

Homework 2

Q1

Suppose data is generated from the model $y_i \mid \mu \stackrel{\text{iid}}{\sim} N(\mu, 1)$ for $i = 1, \dots, n$. Consider a mixture normal prior:

$$\mu \sim .5N(-1, 1) + .5N(1, 1)$$

that is,

$$p(\mu) = \frac{.5}{\sqrt{2\pi}} e^{-\frac{1}{2}(\mu+1)^2} + \frac{.5}{\sqrt{2\pi}} e^{-\frac{1}{2}(\mu-1)^2}$$

Suppose we have observed $y = 1$. Find the posterior distribution of μ .

1. Start by using Bayes theorem: $p(\mu \mid y) \propto p(y \mid \mu)p(\mu) = p(y \mid \mu)\{.5\phi(\mu; -1, 1) + .5\phi(\mu; 1, 1)\}$ and simplify the two components using the following result.

$$\phi(x; \mu_1, \sigma_1^2) \phi(x; \mu_2, \sigma_2^2) = \phi\left(x; \frac{\mu_1/\sigma_1^2 + \mu_2/\sigma_2^2}{1/\sigma_1^2 + 1/\sigma_2^2}, \frac{1}{1/\sigma_1^2 + 1/\sigma_2^2}\right) \phi(\mu_1 - \mu_2; 0, \sigma_1^2 + \sigma_2^2)$$

where $\phi(x; \mu, \sigma^2)$ is the density of a normal distribution with mean μ and variance σ^2 .

2. The posterior is going to be a mixture of two normal distributions. So you will only need to identify the two mean and variance parameters as well as the weights for the two normal distributions.

Q2

Engineering system of type k -out-of- n is operational if at least k out of n components are operational. Otherwise, the system fails. Suppose that a k -out-of- n system consists of n identical and independent elements for which the lifetime has Weibull distribution with parameters r and λ . More precisely, if T is a lifetime of a component,

$$P(T \geq t) = e^{-\lambda t^r}, \quad t \geq 0.$$

Time t is in units of months, and consequently, rate parameter λ is in units $(\text{month})^{-1}$. Parameter r is dimensionless.

Assume that $n = 10, k = 7, r = 1.3$ and $\lambda = 1/20$.

1. Find the probability that a k -out-of- n system is still operational when checked at time $t = 6$.

2. At the check up at time $t = 6$ the system was found operational. What is the probability that at that time exactly 7 components were operational?
3. At the check up at time $t = 6$, the system was found operational. What is the probability that the system would still be operational at the time $t = 9$?

Hint: The probability that a k -out-of- n system is operational corresponds to the tail probability of binomial distribution: $P(X \geq k)$, where X is the number of components working. You can do exact binomial calculations or use `binocdf` in Octave/MATLAB (or `dbinom` in R, or `scipy.stats.binom.cdf` in Python). Be careful with \leq and $<$, because of the discrete nature of the binomial distribution. Part 2 is straightforward Bayes formula. Part 3 is the total probability with hypotheses whose probabilities are obtained as in (b).

Q3

From the first page of Rand's book *A Million Random Digits with 100,000 Normal Deviates*.

31060	10805	45571	82406	35303	42614	86799	07439	23403	09732
85269	77602	02051	65692	68665	74818	73053	85247	18623	88579
63573	32135	05325	47048	90553	57548	28468	28709	83491	25624
73796	45753	03529	64778	35808	34282	60935	20344	35273	88435
98520	17767	14905	68607	22109	40558	60970	93433	50500	73998

The second 50 five-digit numbers form the Rand's "A Million Random Digits with 100,000 Normal Deviates" book (shown above) are rescaled to $[0, 1]$ (by dividing by 100,000) and then all numbers < 0.7 are retained. We can consider the $n = 35$ retained numbers as a random sample from uniform $\mathcal{U}(0, 0.7)$ distribution.

0.3106	0.10805	0.45571	0.35303	0.42614	0.07439	0.23403
0.09732	0.02051	0.65692	0.68665	0.18623	0.63573	0.32135
0.05325	0.47048	0.57548	0.28468	0.28709	0.25624	0.45753
0.03529	0.64778	0.35808	0.34282	0.60935	0.20344	0.35273
0.17767	0.14905	0.68607	0.22109	0.40558	0.60970	0.50500

Pretend now that the threshold 0.7 is not known to us, that is, we are told that the sample is from uniform $\mathcal{U}(0, \theta)$ distribution, with θ to be estimated.

Let M be the maximum of the retained sample u_1, \dots, u_{35} , in our case $M = 0.68665$. The likelihood is

$$f(u_1, \dots, u_{35} | \theta) = \prod_{i=1}^{35} \frac{1}{\theta} \mathbf{1}(\theta > u_i) = \theta^{-35} \mathbf{1}(\theta > M)$$

where $\mathbf{1}(A)$ is 1 if A is true, and 0 if A is false.

Assume noninformative (Jeffreys') prior on θ ,

$$\pi(\theta) = \frac{1}{\theta} \mathbf{1}(\theta > 0)$$

00000	10097 32533	76520 13586	34673 54876	80959 09117	39292 74945
00001	37542 04805	64894 74296	24805 24037	20636 10402	00822 91665
00002	08422 68953	19645 09303	23209 02560	15953 34764	35080 33606
00003	99019 02529	09376 70715	38311 31165	88676 74397	04436 27659
00004	12807 99970	80157 36147	64032 36653	98951 16877	12171 76833
00005	66065 74717	34072 76850	36697 36170	65813 39885	11199 29170
00006	31060 10805	45571 82406	35303 42614	86799 07439	23403 09732
00007	85269 77602	02051 65692	68665 74818	73053 85247	18623 88579
00008	63573 32135	05325 47048	90553 57548	28468 28709	83491 25624
00009	73796 45753	03529 64778	35808 34282	60935 20344	35273 88435
00010	98520 17767	14905 68607	22109 40558	60970 93433	50500 73998
00011	11805 05431	39808 27732	50725 68248	29405 24201	52775 67851
00012	83452 99634	06288 98083	13746 70078	18475 40610	68711 77817
00013	88685 40200	86507 58401	36766 67951	90364 76493	29609 11062
00014	99594 67348	87517 64969	91826 08928	93785 61368	23478 34113
00015	65481 17674	17468 50950	58047 76974	73039 57186	40218 16544
00016	80124 35635	17727 08015	45318 22374	21115 78253	14385 53763
00017	74350 99817	77402 77214	43236 00210	45521 64237	96286 02655
00018	69916 26803	66252 29148	36936 87203	76621 13990	94400 56418
00019	09893 20505	14225 68514	46427 56788	96297 78822	54382 14598
00020	91499 14523	68479 27686	46162 83554	94750 89923	37089 20048
00021	80336 94598	26940 36858	70297 34135	53140 33340	42050 82341
00022	44104 81949	85157 47954	32979 26575	57600 40881	22222 06413
00023	12550 73742	11100 02040	12860 74697	96644 89439	28707 25815

Figure 1: First page of RAND's book.

Posterior depends on data via the maximum M and belongs to the Pareto family, $\mathcal{Pa}(c, \alpha)$, with a density

$$\frac{\alpha c^\alpha}{\theta^{\alpha+1}} \mathbf{1}(\theta > c)$$

1. What are α and c ?
2. Estimate θ in Bayesian fashion. Then calculate the 95% equi-tailed credible set. Is the true value of parameter (0.7) in the credible set?
3. Plot the posterior PDF, adding marks for the regions bound by the above credible set, along with your point estimate, for each plot.
4. Experiment by replacing the Jeffrey's prior with increasingly informative Pareto priors. Start with $c < M$ and very small α . Report what happens to the posterior when varying the Pareto prior parameters and compare them to the Jeffrey's prior model.