Homework 1

Q1

A student answers a multiple choice examination with two questions that have four possible answers each. Suppose that the probability that the student knows the answer to a question is 0.65 and the probability that the student guesses is 0.35. If the student guesses, the probability of guessing the correct answer is 0.25. The questions are independent, that is, knowing the answer on one question is not influenced by the other question.

a) What is the probability that both questions will be answered correctly?

Independence of the questions should make your calculations easier. Let A represent the event that a particular question is answered correctly, and E is the event that both are answered correctly. Use the Law of Total Probability to find $P(A)$. It should follow that $P(E) = [P(A)]^2$

$$
\begin{aligned}
P(A) &= P(A|\text{Knows})P(\text{Knows}) + P(A|\text{Guesses})P(\text{Guesses}) \\
P(A) &= \left[1 \times 0.65\right] + \left[\frac{1}{4} \times 0.35\right] \\
&= 0.7375 \\
P(E) &= [P(A)]^2 = 0.54390625 \\
&\boxed{P(\text{E}) = 0.54390625}
\end{aligned}
$$

Doing it this way is the key to being able to generalize it to "n "questions in part (c). If you happened to break it down into cases, you might have ended up with something like this:

CASE 1: The student knows neither (guesses both)

$$
\begin{aligned}
P(\text{Case 1}) &= (0.35) \times (0.35) = 0.1225 \\
P(\text{Both Correct}|\text{Case 1}) &= (0.25) \times (0.25) = 0.0625 \\
\text{Contribution} &= 0.1225 \times 0.0625 = 0.00765625
\end{aligned}
$$

CASE 2: The student knows Question 1, and guesses Question 2

$$
\begin{aligned}
P(\text{Case 2}) &= (0.65) \times (0.35) = 0.2275 \\
P(\text{Both Correct}|\text{Case 2}) &= (1) \times (0.25) = 0.25 \\
\text{Contribution} &= 0.2275 \times 0.25 = 0.056875
\end{aligned}
$$

CASE 3: The student guesses Question 1, and knows Question 2

$$
\begin{aligned}
P(\text{Case 3}) &= (0.35) \times (0.65) = 0.2275 \\
P(\text{Both Correct}|\text{Case 3}) &= (0.25) \times (1) = 0.25 \\
\text{Contribution} &= 0.2275 \times 0.25 = 0.056875
\end{aligned}
$$

CASE 4: The student knows both (guesses neither)

$$\begin{aligned} P(\text{Case 4}) &= (0.65) \times (0.65) = 0.4225 \\ P(\text{Both Correct}|\text{Case 4}) &= (1) \times (1) = 1 \\ \text{Contribution} &= 0.4225 \times 1 = 0.4225 \end{aligned}$$

$$\begin{aligned} P(\text{E}) &= P(E|C_1)P(C_1) + P(E|C_2)P(C_2) + P(E|C_3)P(C_3) + P(E|C_4)P(C_4) \\ &= \big[(0.0625) \times (0.1225)\big] + \big[(0.25) \times (0.2275)\big] + \\ &\quad \big[(0.25) \times (0.2275)\big] + \big[(1) \times (0.4225)\big] \\ &= 0.00765625 + 0.056875 + 0.056875 + 0.4225 \\ &\boxed{P(\text{E}) = 0.54390625} \end{aligned}$$

And gotten the same answer. This method quickly becomes prohibitive, as the number questions increases.

b) Suppose both questions were answered correctly. What is the probability that the student really knew the correct answer to both questions?

Let A represent the event that both questions were answered correctly. Let B represent the event that the student really knew the answers to both questions. Using Bayes' Rule for Events along with your answer from part (a):

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)} = \frac{(1) \times (0.65)^2}{(0.7375)^2} \boxed{= 0.7768}$$

c) How would you generalize the above from 2 to n questions, that is, what are answers to (a) and (b) if the test has n independent questions? What happens to probabilities in (a) and (b) if $n \to \infty$.

First, define the following events:

$$\begin{aligned} M &: \text{The student knows the answers to all } n \text{ questions} \\ N &: \text{All } n \text{ questions were answered correctly} \end{aligned}$$

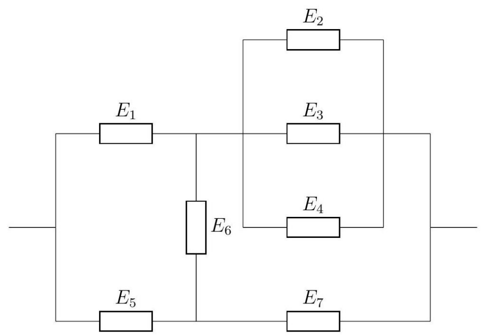
Note that $P(N) = 0.7375^n$ and $P(M|N) = \left(\frac{0.65}{0.7375}\right)^n$. Using Bayes' Rule again:

$$P(M|N) = \frac{P(N|M)P(M)}{P(N)} = \frac{(1) \times (0.65)^n}{(0.7375)^n} = \left(\frac{0.65}{0.7375}\right)^n$$

Noting that $\lim_{n\to\infty} (\frac{0.65}{0.7375})^n = 0$.

Q2

A circuit S consisting of seven independent elements $E_1, \ldots, E_7$ is connected as shown:



The elements are operational during time interval T with probabilities

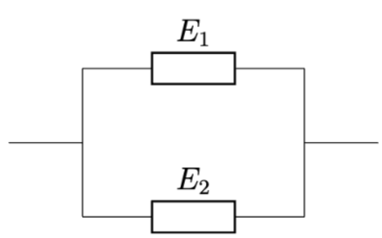
	E_1	E_2	E_3	E_4	E_5	E_6	E_7
Probability of working (p)	0.5	0.4	0.1	0.6	0.9	0.8	0.7

Let the probability of an element E_i of a system S being operational during time interval T as p_{E_i}. Suppose all elements work independently. Elements can be connected in "series" or in "parallel":

- A series of components E_i:

is operational if and only if all components are operational. That is, $p_S = p_{E_1} p_{E_2}$

- A parallel arrangement:



will *fail* if and only if all components *fail*. Noting that S^c represents a system failure, this is $p_{S^c} = p_{E_1^c} p_{E_2^c}$. Then $p_S = 1 - p_{S^c}$. You can verify that this is equivalent to $p_S = p_{E_1 \cup E_2} = p_{E_1} + p_{E_2} - p_{E_1 E_2}$ by using DeMorgan's Laws.

a) Find the probability that the circuit is operational during time interval T.

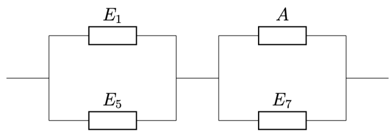
We consider the following two hypotheses:

$$H_1 : E_6 \text{ works during time interval } T$$
$$H_2 : E_6 \text{ fails during time interval } T.$$

We know that $P(H_1) = 0.8$ and $P(H_2) = 0.2$. By the Law of Total Probability, we have

$$P(S) = P(S|H_1)P(H_1) + P(S|H_2)P(H_2)$$

Under hypothesis H_1, S is equivalent to the following circuit where A is a sub-circuit that E_2, E_3 and E_4 are connected in parallel.



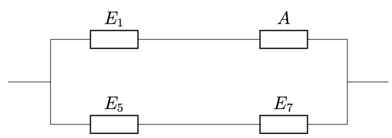
We know that the probability that A is operational during time interval T is

$$p_A = 1 - q_{E2}q_{E3}q_{E4} = 1 - (0.6)(0.9)(0.4) = 0.784$$

Thus, we have

$$p_{S|H1} = (1 - q_{E1}q_{E5})(1 - q_Aq_{E7}) = (1 - (0.5)(0.1))(1 - (0.216)(0.3)) = 0.88844$$

Under hypothesis H_2, S is equivalent to the following circuit where A is a sub-circuit that E_2, E_3 and E_4 are connected in parallel.



We have:

$$\begin{aligned} p_{S|H_2} &= 1 - (1 - p_{E_1}p_A)(1 - p_{E_5}p_{E_7}) \\ &= 1 - (1 - (0.5)(0.784))(1 - (0.9)(0.7)) \\ &= 0.77504 \end{aligned}$$

Hence:

$$P(S) = (0.88844)(0.8) + (0.77504)(0.2) \boxed{= 0.86576}$$

b) If the circuit was found operational at the time T, what is the probability that the element E_6 was operational.

By Bayes' Rule:

$$P(H_1|S) = \frac{P(S|H_1)P(H_1)}{P(S)} = \frac{(0.88844)(0.8)}{0.86576} \boxed{= 0.82096}$$

Q3

A machine has four independent components, three of which fail with probability $q = 1 - p$, and one with probability 0.5. The machine is operational as long as at least three components are working.

a) What is the probability that the machine will fail? Evaluate this probability for $p = 0.75$.

We first define the following:

$$\begin{aligned} M &: \text{The component with a failure probability of } 0.5 \\ A &: \text{The event that the whole machine is working} \\ B &: \text{The event that component } M \text{ is working} \end{aligned}$$

Using the Law of Total Probability, we can condition the overall machine functionality on whether component M is working or not. (E^c *denotes the complement of event* E).

$$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c) \tag{1}$$

If component M is working, then the overall machine will function when at least two of the other three components are functioning:

$$\begin{aligned} P(A|B) &= P(\text{All 3 working}) + P(\text{Any two working}) \\ &= p^3 + \binom{3}{2} p^2(1-p) \end{aligned}$$

Recall that if component M fails when only 2 of the other 3 components are working, the machine fails. Event $A|B^c$ requires all 3 others to be operational

$$P(A|B^c) = P(\text{All non-}M\text{ components operational}) = p^3$$

Returning to line (1):

$$\begin{aligned} P(A) &= P(A|B)P(B) + P(A|B^c)P(B^c) \\ &= \left[\left(p^3 + \binom{3}{2}p^2(1-p)\right) \times \left(\frac{1}{2}\right)\right] + \left[p^3 \times \left(\frac{1}{2}\right)\right] \\ &= \left[\left((0.75)^3 + \binom{3}{2}(0.75)^2(1-(0.75))\right) \times \left(\frac{1}{2}\right)\right] + \left[(0.75)^3 \times \left(\frac{1}{2}\right)\right] \\ P(A) &= 0.6328125 \end{aligned}$$

Thus, the probability of the machine failing is:

$$\begin{aligned} P(A^c) &= 1 - P(A) \\ &= 1 - 0.6328125 \\ &= \boxed{0.3671875} \end{aligned}$$

b) If the machine failed, what is the probability that the component which fails with probability 0.5 actually failed?

Given the machine failing, what is the probability that component M failed, i.e., $P(B^c \mid A^c)$? We use Bayes' rule:

$$P(B^c \mid A^c) = \frac{P(A^c \mid B^c)P(B^c)}{P(A^c)}$$

We already have $P(A^c)$ and $P(B^c)$. The term $P(A^c \mid B^c)$ is the complement of $P(A \mid B^c)$:

$$P(A^c \mid B^c) = 1 - P(A \mid B^c) = 1 - p^3$$

Thus,

$$\begin{aligned} P(B^c \mid A^c) &= \frac{(1-p^3)(0.5)}{1 - 1.5p^2 + 0.5p^3} \\ &= \frac{(1 - 0.421875)(0.5)}{0.3671875} \\ &= \boxed{0.787234} \end{aligned}$$

c) Suppose that after the machine fails, a diagnostic test is performed to determine whether the component which fails with probability 0.5 actually failed. The test is 90% accurate when this component fails (sensitivity) and 80% accurate when it does not fail (specificity). Compute the posterior probability that this component failed, given that the diagnostic test indicates a failure.

Let T^c denote that the test shows a failure in component M (a positive result), and T denote that the test shows no failure in component M (a negative result). Given that:

$$\begin{aligned} P(T^c \mid B^c) &= 0.9 \quad \text{(Sensitivity)} \\ P(T \mid B) &= 0.8 \quad \text{(Specificity)} \end{aligned}$$

We want to find the probability that component M is not working, $P(B^c)$, given that the test is positive and the machine is not working, $P(A^c)$.

$$P(B^c \mid T^c, A^c) = \frac{P(T^c, A^c \mid B^c)P(B^c)}{P(T^c, A^c)}$$

Where:

$$\begin{aligned} P(T^c, A^c \mid B^c) &= P(A^c \mid T^c, B^c)P(T^c \mid B^c) \\ &= P(A^c \mid B^c) \cdot P(T^c \mid B^c) \\ &= (1 - p^3) \cdot 0.9 \end{aligned}$$

$$\begin{aligned} P(T^c, A^c \mid B) &= P(A^c \mid T^c, B)P(T^c \mid B) \\ &= P(A^c \mid B) \cdot P(T^c \mid B) \\ &= \left[1 - \left(p^3 + 3p^2(1-p)\right)\right] \cdot 0.2 \end{aligned}$$

$$\begin{aligned} P(T^c, A^c) &= P(T^c, A^c \mid B^c)P(B^c) + P(T^c, A^c \mid B)P(B) \\ &= (1 - p^3) \cdot 0.9 \cdot 0.5 + \left[1 - \left(p^3 + 3p^2(1-p)\right)\right] \cdot 0.2 \cdot 0.5 \end{aligned}$$

Thus, applying Bayes' Theorem:

$$\begin{aligned} P(B^c \mid T^c, A^c) &= \frac{P(T^c, A^c \mid B^c)P(B^c)}{P(T^c, A^c \mid B^c)P(B^c) + P(T^c, A^c \mid B)P(B)} \\ &= \boxed{0.9433} \end{aligned}$$