MIDTERM EXAM

ISyE6420
Spring 2025

Released February 28, 6:00 PM – due March 2, 6:00 PM. This exam is not proctored and not time limited except the due date. Late submissions will not be accepted.

Use of all course materials, including the class Github site, the textbook, and any personal notes are allowed. Probabilistic programming languages like PyMC or BUGS are not allowed for the midterm. Internet search using exam-related keywords is not allowed during the exam period, nor is any communication with other students relating to the exam. You should start with methods from our course materials to solve all problems and fully explain your reasoning. Your answers must be fully supported by your work. Public Ed Discussion posts about the exam questions are not permitted. If you need any clarification on the questions, please use the private posting function so that your post is only visible to the instructors.

Discussing the exam questions with anyone outside of the course staff and/or posting any portion of the exam to non-GT sites are considered serious violations of the Georgia Tech honor code. AI tools, like ChatGPT/Copilot and other similar ones, are not allowed for any purpose.

Please read and sign (or e-sign) the following honor pledge and submit a copy along with your answers.

I pledge on my honor that I have completed the exam on my own and I have not used any unauthorized materials or taken anyone's help for completing this exam.

Name ________________________________

Problem	1	2	3	Total
Score	/40	/30	/30	/100

Spring 2025 Midterm Exam Problem 1. Consider the following Bayesian model (where the parameters are $N(\text{mean}, \text{variance})$):

$$\begin{aligned} y_i|\beta &\sim^{ind.} N(\beta x_i, \sigma^2),\ i = 1, \cdots, n, \\ \beta &\sim N(\beta_0, \tau^2). \end{aligned}$$

1. Find the posterior distribution of β.
2. Before seeing the data, an expert believes that the true value of β should be around 1. Furthermore, he feels that it will be between 0.8 and 1.2 with 95% confidence. Use this information to choose values for β_0 and τ.
3. Suppose that we obtained the data (given in (x, y) pairs): $(10, 10.5)$, $(12, 12)$, and $(15, 16)$. Then, find the posterior mean and the 95% HPD credible interval for β. (Assume $\sigma^2 = 1$).
4. A new observation is generated from $y = \beta x + \epsilon$, where $\epsilon \sim N(0, \sigma^2)$. Find the posterior distribution of the new observation (that is, the posterior predictive distribution).

Hint for part 4: if $X \sim N(\mu, \sigma^2)$, then for any real numbers a and b,

$$aX + b \sim N(a\mu + b,\ a^2\sigma^2)$$

Spring 2025 Midterm Exam Problem 2. The following data on lifetimes are obtained from a reliability test of light bulbs (measured in days):

$$924, 1165, 988, 1925, 2423, 67, 299, 78, 3813, 897.$$

Assume that the lifetime (y) follows an exponential distribution with mean θ:

$$y_i \overset{iid}{\sim} Exp(\theta).$$

Answer the following.

1. Derive the Jeffreys' prior for θ.
2. Under Jeffreys' prior, estimate the posterior mean time to failure for the light bulbs and its 95% HPD credible interval.
3. The company that produces the light bulbs claims that their product will survive for at least two years. Assuming that the posterior mean of θ computed in the previous question is the true value of θ , what fraction of the light bulbs are likely to fail before two years?

Note: use the scale parameterization of $Exp(\theta)$, the density of which is given by

$$p(y) = \frac{1}{\theta} e^{-y/\theta}.$$

Spring 2025 Midterm Exam Problem 3. Consider the following Bayesian hierarchical model:

$$\begin{aligned} y_i|\theta &\sim^{iid} N(\theta, 1), \quad i = 1, \ldots, 10, \\ \theta|\tau^2 &\sim N(0, \tau^2), \\ \tau^2 &\sim Inv - Gamma(1, 2). \end{aligned}$$

The following data were observed: $y = \{7.54, 8.00, 9.27, 9.20, 5.96, 8.17, 7.63, 9.36, 8.61, 10.10\}$. Use Gibbs sampling to sample from the posterior distribution of θ (generate 100,000 samples and use 1,000 samples as burn-in) and answer the following:

1. Plot the posterior density of θ.
2. Find the posterior mean of θ.
3. Find 95% HPD credible interval of θ.

Note 1: the density of an $Inv - Gamma(a, b)$ is given by

$$p(x) = \frac{b^a}{\Gamma(a)} x^{-a-1} e^{-b/x}$$

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Note 2: τ^2 here represents variance, not precision.