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Part - 1

Law of Large Numbers states that :
$\epsilon > 0$, $n \to \infty$ $P((|\hat{M}| - M) \geq \varepsilon) \to 0$
which can be proved using Chebyshev's Inequality

So As $N \to \infty$, $|\hat{M} - M| < \varepsilon$
As $\epsilon > 0$, we can set $\varepsilon \to 0^+$
So it is proved that $\hat{M} \to M$ as $N \to \infty$

Part - 2

$\mu_n \to$ central moments

$$\hat{V} = \sum_{i=1}^{N} (x_i - \hat{M})^2 / N$$

First we will find $\text{var}(\hat{V})$

$$\hat{V} = s^2$$

As $\text{var}(x) \equiv \langle x^2 \rangle - \langle x \rangle^2$

$$\text{var}(s^2) \equiv \langle s^4 \rangle - \langle s^2 \rangle^2$$

$\langle s^2 \rangle$ is already known

$$\langle s^4 \rangle = \langle \langle s^2 \rangle^2 \rangle$$

$$= \langle \langle \langle x^2 \rangle - \langle x \rangle^2 \rangle^2 \rangle$$

$$= \left\langle \left[\frac{1}{N} \sum x_i^2 - \left(\frac{1}{N} \sum x_i \right)^2 \right]^2 \right\rangle$$

$$= \frac{1}{N^2} \left\langle \left(\sum x_i^2 \right)^2 \right\rangle - \frac{2}{N^3} \left\langle \sum x_i^2 \left(\sum x_i \right)^2 \right\rangle + \frac{1}{N^4} \left\langle \left(\sum x_i \right)^4 \right\rangle$$

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$$\langle (\sum x_i^2)^2 \rangle = \langle \sum x_i^4 + \sum_{i \neq j} x_i^2 x_j^2 \rangle$$

$$= N\langle x_i \rangle^4 + N(N-1)\langle x_i^2 \rangle \langle x_j^2 \rangle$$

$$= N\mu_4 + N(N-1)\mu_2^2$$

$$\langle \sum x_i^2 (\sum x_j)^2 \rangle = N\mu_4 + N(N-1)\mu_2^2$$

$$\langle (\sum x_i)^4 \rangle = N\mu_4 + 3N(N-1)\mu_2^2$$

$$\langle s^4 \rangle = \frac{(N-1)\left[(N-1)\mu_4 + (N^2-2N+3)\mu_2^2\right]}{N^3}$$

$$\text{Var}(s^2) \equiv \text{Var}(\hat{v})$$

$$= \langle s^4 \rangle - \langle s^2 \rangle^2$$

$$= \frac{(N-1)^2}{N^3}\mu_4 - \frac{(N-1)(N-3)}{N^3}\mu_2^2$$

As $N \to \infty$

$$\text{Var}(\hat{v}) = \frac{\mu_4 - \mu_2^2}{N} \Rightarrow \text{Var}(\hat{v}) \to 0 \text{ as } N \to \infty$$

$\text{Var}(\hat{v}) \to 0$

means $\hat{v} \to \text{Var}(x)$ as $N \to \infty$