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On the Variety of Complexes

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INTRODUCTION

This paper is devoted to the characteristic free study of the so-called Buchsbaum–Eisenbud varieties of complexes, which are introduced as follows.

Let $\mathscr{R}$ be any ring with identity and let $V_0,\ldots, V_m$ be a sequence of finite free modules over $\mathscr{R}$, rank $V_i = n_i$.

In the affine space

$$A^N = \bigoplus_{i=0}^{m-1} \operatorname{Hom}(V_{i+1}, V_i)$$

let us consider the variety of $m$ples $(\varphi_1,\ldots, \varphi_m)$ of maps

$$\varphi_i : V_i \to V_{i-1}$$

such that

$$\varphi_i \circ \varphi_{i+1} = 0 \qquad \text{for each} \quad 1 \leqslant i \leqslant m-1. \tag{*}$$

Call W such variety: a point $(\varphi_1,\ldots, \varphi_m) \in W$ represents a complex. If we choose a basis $\{e_1,\ldots, e_{n_i}\}$ for each V_i, we can identify W with the variety of

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$m$ples of matrices $(M_1,\ldots,M_m)$ with entries in $\mathscr{R}$, M_i being an $n_{i-1} \times n_i$ matrix such that

$$M_i \cdot M_{i+1} = 0.$$

Let A be the coordinate ring of the affine space A^N, i.e., the polynomial ring $\mathscr{R}[X^{(i)}_{s_it_i}]$, $s_i = 1 \cdots n_{i-1}$, $i = 1 \cdots m$, and $B = \mathscr{R}[W]$ the reduced coordinate ring of W, and let $\mathscr{E}$ be the ideal in A generated by the elements

$$\sum_{k=1}^{n_i} X^{(i)}_{s_ik} X^{(i+1)}_{kt_{i+1}}$$

with $s_i = 1 \cdots n_{i-1}$, $t_i = 1 \cdots n_{i+1}$, $t \leqslant i \leqslant m-1$.

Let now $W(k_1 \cdots k_m)$ be the subvariety of W consisting of matrices $M_1 \cdots M_m$ such that rank $M_i \leqslant k_i$, where $k_1 \cdots k_m$ are such that $k_i \leqslant \min(n_{i+1}, n_i)$.

The varieties W and $W(k_1 \cdots k_m)$ are called the Buchsbaum–Eisenbud varieties of complexes.

If $\mathscr{E}(k_1 \cdots k_m)$ is the ideal generated by $\mathscr{E}$ and the determinants of the minors of $X^{(i)}$ of size $k_i + 1$ and $B(k_1 \cdots k_m)$ is the reduced coordinate ring of $W(k_1 \cdots k_m)$, the main goal of this work is to prove that $B(k_1 \cdots k_m) \cong A/\mathscr{E}(k_1 \cdots k_m)$, that is to say the equations of the varieties of complexes are given. Moreover we show that such varieties are Cohen–Macaulay and normal, in the case $k_i + k_{i+1} \leqslant n_i$, therefore quite a complete picture of their structure is obtained.

The proofs are performed in various steps.

First in Section 1 we use Young diagrams and Young tableaux to give an explicit basis for $A/\mathscr{E}$, i.e., the basis of what we call "standard multitableaux": the use of such combinatorial devices gets us to the fact tat the ideals involved are reduced.

This enables us to interpret in Section 2 the coordinate ring of $W(k_1 \cdots k_m)$ as an algebra with straightening law and therefore the results contained in [2] can be used to get the Cohen–Macauliness and normality of B–E-varieties.

Varieties of complexes have attracted the attention of various people.

Buchsbaum and Eisenbud [1] have studied them from the point of view of resolutions and they prove the results for the variety $W(k, 1)$.

In characteristic 0, Kempf [11, 12], has proved that the variety $W(k_1 \cdots k_m)$ is Cohen–Macaulay and normal and conjectures our results to be true in any characteristic; Hesselink too asks these questions in [10].

We have been informed by David Eisenbud that Craig Huneke has independently obtained results similar to ours.

Notice that in the case of a sequence of two free modules V_0, V_1, i.e., of one matrix, the variety obtained is the determinantal variety and the results we get are known [8].

In the case of a sequence of three modules V_0, V_1, V_2, i.e., of two matrices, the variety W is the set of null forms for the action of $Gl(V_1)$ on the space $\mathrm{Hom}(V_0, V_1) \times \mathrm{Hom}(V_1, V_2)$ [5, 10].

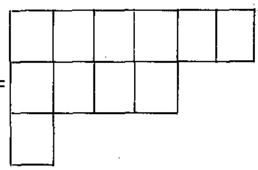
1. Young Tableaux and the Variety of Complexes

The main tools we are going to use of reach our goals are Young diagrams and Young tableaux. For an extensive dissertation on the fundamental facts concerning such objects, the reader can look in the introduction of [3].

Here we recall only that in general a Young diagram σ with k rows is a nonincreasing sequence of positive integers

$$\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_k.$$

One can think of σ as a sequence of rows of "boxes" of length $\sigma_1, \sigma_2 \cdots$. Thus

$\sigma = (6, 4, 1) =$ 

The product of a diagram σ with k rows and a diagram σ with l rows is the diagram of $k + l$ rows, whose rows are the rows of σ together with the rows of σ, arranged in decreasing order.

A Young tableau is a filling of the boxes of a Young diagram σ with integers out of $1, 2, \ldots$.

σ is called the "shape" of the Young tableau.

Given a Young tableau

$$T = \begin{pmatrix} a_{11} & \cdots & & a_{1m_1} \\ a_{21} & \cdots & a_{2m_2} & \\ \vdots & & & \\ a_{s1} & \cdots & a_{sm_s} & \end{pmatrix},$$

where the a_{ij}'s are indices out of $1, 2, \ldots$, and one assumes that

(i) $m_1 \geqslant m_2 \geqslant \cdots \geqslant m_s$

then T is called "standard" if furthermore one has

(ii) $a_{ij} < a_{ik}$ when $k > j$,

(iii) $a_{ij} \leqslant a_{kj}$ when $k \geqslant i$.

In the next, if A and B are two such Young tableaux, by writing

$$\begin{matrix} A \\ B \end{matrix}$$

we mean that we consider the new tableau obtained by arranging the rows and the columns of B under those of A.

EXAMPLE. If

$$A = \begin{array}{|c|c|c|c|c|}\hline 4 & 1 & 1 & 2 & 5 \\ \hline 1 & 3 & 2 & 2 \\ \cline{1-4} 1 \\ \cline{1-1}\end{array} \quad \text{and} \quad B = \begin{array}{|c|c|c|c|c|c|c|}\hline 3 & 1 & 1 & 5 & 7 & 1 & 2 \\ \hline 2 & 2 \\ \cline{1-2}\end{array}$$

then ${}^{A}_{B}$ is the tableau

$$\begin{array}{|c|c|c|c|c|c|c|}\cline{1-5} 4 & 1 & 1 & 2 & 5 \\ \cline{1-5} 1 & 3 & 2 & 2 \\ \cline{1-4} 1 \\ \hline 3 & 1 & 1 & 5 & 7 & 1 & 2 \\ \hline 2 & 2 \\ \cline{1-2}\end{array}.$$

Clearly ${}^{A}_{B}$ is itself a Young tableau if the first row of B is not shorter than the last row of A.

Moreover, given two tableaux

$$H = \begin{pmatrix} i_{11} & \cdots & & i_{1k_1} \\ \vdots & & & \\ i_{s1} & \cdots & i_{sk_s} & \end{pmatrix} \quad \text{and} \quad H' = \begin{pmatrix} j_{11} & \cdots & & j_{1h_1} \\ \vdots & & & \\ j_{t1} & \cdots & j_{th_t} & \end{pmatrix},$$

we say that H is lexicographically less or equal than H', and we write $H \leqslant_{\text{lex}} H'$, if either the sequence $(k_1 \cdots k_s)$ is lexicographically bigger than the sequence $(h_1 \cdots h_t)$, or, whenever $(k_1,..., k_s) = (h_1,..., h_t)$, if the sequence

$(i_{11} \cdots i_{1k_1} \cdots i_{s1} \cdots i_{sk_s})$ lexicographically preceeds the sequence $(j_{11} \cdots j_{1h_1} \cdots j_{t1} \cdots j_{th_t})$.

Let us now introduce the following notation.

The symbol

$$[i_1 \cdots i_s | \hat{\jmath}_1 \cdots \hat{\jmath}_{n_i - s}]_i \ (+)$$

will denote the determinant of the minor of the matrix $X_i = (X^{(i)}_{s_i t_i})$ whose rows are those of indices $i_1 \cdots i_s$ and whose columns are those whose set of indices is the complement $\{h_1 < \cdots < h_s\}$, taken in order, in $\{1 \cdots n_i\}$ of the set of indices $\hat{\jmath} = \{\hat{\jmath}_1 \cdots \hat{\jmath}_{n_i - s}\}$ times $(-1)^t$, t being the sign of the permutation $(\hat{\jmath}_1 \cdots \hat{\jmath}_{n_i - s} h_1 \cdots h_s)$.

In order to use the symbol $(+)$ in an easy way, we are going to write it with the following compact expression:

$$[I | \hat{J}]_i,$$

where $I = (i_1 \cdots i_s)$ and $\hat{J} = (\hat{\jmath}_1 \cdots \hat{\jmath}_{n_i - s})$.

Moreover, if $M = (m_1, ..., m_r)$ and $\Lambda = (\lambda_1, ..., \lambda_q)$, then we shall denote the determinant of a minor whose rows (resp. columns) are those of indices $(m_1, ..., m_r, \lambda_1, ..., \lambda_q)$ in the given order by $[M \cup \Lambda | \hat{J}]$ (resp. $[I | \mathscr{C}(M \cup \Lambda)]$).

With this notation we are now in the position to express some elements of the ideal $\mathscr{E}$.

Proposition 1.1. *Let us fix i and let H and M be subsets of $\{1 \cdots n_i\}$, S a subset of $\{1 \cdots n_{i-1}\}$, $\hat{Q}$ a subset of $\{1 \cdots n_{i+1}\}$.*

Suppose that the cardinality of S, $|S|$, is such that

$$|S| = n_{i+1} + |M| - |\hat{Q}| - |H|;$$

then

$$\sum_{\Gamma} [S | \mathscr{C}(M \cup \Gamma)]_i \, [\Gamma \cup H | \hat{Q}]_{i+1} \in \mathscr{E} \qquad (**)$$

where Γ runs over all subsets of $\{1, ..., n_i\}$ such that

$$|\Gamma| \cup |H| = n_i - |\hat{Q}|.$$

Remark. If $\Gamma \cap H$ (or $M \cap \Gamma$) is not empty, then $[\Gamma \cup H | Q]_{i-1}$ (respectively: $[S | \mathscr{C}(M \cup \Gamma)]_i$) are to be considered identically 0.

Proof of Proposition 1.1. First suppose $|\Gamma| = 1$ and $|S| = 1$ and $H = \varnothing$. Then $(**)$ is just one of the generators of $\mathscr{E}$. Now suppose $M = H = \varnothing$ and let $|\Gamma| = h$. In this case the claim follows since for each i the $h \times h$ minors of X_{1_i} are the entries of the matrix $\Lambda^h X_i$.

Now, since Λ^h is an algebraic functor, then the fact that the entries $X_i X_{i+1}$ lie in $\mathscr{E}$ implies also that the entries of the matrix

$$(\Lambda^h X_i)(\Lambda^h X_{i+1}) = \Lambda^h(X_i X_{i+1})$$

lie in $\mathscr{E}$.

Now suppose that M, $H \neq \varnothing$.

In such case, by using Laplace expansion, we get

$$\sum_{\Gamma} [S|\mathscr{C}(M\cup\Gamma)]_i [\Gamma\cup H|Q]_{i+1}$$
$$= \sum_{\Gamma}\sum_{S'} \pm [S-S'|\mathscr{C}M]_i [S'|\mathscr{C}\Gamma]_i \sum_{Q'} \pm [\Gamma|\hat{Q}']_{i+1} [H|\hat{Q}'']_{i+1},$$

where S' is any subset of S such that $|S'| = |\Gamma|$ and Q', Q'' are subsets in $\{1,..., n_{i+1}\}$, with $Q' \cap Q'' = Q$.

If we now exchange the order of summation, we get

$$\sum_{S'}\sum_{Q'} \pm [S-S'|\mathscr{C}M]_i [H|Q'']_{i+1} \sum_{\Gamma} [S'|\mathscr{C}\Gamma]_i [\Gamma|Q']_{i+1} \in \mathscr{E}$$

by what we have already proved. Q.E.D.

Corollary 1.2. *We keep the notation of Proposition* 1.1. *Let* Λ *be a subset of* $\{1,..., n_i\}$. *Then*

(a) $$\sum_{\substack{\Gamma \\ \Gamma\cap\Lambda=\varnothing}} [S|\mathscr{C}(M\cup\Gamma)]_i [\Gamma\cup H|\hat{Q}]_{i+1} \in \mathscr{E} \qquad \textit{if} \quad |\Lambda| < |\Gamma|,$$

(b) $$[S|\mathscr{C}(M\cup\Lambda)]_i [\Lambda\cup H|\hat{Q}]_{i+1} + (-1)^{|\Lambda|+1} \times \sum_{\substack{\Gamma \\ \Gamma\cap\Lambda=\varnothing}} [S|\mathscr{C}(M\cup\Gamma)]_i [\Gamma\cup H|\hat{Q}]_{i+1} \in \mathscr{E} \qquad \textit{if} \quad |\Lambda| = |\Gamma|.$$

Proof. We shall perform the proof of (a) and (b) together by induction on $|\Lambda|$. If $\Lambda = \varnothing$, then (a) is just one of the relations obtained in Proposition 1.1 and (b) is

$$[S|\mathscr{C}M]_i [H|\hat{Q}]_{i+1} - [S|\mathscr{C}M]_i [H|\hat{Q}]_{i+1} = 0 \in \mathscr{E}.$$

So we can assume the corollary proved for $|\Lambda| \leqslant t-1$.

Let now $|\Lambda| = t$. We have:

$$\mathscr{E} \ni \sum_{\Gamma} [S|\mathscr{C}(M\cup\Gamma)]_i [\Gamma\cup H|\hat{Q}]_{i+1}$$
$$= \sum_{\Delta\subset\Lambda} \sum_{\substack{\Gamma \\ \Gamma\cap\Lambda=\Delta}} [S|\mathscr{C}(M\cup\Gamma)]_i [\Gamma\cup H|\hat{Q}]_{i+1}.$$

Now if $\Delta \neq \varnothing$, using our inductive hypothesis we can conclude that in case (a), i.e., $|\Lambda| < |\Gamma|$ so that $|\Lambda - \Delta| < |\Gamma - \Delta|$ if $\Gamma \supset \Delta$:

$$\sum_{\substack{\Gamma \\ \Gamma \cap \Lambda = \Delta}} [S|\mathscr{C}(M \cup \Gamma)]_i\, [\Gamma \cup H|\hat{Q}]_{i+1} \in \mathscr{E}.$$

This implies

$$\sum_{\substack{\Gamma \\ \Gamma \cap \Lambda = \varnothing}} [S|\mathscr{C}(M \cup \Gamma)]_i\, [\Gamma \cup H|\hat{Q}]_{i+1} \in \mathscr{E}.$$

In case (b), i.e., $|\Lambda| = |\Gamma|$ so that $|\Lambda - \Delta| = |\Gamma - \Delta|$ if $\Gamma \supset \Delta$, our inductive hypothesis gives, for $\Delta \neq \varnothing$

$$[S|\mathscr{C}(M \cup \Lambda)]_i [\Lambda \cup H|\hat{Q}]_{i+1} + (-1)^{|\Lambda - \Delta| + 1} \sum_{\substack{\Gamma \\ \Gamma \cap \Lambda = \Delta}} [S|\mathscr{C}(M \cup \Gamma)]_i \cdot [\Gamma \cup H|\hat{Q}]_{i+1} \in \mathscr{E}.$$

Since

$$\sum_{\Delta \neq \varnothing} (-1)^{|\Lambda - \Delta| + 1} = (-1)^{|\Delta| + 1}$$

we get

$$[S|\mathscr{C}(M \cup \Lambda)]_i [\Lambda \cup H|\hat{Q}]_{i+1} + (-1)^{|\Delta| + 1} \sum_{\Gamma \cap \Lambda = \varnothing} [S|\mathscr{C}(M \cup \Gamma)]_i [\Gamma \cup H|\hat{Q}]_{i+1} \in \mathscr{E}.$$

PROPOSITION 1.3. *Let* $[S|\hat{J}]_i = [s_1 \cdots s_h|\hat{j}_1 \cdots \hat{j}_{n_i - h}]_i$, $[T|Q]_{i+1} = [t_1 \cdots t_k|\hat{q}_1 \cdots \hat{q}_{n_{i+1} - k}]_{i+1}$ *be as above. Then*

(i) *if* $n_i < k + h$

$$[S|\hat{J}]_i\, [T|\hat{Q}]_{i+1} \in \mathscr{E},$$

(ii) *let* $n_i \geqslant k + h$, *let* $\hat{j}_1 < \cdots < \hat{j}_{n_i - h}$; $t_1 < \cdots < t_k$.

Let $1 \leqslant r \leqslant k$ *then if* $\hat{j}_1 \leqslant t_1, \ldots, \hat{j}_{r-1} \leqslant t_{r-1}$, $\hat{j}_r > t_r$ *we have*

$$\sum_{\sigma \in S_{n_i - h + 1}/S_r \times S_{n_i - h + 1 - r}} \pm [s_1 \cdots s_h|\hat{j}_1 \cdots \hat{j}_{r-1} \sigma(\hat{j}_r) \cdots \sigma(\hat{j}_{n_i - h})] \cdot [\sigma(t_1) \cdots \sigma(t_r), t_{r+1} \cdots t_k|\hat{q}_1 \cdots \hat{q}_{n_{i+1} - k}] \in \mathscr{E}, \tag{1.3}$$

where $S_{n_i - h + 1}, S_r, S_{n_i - h + 1 - r}$ *are the symmetric groups on* $n_i - h + 1, r, n_i - k + 1 - r$ *letters, respectively, and* $S_{n_i - h + 1}$ *acts on the set of indices* $\{\hat{j}_r, \ldots, \hat{j}_{n_i - h}, T_1, \ldots, T_r)$ *with the usual action.*

(Notice that the antisymmetry of the determinants implies that each term in (1.3) is well defined up to sign.)

Proof. (i) Let us write $\hat{J} = \mathscr{C}(M \cup \Lambda)$; $T = H \cup \Lambda$ with $\Lambda = \mathscr{C}\hat{J} \cap T$.

Since $h + k > n_i$ and $M \cap H = \varnothing$ we have $|M \cup H \cup \Lambda| = h + k - |\Lambda| > n_i - |\Lambda|$. So if $|\Gamma| = |\Lambda|, \Gamma \cap (M \cup H \cup \Lambda) \neq \varnothing$ and if $\Gamma \subset \mathscr{C}\Lambda$ either $\Gamma \cap M \neq \varnothing$ or $\Gamma \cap H \neq \varnothing$. This implies that relation (b) in Corollary 1.2 becomes

$$[S|\hat{J}]_i\,[T|\hat{Q}]_{i+1} = [S|\mathscr{C}(M \cap \Lambda)]_i\,[\Lambda \cup H'|Q]_{i+1} \in \mathscr{E}.$$

(ii) We have that

$$|\{t_1 \cdots t_r\} \cap \mathscr{C}\hat{J}| = |\{\hat{j}_1 \cdots \hat{j}_{r-1}\} \cap \mathscr{C}T| + 1.$$

In fact we have that $t_r \in \mathscr{C}\hat{J}$ since

$$\hat{j}_1 < \cdots < \hat{j}_{r-1} \leqslant t_{r-1} < t_r < \hat{j}_r < \cdots < \hat{j}_{n_i-h}.$$

Moreover it is clear from our hypothesis that $\{t_1,\dots,t_{r-1}\} \cap \mathscr{C}\hat{J} = \{t_1 \cdots t_{r-1}\} \cap \mathscr{C}\{\hat{j}_1,\dots,\hat{j}_{r-1}\}$ and $\{\hat{j}_1 \cdots \hat{j}_{r-1}\} \cap \mathscr{C}T = \{\hat{j}_1 \cdots \hat{j}_{r-1}\} \cap \mathscr{C}\{t_1 \cdots t_{r-1}\}$.

But then

$$\begin{aligned}|\{\hat{j}_1 \cdots \hat{j}_{r-1}\} \cap \mathscr{C}\{t_1 \cdots t_{r-1}\}| &= r - 1 - |\{\hat{j}_1 \cdots \hat{j}_{r-1}\} \cap \{t_1 \cdots t_{r-1}\}| \\ &= |\{t_1 \cdots t_{r-1}\} \cap \mathscr{C}\{\hat{j}_1 \cdots \hat{j}_{r-1}\}|\end{aligned}$$

and our claim follows.

From this it is a straightforward verification to see that our relation (ii) is just a rewriting of relation (a) in Corollary 1.2, with $\Lambda = \{\hat{j}_1,\dots,\hat{j}_{r-1}\} \cap \mathscr{C}T$, $H = T - (\{t_1,\dots,t_r\} \cap \mathscr{C}\hat{J})$, $M = \mathscr{C}\hat{J} - (\{t_1,\dots,t_r\} \cap \mathscr{C}\hat{J})$. Q.E.D.

Now, we shall write any element in $A/\mathscr{E}$ as a linear combination of "standard multitableaux," where by such an object it is meant the following.

DEFINITION. *Let $1 \leqslant i \leqslant m$. Consider the tableau*

$$T^{(i)} = (H_{(i)}|K_{(i)}) = \left(\begin{array}{cccc|cccc} i_{11} & \cdots & & i_{1h_1} & \hat{j}_{11} & \cdots & \hat{j}_{1n_i-h_1} & \\ i_{21} & \cdots & & i_{2h_2} & \hat{j}_{21} & \cdots & & \hat{j}_{2n_i-h_2} \\ \vdots & & & & \vdots & & & \\ i_{s1} & \cdots & i_{sh_s} & & \hat{j}_{s1} & \cdots & & \hat{j}_{sn_i-h_s} \end{array}\right),$$

where $1 \leqslant i_{lt} \leqslant n_{i-1}$, $1 \leqslant \hat{j}_{rt} \leqslant n_i$.

We can associate to $(H_{(i)}|K_{(i)})$ a polynomial in A which we shall write as follows

$$[i_{11} \cdots i_{1h_1}|\hat{j}_{11} \cdots \hat{j}_{1n_i-h_1}]_i \cdots [i_{s1} \cdots i_{sh_s}|\hat{j}_{s1} \cdots \hat{j}_{sn_i-h_s}]_i. \qquad (^\circ)$$

$T^{(i)}$ will be called "standard", if both the tableau $H_{(i)}$ and the tableau $\hat{K}_{(i)}$, where

$$\hat{K}_{(i)} = \begin{pmatrix} \hat{j}_{s1} & \cdots & \hat{j}_{sn_i-h_s} \\ \vdots & & \\ \hat{j}_{11} & \cdots & \hat{j}_{1n_i-h_1} \end{pmatrix}$$

are standard.

By abuse of notation, we shall denote also the polynomial $(^\circ)$ by $T^{(i)}$ and call it a "double tableau."

Moreover, whenever it is convenient, we shall write $(H|K)_{(i)}$ instead of $(H_{(i)}|K_{(i)})$.

Now suppose $T^{(1)},..., T^{(m)}$ are double tableaux. Their product $T^{(1)} \cdot \cdots \cdot T^{(m)}$ will be called a "multitableau." Such a multitableau is said to be "standard" if for each i

(a) $T^{(i)}$ is standard,

(b) either $T^{(i)}$ or $T^{(i+1)}$ is reduced to 1 or, otherwise, if $T^{(i)} = (H|K)_{(i)}$, and $T^{(i+1)} = (\bar{H}|\bar{K})_{(i+1)}$, then the tableau $\frac{\hat{K}}{\bar{H}}$ is standard.

Let T be a standard multitableau, $T = T^{(1)} \cdot \cdots \cdot T^{(m)}$. We can associate to T an $n+1$-ple of tableaux

$$\left(H_1 \,\middle|\, \begin{matrix} \hat{K}_1 \\ H_2 \end{matrix} \,\middle|\, \begin{matrix} \hat{K}_2 \\ H_3 \end{matrix} \;\cdots \,\middle|\, \hat{K}_m \right).$$

Notice that the entries of H_1 are integers from 1 to n_0, those of

$$\begin{matrix} \hat{K}_i \\ H_{i+1} \end{matrix}$$

integers from 1 to n_i,..., those of $\hat{K}_m$ integers from 1 to n_m.

Furthermore, by the "multishape" Σ of the above objects, we mean the $m+1$-ple of shapes associated to the given multitableau.

We are now going to state the following.

PROPOSITION 1.4. *Each element in $A/\mathscr{E}$ can be written as a linear combination of standard multitableaux.*

Proof of Proposition 1.4. It is clear that in order to prove our proposition, we only need to show that each multitableau $T^{(1)} \cdot \cdots \cdot T^{(m)}$ can be written modulo $\mathscr{E}$ as a linear combination of standard multitableaux.

In fact, each monomial in the $X^{(i)}_{s_i t_i}$ is itself a multitableau.

In order to go through the proof, we introduce a total ordering among multitableaux.

Let $T^{(i)} = (H_{(i)} | K_{(i)})$; $\bar{T}^{(i)} = (\bar{H}_{(i)} | \bar{K}_{(i)})$. We set

$$T^{(i)} \trianglelefteq \bar{T}^{(i)}$$

if either $H_{(i)} \leqslant_{\text{lex}} \bar{H}_{(i)}$, or, in case that $H_{(i)} = \bar{H}_{(i)}$, $\hat{K}_{(i)} \leqslant_{\text{lex}} \hat{\bar{K}}_{(i)}$.

Therefore, given two multitableaux,

$$T = T^{(1)} \cdot \cdots \cdot T^{(m)},$$

$$\bar{T} = \bar{T}^{(1)} \cdot \cdots \cdot \bar{T}^{(m)}$$

we say that T is lexicographically less than $\bar{T}$, if the sequence

$$(T^{(1)}, ..., T^{(m)})$$

is lexicographically smaller than the sequence $(\bar{T}^{(1)}, ..., \bar{T}^{(m)})$, in the ordering just introduced.

Let now T be a multitableau and suppose T is not standard.

We are going to show that we can write T as a linear combination of smaller tableaux in the above ordering. Let $T = T^{(1)} \cdot \cdots \cdot T^{(m)}$.

If one of the $T^{(i)}$ is not standard, then it follows from the standard basis theorem in the polynomial ring [7] that we can write $T^{(i)} = \sum_h a_h T^{(i)}_h$, where $T^{(i)}_h$ is standard and $T^{(i)} \trianglelefteq T^{(i)}$ for any h, so $T = \sum a_h T_h$, where $T_h = T^{(1)} \cdot \cdots \cdot T^{(i)} T^{(i+1)}_h \cdots T^{(m)}$ and $T_h \leqslant T$.

Hence using this we can suppose $T = T^{(1)} \cdot \cdots \cdot T^{(m)}$, where each of the $T^{(i)}$ is itself standard or reduced to 1.

Now suppose that $T^{(i)} T^{(i+1)}$ is not standard.

Let $T^{(i)} = (H_{(i)} | K_{(i)})$, $T^{(i+1)} = (\bar{H}_{(i+1)} | \bar{K}_{(i+1)})$.

If $\frac{\hat{K}_{(i)}}{\bar{H}_{(i+1)}}$ is not a Young tableau, i.e., the last row of $\hat{K}_{(i)}$ is shorter than the first row of $\bar{H}_{(i+1)}$ then, by Proposition 1.3 (i), we immediately get that $T^{(i)} T^{(i+1)} \in \mathscr{E}$, hence also $T^{(1)} \cdot \cdots \cdot T^{(m)} \in \mathscr{E}$.

So we can assume that the last row of $\hat{K}_{(i)}$ is longer than the first row of $\bar{H}_{(i+1)}$.

Now, in order to complete the proof of Proposition 1.4, we can proceed as follows.

Let $[s_1 \cdots s_h | \hat{j}_1 \cdots \hat{j}_{n_i - h}]_i$ be the last row of $T^{(i)}$ and $[t_1 \cdots t_k | \hat{q}_1 \cdots \hat{q}_{n_{i+1} - k}]_{i+1}$ be the last row of $T^{(i+1)}$ and suppose $1 \leqslant r \leqslant k$ is the least index such that $\hat{j}_r > t_r$. Then by applying Proposition 1.3 (ii) we get

$$[s_1,\dots,s_h|\hat{j}_1,\dots,\hat{j}_{n_i-h}]_i\,[t_1,\dots,t_k|\hat{q}_1,\dots,\hat{q}_{n_{i+1}-k}]_{i+1}$$
$$= \sum_{[\sigma]\neq[id]} [s_1\cdots s_h|\hat{j}_1\cdots\hat{j}_{r-1}\sigma(j_r)\cdots\sigma(j_{n_i-h})]$$
$$\cdot\ [\sigma(T_1)\cdots\sigma(T_{r-1})\,T_r\cdots T_k|\hat{q}_1\cdots\hat{q}_{n_{i+1}-k+1}] \bmod \mathscr{E}$$

where $[\sigma]\in S_{n_i-h+1}/S_r\times S_{n_i-h+1-r}$. Since, $s_1<s_2<\cdots<s_r<\hat{j}_r<\cdots<\hat{j}_{n_{i+1}-h}$, this implies for the same reasons as in [3, 7] that up to reordering $[s_1\cdots s_h|\hat{j}_1\cdots\hat{j}_{r-1}\sigma(j_r)\cdots\sigma(j_{n_i-h})]\cdot[s_1\cdots s_h|\hat{j}_1\cdots\hat{j}_{n_i-h}]$ is standard, for each $[\sigma]\neq[id]$. In particular

$$[s_1,\dots,s_h|\hat{j}_1\cdots\hat{j}_{r-1}\sigma(j_r)\cdots\sigma(j_{n_i-h})] \underset{\text{lex}}{\lneq} [s_1\cdots s_h|\hat{j}_1\cdots\hat{j}_{n_i-h}].$$

Using this we can clearly write

$$T^{(i)}T^{(i+1)}=\sum a_i T_t^{(i)}T_t^{(i+1)} \text{ modulo } \mathscr{E},$$

where $T_t^{(i)}\trianglelefteq T^{(i)}$. This by induction in the lexicographic ordering gives our claim. Q.E.D.

Remark. In order to prove our proposition we could have also reasoned in the following way. Let S be the $n_i\times n_i$ matrix

$$S=\begin{pmatrix} 0 & & & 1\\ & & 1 & \\ & \cdot & & \\ & 1 & & \\ 1 & & & 0\end{pmatrix}.$$

Let us consider the $2n_i\times(n_{i+1}+n_{i-1})$ matrix

$$Y^{(i)}=\begin{pmatrix} 0 & S\,{}^tX^i\\ X^{i+1} & 0\end{pmatrix},$$

${}^tX^i$ being the transpose of X^i. Then it is easily seen that the rows of $Y^{(i)}$ span an isotropic subspace with respect to the antisymmetric bilinear form of matrix

$$J=\begin{pmatrix} 0 & S\\ -S & 0\end{pmatrix}.$$

Using this remark one can deduce Proposition 1.4 from the results in [6] (see also [4]).

The following proposition follows immediately from 1.4.

PROPOSITION 1.5. *Every element in $A/\mathscr{E}(k_1\cdots k_m)$ can be written as a* *linear combination of standard multitableaux* $T = T^{(1)} \cdots T^{(m)}$ *such that if* $T^{(i)} = [H|K]_{(i)}$, *the length of the longest row in* H *is less or equal than* k_i.

Let now as before $W(k_1 \cdots k_m)$ be the subvariety of W consisting of matrices $M_1,..., M_m$ such that rank $M_i \leqslant k_i$ and let $B(k_1 \cdots k_m)$ be its reduced coordinate ring. Then

PROPOSITION 1.6. *The standard multitableaux* $T = T^{(1)} \cdots T^{(m)}$ *such that if* $T^{(i)} = (H|K)_{(i)}$, *the length of the longest row in* H *is less or equal than* k_i, *are linearly independent in* $B(k_1 \cdots k_m)$.

Remark. Before proving Proposition 1.5, we note that from it we immediately get

THEOREM 1.7. $B(k_1 \cdots k_m) \cong A/\mathscr{E}(k_1 \cdots k_m)$.

Proof of Proposition 1.6. In order to get linear independence, it is enough to prove it over the rationals. So from now on let $\mathscr{R} = \mathbb{Q}$. Observe that on $W(k_1 \cdots k_m)$ we have a natural action of the group $G = Gl(V_0) \times \cdots \times Gl(V_m)$, defined as follows.

Given $(M_1 \cdots M_m) \in W(k_1 \cdots k_m)$, $(g_0 \cdots g_m) \in G$, we define

$$(g_0 \cdots g_m)(M_1 \cdots M_m) = (g_0 M_1 g_1^{-1}, g_1 M_2 g_2^{-1} \cdots g_{m-1} M_m g_m^{-1}). \quad (\oplus)$$

It is clear that $(\oplus)$ also belongs to $W(k_1 \cdots k_m)$. So the action of G on $W(k_1 \cdots k_m)$ induces an action of G on $B(k_1 \cdots k_m)$. We shall consider $B(k_1 \cdots k_m)$ as a representation of G.

Let us recall for a moment some well-known facts on the representation theory of $Gl(n)$.

Let $B \subset Gl(n)$ be the Borel subgroup of lower triangular matrices and let $w_1 \cdots w_n$ be the fundamental weights with respect to B.

Let $w = \sum_{i=1}^{n} c_i w_i$, $c_i \in \mathbb{Z}^+$.

Then it is well known that there is a unique irreducible polynomial representation $\mathscr{L}(\sigma)$ of $Gl(n)$ whose maximal weight is w and whose dimension is equal to the number of standard tableaux whose shape σ has c_n rows of length n, c_{n-1} rows of length $n-1$, c_{n-2} rows of length $n-2$... and whose entries are integers from 1 to n.

Furthermore, given any irreducible representation $\mathscr{W}$ of $Gl(n)$, $\mathscr{W} \cong \mathscr{L}(\sigma) \otimes L^h$, where $L^h = \det^h$, n is any integer and the maximal weight of $\mathscr{W}$ is $w + nw_n$. Since every irreducible representation for G is a tensor product of an irreducible representation for $Gl(V_0)$ times an irreducible representation for $Gl(V_1)$, times $\cdots$ we immediately get that the number of standard multitableaux of a given multishape $\Sigma = (\sigma_0,..., \sigma_m)$ is equal to the dimension of the representation $\mathscr{L}(\Sigma) = \mathscr{L}(\sigma_0) \otimes \cdots \otimes \mathscr{L}(\sigma_{m-1}) \otimes \mathscr{L}(\sigma_m)$, whose maximal weight is $w^0 + w^1 + \cdots + w^m$, w^i being the maximal weight of $\mathscr{L}(\sigma_i)$.

Notice that $B(k_1 \cdots k_m)$ is naturally graded and that each multitableau is an homogeneous element and that two multitableaux with the same multishape have the same degree: in fact each multitableau is a product of determinants of the matrices $X^{(i)}$ and each such determinant is clearly homogeneous, so that the degree of a multitableau is a sum of degrees of those various determinants.

Let $B(k_1 \cdots k_m)_s$ be the homogeneous component of degree s of $B(k_1 \cdots k_m)$. Then the dimension of $B(k_1 \cdots k_m)_s$ is $\leqslant D$, where D is the number of standard multitableaux $T = T^{(1)} \cdots T^{(m)}$, $T^{(i)} = (H \mid K)_{(i)}$, with the longest row in H of length less or equal than k_i. We shall call such multitableaux "admissible" for $k_1 \cdots k_m$ and their multishape "admissible" multishape for $k_1 \cdots k_m$.

By the above remark, D is the sum of the dimensions of the representations of G whose multishape Σ is admissible for $k_1 \cdots k_m$.

Let us now fix such a multishape $\Sigma = (\sigma_0, \ldots, \sigma_m)$, with $\sigma_{(i)} = (\sigma_1^{(i)} \geqslant \cdots \geqslant \sigma_{h(i)}^{(i)})$.

Let $T_\Sigma = T_{\Sigma_1} \cdots T_{\Sigma_m}$, where

$$T_\Sigma = \left(\begin{array}{ccc|ccc|c|ccc} 1\ 2 & \cdots & \sigma_1^{(0)} & 1\ 2 & \cdots & \sigma_1^{(1)} & \cdots & 1\ 2 & \cdots & \sigma_1^{(m)} \\ \vdots & & & \vdots & & & \cdots & \vdots & & \\ 1\ 2 & \cdots & \sigma_{h(0)}^{(0)} & 1\ 2 & \cdots & \sigma_{h(1)}^{(1)} & & 1\ 2 & \cdots & \sigma_{h(m)}^{(m)} \end{array}\right).$$

Let $\mathscr{U} \subset G$ be the unipotent subgroup which is the product

$$\mathscr{U} = \mathscr{U}_0 \times \cdots \times \mathscr{U}_m,$$

where $\mathscr{U}_i$ is the unipotent subgroup of lower triangular matrices in $Gl(V_1)$. Then each T_i is invariant under $\mathscr{U}$ [3].

It follows that T_Σ is invariant under $\mathscr{U}$. Moreover let τ be the maximal torus in G which is the product of the maximal tori τ_i in $Gl(V_i)$ of diagonal matrices. Then it is straightforward to see [3] that T_Σ is a weight vector for τ whose weight up to an invertible character for G is equal to the maximal weight of $\mathscr{L}(\sum)$. So, if we show that there exists a point $P \in W(k_1 \cdots k_m)$ such that $T(P) \neq 0$, then $B(k_1 \cdots k_m)$ will contain a copy of an irreducible representation whose dimension is equal to that of $\mathscr{L}(\Sigma)$ and since two multishapes have nonisomorphic corresponding irreducible representations, this will imply that $\dim B(k_1 \cdots k_m)_s \geqslant D$.

Since we have already proved that $\dim B(k_1 \cdots k_m)_s \leqslant D$, this will imply our claim.

But now let $P = (M_1 \cdots M_m)$ where $M_i = \left(\begin{smallmatrix} 0 & I \\ 0 & 0 \end{smallmatrix}\right)$, I being the $k_i - 1 \times k_i - 1$ identity matrix.

Then clearly $P \in W(k_1 \cdots k_m)$ and by computation of determinants, we have that $T_\Sigma(P) = 1$ for each multishape Σ. Q.E.D.

Remark. Notice that the above proof implies in particular that we can explicitly determine the decomposition of $B(k_1 \cdots k_m)$ as a G-module. In particular $B(k_1 \cdots k_m)$ has multiplicity one decomposition, i.e., each irreducible G-module appearing in the decomposition of $B(k_1 \cdots k_m)$ occurs with multiplicity one.

Moreover the multitableaux are a basis for $B(k_1 \cdots k_m)$ of weight vectors with respect to $\tau \subset G$.

2. Some Properties of the Varieties of Complexes

In this section we shall prove that when $k_i + k_{i-1} \leqslant n_{i-1}$ for each $i = 1,\dots, m$, then the ideal $\mathbb{E}(k_1 \cdots k_m)$ is perfect and the ring $A/\mathscr{E}(k_1 \cdots k_m)$ is normal. Of course in characteristic 0 these results are an immediate consequence of Kempf's theorem [11, 12] and our results in Section 1.

But here we shall give an alternative proof which works in any characteristic.

Let us first recall some general facts from [2].

Suppose X is a finite simplicial complex together with an equivalence relation $\sim$ on the set of vertices $X^{(0)}$, such that if Y is an equivalence class under $\sim$ and if X_Y denotes the simplicial subcomplex of X whose simplices are spanned by the vertices in Y, then on Y there is a structure of partially ordered set such that X_Y is the simplicial complex of chains of this partially ordered set.

Let us now introduce a total ordering on the ordered sequences, possibly with repetitions of vertices. First let $Y_1,\dots, Y_m$ be the equivalence classes of vertices under $\sim$. Then given two sequences of vertices $(x_1,\dots, x_n)$, $(x'_1,\dots, x'_{n'})$, we say that $(x_1,\dots, x_n) \leqslant_{\text{lex}} (x'_1,\dots, x'_{n'})$ iff either $(x_1 \cdots x_n)$ is contained as an initial subsequence in $(x'_1,\dots, x'_{n'})$ or, if i is the smallest index such that $x_i \neq x'_1$, then either $x_i \in Y_s$, $x'_i \in Y_{s'}$, and $s < s'$ or, if $s = s'$, $x_i \leqslant x'_i$.

Given X as above and given a ring $\mathscr{R}$, we define an $\mathscr{R}$-algebra S to be an algebra with straightening law on X^*, if the following conditions are satisfied:

(1) S is multigraded by $\mathbb{N}^m$, $\mathbb{N}$ being the natural numbers;

(2) we are given a set of distinct algebra generators for S, $\{r_x\}$, for x belonging to the set of vertices of X such that if $x \in Y_i$, degree $r_x = (0 \cdots 0, s, 0 \cdots 0)$, s being at the ith place;

Note added in proof. In the final version of [2] a more general definition of an algebra with straightening law is given which includes the one given here.

(3) given any monomial $\mathscr{M} = r_{x_1}^{h_1} r_{x_2}^{h_2} \cdots r_{x_s}^{h_s}$ with $x_1 \neq x_2 \neq \cdots \neq x_s$, we say that it is standard, if $(x_1 \cdots x_s)$ is a simplex in X. The standard monomials and 1 form a given basis for S over $\mathscr{R}$;

(4) given any monomial $\mathscr{M} = r_{x_1}^{h_1} r_{x_2}^{h_2} \cdots r_{x_m}^{h_m}$, we associate to it the sequence of vertices

$$(\underbrace{x_1, \dots, x_1}_{h_1\text{-times}}, \underbrace{x_2, \dots, x_2}_{h_2\text{-times}}, \dots),$$

then $\mathscr{M} = \Sigma a_i \mathscr{M}_i$, $\mathscr{M}_i$ standard of the same multidegree and the sequence of vertices associated with $\mathscr{M}_i$ is smaller that the sequence associated with $\mathscr{M}$.

Given such an algebra, in [2] it is proved the following:

THEOREM 2.1. *Given a simplicial complex X as above and an algebra with straightening law S on X, there exists a flat deformation whose generic fiber is S and whose special fiber is the algebra $\mathscr{R}\{X\} =_{\text{def}} \mathscr{R}[x]_{x \in X^{(0)}/I}$, where I is generated by the monomials $\mathscr{M}$ which are not standard.*

In particular, if $\mathscr{R}$ is Cohen–Macaulay, it follows from the above that S is Cohen–Macaulay if $\mathscr{R}\{X\}$ is. Furthermore Reisner [14] has proved that $\mathscr{R}\{X\}$ is Cohen–Macaulay iff the following topological condition is satisfied: $\tilde{H}_i(X, \mathscr{R}) = 0$, $i < \dim X$ and $\tilde{H}_i(L, \mathscr{R}) = 0$, $i < \dim L$, for all links L of simplices of X.

In particular if X is a cell, it will satisfy the hypothesis of the theorem of Reisner.

Let X be the symplicial complex whose set of sth-dimensional simplices are the following: $x^{(s)} = \{[h_0|k_0]_{i_0} \cdots [h_s|k_s]_{i_s}$, distinct minors such that $[h_0|k_0]_{i_0} \cdots [h_s|k_s]_{i_s}$ is a standard multitableau$\}$.

Let $X(k_1 \cdots k_m) \subset X$ be the span of the vertices of X corresponding to minors $[h, k]_i$ whose size is $\leqslant k_i$.

Notice that if we fix i, there is a well-defined order relation on the vertices $\{[h, k]_i\}$ defined by $[h, k]_i \leqslant [h', k']_i$ iff the product $[h, k]_i [h', k']_i$ is standard.

So, if we denote by Y_i the set of vertices $\{[h, k]_i\}$, then the span of Y_i in X is the simplicial complex of chains of the poset $(Y_i, \leqslant)$.

Thus X is one of the symplicial complexes considered above, and so is $X(k_1 \cdots k_m)$.

The following result is just a reformulation of our results of Section 1.

THEOREM 2.2. *The coordinate ring of $W(k_1 \cdots k_m)$ is an algebra with straightening law over $X(k_1 \cdots k_m)$.*

It follows that in order to prove that $W(k_1 \cdots k_m)$ is Cohen–Macaulay, we have to study the topology of the simplicial complex $X(k_1 \cdots k_m)$.

We shall need the following well-known result, whose proof we include for completeness.

LEMMA 2.3. *Dim* $W(k_1 \cdots k_m) = \sum_{i=0}^{m} (n_i - k_{i+1})(k_i + k_{i+1})$, *with* $k_{i-1}k_{i+1} \leqslant n_i$, *and* $k_{m+1} = k_0 = 0$.

Proof. Since the variety $W(k_1 \cdots k_m)$ under our hypothesis is irreducible and is the closure of the G-orbit

$$\mathscr{U}^{n_0 \cdots n_m}_{k_1 \cdots k_m} \cong \{(M_1 \cdots M_m) | \operatorname{rank} M_i = k_i\},$$

it is clearly sufficient to prove that dim $\mathscr{U}$ is given by our formula. We shall proceed by induction on m. For $m = 1$, the formula is just the formula for the dimension of a determinantal variety. We claim that

$$\mathscr{U}^{n_0 \cdots n_m}_{k_1 \cdots k_m} \cong \mathscr{U}^{n_{m-1}, n_m}_{k_m} \times \mathscr{U}^{n_0, n_1, \ldots, n_{m-1} - k_m}_{k_1 \cdots k_{m-1}}$$

as algebraic varieties. In fact notice that since $M_{m-1} \cdot M_m = 0$, there exists a unique M'_{m-1} such that the following diagram

$$\begin{tikzcd} V_{m-1} \arrow[rr, "M_{m-1}"] \arrow[dr] & & V_{m-2} \\ & V_{m-1}/\operatorname{Im} M_m \arrow[ur, "M'_{m-1}"'] & \end{tikzcd}$$

commutes and $rk\ M'_{m-1} = k_{m-1}$.

Now we define $\varphi\colon \mathscr{U}^{n_0 \cdots n_m}_{k_1 \cdots k_m} \to \mathscr{U}^{n_{m-1}, n_m}_{k_m} \times \mathscr{U}^{n_0, n_1 \ldots n_{m-1} - k_m}_{k_1 \cdots k_{m-1}}$ by setting $\varphi(M_1 \cdots M_m) = (M_m, (M_1, M_2, \ldots, M'_{m-1}))$.

We leave to the reader the immediate verification that this is an isomorphism.

Thus $\dim \mathscr{U}^{n_0 \cdots n_m}_{k_1 \cdots k_m} = \dim \mathscr{U}^{n_{m-1}, n_m}_{k_m} + \dim \mathscr{U}^{n_0, n_1 \cdots n_{m-1} - k_m}_{k_1 \cdots k_{m-1}} = \sum_{i=0}^{m-2} (n_i - k_{i+1})(k_i + k_{i+1}) + (n_{m-1} - k_m)\, k_{m-1} + (n_{m-1} - k_m)\, k_m + n_m k_m = \sum_{i=0}^{m} (n_i - k_i)(k_{i+1} + k_i)$ where $k_0 = k_{m+1} = 0$. Q.E.D.

THEOREM 2.4. *Let* $k_i + k_{i-1} \leqslant n_{i-1}$ *for each* $i = 1 \cdots m$. *Then the geometric realization of* $X(k_1 \cdots k_m)$ *is homeomorphic to a cell of dimension* $\Sigma(n_i - k_i)(k_{i+1} + k_i) - 1$, *with* $k_0 = k_{m+1} = 0$.

In order to go through the proof of this theorem, we need the following crucial.

LEMMA 2.5. *Fix* $h = (i_1 \cdots i_s)$, *with* $1 < i_1 < i_2 < \cdots < i_s \leqslant n_0$ *and* $s \leqslant k_1$ *and let* X^m_h *be the simplicial subcomplex of* X^m *which is the span of the vertices corresponding to the minors* $[\bar{h}, \bar{k}]_i$, *where, if* $i = 1$, $\bar{h} \geqslant h$, *and if* $i \neq m$, $\bar{h}$, $\bar{k}_i$ *is arbitrary. Then* X^m_h *is a cell.*

Proof. By induction on m.

If $m = 1$, X_h is the span of the vertices corresponding to the minors $[h', k']_1$ with $[h', k']_1 \geqslant [h|s+1,..., n_0]_1$. In such a case the result is well known [2].

Suppose the statement is true for $m - 1$.

Fix $k = j_1 \cdots j_{n_1-s}$ and let $X_{[h,k]_1}$ be the simplicial complex spanned by the vertices corresponding to the minors $[\bar{h}, \bar{k}]_i$, such that, if $i = 1$, $[h, k]_1 \leqslant [\bar{h}, \bar{k}]_1$; if $i > 1$, $[h, k]_1 [\bar{h}, \bar{k}]_1$ is standard.

If such is the situation, then $X_{[h,k]_1}$ is a cell. As a matter of fact one can easily see that $X_{[h,k]_1}$ is the join of the simplicial complexes $Y_{[h,k]_1}$, this being the span of the vertices $[\bar{h}, \bar{k}]_1$ such that

$$[\bar{h}, \bar{k}]_1 \geqslant [h, k]_1$$

and the simplicial complex $X_{k'}^{(m-1)}$, where $k' = j_1 \cdots j_{k_2}$ (notice that as $s \leqslant k_1$ and $k_1 + k_2 \leqslant n_1$, $n_1 - s \geqslant k_2$).

Now, as $Y_{[h,k]_1}$ and, by induction, $X_{k'}^{(m-1)}$ are cells, and as the join of two cells is a cell, it follows that $X_{[h,k]_1}$ is a cell.

Now take the set of all the possible sequences:

$$k = \{j_1 \cdots j_{k_2}, s + k_2 + 1 \cdots n_{m-1}\}$$

ordered lexicographically.

Fixed k, take $Z_{[h,k]_1} = \bigcup_{k' \leqslant k} X_{[h,k]_1}$.

Note that $X_h = Z_{[h,k]_1}$, where $k = (s+1 \cdots n_{m-1})$.

Claim 1. For any k, $Z_{[h,k]_1}$ is a cell.

Such claim is proved like this:

If $k = (1, 2 \cdots n_{m-1} - s)$, then $Z_{[h,k]_1} = X_{[h,k]_1}$ and the lemma has already been proved. We proceed by induction on the lexicographic ordering of the sequences k.

Assume the claim true for $Z_{[h,k']_1}$ and let k be the element immediately following k' in the lexicographic ordering.

We know then that $Z_{[h,k']_1}$ and $X_{[h,k]_1}$ are both cells.

Let us prove that $Z_{[h,k']_1} \cap X_{[h,k]_1}$ is a cell too, contained in the boundary of $Z_{[h,k']_1}$ and of $X_{[h,k]_1}$, with dimension one less of those of $Z_{[h,k']_1}$ and $X_{[h,k]_1}$.

In particular $Z_{[h,k']_1}$ and $X_{[h,k]_1}$ will have the same dimension and their union will be a cell.

As $Z_{[h,k']_1} = \bigcup X_{[h,k'']_1}$, with $k'' \leqslant_{\text{lex}} k'$, we study $X_{[h,k]_1} \cap X_{[h,k'']_1}$ with $k'' \leqslant_{\text{lex}} k'$.

Claim 2. Let $k'' \leqslant_{\text{lex}} k$. Then there exists a cover $k_0 \leqslant k$ of k (i.e., an element k_0 such that for each $\tilde{k}$ with $k_0 \leqslant \tilde{k} \leqslant k$, either $k_0 = \tilde{k}$ or $\tilde{k} = k$) such that

$$X_{[h,k'']_1} \cap X_{[h,k]_1} \subset X_{[h,k_0]_1} \cap X_{[h,k]_1}.$$

In order to obtain the statement of Claim 2, take the sequences s_1, s_2, these being the only sequences with the following property: if μ is a sequence such that $\mu \leqslant k$, $\mu \leqslant k''$, i.e., $\substack{\mu \\ k}$ and $\substack{\mu \\ k''}$ are standard Young tableaux, this implies that $\mu \leqslant s_1$ and $s_1 \leqslant k, k''$; if $\mu \geqslant k, \mu \geqslant k''$, then $\mu \geqslant s_2$ and $s_2 \geqslant k, k''$.

It follows from the definitions that $X_{[h,k'']_1} \cap X_{[h,k]_1}$ is the join of $Y_{[h,s]_1}$ and of $X^{m-1}_{\tilde{s}_2}$, where $\tilde{s}_2$ is the sequence formed by the first k_2 indices of s_2.

In particular, if $k'' \leqslant k$, then $X_{[h,k'']_1} \cap X_{[h,k]_1}$ is the join of $Y_{[h,k'']_1}$ and $X^{m-1}_{\tilde{k}}$.

So, if we take $s_1 \leqslant k_0 \leqslant k$, with k_0 being a cover of k, we obtain that $X_{[h,k_0]_1} \cap X_{[h,k]_1} \supset X_{[h,k'']_1} \cap X_{[h,k]_1}$, which gives Claim 2.

Now, going back to Claim 1, we are reduced to study the intersection $X_{[h,k'']_1} \cap X_{[h,k]_1}$ when $k'' \leqslant k$ and k'' covers k.

But such intersection, being the join of two cells, is itself a cell. Moreover, a simplex of maximal dimension in such cell can be completed to a simplex of maximal dimension in $X_{[h,k]_1}$ only by adding the vertex $[h,k]_1$, and to a simplex of maximal dimension in $X_{[h,k'']_1}$ just adding a suitable vertex $[\tilde{k}', s]_2$, where $\tilde{k}'$ are the first k_2 indices in k'.

So we have that

$$Z_{[h,k']_1} \cap X_{[h,k]_1} = \bigcup T_{k_{0,i}},$$

$k_{0,i}$ being covers of k, $T_{k_{0,i}}$ is the join of $Y_{[h,k_{0,i}]_1}$ and $X^{m-i}_{\tilde{k}}$.

As we know [2] that $\bigcup Y_{[h,k_{0,i}]_1}$ is a cell, $\bigcup T_{k_{0,i}}$ is a cell too.

Finally such cell is certainly contained in the boundary of $X_{[h,k]_1}$, because all its simplexes of maximal dimension are, as we have shown, and moreover it is contained in the boundary of $Z_{[h,k']_1}$ as it follows immediately from what we said in order to show that any simplex of maximal dimension of $X_{[h,k_0]_1} \cap X_{[h,k]_1}$ with k_0 a cover of k, is contained in a unique simplex of maximal dimension of $X_{[h,k_0]_1}$.

We can conclude that $Z_{[h,k]_1}$ is a cell. In particular X^m_h is a cell for any h and this proves Lemma 2.5. Q.E.D.

Having proved Lemma 2.5, the first statement of Theorem 2.4 follows immediately, as one has:

COROLLARY 2.6. *$X(k_1 \cdots k_m)$ is a cell.*

Proof. $X(k_1 \cdots k_m) = X^m_{[1,2,\ldots,k_m]}$. Q.E.D.

Now, to finish the proof of Theorem 2.4, we have to show that dim $X(k_1 \cdots k_m) = \Sigma(n_i - k_i)(k_{i+1} + k_i) - 1$.

But since by Theorems 2.1 and 2.3 we have a flat deformation whose generic fiber is the coordinate ring of $W(k_1 \cdots k_m)$, whose special fiber is $\mathscr{R}\{X(k_1 \cdots k_m)\}$ and since all the fibers in a flat deformation have the same dimension:

$$\begin{aligned}\dim X(k_1 \cdots k_m) &= \mathscr{R}\{X(k_1 \cdots k_m)\} - 1 = \dim W(k_1 \cdots k_m) - 1 \\ &= \Sigma(n_i - k_i)(k_{i+1} + k_i) - 1.\end{aligned}$$

As a corollary we get:

THEOREM 2.7. *If $\mathscr{R}$ is Cohen–Macaulay, then $B(k_1 \cdots k_m)$, coordinate ring of $W(k_1 \cdots k_m)$, is Cohen–Macaulay, equivalently, the ideal $\mathscr{E}(k_1 \cdots k_m)$ is perfect.*

We are left now with the proof of the normality of the variety $W(k_1 \cdots k_m)$ when $k_i + k_{i+1} \leqslant n_i$. From now on $\mathscr{R}$ will be a normal domain.
Let us first recall the following.

LEMMA 2.8 (Hironaka H. [13]). *Let $\mathscr{D}$ be an integral domain and $t \in \mathscr{D}$. If the localization $\mathscr{D}[1/t]$ is normal and $\mathscr{D}/\langle t\rangle$ is reduced, then $\mathscr{D}$ is normal.*

Now consider the element $t = [n_0 | 1\, 2 \cdots n_1 - 1] \in B(k_1 \cdots k_m)$, i.e., the function on $W(k_1 \cdots k_m)$ whose value up to sign on a point $(M_1 \cdots M_m) \in W(k_1 \cdots k_m)$ is the (n_0, n_1)-entry of M_1.

LEMMA 2.9. *$B(k_1 \cdots k_m)/\langle t\rangle$ is reduced.*

Proof. Notice that by our description in Propositions 1.5 and 1.6 of the basis for $B(k_1 \cdots k_m)$ formed by the standard multitableaux, it follows that for any standard multitableau T, the product $T \cdot t$ is also a standard multitableau. So we have that the ideal $\langle T\rangle$, has a basis formed by the standard multitableaux which it contains.

This implies that $B(k_1 \cdots k_m)/\langle t\rangle$ is itself an algebra with straightening law, on the simplicial complex which is the span in $X(k_1 \cdots k_m)$ of all vertices other than t. Hence $B(k_1 \cdots k_m)/\langle t\rangle$ is reduced [2]. Q.E.D.

In order not to create confusion, in the following lemma we shall write ${}_{\mathscr{R}}B(k_1 \cdots k_m, n_0 \cdots n_m)$ instead of $B(k_1 \cdots k_m)$ and ${}_{\mathscr{R}}W(k_1 \cdots k_m, n_0 \cdots n_m)$ instead of $W(k_1 \cdots k_m)$, this in order to point out the fact that we are considering complexes of maps $\varphi_i\colon V_i \to V_{i-1}$, with V_i free $\mathscr{R}$-modules, rank $V_i = n_i$.

LEMMA 2.10. *Let* $\tilde{B} = {}_{\mathscr{R}}B(k_1 \cdots k_m, n_0 \cdots n_m)[1/t]$, $k_i + k_{i+1} \leqslant n_i$. *Then*

$$\tilde{B} \cong {}_{\mathscr{R}'}\mathrm{B}(k_1 - 1, k_2 \cdots k_m, n_0 - 1, n_1 - 1, n_2 \cdots n_m), \tag{*}$$

where $\mathscr{R}' = \mathscr{R}[X_{n_0,1} \cdots X_{n_0,n_1-1}, X_{1,n_1} \cdots X_{n_0-1,n_1}, r, 1/t]$.

Proof. Let $(X_1 \cdots X_m)$, $X_h = (X_{ij}^{(h)})$, $i = 1 \cdots n_{h-1}$, $j = 1 \cdots n_h$, be the generic point of $\mathscr{U}(k_1 \cdots k_m, n_0 \cdots n_m) = W(k_1 \cdots k_m, n_0 \cdots n_m) - \{t = 0\}$.

We concentrate ourselves on the pair (X_1, X_2). There exists (see [3, 15]) $g_1 \in Gl(n_0, \mathscr{R}')$ and $g_2 \in Gl(n_1, \mathscr{R}')$ such that if $g = (g_1, g_2)$, then

$$(X_1, X_2)^g = (g_1 X_1 g_2^{-1}, g_2 X_2)$$

with the following property:

$$g_1 X_1 g_2^{-1} = \begin{pmatrix} X_1' & 0 \\ 0 & 1 \end{pmatrix},$$

where $X_1' = (X_{ij}')$ $i = 1 \cdots n_0 - 1$, $j = 1 \cdots n_1 - 1$ is an $(n_0 - 1) \times (n_1 - 1)$-matrix.

Furthermore, as $X_1 X_2 = 0$, we have that

$$(g_1 X_1 g_2^{-1})(g_2 X_2) = g_1(X_1 X_2) = 0.$$

Therefore the last row of $g_2 X_2$ is identically zero,

$$g_2 X_2 = \begin{pmatrix} X_2' \\ 0 \end{pmatrix}.$$

It is also clear that rank $X_1 = k_1 - 1$ and rank $X_2 = k_2$.

Thus we get an epimorphism

$$\psi\colon {}_{\mathscr{R}'}B(K_i - 1 \cdots k_m, n_0 - 1, n_1 - 1, n_2 \cdots n_m) \to \tilde{B}.$$

We leave to the reader the easy verification that ψ is an isomorphism.

Finally we can state the following:

THEOREM 2.11. *If* $\mathscr{R}$ *is normal, and* $k_i + k_{i+1} \leqslant n_i$, *then* $W(k_1 \cdots k_m)$ *is normal.*

Proof. We work by induction on m. If $m = 0$, here is nothing to prove.

So assume the theorem true for $m - 1$ and work by induction on n_0.

If $n_0 = 0$, then we are in the case $m - 1$. Suppose the statement true for $n_0 - 1$.

But then using Lemma 2.10 and the fact that $\mathscr{R}'$ is normal, we get that $\tilde{B}$ is normal. Since by Lemma 2.9, t generates a radical ideal, we have by Theorem 2.8 that $B(k_1 \cdots k_m)$ is integrally closed. Q.E.D.

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