STOR 635 HW 5

BY AMEER QAQISH

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1 Solutions

1. Show that if X_n, Y_n are submartingales w.r.t $\mathcal{F}_n$ then $X_n \vee Y_n$ is also.

Proof. $X_n \vee Y_n$ is $\mathcal{F}_n$-measurable since both X_n and Y_n are. We have $|X_n \vee Y_n| \leq |X_n| + |Y_n| \in L^1$, so $X_n \vee Y_n \in L^1$. By monotonicity, $E(X_{n+1} \vee Y_{n+1} \,|\, \mathcal{F}_n) \geq E(X_{n+1} \,|\, \mathcal{F}_n)$ and $E(X_{n+1} \vee Y_{n+1} \,|\, \mathcal{F}_n) \geq E(Y_{n+1} \,|\, \mathcal{F}_n)$. Thus

$$\begin{aligned} E(X_{n+1} \vee Y_{n+1} \,|\, \mathcal{F}_n) &\geq E(X_{n+1} \,|\, \mathcal{F}_n) \vee E(Y_{n+1} \,|\, \mathcal{F}_n) \\ &\geq X_n \vee Y_n. \end{aligned}$$

Thus $X_n \vee Y_n$ is a submartingale. □

2. Let $\{\xi_{n,k} : n \geq 1, k \geq 1\}$ be an array of iid nonnegative integer valued random variables with $E(\xi_{n,k}) = \mu$. Define

$$Z_0 = 1 \text{ and } Z_n = \sum_{k=1}^{Z_{n-1}} \xi_{n,k} \quad n \geq 1.$$

Let $\mathcal{F}_n = \sigma(Z_1, \ldots, Z_n)$. Show that $\{Z_n\}$ is an $\mathcal{F}_n$-martingale, submartingale and supermartingale according as $\mu = 1$, $\mu \geq 1$, and $\mu \leq 1$.

Proof. Note that Z_1 is a function of the first row $\{\xi_{1,k} : k \in \mathbb{N}\}$ of the array. Z_2 is a function of Z_1 and the second row $\{\xi_{2,k} : k \in \mathbb{N}\}$ of the array. Hence Z_2 is a function of the first two rows $\{\xi_{n,k} : n \in \{1,2\}, k \in \mathbb{N}\}$ of the array. By an inductive argument, for every $m \in \mathbb{N}$, Z_m is a function of the first m rows $\{\xi_{n,k} : n \in \{1, \ldots, m\}, k \in \mathbb{N}\}$ of the array. In particular, $\xi_{n,k}$ is independent of $\mathcal{F}_{n-1}$ for all $n, k \in \mathbb{N}$. Thus for every $n \in \mathbb{N}$,

$$\begin{aligned} E(Z_n) &= E\left(\sum_{k=1}^{Z_{n-1}} \xi_{n,k}\right) \\ &= E\left(\sum_{k=1}^{\infty} 1_{\{k \leq Z_{n-1}\}} \xi_{n,k}\right) \\ &= \sum_{k=1}^{\infty} E(1_{\{k \leq Z_{n-1}\}} \xi_{n,k}) \\ &= \sum_{k=1}^{\infty} E(1_{\{k \leq Z_{n-1}\}}) E(\xi_{n,k}) \\ &= \mu \sum_{k=1}^{\infty} E(1_{\{k \leq Z_{n-1}\}}) \end{aligned}$$

$$= \mu\, E\left(\sum_{k=1}^{\infty} 1_{\{k\leq Z_{n-1}\}}\right)$$
$$= \mu\, E(Z_{n-1}).$$

Since $E(Z_0)=1$, this means $E(Z_n)=\mu^n$ for all $n\in\mathbb{N}$. Assuming $\mu<\infty$, this means Z_n is integrable for all n. For every $n\in\mathbb{N}_0$,

$$\begin{aligned}
E(Z_{n+1}\,|\,\mathcal{F}_n) &= E\left(\sum_{k=1}^{Z_n} \xi_{n+1,k}\,|\,\mathcal{F}_n\right)\\
&= E\left(\sum_{k=1}^{\infty} 1_{\{k\leq Z_n\}}\,\xi_{n+1,k}\,|\,\mathcal{F}_n\right)\\
&= \sum_{k=1}^{\infty} E(1_{\{k\leq Z_n\}}\,\xi_{n+1,k}\,|\,\mathcal{F}_n)\\
&= \sum_{k=1}^{\infty} 1_{\{k\leq Z_n\}}\,E(\xi_{n+1,k}\,|\,\mathcal{F}_n)\\
&= \sum_{k=1}^{Z_n} E(\xi_{n+1,k})\\
&= \mu\, Z_n.
\end{aligned}$$

Thus Z is a martingale when $\mu=1$, a submartingale when $\mu\geq 1$, and a supermartingale when $\mu\leq 1$. □

3. Suppose that $\{X_n\}$ is a nonnegative $\mathcal{F}_n$-supermartingale and τ is a stopping time. Show that $E(X_\tau)\leq E(X_0)$.

Proof. We first consider the case when τ is bounded. So we assume that there exists $T\in\mathbb{N}$ such that $\tau\leq T$. We have $|X_\tau|\leq\sum_{n=0}^{T}|X_n|\in L^1$, so $X_\tau\in L^1$. We have

$$X_\tau - X_0 = \sum_{m=1}^{T} 1_{\{m\leq\tau\}}\,(X_m - X_{m-1}) = (H\cdot X)_T, \quad H_m = 1_{\{m\leq\tau\}}.$$

Since $\{m\leq\tau\}=\{\tau<m\}^c=\{\tau\leq m-1\}^c\in\mathcal{F}_{m-1}$, H_m is a predictable sequence. H_m is also nonnegative and bounded. Thus $H\cdot X$ is a supermartingale. Thus

$$E(X_\tau - X_0) = E((H\cdot X)_T)\leq E((H\cdot X)_0)=0.$$

Thus $E(X_\tau)\leq E(X_0)$.

Now we consider the case of general τ. In this case, τ may take the value ∞, so we must define X_∞ in order for X_τ to make sense. Since $E(X_n^-)=0$ for all $n\geq 0$, the martingale convergence theorem yields that there exists an $\mathcal{F}_\infty=\sigma(\bigcup_{n=0}^{\infty}\mathcal{F}_n)$-measurable random variable X_∞ with $X_n\to X_\infty$ almost surely as $n\to\infty$ and $E(|X_\infty|)<\infty$.

Note that $X_{\tau\wedge n}\to X_\tau$ almost surely as $n\to\infty$. Since $\tau\wedge n$ is bounded, we have by the previous case that $E(X_{\tau\wedge n})\leq E(X_0)$. Since $X_{\tau\wedge n}\geq 0$ by assumption, Fatou's lemma yields

$$E(X_\tau)\leq\liminf_{n\to\infty} E(X_{\tau\wedge n})\leq E(X_0).$$ □

4. Suppose that S and T are $\mathcal{F}_n$-stopping times. Show that $S \vee T$ and $S+T$ are also $\mathcal{F}_n$-stopping times.

Proof. Since $\{S \vee T \leq n\} = \{S \leq n\} \cap \{T \leq n\} \in \mathcal{F}_n$, $S \vee T$ is a stopping time. We have

$$\{S+T \leq n\} = \bigcup_{j=0}^{n} \{S = j\} \cap \{T \leq n-j\} \in \mathcal{F}_n.$$

Thus $S+T$ is a stopping time. $\square$

5. Suppose that S and T are $\mathcal{F}_n$-stopping times with $S \leq T$. Define:

$$H_n(\omega) = 1_{(S(\omega), T(\omega)]}(n), \quad \omega \in \Omega, n \geq 1.$$

Show that $\{H_n\}$ is predictable and deduce that if $\{X_n\}$ is a supermartingale then

$$E(X_{T \wedge n}) \leq E(X_{S \wedge n}) \text{ for all } n.$$

Proof. We have $H_n = 1_{\{S < n \leq T\}}$. Note that

$$\begin{aligned} \{S < n \leq T\} &= \{S < n\} \cap \{n \leq T\} \\ &= \{S \leq n-1\} \cap \{T > n\}^c \\ &= \{S \leq n-1\} \cap \{T \leq n-1\}^c \in \mathcal{F}_{n-1}. \end{aligned}$$

Thus H is predictable.

Let $(X_n)_{n=0}^{\infty}$ be an arbitrary supermartingale. To show that $E(X_{T \wedge n}) \leq E(X_{S \wedge n})$ for all n, it suffices to assume that T is bounded and show that $E(S) \leq E(T)$. So assume there exists $n_0 \in \mathbb{N}_0$ such that $T \leq n_0$. Since $|X_T| \leq \sum_{n=0}^{n_0} |X_n|$, X_T is integrable. Similarly, X_S is integrable. With $H_n = 1_{\{S < n \leq T\}}$ as above, we have $X_T - X_S = (H \cdot X)_{n_0}$. Since H is predictable, bounded, and nonnegative, $H \cdot X$ is a supermartingale. Thus $E(X_T - X_S) = E((H \cdot X)_{n_0}) \leq E((H \cdot X)_0) = 0$. Thus $E(X_T) \leq E(X_S)$. $\square$

6. Suppose that $\{X_n\}$ is an integrable $\mathcal{F}_n$-adapted sequence such that $E(X_\tau) = E(X_0)$ for every bounded $\mathcal{F}_n$-stopping time τ. Show that $\{X_n\}$ must be a $\mathcal{F}_n$-martingale.

Proof. Let $m, n \in \mathbb{N}_0$ with $m < n$ be arbitrary. To show that $E(X_n \,|\, \mathcal{F}_m) = X_m$, we need to show that $E(X_n 1_A) = E(X_m 1_A)$ for all $A \in \mathcal{F}_m$. Let $A \in \mathcal{F}_m$ be arbitrary. Let $\tau = m\, 1_A + n\, 1_{A^c}$. We have $\{\tau = m\} = A \in \mathcal{F}_m$ and $\{\tau = n\} = A^c \in \mathcal{F}_m \subset \mathcal{F}_n$ and $\{\tau = k\} = \emptyset \in \mathcal{F}_k$ for $k \notin \{m, n\}$. Thus τ is a bounded stopping time. Thus

$$\begin{aligned} E(X_0) &= E(X_\tau) \\ &= E(X_m 1_A + X_n 1_{A^c}) \\ &= E(X_m 1_A + X_n - X_n 1_A) \\ &= E(X_m 1_A) + E(X_n) - E(X_n 1_A) \\ &= E(X_m 1_A) + E(X_0) - E(X_n 1_A). \end{aligned}$$

The equality $E(X_n) = E(X_0)$ used above follows from the fact that the constant random variable n is a bounded stopping time. Thus $E(X_n 1_A) = E(X_m 1_A)$. Thus $E(X_n \mid \mathcal{F}_m) = X_m$. $\square$

7. Let $\xi_1, \xi_2, \ldots$ be i.i.d. $N(0,1)$. Let $X_n = \sum_{i=1}^n \xi_i$, $n \geq 1$; and let $X_0 = 0$. Let $\mathcal{F}_n = \sigma\{\xi_i, i = 1, \ldots, n\}$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Let $\{H_n\}_{n=1}^\infty$ be a $\mathcal{F}_n$-predictable sequence of bounded random variables. Let

$$Z_n = \exp\left(\sum_{i=1}^n H_i (X_i - X_{i-1}) - \frac{1}{2}\sum_{i=1}^n H_i^2\right), n \geq 1, \quad Z_0 = 1.$$

Show that $\{Z_n\}$ is an $\mathcal{F}_n$-martingale.

Solution. First we define the conditional expectation for nonnegative random variabels which are not necessarily integrable.

Theorem 1. *Let $(\Omega, \mathcal{F}, P)$ be a probability space, and let $X: \Omega \to [0, \infty)$ be an $\mathcal{F}$-measurable random variable. Suppose $\mathcal{G} \subset \mathcal{F}$ is a σ-algebra. Then there is a unique $\mathcal{G}$-measurable random variable $E(X \mid \mathcal{G}): \Omega \to [0, \infty)$ such that $E(1_A X) = E(1_A E(X \mid \mathcal{G}))$ for all $A \in \mathcal{G}$.*

Proof. Define a measure ν on $\mathcal{G}$ by $\nu(A) = E(1_A X)$. Clearly $\nu \ll P|_{\mathcal{G}}$. Note that $\Omega = \bigcup_{n=0}^\infty \{X \leq n\}$ and that $\nu(X \leq n) = E(1_{\{X \leq n\}} X) \leq n$. Thus ν is σ-finite. By the Radon Nikodym theorem, there is a unique $\mathcal{G}$-measurable random variable $E(X \mid \mathcal{G}): \Omega \to [0, \infty)$ such that for every $A \in \mathcal{G}$,

$$\nu(A) = \int_A E(X \mid \mathcal{G})\, dP|_{\mathcal{G}} = \int_A E(X \mid \mathcal{G})\, dP = E(1_A E(X \mid \mathcal{G})). \qquad \square$$

Proof of Problem 7. For each $n \in \mathbb{N}_0$, Z_n is $\mathcal{F}_n$-measurable. Now we show that $E(Z_{n+1} \mid \mathcal{F}_n) = Z_n$. Let $n \in \mathbb{N}_0$ be arbitrary. Note that $E(Z_{n+1} \mid \mathcal{F}_n) = Z_n$ if and only if $\frac{1}{Z_n} E(Z_{n+1} \mid \mathcal{F}_n) = E\left(\frac{Z_{n+1}}{Z_n} \mid \mathcal{F}_n\right) = 1$. Thus to show that $E(Z_{n+1} \mid \mathcal{F}_n) = Z_n$, it suffices to show that $E\left(\frac{Z_{n+1}}{Z_n} \mid \mathcal{F}_n\right) = 1$. We have

$$\begin{aligned} E\left(\frac{Z_{n+1}}{Z_n} \mid \mathcal{F}_n\right) &= E\left(\exp\left(H_{n+1}\, \xi_{n+1} - \frac{1}{2} H_{n+1}^2\right) \mid \mathcal{F}_n\right) \\ &= E\left(\exp(H_{n+1}\, \xi_{n+1}) \exp\left(-\frac{1}{2} H_{n+1}^2\right) \mid \mathcal{F}_n\right) \\ &= E(\exp(H_{n+1}\, \xi_{n+1}) \mid \mathcal{F}_n) \exp\left(-\frac{1}{2} H_{n+1}^2\right). \end{aligned}$$

Thus it suffices to show that $E(\exp(H_{n+1}\, \xi_{n+1}) \mid \mathcal{F}_n) = \exp\left(\frac{1}{2} H_{n+1}^2\right)$. Since H_{n+1} is $\mathcal{F}_n$-measurable, there exists a Borel measurable function $f: \mathbb{R}^n \to \mathbb{R}$ such that $H_{n+1} = f(\xi_1, \ldots, \xi_n)$. For $c_1, \ldots, c_n \in \mathbb{R}$,

$$\begin{aligned} E(\exp(H_{n+1}\, \xi_{n+1}) \mid \xi_1 = c_1, \ldots, \xi_n = c_n) &= E(\exp(f(c_1, \ldots, c_n)\, \xi_{n+1}) \mid \xi_1 = c_1, \ldots, \xi_n = c_n) \\ &= E(\exp(f(c_1, \ldots, c_n)\, \xi_{n+1})). \end{aligned}$$

For $t \in \mathbb{R}$,

$$E(\exp(t\,\xi_{n+1})) = \int_{-\infty}^{\infty} e^{tx}\, e^{-x^2/2}\, dx.$$

We have

$$-\frac{x^2}{2} + t\,x = -\frac{1}{2}\,(x^2 - 2\,t\,x) = -\frac{1}{2}\,((x-t)^2 - t^2) = -\frac{1}{2}\,(x-t)^2 + \frac{1}{2}\,t^2.$$

Thus

$$\int_{-\infty}^{\infty} e^{tx}\, e^{-x^2/2}\, dx = e^{\frac{1}{2}t^2} \int_{-\infty}^{\infty} e^{-\frac{(x-t)^2}{2}}\, dx = e^{\frac{1}{2}t^2} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} = e^{\frac{1}{2}t^2}.$$

Thus $E(\exp(t\,\xi_{n+1})) = e^{\frac{1}{2}t^2}$. Thus

$$E(\exp(f(c_1, \ldots, c_n)\,\xi_{n+1})) = e^{\frac{1}{2} f(c_1, \ldots, c_n)^2}.$$

Thus

$$E(\exp(H_{n+1}\,\xi_{n+1}) \,|\, \xi_1, \ldots, \xi_n) = \exp\!\left(\frac{1}{2}\, f(\xi_1, \ldots, \xi_n)^2\right) = \exp\!\left(\frac{1}{2}\, H_{n+1}^2\right).$$

This finishes the proof that $E(Z_{n+1} \,|\, \mathcal{F}_n) = Z_n$.

To establish integrability of Z_n, note that

$$E(Z_n) = E(E(Z_n \,|\, \mathcal{F}_{n-1})) = E(Z_{n-1}) = \cdots = E(Z_0) = 1.$$

Thus Z is a martingale. $\square$