Space, Time and Space-time

Time and space concept is absolute. According to Newton's theory, time flows with perfect uniformity forever and space is a limitless container. Nothing in the universe affects the time's flow.

Space

The term 'Space' has different meanings to different people and can be used in different ways. Space is defined as a boundless, three-dimensional (3-D) extent in which objects can occupy and where events occur. It has relative position and direction. Also, it can be referred to the unlimited 3-D expanse in which all material objects are located.

In astronomy - **space** is defined as the region beyond the earth's atmosphere containing the other planets of the solar system, stars, galaxies etc, i.e. the universe.

Space can also be defined as a region between two points. An interval of distance or time between two points, objects or events. The most important concept about space and time is that in Newtonian mechanics, space is time and time is space.

When we talk about the length of a space between two points – this implies ***distance***. Hence, space has dimension of length / distance i.e. meter.

Space is represented in spatial coordinate system which can be 1-D, 2-D or 3-D. The Cartesian coordinate system i.e. xyz plane like others coordinate systems can be used as reference frame to represent space.

Reference Frame: This is an abstract coordinate system and a set of physical reference points that uniquely fix the coordinate system and standardize measurements within that frame.

Figure 1 is an illustration of reference frames i.e. static 1-D coordinate system and moving 1-D reference frame. To object A, object B is static though the container i.e. the moving reference frame is moving at constant velocity V. Whereas to object C, object B is moving with velocity V in the positive x-direction.

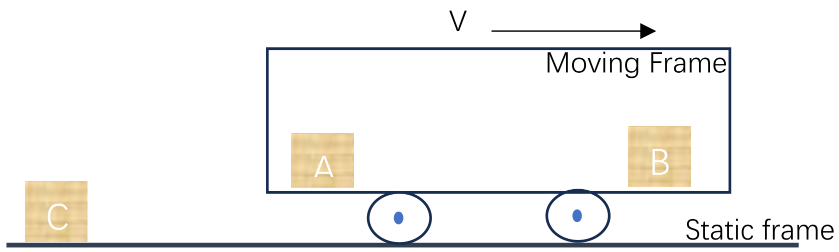


Figure 1: Moving and static frame of reference

Inertial reference frame: This is a frame of reference that is not accelerating and the Newton's law holds true. The two reference frames in Figure 1 are inertial reference frames. The implication is that if there is no external force is acting on a body in such a reference frame, the object will stay at rest or remain in uniform motion.

Non-inertial reference frame: This is a frame of reference that is accelerating and the Newton's law does not hold true. A good example of a non-inertial reference frame is a motion of a car round a corner or a round-about OR a car that accelerating or decelerating. In this scenario, an occupant of such a car will experience a change of state as if an external force is acting on it.

Time

Time is defined as the interval or space between two events. The SI unit of time is in second.

In classical mechanics, **space** and **time** are related concepts. Just as stated above, *space is time and time is space.* Time is required to cover the space between two points. The reason we use it loosely e.g. Lagos to Ibadan is 1 hour 30 minutes. The implication of a statement like this is that 1 hour 30 minutes is required to travel from Lagos to Ibadan. Leading to the well-known equation:

$$\text{Speed} = \frac{\text{Distance}}{\text{Time}} \tag{1}$$

i.e.

$$\text{Distance} = \text{Speed} \times \text{Time} \tag{2}$$

If the speed at which an object is traveling is known, one can calculate the time it will take that object to cover the space between two points. Equation 2, so to say, links space to time in mechanics and based on this one can talk about the space between two points in terms of time.

Note:

i. *We talk about the length of a space between two events – this implies time, and time is measured in seconds.*

ii. *Space and time are both observable and measurable physical quantities. Their combination gives a new coordinate system called* ***space-time****.*

Space-time

In physics, space-time, also called the space-time continuum, is a mathematical model that fuses the three dimensions of space and the one dimension of time into a single four-dimensional continuum.

Space-time is a four (4) coordinate system in which the space coordinate is combined with the time coordinate to make a single coordinate system e.g. (x, y, z, t). This 4-D (space-time) is used to specified events. Figure 1 shows a 4-D coordinate system in which the coordinate (x,y.z) is represented by a single axis as indicated and the time axis by an axis. An event can be specified as a point on this (x, y, z, t) coordinate. The events A and B occurred at the same time but at different locations. This is because the two occurred at the same time t_1 but at different locations i.e. event A occurred at location P_1 and event B at location P_2. Similarly, event A and C occurred at the same location P_1 but at different time, i.e. event A occurred at time t_1 and B at time t_2.

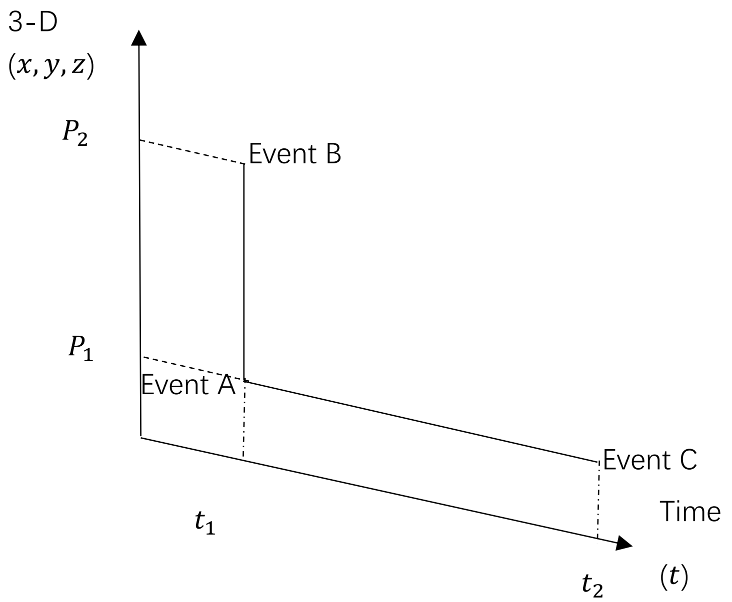


Figure 1: Space-time coordinate

Talk about the concept of relativity ----- here.

Lorentz transformations – contraction and dilation

Length contraction applies ***when you are talking about a distance that is independent of time*** (e.g. the distance between two objects that are fixed relative to one another or the distance between two ends of a single object). The "proper distance" in the formula is the distance in a frame where the two ends of the distance are not moving, and the formula says that the observed distance in another frame is smaller by a factor of gamma.

Note that length contraction ***does not apply*** if we are talking about the distance between two things that are moving relative to one another or to the distance between two events (rather than physical objects).

Time dilation applies ***when you are talking about two events that are at the same place in some frame*** (e.g. two ticks on the same clock). In this case, the "proper time" in the formula is the time in this frame where the two events are at the same place, and the formula says that the observed time between these two events in some other frame is larger by a factor of gamma.

Note that time dilation ***does not apply*** if we are talking about the time between two events that are not in the same place in either frame. When length contraction and time dilation do not apply (or even when they do), we can use **Lorentz transformations.**

1) Lorentz transformations relate the position and time of a SINGLE EVENT in some frame S to the position and time in another frame S'. If you are applying the LT formula, ask yourself whether the x and t you are plugging in correspond to some specific event.

2) The main steps to using Lorentz transformations to solve a problem:

a) Identify one frame of reference and call it S. It is your choice, but usually it is simplest to call S the frame of reference in which you have the most information.

b) Identify another frame of reference and call it S'. Usually it is convenient to make this the reference frame corresponding to the information that you are trying to find in the problem.

c) Determine the velocity of frame S' relative to frame S. The velocity v that appears in the Lorentz transformation formula or the velocity transformation formula is ***always*** this velocity. Make sure that you have defined the x direction so that this velocity is either in the positive or negative x direction.

d) Decide which events are relevant for the problem. Pick some event to be the origin of space and time coordinates (if the question doesn't do this for you). Determine the coordinates and times of the other events in one frame of reference. This might be information you are given, or information that you need to work out using ordinary kinematics formulae (e.g. distance = speed * time). Finally, you can use the Lorentz transformation formulae to determine the positions and times of these events in the other frame of reference. This should be enough information to allow you to solve the problem.

PHY 115 UNITS AND DIMENSION

Observation and measurement of physical quantities / parameters are the core of scientific activities. Measurement is the act or process of assigning size, or value to a physical quantity.

Quantities are measured when we are able to quantify it by assigning value and unit to such quantities or parameters. This is done by the use of measuring equipment- which are mostly based on one physical law or the other. Quantities are categorised into two namely: fundamental quantities and derived quantities (likewise the Units- fundamental units and derived units).

Fundamental quantities are quantities upon which other quantities are based while derived quantities are quantities that are obtained from fundamental quantities.

Fundamental quantities, units and dimensions

Quantity	**Unit**	**Dimension**
Length	Meter(m)	L
Time	Second(s)	T
Mass	Kilogramme(kg)	M
Temperature	Degree Kelvin(ºK)	Θ or K
Electric current	Ampere(A)	I or A
Electric charge	Coulomb(C)	Q
Mole(Amount of substance)	Mole(mol)	N
Luminous intensity	Candela(cd)	J or C

The underlined dimensions are the one adopted for this course.

Derived quantities, Units and Dimension

Quantity	**Equation**	**Unit**	**Dimension**
Velocity	$\frac{displacement}{time}$	m/s	LT^{-1}
Acceleration	$\frac{velocity}{time}$	m/s^2	LT^{-2}
Momentum	$mass \times velocity$	kgm/s	MLT^{-1}
Force	$mass \times acceleration$	kgm/s^2	MLT^{-2}

Example

Electrostatic Force = $F = k\frac{q_1 q_2}{r^2}$

F=[MLT $^{-2}$]=$[kQ^2L^{-2}]$

This implies: $[kQ^2]$ = $[ML^3\,T^{-2}]$

[k] = $ML^3T^{-2}Q^{-2}$

Work done = $F \times d = (Nm) = ML^2T^{-2}$

The relationship between Work done and voltage due to separation of charges is

$Work\ \ done = qV$, but the dimension of work is $[ML^2T^{-2}]$

Therefore, $[qV] = [ML^2T^{-2}]$

Implying [V] = $[ML^2T^{-2}Q^{-1}]$.

The quantity of heat energy transfer $Q = mcT$, obtain the dimension of the specific heat capacity (c). (Please note that Q is used as the dimension of charge above, quantity of heat is also represented with symbol Q - but not the same as the dimension of charge)

The quantity of heat is in Joules → dimension of Q, $[Q] = [ML^2T^{-2}]$

Using $Q = mcT,$ implies $Q = [McK] = [ML^2T^{-2}]$, or

[c] = $\frac{ML^2T^{-2}}{MK} = L^2T^{-2}K^{-1}$

Optics

Refractive index $n = \frac{Sin\,i}{Sin\,r}$ is dimensionless.

Wavelength λ [L] has the dimension of length and the unit is meter.

Questions

1. The gravitational force F is given by $F = \frac{GmM}{R^2}$, obtain the dimension of G.
2. Acceleration due to gravity (g), below the earth surface, is given by $g = \frac{bGM}{r^3}$, obtain the dimension of b.

3. The force a current carrying wire will experience when placed in a magnetic field of magnitude B is $F_B = BIL$, where I is the current passing through the wire and L is the length of the wire that is inside the field. Obtain the dimension of B.

SCALAR AND VECTOR QUANTITIES

Physical quantities can be grouped into two namely: scalar and vector quantities.

SCALAR – A quantity defined only by magnitude e.g. Distance, Speed, Mass.

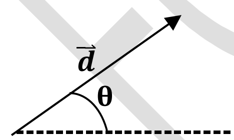
Distance – Shortest space dx or d between two points. The unit is in meter or foot.

Speed(s) – Rate of change of distance with time and is equal to (distance divided by time)

Mass – Quantity of matter in a body and is measured in kg

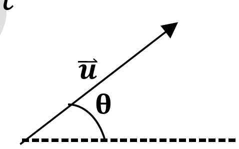
VECTOR – A quantity defined by both magnitude and direction e.g. displacement, velocity, acceleration, force.

Displacement – Distance in a defined direction.



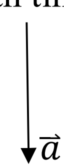
Velocity – Time rate of chance of displacement or change of distance with time in a given direction.

$v = d/t = ds/dt = dx/dt$



Acceleration - rate of change of velocity with time.

$a = dv/dt$

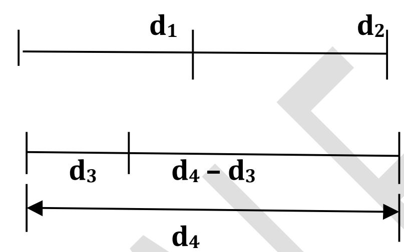


Force – Rate of change of momentum with time. This can be shown (later) to be the product of mass and acceleration.

Addition and Subtraction of Scalars **-** Addition and subtraction of scalar quantity is done like numbers.

Sum of d_1 and $d_2 = d_1 + d_2$

Difference between d_4 and d_3 is equal to d_4-d_3



Note – *Scalars* have magnitude only e.g. mass, length, time, density, energy while *Vectors* have magnitude and direction.

Displacement is characterized by length and direction. $\overrightarrow{AB}$ is net the effect and is independent of the path taken to go from A to B.

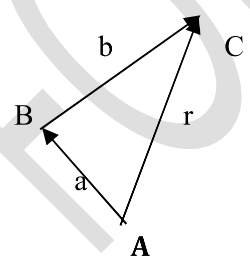
Vectors: Considering the net displacement say from A to C through AB followed by BC. The sum is not an algebraic sum

Other example includes:

Force, Velocity, Acceleration

E Electric field strength

B Magnetic induction



The symbols for vectors are bold face letters or letters with arrow, i.e., $\mathbb{A}$ or $\vec{A}$.

The magnitude is represented by the modulus, i.e., the magnitude of vector $\vec{A}$ is

$|\vec{A}|$

Vectors can be expressed in vector notation or in magnitude and the angle the vector makes with positive x-axis.

In vector notation, a 2D vector, e.g., $\vec{A}$ can be written as

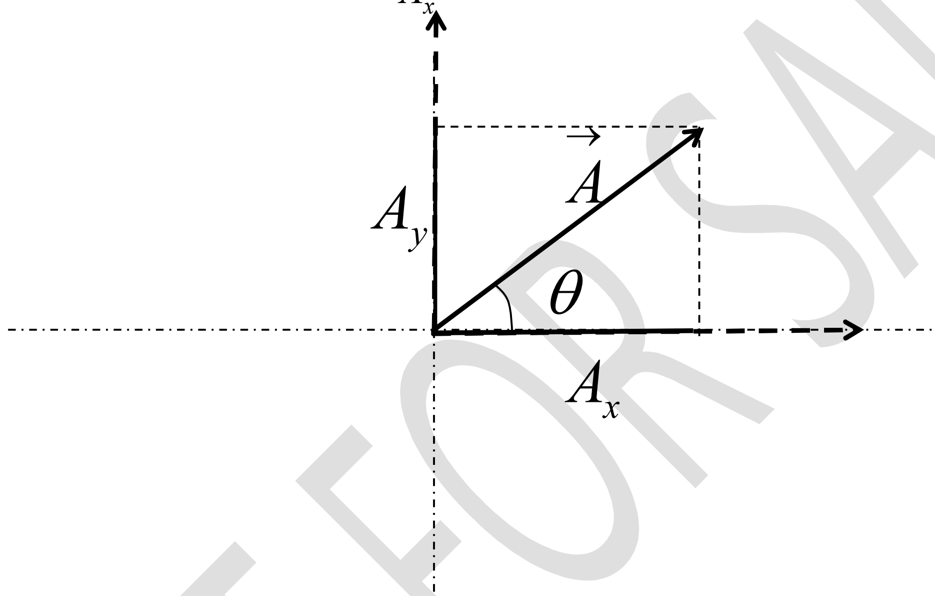
$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath}$$

where A_x is the x-component of the vector, A_y is the y-component of the vector while $\hat{i}$ and $\hat{j}$ are the unit vectors.

In terms of magnitude and angle / direction, vector $\vec{A}$ can be expressed as $|\vec{A}|$, θ.

Magnitude of vector $\vec{A}$

$$\left|\vec{A}\right| = \sqrt{A_x^2 + A_y^2}, \quad \theta = \tan^{-1}\frac{A_y}{A_x}$$



x- and y- components of a vector can be obtained from the magnitude and angle representation of the vector.

$$A_x = A\cos\theta, \quad A_y = A\sin\theta$$

Therefore, vector **A** can also be expressed as

$$\vec{A} = A\cos\theta\,\hat{i} + A\sin\theta\,\hat{j}$$

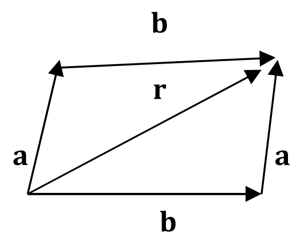
Addition of Vectors

The addition of two vectors can be written as

$$\vec{r} = \vec{a} + \vec{b}$$

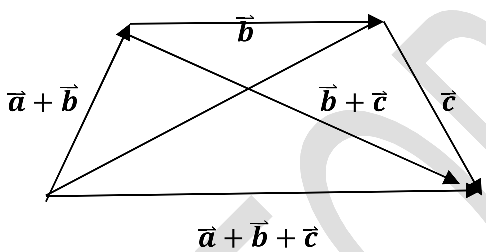
PROPERTIES OF VECTORS

Cumulative Law: $\vec{\boldsymbol{a}} + \vec{\boldsymbol{b}} = \vec{\boldsymbol{b}} + \vec{\boldsymbol{a}} = \vec{\boldsymbol{r}}$

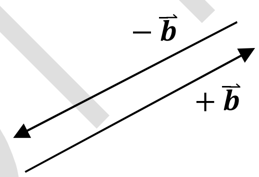


Association Law
$- \ i.e. independent\ of\ order\ of\ grouping, the\ sum\ of\ vectors\ is\ thesame$

$$\vec{\boldsymbol{a}} + (\vec{\boldsymbol{b}} + \vec{\boldsymbol{c}}) = (\vec{\boldsymbol{a}} + \vec{\boldsymbol{b}}) + \vec{\boldsymbol{c}}$$



<u>Subtraction</u> - For a vector $\vec{b}$ one can define another vector $-\vec{b}$ that is of the same magnitude with $\vec{b}$ but of opposite direction as shown below.



Then, $\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$

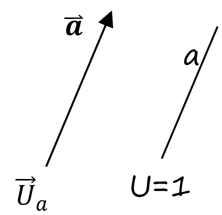
$Then\ \vec{a} - \vec{b} = \vec{a} + (-\vec{b})$

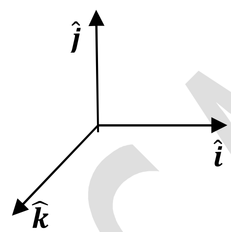
$Vector\ is\ horizontal\ and\ its\ magnitude\ 5\ units\ vector\ b\ is\ 45^0 from\ the\ horizontal$

Example: vector $\vec{a}$ is along the horizontal and its magnitude is 5 units, vector $\vec{b}$ makes angle 45° with the horizontal and its of magnitude 4 units while vector $\vec{c}$ makes angle 30° with the vertical axis and its magnitude is 3 units.

Compute $\vec{a} + \vec{b} - \vec{c}$.

<u>Unit Vector</u>: This is a vector of magnitude 1 in a particular direction. For example, in 1D, vector $\vec{a} = 2.5\hat{a} = 2.5\hat{i}$, where 2.5 is the magnitude and $\hat{a}$ or $\hat{i}$ is the unit vector.





<u>Sum of Two Vectors (</u>$\vec{a}$ *and* $\vec{b}$)

$\vec{r} = \vec{a} + \vec{b}$

Two vector $(\vec{a} + \vec{b})$***are equal if their corresponding components are equal***

and its resultant vector is $\vec{r}$**, i.e.,**

$$r_x = a_x + b_x\ , \quad r_y = a_y + b_y$$

$$r = \sqrt{r_x^2 + r_y^2},\ \ tan\theta = \frac{r_y}{r_x} \text{ (in magnitude and direction / angle)}$$

$$\vec{r} = r_x\hat{i} + r_y\hat{j} \text{ (in vector notation)}$$

<u>EXAMPLE</u> – Three coplanar vectors are expressed with respect to a certain rectangular coordinates system of a given reference frame as

$\vec{a} = 4\vec{i} - j$

$\vec{b} = -3i + 2\vec{j}$

$\vec{c} = -3\vec{j}$

The components are given in arbitrary units.

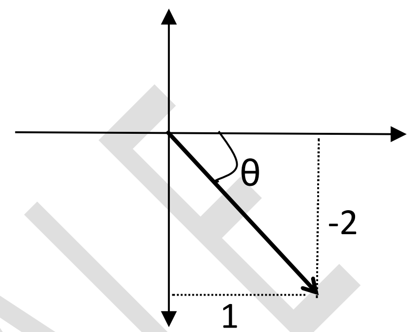
Find the vector r which is the sum of these vectors.

$$r_x = a_x + b_x + c_x = 4 - 3 + 0 = 1$$

$r_y = a_y + b_y + c_y = -1 + 2 - 3 = -2$

$\vec{r} = \vec{i}\, r_x + \vec{j}\, r_y$

$\vec{r} = \vec{i} - 2\vec{j}$ and can be represented as shown below



$Since\ magnitude\ r = \sqrt{r_x^2 + r_y^2}$

$= \sqrt{1 + 4}$

$= \sqrt{5} = 2.24$

The angle the vector makes with positive x-axis is $360° - \theta$.

$Angle\ made\ with\ positive\ x - axis\ measured\ counterclockwise$

$Tan^{-1}(-\frac{2}{1}) = 296.57°$

MULTIPLICATION OF VECTORS

One can not add vector and scalar in each task. One can add and multiply vectors with vectors. One can also multiple a vector by a scalar.

Three Kinds –

a) Multiplication of a vector by scalar
b) Multiplication of 2 vectors yielding scalar (dot product)
c) Multiplication of 2 vectors yielding vectors (cross product), etc.

VECTOR-SCALAR MULTIPLICATION

Scalar k, can be used to multiply vector $\vec{a}$ to have

Product $= k\,\vec{a}$

A new vector with magnitude $(k\,|\vec{a}|)$ will have the same direction as $\vec{a}$

$(1) - a\ new\ vector\ with\ magnitude\ k\ times\ as\ \vec{a}.\ Same\ direction\ as\ \vec{a}.$

$Check - if\ k\ is + ve$

To divide by scalar, multiply by reciprocal of k

$$\frac{1}{k}.\vec{a}$$

VECTOR-VECTOR MULTIPLICATION

I. Scalar (or dot .)
II. Vector (or cross ×)

SCALAR PRODUCTS

$\vec{a}.\vec{b} = ab\cos\theta$

$where\ a\ is\ the\ magnitude\ of\ \vec{a},\ \ b\ is\ the\ magnitude\ of\ \vec{b}\ \ and \cos\theta\ is\ the\ cosine\ of\ the$

$angle\ between\ the\ two\ vectors.$

$Thus,\ scalar\ product\ can\ be\ regarded\ as\ the\ product\ of\ the\ magnitude\ of\ one\ vector$

$and\ the\ component\ of\ the\ \ other\ in\ the\ direction\ of\ the\ first.$

For example, the dot product of vector $\vec{a}$ and vector $\vec{b}$, i.e., $c = \vec{a}.\vec{b}$ can be obtained if $\vec{a}$ and $\vec{b}$ are explicitly defined.

For $\vec{a} = a_x i + a_y j \quad and \quad \vec{b} = b_x i + b_y j$,

$$\vec{a}.\vec{b} = (a_x i + a_y j).(b_x i + b_y j) = a_x b_x + a_y b_y$$

Since, $i.i = j.j = 1,\ and \quad i.j = j.i = 0$

This implies that $a_x b_x + a_y b_y = a.b\cos\theta$

The equation above can be used to obtain the angle between two vectors.

Example 1:

Calculate the angle between two vectors $-3i + 4j \quad and \quad 2i + 3j$

Solution: Using $a_x b_x + a_y b_y = a.b\cos\theta$

$$-6+12=\sqrt{3^2+4^2}.\sqrt{2^2+3^2}\cos\theta$$

$$6=5\times\sqrt{13}\cos\theta;\ \theta=\cos^{-1}\frac{6}{18.0277}=\cos^{-1}0.3328$$

$$\theta=70.56°$$

VECTOR PRODUCT

$\vec{c}=\vec{a}\times\vec{b}$

Magnitude of vector $\vec{c}$ can be written as $c=ab\sin\theta$

where θ is the angle between $\vec{a}$ *and* $\vec{b}$

Example

Calculate the angle between two vectors $-3i+4j$ *and* $2i+3j$ using magnitude $c=ab\sin\theta$.

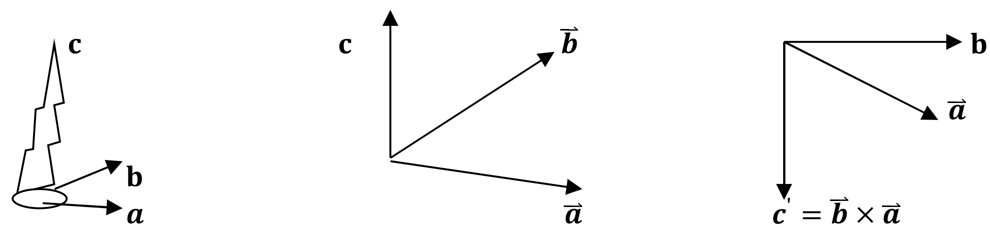
Solution

$$(-3i+4j)\times(2i+3j)=\begin{vmatrix} i & j & k \\ -3 & 4 & 0 \\ 2 & 3 & 0 \end{vmatrix}=i(0)-j(0)+k(-9-8)=-17k$$

Therefore using $c=ab\sin\theta$, note that magnitude of $-17k$ is 17.

$$17=18.0277\sin\theta,\ \theta=\sin^{-1}\frac{17}{18.0277}=\sin^{-1}0.9430=70.56$$

θ is the angle between $\vec{a}$ *and* $\vec{b}$. By definition, the direction of $\vec{c}$, the cross product of $\vec{a}$ *and* $\vec{b}$ is perpendicular to the plane formed by $\vec{a}$ *and* $\vec{b}$ or the plane that contains $\vec{a}$ *and* $\vec{b}$



$\vec{a} \times \vec{b}$ is pronounced as "$\vec{a}$ cross $\vec{b}$"

$\vec{b} \times \vec{a} \neq \vec{a} \times \vec{b}$

$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

For cross product,
$i \times j = k,\ j \times k = i,\ k \times i = j,\ i \times k = -j,\ j \times i = -k,\ k \times j = -i$

because magnitude of $absin\theta = basin\theta$ *but the directions are opposite*.

Examples of such cross products include – Torque, angular momentum, force of a moving charge in a magnet, flow of electromagnetic energy.

Vector Products

Tensor – generated by multiplying each of three components of one vector by the three components of another vector.

Tensor 2nd Rank – Has nine numbers associated with it.

Vector – Three numbers

Scalar – One

Example of Tensors – Mechanical and Electrical Stress, Moments, Products of Inertia, Strain.

PROPERTIES OF VECTORS

What happens to the laws of physics when simple operations such as translation and rotation of coordinates are performed?

Coordinate system x,y,z

Vectors $\vec{a}, \vec{b}, \vec{r}$

$Relationship\ between\ them$

$\vec{r} = \vec{a} + \vec{b}$

$By\ earlier\ definition\ of\ sum$

$r_x = a_x + b_x$

$r_y = a_y + b_y$

$r_z = a_z + b_z$

Consider a New coordinate system x,y,z with properties

I. Origin does not coincide with the origin of the first coordinate system x,y,z – Translation
II. Its three axes are not parallel to the corresponding axes in the first system – Rotation

Representation of vectors x,y,r in the new system would in general prove to be different. Let's put them in primes. The relationship between them however would be:

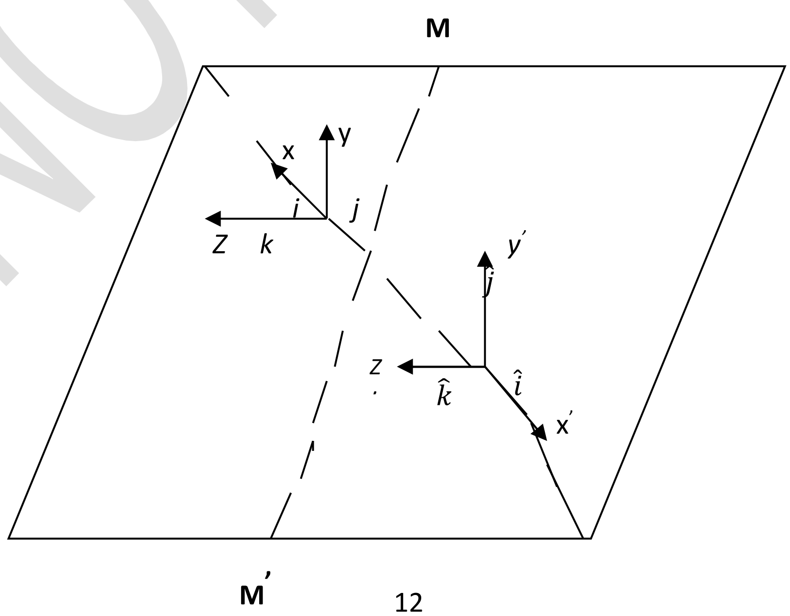
$r_x{}' = a_x{}' + b_z{}'$

$r_y{}' = a_y{}' + \ b_y{}'$

$r_z{}' = \ a_z{}' + b_z{}'$

And, the relationship $\vec{r} = \vec{a} + \vec{b}\ still\ holds$

Consequently, it may be said that relations among vectors are invariant (unchanged) with respect to translation or rotation of coordinates. I.e. the laws of physics are unchanged when we rotate or translate the reference system.



(a) is left handed (b) is right handed

The other is a mirror image of one

$in \quad (a)\ i \times j = -k$

$(b)\ i \times j = k$

VIOLATION (1956)

Decay of some elementary particles showed that the result was independent of the handedness whether left or right.

i.e., the experiment and its mirror image would yield different results.

This leads to question on the symmetry of physical laws.

PHY 115: Mechanics and Properties of Matter	Hamattan Semester 2014
Lecture 6: November-December, 2014	
Lecturer: Ibrahim T. T. Department: Physics	Institution: Unilorin
Modified: O.A. Oladipo (Dec. 2016)	

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6.1 Particle Kinematics

Particles may be described as point-like objects with mass but no size i.e. they exhibit negligible size and internal structure. Kinematics is the mathematical description of motion of an object without referring to the cause of the motion. Here we shall be concerned with the study of one-dimensional straight line motion of a particle and its generalization to two- and three-dimensional motion.

6.1.1 One-Dimensional Rectilinear Motion

A rectilinear motion is the motion of a particle along a straight line. It may, for example, be the motion of a ball rolling across the floor, the motion of a car on a straight road, the motion of a ball tossed straight up or that of an object dropped from a height.

Consider a particle moving from an initial position X_i to a final position X_f along a straight line, the change in position $\Delta X = X_f - X_i$, is equal to the DISTANCE covered by the particle. If the direction of motion is specified along with the magnitude ΔX, then the change in position is a vector quantity referred to as the DISPLACEMENT. If the corresponding initial time t_i at the start of the motion and the final time t_f are noted, then the time interval for the duration of the motion is given by $\Delta t = t_f - t_i$. The ratio $\Delta X / \Delta t$ therefore defines the SPEED of the particle. If the direction is specified then the ratio is referred to as the VELOCITY. In summary, Speed is the rate of change of distance covered with time, and Velocity is the rate of change of displacement with time.

When the rate of change of distance covered (or displacement) with time is constant the Speed (Velocity) is said to be UNIFORM. This means that the particle travels equal distance in equal time interval.

Similarly the rate of change of the velocity of a particle, say from an initial value U to a final value V over a time interval Δt, defines the ACCELERATION of the particle i.e. $\sim a = \Delta V / \Delta t$.

6.1.1.1 Average Speed and Instantaneous Speed

Average speed is defined as the ratio of the total distance covered with time. Instantaneous Speed on the other hand is the speed of the particle at an instant or a particular time t.

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6.1.1.2 1-D Motion plus Uniform Acceleration

Uniform acceleration is therefore a constant rate of change of velocity with time. Positive acceleration is associated with increasing velocity, while negative acceleration or DECELERATION refers to a decreasing velocity.

Consider a uniformly accelerated particle whose velocity changes from an initial velocity U to a final velocity V over a time interval t, the acceleration of the particle is given by

$$a = \frac{V-U}{t} \qquad (6.1)$$

For the motion, the area on the velocity-time $(V-t)$ corresponds to the total distance covered by the particle and the slope or the gradient gives the acceleration a. For a uniformly accelerated body, the $(V-t)$ graph is a straight line graph whose area is given by the area of the trapezium formed by the line and the two axes. Thus the total distance is given by either,

$$S = \int V dt \qquad (6.2)$$

$$= Ut + \frac{1}{2}at^2 \qquad (6.3)$$

or

$$S = \int V dt \qquad (6.4)$$

$$= \frac{(U+V)t}{2} \qquad (6.6)$$

A quick look at equations 6.1, 6.3 and 6.5 show that each contains four of the five parameters a, V, U, S, and t, describing the particle motion. Eliminating the time t from equation 6.3 or 6.5 using equation 6.1 gives the fourth equation

$$V^2 = U^2 + 2aS. \qquad (6.6)$$

The four equations are together referred to as the Newtons's equations of motion.

Note, objects falling freely on the earth's surface experiences the force of attraction towards the earth's surface. The associated acceleration is a constant called the acceleration due to gravity = $9.8\, m/s^2$. The values of g is a function of the radius of the earth but is independent on the mass m of the falling body. For example, the values of g at the pole is greater than that at the equator.

Examples

Q1.) A car starts from rest and travels 0.25 Km in 25 s at constant acceleration.
a.)Calculate the acceleration of the car, b.) Determine the final speed of the car.
Soln:
Rest implies the initial velocity U = 0 Distance covered S = 0.25 Km = 250 m, time interval t = 25 s.
Require to calculate the velocity V =?, and the acceleration a =?

$S = Ut + 1/2at^2$, Use an equation that does not contain the unknown V (6.7)

$$a = 2(S - Ut)/t^2 \qquad (6.8)$$

$$a = 2 * (250 - 0)/625 = 0.8 m/s^2 \qquad (6.9)$$

$$S = (U + V)t/2, \qquad (6.10)$$

$$V = 2S/t - U = 2 * 250/25 - 0 = 20 m/s. \qquad (6.11)$$

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Q2.) A man sitting on the ground throws a ball vertically upward with an initial speed 30 m/s.
a.) How long does it take the ball to return to the ground?
b.)What is its velocity on striking the ground?
Soln:
U = 30 m/s, deceleration $a = g = -9.8$ m/s^2, t=? V=?
If the time required to travel up to the maximum height is t_1, at this height the particle is momentarily at rest and as such the final velocity V=0, then

$$V = U - gt_1 \tag{6.12}$$
$$t_1 = (V - U)/-g = (0 - 30)/-9.8 = 3.06\,s \tag{6.13}$$

therefore the actual time to return to the ground $t = 2 * t_1 = 6.12$ s
For final velocity on striking the ground V=?

$$V = U - gt_1 \tag{6.14}$$
$$V = 30 - 9.8 * 6.12 = -29.98\,m/s \tag{6.15}$$

The negative sign indicates the direction of motion.

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~~Free~~ Fall Acceleration

If an object is thrown up or down and we eliminate the effects of air on its flight, the object accelerates downward at a certain constant rate. That rate is called the free-fall acceleration, and its magnitude is represented by g. The acceleration is independent of the object's ~~characteristics~~ such as mass, density, or shape. i.e. g is the same ~~fall~~ for all objects.

The value of g varies slightly with latitude and with elevation. g is taken as 9.8m/s which is ~~the~~ its value at sea level in Earth's midlatitudes.

All the equations under constant acceleration also apply ~~to~~ free-fall near Earth's surface.

Note that the directions of motion are now along a vertical y axis instead of the x axis, with the +ve direction of y upward. The free-fall acceleration is -ve for downward motion on the y axis, ~~So the~~ it has the value -g in the eqns. (eqn 5).

eq: $y - y_0 = V_0 t \cancel{+} \frac{1}{2} g t^2$

Example 1

A man fell from the top of a building which is 48m above the sea level, assuming that the initial velocity was ~~is~~ zero ~~and the~~ ~~effect of air resistance is~~ neglecting the effect of the air on the man during the fall.

(a) How long did he fall to reach the sea level.

(b) What was his velocity as he reached the sea level.

2. A mass throws a ball ~~ball was thrown~~ up along a y axis with an initial speed of 12 m/s.

(a) How long does the ball take to reach its maximum height

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(b) What is the ball's maximum height above its release point?

3. Raindrops fall 1700m from a cloud to the ground (a)
(a) If they were not slowed by air resistance, how fast would the drops be moving when they struck the ground?
(b)

4. At a construction site a pipe wrench struck the ground with a speed of 24 m/s. (a) From what height was it inadvertently dropped? (b) How long ~~does it take~~ was it falling?

Kinematics in Two Dimensions (3)

~~x Component~~

To Illustrate how this.

Equations of Kinematics for Constant Acceleration In Two-Dimensional Motion

x-component	Variable	y-component
x	Displacement	y
a_x	Acceleration	a_y
v_x	final velocity	v_y
v_{ox}	Initial velocity	v_{oy}
t	Elapsed time	t

$v_x = v_{ox} + a_x t$ $\qquad$ $v_y = v_{oy} + a_y t$

$x = \frac{1}{2}(v_{ox} + v_x)t$ $\qquad$ $y = \frac{1}{2}(v_{oy} + v_y)t$

$x = v_{ox}t + \frac{1}{2}a_x t^2$ $\qquad$ $V = v_{oy}t + \frac{1}{2}a_y t^2$

$v_x^2 + v_{ox}^2 + 2a_x x$ $\qquad$ $v_y^2 = v_{oy}^2 + 2a_y y$

E.g (1) In the x direction, the spacecraft in the figure below has an initial velocity of $v_{ox} = +25 m/s$ and an acceleration of $a_x = +24 m/s^2$. In the y direction, the analogous quantities are $v_{oy} = +14 m/s$ and $a_y = +12 m/s$. After a time of 7.0s, find (a) x and v_x (b) y and v_y and (c) the final velocity (magnitude and direction) of the spacecraft.

(a) x-direction data

x	a_x	v_x	v_{0x}	t
?	+24m/s	?	+25m/s	7.0s

$$x = v_{0x}t + \frac{1}{2}a_x t^2$$
$$= (25)(7) + \frac{1}{2}(24)(7)^2$$
$$= +760\,m$$

$$v_x = v_{0x} + a_x t = 25 + (24)(7)$$
$$= +190\,m/s$$

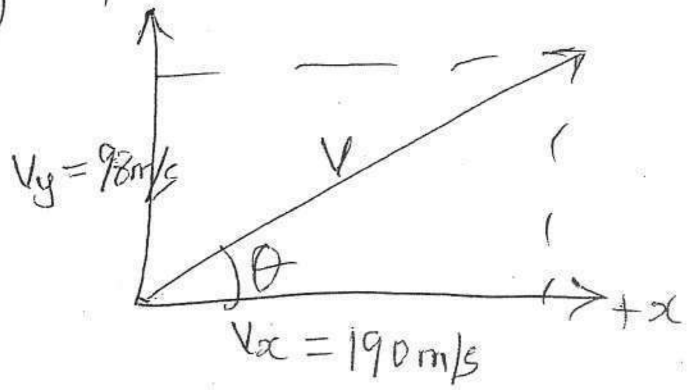
(b) y-direction data

y	a_y	v_y	v_{0y}	t
?	$+12m/s^2$	?	+14m/s	7.0s

$$y = v_{0y}t + \frac{1}{2}a_y t^2$$
$$= +390\,m$$

$$v_y = v_{0y} + a_y t$$
$$= +98\,m/s$$

(c)



$$v = \sqrt{v_x^2 + v_y^2}$$
$$= \sqrt{(190)^2 + (98)^2}$$
$$= 210\,m/s$$

direction of the velocity

$$\tan\theta = \frac{v_y}{v_x} = \frac{98\,m/s}{190\,m/s}$$
$$= 0.52$$
$$\theta = \tan^{-1}(0.52)$$
$$= 27^\circ$$

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Motion in two dimension – Projectile.

We are going to extend the motn treated under one dimensional space to two or three dimension. And many of the ideas used ~~under~~ previously, such as position, velocity, and acceleratn are used here also but in a little more complex because of the extra dimensions.

To locate a ptc, one needs to find its position vector $\vec{r}$, which is a vector that extends from a reference point (origin of a coordinate system) to the ptc. In a unit-vector notation $\vec{r}$ can be written as

$$\vec{r} = xi + yj + zk$$

where x, y, z are components and $i\ j\ k$ are unit vectors.

e.g $\vec{r} = \cancel{3i} (-3m)i + (2m)j + (5m)k$.

As a ptc moves, its position vector changes. If the positn vector changes, say, from $\vec{r_1}$ to $\vec{r_2}$ during a certain time interval, then the ptc's displacement $\Delta\vec{r}$ during the time interval is

$$\Delta\vec{r} = \vec{r_2} - \vec{r_1}$$

Using the unit vector notation,

$$\Delta\vec{r} = (x_2 i + y_2 j + z_2 k) - (x_1 i + y_1 j + z_1 k)$$

$$\Delta\vec{r} = (x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)k$$

also we can rewrite the displacement as

$$\Delta\vec{r} = \Delta x i + \Delta y j + \Delta z k$$

Example 1

The position vector for a ptc is initially

$r_1 = (-3.0m)i + (2.0m)j + (5.0m)k$ and the later is

$r_2 = (9.0m)i + (2.0m)j + (8.0m)k$.

What is the particle's displacement $\Delta\vec{r}$ from $\vec{r_1}$ to $\vec{r_2}$.

Example 2

A rabbit runs across a parking lot on which a set of coordinate axes has, strangely enough, been drawn. The coordinates of the rabbit's position as a fxs of time t are given by

$$x = 0.31t^2 + 7.2t + 28$$

and $y = 0.22t^2 \neq 9.1t + 30$.

with t in seconds and x and y in meters.

(a) At t = 15 s, what is the rabbit's position vector $\vec{r}$ in unit-vector notation and as a magnitude and an angle?

Also Average velocity and instantaneous velocity can be written as

$$\vec{V}_{av} = \frac{\text{displacement}}{\text{time interval}} = \frac{\Delta \vec{r}}{\Delta t}$$

$$\vec{V}_{av} = \frac{\Delta x i + \Delta y j + \Delta z k}{\Delta t} = \frac{\Delta x}{\Delta t} i + \frac{\Delta y}{\Delta t} j + \frac{\Delta z}{\Delta t} k$$

instantaneous velocity $\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(xi + yj + zk)$

$$\vec{v} = \frac{dx}{dt} i + \frac{dy}{dt} j + \frac{dz}{dt} k.$$

The eqn can be simplified as

$$\vec{V} = V_x i + V_y j + V_z k.$$

where the component of $\vec{v}$ are

$$V_x = \frac{dx}{dt} \quad V_y = \frac{dy}{dt} \quad V_z = \frac{dz}{dt}.$$

Example (3)

In example 2. find the velocity $\vec{V}$ at time t = 15 in unit vector notation and as magnitude and an angle

Average acceleration and instantaneous acceleration.

When pts's velocity changes from $\vec{V_1}$ to $\vec{V_2}$ in time interval Δt, ~~avera~~ its average acceleration $\vec{a}_{av}$ during Δt is

$$\vec{a}_{av} = \frac{\text{change in velocity}}{\text{time interval}}$$

$$= \frac{\vec{V_2} - \vec{V_1}}{\Delta t} = \frac{\Delta \vec{V}}{\Delta t}$$

Also, instantaneous acceleration $\vec{a}$ at that instant is

$$\vec{a} = \frac{d\vec{v}}{dt}$$

$$\vec{a} = \frac{d}{dt}(V_x i + V_y j + V_z k)$$

$$= \frac{dV_x}{dt} i + \frac{dV_y}{dt} j + \frac{dV_z}{dt} k$$

i.e $\vec{a} = a_x i + a_y j + a_z k$

The components of $\vec{a}$ are

$$a_x = \frac{dv_x}{dt}, \quad a_y = \frac{dV_y}{dt}, \text{ and } a_z = \frac{dV_z}{dt}$$

Example (4)

In example 2, find the acceleration $\vec{a}$ at time $t = 15s$ unit-vector notation and as a magnitude and an angle.

Example (5): A particle with velocity $\vec{V_0} = -2.0i + 4.0j$ (in meters/s) at $t = 0$ undergoes a constant acceleration $\vec{a}$ of magnitude $a = 3.0 m/s^2$ at an angle $\theta = 130°$ from the positive direction of the x axis. What is the particle's velocity at $t = 5.0s$

Because a is constant, $V = V_0 + at$

$V_x = V_{0x} + a_x t$ $\qquad$ $V_y = V_{0y} + a_y t$

from $\vec{V_0} = -2.0i + 4.0j$, $V_{0x} = -2.0 m/s$ and $V_{0y} = 4.0 m/s$

$a_x = a\cos\theta = (3.0)(\cos 130) = -1.93 m/s^2$

$a_y = a\sin\theta = (3.0)(\sin 130) = +2.30 m/s^2$

$V_x = -2.0 + (-19.3)(5.0) = -11.65 m/s$, $V_y = 15.50 m/s$, $\quad V = \sqrt{12^2 + 16^2} = 19 m/s.$

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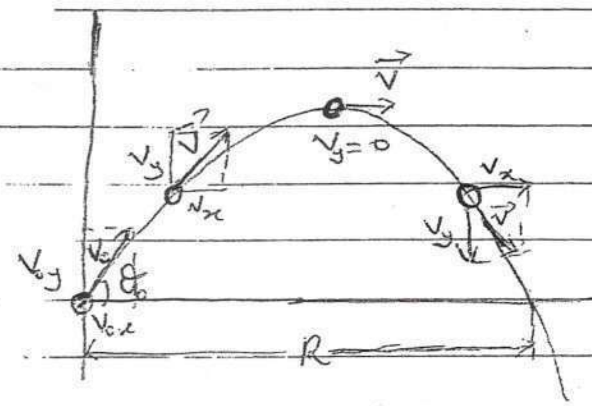
Projectile

Now, we consider a special case of two-dimensional motion: let A ptc moving in a vertical plane with some initial velocity $\vec{V_0}$ but its acceleration is always the free-fall acceleration $\vec{g}$, which is downward. Such a ptc is called a projectile (meaning that it is projected or launched) and its motion is called projectile motion.



Consider a projectile launched with an initial velocity $\vec{V_0}$, $\vec{V_0}$ can be written as

$$\vec{V_0} = V_{0x}\hat{i} + V_{0y}\hat{j}$$

And the component V_{0x} and V_{0y} can be found if the angle θ_0 btw $\vec{V_0}$ and the positive x direction is known.

$$V_{0x} = V_0 \cos\theta_0 \quad \& \quad V_{0y} = V_0 \sin\theta_0$$

During the motion, the projectile's position vector $\vec{r}$ and velocity vector $\vec{V}$ change continuously, but its acceleration vector $\vec{a}$ is constant and always directed vertically downward. The projectile has no horizontal acceleration.

Note

In Projectile motion, the horizontal motion and the vertical motion are independent of each other, i.e. neither motion affects the other.

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Total time of flight

The time taken for projectile to return to its initial level. ~~At~~ At this level $y = y_0$ eqn (2) becomes

$$0 = V_0 \sin\theta_0 t - \frac{1}{2}gt^2$$

$$\frac{1}{2}gt^2 = V_0 \sin\theta_0 t$$

$$t = \frac{2V_0 \sin\theta_0}{g}$$

Maximum height

Time taken to reach maximum height H or y_{max} is T

$$T = \frac{1}{2}t = \frac{V_0 \sin\theta_0}{g}$$

From eqn (2).

$$y_{max} - y_0 = V_0 \sin\theta_0 T - \frac{1}{2}gT^2$$

replace T by $V_0 \sin\theta_0 / g$ and y_0 with zero.

$$y_{max} = V_0 \sin\theta_0 \cdot \frac{V_0 \sin\theta_0}{g} - \frac{1}{2}g\left(\frac{V_0 \sin\theta_0}{g}\right)^2$$

$$= \frac{V_0^2 \sin^2\theta_0}{g} - \frac{1}{2}\frac{V_0^2 \sin^2\theta_0}{g}$$

$$= \frac{1}{2}\frac{V_0^2 \sin^2\theta_0}{g}$$

Horizontal Range R.

To find the range R multiple the total time of flight t with the horizontal component of velocity.

$$R = V_{0x} t = V_0 \cos\theta_0 t$$

but $$t = \frac{2V_0 \sin\theta_0}{g}$$

$$R = \cancel{2} V_0 \cos\theta_0 \cdot \frac{2 V_0 \sin\theta_0}{g}$$

$$= \frac{2 V_0^2 \sin\theta_0 \cos\theta_0}{g}$$

but $\sin\theta_0 \cos\theta_0 = \frac{1}{2}\sin 2\theta_0$

$$R = \frac{V_0^2}{g}\sin 2\theta_0$$

Note.

This ~~distance~~ eqn does not give the horizontal distance travelled by a projectile when the final height is not the launch height. R has its maximum value when $\sin 2\theta_0 = 1$, $\theta_0 = 45°$

Q. A rifle is aimed horizontally at a ~~given~~ target 30m away. The bullet hits the target 1.9cm below the aiming point. What are (a) the bullet's time of flight and (b) its speed as it emerges from the rifle?

Q. A small ball rolls horizontally off the edge of a tabletop that is 1.20m high. It strikes the floor at a point 1.52m horizontally away from the edge of the table. (a) How long is the ball in the air? (b) What is its speed at the instant it leaves the table?

PHY 115: Mechanics and Properties of Matter I

(Newton's laws of Motion, impulse and Momentum)

Newton's Laws of Motion

Newton's laws deal with force and motion / acceleration.

Newton's First law of Motion

The law states that every object will continue in its state of rest, or of uniform motion in a straight line, unless an external force acts on it. But it is possible to apply a force to an object (i.e. an external force acting on an object) without producing motion. A good example of this is when a car is to be pushed to start the engine by gradually increasing the applied force. It will get to a particular value of the applied force that the car will start to move, that is any values of the applied force below this value is not strong enough to cause the car to move. Therefore it can be said that though an external force is applied, there is no movement or change of state of the car.

$$a = 0\text{, then } F_{applied} = F_{friction}$$

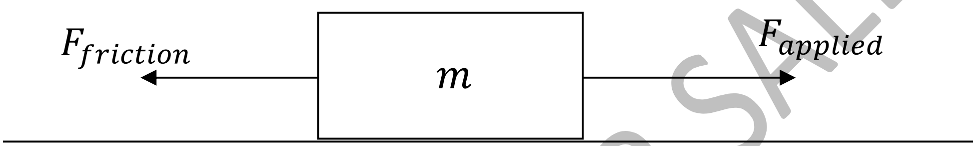


Fig. 1: Applied force is equal to the frictional force between the box of mass m and the surface, only when acceleration $a = 0$.

Our basic assumption here, for now, is that there is no friction between the surfaces (i.e. the object and the horizontal plane) in contact. The explanation above is to introduce a new concept called Net Force, F_{Net}, to replace the external force $F_{applied}$. Therefore we can state the Newton's first law of motion as follow:

Every object will continue in its state of rest, or of uniform motion in a straight line, unless a net force acts on it. This tendency of an object to resist change of state of motion is called as inertia.

Inertia: the resistance an object has to a change in its state of motion.

If $\boldsymbol{F_{Net} = 0}$, then $\boldsymbol{a = 0}$ statement of the 1st law.

Consider an object of mass m under the influence of three forces as shown in Fig, 2 below, the net force F_{Net} is the vector sum of all the three forces.

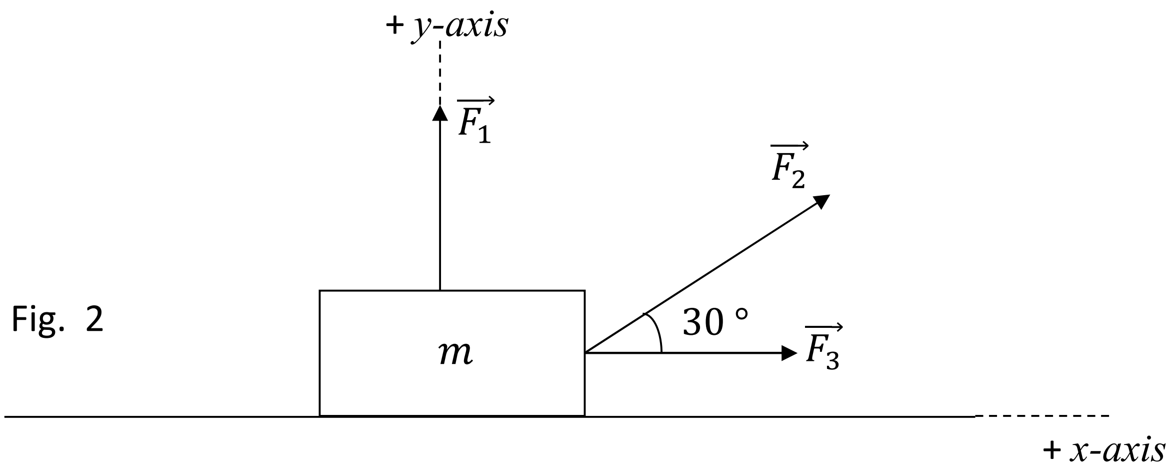


Fig. 2

$$\vec{F}_{Net} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3$$

Remember that a vector, $\vec{V}$, can expressed as $\vec{V} = V cos\theta\ i + V sin\theta\ j = V_x\ i + V_y\ j$

where V= magnitude of the vector and θ = angle the direction of the vector makes with the positive x-axis.

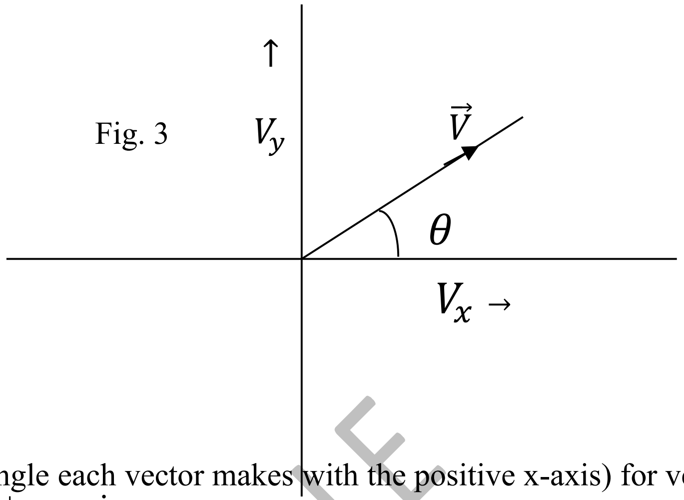
Y-component of vector $\vec{V}$,

$V_y = V sin\theta$

X-component of vector $\vec{V}$,

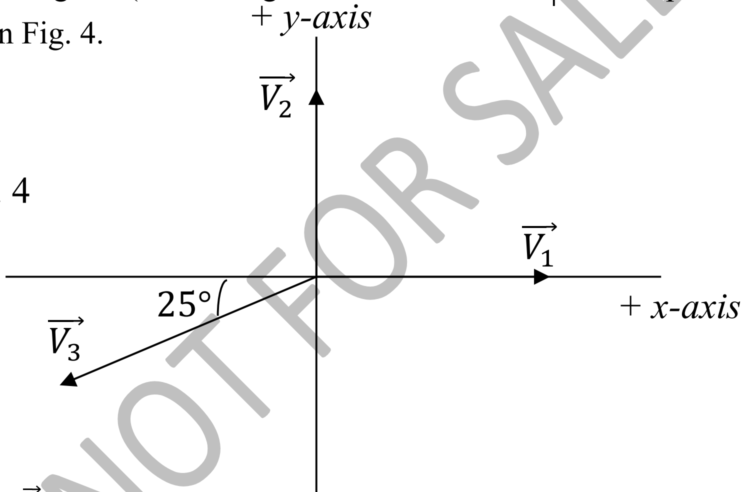
$V_x = V cos\theta$

Fig. 3



Example 1: Determine angle θ (i.e. the angle each vector makes with the positive x-axis) for vectors $\vec{V}_1, \vec{V}_2$ and $\vec{V}_3$ shown in Fig. 4.

Fig. 4



Solution:

The direction of vector $\vec{V}_1$, as indicated the diagram, is along the positive x-axis

Therefore, the angle $\vec{V}_1$ makes with the positive *x-axis* $\theta_1 = 0\ °$.

The direction of vector $\vec{V}_2$, as indicated in the diagram, is along the positive *y-axis*, and the angle between the positive *y-axis* and positive *x-axis* is 90 °.

Therefore, the angle $\vec{V}_2$ makes with the positive *x-axis* $\theta_2 = 90\ °$.

For vector $\vec{V}_3$, the angle θ_3 that $\vec{V}_3$ makes with positive *x-axis* is $180\ ° + 25\ °$,

therefore $\theta_3 = 205\ °$.

Example 2: Using fig. 4, if the magnitude of vectors $\vec{V}_1, \vec{V}_2$ and $\vec{V}_3$ is 10 N, 15 N and 20 N respectively, calculate the *y*- and *x*-components of each of the vectors.

Solution:

Vector $\vec{V}_1$, magnitude of the vector $V_1 = 10\ N\ and\ \theta_1 = 0\ °$.

x-component $V_{1,x} = 10\cos 0° = 10\,N$, and

y-component $V_{1,y} = 10\sin 0° = 0\,N$

For $\overrightarrow{V_2}$, magnitude V_2 = 15N, $\theta_2 = 90^0$

$V_{2,x}$ = 15 Cos90^0 = 0N x – component

$V_{2,y}$ = 15 Sin90^0 = 15N y - component

For $\overrightarrow{V_3}$, V_3 = 20N, $\theta_3 = 205^0$

$V_{3,x}$ = 20 Cos205^0 = 20 x (-0.9063) = - 18.13N x – component

$V_{3,y}$ = 20 Sin205^0 = 20 x (-0.4226) = - 8.45N y – component

Note: If the direction of a vector is along *x-axis* or *y-axis*, the vector will have only one component e.g. $\overrightarrow{V_2}$ is along *y-axis*, $V_{2,y}$ = 15N while $V_{2,x}$ = 0N.

Now to $\overrightarrow{F_{Net}}$ in Fig. 2, we can write each of the forces in a vector equation and the net force will be the vector sum of all the three forces.

$$\overrightarrow{F_{Net}} = F_1 Cos90^0 i + F_1 Sin90^0 j$$
$$+ F_2 Cos30^0 i + F_2 Sin30^0 j$$
$$+ F_3 Cos0^0 i + F_3 Sin0^0 j$$

$$\overrightarrow{F_{Net}} = (0 + F_2 Cos30^0 + F_3)i + (F_1 + F_2 Sin30^0 + 0)j$$

NEWTON'S SECOND LAW OF MOTION

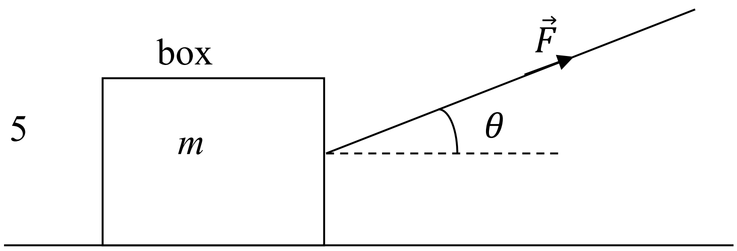
The net force $\overrightarrow{F_{Net}}$ on a body is equal to the product of the body's mass (m) and its acceleration.

$$\overrightarrow{F_{Net}} = m\vec{a}$$

$\overrightarrow{F_{Net}}$ can also be written in form of its components; $F_{net,x} = ma_x$ & $F_{net,y} = ma_y$

The implication of these two equations above is that x-component of the resultant or net force is responsible for acceleration along *x-axis* while y-component is responsible for acceleration along *y-axis*. A good illustration of this is shown in Fig. 5 below.

Fig. 5



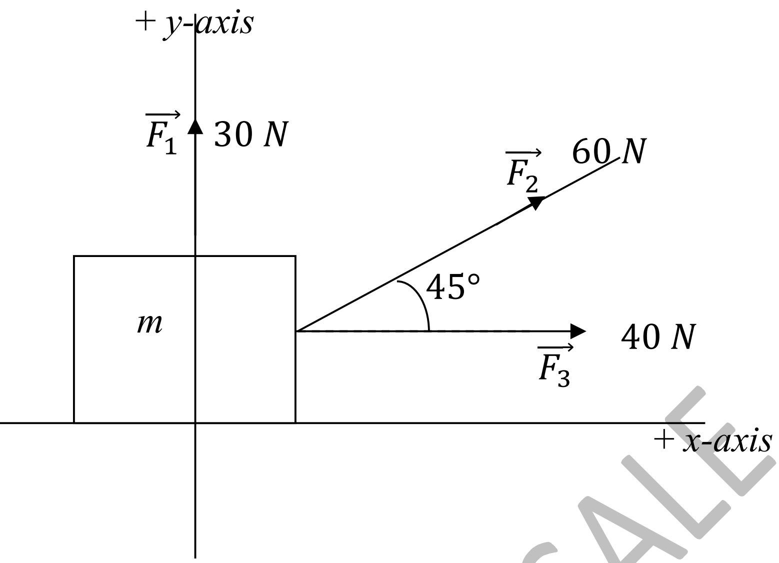
If the value of the applied or net force, in this case F, is gradually increased from zero upward, it will get to a particular value of $\vec{F}$ at which the box will be lifted (or just about to be lifted) off the surface. At this value of $\vec{F}$, in the direction shown, the y-component (i.e. F_y) is equal to the weight of the box. To keep the box moving along the horizontal surface, the applied force needs to be reduced.

Therefore, from this illustration, the x-component of the net-force ($F_{net,x}$) will drag the box along the horizontal surface while the y-component ($F_{net,y}$) will try to lift the box off the surface.

We can also state the Newton's second law of motion as: "The effect of a net force is to change the state of motion of the object on which it acts".

Example 3: Find the magnitude and direction of the net force acting on the object shown in Fig. 6 below.

Fig. 6



Solution:

$\overrightarrow{F_1}$, $F_1 = 30N$, $\theta_1 = 90^0$

$\overrightarrow{F_2}$, $F_2 = 60N$, $\theta_2 = 45^0$

$\overrightarrow{F_3}$, $F_3 = 40N$, $\theta_3 = 0^0$

The net force $\overrightarrow{F_{Net}} = \overrightarrow{F_1} + \overrightarrow{F_2} + \overrightarrow{F_3}$

$= 30Cos90^0 i + 30Sin90^0 j$

$+ 60Cos45^0 i + 60Sin45^0 j$

$+ 40Cos0^0 i + 40Sin0^0 j$

$= (0 + 42.4 + 40.0)i + (30 + 42.4 + 0)j$

$\Longrightarrow$

$\overrightarrow{F_{Net}} = 0i + 30j$

$+42.4i + 42.42j$

$+40.0i + 0j$

$\overrightarrow{F_{Net}} = 82.42i + 72.42j$

The magnitude of $\overrightarrow{F_{Net}}$, $F_{Net} = \sqrt{82.42^2 + 72.42^2} = 109.72\ N$

Angle θ that $\overrightarrow{F_{Net}}$ makes with positive x-$axis$,

$$\theta = \tan^{-1}\left(\frac{72.42}{82.42}\right) =$$

The magnitude $F_{net} = 109.72N\ and\ \theta$ which indicates direction =

Example 4: If the mass of the object in Fig. 6 is 15Kg, find the acceleration (in magnitude and direction).

Solution:

$\overrightarrow{F_{Net}} = m\vec{a}$, From example 3, $\overrightarrow{F_{Net}} = 82.42i + 72.42j$

Therefore, $\vec{a} = \frac{F_{Net}}{m} = \frac{82.42i+7\ \ .42j}{15}$

$$\vec{a} = 5.495i + 4.828j$$

Magnitude of $\vec{a}$, $a = \sqrt{5.495^2 + 4.828^2} = 7.31 m/s^2$

The angle θ, $\vec{a}$ makes with positive x-axis, $\theta = \tan^{-1}\frac{4.828}{5.495}$

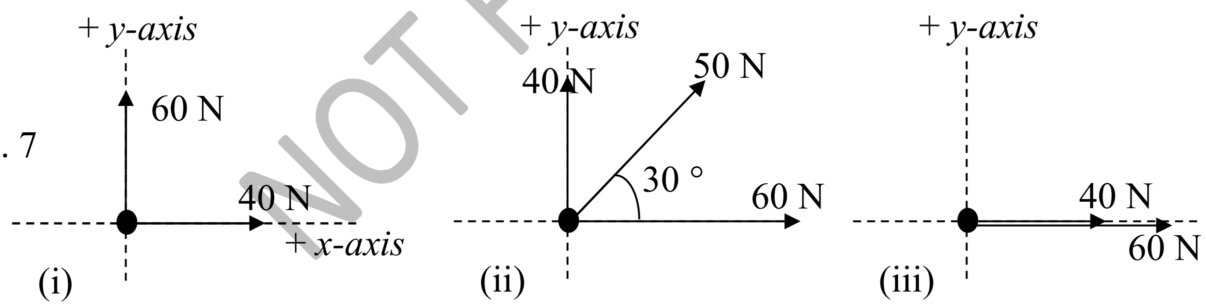
The magnitude of acceleration $a = 7.31\ m/s^2$, and $\theta =$

Alternative Solution: From $\overrightarrow{F_{Net}} = m\vec{a}$, the direction of the net force $\overrightarrow{F_{Net}}$ is the direction of the resulting acceleration. For the magnitude of acceleration, one can divide the magnitude of net force F_{net} by the mass i.e. $a = \frac{F_{net}}{m} = \frac{109.72}{15} = 7.31 m/s^2$

Problems

1. A 1580Kg car is travelling at a speed of 15.0m/s. What is the magnitude of the horizontal net force that is required to bring it to a halt in a distance of 50.0m? Ans = 355N.
2. A 900Kg car travelling at 25m/s brakes hard and comes to rest in 5s. What is the average breaking force? (Ignore drag) Ans = 4500N.
3. A net force acts on mass m_1 and creates an acceleration. A mass m_2 is added to mass m_1. The same net force acting on the two masses together creates one-third the acceleration. Determine the ratio $\frac{m_2}{m_1}$. Ans = 2:1 or 2.
4. An empty plane whose mass is 30, 400Kg has a maximum takeoff acceleration of 1.20m/s^2. What is its maximum acceleration when it is carrying a load of 8200Kg? Ignore friction. Ans = 0.945m/s^2.
5. Find the magnitude and direction of the net force acting on each of the three objects shown in Fig. 7 below.

Fig. 7



Ans: 72.11 N, 56.31 ^{0}, Ans: 122.05 N, 32.18 ^{0}, Ans: 100 N, 0 ^{0}

6. If the masses of the objects in Fig. 7 above are 7.5Kg, find their accelerations (both magnitude and direction). Ans: 9.61 m/s^2, 56.31 ^{0}; 16.27 m/s^2, 32.18 ^{0}; 13.33 m/s^2, 0 ^{0}

Example 4

In fig. 8 below, If M = 3.3Kg and m=2.1K and assuming that the cord and pulley have negligible mass, find

a. The acceleration of the sliding block
b. The acceleration of the hanging block
c. The tension in the cord

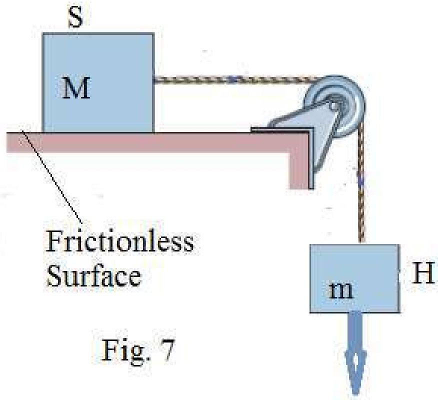


Fig. 7

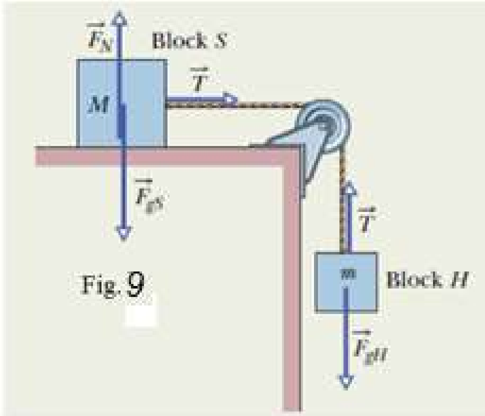
Solution:

The forces and their direction acting on block S and block H are shown. Since block S is on a frictionless surface, the net force on it is T. The forces on block H are T and F_g (Please, take note of the direction of T and F_g).

Applying second law of motion, $\overrightarrow{F_{Net}} = m\vec{a}$

On block S, T = ma (1)

On block H, mg – T = ma (2)



Using eqn (1), T = ma, replace T in (2) with Ma

Mg – Ma = ma

Mg = Ma + ma = (M + m)a

a = $\frac{M}{M+m} g = \frac{2.1}{3.3+2.1} x10 = 3.89 m/s^2$

Since the cord is inextensible, bloth blocks (S and H) will accelerate at a= 3.89m/s^2.

(c) Using equation1, T = 3.3 x 3.90 = 12.83N

SECOND LAW OF MOTION (MOMENTUM)

Force is equal to the rate of change of momentum,

$F = \frac{dp}{t}$ = force acting on a body in motion is equal to the rate of change of momentum of the body.

Where dp is change in momentum and t, the time for the change. Suppose an external force F acts on an object of mass m moving with initial velocity v_1 to change its velocity to v_2 at time interval t.

Then F = $F = \frac{dp}{t} = \frac{mv_2 - mv_1}{t} = \frac{m(v_2 - v_1)}{t}$

But $F = ma = \frac{m(v_2 - v_1)}{t}$,

$a = \frac{(v_2 - v_1)}{t}$ - definition of acceleration.

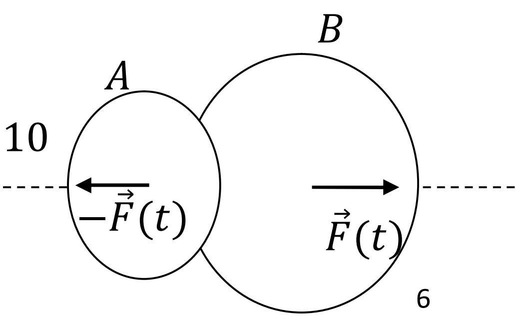
This is another statement of Newton's second law of motion

COLLISION

Collision is an isolated event in which two or more bodies exert relatively strong forces on each other for a relatively short time.

IMPULSE AND LINEAR MOMENTUM

$Fig. 10$



The diagram in Fig. 10 above shows the collision between two objects A and B of different masses. If the collision is head-on, the direction of their final velocity will be along x-axis. The two forces $\overrightarrow{F(t)}$ and $-\overrightarrow{F(t)}$ will change the linear momentum of both bodies. The amount of change dp is

$d\vec{p} = \overrightarrow{F(t)}dt$ according to Newton's second law of motion

Integrating over the interval Δt – from initial time t_i to final time t_f

$$\int_{t_i}^{t_f} d\vec{p} = \overrightarrow{F(t)}dt$$

$\int_{t_i}^{t_f} d\vec{p} = \vec{p}_f - \vec{p}_i$, the change in linear momentum of the body.

$\int_{t_i}^{t_f} \overrightarrow{F(t)}dt = \vec{J}$ i.e. the impulse

For each body, change in the linear momentum is equal to the impulse that acts on the body.

MOMENTUM AND KINETIC ENERGY IN COLLISIONS

Elastic Collision: Both momentum and Kinetic energy are conserved i.e. change in momentum dp = constant and $k_i = k_f$.

Inelastic Collision: Momentum is conserved by Kinetic energy decreases.

i.e. dp = constant but $k_i \neq k_f$.

LAW OF CONSERVATION OF LINEAR MOMENTUM

In a closed, isolated system containing a collision, the linear momentum of each colliding body may change but the total linear momentum $\vec{p}$ of the system cannot change, whether the collision is elastic or inelastic.

Collision in one dimension: For objects moving along x-axis or y-axis, velocity has a single component. Under **in-elastic collision**, the two bodies join together after collision and move with common velocity v. Therefore, we can write;

$$\mathrm{m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2} \qquad \text{i.e. } \mathrm{dp = constant}$$

$$\frac{1}{2}\mathrm{m_1u_1}^2 + \frac{1}{2}\mathrm{m_2u_2}^2 = \frac{1}{2}\mathrm{m_1v_1}^2 + \frac{1}{2}\mathrm{m_2v_2}^2 \qquad \mathrm{k_i = k_f}$$

Elastic collision in one dimension:

The two bodies of masses m_1, m_2 and moving with initial velocities u_1, u_2 respectively, if after collision the velocities change to V_1, V_2 respectively, then we can write

$$m_1u_1 + m_2u_2 = m_1V_1 + m_2V_2 \quad \text{i.e. } dP = Constant$$

$$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1V_1^2 + \frac{1}{2}m_2V_2^2 \quad \text{i.e. } k_i = k_f$$

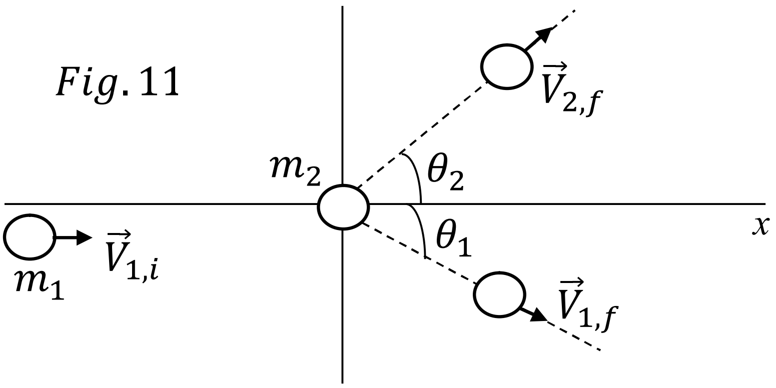
Collision in Two Dimensions

When collision between two bodies is not head-on, the direction of their final velocities is not along their initial axis. For this type of collision, linear momentum is still conserved.

$$\vec{p}_{1i} + \vec{p}_{2i} = \vec{p}_{1f} + \vec{p}_{2f}$$

If collision is elastic, total kinetic energy is conserved.

$$k_{1i} + k_{2i} = k_{1f} + k_{2f}$$



From Fig. 11 above, after collision, the impulse between the bodies sends the bodies off at angles θ_1 and θ_2 to the *x axis*.
Momentum is a vector quantity, so the components along x-axis is

$$m_1V_{1i} = m_1V_{1f}\cos\theta_1 + m_2V_{2f}\cos\theta_2$$

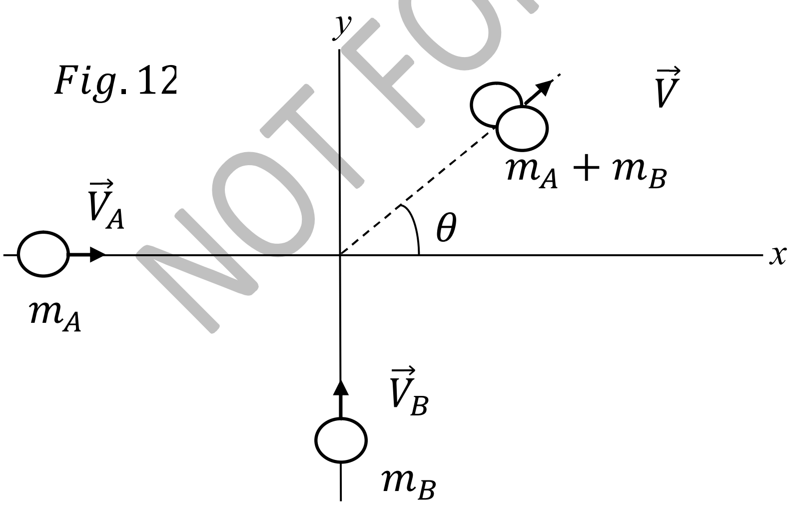
And along y-axis is

$$0 = -m_1V_{1f}\mathrm{Sin}\theta_1 + m_2V_{2f}\mathrm{Sin}\theta_2$$

Also, kinetic energy is conserved.

$$\frac{1}{2}m_1{V_{1i}}^2 = \frac{1}{2}m_1{V_{1f}}^2 + \frac{1}{2}m_2{V_{2f}}^2$$

Example



Two objects A and B collide and embrace, in a completely inelastic collision. Thus, they stick together after impact as shown in the diagram above, where the origin is placed at the point of collision. A is of mass 83 Kg originally moving east with speed 6.2 Km/h and B is of mass 55 Kg and is originally moving north with speed 7.8 Km/h. What is the common velocity $\vec{v}$.

Questions

1. A 1600Kg van moving at $20ms^{-1}$ crashes into the back of a small car of mass 700Kg traveling in the same direction at $15ms^{-1}$.
   a) What is the velocity of the two vehicles immediately after the collision of the lock together?
   b) How much kinetic energy is transferred to other forms in the collision?

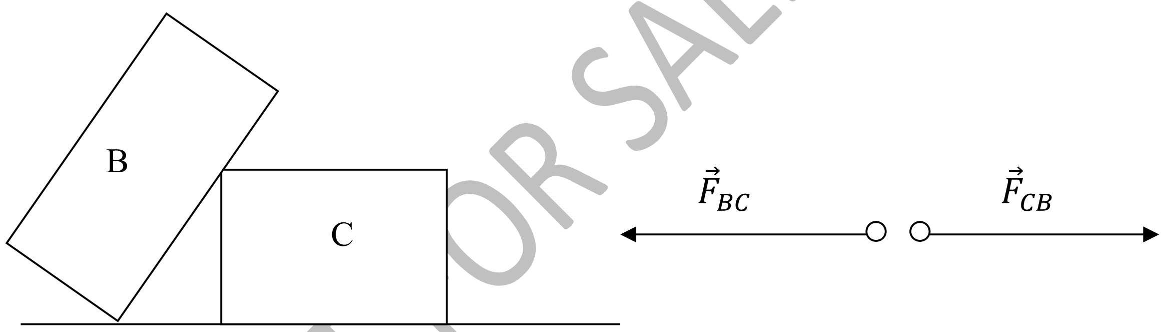
Ans: (a) 18.478 m/s (b) 6098.04 J

2. In an experiment using a linear air track a, 0.5Kg rider travelling to the right at $2ms^{-1}$ collides with a stationary rider of mass 1Kg. Find the velocities of the two riders after the collision if:
   a. It is perfectly elastic Ans: V1 = -0.667 m/s V2 = 1.33 m/s
   b. It is totally inelastic. Ans: 0.667 m/s
3. The National Transportation Safety Board is testing the crash-worthiness of a new car. The 2300Kg vehicle moving at 15 m/s is allowed to collide with a bridge abutment, which stops it in 0.56 s. What is the magnitude of the average force that acts on the car during the impact? Ans: 61607.1 N
4. A 150 g baseball pitched at a speed of 40 m/s is hit straight back to the pitcher at a speed of 60 m/s. What is the magnitude of the average force on the ball if the bat is in contact with the ball for 5.0 m/s? Ans: 3,000 N
5. Two 2.0Kg bodies, A and B collide. The velocities before the collision are $\overrightarrow{V_A} = 15i + 30j$ and $\overrightarrow{V_B} = -10i + 5.0j$. After the collision, $\overrightarrow{V_A} = -5.0j + 20j$. All speeds are given in meters per second. (a) What is the final velocity B? (b) How much Kinetic energy is gained or lost in the collision? Ans: (a) 10i + 15j (b) 525 J

NEWTON'S THIRD LAW OF MOTION

When two objects interact, the forces on the objects from each other are always equal in magnitude and opposite in direction.

Fig. 13



Consider the two objects shown in Fig. 13 box B is leaning on crate C, the third law can be written for this arrangement as a scalar relation.

$F_{BC} = F_{CB}$ i.e. equal magnitude

As a vector relation,

$\overrightarrow{F_{BC}} = \overrightarrow{F_{CB}}$ i.e. equal magnitude and opposite direction

Where F_{BC} is the force on the box from the crate

F_{CB} is the force on the crate from the box.

RELATIVE MOTION

Department of Physics, University of Ilorin

Motion is the change in position of an object with respect to the surrounding at a given time interval. Therefore, any object that its position in changing as a function of time with respect to the surrounding is said to be in motion. Going by the definition above, motion does not happen in isolation – hence the statement 'with respect to the surrounding'. Just as a frame of reference (coordinate system) is required to specify a position – so also it is for motion. Therefore, motion is defined as change in position of an object as a function of time relative to a coordinate system (i.e. the surrounding). ***Motion is always relative***; there is no such thing like absolute motion or absolute rest. Motion is always defined with respect to an observer or a reference frame.

Reference frame: Reference frame is a coordinate system from which motion is observed along with a clock attached to the axis, to measure time. Reference frame can be stationary or moving.

i. Suppose there are two persons A and B inside a car (as shown in Figure 1) moving at constant speed. Two stationary persons C and D observe them from the outside of the car.

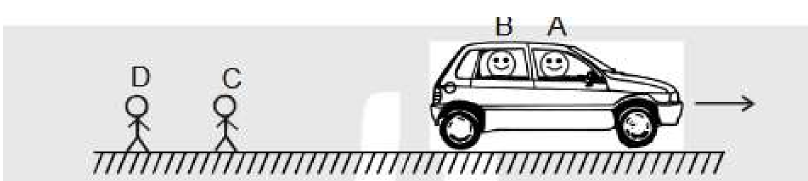


Figure 1

Here, to the observers C and D - B and A are in motion whereas B appears to be at rest for A. Similarly, C appears to be at rest for D but moving backward for A and B.

Relative motion in one dimension

1. **Relative Position**: It is the position of an object w.r.t. an observer.

In general, considering Figure 2, if position of A w.r.t. to origin is x_A and that of B w.r.t. origin is x_B, then "Position of A w.r.t. B x_{AB} is $x_{AB} = x_A - x_B$

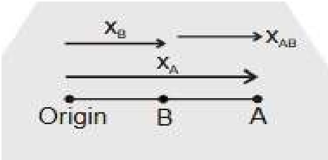
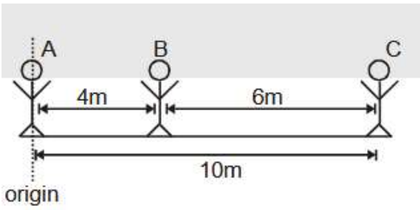


Figure 2

Solved Example:

Example 1. See the figure below (take +ve direction towards right and –ve towards left). Find x_{BA}, x_{CA}, x_{CB}, x_{AB} and

xAC.



Here,

Position of B w.r.t. A is 4 m towards right. (x_{BA} = +4m)

Position of C w.r.t. A is 10 m towards right. (x_{CA} = +10m)

Position of C w.r.t. B is 6 m towards right (x_{CB} = +6m)

Position of A w.r.t. B is 4 m towards left. (x_{AB} = –4 m)

Position of A w.r.t. C is 10 m towards left. (x_{AC} = –10m)

2. **Relative Velocity**:

Definition: Relative velocity of an object A with respect to B is defined as the velocity with which A appears to move if B is considered to be at rest. In other words, it is the velocity with which A appears to move as seen by B considering itself to be at rest.

POINT 1 *: All velocities are relative and have no significance unless observer is specified. However, when we say "velocity of A", what we mean is, velocity of A w.r.t. ground which is assumed to be at rest.*

Relative velocity in one dimension -

If x_A is the position of A w.r.t. ground, x_B is position of B w.r.t. ground and x_{AB} is position of A w.r.t. B

then we can say v_A = velocity of A w.r.t. ground = dx_A / dt

v_B = velocity of B w.r.t. ground = $\frac{dx_B}{dt}$

and v_{AB} = velocity of A w.r.t. B = $\frac{dx_{AB}}{dt} = \frac{d}{dt}(x_A - x_B) = \frac{dx_A}{dt} - \frac{dx_B}{dt}$

Thus $\boxed{v_{AB} = v_A - v_B}$

POINT 2: *Velocity of an object w.r.t. itself is always zero.*

Solved Example

Example 2. An object A is moving with 5 m/s and B is moving with 20 m/s in the same direction. (Positive x-axis)

(i) Find velocity of B with respect to A.

(ii) Find velocity of A with respect to B

Solution :

(i) $v_B = +20$ m/s, $v_A = +5$ m/s,

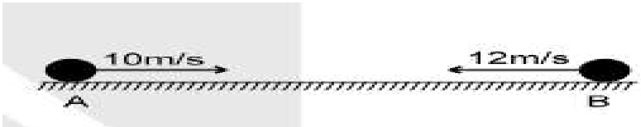
$v_{BA} = v_B - v_A = +15$ m/s

(ii) $v_B = +20$ m/s, $v_A = +15$ m/s

$v_{AB} = v_A - v_B = -15$ m/s

Note : $v_{BA} = -v_{AB}$

Example 3. Two objects A and B are moving towards each other with velocities 10 m/s and 12 m/s respectively as shown. (i) Find the velocity of A with respect to B. (ii) Find the velocity of B with respect to A



Solution : $v_A = +10$, $v_B = -12$

(i) $v_{AB} = v_A - v_B = (10) - (-12) = 22$ m/s.

(ii) $v_{BA} = v_B - v_A = (-12) - (10) = -22$ m/s.

3. **Relative Acceleration**: It is the rate at which relative velocity is changing.

$$a_{AB} = \frac{dv_{AB}}{dt} = \frac{dv_A}{dt} - \frac{dv_B}{dt} = a_A - a_B$$

Equations of motion when relative acceleration is constant.

$v_{rel} = u_{rel} + a_{rel}t$

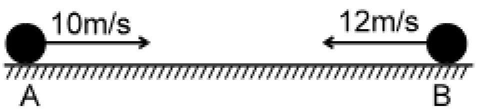
$s_{rel} = u_{rel}t + \frac{1}{2} a_{rel}t^2$

$v^2_{rel} = u^2_{rel} + 2a_{rel}\, s_{rel}$

4. **Velocity of Approach / Separation**: It is the component of relative velocity of one particle w.r.t. another, along the line joining them. *If the separation is decreasing, we say it is velocity of approach and if separation is increasing, then we say it is velocity of separation.* In one dimension, since relative velocity is along the line joining A and B, hence velocity of approach / separation is simply equal to magnitude of relative velocity of A w.r.t. B.

Solved Examples

Example 4. A particle A is moving with a speed of 10 m/s towards right and another particle B is moving at speed of 12 m/s towards left. Find their velocity of approach.



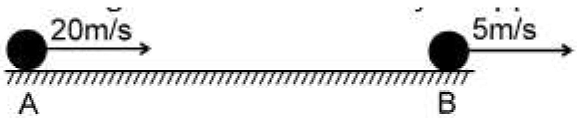
Solution:

$V_A = +10$, $V_B = -12 \Rightarrow V_{AB} = V_A - V_B \Rightarrow 10 - (-12) = 22$ m/s

since separation is decreasing hence $V_{app} = |V_{AB}| = 22$ m/s

Ans. 22 m/s

Example 5. A particle A is moving with a speed of 20 m/s towards right and another particle B is moving at a speed of 5 m/s towards right. Find their velocity of approach.



Solution : $V_A = +20$, $V_B = +5$

$V_{AB} = V_A - V_B$

$20 - (+5) = 15$ m/s

since separation is decreasing hence $V_{app} = |V_{AB}| = 15$ m/s

Ans. 15 m/s

Example: A car is moving with a speed of 46 m/s in the same direction as a truck ahead with a speed of 35 m/s. How long will it take the car to overtake the truck - if the truck is 12 m long?

Hint: To overtake the truck means the time it will take for the car to cover a relative distance of 12 m.

Relative motion in two dimension

$\vec{r}_A$ = position of A with respect to O

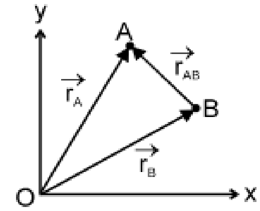
$\vec{r}_B$ = position of B with respect to O

$\vec{r}_{AB}$ = position of A with respect to B.

$\vec{r}_{AB} = \vec{r}_A - \vec{r}_B$ (The vector sum $\vec{r}_A - \vec{r}_B$ can be done by Δ law of addition or resolution method)

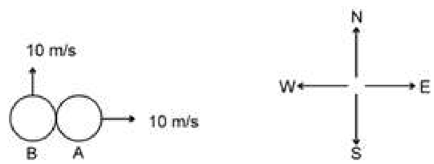
$$\therefore \quad \frac{d(\vec{r}_{AB})}{dt} = \frac{d(\vec{r}_A)}{dt} - \frac{d(\vec{r}_B)}{dt}.$$

$$\Rightarrow \quad \vec{v}_{AB} = \vec{v}_A - \vec{v}_B \quad ; \quad \frac{d(\vec{v}_{AB})}{dt} = \frac{d(\vec{v}_A)}{dt} - \frac{d(\vec{v}_B)}{dt} \quad \Rightarrow \quad \vec{a}_{AB} = \vec{a}_A - \vec{a}_B$$



Solved Example

Example 6. Object A and B both have speed of 10 m/s. A is moving towards East while B is moving towards North starting from the same point as shown. Find velocity of A relative to B



Solution :

Method 1 : $\vec{v}_{AB} = \vec{v}_A - \vec{v}_B$ $\quad \therefore \ |\vec{v}_{AB}| = 10\sqrt{2}$

Example 7. A and B are thrown vertically upwards with velocity, 10 m/s and 20 m/s respectively (g = 10 m/s^2) . Find the separation between them after 1 second.

Solution:

The relevant equation is $S = Ut - \frac{1}{2}gt^2$, note that g is negative for upward motion and positive for downward motion.

Applications of Relative motion: Some of the applications of relative motion are below.

1. Useful in predicting occurrence of a solar eclipse and lunar eclipse
2. Useful in predicting when to sight a new moon

EQUATION OF MOTION AND APPLICATIONS OF NEWTONIAN MECHANICS

Newton's second law of motion relates resultant force or net force to acceleration i.e. $F_{net} = ma$

The implication is that for any object that is under the influence of a net force, its acceleration **a** is

$\mathbf{a} = \frac{\boldsymbol{F_{net}}}{m} \rightarrow a = \frac{F_{net}}{m}$ ------- (1)

where m is the mass of the object

Remember that the instantaneous velocity V is given by $V = \frac{dx}{dt}$

and also that of acceleration a is $a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$

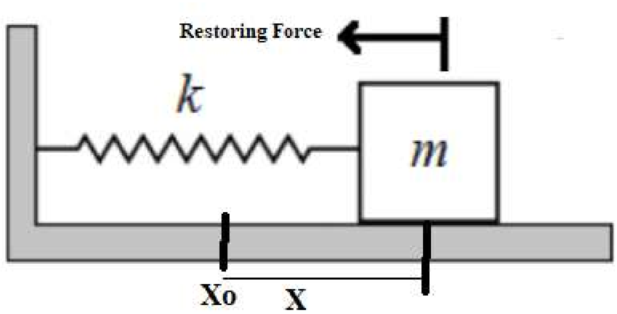
$\rightarrow \frac{d^2x}{dt^2} = F_{net}$ ------- (2)

Equation 2 is the equation of motion – it is a second order differential equation. The solution of the equation of motion i.e. x(t) gives the position of an object as a function of time.

Examples:-

1. <u>Simple Harmonic Oscillator</u>

   A mass-spring system of mass m displaced by a distance x from the equilibrium position (see the figure below)



In the mass-spring system above, the restoring force (i.e. the net force) on a mass displaced by a distance x from the equilibrium position (i.e. x_0) is

$F_r = - kx$ -------- (3)

Therefore, the equation of motion is

$m\frac{d^2x}{dt^2} = -kx$ ------- (4)

Assuming a solution of the form x(t) = A$\cos\omega t$

$\rightarrow \frac{dx}{dt}$ = - $\omega A\sin\omega t$ **&** $\frac{d^2x}{dt^2} = -\omega^2 A\cos\omega t$ } ----- (5)

Put (5) in (4): $m\,(-\omega^2 A\cos\omega t)$ = -kA$\cos\omega t,$

$$-\omega^2 = -\frac{k}{m}$$

$$\rightarrow \omega = \sqrt{\frac{k}{m}} \;,$$

$$therefore\ x(t) = A\cos\left(\sqrt{\frac{k}{m}}\right)t \qquad \text{------ (6)}$$

The implication of equation (6) is that, if the maximum displacement A is known at t =0, then the position as a function of time of the oscillator can be obtained. Remember that $\omega = \sqrt{\frac{k}{m}}$ is a parameter of the system i.e. spring-mass system

2. Motion of satellite

   For a satellite orbiting the earth about a central axis, the gravitation m of the satellite is

   $$\mathbf{F} = \frac{-GM_E m_r}{r^3} \qquad \textbf{(7)}$$

   Then the equation of motion is

   $$m\frac{d^2r}{dt^2} = \frac{-GM_E m_r}{r^3} \qquad \textbf{(8)}$$

   $$\frac{d^2r}{dt^2} = -\frac{r}{r^3}\mu \quad \text{, where } \mu = GM_E$$

   Therefore,

   $$\frac{d^2r}{dt^2} + \frac{r}{r^3}\mu = 0 \qquad (9)$$

   This is a second order differential equation. Its solution involves six constants to be determined and they are called the orbital elements.

3. Molecular Dynamics

This is a technique for computer simulation of complex systems, modelled at the atomic level. Many particles are involved. The position of ith particle is written as

$r_i(t) = (x_i(t), y_i(t), z_i(t))$

and the force acting upon ith particle at time t is $F_i = m_i \frac{d^2 r(t)}{dt^2}$ where m_i is the mass of the ith particle.

The solution of the second order differential equation can be solved.

Numerically, the solution is

$$r_i(t + \Delta t) \cong 2\, r_i(t) - r_i(t - \Delta t) + \frac{F_i(t)}{m_i} \Delta t^2 \qquad (10)$$

Equation 10 gives the position of ith particle as a function of time

<u>Conservation principles in Physics</u>

Department of Physics, University of Ilorin

The rock climber does work as she ascends the vertical cliff. So does the mover, as he pushes a heavy chest across the floor. The difference is that if the rock climber lets go, down she goes; the work she put into the climb comes back as the kinetic energy of her fall, and if the mover lets go of the chest, though, he and the chest stay right where they are.

FIGURE 1

This contrast highlights a distinction between two types of forces, called conservative and nonconservative. From that distinction we will develop the **conservation of energy**. M**echanical energy**, which includes kinetic energy and **potential energy**, are very important, here.

Potential Energy

Work done against a conservative force is somehow "stored," in the sense that we can get it back again in the form of kinetic energy. The climber is acutely aware of that "stored work"; it gives her the potential for a dangerous fall. *Potential* is an appropriate word here: We can consider the "stored work" as **potential energy *U***, in the sense that it can be released as kinetic energy.

We define potential energy formally in terms of the work done by a conservative force.

Specifically:

The change in potential energy ΔU_{AB} associated with a conservative force is the negative of the work done by that force as it acts over any path from point *A* to point *B*:

$$\Delta U_{AB} = -\int_A^B \bar{F}.d\bar{r} \qquad \text{(potential energy)} \qquad (1)$$

If a conservative force does *positive* work (as does gravity on a falling object), then potential energy must decrease—and that means ΔU must be *negative*. Conversely, if a conservative force does *negative* work (as does gravity on a weight being lifted), then energy is stored, and ΔU must be *positive*. The minus sign in Eqn. (1) handles both these cases. We'll often drop the subscript *AB* and write simply ΔU for potential-energy change. Keeping the subscript is important, though, when we need to be clear about whether we're going from *A* to *B* or from *B* to *A*.

Eqn. (1) is a completely general definition of potential energy, applicable in all circumstances. Often, though, we can consider a path where force and displacement are parallel (or antiparallel). Then Eqn. (1) simplifies to

$$\Delta U = -\int_{x_1}^{x_2} F(x).dx \tag{1a}$$

Where x_1 and x_2 are the starting and ending points on the *x*-axis, taken to coincide with the path. When the force is constant, this equation simplifies further to

$$\Delta U = -F(x_2 - x_1) \tag{1b}$$

Eqn. (1b) provides a very simple expression for potential-energy changes, but it applies *only* when the force is constant. Eqn. (1b) is a special case of Eqn. (1a) that follows because a constant force can be taken outside the integral.

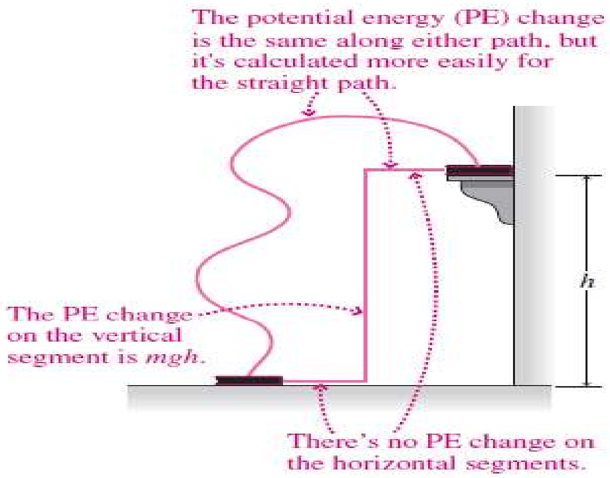
Gravitational Potential Energy

Things are frequently moved up and down, causing changes in potential energy. Figure 2 shows two possible paths for a book lifted from the floor to a shelf of height *h*. Since the gravitational force is conservative, we can use either path to calculate the potential-energy change. It's easiest to use the path consisting of straight segments. No work or potential-energy change is associated with the horizontal motion. For the vertical lift, the force of gravity is constant and Eqn. (1b) becomes $\Delta U = mgh$ where the minus sign in Eqn. (1b) cancels with the minus sign associated with the *downward* direction of gravity. This result is quite general: When a mass *m* undergoes a vertical displacement Δy near Earth's surface, its gravitational potential energy changes by

$$\Delta U = mg\Delta y \qquad \text{(gravitational potential energy)} \tag{2}$$

The quantity Δy can be positive or negative, depending on whether the object moves up or down; correspondingly, the potential energy can either increase or decrease. We emphasize that Eqn. (2) applies *near Earth's surface*—that is, for distances small compared with Earth's radius. That assumption allows us to treat the gravitational force as constant over the path.

We've found the *change* in the book's potential energy, but what about the potential energy itself? That depends on where we define the zero of potential energy. If we choose $U = 0$ at the floor, then $U = mgh$ on the shelf. But we could just as well take $U = 0$ at the shelf; then the book's potential energy on the floor would be $U = -mgh$. Negative potential energies arise frequently, and that's OK because only *differences* in potential energy really matter. Figure 3 shows a plot of potential energy versus height with $U = 0$ taken at the floor. The *linear* increase in potential energy with height reflects the *constant* gravitational force.



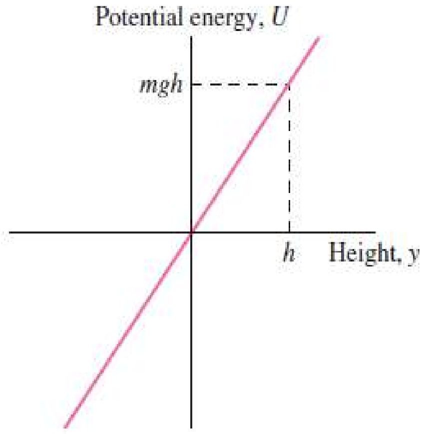


FIGURE 3

FIGURE 2

Example 1

A 55-kg engineer leaves her office on the 33rd floor of a skyscraper and takes an elevator up to the 59th floor. Later she descends to street level. If the engineer chooses the zero of potential energy at her office and if the distance from one floor to the next is 3.5 m, what is the engineer's potential energy (a) in her office, (b) on the 59th floor, and (c) at street level?

Soln

This is a problem about gravitational potential energy relative to a specified point of zero energy—namely, the engineer's office.

Change in gravitational energy associated with a change in vertical position is given by $\Delta U = mg\Delta y$. We're given positions in floors, not meters, so we need to convert using the given factor 3.5 m per floor.

(a) In her office, the engineer's potential energy is zero, since she defined it that way.

(b) The 59th floor is 59 - 33 = 26 floors higher, so the potential energy there is

$$U_{59} = mg\Delta y = (55\,kg)(9.8\,m/s^2)(26\,floors)(3.5\,m/floor) = 49kJ$$

Note that we can write U rather than ΔU because we're calculating the potential-energy change from the place where $U = 0$.

(c) The street level is 32 floors below the engineer's office, so

$$U_{street} = mg\Delta y = (55\,kg)(9.8\,m/s^2)(-32\,floors)(3.5\,m/floor) = -60kJ$$

Elastic Potential Energy

When you stretch or compress a spring or other elastic objects, you do work against the spring force, and that work ends up stored as elastic potential energy. For an ideal spring, the force is $F = -kx$, where x is the distance the spring is stretched from equilibrium, and the minus sign shows that the force opposes the stretching or compression. Since the force varies with position, we use Eqn. (1a) to evaluate the potential energy:

$$\Delta U = -\int_{x_1}^{x_2} F(x)\,dx = -\int_{x_1}^{x_2}(-kx)\,dx = \frac{1}{2}kx_2^2 - \frac{1}{2}kx_1^2$$

Where x_1 and x_2 are the initial and final values of the stretch. If we take $U = 0$ when $x = 0$ — that is, when the spring is neither stretched nor compressed—then we can use this result to write the potential energy at an arbitrary stretch (or compression) x as

$$U = \frac{1}{2}kx^2 \qquad \text{(elastic potential energy)} \qquad (3)$$

Comparing Eqn. (3) with workdone, $W = \frac{1}{2}kx^2$, shows that this is equal to the work done in stretching the spring. That work gets stored as potential energy.

Example 2

A car's suspension consists of springs with an overall effective spring constant of 120 k/N m. How much would you have to compress the springs to store the same amount of energy as in 1 gram of gasoline if the energy content of gasoline is 44ML/kg?

Soln

This problem is about the energy stored in a spring, as compared with the chemical energy of gasoline. $U = \frac{1}{2}kx^2$, gives a spring's stored energy when it's been compressed a distance x. Here we want that energy to equal the energy in 1 gram of gasoline.

At 44 MJ/kg, the energy in 1 g of gasoline is 44 kJ. Setting this equal to the spring energy and solving for x, we get

$$x = \sqrt{\frac{2U}{k}} = \sqrt{\frac{(2)(44kJ)}{120\ kN/m}} = 86\ cm$$

This answer is absurd. A car's springs couldn't compress anywhere near that far before the underside of the car hit the ground. And 1 g is not much gasoline. This example shows that springs, though useful energy-storage devices, can't possibly compete with chemical fuels.

Example 3

Ropes used in rock climbing are "springy" so that they cushion a fall. A particular rope exerts a force $F = -kx + bx^2$, where k = 223 N/m, $b = 4.10\ N/m^2$, and x is the stretch. Find the potential energy stored in this rope when it's been stretched 2.62 m, taking $U = 0$ at $x = 0$.

Soln

This one isn't so easy because the rope isn't a simple $F = -kx$ spring for which we already have a potential-energy formula.

Because the rope force varies with stretch, we'll have to integrate. Since force and displacement are in the same direction, we can use Eqn. (1a), $\Delta U = -\int_{x_1}^{x_2} F(x).\,dx$. But that's not so much a formula as a strategy for deriving one.

Applying Eqn. (1) to this particular rope, we have

$$U = -\int_{x_1}^{x_2} F(x).\,dx = -\int_0^x (-kx + bx^2)dx = \frac{1}{2}kx^2 - \frac{1}{3}bx^3\Big|_0^x$$

$$= \frac{1}{2}kx^2 - \frac{1}{3}bx^3 = \frac{1}{2}\left(223\,\frac{\mathrm{N}}{\mathrm{m}}\right)(2.62\,m)^2 - \frac{1}{3}\left(4.10\,\frac{N}{m^2}\right)(2.62\,m)^3 = 741\,J$$

This result is about 3% less than the potential energy $U = \frac{1}{2}kx^2$ of an ideal spring with the same spring constant. This shows the effect of the extra term $+bx^2$, whose positive sign reduces the restoring force and thus the work needed to stretch the spring.

Conservation of Mechanical Energy

The work-energy theorem shows that the change in a body's kinetic energy ΔK is equal to the net work done on it:

$$\Delta K = W_{net}$$

Consider separately the work W_c done by conservative forces and the work W_{nc} done by nonconservative forces. Then

$$\Delta K = W_c + W_{nc}$$

We've defined the change in potential energy ΔU as the negative of the work done by conservative forces. So we can write

$$\Delta K = -\Delta U + W_{nc}$$

Or $$\Delta K + \Delta U = W_{nc} \qquad (4)$$

We define the sum of the kinetic and potential energy as the mechanical energy. Then Eqn. (4) shows that the change in mechanical energy is equal to the work done by nonconservative forces. In the absence of nonconservative forces, the mechanical energy is unchanged:

$$\Delta K + \Delta U = 0 \qquad (5)$$

and, equivalently , $(conservation\ of\ mechanical\ energy)$

$$K + U = constant = K_0 + U_0 \qquad (6)$$

where K_0 and U_0 are the kinetic and potential energy of a body at some point, and K and U are their values when the body is at any other point. Eqns. (5) and (6) express the law of conservation of mechanical energy. They show that, in the absence of nonconservative forces, the mechanical energy $K + U$ remains always the same. The kinetic energy K may change, but that change is always compensated by an equal but opposite change in potential energy.

Conservation of mechanical energy is a powerful principle. Throughout physics, from the subatomic realm through practical problems in engineering and on to astrophysics, the principle of energy conservation is widely used in solving problems that would be intractable without it.

Example 4

A biologist uses a spring-loaded gun to shoot tranquilizer darts into an elephant. The gun's spring has $k = 5\,940\ \mathrm{N/m}$ and is compressed a distance $x_0 = 25\ \mathrm{cm}$ before firing a 38-g dart. Assuming the gun is pointed horizontally, at what speed does the dart leave the gun?

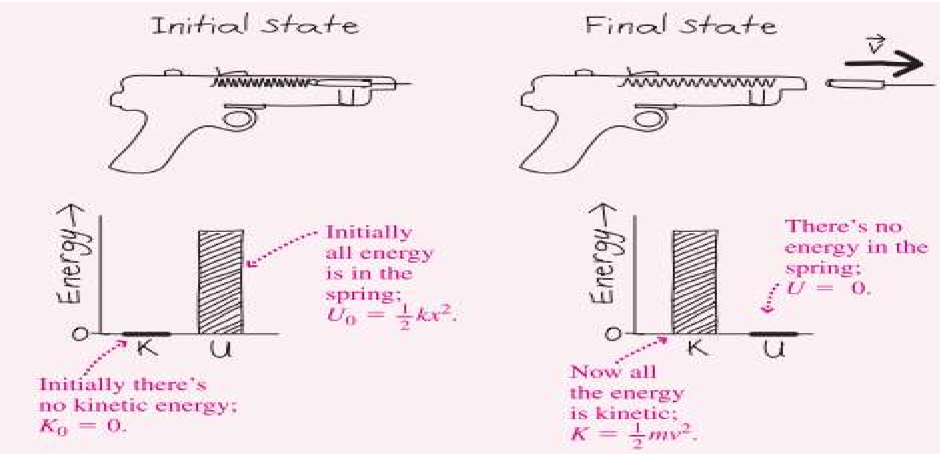


FIGURE 4 Our sketches for Example 4, showing bar charts for the initial and final states.

Soln

In Fig. 4 we've sketched the two states, giving the potential and kinetic energy for each. We've also sketched bar graphs showing the relative sizes of the energies. To use the statement of energy conservation, Eqn. (6), we also need expressions for the kinetic energy ($\frac{1}{2}mv^2$) and the spring potential energy ($\frac{1}{2}kx^2$), Eqn. (3)). Incidentally, using Eqn. (3) implicitly sets the zero of elastic potential energy when the spring is in its equilibrium position. We might as well set the zero of gravitational energy at the height of the gun, since there's no change in the dart's vertical position between our initial and final states.

We know three of the terms in Eqn. (6),

$$K + U = K_0 + U_0.$$

The initial kinetic energy K_0 is 0, since the dart is initially at rest. The initial potential energy is that of the compressed spring, $U_0 = \frac{1}{2}kx_0^2$. The final potential energy is $U = 0$ because the spring is now in its equilibrium position and we've taken the gravitational potential energy to be zero. What we don't know is the final kinetic energy, but we do know that it's given by $K = \frac{1}{2}mv^2$

So Eqn. (6) becomes

$\frac{1}{2}mv^2 + 0 = 0 + \frac{1}{2}kx_0^2$, which solves to give

$$v = \sqrt{\frac{k}{m}}x_0 = \left(\sqrt{\frac{940\ \mathrm{N/m}}{0.038\ kg}}\right)(0.25\ m) = 39\ m/s$$

Example 5

The spring in Fig. 5 has $k = 140\ \mathrm{N/m}$. A 50-g block is placed against the spring, which is compressed 11 cm. When the block is released, how high up the slope does it rise? Neglect friction.

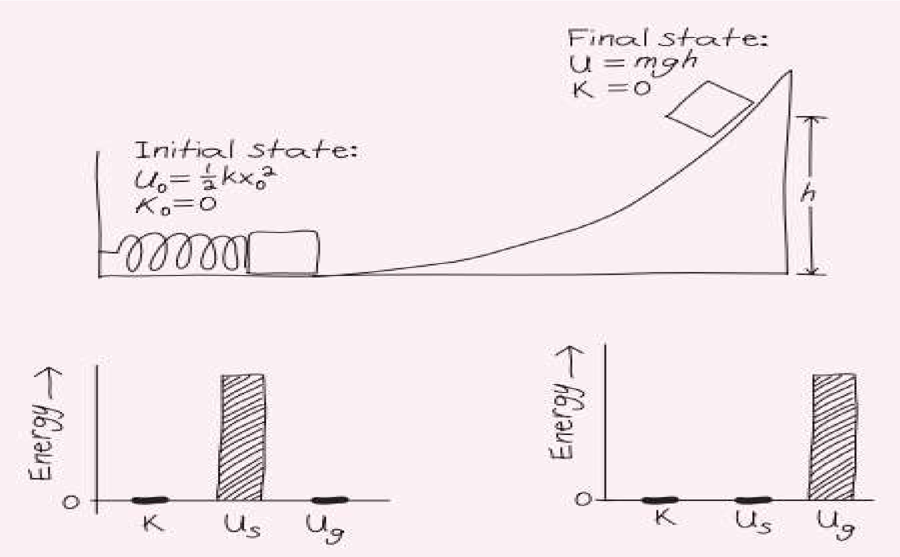


FIGURE 5 Our sketches for Example 5.

Soln

Figure 5 shows the initial and final states, along with bar charts for each. We've drawn separate bars for the spring and gravitational potential energies, U_s and U_g. Now apply Eqn. (6),

$$K + U = K_0 + U_0$$

In both states the block is at rest, so kinetic energy is zero. In the initial state we know the potential energy U_0: It's the spring energy $\frac{1}{2}kx^2$. We don't know the final-state potential energy, but we do know that it's gravitational energy—and with the zero of potential energy at the bottom, it's $U = mgh$. With

$$K = K_0 = 0,\ U_0 = \frac{1}{2}kx^2, \text{ and } U = mgh,$$

Eqn. 7 reads $0 + mgh = 0 + \frac{1}{2}kx^2$.

We then solve for the unknown h to get

$$h = \frac{kx^2}{2mg} = \frac{(140\ \mathrm{N/m})(0.11\ m)^2}{2(0.05\ m)(9.8\ m/s^2)} = 1.7\ m$$

Conservation of Linear Momentum

Linear momentum is a useful quantity for cases where we have a few particles (objects) which interact with each other but not with the rest of the world. Such a system is called an isolated system.

We often have reason to study systems where a few particles interact with each other very briefly, with forces that are strong compared to the other forces in the world that they may experience. In those situations, and for that brief period of time, we can treat the particles as if they were isolated. We can show that when two particles interact only with each other (i.e. they are isolated) then their total momentum remains constant:

$$p_1 i + p_2 j = p_1 f + p_2 f \qquad (7)$$

or, in terms of the masses and velocities,

$$m_1 v_1 i + m_2 v_2 i = m_1 v_1 f + m_2 v_2 f \qquad (8)$$

Or, abbreviating $p_1 + p_2 = P$ (total momentum), this is: $P_i = P_f$. It is important to understand that Eqn. (7) is a vector equation; it tells us that the total x component of the momentum is conserved, and the total y component of the momentum is conserved.

Example 6

A 3.00 kg particle has a velocity of (3.0i − 4.0j) m/s. Find its x and y components of momentum and the magnitude of its total momentum.

Soln

Using the definition of momentum and the given values of m and v we have:

$$: P = mv = \frac{(3.0\ kg)(3.0\mathrm{i} - 4.0\mathrm{j})\mathrm{m}}{\mathrm{s}} = (9.0\mathrm{i} - 12.0\mathrm{j})\ \mathrm{kg.m/s}$$

So the particle has momentum components $p_x =$ +9.0 kg·m/s and $p_y =$ −12. kg·m/s .

The magnitude of its momentum is

$$p = \sqrt{p_x^2 + p_y^2} = \sqrt{(9.0)^2 + (-12.0)^2}\ \mathrm{kg} \cdot \frac{\mathrm{m}}{\mathrm{s}} = 15.0\mathrm{kg} \cdot \mathrm{m/s}$$

Conservation of angular momentum

We have already learned about conservation of linear momentum: $\sum \bar{p}_i = \sum \bar{p}_f$. Where $\bar{p}_i$ is the initial linear momentum and $\bar{p}_f$ is the final linear momentum.

As you can imagine, the equation for conservation of angular momentum is similar: $\sum \bar{L}_i = \sum \bar{L}_f$. Where $\bar{L}_i$ is the initial angular momentum and $\bar{L}_f$ is the final angular momentum.

If I am spinning on a stool holding two masses at arm's length straight out from my body and then pull in my arms, what will happen? From general knowledge of sports like figure skating, gymnastics, dance, and diving, you probably already know that my angular velocity will increase. It is important to know why. The equation for angular momentum of a rigid object with shape is:

$\bar{L} = I\bar{\omega}$. L is the angular momentum, I is the moment of inertia and ω is the angular velocity. Therefore, the equation for conservation of angular momentum in the example of me spinning on a stool is:

$$\sum \bar{L}_i = \sum \bar{L}_f \rightarrow I_i\overline{\omega_i} + I_f\overline{\omega_f}$$

It is important to recognize that the axis of rotation for the angular momenta and rotational inertias in these equations is the vertical axis through the center of the stool.

The equation for the rotational inertia of a system of particles is:

$$I_{particle} = \sum m_i r_i^2 \quad \text{m = mass and r = distance from the center of the body}$$

In other words, bringing in my arms decreases the average distance the particles of the system are from the axis of rotation, which decreases the rotational inertia of the system, therefore, because angular momentum is conserved, the angular velocity of the system must increase.

$$r \downarrow \rightarrow I \downarrow \rightarrow \bar{\omega} \uparrow$$

Now let’s talk about when angular momentum is conserved. As you recall linear momentum is conserved when the net force on the system equals zero.

$$\sum \bar{F}_{external\ on\ system} = 0 = \frac{\Delta\bar{p}_{system}}{\Delta t} \rightarrow 0.\Delta t = \left[\frac{\Delta\bar{p}_{system}}{\Delta t}\right]\Delta t$$

$$\rightarrow 0 = \Delta\bar{p}_{system} = \Delta\bar{p}_{i\ system} - \Delta\bar{p}_{f\ system}$$

$$\rightarrow \Delta\bar{p}_{i\ system} = \Delta\bar{p}_{f\ system}$$

Angular momentum is conserved when the net external torque acting on the system equals zero.

$$\sum \bar{\tau}_{external\ on\ system} = 0 = \frac{\Delta\bar{L}_{system}}{\Delta t} \rightarrow 0.\Delta t = \left[\frac{\Delta\bar{L}_{system}}{\Delta t}\right]\Delta t$$

$$\rightarrow 0 = \Delta\bar{L}_{system} = \Delta\bar{L}_{i\ system} - \Delta\bar{L}_{f\ system}$$

$$\rightarrow \Delta\bar{L}_{i\ system} = \Delta\bar{L}_{f\ system}$$

Returning back to me sitting on the stool. Notice that, about the vertical axis through the center of the stool, the net external torque acting on the system of me and the stool is zero, therefore angular momentum of the system will stay constant. The initial angular velocity of the system is zero, therefore, I can wave my arms around all I want, but doing so will not change the angular momentum of the system. However, if I push on something external to the system, I can cause a net torque on the system, angular momentum is no longer conserved, and I can increase my angular velocity. Angular momentum is a vector, therefore, when angular momentum is conserved, its direction is conserved as well. This is why a spinning top will maintain its vertical position. Its angular momentum will be vertical and therefore, as long as the top continues to spin, the angular momentum will be conserved and the top will stay vertical. However, a top which is not spinning, has no angular momentum, and will not stay vertical. We can also apply this concept to a moving bicycle. The wheels of the bike, while the bike is moving, are spinning and have angular momentum. While you are moving forward, the direction of the angular momentum of the wheels will be to the left. Conservation of angular momentum will try to maintain the direction of the angular momentum of the wheels and therefore will help keep the bicycle vertical. If the bike is not moving, the wheels have no angular momentum and therefore do not help keep the bicycle vertical. Conservation of angular momentum is why it is easier to balance on a bike while it is moving.

Example 7

A flywheel rotates without friction at an angular velocity ω_0 = 600 rev/min on a frictionless, vertical shaft of negligible rotational inertia. A second flywheel, which is at rest and has a moment of inertia three times that of the rotating flywheel, is dropped onto it. Because friction exists between the surfaces, the flywheels very quickly reach the same rotational velocity, after which they spin together. (a) Use the law of conservation of angular momentum to determine the angular velocity ω of the combination. (b) What fraction of the initial kinetic energy is lost in the coupling of the flywheels?

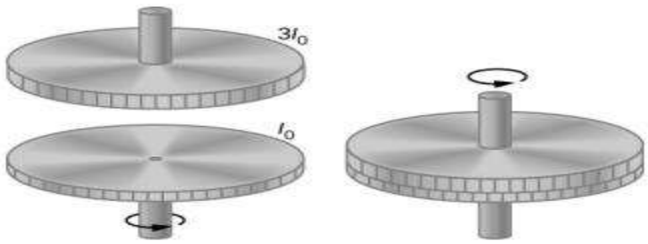


Fig. 6

Soln

(a). No external torques act on the system. The force due to friction produces an internal torque, which does not affect the angular momentum of the system. Therefore conservation of angular momentum gives

$$I_0\omega_0 = (I_0 + 3I_0)\omega,$$

$$\omega = (1/4)\ \omega_0 = 150 \text{ rev/min}$$

in rad/s, $\omega = 150 \text{ rev/min x } 2\pi = 15.7 \text{ rad/s}.$

(b). Before contact, only one flywheel is rotating. The rotational kinetic energy of this flywheel is the initial rotational kinetic energy of the system, $\frac{1}{2}I_0\omega_0^2$.

The final kinetic energy is

$$\frac{1}{2}(4I_0)\omega^2 = \frac{1}{2}(4I_0)\left(\frac{\omega_0}{4}\right)^2 = \frac{1}{8}I_0{\omega_0}^2.$$

Therefore, the ratio of the final kinetic energy to the initial kinetic energy is

$$\frac{\frac{1}{8}I_0{\omega_0}^2}{\frac{1}{2}I_0{\omega_0}^2} = \frac{1}{4}.$$

Thus, 3/4 of the initial kinetic energy is lost to the coupling of the two flywheels.

Conservation of mass

"In any kind of physical or chemical process, mass is neither created nor destroyed - the mass before the process equals the mass after the process." - the total mass of the system does not change, the total mass of the products of a chemical reaction is always the same as the total mass of the original materials.

Example 8

1. When wood burns, mass seems to disappear because some of the products of reaction are gases; if the mass of the original wood is added to the mass of the oxygen that combined with it and if the mass of the resulting ash is added to the mass o the gaseous products, the two sums will turn out exactly equal.
2. Iron increases in weight on rusting because it combines with gases from the air, and the increase in weight is exactly equal to the weight of gas consumed. Out of thousands of reactions that have been tested with accurate chemical balances, no deviation from the law has ever been found.

Consider a collection of matter located somewhere in space. This quantity of matter with well-defined boundaries is termed a system. The law of conservation of mass then implies that the mass of this given system remains constant,

$$\frac{\Delta m}{\Delta t} = 0 \qquad \text{(Equation of conservation of mass)} \qquad (9)$$

The volume occupied by the matter may be changing and the density of the matter within the system may be changing, but the mass remains constant.

Considering a differential mass element at position X in the reference configuration and at x in the current configuration, Eqn. (9) can be rewritten as

$$dm(\mathrm{X}) = dm \qquad (10)$$

Conservation of Charge:

The total electric charge of an isolated system never changes.

What is an isolated system? We could start with the universe. In other words, the net electric charge of the universe never changes. Add up all the positive charges and subtract all the negative charges and you will always get the same number.

Or we could have a smaller isolated system, like the conductive metal pieces of an electroscope which are electrically isolated from the rest of the universe by the rubber and glass insulators. In other words, the net electric charge of the electroscope will remain constant, as long as it remains isolated.

Example 9

Two charged, conducting objects collide and separate. Before colliding, the charges on the two objects are +3e and -6e. Which of the following are possible values for the final charges on the two objects? Choose all possible answers. (a) +4e, -7e (b) +2e, -2e (c) -1.5e, -1.5e (d) -3.5e, +2.5e (e) +e, -4e

Soln

$$q_{1i} = +3e; q_{2i} = -6e; q_{1f} = ?; q_{2f} = ?$$

$$q_{total\ i} = q_{1i} + q_{2i} = \big(+3e + (-6e)\big) = -3e = q_{total\ f}$$

a) $q_{1f} + q_{2f} = \big(+4e + (-7e)\big) = -3e = q_{total\ f}$

b) $q_{1f} + q_{2f} = \big(+2e + (-2e)\big) = 0 \neq q_{total\ f}$

c) $q_{1f} + q_{2f} = \big(-1.5e + (-1.5e)\big) = -3e = q_{total\ f}$

d) $q_{1f} + q_{2f} = (-3.5e + 2.5e) = -e = q_{total\ f}$

e) $q_{1f} + q_{2f} = \big(+e + (-4e)\big) = -3e = q_{total\ f}$

Correct answers are (a) and (e) because those are the only two options which have a total final charge equal to the total initial charge and are integer multiples of the fundamental charge e.

Example 10

Two identical conducting are held using insulating gloves a distance x apart. Initially the charges on each sphere are +3.0 pC and +6.0 pC. The two spheres are touched together and returned to the same distance x apart. You may assume x is the distance between their centers of charge. What is the final charge on each sphere?

Soln

$$q_{1i} = +3.0\ pC; q_{2i} = +6.0\ pC; r_i = r_f = x; q_{1f} = ?; q_{2f} = ?$$

$$q_{total\ i} = q_{total\ f} = q_{1i} + q_{2i} = +3.0\ pC + 6.0\ pC = +9.0\ pC$$

Because the two spheres are identical, after touching, the spheres will have equal charge.

$$q_{1f} = q_{2f} = q_f \rightarrow q_{total\ f} = q_{1f} + q_{2f} = 2q_f = +9.0\ pC \rightarrow q_f = +4.5\ pC$$

Both charges end with 4.5 pC of charge. This is 4.5 picocoulombs of charge or 4.5 x 10^{-9} C which an object is physically able to have. Because then it will have:

$$q_f = n_f e$$

$$\frac{q_f}{e} = \frac{4.5 \text{ x } 10^{-9} C}{1.6 \text{ x } 10^{-19}\ C/charge\ carrier} = 2.8125 \text{ x } 10^{10} charge\ carriers$$

$$\approx 2.8 \text{ x } 10^{10} excess\ protons$$

Imagine that. 28 billion more protons than electrons on each sphere. Each sphere will have a lot more total protons and electrons, however, it has a deficit of 28 billion electrons and therefore has a net charge of 4.5 pC.

Exercise

1. Find the potential energy of a 70-kg hiker (a) atop a Mountain, 1900 m above sea level, and (b) in a Valley, 86 m below sea level. Take the zero of potential energy at sea level.
2. You fly from an Airport A, at sea level, to another Airport B, altitude 1.6 km. Taking your mass as 65 kg and the zero of potential energy at A, what's your gravitational potential energy (a) at the plane's 11-km cruising altitude and (b) in B?
3. A 60-kg hiker ascending 1250-m-high Mountain in Vermont has potential energy 2240 kJ; the zero of potential energy is taken at the mountaintop. What's her altitude?
4. How much energy can be stored in a spring with $k = 320\ \mathrm{N/m}$ if the maximum allowed stretch is 18 cm?
5. How far would you have to stretch a spring with $k = 1.4\ \mathrm{kN/m}$ for it to store 210 J of energy?
6. A 10,000-kg Navy jet lands on an aircraft carrier and snags a cable to slow it down. The cable is attached to a spring with $k = 40\ \mathrm{kN/m}$. If the spring stretches 25 m to stop the plane, what was its landing speed?
7. A 120-g arrow is shot vertically from a bow whose effective spring constant is 430 N/m. If the bow is drawn 71 cm before shooting, to what height does the arrow rise?
8. In a railroad yard, a 35,000-kg boxcar moving at 7.5 m/s is stopped by a spring-loaded bumper mounted at the end of the level track. If $k = 2.8\ \mathrm{MN/m}$, how far does the spring compress in stopping the boxcar?
9. You work for a toy company, and you're designing a spring-launched model rocket. The launching apparatus has room for a spring that can be compressed 14 cm, and the rocket's mass is 65 g. If the rocket is to reach an altitude of 35 m, what should you specify for the spring constant?

6.2 Center of Mass (CM)

The center of mass concept is central to the study of the motion of many-particle systems. Here we shall not be concerned with the study of the motion of such systems but with their mass distribution. Recall that particles are bodies with NEGLIGIBLE spatial extent and internal structure. Real objects on the other hand are many-body system consisting of very large number of particles. At the extreme, these objects may either be a RIGID body or a FLUID.

The Center of mass of a many-body system is defined as the point in the system which moves as if the total mass is concentrated at that point. For an isolated n-body system with no external force, the total momentum $\tilde{P}$ is given by

$$\tilde{P} = m1\vec{v}1 + m2\vec{v}2 + m3\vec{v}3 + m4\vec{v}4 + m5\vec{v}5 + \cdots + m_n\vec{v}_n \quad (6.16)$$

$$= m_1\frac{d\vec{r}_1}{dt} + m_2\frac{d\vec{r}_2}{dt} + ----+m_n\frac{d\vec{r}_n}{dt} \quad (6.17)$$

$$= \frac{d}{dt}(m1\vec{r}1 + m2\vec{r}2 + ---- +m_n\vec{r}_n) \quad (6.18)$$

where $r_i\,(i = 1,2,\ldots,n)$ are the position vectors of each particle in the system. Note that the masses m_i are time independent ($\frac{dm}{dt} = 0$). The centre of mass of the system is then defined by the position vector

$$\vec{R} = \frac{m_{1\vec{r}_1}+m_{2\vec{r}_2}+m_{3\vec{r}_3}+-----+m_{n\vec{r}_n}}{m_1+m_2+m_3+ ----- +m_n} \quad (6.19)$$

$$= \frac{\sum_{i=1}^{n} m_i\vec{r}_i}{\sum m_i} \quad (6.20)$$

$$= \frac{\sum_{i=1}^{n} m_i\vec{r}_i}{M} \quad (6.21)$$

Where $M = \sum m_i$ is the total mass of the system. The corresponding Cartesian coordinate of the position vector $\vec{R}$ is given by

$$\mathbf{X} = \frac{\mathbf{m_1x_1+m_2x_2+m_3x_3+ ----- +m_nx_n}}{\mathbf{m_1+m_3+m_3+ ------ + m_n}} \quad (6.22)$$

$$\mathbf{Y} = \frac{\mathbf{m_1y_1+m_2y_2+m_3x_3+ ----- +m_ny_n}}{\mathbf{m_1+m_3+m_3+ ------ + m_n}} \quad (6.23)$$

$$\mathbf{Z} = \frac{\mathbf{m_1z_1+m_2z_2+m_3z_3+ ----- +m_nz_n}}{\mathbf{m_1+m_3+m_3+ ------ + m_n}} \quad (6.24)$$

6.2.1 Velocity and Acceleration of CM

Since the velocity V is defined as the rate of change of displacement with time then

$$\vec{V} = \frac{d\vec{R}}{dt} = \frac{m_{1\vec{v}_1}+m_{2\vec{v}_2}+m_{3\vec{v}_3}+-----+m_{n\vec{v}_n}}{m_1+m_2+m_3+ ----- +m_n} \quad (6.25)$$

$$= \vec{M} = \frac{d\overrightarrow{p_1}}{dt} + \frac{d\overrightarrow{p_2}}{dt} + \frac{d\overrightarrow{p_3}}{dt} + \cdots + \frac{d\overrightarrow{p_n}}{dt}\frac{\sum\vec{p}_i}{\sum m_i} = \frac{\vec{P}_{total}}{M} \quad (6.26)$$

The acceleration A is the rate of change of velocity with time

$$\vec{A} = \frac{d\vec{V}}{dt} = \frac{d^2R}{dt^2} \qquad (6.27)$$

$$M\vec{A} = \frac{d\vec{P}_1}{dt} + \frac{d\vec{P}_2}{dt} + \frac{d\vec{P}_3}{dt} + ------- + \frac{d\vec{P}_n}{dt} \qquad (6.28)$$

$$= \frac{d\vec{P}_{total}}{dt} = \vec{F}_{ext.} \qquad (6.29)$$

We see from the above that the CM moves like a single point mass i.e. it moves with constant velocity in the absence of the external force and accelerates when the resultant force is non-zero. The CM therefore exhibit a simple motion irrespective of the motion of the constituent bodies.

Examples

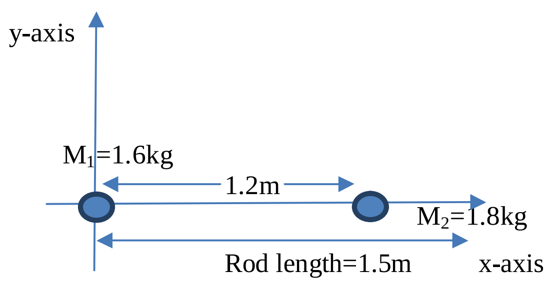
Q 1.)Two masses are placed on a 1.5 m long massless rod. The masses are arranged such that a 1.6 kg mass is placed at the left end of the rod and a 1.8 kg mass is placed at 1.2 m from the left end.

a.) What is the location of the CM ?

b.) By moving the 1.8 kg mass, can you arrange to have the CM in the middle of the rod?

Soln:

a.) Arrange the masses on a 1-D coordinate such that the left most mass is placed at the origin (see Fig. 1)



Fig,1

$$m_1 = 1.6\,kg, \quad x_1 = 0, \quad m_2 = 1.8\,kg, \quad x_2 = 1.2\,m, \qquad (6.30)$$

$$X = \frac{m_1x_1 + m_2x_2}{m_1 + m_2} \qquad (6.31)$$

$$X = \frac{1.6\times0 + 1.8\times2}{1.6+1.8} = 0.64m \qquad (6.32)$$

b.) Let the 1.8 kg mass be at x_2 and the mid-point be $x_m = 0.75$ m. If the CM is located at x_m, then $X = 0.75m$

$$m_1 = 1.6\ kg, \quad x_1 = 0, \quad m_2 = 1.8\ kg, \quad x_2 = ? \qquad (6.33)$$

$$X = \frac{m_1x_1 + m_2x_2}{m_1 + m_2} \qquad (6.34)$$

$$x_2 = \frac{(m_1 + m_2)X - m_1x_1}{m_2} \qquad (6.35)$$

$$x_2 = \frac{0.75\times3.4 + 1.6\times0}{1.8} = 1.4m \ \ it\ is\ possible \qquad (6.36)$$

Q 2.)In the question above, determine the CM of the new system if a third mass of 2.3 kg is placed at a distance of 1.1 m vertically above the 1.6 kg such that a straight line connects the CM of the 1.6 kg and that of 2.3 kg.
Soln:
This is now a 2-D configuration, and the x,y coordinate system will be considered (see Fig. 2).

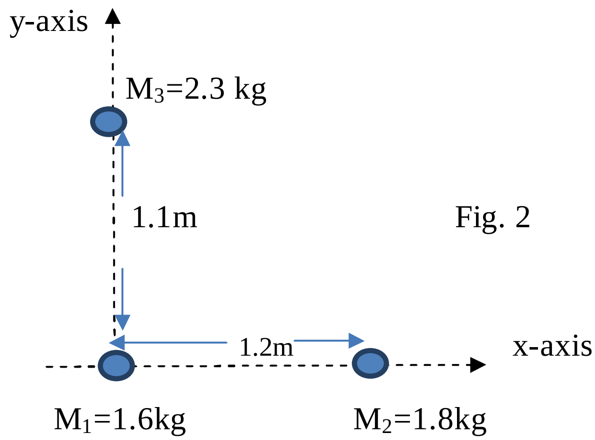


Fig. 2

$m_1 = 1.6kg, x_1 = 0,\ y_1 = 0, m_2 = 1.8\ kg,\ x_2 = 1.2m, y_2 = 0\ m_3 = 2.3kg, x_3 = 0, y_3 = 1.1m.$ (6.37)

$$X = \frac{m_1x_1+m_2x_2+m_3x_3}{m_1+m_3+m_3} \quad Y = \frac{m_1y_1+m_2y_2+m_3x_3}{m_1+m_3+m_3} \tag{6.38}$$

$$X = \frac{1.6\times0+1.8\times1.2+2.3\times0}{1.6+1.8+2.3} = 0.38m \tag{6.39}$$

$$Y = \frac{1.6\times0+1.8\times1.2+2.3\times1.1}{1.6+1.8+2.3} = 0.44m \tag{6.40}$$

$$R = 0.38\hat{\imath} + 0.44\hat{\jmath}\,, |R| = \sqrt{0.38^2 + 0.44^2} = 0.58m, \theta = \tan^{-1}(0.44/0.38)\ \mathit{From\ the\ origin.} \tag{6.41}$$

Exercise 1

1.)Two masses $m_1 = 15$ kg amd $m_2 = 25$ kg are joined by connecting a rod of length 0.8 m. Determine the distance of the CM of the system from the m_1 if a.) the connecting rod is massless, and b.) the connecting rod is a uniform rod of mass 15 kg.

2.)Find the location of the CM of a system of three particles arranged such that two of the particles with mass m are separated by distance l along the y-axis. The thimrd mass 2m lies on the x-axis and is separated by a distance l from one of the masses. (m = 12kg and L=1.5 m)

HINTS

1. You are asked to determine CM of the system from m1,

a. This is similar to what we have in Fig.1, which implies *that* $m_1 = 15\ kg, x_1 = 0.0m,\ m_2 = 25\ kg$ *and* $x_2 = 0.8m$ (i.e. the length of the rod).

b. If the rod is a uniform rod of mass 15kg , the CM of a uniform object(of any shape) is at the center. Therefore, it will appear as if there is a third mass on the system located at the middle I.e. 0.4m from m1
$m_1 = 15\ kg, x_1 = 0.0m,\ m_2 = 25\ kg$ *and* $x_2 = 0.8m$

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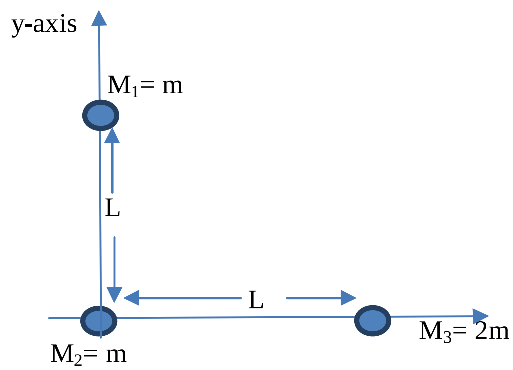
$m_{rod} = 15\ kg, x_{rod} = 0.4m$
then you can obtain CM(i.e. x)for this system.

2. Diagram



Taking m_2 as the origin

$m_1 = m = 12\ kg \quad x_1 = 0 \quad y_1 = 0$
$m_2 = m = 12\ kg \quad x_2 = 0 \quad y_2 = L = 1.5\ m$
$m_3 = 2m = 24\ kg \quad x_3 = l = 1.5\ m \quad y_3 = 0$

Then follow the steps in example 2 to obtain $\vec{R}$ *and* θ.

6.2.2 Continous Mass Distribution

A solid object is a many-particle system consisting of tiny particles that are not visible to our naked eyes. Such systems are considered as CONTINUUM OF MATTER. The CM of a continous distribution of mass may be defined by considering a rod of length L (with ends a and b) divided into equal length ΔX but not necessarily of equal mass unless the mass distribution is uniform. If the i-th segment has a mass Δm_i, the local mass density is

given by $\frac{\Delta m_i}{\Delta x}$. The total mass of the rod M may then be expressed as

$$M = \Delta m_1 + \Delta m_2 + \Delta m_3 + \cdots + \Delta m_n = \left(\frac{\Delta m_1}{\Delta x} + \frac{\Delta m_2}{\Delta x} + \frac{\Delta m_3}{\Delta x} + \cdots + \frac{\Delta m_n}{\Delta x}\right)\Delta x. \quad (6.42)$$

$$= \sum_{i=1}^{n} \frac{\Delta m_i}{\Delta x}\ \Delta x. \quad (6.43)$$

$$= \int_a^b \left(\frac{dm}{dx}\right) dx \qquad For\ infinitesimal\ \ \Delta x \quad (6.44)$$

The corresponding CM of the linear mass distribution may then be written as

$$X = \frac{\Delta m_1 x_1 + \Delta m_2 x_2 + \Delta m_3 x_3 + \cdots + \Delta m_n x_n}{M} \quad (6.45)$$

$$X = \frac{1}{M}\left[\frac{\Delta m_1}{\Delta x}x_1 + \frac{\Delta m_2}{\Delta x}x_2 + \frac{\Delta m_3}{\Delta x}x_3 + \cdots + \frac{\Delta m_n}{\Delta x}x_n\right]\Delta x. \quad (6.46)$$

$$= \frac{1}{M}\int_a^b \left(\frac{dm}{dx}\right) x dx\ . \qquad \text{For infinitesimal } \Delta x \quad (6.47)$$

$$= \frac{1}{M}\int_a^b \lambda x dx\ . \quad (6.48)$$

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where $\lambda = \frac{dm}{dx}$ is the mass density.

Example

Q.1) Consider a rod of length L, whose mass density is given by $\lambda = C(1 + ax^2)$, where x is the distance from the light end and C is a constant with a dimension of mass per unit length. Obtain the expression for CM of the rod.

Solution:

Require to determine the CM X given eqns 4.29 and 4.33.

$$\lambda = C(1 + ax^2), \quad X = ? \tag{6.49}$$

$$M = \int_0^l \left(\frac{dm}{dx}\right) dx \, . \tag{6.50}$$

$$= C|x + a\,x^3/3|_0^l = C(L + a\,L^3/3) \tag{6.51}$$

$$X = \frac{1}{M}\int_0^l \lambda x dx = \quad \frac{C}{M}\int_0^l x(1 + ax^2)dx \, . \tag{6.52}$$

$$= \frac{C}{M}|x^2/2 + a\,x^4/4|_0^l = \frac{[L^2/2+aL^4/4]}{[L+aL^3/3]} = \left(\frac{L}{2}\right)\frac{1+aL^2/2}{1+aL^2/3} \tag{6.53}$$

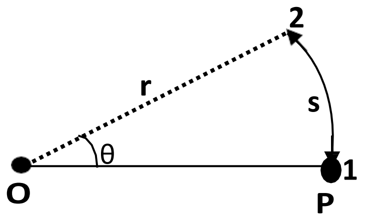
PHY 101: MECHANICS & PROPERTIES OF MATTER I

1.0 DYNAMICS OF A RIGID BODY

The study of motion which describes the way rigid objects rotate to determine their velocity, their acceleration, and their other parameters. Dynamics consider the forces that affect the motion of rotating objects. A **rigid** object is one that is non-deformable, that is, the relative locations of all particles of which the object is composed remain constant under the influence of applied force.

1.1 ROTATION OF A RIGID BODY

Suppose a point P (figure below) moves along a circular path from 1 to 2, in time Δt (at point 1, $t = 0$),



r is the radius of the circle and ***θ*** is the angle line OP moved through, from point 1 to 2,

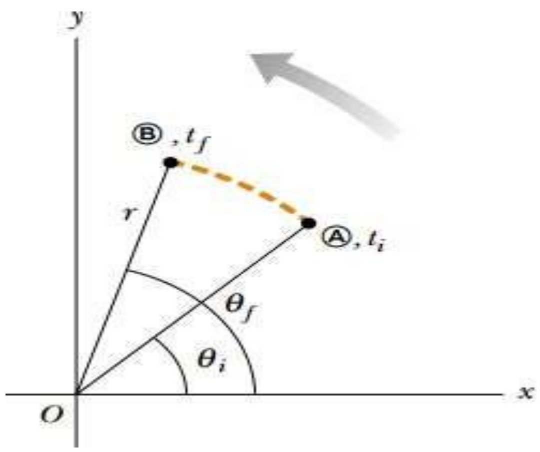
s is the length of the circular arc travelled by P.

Generally, $\theta = \frac{s}{r}\ (radians)$ 1.1

Because θ is the ratio of an arc length and the radius of the circle, it is a pure number. However, we commonly give θ the artificial unit radian (rad), where one radian is the angle subtended by an arc length equal to the radius of the arc, i.e.,

$1\ \text{revolution} = 360^{\circ} = 2\pi\ rad;\ 1\ rad = \frac{360^{\circ}}{2\pi} = 57.3^{\circ} = 0.159\ \text{rev.}$

Figure of a particle on a rotating rigid object moves from A to B along the arc of a circle. In the time interval $\Delta t = t_f - t_i$, the radius vector moves through an angular displacement $\Delta\theta = \theta_f - \theta_i$:



Angular displacement ($\Delta\theta$): For a particle on a rotating rigid object such that it moves from point 1 to 2 along the arc of a circle (Figure above) from an initial angle θ_i to θ_f the angular displacement is

$$\Delta\theta = \theta_f - \theta_i \qquad 1.2$$

The **average angular velocity $\bar{\omega}$** of a rotating rigid object is the ratio of the angular displacement $\Delta\theta$ to the time interval Δt.

That is, $$\bar{\omega} = \frac{\Delta\theta}{\Delta t} = \frac{\theta_f - \theta_i}{t_f - t_i} \qquad 1.3$$

The unit is radians per second (*rad/s*).

Note : **ω** is positive for counterclockwise motion and negative for clockwise motion.

The **average angular acceleration $\bar{\alpha}$** of an object is defined as the ratio of the change in the angular speed to the time Δt it takes the object to undergo the change.

That is, $$\bar{\alpha} = \frac{\Delta\omega}{\Delta t} = \frac{\omega_f - \omega_i}{t_f - t_i} \qquad 1.4$$

The unit is radians per square second (rad/s^2).

The quantities θ, ω, and α characterize the rotational motion of the entire rigid object as well as individual particles in the object. Using these quantities, we can greatly simplify the analysis of rigid-object rotation.

Angular position (θ), angular speed (ω), and angular acceleration (α) are analogous to linear position (x), linear speed (v), and linear acceleration (a). The variables θ, ω, and α differ dimensionally from the variables x, v, and a only by a factor having the unit of length.

The relationship between linear velocity and the angular velocity of a particle, moving along a circular path, at a point is $v = \omega r$. 1.5a

The relationship between the linear acceleration and the angular acceleration given by:

$$a = r\alpha \qquad 1.5b$$

Rotational Motion About Fixed Axis	Linear Motion
$\omega_f = \omega_i + \alpha t$	$v_f = v_i + at$
$\theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2$	$x_f = x_i + v_i t + \frac{1}{2}at^2$
$\omega_f^2 = \omega_i^2 + 2\alpha(\theta_f - \theta_i)$	$v_f^2 = v_i^2 + 2a(x_f - x_i)$
$\theta_f = \theta_i + \frac{1}{2}(\omega_i + \omega_f)t$	$x_f = x_i + \frac{1}{2}(v_i + v_f)t$

1.1.1 ROTATION OF A RIGID BODY WITH UNIFORM ANGULAR ACCELERATION

The tangential component of the linear acceleration of a point on a rotating rigid object equals the point's distance from the axis of rotation multiplied by the angular acceleration.

$$a_t = r\alpha \qquad 1.6a$$

The radial acceleration: we can also express the radial acceleration at that point in terms of angular speed as

$$a_r = \frac{v^2}{r} = \omega^2 r \qquad 1.6b$$

The total linear acceleration vector at the point is **a = $\mathbf{a_t}$ + $\mathbf{a_r}$**, where the magnitude of $\mathbf{a_r}$ is also the centripetal acceleration $\mathbf{a_c}$. Because a is a vector having a radial and a tangential component, the magnitude of **a** at the point P on the rotating rigid object is

$$a = \sqrt{a_t^2 + a_r^2} = r\sqrt{\alpha^2 + \omega^4} \qquad 1.6c$$

Example: A wheel rotates with a constant angular acceleration of 3.50 rad/s^2.

(a) If the angular speed of the wheel is 2.00 rad/s at t_i = 0, (i) through what angular displacement does the wheel rotate in 2.00 s?

Solution:

$$\Delta\theta = \theta_f - \theta_i = \omega_i t + \frac{1}{2}\alpha t^2$$

$$= (2rad/s)(2\ \cancel{s}) + \frac{1}{2}(3.5rad/s^2)(2s)^2$$

= 11.0 rad

(ii) Convert 11 radians to degrees

Solution:

If 1 $rad = 57.3^o$, then 11.0 rad will be $= 11.0 \times 57.3^o = 630.3^o$

(b) Through how many revolutions has the wheel turned during this time interval?

Solution: We multiply the angular displacement found in part a(i) by a conversion factor to find the number of revolutions: (note that 1 rev = 360^o)

$$\Delta\theta = 630.3^o \times \left(\frac{1\,rev}{360^o}\right) = 1.75\,rev$$

(c) What is the angular speed of the wheel at t = 2.00 s?

Solution:

$$\omega_f = \omega_i + \alpha t = 2rad/s + (3.50\,rad/s^2)(2\,s)$$

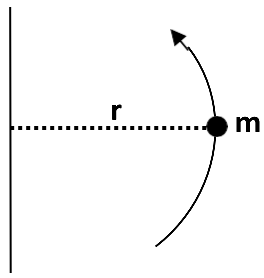
$= 9.00$ rad/s

1.2 MOMENT OF INERTIA OF A RIGID BODY

The resistance of an object to change its state of rotational motion about an axis is known as moment of inertia (or rotational inertia). It depends on the mass of the object and the distribution of its mass from the axis of rotation.

Consider a particle of mass m rotating about an axis through a radius r :

axis of rotation



The moment of inertia of the particle, $I = mr^2$ 1.7a

However, if we have a rigid body that consists of n particles of masses $m_1, m_2, m_3, \ldots\ldots m_n$ at distances $r_1, r_2, r_3, \ldots\ldots r_n$ from the axis of rotation,

moment of inertia, $I = (m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + \cdots\cdots\cdots + m_nr_n^2)$

$$= \sum_{i=1}^{n} m_i r_i^2 \qquad 1.7b$$

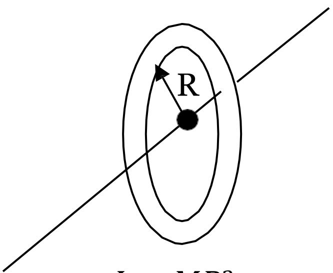
In integral form, $I = \int_{i=1}^{n} r^2\, dm$ 1.7c

i

1.2.1 MOMENT OF INERTIA OF SOME BODIES

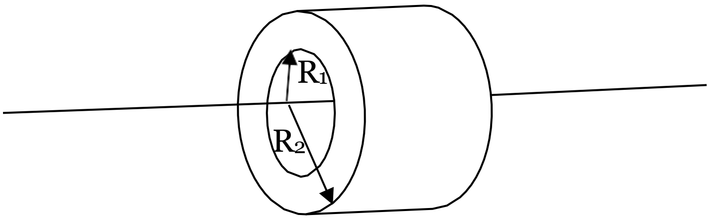
i. Hoop (Ring) about central axis

axis of rotation



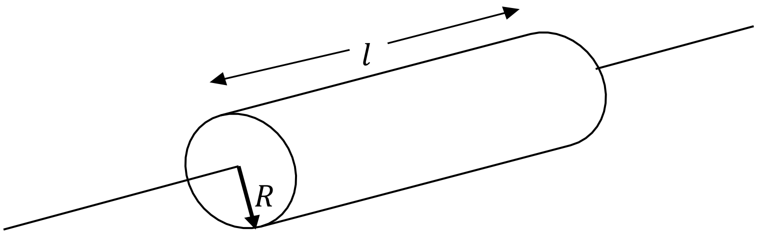
$$I = MR^2 \qquad 1.7d$$

ii. Annular cylinder about central axis



$$I = \frac{1}{2}M(R_1^2 + R_2^2) \quad 1.7e$$

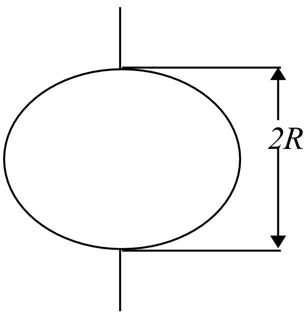
iii. Solid cylinder (or disk) about central axis



$$I = \frac{1}{2}MR^2 \quad 1.7f$$

iv. Solid sphere about any diametrical axis

Axis of rotation

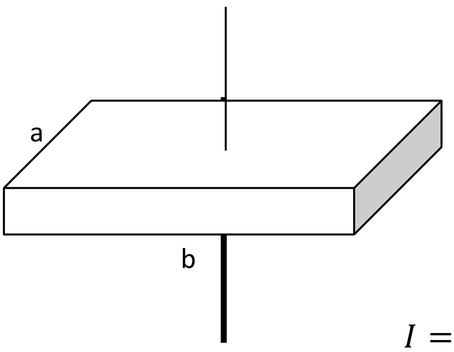


$$I = \frac{2}{5}MR^2 \quad 1.7g$$

Note: For a Thin spherical shell, the moment of inertia is $I = \frac{2}{3}MR^2$ 1.7h

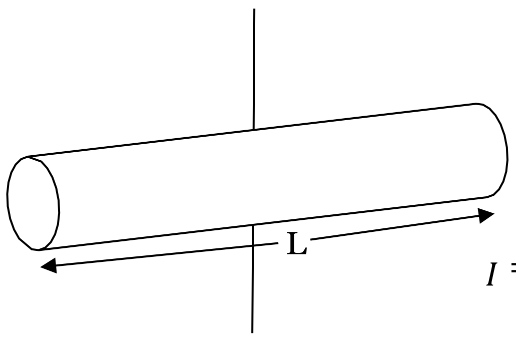
v. Rectangular slab about perpendicular axis through centre

Axis of rotation



$$I = \frac{1}{12}M(a^2 + b^2) \quad 1.7i$$

vi. Thin rod about axis through its centre perpendicular to length



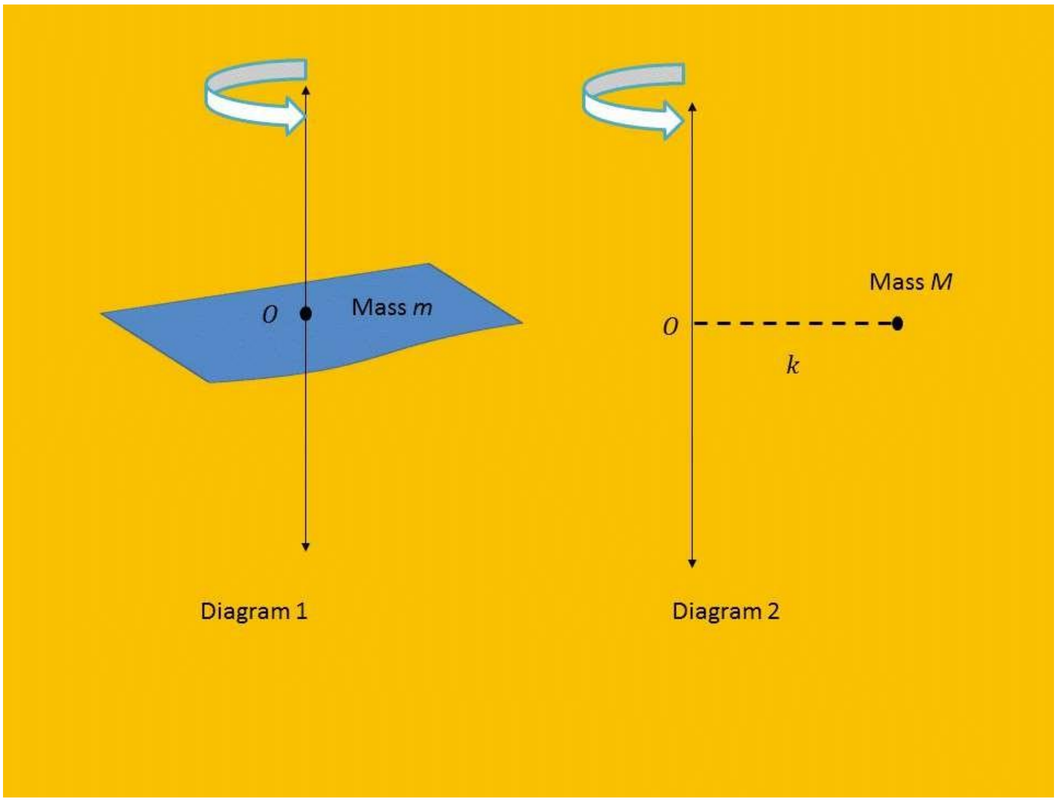
$$I = \frac{1}{12}ML^2 \qquad 1.7j$$

Note: The moment of inertia of the thin rod about axis through its end is

$$I = \frac{1}{3}ML^2. \qquad 1.7k$$

1.3 RADIUS OF GYRATION

The moment of inertia of a body about an axis is sometimes represented using the radius of gyration. Radius of gyration is defined as the imaginary distance from the centroid at which the area of cross section is imagined to be focused at a point in order to obtain the same moment of inertia. Simply, gyration is the distribution of the components of an object.



Consider the two diagrams above.

In diagram 1, we have a shape rotating about an axis. If the mass is m then its moment of inertia is

$$I = mr^2 \qquad 1.8$$

In diagram 2, we have the same axis of rotation but this time the same mass is concentrated at a point. Its moment of inertia is

$$I = Mk^2 \qquad 1.9a$$

If the two values of the moments of inertia are the same, then k is called the radius of gyration

In other words, the radius of gyration is the distance from the axis of rotation to a point where the total mass of the body is supposed to be concentrated, so that the moment of inertia about the axis may remain the same. Or, the distance from the axis of rotation at which the total mass of a body is assumed concentrated and at which the moment of inertia will be equal to the actual moment of inertia of the body.

So, $\quad I = Mk^2$

$$\text{k} = \sqrt{\frac{I}{M}} \qquad 1.9b$$

Suppose a body consists of **n** particles each of mass **m**. Let $r_1, r_2, r_3, \dots, r_n$ be their perpendicular distances from the axis of rotation. Then, the moment of inertia **I** of the body about the axis of rotation is

$$I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \cdots + m r_n^2 \qquad 1.10a$$

If all the particles are of same mass **m**, then

$$I = m(r_1^2 + r_2^2 + r_3^2 + \cdots + r_n^2)$$

$$= \frac{nm(r_1^2+r_2^2+r_3^2+\cdots+r_n^2)}{n} \qquad 1.10b$$

Since $\boldsymbol{nm} = \boldsymbol{M}$, total mass of the body,

$$\text{I} = \frac{\text{M}(r_1^2+r_2^2+r_3^2+\cdots+r_n^2)}{\text{n}} \qquad 1.10c$$

From the above equations, we have

$$MK^2 = \frac{M(r_1^2+r_2^2+r_3^2+\cdots+r_n^2}{n}$$

$$K = \sqrt{\frac{(r_1^2+r_2^2+r_3^2+\cdots+r_n^2}{n}} \qquad 1.10d$$

$=$ root mean square distance of the particles from the axis of rotation

Therefore, the radius of gyration of a body about a given axis may also be defined as the root mean square distance of the various particles of the body from the axis of rotation.

Radius of gyration can be applied to various rigid objects using their moments of inertia as it is determined with some examples below:

(a) Radius of Gyration of Uniform Thin rod

The moment of inertia of uniform thin rod of mass **M** and length **L** about an axis through its centre and perpendicular to its length is given by

$$I = \frac{1}{12}ML^2$$

if **k** is the radius of gyration of the rod about the axis, then we have

$$I = Mk^2$$

From the above equations, we have

$$Mk^2 = \frac{1}{12}ML^2$$

$$k = \frac{L\sqrt{3}}{6} \qquad 1.11a$$

(b)Radius of Gyration of a Solid Sphere

The moment of inertia for a solid sphere of radius **R** and mass **M** is given by

$$I = \frac{2}{5}MR^2$$

If **k** is the radius of gyration of the solid sphere, then

$$I = Mk^2$$

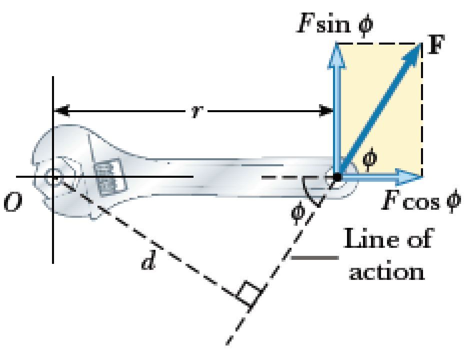
Or, $Mk^2 = \frac{2}{5}MR^2$

So that, $k = R\sqrt{\frac{2}{5}}$ 1.11b

1.4 TORQUE

Force is known to be the cause of linear acceleration while torque causes rotational acceleration. An object remains in a state of uniform rotational motion unless acted on by a **net torque**. This principle is the equivalent of Newton's first law.

The tendency of a force to rotate an object about some axis is measured by a vector quantity called torque ($\boldsymbol{\tau}$). The direction of the torque vector depends on the direction of the force on the axis.



Rotating wrench about point O

Consider the wrench pivoted on the axis through O as shown in the figure.

Here, **r** is the length of the position vector, **F** is the magnitude of the applied force and **d** is the

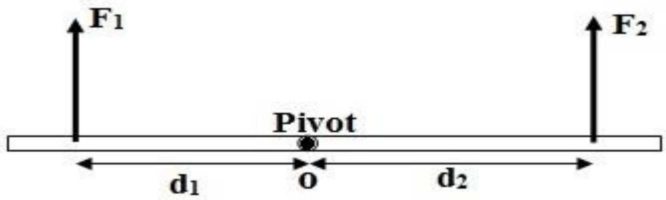
perpendicular distance from the pivot point to the line of action of F. The applied force **F** acts at an angle **ϕ** to the horizontal.
From the figure, the quantity **d** is equal to rsinϕ and it is called the MOMENT ARM (or LEVER ARM).
The magnitude of the torque is given by

$$\tau = r \times F \sin\phi = F.d = r \times F \qquad 1.12$$

S.I unit of torque is Newton meter (N.m)

It can be seen from the figure that the only component of **F** that tends to make the wrench rotate is Fsinϕ. Looking at the horizontal component, it has no tendency to produce rotation. From the definition of torque, the rotating tendency increases as F increases and as d increases. This explains the observation that it is easier to close a door if we push at the door knob rather than at a point close to the hinge. If a force **F** is applied to a door, there are three factors that determine the effectiveness of the force in opening the door: the magnitude of the force, the position of application of the force, and the angle at which it is applied.

If two or more forces are acting on a long rod as shown in figure 2; each tend to produce rotation about the pivot at O



A long rod being rotated by two forces F_1 and F_2

By convention, when an applied force causes an object to rotate counterclockwise, the torque on the object is positive. When the force causes the object to rotate clockwise, the torque on the object is negative.
In the figure above, F_1 rotates the rod clockwise and F_2 rotates it counter-clockwise.
The torque from $F_1 = -F_1d_1$ and from $F_2 = F_2d_2$. Hence the net torque about O is

$$\sum\tau = \tau_1 + \tau_2 = F_1d_1 - F_2d_2$$

Torque and the two conditions for equilibrium

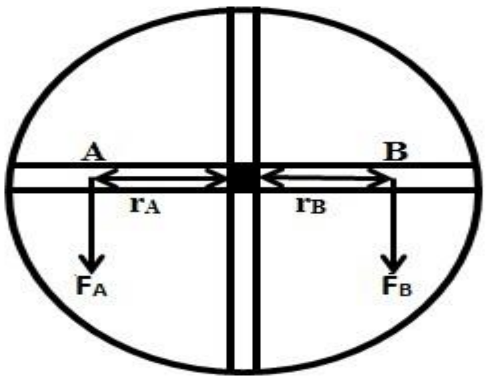
For an object to be in mechanical equilibrium it must satisfy the following two conditions;
1. The net external force must be zero $\sum F = 0$
2. The net external torque must be zero $\sum\tau = 0$

The first condition is a statement of translational equilibrium while the second condition is a statement of rotational equilibrium
For the first condition; translational acceleration, **a** = 0

For the second condition; angular acceleration, $\boldsymbol{\alpha}$ = 0

For an object to be in equilibrium, it must move through space at constant speed and rotate a constant speed

Example: Two men at point A and B are attempting to use a revolving door as shown in the figure below. The person at A exerts a force of 700 N perpendicular to the door and 0.20 m from the hub's centre, while the second person (B) exerts a force of 500 N perpendicular to the door and 0.60 m from the hub's centre. Find the net torque on the door



Solution

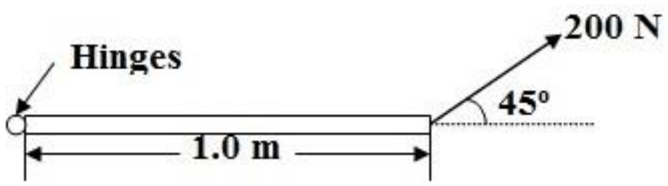
$$\tau_A = r_A F_A = 0.20 \times 700 = 1.40 \times 10^2\ Nm$$

$$\tau_B = -r_B F_B = 0.60 \times 500 = -3.0 \times 10^2\ Nm$$

Net torque $\tau_{net} = \tau_A + \tau_B = 1.40 \times 10^2\ Nm - 3.0 \times 10^2 = -1.60\ \times 10^2\ Nm$

Negative results indicate thatτ_{net} will produce a clockwise rotation

Example: A force of 200 N is applied by a man to a door 1 m from the hinges as shown in the figure below. Choosing the position of the hinges as the axis of rotation, find the torque on the door.

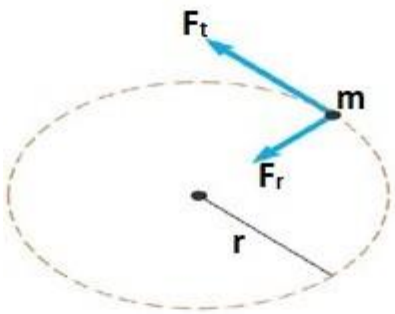


Solution: Applying the torque equation i.e $\tau = rFsin\phi$

$\tau = (1)(200)sin45^o$ = 141.42 N.m

Relationship between torque and angular acceleration

In the figure shown below a particle of mass **m** rotating in a circle of radius **r**. Two forces keep the particle moving along the circular path i.e tangential force $\mathbf{F_t}$ and radial force $\mathbf{F_r}$.



A particle of mass m rotating in a circle under the influence of two forces F_t and F_r

The tangential force (F_t) = ma_t, where a_t is the tangential acceleration.

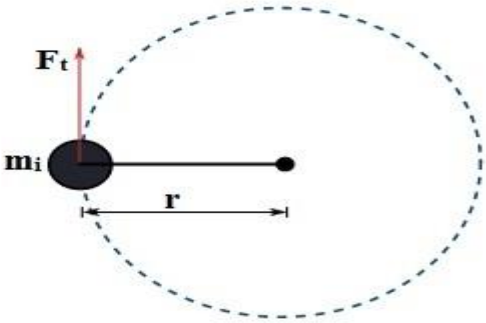
The torque about the centre of the circle due to F_t is:

$$\tau = r.F_t = (ma_t)r \qquad 1.13a$$

but $a_t = r\alpha$ (relationship between linear acceleration and angular acceleration)

Therefore $$\tau = (mr\alpha)r = (mr^2)\alpha = I\alpha \qquad 1.13b$$

The quantity mr^2 is known as the **moment of inertia (I)** of a rotating particle about an axis through the origin.



A rigid body with many particles rotating in a circle

In case of a rigid body having particles with masses m_1, m_2…... m_i as represented in the figure, the net torque on the body is given by the sum of the individual torques on all the particles:

$$\sum\tau = (\sum mr^2)\alpha \qquad 1.13c$$

Because the body is rigid, all of its particles have the same angular acceleration, so **α** is not included in the sum.

$$\sum mr^2 = m_1r_1^2 + m_2r_2^2 \dots\dots\dots\dots\dots \qquad 1.13d$$

The moment of inertial for the whole rigid body is

$$I = \sum mr^2 \qquad 1.13e$$

Hence,

$$\sum\tau = I\alpha \qquad 1.13f$$

Equation 1.13f is the rotational equivalent of Newton's second law of motion, with torque replacing force, moment of inertia replacing mass, and angular acceleration replacing linear acceleration. Even though the moment of inertia **I** of an object is related to its mass **m**, there is an important difference between them. The mass **m** depends only on the quantity of matter in an object, whereas the moment of inertia, **I**, depends on both the quantity of matter and its distribution (through the r^2 in equation in the rigid body.

Example

A motorcycle wheel is found to rotate with constant acceleration when a torque of 1.02 Nm is applied to it. If the mass and radius of the wheel is 0.50 kg and 0.30 m respectively, what is the value of the angular acceleration and the tangential force?

Solution:

From, $\tau = I\alpha$; $\alpha = \frac{\tau}{I}$ but $I = mr^2$

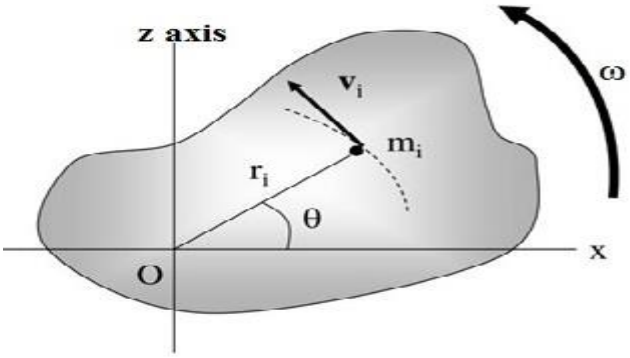
Therefore, $\alpha = \frac{\tau}{mr^2} = \frac{1.02}{(0.5)(0.3)^2} = 22.67$ rad/s^2

1.5 ROTATIONAL KINETIC ENERGY

Consider a particle of mass $\mathbf{m_i}$ rotating about a fixed axis with an angular speed $\boldsymbol{\omega}$ and a tangential speed $\mathbf{v_i}$ as shown in the figure shown below.

The kinetic energy of the particle is giving as;

$$K.E_i = \frac{1}{2} m_i v_i^2 \qquad 1.14a$$



A rigid plate rotating about the z-axis with angular speed

The total kinetic energy of the rigid object is the sum of the kinetic energies of the individual particles

$$K.E_T = \sum K.E_i = \sum_i \frac{1}{2} m_i v_i^2 \qquad 1.14b$$

$$v_i = \omega r_i$$

Hence;

$$\sum K.E_T = \frac{1}{2} \sum_i m_i {r_i}^2 \omega^2 \qquad 1.14c$$

But, $\sum_i m_i {r_i}^2 = I$

Therefore,

$$K.E_T = \frac{1}{2} I\omega^2 \qquad 1.14d$$

Equation 11 is analogous to $\frac{1}{2}mv^2$; the kinetic energy of a particle moving through space with a linear speed **v**.

1.6 ROTATIONAL WORKDONE

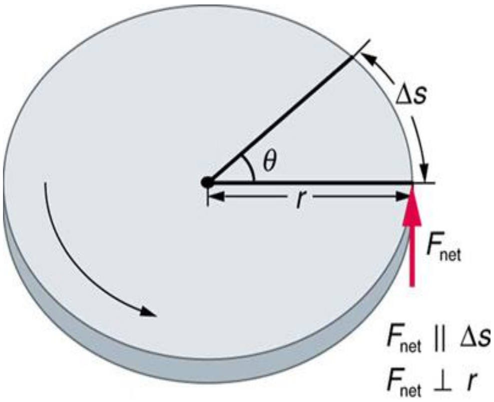
Consider a simple rotational situation in which the net force F_{net} is exerted perpendicular to the radius of a disk (as shown in the figure) and remains perpendicular as the disk starts to rotate. If the disk travels through arc length Δs, then,

$$W_{net} = F_{net} \times \Delta s \qquad 1.15a$$

But $\Delta s = r\Delta\theta$ and $\tau = rF_{net}$

where θ is the angular distance and τ is the torque

Hence,
$$W_{net} = \tau\Delta\theta \qquad 1.15b$$



A disk rotated by a force F_{net} perpendicular to its radius

1.7 ROTATIONAL POWER

The power is defined as the rate at which work is been done

i.e
$$P = \frac{\Delta W}{\Delta t} \qquad 1.16a$$

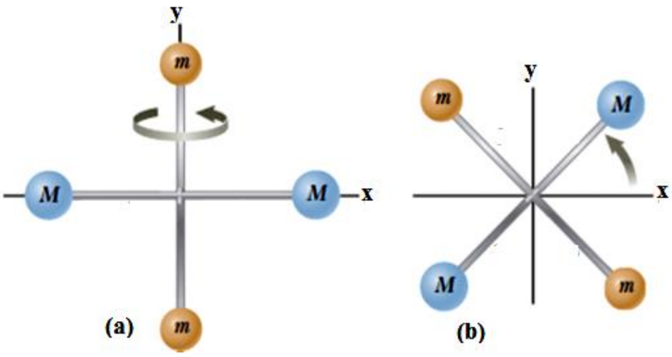
$$P = \frac{\tau\Delta\theta}{\Delta t} \qquad 1.16b$$

$$P = \tau\omega \qquad 1.16c$$

where ω is the angular velocity.

Example

The figures below show a system consist of four tiny spheres with masses M = 4.0 kg and m = 3.0 kg, fastened to the end of very light rods, each with length 2.0 m and arranged in different configuration. In each case, (i) find the moment of inertia of the system about the axis perpendicular to the page of the paper and passing through the point where the rods cross. (ii) find the rotational kinetic energy if the angular speed of the system is 12 rad/s.



Solution

(i) For figure **(a),**

Moment of inertia, I = $\sum mr^2 = mr_1{}^2 + mr_2{}^2 + Mr_3{}^2 + Mr_4{}^2$

= (0.30) $(0)^2$ + (0.30)$(0)^2$ + (0.40)$(1.0)^2$ + (0.40)$(1.0)^2$

I = 0.80 kgm^2

For figure **(b),**

Moment of inertia, I = $\sum mr^2 = mr_1{}^2 + mr_2{}^2 + Mr_3{}^2 + Mr_4{}^2$

= (0.30)$(1.0)^2$+(0.30)$(1.0)^2$+(0.40)$(1.0)^2$+(0.40)$(1.0)^2$

= 0.3 + 0.3 + 0.4 + 0.4

I = 1.40 kgm^2

(ii) Using the equation, $K.E_T = \frac{1}{2}I\omega^2$

For figure (a), $K.E_T = \frac{1}{2} \times 0.80 \times 12^2 = 115$ J

For figure (b), $K.E_T = \frac{1}{2} \times 1.40 \times 12^2 = 201$ J

Exercise: If each of the four tiny spheres has a mass 0.4 kg but with one of the rods having a length of 2 m and the other with length 2.5m. Calculate the moment of inertia of the system (a) when oriented as figure **a** with the shorter rod vertical (b) when oriented as in figure **a** but with longer rod vertical (c) oriented as figure **b**

Example: A round grindstone with a moment of inertia 1200 kgm^2 rotates at an angular velocity of 10 $rads^{-1}$. What is grindstone's rotational kinetic energy?

Solution

Moment of inertia = 1200 kgm^2

Angular velocity = 10 $rads^{-1}$

From the rotational kinetic energy formula,

$$K.E = \frac{1}{2}I\omega^2 = \frac{1}{2} \times 1200 \times 10^2 = 60000\,J$$

Example: A boat engine operating at 6.0×10^5 W is running at 400 rev/min. What is the torque on the propeller shaft?

Solution

Angular velocity ω = 400 rev/min = $\frac{400\times2\times3.142}{60} = \frac{2513}{60} = 41.89\ rad/s$

P = $\tau\omega$; $\tau = \frac{P}{\omega}$

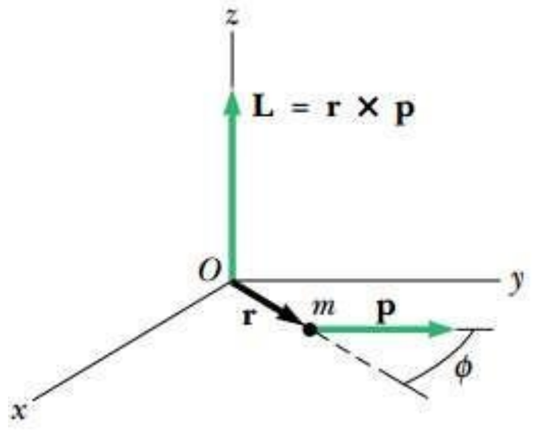
$$\tau = \frac{6\times10^5}{41.89}$$

τ = 14323.23 N.m

1.8 ANGULAR MOMENTUM

The angular momentum $\vec{L}$ of a particle of mass *m* moving along a circular path of radius *r* is defined as

$$\vec{L} = \vec{r}\times\vec{p} = m(\vec{r}\times\vec{v}) \qquad 1.17$$



Because **p** = *m***v**, the magnitude of **L** is **L** = m**vr** sin ϕ, where ϕ is the angle between **r** and **p**.

Therefore, **L** = 0 when **r** is parallel to **p** ($\emptyset = 0\ or\ 180^o$). On the other hand, if **r** is perpendicular to **p** ($\emptyset = 90^o$), then

L = mvr 1.18

The S.I. unit of angular momentum is kilogram square metre per second (kgm^2s^{-1}).

Notice that the angular momentum for a particle moving in a circular path has been defined. Even a particle moving in a straight line has angular momentum about any axis displaced from the path of the particle.

A particle in uniform circular motion has a constant angular momentum about an axis through the centre of its path.

Starting with equation 1.18, we can write

$$L = m\omega r^2 \quad \text{(since } v = \omega r\text{)} \qquad 1.19a$$

But $I = mr^2$

Hence, $L = I\omega$ 1.19b

From equation 1.13f,

$$\sum \tau = I\alpha = I\frac{d\omega}{dt} \qquad 1.19c$$

$$\sum \tau = \frac{dL}{dt} \qquad 1.19d$$

The net torque on a system equals time rate of change of angular momentum of the system

Torque causes the angular momentum L to change just as force causes linear momentum $\boldsymbol{p}$ to change, i.e., the torque acting on a particle is equal to the time rate of change of the particle's angular momentum.

Let L_i and L_f be the initial and final angular momenta of a system, and suppose there is no net external torque, so $\sum \tau = 0$. Then,

$$L_i = L_f \qquad 1.20$$

Equation 1.20 shows that the angular momentum of a system is said to be conserved.

Therefore,

$$I_i\omega_i = I_f\omega_f \qquad \text{if} \sum \tau = 0 \qquad 1.21$$

The total angular momentum of a system is constant in both magnitude and direction if the resultant external torque acting on the system is zero, that is, if the system is isolated

Example: Estimate the magnitude of the angular momentum of a bowling ball with a mass of 6 kg and radius of 12 cm spinning at 10 rev/s.

$$I = \tfrac{2}{5}MR^2 = \tfrac{2}{5}(6.0\text{ kg})(0.12\text{ m})^2 = 0.035\text{ kg}\cdot\text{m}^2$$

Therefore, the magnitude of the angular momentum is

$$L_z = I\omega = (0.035\text{ kg.m}^2)(10\text{ rev/s})(2\pi\text{ rad/rev})$$
$$= 2.2\text{ kg.m}^2/s$$

Example: A car of mass 1200 Kg moves with a linear speed of 40 m/s on a circular race track of radius 40 m. What is the magnitude of its angular momentum relative to the center of the track?

Solution

$$L = mvr = 1200 \times 40 \times 40 = 1920000\text{ Kgm}^2/\text{s}$$

$$L = 1.92 \times 10^6\text{ Kgm}^2/\text{s}$$

1.9 A Couple

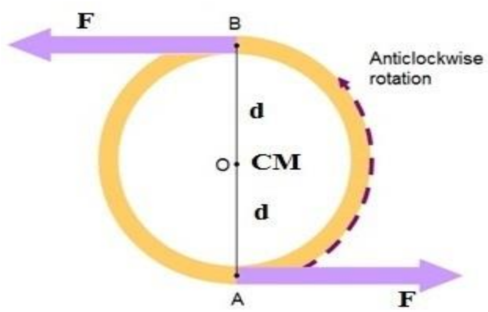


Diagram representing a couple

Suppose an object is pivoted about an axis through its centre of mass **CM** and two forces of equal magnitude act in opposite directions along parallel lines of action as shown in the figure. A pair of forces acting this way is referred to as a **COUPLE**. The net torque about the CM is 2Fd. The net torque will give rise to an angular acceleration α

$$\sum\tau = 2Fd = I\alpha \qquad 1.22$$

Useful Equations in Rotational and Linear Motion

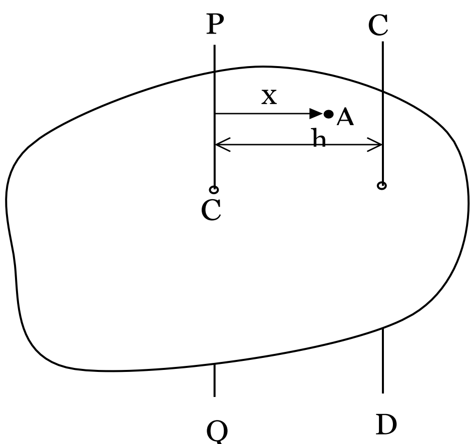
Rotational Motion About a Fixed Axis	Linear Motion
Angular speed $\omega = d\theta/dt$	Linear speed $v = dx/dt$
Angular acceleration $\alpha = d\omega/dt$	Linear acceleration $a = dv/dt$
Net torque $\Sigma\tau = I\alpha$	Net force $\Sigma F = ma$
If $\alpha =$ constant $\begin{cases} \omega_f = \omega_i + \alpha_t \\ \theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2 \\ \omega_f^2 = \omega_i^2 + 2\alpha(\theta_f - \theta_i) \end{cases}$	If $a =$ constant $\begin{cases} v_f = v_i + at \\ x_f = x_i + v_i t + \frac{1}{2}at^2 \\ v_f^2 = v_i^2 + 2a(x_f - x_i) \end{cases}$
Work $W = \int_{\theta_i}^{\theta_f} \tau\, d\theta$	Work $W = \int_{x_i}^{x_f} F_x\, dx$
Rotational kinetic energy $K_R = \frac{1}{2}I\omega^2$	Kinetic energy $K = \frac{1}{2}mv^2$
Power $\mathcal{P} = \tau\omega$	Power $\mathcal{P} = Fv$
Angular momentum $L = I\omega$	Linear momentum $p = mv$
Net torque $\Sigma\tau = dL/dt$	Net force $\Sigma F = dp/dt$

2.0 AXES THEOREM OF A RIGID BODY

The axes theorem responsible for the moment of inertia of a rigid body of mass M about an axis through its centre of gravity G when it rotates. These axes theorem, as enumerated below, will be treated with detail explanation as follows:

2.1.1 PARALLEL AXIS THEOREM

The moment of inertia of a body about any axis is equal to the moment of inertia I_G about a parallel axis through the centre of gravity of the body plus Mh^2, where M is the mass of the body and h is the distance between the two axes, i.e., $I = I_G + Mh^2$.



Let I be the MI about CD

I_G be the MI about PQ, axis through point of the centre of gravity CG,

PQ is parallel to CD and h is perpendicular distance away from CD

Let a unit particle A of mass m be at distance x from PQ axis.

Moment of inertia I of A about CD = $m(h - x)^2$

i.e., $I = \sum m(h - x)^2$

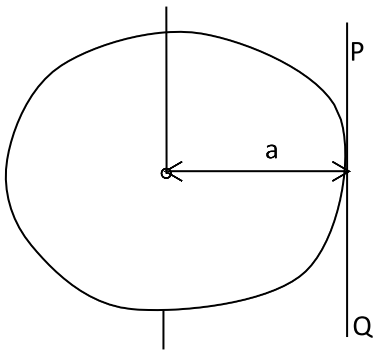
$I = \sum mh^2 + \sum mx^2 + \sum 2mhx$

But $\sum mh^2 = h^2 \sum m = Mh^2$ and $I_G = \sum mx^2$, where M is the total mass of the body.

Also, $\sum 2mhx = 2h \sum mx = 0$ since $\sum mx$ is the sum of moment about CG which is equal to 0. Using these simplified equations in combination, we have the parallel axis law proved:

$$I = I_G + Mh^2 \qquad 1.15$$

Example: What is the moment of I of a disc, whose radius is 'a' and mass is m, about an axis PQ through a point on its circumference perpendicular to its plane (Its $I_G = \frac{1}{2}Ma^2$).



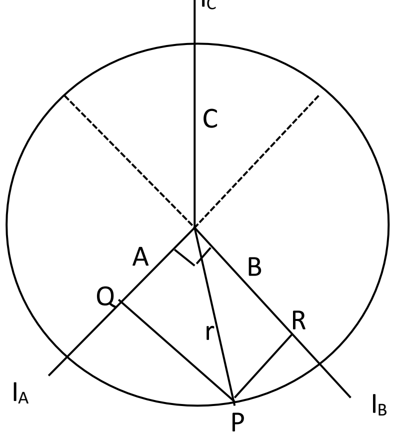
$I_{PQ} = I_G + Mh^2$ ....parallel axis theorem

So, $I = \frac{1}{2}Ma^2 + Ma^2$

i.e., $I_{PQ} = \frac{3}{2}Ma^2$

2.1.2 PERPENDICULAR AXIS THEOREM

The moment of inertia I of a body about any axis is the sum of the moment of inertia about any mutually perpendicular axes. This theorem is most useful when considering a body which is of regular form, i.e., symmetrical about two out of three axes. If the moment of inertia about these axes is mutually perpendicular to each other, then the third may be calculated.



Axes A and B are mutually perpendicular and are through symmetry of the body.

From the diagram, $I_C = \sum Mr^2$

But, $r^2 = (PQ)^2 + (PR)^2$

We now have the third one as

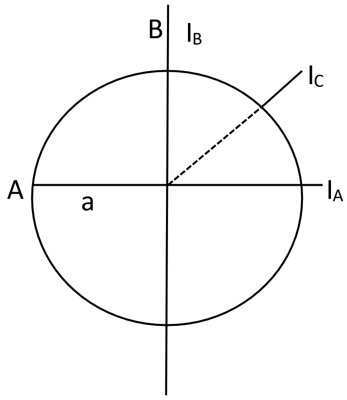
$I_C = \sum M[(PQ)^2 + (PR)^2]$

$I_C = \sum M(PQ)^2 + \sum M(PR)^2$

Using the simplified form, we have the perpendicular axis law proved:

$I_C = I_A + I_B$ 1.16

Example: Find the moment of inertia I of disc about axis through its diameter.



Using the perpendicular axis law expression:

$I_C = I_A + I_B$

where I_C = M.I. through its centre perpendicular to its plane, i.e., through its centre of mass.

Therefore, $I_C = \frac{1}{2}Mr^2$

But, M.I. about A is $\frac{1}{2}$ of $\frac{1}{2}Mr^2$ and the rest half is M.I. about B.

Axes A and B are parallel to the plane of the disc.

Thus M.I through diameter = $\frac{1}{4}Mr^2$

EXERCISES

1. A diver makes 5 revolutions on the way from a 12 m high platform to the water. Assuming zero initial vertical velocity, find the diver's average angular velocity during the dive.

2. A body of mass 1.5 kg is whirled in a circle by a string of length 0.5 m. If the period of revolution is π sec. Calculate the force on the body by the string.

3. The SUG bus goes half-way around the round-about in 1 s. Calculate the bus's angular velocity.

4. If the mass of a bus is 5,000 kg and the distance between the bus and the centre of the round-about is 4 m, calculate the moment of inertia of the bus.

5. A flywheel of a gasoline engine is required to give up 300 J kinetic energy while its angular velocity decreases from 600 rev/min to 540 rev/min. What is the moment of

inertia required?

6. Find the radial acceleration of an object at the equator due to the rotation of the earth.

7. An object of mass 4 kg moves round a circle of radius 6 m with a constant speed of 12 m/s. What is the angular speed of the object?

8. Compute the angular speed and angular acceleration of an automobile wheel 356 mm in radius when the car is moving at 72 km/h in 11s.

9. A stone of mass 20 kg is whirled round in a vertical circular pattern by a rope of negligible mass of length 0.8 m. That is the moment of inertia of the stone?

10. A curve in a road forms part of a horizontal circle. As a car of mass 1,500 kg goes around it at constant speed 14.0m/s, the total force on the driver has magnitude 130 N. What is the total vector force on the driver if the speed is 18.0m/s instead?

11. A hawk flies in a horizontal arc of radius 12.0 m at a constant speed of 4.0 m/s. Find its centripetal acceleration.

12. A roller coaster car has a mass of 500 kg when fully loaded with passengers. (a) If the vehicle has a speed of 20.0 m/s at point A as it rolls at 10 m, what is the force exerted by the track on the car at this point? (b) What is the maximum speed the vehicle can have at B as it now rolls at 15 m and still remain on the track?

13. A small container of water is placed on a carousel inside a microwave oven, at a radius of 12.0 cm from the center. The turntable rotates steadily, turning through one revolution in each 7.25 s. What angle does the water surface make with the horizontal?

14. During a certain period of time, the angular position of a swinging door is described by $\theta = 5.00 + 10.0t + 2.00t^2$, where θ is in radians and t is in seconds. Determine the angular position, angular speed, and angular acceleration of the door (a) at t = 0 (b) at t = 3.0 s.

15. A wheel starts from rest and rotates with constant angular acceleration to reach an angular speed of 12.0 rad/s in 3.0 s. Find (a) the magnitude of the angular acceleration of the wheel and (b) the angle in radians through which it rotates in this time.

16. A racing car travels on a circular track of radius 250 m. If the car moves with a constant linear speed of 45.0 m/s, find (a) its angular speed and (b) the magnitude and direction of its acceleration.

17. A uniform thin solid door has height 2.20 m, width 0.870 m, and mass 23.0 kg. Find its moment of inertia for rotation on its hinges.

18. A grind-wheel is started from rest given an angular acceleration of 3 rad/s^2. Calculate its (a) angular velocity, and (b) angular displacement 8 s after starting to rotate.

19. Calculate the torque needed to bring a disc, which is rotating at 78 rev/min to a stop within two revolutions after the torque is applied. The disc has a moment of inertia

280 kg.m^2.

20. A playground merry-go-round of radius R = 2.0 m has a moment of inertia I = 250 kg·m^2 and is rotating at 10.0 rev/min about a frictionless vertical axle. Facing the axle, a 25.0 kg child hops onto the merry-go-round and manages to sit down on the edge. What is the new angular speed of the merry-go-round?

21. A horizontal platform in the shape of a circular disk rotates freely in a horizontal plane about a frictionless vertical axle. The platform has a mass M=100 kg and a radius R = 2.0 m. A student whose mass is m = 60 kg walks slowly from the rim of the disk toward its centre. If the angular speed of the system is 2.0 rad/s when the student is at the rim, (a) what is the angular speed when he reaches a point r = 0.50 m from the centre? (b) What is the kinetic energy of the system before and after the student walks inward?

22. A uniform sphere with mass 28.0 kg and radius 0.380 m is rotating at constant angular velocity about a stationary axis that lies along a diameter of the sphere. If the kinetic energy of the sphere is 176 J, what is the tangential velocity of a point on the rim of the sphere?

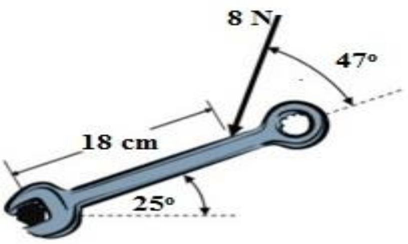
23. One force acting on a machine part is $\boldsymbol{F} = (-5.0)\hat{\imath} + (4.0)\hat{\jmath}$. The vector position from the origin to the point where the force is applied is $\boldsymbol{r} = (-0.45)\hat{\imath} + (0.15)\hat{\jmath}$.
(a) Determine the direction of the torque. (b) Calculate the vector torque for an axis at the origin produced by this force.

24. A playground merry-go-round has radius 2.40 m and moment of inertia 2100 kgm^2 about a vertical axle through its center, and it turns with negligible friction. (a) A child applies an 18.0 N force tangentially to the edge of the merry-go-round for 15.0 s. If the merry-go-round is initially at rest, what is its angular speed after this 15.0-s interval? (b) How much work did the child do on the merry-go-round? (c) What is the average power supplied by the child?

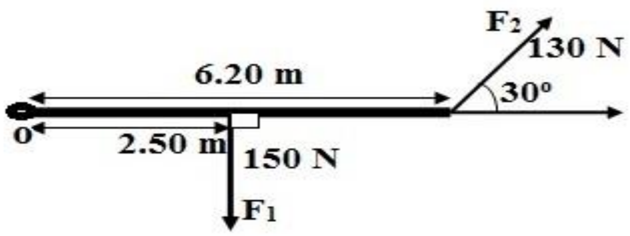
25. A 1.50 kg grinding wheel is in the form of a solid cylinder of radius 0.10 m. (a) What constant torque will bring it from rest to an angular speed 1200 $rev/mins$of in 2.5 s? (b) Through what angle has it turned during that time? (c) Calculate the work done by the torque. (d) What is the grinding wheel's kinetic energy when it is rotating at 1200 $rev/mins$.

26. What is the moment of I of cylinder about an axis along its height through its centre perpendicular to its plane, if $I_G = Mr^2/2$?

27. Calculate the torque supplied by the wrench when 8 N force is applied as shown in the figure below.



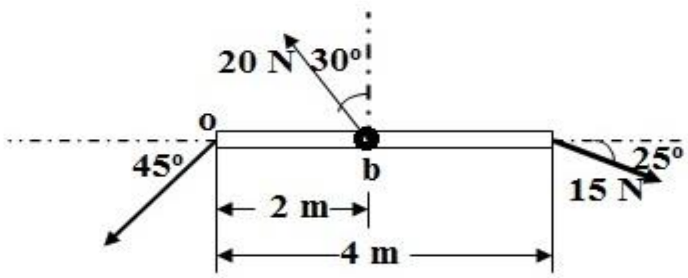
28. Find the net torque produced by the two forces (F_1 and F_2) shown in the figure below about the point O.



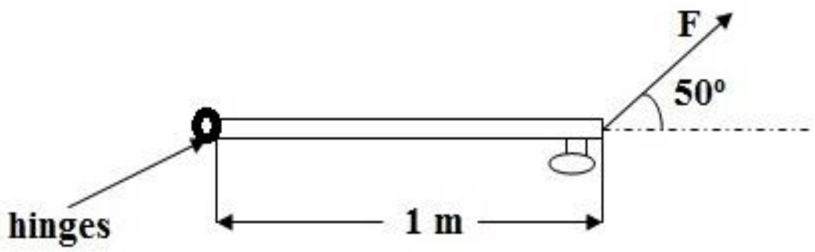
29. Calculate the net torque (magnitude and direction) on the beam shown in the figure about

i. An axis through o perpendicular to the page

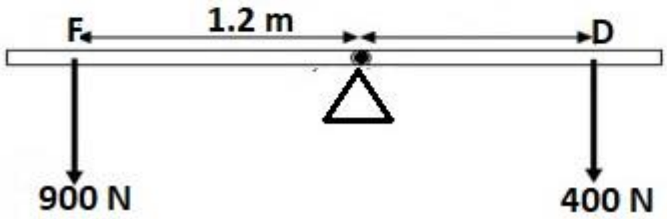
ii. An axis through b perpendicular to the page



30. A boy applies a force of 2 x10^2 N at an angle of 50° to a door 1 m from the hinges as shown in the figure. Find the torque on the door choosing the position of the hinges as the axis of rotation.



31. A uniform 40 N board supports a father **(F)** and her daughter (**D**) weighing 900 N and 400 N respectively. If the support (fulcrum) is at the centre of gravity of the board and the father is 1.2 m from the centre, as shown in the figure (a) determine the magnitude of the upward force exerted on the board by the support (b) determine where the daughter should sit to balance the system

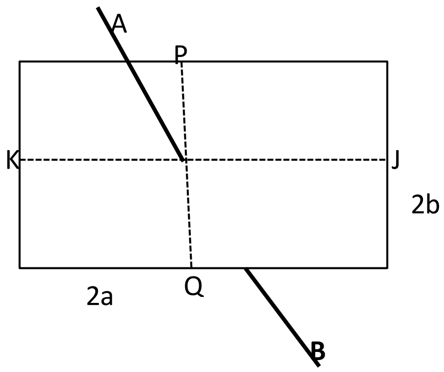


32. 6. A long rod of mass 20 kg is rotating about an axis perpendicular to the rod and passing through one end. Calculate the moment of inertia of the rod if its diameter is 6 cm the length of the rod is 2 m.

33. 7. A uniform solid sphere rotating through its centre and having a mass of 8 Kg and a radius of 10 cm spins at 12 rev/s. Calculate the magnitude of the angular momentum of the solid sphere.

34. Find the moment of inertia of lamina shown about axis AB through its centre if

$I_{JK} = \frac{mb^2}{3}$ and $I_{PQ} = \frac{ma^2}{3}$.



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2.0 GRAVITATION

2.1 UNIVERSAL GRAVITATION

Newton's law of universal gravitation states that "every particle in the universe attracts every other particle with a force that is directly proportional to the product of the masses and inversely proportional to the square of the distance between them".

If the masses of the two particles are m_1 and m_2 , the law can be expressed thus:

$$F = \frac{Gm_1m_2}{r^2} \qquad 2.1$$

where r is the distance between the two particles, G is a universal constant called the universal gravitation constant with a value of $6.673 \times 10^{-11} Nm^2/kg^2$.

Let us asssume the earth is a uniform sphere of mass M, the magnitude of the gravitational force from the earth on a particle of mass m, located on its surface is

$$F_g = \frac{GMm}{R^2} \qquad 2.2$$

where R is the radius of the earth with a value of $6.37 \times 10^6 m$.

But, Newton's second law of motion, the gravitational force can be expressed as

$$F_g = mg$$

$$=> \quad mg = \frac{GMm}{r^2}$$

$$\therefore \quad g = \frac{GM}{r^2} \qquad 2.3$$

The gravitational potential energy U_g possessed by each of two particles of masses m and M, separated by a distance r is given by

$$U_g = \frac{-GMm}{r} \qquad 2.4$$

This equation shows that the acceleration of free fall varies with attitude. See the table below.

The variation of g_o with Altitude

Altitude (Km)	g_o (m/s^2)
0	9.83
5	9.81
10	9.80
50	9.68
100	9.53
400[a]	8.70
35,700[b]	0.225
380,000[c]	0.0027

a. A typical space shuttle altitude.

b. The altitude of communication satellite.

c. The distance of the moon.

Example 1: A 425 kg satellite revolves around the sun at a distance $1.6 \times 10^6\ km$ from the earth. It is observed that the satellite orbits between the earth and the sun. Determine the attractive force of the satellite by the earth. Assume gravitational constant is $6.67 \times 10^{-11}\ Nm^{-2}Kg^{-2}$.

Solution:

$$F_{E-S} = \frac{GM_E m_s}{R^2} = \frac{6.67 \times 10^{-11} \times 5.98 \times 10^{24} \times 425}{(1.6 \times 10^9)^2} = 6.62 \times 10^{-2}\ N$$

Example 2: Calculate the pull of the sun on the satellite in the example 1 above since the distance between the earth and the sun is $1.5 \times 10^8\ km$.

Solution:

$R_{Sun-sat} = R_{Sun-earth} - R_{earth-sat} = 1.5 \times 10^{11} - 1.6 \times 10^9 = 1.484 \times 10^{11}\ m$

$$F_{S-S} = \frac{GM_S m_s}{R^2} = \frac{6.67 \times 10^{-11} \times 1.99 \times 10^{30} \times 425}{(1.484 \times 10^{11})^2} = 2.56\ N$$

Example 3: Find the acceleration of the magnitude of the resultant force on the satellite in the example 2 above.

Solution:

Resultant force, $F_R = F_{S-S} - F_{E-S} = 2.56 - 0.066 = 2.49\ N$

$F_R = m_s a \quad \Rightarrow \quad 2.49 = 425a$

$a = \frac{2.49}{425} = 5.8 \times 10^{-3} m/s$

Example 4: At what angular speed does it take the earth to revolve around the sun at distance $1.5 \times 10^8\ km$ from the sun?

Solution:

$F = M_E \omega^2 R = \frac{GM_E M_S}{R^2}$

$\omega = \sqrt{\frac{GM_S}{R^3}} = \sqrt{\frac{6.67\times10^{-11}\times1.99\times10^{30}}{(1.5\times10^{11})^3}} = 1.98 \times 10^{-7} rad/s$

Example 5: If the mass of the planet and its radius are 1.91×10^{27}kg and 7.15×10^4kmrespectively, find the free fall acceleration of a body on or near the surface of the planet.

Solution:

$$g_\mathrm{p} = \frac{GM_\mathrm{p}}{r^2} = \frac{6.67\times10^{-11}\times1.91\times10^{27}}{(7.15\times10^7)^2} = 24.92\mathrm{m/s^2}$$

2.2 ESCAPE SPEED

This is the minimum speed required by a body travelling away from the earth's surface, to break away from the effect of the earth's gravitational pull.

Let us consider a body of mass m, projected from the surface of the earth with a velocity v,

Kinetic energy, $K.E. = \frac{1}{2}mv^2$ and, Potential energy, $P.E. = \frac{-GMm}{R}$, where M is the mass of the earth and R, the radius of the earth

At the surface of the earth, the total energy must be zero.

That is, $\quad K.E. + P.E. = 0$

=> $\quad \frac{1}{2}mv^2 + \frac{-GMm}{R} = 0$

$\therefore \quad Escape\ velocity, v = \sqrt{\frac{2GM}{R}} \approx 11.2\ km/s = 11{,}200\ m/s \qquad 2.5$

Taking $M \approx 5.98 \times 10^{24}\ kg$ as the mass of the earth and $R \approx 6.37 \times 10^6\ m$ as the radius of the earth.

Example 6: If the mass of the moon 7.36×10^{22}kg while its radius 1.74×10^6m revolves at a distance of 4.05×10^8m from the Jupiter planet of mass 1.89×10^{27}kg and radius 7.15×10^7m. Determine the gravitational potential energy of the moon with respect to the Jupiter.

Solution:

$$E_{\mathrm{p(grav)}} = \frac{-GM_{\mathrm{J}}\mathrm{M_m}}{\mathrm{R}}$$

R is the separation between the Jupiter and the moon, i.e.,$\mathrm{R} = 4.05 \times 10^{8}\mathrm{m}$

$$E_{\mathrm{p(grav)}} = \frac{-6.67 \times 10^{-11} \times 1.89 \times 10^{27} \times 7.36 \times 10^{22}}{4.05 \times 10^{8}} = -2.29 \times 10^{31}\mathrm{J}$$

Example 7: What is the escape velocity from the surface of planet Jupiter of mass $1.89 \times 10^{27}\mathrm{kg}$ and radius $7.15 \times 10^{7}\mathrm{m}$?

Solution:

$$v = \sqrt{\frac{2GM_{\mathrm{J}}}{R_{\mathrm{J}}}} = \sqrt{\frac{2 \times 6.67 \times 10^{-11} \times 1.89 \times 10^{27}}{7.15 \times 10^{7}}} = 59{,}382.1\ \mathrm{m/s^2} = 59.4\ \mathrm{km/s^2}$$

2.3 MOTION OF SATELLITES

Artificial satellites launched into space revolve around the earth in nearly circular orbits. The gravitational force of attraction between a satellite of mass $m_s$and the earth brings about the centripetal acceleration of the satellite. Newton's second law effect on the satellite in circular orbit with radius r gives:

$$\frac{m_s v^2}{r} = \frac{GM_E m_s}{r^2} \qquad 2.6$$

Solving for v, we find

$$v = \sqrt{\frac{GM_E}{r}} \qquad 2.7$$

This relationship shows that the orbit radius r and the speed v is chosen independently; for a given radius r, the speed v for a circular orbit is determined.

If the motion of the satellite is uniform, i.e., the magnitude of the velocity does not change with time, the time (T) taken for it to complete one revolution around the circular orbit of radius r is the distance $2\pi r$ (as the circumference of the circle) divided by the velocity v:

$$T = \frac{2\pi r}{v} = 2\pi\sqrt{\frac{r^3}{GM_E}} \qquad 2.8$$

Example 8: A space shuttle orbited the moon at an altitude of 230 km in a circular orbit. Find its velocity and the time required to a complete circle.

Solution:

$$r = \mathrm{R_m} + \mathrm{h} = 1.74 \times 10^{6} + 0.23 \times 10^{6} = 1.97 \times 10^{6}\mathrm{m}$$

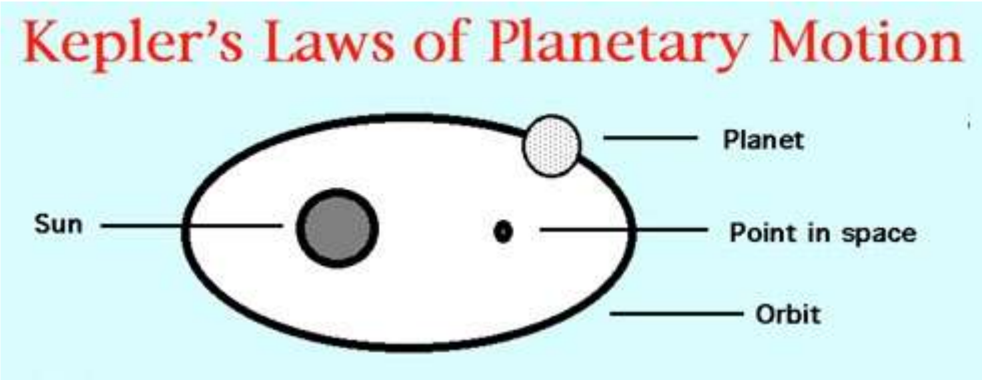
$$v = \sqrt{\frac{GM_{\mathrm{m}}}{r}} = \sqrt{\frac{6.67 \times 10^{-11} \times 7.36 \times 10^{22}}{1.97 \times 10^{6}}} = 2.49 \times 10^{6}\ \mathrm{m/s}$$

For the time required to complete one orbit,

$$T = \frac{2\pi r}{v} = \frac{2\pi \times 1.97 \times 10^{6}}{2.49 \times 10^{6}} = 4.97\ \mathrm{s} \approx 5.0\ \mathrm{s}$$

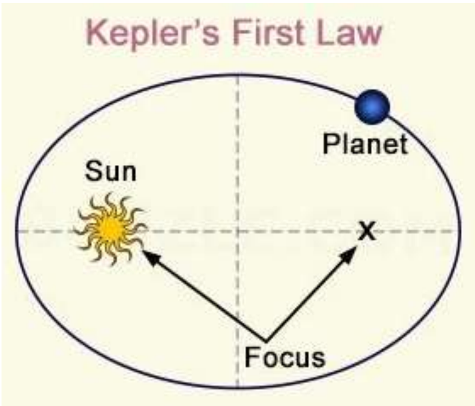
2.4 KEPLER'S LAWS OF PLANETARY MOTION

Kepler's planetary laws of motion describes the motion of satellites around the planets, such as, the motion of satellites like the moon around the planets like the earth.

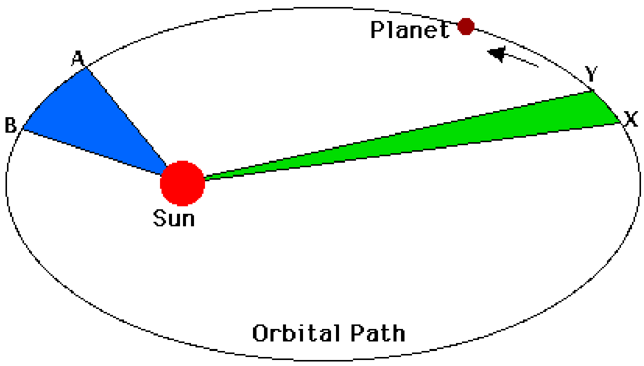


Kepler's analysis first showed that the concept of circular orbits about the Sun had discovered that the orbit of planets could be accurately described by an ellipse with the Sun at one focus. He then generalized this analysis to include the motions of all planets. The complete analysis is summarized in three statements that govern planetary motion as established by Kepler:

First law: All planets move in elliptical orbits with the sun at one of the focal points. Kepler's first law - sometimes referred to as the law of ellipses - means that the orbit or path of a planet around the sun is an ellipse i.e. an oval-shaped and not an exact circle. An elliptical path has two foci and the sun is at one of the two foci of the elliptical path. This law is important for us as it helps us discover if other stars have planets! We cannot see those planets but if the star wiggles back and forth in a complicated way, it may be because a planet makes it move so.



Second law: A line drawn from the sun to any planet sweeps out equal areas in equal time intervals. Kepler's second law - sometimes referred to as the law of equal areas - describes the speed at which any given planet will move while orbiting the sun. The speed at which any planet moves through space is constantly changing. It is known that a planet moves around the sun in an elliptical orbit with sun at one of its focus. Now, since the line joining the planet and the sun sweeps over equal areas in equal intervals of time, a planet moves fastest when it is closest to the sun and slowest when it is furthest from the sun.



Third law: The square of the orbital period of any planet is proportional to the cube of the average distance from the planet to the sun. Kepler's third law - sometimes referred to as the law of periods - compares the orbital period and radius of orbit of a planet to those of other planets. Unlike Kepler's first and second laws that describe the motion characteristics of a single planet, the third law makes a comparison between the motion characteristics of different planets. The comparison being made is that the ratio of the squares of the periods to the cubes of their average distances from the sun is the same for every one of the planets. As an illustration, consider the orbital period and average distance from Sun (orbital radius) for Earth and Mars as given in the table below:

Planet	**Period (s)**	**Average Distance (m)**	**T^2/r^3 (s^2/m^3)**
Earth	3.156×10^{7} s	1.4957×10^{11}	2.977×10^{-19}
Mars	5.93×10^{7} s	2.278×10^{11}	2.975×10^{-19}

Observe that the T^2/r^3 ratio is the same for Earth as it is for mars. In fact, if the same T^2/r^3 ratio is computed for the other planets, it can be found that this ratio is nearly the same value for all the planets (see table below). Amazingly, every planet has the same T^2/r^3 ratio.

Planet	**Period (yr)**	**Average Distance (au)**	**T^2/r^3 (yr^2/au^3)**
Mercury	0.241	0.39	0.98
Venus	0.615	0.72	1.01
Earth	1.00	1.00	1.00
Mars	1.88	1.52	1.01

Jupiter	11.8	5.20	0.99
Saturn	29.5	9.54	1.00
Uranus	84.0	19.18	1.00
Neptune	165	30.06	1.00
Pluto	248	39.44	1.00

NOTE: The average distance value is given in astronomical units where 1 a.u. is equal to the distance from the earth to the sun - 1.4957 x 10^{11} m. The orbital period is given in units of earth-years where 1 earth year is the time required for the earth to orbit the sun - 3.156 x 10^7 seconds.

From the third law, the statement is expressed in a mathematical form, so that, we have:

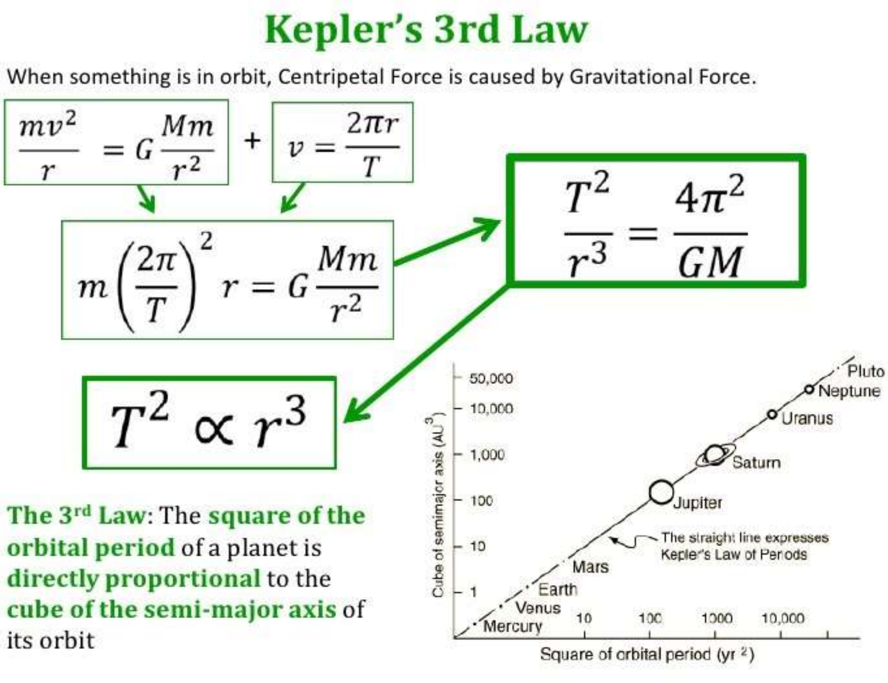
$$T^2 = \left(\frac{4\pi^2}{GM_S}\right) r^3 = k_s r^3 \qquad 2.9a$$

$$\frac{T^2}{r^3} = k_s \qquad 2.9b$$

where M_s is mass of the sun, r is the distance between the planet and the sun, and T is the period of the orbit of the planet.

$$k_s = \frac{4\pi^2}{GM_S} = 2.97 \times 10^{-19} s^2/m^3 \qquad 2.10$$

It is further illustrated in a graphical diagram below:



Kepler's third law provides an accurate description of the period and distance for a planet's orbits about the sun. Additionally, the same law that describes the T^2/r^3 ratio for the planets' orbits about the sun also accurately describes the T^2/r^3 ratio for any satellite (whether a moon or a man-made satellite) about any planet.

In conclusion, with the help of Kepler's third law of planetary motion, we can show how long does it takes to reach Mars, how long would it take for a spacecraft from earth to reach the Sun, how far from the center of Earth do synchronous satellites orbit, how far does Halley's comet go, etc.

Example 9: A moon at a distance above a point on the earth rotates about the earth of mass 5.98×10^{24}kg with a period of 27.5 days. Determine the radius of such an orbit.

Solution:

Orbital velocity of moon

$$v = \sqrt{\frac{GM_E}{r}}$$

Period of the moon

$$T = 27.5 \text{ days} = 27.5 \times 24 \times 3600 = 2{,}376{,}000 \text{ s}$$

But, $T = \frac{2\pi r}{v} = 2\pi\sqrt{\frac{r^3}{GM_E}}$

$$\Longrightarrow 2\pi\sqrt{\frac{r^3}{6.67\times10^{-11}\times5.98\times10^{24}}} = 2.376 \times 10^6$$

$\therefore\ r = 3.85 \times 10^5$ km

Example 10: Mars is an average distance 2.28×10^8 km from the sun. Compute the length of Marsian year using the fact that the Uranus is 84 years and 2.87×10^9 km from the sun on the average for the period and distance respectively.

Solution:

$$\frac{T_m^2}{(84)^2} = \frac{(2.28\times10^{11})^3}{(2.87\times10^{12})^3}$$

$\therefore\ T_m = 1.88$ years

EXERCISES

1. What force of attraction exists between Cynthia and Femi, if their masses are $65\ kg\ and\ 70\ kg$, respectively, and they are standing $6\ m$ from each other?
2. Two pieces of stone, each weighing $100\ kg$ are separated by a distance of $5\ m$. Calculate the gravitational force of attraction between them.
3. Consider a neutron star with a mass M equal to the mass of the sun, $1.98\ \times 10^{30}\ kg$, and a radius of $12\ \times 10^3\ m$. What is the free-fall acceleration at its surface?

4. The asteroid has a mass of $1.2 \times 10^{21}\ kg$ and a radius of $470\ km$. What is the free-fall acceleration at its surface?

5. What is the gravitational potential energy of the moon – earth system? The masses of the earth and the moon are $5.98 \times 10^{24}\ kg$ and $7.36 \times 10^{22}\ kg$, respectively, and their mean separation distance d is $3.82 \times 10^{8}\ m$.
6. A satellite of Mars has an orbital radius of $9.43 \times 10^{6}\ m$ and a period of $2.83 \times 10^{4}\ s$. Assuming the orbit is circular, determine the mass of Mars.

7. A 600 kg satellite is in a circular orbit about Earth at a height above Earth equal to Earth's mean radius. Find (a) the satellite's orbital speed, (b) the period of its revolution, and (c) the gravitational force acting on it.

8. A satellite of mass m circles the earth in an orbit at a distance R from the centre of the earth. If the radius of the earth is $6.4 \times 10^{6}\ m$, calculate the height above the earth's surface of the parking orbit and the velocity of the satellite in orbit.

9. Ten days after it was launched toward Mars in December 1998, the Mars Climate Orbiter spacecraft (mass 629 kg) was $2.87 \times 10^{6}\ km$ from the earth and traveling at $1.4 \times 10^{4}\ km/h$ relative to the earth. At this time, what were (a) the spacecraft's kinetic energy relative to the earth and (b) the potential energy of the earth–spacecraft system?

10. Two satellites are in circular orbits around a planet that has radius $9.0 \times 10^{6}\ m$. One satellite has mass 68.0 kg, orbital radius $5.0 \times 10^{7}\ m$, and orbital speed $4.8 \times 10^{3}\ m/s$. The second satellite has mass 84.0 kg and orbital radius $3.0 \times 10^{7}\ m$. What is the orbital speed of this second satellite?