Optimization

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Say you want to...

- Choose how many bottles of holiday rum to make to maximize expected profit given uncertain demand for said rum at the beginning of the holiday season.

You might want to learn about...

Optimization

You mean like code optimization, right?

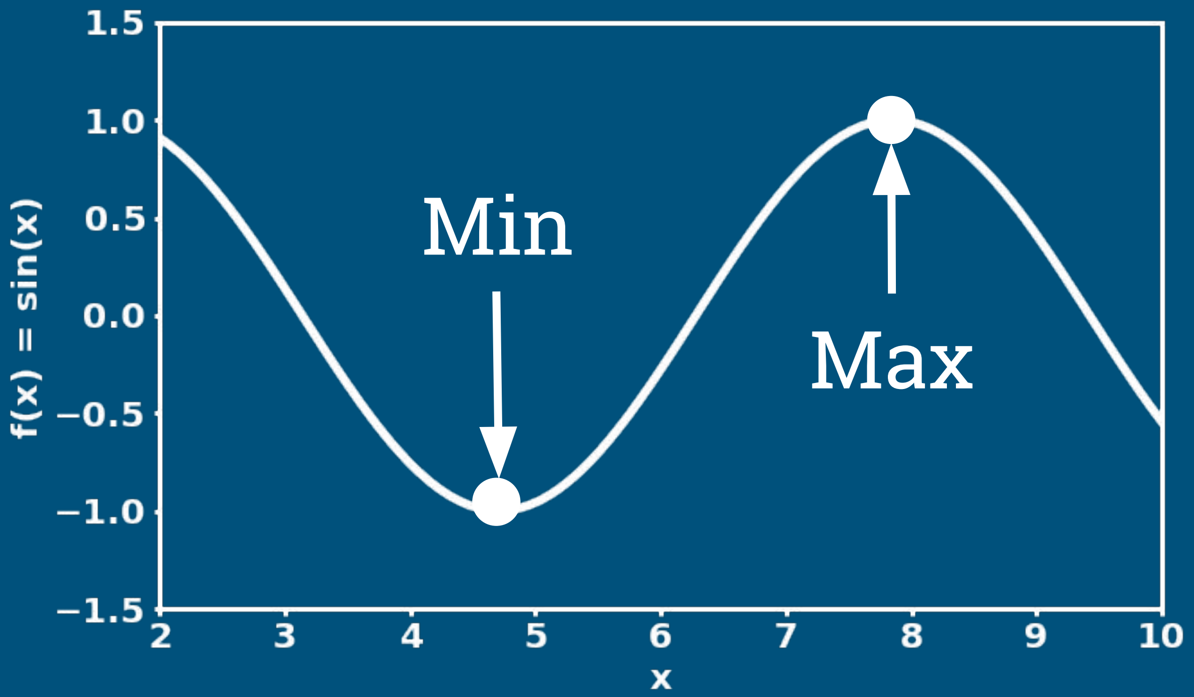
NO

I mean like MATHEMATICAL optimization.

That allows you to find the best option from a set of alternatives.

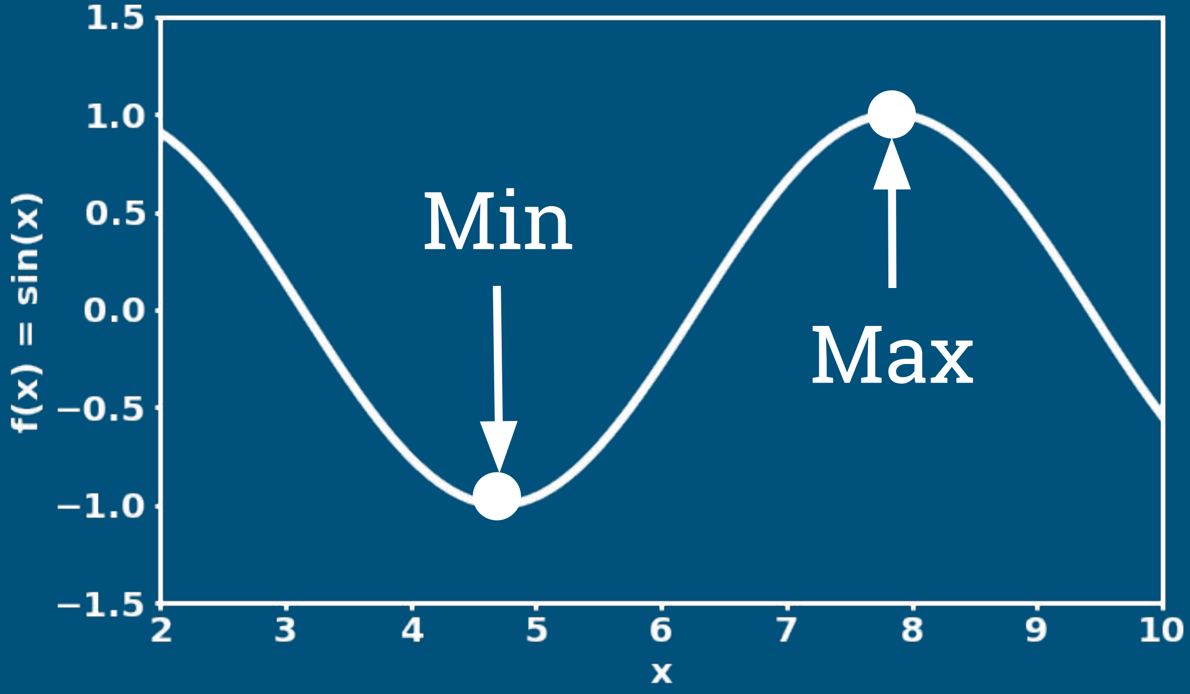
And puts the "learning" in machine learning.
(OK, statistics helps too.)

All optimization problems find..



Of an..

Objective Function

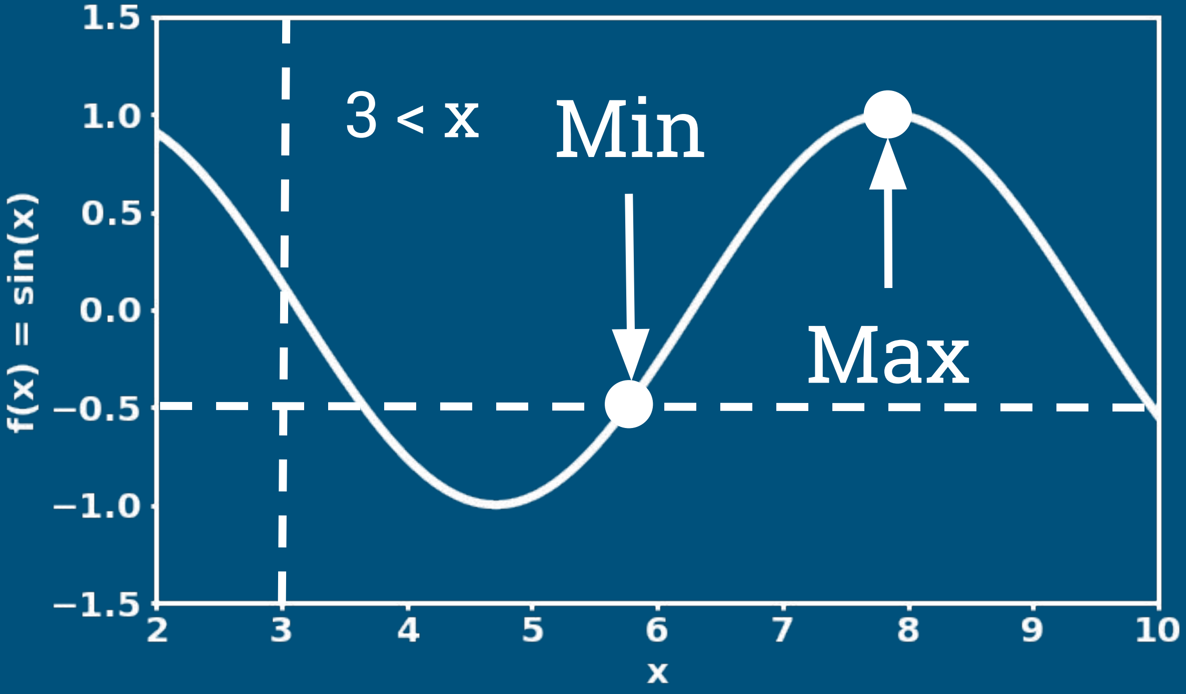


w.r.t. Decision Variables

Subject to..

Constraints

Objective Function



f(x) ≥ -0.5

w.r.t. Decision Variables

Formulating optimization problems

- Step 1 - Define an **objective function** to minimize or maximize.

$$\underset{x}{\text{Minimize}} \quad f(x) = \sin(x)$$

- Step 2 - Define **decision variables** to optimize, e.g. x.
- Step 3 - Define **constraints** to be satisfied.

$$\text{Subject to} \quad 3 < x$$
$$f(x) \geq -0.5$$

QUESTIONATOR

- In our rum example, what was our **objective function**?
- What was our **decision variable**?
- What are some business **constraints** we might wish to satisfy?
- In machine learning, what is our **objective function**?
- What are our **decision variables**?

BREAKOUT

- Work in pairs to formulate our rum example as an optimization problem.
- Remember, we want to choose how many bottles of holiday rum, q, to make to minimize expected cost, $\mathbb{E}(C(q, D))$, given uncertain demand, D, at the beginning of the holiday season. Holiday rum has price, p, and cost, c. Bottles not sold have salvage value, s.
- HINT: We have a Newsvendor Problem (see next slide).

Newsvendor Problem

- Choose **quantity**, q, to minimize **expected cost**, $\mathbb{E}(C(q, D))$, given **uncertain demand**, D.

$$C(q, D) = c_o * \max(q - D, 0) + c_u * \max(D - q, 0)$$

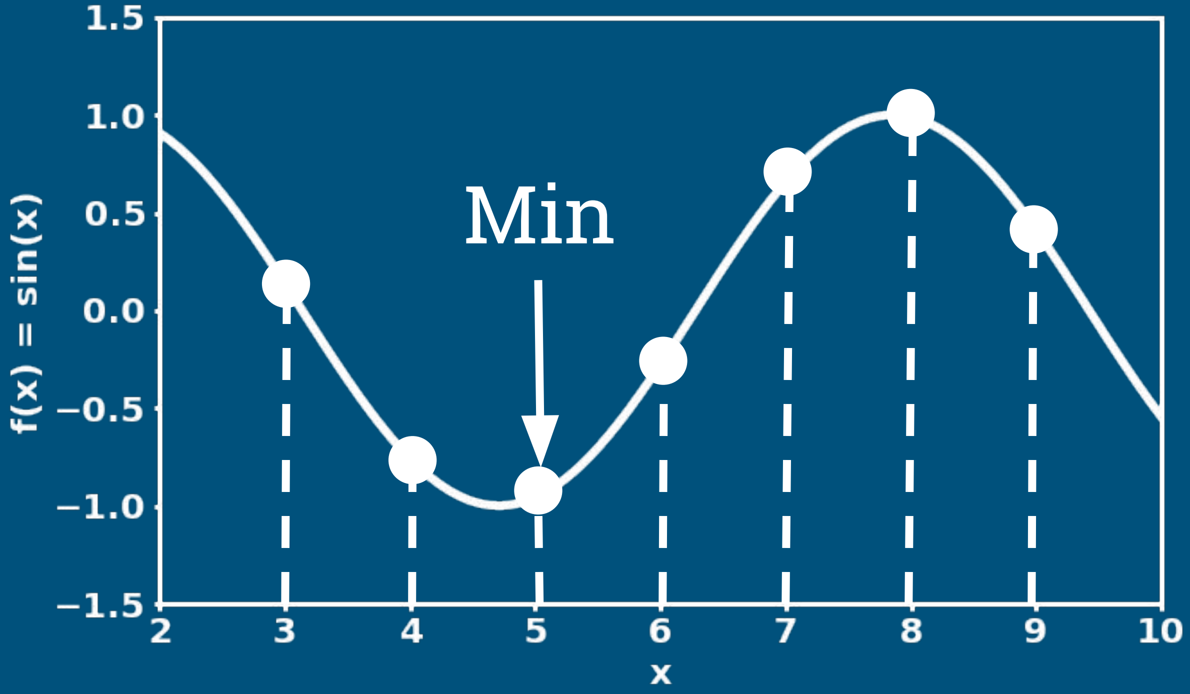
$$\underset{q}{\text{Minimize}} \;\; \mathbb{E}(C(q, D)) \quad \text{Subject to} \;\; q \geq 0$$

c_o - overage cost per unit of demand exceeded ← ?
c_u - underage cost per unit of demand not met ← ?

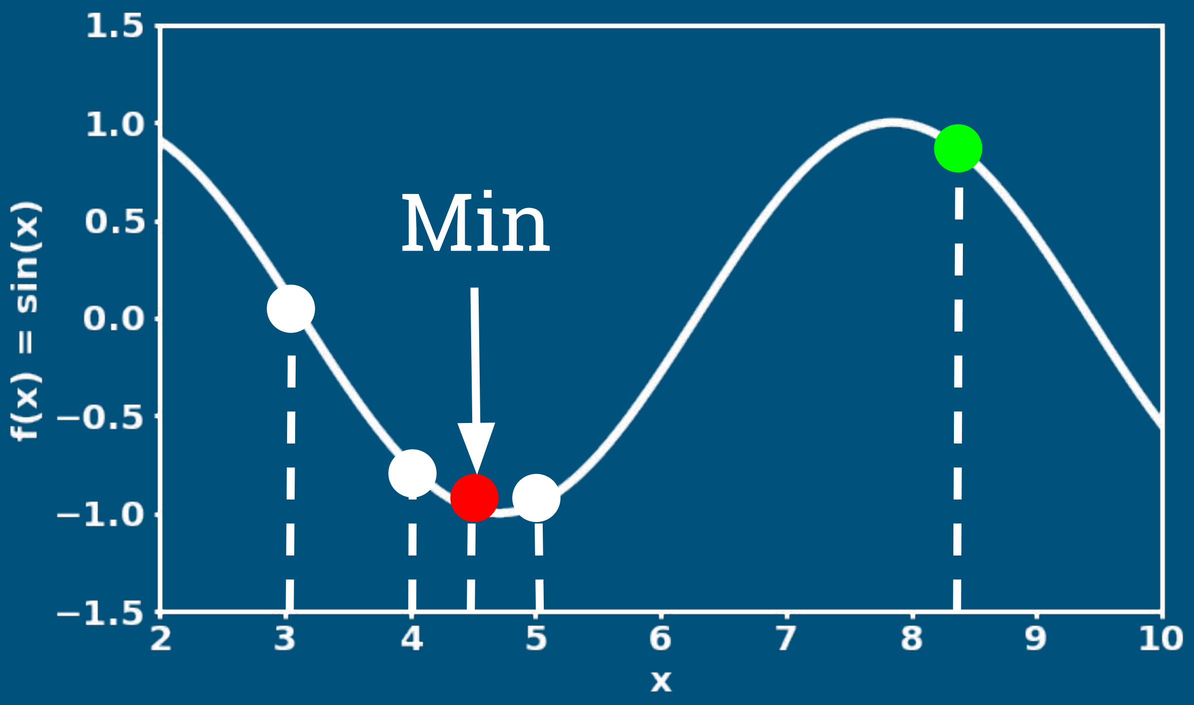
Newsvendor Paper

http://warrington.ufl.edu/departments/isom/docs/vakharia/2011_EJOR.pdf

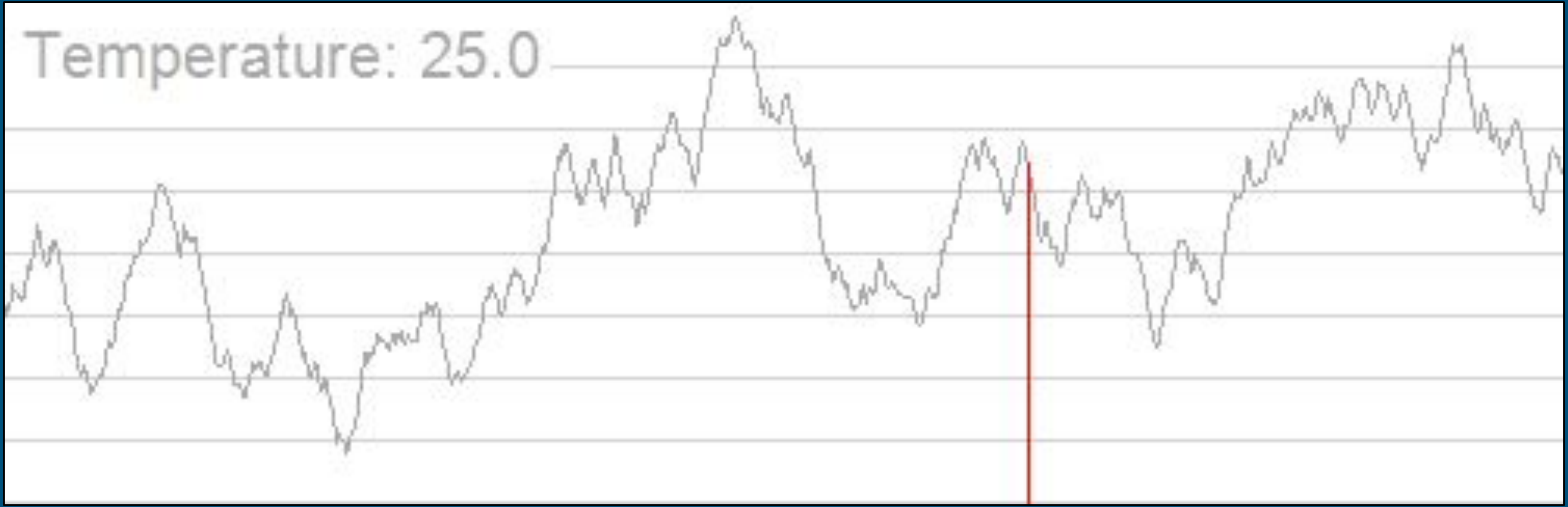
Solving optimization problems (grid search)



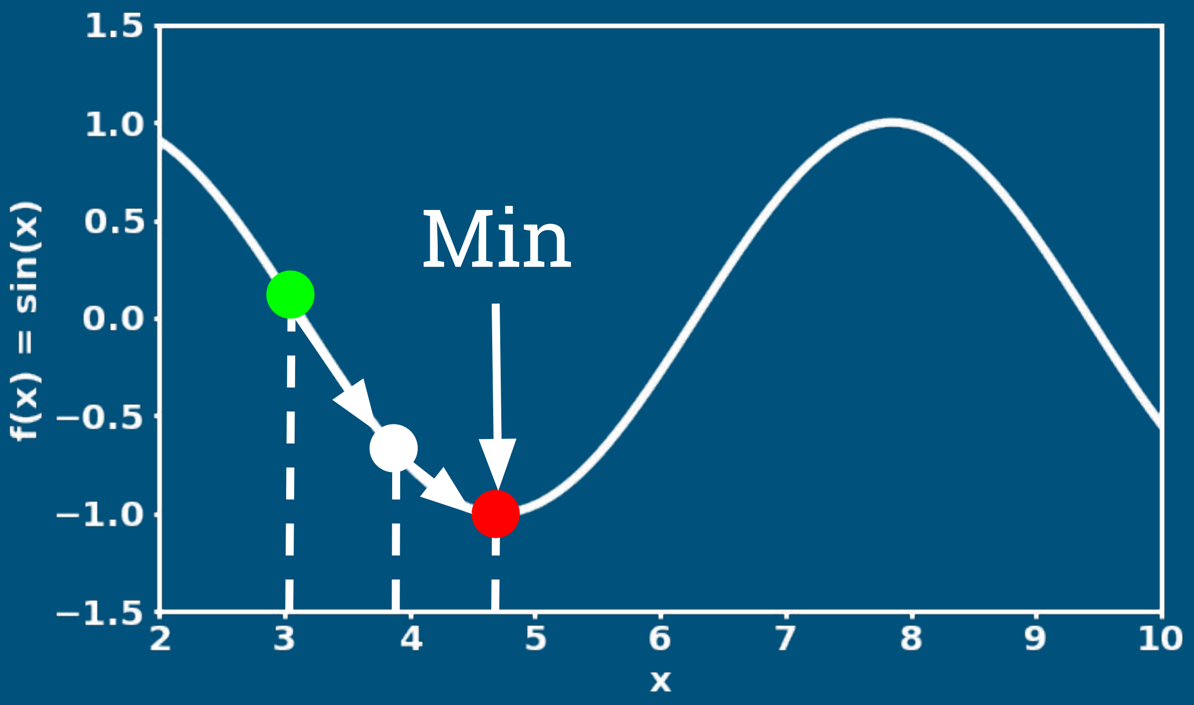
Solving optimization problems (discrete)



Simulated annealing



Solving optimization problems (continuous)



Steepest Descent

<u>Step 1 - Direction</u>

$\mathbf{d}_k = -\nabla_x f(x)$

<u>Step 2 - Step</u>

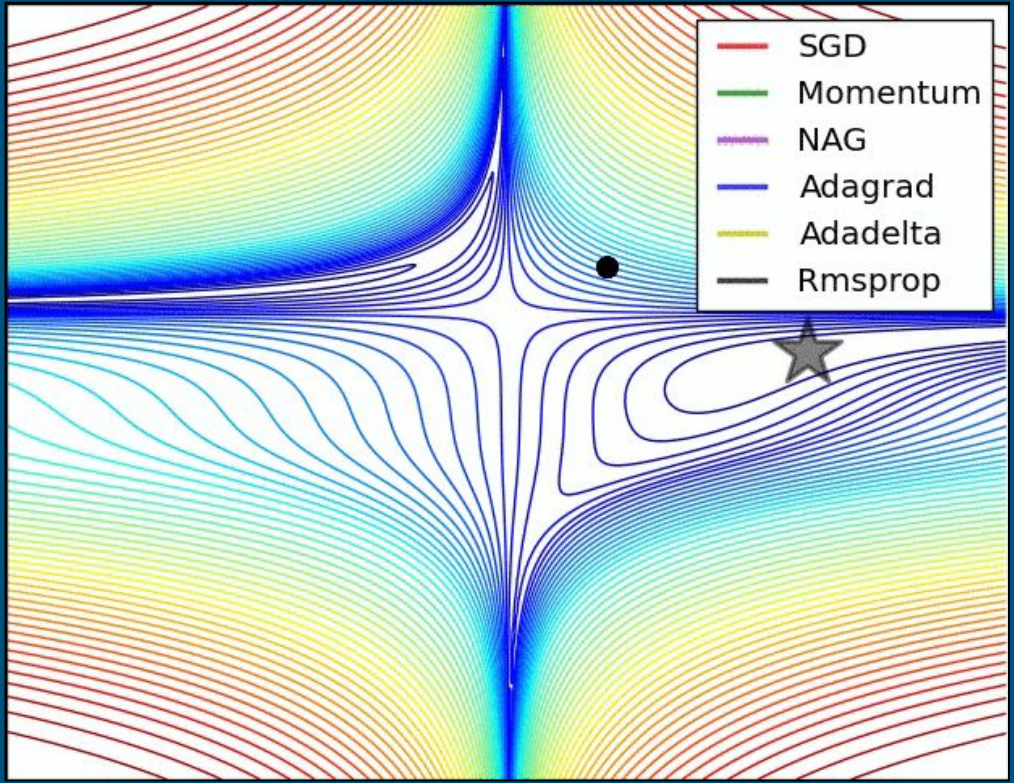
$\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha\, \mathbf{d}_k$

<u>Step 3 - Repeat</u>

$k = k + 1$

Go to Step 1

Gradient descent in TensorFlow



Steepest descent not the only fish in the sea

Problem	Variables	SD Iter (F)	L-BFGS Iter (F)
HS1	2	18,291 (257,000)	18 (67)
HS2	2	4,132 (62,543)	13 (63)
HS4	2	1 (20)	1 (4)
HS5	2	25 (93)	7 (17)
HS110	10	26 (120)	7 (18)

Optimization Taxonomy

https://neos-guide.org/content/optimization-taxonomy

QUESTIONATOR

- In our rum example, what type of optimization problem would we have given **certain demand**?
- What type of optimization problem would we have given **uncertain demand**?
- What type of optimization problem would we have given we can can only make 1,000 bottle batches?
- In machine learning, what are some types of optimization problems we have?

Optimization Software

https://en.wikipedia.org/wiki/List_of_optimization_software

Newsvendor by hand (deterministic)

- Choose quantity, q, to minimize expected cost, $\mathbb{E}(C(q, d))$, given **certain** demand, d.

  $$C(q, d) = c_o * \max(q - d, 0) + c_u * \max(d - q, 0)$$

  $$\underset{q}{\text{Minimize}}\ \mathbb{E}(C(q, d)) = C(q, d) \qquad \text{Subject to } q \geq 0$$

- Solution is trivial. Choose quantity equal to demand, q = d.

Newsvendor as LP (deterministic)

- Choose quantity, q, to minimize expected cost, $\mathbb{E}(C(q, d))$, given **certain** demand, d.

$$
\begin{aligned}
\underset{q}{\text{Minimize}} \quad & C(q, d) = c_o * s + c_u * t \\
\text{Subject to} \quad & s \geq (q - d) \\
& t \geq (d - q) \\
& s, t, q \geq 0
\end{aligned}
$$

Newsvendor by hand (stochastic)

- Choose quantity, q, to minimize expected cost, $\mathbb{E}(C(q, D))$, given **uncertain** demand, D.

  $C(q, D) = c_o * \max(q - D, 0) + c_u * \max(D - q, 0)$

  $$\underset{q}{\text{Minimize}} \quad \mathbb{E}(C(q, D)) = \int_0^q c_o\,(q - x)\,\underbrace{f(x)}_{\text{PDF}}\,dx + \int_q^\infty c_u\,(x - q)\,\underbrace{f(x)}_{\text{PDF}}\,dx$$

  Subject to $\quad q \geq 0$

- Not trivial, but don't panic. We have a few tricks up our sleeve.

Newsvendor by hand (stochastic)

$$\mathbb{E}(C(q, D)) = c_o\, q \int_0^q f(x)\, dx - c_o \int_0^q x\, f(x)\, dx + c_u \int_q^\infty x\, f(x)\, dx - c_u\, q \int_q^\infty f(x)\, dx$$

Newsvendor by hand (stochastic)

$\mathbb{E}(C(q, D)) = c_o\, q \int_0^q f(x)\, dx - c_o \int_0^q x\, f(x)\, dx + c_u \int_q^\infty x\, f(x)\, dx - c_u\, q \int_q^\infty f(x)\, dx$

Min of $\mathbb{E}(C(q, D))$ will occur where derivative w.r.t. q equals 0.

$d\, \mathbb{E}(C(q, D)) / dq = c_o \int_0^q f(x)\, dx - c_u \int_q^\infty f(x)\, dx = 0$

Newsvendor by hand (stochastic)

$\mathbb{E}(C(q, D)) = c_o\, q \int_0^q f(x)\, dx - c_o \int_0^q x\, f(x)\, dx + c_u \int_q^\infty x\, f(x)\, dx - c_u\, q \int_q^\infty f(x)\, dx$

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$d\,\mathbb{E}(C(q, D)) / dq = c_o \int_0^q f(x)\, dx - c_u \int_q^\infty f(x)\, dx = 0$

Remember, integrating is just finding the area under the curve.

$d\,\mathbb{E}(C(q, D)) / dq = c_o\, \underbrace{F(q)}_{\text{CDF}} - c_u\, (1 - \underbrace{F(q)}_{\text{CDF}}) = 0$

Newsvendor by hand (stochastic)

$\mathbb{E}(C(q, D)) = c_o\, q \int_0^q f(x)\, dx - c_o \int_0^q x\, f(x)\, dx + c_u \int_q^\infty x\, f(x)\, dx - c_u\, q \int_q^\infty f(x)\, dx$

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$d\,\mathbb{E}(C(q, D)) / dq = c_o \int_0^q f(x)\, dx - c_u \int_q^\infty f(x)\, dx = 0$

Remember, integrating is just finding the area under the curve.

$d\,\mathbb{E}(C(q, D)) / dq = c_o\, F(q) - c_u\, (1 - F(q)) = 0$

$\Rightarrow\ F(q) = c_u / (c_o + c_u)\ \Rightarrow\ q = F^{-1}(c_u / (c_o + c_u)) = \mathrm{PPF}(c_u / (c_o + c_u))$

Newsvendor as LP (stochastic)

- Choose quantity, q, to minimize expected cost, $\mathbb{E}(C(q, D))$, given **uncertain** demand, D.

Minimize $\mathbb{E}(C(q, D)) = \sum_{i=1}^{n} (c_o s_i + c_u t_i)\, P(D_i)$
q

Subject to $s_i \geq (q - D_i)$

$t_i \geq (D_i - q)$

$s_i, t_i, q \geq 0$

Why bother with an optimization approach when we have a closed form solution?