STOR 635 HW 8

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1 Solutions

1. Let $\{X_n\}$ be a $\mathcal{F}_n$- martingale on $(\Omega, \mathcal{F}, P)$ such that for some $M < \infty$, $|X_{n+1} - X_n| \le M$ for all n. Suppose $X_0 = 0$. Let

$$A \doteq \{\lim X_n \text{ exists and is finite}\}, \quad B = \{\limsup X_n = \infty \text{ and } \liminf X_n = -\infty\}.$$

Show $P(A \cup B) = 1$.

Proof. Let $S = \left\{\limsup_{n\to\infty} X_n < \infty\right\}$. We will show that $A =_P S$, i.e. that $1_A = 1_S$ a.s.. We have $A \subset S$. Now it remains to show that $S \subset_P A$, i.e. that $1_S \le 1_A$ a.s., or equivalently, that almost every $\omega \in S$ is also in A. For $K > 0$, define the stopping time

$$\tau_K = \inf\{n \in \mathbb{N}_0 : X_n > K\}.$$

Note that $\tau_K \ge 1$ since $X_0 = 0$. For $\omega \in \Omega$, we have $X_n(\omega) \le K$ for $n < \tau_K(\omega)$, and if $\tau_K(\omega) < \infty$ we have $X_{\tau_K(\omega)}(\omega) \le X_{\tau_K(\omega)-1}(\omega) + |X_{\tau_K(\omega)}(\omega) - X_{\tau_K(\omega)-1}(\omega)| \le K + M$. Thus $X_{\tau_K \wedge n} \le K + M$. Thus $\sup_{n\in\mathbb{N}_0} X^+_{\tau_K \wedge n} \le K + M < \infty$. By the martingale convergence theorem, there is an L^1 real valued random variable $X^{\tau_K}_\infty$ such that $X_{\tau_K \wedge n} \to X^{\tau_K}_\infty$ a.s.. Thus

$$\{\tau_K = \infty\} \subset_P \{X_n \to X^{\tau_K}_\infty\} \subset A.$$

Thus

$$\{\tau_K = \infty\} = \left\{\sup_{n\in\mathbb{N}_0} X_n \le K\right\} \subset_P A.$$

Taking the union over $K \in \mathbb{N}$ yields

$$\left\{\sup_{n\in\mathbb{N}_0} X_n < \infty\right\} \subset_P A.$$

For $k \in \mathbb{N}$, we have $\{\sup_{n\in\mathbb{N}_0} X_n < \infty\} = \{\sup_{n\ge k} X_n < \infty\}$. Thus

$$\left\{\sup_{n\in\mathbb{N}_0} X_n < \infty\right\} = \bigcup_{k=1}^{\infty} \left\{\sup_{n\ge k} X_n < \infty\right\} = \left\{\inf_{k\ge1} \sup_{n\ge k} X_n < \infty\right\} = \left\{\limsup_{n\to\infty} X_n < \infty\right\}.$$

Thus $S \subset_P A$. Thus $S =_P A$.

Applying the previous result to the martingale $(-X_n)_{n\in\mathbb{N}_0}$, we get

$$C := \left\{\liminf_{n\to\infty} X_n > -\infty\right\} =_P A.$$

We have $B = S^c \cap C^c =_P A^c \cap A^c = A^c$. Thus $P(A \cup B) = 1$. $\square$

2. Let $\{\mathcal{F}_n\}_{n \in \mathbb{N}_0}$ be a filtration with $\mathcal{F}_0 \doteq \{\emptyset, \Omega\}$. Let $\{A_n\}$ be a sequence of events with $A_n \in \mathcal{F}_n$. Show that

$$\{A_n \text{ i.o}\} =_P \left\{ \sum_{n=1}^{\infty} P(A_n \,|\, \mathcal{F}_{n-1}) = \infty \right\}.$$

Proof. Let $1_{A_n} = M_n + C_n$ be the Doob's decomposition of 1_{A_n} so that for $n \in \mathbb{N}_0$,

$$M_n = \sum_{i=1}^{n} (1_{A_i} - P(A_i \,|\, \mathcal{F}_{i-1}))$$

is a martingale. Since $\{A_n \text{ i.o}\} = \{\sum_{i=1}^{\infty} 1_{A_i} = \infty\}$, it suffices to show that

$$\left\{ \sum_{i=1}^{\infty} 1_{A_i} < \infty \right\} =_P \left\{ \sum_{i=1}^{\infty} P(A_i \,|\, \mathcal{F}_{i-1}) < \infty \right\}$$

For each $i \in \mathbb{N}$, since $P(A_i \,|\, \mathcal{F}_{i-1}) \geq 0$ a.e. we can let $P(A_i \,|\, \mathcal{F}_{i-1})$ be a version of itself such that $P(A_i \,|\, \mathcal{F}_{i-1}) \geq 0$ everywhere. Suppose $\omega \in \{\sum_{i=1}^{\infty} 1_{A_i} < \infty\}$. For $i \in \mathbb{N}$, let

$$a_i = 1_{A_i}(\omega) \geq 0,$$
$$b_i = P(A_i \,|\, \mathcal{F}_{i-1})(\omega) \geq 0.$$

We have

$$M_n(\omega) = \sum_{i=1}^{n} (a_i - b_i).$$

Since $\sum_{i=1}^{\infty} a_i = r < \infty$,

$$\begin{aligned} \limsup_{n \to \infty} M_n(\omega) &= \limsup_{n \to \infty} \left(\sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i \right) \\ &= r - \sum_{i=1}^{\infty} b_i \\ &\leq r \\ &< \infty. \end{aligned}$$

Thus

$$\left\{ \sum_{i=1}^{\infty} 1_{A_i} < \infty \right\} \subset \left\{ \limsup_{n \to \infty} M_n < \infty \right\}.$$

Note that M is a martingale with $|M_{n+1} - M_n| \leq 2$ for all n. By the proof of problem 1,

$$\left\{ \limsup_{n \to \infty} M_n < \infty \right\} =_P \left\{ \lim_{n \to \infty} M_n \text{ exists in } \mathbb{R} \right\}.$$

Thus

$$\left\{ \sum_{i=1}^{\infty} 1_{A_i} < \infty \right\} \subset_P \left\{ \lim_{n \to \infty} M_n \text{ exists in } \mathbb{R} \right\}.$$

On the set $\{\sum_{i=1}^\infty 1_{A_i} < \infty\} \cap \{\lim_{n\to\infty} M_n \text{ exists in } \mathbb{R}\}$ we have

$$\mathbb{R} \ni \lim_{n\to\infty} M_n - \sum_{i=1}^\infty 1_{A_i} = \sum_{i=1}^\infty (1_{A_i} - P(A_i \,|\, \mathcal{F}_{i-1})) - \sum_{i=1}^\infty 1_{A_i} = \sum_{i=1}^\infty -P(A_i \,|\, \mathcal{F}_{i-1}).$$

Hence

$$\left\{\sum_{i=1}^\infty 1_{A_i} < \infty\right\} \cap \left\{\lim_{n\to\infty} M_n \text{ exists in } \mathbb{R}\right\} \subset \left\{\sum_{i=1}^\infty P(A_i \,|\, \mathcal{F}_{i-1}) < \infty\right\}.$$

Thus

$$\begin{aligned}\left\{\sum_{i=1}^\infty 1_{A_i} < \infty\right\} &=_P \left\{\sum_{i=1}^\infty 1_{A_i} < \infty\right\} \cap \left\{\lim_{n\to\infty} M_n \text{ exists in } \mathbb{R}\right\} \\ &\subset_P \left\{\sum_{i=1}^\infty P(A_i \,|\, \mathcal{F}_{i-1}) < \infty\right\}.\end{aligned}$$

Now suppose $\omega \in \{\sum_{i=1}^\infty P(A_i \,|\, \mathcal{F}_{i-1}) < \infty\}$. Let a_i and b_i be defined as before. Since $\sum_{i=1}^\infty b_i = s < \infty$,

$$\begin{aligned}\liminf_{n\to\infty} M_n(\omega) &= \liminf_{n\to\infty} \left(\sum_{i=1}^n a_i - \sum_{i=1}^n b_i\right) \\ &= \sum_{i=1}^\infty a_i - s \\ &\geq -s \\ &> -\infty.\end{aligned}$$

Thus

$$\left\{\sum_{i=1}^\infty P(A_i \,|\, \mathcal{F}_{i-1}) < \infty\right\} \subset \left\{\liminf_{n\to\infty} M_n > -\infty\right\}.$$

By the proof of problem 1,

$$\left\{\liminf_{n\to\infty} M_n > -\infty\right\} =_P \left\{\lim_{n\to\infty} M_n \text{ exists in } \mathbb{R}\right\}.$$

Thus

$$\left\{\sum_{i=1}^\infty P(A_i \,|\, \mathcal{F}_{i-1}) < \infty\right\} \subset_P \left\{\lim_{n\to\infty} M_n \text{ exists in } \mathbb{R}\right\}$$

By a similar argument as before,

$$\begin{aligned}\left\{\sum_{i=1}^\infty P(A_i \,|\, \mathcal{F}_{i-1}) < \infty\right\} &=_P \left\{\sum_{i=1}^\infty P(A_i \,|\, \mathcal{F}_{i-1}) < \infty\right\} \cap \left\{\lim_{n\to\infty} M_n \text{ exists in } \mathbb{R}\right\} \\ &\subset \left\{\sum_{i=1}^\infty 1_{A_i} < \infty\right\}.\end{aligned}$$

Thus we have shown that $\{\sum_{i=1}^{\infty} 1_{A_i} < \infty\} \subset_P \{\sum_{i=1}^{\infty} P(A_i \mid \mathcal{F}_{i-1}) < \infty\}$ and $\{\sum_{i=1}^{\infty} P(A_i \mid \mathcal{F}_{i-1}) < \infty\} \subset_P \{\sum_{i=1}^{\infty} 1_{A_i} < \infty\}$. Hence

$$\left\{\sum_{i=1}^{\infty} 1_{A_i} < \infty\right\} =_P \left\{\sum_{i=1}^{\infty} P(A_i \mid \mathcal{F}_{i-1}) < \infty\right\}. \qquad \square$$

3. Let $n \in \mathbb{N}$ and $X_1, \ldots, X_n$ be real valued random variables. Show that

$$\frac{1}{n!}\sum_{\rho \in S(n)} \left(\frac{1}{n-1}\sum_{i=1}^{n-1} X_{\rho(i)}\right) = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

Proof. It is enough to show that for every $x_1, \ldots, x_n \in \mathbb{R}$,

$$\frac{1}{n!}\sum_{\rho \in S(n)} \left(\frac{1}{n-1}\sum_{i=1}^{n-1} x_{\rho(i)}\right) = \frac{1}{n}\sum_{i=1}^{n} x_i.$$

For $f : \mathbb{R}^n \to \mathbb{R}$, define $A_n f : \mathbb{R}^n \to \mathbb{R}$ by

$$A_n f = \frac{1}{n!}\sum_{\rho \in S(n)} f \circ \rho,$$

where for $\rho \in S(n)$, we define $\rho : \mathbb{R}^n \to \mathbb{R}^n$ by $\rho(x) = (x_{\rho(1)}, \ldots, x_{\rho(n)})$. Since A_n is linear,

$$\begin{aligned}
\frac{1}{n!}\sum_{\rho \in S(n)} \left(\frac{1}{n-1}\sum_{i=1}^{n-1} x_{\rho(i)}\right) &= A_n\left(\frac{1}{n-1}\sum_{i=1}^{n-1} x_i\right) \\
&= \frac{1}{n-1}\sum_{i=1}^{n-1} A_n x_i.
\end{aligned}$$

For $i \in \{1, \ldots, n-1\}$ we have

$$\begin{aligned}
A_n x_i &= \frac{1}{n!}\sum_{\rho \in S(n)} x_{\rho(i)} \\
&= \frac{1}{n!}\sum_{j=1}^{n} \sum_{\rho \in S(n) : \rho(i) = j} x_j \\
&= \frac{1}{n!}\sum_{j=1}^{n} \#\{\rho \in S(n) : \rho(i) = j\}\, x_j \\
&= \frac{1}{n}\sum_{j=1}^{n} x_j.
\end{aligned}$$

Thus

$$\frac{1}{n-1}\sum_{i=1}^{n-1} A_n x_i = \frac{1}{n}\sum_{j=1}^{n} x_j.$$

This completes the proof. $\square$

4. Recall the σ-field $\mathcal{E}_n$ introduced in the class. Show that $A \in \mathcal{E}_n$ if and only if for some $B \in \mathcal{B}(E)^{\otimes\infty}$, with the property $B^\rho = B$ for all $\rho \in S(n)$,

$$A = X^{-1}(B) = \{\omega : (X_1(\omega), \dots) \in B\}$$

where $B^\rho = \{x = (x_1, \dots) \in E^\infty : (x_{\rho(1)}, \dots) \in B\}$.

Proof. On $E^{\mathbb{N}}$ we use the σ-algebra $\mathcal{B}(E)^{\otimes\mathbb{N}} = \mathcal{B}(E^{\mathbb{N}})$. We have the definitions

$$\mathcal{E}'_n = \sigma\{F \mid F : E^{\mathbb{N}} \to \mathbb{R} \text{ is measurable and } n\text{-symmetric}\},$$

$$\mathcal{E}_n = X^{-1}(\mathcal{E}'_n) = \{\{X \in A\} : A \in \mathcal{E}'_n\}.$$

For $\rho \in S(n)$, define $\rho : E^{\mathbb{N}} \to E^{\mathbb{N}}$ by $\rho(x) = (x_{\rho(1)}, \dots, x_{\rho(n)}, x_{n+1}, \dots)$. Hence "$F$ is n-symmetric" means $F \circ \rho = F$ for all $\rho \in S(n)$. To solve the problem, it suffices to show that

$$\mathcal{E}'_n = \mathcal{F} := \{B \in \mathcal{B}(E^{\mathbb{N}}) : B^\rho = B \text{ for all } \rho \in S(n)\}.$$

Since $B^\rho = \rho^{-1}(B)$ and inverse image commutes with unions and complements, it follows that $\mathcal{F}$ is a σ-algebra. Let $F : E^{\mathbb{N}} \to \mathbb{R}$ be measurable and n-symmetric and $A \in \mathcal{B}(\mathbb{R})$. Then for every $\rho \in S(n)$,

$$F^{-1}(A)^\rho = \rho^{-1}(F^{-1}(A)) = (F \circ \rho)^{-1}(A) = F^{-1}(A).$$

Thus $F^{-1}(A) \in \mathcal{F}$. Since such sets $F^{-1}(A)$ generate $\mathcal{E}'_n$, it follows that $\mathcal{E}'_n \subset \mathcal{F}$. Now let $B \in \mathcal{F}$. We have

$$1_B = 1_{B^\rho} = 1_{\rho^{-1}(B)} = 1_B \circ \rho.$$

Thus 1_B is measurable and n-symmetric. Thus $B \in \mathcal{E}'_n$. Thus $\mathcal{F} \subset \mathcal{E}'_n$. Thus $\mathcal{E}'_n = \mathcal{F}$. $\square$

5. Recall the exchangeable σ-field $\mathcal{E}$ and the tail σ-field $\mathcal{T}$. Show that $\mathcal{T} \subset \mathcal{E}$ and the inclusion is proper.

Proof. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of $(E, \mathcal{B})$-valued random variables. Let $n \in \mathbb{N}$ be arbitrary. For $A \in \mathcal{B}$ and $m \geq n+1$, we have that $x \to x_m$ is measurable and n-symmetric, and hence $X_m^{-1}(A) \subset \mathcal{E}_n$. Thus $\sigma(\{X_m : m \geq n+1\}) = \sigma\{X_m^{-1}(A) : m \geq n+1, A \in \mathcal{B}\} \subset \mathcal{E}_n$. Taking the intersection over $n \in \mathbb{N}$ gives $\mathcal{T} \subset \mathcal{E}$.

For an example where the inclusion is proper, assume that E has more than one element, and let $(X_n)_{n\in\mathbb{N}}$ be the coordinate functions on $(E^{\mathbb{N}}, \mathcal{B}^{\otimes\mathbb{N}})$, so that $X_n(x) = x_n$. Define $f : E^{\mathbb{N}} \to \{0,1\}$ by $f(x) = 1_{\{x_1 = x_2 = \cdots\}}$. Since $f(x) = \prod_{n=1}^\infty 1_{\{x_n = x_{n+1}\}}$, f is measurable. f is invariant under permutations of $\mathbb{N}$, so f is $\mathcal{E}$-measurable. For contradiction, suppose f is $\mathcal{T}$-measurable. Since f is $\sigma\{X_m : m \geq 2\}$-measurable, there exists a measurable function $g : E^{\mathbb{N}} \to \mathbb{R}$ such that $f(x) = g(x_2, x_3, \dots)$ for all $x \in E^{\mathbb{N}}$. But then for every $x, y \in E$,

$$1 = f(x, x, x, \dots) = g(x, x, \dots) = f(y, x, x, \dots),$$

and hence $x = y$. Since E has more than one element, this is a contradiction. Thus f is not $\mathcal{T}$-measurable. $\square$

6. Let $\{X_n\}_{n\in\mathbb{N}}$ be exchangeable with values in a a Polish space E. Assume that for some $k\in\mathbb{N}$, $\varphi: E^k \to \mathbb{R}$ is measurable and satisfies $E(|\varphi(X_1,\ldots,X_k)|)<\infty$. Let

$$A_n(\varphi) \doteq \frac{1}{n!}\sum_{\rho\in S(n)} \varphi(X_{\rho(1)},\ldots,X_{\rho(k)}).$$

Recall the σ-field $\mathcal{E}_n$ introduced in the online lecture. Show that $(A_n(\varphi),\mathcal{E}_n)_{n\geq k}$ is a backward martingale.

Proof. For $n\in\mathbb{N}$ we have $E(\varphi(X)\,|\,\mathcal{E}_n)=A_n(\varphi)$ (this was proven in class). For $n\geq 1$ we have by the tower property that

$$E(A_{n-1}(\varphi)\,|\,\mathcal{E}_n)=E(E(\varphi(X)\,|\,\mathcal{E}_{n-1})\,|\,\mathcal{E}_n)=E(\varphi(X)\,|\,\mathcal{E}_n)=A_n(\varphi).$$

Thus $(A_n(\varphi),\mathcal{E}_n)_{n\geq 1}$ is a backward martingale. $\square$