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The Invariant Theory of $n \times n$ Matrices

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INTRODUCTION

This paper has arisen out of a set of problems that I will now describe. Most of these problems arose out of a beautiful paper by Artin [1], they are all related to the same root: Describe the invariant theory of n-tuples of matrices.

The first problem was a conjecture made by Artin on the nature of the invariants of m $n \times n$ matrices $X_1, \ldots, X_m$ under simultaneous conjugation in characteristic 0. He conjectured that any such invariant is a polynomial in the elements $\mathrm{Tr}(X_{i_1}, X_{i_2} \cdots X_{i_R})$. This fact was classical for $n = 2$ [5], and proved by Spencer and Rivlin [13–15] for orthogonal invariants of symmetric 3×3 matrices. The theory developed by them also contains a finiteness statement and some discussion of the relations among such invariants. They were also interested in various kinds of concomitats always for $n = 3$ and the orthogonal groups. A complete account of their theory can be found in [12]. In this paper, we first solve Artin's conjecture (Theorem 1.3). Next, we take the problem of deducing a finite set of generators. This is accomplished in Section 3, where we give a general finiteness theorem for graded algebras from which we deduce that one may restrict to elements of type, $\mathrm{Tr}(X_{i_1} \cdots X_{i_k})$, where $k \leqslant 2^n - 1$ (Theorem 3.4a).

In view of the results of Spencer and Rivlin, we consider the problem of finding the matrix valued concomitants. This is a noncommutative algebra, which we also describe, that is generated over the ring of invariants T by the "coordinates" X_i. Here too, we have a finiteness statement. The monomials in the X_i's of degree $\leqslant 2^n - 2$ span this algebra over T. Both estimates are sharp (possibly both estimates should give $2^n - 2$, cf., [12], for $n = 3$) at least in the sense that they are equivalent, 4.7, to the known estimates for the theorem of Nagata–Higman on nil algebras (cf. [6 p. 274]).[1]

[1] *Note added in proof.* In a recent paper, this estimate is sharpened: J. Rasmyslev, Trace identities of full matrix algebras over a field of characteristic zero, *Izv. Akad. Nauk USSR* (1974), No. 4.

The theorem just cited may be said to be the "first fundamental theorem" for invariants and matrix concomitants of m matrices (in characteristic 0). In the spitit of Weyl's book [16], we then take the problem of describing the relations among such invariants and concomitants. The result is quite striking in that it basically says that any relation among invariants and matrix concomitants is a consequence of the theorem of Hamilton–Cayley (Theorem 4.6).[2]

As a consequence, we tie the theory thus far obtained to the theory of polynomial identities of algebras. We have two results that seem quite interesting:

(1) If an algebra over a field of characteristic 0 satisfies the identity $X^n = 0$, then it satisfies all the identities of $n \times n$ matrices (Corollary 4.8).

(2) The space of multilinear identities of degree m of $n \times n$ matrices can be described completely in terms of Young diagrams (Theorem 6.1).

The technique of the proofs is quite simple. It is based on the remark that in theorems on invariants, we may analyze only the multilinear ones. If μ is a multilinear invariant depending on m matrices $X_1, \ldots, X_m$, we may think of μ as the linear invariant map

$$\mu: (K)_n \otimes (K)_n \otimes \cdots \otimes (K)_n \to K,$$

($(K)_n$ the ring of matrices). Now, if $V = K^n$, the basic vector space, identify $(K)_n$ with $V \otimes V^*$, and $\mu: V^{\otimes m} \otimes V^{*\otimes m} \to K$ is an invariant map. Next, identify $(V^{\otimes m} \otimes V^{*\otimes m})^*$ with $\mathrm{End}(V^{\otimes m})$: $\lambda \in \mathrm{End}(V^{\otimes m})$ induces the form $u \otimes \varphi \to \langle \varphi, \lambda(u) \rangle$. Thus, μ corresponds to an element $\bar{\mu}$ of $\mathrm{End}(V^{\otimes m})$ commuting with $Gl(V)$. At this point, one invokes the classical theory that implies $\bar{\mu} = \sum \alpha_\sigma \sigma$, σ in the symmetric group of m letters. Finally, we have to interpret a permutation σ as an invariant. The computation is easy (Theorem 1.2) if $\sigma = (i_1 \cdots i_k)(j_1 \cdots j_h) \cdots (t_1 \cdots t_z)$ is the decomposition in cycles, the invariant associated is:

$$\mathrm{Tr}(X_{i_1} X_{i_2} \cdots X_{i_k}) \, \mathrm{Tr}(X_{j_1} \cdots X_{j_h}) \cdots \mathrm{Tr}(X_{t_1} \cdots X_{t_z}).^3$$

[2] *Note added in proof.* This computation appears already in B. Kostant, a theorem of Frebenius, a theorem of Amitsur Levitski and cohomology theory, *J. Math. Mech.* **7** (1958), 237–264.

[3] *Note added in proof.* This result has been obtained independently by Rasmyslev in the paper cited in footnote 1.

With this dictionary now in hand, it is just a question of translating the usual theorems of invariants into this language of matrices.

Sections 7 through 10 of Part I are dedicated to invariants over the other classical groups. The first fundamental theorem for $O(n)$ and $Sp(n)$ is an easy generalisation of Artin's conjecture. In both cases, we get that the invariants are generated by the elements $\mathrm{Tr}(U_{i_1} \cdots U_{i_k})$, where $U_i = X_i$ or $X_i{}^t$ for $O(n)$, $U_i = X_i$ or $X_i{}^*$, the symplectic transpose, for $Sp(n)$. One has similar results for matrix concomitants with all the necessary finiteness theorems. The second fundamental theorem also can be proved, but it is somewhat more mysterious than for $Gl(n, K)$, in that strange new identities appear, whose natures are not fully clarified. In Section 11, we describe the unitary invariants with complete results. In Section 12, we study mixed invariants and concomitants for $Gl(n, K)$ (for simplicity).

This finishes what might be called the quantitative part of invariant theory, i.e., the explicit description of invariants and their relations.

In Part II, we develop the qualitative results, basically, the theory of the quotient varieties associated to the invariant problems considered in Part I.

In this part, we basically develop and adopt to the other classical groups the ideas and techniques given by Artin in [1]. The results are in all cases parallel to the theory for $Gl(n, K)$. The ring of orthogonal invariants of m $n \times n$ matrices is the coordinate ring of a variety whose points are the equivalence classes under $O(n)$ of orthogonal representations of the free *-algebra in m-variables (Theorem 15.3). The irreducible representations are simple points of the quotient variety (Theorem 20.2), and on this set, the quotient map is a principal fibration (Theorem 19.4). Some of these results are proved in a characteristic free approach. The possibility of a full generalization to characteristic $p \neq 0$ is still subject to unsolved obstacles, although many new developments in this direction have occurred. Hopefully, the state of affairs of invariant theory in char $p > 0$, in the next five years will change completely. At present Doubilet, Rota, and Stein have proved the first fundamental theorem for $Gl(n, K)$ in char $p > 0$ (and even over Z) [2]. The Mumford conjecture has been solved by Haboush (and by the author jointly with Formanek for $Gl(n, k)$). (This implies that, except for the explicit computation of the invariant rings, the qualitative theory of Part II is valid in every characteristic ($\neq 2$ for the moment).

We thus formulate a conjecture analogous to Artin's conjecture for char $p > 0$.

The invariants of m $n \times n$ matrices are generated by the elements $\sigma_i(p(X_1, \ldots, X_m))$, where p is a noncommutative polynomial in the X_i's, and σ_i is a coefficient of the characteristic polynomial. It can be proved that:

(1) The ring A generated by such elements is finitely generated [8].

(2) The variety associated to A classifies equivalence classes of semisimple representations of the free algebra $K\{X_1, \ldots, X_m\}$. [8].

(3) The ring of invariants B is integral over A and the map $\operatorname{Spec} B \to \operatorname{Spec} A$ is a homeomorphism. (This follows from Mumford's conjecture).

We may add that, especially in Part II, some more or less known theorems have been developed anew to put them in a suitable form for our purposes. Moreover, many well-known special theorems on orthogonal and unitary equivalence of matrices are consequences of the theory developed, but we do not go into this for reasons of space.

Finally, I would like to express my admiration to M. Artin for his discovery of the deeper relations between the theory of polynomial identities and invariant theory. The ties between noncommutative algebra on one hand, and algebraic geometry and arithmetic on the other should be made stronger by these ideas. They have already yielded many interesting results and promise to give more.

Contents

I. Algebraic Invariants

1. *Invariants of $n \times n$ Matrices*

Let us fix the following notations: K denotes a field of characteristic 0, $V \simeq K^n$ is an n-dimensional vector space, $(K)_n \simeq \mathrm{End}(V)$ is the full ring of $n \times n$ matrices, V^* is the dual space of V, and $G = Gl(n, K)$ is the group of invertible matrices. We wish to study the following problem; consider the space $W = (K)_n^i$ of i-tuples of $n \times n$ matrices. The group G acts rationally on W according to the formula:

$$\text{If} \quad A \in G,\ B_j \in (K)_n,$$
$$\text{then} \quad A \cdot (B_1, B_2, \ldots, B_i) = (AB_1A^{-1}, AB_2A^{-1}, \ldots, AB_iA^{-1}).$$

We want to describe the ring $T_{i,n}$ of polynomial functions on W, invariant under the action of G. According to the general theory, we will split the description into two steps. The so called "first fundamental theorem," i.e., a list of generators for $T_{i,n}$, and then the "second fundamental theorem," i.e., a list of relations among the previously found generators. Of course, it would be very interesting to continue the process by giving the "ith fundamental theorem," i.e., the full theory of syzigies; unfortunately, this seems to be still out of the scope of the theory as presented in this paper.

To obtain the first fundamental theorem, we recall a part of the classical theory of invariants (cf. [16]).

First, we have the identification of the ith tensor power $(K)_n^{\otimes i}$ of $(K)_n$, with $\mathrm{End}(V^{\otimes i})$. The group $G = Gl(n, K)$ is embedded in $\mathrm{End}(V^{\otimes i})$ using the diagonal action $A \cdot (v_1 \otimes v_2 \otimes \cdots \otimes v_i) = Av_1 \otimes Av_2 \otimes \cdots \otimes Av_i$; finally, the centralizer of G in $\mathrm{End}(V^{\otimes i})$, i.e., the algebra of G-linear transformations of $V^{\otimes i}$, is spanned, as a vector space, by the endomorphisms λ_σ, σ an element of the symmetric group $\mathscr{S}_i$ on i letters, defined by the formula:

$$\lambda_\sigma(v_1 \otimes v_2 \otimes \cdots \otimes v_i) = v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes \cdots \otimes v_{\sigma^{-1}(i)}.$$

We have, furthermore, a canonical identification

$$\pi\colon (V^{*\otimes i} \otimes V^{\otimes i})^* \simeq \mathrm{End}(V^{\otimes i}),$$

where π is obtained from the nondegenerate pairing $\mathrm{End}(V^{\otimes i}) \times V^{*\otimes i} \otimes V^{\otimes i} \to K$ given by the formula:

$$\text{(a)} \quad \langle \lambda, \varphi_1 \otimes \varphi_2 \otimes \cdots \otimes \varphi_i \otimes x_1 \otimes x_2 \otimes \cdots \otimes x_i \rangle$$
$$= \langle \varphi_1 \otimes \cdots \otimes \varphi_i, \lambda(x_1 \otimes x_2 \otimes \cdots \otimes x_i) \rangle.$$

where, $\varphi_j \in V^*$, $x_j \in V$, $j = 1,..., i$, and the right side of (a) is the evaluation of the form $\varphi_1 \otimes \cdots \otimes \varphi_i \in V^{*\otimes i} \simeq (V^{\otimes i})^*$ on the vector $\lambda(x_1 \otimes x_2 \otimes \cdots \otimes x_i)$. It is easily verified that π is an isomorphism of G spaces with their canonical G structures. Hence, the space of G invariant vectors of $(V^{*\otimes i} \otimes V^{\otimes i})^*$, which is the space of linear maps $V^{*\otimes i} \otimes V^{\otimes i} \to K$ invariant under G is identified under π to the space of G linear endomorphisms of $V^{\otimes i}$. We already know that this last space is spanned by the elements λ_σ, $\sigma \in \mathscr{S}_i$. Thus we want to find an explicit expression for the linear invariant μ_σ corresponding to λ_σ under π.

One easily computes

$$\begin{aligned}
&\langle \lambda_\sigma, \varphi_1 \otimes \cdots \otimes \varphi_i \otimes X_1 \otimes \cdots \otimes X_i \rangle \\
&= \langle \varphi_1 \otimes \cdots \otimes \varphi_i, X_{\sigma^{-1}(1)} \otimes \cdots \otimes X_{\sigma^{-1}(1)} \rangle \\
&= \prod_j \langle \varphi_j, X_{\sigma^{-1}(j)} \rangle = \prod_j \langle \varphi_{\sigma(j)}, X_j \rangle \\
&= \mu_\sigma(\varphi_1 \otimes \cdots \otimes \varphi_i \otimes X_1 \otimes \cdots \otimes X_i).
\end{aligned}$$

In this way, we recover a part of the first fundamental theorem:

THEOREM 1.1. *Any multilinear invariant* γ: $V^{*\otimes i} \otimes V^{\otimes i} \to K$ *is a linear combination of the invariants*

$$\mu_\sigma(\varphi_1 \otimes \cdots \otimes \varphi_i \otimes X_1 \otimes \cdots \otimes X_i) = \prod_j \langle \varphi_{\sigma(j)}, X_j \rangle.$$

The next step now should be to determine the exact relations among the λ_σ's (or the μ_σ's). Rahter than doing this now, we want to interpret the μ_σ's in a different form.

We recall the canonical isomorphism between $\mathrm{End}(V)$ and $V^* \otimes V$ given by the formula:

$$(\varphi \otimes v)(u) = \langle \varphi, u \rangle v.$$

This is a G-isomorphism and we will use it in systematically identifying the two spaces. For instance, we will refer to a decomposable endomorphism as one corresponding to a decomposable tensor $\varphi \otimes v$, notice that an endomorphism is decomposable if and only if it is of rank $\leqslant 1$.

We recall, for completeness, how multiplication of endomorphisms and the trace map are obtained using the previous identification.

(b) $\varphi \otimes v \cdot \psi \otimes u = \varphi \otimes \langle \psi, v \rangle u$

(c) $\mathrm{tr}(\varphi \otimes v) = \langle \varphi, v \rangle$.

We can use the previous isomorphism to obtain an isomorphism of G-spaces:

$$(K)_n^{\otimes i} \simeq (V^* \otimes V)^{\otimes i} \simeq V^{*\otimes i} \otimes V^{\otimes i}.$$

The description of the linear invariants of $V^{*\otimes i} \otimes V^{\otimes i}$ obtained in 1.1 yields, therefore, a description for the linear invariants of $(K)_n^{\otimes i}$. Choose, therefore, a $\sigma \in \mathscr{S}_i$, and consider the linear invariant μ_σ : $(K)_n^{\otimes i} \to K$. We are going to give a new explicit formula for μ_σ in terms of the "matrix" variables. Decompose σ in disjoint cycles, including the ones of length 1,

$$\sigma = (i_1 i_2 \cdots i_k)(j_1 j_2 \cdots j_n) \cdots (t_1 t_2 \cdots t_e).$$

THEOREM 1.2. *Given* $A_1, A_2, \ldots, A_i \in (K)_n$, *we have*:

$$\begin{aligned}&\mu_\sigma(A_1 \otimes A_2 \otimes \cdots \otimes A_i)\\ &\quad = \operatorname{tr}(A_{i_1} A_{i_2} \cdots A_{i_k}) \operatorname{tr}(A_{j_1} A_{j_2} \cdots A_{j_n}) \cdots \operatorname{tr}(A_{t_1} A_{t_2} \cdots A_{t_e})\end{aligned}$$

Proof. Since the two sides of the equality are multilinear maps, it is sufficient to prove it when $A_1, \ldots, A_i$ are decomposable, i.e., $A_j = \varphi_j \otimes X_j$. Thus,

$$\begin{aligned}&\mu_\sigma(A_1 \otimes \cdots \otimes A_i)\\ &\quad = \mu_\sigma(\varphi_1 \otimes \varphi_2 \otimes \cdots \otimes \varphi_i \otimes X_1 \otimes \cdots \otimes X_i)\\ &\quad = \prod_{j=1}^{i} \langle \varphi_{\sigma(i)}, X_i \rangle = \langle \varphi_{i_2}, X_{i_1} \rangle \langle \varphi_{i_3}, X_{i_2} \rangle \cdots \langle \varphi_{i_k}, X_{i_{k-1}} \rangle \langle \varphi_{i_1}, X_{i_k} \rangle\\ &\quad\quad \cdot \langle \varphi_{j_2}, X_{j_1} \rangle \cdots \langle \varphi_{j_1}, X_{j_n} \rangle \cdots \langle \varphi_{t_1}, X_{t_e} \rangle.\end{aligned}$$

Consider, for instance, the product

$$M = \langle \varphi_{i_2}, X_{i_1} \rangle \langle \varphi_{i_3}, X_{i_2} \rangle \cdots \langle \varphi_{i_k}, X_{i_{k-1}} \rangle \langle \varphi_{i_1}, X_{i_k} \rangle.$$

One verifies immediately from the formulas (b), (c) that

$$\begin{aligned}&\varphi_{i_1} \otimes X_{i_1} \cdot \varphi_{i_2} \otimes X_{i_2}, \ldots, \varphi_{i_k} \otimes X_{i_k}\\ &\quad = \langle \varphi_{i_2}, X_{i_1} \rangle \langle \varphi_{i_3}, X_{i_2} \rangle \cdots \langle \varphi_{i_k}, x_{i_{k-1}} \rangle \, \varphi_{i_1} \otimes X_{i_k}.\end{aligned}$$

Therefore, $M = \operatorname{tr}(A_{i_1} A_{i_2} \cdots A_{i_k})$, and the theorem follows.

We are now in a position to state and prove the first fundamental theorem for invariants of $n \times n$ matrices.

THEOREM 1.3. *Any polynomial invariant of i $n \times n$ matrices $A_1,..., A_i$ is a polynomial in the invariants* $\operatorname{tr}(A_{i_1}A_{i_2} \cdots A_{i_j})$; $A_{i_1}A_{i_2} \cdots A_{i_j}$ *running over all possible (noncommutative) monomials.*

Proof. The theorem has already been proved for multilinear invariants. We claim that the general case follows immediately. In fact, one can fully polarize an invariant to obtain a multilinear one, and then recover the original invariant by identifying the variables. Now clearly, this last restitution process carries the invariant $\operatorname{tr}(A_{i_1}A_{i_2} \cdots A_{i_j})$ in invariants of the same type, and so the theorem is proved.

2. *Matrix Concomitants*

We recall that, given a group G and two G-spaces V, W, a polynomial concomitant is a polynomial map ψ: $V \to W$ that is compatible with the G-structures, i.e., $\psi(gv) = g\psi(v)$, $\forall g \in G$, $v \in V$.

We want to describe the concomitants in the case that $V = (K)_n^i$ and $W = (K)_n$, $G = Gl(n, K)$, with the usual action. We will refer to these as matrix valued concomitants and denote such a set by $S_{i,n}$.

The first remark is that $S_{i,n}$ is a noncommutative ring under pointwise sum and multiplication, in fact, $S_{i,n}$ is the subring of the ring $P_{i,n}$ of polynomial maps from $(K)_n^i$ to $(K)_n$ formed by those elements left fixed by the group $G = Gl(n, K)$, acting on $P_{i,n}$ as follows

$$g \in G, \quad \psi \in P_{i,n}, \qquad \text{then} \quad (g\psi)(u) = g \cdot (\psi(g^{-1}u)).$$

On the other hand, $P_{i,n}$ is easily identified as a ring isomorphic to the full ring of $n \times n$ matrices over the ring of polynomial functions on $(K)_n^i$, i.e., a ring of polynomials in $i \cdot n^2$ variables. If one identifies the scalars K with the center of $(K)_n$, one sees that the ring $T_{i,n}$ of invariants of $(K)_n^i$ is a subring of the center of $S_{i,n}$, which is, therefore, a $T_{i,n}$ algebra. It is easy to show (cf. [9, p. 94]) that $T_{i,n}$ is exactly the center of $S_{i,n}$ as soon as $i > 1$ and $n > 1$, otherwise, $S_{i,n}$ is commutative.

To complete these preliminary remarks, we notice that, among the matrix concomitants, we can consider the j-coordinate maps, indicated by X_j and given by:

$$X_j\colon (A_1, A_2, ..., A_i) \to A_j\,.$$

The first fundamental theorem for matrix concomitants now can be formulated.

THEOREM 2.1. *The ring $S_{i,n}$ is generated, as an algebra over $T_{i,n}$, by the elements X_j.*

Proof. Given a concomitant $f\colon (K)_n^i \to (K)_n$, we build an invariant $\bar{f}\colon (K)_n^{i+1} \to (K)_n$ defined as follows

$$\bar{f}(A_1, A_2, \ldots, A_{i+1}) = \operatorname{tr}(f(A_1, A_2, \ldots, A_i) \cdot A_{i+1}).$$

We claim that, if $f, g\colon (K)_n^i \to (K)_n$ are two concomitants and $\bar{f} = \bar{g}$, then $f = g$. In fact, if $\operatorname{tr}(f(A_1, A_2, \ldots, A_i) \cdot A_{i+1}) = \operatorname{tr}(g(A_1, A_2, \ldots, A_i) A_{i+1})$ for all $A_1, A_2, \ldots, A_{i+1}$, we have, by the nondegeneracy of the form $\operatorname{tr}(xy)$, that $f(A_1, A_2, \ldots, A_i) = g(A_1, A_2, \ldots, A_i)$, as claimed.

According to the classification theorem, 1.3, for the invariants, we have that, if f is a matrix concomitant, $\bar{f}$ is a polynomial in the elements $\operatorname{tr}(A_{i_1} A_{i_2} \cdots A_{i_j})$, which is linear in A_{i+1}. Therefore,

$$\bar{f} = \sum \lambda_{i_1 \ldots i_j} \operatorname{tr}(A_{i_1} A_{i_2} \cdots A_{i_j} A_{i+1}),$$

with $\lambda_{i_1 \cdots i_j} \in T_{i,n}$, and $i_1, i_2, \ldots, i_j \neq i + 1$. (If A_{i+1} appears in the middle of a monomial, we can shift it to the end by a cyclic permutation.) We have, thus,

$$\bar{f}(A_1, \ldots, A_{i+1}) = \operatorname{tr}\Big(\sum \lambda_{i_1 \cdots i_j} A_{i_1} A_{i_2} \cdots A_{i_j} \cdot A_{i+1}\Big),$$

therefore, $f = \sum \lambda_{i_1 \cdots i_j} X_{i_1} X_{i_2} \cdots X_{i_j}$, as announced.

3. *Finiteness Theorems for Graded Algebras*

The theorems proved in the previous sections still lack some necessary features for explicit computations. One is the lack of the necessary finiteness statements. We are going to provide them now. First, we make some general remarks on $T_{i,n}$ and $S_{i,n}$.

Given $f \in S_{i,n}$, we can consider its characteristic polynomial $\chi_f(X) = X^n + \sum_{i=1}^n \sigma_i(f) X^{n-i}$. Here, $\sigma_i(f)$ is an invariant, for instance, $\sigma_1(f) = -\operatorname{tr}(f)$, the others can be described using the expression of the coefficients of the characteristic polynomial of a matrix A in term of the invariants $\operatorname{tr}(A^i)$, (these formulas are the expressions of the Newton functions in terms of the elementary symmetric polynomials). We clearly have the Hamilton–Cayley theorem $\chi_f(f) = 0$.

Furthermore, both $S_{i,n}$ and $T_{i,n}$ are graded algebras, the degree being the usual one of polynomial maps, if $f \in S_{i,n}$ is homogeneous of degree h, then $\sigma_i(f)$ is homogeneous of degree $h \cdot i$. Finally, both $S_{i,n}$ and $T_{i,n}$ are connected, i.e., $(S_{i,n})_0 = (T_{i,n})_0 = K$.

Based on the previous remarks and having in mind Theorems 1.3 and 2.1, we develop here a general approach to finiteness theorems that will apply for the algebras $T_{i,n}$ and $S_{i,n}$ as well as for the other algebras that we will encounter in the study of the other classical groups.

Let us fix the following notation, $A = \bigoplus_{i=0}^{\infty} A_i$ will be a connected commutative graded algebra over a field $K = A_0$. We will set $A^+ = \sum_{i=1}^{\infty} A_i$ and recall the following well-known and easy lemma (Nakayama's lemma for graded module).

LEMMA 3.1. *Let $M = \bigoplus_{i=0}^{\infty} M_i$ be a graded A module and let N be a graded submodule. If $M = N + A^+M$, then $M = N$.*

Let us consider now an associative, not necessarily commutative, graded algebra R over A and a subset X of R^+ such that: (i) $R_0 = A_0 = K$ (ii) R is generated as an A algebra by 1 and X.

THEOREM 3.2. *If every element $r \in R^+$ satisfies a monic polynomial of degree n (depending on r) $x^n + \sum_{i=1}^{n} \alpha_i x^{n-i}$ with $\alpha_i \in A^+$ and* char $K = 0$ *or* char $K > n$, *then R is spanned over A by the monomials in the elements of X of degree $\leqslant 2^n - 2$.*

Proof. Consider the algebra $U = R^+/A^+R$. By 3.1, it is enough to show that the monomials of degree $\leqslant 2^n - 2$ (and $\geqslant 1$) in the elements of $\bar{X}$ (image of X in U), span U as a vector space over K. Now, the hypothesis implies that U is generated, as a K algebra, by $\bar{X}$, and so it is sufficient to show that U is nilpotent of degree $\leqslant 2^n - 1$. If $r \in R^+$, we have $r^n + \sum \alpha_i r^{n-i} = 0$ with $\alpha_i \in A^+$, therefore, every element $\bar{r}$ of U satisfies the equation $\bar{r}^n = 0$. We are, therefore, in the position to apply the theorem of Nagata–Higman [6, p. 274] to conclude the proof.

Assume now that the algebra R is equipped with an A linear map $t\colon R \to A$ preserving degrees. Furthermore, assume that, if T denotes the K algebra generated by X, the elements $t(T^+)$ generate A^+ as an ideal.

THEOREM 3.3. *In the previous hypotheses, A is generated as a K algebra by the elements $t(r)$, where r is a monomial in the elements of X of degree $\leqslant 2^n - 1$.*

Proof. By 3.2, R is spanned, as an A module, by the monomials in the elements of X of degree $\leqslant 2^n - 2$, let us call this set of monomials S. Let B denote the K subalgebra of A generated by the elements $t(r)$, where r runs on the set S' of monomials in the elements of X of degree $\leqslant 2^n - 1$. We must show that $B = A$, since B is a graded algebra and A is a graded B module, it is sufficient to show, by 3.1, that $B + B^+A = A$. Now, A^+ is generated by $t(T^+)$ as an ideal, $R = AS$, and $T^+ \subseteq T \cdot X$ therefore $t(T^+) \subseteq A \cdot t(SX) \subseteq AB^+$ (since SX is the set of monomials of positive degree $\leqslant 2^n - 1$ in the elements of X). Hence, $A^+ = A \cdot t(T^+) \subseteq AB^+$, and so $A = B + B^+A$.

We can apply these theorems in the case in which A is already known to be generated, as a K-algebra, by the elements $t(r)$, $r \in T$.

In particular, we can apply the previous theorems to obtain the finiteness theorems for the rings of invariants.

THEOREM 3.4. (a) *The ring $T_{i,n}$ is generated over K by the elements* $\operatorname{tr}(A_{i_1} A_{i_2} \cdots A_{i_j})$, *with $j \leqslant 2^n - 1$.*

(b) *$S_{i,n}$ is spanned, as a $T_{i,n}$ module, by the elements $X_{i_1} X_{i_2} \cdots X_{i_j}$ with $j \leqslant 2^n - 2$.*

Proof. This is just a special case, in light of the remarks at the beginning of the section, of the previous theorems. We use for t the trace map, and for X the set $X_1, X_2, \dots, X_i$.

4. *Trace Identities*

We consider now the problem of finding, in a systematic way, all relations among the elements $\operatorname{tr}(M)$ and M, M varying on the monomials in the $n \times n$ matrix variables $X_1, X_2, \dots, X_i, \dots$, $i = 1, \dots, \infty$.

We construct, for this purpose, the formal polynomial ring T generated by the symbols $\operatorname{Tr}(X_{i_1} X_{i_2} \cdots X_{i_n})$, with the convention that $\operatorname{Tr}(M) = \operatorname{Tr}(N)$ if and only if N is obtained from M by a cyclic permuation.

We will call an element $f \in T$ a commutative trace polynomial in the variables X_i and write it $f(X_1, X_2, \dots, X_i, \dots)$. Furthermore, we consider the free algebra $S = T\{X_i\}_{i=1,\dots,\infty}$ over T in the variables X_i.

We will refer to the elements of S as noncommutative trace polynomials S and T are equipped with some extra structure:

(a) *A* T-linear map $\operatorname{Tr}\colon S \to T$ defined by the formula

$$\operatorname{Tr}\left(\sum \lambda_{i_1 i_2 \cdots i_s} X_{i_1} X_{i_2} \cdots X_{i_s}\right) = \sum \lambda_{i_1 i_2 \cdots i_s} \operatorname{Tr}(X_{i_1} X_{i_2} \cdots X_{i_s}), \qquad \lambda_{i_1 i_2 \cdots i_s} \in T.$$

(b) For all choices $g_1, g_2, \dots, g_i, \dots$; $i = 1, \dots, \infty$ of elements $g_i \in S$, a formal substitution $f \to f(g_1, g_2, \dots, g_i, \dots)$, which is the uniquely determined endomorphism of the ring S mapping X_i into g_i and compatible with the map Tr.

The substitution is easily defined on the generators: it sends a monomial $X_{i_1}X_{i_2}\cdots X_{i_s}$ into $g_{i_1}g_{i_2}\cdots g_{i_s}$ and an element $\operatorname{tr}(X_{i_1}X_{i_2}\cdots X_{i_s})$ into $\operatorname{Tr}(g_{i_1}\cdots g_{i_s})$, one verifies easily that this is well defined and unique.

Having these structures in S, we can define the notion of a T-ideal.

DEFINITION 4.1. (a) A T-ideal I of S is an ideal that is closed under Tr and under all substitutions.

(b) A T-ideal J of T is an ideal closed under substitutions.

One easily verifies that, given any set $A \subseteq S$ (resp. $B \subseteq T$, there is a minimal T-ideal of S containing A and it is the ideal generated by the elements obtained from A by making all possible substitutions and taking the Tr values, similarly for $B \subseteq T$. We will refer to this T-ideal as to the T-ideal generated by A in S, (resp. by B in T).

The meaning of the previous definitions is made more explicit by associating, to the formal trace polynomials, actual functions. One chooses an integer n and considers the space $(K)_n^\infty$ of sequences $(A_1, A_2, ..., A_i, ...)$ of $n \times n$ matrices almost all zero.

Given an element $f(X_1, ..., X_i, ...) \in T$, one associate to it a polynomial map, in fact an invariant, $\bar{f}$: $(K)_n^\infty \to K$ by the obvious formula, if $f = \operatorname{Tr}(X_{i_1}\cdots X_{i_K})$, we set $\bar{f}(A_1, A_2, ..., A_i, ...) = \operatorname{tr}(A_{i_1}A_{i_2}\cdots A_{i_k})$ and then extend the definition on all T. Furthermore, if $g(X_1, ..., X_i, ...) \in S$, one associates to it a polynomial map, in fact, a matrix concomitant $\bar{g}$: $(K)_n^\infty \to (K)_n$ by the obvious formulas, on T, it is the already defined evaluation and to the monomials $X_{i_1}\cdots X_{i_k}$, one associates the map $(A_i, ..., A_i, ...) \to A_{i_1}\cdots A_{i_k}$.

If we indicate by $T_{\infty,n}$, $S_{\infty,n}$ the rings of invariants and concomitants of infinitely many matrices, we have, thus, two onto maps:

$$\pi\colon T \to T_{\infty,n}, \qquad \tau\colon S \to S_{\infty,n}.$$

If we consider $T \subseteq S$ and $T_{\infty,n} \subseteq S_{\infty,n}$, we remark that π is the restriction of τ to T. We are now able to state the problem of finding the relations among invariants and concomitants, it consists of describing the Kernels of the two maps π and τ.

An important remark, before continuing, is that we have the compatibylity of τ, π with the extra trace operators, i.e., the diagram:

$$\begin{tikzcd} S \arrow[r, "\pi"] \arrow[d, "\mathrm{Tr}"'] & S_{\infty,n} \arrow[d, "\mathrm{tr}"] \\ T \arrow[r, "\tau"] & T_{\infty,m}, \end{tikzcd}$$

is commutative, where Tr is the formal trace in S, and tr is the usual trace.

Finally, π is compatible with the substitution, which becomes in $S_{\infty,n}$ composition of map; i.e., $\pi(f(g_1,..., g_i,...)) = (\pi f)(\pi(g_1),..., \pi(g_i),...)$.

Having made these remarks, the next proposition follows immediately:

PROPOSITION 4.2. *The ideals* ker π, ker τ *are T-ideals.*

We will refer to the elements of ker π as commutative trace identities of $n \times n$ matrices and to the elements of ker τ as noncommutative trace identities of $n \times n$ matrices. We remark that we have already established, in the proof of 2.1, a strict relationship between the two concepts.

We come now to the basic theorem from which all our results will follow.

Establish the following notation, given a permutation $\sigma \in \mathcal{S}_m$, we define an element $\phi_\sigma \in T$ as follows. Decompose ϕ in disjoint cycles, including the ones of length 1:

$$\sigma = (i_1 \cdots i_k)(j_1 \cdots j_h) \cdots (t_1 \cdots t_e)$$

and set $\Phi_\sigma(X_1,..., X_m) = \mathrm{Tr}(X_{i_1}X_{i_2} \cdots X_{i_k}) \mathrm{Tr}(X_{j_1} \cdots X_{j_h}) \cdots \mathrm{Tr}(X_{t_1} \cdots X_{t_e})$. ϕ_σ is a multilinear trace monomial of degree m. With the notations of 1.1 and 1.2 we have, if $A_1,..., A_m \in (K)_n$, that

$$\pi(\phi_\sigma) = \phi_\sigma(A_1,..., A_m) = \mu_\sigma(A_1 \otimes A_2 \otimes \cdots \otimes A_m).$$

From the theory of Young diagrams we obtain immediately.

THEOREM 4.3. (a) *An element* $\sum_{\sigma\in\mathcal{S}_m} \alpha_\sigma \Phi_\sigma$ *is a trace identity for* $n \times n$ *matrices if and only if the element* $\sum \alpha_\sigma \sigma$ *belongs to the ideal of the group algebra of* $\mathcal{S}_m$ *spanned by the Young symmetrizers relative to diagrams with at least* $n + 1$ *rows.*

(b) *In particular we have the fundamental trace identity* $F(X_1,..., X_{n+1}) = \sum_{\sigma\in\mathcal{S}_{n+1}} \epsilon(\sigma)\Phi_\sigma$ ($\epsilon(\sigma)$ *the signature of* σ), *corresponding to the Young diagram with one column and* $n + 1$ *rows.*

Proof. Clearly, $\sum \alpha_\sigma \Phi_\sigma$ is a trace identity if and only if the corresponding element $\sum \alpha_\sigma \mu_\sigma$ is zero. This is zero if and only if the endomorphism $\sum \alpha_\sigma \lambda_\sigma$ on $V^{\otimes m}$ induced by $\sum \alpha_\sigma \sigma$ is zero. Hence, (a) and (b) follow from the theory of Young diagrams and the representation of $\mathcal{S}_m$ on $V^{\otimes m}$ (cf. [16]).

Since $F(X_1, ..., X_{n+1})$ is multilinear in all variables we can apply to it the same process used in the proof of 2.1 to write it formally as

$$F(X_1, ..., X_{n+1}) = \operatorname{Tr}(G(X_1, ..., X_n)\, X_{n+1}),$$

where $G(X_1, ..., X_n) \in S$.

The explicit form of $G(X_1, ..., X_n)$ is the following

$$G(X_1, ..., X_n) = \sum_{k=0}^{n} (-1)^{k+1} \sum_{i_1 \neq i_2 \neq \cdots \neq i_k} X_{i_1} X_{i_2} \cdots X_{i_k} \sum_{\sigma \in \mathscr{S}_{n-k}} \epsilon(\sigma) \Phi_\sigma ,$$
$$\text{on } \{1, ..., m\} - \{i_1, ..., i_k\}.$$

In different notations, let us write, for a set $S = (s_1, ..., s_k)$

$$F_S = F(X_{s_1}, X_{s_2}, ..., X_{s_k}),$$
$$P_S = \sum_{\sigma \in \mathscr{S}_S} X_{\sigma(s_1)} X_{\sigma(s_2)} \cdots X_{\sigma(s_k)} .$$

Then, we have, setting $M = 1, 2, ..., n$ and $|S|$ the cardinality of S

$$G(X_1, ..., X_n) = \sum_{S \subseteq M} (-1)^{|M-S|+1} F_S \cdot P_{M-S} .$$

An easy consequence of 4.3 is the following.

COROLLARY 4.4. (a) *A multilinear commutative trace identity of degree $n+1$ in $n+1$ variables is a scalar multiple of $F(X_1, ..., X_{n+1})$.*

(b) *A multilinear noncommutative trace identity of degree n in n variables is a scalar multiple of $G(X_1, ..., X_n)$.*

(c) *$G(X_1, ..., X_n)$ is obtained by full polarization of the "characteristic polynomial" of X, times $(-1)^{n+1}$.*

Proof. (a) This is a consequence of 4.3(a), since the ideal described there, for $m = n+1$, is just the scalar multiples of $\sum_{\sigma \in \mathscr{S}_{n+1}} \epsilon(\sigma)\sigma$.

(b) This is a consequence of (a) and the relation established in the proof of 2.1, that $f(X_1, ..., X_n) \in \operatorname{Ker} \tau$ if and only if

$$\operatorname{Tr}(f(X_1, ..., X_n)\, X_{n+1}) \in \operatorname{Ker} \pi.$$

(c) First, one has to explain the meaning of characteristic polynomial of an element of S. We know that if A is an $n \times n$ matrix, its characteristic polynomial $\chi_A(X)$ is a polynomial whose coefficients can be expressed in a formal way via the elements $\operatorname{tr}(A^i)$. We use the same formulas to construct a formal characteristic polynomial of an element of S, using Tr instead of the usual trace. Now, if X is a variable, we consider $\chi_X(X)$, it is an element of S homogeneous of degree n. By the Hamilton–Cayley theorem, $\chi_X(X) \in \operatorname{Ker} \tau$, hence, if we fully polarize $\chi_X(X)$, we have a multilinear trace identity of degree n.

By (b), it is a scalar multiple of $G(X_1, ..., X_n)$. On the other hand, its leading term (the one without coefficients in T) is $\sum_{\sigma \in \mathscr{S}_n} X_{\sigma(1)} \cdots X_{\sigma(n)}$, therefore, (c) follows by inspection.

We are now ready to prove the second fundamental theorem.

THEOREM 4.5. (a) *The ideal* Ker π *is generated by the elements* $F(M_1, ..., M_{n+1})$. *The M_i's running over all possible monomials.*

(b) *The ideal* Ker τ *is generated by the elements* $F(M_1, ..., M_{n+1})$, $G(N_1, ..., N_n)$.

The M_i's and N_j's running over all possible monomials.

Proof. (a) First, we want to reduce ourselves to the analysis of multilinear identities. This is possible by the processes of polarization and restitution. In fact, if $f \in \operatorname{Ker} \pi$, and we fully polarize it, the result f' is still in Ker π. If we show that f' is in the ideal described by (a), the same will follow for f, since the restitution maps this ideal into itself.

Therefore, let $f \in \operatorname{Ker} \pi$ be multilinear and of degree m. A priori f may depend on more than m variables, but we can separate f as a sum of polynomials f_i each depending on m variables, such that f_i and f_j do not depend on the same variables if $i \neq j$. One easily sees, by setting some of the variables equal to zero, that each f_i is a trace identity. Therefore, we may assume that f is multilinear of degree m and depends on the variables

$$X_1, X_2, ..., X_m, \qquad \text{then } f = \sum_{\sigma \in \mathscr{S}_m} \alpha_\sigma \phi_\sigma .$$

We know, by 4.3, that $\sum \alpha_\sigma \sigma$ is in the ideal relative to the Young diagrams with at least $n + 1$ rows. This ideal is generated by the antisymmetrizer $\sum_{\sigma \in \mathscr{S}_{n+1}} \epsilon(\sigma)\sigma$, under the embedding of $\mathscr{S}_{n+1}$ in $\mathscr{S}_m$.

Therefore,

$$\sum \alpha_\sigma \sigma = \sum_{\tau_i, \tau_j \in \mathscr{S}_m} \alpha_{ij} \tau_i \Big(\sum_{\sigma \in \mathscr{S}_{n+1}} \epsilon(\sigma)\sigma \Big) \tau_j .$$

Let us analyze, therefore, the form of the trace identity associated to the element

$$\tau \Big(\sum_{\sigma \in \mathscr{S}_{n+1}} \epsilon(\sigma)\sigma \Big) \zeta; \qquad \tau, \zeta \in \mathscr{S}_m.$$

We make a series of remarks.

Remark 1. $\sum_{\sigma\in\mathscr{S}_{n+1}} \epsilon(\sigma)\sigma$ thought of as an element of the group algebra over $\mathscr{S}_m$ correspond to the identity:

$$F(X_1, ..., X_{n-1}) \cdot \mathrm{Tr}(X_{n+2}) \, \mathrm{Tr}(X_{n+3}) \cdots \mathrm{Tr}(X_m)$$

Remark 2. If $\sum_{\lambda\in\mathscr{S}_m} \alpha_\lambda\lambda$ corresponds to a trace polynomial $H(X_1, ..., X_m)$, and $\tau \in \mathscr{S}_m$, then $\tau(\sum \alpha_\lambda\lambda)\tau^{-1}$ corresponds to the trace polynomial $H(X_{\tau(1)}, X_{\tau(2)}, ..., X_{\tau(m)})$.

Remark 3. $\tau(\sum \epsilon(\sigma)\sigma)\zeta = \tau(\sum \epsilon(\sigma)\sigma)\zeta\, \tau \cdot \tau^{-1}$ hence, to prove that the trace polynomial associated to $\tau(\sum \epsilon(\sigma)\sigma)\zeta$ is obtained from $F(X_1, ..., X_{n+1})$, substituting for the X_i's some monomials (and changing sign if necessary) it is sufficient, it light of Remark 2, to do it for $\sum_{\sigma\in\mathscr{S}_{n+1}} \epsilon(\sigma)\sigma \cdot \zeta\tau$. Let us call $\zeta\tau = \eta$.

Remark 4. $\sum_{\sigma\in\mathscr{S}_{n+1}} \epsilon(\sigma)\sigma \cdot \eta = \pm \sum_{\sigma\in\mathscr{S}_{n+1}} \epsilon(\sigma)\sigma \cdot \gamma$, where γ is a permutation of $\mathscr{S}_m$ containing, in each cycle, at most one of the elements $1, 2, ..., n+1$.

Proof. It is sufficient to show that we can write $\eta = \lambda \cdot \gamma$, with γ of the desired type, and $\lambda \in \mathscr{S}_{n+1}$, since in this case $\sum \epsilon(\sigma)\,\sigma\lambda = \epsilon(\lambda) \cdot \sum \epsilon(\sigma)\sigma$. The possibility of writing $\eta = \lambda\gamma$ is obtained by a simple induction of which we explain the first step. Assume that η contains in a cycle two elements of $1, 2, ..., n+1$, say, 1 and 2: $\eta = (1\, i_1 i_2 \cdots i_k\, 2\, j_1 \cdots j_l) \cdots (\ )$.

We have $(12) \cdot \eta = (1\, i_1 \cdots i_k)\,(2 \cdots j_l) \cdots (\ )$.

Remark 5. If $\sigma \in \mathscr{S}_{n+1}$ and γ is as in 4, the cycle decomposition of σ is obtained by formally substituting; in each cycle of σ, in place of the elements $1, 2, ..., n+1$, the strings $1\, i_1, ..., i_k, 2\, j_1, ..., j_s, ..., n+1\; t_1 \cdots t_h$ appearing in the cycle decomposition of η and finally adjoining the cycles of η in which the elements $1, 2, ..., n+1$ do not appear.

We are now in a position to finish, it is clear, by 5, that if $\gamma = (1 i_1 \cdots i_k)(2 j_1 \cdots j_s) \cdots (n+1 t_1 \cdots t_n)(\lambda_1 \cdots \lambda_r)(\mu_1 \cdots \mu_v) \cdots (\rho_1 \cdots \rho_z)$, the trace polynomial corresponding to $\sum_\sigma \epsilon(\sigma) \cdot \sigma \cdot \gamma$ is

$$F(X_1 X_{i_1} X_{i_2} \cdots X_{i_k}, X_2 X_{j_1} \cdots X_{j_s}, ..., X_{n+1} X_{t_1} X_{t_2} \cdots X_{t_h})$$
$$\cdot \mathrm{Tr}(X_{\lambda_1} X_{\lambda_2} \cdots X_{\lambda_r}) \, \mathrm{Tr}(X_{\mu_1} X_{\mu_2} \cdots X_{\mu_v}) \cdots (\mathrm{Tr}(X_{\rho_1} \cdots X_{\rho_z})).$$

Therefore, from the various reductions operated, the theorem follows.

(b) Let $H(X_1, ..., X_m) \in \operatorname{Ker} \tau$ be a noncommutative trace identity of $n \times n$ matrices. We know then that the trace polynomial $\mathrm{Tr}(H(X_1, ..., X_m) \cdot X_{m+1})$ is in $\mathrm{Ker}\,\pi$.

By (a), we know that such an element has the form:

$$\sum \lambda_{i_1 \cdots i_{n+1}} F(M_{i_1}, M_{i_2}, ..., M_{i_{n+1}}),$$

and it is linear in X_{m+1}. Therefore, we may assume that X_{m+1} appears in each term of the sum, and for each term it appears linearly. Let us consider one such term $\lambda \cdot F(M_1, ..., M_{n+1})$, X_{m+1} will either appear in λ or in one of the monomials M_i's. If X_{m+1} appears in λ, we can write $\lambda = \mathrm{Tr}(\bar{\lambda} \cdot X_{m+1})$, otherwise, permuting the monomials if necessary, we may assume that $M_{n+1} = A \cdot X_{m+1} \cdot B$; A, B two monomials. Then, $\lambda F(M_1, ..., M_{n+1}) = \mathrm{Tr}(\lambda \cdot BG(M_1, ..., M_n) \cdot A \cdot X_{m+1})$.

Finally, we see that

$$\mathrm{Tr}(H(X_1, ..., X_m)\, X_{m+1}) = \mathrm{Tr}\left(\left[\sum \bar{\lambda}_{i_1 \cdots i_{n+1}} F(M_{i_1}, ..., M_{i_{n+1}})\right.\right.$$
$$\left.\left. + \sum \lambda_{j_1 \cdots j_{n+1}} \cdot B_{j_{n+1}} G(N_{j_1}, ..., N_{j_n})\, A_{j_{n+1}}\right] \cdot X_{m+1}\right).$$

Now, one should note that, in the formal ring S, if $H(X_1, ..., X_m)$, $K(X_1, ..., X_m)$ are two polynomials such that $\mathrm{Tr}(H(X_1, ..., X_n)\, X_{m+1}) = \mathrm{Tr}(K(X_1,...,X_m) X_{m+1})$, one has necessarily $H(X_1,...,X_m) = K(X_1,...,X_m)$, therefore, (b) is also proved.

We may express the previous theorem in a more suggestive form by using the nonhomogeneous analogs of F and G.

We already know that G is obtained up to sign from the characteristic polynomial by means of full polarization. As for F, we have a similar result. The analog of the characteristic polynomial is the expression of $\mathrm{Tr}(A^{n+1})$ in terms of the elements $\mathrm{Tr}(A^i)$, $i \leqslant n$. This, of course, can be realized by the equation $\mathrm{tr}(\chi_A(X) \cdot X) = 0$. Let us call the formal expressions associated to $\chi_X(X)$ and to $\mathrm{tr}(\chi_X(X) \cdot X)$, $G(X)$ and $F(X)$. Thus, $G(X)$ is the "characteristic polynomial of X" and $F(X)$ is the "expression" of $\mathrm{Tr}(X^{n+1})$ in terms of $\mathrm{Tr}(X^i)$, $i \leqslant n$. (Of course, this expression holds only if we evaluate X in $(K)_n$.)

We then have

THEOREM 4.6. (a) $\mathrm{Ker}\,\pi$ *is the* T*-ideal of* T *generated by* $F(X)$.

(b) $\mathrm{Ker}\,\tau$ *is the* T*-ideal of* S *generated by the characteristic polynomial* $G(X)$.

Proof. (a) and (b) will follow from 4.5 and the definition of T-ideals for T and S once we prove that the fully polarized forms of $F(X)$ and $G(X)$ lie in the T-ideal that they generate. Now this is truc since we may replace the process of polarization with the process of multilinearization. whose first step is to replace, for instance, $G(X)$, by $G(X+Y) - G(X) - G(Y)$. The final result of multilinearization is the same as of full polarization and so the claim follows, since clearly, multilinearizing a polynomial in a T-ideal, we remain in the same ideal.

We can deduce now a rather intriguing corollary that ties completely theorem 3.4(b) with the Nagata–Higman theorem.

COROLLARY 4.7. *The ring $S_{i,n}^+/T_{i,n}^+S_{i,n}$ is isomorphic to the free algebra without* 1, $\{X_1, ..., X_i\}$ *in n-variables modulo the T-ideal defined by the polynomial identity* $Z^n = 0$.

Proof. We know that $S_{i,n}^+/T_{i,n}^+S_{i,n}$ satisfies the identity $Z^n = 0$ and it is generated by the classes of the elements X_j, $j = 1, ..., i$ over K. Therefore, the canonical map $\{X_1, ..., X_i\} \to S_{i,n}^+/T_{i,n}^+S_{i,n}$ factors through the T-ideal J defined by the polynomial identity $Z^n = 0$. We have to show that the induced map $\psi: \{X_1, ..., X_i\}/J \to S_{i,n}^+/T_{i,n}^+S_{i,n}$ is an isomorphism.

Now, ψ is onto by construction, and we have to show that the only relations among the classes of the elements X_j in $S_{i,n}^+/T_{i,n}^+S_{i,n}$ are deducible from the polynomial identity $Z^n = 0$.

We have a presentation of $S_{i,n}^+$ and $T_{i,n}^+$ given by the Theorems 4.5 or 4.6. To have a presentation for $S_{i,n}^+/T_{i,n}^+S_{i,n}$, we have to add to the relations given in 4.5, 4.6 the relation $\mathrm{Tr}(M) = 0$ for all monomials M of positive degree. If we start from these relations, i.e., we construct S^+/T^+S, we just get the free algebra without 1 over K, since $T/T^+ = K$.

Now, if we read in this algebra the relations given, for instance, by 4.6, we see that the characteristic polynomial $G(X)$ becomes X^n, the trace map now is 0 (as well as $F(X)$), and so the T-ideal generated by $G(X)$ becomes modulo T^+, exactly the T-ideal generated by the identity $X^n = 0$ as announced.

There is another way of formulating the preceding corollary, which we state for completeness.

COROLLARY 4.8. *If R is an associative algebra over a field of characteristic* 0, *and R satisfies the polynomial identity $X^n = 0$, then R satisfies all the polynomial identities of $n \times n$ matrices.*

5. *Trace* 0 *Matrices*

It may be useful to recall that the representation of $Gl(n, K)$ on $(K)_n$, which we have been considering all along, is not irreducible. In fact, it decomposes as $(K)_n = K \cdot 1 + (K)_n^0$ where $(K)_n^0$ stands for the subspace of matrices with trace 0, and $K \cdot 1$ is the center of $(K)_n$.

This decomposition corresponds to the possibility of giving to each invariant a double degree in the scalar variables and in the trace 0 variables. In fact, if we write, for a matrix A, $A = (\mathrm{tr}(A)/n) \cdot 1 + A_0$, where $\mathrm{Tr}(A_0) = 0$, we have, for a given monomial $A_{i_1} A_{i_2} \cdots A_{i_k}$, that

$$\mathrm{tr}(A_{i_1} A_{i_2} \cdots A_{i_k}) = \sum_{h_1 \cdots h_t; j_1 \cdots j_s} n^{-t} \,\mathrm{tr}(A_{h_1})\,\mathrm{tr}(A_{h_2}) \cdots \mathrm{tr}(A_{h_t})\,\mathrm{tr}(A^0_{j_1} A^0_{j_2} \cdots A^0_{j_s});$$

where $h_1 \cdots h_t j_1 \cdots j_s$ is a shuffle of the indices $i_1 i_2 \cdots i_k$.

It is easy to see that, if we denote by $T^0_{i,n}$ the ring of invariants of the space $((K)^0_n)^i$ of i-tuples of trace 0 matrices, we have that $T_{i,n}$ is the polynomial ring over $T^0_{i,n}$ in the variables $\mathrm{tr}(A_1), \mathrm{tr}(A_2),\ldots, \mathrm{tr}(A_i)$. In particular, the ring $T^0_{i,n}$ is obtained from $T_{i,n}$ setting $\mathrm{tr}(A_j) = 0$, $j = 1,\ldots, i$.

6. *Relations with Polynomial Identities and Central Polynomials*

We sketch here some consequences of the previous theorems, which should be expanded more. If we consider the free algebra $K\{X_1, \ldots, X_i, \ldots\}$ over K and embed it in the free algebra S over T, we see immediately that $\mathrm{Ker}\,\tau \cap K\{X_i, \ldots, X_i, \ldots\}$ is the ideal of polynomial identities of $n \times n$ matrices. As for central polynomials, an element $f \in K\{X_i\}$ is a central polynomial for $n \times n$ matrices if and only if there is a $g \in T$ such that $f - g \in \mathrm{Ker}\,\tau$.

We know that $\mathrm{Ker}\,\tau$ is generated as a T-ideal from the characteristic polynomial $G(X)$, or its linearized form $G(X_1, \ldots, X_n)$; therefore, every polynomial identity or central polynomial is deducible from $G(X_1, \ldots, X_n)$ in an explicit way. On the other hand, the task of describing all polynomial identities is still quite far away. In a separate paper, Formanek [3][4] shows how to deduce very simply the Amitsur–Levitzki identity $S_{2n}(X_1, \ldots, X_{2n})$ from the polynomial $G(X_1, \ldots, X_n)$, a similar approach to central polynomials also should be possible. Nevertheless, we can give a description of polynomial identities in terms of Young diagrams according to 4.3(a). Let I be the ideal of the group algebra of $\mathscr{S}_{m+1}$

[4] *Note added in proof.* This description has been found independently by Rasmyslev in the paper cited in footnote 1.

generated by the antisymmetrizer on $n + 1$ letters (we assume $m \geqslant n$). We know that I can be identified to the space of multilinear trace identities of $n \times n$ matrices in $m + 1$ variables. Let P be the subspace of the group algebra of $\mathscr{S}_{m+1}$ spanned by the $m + 1$ cycles, we have

THEOREM 6.1. *The space of multilinear polynomial identities of $n \times n$ matrices of degree m in m variables can be identified to the space $I \cap P$.*

Proof. We fix a variable, say, X_{m-1}, then writing a commutative trace identity in the form $\mathrm{Tr}(g \cdot X_{m+1})$, we establish a 1-1 correspondance between the space I and the space of noncommutative trace identities. Now, a noncommutative trace identity is a polynomial identity if and only if it does not contain coefficients $\mathrm{Tr}(M)$. This is easily seen to be equivalent to the fact that the element of I considered is a sum of $m + 1$ cycles.

The situation for central polynomials is slightly more complicated. We take again an element $\sum \alpha_\sigma \sigma$ of the group algebra over $\mathscr{S}_{m+1}$ that is a trace identity, i.e., it is in I, and we associate to it the element g as before. The g will be, in general, a sum of monomials in the X_j's, times products of trace monomials $\mathrm{tr}(M)$. We can recover from g a central polynomial exactly when g splits as the sum of pure monomials (without factors of type $\mathrm{tr}(M)$) and scalars, i.e., products of factors $\mathrm{tr}(M)$ only.

This, of course, can be read in the group algebra, it means that the permutations σ appearing in $\sum \alpha_\sigma \sigma$ are only of two types: $m + 1$ cycles, and permutations that fix $m + 1$. In this case, the corresponding noncommutative trace polynomial is of the form

$$\sum \alpha_{i_1 i_2 \cdots i_m} X_{i_1} X_{i_2} \cdots X_{i_m} + \sum \alpha_{j_1 \cdots j_t} \mathrm{tr}(M_{j_1})\, \mathrm{tr}(M_{j_2}) \cdots \mathrm{tr}(M_{j_t}),$$

where the left sum comes from the $m + 1$ cycles, and the right sum comes from the permutations fixing $m + 1$. In this case, the left sum is a central polynomial, and the opposite of the right sum is the scalar value taken by the central polynomial.

Again, we can use these remarks to characterize multilinear central polynomials in terms of Young diagrams.

7. *Orthogonal Invariants*

We consider now the same type of questions that we have treated in the previous sections for the other classical groups. We study now the orthogonal group. The set up is the following: Consider the algebra $(K)_n$ of $n \times n$ matrices equipped with the standard involution given by transposition. This involution is associated to the canonical bilinear from on $V = K^n$ given by $\sum_{i=1}^{n} x_i y_i$.

The relation between the form and the involution is, of course, the usual: $(Av, w) = (v, A^t w)$ if $v, w \in V$, $A \in (K)_n$.

The scalar product allows us to identify canonically V with its dual V^* and hence, $\mathrm{End}(V) \simeq V^* \otimes V$ with $V \otimes V$.

If we want to translate the multiplication, the trace map, and the involution of $(K)_n$ in terms of its identification with $V \otimes V$, we obtain the following formulas:

(a) $v \otimes w \cdot u \otimes t = v \otimes (u, w)t$,

(b) $\mathrm{tr}(v \otimes w) = (v, w)$,

(c) $(v \otimes w)^t = w \otimes v$.

As for the groups involved, the orthogonal group $O(n, k)$ is, by definition, the group of automorphisms of V with its structure of scalar product. As for $(K)_n$ with its involution, assume that $\psi: (K)_n \to (K)_n$ is an automorphism of algebras with involution. We must have $\psi(B) = A \cdot BA^{-1}$ for some $A \in Gl(n, K)$. On the other hand, the hypothesis implies that $\psi(B^t) = \psi(B)^t$, hence, $A \cdot B^t \cdot A^{-1} = (ABA^{-1})^t$, for all $B \in (K)_n$.

This implies easily $A^t \cdot A \in K \cdot 1$, let $A^t \cdot A = \alpha \in K$.

If $\alpha = \beta^2$ setting $A' = A/\beta$ we see that $A' \in O(n, K)$, and A' defines the same inner automorphism as A. Hence, if we take K algebraically closed, we see that the automorphism group of $(K)_n$ with its transpose involution is $O(n, K)\{1, -1\}$.

We recall now the theorems on invariants of $2i$ vectors under the orthogonal group (cf. [16, p. 53, 75]).

First Fundamental Theorem. *Any multilinear orthogonal invariant of $2i$ vectors:*

$$\psi: V \otimes V \otimes \underset{2i}{\cdots} \otimes V \to K$$

is a linear combination of "contraction maps" i.e., maps of type:

$$\psi_\sigma: v_1 \otimes v_2 \otimes \cdots \otimes v_{2i} \to (v_{j_1}, v_{j_2})(v_{j_3}, v_{j_4}) \cdots (v_{j_{2i-1}}, v_{j_{2i}})$$

$\sigma = (j_1, j_2, \ldots, j_{2i})$ *a permutation of* $1, 2, \ldots, 2i$.

SECOND FUNDAMENTAL THEOREM. *Any relation among the scalar products* (u_i, v_j) *is a consequence of relations of the following kind:*

$$\begin{vmatrix} (u_1, v_1) & (u_1, v_2) & \cdots & (u_1, v_{n+1}) \\ (u_2, v_1) & (u_2, v_2) & \cdots & (u_2, v_{n+1}) \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ (u_{n+1}, v_1) & (u_{n+1}, v_2) & \cdots & (u_{n+1}, v_{n-1}) \end{vmatrix} = 0.$$

We are now going to extract, from these theorems, the analogous ones for matrix invariants and concomitants.

THEOREM 7.1. *Every orthogonal invariant of i matrices $(A_1, \ldots, A_i)$ is a polynomial in the elements* $\operatorname{tr}(U_{i_1} U_{i_2} \cdots U_{i_k})$, *where* $U_j = A_j$, *or* $U_j = A_j{}^t$.

Proof. We can reduce ourselves, as in the proof of 1.3, to the multilinear case.

Consider, therefore, a multilinear orthogonal invariant

$$\psi\colon (K)_n \otimes (K)_n \otimes \cdots \otimes (K)_n \to K.$$

We identify $(K)_n$ with $V \otimes V$, and this is compatible with the $O(n, K)$ structure, then, ψ is a linear combination of the maps ψ_σ previously described. Let us consider decomposable matrix variables $A_j = u_j \otimes v_j$, $j = 1, \ldots, i$. Then,

$$\begin{aligned} &\psi_\sigma(A_1 \otimes A_2 \otimes \cdots \otimes A_i) \\ &\quad = \psi_\sigma(u_1 \otimes v_1 \otimes u_2 \otimes v_2 \otimes \cdots \otimes u_i \otimes v_i) \\ &\quad = (w_{i_1}, \bar{w}_{i_2})(w_{i_2}, \bar{w}_{i_3}) \cdots (w_{i_k}, \bar{w}_{i_1}) \cdot (w_{j_1}, \bar{w}_{j_2})(w_{j_2}, \bar{w}_{j_3}) \cdots (w_{j_s}, \bar{w}_{j_1}) \cdots \end{aligned}$$

where we use the following convention:

w_j stands for u_j or v_j, and by definition, $\bar{u}_j = v_j$, $\bar{v}_j = u_j$.

It is now easy to verify that:

$$(*) \quad \psi_\sigma(A_1 \otimes A_2 \otimes \cdots \otimes A_i) = \operatorname{tr}(U_{i_1} U_{i_2} \cdots U_{i_k}) \operatorname{tr}(U_{j_1} \cdots U_{j_s}) \cdots,$$

where $U_j = A_j$, or $U_j = A_j{}^t$, according to the following rule:
Let us say that u_j, u_k are of the same type, as well as v_j, v_k, while u_j, v_k are of different type. Let us say, furthermore, that A_j, A_k are of the same type as well as $A_j{}^t$, $A_k{}^t$, while A_j and $A_k{}^t$ are of different type.

Then, we define inductively:

(a) $U_{i_1} = A_{i_1}$, if $w_{i_1} = v_{i_1}$; $U_{i_1} = A_{i_1}^t$, if $w_{i_1} = u_{i_1}$.

(b) Set $U_{i_{t+1}}$ of the same type as U_{i_t} if and only if w_{i_t}, $w_{i_{t+1}}$ have the same type.

The equality (*) has been proved for decomposable A_j's, but it is multilinear on both sides, hence, it holds for any choice of the A_j's.

Having classified the orthogonal invariants, we pass now to the matrix valued concomitants. Let us fix some notations, let us indicate $TO_{i,n}$ the ring of orthogonal invariants of i $n \times n$ matrices and $SO_{i,n}$ the matrix valued concomitants. We see immediately that $SO_{i,n}$ is not a necessarily commutative algebra over $TO_{i,n}$; furthermore, $SO_{i,n}$ is equipped with an involution, that we will still call transposition: $f^t(A_1,..., A_i) = [f(A_1,..., A_i)]^t$.

We have, as in 2.1, the basic concomitants

$$X_j: (A_1, A_2,..., A_j) \to A_j,$$

and also

$$X_j^t: (A_1, A_2,..., A_j) \to A_j^t.$$

THEOREM 7.2. *$SO_{i,n}$ is generated, as a $TO_{i,n}$ algebra, by the elements X_j, X_j^t.*

Proof. We follow the lines of 2.1. We introduce an extra variable X_{i+1} and associate, to any $g \in SO_{i,n}$ the orthogonal invariant $f = \operatorname{tr}(g \cdot X_{i+1})$. By 7.1, f is a polynomial in the elements $\operatorname{tr}(U_{j_1} \cdots U_{j_i})$ with $U_j = X_j$ or X_j^t. Since f is linear in X_{i+1}, it is a sum of monomials, in each of which, it appears either X_{i+1} or X_{i+1}^t, and only once. Since $\operatorname{tr}(M) = \operatorname{tr}(M^t)$, we can rewrite the monomials, if necessary, to contain always X_{i+1}, and not X_{i+1}^t. At this point, we can write $f = \operatorname{tr}(h \cdot X_{i+1})$, where h is a polynomial in X_j, X_j^t, $j = 1,..., i$, with coefficients in $TO_{i,n}$. Clearly, $h = g$, and so the proof is complete.

Clearly, if $g \in SO_{i,n}$, we can compute its characteristic polynomial, which has coefficients in $TO_{i,n}$, and g satisfies $\chi_g(X)$. Therefore, we can apply the results of Section 3 to obtain the necessary finiteness statements for $TO_{i,n}$ and $SO_{i,n}$. We reproduce the statements for completeness.

THEOREM 7.3. (a) *The ring $TO_{i,n}$ is generated, as a K algebra, by the* *elements* $\operatorname{tr}(M)$, *where* M *is a monomial in* X_j, X_j^t, $j = 1,\ldots, i$ *of degree* $\leqslant 2^n - 1$.

(b) $SO_{i,n}$ *is spanned, as a* $TO_{i,n}$ *module, by the monomials in* X_j, X_j^t, $j = 1,\ldots, i$ *of degree* $\leqslant 2^n - 2$.

8. *Orthogonal Trace Identities*

We pass now to the second fundamental theorem. As usual, we will fix our attention mainly on multilinear identities. To make the formalism complete, we must introduce, as in Section 4, the formal algebras T^* and S^*. T^* is the commutative algebra in the variables $\operatorname{Tr}(M)$, M is a monomial in X_j, X_j^t, $j = 1,\ldots, \infty$, with the obvious identifications and S^* is the free algebra over T^* in the variables X_j, X_j^*. S^* is equipped with the trace map $\operatorname{Tr}: S^* \to T^*$, and also with an involution. linear over T^*, mapping X_j into X_j^t. In this case also, we have the operation of substitution of the variables. Of course, if we substitute X_j with g_j, we must substitute X_j^t with g_j^t. Thus, we can introduce the notion of T-ideals as in Section 4, recalling that now we have also the structure of algebra with involution, under which the T-ideal must be closed. We will refer to the elements of T^* as commutative trace polynomials, and to the ones of S^* as noncommutative trace polynomial. Finally, we have, as in Section 4, the two maps $\pi_0: T^* \to TO_{\infty,n}$, $\tau_0: S^* \to SO_{\infty,n}$, which are compatible with all the operations defined. The involution on $SO_{\infty,n}$ being transposition.

Theorems 7.1 and 7.2 state that π_0, and τ_0 are onto, for every n. The determination of their Kernels is the object of the second fundamental theorem. To make the theory reasonably smooth, we must find an analog of the interpretation, given in Section 3, of elements of the group algebra as multilinear trace polynomials. We proceed now as follows: We introduce two infinite sequences of symbols $u_1,\ldots, u_m,\ldots$; $v_1,\ldots, v_m,\ldots$, and construct, from these symbols, the symbols (u_i, u_j), (v_i, v_j) $i \neq j$, and (u_i, v_j) any i and j. Consider next the polynomial ring P in the "variables" (u_i, u_j), (v_i, v_j), (u_i, v_j), (v_i, u_j). We make the conventional identifications: $(u_i, u_j) = (u_j, u_i)$, $(v_i, v_j) = (v_j, v_i)$, $(u_i, v_j) = (v_j, u_i)$. For every m, we consider now the subspace I_m of this polynomial ring spanned by the monomials of degree m in which appear all the symbols $u_1, u_2,\ldots, u_m$; $v_1, v_2,\ldots, v_m$.

Reasoning as in the proof of 7.1, we see that we can associate to each such monomial, a formal monomial in the elements $\operatorname{Tr}(U_{i_1} U_{i_2} \cdots U_{i_j})$, where $U_t = X_i$ or X_i^t. If one follows closely the proof of 7.1, one sees that this monomial is uniquely determined up to: (1) cyclic permutation of the factors of a monomial $U_{i_1} \cdots U_{i_j}$, (2) replacing $U_{i_1} \cdots U_{i_j}$, with $U^t_{i_j} \cdots U^t_{i_1}$.

The formal association becomes an actual functional relation once we substitute:

(1) For the symbols u_i, v_i, vectors.

(2) For the symbols (u_i, v_j) etc., the scalar products among such vectors.

(3) For the matrix variables X_i, the actual matrices $u_i \otimes v_i$.

We can sum up the results in a proposition.

PROPOSITION 8.1. (a) *The space I_m is isomorphic to the subspace of T^* formed of multilinear trace polynomials in the variables $X_1, \ldots, X_m$ (eventually transposed).*

(b) *Given vectors u_j, v_j, $j = 1, \ldots,$ and the matrices $A_j = u_j \otimes v_j$, we have the compatibility of the evaluations:*

$$\begin{tikzcd} I_m \arrow[r] \arrow[d] & P \arrow[drr, "\rho"] & & \\ T^* \arrow[r] & TO_{\infty, n} \arrow[rr, "\sigma"] & & K, \end{tikzcd}$$

where σ consists in evaluating the invariants in A_j, and ρ the symbols (u_i, u_j) etc. *in the scalar products.*

From this proposition and the multilinearity of the trace polynomials in the image of I_m, we deduce that the trace identities of degree m, multilinear in $X_1, \ldots, X_m$, correspond to the subspace of I_m of multilinear relations among the scalar products.

We are now in a position to apply the second fundamental theorem for the scalar products. We introduce a notation for convenience.

If $w_1, w_2, \ldots, w_k$; $z_1, z_2, \ldots, z_k$, are vectors, we indicate by $\langle w_1, \ldots, w_k \mid z_1, \ldots, z_k \rangle$ the determinant:

$$\begin{vmatrix} (w_1, z_1) & (w_1, z_2) & \cdots & (w_1, z_k) \\ \cdots & \cdots & \cdots & \cdots \\ (w_k, z_1) & (w_k, z_2) & \cdots & (w_k, z_k) \end{vmatrix}.$$

We know that, if we are working on the vector space K^n, we have the fundamental relations: $\langle w_1, \ldots, w_{n+1} \mid z_1, \ldots, z_{n+1} \rangle = 0$ for any choice of vectors w_j, z_j.

Let us look at the occurrence of this relation in the space I_{n+1}. We obtain one such relation if we divide the vectors $u_1, u_2, \ldots, u_{n+1}, v_1, v_2, \ldots, v_{n+1}$ in two subsets of $n+1$ elements each.

In particular, consider the following ones (which we will show to be the relevant ones): Given $k \leqslant (n+1)/2$ setting $n+1 = 2k+s$, we form

$$\langle u_1, u_2, \ldots, u_{k+s}, v_1, v_2, \ldots, v_k \mid u_{k+s+1}, u_{k+s+2}, \ldots, u_{n+1}, v_{k+1}, \ldots, v_{n+1} \rangle.$$

To this element of I_m corresponds, by the general theory, a trace polynomial that we will denote $F_{k,n+1}(X_1, \ldots, X_{n+1})$. This polynomial is clearly a trace identity for $n \times n$ matrices.

Let us show now that these polynomials are, in a certain sense, the only multilinear polynomials of degree $n+1$ to be considered.

PROPOSITION 8.2. *Let $w_1, w_2, \ldots, w_{n+1}, z_1, \ldots, z_{n+1}$ be a permutation of $u_1, u_2, \ldots, u_{n+1}, v_1, v_2, \ldots, v_{n+1}$. The trace polynomial corresponding to $\langle w_1, w_2, \ldots, w_{n+1} \mid z_1, z_2, \ldots, z_{n+1} \rangle$ can be obtained, up to sign, from exactly one of the $F_{k,n+1}(X_1, \ldots, X_{n+1})$ by permuting the variables and substituting to some of the X_j's their transposes X_j^t.*

Proof. This is one of those statements that are hard only to write in detail, so we sketch the steps. To change X_j with X_j^t means in I_m to exchange u_j with v_j, to permute the variables corresponds to permuting simultaneously the u_j's and the v_j's in the same fashion. Finally, by the properties of determinants we can rearrange in $\langle w_1, \ldots, w_{n+1} \mid z_1, \ldots, z_{n+1} \rangle$ either the w_j's or the z_j's, and we only change eventually the sign. Now, the proof of the proposition is immediate. First, one exchanges the u_j's and the v_j's (i.e., substitutes X_j with X_j^t) to obtain that the symbols $w_1, w_2, \ldots, w_{n+1}$ consist of a certain number $k+s$ of u's, and k of v's having the same indices of some of the u's listed. Then, one permutes the variables to make sure that one has on the left side, the symbols $u_1 \cdots u_{k+s}$ $v_1 \cdots v_k$ in some order, then, one rearranges the left and the right side to obtain the desired expression. One can be easily convinced that the number k is uniquely determined as the number of indices appearing twice (once in a u and once in a v) in the left side of $w_1, \ldots, w_{n+1}\, z_1, \ldots, z_{n+1}$.

The following properties of the polynomials $F_{k,n+1}(X_1, \ldots, X_{n+1})$ are easily verified by the same ideas developed in proving 8.2.

(a) $F_{k,n+1}(X_1, X_2, \ldots, X_i, \ldots, X_{n+1}) = -F_{k,n+1}(X_1, X_2, \ldots, X_i{}^t, \ldots, X_{n+1})$ if $i \leqslant k$, or $i > k + s$.

(b) $F_{k,n+1}(X_1, X_2, \ldots, X_{n+1}) = F_{k,n+1}(X_{\sigma(1)}, X_{\sigma(2)}, \ldots, X_{\sigma(n+1)})$

If σ is a permutation of the following type: (i) $\sigma(\{1, 2, \ldots, k\}) = \{1, 2, \ldots, k\}$, $\sigma(\{k+1, \ldots, k+s\}) = \{k+1, \ldots, k+s\}$ and $\sigma = (\{k+s+1, \ldots, n+1\}) = \{k+s+1, \ldots, n+1\}$.

(c) $F_{k,n+1}(X_1{}^t, X_2{}^t, \ldots, X_{n+1}^t) = F_{k,n+1}(X_{k+s+1}, \ldots, X_{n+1}, X_{k+1}, \ldots, X_{k+s}, X_1, \ldots, X_k)$.

Parts (a) and (b) are easily proved from the determinant expression of $F_{k,n+1}$. Part (c) is also easy, we have

$$\begin{aligned} &F_{k,n+1}(X_1{}^t, X_2{}^t, \ldots, X_{n+1}^t) \\ &\quad = \langle v_1, v_2, \ldots, v_{k+s}, u_1, u_2, \ldots, u_k \mid v_{k+s+1}, \ldots, v_{n+1}, u_{k+1}, \ldots, u_{n+1} \rangle \\ &\quad = \langle v_{k+s+1}, \ldots, v_{n+1}, u_{k+1}, \ldots, u_{n+1} \mid v_1, \ldots, v_{k+s}, u_1, \ldots, u_k \rangle \\ &\quad = \langle u_{k+s+1}, \ldots, u_{n+1}, u_{k+1}, \ldots, u_{k+s}, v_{k+s+1}, \ldots, v_{n+1} \mid u_1, \ldots, u_k, \\ &\qquad\quad v_{k+1}, \ldots, v_{k+s}, v_1, \ldots, v_k \rangle \\ &\quad = F_{k,n+1}(X_{k+s+1}, \ldots, X_{n+1}, X_{k+1}, X_{k+s}, X_1, \ldots, X_k). \end{aligned}$$

Property (b) suggests to extract two noncommutative trace polynomials out of each $F_{k,n+1}$ when s and k are nonzero, by taking out of the trace a variable in the set $X_1, \ldots, X_k, X_{k+s+1}, X_{k+s+2}, \ldots, X_{n+1}$, or in the set $X_{k+1}, \ldots, X_{k+s}$.

Since we want to have the two polynomials depending on $X_1, \ldots, X_n$ for convenience, we define them as follows, implicitely:

$$\begin{aligned} &F_{k,n+1}(X_1, \ldots, X_{n+1}) = \operatorname{Tr}(G_{k,n}(X_1, \ldots, X_n) \cdot X_{n+1}) \\ &F_{k,n+1}(X_1, \ldots, X_{k+s-1}, X_{n+1}, X_{k+s+1}, \ldots, X_n, X_{k+s}) \\ &\qquad = \operatorname{Tr}(H_{k,n}(X_1, \ldots, X_n) \cdot X_{n+1}). \end{aligned}$$

When $s = 0$, only G is defined, while when $k = 0$, only H is defined.

From the properties (a)–(c) of $F_{k,n+1}$, we can deduce properties of G and H. In particular, we will need the computation:

(c)′ $\operatorname{Tr}(G_{k,n}(X_1, \ldots, X_n)^t \cdot X_{n+1}) = \operatorname{Tr}(G_{k,n}(X_1, \ldots, X_n)\, X_{n+1}^t) = F_{k,n+1}(X_1, \ldots, X_n, X_{n+1}^t) = F_{k,n+1}(X_{k+s+1}^t, \ldots, X_n{}^t, X_{n+1}, X_{k+1}^t, \ldots, X_k{}^t, X_1{}^t, \ldots, X_k{}^t)$.

The meaning of these definitions is made clear by the following:

PROPOSITION 8.3. (a) *The polynomials $F_{k,n+1}$ are commutative orthogonal trace identities for $n \times n$ matrices.*

(b) *The polynomials $G_{k,n}$, $H_{k,n}$ are noncommutative trace identities for $n \times n$ matrices.*

(c) *Any commutative (resp. noncommutative), orthogonal trace identity has degree $\geqslant n+1$ (resp. $\geqslant n$).*

(d) *Any multilinear commutative (resp. noncommutative) trace identity of degree $n+1$ (resp. n) is a linear combination of polynomials deduced from the $F_{k,n+1}$ (resp. the $G_{k,n}$ and $H_{k,n}$) by substitution of the variables (and eventual transposition of the final result).*

Proof. First we remark that substituting the variables means to substitute for $X_1, \ldots, X_{n+1}$ variables $W_{i_1}, \ldots, W_{i_{n+1}}$, where $W_j = X_j$, or X_j'. The proof now is a consequence of 8.1, 8.2, and the second fundamental theorem, plus the usual remark that noncommutative trace identities in the variables $X_1, \ldots, X_i$ correspond bijectively to commutative trace identities in $X_1, \ldots, X_i, X_{i+1}$; linear in X_{i+1}, via the map

$$g \to \operatorname{Tr}(g \cdot X_{i+1}).$$

In this case, one might have to transpose and use (c)' if X_{i+1} appears among the first k variables.

Having described the minimal identities, we proceed now to prove the second fundamental theorem for matrix invariants and concomitants:

THEOREM 8.4. (a) $\operatorname{Ker} \pi_0$ *is the T-ideal of T^* generated by the polynomials $F_{k,n+1}$.*

(b) $\operatorname{Ker} \tau_0$ *is the T-ideal of S^* generated by the polynomials $G_{k,n}$, $H_{k,n}$.*

Proof. (a) We proceed as in 4.5 to reduce ourselves to analyze multilinear identities only, of degree m and depending on the variables $X_1, \ldots, X_m$. We can apply now Proposition 8.1 and the second fundamental theorem to reduce ourselves to analyze a trace identity corresponding to an element

$$f = \langle p_1, \ldots, p_{n-1} \mid q_1, \ldots, q_{n+1} \rangle (p_{n+2}, q_{n+2}) \cdots (p_m, q_m),$$

where $p_1, \ldots, p_m$, $q_1, \ldots, q_m$ is a permutation of $u_1, \ldots, u_m$, $v_1, \ldots, v_m$.

We are clearly allowed to permute the variables if necessary and to exchange X_j with $X_j{}^t$, i.e., to permute simultaneously the u_i's, v_i's, and exchange u_j with v_j. Finally, changing possibly the sign, we can rearrange separately the elements $p_1, \dots, p_{n+1}$ and $q_1, \dots, q_{n+1}$.

If we operate this way, it is easy to convince oneself that we may assume that $p_1, \dots, p_{n+1}$ is the sequence $u_1, u_2, \dots, u_{k+s}, v_1, \dots, v_k$ for some k such that $n + 1 = 2k + s$. Furthermore, $q_1, \dots, q_{n+1}$ may be taken as the sequence $u_{k+s+1}, u_{k+s+2}, \dots, u_{k+s+m}, v_{k+1}, \dots, v_{k+l}, v_{j_1}, v_{j_2}, \dots, v_{j_r}, v_{k+s+1}, v_{k+s+2}, \dots, v_{n+1}$, with $l \leqslant s$, $k + s + m \leqslant n + 1$ and the indices $j_1, j_2, \dots, j_r > n + 1$.

Consider now u_{j_1}, it appears as one of the elements p_i or q_i with $i > n + 1$ in a scalar product (u_{j_1}, w_{h_1}) outside the determinant, $w_{h_1} = u_{h_1}$ on $w_{h_1} = v_{h_1}$.

Let us separate some cases, introducing new formal symbols:

(a) If $w_{h_1} = v_{h_1}$, set $\bar{v}_{h_1} = (u_{j_1}, v_{h_1})\, v_{j_1}$, $\bar{u}_{h_1} = u_{h_1}$.

(b) If $w_{h_1} = u_{h_1}$ and $h_1 > n + 1$, set $\bar{v}_h = (u_{j_1}, u_{h_1})\, v_{j_1}$ and $\bar{u}_{h_1} = v_{h_1}$.

(c) If $w_{h_1} = u_{h_1}$ and $h_1 \leqslant n + 1$ set $\bar{u}_{h_1} = (u_{j_1}, u_{h_1}) \cdot v_{j_1}$, and $\bar{v}_{h_1} = u_{h_1}$.

Furthermore, we can take the scalar product (u_{j_1}, v_{h_1}) inside the determinant, and using the symbols introduced, rewrite f as:

$$f = \langle \bar{p}_1, \dots, \bar{p}_{n+1} \mid \bar{q}_1, \dots, \bar{q}_{n+1} \rangle (\bar{p}_{n+2}, \bar{q}_{n+2}) \cdots (\bar{p}_{m-1}, \bar{q}_{m-1}),$$

where $\bar{p}_1, \dots, \bar{p}_{m-1}, \bar{q}_1, \dots, \bar{q}_{m-1}$ is a permutation of

$$u_1 u_2 \cdots \bar{u}_{h_1} \cdots \check{u}_{j_1} \cdots u_m \qquad v_1 \cdots \bar{v}_{h_1} \cdots \check{v}_{j_1} \cdots v_m$$

(where $\check{}$ means omitted).

We apply now induction and assume that f, as a trace polynomial in the variables $X_1, X_2, \dots, \bar{X}_{h_1}, \dots, X_{j_1}, \dots, X_m$, is of the form

$$\pm F_{k,n+1}(M_1, M_2, \dots, M_{n+1}) \cdot P.$$

Now, we see immediately that we can substitute to the variable $\bar{X}_{h_1}$, respectively: In case (a), $\bar{X}_{h_1} = X_{h_1} X_{j_1}$, in case (b) $\bar{X}_{h_1} = X_{h_1}^t X_{j_1}$, and in case (c) $\bar{X}_{h_1} = X_{j_1}^t X_{h_1}^t$ using the formal multiplication rule for the symbols $p \otimes q$ given by $p \otimes q \cdot r \otimes s = p \otimes (r, q) s$, and thinking $\bar{X}_{h_1} = \bar{u}_{h_1} \otimes \bar{v}_{h_1}$.

All these formal operations are compatible and give

$$f = \pm F_{k,n+1}(M_1', M_2', \dots, M_{n+1}')\, P',$$

where the M_j''s and P' are obtained from the M_j's and P substituting for $\bar{X}_{h_1}$ the corresponding expression. This completes case (a).

(b) This part is easy having developed (a). Let $K(X_1, \dots, X_t) \in \operatorname{Ker} \tau_0$, add a new variable X_{t+1}, and consider $\mathrm{Tr}(K(X_1, \dots, X_t)\, X_{t+1}) \in \operatorname{Ker} \pi_0$. By part (a) we have

$$\mathrm{Tr}(K(X_1, \dots, X_t)\, X_{t+1}) = \sum_i P_i F_{k,n+1}(M_1^{(i)}, \dots, M_{n+1}^{(i)}),$$

and we may assume that X_{t+1} appears linearly in each term of the sum. Take a term $PF_{k,n+1}(M_1, \dots, M_{n+1})$. We have, then, two cases, either X_{t+1} appears in P, or X_{t+1} appears in one of the monomials M_j's. Further, X_{t+1} may appear transposed.

If X_{t+1} appears in P (transposed or not) we can write $P = \mathrm{Tr}(P'X_{t+1})$, if X_{t+1} appears in one of the M_j's, then, using property (b) and (c) of the polynomial $F_{k,n+1}$, we may permute the variables and assume that X_{t+1} appears in M_{n+1} or in M_{k+s}. Since both cases are similar (one gives rise to the appearence of $G_{k,n}$ the other of $H_{k,n}$), we treat the first. We might have $M_{n+1} = A_{n+1}X_{t+1}B_{n+1}$, or $M_{n+1} = A_{n+1}X_{t+1}^t B_{n+1}$.

In the first case,

$$F_{k,n+1}(M_1, \dots, M_{n+1}) = \mathrm{Tr}(B_{n+1}G_{k,n}(M_1, \dots, M_n)\, A_{n+1} \cdot X_{t+1}).$$

In the second case, we have

$$\begin{aligned} F_{k,n+1}(M_1, \dots, M_{n+1}) &= \mathrm{Tr}(B_{n+1}G_{k,n}(M_1, \dots, M_n)\, A_{n+1}X_{n+1}^t \\ &= \mathrm{Tr}(A_{n+1}^t G_{k,n}(M_1, \dots, M_n)^t\, B_{n+1}^t X_{n+1}). \end{aligned}$$

From these computations, it follows that $\operatorname{Ker} \tau_0$ is generated by the elements $G_{k,n}(M_1, \dots, M_n)$, $G_{k,n}(M_1, \dots, M_n)^t$, $H_{k,n}(M_1, \dots, M_n)$, $F_{k,n+1}(M_1, \dots, M_{n+1})$, $H_{k,n}(M_1, \dots, M_n)^t$.

When we look, therefore, at the T-ideal generated by the polynomials $G_{k,n}$ and $H_{k,n}$ we obtain, closing the ideal under substitutions, the polynomials $G_{k,n}(M_1, \dots, M_n)$, $H_{k,n}(M_1, \dots, M_n)$ and, closing under the operator Tr and under the involution, the polynomials $F_{k,n+1}(M_1, \dots, M_{n+1})$ $G_{k,n}(M_1, \dots, M_n)^t$, $H_{k,n}(M_1, \dots, M_n)^t$, as desired.

9. *Explicit Computations of Orthogonal Trace Identities*

We want to make now a few explicit computations of the polynomials introduced in Section 8.

The first remark is that the polynomial $F_{o,n+1}(X_1,\dots,X_{n+1})$ is exactly the trace polynomial $F(X_1,\dots,X_{n+1})$ introduced in 4.3 and obtained by full polarization of the expression of $\operatorname{Tr}(X^{n+1})$ in terms of $\operatorname{Tr}(X^i)$, $i \leqslant n$.

Similarly, $H_{o,n}(X_1,\dots,X_n)$ is $G(X_1,\dots,X_n)$ of Section 4. As for the other polynomials, they are not fully symmetric in all variables, hence, cannot be obtained by full polarization of a polynomial in one variable. Rather, we could see $F_{k,n+1}$, $n+1 = 2k+s$ as obtained from full polarization of a polynomial in 3 variables if $k, s \neq 0$, in two variables if $s = 0$, by identifying the variables $X_1,\dots,X_k$, or $X_{k+1}\cdots X_{k+s}$, or $X_{k+s+1},\dots,X_{n+1}$, with respect to which the polynomial is fully symmetric.

Furthermore, from the remark (a) on the properties of such polynomials we have:

$$
\begin{aligned}
&F_{k,n+1}(X_1,\dots,X_{n+1})\\
&\quad= \tfrac{1}{2}(F_{k,n+1}(X_1,\dots,X_i,\dots,X_{n+1}) - F_{k,n+1}(X_1,\dots,X_i^t,\dots,X_{n+1}))\\
&\quad= F_{k,n+1}(X_1,\dots,(X_i - X_i^t)/2,\dots,X_{n+1}),
\end{aligned}
$$

when $i \leqslant k$ or $i > k+s$. Setting $(X_i - X_i^t)/2 = X_i^-$, we have that X_i^- is the antisymmetric part of the variable X_i, and so $F_{k,n+1}$ depends on $2k$ antisymmetric variables X_i^-, $i \leqslant k$, $i > k+s$, plus the other s variables X_j, $k < j < k+s$,

$$
\begin{aligned}
&F_{k,n+1}(X_1,\dots,X_{n+1})\\
&\quad= F_{k,n+1}(X_1^-, X_2^-,\dots,X_k^-, X_{k+1},\dots,X_{k+s}, X_{k+s+1}^-,\dots,X_{n+1}^-).
\end{aligned}
$$

We have similar results for the polynomials $G_{k,n}$ and $H_{k,n}$. In particular, we have that for $n = 2k-1$, the polynomials $F_{k,2k}$ and $G_{k,2k-1}$ depend only on skew symmetric arguments. For $n = 2k$. the polynomial $H_{k,2k}$ depends only on skew symmetric variables, while $F_{k,2k+1}$ depends on $2k$ skew symmetric variables plus the variable X_{k+1}.

Let us compute explicitely $F_{k,2k}$. Its formal expression in I_{2k} (4.8.1) is given by

$$(*)\quad \langle u_1u_2\cdots u_kv_1v_2\cdots v_k \mid u_{k+1}\cdots u_{2k}v_{k+1}\cdots v_{2k}\rangle.$$

Let us analyze the monomials appearing in the expression of (*). Let us call a monomial M pure if, in the scalar products, each u is paired with a v and conversely. Such a pure monomial can be written

$$M = (u_{i_1}, v_{j_1})(u_{j_1}, v_{i_2})(u_{i_2}, v_{j_2}) \cdots (u_{j_k}, v_{i_1})$$
$$\cdot (u_{s_1}, v_{t_1})(u_{t_1}, v_{s_2}) \cdots (u_{t_h}, v_{s_1}) \cdots (u_{z_1}, v_{w_1}) \cdots (u_{w_e}, v_{z_1}),$$

where the indices $i, s, \ldots, z$ runs from 1 through k, and the indices $j, t, \ldots, w$, from $k + 1$ to $2k$.

This monomial corresponds to a permutation $\sigma \in \mathscr{S}_{2k}$, which in cycle form is

$$(i_1 j_1 i_2 j_2 \cdots i_k j_k)(s_1 t_1 s_2 t_2 \cdots s_h t_h) \cdots (z_1 w_1 z_2 w_2 \cdots z_e w_e).$$

Any such permutation can be built from two bijective maps α: $\{1, \ldots, k\} \to \{k + 1, \ldots, 2k\}$, and β: $\{k + 1, \ldots, 2k\} \to \{1, \ldots, k\}$. σ is just the sum $\alpha + \beta$ of the two maps. The monomial M, being one of the expansion of the determinant, corresponds to a map $\bar{\sigma}$ between rows and columns. One easily verifies that $\bar{\sigma}$ is equal to α on the first k rows and to $\mu\beta^{-1}\mu$ on the last k rows, where μ: $\{k + 1, \ldots, 2k\} \to \{1, 2, \ldots, k\}$ is the map $i \to i - k$. The sign of M in the expansion of (*) is the signature of $\bar{\sigma}$, but we claim that σ and $\bar{\sigma}$ have the same signature. In fact, compose σ and $\bar{\sigma}$ with $\tau = \mu + \mu^{-1}$, then, $\tau\sigma = \mu\alpha + \mu^{-1}\beta$, with $\mu\alpha$, $\mu^{-1}\beta$ permutations on $1, \ldots, k$, $k + 1, \ldots, 2k$, respectively, while $\tau\bar{\sigma} = \mu\alpha + \beta^{-1}\mu = \mu\alpha + (\mu^{-1}\beta)^{-1}$.

Therefore, $\epsilon(\tau)\,\epsilon(\sigma) = \epsilon(\mu\alpha) \cdot \epsilon(\mu^{-1}\beta) = \epsilon(\mu\alpha)\,\epsilon(\beta^{-1}\mu) = \epsilon(\tau)\,\epsilon(\bar{\sigma})$.

Let us return to M and write it as a trace monomial

$$M = \operatorname{Tr}(X_{i_1}^t X_{j_1} X_{i_2}^t X_{j_2} \cdots X_{j_k}) \operatorname{Tr}(X_{s_1}^t X_{t_1} X_{s_2}^t \cdots X_{t_h}) \cdots \operatorname{Tr}(X_{z_1}^t X_{w_1} \cdots X_{w_e}).$$

In the expansion of (*), all the impure monomials are obtained from pure monomials upon exchanging some u_j's with the corresponding v_j's, conversely given a pure monomial any monomial obtained by such exchanges appears; any time such an exchange is made the sign changes, if we exchange the u_j's and v_j's for all the indices in a cycle α of the permutation σ associated to a pure monomial, we obtain again a pure monomial. Its permutation σ' is obtained from σ exchanging the cycle α with α^{-1}.

Therefore, given the pure monomial M, let us compute the contribution to (*) given by M and all the monomials obtained from M upon exchanges. Making all possible exchanges except for the indices $i_1, s_1, \ldots, z_1$, we obtain the sum

$$\epsilon_\sigma \cdot 2^{N_\sigma} \operatorname{Tr}(X^t_{i_1} X^-_{j_1} X^-_{i_2} \cdots X^-_{j_k}) \operatorname{Tr}(X^t_{s_1} X^-_{t_1} X^-_{s_2} \cdots X^-_{t_h}) \cdots \operatorname{Tr}(X^t_{z_1} X^-_{w_1} \cdots X^-_{w_e}),$$

where $N_\sigma = 2k - \lambda$, where λ is the number of cycles in which σ decomposes. If we want now to complete the exchanges, we see that we obtain also trace monomials relative to permutation σ' obtained by substituting a cycle of σ with its inverse. Since we want eventually to sum over all allowable σ's, we should not count twice the same term. This is achieved by substituting for $X^t_{i_1}, X^t_{s_1}, \ldots, X^t_{z_1}$ the variables $-X^-_{i_1}, -X^-_{s_1}, \ldots, -X^-_{z_1}$ without changing the coefficient 2^{N_σ}. Noting that if a cycle is a transposition, hence, equal to its inverse, we have the formal equality $\operatorname{Tr}(X_i{}^t X_j{}^-) = \operatorname{Tr}(X_i{}^- X_j{}^-)$. To sum up, we have

$$F_{k,2k} = \sum_{\sigma \in \bar{\mathscr{S}}_{2n}} \epsilon_\sigma (-2)^{N_\sigma} \operatorname{Tr}(X^-_{i_1} X^-_{j_1} \cdots X^-_{j_k}) \cdots \operatorname{Tr}(X^-_{z_1} \cdots X^-_{w_e}),$$

where $\bar{\mathscr{S}}_{2n}$ stands for the set of special permutations of type $\alpha + \beta$, as above, e.g.,

$$F_{1,2} = 2 \cdot \operatorname{Tr}(X_1{}^- X_2{}^-)$$

$$\begin{aligned} F_{2,4} = {} & 8 \operatorname{Tr}(X_1{}^- X_3{}^- X_2{}^- X_4{}^-) + 8 \operatorname{Tr}(X_1{}^- X_4{}^- X_2{}^- X_3{}^-) \\ & - 4 \operatorname{Tr}(X_1{}^- X_3{}^-) \operatorname{Tr}(X_2{}^- X_4{}^-) - 4 \operatorname{Tr}(X_1{}^- X_4{}^-) \operatorname{Tr}(X_2{}^- X_3{}^-). \end{aligned}$$

Of course, by lumping into a unique term all the monomials obtained from a permutation σ and the σ's gotten from σ inverting the cycles, we can simplify the expressions:

$$\begin{aligned} F_{2,4} = {} & 16 \operatorname{Tr}(X_1{}^- X_3{}^- X_2{}^- X_4{}^-) - 4 \operatorname{Tr}(X_1{}^- X_3{}^-) \operatorname{Tr}(X_2{}^- X_4{}^-) \\ & - 4 \operatorname{Tr}(X_1{}^- X_4{}^-) \operatorname{Tr}(X_2{}^- X_3{}^-). \end{aligned}$$

Let us compute now $F_{k,2k+1}$; rather than repeating the full reasoning we use a trick; introduce a new variable $\bar{X}_{2k+2} = u'_{2k+2} \otimes v'_{2k+2}$, where $v'_{2k+2} = v_{k+1}$ and u'_{2k+2} satisfies (formally) the relations $(u_i, u'_{2k+2}) = 0$; $i = 1, \ldots, k+1$ and $(v_i, u'_{2k+2}) = 0, i = 1, \ldots, k$, while $(v_{k+1}, u'_{2k+2}) = 1$.

Then,

$$\begin{aligned}
&F_{k+1,2k+2}(X_1,\dots,X_{2k+1},\bar{X}_{2k+2}) \\
&\quad = \langle u_1,\dots,u_{k+1},v_1,\dots,v_{k+1} \mid u_{k+2}\cdots u_{2k+1}u'_{2k+2}v_{k+2}\cdots v_{2k+1}v'_{2k+2}\rangle \\
&\quad = \langle u_1u_2\cdots u_{k+1}v_1\cdots v_k \mid u_{k+2}\cdots u_{2k+1}v_{k+1}\cdots v_{2k+1}\rangle \\
&\quad = F_{k,2k+1}(X_1,\dots,X_{2k+1}).
\end{aligned}$$

Now, we must eliminate all monomials in $F_{k+1,2k+2}$ in which we do not have the scalar product $(v_{k+1}, u'_{2k+2}) = 1$. Hence, we sum over all special permutations of $2k+2$ indices, where $k+1$ is paired with $2k+2$. Expanding the relative monomials, we use the computations

$$\operatorname{Tr}(X_{i_1}^- X_{j_1}^- \cdots X_{i_s}^- X_{j_s}^- X_{k+1}\bar{X}_{2k+2}) = \operatorname{Tr}(X_{i_1}^- X_{j_1}^- \cdots X_{j_s}^- X_{k+1}),$$

$$\operatorname{Tr}(X_{k+1}^j X_{j_1}^- \cdots X_{i_s}^- \bar{X}_{2k+2}) = \operatorname{Tr}(X_{i_s}^- \cdots X_{j_1}^- X_{k+1}),$$

to eliminate the variable $\bar{X}_{2k+2}$, and we obtain an explicit formula.

Let us be content now to extract the trace identities for $n \times n$ matrices $n \leqslant 3$.

We already know that $F_{0,n} = F$, of which we know the explicit formula. We must compute $F_{1,2}$, $F_{1,3}$, $F_{1,4}$, $F_{2,4}$ we have already computed $F_{1,2}$ and $F_{2,4}$; $F_{1,3}$ can be computed using the previous ideas

$$F_{1,3} = 4\operatorname{Tr}(X_1^- X_3^- X_2) - 2\operatorname{Tr}(X_1^- X_3^-)\operatorname{Tr}(X_2)$$

As for $F_{1,4}$ we have

$$\begin{aligned}
F_{1,4} = {} & 4[\operatorname{Tr}(X_4^- X_2 X_1^-)\operatorname{Tr}(X_3) + \operatorname{Tr}(X_1^- X_4^- X_3)\operatorname{Tr}(X_2) \\
& - \operatorname{Tr}(X_1^- X_4^- X_3 X_2) - \operatorname{Tr}(X_1^- X_4^- X_2 X_3) - \operatorname{Tr}(X_2^t X_4^- X_3 X_1^-)] \\
& - \operatorname{Tr}(X_1^- X_4^-)\operatorname{Tr}(X_2)\operatorname{Tr}(X_3) + \operatorname{Tr}(X_1^- X_4^-)\operatorname{Tr}(X_2 X_3)
\end{aligned}$$

From these computations it is easy to extract the formulas fo $G_{k,n}$, $H_{k,n}$ for $n \leqslant 3$, and we leave it to the reader.

10. *Symplectic Invariants*

We consider now the vector space $V = K^{2n}$ endowed with the canonical alternating form:

$$\langle (u_1, u_2, \dots, u_n, v_1, \dots, v_n), (p_1 p_2, \dots, p_n, q_1, \dots, q_n)\rangle = \sum (u_i q_i - p_i v_i).$$

We can proceed exactly as for the orthogonal case in almost all points and, therefore, we will only sketch the steps involved. First, the alternating form identifies V with its dual V^*. Therefore, we identify End(V) with its dual V^*. Therefore, we identify End(V) with $V \otimes V$. Under this identification, the product, trace, and adjoint involution are expressed as follows:

(a) $a \otimes b \cdot c \otimes d = a \otimes \langle b, c\rangle d$,

(b) $\mathrm{Tr}(a \otimes b) = \langle b, a\rangle$,

(c) $(a \otimes b)^* = -b \otimes a$.

The automorphism group of V with its form is the symplectic group $Sp(2n, k)$, and the automorphism group of End(V) with its involution $*$ is $\mathrm{Sp}(2n, K)/\{1, -1\}$.

The theorems on invariants of the symplectic group that are necessary to carry the theory through as in the orthogonal case are the following [16, pp. 167, 168].

FIRST FUNDAMENTAL THEOREM. *Every multilinear symplectic invariant $\mu: V \otimes V \otimes \cdots \otimes V \to K$ of $2k$ vectors is a linear combination of invariants obtained by contractions*

$$v_1 \otimes \cdots \otimes v_{2k} \to \langle v_{i_1}, v_{i_2}\rangle \cdots \langle v_{i_{2k-1}}, v_{i_{2k}}\rangle.$$

SECOND FUNDAMENTAL THEOREM. *Every relation among the invariants $\langle v, w\rangle$ is a consequence of the following basic relations*

$$J_1 = \sum_{\sigma \in \mathscr{S}_{2n+1}} \epsilon(\sigma)\langle x_{\sigma(1)}, y_1\rangle\langle x_{\sigma(2)}, x_{\sigma(3)}\rangle \cdots \langle x_{\sigma(2n)}, x_{\sigma(2n+1)}\rangle$$

(A)
$$J_2 = \sum_{\sigma \in \mathscr{S}_{2n+1}} \epsilon(\sigma)\langle x_{\sigma(1)}, y_1\rangle\langle x_{\sigma(2)}, y_2\rangle\langle x_{\sigma(3)}, y_3\rangle\langle x_{\sigma(4)} x_{\sigma(5)}\rangle \cdots \langle x_{\sigma(2n)}, x_{\sigma(2n+1)}\rangle.$$

. . . .

$$J_{n+1} = \sum_{\sigma \in \mathscr{S}_{2n+1}} \epsilon(\sigma)\langle x_{\sigma(1)}, y_1\rangle\langle x_{\sigma(2)}, y_2\rangle \cdots \langle x_{\sigma(2n+1)}, y_{2n+1}\rangle.$$

One deduces immediately from the first fundamental theorem and the theory already developed the following results.

Call $T(Sp)_{i,2n}$ the ring of invariants of i $2n \times 2n$ matrices under the symplectic group and $S(Sp)_{i,n}$ the $*$ algebra, over $T(Sp)_{i,2n}$, of matrix valued concomitants we have:

THEOREM 10.1. (a) *The ring $T(Sp)_{i,2n}$ is generated, as a K algebra, by the elements* $\operatorname{Tr}(M)$, where *M is a monomial in X_j, X_j^*, $j = 1, \ldots, i$ of degree $\leqslant 2^n - 1$.*

(b) *The algebra $S(Sp)_{i,2n}$ is spanned, as a $T(Sp)_{i,2n}$ module, by the monomials of degree $\leqslant 2^n - 2$ in the X_j, X_j^*, $j = 1, \ldots, i$.*

As for the second fundamental theorem, one proceeds as in Section 8. First, one constructs the formal algebras T^* and S^* endowed with Tr and $*$. These are the same as the ones considered in Section 8 except that now, we will write $X \to X^*$ for the involution to remind ourselves that eventually, we want to compute the formal polynomials into matrices endowed with simplectic involution.

Next, one defines the maps π_s: $T^* \to T(Sp)_{\infty,2n}$, and τ_s: $S^* \to S(Sp)_{\infty,2n}$, of which one wants to compute the kernels. Next, one constructs the polynomial ring in the variables $\langle x_i, x_j \rangle$, $\langle x_i, y_j \rangle$, and $\langle y_i, y_j \rangle$, subject to the antisymmetry laws $\langle x_i, x_j \rangle = -\langle x_j, x_i \rangle$ etc. One consider the space I_m spanned by the monomials of degree m in which $x_1, \ldots, x_m$, $y_1, \ldots, y_m$ appear, and identifies this space with the commutative trace polynomials in the variables $X_j = x_j \otimes y_j$, $j = 1, \ldots, m$. One should note that this identification is not the same as in Section 8, due to the new laws of the symbols $\langle z, w \rangle$.

To establish the second fundamental theorem, we have to translate the relations (A) into trace polynomials, i.e., into elements of I_m. This clearly can be done in various inequivalent ways. The way to describe these translations is to stick to the notations of (A) and use the vectors $y_1, \ldots, y_{2i-1}$, $x_1, \ldots, x_{2n+1}$ to form in all possible ways $n + 1$ decomposable matrices $X_1, \ldots, X_{n-1}$.

Due to the internal symmetries of the two sets of variables y's and x's in the polynomial J_i, it is easy to see that it is sufficient to choose a number h with $0 \leqslant h < i$, write $2i - 1 = 2h + s$ and set

$$X_1 = x_1 \otimes y_1,\ X_2 = x_2 \otimes y_2, \ldots, X_s = x_s \otimes y_s,\ X_{s+1} = y_{s+1} \otimes y_{s+2},$$

$$X_{s+2} = y_{s+3} \otimes y_{s+4}, \ldots, X_{s+h} = y_{s-2h-1} \otimes y_{s+2h},\ X_{s+h+1} = x_{s+1} \otimes x_{s+2},$$

$$X_{s+h-2} = x_{s+3} \otimes x_{s+4}, \ldots, X_{n+i} = x_{2n} \otimes x_{2n-1}.$$

With these notations, J_i gives rise to a trace polynomial denoted $F^i_{h,n}(X_1, \ldots, X_{n+i})$.

The polynomials thus defined have the following properties:

(a) $F^i_{h,n+1}$ is invariant under a permutation of the variables acting separately on the three sets $X_1, \ldots, X_s$; $X_{s+1}, \ldots, X_{s+h}$; $X_{s+h+1}, \ldots, X_{n+i}$.

(b) $F^i_{h,n+1}$ is invariant under the exchange of X_j with X_j^* for $j > s$.

The proof of (b) is obtained by noticing that $(a \otimes b)^* = -b \otimes a$. Hence, the exchange of X_j with X_j^* involves a transposition and a change of sign, the formula for J_i gives then the result.

Therefore, we can write

$$F^i_{h,n}(X_1, \ldots, X_{n+1}) = F^i_{h,n}(X_1, \ldots, X_s, X^+_{s+1}, \ldots, X^+_{n+i}),$$

where as usual, $X^+ = (X + X^*)/2$.

Finally, reasoning as we did for the orthogonal group, we deduce from each $F^i_{h,n}$ three different noncommutative trace identities by singling out a variable out of each of the three sets $X_1, \ldots, X_s$; $X_{s+1}, \ldots, X_{s+h}$; $X_{s+h+1}, \ldots, X_{n+i}$.

$$F^i_{h,n} = \mathrm{Tr}(G^i_{h,n} \cdot X_s) = \mathrm{Tr}(H^i_{h,n} \cdot X_{s+h}) = \mathrm{Tr}(K^i_{h,n} \cdot X_{n+i}).$$

Finally, a procedure totally similar to the one of Section 8, 8.2, and 8.4 gives the second fundamental theorem for symplectic invariants.

THEOREM 10.2. (a) Ker π_s *is the T-ideal of* T^* *generated by the polynomials*

$$F^i_{h,n}(X_1, \ldots, X_{n+i})$$

(b) Ker τ_s *is the T-ideal of* S^* *generated by the polynomials*

$$G^i_{h,n}, H^i_{h,n}, K^i_{h,n}.$$

We want to make only one explicit computation for the minimal identities. The minimal noncommutative identities are, of course, of degree n, and obtained from $F^1_{o,n}$. A priori, one has two identities $G^1_{o,n}$ and $K^1_{o,n}$, in reality, for this case, one sees that the variable y_1 is not privileged, since it may be paired with any other variable, hence, we deduce a unique noncommutative identity of degree n. Its computation is possibly made simpler by changing notations and writing $X_1 = x_1 \otimes y_1$, $X_2 = x_2 \otimes y_2, \ldots, X_{n+1} = x_{n+1} \otimes y_{n+1}$, and $J_1 = \sum \epsilon(\sigma)\langle\ , y_{n+1}\rangle\langle\ ,\ \rangle \cdots \langle\ ,\ \rangle$. the blanks being filled in all possible ways by the remaining symbols. First, transposing two symbols appearing in a single bracket, one obtains the same term, hence, collecting all such terms a factor 2^n.

Normalizing by dividing for this factor, one has a polynomial that is obtained as a sum of pure monomials in which in each braket, an x is always paired with a y in the order $\langle x, y\rangle$.

The impure monomials are obtained from these on substituting some variables X_j, $j < n + 1$ with X_j^*. Finally, one sees that, after taking out X_{n+1}, we obtain the polynomial that is formally equal to $G(X_1,\ldots, X_n)$ of Section 4, except now, the variables are taken symmetric; i.e., G is in the variables $X_j^+ = (X_j + X_j^*)/2$, $j = 1,\ldots, n$. Therefore, we obtain the analog of 4.4(c).

PROPOSITION 10.3. *The minimal noncommutative trace identity for $2n \times 2n$ symplectic matrices is $G(X_1^+,\ldots, X_n^+)$. It is obtained by full polarization of the Pfaffian polynomial of a symmetric matrix.*

Proof. We recall that the Pfaffian polynomial of a is obtained as the characteristic polynomial by the expression $Pf(x \cdot 1 - a)$. The proof of 10.3 is trivial by the uniqueness of the minimal noncommutative trace identity, as in 4.4.

11. *Unitary Invariants*

Let $V = C^n$ be the n-dimensional vector space over the complex numbers, and endow V with the canonical Hilbert space structure:

$$((a_1,\ldots, a_n), (b_1,\ldots, b_n)) = \sum a_i \bar{b}_i .$$

The group of linear transformations compatible with the given Hermitian form is the unitary group $U(n)$. The algebra End(V) is endowed with the canonical involution $A \to A^*$, which in matrix form is $A^* = \bar{A}^t$. The theory of invariants for $U(n)$ is slightly twisted with respect to the orthogonal group theory because of the two following facts:

(i) $U(n)$ is not an algebraic group.

(ii) The involution $*$ on End(V) is of the 2nd kind, in fact $\alpha^* = \bar{\alpha}$ for any scalar α.

The way to deduce the invariant theory for $U(n)$ from the one already developed is based on the following remarks.

(iii) The automorphism group of End(V), as algebra with involution over C, is $\Gamma = U(n)/T$, $T = \{\alpha \in C^* \mid |\alpha| = 1\}$.

(iv) The automorphism group of End(V) as algebra with involution over R is the semidirect product of Γ with $Z/(2) = \{1, \tau\}$ acting on Γ as $\{a\}^\tau = \{a^t\}^{-1}$ ($\{a\}$ the class of $a \in U(n)$ modulo T).

(v) The Zariski closure of $U(n)$ in the algebraic variety $Gl(n, C)$ is $Gl(n, C)$ itself [16, p. 177].

This last assertion is essentially the unitarian trick of H. Weyl.

We consider $\mathrm{End}(V)$ as a real vector space and construct $\mathrm{End}(V) \otimes_R C$ with the involution $(a \otimes \alpha)^* = a^* \otimes \alpha$.

Then, $\mathrm{End}(V) \otimes_R C$ as a $*$ algebra is isomorphic, over C, to $\mathrm{End}(V) \oplus \mathrm{End}(V) = (C)_n \oplus (C)_n$ with the involution $(a, b)^* = (b^t, a^t)$, its automorphism group (as $*$ algebra over C) is easily described. It is the semidirect product of $Z/(2)$ with $PGl(n, C)$; an element $c \in Gl(n, C)$ induces the inner automorphism $(a, b) \to (cac^{-1}, (c^t)^{-1}bc^t)$ and $Z/(2)$ acts mapping (a, b) into (b, a) (and on $PGl(n, C)$ sending c into $(c^t)^{-1}$).

In particular, we consider $U(n) \subset Gl(n, C)$.

Let us restrict the action of $Gl(n, C)$ on $(C)_n \otimes_R C \simeq (C)_n \oplus (C)_n$ to $U(n)$. Then, the map of $(C)_n \to (C)_n \oplus (C)_n$, sending a into $(a, \bar{a})$, is compatible with the $U(n)$ actions, where $(C)_n$ has the usual action of conjugation: $u \in U(n)$, $a \in (C)_n$, u acts on a giving uau^{-1}.

If we sum all the remarks, we see that:

(a) The ring of complex valued polynomials on $\oplus_j ((C)_n \otimes_R C)$ is isomorphic to the ring of polynomials on the real vector space $\oplus_j (C)_n$.

(b) The invariants under $U(n)$ coincide with the invariants under $Gl(n, C)$.

To conclude, we have to see what is the $Gl(n, C)$ module $M = \oplus_j ((C)_n \oplus (C)_n)$.

LEMMA 11.1. *The given $Gl(n, C)$ representation on M is isomorphic to the canonical representation, $(C)_n^{2j}$ on $2j$ tuples of matrices, by sending $((a_1, b_1), (a_2, b_2), \ldots, (a_j, b_j))$ into*

$$((a_1, b_1^t), (a_2, b_2^t), \ldots, (a_j, b_j^t)).$$

Proof. This is trivial by the formula

$$((c^{-1})^t bc^t)^t = cb^t c^{-1}.$$

Summing up all these remarks and recalling the formula for the embedding of $(C)_n$ into $(C)_n \otimes C \simeq (C)_n \oplus (C)_n$, we have:

THEOREM 11.2. (a) *The unitary invariants $TU_{j,n}$ of j complex matrices $X_1, \ldots, X_j$ are polynomials in the elements* $\mathrm{Tr}(W_{i_1}, \ldots, W_{i_s})$, *where $W_j = X_j$, or $W_j = X_j^*$. As a ring, it is isomorphic to the ring of $Gl(n, C)$ invariants of $2j$ matrices $X_1, \ldots, X_j, X_1^*, \ldots, X_j^*$.*

(b) *The unitary matrix valued concomitants $SU_{j,n}$ of j complex matrices $X_1, \ldots, X_j$ are an algebra generated over $TU_{j,n}$ by the elements X_i, X_i^*, $i = 1, \ldots, j$. As an algebra, it is isomorphic to the algebra of $Gl(n, C)$ concomitants of $2j$ matrices X_i, X_i^*.*

We remark that this theorem contains implicitely also the second fundamental theorem, it describes formally the rings TU and SU as isomorphic to rings already considered.

The involution that, as a function, is induced by $*$, can be thought of formally as the involution of the 2nd kind obtained as $*$ on C and as the exchange of the two distinct variables X_i with X_i^*, $i = 1, \ldots, j$.

12. *Mixed Invariants and Concomitants*

We consider now the problem of studying the simultaneous invariants of matrices, vectors and covectors. For simplicity, we will restrict ourselves to $Gl(n, K)$ the other cases being similar. Since the study of relative invariants of $Gl(n, K)$ is equivalent to the study of absolute invariants of $Sl(n, K)$, we will refer to this group when talking about invariants.

We consider, thus, the ring $T_{k,h,t}$ of invariants of k $n \times n$ matrices, h, n-vectors, and t, n-covectors, and we have:

THEOREM 12.1. *$T_{k,h,t}$ is generated by the following elements:*

(a) *Invariants of k matrices alone.*

(b) *Scalar products $\langle \varphi_j, Mv_i \rangle$, where M is a monomial in the given matrices, φ_j a covector, v_i a vector.*

(c) *Brackets $[M_1 v_{i_1}, M_2 v_{i_2}, \ldots, M_n v_{i_n}]$, where the M_i's are monomials in the matrices and the v_j's are vectors.*

(d) *Brackets $[M_1 {}^t\varphi_{i_1}, M_2 {}^t\varphi_{i_2}, \ldots, M_n {}^t\varphi_{i_n}]$, where the M_i's are monomials in the matrices and the φ_j's are covectors.*

Proof. It is immediately verified that, upon polariztion and restitution of variables, an expression involving the elements of the previous type remains of the same type. Therefore, it is sufficient to deal with multilinear invariants.

Assume that the given multilinear invariant depends on i matrices

$A_1, A_2, \ldots, A_i$, j vectors and z covectors $\varphi_1, \ldots, \varphi_z$. Hence, it is described by a multilinear map

$$\operatorname{End}(V) \otimes \underbrace{\operatorname{End}(V) \otimes \cdots \otimes \operatorname{End}(V)}_{i\text{-Times}} \otimes \underbrace{V \otimes \cdots \otimes V}_{j\text{-Times}} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{z\text{-Times}} \to K.$$

By the identification, $\operatorname{End}(V) \simeq V^* \otimes V$, we have an invariant

$$\psi\colon V^{\otimes i+j} \otimes V^{*\otimes i+z} \to K.$$

The classification of such invariants [16, p. 45] ensures that ψ is a linear combination of invariants of one of the following two types:

(1) $[v_{i_1}, \ldots, v_{i_n}][v_{h_1}, \ldots, v_{h_n}] \cdots [v_{t_1}, \ldots, v_{t_n}]\langle \varphi_{u_1}, v_{s_1}\rangle \cdots \langle \varphi_{u_r}, v_{s_r}\rangle$, where $i_1, \ldots, i_n$ $h_1, \ldots, h_n \cdots t_1, \ldots, t_n$ $s_1, \ldots, s_r$ is a permutation of the $i + j$ vector indices and $u_1, \ldots, u_r$ is a permutation of the $i + z$ covector indices.

(2) The same expression where the brackets of vectors are replaced by brackets of covectors.

One can assume that one does not have at the same time brackets of vectors and of covectors due to the relation:

$$[v_1, v_2, \ldots, v_n][\varphi_1, \ldots, \varphi_n] = \det(\langle \varphi_j, v_i\rangle).$$

Let us consider the type (1), type (2) being perfectly similar.

Let us analyze the factor $[v_{i_1}, v_{i_2}, \ldots, v_{i_n}]$. If all the indices $i_1, \ldots, i_n$ are vector indices (and not matrix indices), we save this factor and pass to the rest of the product; otherwise, say i_1 is one of the matrix indices. Let us write $A_{i_1} = \varphi_{i_1} \otimes v_{i_1}$, φ_{i_1} is necessarily paired in a scalar product with a vector $\langle \varphi_{i_1}, v_{t_1}\rangle$.

If t_1 is a vector index we have, setting $v'_{i_1} = \langle v_{t_1}, \varphi_{i_1}\rangle v_{i_1}$ that $v'_{i_1} = A_{i_1} v_{t_1}$ and

$$\langle \varphi_{i_1}, v_{t_1}\rangle [v_{i_1}, \ldots, v_{i_n}] = [v'_{i_1}, v_{i_2}, \ldots, v_{i_n}],$$

and we continue on the other indices.

Otherwise, we keep matching the matrix indices

$$\langle \varphi_{i_1}, v_{t_1}\rangle\langle \varphi_{t_1}, v_{t_2}\rangle \cdots$$

until we hit a vector index t_k and then we have

$$\langle \varphi_{i_1}, v_{t_1}\rangle\langle \varphi_{t_1}\ \ v_{t_2}\rangle \cdots \langle \varphi_{t_{k-1}}, v_{t_k}\rangle [v_{i_1}, v_{i_2}, \ldots, v_{i_n}] = [M v_{t_k}, v_{i_r}, \ldots, v_{i_n}],$$

where $M = A_{i_1} A_{t_1} \cdots A_{t_{k-1}}$.

One preceeds similarly for all the vectors in the brackets. As for the remaining scalar products, we have a similar procedure of matching the matrix indices and one easily sees that in doing this, one obtains the factors of type $\langle \varphi, Mv \rangle$ and $\operatorname{Tr}(M)$, where M is a monomial in the matrices. The second case is, as we have said, similar except that matching the matrix indices with a covector φ, one obtains a monomial in the transposed matrices. (Of course one could have considered invariants $\langle M^t\varphi, Nv \rangle$, but they are just $\langle \varphi, MNv \rangle$.)

We describe now the concomitants.

Remark first that, over $Sl(n, K)$, we have the identifications $\wedge^n V \simeq K$, $\wedge^n V^* \simeq K$, $\wedge^{n-1} V^* \simeq V$, $\wedge^{n-1} V \simeq V^*$. With these identifications the "brackets" $(\wedge^{n-1} V) \otimes V \to K$, $(\wedge^{n-1} V^*) \otimes V^* \to K$, are identified with the usual brackets $\langle\ ,\ \rangle$, $V \otimes V^* \to K$. Keeping these identifications in mind, plus the usual identification $\operatorname{End}(V) \simeq V^* \otimes V$, we formulate the main result.

THEOREM 12.2 (a) *The vector valued concomitants form a module over* $T_{k,h,t}$ *spanned by the elements* $M_i v_j$, $M_1{}^t\varphi_{i_1} \wedge M_2{}^t\varphi_{i_2} \wedge \cdots \wedge M^t_{n-1}\varphi_{i_{n-1}}$ *where the* M_j*'s are monomials in the matrices, the* v_j*'s vectors and the* φ_i*'s are covectors.*

(b) *The covector valued concomitants form a module over* $T_{k,h,t}$ *spanned by the elements* $M_i{}^t\varphi_j$, $M_1 v_{i_1} \wedge M_2 v_{i_2} \wedge \cdots \wedge M_{n-1} v_{i_{n-1}}$, *where the* M_j*'s are monomials in the matrices, the* φ_j*'s are covectors, and the* v_i*'s are vectors.*

(c) *The matrix valued concomitants form an algebra over* $T_{k,h,t}$ *generated by the elements* A_i, $\varphi_h \otimes v_j$, $(M_1 v_{i_1} \wedge \cdots \wedge M_{n-1} v_{i_{n-1}}) \otimes v_j$ *and* $\varphi_h \otimes (M_1{}^t\varphi_{i_1} \wedge M_2{}^t\psi_{i_2} \wedge \cdots \wedge M^t_{n-1}\varphi_{i_{n-1}})$, *where, as usual, the* A_i*'s are the matrices, the* M_j*'s monomials in the matrices, the* v_i*'s are vectors, and the* φ_j*'s are covectors.*

Proof. (a) Let us introduce an extra variable covector φ. If g is a vector valued concomitant, then $\langle \varphi, g \rangle$ is an invariant linear in φ.

Furthermore, $\langle \varphi, g \rangle = 0$ if and only if $g = 0$ since φ is an independent variable.

By the classification theorem (12.1), $\langle \varphi, g \rangle$ is a polynomial in the invariants of types (a)–(d).

Now we make a case analysis. Clearly, φ appears either in a factor of type (b) or in one of type (d). In the first case, we pull out the factor Mv and see g as this factor times an invariant; in the second case, we arrive to the factor $M_1{}^t\varphi_{i_1} \wedge M_2{}^t\varphi_{i_2} \wedge \cdots \wedge M_{n-1}^t\varphi_{i_{n-1}}$ using the identifications

$$\begin{aligned}[M_1{}^t\varphi_{i_1} \wedge \cdots \wedge M_{n-1}^t\varphi_{i_{n-1}} \wedge M_n{}^t\varphi] &= \langle M^t\varphi, M_1{}^t\varphi_{i_1} \wedge \cdots \wedge M_{n-1}^t\varphi_{i_{n-1}}\rangle \\ &= \langle \varphi, M(M_1{}^t\varphi_{i_1} \wedge \cdots \wedge M_{n-1}^t\varphi_{i_{n-1}})\rangle \\ &= \langle \varphi, (M^*M_1{}^t)\,\varphi_{i_1} \wedge \cdots \wedge (M^*M_{n-1}^t)\,\varphi_{i_{n-1}}\rangle.\end{aligned}$$

M^* the matrix defined formally by $(\det M)\cdot(M^t)^{-1}$. The Cayley–Hamilton theorem for M^t shows that M^* is a polynomial in M^t and $\mathrm{Tr}((M^t)^i)$.

(b) This is dual to (a), so we do not repeat the proof.

(c) In this case, we introduce an extra matrix variable as in 2.1, and associate to the matrix concomitant g the invariant $\mathrm{Tr}(gz)$, from which g can be recovered. Then, we make, as in (a), a case analysis of the expression of $\mathrm{Tr}(gz)$ in terms of the basic invariants.

If z appears in a factor $\mathrm{Tr}(M)$, we single out in g a term $\alpha \cdot N$, α an invariant, N a monomial in the matrix variables. If z appears in a monomial M in a factor $\langle\varphi, Mv\rangle$, we single out in g a term $\alpha \cdot N$, α an invariant and N a monomial in the matrix variable plus the matrix concomitant $\varphi \otimes v$. If z appears in a monomial, say M_1, in a bracket $[M_1v_{i_1}, M_2v_{i_2}, \ldots, M_nv_{i_n}]$, we can set

$$[M_1v_{i_1}, M_2v_{i_2}, \ldots, M_nv_{i_n}] = \mathrm{Tr}(Nz),$$

where N is a monomial in the matrix variables plus the matrix concomitant

$$(M_2v_{i_1} \wedge \cdots \wedge M_nv_{i_n}) \otimes v_{i_1}.$$

We have not bothered to write explicitely the finiteness statements for all the various cases, since they are obvious consequences of Theorem 3.4.

Similarly, one can write the second fundamental theorem in all cases considered by taking the second fundamental theorem for vectors and covectors and using the method of matching the matrix indices.

The resulting theorem would be very messy, in reality, what one proves, is that we have the following "free additing category with exterior product."

The objects are: The vectors M, the covectors M^*, the exterior powers of them: $\bigwedge^i M$, $\bigwedge^j M^*$. Freely generated by: vectors $\{u_i \in M\}_{i=1,\ldots,\infty}$; covectors $\{\xi_j \in M^*\}_{j=1,\ldots,\infty}$; matrices $\{X_k \in \hom(M, M)\}_{k=1,\ldots,\infty}$, with all the formal operators:

(a) The pairing: $(x, \xi) \to x \otimes \xi$, $M \times M^* \to \hom(M, M)$;

(b) Transposition: $X \to X^*$, $\hom(M, M) \to \hom(M^*, M^*)$.

(c) Trace: $\hom(M, M) \to \Lambda^0 M$.

(d) Evaluation: $(x, \xi) \to \langle x, \xi \rangle$; $M \times M^* \to \Lambda^0 M$.

(e) Object isomorphism: $\Lambda^h M \simeq \Lambda^{n-h} M^*$, $\Lambda^0 M \simeq \Lambda^0 M^*$.

Then, one has to write the usual formal axioms and then, the resulting rings and modules are the ones under consideration.

This is the form that the second fundamental theorem takes. The method of proof is by full polarization and the method of matching the matrix indices in the resulting formulas taken from the second fundamental theorem as proved by Weyl. We hope to give a more detailed account of these ideas elsewhere, they seem to be related to the concept of Cayley algebra introduced in [2].

II. Representations of *-Algebras

13. **-Algebras and Representations*

We want to extend here the theory of Artin on representations of algebras [1] (see also [7, 8]). The case treated by Artin is the one relative to the group $GL(n, K)$. Here, we treat the other classical groups, for which a similar theory can be developed.

We are not going to discuss the theory in the generality that perhaps it deserves. In particular, we will stick often to our assumption that the fields under consideration have characteristic 0, although it will be apparent from the proofs that most results extend to characteristic $p > 0$ or even to the characteristic free case.

We hope to return to this point elsewhere. We recall that a ring with involution is, a ring R, with a map $a \to a^*$ satisfying:

(i) $(a + b)^* = a^* + b^*$,

(ii) $(ab)^* = b^*a^*$,

(iii) $a^{**} = a$.

Rings with involutions form a category, if we insist that a map $\varphi: R \to S$ between such rings is a homomorphism preserving *.

We will consider, rather than rings, algebras with involutions over a field K (or more generally over a commutative ring Λ). We will refer to such rings as * algebras. The map * may be either the identity or an automorphism of order 2 on K. In practice, we will consider this second case only when $K = C$ and $\alpha^* = \bar{\alpha}$ on C.

The relation among involutions and forms is the usual one. Given a finite-dimensional vector space V over a field K, and a nondegenerate ϵ symmetric form ($\epsilon = \pm 1$), the algebra $\mathrm{End}(V)$ is equipped with a canonical involution having the property that $(a^*v, w) = (v, aw)$.

If $\epsilon = 1$, we will refer to the involution as transposition otherwise as symplectic involution.

Similarly, when $K = C$, and V is equipped with a nondegenerate Hermitian form, $\mathrm{End}(V)$ is endowed with an involution called adjoint.

We will study the following objects.

Given a vector space V with a form of one of the previous three types, and a * algebra R, a * representation of R in V will be a * map φ: $R \to \mathrm{End}(V)$.

In the language of modules, φ gives rise to an R module structure on V, the hypothesis that φ is a * map becomes

$$(r^*v, w) = (v, r\,w), \qquad \text{for all } v, w \in V, r \in R.$$

We will speak, respectively, of an orthogonal, symplectic, or unitary representation according to the nature of the form on V symmetric, antisymmetric, or Hermitian.

We have the natural notion of equivalence, for representations of the same type.

Two representations φ: $R \to \mathrm{End}(V)$, ψ: $R \to \mathrm{End}(W)$ will be called equivalent, if there exists an isometry u: $V \to W$ for which

$$u(\varphi(r)v) = \psi(r)\,u(v).$$

In the language of R modules, u is an R-linear isometry. In this situation, we will also say that the two modules are isometric.

In particular the group of isometries of a space V acts on the set of * representations of R and the orbits of this action are the equivalence classes of representations.

To give a concrete example, which is also fundamental for the theory, we specialize to the case that R is a free *-algebra.

We will consider three different types of free algebras. Given a set I, construct on the category of * algebras the three set valued functors:

$$R \hookrightarrow R^I, \qquad R \hookrightarrow R^{+I}, \qquad R \hookrightarrow R^{-I}.$$

R^+ and R^- will always denote the sets of symmetric, repectively, antisymmetric elements of R.

Each of the three given functors is representable.

The representing algebras are constructed in this way.

(a) The free algebra $K\{x_i, y_i\}_{i\in I}$ with the involution assigned by the rule $x_i^* = y_i$. This will be called the free *-algebra in the variables x_i, we will write x_i^* rather than y_i and thus, $K\{x_i, y_i\}$ will be denoted $K\{x_i, x_i^*\}$.

(b) The free algebra $K\{x_i\}_{i\in I}$ with the involution defined by $x_i^* = x_i$. This will be called the free *-algebra in the symmetric variables x_i. For convenience of notation, we will write s_i rather than x_i.

(c) The free algebra $K\{x_i\}_{i\in I}$ with the involution defined by $x_i^* = -x_i$. This will be called the free *-algebra in the antisymmetric variables x_i. For convenience of notations, we will write t_i rather than x_i.

There is a very simple relation between these algebras as soon as $\frac{1}{2} \in K$. In this case, the canonical decomposition $R = R^+ \oplus R^-$ gives rise to the canonical isomorphism

$$K\{x_i, x_i^*\} = K\{s_i\} \coprod K\{t_i\}$$

where $\coprod$ denotes the free product and

$$s_i = (x_i + x_i^*)/2, \qquad t_i = (x_i - x_i^*)/2.$$

Giving a representation φ of $K\{x_i, x_i^*\}$ in $\mathrm{End}(V)$ is equivalent to assigning the elements $a_i = \varphi(x_i) \in \mathrm{End}(V)$.

The equivalence of two such representations $\{a_i\}$, $\{h_i\}$ corresponds to the existence of an isometry u with $ua_iu^{-1} = h_i$, for all i.

We see, therefore, that we are considering exactly the actions studied in Part I, for which the invariant theory has been developed.

Thus, our task will be to relate the invariant theory to the problem of equivalence of representations. This is our final goal.

We need some further notations and generalities.

Let us give a vector space V, without any form, and a map $\varphi: R \to \mathrm{End}(V)$, with R a *-algebra.

We can deduce, from this map, an orthogonal and a symplectic representation as follows.

(a) Construct the dual representation $\varphi^*: R \to \mathrm{End}(V^*)$ by the formula $\varphi^*(r) = \varphi(r^*)^t$.

(b) Construct the space $W = V \oplus V^*$ and the direct sum representation, $\varphi: R \to \mathrm{End}(W)$, of φ and φ^*.

(c) Equip W with the canonical ϵ-symmetric form:

$$\langle (v, \xi), (w, \zeta) \rangle = \tfrac{1}{2}[\langle v, \zeta \rangle + \epsilon \langle w, \xi \rangle].$$

One readily verifies that φ is in each case a $*$-map. We will denote the orthogonal and symplectic representation so defined by φ^h, φ^s, respectively.

In our future work, it will be important to endow, in the previous situation, directly V with a form for which φ is a $*$ representation. Such a form will be called a compatible form. In some cases, such a form may not exist at all, in other cases, many inequivalent compatible forms may be constructed.

In the next section, an important special case will be studied for which one has existence and uniqueness. For the moment, let us make one further general remark. Given a left R module V we construct a right R module $V^{\#}$ by the formula:

$$vr = r^* v.$$

Next consider the abelian group $V^{\#} \otimes_R V = T$.

If $B: V \times V \to K$ is a compatible form we have an induced map $\bar{B}: T \to K$ making the diagram

$$\begin{tikzcd} V \times V \arrow[rr] \arrow[dr, "B"'] & & V^{\#} \otimes_R V \arrow[dl, "\bar{B}"] \\ & K & \end{tikzcd}$$

commutative.

The group $V^{\#} \otimes_R V$ is equipped with an involutory map $\tau: v \otimes w \to w \otimes v$; B will be ϵ-symmetric if and only if $\bar{B} \cdot \tau = \epsilon \bar{B}$.

14. *Semisimple *-Algebras*

We want to specialize the general discussion of the previous paragraph, to the case that R is a semisimple $*$-algebra. The theorems that we will prove are mostly well known, in some cases even in a much greater generality, we have chosen to reproduce them here for lack of a coherent reference taking our point of view of representation theory.

Let R be a semisimple Artinian $*$-algebra,

$$R = \bigoplus_{i=1}^{m} R_i\,, \qquad R_i \text{ a simple algebra.}$$

The involution $*$ induces a map of order 2 on the set of simple factors R_i; therefore, we can subdivide this set in the factors R_i that are fixed under $*$, and the remaining ones exchanged:

$$R = \bigoplus_{i=1}^{h} R_i \bigoplus_{i=1}^{t} (S_j \oplus S_j{}^*), \qquad (R_i = R_i{}^*).$$

$S_j{}^*$ is isomorphic, via $*$, to $S_j{}^0$ (the opposite algebra).

The $*$ algebra $S_j \oplus S_j{}^*$ is thus isomorphic to the $*$ algebra $S_j \oplus S_j{}^0$ with the exchange involution $(a, b)^* = (b, a)$.

We want to analyze modules over R.

THEOREM 14.1. *Let R be a simple $*$-algebra, let V be an irreducible R module. $\Delta = \mathrm{End}_R(V)$, the centralizer of R.*

(a) *There exists an involution $*$ on Δ and a nonzero biadditive map $B\colon V \times V \to \Delta$ such that:*

(i) $B(r^*v, w) = B(v, rw)$ *for all* $v, w \in V$, *and* $r \in R$.

(ii) $B(dv, w) = dB(v, w)$; $B(v, dw) = B(v, w)\, d^*$ *for* $v, w \in V$, $d \in \Delta$.

(iii) $B(v, w) = \epsilon B(w, v)^*$, ϵ *fixed, and* $\epsilon = \pm 1$.

(b) *Condition* (i) *implies that B is nondegenerate, i.e.,* $B(v, w) = 0$ *for all* $v \in V$ *implies* $w = 0$ *and symmetrically for all* $w \in V$ *implies* $v = 0$.

(c) *The involution on Δ and the form B are unique up to the following changes.*

If $*$, $\#$ are two involutions on Δ, B_1, B_2 the corresponding forms on V, there is an element $a \in \Delta$, $a \neq 0$ and $a^* = \epsilon a$ ($\epsilon = \pm 1$), such that:

(iv) $b^\# = ab^*a^{-1}$, $B_2(v, w)a = B_1(v, w)$ *for all* $v, w \in V$.

(d) *If Δ is finite dimensional over its center K, every involution on Δ, coinciding on K with the automorphism induced by the involution on R, is* *obtained in the way described before. Provided that, given* $\epsilon \in K$, *if* $\epsilon\epsilon^* = 1$, *then* $\epsilon = \pm(\alpha/\alpha^*)$, $\alpha \in K$.

(e.g., if * is the identity of K or conjugation of the complex numbers.)

Proof. (a) With the notations of Section 13, consider the group $V^\# \otimes_R V$ with its involutory map τ.

$$V \simeq R/I, \qquad I \text{ a maximal left ideal, as an } R \text{ module.}$$

Similarly, $V^\# \simeq R/I^*$ and I^* is a maximal right ideal. Thus, $V^\# \otimes_R V \simeq V/I^*V \simeq V^\#/V^\# I$ as abelian groups. Consider V as a vector space over Δ, we have an induced vector space structure on $V^\# \otimes_R V$, which makes it isomorphic to V/I^*V. Since I^* is a maximal right ideal, V/I^*V is one-dimensional over Δ.

Similarly, we can act on the first factor $V^\#$.

We obtain, therefore, two structures of vector space over Δ for $V^\# \otimes_R V$, both one-dimensional.

Denoting $\circ$ and $\square$ the previously defined operations, one has:

$$d \circ (v \otimes w) = v \otimes dw, \qquad d \square (v \otimes w) = dv \otimes w$$

$$\tau(d \circ (v \otimes w)) = d \square \tau(v \otimes w)$$

$$d_1 \circ (d_2 \square (v \otimes w)) = d_2 v \otimes d_1 w = d_2 \square (d_1 \circ (v \otimes w)).$$

Let us now analyze τ. We may have three possibilities:

(1) $\tau(m) = m$ for every $m \in V^\# \otimes_R V$.

(2) $\tau(m) = -m$ for every $m \in V^\# \otimes_R V$.

(3) None of the above.

We remark first: In case (1) and (2), Δ is commutative and * is the identity on the center Δ of R.

Let us show it for (1), (2) being similar.

From $\tau(m) = m$ for all m, we deduce $d \circ m = \tau(d \circ m) = d \square \tau(m) = d \square m$. The two operations $\circ$, $\square$ coincide, and we denote them by dm. From $d_1 \circ (d_2 \square m) = d_2 \square (d_1 \circ m)$ we deduce $d_1 d_2 m = d_2 d_1 m$, and so Δ is commutative and thus, identified to the center of R. Finally, if $r \in R$, $(r^* v) \otimes w = v \otimes rw$, if, furthermore, $r \in \Delta$, $(r^* v) \otimes w = r^* \cdot (v \otimes w)$, and $v \otimes rw = r \cdot (v \otimes w)$, hence, $r^* = r$.

Any choice of a basis element $m \neq 0$ in $V \otimes_R V$, identifies this space to Δ and thus, the canonical map $V \times V \to V^{\#} \otimes_R V$ gives rise to a form $B: V \times V \to \Delta \simeq V \otimes_R V$.

In case (1), B is symmetric, while in case (2), it is antisymmetric. The involution on Δ is, as we have noticed, the identity. In these two cases, the uniqueness of B is clear, every compatible form factors through $V^{\#} \otimes_R V$, so it amounts to chosing an isomorphism of $V^{\#} \otimes_R V$ with Δ. This is unique up to a nonzero scalar $a \in \Delta$, $a = a^*$, since $*$ is the identity on Δ.

For case (3), consider the elements $w = \tau(m) + m$ and $v = \tau(m) - m$ in $V^{\#} \otimes_R V$. We have $\tau(w) = w$, while $\tau(v) = -v$. In our present case, we can choose w and v to be nonzero.

Let us restrict our attention to the choice of w with $\tau(w) = w \neq 0$.

Define a map $*$ on Δ by the formula: $\tau(d \circ w) = d^* \circ w$. We have:

$$d \circ w = \tau^2(d \circ w) = \tau(d^* \circ w) = d^{**} \circ w$$

$$\begin{aligned}(bd)^* \circ w &= \tau((bd) \circ w) = b \square \tau(d \circ w) = b \square (d^* \circ w)\\ &= d^* \circ (b \square w) = d^* \circ (b \square \tau(w)) = d^* \circ (\tau(b \circ w)) = d^* \circ (b^* \circ w).\end{aligned}$$

Thus, $d^{**} = d$ and $(bd)^* = d^*b^*$ for all $b, d \in \Delta$. The $*$ is, therefore, an involution.

We use w finally to identify $V \otimes_R V$ with Δ by the map $d \to dw$. Let us call j the inverse of this map, one readily verifies that $j(d \circ u) = dj(u)$ and $j(d \square u) = j(u)d^*$, this proves (i) and (ii) for the form $V \times V \to V \otimes_R V \xrightarrow{j} \Delta$.

Part (iii) is also clear with $\epsilon = 1$ by the fact that $\tau(w) = w$ and the definition of $*$ on Δ.

Everything we have done for the choice of a τ symmetric element w could have been repeated with the choice of v with $\tau(v) = -v$.

In this case, the involution is defined by $\tau(d \circ w) = -d^* \circ w$, the resulting ϵ is -1.

(b) If $B(v, w) = 0$ for all $w \in V$ and we had $v \neq 0$, choosing $v' \in V$, we can find, by the irreducibility of V, an $r \in R$ with $v' = rv$. Hence, $B(v', w) = B(v, r^*w) = 0$. We could deduce that B is 0.

(c) Assume that B is a form with the prescribed requirements. By the universal property of $V \otimes_R V$, we have a map $\bar{B}$: $V^{\#} \otimes_R V \to \Delta$ such that

(1) The diagram

$$\begin{tikzcd} V \times V \arrow[rr] \arrow[dr, "B"'] & & V^{\#} \otimes_R V \arrow[dl, "\bar{B}"] \\ & \Delta & \end{tikzcd}$$

is commutative.

(2) $\bar{B} \cdot \tau = \epsilon \bar{B}$

(3) $\bar{B}(d \circ u) = d\bar{B}(u)$, $\bar{B}(d \square u) = \bar{B}(u)d^*$.

In particular, since $B \neq 0$, $\bar{B}$ is an isomorphism between the vector space $V^{\#} \otimes_R V$ with the operation $\circ$ and Δ. Let $w \in V^{\#} \otimes_R V$ be such that $\bar{B}(\bar{w}) = 1$.

Since $\bar{B}(\tau(\bar{w})) = \epsilon\bar{B}(\bar{w}) = \epsilon$, we have $\tau(\bar{w}) = \epsilon\bar{w}$.

Formula (3) tells us that the involution on Δ is necessarily the one deduced from $\bar{w}$, according to the procedure used in (a). Finally, to compare it with the previously chosen involution and form relative to w, let us indicate $*$, $\#$ the involutions relative to w and $\bar{w}$, respectively. Let us assume $\tau(w) = \epsilon w$, $\tau(\bar{w}) = \bar{\epsilon}\bar{w}$, ϵ, $\bar{\epsilon}$ being 1 or -1.

Then, $\bar{w} = a \circ w$, and $\bar{\epsilon}\bar{w} = \tau(\bar{w}) = \epsilon a^* \circ w$; hence, $a = (\bar{\epsilon}\epsilon)\, a^*$. Moreover, $\tau(d \circ \bar{w}) = \bar{\epsilon} d \circ \bar{w}$ and $\tau(d \square \bar{w}) = \tau((da) \circ w) = \epsilon(da)^* \circ w = \epsilon a^* d^* a^{-1}\bar{w}$, hence, $d^* = ad^*a^{-1}$. Finally, calling the two forms B_1, B_2, we have $B_2(v, z)\bar{w} = v \otimes z = B_1(v, z)w$, thus $B_1(v, z) = B_2(v, z)a$.

(d) Let $\#$ be any involution on Δ, coinciding with $*$ on K. The map φ: $d \to (d^{\#})^*$ is then an automorphism of Δ, which is the identity on K. Therefore, φ is an inner automorphism and there is a $c \in \Delta$ with $(d^{\#})^* = cdc^{-1}$ for every $d \in \Delta$.

Hence, $d^{\#} = (c^{-1})^* d^* c^*$. Set $a = (c^{-1})^*$, the only thing that remains to be proved is that $a^* = \epsilon a$ with $\epsilon = +1$. Since $\#$ is an involution, we have $d = d^{\#\#} = (c^{-1})^*((c^{-1})^* d^* c^*)^* \cdot c^*$, or $d = aa^{*-1}d(aa^{*-1})^{-1}$. From this, we deduce that $aa^{*-1} \in K$. Set $\epsilon = aa^{*-1}$. Thus, $a^*\epsilon = a$. Applying $*$, we have $a\epsilon^* = a^*$, and using the two equalities, $\epsilon\epsilon^* = 1$. By hypothesis, $\epsilon = \pm(\alpha/\alpha^*)$, set $\bar{a} = a\alpha$, $\bar{a}$ satisfies the conclusion.

Let us now restrict our attention to finite-dimensional semisimple $*$-algebras over an algebraically closed field K, with the further restriction that $*$ is the identity on K.

COROLLARY 14.2. *If R is a simple $*$-algebra over K, as before, R is isomorphic to one of the two algebras*:

(a) *$n \times n$ matrices with transposition.*

(b) *$2n \times 2n$ matrices with symplectic involution.*

Furthermore, any irreducible module has a unique compatible form up to a scalar multiple.

In particular, in case (a), *every irreducible $*$ representation of R is orthogonal, and any two such representations are isometric. In case* (b), *every irreducible $*$ representation of R is symplectic and any two such representations are isometric.*

Proof. This is a special case of 14.1, once one recalls that, over an algebraically closed field K of characteristic not 2, a nondegenerate ϵ symmetric form on a space V is unique up to isometry.

Remark. If S is a central simple algebra with involution of the first kind, finite dimensional over its center K, and $\bar{K}$ is the algebraic closure of K; $S \otimes_K \bar{K}$ with the involution $(a \otimes \alpha)^* = a^* \otimes \alpha$ is one of the two previous types.

We will say accordingly that S has transpose or symplectic involution. If $\dim_K S = n^2$, S has transpose involution if and only if $\dim_K S^+ = n(n+1)/2$, and $\dim_K S^- = n(n-1)/2$; if S has symplectic involution, then of course, $n = 2m$ is even and $\dim_K S^+ = n(n-1)/2$, $\dim_K S^- = n(n+1)/2$.

If a is an invertible symmetric element, we can define a new involution on S by the formula $d^\# = ad^*a^{-1}$, $\#$ has the same type as $*$.

If a is an invertible antisymmetric element, the formula $d^\# = ad^*a^{-1}$ defines also an involution on S of type opposite to the one of $*$. Such an element exists if and only if n is even.

The same reasoning as in 14.1 (d) shows that every involution on S is deduced from $*$ in the previous way. An easy computation shows finally that, if $\#$ and b are two involutions deduced from $*$ by the elements a, a', respectively, $\#$ and b give rise to isomorphic $*$-algebra structures on S if and only if there is an invertible element $c \in S$ with $c^*c = a'a^{-1}$.

LEMMA 14.3. *Let $R = S \oplus S^0$ be a $*$-simple* (*but not simple*) *algebra. Let V,* (*resp. V^0*) *be an irreducible S module* (*resp. S^0 module*).

(a) *$W = V \oplus V^0$ has a nonzero symmetric compatible form B. B is unique up to a scalar multiple and necessarily nondegenerate.*

(b) *$W = V \oplus V^0$ has a nonzero antisymmetric compatible form $\tilde{B}$. $\tilde{B}$ is unique up to a scalar multiple and necessarily nondegenerate.*

Proof. Let us consider $W^{\#} \otimes_R W$; we easily see that $W^{\#} \otimes_R W \simeq (V^0 \otimes_S V) \oplus (V \otimes_{S^0} V^0)$, where we consider V^0 as a right S module and V as a right S^0 module in the natural way. The two summands are both one-dimensional over K and τ exchanges them.

Let us choose a nonzero vector $u \in V^0 \otimes_S V$ and set

$$\bar{u} = \tau(u) \in V \otimes_{S^0} V^0.$$

If $\bar{B}$: $W^{\#} \otimes_R W \to K$ is a linear map, $\bar{B}$ corresponds to a symmetric compatible form B if and only if $\bar{B}(u) = \bar{B}(\bar{u})$; $\tilde{B}$ corresponds to a compatible antisymmetric form $\tilde{B}$ if and only if $\tilde{B}(u) = -\tilde{B}(\bar{u})$. Therefore, the existence and uniqueness of B in both cases is ensured. The nondegeneracy of B is proved as in 14.1; one remarks that V and V^0 are isotropic subspaces of W that are set in duality by B.

We can read the previous lemma in the language of representations, using the notations of Section 13.

Let us call φ, φ^0, ψ the representations of $S \oplus S^0$ on the vector spaces V, V^0, $V \oplus V^0$, respectively.

COROLLARY 14.4. (a) *φ^0 is isomorphic to φ^*.*

(b) *ψ is equivalent to φ^h and φ^s, respectively.*

(c) *Any * representation isomorphic to ψ is isometric to φ^h or to φ^s.*

We plan to extend now the previous results to not necessarily irreducible representations.

We deal first with the simple case, then with the case $R = S \oplus S^0$.

Let R be a simple *-algebra and V an irreducible R module. Any R module is isomorphic to $V \otimes_K U$, where U is a vector space and R acts on the first factor.

LEMMA 14.5. *Let R be simple with transpose involution (resp. symplectic involution).*

(a) *If U is odd-dimensional, $W = V \otimes_K U$ possesses a symmetric nondegenerate compatible form B and no antisymmetric nondegenerate compatible form (resp. W possesses an antisymmetric and no symmetric compatible form). B is unique up to R linear isometries.*

(b) *If U is even-dimensional, W possesses both a symmetric and an* *antisymmetric compatible form. Both forms are unique up to R linear isometries.*

Proof. Consider $W^{\#} \otimes_R W$, it is isomorphic to $(V^{\#} \otimes_R V) \otimes_K (U \otimes_K U)$. This implies that a compatible nondegenerate form on W is the tensor product of a compatible form on V with a nondegenerate form on U. The form on V is, by 14.2, uniquely determined up to a scalar and it is symmetric or antisymmetric, according to the type of involution on R.

The form on U can be chosen only symmetric if U is odd dimensional, otherwise it can be chosen both symmetric or antisymmetric. In each case, such a form is unique up to linear isometries of U. The claims follow immediately.

For the algebra $R = S \oplus S^0$, we have a similar result, a typical R module is of type $W = V \otimes_K U_1 \oplus V^0 \otimes_K U_2$.

LEMMA 14.6. *A compatible nondegenerate form on W exists if and only if* $\dim U_1 = \dim U_2$.

In this case, one can choose both a symmetric or an antisymmetric compatible form on W.

Such a form is unique up to R linear isometries.

Proof. Assume that B is a nondegenerate compatible form on W. Let e denote the unit element of S, e^* is the unit element of S^0.

If $w, v \in V \otimes U_1$ we have

$$B(w, v) = B(ew, v) = B(w, e^*v) = B(w, 0) = 0.$$

Thus, $V \otimes U_1$ is an isotropic space, similarly, $V^0 \otimes U_2$ is isotropic.

The nondegeneracy of B implies that $\dim V \otimes U_1 = \dim V^0 \otimes U_2$, hence, $\dim U_1 = \dim U_2$. Set $U_1 = U_2 = U$ and construct $W^{\#} \otimes_R W = [(V^0 \otimes_S V) \oplus (V \otimes_{S^0} V^0)] \otimes (U \otimes_K U)$.

As in 14.5, we see that a compatible symmetric nondegenerate form is the tensor product of a compatible symmetric form on $V \oplus V^0$ with a nondegenerate symmetric form on U, or the tensor product of a compatible antisymmetric form on $V \oplus V^0$ with a nondegenerate antisymmetric form on U. Similarly, for a compatible antisymmetric form. Clearly, the forms obtained in each case are isometric over R. Thus, to conclude, we have only to compare any form obtained as a tensor product of two symmetric ones with one obtained as a tensor product of two antisymmetric ones (similarly in the antisymmetric case). We have, in each case, complete freedom of choice by the previous remarks.

Let us choose a compatible symmetric form given by a linear map B on

$(V^0 \otimes_S V) \oplus (V \otimes_{S^0} V^0)$. A compatible antisymmetric form $\bar{B}$ is obtained from B by defining $\bar{B} = B$ on $V^0 \otimes_S V$, $\bar{B} = -B$ on $V \otimes_{S^0} V^0$. Now, write $U \simeq P \oplus P^*$, P a vector space, equip U with the canonical hyperbolic form H and the canonical symplectic form S, given in Section 13 (i), (ii).

It is immediately verified that $B \otimes H = \bar{B} \otimes S$, and so the lemma is proved.

We are now ready to prove the conclusive theorem on semisimple modules. We need to fix our notations. Let

$$R = \left(\bigoplus_{i=1}^{s} R_i\right) \oplus \left(\bigoplus_{j=1}^{t} S_j\right) \oplus \left(\bigoplus_{k=1}^{u} (T_k + T_k^0)\right)$$

be a semisimple $*$-algebra. The terms R_i are the ones with transpose involution, the S_j the ones with symplectic involution, and T_k is exchanged with T_k^0. Let V_i, $i = 1,..., s$; W_j, $j = 1,..., t$; Z_k, Z_k^0, $k = 1,..., u$ be the irreducible modules over R_i, S_j, T_k, and T_k^0, respectively.

Consider an R module M, M is isomorphic to $\sum n_i V_i + \sum m_j W_j + \sum p_k Z_k + \sum q_k Z_k^0$.

THEOREM 14.7. (a) *M has a compatible symmetric form if and only if m_j is even for $j = 1,..., t$, and $p_k = q_k$ for $k = 1,..., u$. Any two such forms are isometric over R.*

(b) *M has a compatible antisymmetric form if and only if n_i is even, $i = 1,..., s$, and $p_k = q_k$ for $k = 1,..., u$. Any two such forms are isometric over R.*

Proof. Let e_i, f_j, g_k, g_k^0, respectively, be the unit element of R_i, S_j, T_k, and T_k^0. We have

$$e_i^* = e_i, \qquad f_j^* = f_j, \qquad g_k^* = g_k^0, \qquad g_k^{0*} = g_k.$$

Let us carry out the proof of (a), (b) is perfectly similar. If B is a compatible form on M, then, since the idempotents e_i, f_j, $g_k + g_k^0$ are symmetric and orthogonal, we have the decomposition of M in the orthogonal subspaces e_iM, f_jM, $(g_k + g_k^0)M$. Thus, B is the direct sum of compatible forms on such subspaces. Now, $e_iM = n_iV_i$, $f_jM = m_jW_j$ and $(g_k + g_k^0)M = p_kZ_k + q_kZ_k^0$. Therefore, we are reduced to the case studied in the previous lemmas, and the theorem is proved.

15. *Equivalence of Representations under $O(n)$ and $Sp(n)$*

We start with some definitions.

Recall that a flag on a vector space V is a sequence

$$V = V_n \supset V_{n-1} \supset \cdots \supset V_0 = 0$$

of subspaces. If $V_k \supset V_r$ are two subspaces of the flag, we have an induced flag on V_k/V_r by $V_k/V_r \supset V_{k-1}/V_r \supset \cdots \supset V_r/V_r = 0$.

If $V = V_n \supset V_{n-1} \supset \cdots \supset V_0 = 0$ is a flag on V and W is another space, we can define the direct sum flag on $V \oplus W$ by

$$V \oplus W \supset V_{n-1} \oplus W \supset \cdots \supset V_0 \oplus W = W \supset 0.$$

Assume that V is now endowed with an ϵ-symmetric nondegenerate form, we make an inductive definition:

DEFINITION 15.1. A flag $V = V_{n-1} \supset \cdots \supset V_0 = 0$ is a compatible flag if one of the two possibilities are verified:

(a) V_1 is isotropic, $V_{n-1} = V_1^{\perp}$, and the induced flag on V_1/V_{n-1} is compatible.

(b) The form is nondegenerate on V_1 and the flag is the sum of a compatible flag on $V_1^{\perp}$ with V_1.

We are implicitly using the fact that, if $W \subset V$ is isotropic $W^{\perp}/W$ inherits from V a natural nondegenerate form, and if the form, on the other hand, is nondegenerate on W, then it is so on $W^{\perp}$ and $V = W \oplus W^{\perp}$.

The meaning of a compatible flag is understood considering the graded space associated $\operatorname{gr} V = \oplus V_{i+1}/V_i$. If the flag is compatible, $\operatorname{gr} V$ inherits a nondegenerate form defined inductively as follows. If we are in case (b), $V = V_1 \oplus V_1^{\perp}$, and canonically, $\operatorname{gr} V \simeq W_1 \oplus \operatorname{gr} V_1^{\perp}$ (we use the induced compatible flag on V_1), the form is the orthogonal sum. In case (a), we have

$$\operatorname{gr} V \simeq V_1 \oplus V/V_1^{\perp} \oplus \operatorname{gr}(V_1^{\perp}/V_1) = V_1 \oplus V/V_{n-1} \oplus \operatorname{gr}(V_{n-1}/V_1).$$

The form is the sum of the inductively defined form on $\operatorname{gr}(V_1^{\perp}/V_1)$ with the canonical hyperbolic form induced by the form on V on the soace $V_1 \oplus V/V_1^{\perp}$.

Let us now consider an orthogonal representation $\varphi: R \to \operatorname{End}(V)$ (the symplectic case is similarly treated).

THEOREM 15.2. (a) *There is a compatible flag that is a composition series for the R module structure.*

(b) *The quotient semisimple representation φ^σ on* gr V, *with the quotient form, is orthogonal.*

(c) *φ^σ is in the "closure of the orbit" of φ under $O(V)$.*

Proof. (a), (b) The orthogonal and symplectic case are essentially similar, therefore, let us treat the first one. We proceed by induction on dim V. Let V_1 be an irreducible submodule of V. Since φ is orthogonal $V_1{}^\perp$ is also an R submodule. Therefore, we either have $V_1 \subset V_1{}^\perp$, or $V_1 \cap V_1{}^\perp = 0$. In the last case, $V = V_1 \oplus V_1{}^\perp$, and φ is the orthogonal sum of two representations, by induction $V_1{}^\perp$ has a compatible composition series and we take the direct sum. In the first case, the form B induces a nondegenerate bilinear pairing $\bar{B}$: $V/V_1{}^\perp \times V_1 \to K$, identifying $V/V_1{}^\perp$ with $V_1{}^*$. $V/V_1{}^\perp$ is also an R module, if $v \in V$, $\bar{v}$ its class modulo $V_1{}^\perp$ and $w \in V_1$, we have: $\bar{B}(r\bar{v}, w) = \bar{B}(rv, w) = B(rv, w) = B(v, r^*w)$.

Hence, $V/V_1{}^\perp$ is identified to $V_1{}^*$ as R module and the quotient representation of R on $V/V_1{}^\perp \oplus V_1$ is isomorphic to $\varphi_1{}^h$, φ_1 being the induced representation on V_1.

The representation $V/V_1{}^\perp$ is also clearly irreducible. The induced representation on $V_1{}^\perp/V_1$ is also orthogonal, and one proceeds by induction.

(c) We do not want to formalize the meaning of this phrase in general; in case R is the free algebra on j variables, the set of representations is an affine space, the closure is intended in the Zariski topology. In general, one can define it using the language of schemes.

We proceed by induction. In case (a) $V = V_1 \oplus V_1{}^\perp$, there is nothing to prove, we pass directly to $V_1{}^\perp$ and use induction.

If $V_1 \subset V_1{}^\perp$, we must show that the representation φ' on $V/V_1{}^\perp \oplus V_1 \oplus V_1{}^\perp/V_1$ is in the closure of the orbit of φ.

Consider a basis of V, $v_1, v_2, \ldots, v_k, v_{k+1}, \ldots, v_{k+s}, w_1, \ldots, w_k$, where $v_1, \ldots, v_k$ is a basis of V_1, $v_1, \ldots, v_{k+s}$ is a basis of $V_1{}^\perp$, and $B(v_i, w_j) = \delta_j{}^i$, $i, j = 1, \ldots, k$. Write the representation in block form:

$$\begin{pmatrix} A & B & C \\ 0 & D & E \\ 0 & 0 & F \end{pmatrix}.$$

Consider the matrix in block form:

$$A(t) = \begin{pmatrix} t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t^{-1} \end{pmatrix},$$

$A(t)$ is an isometry of V, as is easily verified,

$$A(t) \begin{pmatrix} A & B & C \\ 0 & D & E \\ 0 & 0 & F \end{pmatrix} A(t)^{-1} = \begin{pmatrix} A & tB & t^2C \\ 0 & D & tE \\ 0 & 0 & F \end{pmatrix}.$$

This describes a piece of the orbit of φ, setting $t = 0$, we stay in the closure of the orbit and obtain the representation in block form:

$$\begin{pmatrix} A & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & F \end{pmatrix},$$

which is exactly φ'.

We are now ready to conclude our work.

THEOREM 15.3. *Let R be a finitely generated * algebra. Let W be the space of n-dimensional orthogonal representations. Let $O(n)$ be the orthogonal group acting on W, Then*:

The quotient variety $W/O(n)$ classifies isomorphism classes of orthogonal semisimple representations.

More precisely, the map π: $W \to W/O(n)$ is onto and $\pi(\varphi) = \pi(\psi)$ if and only if φ^σ is isomorphic to ψ^σ.

Proof. First, let us assume that R is the free *-algebra in j variables. W is then the $(n^2 \cdot j)$-dimensional affine space.

We already know, from 15.2(c), that φ and φ^σ have the same invariants.

From 14.7, we deduce that φ^σ is uniquely determined by φ (and not by the composition series).

If φ^σ is isomorphic to ψ^σ, we have $\pi(\varphi) = \pi(\psi)$. π is onto from the theory of reductive groups, since we are in characteristic 0. (This hypothesis now can be removed because Mumford's conjecture has been proved.)

The only part that has to be checked is : If φ_1 , φ_2 are nonisomorphic semisimple representations, they have different invariants.

First, since φ_1 is semisimple, the ideal ker φ_1 is detected as the maximal ideal I of R such that $\varphi_1(I)$ is nilpotent. This is equivalent to $\mathrm{Tr}(a) = 0$ for all $a \in \varphi_1(I)$, hence, this condition is expressed by invariant equations. If $\mathrm{Ker}\,\varphi_1 = \mathrm{Ker}\,\varphi_2 = I$, let us consider the semisimple algebra R/I

$$R/I = \bigoplus R_i \oplus S_j \oplus (T_k \oplus T_k^0),$$

with the notations of 14.7. By the same theorem, an orthogonal R/I module is specified by the ranks of the idempotents e_i, f_j, $g_t + g_t^0$. These ranks are in turn determined by the values of the coefficients of their characteristic polynomial, again an invariant condition.

To pass from the free *-algebra to a general finitely generated algebra, we present such an algebra as the quotient R/I of a free *-algebra modulo a *-invariant ideal.

The representations of R/I form a subvariety of the representations of R, invariant under $O(n)$. A semisimple representation φ of R factors through I if and only if the invariants $\mathrm{Tr}(\varphi(a))$ vanish for $a \in I$.

The theorem is thus completed.

We have clearly a similar theorem for symplectic representations, which is proved exactly in the same way.

THEOREM 15.4. *Let R be a finitely generated *-algebra. Let Z be the space of symplectic $2n$-dimensional representations. The quotient variety $Z/Sp(n)$ classifies isomorphism classes of semisimple symplectic representations.*

16. *Positive Involutions and Real Points*

We want to consider new *-algebras over the real numbers R and "real" representations of them. For the three classical groups $O(n, C)$, $Sp(n, C)$, $Gl(n, C)$, we take the real compact groups $O(n, R)$, $Sp(n)$, $U(n)$, where $Sp(n)$ now will stand for unitary quaternionic matrices. In each case, we may consider the relative notion of real representation and their equivalence under the previously defined groups. Thus, we have orthogonal representation on the vector space R^n with its form $\sum_{i=1}^n x_i^2$, quaternionic representations in H^n with its form $\sum q_i\bar{q}_i$ and unitary representation in C^n with the Hermitian form $\sum \alpha_i\alpha_i$. The theory in these cases in particularly pleasing, the groups are compact, thus, the orbits are closed and the quotient is the orbit space on one hand. On the other hand, this is reflected at the level of representations by the fact that every representation is semisimple, since every subspace has an orthogonal complement. We come now to the details.

Let S be a semisimple $*$-algebra over the real numbers R. Consider the reduced trace tr: $S \to R$ and the associated form on S, $\operatorname{tr}(aa^*)$.

PROPOSITION 16.1. *The following conditions are equivalent*:

(a) $\operatorname{tr}(aa^*) \geqslant 0, \quad \forall a \in S.$

(b) $\operatorname{tr}(aa^*) > 0, \quad \forall a \in S, \quad a \neq 0.$

(c) *S is the direct sum of $*$-algebras of the following three types*:

(1) *real matrices with transposition*;

(2) *quaternionic matrices with the involution* $(q_{ij})^* = (\bar{q}_{ji})$;

(3) *complex matrices with adjoint involution* $(a_{ij})^* = (\bar{a}_{ji})$.

(d) *If S is simple, any irreducible module V has a positive symmetric compatible form with values in R. Such a form is unique up to positive multiples.*

Proof. (c) $\Rightarrow$ (b) $\Rightarrow$ (a) trivially. Assume (a) holds; let $S = \bigoplus S_i$. If we had two terms exchanged by $*$, S_i and S_j, consider their unit elements e_i and e_j. We have $e_i{}^* = e_j$. Hence,

$$\operatorname{tr}((e_i - e_j)(e_i - e_j)^*) = \operatorname{tr}(-e_i - e_j) < 0.$$

Therefore, every terms is fixed under $*$, furthermore, each simple $*$-algebra S_i has an involution satisfying (a), hence, (b) since the form is nondegenerate. Then, it follows from [10, Theorem 0] that S_i is of the desired type.

We have to prove now the equivalence of (d) with the remaining assertions. Clearly, if S is of the type described in (c), there is a positive form on any irreducible module.

Conversely, let V be irreducible and consider $V^\# \otimes_S V$. A compatible form of the type described in (d) gives rise to a map B: $V^\# \otimes_S V \to R$.

Let $w \in V^\# \otimes_S V$ be a τ symmetric element mapping into 1. We identify $V^\# \otimes_S V$ with the centralizer Δ of S in V, $d \to d \cdot w$, and consider the induced map B: $\Delta \to R$. The map B induces a quadratic form on Δ, $\bar{B}(a, b) = B(ab^*)$. The axioms on the original form imply that $\bar{B}$ satisfies the properties:

$$\bar{B}(d, b) = \bar{B}(b, d), \qquad \bar{B}(da^*, b) = \bar{B}(d, ab), \qquad B(d, d) > 0,$$

if $d \neq 0$.

Then, it is easily checked (e.g., $0 < \bar{B}(i, i)$, while $\bar{B}(-1, 1) < 0$) that we must be in one of the three cases (1) $\Delta = R$, (2) $\Delta = C$ and the involution is conjugation, (3) $\Delta = H$ and the involution is conjugation. In each case, one sees immediately that B is a positive multiple of the reduced trace $\operatorname{tr}: \Delta \to R$.

As for the uniqueness, one remarks that in each of the previous cases, the symmetric elements are the elements of R, therefore, the choice of w, giving the same involution on Δ, is unique up to a scalar multiple in R (14.1(c)). Hence, a new form on V induces a new map $B'\colon V^* \otimes_S V \to R$ differing from B by a scalar $\alpha \in R$, since both forms are positive, we must have $\alpha > 0$.

DEFINITION 16.2. If S is a semisimple *-algebra over R, satisfying the hypotheses of 16.1, S will be called a positive *-algebra.

LEMMA 16.3. *Let S be a positive *-algebra over R, let V be a complex orthogonal module over $S \otimes_R C$.*

V is the complexification of a unique real orthogonal S module with a positive form.

Proof. From 16.1, $S = \oplus R_i \oplus S_j \oplus T_k$, where R_i is a ring of real matrices with transposition, S_j quaternionic matrices and T_k complex matrices with their involution $(a_{ij})^* = (\bar{a}_{ji})$.

Thus, $R_i \otimes_R C$ is complex matrices with transposition, $S_j \otimes_R C$ complex matrices with symplectic involution and $T_k \otimes_R C$ is $(C)_n \oplus (C)_n$ with exchange involution.

We use the classification of orthogonal modules over such a ring (14.7), then we see immediately that such a module is the complexification of a direct sum of the standard real modules described at the beginning of this paragraph. The uniqueness has been proved in part (d) of 16.1, remarking that a positive compatible form is always the direct sum of the form restricted on orthogonal irreducible submodules, since no subspace is isotropic (cf. 15.2).

We sum up our work for $O(n, R)$.

THEOREM 16.4. *Let $S = R\{x_i, x_i^*\}$ be a finitely generated *-algebra.*

The variety W of equivalence classes of semisimple orthogonal representations of $S \otimes_R C$ has a real structure.

The equivalence classes of real orthogonal representations of S correspond to the real points of W, where the invariant functions $\operatorname{tr}(aa^*)$, $a \in S$, are ≥ 0.

Proof. The real structure on W is clear from the form of the invariant ring, generated by the elements $\mathrm{tr}(a_{i_1} \cdots a_{i_s})$, $a_i \in S$.

If φ is a real representations, it is semisimple and its invariants $\mathrm{tr}(\varphi(a_{i_1}) \cdots \varphi(a_{i_s}))$ are real, hence, it gives a real point of W, clearly, in this point, we have the desired inequalities $\mathrm{tr}(aa^*) \geqslant 0$.

Conversely, let P be a real point, with $\mathrm{tr}(aa^*) \geqslant 0$. Let $\bar{S}$ be the semisimple algebra of operators obtained evaluating S in P.

It is easily verified that the reduced trace $\mathrm{tr}\colon \bar{S} \to R$ is strictly related to the trace Tr of the representation.

In fact, if $\bar{S} = \bigoplus \bar{S}_i$, $\bar{S}_i$ simple, and tr_i is the reduced trace in $\bar{S}_i$, we have $\mathrm{tr} = \sum \mathrm{tr}_i$, while $\mathrm{Tr} = \sum n_i \,\mathrm{tr}_i$, $n_i > 0$ some integers. Therefore, from the condition $\mathrm{Tr}(aa^*) \geqslant 0$, we deduce that $\bar{S}$ is a positive $*$-algebra and the claim follows from 16.3.

We pass now to unitary representations. Consider an algebra S over C with an involution inducing conjugation on C.

We want to classify unitary representations of S in $(C)_n$. The first remark is that every such representation is semisimple and, if $\bar{S}$ denotes the induced algebra of operators, $\bar{S}$ is a semisimple positive $*$-algebra.

The converse is clear, moreover, 16.1 (d) implies that two unitary isomorphic representations are isometric, over $U(n)$. Now, we have to read these results geometrically. This is done as follows, let S for simplicity be $C\{x_1, \dots, x_n, x_1^*, \dots, x_n^*\}$.

The representations of S in $(C)_m$ are $2n$-tuples of matrices and are classified by the space $[\bigoplus_n (C)_m] \otimes_R C$ with real structure given by the first factor. A real point is a $2n$-tuple $(a_1, a_1^*, a_2, a_2^*, \dots, a_n, a_n^*)$.

Thus, the real points of the variety of representations are exactly the unitary representations.

The initial remarks show:

THEOREM 16.5. *The equivalence classes, under $U(n)$, of n-dimensional unitary representations of S are classified by those real points, of the variety of equivalence classes, under $Gl(n, C)$, of semisimple representations of S, where the invariants* $\mathrm{Tr}(aa^*)$, $a \in S$, *are* $\geqslant 0$.

Finally, we turn our attention to quaternionic representations. If V is a $2n$-dimensional complex space, a quaternionic structure on V is a structure of right vector space over H inducing the C vector space structure when the scalars are restricted to $C \subsetneq H$.

Clearly, to give a quaternionic structure on V, one must only give a map $j\colon V \to V$, with $j^2 = -1$ and j antilinear over C, i.e., $j(\alpha v) = \bar{\alpha} j(v)$.

We will sketch the main points.

A nondegenerate quaternionic form B on V can be defined as in 14.1, and it has the form $\sum q_i \bar{q}_i{}'$.

Considering $H = C \oplus jC$, the form B gives rise to two forms, K and A, with values in C such that

$$B(v, w) = K(v, w) + jA(v, w).$$

Since $B(w, v) = \overline{B(v, w)}$ we have

$$K(v, w) = \overline{K(w, v)} \qquad \text{and} \qquad A(w, v) = -A(v, w).$$

Thus, K is Hermitian and A is alternating.

They are nondegenerate on V. If S is a *-algebra over R, and V is a symplectic module over $S \otimes_R C$, we will say that V is quaternionic if it possesses a quaternionic structure with a quaternionic form, compatible with the involution on S, inducing the given alternating form. It is clear that, if $\bar{S}$ is the operator algebra induced by S on V, and V is quaternionic, then $\bar{S}$ is a positive *-algebra and the representation is semisimple.

The representation is identified, up to equivalence relative to $Sp(n)$ ($Sp(n) \subseteq (H)_n$ in this case), by the irreducible quaternionic subspaces.

These subspaces are easily analyzed for the various terms R_i, S_j, T_k of the direct sum decomposition of S. In each case, one has the indecomposable symplectic representation of the relative algebra $R_i \otimes C$, $S_j \otimes C$, $T_k \otimes C$ a case analysis similar to the one carried in the previous case, shows that the way to make this representation quaternionic is essentially unique up to positive real numbers (one has to reduce the discussion to $V^{\#} \otimes V$), thus, one obtains, as in the previous case, the theorem:

THEOREM 16.6. *The equivalence classes, under $Sp(n)$, of $2n$-dimensional complex symplectic representations of $S \otimes_R C$ with quaternionic structure are classified by the real points of the variety of equivalence classes of semisimple symplectic representations of $S \otimes_R C$, where the invariants* $\mathrm{tr}(aa^*)$, $a \in S$, *are* ≥ 0.

Remark. We have, in the three cases considered, the following set up. An algebraic group G defined over R. The real points of G form a compact group G_R. A variety W defined over R, and a group action $G \times W \to W$ defined also over R, the quotient variety W/G defined over R. The real points W_R of W and $(W/G)_R$ of W/G. The quotient map $W \to W/G$ restricted to the real points W_R gives a map π_R: $W_R \to (W/G)_R$. Then, the fibers of π_R are exactly the orbits under the compact group G_R, and the image of π_R is the semianalytic subset of $(W/G)_R$ defined by the fact that the symmetric elements xx^* of a certain non-commutative algebra have trace $\geqslant 0$.

17. *Azumaya Algebras*

We want to develop now the notions necessary to deal with irreducible representations.

We will follow the theory for $Gl(n, K)$ very closely (cf. [1, 7, 8]).

Let R be an Azumaya algebra over its center A. Assume that R has an involution $*$.

DEFINITION 17.1. We say that the involution is of the first kind if $*$ is the identity on A.

Otherwise we say that it is of the second kind.

Let us assume now that 2 is invertible in A and $*$ of the first kind.

If $A \to K$ is a map in an algebraically closed field (a geometric point P of Spec A), the $*$ algebra $R \otimes_A K$ is either $n \times n$ matrices with transposition or $2n \times 2n$ matrices with symplectic involution (14.2). We will say that R is of transpose type, resp. of symplectic type in P. We notice that the type depends only on the point of Spec A over which the given geometric point lies (cf. remark after 14.2).

PROPOSITION 17.2. *Let R be an Azumaya algebra over A with involution of the first kind and $p \in$ Spec A.*

(a) *If* rk $R = n^2$ *and R has transposition type in p, then there exists an etale neighborhood* Spec U *of p such that $R \otimes_A U$ is $n \times n$ matrices with transposition.*

(b) *If* rk $R = (2n)^2$ *and R has symplectic involution in p, there is an etale neighborhood* Spec U *of p such that $R \otimes_A U$ is $2n \times 2n$ matrices with symplectic involution.*

Proof. Let us do case (a), (b) is analogous.

Let P be a geometric point centered in p and $\bar{A}$ the strict henselization of A in P.

$R \otimes_A \bar{A}$ is an Azumaya algebra of rank n^2 over A with involution of the first kind. If K is the residue field of $\bar{A}$, $R \otimes_A K$ is $n \times n$ matrices over K with transpose involution.

We choose matrix units in $R \otimes_A K$ for which the involution is the usual transposition, then, by the theory of Azumaya algebras over Hensel rings, such matrix units lift uniquely to units e_{ij}.

Since e_{ij}^* reduces to the same element as e_{ji}, we must have, for the same theorem, $e_{ij}^* = e_{ji}$, and the claim follows for $R \otimes \bar{A}$. To reduce from $\bar{A}$ to an etale neighborhood is now a standard direct limit argument.

Let us work now in the category ${}_{\Lambda}\mathscr{C}^*$ of * algebras over a fixed ring Λ, with * the identity on Λ.

If A is a commutative algebra, define $(A)_n^t$, $(A)_{2n}^s$ to be the algebra of $n \times n$ matrices with transposition and of $2n \times 2n$ matrices with symplectic involution. The two previously defined algebras are in fact functors in A, from the category of commutative algebras to ${}_{\Lambda}\mathscr{C}^*$.

Let us denote them $F_{t,n}$ and $F_{s,n}$.

Consider the free algebra $R = \Lambda\{x_i, x_i^*\}$ in infinitely many variables.

An element $f \in R$ will be called a polynomial identity (briefly P.I.) of $n \times n$ matrices with transposition, if f vanishes when computed in all the rings $(A)_n^t$ (similarly for symplectic P.I.'s).

Any set of polynomials in R determines a variety in ${}_{\Lambda}\mathscr{C}^*$ formed by those algebras on which all these identities vanish.

PROPOSITION 17.3. *Let R be a rank n^2 Azumaya algebra with involution. Assume that R satisfies the* P.I.'s *of $n \times n$ matrices with transposition.*

Then, the involution is of the first kind and R is of transpose type at each point.

Similarly, for symplectic involution.

Proof. Let A be the center of R. It is known that $A = F(R)$, the Formanek center [8].

If $f(x_1, \ldots, x_k)$ is a central polynomial for $n \times n$ matrices, and we evaluate f in $(A)_n^t$ to obtain $f(x_1, \ldots, x_k)^t = f(x_1, \ldots, x_k)$. This is, therefore an identity of $n \times n$ matrices with transposition hence it holds in R and so the involution is of the first kind.

The rest of the statement follows from the fact that, if $n = 2k$, there are different multilinear identities for $n \times n$ matrices with transposition, and for $n \times n$ matrices with symplectic involution (cf. Section 20).

18. *Universal Maps*

We return now to the two functors $F_{t,n}$ and $F_{s,n}$ considered in Section 17.

PROPOSITION 18.1. *Both functors $F_{t,n}$ and $F_{s,n}$ possess a left adjoint.*

Proof. Let us do it, for instance, for $F_{t,n}$. Given a * algebra, R we must find a universal map $\gamma: R \to (A)_n^t$. This follows of course by the general theorem on the existence of adjoint functors. On the other hand, it can be easily accomplished building A by generators and relations.

In this case, the essential point is to do it for $R = \Lambda\{x_i, x_i^*\}_{i \in I}$ the free algebra.

Construct $A = \Lambda[x_{s,t}^{(i)}]$, $i \in I$, $s, t = 1,..., n$ the polynomial ring. Map x_i in the matrix $\xi_i = (x_{s,t}^{(i)})$, while x_i^* is mapped in ξ_i^t.

The formal properties of these functors are easily checked and follow the ones given in [9, Chap. 4] for rings without *.

Given a commutative algebra A consider: $G_n(A)$ the group of A automorphisms of the *-algebra $(A)_n^t$; $H_n(A)$ the group of A automorphisms of the *-algebra $(A)_{2n}^s$.

Both $G_n(\ )$ and $H_n(\ )$ are group valued functors on the category of commutative algebras.

It is immediately verified that they are both representable by Hopf algebras finitely presented over Λ.

Indicate the two algebras $\mathscr{G}_n$ and $\mathscr{H}_n$.

Consider, furthermore, the orthogonal group

$$O_n(A) = \{a \in (A)_n^t \mid aa^t = 1\},$$

and the symplectic group $Sp_n(A) = (b \in (A)_{2n}^s \mid bb^* = 1\}$.

These group valued functors are also clearly representable by finitely presented Hopf algebras over Λ.

Call these algebras $\mathscr{U}_n$ and $\mathscr{S}_n$.

We have two natural transformations:

$$\pi: O_n(A) \to G_n(A),$$

$$\tau: Sp_n(A) \to H_n(A),$$

given by $\pi(a)(b) = aba^{-1}$ (similarly for τ).

LEMMA 18.2. *Let A be a local ring. We have two exact sequences:*

(i) $0 \to {}_2A^* \to O_n(A) \to G_n(A) \to A^*/(A^*)^2$

(ii) $0 \to {}_2A^* \to Sp_n(A) \to H_n(A) \to A^*/(A^*)^2 \to 0,$

where ${}_2A^* = \{\alpha \in A \mid \alpha^2 = 1\}$.

If n is odd, $O_n(A) \to G_n(A)$ is onto; if n is even and A^n is hyperbolic, $G_n(A) \to A^*/(A^*)^2$ is onto.

Proof. Some parts are trivial; in particular, the definition and exactness of

$$0 \to {}_2A^* \to O_n(A) \to G_n(A),$$

and

$$0 \to {}_2A^* \to Sp_n(A) \to H_n(A).$$

Let $\sigma \in G_n(A)$, since σ is an automorphism of $(A)_n$ and A is local, σ is inner.

Thus, $\sigma(b) = aba^{-1}$, $a \in Gl(n, A)$. We have $\sigma(b^t) = \sigma(b)^t$, hence, $ab^ta^{-1} = (aba^{-1})^t$, and so $a\,ab^t = b^ta^ta$ for all $b \in (A)_n$.

We deduce $a^ta \in A^*$, the invertible scalars. If $c \in Gl(n, A)$ is another element such that $\sigma(b) = cbc^{-1}$, we have $c = \alpha a$, $\alpha \in A^*$, thus, $c^tc = \alpha^2a^ta$. Therefore, the scalar a^ta is defined up to elements of $(A^*)^2$. If σ comes from $O_n(A)$, we can choose $a \in O_n(A)$, hence, $a^ta = 1$. Finally, if $aa^t = \beta^2$, set $c = a/\beta$, we have $c \in O_n(A)$, and the exactness of (i) is proved. For (ii), the steps are similar.

If n is odd and $\alpha = a^t \cdot a$, we have $\alpha^n = \det(a)^2$, and so $\alpha \in (A^*)^2$.

For the other assertion, if A^{2k} is hyperbolic (and in the symplectic case), choose a hyperbolic basis and consider, in that basis, the block matrix

$$a = \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}.$$

We have $a^t = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \alpha \end{smallmatrix}\right)$ and $a^ta = \alpha$.

Consider now, the maps of Hopf algebras

$$i\colon \mathscr{G}_n \to \mathscr{U}_n, \qquad j\colon \mathscr{H}_n \to \mathscr{S}_n$$

induced by the natural transformations π, τ.

PROPOSITION 18.3. *Both i and j are injective.*

Proof. Let us do it for i. Let $u \in \operatorname{Ker} i$, if $\mathbf{m}$ is any maximal ideal of $\mathscr{G}_n$, localize at $\mathbf{m}$ and consider the map $\mathscr{G}_n \to (\mathscr{G}_n)_{\mathbf{m}}$.

This map corresponds to a point $\sigma \in G_n((\mathscr{G}_n)_{\mathbf{m}})$, associate to σ the scalar $\alpha \in (\mathscr{G}_n)^*_{\mathbf{m}}$ modulo squares. Let $A = (\mathscr{G}_n)_{\mathbf{m}}[x]/(x^2 - \alpha)$. We have an induced map $\mathscr{G}_n \to A$ classifying a $\sigma \in G_n(A)$ induced, as inner automorphism, by some element $a \in O_n(A)$. We can factor:

$$\begin{tikzcd} & \mathscr{U}_n \arrow[dr, dashed] & \\ \mathscr{G}_n \arrow[ur] \arrow[rr] & & A. \end{tikzcd}$$

Hence, u is mapped to zero in $(\mathscr{G}_n)_{\mathfrak{m}}$. Since this is true for all maximal ideals, we have $u = 0$.

One can derive some consequences on the structure of $\mathscr{G}_n$, $\mathscr{U}_n$, when A is an integral domain.

In this case, $\mathscr{S}_n$ is an integral domain while $\mathscr{U}_n$ has two minimal primes relative to the decomposition of O_n in SO_n and its complement.

For $\mathscr{G}_n$, $\mathscr{H}_n$, we easily see: $\mathscr{H}_n$ is always a domain, $\mathscr{G}_n$ is a domain when n is odd, and has two minimal primes when n is even.

This last assertion comes from the fact that, -1 has determinant -1 if n is odd, $O_n = SO_n \times \{1, -1\}$, -1 is in the kernel of the map $O_n \to G_n$.

Finally, consider $A = Z$. In this case, $\mathscr{G}_n$ and $\mathscr{H}_n$ are torsion free, hence, flat, in each case there is an integer valued point (e.g., the one corresponding to the identity of $(Z)_n^t$, $(Z)_{2n}^s$). Therefore, $\mathscr{G}_n$ and $\mathscr{H}_n$ are faithfully flat over Z.

Therefore, the same is true for any A by base change.

Let us call for simplicity $\Gamma_n = \operatorname{Spec} \mathscr{G}_n$, the group scheme associated to $\mathscr{G}_n$, similarly, $\Pi_n = \operatorname{Spec} \mathscr{H}_n$.

Consider again the two functors $F_{t,n}$ and $F_{s,n}$; we analyze $F_{t,n}$ since the other case is similar.

If R is a $*$-algebra, we know that the functor $\operatorname{Maps}_A\mathscr{C}^*(R, (A)_n^t)$ is representable. Call the representing object $B_{R,n}$ and $A_{R,n} = \operatorname{Spec} B_{R,n}$.

Since the group valued functor $G_n(A)$ acts on $\operatorname{Maps}_A\mathscr{C}^*(R, (A)_n^t)$, we have a group scheme action:

$$\mu: \Gamma_n \times A_{R,n} \to A_{R,n}.$$

This is clearly functorial in R.

19. *Irreducible Representations*

We wish to study irreducible representations over an algebraically closed field.

We find it convenient to generalize the concept to deal with generic points applying the functorial language.

We will restrict our analysis to the orthogonal case, the symplectic case being absolutely similar.

DEFINITION 19.1. Let S be a rank n^2 Azumaya algebra over a commutative ring A with an involution of transposition type.

(i) A * map $\varphi: R \to S$ is an orthogonal irreducible representation if $\varphi(R)A = S$.

(ii) Two representations $\varphi_1: R \to S_1$, $\varphi_2: R \to S_2$ are equivalent if there is a *-isomorphism $\eta: S_1 \to S_2$ with $\varphi_2 = \eta\varphi_1$.

Consider now the subset $I_n(A) \subseteq \text{Maps}\ {}_A\mathscr{C}^*(R, (A)_n^t)$ of irreducible representations. We have:

PROPOSITION 19.2. *$I_n(A)$ corresponds to an open subscheme $\Lambda_{R,n}^*$ of $\Lambda_{R,n}$ invariant under Γ_n.*

Proof. The invariance of $I_n(A)$ under $G_n(A)$ is clear. As for the open condition, we use the criterion of [7, 8]. Let $\eta: R \to (B_R)_n^t$ be the universal map and $\bar{R} = \operatorname{Im}\gamma$, let $F(\bar{R})$ be the Formanek center of $\bar{R}$. We have $F(\bar{R}) \subseteq B_R$ and a point φ Maps ${}_A\mathscr{C}^*(R, (A)_n^t)$ is in $I_n(A)$ if and only if the classifying map $\bar{\varphi}: B_R \to A$ gives $\bar{\varphi}(F(\bar{R})) \neq 0$.

We want to construct now the quotient of $\Lambda_{R,n}^*$ under Γ_n. Consider the previous set up

$$\gamma: R \to (B_R)_n^t,$$

$\bar{R} = \gamma(R)$, L is the center of $\bar{R}$, $F(\bar{R}) \subseteq L$ is the Formanek center. If $f \in F(\bar{R})$, the ring $\bar{R}_f = \bar{R}[1/f]$ is an Azumaya algebra with transpose type involution (cf. 17.3). Let $\bar{\Lambda}_{R,n}^*$ be the open subscheme of $\operatorname{Spec}(L)$, where $F(\bar{R})$ is not identically zero. By the previous remarks, $\Lambda_{R,n}^*$ is equipped with a coherent sheaf $\mathscr{R}$ of Azumaya algebras with transposition.

PROPOSITION 19.3. *The scheme $\bar{\Lambda}_{R,n}^*$ represents the following functor: equivalence classes of irreducible representations.*

Proof. The proof is similar to the one in [7]. We clearly have a map $\lambda: R \to \Gamma(\bar{\Lambda}_{R,n}^*, \mathscr{R})$. To a map $\varphi: \operatorname{Spec} A \to \bar{\Lambda}_{R,n}^*$, we associate the composition

$$\lambda: R \to \Gamma(\bar{\Lambda}_{R,n}^*, \mathscr{R}) \to \Gamma(\operatorname{Spec} A, \gamma^*(\mathscr{R})).$$

Conversely, if $\lambda: R \to S$ is an orthogonal irreducible representation, we split S be a faithfully flat extension T of its center A (17.2) to make it $n \times n$ matrices with transposition.

Then, we complete the diagram

$$\begin{tikzcd} R \arrow[r, "\gamma"] \arrow[dr, "\lambda"'] & (B_R)_n \arrow[dr, "\bar{\lambda}_n"] & \\ & S \arrow[r] & (S \otimes_A T) = (T)_n . \end{tikzcd}$$

We verify, since λ is irreducible, $\bar{\lambda}$ induces a map from Spec T to Spec $B_R - V(F(\bar{R}))$.

Then, we use faithfully flat descent to see that the map factors through $\bar{\Lambda}^*_{R,n}$.

We finally tie the previous discussion with the classification of irreducible representations in the sense of 15.3, 15.4.

We state the Main Theorem for the orthogonal case, the other case is similar.

Recall the action $\mu: \Gamma_n \times \Lambda_{R,n} \to \Lambda_{R,n}$ defined at the end of Section 18.

THEOREM 19.4. *The action μ is scheme theoretically free on $\Lambda^*_{R,n}$. The quotient scheme is $\bar{\Lambda}^*_{R,n}$. The resulting quotient map $p: \Lambda^*_{R,n} \to \bar{\Lambda}^*_{R,n}$ is a principal fibration over Γ_n locally trivial in the etale topology of $\bar{\Lambda}^*_{R,n}$.*

Proof. We follow the lines of [7], to which we refer for details. Take $f \in F(\bar{R})$ and consider $\Lambda(f)$, $\bar{\Lambda}(f)$ the open subschemes of $\Lambda^*_{R,n}$, $\bar{\Lambda}^*_{R,n}$, where f is invertible. We will work locally on these subschemes, which cover $\Lambda^*_{R,n}$ and $\bar{\Lambda}^*_{R,n}$. The universal map $\bar{R} \subseteq (B_R)^t_n$ gives by localization at f the universal map $\bar{R} \to (B_R[1/f])^t_n$. $\bar{R}_f$ is a rank n^2 Azumaya algebra with transpose involution.

Let us analyze thus the universal map in this case.

If S is a rank n^2 Azumaya algebra with transpose involution over its center A, $S \to (T)^t_n$ is the universal map and $A \to B$ is a map of commutative rings we obtain, by base change, the universal map $S_B \to (T_B)^t_n$. In particular, we will apply this remark when B splits S.

Thus, we are led to study the split case $S = (A)^t_n$. In this case, reasoning as in [7], we see that the universal map is obtained as follows:

Consider $A \otimes_A \mathscr{G}_n$ and the two maps

$$i: A \to A \otimes_A \mathscr{G}_n, \qquad j: \mathscr{G}_n \to A \otimes_A \mathscr{G}_n$$

given by

$$i(a) = a \otimes 1, \qquad j(b) = 1 \otimes b.$$

The map j classifies an automorphism σ of the algebra $(A \otimes \mathscr{G}_n)_n^t$. The composition

$$(A)_n^t \xrightarrow{i_n} (A \otimes \mathscr{G}_n)_n^t \xrightarrow{\sigma} (A \otimes \mathscr{G}_n)_n^t$$

is the universal map for $(A)_n^t$.

In the scheme language of Section 18,

$$\Lambda_{(A)_{n,n}^t} = \operatorname{Spec}(A \otimes \mathscr{G}_n) = \operatorname{Spec} A \times \Gamma_n ,$$

the action $\mu: \Gamma_n \times \Lambda_{(A)_{n,n}^t} \to \Lambda_{(A)_{n,n}^t}$ is the canonical action of Γ_n on itself as second factor.

We now collect all the previous steps.

The map $\bar{R}_f \to (B_R[1/f])_n^t$ induces a map $L_f \to B_R[1/f]$ that dually gives the projection $p_f: \Lambda(f) \to \bar{\Lambda}(f)$.

These maps glue together to give the required projection $p: \Lambda_{R,n}^* \to \bar{\Lambda}_{R,n}^*$. Take an etale covering $U_\alpha \to \bar{\Lambda}(f)$ over which the Azumaya algebra $\bar{R}_f$ splits. By pull back and all the previous propositions we have

$$\begin{tikzcd} \Lambda(f) \times_{\bar{\Lambda}(f)} U_\alpha \arrow[r] \arrow[d, "p_\alpha'"'] & \Lambda(f) \arrow[d] \\ U_\alpha \arrow[r] & \bar{\Lambda}(f) \end{tikzcd}$$

and $\Lambda(f) \times_{\bar{\Lambda}(f)} U_\alpha = U_\alpha \times \Gamma_n$, p_α' is the first rejection and the action of Γ_n is the canonical action on the second factor.

This is sufficient for all the statements of our theorem.

Remark. It follows from the proof that the universal map $R \to (B_R)_n^t$, when R is a rank n^2 Azumaya algebra with transposition over its center A, is injective and furthermore, B_R is faithfully flat over A.

20. *Qualitative Results for Rational Concomitants*

We want to conclude our work describing various qualitative results that apply to the rings of matrix concomitants. Many of these results are, of course, well known (cf. [9]).

We consider a field K of characteristic not 2. Let $K\{x_i, x_i^*\}$, $K\{s_{ij}\}$, $K\{t_{ij}\}$ be the free algebras described in Section 13. Let D be a simple * algebra with center an infinite field $F \supseteq K$. Assume $\dim_F D = n^2$. We have the usual two possibilities: * is of the first kind or of the second kind. In the first case, we have two subcases, transpose or symplectic type. In each case one easily shows, as in [9, p. 20], that the ideal of polynomial identities in each free algebra does not depend on D. We have in fact only: the identities of $(F)_n^t$, the identities of $(F)_{2n}^s$, the identities of $(F)_n \oplus (F)_n^0$. The last case does not give anything new, one easily can verify that the identities of $(F)_n \oplus (F)_n^0$ as a *-algebra are the same as the ordinary identities of $(F)_n$, thinking x_i and $x_i{}^*$ are distinct variables.

In fact, one can by the same argument used in [9, p. 66] show that:

(i) every prime T-ideal is one of the ideals of identities previously described.

(ii) If K is infinite, the radical of a T-ideal is a T-ideal intersection of at most 3 prime T-ideals relative to the 3 types.

(iii) There are various inclusions among the various T-ideals deducible from the fact that:

$$(F)_n^t \otimes (F)_2^s = (F)_{2n}^s, \qquad (F)_{2n}^s \otimes (F)_2^s = (F)_{4n}^t,$$

$$(F)_n^t \oplus (F)_n^0 \subseteq (F)_{2n}^s, \qquad (F)_n \oplus (F)_n^0 \subseteq (F)_{2n}^t,$$

$(F)_n^t \subseteq (F)_{n+1}^t$, $(F)_{2n}^s \subseteq (F)_{2(n+1)}^s$ (the inclusions do not preserve 1).

Let us call $K[\xi_i, \xi_i{}^t]_n$, $K[\xi_i, \xi_i{}^*]_n$ the free algebra modulo the ideal of identities of $(F)_n^t$, respectively, of $(F)_{2n}^t$.

Similarly, taking the identities in the free algebras $K\{s_i\}$, $K\{t_i\}$ we will obtain algebras $K[\bar{s}_i]_n^t$, $K[\bar{s}_i]_n^s$, $K[\bar{t}_i]_n^t$, $K[\bar{t}_i]_n^s$.

We list now the properties of these rings and sketch the proofs.

THEOREM 20.1. *All the algebras constructed are prime rings with polynomial identities.*

Indicate by $K\langle \xi_i, \xi_i{}^t\rangle_n$, $K\langle \xi_i, \xi_i{}^*\rangle_n$, $K\langle \bar{s}_i\rangle_n^t$, $K\langle \bar{s}_i\rangle_n^s$, $K\langle \bar{t}_i\rangle_n^t$, $K\langle \bar{t}_i\rangle_n^s$, $K\langle \bar{t}_i\rangle_n^s$ *their respective rings of quotients.*

(1) (i) $K\langle \xi_i, \xi_i{}^t\rangle_n$ *is a central simple algebra of rank* n^2 *over its center* Z.

(ii) $K\langle \xi_i, \xi_i{}^t\rangle_n$ *is a *-algebra of transpose type,* $K\langle \xi_i, \xi_i{}^t\rangle_n = (D)_h$, $h \times h$ *matrices over a division ring* D *of degree* 2^r, *where* $n = 2^r \cdot h$ *and* $2 \nmid h$.

(iii) $K\langle \xi_i, \xi_i{}^t\rangle_n$ *is the ring of rational concomitants, for the* *orthogonal group, from I-tuples of matrices to matrices. Z is the field of orthogonal invariants of matrices.*

(iv) *If I consists of m elements, we have* $\mathrm{Tr}\deg Z/K = mn^2 - (n^2 - n)/2$.

(v) *If A is the center of $K[\xi_i, \xi_i^t]_n$, Z is the quotient field of A and $K\langle \xi_i, \xi_i^t\rangle_n = K[\xi_i, \xi_i^t]_n \otimes_A Z$.*

(2) (i) *$K\langle \xi_i, \xi_i^*\rangle_n$ is a central simple algebra of rank $(2n)^2$ over its center W.*

(ii) *$K\langle \xi_i, \xi_i^*\rangle_n$ is a *-algebra of symplectic type $K\langle \xi_i, \xi_i^*\rangle_n = (\Delta)_h$, $h \times h$ matrices over a division ring Δ of degree 2^r; with $2n = 2^r \cdot h$, $2 \nmid h$.*

(iii) *$K\langle \xi_i, \xi_i^*\rangle_n$ is the ring of rational concomitants, for the symplectic group, from I-tuples of matrices to matrices. W is the field of symplectic invariants of I-tuples of matrices.*

(iv) *If I is finite, with m elements, we have*

$$\mathrm{Tr}\deg W/K = m \cdot (2n)^2 - ((2n)^2 + 2n)/2.$$

(v) *If B is the center of $K[\xi_i, \xi_i^*]_n$, W is the field of fractions of B and $K\langle \xi_i, \xi_i^*\rangle_n \simeq K[\xi_i, \xi_i^*]_n \otimes_B W$.*

(3) *Similar results for the algebras $K\langle \bar{s}_i\rangle_n^t$, $K\langle \bar{t}_i\rangle_n^t$, $K\langle \bar{s}_i\rangle_n^s$, $K\langle \bar{t}_i\rangle_n^s$ with the following exceptions:*

(a) *I has one element, $n > 1$ for the transpose type; every n for the symplectic type. In this case the algebras are commutative.*

(b) *$K\langle \bar{t}_i\rangle_2^t$ is commutative.*

(c) *$K\langle \bar{s}_i\rangle_1^s$ is commutative.*

(d) *$K\langle \bar{s}_1, \bar{s}_2\rangle_2^s$ is a quaternion algebra.*

In the other cases, the results are parallel to cases (1) and (2) with the exception of the computation of the transcendence degree which is, respectively;

$$m\frac{n^2+n}{2} - \frac{n^2-n}{2}, \qquad m\frac{n^2-n}{2} - \frac{n^2-n}{2},$$

$$m\frac{(2n)^2-2n}{2} - \frac{(2n)^2+2n}{2}, \qquad m\frac{(2n)^2+2n}{2} - \frac{(2n)^2+2n}{2}.$$

Proof. Let us give the main ideas of the proof, for instance, in the transposition case.

It is clear that $K[\xi_i, \xi_i^t]_n$ is the image of the free algebra $K\{x_i, x_i^*\}$ under the universal map γ: $K\{x_i, x_i^*\} \to (K[x_{s,t}^{(i)}])_n$, $i \in I$, $s, t = 1,\dots, n$, (cf. 18.1). All the statements, except for the last part of (ii), will be a consequence of the theory of rings with polynomial identities and of Theorem 19.4 once we show that the representation $K\{x_i, x_i^*\} \to (K(x_{s,t}^{(i)}))_n^t$ is irreducible (here, $K(x_{s,t}^{(i)})$ denotes the field of rational functions in the variables $x_{s,t}^{(i)}$).

To prove that the representation is irreducible, one may proceed in various ways. If I has more than one element, we already have two generic matrices, and so the claim follows. Otherwise, we must show that a generic matrix and its transpose are irreducible, this can be checked by specializing to a matrix that with its transpose is irreducible, such matrices are readily found.

One can proceed similarly in all the other cases, of course, we have the exceptional cases described in (3) for which the universal map is not irreducible (at the generic point).

To complete the theorem, one has to prove the last part of (ii). Let us do it for the transpose type. First of all, any central simple algebra with involution of first kind is the full matrix ring over a division ring of degree a power of 2.

In our case, setting $K\langle \xi_i, \xi_i^t \rangle_n = (D)_h$, we must only prove that h is odd.

Let $n = 2^r \cdot k$, thus, we have to show that D has degree 2^r and $h = k$.

It is known that, given a field K and a number 2^r, there is a division ring E with involution of degree 2^r with center a field F containing K. The involution can be fixed to be of transposition type (cf. the remark after 14.2).

Consider the simple algebra $(E)_k$, of degree n and of transposition type. Let $a_1, \dots, a_{n^2}$ be a basis of $(E)_k$ over the center F. Introduce variables $x_j^{(i)}$, $i \in I$, $j = 1, \dots, n^2$ over F and construct the generic elements

$$\eta_i = \sum_{j=1}^{n^2} x_j^{(i)} a_j \in (E)_k \otimes_F F(x_j^{(i)}).$$

By the initial remarks, it is clear that the kernel of the map λ: $K\{x_i, x_i^*\} \to (E \otimes_F F(x_j^{(i)}))_k$ is the ideal of polynomial identities of $n \times n$ matrices with transposition. Thus, $K[\xi_i, \xi_i^t]_n$ is isomorphic to the algebra $K[\eta_i, \eta_i^t]$ generated by the elements η_i, η_i^t. Furthermore, λ is irreducible. Hence, we have an embedding

$$K\langle \xi_i, \xi_i^t \rangle_n \subseteq (E \otimes_F F(x_j^{(i)}))_k$$

and an isomorphism of *-algebras:

$$K\langle \xi_i , \xi_i^t \rangle_n \otimes_Z F(x_j^{(i)}) \simeq (E \otimes_F F(x_j^{(i)}))_k .$$

If the division ring constituent of $K\langle \xi_i , \xi_i^t \rangle_n$ had degree 2^s with $s < r$, we would have a contradiction since $E \otimes_F F(x_j^{(i)})$ is clearly a division ring of degree 2^r.

We finish with a consequence of Theorem 19.4 for the rings of invariants. It can be derived in characteristic $\neq 0$, but we will limit ourselves to the rings $TO_{i,n}$, $SO_{i,n}$, $T(Sp)_{i,2n}$, $S(Sp)_{i,2n}$.

We can consider the elements of $TO_{i,n}$ that are expressible, as elements of $SO_{i,n}$, as polynomials in the variables X_i, X_i^t with coefficients in K (i.e., the central polynomials for $n \times n$ matrices with transposition). The variety of points of $TO_{i,n}$ represents equivalence classes of semisimple orthogonal representations of the free algebra, 15.3, and the points of this variety on which some central polynomial does not vanish, represent the irreducible representations, by 19.3. This set is nonempty, by 20.1, and we may apply Theorem 19.4 to obtain:

THEOREM 20.2, *The irreducible representations of the free algebra are simple points of the variety of semisimple representations.*

Proof. The irreducible representations are the points of the base of a principal fibration over a reduced algebraic group with the total space nonsingular.

Remark. The discussion before 20.2 was done for the orthogonal case for convenience of language, but it clearly holds in both the orthogonal and the symplectic case. In both cases, one has 20.2 (and also in the case of $Gl(n, K)$, from which one has in fact started to obtain all these generalizations).

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