Proofs of Geometric Formulas for Dot Product and Cross Product

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For $u, v \in \mathbb{R}^2$, we define $\theta(u, v)$ to be the smaller of the two angles between u and v, so that $0 \leq \theta(u, v) \leq \pi$.

Equivalence of the geometric and algebraic angle. Let u, v be nonzero vectors in $\mathbb{R}^2$. For $x, y \in \mathbb{R}^2$, let $(x, y) = x_1 y_1 + x_2 y_2$ denote the dot product of x and y, and let $\|x\| = \sqrt{(x, x)}$ denote the norm of x. Then

$$\theta(u, v) = \cos^{-1}\left(\frac{(u, v)}{\|u\|\,\|v\|}\right). \tag{1}$$

Proof: Since $0 \leq \theta \leq \pi$ by definition, it suffices to show that

$$\cos(\theta(u, v)) = \frac{(u, v)}{\|u\|\,\|v\|}. \tag{2}$$

Since $\theta(u, v) = \theta(\frac{u}{\|u\|}, \frac{v}{\|v\|})$ and $\frac{(u,v)}{\|u\|\|v\|} = (\frac{u}{\|u\|}, \frac{v}{\|v\|})$, it suffices to prove (2) when $\|u\| = \|v\| = 1$, that is, to prove that

$$\cos(\theta(u, v)) = (u, v). \tag{3}$$

Let $R = \begin{bmatrix} \cos(\tau) & -\sin(\tau) \\ \sin(\tau) & \cos(\tau) \end{bmatrix}$, the linear transformation that rotates vectors τ degrees counterclockwise, where τ is chosen so that $Ru = e_1$. Since R is a rotation, $\theta(Ru, Rv) = \theta(u, v)$. Also, it is easily verified that $(Rx, Ry) = (x, y)$ for all $x, y \in \mathbb{R}^2$. Thus $(Ru, Rv) = (u, v)$. Thus it suffices to prove (3) when $u = e_1$, that is, to prove that

$$\cos(\theta(e_1, v)) = (e_1, v). \tag{4}$$

But $(e_1, v) = v_1$, the x component of v, so (4) follows from the unit circle definition of cos. $\square$

If V is a real inner product space, and $u, v \in V$, we define $\theta(u,v) = \cos^{-1}\left(\frac{(u,v)}{\|u\|\|v\|}\right)$. Note that if $A \in O(V)$ (meaning $A^T = A^{-1}$), then $\theta(Au, Av) = \theta(u, v)$ for all $u, v \in V$ since $(Au, Av) = (u, A^T Av) = (u, v)$.

Equivalence of the algebraic and geometric cross product. Fix $u, v \in \mathbb{R}^3$. Define a linear map $T\colon \mathbb{R}^3 \to \mathbb{R}$ by

$$Tw = \det(w, u, v). \tag{5}$$

Since T is a linear map from $\mathbb{R}^3 \to \mathbb{R}$, there exists a unique $x \in \mathbb{R}^3$ such that for every $w \in \mathbb{R}^3$,

$$Tw = (w, x). \tag{6}$$

We define $u \times v := x$, so $(u \times v)$ is the unique vector in $\mathbb{R}^3$ such that for all $w \in \mathbb{R}^3$,

$$\det(w, u, v) = (w, u \times v). \tag{7}$$

From (7) it is easily verified that $v \times u = -(u \times v)$, that the cross product is linear in each argument, and that $e_1 \times e_2 = e_3$, $e_2 \times e_3 = e_1$, $e_3 \times e_1 = e_2$.

Proposition 1. Suppose $A \in SO(3)$ (meaning $A^T = A^{-1}$ and $\det(A) = 1$) and $u, v \in \mathbb{R}^3$. Then

$$Au \times Av = A(u \times v). \tag{8}$$

Proof: We have

$$\begin{aligned}\det(w, Au, Av) &= \det(A)\det(A^{-1}w, u, v)\\ &= (A^{-1}w, u \times v)\\ &= (w, A(u \times v)).\end{aligned}$$

Thus $Au \times Av = A(u \times v)$. $\square$

Corollary. If $A \in O(3)$ and $\det(A) = -1$, then $Au \times Av = -Au \times -Av = -A(u \times v)$.

Proposition 2: If $u, v \in \mathbb{R}^3$, then

$$\|u \times v\| = \|u\|\,\|v\| \sin(\theta(u, v)). \tag{9}$$

Proof: The proof of (9) is similar to the proof of (1). First we note that since $\theta(u,v) = \theta(\frac{u}{\|u\|}, \frac{v}{\|v\|})$, by dividing both sides of (9) by $\|u\|\,\|v\|$ we see that it suffices to prove (9) when $\|u\| = \|v\| = 1$, that is, to prove that

$$\|u \times v\| = \sin(\theta(u,v)). \tag{10}$$

Pick $A \in O(3)$ such that $Au = e_1$, and $Av \in \mathrm{span}(e_1, e_2)$, in effect rotating u and v onto the x-y plane. We have by proposition 1 that $\|Au \times Av\| = \|A(u \times v)\| = \|u \times v\|$. Also $\theta(Au, Av) = \theta(u,v)$. Thus it suffices to prove (10) when $u = e_1$ and $v \in \mathrm{span}(e_1, e_2)$, that is, to prove that

$$\|e_1 \times v\| = \sin(\theta(e_1, v)). \tag{11}$$

We have $e_1 \times v = v_2 e_3$, and by the unit circle definition of sin (or by the algebraic definition of $\theta(e_1, v)$), $\sin(\theta(e_1, v)) = |v_2|$. Since $\|v_2 e_3\| = |v_2|$, the proposition is proved. $\square$