Homework 3

Q1

Marietta Traffic Authority is concerned about the repeated accidents at the intersection of Canton and Piedmont Roads. Bayes-inclined city-engineer would like to estimate the accident rate, even better, find a credible set.

A well-known model for modeling the number of road accidents in a particular location/time window is the Poisson distribution. Assume that X represents the number of accidents in a 3 month period at the intersection of Canton and Piedmont Roads.

Assume that $[X \mid \theta] \sim \mathcal{P}oi(\theta)$. Nothing is known a priori about θ, so it is reasonable to assume the Jeffreys prior

$$\pi(\theta) = \frac{1}{\sqrt{\theta}} \mathbf{1}(0 < \theta < \infty)$$

In the four most recent three-month periods the following realizations for X are observed: $1, 2, 0$, and 2 .

(a) Compare the Bayes estimator for θ with the MLE (For Poisson, recall, $\hat{\theta}_{MLE} = \bar{X}$).

(b) Compute the 95% equitailed credible set.

(c) Compute (numerically) the 95% HPD credible set. Use an optimization method, not a sampling method.

(d) Numerically find the mode of the posterior, that is, MAP estimator of θ. Make sure it matches the result of the known equation for the posterior mode.

(e) If you test the hypotheses

$$H_0 : \theta \geq 1 \quad \text{vs} \quad H_1 : \theta < 1$$

based on the posterior, which hypothesis will be favored?

(f) Derive the posterior predictive distribution. Based on this, how many accidents do you predict for the next year?

Q2

Waiting time. The waiting time for a bus at a given corner at a certain time of day is known to have a $U(0, \theta)$ distribution. It is desired to test $H_0 : 0 \leq \theta \leq 15$ versus $H_1 : \theta > 15$. From other similar routes, it is known that θ has a Pareto $(5, 3)$ distribution. If waiting times of $10, 8, 10, 5,$ and 14 are observed at the given corner, calculate the posterior odds ratio and the Bayes factor. Which hypothesis would you favor?

Note: the density of a Pareto distribution with parameters (c, α) is given by

$$\frac{\alpha c^{\alpha}}{\theta^{\alpha+1}} \mathbf{1}(\theta > c)$$

Q3

The Maxwell distribution with parameter $\alpha > 0$, has a probability density function for $x > 0$ given by

$$p(x \mid \alpha) = \sqrt{\frac{2}{\pi}} \alpha^{3/2} x^2 \exp\left(-\frac{1}{2}\alpha x^2\right)$$

(a) Find the Jeffreys prior for α.

(b) Find a transformation of this parameter in which the corresponding prior is uniform. (Hint: See Section 5.7 of *Engineering Biostatistics*).

(c) Find the posterior distribution for n independent and identically distributed datapoints $x_1, \ldots, x_n$ from the Maxwell distribution, assuming the Jeffreys prior on α from part (a).