Report for CS215 Assignment 3

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Question 2

Given $y = -\frac{1}{\lambda}log(x)$ with $\lambda = 5$ and x = U(0,1), the pdf q(y) can be analytically derived as follows, Since, $g(x) = -\frac{1}{\lambda}log(x)$ is a monotonically decreasing function for $x > 0$, the transformation of random variables formula can be applied.

$$y = -\frac{1}{\lambda}log(x)$$

$$x = exp(-\lambda y)$$

$$g^{-1}(y) = exp(-\lambda y)$$

$$\frac{d}{dy}g^{-1}(y) = \lambda exp(-\lambda y)$$

Let PDF of X be represented by p(x) and PDF of Y by q(y):

$$p(x) = \begin{cases} 1 & \text{if x} \in (0,1) \\ 0 & \text{otherwise.} \end{cases}$$

$$q(y) = p(g^{-1}(y))\frac{d}{dy}g^{-1}(y) = \begin{cases} \lambda\ exp(-\lambda y) & \text{if y} > 0 \\ 0 & \text{otherwise.} \end{cases}$$

It can be clearly seen that Y is an exponential Random Variable.

A Posterior Mean

Given data $y_1, y_2, y_n, n \geq 1$. Since these samples are drawn from a exponential(λ) distribution with unknown λ and λ = Gamma(α, β) with $\alpha = 5.5$ and $\beta = 1$ (Gamma prior on λ),

- Joint Likelihood:

$$P(y_1, y_2, ..., y_n|\lambda) = \begin{cases} \lambda^n exp\{(-\lambda\Sigma_{i=1}^n y_i)\} & \text{if x} \in (0,1) \\ 0 & \text{otherwise.} \end{cases}$$

- Prior:

$$P(\lambda|\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)}\lambda^(\alpha - 1)e^{-\beta\lambda}$$

- Posterior:

$$P(\lambda|y_1, y_2, ..., y_n) = P(y_1, y_2, ..., y_n|\lambda).P(\lambda|\alpha, \beta)$$

$$P(\lambda|y_1, y_2, ..., y_n) = \lambda^n exp\left(-\lambda\sum_{i=1}^{n} y_i\right).\lambda^{\alpha-1}exp(-\beta\lambda)$$

$$P(\lambda|y_1, y_2, ..., y_n) = \lambda^{\alpha+n-1}exp\left(-\lambda\left(\sum_{i=1}^{n} y_i + \beta\right)\right)$$

$$P(\lambda|y_1, y_2, ..., y_n) = Gamma\left(\alpha + n, \sum_{i=1}^{n} y_i + \beta\right)$$

Since the mean of Gamma$(\alpha, \beta) = \frac{\alpha}{\beta}$,

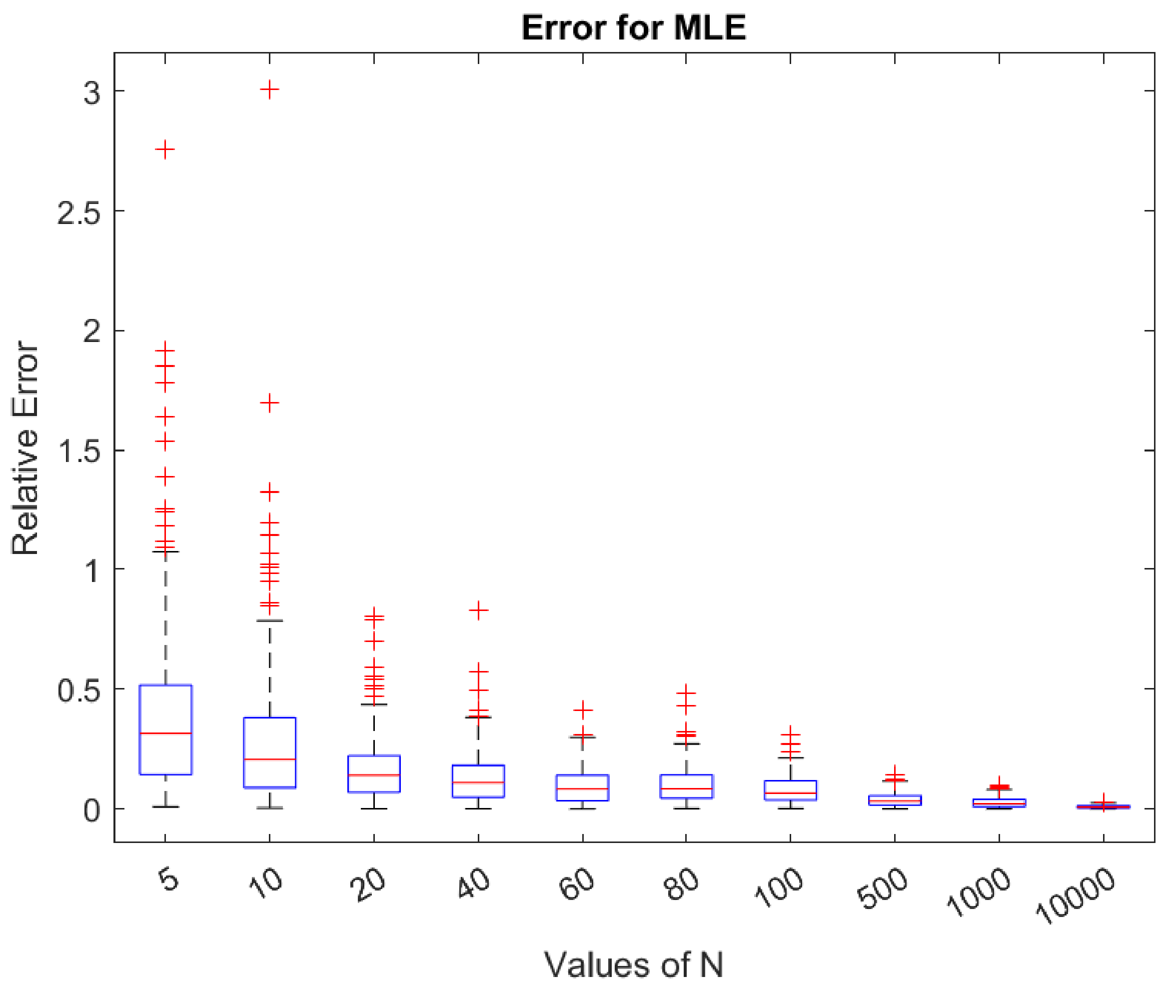
$$\text{Posterior Mean, } \hat{\lambda}^{PosteriorMean} = \frac{\alpha + n}{\Sigma_{i=1}^{n} + \beta}$$

It is known that for exponential(λ) distribution with n samples y_1, y_2, y_n, the ML estimate is

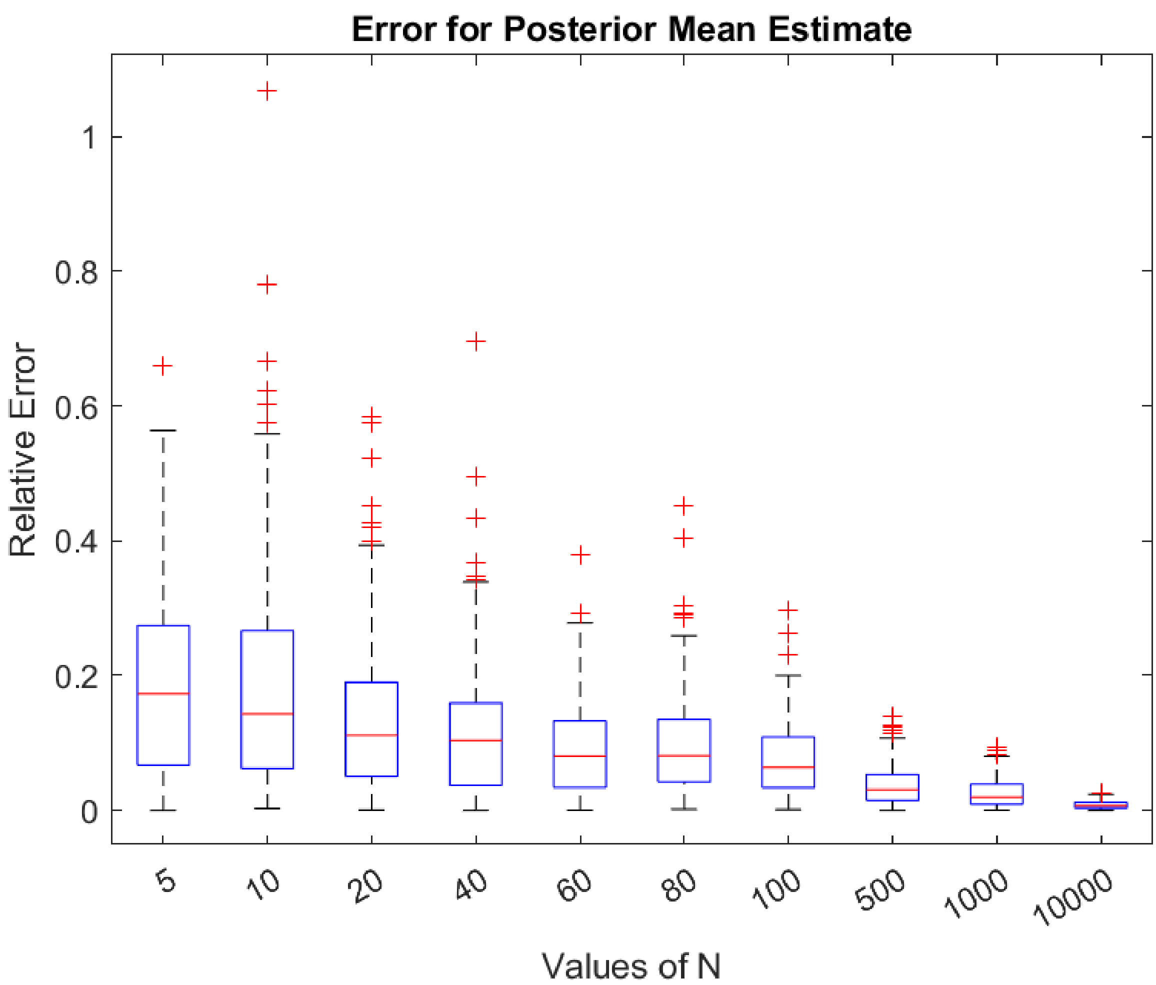
$$\hat{\lambda}^{ML} = \frac{n}{\Sigma_{i=1}^{n} y_i}$$

B Interpretation of BoxPlots

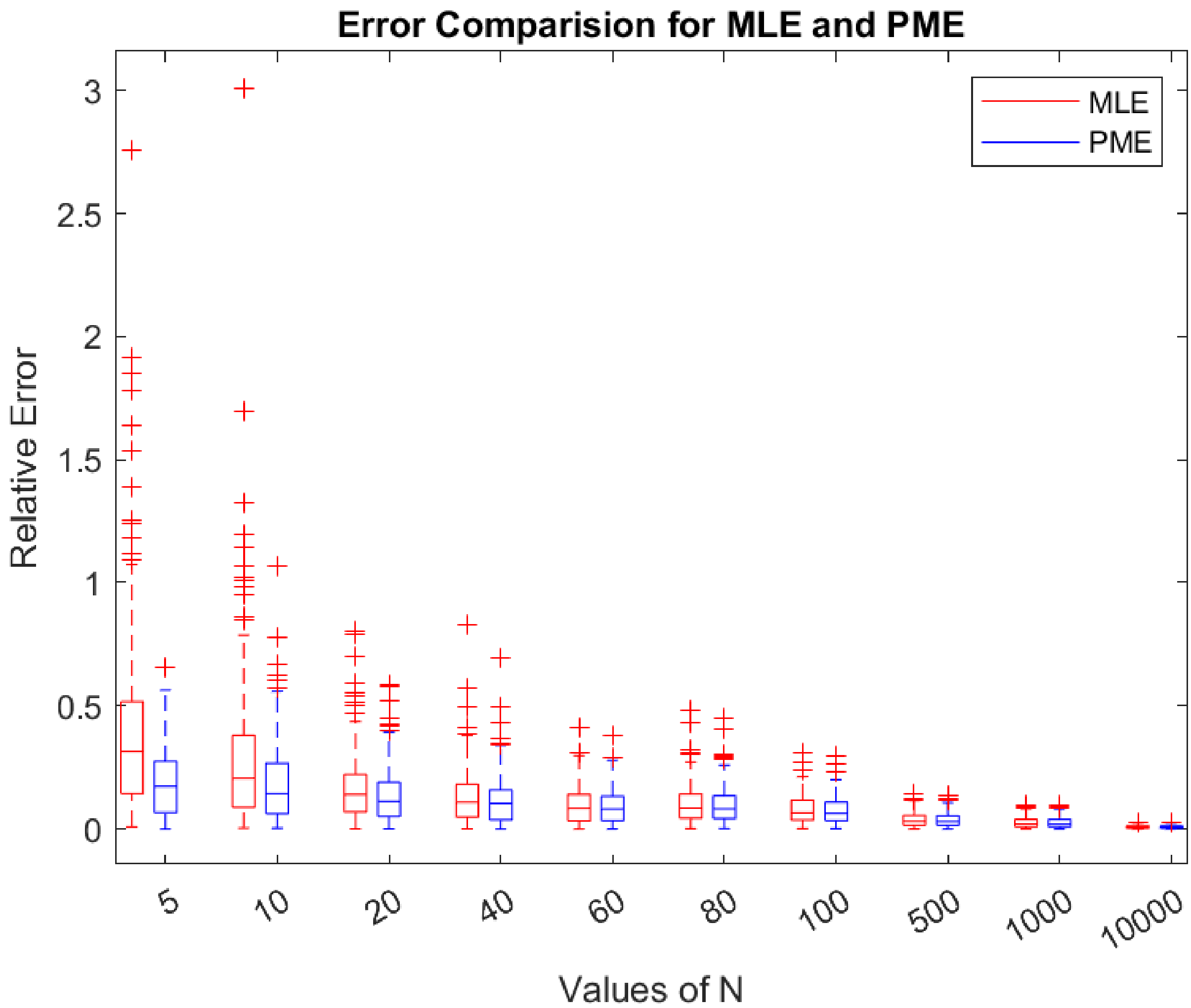
- As the value of N keeps on increasing, the error keeps on decreasing and approaches to zero in both the cases.
- The posterior mean estimate is more preferable since since it has a smaller values of relative error than the ML estimate for small values of N. Since, in practice we have only finite data, the posterior mean estimate is a better choice. Also for large values of N, the posterior mean estimate converges to the ML estimate and hence has all the desired asymptotic properties(consistency, asymptotic normality, efficiency) of the ML estimate.



Error for MLE
Relative Error
0
0.5
1
1.5
2
2.5
3
5
10
20
40
60
80
100
500
1000
10000
Values of N



Error for Posterior Mean Estimate
Relative Error
0
0.2
0.4
0.6
0.8
1
5
10
20
40
60
80
100
500
1000
10000
Values of N



Error Comparision for MLE and PME
MLE
PME
Relative Error
0
0.5
1
1.5
2
2.5
3
5
10
20
40
60
80
100
500
1000
10000
Values of N