Proof of Divergence Theorem

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September 2022

1 Introduction

In this note we prove the divergence theorem.

Divergence Theorem Let $\overline{\Omega}$ be a C^2 compact manifold with boundary with C^1 metric tensor g. Let Ω denote the manifold interior of $\overline{\Omega}$ and let $\partial\Omega$ denote the manifold boundary of $\overline{\Omega}$. Let $(\cdot,\cdot)$ denote L^2 inner products of functions and $\langle\cdot,\cdot\rangle$ denote inner products of vectors. Suppose $u \in C^1(\overline{\Omega}, \mathbb{R})$ and X is a C^1 (real valued) vector field on $\overline{\Omega}$. Then

$$(\operatorname{grad} u, X) = -(u, \operatorname{div} X) + \int_{\partial\Omega} u\langle X, N\rangle \, dS,$$

where N is the outward normal vector field on $\partial\Omega$.

Proof. We use the Einstein summation convention. By using a partition of unity, we may assume that u and X have compact support in a coordinate patch $O \subset \overline{\Omega}$. First consider the case where the patch is disjoint from $\partial\Omega$. Then O is identified with an open subset of $\mathbb{R}^n$ and integration by parts produces no boundary terms:

$$
\begin{aligned}
(\operatorname{grad} u, X) &= \int_O \langle \operatorname{grad} u, X\rangle \sqrt{g}\, dx && (1)\\
&= \int_O \partial_j u X^j \sqrt{g}\, dx && (2)\\
&= -\int_O u \partial_j(\sqrt{g} X^j)\, dx && (3)\\
&= -\int_O u \frac{1}{\sqrt{g}} \partial_j(\sqrt{g} X^j) \sqrt{g}\, dx && (4)\\
&= (u, -\frac{1}{\sqrt{g}} \partial_j(\sqrt{g} X^j)) && (5)\\
&= (u, -\operatorname{div} X). && (6)
\end{aligned}
$$

Here we used the Voss-Weyl formula for the divergence (alternatively, we can use this identity to define $-\operatorname{div}$ invariantly as the formal adjoint of grad). Now suppose O intersects $\partial\Omega$. Then O is identified with an open set in $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n \geq 0\}$. We zero extend u and X to $\mathbb{R}^n_+$ and perform integration by parts to obtain

$$
\begin{aligned}
(\operatorname{grad} u, X) &= \int_O \langle \operatorname{grad} u, X\rangle \sqrt{g}\, dx && (7)\\
&= \int_{\mathbb{R}^n_+} \partial_j u X^j \sqrt{g}\, dx && (8)\\
&= (u, -\operatorname{div} X) - \int_{\mathbb{R}^{n-1}} u(x', 0) X^n(x', 0) \sqrt{g(x', 0)}\, dx', && (9)
\end{aligned}
$$

where $dx' = dx_1 \dots dx_{n-1}$. By a variant of the straightening out theorem, we may choose O so that $\frac{\partial}{\partial x_n}$ is the inward unit normal $-N$ at $\partial\Omega$. In this case $\sqrt{g(x', 0)}\, dx' = \sqrt{g_{\partial\Omega}(x')}\, dx' = dS$ is the volume element on $\partial\Omega$ and the above formula reads

$$
(\operatorname{grad} u, X) = (u, -\operatorname{div} X) + \int_{\partial\Omega} u \langle X, N\rangle\, dS.
$$

This completes the proof.