STOR 635 HW 7

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1 Solutions

1. Let X_n and Y_n be positive integrable and adapted to $\mathcal{F}_n$. Suppose that $E(X_{n+1}\,|\,\mathcal{F}_n) \le X_n + Y_n$ with $\sum Y_n < \infty$ a.s. Prove that X_n converges a.s. to a finite limit.

Proof. For $n \in \mathbb{N}_0$, let $Z_n = X_n - \sum_{i=0}^{n-1} Y_i$. Clearly Z_n is integrable and $\mathcal{F}_n$-measurable. For $n \in \mathbb{N}$,

$$\begin{aligned} E(Z_n\,|\,\mathcal{F}_{n-1}) &= E(X_n\,|\,\mathcal{F}_{n-1}) - \sum_{i=0}^{n-1} Y_i \\ &\le X_{n-1} + Y_{n-1} - \sum_{i=0}^{n-1} Y_i \\ &= X_{n-1} + \sum_{i=0}^{n-2} Y_i \\ &= Z_{n-1}. \end{aligned}$$

Thus $(Z_n)_{n\in\mathbb{N}_0}$ is a supermartingale. For $K > 0$, let

$$\tau_K = \inf\left\{ n \in \mathbb{N}_0 : \sum_{i=0}^{n} Y_i > K \right\}.$$

τ_K is a stopping time since $\sum_{i=1}^{n} Y_i$ is $\mathcal{F}_n$-measurable. Since $(\tau_K \wedge n) - 1 < \tau_K$, we have $\sum_{i=0}^{(\tau_K\wedge n)-1} Y_i \le K$. Hence

$$Z_{\tau_K\wedge n} = X_{\tau_K\wedge n} - \sum_{i=0}^{(\tau_K\wedge n)-1} Y_i \ge -\sum_{i=0}^{(\tau_K\wedge n)-1} Y_i \ge -K.$$

Thus $Z_{\tau_K\wedge n}^- \le K$. By the martingale convergence theorem, there is an integrable (hence a.s. finite) random variable $Z_\infty^{\tau_K}$ such that $Z_{\tau_K\wedge n} \to Z_\infty^{\tau_K}$ a.s.. On the set $\{\tau_K = \infty\}$ we have $Z_n = Z_{\tau_K\wedge n} \to Z_\infty^{\tau_K}$ a.s.. Thus Z_n converges to a finite limit Z a.s. on the set

$$S = \bigcup_{K=1}^{\infty} \{\tau_K = \infty\} = \bigcup_{K=1}^{\infty} \left\{ \sum_{i=0}^{\infty} Y_i \le K \right\} = \left\{ \sum_{i=0}^{\infty} Y_i < \infty \right\}.$$

Since $\sum_{i=0}^{\infty} Y_i < \infty$ on S, it follows that $X_n = Z_n + \sum_{i=0}^{n-1} Y_i \to Z + \sum_{i=0}^{\infty} Y_i$ a.s. on S. By assumption, $P(S) = 1$. $\square$

2. Let $X_1, X_2, \ldots$ be i.i.d. random variables with

$$P(X_1 = 1) = P(X_1 = -1) = \frac{1}{2}.$$

Let $S_0 = 0$ and $S_n = X_1 + \cdots + X_n$ for $n \geq 1$. Let

$$\tau = \inf\{n \geq 1 : S_n = 0\}.$$

Show that the martingale $S_{n\wedge\tau}$ converges almost surely but not in L^1.

Proof. By conditioning and symmetry,

$$\begin{aligned} P(\tau < \infty) &= P(X_1 = -1)\, P(\tau < \infty \mid X_1 = -1) + P(X_1 = 1)\, P(\tau < \infty \mid X_1 = 1) \\ &= \frac{1}{2}\left(P(S_n = 1 \text{ for some } n \in \mathbb{N}_0) + P(S_n = -1 \text{ for some } n \in \mathbb{N}_0)\right) \\ &= \frac{1}{2}(1+1) \\ &= 1. \end{aligned}$$

If $\omega \in \Omega$ and $\tau(\omega) < \infty$, then $\tau(\omega) \wedge n = \tau(\omega)$ for n large, so $S_{\tau(\omega)\wedge n}(\omega) \to S_{\tau(\omega)}(\omega)$. Thus $S_{\tau\wedge n} \to S_\tau = 0$ a.s.

Since $S_{\tau\wedge n}$ is a martingale, $E(S_{\tau\wedge n}) = E(S_{\tau\wedge 0}) = E(S_0) = 0$. On the other hand, $S_\tau = 1$, so $E(S_\tau) = 1$. Since $E(S_{\tau\wedge n}) \nrightarrow E(S_\tau)$, it follows that $S_{\tau\wedge n}$ does not converge to S_τ in L^1. Since any L^1 limit of $S_{\tau\wedge n}$ must agree with the a.s. limit S_τ almost everywhere, it follows that $S_{\tau\wedge n}$ does not converge in L^1. □

3. Let $\{X_n\}$ be a martingale on $(\Omega, \mathcal{F}, P)$ with respect to a filtration $\{\mathcal{F}_n\}$. Then show that the following are equivalent.

(i) There exists a random variable X_∞ in L^1 such that $X_n \to X_\infty$ in L^1.

(ii) $\{X_n\}_{n\geq 1}$ is uniformly integrable.

(iii) For some integrable random variable X, $X_n = E(X \mid \mathcal{F}_n)$.

Proof. (i) $\Longrightarrow$ (ii): Suppose there exists a random variable X_∞ in L^1 such that $X_n \to X_\infty$ in L^1. Recall the theorem that if $f_n \in L^1$ and $f_n \to f$ in measure, then $f_n \to f$ in L^1 if and only if $\{f_n : n \in \mathbb{N}_0\}$ is uniformly integrable. Since $X_n \to X_\infty$ in L^1, it follows that $X_n \to X_\infty$ in measure. By the theorem, $\{X_n : n \in \mathbb{N}_0\}$ is uniformly integrable.

(ii) $\Longrightarrow$ (iii): Suppose $\{X_n\}_{n\geq 1}$ is uniformly integrable. By the martingale convergence theorem for uniformly integrable martingales, there exists a random variable $X_\infty \in L^1$ such that $X_n \to X_\infty$ a.s. and in L^1 and $E(X_\infty \mid \mathcal{F}_n) = X_n$ for all $n \in \mathbb{N}_0$. Hence taking $X = X_\infty$ proves the claim.

(iii) $\Longrightarrow$ (i): Suppose that there is a random variable $X \in L^1$ such that $X_n = E(X \mid \mathcal{F}_n)$ for all $n \in \mathbb{N}_0$. X_n is indeed a martingale since if $m < n$, the tower property yields $E(E(X \mid \mathcal{F}_n) \mid \mathcal{F}_m) = E(X \mid \mathcal{F}_m)$. Since $\{X\}$ is uniformly integrable, there exists a measurable increasing convex function $H : [0,\infty) \to [0,\infty)$ with $\lim_{x\to\infty} \frac{H(x)}{x} = \infty$ such that $E(H(|X|)) < \infty$. For any $n \in \mathbb{N}_0$,

$$H(|E(X \mid \mathcal{F}_n)|) \leq H(E(|X| \mid \mathcal{F}_n)) \leq E(H(|X|) \mid \mathcal{F}_n).$$

Hence $E(H(|E(X \mid \mathcal{F}_n)|)) \leq E(H(|X|))$. Hence $\sup_{n\in\mathbb{N}_0} E(H(|E(X \mid \mathcal{F}_n)|)) \leq E(H(|X|)) < \infty$. Thus $\{E(X \mid \mathcal{F}_n) : n \in \mathbb{N}_0\}$ is uniformly integrable. By the martingale convergence theorem for uniformly integrable martingales, there exists a random variable X_∞ in L^1 such that $X_n \to X_\infty$ a.s. and in L^1. □

4. Let $\{Z_n\}$ be iid sequence of $N(0,1)$ random variables. Let θ be an integrable random variable independent of $\{Z_i\}$. Let $Y_n \doteq Z_n + \theta$, $n \geq 1$. Show that

$$E(\theta \mid \sigma\{Y_1, \ldots, Y_n\}) \to \theta$$

a.s. and in L^1.

Solution. First we need a theorem

Theorem 1. *Let $(\mathcal{F}_n)_{n\in\mathbb{N}_0}$ be a filtration, and let $X \in L^1$. Then $E(X \mid \mathcal{F}_n) \to E(X \mid \mathcal{F}_\infty)$ a.s. and in L^1, where $\mathcal{F}_\infty = \sigma(\bigcup_{n=0}^{\infty} \mathcal{F}_n)$.*

Proof. By problem 3, $(E(X \mid \mathcal{F}_n))_{n\in\mathbb{N}_0}$ is a uniformly integrable martingale. By the martingale convergence theorem for uniformly integrable martingales, there is an $\mathcal{F}_\infty$-measurable random variable $Y \in L^1$ such that

$$E(X \mid \mathcal{F}_n) \to Y \text{ a.s. and in } L^1.$$

To show that $Y = E(X \mid \mathcal{F}_\infty)$, we will show that $E(X\,1_A) = E(Y\,1_A)$ for all $A \in \mathcal{F}_\infty$. Let

$$\mathcal{D} = \{A \in \mathcal{F}_\infty : E(X\,1_A) = E(Y\,1_A)\}.$$

We need to show that $\mathcal{D} = \mathcal{F}_\infty$. First we show that $\bigcup_{n=0}^{\infty} \mathcal{F}_n \subset \mathcal{D}$. Let $n \in \mathbb{N}_0$ and $A \in \mathcal{F}_n$ be arbitrary. By the martingale convergence theorem for uniformly integrable martingales, $E(Y \mid \mathcal{F}_n) = E(X \mid \mathcal{F}_n)$. Hence $E(X\,1_A) = E(Y\,1_A)$. Thus $\bigcup_{n=0}^{\infty} \mathcal{F}_n \subset \mathcal{D}$.

Next note that the π-system generated by $\bigcup_{n=0}^{\infty} \mathcal{F}_n$ is $\bigcup_{n=0}^{\infty} \mathcal{F}_n$ since if $A_1, \ldots, A_N \in \bigcup_{n=0}^{\infty} \mathcal{F}_n$ with $A_j \in \mathcal{F}_{n_j}$, then $A_1 \cap \ldots \cap A_N \in \mathcal{F}_{\max(n_1, \ldots, n_N)}$. Thus, by the π-λ theorem, it will suffice to show that $\mathcal{D}$ is a λ-system.

To show that $\mathcal{D}$ is a λ-system, we verify the three axioms:

(i) Clearly $\emptyset \in \mathcal{D}$.

(ii) Let $A \in \mathcal{D}$ be arbitrary. We have

$$E(X\,1_{A^c}) = E(X) - E(X\,1_A).$$

Since $E(E(X \mid \mathcal{F}_n)) = E(X)$ for all n, it follows from the L^1 convergence $E(X \mid \mathcal{F}_n) \to Y$ that $E(Y) = E(X)$. Hence

$$\begin{aligned} E(X\,1_{A^c}) &= E(X) - E(X\,1_A) \\ &= E(Y) - E(Y\,1_A) \\ &= E(Y\,1_{A^c}). \end{aligned}$$

Thus $A^c \in \mathcal{D}$.

(iii) Suppose $A_1, A_2, \ldots \in \mathcal{D}$ are disjoint. By the dominated convergence theorem,

$$\begin{aligned} E(X\,1_{\bigcup_{n=1}^{\infty} A_n}) &= E\left(\sum_{n=1}^{\infty} X\,1_{A_n}\right) \\ &= \sum_{n=1}^{\infty} E(X\,1_{A_n}) \\ &= \sum_{n=1}^{\infty} E(Y\,1_{A_n}) \\ &= E(Y\,1_{\bigcup_{n=1}^{\infty} A_n}). \end{aligned}$$

Thus $\bigcup_{n=1}^{\infty} A_n \in \mathcal{D}$.

Thus $\mathcal{D}$ is a λ-system. This completes the proof. $\square$

Proof of Problem 4. By Theorem (1), $E(\theta \mid \sigma(Y_1, \ldots, Y_n)) \to E(\theta \mid \mathcal{F}_\infty)$ a.s. and in L^1, where $\mathcal{F}_\infty = \sigma(\bigcup_{n=1}^{\infty} \sigma(Y_1, \ldots, Y_n))$. Note that for each $n \in \mathbb{N}$, Y_n is $\mathcal{F}_\infty$-measurable. Thus $\sigma\{Y_n : n \in \mathbb{N}\} \subset \mathcal{F}_\infty$. Note that for each $n \in \mathbb{N}$, $\sigma(Y_1, \ldots, Y_n) \subset \sigma\{Y_n : n \in \mathbb{N}\}$. Thus $\bigcup_{n=1}^{\infty} \sigma(Y_1, \ldots, Y_n) \subset \sigma\{Y_n : n \in \mathbb{N}\}$, and therefore $\mathcal{F}_\infty \subset \sigma\{Y_n : n \in \mathbb{N}\}$. Thus $\mathcal{F}_\infty = \sigma\{Y_n : n \in \mathbb{N}\}$.

The strong law of large numbers yields

$$\frac{Y_1 + \cdots + Y_n}{n} = \theta + \frac{Z_1 + \cdots + Z_n}{n} \to \theta \text{ a.s.}$$

Let

$$Y = \left(\limsup_{n\to\infty} \frac{Y_1 + \cdots + Y_n}{n}\right) \mathbb{1}_{\left\{\limsup\limits_{n\to\infty} \frac{Y_1 + \cdots + Y_n}{n} = \liminf\limits_{n\to\infty} \frac{Y_1 + \cdots + Y_n}{n}\right\} \cap \left\{\left|\limsup\limits_{n\to\infty} \frac{Y_1 + \cdots + Y_n}{n}\right| < \infty\right\}}.$$

Since $\limsup_{n\to\infty} \frac{Y_1 + \cdots + Y_n}{n}$ and $\liminf_{n\to\infty} \frac{Y_1 + \cdots + Y_n}{n}$ are $\mathcal{F}_\infty$ measurable, it follows that Y is $\mathcal{F}_\infty$-measurable. Since $\frac{Y_1 + \cdots + Y_n}{n} \to \theta$ a.s., it follows that $Y = \theta$ a.s. Hence

$$E(\theta \mid \mathcal{F}_\infty) = E(Y \mid \mathcal{F}_\infty) = Y = \theta.$$ $\square$

5. (i) Let $\{X_n\}$ be a square integrable martingale and let τ be a stopping time. Show that $\{X_n^\tau\}_{n \in \mathbb{N}_0}$ is a square integrable martingale and $\langle X^\tau \rangle_n = \langle X \rangle_{\tau \wedge n}$.

(ii) Let $\{X_n\}$ be a square integrable martingale for which $\sup_{n \in \mathbb{N}_0} \langle X \rangle_n < \infty$ a.s.. Then X_n converges a.s. to a r.v. that is finite a.s.

Proof. (i) By the optional stopping theorem, X^τ is a martingale. It is square integrable since $|X_{\tau\wedge n}| \le |X_1| + \cdots + |X_n| \Rightarrow \|X_{\tau \wedge n}\|_{L^2} \le \sum_{k=0}^{n} \|X_k\|_{L^2} < \infty$. We have

$$\langle X^\tau \rangle_n = \sum_{i=1}^{n} E((X_{\tau\wedge i} - X_{\tau \wedge (i-1)})^2 \mid \mathcal{F}_{i-1}).$$

Note that when $\tau \le i-1$, we have $\tau \wedge i = \tau \wedge (i-1) = \tau$. Thus when $i \ge \tau + 1$, we have $E((X_{\tau\wedge i} - X_{\tau\wedge(i-1)})^2 \mid \mathcal{F}_{i-1}) = 0$. Hence

$$\begin{aligned}
\langle X^\tau \rangle_n &= \sum_{i=1}^{\tau\wedge n} E((X_{\tau\wedge i} - X_{\tau\wedge(i-1)})^2 \mid \mathcal{F}_{i-1}) \\
&= \sum_{i=1}^{\tau\wedge n} E((X_i - X_{i-1})^2 \mid \mathcal{F}_{i-1}) \\
&= \langle X \rangle_{\tau\wedge n}.
\end{aligned}$$

(ii) Let $(\mathcal{F}_n)_{n \in \mathbb{N}_0}$ be the filtration with respect to which $(X_n)_{n \in \mathbb{N}_0}$ is a martingale. For $K \ge 0$, let

$$\tau_K = \inf\{n \in \mathbb{N}_0 : \langle X \rangle_{n+1} > K\}.$$

Since $(\langle X\rangle_{n+1})_{n\in\mathbb{N}_0}$ is $\mathcal{F}_n$-adapted, it follows that τ_K is a stopping time. Let

$$X_n^2 = X_0^2 + M_n + \langle X\rangle_n$$

be the Doob decomposition of $(X_n^2)_{n\in\mathbb{N}_0}$. Then

$$X_{\tau_K\wedge n}^2 = X_0^2 + M_{\tau_K\wedge n} + \langle X\rangle_{\tau_K\wedge n}.$$

Since $\langle X\rangle_{\tau_K\wedge n} \le K$, the optional sampling theorem gives

$$E(X_{\tau_K\wedge n}^2) = E(X_0^2) + E(\langle X\rangle_{\tau_K\wedge n}) \le E(X_0^2) + K.$$

Thus $\sup_{n\in\mathbb{N}_0} E(X_{\tau_K\wedge n}^2) < \infty$. By the L^2 martingale convergence theorem, there is a random variable $X_\infty^{\tau_K}\in L^2$ such that $X_{\tau_K\wedge n}\to X_\infty^{\tau_K}$ a.s. and in L^2. Since $X_\infty^{\tau_K}\in L^2$, $X_\infty^{\tau_K}$ is a.s. finite. On the set $\{\tau_K=\infty\}$, we have $X_{\tau_K\wedge n} = X_n \to X_\infty^{\tau_K}$ a.s.. Thus X_n converges to a finite limit a.s. on the set

$$\bigcup_{K=1}^{\infty}\{\tau_K=\infty\} = \bigcup_{K=1}^{\infty}\left\{\sup_{n\in\mathbb{N}_0}\langle X\rangle_n \le K\right\} = \left\{\sup_{n\in\mathbb{N}_0}\langle X\rangle_n < \infty\right\}.$$

By hypothesis, $P(\sup_{n\in\mathbb{N}_0}\langle X\rangle_n<\infty)=1$. Thus X_n converges to a finite limit a.s.. $\square$

6. Let $\{\xi_i\}$ be iid with mean 0 and variance σ^2. Let $S_0=0$, $S_n=\sum_{i=1}^n \xi_i$, $n\ge 1$. If N is a stopping time with $E(N)<\infty$, show that $E(S_N^2)=\sigma^2 E(N)$.

Proof. For $n\in\mathbb{N}_0$, let $\mathcal{F}_n=\sigma(\xi_1,\dots,\xi_n)=\sigma(S_1,\dots,S_n)$. Then $(S_n)_{n\in\mathbb{N}_0}$ is a martingale with respect to the filtration $(\mathcal{F}_n)_{n\in\mathbb{N}_0}$. We assume that N is a stopping time with respect to $(\mathcal{F}_n)_{n\in\mathbb{N}_0}$. Let $S_n^2 = M_n + \langle S\rangle_n$ be the Doob decomposition of $(S_n^2)_{n\in\mathbb{N}_0}$. We have

$$\begin{aligned}\langle S\rangle_n &= \sum_{i=1}^n E((S_i - S_{i-1})^2 \mid \mathcal{F}_{i-1})\\ &= \sum_{i=1}^n E(\xi_i^2 \mid \mathcal{F}_{i-1})\\ &= \sum_{i=1}^n E(\xi_i^2)\\ &= n\,\sigma^2.\end{aligned}$$

First consider the case when N is bounded. By the optional sampling theorem,

$$\begin{aligned}E(S_N^2) &= E(M_N + N\sigma^2)\\ &= E(M_N) + E(N)\,\sigma^2\\ &= E(N)\,\sigma^2.\end{aligned}$$

Now conisder the general case where only $E(N)<\infty$. Since $E(N)<\infty$, N is a.s. finite. Thus $S_{N\wedge n}\to S_N$ a.s.. By the bounded case, for each $n\in\mathbb{N}_0$,

$$E(S_{N\wedge n}^2) = E(N\wedge n)\,\sigma^2 \le E(N)\,\sigma^2 < \infty.$$

Hence the martingale $(S_{N\wedge n})_{n\in\mathbb{N}_0}$ is L^2 bounded. By the martingale convergence theorem, $S_{N\wedge n}$ converges a.s. and in L^2. Thus $S_{N\wedge n}\to S_N$ a.s. and in L^2. In particular,

$$\begin{aligned} E(S_N^2) &= \lim_{n\to\infty} E(S_{N\wedge n}^2) \\ &= \lim_{n\to\infty} E(N\wedge n)\,\sigma^2 \\ &= E(N)\,\sigma^2, \end{aligned}$$

where the last equality is by the monotone convergence theorem. □