L^p spaces, II

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Abstract

These are some notes on L^p spaces, filling in the gaps in chapter 9 of [1].

1 Filling in Gaps

The Fourier transform of a function $f \in L^1(\mathbb{R}^n)$ is provisionally defined by

$$\mathcal{F}f(\xi) = \hat{f}(\xi) \int_{\mathbb{R}^n} f(x)\, e^{-ix\cdot\xi}\, dx.$$

Later $\mathcal{F}$ will be multiplied by a constant to make it unitary from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$. By the dominated convergence theorem, $\mathcal{F}: L^1(\mathbb{R}^n) \to C_b(\mathbb{R}^n)$.

Theorem 1. *Let $f \in \mathcal{S}(\mathbb{R}^n)$. Then*

$$\partial_\xi^\alpha \mathcal{F}f = \mathcal{F}((-i\,x)^\alpha f) = (-i)^{|\alpha|}\, \mathcal{F}(x^\alpha f).$$

Proof. First assume $\alpha = j$ for some $j \in \{1, \ldots, n\}$. We have

$$\begin{aligned}
\partial_{\xi_j} \mathcal{F}f(\xi) &= \partial_{\xi_j}\left(\int_{\mathbb{R}^n} f(x)\, e^{-ix\cdot\xi}\, dx\right) \\
&= \int_{\mathbb{R}^n} \partial_{\xi_j}(f(x)\, e^{-ix\cdot\xi})\, dx \\
&= \int_{\mathbb{R}^n} f(x)\, e^{-ix\cdot\xi}\,(-i\,x_j)\, dx \\
&= \mathcal{F}(-i\,x_j\, f)(\xi).
\end{aligned}$$

The swapping of the derivative and integral in the second equality is justified by the mean value theorem and the dominated convergence theorem since

$$|\partial_{\xi_j}(f(x)\, e^{-ix\cdot\xi})| = |f(x)\, e^{-ix\cdot\xi}(-i\,x_j)| = |x_j\, f(x)|$$

and $x_j\, f \in \mathcal{S}(\mathbb{R}^n)$. By induction on $|\alpha|$, we get the result for every α. □

Theorem 2. *Let $f \in \mathcal{S}(\mathbb{R}^n)$. Then*

$$\mathcal{F}\partial^\alpha f = (i\,\xi)^\alpha\, \mathcal{F}f = i^{|\alpha|}\, \xi^\alpha\, \mathcal{F}f.$$

Proof. First suppose $\alpha = j$ for some $j \in \{1, \ldots, n\}$. By Fubini's theorem,

$$\mathcal{F}\partial_{x_j} f(\xi) = \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \partial_{x_j} f(x)\, e^{-ix\cdot\xi}\, dx_j\, dx'.$$

To compute $\int_{\mathbb{R}} \partial_{x_j} f(x)\, e^{-i x\cdot\xi}\, d x_j$, since $|\partial_{x_j} f(x)\, e^{-i x\cdot\xi}| = |\partial_{x_j} f(x)| \leq \frac{C}{1+|x_j|^N} \in L^1(\mathbb{R})$ we can use integration by parts with $u = e^{-i x\cdot\xi}$, $d v = \partial_{x_j} f(x)$ to get

$$\begin{aligned}\int_{\mathbb{R}} \partial_{x_j} f(x)\, e^{-i x\cdot\xi}\, d x_j &= e^{-i x\cdot\xi} f(x)|_{x_j=-\infty}^{\infty} - \int_{\mathbb{R}} f(x)\, e^{-i x\cdot\xi} - i\,\xi_j\, d x_j \\ &= i\,\xi_j \int_{\mathbb{R}} f(x)\, e^{-i x\cdot\xi}\, d x_j.\end{aligned}$$

Thus

$$\begin{aligned}\mathcal{F}\partial_{x_j} f(\xi) &= \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \partial_{x_j} f(x)\, e^{-i x\cdot\xi}\, d x_j\, d x' \\ &= i\,\xi_j \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} f(x)\, e^{-i x\cdot\xi}\, d x_j\, d x' \\ &= i\,\xi_j\, \mathcal{F} f(\xi).\end{aligned}$$

The result follows by induction on $|\alpha|$. □

Theorem 3. $\mathcal{F}: \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$.

Proof. Let $f \in \mathcal{S}(\mathbb{R}^n)$. Theorem 1 implies that $\mathcal{F} f \in C^\infty(\mathbb{R}^n)$. By theorems 1 and 2, if $f \in \mathcal{S}(\mathbb{R}^n)$, then

$$\begin{aligned}\xi^\beta\, \partial_\xi^\alpha\, \mathcal{F} f &= \xi^\beta\, (-i)^{|\alpha|}\, \mathcal{F}(x^\alpha f) \\ &= (-i)^{|\beta|} (-i)^{|\alpha|}\, \mathcal{F}\partial^\beta (x^\alpha f),\end{aligned}$$

where in the second inequality we use the fact that $x^\alpha f \in \mathcal{S}(\mathbb{R}^n)$. Since $x^\alpha f \in \mathcal{S}(\mathbb{R}^n)$, it follows that $\partial^\beta (x^\alpha f) \in \mathcal{S}(\mathbb{R}^n)$. In particular, $\partial^\beta (x^\alpha f) \in L^1(\mathbb{R}^n)$. Thus $\mathcal{F}\partial^\beta (x^\alpha f) \in C_b(\mathbb{R}^n)$. Thus $\xi^\beta\, \partial_\xi^\alpha\, \mathcal{F} f$ is bounded. □

Theorem 4. *For* $f \in \mathcal{S}(\mathbb{R}^n)$, $\mathcal{F}^* \mathcal{F} f = (2\,\pi)^n f$.

Proof. By the dominated convergence theorem (note $\mathcal{F} f \in \mathcal{S}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$) and Funini's theorem,

$$\begin{aligned}\mathcal{F}^* \mathcal{F} f(x) &= \int \mathcal{F} f(\xi)\, e^{i x\cdot\xi}\, d\xi \\ &= \lim_{\varepsilon \searrow 0} \int \mathcal{F} f(\xi)\, e^{-\varepsilon|\xi|^2}\, e^{i x\cdot\xi}\, d\xi \\ &= \lim_{\varepsilon \searrow 0} \iint f(y)\, e^{-i y\cdot\xi}\, d y\, e^{-\varepsilon|\xi|^2}\, e^{i x\cdot\xi}\, d\xi \\ &= \lim_{\varepsilon \searrow 0} \iint f(y)\, e^{-\varepsilon|\xi|^2 + i(x-y)\cdot\xi}\, d y\, d\xi \\ &= \lim_{\varepsilon \searrow 0} \int f(y) \int e^{-\varepsilon|\xi|^2 + i(x-y)\cdot\xi}\, d\xi\, d y \\ &= \lim_{\varepsilon \searrow 0} \int f(y)\, p(\varepsilon, x-y)\, d\xi\, d y, \\ &= \lim_{\varepsilon \searrow 0} (f * p(\varepsilon, \cdot))(x),\end{aligned}$$

where

$$p(\varepsilon, x) = \int e^{-\varepsilon|\xi|^2 + i x\cdot\xi}\, d\xi.$$

Making the substitution $v = \sqrt{\varepsilon}\,\xi$ yields

$$\begin{aligned} p(\varepsilon, x) &= \varepsilon^{-n/2} \int e^{-|v|^2 + i\varepsilon^{-1/2} x\cdot v}\, dv \\ &= \varepsilon^{-n/2}\, q(x/\sqrt{\varepsilon}), \end{aligned}$$

where $q(x) = p(1, x)$. We proceed to compute $q(x)$. By Fubini's theorem,

$$\begin{aligned} q(x) &= \int_{\mathbb{R}^n} e^{-|\xi|^2 + i x\cdot\xi}\, d\xi \qquad (1) \\ &= \left(\int_{\mathbb{R}} e^{-\xi_1^2 + i x_1 \xi_1}\, d\xi_1\right) \cdots \left(\int_{\mathbb{R}} e^{-\xi_n^2 + i x_n \xi_n}\, d\xi_n\right) \\ &= \hat{g}(-x_1) \cdots \hat{g}(-x_n), \end{aligned}$$

where $g(x) = e^{-x^2}$. Note that $g \in \mathcal{S}(\mathbb{R})$. We proceed to compute $\hat{g}$. Note that $g'(x) = -2\,x\,e^{-x^2} = -2\,x\,g(x)$. Thus

$$g' + 2\,x\,g = 0.$$

Taking the Fourier transform of both sides using theorems (1) and (2) yields $i\,\xi\,\hat{g} + 2\,i\,\hat{g}' = 0$. Thus

$$2\,\hat{g}' + \xi\,\hat{g} = 0.$$

Thus $2\frac{d\hat{g}}{d\xi} + \xi\,\hat{g} = 0$. Thus $2\,d\hat{g} + \xi\,d\xi\,\hat{g} = 0$. Thus $\frac{1}{\hat{g}}\,d\,\hat{g} = -\frac{\xi}{2}\,d\xi$. Thus $\ln(\hat{g}) = \frac{-\xi^2}{4} + C$. Thus

$$\hat{g} = C\,e^{-\xi^2/4}.$$

We have

$$\hat{g}(0) = \int_{\mathbb{R}} e^{-x^2}\, dx = \pi^{1/2}.$$

Thus

$$\hat{g}(\xi) = \pi^{1/2}\, e^{-\xi^2/4}.$$

Thus by (4),

$$q(x) = \hat{g}(-x_1) \cdots \hat{g}(-x_n) = \pi^{n/2}\, e^{-|x|^2/4}.$$

We have

$$\begin{aligned} \int q(x)\, dx &= \pi^{n/2} \int e^{-|x|^2/4}\, dx \\ &= \pi^{n/2}\, 2^n \int e^{-|y|^2}\, dy \\ &= 2^n\, \pi^{n/2}\, \pi^{n/2} \\ &= (2\,\pi)^n. \end{aligned}$$

Thus since f is bounded and continuous, $f * p(\varepsilon, \cdot) \to (2\pi)^n f$ pointwise as $\varepsilon \searrow 0$, so

$$\begin{aligned}\mathcal{F}^* \mathcal{F} f(x) &= \lim_{\varepsilon \searrow 0} (f * p(\varepsilon, \cdot))(x) \\ &= (2\pi)^n f(x).\end{aligned}$$

This completes the proof. □

Thus if we redefine

$$\mathcal{F} f(\xi) = (2\pi)^{-n/2} \int f(x)\, e^{-i x\cdot\xi}\, dx,$$

then $\mathcal{F}^* \mathcal{F} = I$. Note that $\mathcal{F}^* = R\mathcal{F} = \mathcal{F}R$, where $Rf(x) = f(-x)$. Thus $\mathcal{F}\mathcal{F}^* = I$, so $\mathcal{F}: \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is unitary.

2 Exercises

Exercises from chapter 9.

1. Let (X, μ) and (Y, ν) be σ-finite measure spaces, and let $k(x, y)$ be measurable on $(X \times Y, \mu \times \nu)$. Let $p \in (1, \infty)$, and let q be the conjugate exponent to p. Assume that there are measurable functions $A(x)$, $B(y)$, positive a.e. on X and Y, respectively, such that

$$\int_X |k(x, y)|\, A(x)^p\, d\mu(x) \leq C_1\, B(y)^p,$$

$$\int_Y |k(x, y)|\, B(y)^q\, d\nu(y) \leq C_2\, A(x)^q.$$

Then $Ku(x) = \int_Y k(x, y)\, u(y)\, d\nu(y)$ defines a bounded operator

$$K: L^p(Y, \nu) \to L^p(X, \nu), \quad \|K\| \leq C_1^{1/p}\, C_2^{1/q}.$$

Proof. Let $f \in L^p(Y, \nu)$ be arbitrary. We use the dual characterization of the L^p norm to show that the function $x \mapsto \int_Y |k(x, y)|\, |f(y)|\, dy$ has finite L^p norm. This will show that the integral $K f(x)$ converges absolutley for a.e. x, so $K f$ is well defined. In the process, we will also establish the bound $\|K\| \leq C_1^{1/p}\, C_2^{1/q}$. Let $g \in L^q(X, \mu)$ be arbitrary. We have

$$\int_X \left| \int_Y |k(x, y)|\, |f(y)|\, d\nu(y)\, g(x) \right| d\mu(x) \;\leq\; \int_X \int_Y |k(x, y)|\, |f(y)|\, |g(x)|\, d\nu(y)\, d\mu(x).$$

Using the inequality $a\, b \leq \frac{a^p}{p} + \frac{b^q}{q}$ for $a, b \geq 0$, we estimate

$$\begin{aligned}|f(y)|\, |g(x)| &= \frac{A(x)}{B(y)} |f(y)| \frac{B(y)}{A(x)} |g(x)| \\ &\leq \frac{1}{p} \frac{A(x)^p}{B(y)^p} |f(y)|^p + \frac{1}{q} \frac{B(y)^q}{A(x)^q} |g(x)|^q.\end{aligned}$$

By Fubini's theorem,

$$\int_X \int_Y |k(x,y)|\,|f(y)|\,|g(x)|\,d\nu(y)\,d\mu(x) \;\leq\; \frac{1}{p} C_1 \,\|f\|_{L^p}^p + \frac{1}{q} C_2 \,\|g\|_{L^q}^q.$$

Replacing f with $t\,f$ and g with $t^{-1}\,g$ yields

$$\int_X \int_Y |k(x,y)|\,|f(y)|\,|g(x)|\,d\nu(y)\,d\mu(x) \leq \frac{1}{p} C_1 \,\|f\|_{L^p}^p\, t^p + \frac{1}{q} C_2 \,\|g\|_{L^q}^q\, t^{-q}.$$

Minimizing over t yields

$$\int_X \int_Y |k(x,y)|\,|f(y)|\,|g(x)|\,d\nu(y)\,d\mu(x) \leq C_1^{1/p}\, C_2^{1/q}\, \|f\|_{L^p}\,\|g\|_{L^q}.$$

Thus $\|K\,f\|_{L^p} \leq C_1^{1/p}\, C_2^{1/q}\,\|f\|_{L^p}$. Thus $\|K\| \leq C_1^{1/p}\, C_2^{1/q}$. □

3. Given $u \in L^1_{\text{loc}}(\mathbb{R}^n)$, $p \in [1,\infty]$, and $\varphi \in C_c^\infty(\mathbb{R}^n)$, show that $\varphi * u \in C^\infty(\mathbb{R}^n)$.

Proof. We first show that $\varphi * u$ is continuous. Let $x \in \mathbb{R}^n$ be arbitrary, and let (x_k) be an arbitrary sequence converging to x. We need to show that

$$\lim_{k\to\infty} \int \varphi(x_k - y)\, u(y)\, dy = \int \varphi(x - y)\, u(y)\, dy.$$

Since x_k is convergent, there is a ball $B \subset \mathbb{R}^n$ such that $x_k \in B$ for all $k \in \mathbb{N}$. Let $K = \operatorname{supp}(\varphi)$. Note that $x_n - y \in K$ if and only if $y \in x_n - K \subset B - K$ and that $B - K$ is bounded. Thus

$$|\varphi(x_k - y)\, u(y)| = |\varphi(x_k - y)\, u(y)\, \chi_{B-K}| \leq \left(\sup_{z\in\mathbb{R}^n} |\varphi(z)|\right) |u(y)|\, \chi_{B-K} \in L^1(\mathbb{R}^n).$$

By the dominated convergence theorem,

$$\lim_{k\to\infty} \int \varphi(x_k - y)\, u(y)\, dy = \int \varphi(x - y)\, u(y)\, dy.$$

Thus $\varphi * u$ is continuous. Now for any $j \in \{1,\ldots,n\}$,

$$\begin{aligned}
\partial_{x_j}(\varphi * u) &= \lim_{h\to 0} \int \frac{\varphi(x + h\,e_j - y) - \varphi(x - y)}{h}\, u(y)\, dy \\
&= \lim_{h\to 0} \int \partial_{x_j}\varphi(x + \theta(h)\,h\,e_j - y)\, u(y)\, dy \\
&= \lim_{h\to 0} (\partial_{x_j}\varphi * u)(x + \theta(h)\,h\,e_j) \\
&= (\partial_{x_j}\varphi * u)(x),
\end{aligned}$$

where the second equality is by the mean value theorem, and in the last equality we use the fact that $\partial_{x_j}\varphi \in C_c^\infty$ and the continuity result previously established. By induction, $\partial^\alpha(\varphi * u) = \partial^\alpha \varphi * u$ for every multi-index α. □

4. If $u \in L^p(\mathbb{R}^n)$ and $w \in L^q(\mathbb{R}^n)$, $p \in [1,\infty]$, $\frac{1}{p} + \frac{1}{q} = 1$, $u * w$ is uniformly continuous on $\mathbb{R}^n$.

Proof. By Holder's inequality, $\int |u(x-y)\, w(y)|\, dy \le \|u\|_{L^p}\, \|w\|_{L^q}$ for all $x \in \mathbb{R}^n$. Thus $(u*w)(x)$ is well defined for every $x \in \mathbb{R}^n$. Let $x_1, x_2 \in \mathbb{R}^n$ be arbitrary. We have

$$\begin{aligned} |(u*w)(x_1) - (u*w)(x_2)| &\le \int |u(x_1-y) - u(x_2-y)|\, |w(y)|\, dy \\ &\le \|u(x_1-\cdot) - u(x_2-\cdot)\|_{L^p}\, \|w\|_{L^q}. \end{aligned}$$

By translation and reflection invariance of the integral,

$$\begin{aligned} \|u(x_1-\cdot) - u(x_2-\cdot)\|_{L^p} &= \|u(x_1+\cdot) - u(x_2+\cdot)\|_{L^p} \\ &= \|u(x_1-x_2+\cdot) - u(\cdot)\|_{L^p} \\ &= \|\tau_{x_2-x_1} u - u\|_{L^p} \end{aligned}$$

Since $\|\tau_z u - u\|_{L^p} \to 0$ as $z \to 0$, $\omega(\delta) = \sup_{\|z\| \le \delta} \|\tau_z u - u\|_{L^p}$ is a modulus of continuity for $u*w$, so $u*w$ is uniformly continuous. □

In excercises 5-6, let $(X, \mathfrak{F}, \mu)$ be a measure space and $\mathcal{A}$ and algebra of subsets of X such that $\sigma(\mathcal{A}) = \mathfrak{F}$. Let

$$\Lambda = \mathrm{span}\{\chi_A : A \in \mathcal{A}\}.$$

5. If $f \in L^1(X, \mu)$ and $\int_A f\, d\mu = 0$ for all $A \in \mathcal{A}$, then $f = 0$.

Proof. Let $\mathcal{C} = \{B \in \mathfrak{F} : \int_B f\, d\mu = 0\}$. Clearly $\mathcal{A} \subset \mathcal{C}$. If $B_n \in \mathcal{C}$ and $B_n \nearrow B$, then $B \in \mathfrak{F}$ and by the dominated convergence theorem, $\int_B f\, d\mu = \lim_{n\to\infty} \int_{B_n} f\, d\mu = 0$, so $B \in \mathcal{C}$. Similarly, if $B_n \in \mathcal{C}$ and $B_n \searrow B$, then $B \in \mathcal{C}$. Thus $\mathcal{C}$ is a monotone class. Since $\mathcal{C} \supset \mathcal{A}$, the monotone class lemma implies $\mathcal{C} \supset \sigma(\mathcal{A}) = \mathfrak{F}$. Thus $\int_B f\, d\mu = 0$ for all $B \in \mathfrak{F}$. For all $n \in \mathbb{N}$, $0 = \int_{\left\{f \ge \frac{1}{n}\right\}} f\, d\mu \ge \frac{1}{n}\, \mu\big(f \ge \frac{1}{n}\big)$. Thus $\mu\big(f \ge \frac{1}{n}\big) = 0$ for all $n \in \mathbb{N}$. Thus $\mu(f > 0) = 0$. Applying the same argument to $-f$ yields $\mu(-f > 0) = 0$, so $\mu(f < 0) = 0$. Thus $f = 0$ a.e. □

6. If $\mu(X) < \infty$, Λ is dense in $L^p(X, \mu)$ for all $p \in [1, \infty)$.

Proof. Let $\mathcal{C} = \{B \in \mathfrak{F} : \chi_B \in \overline{\Lambda}\}$. Clearly $\mathcal{A} \subset \mathcal{C}$. Suppose $B_n \in \mathcal{C}$ and $B_n \nearrow B$. Obviously $B \in \mathfrak{F}$. Note that $\chi_{B_n} \to \chi_B$ pointwise and $|\chi_{B_n}| \le 1 \in L^p(X, \mu)$. By the dominated convergence theorem, $\chi_{B_n} \to \chi_B$ in $L^p(X, \mu)$. Thus $\chi_B \in \overline{\Lambda}$. Similarly, if $B_n \in \mathcal{C}$ and $B_n \searrow B$, then $B \in \mathcal{C}$. Thus $\mathcal{C}$ is a monotone class, so $\mathcal{C} = \mathfrak{F}$. It follows that any simple function is in $\overline{\Lambda}$. By density of simple functions in $L^p(X, \mu)$, it follows that $\overline{\Lambda} = L^p(X, \mu)$. □

Bibliography

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