Notes and calculations

A. Foroozande Nejad
Student ID: deducted

1 The Overlapping generation model:

This document is a compilation of my notes on Eggertsson, G. B., Mehrotra, N. R., & Robbins, J. A. (2019). A model of secular stagnation: Theory and quantitative evaluation. American Economic Journal: Macroeconomics, 11(1), 1-48.. This is purely for my personal use. Thus, I will not cite sources throughout the text.

1.1 Household problem

They layout an OLG model with three generations (young, middle age, and old) to analyze the determination of interest rate. The model does not allow for aggregate saving, so the young can only finance consumption through intergenerational loans. The middle-age are going to finance the loans and will be saving for retirement. Note that households are hand to mouth in the last period, and there is no altruistic or bequest motive in the baseline model. Thus, the households utility function is:

$$\max \mathbb{E}\left\{\log(C_t^y) + \beta \log(C_{t+1}^m) + \beta^2 \log(C_{t+2}^o)\right\} \tag{1}$$

The budget constraints facing households born at time t in each period is:

$$C_t^y = B_t^y \tag{2}$$

$$C_{t+1}^m = Y_{t+1}^m - (1+r_t)\, B_t^y + B_{t+1}^m \tag{3}$$

$$C_{t+2}^o = Y_{t+2}^o - (1+r_{t+1})\, B_{t+1}^m \tag{4}$$

$$(1+r_t)\, B_t^i \leq D_t \tag{5}$$

The collateral constraint (5) is assumed to be binding. Thus we can write it down as:

$$C_t^y = B_t^y = \frac{D_t}{(1+r_t)} \tag{6}$$

Considering that the young generation is limited by (6) and consumption in the older generation is determined by saving in middle age, then we can write:

$$L = \max_{B_{t+1}^m} \mathbb{E}\left\{\log(B_t^y) + \beta \log(Y_{t+1}^m - (1+r_t)\, B_t^y + B_{t+1}^m) + \beta^2 \log(Y_{t+2}^o - (1+r_{t+1})\, B_{t+1}^m)\right\}$$

Now, we can write down the first order condition for the household side:

$$
\begin{aligned}
\frac{\partial L}{\partial B_{t+1}^{m}} &= \frac{\beta}{Y_{t+1}^{m} - (1+r_t)\, B_t^{y} + B_{t+1}^{m}} + \beta^2 \mathbb{E} \frac{-\,(1+r_{t+1})}{Y_{t+2}^{o} - (1+r_{t+1})\, B_{t+1}^{m}} = 0 \\
\implies & \frac{1}{Y_{t+1}^{m} - (1+r_t)\, B_t^{y} + B_{t+1}^{m}} = \beta \mathbb{E} \frac{1+r_{t+1}}{Y_{t+2}^{o} - (1+r_{t+1})\, B_{t+1}^{m}}
\end{aligned}
$$

For middle age households at period t, we can write down the Euler equation as:

$$
\frac{1}{C_t^{m}} = \beta \mathbb{E} \frac{1+r_t}{C_{t+1}^{o}} \tag{7}
$$

1.2 Equilibrium in the asset market

In the model, the demand arises from young borrowing and middle age supplying loans. In the equilibria, The aggregate demand is equal population of young household multiplied by individual demand, and aggregate supply is equality to middle age household saving multiplied by their population. If we denote each generation population by N_t and growth by $1+g_t$, then we can write:

$$
\begin{aligned}
N_t B_t^{y} &= -\, N_{t-1} B_t^{m} \\
(1+g_t) B_t^{y} &= -\, B_t^{m}
\end{aligned} \tag{8}
$$

By using equation (6), aggregate demand is equal to:

$$
\begin{aligned}
L_t^{d} &= (1+g_t) B_t^{y} \\
&= (1+g_t) \frac{D_t}{(1+r_t)}
\end{aligned} \tag{9}
$$

Loan supply is determined by relative income in middle age versus old age. Assuming perfect foresight, the aggregate demand is:

$$
\begin{aligned}
\frac{1}{Y_t^m - (1+r_{t-1})\, B_{t-1}^y + B_t^m} - \beta \frac{1+r_t}{Y_{t+1}^o - (1+r_t)\, B_t^m} &= 0 \\
Y_t^m - D_{t-1} + B_t^m - \frac{Y_{t+1}^o - (1+r_t)\, B_t^m}{\beta(1+r_t)} &= 0 \\
Y_t^m - D_{t-1} + \frac{1+\beta}{\beta} B_t^m - \frac{Y_{t+1}^o}{\beta(1+r_t)} &= 0
\end{aligned}
$$

We can rearrange it as to determine supply:

$$
L_t^s = \frac{\beta}{1+\beta}(Y_t^m - D_{t-1}) - \frac{1}{1+\beta}\frac{Y_{t+1}^o}{1+r_t} \tag{10}
$$

The real interest rate is when loan demand (9) intersect loan supply (10).

$$
\begin{aligned}
(1+g_t)\frac{D_t}{(1+r_t)} &= \frac{\beta}{1+\beta}(Y_t^m - D_{t-1}) - \frac{1}{1+\beta}\frac{Y_{t+1}^o}{1+r_t} \\
\frac{1}{1+r_t}((1+g_t)D_t + \frac{Y_{t+1}^o}{1+\beta}) &= \frac{\beta}{1+\beta}(Y_t^m - D_{t-1}) \qquad (11) \\
\implies 1+r_t &= \frac{1+\beta}{\beta}\frac{(1+g_t)\, D_t}{Y_t^m - D_{t-1}} + \frac{1}{\beta}\frac{Y_{t+1}^o}{Y_t^m - D_{t-1}}
\end{aligned}
$$

1.3 Price Level Determination

By introducing perfectly flexible nominal prices, "the saver in our economy (the middle-generation household) now has access to risk-free nominal debt that is indexed in dollars, in addition to one-period risk-free real debt". For simplicity, we assume that this asset trades in zero net supply, so that, in equilibrium, the budget constraints already considered are unchanged. This would change the (4) to:

$$
C_{t+2}^o = Y_{t+2}^o - \frac{P_{t+1}\,(1+i_{t+1})}{P_{t+2}} B_{t+1}^m \tag{12}
$$

The new old-generation budget constraint changes the consumption Euler equation.

$$
\frac{1}{C_t^m} = \beta \mathbb{E} \frac{1}{C_{t+1}^o} \frac{P_t\,(1+i_t)}{P_{t+1}} \tag{13}
$$

Equations (7) and (13) imply that:

$$\begin{aligned} \beta\mathbb{E}\frac{1+r_t}{C^o_{t+1}} &= \beta\mathbb{E}\frac{1}{C^o_{t+1}}\frac{P_t\left(1+i_t\right)}{P_{t+1}} \\ \text{perfect foresight} \implies 1+r_t &= \frac{P_t\left(1+i_t\right)}{P_{t+1}} \end{aligned} \tag{14}$$

In addition, the nominal interest rate is limited by a non-negativity constraint.

$$i_t \geq 0 \tag{15}$$

By introducing money into the economy, if the real interest is negative then a no inflation environment is not possible at equilibria. Further, let us denote the growth rate of the price level (inflation) by $\Pi_t = \frac{P_{t+1}}{P_t} = \bar{\Pi}$. The zero bound and the Fisher equation then imply that, for an equilibrium with constant inflation to satisfy the ZLB, there is a bound on the inflation rate given by $\Pi(1+r) = 1+i \geq 1$ or

$$\bar{\Pi} \geq \frac{1}{1+r} \tag{16}$$

1.4 Rigidities

They incorporate a permanent upward Philips-curve in a form of long-run rigidity. This change implies that high enough inflation does not have an effect on the output as expectation adjust in the long-run. However, a fail attempt to meet the inflation target, i.e. low inflation/deflation environment, results in long-run trade-off between inflation and unemployment. Further, labor is now endogenous, which changes (3) and (4).

$$C^m_{t+1} = \frac{W_{t+1}}{P_{t+1}}L_{t+1} + \frac{Z_{t+1}}{P_{t+1}} - \left(1+r_t\right)B^y_t + B^m_{t+1} \tag{17}$$

$$C^o_{t+2} = -\left(1+r_{t+1}\right)B^m_{t+1} \tag{18}$$

We assume that the middle-aged generation will supply a constant level of labor $\bar{L}$ inelastically. Note that if the firms do not hire all available labor supplied, then labor demand L_t may be lower than labor supply $\bar{L}$ due to rationing. Under these assumptions, each of the generations' consumption-saving decisions remains the same as before. On the firm side, we assume that firms are perfectly competitive and take prices as given. They hire labor to maximize period-by-period profits. Their problem is given by

$$Z_t = \max_{L_t} P_t Y_t - W_t L_t \tag{19}$$

$$\text{s.t. } Y_t = L_t^\alpha \tag{20}$$

The firms' labor demand condition is then given by

$$\begin{aligned}
\tilde{Z}_t &= \max_{L_t} P_t L_t^\alpha - W_t L_t \\
\frac{\partial \tilde{Z}_t}{\partial L_t} &= \alpha P_t L_t^{\alpha-1} - W_t \\
\implies L_t &= (\frac{W_t}{\alpha P_t})^{\frac{1}{\alpha-1}}
\end{aligned} \tag{21}$$

In a perfectly frictionless production side, the model would be analogous to what we have already considered in the endowment economy. consider a world in which households will never accept working for wages that fall below their wage in the previous period, so nominal wages at time t cannot be lower than what they were at time $t-1$. Or, slightly more generally, imagine that the household would never accept lower wages than a wage norm given by $\tilde{W}_t = \gamma W_{t-1} + (1-\gamma) W_t^{flex}$ where

$$W_t^{flex} = P_t \alpha \bar{L}^{\alpha-1} \tag{22}$$

If $\gamma = 1$, the wage norm is simply last period's nominal wages and wages are perfectly downwardly rigid; if $\gamma = 0$, we obtain the flex-price nominal wage. Formally, wages in our economy are given by

$$W_t = \max \left\{ \tilde{W}_t, W_t^{\text{flex}} \right\} \text{ where } \tilde{W}_t = \gamma W_{t-1} + (1-\gamma) W_t^{\text{flex}} \tag{23}$$

Nominal wages in our economy can never fall below the wage norm $\tilde{W}_t$. If labor market clearing requires nominal wages lower than the previous nominal wage rate, then the labor market will not clear, and labor is rationed.

To close the model, we specify a monetary policy rule. We posit that the central bank sets the nominal rate according to a standard Taylor rule:

$$1 + i_t = \max\left(1, (1 + i^*)\left(\frac{\Pi_t}{\Pi^*}\right)^{\phi_\pi}\right) \tag{24}$$

where $\phi_\pi > 1; \Pi^*$ and i^* are parameters of the policy rule that we hold constant. This rule states that the central bank attempts to keep inflation at the inflation target Π^* and the nominal rate at i^* so long as the nominal rate implied by the rule is not constrained by the zero bound.

1.5 The aggregates

The aggregate supply specification of the model consists of two regimes: one in which the real wage equals the market-clearing real wage (if $\Pi \geq 1$) and the other when the bound on nominal wages is binding (Π ¡ 1). Intuitively, positive inflation in the steady state means that wages behave as if they are flexible, since nominal wages must rise to keep real wages constant. If $\Pi \geq 1$, then labor demand equals the exogenous level of labor supply $\bar{L}$, defining the full-employment level of output Y^f :

$$Y = \bar{L}^\alpha = Y^f \text{ for } \Pi \geq 1 \tag{25}$$

If there is deflation in the steady state ($\Pi < 1$), the wage norm binds (W = $\tilde{W}$) and the real wage exceeds the market-clearing real wage. If we denote real wage by $w = \frac{W}{P}$ then:

$$\begin{aligned}
\tilde{W}_t &= \gamma \tilde{W}_{t-1} + (1-\gamma) W_t^{\text{flex}} \text{ for } \Pi < 1 \\
P_t \tilde{w}_t &= \gamma P_{t-1} \tilde{w}_{t-1} + (1-\gamma) P_t \alpha \bar{L}^{\alpha-1} \\
\tilde{w}_t &= \gamma \tilde{w}_{t-1} \Pi^{-1} + (1-\gamma) \alpha \bar{L}^{\alpha-1} \\
w &= \frac{(1-\gamma)\alpha \bar{L}^{\alpha-1}}{1-\gamma \Pi^{-1}}
\end{aligned} \tag{26}$$

This raises an issue as (26) is not continuous for all reasonable values of Π^{-1}, which needs further investigation. We can then derive a relationship between output and inflation tracing the lower segment of the AS curve by substituting (26) into equation (21) and using the production function (20) to express the result in terms of output:

$$L_t = (\frac{W_t}{\alpha P_t})^{\frac{1}{\alpha-1}}$$
$$L_t = (\frac{(1-\gamma)\alpha \bar{L}^{\alpha-1}}{\alpha(1-\gamma\Pi^{-1})})^{\frac{1}{\alpha-1}}$$
$$Y_t^{\frac{\alpha-1}{\alpha}} = (\frac{(1-\gamma)\alpha Y_t^{f\ \frac{\alpha-1}{\alpha}}}{\alpha(1-\gamma\Pi^{-1})})$$

$$\frac{\gamma}{\Pi} = 1-(1-\gamma)\left(\frac{Y}{Y^f}\right)^{\frac{1-\alpha}{\alpha}} \text{ for } \Pi < 1 \tag{27}$$

Turning to the aggregate demand relation, we again have two regimes: one in which the zero bound is not binding and the other in which it is binding. When the nominal rate is unconstrained, demand is obtained by summing up the consumption of the three generations and substituting out for the nominal interest rate with the policy reaction function (24). Combining the real interest rate equation, Fisher relation, and monetary policy rule — equations (11), (14) and (24) — yields:

$$Y = D + \tfrac{(1+\beta)(1+g)D\Gamma^*}{\beta}\tfrac{1}{\Pi^{\phi_\pi - 1}} \text{ for } i > 0 \tag{28}$$

where $\Gamma^* \equiv (1+i^*)^{-1}(\Pi^*)^{\phi_\pi}$ is the policy parameter given in the policy reaction function. The logic for this relationship is if inflation increases, then the central bank raises the nominal interest rate by more than one for one (since $\phi_\pi > 1$), which in turn increases the real interest rate and reduces demand.

Consider now the situation in which $i = 0$. In this case, again combining equations (11), (14) and (24) and assuming $i = 0$, we get:

$$Y = D + \frac{(1+\beta)(1+g)D}{\beta}\Pi \text{ for } i = 0 \tag{29}$$

1.6 Equilibrium and market clearing

There is no capital in the baseline model. Thus, two market must clear — the labor market, and the goods market. Further, A competitive equilibrium is a set of aggregate allocations $\left\{Y_t, C_t^m, B_t^m, L_t^{flex}\right\}_{t=0'}^{\infty}$ price processes $\left\{i_t, \Pi_t, w_t, w_t^{flex}\right\}_{t=0^0}^{\infty}$ exogenous processes $\{D_t, g_t\}_{t=0}^{\infty}$ and initial values of household saving, nominal interest rate, real wage, and the public debt $\left\{B_{-1}^m, i_{-1}, w_{-1}, B_{-1}\right\}$ that jointly satisfy:

1. Household Euler equation (13)
2. Household budget constraints (17) and (18)
3. Asset market clearing (11)
4. Monetary policy rule (24)
5. Full-employment labor supply
6. Labor demand condition: $w_t = \alpha\left(Y_t\right)^{\frac{\alpha-1}{\alpha}}$
7. Wage process: $w_t = \max\left\{\tilde{w}_t, w_t^{\text{flex}}\right\}$, where $\tilde{w}_t = \gamma\frac{w_{t-1}}{\Pi_t} + (1-\gamma)w_t^{\text{flex}}$

1.7 Review of important equations

Household

$$C_t^y = B_t^y$$

$$C_{t+1}^m = \frac{W_{t+1}}{P_{t+1}}L_{t+1} + \frac{Z_{t+1}}{P_{t+1}} - \left(1+r_t\right)B_t^y + B_{t+1}^m$$

$$C_{t+2}^o = -\left(1+r_{t+1}\right)B_{t+1}^m$$

$$C_t^y = B_t^y = \frac{D_t}{\left(1+r_t\right)}$$

$$\frac{1}{C_t^m} = \beta\mathbb{E}\frac{1+r_t}{C_{t+1}^o}$$

$$(1+g_t)B_t^y = -B_t^m$$

$$\frac{1}{C_t^m} = \beta\mathbb{E}\frac{1}{C_{t+1}^o}\frac{P_t\left(1+i_t\right)}{P_{t+1}}$$

$$1+r_t = \frac{P_t\left(1+i_t\right)}{P_{t+1}}$$

$$i_t \geq 0$$

$$Z_t = \max_{L_t} P_tY_t - W_tL_t$$

$$\text{s.t. } Y_t = L_t^{\alpha}$$

$$\implies L_t = (\frac{W_t}{\alpha P_t})^{\frac{1}{\alpha-1}}$$

$$W_t^{flex} = P_t \alpha \bar{L}^{\alpha-1}$$

$$W_t = \max\left\{\tilde{W}_t, W_t^{\text{flex}}\right\} \text{ where } \tilde{W}_t = \gamma W_{t-1} + (1-\gamma) W_t^{\text{flex}}$$

$$1 + i_t = \max\left(1, (1+i^*)\left(\frac{\Pi_t}{\Pi^*}\right)^{\phi_\pi}\right)$$

1.8 Fiscal policy

If we denote taxation for each generation by T_t^i then we can write the household budget constraint as:

$$C_t^y = B_t^y \tag{30}$$

$$C_{t+1}^m = Y_{t+1}^m - (1+r_t) B_t^y + B_{t+1}^m - T_{t+1}^m \tag{31}$$

$$C_{t+2}^o = Y_{t+2}^o - (1+r_{t+1}) B_{t+1}^m - T_{t+2}^o \tag{32}$$

Beside taxing that could change the supply side, the government can also borrow from the middle-age household. This changes the asset market clearing (8) to:

$$\begin{aligned} N_t B_t^y + N_{t-1} B_t^g &= -\, N_{t-1} B_t^m \\ (1+g_t) B_t^y + B_t^g &= -\, B_t^m \end{aligned} \tag{33}$$

B^g denote the government debt. Government ability to borrow could be a new leverage to raise the real interest rate as it increases loan demand.

$$\begin{aligned} L_t^d &= (1+g_t) B_t^y + B_t^g \\ &= (1+g_t) \frac{D_t}{(1+r_t)} + B_t^g \end{aligned} \tag{34}$$

$$L_t^s = \frac{\beta}{1+\beta}(Y^m - D - T^m) - \frac{1}{1+\beta}\frac{Y^o - T^o}{1+r} \tag{35}$$

In this environment government generate revenue for itself by taxing and borrowing from households.

$$T^m + B^g + \frac{1}{1+g} T^o + (1+g) T^y = G + (1+r) \frac{1}{1+g} B^g \tag{36}$$

To determine the full employment real interest rate, we have to define a fiscal policy and substitute it in (36) and then equate loan supply and demand equations.

1.9 Introducing capital

In this section, consider an economy similar to previous sections, however, with two main extensions. First, the government tax middle-age and old household, and pay young ones in a form of lump-sum transfer. Further, middle generation households can invest part of their income in firms, which means households can transfer wealth across generations.

$$
\begin{aligned}
&\max_{C_t(i),C_{t+1}(i),C_{t+2}(i)} \mathbb{E}_t \left\{\log\left(C_t(i)\right) + \beta \log\left(C_{t+1}(i)\right) + \beta^2 \log\left(C_{t+2}(i)\right)\right\} \\
&\text{s.t.} \quad C_t(i) = B_t(i) \\
&\quad C_{t+1}(i) = w_{t+1}L_{t+1} + r^k_{t+1}K_{t+1}(i) + B_{t+1}(i) - p^k_{t+1}K_{t+1}(i) - \frac{(1+i_t)}{\Pi_{t+1}}B_t(i) \\
&\quad C_{t+2}(i) = p^k_{t+2}K_{t+1}(i)(1-\delta) - \frac{(1+i_{t+1})}{\Pi_{t+2}}B_{t+1}(i) \\
&\quad B_{t+j}(i) \leq \mathbb{E}_{t+j}\left(1+r_{t+j+1}\right)D_{t+j} \text{ for } j = 0,1
\end{aligned}
$$

where p^k_t is an exogenous relative price of capital goods and $K_t(i)$ are households purchases of the physical capital good.

The two Euler equations determine the optimal choice for supplying loan and capital.

$$\frac{1}{C^m_t} = \beta \mathbb{E}_t \frac{1+i_t}{\Pi_{t+1}C^o_{t+1}} \tag{37}$$

$$\frac{p^k_t - r^k_t}{C^m_t} = \beta \mathbb{E}_t \frac{p^k_{t+1}(1-\delta)}{C^o_{t+1}} \tag{38}$$

Assuming perfect foresight, we can combine the two Euler equations to derive the relation between capital price and real interest rate.

$$\frac{p^k_t - r^k_t}{p^k_{t+1}(1-\delta)} = \frac{\Pi_{t+1}}{1+i_t} \tag{39}$$

The budget constraints for each type of household at any point in time is given below:

$$C_t^y = \mathbb{E}_t \Pi_{t+1} \frac{D_t}{1+i_t} \tag{40}$$

$$C_t^m = Y_t - D_{t-1} - p_t^k K_t - B_t^m \tag{41}$$

$$C_t^o = p_t^k K_{t-1}(1-\delta) + B_{t-1}^m \frac{1+i_{t-1}}{\Pi} \tag{42}$$

On the firm side, it chooses optimal level of capital and labor to maximize its profit.

$$Z_t = \max_{L_t} P_t Y_t - W_t L_t - P_t r_t^k K_t \tag{43}$$

$$\text{s.t. } Y_t = A_t K_t^{1-\alpha} L_t^{\alpha} \tag{44}$$

The real rental rate on capital and the real wage is equal to:

$$r_t = (1-\alpha)\frac{Y_t}{K_t} \tag{45}$$

$$w_t = \alpha \frac{Y_t}{L_t} \tag{46}$$

In the deflationary environments, when the real wage does not clear the market, equation (23) is going to determine the wage.

$$\begin{aligned} \tilde{W}_t &= \gamma \tilde{W}_{t-1} + (1-\gamma) W_t^{\text{flex}} \quad \text{for } \Pi < 1 \\ P_t \tilde{w}_t &= \gamma P_{t-1} \tilde{w}_{t-1} + (1-\gamma) P_t \alpha A_t K_t^{1-\alpha} L_t^{\alpha-1} \\ \tilde{w}_t &= \gamma \tilde{w}_{t-1} \Pi^{-1} + (1-\gamma) \alpha A_t K_t^{1-\alpha} L_t^{\alpha-1} \\ w &= \frac{(1-\gamma)\alpha A_t K_t^{1-\alpha} \bar{L}_t^{\alpha-1}}{1-\gamma \Pi^{-1}} \end{aligned} \tag{47}$$

Equilibrium: there are two main differences in this section compare to the previous ones. First, AD and AS curves can have multiple kink points with introduction of capital. Thus, we assume that target inflation and interest rate are both zero to simplify the model. Second, we need to check for equilibrium in two markets now.
The policy assumption divides the economy into two different environments. When inflation is above policy values, we have the following relation between variables in the economy.

According to equation (23) (Taylor rule), if inflation is above zero then ZLB is not binding, which result in a flexible wages($w_t = \alpha\frac{Y_t}{L_t}$). This implies full-employment in the economy ($L_t = \bar{L}$).

$$\begin{aligned}
\frac{p_t^k - r_t^k}{p_{t+1}^k(1-\delta)} &= \frac{\Pi_{t+1}}{1+i_t} \\
\frac{p_t^k - r_t^k}{p_{t+1}^k(1-\delta)} &= \frac{1}{1+r_t} \\
r_t^k &= p_t - \frac{p_{t+1}^k(1-\delta)}{1+r_t}
\end{aligned} \tag{48}$$

Then we can (45) to determine optimal capital and then wage:

$$\begin{aligned}
r_t^k &= (1-\alpha)\frac{A_t K_t^{1-\alpha} L_t^{\alpha}}{K_t} \\
K_t &= ((1-\alpha)\frac{A_t}{r_t^k})^{-\alpha}
\end{aligned} \tag{49}$$

$$\begin{aligned}
w_t &= \alpha\frac{A_t K_t^{1-\alpha} L_t^{\alpha}}{L_t} \\
w_t &= \alpha A_t K_t^{1-\alpha} \\
w_t &= \alpha A_t \left(((1-\alpha)\frac{A_t}{r_t^k})^{-\alpha} \right)^{1-\alpha}
\end{aligned} \tag{50}$$

The demand for loans is determined by (34), and loan demand by $L_t^d = -B_t^m$. However, B_t^m is derived by below equation:

$$L_t^s = \frac{\beta}{1+\beta}\left(Y_t - D_{t-1} - T_t^m\right) - \frac{\beta}{1+\beta}\left(p_t^k + \frac{p_{t+1}^k(1-\delta)}{\beta\left(1+r_t\right)}\right)K_t + \frac{T^o}{(1+r)(1+\beta)} \tag{51}$$

Above equations help us to derive the aggregate supply and demand equations similar to previous parts.

$$Y_t^s = A_t K_t^{1-\alpha} L_t^{\alpha} \tag{52}$$

$$Y_t^d = D + \frac{1+\beta}{\beta}B_g + \frac{1+\beta}{\beta}\frac{(1+g)}{1+r}D + p^k K\left(1 + \frac{1}{\beta}\frac{(1-\delta)}{1+r}\right) \tag{53}$$

When inflation is below unity, under the assumptions made, the zero lower bound and the bound on nominal wages is binding so that labor is rationed. Similar to the previous part, bellow equations determine the steady state.

$$
\begin{aligned}
\frac{p_t^k - r_t^k}{p_{t+1}^k(1-\delta)} &= \frac{\Pi_{t+1}}{1+i_t} \\
\frac{p_t^k - r_t^k}{p_{t+1}^k} &= (1-\delta)\Pi_{t+1} \\
r^k &= p^k(1-\Pi(1-\delta))
\end{aligned}
\tag{54}
$$

$$
\begin{aligned}
L_t &= \alpha \frac{Y_t}{w_t} \\
L_t &= \alpha \frac{Y_t}{\frac{(1-\gamma)\alpha A_t K_t^{1-\alpha} \bar{L}_t^{\alpha-1}}{1-\gamma\Pi^{-1}}} \\
L &= \left(\frac{1-\frac{\gamma}{\Pi}}{1-\gamma}\right)^{\frac{1}{1-\alpha}} \bar{L}
\end{aligned}
\tag{55}
$$

$$
K_t^{\alpha} = (1-\alpha)\frac{A_t L_t^{\alpha}}{r^k} \tag{56}
$$

$$
Y_t^s = AK^{1-\alpha}L^{\alpha} \tag{57}
$$

$$
Y_t^d = D + \frac{1+\beta}{\beta}B_g + \frac{1+\beta}{\beta}\Pi(1+g)D + p^k K\left(1+\frac{\Pi}{\beta}(1-\delta)\right) \tag{58}
$$