(\mathbb{R}^n)$. Since $V\subset U$,

$$\int|f-1_E|^p=\int_{V\setminus K}|f-1_E|^p\le 2^p\mu(V\setminus K)\le 2^p\mu(U\setminus K)\le 2^p\varepsilon.$$

Since ε was arbitrary, this completes the proof. □

(ii) To prove (ii), we claim that it suffices to prove the following result:

Theorem 7. *Let $(X,\mathcal{F},\mu)$ be a measure space and let $p\in(0,\infty)$. Assume that μ is σ-finite. Assume that there is a countable collection $H\subset\mathcal{F}$ such that $\mathcal{F}$ is the σ-algebra generated by H. Then $L^p(X,\mu)$ is separable.*

To show that Theorem 7 implies (ii), we just note that $\mathcal{B}(\mathbb{R}^n)$ is generated by the countable set $H=\{\prod_{i=1}^n[a_i,b_i]: a_i,b_i\in\mathbb{Q}\}$ since every open $U\subset\mathbb{R}^n$ is a countable union of closed dyadic cubes, and closed dyadic cubes are in H. To prove Theorem 7, we need some auxiliary theorems.

Theorem 8. *Let $(X,\mathcal{F},\mu)$ be a measure space. Assume $\mu(X)<\infty$. Suppose $\mathcal{A}$ is an algebra of sets such that $\mathcal{F}$ is the σ-algebra generated by $\mathcal{A}$. Let*

$$\Lambda=\{1_A: A\in\mathcal{A}\},\quad V=\mathrm{span}(\Lambda)=\{a_1 1_{A_1}+\cdots+a_n 1_{A_n}: a_i\in\mathbb{C}, A_i\in\mathcal{A}\}.$$

Then V is dense in $L^p(X,\mu)$, that is, $\overline{V}=L^p(X,\mu)$.

Proof. Since simple functions are dense in $L^p(X,\mu)$ and $\overline{V}$ is a linear subspace of $L^p(X,\mu)$, it suffices to show that if $E\in\mathcal{F}$, then $1_E\in\overline{V}$. We will establish the stronger result that $\{1_E: E\in\mathcal{F}\}\subset\overline{\Lambda}$. To establish this, we make use of the monotone class theorem. Let

$$\mathcal{C}=\{E\in\mathcal{F}: 1_E\in\overline{\Lambda}\}.$$

We need to show that $\mathcal{C}=\mathcal{F}$. We verify that $\mathcal{C}$ is a monotone class. Suppose $E_j\in\mathcal{C}$ and $E_j\nearrow E$. Since $E_j\in\mathcal{F}$, it follows that $E\in\mathcal{F}$. We have $1_{E_j}\nearrow 1_E$ pointwise and $1_{E_j}\leq 1\in L^p$, since $\mu(X)<\infty$. By the dominated convergence theorem (Theorem 3), it follows that $1_{E_j}\to 1_E$ in L^p. Thus $1_E\in\overline{\Lambda}$. Thus $E\in\mathcal{C}$. If $E_j\in\mathcal{C}$ and $E_j\searrow E$, a similar argument shows that $E\in\mathcal{C}$. Thus $\mathcal{C}$ is a monotone class. Since $\mathcal{C}$ contains $\mathcal{A}$, it follows that $\mathcal{C}$ contains the monotone class generated by $\mathcal{A}$. By the monotone class theorem, the monotone class generated by $\mathcal{A}$ is equal to the σ-algebra generated by $\mathcal{A}$, which is $\mathcal{F}$. Thus $\mathcal{C}$ contains $\mathcal{F}$ and the proof is complete. □

Theorem 9. *Let X be a set. If H is a countable collection of subsets of X, then the algebra $\mathcal{A}$ generated by H is countable.*

Proof. Let

$$\mathcal{A}'=\left\{\bigcup_{i=1}^{N}\bigcap_{j=1}^{n_i}A_{i,j}:N\geq 0,n_i\geq 0,A_{i,j}\in H\ \text{ or }\ A_{i,j}^c\in H\right\}.$$

We claim that $\mathcal{A}=\mathcal{A}'$. To see this, note that $\mathcal{A}'$ is an algebra that contains H and that any other algebra that contains H must contain $\mathcal{A}'$. Using the fact that $\mathbb{N}^n$ is countable for any $n\in\mathbb{N}$ and the fact that a countable union of countable sets is countable, it is simple to check that $\mathcal{A}'$ is countable. □

Now we prove Theorem (7)

Proof of Theorem 7. First assume $\mu(X)<\infty$. Let $\mathcal{A}$ be the algebra generated by H. By Theorem 9, $\mathcal{A}$ is countable. Thus the set

$$W=\operatorname{span}\{a_1\,1_{E_1}+\cdots+a_n\,1_{E_n}:a_j\in\mathbb{Q}+i\,\mathbb{Q},E_j\in\mathcal{A}\}$$

is countable. Since $\mathcal{F}$ is the σ-algebra generated by H, it follows that $\mathcal{F}$ is the σ-algebra generated by $\mathcal{A}$. Since vector space operations on a normed space are continuous and $\mathbb{Q}$ is dense in $\mathbb{R}$, $\overline{W}$ contains $V=\{a_1\,1_{E_1}+\cdots+a_n\,1_{E_n}:a_j\in\mathbb{C},E_j\in\mathcal{A}\}$. It follows from Theorem 8 that $\overline{V}=L^p(X,\mu)$, so $\overline{W}=L^p(X,\mu)$. Thus W is a countable dense subset of $L^p(X,\mu)$. This finishes the proof when $\mu(X)<\infty$.

Now suppose μ is an arbitrary σ-finite measure on $(X,\mathcal{F})$. Take measurable $E_n\nearrow X$ with $\mu(E_n)<\infty$. Since H generates $\mathcal{F}$, it follows that for each $n\in\mathbb{N}$, the induced σ-algebra on E_n is generated by $\{A\cap E_n:A\in H\}$, which is countable since H is. Thus by the case of the theorem proved for finite measures, for each $n\in\mathbb{N}$, there is a countable dense set $W_n\subset L^p(E_n,\mu)$. We identify $L^p(E_n,\mu)$ with the linear subspace of $L^p(X,\mu)$ consisting of functions that vanish off E_n. Let $W=\bigcup_{n=1}^{\infty}W_n$. W is countable since it is a countable union of countable sets. We claim that W is dense in $L^p(X,\mu)$. Let $f\in L^p(X,\mu)$ be arbitrary. Let $\varepsilon>0$ be arbitrary. By Theorem 3, $1_{E_n}f\to f$ in L^p. Pick $N\in\mathbb{N}$ such that $\|1_{E_N}f-f\|_{L^p(X,\mu)}<\varepsilon$. Pick $g\in W_N$ such that $\|g-1_{E_N}f\|_{L^p(E_N,\mu)}<\varepsilon$. We have

$$\begin{aligned}\|g-f\|_{L^p(X)} &\leq \|g-1_{E_N}f\|_{L^p(X)}+\|1_{E_N}f-f\|_{L^p(X)}\\ &= \|g-1_{E_N}f\|_{L^p(E_N)}+\|1_{E_N}f-f\|_{L^p(X)}\\ &\leq 2\,\varepsilon.\end{aligned}$$

This completes the proof. □

3. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $g \in L^1(\mu)$ such that $g \geq 0$. Show that

$$\nu_g(A) \doteq \int_A g\, d\mu, \quad A \in \mathcal{F}$$

defines a finite measure on $(\Omega, \mathcal{F})$. Further show that $h \in L^1(\nu_g)$ if and only if $h \cdot g \in L^1(\mu)$ and in that case

$$\int h\, d\nu_g = \int h\, g\, d\mu. \tag{1}$$

Proof. We have

$$\nu_g(\emptyset) = \int_\emptyset g\, d\mu = \int g\, 1_\emptyset\, d\mu = \int 0\, d\mu = 0.$$

Suppose $A_1, A_2, \ldots \in \mathcal{F}$ are disjoint. Note that $1_{\bigcup_{n=1}^\infty A_n} = \sum_{n=1}^\infty 1_{A_n}$. Thus

$$\begin{aligned}
\nu_g\left(\bigcup_{n=1}^\infty A_n\right) &= \int_{\bigcup_{n=1}^\infty A_n} g\, d\mu \\
&= \int g\, 1_{\bigcup_{n=1}^\infty A_n}\, d\mu \\
&= \int g \sum_{n=1}^\infty 1_{A_n}\, d\mu \\
&= \int \sum_{n=1}^\infty g\, 1_{A_n}\, d\mu && (2) \\
&= \sum_{n=1}^\infty \int g\, 1_{A_n}\, d\mu && (3) \\
&= \sum_{n=1}^\infty \int_{A_n} g\, d\mu \\
&= \sum_{n=1}^\infty \nu_g(A_n).
\end{aligned}$$

The equality of (2) and (3) follows from the monotone convergence theorem. Thus ν_g is a measure. We have $\nu_g(\Omega) = \int g\, d\mu < \infty$ since $g \in L^1(\mu)$.

Now we prove that (1) holds for all measurable $h: \Omega \to [0, \infty]$. By definition of ν_g, (1) holds whenever $h = 1_A$ for some $A \in \mathcal{F}$. Recall that for any measure θ on $(\Omega, \mathcal{F})$,

$$a \in [0, \infty], h: \Omega \to [0, \infty] \text{ measurable} \quad \Longrightarrow \quad \int a\, h\, d\theta = a \int h\, d\theta,$$

$$h_1, h_2: \Omega \to [0, \infty] \text{ measurable} \quad \Longrightarrow \quad \int (h_1 + h_2)\, d\theta = \int h_1\, d\theta + \int h_2\, d\theta.$$

From these facts, we deduce that (1) holds for nonnegative simple functions h. Now let $h: \Omega \to [0, \infty]$ be an arbitrary measurable function. By Theorem (1), there exist nonnegative simple functions $h_n \nearrow h$. For each $n \in \mathbb{N}$,

$$\int h_n\, d\nu_g = \int h_n\, g\, d\mu.$$

Letting $n \to \infty$ and using the monotone convergence theorem yields

$$\int h\, d\nu_g = \int h\, g\, d\mu.$$

Thus 1 holds for all measurable $h : \Omega \to [0, \infty]$.

If $h : \Omega \to \mathbb{C}$ is measurable, then applying this result to $|h|$ shows that

$$h \in L^1(\nu_g) \quad \Longleftrightarrow \quad h\, g \in L^1(\mu).$$

If $h \in L^1(\nu_g)$, then $\mathrm{Re}(h)^+, \mathrm{Re}(h)^-, \mathrm{Im}(h)^+, \mathrm{Im}(h)^-$ are all nonnegative and in $L^1(\nu_g)$. Since (1) holds for these four functions, linearity of the integral on $L^1(\nu_g)$ and $L^1(\mu)$ implies that (1) holds for h. □

4. Show that the Lebesgue Decomposition Theorem for finite measures implies the theorem for σ-finite measures.

Proof. Let μ and ν be arbitrary σ-finite measures on a measure space $(X, \mathcal{F})$. Write $X = \bigcup_{i=1}^{\infty} A_i$ with $\{A_i\}$ disjoint and $\mu(A_i) < \infty$. Similarly, write $X = \bigcup_{j=1}^{\infty} B_j$ with $\{B_j\}$ disjoint and $\nu(B_j) < \infty$. We have $X = \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} A_i \cap B_j$. Thus X is a countable union of disjoint sets $E_n, n \in \mathbb{N}$ with $\mu(E_n) < \infty$ and $\nu(E_n) < \infty$.

For each $n \in \mathbb{N}$, define finite measures μ_n, ν_n on $(X, \mathcal{F})$ by

$$\mu_n(A) = \mu(A \cap E_n),$$
$$\nu_n(A) = \nu(A \cap E_n).$$

By the Lebesgue decomposition theorem for finite measures, for each $n \in \mathbb{N}$ there exist measures $\nu_{n,a}$, $\nu_{n,s}$ on $(X, \mathcal{F})$ such that

$$\nu_n = \nu_{n,a} + \nu_{n,s}, \quad \nu_{n,a} \ll \mu_n, \quad \nu_{n,s} \perp \mu_n$$

We have

$$\nu = \sum_{n=1}^{\infty} \nu_n = \sum_{n=1}^{\infty} \nu_{n,a} + \sum_{n=1}^{\infty} \nu_{n,s} = \nu_a + \nu_s,$$

where

$$\nu_a = \sum_{n=1}^{\infty} \nu_{n,a},$$

$$\nu_s = \sum_{n=1}^{\infty} \nu_{n,s}.$$

Since $\mu = \sum_{n=1}^{\infty} \mu_n$,

$$\mu(A) = 0 \quad \Rightarrow \quad \mu_n(A) = 0 \text{ for all } n \quad \Rightarrow \quad \nu_{n,a}(A) = 0 \text{ for all } n \quad \Rightarrow \quad \nu_a(A) = 0.$$

Thus $\nu_a \ll \mu$. Let $n \in \mathbb{N}$. We have $\nu_{n,s}(E_n^c) \le \nu_n(E_n^c) = 0$. Thus $\nu_{n,s}$ is supported on E_n. Since μ_j is supported on E_j and $E_j \cap E_n = \emptyset$ for $j \neq n$, it follows that $\nu_{n,s} \perp \mu_j$ for all $j \neq n$, and therefore for all $j \in \mathbb{N}$. Thus $\nu_{n,s} \perp \sum_{j=1}^{\infty} \mu_j = \mu$. Since this holds for every n, $\nu_s = \sum_{n=1}^{\infty} \nu_{n,s} \perp \mu$. This proves existence of the Lebesgue decomposition.

Now we show ν_a has a density with respect to μ. By the Lebesgue decomposition theorem for finite measures, there exists $f: X \to [0, \infty)$ such that $d\,\nu_{n,a} = f_n\, d\mu_n$. Note that $d\mu_n = 1_{E_n}\, d\mu$ since both sides are measures that agree on every set in $\mathcal{F}$. Thus for $A \in \mathcal{F}$,

$$\begin{aligned} \nu_a(A) &= \sum_{n=1}^{\infty} \nu_{n,a}(A) \\ &= \sum_{n=1}^{\infty} \int_A f_n\, d\mu_n \\ &= \sum_{n=1}^{\infty} \int_A f_n\, 1_{E_n}\, d\mu \\ &= \int_A \sum_{n=1}^{\infty} f_n\, 1_{E_n}\, d\mu. \end{aligned}$$

Thus $f = \sum_{n=1}^{\infty} f_n\, 1_{E_n}$ is the density of ν_a with respect to μ.

Now we show uniqueness. Suppose there is another decomposition $\nu = \tilde{\nu_a} + \tilde{\nu_s}$ with $\tilde{\nu_a} \ll \mu$ and $\tilde{\nu_s} \perp \mu$. Let $E \in \mathcal{F}$ be an arbitrary set with $\mu(E) < \infty$ and $\nu(E) < \infty$. Then for any $n \in \mathbb{N}$,

$$1_E\, d\nu = 1_E\, d\,\tilde{\nu_a} + 1_E\, d\,\tilde{\nu_s} = 1_E\, d\nu_a + 1_E\, d\nu_s.$$

Since $1_E\, d\nu_a \leq \nu_a \ll \mu$ and $1_E\, d\nu_s \leq \nu_s \perp \mu$, it follows that $1_E\, d\nu_a \ll \mu$ and $1_E\, d\nu_s \perp \mu$. Similarly, $1_E\, d\,\tilde{\nu_a} \ll \mu$ and $1_E\, d\,\tilde{\nu_s} \perp \mu$. By the uniqueness part of the Lebesgue decomposition theorem for finite measures, $1_E\, d\nu_a = 1_E\, d\,\tilde{\nu_a}$ and $1_E\, d\nu_s = 1_E\, d\,\tilde{\nu_s}$. Since X is a countable disjoint union of such sets E, it follows that $\nu_a = \tilde{\nu_a}$ and $\nu_s = \tilde{\nu_s}$. $\square$

Bibliography

[1] G. B. Folland. *Real analysis: modern techniques and their applications*. Pure and applied mathematics. Wiley, New York, 2nd ed edition, 1999.