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Young Diagrams, Standard Monomials and Invariant Theory

Claudio Procesi

The topics I want to treat have various sources: (1) the fundamental theorems of invariant theory of the classical groups, (2) the representation theory of classical groups, (3) the geometry of the Grassmann and flag varieties.

The initial interest in these questions came for me from the solution given by Formanek and myself, **[8]**, of Mumford's conjecture for the linear group, **[25]**, obtained independently from the general solution due to Haboush **[9]**.

Our approach consisted in relating the geometric reductivity of the group $\mathrm{Gl}(V)$ with a property of the symmetric group S_m acting on the tensor space $V^{\otimes m}$.

In fact, in characteristic zero, one classical approach to the representation theory of $\mathrm{Gl}(V)$, and in particular to the proof that it is linearly reductive, is the following [27], **[29]**:

The algebras of operators spanned by the symmetric group and by $\mathrm{Gl}(V)$ on $V^{\otimes m}$ are each the full centralizer of the other. In characteristic zero then one can apply the theory of semisimple algebras.

In positive characteristic, if we consider the algebra Σ_m spanned by S_m on $V^{\otimes m}$, we have that Σ_m is no more semisimple but nevertheless the limit $\Sigma = \lim_{m\to\infty} \Sigma_m$ is semisimple in the sense of Jacobson and this suffices to prove the geometric reductivity of $\mathrm{Gl}(V)$.

This analysis led naturally to the question of extending the classical relations between $\mathrm{Gl}(V)$ and S_m to all characteristics; in particular to prove that $\Sigma_m = \mathrm{End}_{\mathrm{Gl}(V)}(V^{\otimes m})$.

It was well known that this equality is a simple consequence of a statement in invariant theory, the so-called first fundamental theorem, to which it is equivalent in characteristic zero. Similar questions arise for the other classical groups which require a similar (or sometimes more complicated) analysis **[29]**.

The relation with invariant theory comes from the identification $\mathrm{End}(V^{\otimes m}) \cong (V^{\otimes m} \otimes V^{*\otimes m})^*$ and hence the identification of $\mathrm{End}_{\mathrm{Gl}(V)}(V^{\otimes m})$ with the multilinear invariant polynomials in m vector and m covector variables.

It turns out, in positive characteristic, that it is possible to study first all the invariant polynomials and then deduce the form of the multilinear ones.

1. Fundamental theorems of classical invariant theory. Let us state, in a form slightly different from the usual one, the fundamental theorems of invariant theory of the classical groups, (see **[29]** for the theory in char 0).

(i) $G=\mathrm{Gl}(n, k)$. Consider multiplication of matrices $\pi: M_{k,n} \times M_{n,h} \to M_{k,h}\{(A, B) \to AB\}$, with image the variety V_n of $k \times h$ matrices of rank $\leq n$.

FIRST FUNDAMENTAL THEOREM. *The coordinate ring of V_n is the ring of invariants of G acting on $M_{k,n} \times M_{n,h}$ by $X \cdot (A, B)=(AX^{-1}, XB)$.*

SECOND FUNDAMENTAL THEOREM. *The ideal of functions on $M_{k,h}$ vanishing on V_n is generated by the $n+1 \times n+1$ minors.*

We should remark that there exists now in characteristic 0 a beautiful theory of the higher syzygies due to A. Lascoux **[15]**, **[22]**, **[23]**. Similar theorems hold for the other classical groups.

These theorems, with the exception of the very last, are all classical in characteristic zero and can be found in H. Weyl's book **[29]**.

The second fundamental theorems and various qualitative results on the determinantal varieties have been the object of intensive study by many authors both in positive and zero characteristics **[10]**, **[11]**, **[21]**, **[26]**.

In a joint paper with De Concini **[5]** we attacked all these problems in the spirit of Igusa's proof of the projective normality for the Grassmann variety **[14]**, again a classical theorem of invariant theory for $\mathrm{Sl}(n, K)$. His approach was through a careful analysis of the projective coordinate ring of the Grassmann variety. Such ring, as the one for the flag variety, were always fundamental objects in invariant theory **[1]** (the classical primary covariants of the Capelli–Deruyts expansion), the explicit bases (standard bases) for such rings were popularized by Hodge **[12]**, **[13]** (although they seem to have been known to Young **[30]**).

They give, in algebra, very explicit descriptions of the irreducible representations of the linear groups and the symmetric group (in characteristic zero), in geometry a very thorough understanding of the cellular decomposition of the Grassmann and flag varieties (by Schubert cells).

In char 0 the theory is based on the fact that the irreducible polynomial representations of $\mathrm{Gl}(V)$ can be indexed, as $L_\sigma(V)$, by partitions σ. One has the classical plethysm formulas:

$$S[V\otimes W]=\bigoplus_{\sigma} L_\sigma(V)\otimes L_\sigma(W),$$

$$S[S^2V]=\bigoplus L_\sigma(V),\ \sigma \text{ has even columns},$$

$$S[\Lambda^2V]=\bigoplus L_\sigma(V),\ \sigma \text{ has even rows}$$

(nevertheless the space $L_\sigma(V)$ is defined over Z!).

2. Standard bases. If $Y=(y_{ij})$ is an $n\times m$ matrix, we indicate by $p=(i_k\dots i_2i_1|j_1j_2\dots j_k)$ the determinant of the minor with rows i_t's, columns j_u's. If $p_1, p_2, \dots, p_k$ are minors of size $\sigma: h_1\geqslant h_2\geqslant\dots\geqslant h_k$, we display their product M as a double tableau with rows the p_i's. We say that M is a standard monomial (of shape σ) if the indices are strictly increasing on each row, non decreasing on each column, (separately on the right and on the left) [4], [7]. A similar definition holds for Y symmetric or antisymmetric.

THEOREM. *The standard monomials (in each case) are a Z basis of* $Z[y_{ij}]$. (Cf. [4], [7], [12].)

The theorems stated are very convenient to study the coordinate rings of the various determinantal varieties appearing in the fundamental theorems of invariant theory, in fact such theorems can be proved in a characteristic free way [5], precisely by using such standard bases.

In fact the previous form of the theorems contains explicit algorithms, by use of quadratic equations, to express a non standard product in terms of standard ones.

To a partition $\sigma: k_1\geqslant k_2\geqslant\dots\geqslant k_r$ associate the dual partition $\check{\sigma}: h_1\geqslant h_2\geqslant\dots\geqslant h_t$ with $h_i=\{\# j|k_j\geqslant i\}$ and a sequence $\gamma_i(\sigma)=\sum_{j\geq i}h_j$; then set $\sigma\geqslant\tau$ if $\gamma_i(\sigma)\geqslant\gamma_i(\tau)$ for all i.

A filtration of $Z[y_{ij}]$ is defined setting A_σ=span of all standard monomials of shape $\geqslant\sigma$. It has the following geometric interpretation [3]: A_0 is exactly the ideal of functions vanishing, for each i, on the variety P_i of matrices of rank i to order $\geqslant\gamma_{i+1}(\sigma)$.

The graded space of this G invariant filtration has again the direct sum decomposition as in the Plethysm formulas.

3. Admissible pairs. More or less at the same time that these standard bases were studied for the invariant theory of classical groups, Musili [26] and Seshadri [28] analyzed the relationship between standard bases and Schubert cells having in mind a better understanding of the vanishing theorems (proved in general by Kempf [16]; see also [17]) for the cohomology of line bundles in the positive chamber on a variety G/P, G a reductive group, P a parabolic subgroup.

If the parabolic P is associated to a fundamental weight ω, with L_ω the corresponding line bundle, the purpose of the analysis is first of all to understand $H^0(G/P, L_\omega^m)$. This was done, first of all, when ω is minuscule. In this case one has a basis of $H^0(G/P, L_\omega)$ by extremal weight vectors which index the Bruhat cells of G/P.

The face ordering of Bruhat cells induces a partial ordering on these generalized Plücker coordinates which also satisfy quadratic equations, so that:

THEOREM. *The standard monomials of degree m in such coordinates are a basis of $H^0(G/P, L_\omega^m)$.*

This theorem includes, by suitable interpretation, the standard basis theory for $S[V\otimes W]$ and $S[\Lambda^2 V]$.

The case $S[S^2V]$ can be interpreted as a theorem on $H^0(G/P, L_\omega^m)$ (G the symplectic group) but it is related to a non minuscule weight.

From this example Lakshmibai, Musili and Seshadri have been able to formulate **[20]** and prove **[19]** a general theorem. The hypothesis is that ω is a "classical weight", this means that intersecting a Schubert variety (closure of a Bruhat cell) with the hyperplane class one obtains the Schubert varieties σ_i faces of σ with multiplicity $\leqslant 2$. We can draw a diagram of the ordered set of Bruhat cells, with a double bond each time that the intersection multiplicity is 2.

One defines an admissible pair of cells as one which can be joined by a sequence of double bonds. The pairs (τ, τ) are considered admissible and called trivial pairs.

THEOREM. (LAKSHMIBAI, MUSILI, SESHADRI **[19]**.)

(a) *There is a basis $P_{\tau,\sigma}$ of $H^0(G/P, L_\omega)$ indexed by admissible pairs. $P_{\tau,\sigma}$ is a weight vector of weight $-\frac{1}{2}(\tau(\omega)+\sigma(\omega))$.*

(b) *Define a product $P_{\tau_1\sigma_1}P_{\tau_2\sigma_2}\dots P_{\tau_k\sigma_k}$ to be standard if $\tau_1\geqslant\sigma_1\geqslant\tau_2\geqslant\sigma_2\geqslant\dots\geqslant\tau_k\geqslant\sigma_k$. The standard monomials of degree m are a basis of $H^0(G/P, L_\omega^m)$.*

A similar but more complicated analysis gives, for classical groups, standard bases of $H^0(G/B, L)$, B a Borel subgroup and L any positive line bundle.

4. Combinatorial theory of invariant ideals. We go back to the decomposition $A=S[V\otimes W]=\oplus L_\sigma(V)\otimes L_\sigma(W)$. The module $M_\sigma=L_\sigma(V)\otimes L_\sigma(W)$ is irreducible under $G=\mathrm{Gl}(V)\times\mathrm{Gl}(W)$. Any invariant subspace I of A is thus of the form $\oplus_{\sigma\in\mathscr{I}}M_\sigma$ for some set $\mathscr{I}$ of partitions. In particular one may study G invariant ideals; this has been accomplished in joint work with De Concini and Eisenbud **[3]**. We summarize the results.

DEFINITION. A set $\mathscr{I}$ of partitions is an "ideal" if: $\sigma\in\mathscr{I}$ and $\tau\supseteq\sigma$ implies $\tau\in\mathscr{I}$. (If $\sigma: k_1\geqslant k_2\geqslant\dots\geqslant k_r$, $\tau: m_1\geqslant\dots\geqslant m_r$ we say $\tau\supseteq\sigma$ if $m_i\geqslant k_i$ for all i.)

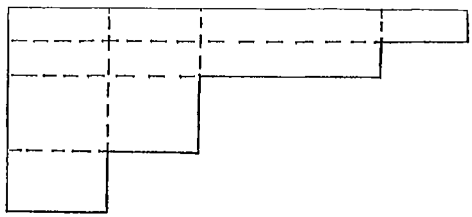
THEOREM. *The decomposition $I=\oplus_{\sigma\in\mathscr{I}}M_\sigma$ establishes a 1–1 correspondence between G-invariant ideals of A and "ideals" of partitions.*

DEFINITION. $\sigma\cdot\tau$ is the partition having as columns the sum of the corresponding columns.

DEFINITION. (a) An "ideal" $\mathscr{I}$ of partitions is prime if $\sigma\tau \in \mathscr{I}$, $\sigma \notin \mathscr{I}$ implies $\tau \in \mathscr{I}$.
(b) $\sqrt{\mathscr{I}} = \{\sigma | \sigma^k \in \mathscr{I}$ for some $k\}$.
(c) $\mathscr{I}$ is primary if $\sigma\tau \in \mathscr{I}$, $\sigma \notin \sqrt{\mathscr{I}}$ implies $\tau \in \mathscr{I}$.

THEOREM. *The* 1–1 *correspondence previously given preserves the notions of prime, primary and radical.*

As an example let us define I_σ as the (G-invariant) ideal generated by M_σ. I_σ corresponds to the "principal ideal" of partitions $(\sigma) = \{\tau | \tau \supseteq \sigma\}$. Then I_σ has a nice primary decomposition: e.g.

$\sigma =$ 

$\sigma_1, \sigma_2, \sigma_3, \sigma_4$ the diagrams of maximal rectangles in σ, then: $I_\sigma = I_{\sigma_1} \cap I_{\sigma_2} \cap I_{\sigma_3} \cap I_{\sigma_4}$.

THEOREM. (i) $A_\sigma = I_{k_1} I_{k_2} \ldots I_{k_r}$ $(\sigma: k_1 \geqslant k_2 \geqslant \ldots \geqslant k_r)$.
(ii) $A_\sigma = I_1^{(\gamma_1(\sigma))} \cap I_2^{(\gamma_2(\sigma))} \cap \ldots \cap I_n^{(\gamma_n(\sigma))}$
(where $P^{(m)}$ *means the symbolic power) is a primary decomposition.*
(iii) A_σ *is the integral closure of* I_σ.

THEOREM. *An integrally closed G-invariant ideal is of the form* ΣA_{σ_i} *with the restriction:*
If $\gamma(\tau) \geqslant$ *convex combination of* $\gamma(\sigma_i)$, *then* $\gamma(\tau) \geqslant$ *some* $\gamma(\sigma_i)$.

We want to mention a last result which can be obtained by using all the ingredients of the theory [4]: Let $I_k^{(n)}$ denote the nth symbolic power of I_k. The algebra $B = \bigoplus_n I_k^{(n)} / I_k^{(n+1)}$ has a very explicit theory of standard monomials and:
(i) B is a finitely generated algebra;
(ii) B is normal and Cohen–Macaulay;
(iii) B is the ring of global functions on the normal bundle (in the affine space of matrices) of the variety of matrices of rank $k-1$.

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UNIVERSITY OF ROME
00100 ROME, ITALY