Geheimtun kann auch aus einer andern Quelle kommen. Das wirkliche Geheimnis nämlich tut zwar nicht geheim; es spricht aber geheim: es deutet sich an durch vielerlie Bilder, die auf sein Wesen hinweisen. Ich meine damit nun nicht ein von jemand persönlich gehütetes Geheimnis, das einem dem Besitzer bekannten Inhalt hat, sondern eine Sache oder Angelegenheit welche "geheim", d.h. nur aus Andeutungen bekannt im wesentlichen aber unbekannt ist.

C. G. Jung, *-Alchemie und Psychologie Teil III, Kapitel II.1*

AMELIE SCHREIBER

SURFACE ALGEBRAS

Contents

Introduction

This is an introduction to **surface algebras**, a class of (often infinite dimensional) associative algebras, arising naturally from the medial quiver of a graph that has been cellularly embedded into a Riemann surface.

Surface algebras are a class of infinite dimensional *gentle algebras*, which provide a link between many classes of *special biserial algebras*, as well as the *skew-gentle* and *clannish* algebras. They are introduced in C. M. Ringel's Shanghai Lectures as *complete gentle algebras*, and are used in [1] to study the invariant theory of special biserial algebras. The first examples of such algebras go back to G'elfand and Ponomarev's work "Indecomposable Representations of the Lorentz Groups," and subsequently Ringel's "Indecomposable Representations of the Dihedral 2-Group". They occur implicitly in many places, but seem to have almost no explicit published material written about them as a whole, and there are no references which presents them as "surface algebras" per se, although constructing algebras from quivers on surface triangulations is currently quite popular among those working on *cluster algebras*, and was implicit in the Shanghai Lectures. All current constructions of algebras from triangulated surfaces seem to be modifications of the construction provided here, so it is likely many are using these ideas without explicitly stating them. It is my hope that what follows will open the door for others to study *surface algebras*, especially those who are newcomers to quivers and their representations and wish to apply them to other areas.

[1]

Part I

Combinatorics and Compact Gentle Surface Algebras

Graphs, Quivers, and Combinatorial Topology

Graphs, Quivers, Combinatorial Maps, and Constellations

We begin by introducing the notation and basic terminology needed from combinatorial topology that we will need throughout. Much of the material throughout can be conveniently and simply phrased in terms of some basic notions from combinatorial topology for (possibly directed) graphs, and two dimensional surfaces given as a CW-complex. Many of the results and statements are then able to be illustrated by very concrete examples which can be calculated by hand and are conveniently encoded in the usual pictures seen in an introductory combinatorial topology with polygon presentations of surfaces. The intent is to set up most if not all of the representation theory in this language so that the prerequisites are minimal beyond a basic course in algebraic topology, and an understanding of rings and modules. In particular, the material from [2] and [3] should be more than sufficient preparation.

[2]

[3]

Graphs and Quivers

We will define a **graph** G to be a set V of **vertices**, a set E of **edges**, and an **incidence map** $\partial : E \to V \times V$, sending an edge e to its two (possibly equal) incident vertices $\partial e = \{partial_1 e, \partial_2 e\} = \{v, w\}$. We define a **directed graph** or a **quiver** to be a graph with *vertices* Q_0, *arrows* Q_1, and incidence map $\partial a = \{partial_1 a, \partial_2 a\} = \{ta, ha\}$ which gives an *ordered* set $ta < ha$, with ta the *tail* of the arrow a, and ha the *head* of the arrow a.

We will define a **graph map** $\theta : G \to G'$, where $G = (V, E)$ and $G' = (V', E')$ to be a set map on vertices, $\theta_0 : V \to V'$, and edges $\theta_1 : E \to E'$ are mapped according to the following diagram for every edge $e \in E$ such that $\partial e = \{v, w\}$,

$$\begin{tikzcd}
v \arrow[r, dash, "e"] \arrow[d, "\theta_0"'] & w \arrow[d, "\theta_0"] \\
\theta_0(v) \arrow[r, dash, "\theta_1(e)"'] & \theta_0(w)
\end{tikzcd}$$

In other words, we say graph maps *"preserve adjacency"*. Similarly, if we have a quiver $Q = (Q_0, Q_1)$, a **quiver map** will be one which not only preserves adjacency by also the order of the incidence map, as in the following diagram,

$$\begin{tikzcd}
v = ta \arrow[r, "a"] \arrow[d, "\theta_0"'] & w = ha \arrow[d, "\theta_0"] \\
\theta_0(v) = t(\theta_1 a) \arrow[r, "\theta_1(a)"'] & \theta_0(w) = h(\theta_1 a)
\end{tikzcd}$$

We will only work with **locally finite** graphs (resp. quivers), i.e. there will be countably many vertices and edges (resp. arrows), and between any two vertices there will be finitely many edges (resp. arrows).

Combinatorial Maps and Constellations

Let S_n be the symmetric group on $[n] = \{1, 2, 3..., n\}$. Permutations will act on the left, so if $\sigma \in S_n$, we will say $\sigma \cdot i = \sigma(i)$. For example, for $\sigma = (1, 3, 2) \in S_3$ we have

$$\sigma(1) = 3, \quad \sigma(2) = 1, \quad \sigma(3) = 2.$$

We will define a k-**constellation** to be a sequence $C = [g_1, g_2, ..., g_k]$, $g_i \in S_n$, such that:

1. The group $G = \langle g_1, g_2, ..., g_k \rangle$ generated by the g_i acts *transitively* on $[n]$.
2. The product $\prod_i^k g_i = \mathbf{id}$ is the identity.

The constellation C has "**degree** n" in this case, and "**length** k". Our main interest will be in 3-constellations $C = [\sigma, \alpha, \phi]$, which we will describe in detail momentarily. The group $G = \langle g_1, g_2, ..., g_k \rangle$ will be called the **cartographic group** or the **monodromy group** generated by C.

Combinatoral Maps as CW-Complexes

There is a correspondence between 3-constellations $C = [\sigma, \alpha, \phi]$ such that α is a fixed point free involution, and graphs which are cellularly embedded in closed Riemann surface. In particular, such constellations give a CW-complex structure on the surface Σ.

The Clockwise Cyclic Vertex Order Construction

There are many equivalent ways of defining a graph on a Riemann surface. One of the simplest and probably the most combinatorial ways is by constellations. There are at least two ways of viewing this construction. We present two here, which are in some sense dual to one another. Intuitively, we follow the recipe:

1. First choose some positive integer $r \in \mathbb{N}$ to be the number of *vertices* of the graph, say $\Gamma_0 = \{x_1, x_2, ..., x_r\}$.
2. Then, to each vertex x_i, we choose some number k_i, of *"half edges"* to attach to it, with the rule that once we have chosen k_i for each x_i, the sum $\sum_{i=1}^{r} k_i = 2n$, must be some positive even integer.
3. We then choose a *clockwise cyclic ordering* of the *"half-edges"* around each vertex x_i, i.e. some cyclic permutation σ_i of $[k_i] = \{1, 2, ..., k_i\}$ for each x_i. The cyclic permutations σ_i must all be disjoint from one another, and together they form a permutation of $[2n]$.
4. Once such a *cyclic ordering* is chosen, we then define a *gluing* of all of the *"half edges"*. In particular, we choose some fixed-point free involution on the collection of all half edges, which is a permutation in S_{2n} given by $|\Gamma_1|$ many 2-cycles. This defines α.

   We then have the usual **Euler formula**,

$$|\phi| - |\alpha| + |\phi| = F - E + V = \chi(\Sigma).$$

Said a slightly different way, we define a pair $[\sigma, \alpha]$, where $\sigma, \alpha \in S_{2n}$. The permutation

$$\sigma = \sigma_1 \sigma_2 \cdots \sigma_r$$

is a collection of cyclic permutations, one σ_i for each vertex x_i of our graph $\Gamma = (\Gamma_0, \Gamma_1)$. So, perhaps in better notation, each

$$\sigma_x = (e(x)_1, e(x)_2, ..., e(x)_{k(x)}),$$

can be thought of as giving a cyclic ordering of the **half edges**[4],

$$\Gamma_1(x) = \{e(x)_1, e(x)_2, ..., e(x)_{k(x)}\},$$

attached to each vertex $x \in \Gamma_0$ in our graph. The cycles σ_x are all necessarily disjoint. Then we define how to glue pairs of *half edges*, in order to get a connected graph Γ, via the permutation α. The permutation α is the form [5],

$$\begin{aligned}\alpha &= \alpha^1 \alpha^2 \cdots \alpha^t \\ &= (\alpha_1, \alpha_2)(\alpha_3, \alpha_4) \cdots (\alpha_{2n-1}, \alpha_{2n}) \\ &= \prod_{e \in \Gamma_1} \alpha(e)\end{aligned}$$

and each (α_i, α_{i+1}) tells us to glue the two corresponding *half-edges*. Let us illustrate this by a simple example.

[4] We will use the notation $\Gamma_1(x)$ to denote the *"half-edges"* around the vertex $x \in \Gamma_0$. The term *"half-edges"* is a commonly used term in the literature on combinatorial maps, and is a nice conceptual way of remembering the formalities of the construction of a graph cellularly embedded into a surface.

[5] Here we view

$$\alpha : \Gamma_1 \to \amalg_{x \in \Gamma_0} \Gamma_1(x)$$

as a map from the edges Γ_1, to the *half-edges* $\amalg_{x \in \Gamma_0} \Gamma_1(x)$. So $\alpha(e) = (e(x)_p, e(y)_q)$.

As a permutation on the set of all *half edges*,

$$\Gamma_1(x) \amalg \Gamma_1(y) = \{e(x)_1, e(x)_2, e(x)_3, e(y)_1, e(y)_2, e(y)_3\},$$

around two vertices $\Gamma_0 = \{x, y\}$ we may identify $\sigma, \alpha \in \mathbf{Perm}(E_x \amalg E_y)$ with permutations in S_6. Namely, let us define the identification

$$\begin{aligned} \sigma &= \sigma_x \sigma_y \\ &= (e(x)_1, e(x)_2, e(x)_3) \cdot (e(y)_1, e(y)_2, e(y)_3) \leftrightarrow (1,2,3)(4,5,6) \in S_6 \end{aligned}$$

and let

$$\alpha = (e(x)_1, e(y)_1)(e(x)_2, e(y)_2)(e(x)_3, e(y)_3) \leftrightarrow (1,4)(2,5)(3,6) \in S_6.$$

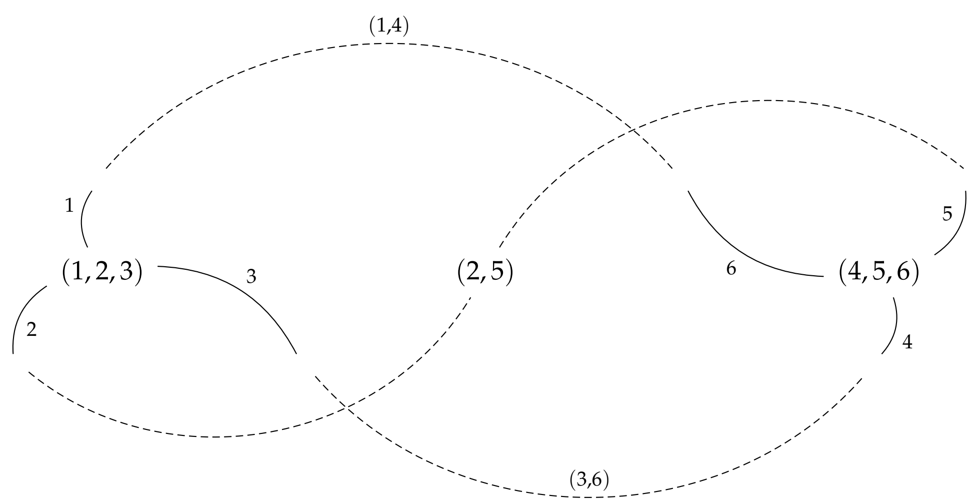
Then under this identification, the graph with

- two vertices $\Gamma_0 = \{x, y\} \leftrightarrow \{\sigma_x, \sigma_y\} = \{(1,2,3), (4,5,6)\}$,
- and six half edges

$$\Gamma_1(x) \amalg \Gamma_1(y) \leftrightarrow \{1,2,3\} \amalg \{4,5,6\}.$$

may be represented by the following picture to help visualize this,

$$\sigma = (1,2,3)(4,5,6) \qquad\qquad \alpha = (1,4)(2,5)(3,6)$$



It may be prudent to have more of the notation gathered into one place to refer back to later.

Notation	Meaning
$\Gamma \hookrightarrow \Sigma$	cellularly embedded graph
Σ	a Riemann surface, generally closed
Γ_0	the vertex set of a graph Γ
Γ_1	the edge set of a graph Γ
$\Gamma_1(x)$	the *half-edges* around a vertex
$e(x)_i$	a *half-edge* in $\Gamma_1(x)$ attached to $x \in \Gamma_0$
$\partial e = \{\partial_1 e, \partial_2 e\}$	the vertices adjacent to $e \in \Gamma_1$
$\alpha^k = (\alpha_i, \alpha_{i+1})$	a 2-cycle of α
$(\alpha_i, \alpha_{i+1}) = (e(x)_p, e(y)_q)$	glued *half-edges* $e(x)_p$ and $e(y)_q$
$\alpha(e) = (e(x)_p, e(y)_q)$	α as a map $\Gamma_1 \to \amalg_{x\in\Gamma_0}\Gamma_1(x)$

Table 1: There are now a mildly disturbing amount of notations for describing a 2-cycle

$$\alpha^k = (\alpha_i, \alpha_{i+1}) = (e(x)_p, e(y)_q).$$

It indicates a gluing of two *"half-edges"*, and so represents a single edge in Γ_1 It can also be thought of as a component of the map

$$\alpha : \Gamma_1 \to \amalg_{x\in\Gamma_0}\Gamma_1(x),$$

identifying the edge e, with its two half-edges $\alpha(e) = (e(x)_p, e(y)_q)$.

The Polygon Construction

It is important at this point to make a few comments. Not every graph is planar, i.e. there may be no embedding on the sphere $S^2 = \mathbb{P}^1$ without edge crossings. To see a second way this plays out with constellations, we now turn to the dual construction on faces. In the last section, the permutation $\phi = \alpha\sigma^{-1}$, defining the constellation $C = [\sigma, \alpha, \phi]$[6], were quite neglected in the construction. This is partially because they are not strictly needed since $\sigma\alpha\phi = \mathbf{id} \implies \alpha\sigma^{-1} = \phi$.

[6] Recall: 3 constellations $C = [\sigma, \alpha, \phi]$ giving a graph cellularly embedded into a Riemann surface, must be defined by permutations $\sigma, \alpha, \phi \in S_{2n}$ such that $\sigma\alpha\phi = \mathbf{id}$, the permutation α is a fixed-point free involution, and the *cartographic group* $G = \langle\sigma, \alpha, \phi\rangle = \langle\sigma, \alpha\rangle$ acts transitively on $[2n] = \{1, 2, ..., 2n-1, 2n\}$.

Perhaps more importantly though, the previous construction focused on *"cyclic orderings"* of the *half edges* around each vertex, and gluings of those half edges to obtain a connected graph Γ. There is another way of constructing cellular embeddings which comes from polygon presentations of surfaces. This is likely more familiar to the reader, and therefore more intuitive. The question might be asked, "why not just use this more typical example." One answer would be, the former construction is actually quite standard in the literature on combinatorial maps. A better answer however is, the combinatorics and the notation involved in the previous (clockwise) *"cyclic vertex ordering"* construction is much more convenient for later constructions involving *medial quivers*, *surface algebras*, and the representation theory that follows. It will be useful, and sometimes more intuitive to have this second construction though. Let us begin with the following recipe:

1. Write ϕ as a product of disjoint cycles $\phi_1\phi_2 \cdot \phi_p$

2. To each cycle ϕ_i of length m_i we associate a m_i-gon, oriented *counterclockwise*.

3. Then we glue the sides of each polygon according to α so that the sides which are glued have opposite orientation.

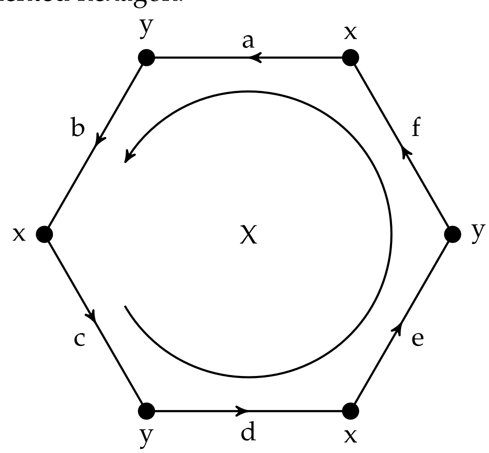
4. From this gluing we obtain a cyclic order of edges $\sigma = \phi^{-1}\alpha$ around each vertex.[7]

[7] Note: $\alpha = \alpha^{-1}$ since it is required to be an involution. Also, the vertices with cyclic orderings are the corners of the polygons after gluing.

Let us once again illustrate by example. Take $C = [\sigma, \alpha, \phi]$ from the previous construction where

$$\sigma = \sigma_x \sigma_y = (1,2,3)(4,5,6), \quad \alpha = \alpha^1 \alpha^2 \alpha^3 = (1,4)(2,5)(3,6).$$

This implies $\phi = (162435)$. This is represented by a counterclockwise oriented hexagon:



The *"word"* associated to the polygon given by ϕ in most standard texts containing material on polygon presentations of surfaces is

$$abcdef \leftrightarrow (162435).$$

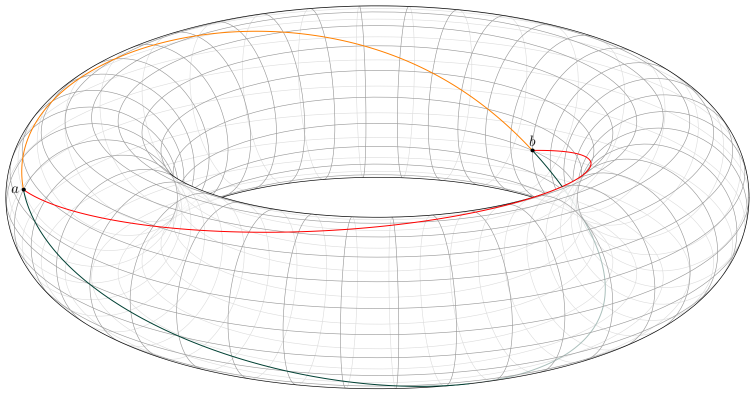
The gluing $\alpha = (1,4)(2,5)(3,6)$, then says we must glue the faces:

$$a \leftrightarrow d, \quad b \leftrightarrow e, \quad c \leftrightarrow f.$$

Care must be taken to glue sides so that their orientations "appose" one another so that the surface obtained is *oriented* according to the *counterclockwise* oriented face. The cellularly embedded graph that we obtain lives on a torus $\mathbb{T}^2$. We can determine this purely via the combinatorics by computing

$$\chi(\Sigma) = |phi| - |\alpha| + |\sigma| = |\phi| - |\Gamma_1| + |\Gamma_0| = 1 - 3 + 2 = 0,$$

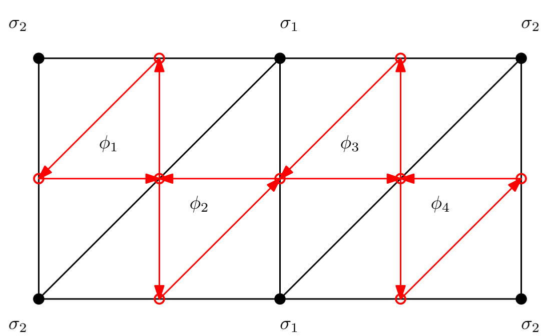
and since $\chi(\Sigma) = 2g(\Sigma) - 2$ we have that the genus of Σ is $g = 1$.



Medial Quivers of Combinatorial Maps and Constellations

There is a very natural way of associating a cellularly embedded graph to a quiver, and a quiver to a cellularly embedded graph. In particular, we can define a bijection of such objects. The way we do this is by choosing the quiver to be the directed medial graph of the cellularly embedded graph. In particular, for each face ϕ_j of $C = [\sigma, \alpha, \phi]$, we place a vertex on the interior of each edge of the boundary of ϕ_j. We then connect the vertices counter-clockwise with arrows. This forms the **medial quiver** of the constellation, or equivalently of the cellularly embedded graph.

Example 0.0.1.



As an example, we have the medial quiver for a triangulation of a torus given by the constellation

$$\phi = (1,2,3)(4,5,6)(7,8,9)(10,11,12), \quad \alpha = (1,5)(2,12)(3,4)(6,7)(8,10)(9,11).$$

It has face cycles given by $\phi = \phi_1\phi_2\phi_3\phi_4$, and the gluing α identifies the top edges and bottom edges, as well as the left and right side edges, in the typical way.

Covering Theory of Combinatorial Maps and Constellations

Covering Theory of Medial Quivers

Linear Path Categories of Quivers

Category Theory for Pedestrians

Combinatorial Topology Approach

We may give any category, in particular "*linear categories*", the structure of a *directed graph* with *path multiplication*, which is *extended linearly* (i.e. a *quiver path algebra*) [8] where objects are vertices, and morphisms are arrows. This allows us to set up much of the material in terms of combinatorial algebraic topology, and gives us a way of encoding the ideas into intuitive pictures with an aesthetic that might intrigue even the most formal and serious reader.

[8] As an example, say we identify some category with two objects and a single morphism $\mathcal{C}$, with the quiver Q,

$$x \xrightarrow{a} y$$

and the paths of Q are $\{e_x, e_y, ae_x = a = e_y a\}$, where we read paths from right to left like composition of linear maps, and e_x, e_y are the **trivial paths**. The path category $\mathbf{P}(Q)$ has objects $Q(x,x) = \{e_x\}$, $Q(y,y) = \{e_y\}$, and $Q(x,y) = \{a\}$.

Under this construction, we may define the **category of categories**, denoted by $\mathfrak{C}$. We may consider concatenation of two consecutive arrows in the quiver Q of the category $\mathcal{C}$ as a composition of two morphisms. In this way, we may define a category as a **path category** $\mathbf{P}(Q)$, of a quiver Q. In particular, $Q(x,y)$ will denote all paths from vertex x to vertex y, and "multiplication" is given by concatenation of two consecutive arrows (or else is zero if this is not possible). This is an associative operation, and the collection of all paths in the quiver Q identified with the category $\mathcal{C}$ will be what we have called path category $\mathbf{P}(Q)$, with Q_0 as objects, and the collection of all $Q(x,y)$ the morphisms. The morphisms in the category of all categories $\mathfrak{C}$, will be called **functors**, and will be given by directed graph maps (i.e. quiver maps) which preserve products (of arrows) and the *local identities* $e_i, i \in Q_0$, at each vertex $i \in Q_0$. We will call a category a **linear category**[9], or a $\mathbb{K}$-**category**, if all morphism sets $\mathcal{C}(A,B)$ have a $\mathbb{K}$-linear structure, and all composition maps are $\mathbb{K}$-bilinear, for some field $\mathbb{K}$.

[9] So, we define the linear structure on Q to be "formal linear combinations of paths". If Q is

$$x \xrightarrow{a} y$$

$\mathbb{K}\,Q(x,y) = \mathbb{K}\,a \cong \mathbb{K}$, is a one dimensional vector space. Similarly $\mathbb{K}\,Q(x,x) = \mathbb{K}\,x \cong \mathbb{K}\,y = \mathbb{K}\,Q(y,y)$. We may define a vector space $\mathbb{K}\,Q$ with basis $\{x, y, a\}$, which is isomorphic to $\mathbb{K}^3$ as a vector space.

A **linear functor** $\theta : \mathcal{A} \to \mathcal{B}$ between two $\mathbb{K}$-categories if the associated graph maps are $\mathbb{K}$-linear[10]. In pictures, this means all arrows in the following diagram are $\mathbb{K}$-linear,

$$\begin{tikzcd} v = ta \arrow[r, "a"] \arrow[d, "\theta_0"'] & w = ha \arrow[d, "\theta_0"] \\ \theta_0(v) = t(\theta_1 a) \arrow[r, "\theta_1(a)"'] & \theta_0(w) = h(\theta_1 a) \end{tikzcd}$$

We define an **ideal** $\mathcal{I}$ of a $\mathbb{K}$-category $\mathcal{C}$, to be a family of subspaces $\mathcal{I}(v,w) \subset \mathcal{C}(v,w)$, such that if $f \in \mathcal{C}(u,v)$ and $g \in \mathcal{C}(w,x)$, then $g \cdot \mathcal{I}(v,w) \cdot f \subset \mathcal{I}(u,x)$. In terms of quivers, we may visualize this as follows. Suppose we have the following diagram, with $a \in \mathcal{I}(v,w)$:

$$u \xrightarrow{f} v \xrightarrow{a} w \xrightarrow{g} x$$

Then $a \cdot f \in \mathcal{I}(u,w)$, $g \cdot a \in \mathcal{I}(v,x)$, and $g \cdot a \cdot f \in \mathcal{I}(u,x)$. We will define the **quotient category** $\mathcal{C}/\mathcal{I}$ to be the category with the same objects as $\mathcal{C}$, but with morphisms $(\mathcal{C}/\mathcal{I})(v,w) = \mathcal{C}(v,w)/\mathcal{I}(v,w)$, and composition of morphisms will be the residue class of the composition of some chosen representative morphisms. We will define the $\mathbb{K}$-linear path category $\mathbb{K}\,Q$, of a quiver Q, to have objects Q_0, the vertices of the quiver, morphisms all paths $p \in Q(u,v)$ endowed with a linear vector space structure $\mathbb{K}\,Q(v,w)$, treating the paths as formal basis elements. In other words, $\mathbb{K}\,Q(u,v)$ is the vector space of formal linear combinations of paths from u to v, for vertices $u, v \in Q_0$. All ideals we will deal with will be generated by linear combinations of paths in various $Q(u,v)$.

[10] Take Q to be the category with two objects and one morphism,

$$x \xrightarrow{a} y \,.$$

For a second category, say

$$x' \xrightarrow{a'} y' \xrightarrow{b'} z'$$

we have $\mathbb{K}\,Q'$ has basis $\{x', y', z', a', b'b'a'\}$ and is isomorphic to $\mathbb{K}^6$. Then we must define a $\mathbb{K}$-linear map $V : \mathbb{K}^3 \to \mathbb{K}^6$ and then we must impose some "graph map structure" onto it that makes sense. On basis elements we could order the bases as above, then define V to be $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. This would take $(x \to y) \mapsto (x' \to y')$ mapping the arrow $a \mapsto a'$. We could also choose $Va = b'$ which would be given by the matrix $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

The Typical Definition of Category, Functor, and Natural Transformation

The following definition is the usual definition of a category, however, we will interpret categories in a more intuitive way. A **category** $\mathcal{C}$ can be defined as

1. A collection of **objects** $A, B, C, \ldots$, denoted $(Ob)((C)$;

2. a family of disjoint sets $\mathcal{C}(A,B)$, one for each pair (A,B) of objects, called **morphisms**, with a rule of composition that is associative;

3. a distinguished element called a **local identity** e_A of $\mathcal{C}(A,A)$ for each object A, such that if $\theta : A \to B$ is a morphism of $\mathcal{C}(A,B)$, then $e_B\theta = \theta = \theta e_A$; and

The Linear Path Category of a Medial Quiver

Now, let us suppose that $Q = Q(\Sigma)$ for some combinatorially embedded graph $\Sigma = [\sigma, \alpha, \phi]$. We will define a representation V of Q to be:

1. For every vertex $x \in Q_0$ of the quiver, an assignment of some $\mathbb{K}$-vector space $V(x)$, and

2. for every arrow $a \in Q_1$ the assignment of some linear map $V(a)$.

This can be thought of as a linear functor [11] from a small linear category $\mathbb{K}\,Q$, to the category of $\mathbb{K}$-vector spaces $\mathbf{Mod}(\mathbb{K})$, so that for each vertex and arrow we have the following diagram,

$$\begin{tikzcd} x \arrow[r, "a"] \arrow[d, "V_0"'] & y \arrow[d, "V_0"] \\ V_0(x) \arrow[r, "V_1(a)"'] & V_0(y) \end{tikzcd}$$

[11] i.e. a map of quivers (equivalently a directed graph map), $V : Q \to Q'$ in the category of categories $\mathfrak{C}$, "extended linearly". Here, Q can be thought of as a subcategory of the category $\mathcal{D}$ of all directed graphs. Thinking of the category $\mathbf{Mod}(\mathbb{K})$ of all $\mathbb{K}$-vector spaces as a directed graph can be more intuitive, but after a moment of thought it is potentially a bit unsettling considering the absurdity of vertices and arrows that follow. Try to use the picture as a mnemonic tool, and a way of encoding a large amount of abstract information into a picture which can then be filtered by all of the formalisms later on.

Here we have $V_0 : Q_0 \to Q_0'$ is the assignment of the vector spaces to the vertices, and $V_1 : Q_1 \to Q_1'$ is the assignment of a linear map. The functor $V = (V_0, V_1)$ is extended to a linear functor in the obvious way. Representation of a quiver Q are often identified with "*modules over the path algebra*", which we will discuss later.

The Path Category as an Associative Algebra

For the sake of completeness, let us introduce some standard terminology used in the literature.

path algebra
ideal
path algebra of quiver with relations
modules over the path algebra

Definition 0.0.2. Let $Q = (Q_0, Q_1)$ be a finite connected quiver. Then we say the bound path algebra $\Lambda = kQ/I$ is a **compact gentle surface algebra** if the following properties hold:

1. For every vertex $x \in Q_0$ there are exactly two arrows $a, a' \in Q_1$ with $ha = x = ha'$, and exactly two arrows $b, b \in Q_1$ such that $tb = x = tb'$.

2. For any arrow $a \in Q_1$ there is exactly one arrow $b \in Q_1$ such that $ba \in I$, and there is exactly one arrow $c \in Q_1$ such that $ac \in I$.

3. For any arrow $a \in Q_1$ there is exactly one arrow $b' \in Q_1$ such that $b'a \notin I$, and there is exactly one arrow $c' \in Q_1$ such that $ac' \notin I$.

4. The ideal I is generated by paths of length 2.

We let e_x denote the primitive idempotent corresponding to $x \in Q_0$ and $P(x) = \Lambda e_x$ denote the (infinite- dimensional) projective left Λ-module. Similarly we let $S(x) = P(x)/\,\mathbf{rad}\,P(x)$ denote the simple Λ-module corresponding to $x \in Q_0$, and $I(x) = e_x\Lambda$ is the injective left Λ-module corresponding to x.

Thus, given a quiver such that every vertex has in-degree and out-degree exactly 2, we may choose several ideals I such that kQ/I is a compact gentle surface algebra. Such algebras are always infinite dimensional, but they retain many of the nice combinatorial and representation theoretic properties of finite dimensional gentle algebras.

Equivalence of Modules and Representations

$\mathbf{Mod}(\Lambda)$ and $\mathbf{Rep}(Q)$ $\mathbf{mod}(\Lambda)$ and $\mathbf{rep}(Q)$ $\mathbf{Mod}(\Lambda)$ and $\mathbf{Rep}(Q/I)$ $\mathbf{mod}(\Lambda)$ and $\mathbf{Rep}(Q/I)$

For a quiver Q, we will define a **dimension vector** to be a function $\mathbf{d} : Q_0 \to \mathbb{N}_{\geq 0}$. For a finite quiver this can be written as a vector in $\mathbb{N}^{Q_0}$. In particular, if we have a representation V of the quiver Q, then $\mathbf{d}\,V(x) = \dim_{\mathbb{K}} V(x)$.

Descriptions and Examples of Objects in the Path Category

projective objects

injective objects

free objects

irreducible or **simple objects**

indecomposable objects

Examples

The Riemann Surfaces

Let $\tilde{A}(n) = kQ/I$ be a double simple cycle algebra of length n. Then $I = \langle a_{i+1}a_i, b_{i+1}b_i \rangle_{i \in \mathbb{Z}/n\mathbb{Z}}$. In particular, there are two cycles, up to cyclic permutation, "R" and "B" of length n, such that any two consecutive arrows lies in I, and there is a single unique non-zero cycle (not in I) of length $2n$,

$$C = (a_1 b_2 a_3 b_4 \cdots b_1 a_2 b_3 a_4 b_5 \cdots a_{n-1} b_n)$$

up to a cyclic permutation of the arrows if n is even, and there are two

$$C_1 = (a_1 b_2 a_3 \cdots a_{n-1} b_n), \quad C_2 = (b_1 a_2 b_3 \cdots b_{n-1} a_n)$$

if n is odd. Let $\varphi = \varphi_R \varphi_B \in S_{Q_1}$ be two cyclic permutations in the permutation group $S_{Q_1} \cong S_{2n}$. Let $\varphi_R = (a_1, a_2, ..., a_n) \mapsto (1, 2, ..., n)$ and $\varphi_B = (b_1, b_2, ..., b_n) \mapsto (n+1, n+2, ..., 2n)$ given by identifying Q_1 with $[2n]$ by fixing a labeling of the arrows. Let

$$\sigma = \begin{cases} (a_1 b_2 a_3 b_4 \cdots b_1 a_2 b_3 a_4 b_5 \cdots a_{n-1} b_n) & \text{if } |Q_0| = 2k+1 \\ (a_1, b_2, a_3, ..., b_{n-2}, a_{n-1}, b_n)(b_1, a_2, b_3, ..., a_{n-2} b_{n-1} a_n) & \text{if } |Q_0| = 2k \end{cases}$$

In the above cases we may identify,

$$(a_1 b_2 a_3 b_4 \cdots b_1 a_2 b_3 a_4 b_5 \cdots a_{n-1} b_n) \mapsto (1, n+1, 2, n+2, ..., n, 2n) \in S_{2n}$$

and

$$\begin{aligned}(a_1, b_2, a_3, ..., b_{n-2}, a_{n-1}, b_n)&(b_1, a_2, b_3, ..., a_{n-2} b_{n-1} a_n) \\ &\mapsto (1, n+2, 3, ..., n-1, 2n)(n+1, 2, n+3, ..., 2n-1, n) \in S_{2n}.\end{aligned}$$

for $k = 0, 1, 2, ...$; finally, let $\alpha = (1, n+1)(2, n+2) \cdots (n, 2n)$. If we restrict I to R or B, we have an algebra $\Lambda_R = kR/I \cong k\tilde{\mathbb{A}}_n / \mathbf{rad}(k\tilde{\mathbb{A}}_n)^2 \cong kB/I = \Lambda_B$. We may then view α as the gluing that identifies $ta'_i \leftrightarrow tb'_i$, where $a'_i \in \Lambda_R$ and $b'_i \in \Lambda_B$ are arrows in the respective algebras. It is not hard to verify $\sigma\alpha = \varphi^{-1}$, and thus the triple $[\sigma, \alpha, \varphi]$ may be seen as a combinatorial embedding of a graph into a Riemann surface. This map $\Gamma(n) \hookrightarrow X$ has either one vertex if n is even, or two vertices if n is odd, corresponding to σ. It has two faces, corresponding to φ_R and φ_B; it also has exactly $n = |Q_0|$ edges corresponding to α. Thus we may compute the surface associated to the algebra $\tilde{A}(n)$ via

$$\chi(\tilde{A}(n)) = |\sigma| - |\alpha| + |\varphi| = \begin{cases} 2 - 2k & \text{if } |Q_0| = n = 2k+1 \\ 4 - 2k & \text{if } |Q_0| = n = 2k \end{cases} .$$

for $k = 0, 1, 2, ...$, and $n \geq 1$, so that we have the genus function

$$g(\tilde{A}(n)) = \begin{cases} k & \text{if } n = 2k+1 \\ k-1 & \text{if } n = 2k \end{cases}$$

Genus 0 Examples

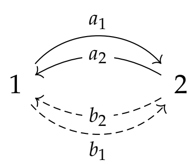
The Dihedral Ringel Algebra $\tilde{A}(1)$

Let $\sigma = (1, 2), \alpha = (1, 2), \phi = (1)(2)$. Then the closed surface algebra $\Lambda(\mathfrak{c})$ given by the constellation $\mathfrak{c}_1 = [\sigma, \alpha, \varphi]_0$ is given by the graph with one vertex and one loop embedded in the sphere. In particular $\Lambda(\mathfrak{c}) = \tilde{A}(1)$ is given by the quiver

and is isomorphic to $k\langle x, y\rangle / \langle x^2, y^2\rangle$. Any representation of this quiver is a pair $(X, Y) \in \mathbf{End}(V) \times \mathbf{End}(V)$ of nilpotent operators on a vector space V

$\tilde{A}(2)$

Let $\mathfrak{c}_2 = [\sigma, \alpha, \varphi]_2$ be given by $\sigma = (1,4)(2,3)\alpha = (1,3)(2,4), \varphi = (1,2)(3,4)$, then $\chi(\mathfrak{c}_2) = 2$ and $g(\mathfrak{c}_2) = 0$. The embedded graph can be represented by the equator of the sphere with two vertices on it. The quiver which comes from this graph is

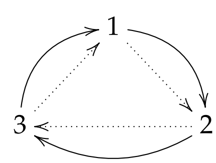


Genus 1 Examples

$\tilde{A}(3)$

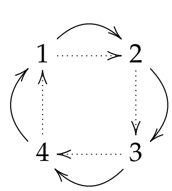
Let $\mathfrak{c}_3 = [\sigma, \alpha, \varphi]_3$ be defined by $\sigma = (1,6,2,4,3,5), \alpha = (1,4)(2,5)(3,6), \varphi = (1,2,3)(4,5,6)$. Then we have $\chi(\mathfrak{c}_3) = 1 - 3 + 2 = 0$ so $g(\mathfrak{c}_3) = 1$. The graph embedded on the torus can be obtained by the following gluing of the square to obtain the torus,

the quiver is then,



$\tilde{A}(4)$

Let $\mathfrak{c}_4 = [\sigma, \alpha, \varphi]_4$ be given by $\sigma = (1,8,3,6)(2,5,4,7), \alpha = (1,5)(2,6)(3,7)(4,8), \varphi = (1,2,3,4)(5,6,7,8)$ $\chi(\mathfrak{c}_4) = 2 - 4 + 2$ so $g(\mathfrak{c}_4) = 1$.



Genus 2 Examples

$\tilde{A}(5)$

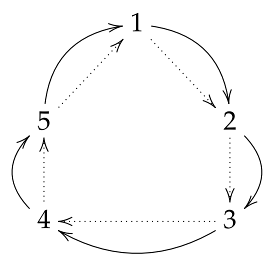
The constellation $\mathfrak{c}_5 = [\sigma, \alpha, \varphi]_5$ is defined by,

- $\sigma = (1, 10, 4, 8, 2, 6, 5, 9, 3, 7)$,

- $\alpha = (1, 6)(2, 7)(3, 8)(4, 9)(5, 10)$,

- $\varphi = (1, 2, 3, 4, 5)(6, 7, 8, 9, 10)$.

$\chi(\mathfrak{c}_5) = 1 - 5 + 2 = -2$, so $g(\mathfrak{c}_5) = 2$.



$\tilde{A}(6)$

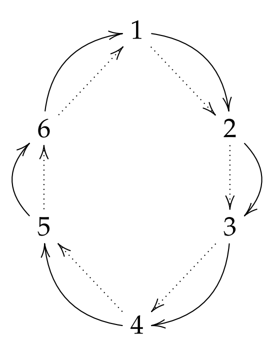
The constellation $\mathfrak{c}_6 = [\sigma, \alpha, \varphi]_6$ is defined by,

- $\sigma = (1, 12, 5, 10, 3, 8)(2, 7, 6, 11, 4, 9)$,

- $\alpha = (1, 7)(2, 8)(3, 9)(4, 10)(5, 11)(6, 12)$,

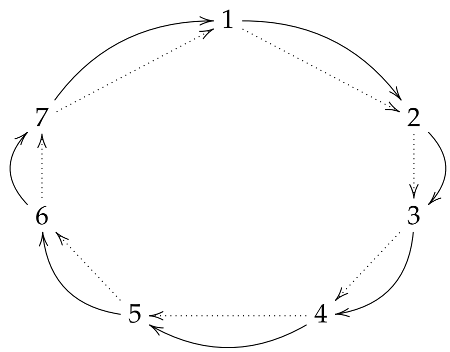
- $\varphi = (1, 2, 3, 4, 5, 6)(7, 8, 9, 10, 11, 12)$.

The quiver of $\tilde{A}(6)$ is then,

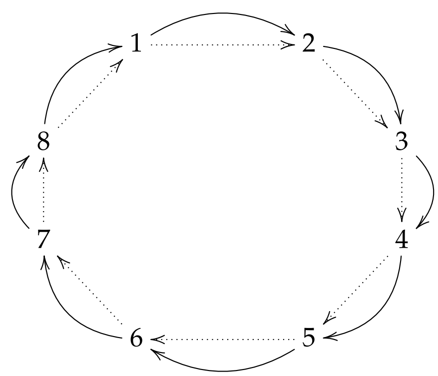


Genus 3 Examples

$\tilde{A}(7)$



$\tilde{A}(8)$



Part II

Representation Theory of Compact Gentle Surface Algebras

Linear Relations on Vector Spaces and Functorial Filtrations

Next we will study the indecomposable modules for compact gentle surface algebras. Because these algebras are infinite dimensional, the usual techniques of Auslander-Reiten theory are sometimes less helpful or do not apply. We can still however define a linear order on the finite dimensional modules, and describe the **Hom**-spaces between them. We can also compute projective resolutions of finitely presented modules, and many of the combinatorics still hold. We will also be able to define a kind of Auslander-Reiten translate on some of the modules. One might want to understand such things for their own sake and such questions are standard in the representation theory of quivers, but we are interested in them for other reasons as well. In particular, we would like to have a *"categorification"* of the relations between the generators of rings of relative invariants under the action of an algebraic group on the representations varieties. The word "categorification" is a popular one currently, and means many different things to many different people. Here it will simply mean a correspondence between generators of relative invariants and modules over the algebra, along with some hope of explaining the relations between the generators in terms of the representation theory of the quivers and the module categories.

Linear Relations on Vector Spaces

We begin by defining *linear relations* on vector spaces, a technique used by Gel'fand and Ponomarev in their study of the Lorentz group.[12] Later, the method was applied by Ringel[13] to study the dihedral 2-group, and subsequently by Butler and Ringel for string algebras, and Crawley-Boevey[14,15,16] in the study of clannish algebras. The method was most recently applied in a preprint to infinite- dimensional string algebras by Crawley-Boevey[17].

[12] I.M Gel'fand, V.A. Ponomarev. *Indecomposable Representations of the Lorentz Group*.

[13] C.M. Ringel. *Indecomposable Representations of the Dihedral 2-Group*.

[14] W. W. Crawley-Boevey. *Functorial Filtrations I: The Problem of an Idempotent and a Square-Zero Matrix*. J. London Math. Soc. (2) 38 (1988) 385-402

[15] W. W. Crawley-Boevey. *Functorial Filtrations II: Clans and the Gelfand Problem*. J. London Math. Soc. (2) 40 (1989) 9-30.

[16] W. W. Crawley-Boevey. *Functorial Filtrations III: Semi-dihedral Algebras*. J. London Math. Soc. (2) 40 (1989) 31-39

[17] W. W. Crawley-Boevey. *Indecomposable Representations of Infinite Dimensional String Algebras*. Preprint

Definitions and Examples

We define a **relation** on a vector space V, to be a subspace C of $V \times V$. So, for example, we have that any endomorphism $f : V \to V$ defines the relation $\{(x, fx) : x \in V\}$. We define the relation C^{-1} as the relation $\{(y, x) : (x, y) \in C\}$ for any relation C. We may compose relations,

$$C_2C_1 = \{(x,z) | \exists\, y \in V : (x,y) \in C_1, (y,z) \in C_2\}.$$

For any $x \in V$, we define

$$Cx = \{y \in V : (x,y) \in C\},$$

and for any subset $U \subset V$,

$$CU = \{y \in V | \ \exists \ x \in U : (x,y) \in C\}.$$

All of our relations will come from endomorphisms $f : V \to V$, and compositions of such relations. As an example, take the endomorphism $f, g : \mathbb{K}^3 \to \mathbb{K}^3$ given in the standard basis $\{e_1, e_2, e_3\}$, by the matrices

$$f = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad g = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The relation given by f is then the subspace of $\mathbb{K}^3 \times \mathbb{K}^3$ generated by $\mathbb{K}\{(e_1, e_2), (e_2, e_3)\}$, and the relation given by g is the subspace of $\mathbb{K}^3 \times \mathbb{K}^3$ generated by $\mathbb{K}\{(e_2, e_1), (e_3, e_2)\}$. We may visualize the composition $f \cdot f$, of the relation induced by f on $V \times V$ as follows,

$$\begin{tikzcd}
e_1 \arrow[dr, "f"] & e_1 & e_1 \\
e_2 \arrow[dr, "f"] & e_2 \arrow[dr, "f"] & e_2 \\
e_3 & e_3 & e_3
\end{tikzcd}$$

This is the subspace of $\mathbb{K}^3 \times \mathbb{K}^3$ generated by $\mathbb{K}\{(e_1, e_3)\}$. As another example, let f and g be as before. Then we may view the relation $g \cdot f \cdot f$ as,

$$\begin{tikzcd}
e_1 \arrow[dr, "f"] & e_1 & e_1 & e_1 \\
e_2 \arrow[dr, "f"] & e_2 \arrow[dr, "f"] & e_2 & e_2 \\
e_3 & e_3 & e_3 \arrow[ur, "g"] & e_3
\end{tikzcd}$$

This is the subspace generated by $\mathbb{K}\{(e_1, e_2)\}$. For any relation C on V, define the **stable kernel of** C^{-1}, to be $\kappa(C) = \bigcup_{n \geq 0} C^n 0_V$. Define the **stable image of** C, to be $\iota(C) = \bigcap_{n \geq 0} C^n V$. As an example, let us return to our endomorphisms f and g and the following visualizations,

$$
\begin{tikzcd}
e_1 \arrow[dr, "f"] & e_1 & e_1 & \boxed{e_1} \\
e_2 \arrow[dr, "f"] & e_2 \arrow[dr, "f"] & e_2 & \boxed{e_2} \\
e_3 & e_3 & e_3 & \boxed{e_3}
\end{tikzcd},
\qquad
\begin{tikzcd}
e_1 & e_1 & e_1 & \boxed{e_1} \\
e_2 \arrow[ur, "g"] & e_2 \arrow[ur, "g"] & e_2 & \boxed{e_2} \\
e_3 \arrow[ur, "g"] & e_3 & e_3 & \boxed{e_3}
\end{tikzcd}
$$

$$
\begin{tikzcd}
e_1 \arrow[dr, "f"] & e_1 & e_1 \arrow[dr, "f"] & e_1 & e_1 & \cdots \\
e_2 \arrow[dr, "f"] & e_2 \arrow[ur, "g"] & e_2 \arrow[dr, "f"] & e_2 \arrow[ur, "g"] & e_2 & \cdots \\
e_3 & e_3 \arrow[ur, "g"] & e_3 & e_3 \arrow[ur, "g"] & e_3 & \cdots
\end{tikzcd}
$$

Relation	Stable Spaces
$\kappa(f)$	$\mathbb{K}\{(e_1, e_2), (e_2, e_3), (e_1, e_3)\}$
$\iota(f)$	0
$\kappa(g)$	$\mathbb{K}\{(e_2, e_1), (e_3, e_2), (e_3, e_1)\}$
$\iota(g)$	0
$\kappa(gf)$	$\mathbb{K}\{(e_1, 0)\}$
$\iota(gf)$	$\mathbb{K}\{(e_1, e_1), (e_2, e_2)\}$
$\kappa(fg)$	$\mathbb{K}\{(e_3, 0)\}$
$\iota(fg)$	$\mathbb{K}\{(e_2, e_2), (e_3, e_3)\}$

Table 2: Stable kernels of $f^{-1}, g^{-1}, (fg)^{-1}$, and $(gf)^{-1}$; and stable images of f, g, fg, and gf.

Properties of Relations

Suppose we have a relation on V identified with an endomorphism $f : V \to V$. Then there are subspaces $U_1, U_2 \subset V$, such that

1. $V = U_1 \oplus U_2$,
2. $f = (f \cap (U_1 \times U_1)) \oplus (f \cap (U_2 \times U_2))$,
3. $f \cap (U_1 \times U_1)$ is the graph of an automorphism $f|_{U_1}$,
4. $\kappa(f) \oplus U_1 = \iota(f)$,
5. f induces an automorphism φ on $\iota(f)/\kappa(f)$ such that $\varphi(x + \kappa(f)) = (fx \cap \iota(f)) + \kappa(f)$, for all $x \in \iota(f)$,
6. the relation ϕ on $\iota(f)/\kappa(f)$ is the **regular part** of f, and splits off,
7.
8.

Words on the Quiver

Strings

Bands

The Functorial Filtration

Suppose we have some index set I, and Q and $\mathcal{Q}_i$, $i \in I$ abelian categories[18]. Suppose $S_i : \mathcal{Q}_i \to Q$ and $F_i : Q \to \mathcal{Q}_i$ are additive functors[19]. Suppose further that[20]

[18] Recall: an abelian category can be thought of as the path category of a quiver Q, such that we have addition on Q_0 (vertices) and also on Q_1 (arrows). In particular, Q_0, and $Q(x, y)$ are abelian groups.

[19] Identifying quivers with categories, an additive functor can be thought of as a quiver map (directed graph map) $\theta : Q \to Q'$, so that there is an additive structure on the maps $x \to \theta(x)$, $x \in Q_0$, $\theta(x) \in Q'_0$.

[20] We say the functor (quiver map) $F_i : Q \to \mathcal{Q}_i$ is **locally finite** if only finitely many $F_i(x) \in \mathcal{Q}_{i,0}$ are nonzero, for $x \in Q_0$. We say it **reflects isomorphisms** if $F_i(a) \in \mathcal{Q}_i(F_i(x), F_i(y))$ is an isomorphism for all $i \in I$, then $a \in Q(x, y)$ is an isomorphism. Here of course $x = ta$ and $y = ha$.

1. $F_i S_i \cong \mathbf{id}_{\mathcal{Q}_i}$.
2. $F_i S_j \cong 0$ if $i \neq j$.
3. The set $\{F_i : i \in I\}$ is **locally finite**, and **reflects isomorphisms**.
4. For every $M \in \mathcal{M}$, and $i \in I$, there is a map $\gamma_{i,M} : S_i F_i(M) \to M$ such that $F_i(\gamma_{i,M})$ is an isomorphism.

How should one think of such a definition? What is a "good" picture to have in mind? First observe the following diagrams,

$$\begin{tikzcd} x' \in \mathcal{Q}_{i,0} \arrow[rr, "\mathbf{id}_{\mathcal{Q}_i}"] \arrow[dr, "S_i"'] & & x' \in \mathcal{Q}_i \\ & S_i(x') \in Q_0 \arrow[ur, "F_i"'] & \end{tikzcd}$$

$$\begin{tikzcd} & F_i(x) \in \mathcal{Q}_{i,0} \arrow[dr, "S_i"] & \\ x \in Q_0 \arrow[ur, "F_i"] \arrow[rr, "\gamma_{i,x}"'] & & S_i F_i(x) \in Q_0 \end{tikzcd}$$

If all of the above properties are satisfied, then the **indecomposable objects** in Q are all of the form $S_i(m)$, with m an indecomposable objects in Q_i. All objects of the form $S_i(m)$ are indecomposable. Moreover, $S_i(V) \cong S_j(V')$ if and only if $i = j$ and $V \cong V'$ in Q_i. Now, framing this in terms of *representations* V in $\mathbf{rep}(\Lambda_i) = Q_i$[21], and idenitifying the category $\mathbf{rep}(\Lambda_i)$ of representations of the quiver (or quiver with relations) Λ_i, with the quiver Q_i, the above construction says we have the following diagrams of categories of representations,

[21] Recall: a representation of a quiver Λ_i, denoted $V : \Lambda_i \to \mathbf{Mod}(k)$, is a functor which assigns vector spaces $V(x)$ to each vertex $x \in \Lambda_{i,0}$, and linear maps $V(a)$ to each arrow $a \in \Lambda_{i,1}$.

$$\begin{tikzcd} & F_i(V) \in \mathbf{rep}(\Lambda_i) \arrow[dr, "S_i"] & \\ V \in \mathbf{rep}(\Lambda) \arrow[ur, "F_i"] \arrow[rr, "\gamma_{i,V}"'] & & S_i F_i(V) \in \mathbf{rep}(\Lambda) \end{tikzcd}$$

Morphisms in the Module Category

In this section we follow Crawley-Boevey [22] and Krause [23] and describe nice bases for the **Hom**-spaces between modules over a fixed complete gentle algebra using covering theory. It has recently been established by Crawley-Boevey in [24] that the indecomposable modules are again string and band modules as in the finite dimensional case, but we may also have infinite string modules.

[22] ; ; and

[23]

[24]

First let us state some basic definitions and results on the indecomposable modules for a complete gentle algebras which follow from [25].

[25]

Definition 0.0.3. We say a module $M \in \mathbf{Mod}(\Lambda)$ over a complete gentle algebra Λ is **finitely generated** if $M = \Lambda m_1 + \cdots \Lambda m_n$ for some elements $m_i \in M$. If Λ is a complete gentle algebra without identity associated to an (infinite) quiver with relations (Q, I), then we say that $M \in \mathbf{Mod}(\Lambda)$ is **finitely controlled** if for every $x \in Q_0$, the set $e_x M$ is contained in a finitely generated submodule of M. We say that M is a **pointwise artinian module** if for any descending chain of submodules $M_1 \supset M_2 \supset M_3 \supset \cdots$, and any $x \in Q_0$, the chain of subspaces $e_x M_1 \supset e_x M_2 \supset e_x M_3 \supset \cdots$ stabilizes. We say M is **finitely presented** if there is a presentation $P \to M$ such that P is a projective module with finitely many indecomposable summands.

Example 0.0.4. Take Q to be the quiver

$$\begin{tikzcd}
 & \cdots & \bullet \arrow[loop above, "c_{i-1}"] \arrow[dr, "a'_{i-1}"] & & \bullet \arrow[loop above, "c_i"] \arrow[dr, "a'_i"] & & \bullet \arrow[loop above, "c_{i+1}"] & \cdots \\
\cdots & \bullet \arrow[ur, "a_{i-1}"] \arrow[dr, "b_{i-1}"'] & & \bullet \arrow[ur, "a_i"] \arrow[dr, "b_i"'] & & \bullet \arrow[ur, "a_{i+1}"] \arrow[dr, "b_{i+1}"'] & \cdots & \\
 & \cdots & \bullet \arrow[loop below, "d_{i-1}"] \arrow[ur, "b'_{i-1}"'] & & \bullet \arrow[loop below, "d_i"] \arrow[ur, "b'_i"'] & & \bullet \arrow[loop below, "d_{i+1}"] & \cdots
\end{tikzcd}$$

with $i \in \mathbb{Z}$ and relations $\langle a_{i+1}a'_i, a'_i a_i, d_i b_i, b'_i d_i, b_i b'_{i-1}, c_i^2 \rangle_{i \in \mathbb{Z}}$. Then $\Lambda = kQ/I$ is a complete gentle algebra without identity. If we identify a_i and a_{i+n}, a'_i and a'_{i+1}, b_i and b_{i+n}, b'_i and b'_{i+n} c_i and c_{i+n}, and d_i and d_{i+n} for all $i \in \mathbb{Z}$, we can think of Q' as being embedded on a cylinder. With the ideal $I' = \langle a_{i+1}a'_i, a'_i a_i, d_i b_i, b'_i d_i, b_i b'_{i-1}, c_i^2 \rangle_{i \in \mathbb{Z}/n\mathbb{Z}}$, we have a unital complete gentle algebra $\Lambda' = kQ'/I'$. For any projective $P(i)$ in Λ or Λ' we will always have that $\mathbf{rad}\, P(i)$ is a sum of two uniserial submodules given by infinite words beginning at i (this is the case for any complete gentle algebra). They will correspond to subwords of $\omega_1 = \cdots b'_{i+1} b_{i+1} a'_i c_i a_i \cdots$, and $\omega_{2,i} = d_i^{\infty}$, which extend infinitely to the left (for $i \in \mathbb{Z}$ or $i \in \mathbb{Z}/n\mathbb{Z}$).

Definition 0.0.5. If $I \in \{\mathbb{N}, -\mathbb{N}, \mathbb{Z}, [n] : n \in \mathbb{N}\}$, we say an I word in the letters $\{Q_0, Q_1\}$ is a **string** if it is given by a walk in the quiver not passing through any relations and it is not periodic, and it is a **band** if it is periodic. If all letters are *direct* we call the word a **direct word** and if all the letters are *inverse* we call it an **inverse word**. The usual equivalences on words holds as in the finite dimensional string modules. For a periodic word ω which is direct or inverse and an injective envelope V of a simple $k[T, T^{-1}]$-module, we will call the associated band module a **primitive injective band module**. More formally, let $l \in Q \amalg Q^{-1}$ be a letter, where Q^{-1} is the set of formal inverses of elements in Q. For the *head* and *tail* of a letter, we use the notation hl and tl respectively, and we define $tl^{-1} = hl$ and $hl^{-1} = tl$. Let I be either $\{1, 2, 3, ..., n\}$, $\mathbb{N} = \{0, 1, 2, ...\}$, $-\mathbb{N}$, or $\mathbb{Z}$, and define an I-**word** ω as follows: If $I \neq \{0\}$, then ω consists of a sequence of letters ω_i for all $i \in I$ with $i - 1 \in I$, so

$$\omega = \begin{cases} \omega_1\omega_2 \cdots \omega_n & \text{if } I = \{0, 1, 2, ..., n\} \\ \omega_1\omega_2\omega_3 \cdots & \text{if } I = \mathbb{N} \\ \cdots \omega_{-2}\omega_{-1}\omega_0 & \text{if } I = -\mathbb{N} \\ \cdots \omega_{-1}\omega_0|\omega_1\omega_2 \cdots & \text{if } I = \mathbb{Z}\,. \end{cases}$$

where the verticle bar shows the position of ω_0 and ω_1 if $I = \mathbb{Z}$, satisfying

1. if ω_i and ω_{i+1} are consecutive letters, then the tail of ω_i is equal to the head of ω_{i+1};
2. if ω_i and ω_{i+1} are consecutive letter then $\omega_i^{-1} \neq \omega_{i+1}$; and
3. no zero relation $\rho \in I$ nor its inverse (given by the formal inverse of the word ρ) occurs as a sequence of consecutive letters in ω.

If $I = \{0\}$ then there are trivial words $1_{x,\epsilon}$ for every $x \in Q_0$, and $\epsilon = \pm 1$. By a **word** we mean an I-word for some I. If I is finite, then the word is finite and of length $|I|$. If ω is an I-word, then for each $i \in I$ there is an associated vertex $x_i(\omega)$ the tail of ω_i or the head of ω_{i+1}, or x for $1_{x,\epsilon}$. We call ω direct if all letters are direct and inverse if they are all inverse. The inverse of ω, denoted ω^{-1} is given by formally inverting all letters of ω and reversing their order. By convention $1_{x,\epsilon}^{-1} = 1_{x,-\epsilon}$, and the inverse of a $\mathbb{Z}$-word is indexed so that $(\cdots \omega_0|\omega_i \cdots)^{-1} = \cdots \omega_1^{-1}|\omega_0^{-1} \cdots$. If ω is a $\mathbb{Z}$-word, and $n \in \mathbb{Z}$ we define the **shift** of ω, denoted $\omega[n]$, as the word $\cdots \omega_n|\omega_{n+1}$. A word is **periodic** if it is a $\mathbb{Z}$-word and $\omega = \omega[n]$ for some $n > 0$. The minimal such n is called the **period**. If we let $\omega[n] = \omega$ for all $I \neq \mathbb{Z}$, we may extent the shift. We say two words ω and ω' are equivalent if and only if $\omega = \omega'[n]$ or $\omega = \omega'^{-1}[n]$.

Given any I-word ω, and any complete gentle algebra Λ, we define a Λ-module $M(\omega)$ with basis $\{e_i\}_{i\in I}$ as a vector space, and the action of Λ is given by

$$x_j \cdot e_i = \begin{cases} e_i & \text{if } \ x_j(\omega) = x_i \\ 0 & \text{otherwise} \end{cases}$$

for a trivial path x_j in Q (where $\Lambda = kQ/I$ and $x_j \in Q_0$), and

$$a \cdot e_i = \begin{cases} e_{i-1} & \text{if } \ i-1 \in I \text{ and } \ \omega_i = a \\ e_{i+1} & \text{if } \ i+1 \in I \text{ and } \ \omega_{i+1} = a^{-1} \\ 0 & \text{otherwise} \end{cases}$$

for an arrow $a \in Q_1$. There is an isomorphism $M(\omega) \cong M(\omega^{-1})$, and $M(\omega) \cong M(\omega[n])$ given by $e_i \mapsto e_{i-n}$. Therefore, modules given by equivalent words are isomorphic. If ω is a periodic word of period n, then $M(\omega)$ is a $\Lambda - k[T, T^{-1}]$-bimodule, with T acting as the bijection $e_i \mapsto e_{i-n}$ on basis elements, and we define

$$M(\omega, V) = M(\omega) \otimes_{k[T,T^{-1}]} V$$

for V a $k[T, T^{-1}]$-module. Such a module is finite dimensional if and only if V is.

Definition 0.0.6. We define **string modules** to be those modules $M(\omega)$ such that ω is non-periodic. We define **band modules** to be those modules $M(\omega, V)$. A **primitive injective band module** will be one of the form $M(\omega, V)$ such that ω is a direct (periodic) word and V is the injective envelope of a simple $k[T, T^{-1}]$-module.

For Λ a complete gentle algebra, the indecomposable projective (left) module $P(x) = e_x\Lambda$ is a string module and has the form

$$\begin{tikzcd}
 & & & ke_x \arrow[dl, "\mathbf{id}"'] \arrow[dr, "\mathbf{id}"] & & & \\
 & & ke_{i_1} \arrow[dl, "\mathbf{id}"'] & & ke_{i_2} \arrow[dr, "\mathbf{id}"] & & \\
 & ke_{i_3} \arrow[dl, "\mathbf{id}"'] & & & & ke_{i_4} \arrow[dr, "\mathbf{id}"] & \\
 {} & & & & & & {}
\end{tikzcd}$$

where e_i and e_j may be equal in Q_0. Denote the one dimensional simple modules $S(x) := \textbf{top}\, P(x) = P(x)/\,\textbf{rad}\, P(x)$, and the indecomposable injective modules $I(x)$, which are again string modules and have the form,

$$\begin{tikzcd}
{} \arrow[dr, "\mathbf{id}"] & & & & & & {} \arrow[dl, "\mathbf{id}"'] \\
& ke_{i_1} \arrow[dr, "\mathbf{id}"] & & & & ke_{i_2} \arrow[dl, "\mathbf{id}"'] & \\
& & ke_{i_3} \arrow[dr, "\mathbf{id}"] & & ke_{i_4} \arrow[dl, "\mathbf{id}"'] & & \\
& & & ke_x & & &
\end{tikzcd}$$

where $ke_j \cong k$ is a copy of the base field k indexed by the primitive othogonal idempotents e_i corresponding to $i \in Q_0$, and e_i may again be equal to some e_j in the diagram.

Definition 0.0.7. [$\star$ Amelie: put definitions of substring and factor strings here. $\star$]

Theorem 0.0.8. *([26]):String modules, finite-dimensional band modules and prim- itive injective band modules are indecomposable. Moreover, there only exist isomorphisms between such modules when the corresponding words are equiv- alent: there are no isomorphisms between string modules and modules of the form $M(\omega, V); stringmodules M(\omega) and M(\omega'))$ are isomorphic if and only if $\omega \equiv \omega'$; and $M(\omega, V) \equiv M(\omega, W)$ if and only if $\omega = \omega'[n]$ and $W \cong V$ or $\omega = (\omega'^{-1})[n]$ and $W \cong res_\iota V$ for some m, where ι is the automorphism $T \leftrightarrow T^{-1}$, of $k[T, T^{-1}]$, and res is the restriction via ι.* 26

Theorem 0.0.9. *([27]): Every finitely controlled module is isomorphic to a direct sum of copies of string modules and finite-dimensional band modules.* 27

Theorem 0.0.10. *([28]): The category of finitely controlled modules, and the category of pointwise artinian modules have the Krull-Remak-Schmidt property.* 28

Note that string modules may be given by infinite words, but that not all such words give finitely controlled or finitely generated modules.

Theorem 0.0.11. *([29]): Every pointwise-artinian module is isomorphic to a direct sum of copies of string modules and finite-dimensional band modules and primitive injective band modules.* 29

Further we have that $\mathbf{Mod}(A)$ has the Krull-Remak-Schmidt property if A is a complete gentle algebra.

Proposition 0.0.12. *The universal cover $(\overline{Q}, \overline{I})$, of all complete gentle algebras with finitely many simple modules is the four-regular tree quiver,* *i.e. the infinite quiver which has underlying graph the infinite four-regular tree, and such that at every vertex there are exactly two arrows entering, two arrows leaving, and the relations are gentle. We will sometimes abuse notation and simply write $\overline{Q}$ for the universal cover of the complete gentle algebras.*

Let (Q, I) be a complete gentle quiver with relations and let $\Lambda = kQ/I$. Let T be a quiver which has underlying graph a tree or simple cycle, i.e. T is of type A_n or $\tilde{A}_n$. If T is of type A_n, denote by $V_{T,1,1}$ the kT module such that $V_T(x) = k$ for all $x \in T_0$, and $V_T(a) = \mathbf{id}$, for all arrows $a \in T_1$. If T is of type $\tilde{A}_n$ then let $V_{T,n,\lambda}$, $(\lambda \in k^\times)$, be the kT-module such that $V_{T,n,\lambda}(x) = k^n$ for all $x \in T_0$, $V_{T,n,\lambda}(a) = \mathbf{id}_n$ for all arrows $a \in T_1$ except a distinguished arrow a_T for which

$$V_{T,n,\lambda}(a_T) = \begin{pmatrix} \lambda & 1 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & \lambda \end{pmatrix},$$

is the $n \times n$ Jordan block $J_{n,\lambda}$.

We call a morphism of quivers $F : T \to Q$ a **winding** if,

1. F is injective on sinks and sources, i.e. there are no subquivers of the form

$$\bullet \xleftarrow{a} \bullet \xrightarrow{b} \bullet \qquad \text{or} \qquad \bullet \xrightarrow{a} \bullet \xleftarrow{b} \bullet$$

   in T such that $F(a) = F(b)$.

2. If T is of type $\tilde{A}_n$ then F is non-periodic (so there is no nontrivial $\sigma \in \mathbf{Aut}(T)$ such that $F \circ \sigma = F$).

3. There is no path p in T such that $F(p) \in I$.

Morphisms between windings are commutative triangles

$$\begin{tikzcd} T \arrow[rr, "f"] \arrow[dr, "F"'] & & T' \arrow[dl, "F'"] \\ & Q & \end{tikzcd}$$

For such a covering there us a push-down functor $F_\lambda : \mathbf{Mod}(kT) \to \mathbf{Mod}(A)$ given by

$$(F_\lambda V)(x) = \bigoplus_{y \in F^{-1}(x)} V(y)$$

for all $x \in Q_0$, and similarly $(F_\lambda V)(a) = \bigoplus_{b \in F^{-1}(a)} V(b)$, for all $a \in Q_1$. The modules $F_\lambda V_{T,n,\lambda}$ are called **tree modules** if T is of type

A_n, and **band modules** if T is of type $\tilde{A}_n$. Such push-down functors are restrictions of functors $\overline{F}_\lambda : \mathbf{Mod}(k\overline{Q}/\overline{I}) \to \mathbf{Mod}(kQ/I)$ which is induced by the covering $\overline{F} : \overline{Q} \to Q$ which extends F, i.e.

$$\begin{tikzcd} T \arrow[rr, hook] \arrow[dr, "F"'] & & \overline{Q} \arrow[dl, "\overline{F}"] \\ & Q & \end{tikzcd} \qquad \begin{tikzcd} \mathbf{Mod}(kT) \arrow[rr, hook] \arrow[dr, "F_\lambda"'] & & \mathbf{Mod}(k\overline{Q}/\overline{I}) \arrow[dl, "\overline{F}_\lambda"] \\ & \mathbf{Mod}(kQ/I) & \end{tikzcd}$$

where $\overline{F} : \overline{Q} \to Q$ is the universal cover as in Proposition 0.0.12 if T is of type A_n. Otherwise, interpret F as an element $\omega_F \in \pi(Q, x)$ in the fundamental group ($x \in F(T)$), acting freely on $\overline{Q}$, and choose $\overline{F} : \overline{Q}/\langle\omega_F\rangle \to Q$ to be the covering. Since F is injective on sinks and sources, $\overline{F}$ extends F. Further, $\overline{F}$ is Galois, and the push-down functor F_λ preserves indecomposables.

Let $A = kQ/I$ be a complete gentle algebra and let $V = F_\lambda V_{T,n,\lambda}$ and $V' = F_\mu V_{T',m,\mu}$ be tree or band modules. Then the (finite dimensional) space $\mathbf{Hom}_A(V, V')$ has a basis given by triples (U, f, g)

$$\begin{tikzcd} T \arrow[dr, "F"'] & U \arrow[l, "f"'] \arrow[r, "g"] \arrow[d, "F''"] & T' \arrow[dl, "F'"] \\ & Q & \end{tikzcd}$$

where $F'' : U \to Q$ is a winding, and $f : U \to T$ and $g : U \to T'$ are winding morphisms. A morphisms of triples $h : (U, f, g) \to (U', f', g')$ is a winding morphism $h : U \to U'$ such that $f' = f \circ h$ and $g' = g \circ h$. Let $\mathcal{U}(F, F')$ be the isomorphism classes of triples (U, f, g). Inclusions $h : (U', f', g') \hookrightarrow (U, f, g)$ induce a partial order on $\mathcal{U}(F, F')$. Call (U, f, g) **admissible** if

1. If $x \in U_0$ and $a \in T_1$ with $ha = f(x)$, then there is an arrow a' such that $ha' = x$ and $f(a') = a$.
2. If $y \in U_0$ and $b \in T'_1$ with $tb = g(x)$, then there is an arrow b' with $tb' = x$ and $g(b') = b$.

Let i, j be a pair of vertices in T and T' respectively. Then, there is at most one admissible triple in

$$\mathcal{U}_{ij} = \{(U, f, g) : \exists\, x \in U_0 : \ f(x) = i, \ g(x) = j\}.$$

[$\star$ Amelie: Should finish filling in a few details from Krause paper here. $\star$]

Definition 0.0.13. Let $s = s_n \cdots s_2 s_1$ be a string, and let $t = s_i \cdots s_j$ be a substring of s. Then t is an **image substring** of s if s_{i+1} is inverse or $i = n$, and s_{j-1} is direct or $j = 1$. If $x + x(s_i) = ts_i$ is a vertex which is a sink in the string s, then $1_{ts_i,\epsilon(x)}$ is a an image substring. We call t a **factor substring** if s_{i+1} is direct or $i = n$ and s_{j-1} is inverse, or $j = 1$. Similarly if $x = x(s_i)$ is a source in the string s, then $1_{ts_i,\epsilon(x)}$ is a factor substring of s. If t is a factor sbstring of s then there is an epimorphism $f : M(s) \to M(t)$, and if t is an image substring then there is a monomorphism $g : M(t) \to M(s)$. Given two strings ω and ω', a composition of the form $gf : M(\omega) \to M(\omega')$ is called a **graph map**, and we will associate the image/factor substring t with gf. Since for any string s, we have $M(s) \cong M(s^{-1})$, we will also define maps of the form

$$M(\omega) \to M(t) \cong M(t^{-1}) \to M(\omega')$$

to be graph maps.

From the results of [30] and [31] we get the following.

[30]

[31]

Theorem 0.0.14. *If Λ is a complete gentle algebra (indeed it can be any monomial algebra), and if ω, ω' are finite strings, then the collection $\mathfrak{g}(\omega, \omega')$ os all graph maps $M(\omega) \to M(\omega')$ is a k-linear basis for $\mathbf{Hom}_\Lambda(M(\omega), M(\omega'))$.*

Let s is a periodic string (so it is infinite on the left and right, and is repeating in both directions) and $M = M(s, V)$ is a finite dimensional band module. Let $t \subset s$ be a finite factor substring of s, and let $\phi \in \mathbf{Hom}_k(V, k) = V^*$. Then we define a morphism $f : M(s, V) \to M(t)$, and composition with a graph map $g : M(t) \to M(\omega)$ of finite string modules gives a map $gf : M(s, V) \to M(\omega)$ of a band module to a string module. We will include these in the collection of maps we call graph maps. We will associate gf with the substring t. Dually, let t be a finite image substring of the periodic string s, and let $\phi \in \mathbf{Hom}_k(k, V)$, and define a map $f : M(t) \to M(s, V)$. composing with any graph map $g : M(\omega) \to M(t)$ yields a map $gf \in \mathbf{Hom}_\Lambda(M(\omega), M(s, V))$. We will also call such maps graph maps, and we will associate the image substring t with gf. Again, direct applications of results of [32] and [33] give the following.

[32]

[33]

Theorem 0.0.15. *Fixing a basis of the finite dimensional vector spaces $\mathbf{Hom}_k(V, k)$ and $\mathbf{Hom}_k(k, V)$ for V a $k[T, T^{-1}]$-module, the graph maps $M(\omega) \to M(s, V)$ and $M(s, v) \to M(\omega)$ form bases of $\mathbf{Hom}_\Lambda(M(\omega), M(s, V))$ and $\mathbf{Hom}_\Lambda(M(s, V), M(\omega))$ respectively.*

Further, if ω and ω' are periodic, fixing a basis of two finite dimensional $k[T, T^{-1}]$-modules U and V, and fix bases for $\mathbf{Hom}_k(U, V)$ and $\mathbf{Hom}_{k[T,T^{-1}]}(U, V)$. Let $\mathfrak{g}_k(\omega, \omega')$ be the set of graph maps $M(\omega, U) \to M(\omega', V)$ determined by the basis of $\mathbf{Hom}_k(U, V)$, and let $\mathfrak{g}_{k[T,T^{-1}]}(\omega, \omega')$ be the graph maps determined by the basis of $\mathbf{Hom}_{k[T,T^{-1}]}(U, V)$. Then $\mathfrak{g}_k(\omega, \omega') \cap \mathfrak{g}_{k[T,T^{-1}]}(U, V)$ is a k-linear basis of $\mathbf{Hom}_\Lambda(M(\omega), M(\omega'))$.

Now, we have described all *Hom*-spaces for maps between finite dimensional indecomposable modules over any complete gentle algebra Λ. We are left to describe maps to and from infinite dimensional string modules.

Theorem 0.0.16. *Given an infinite non-periodic string ω, and a finite string s, each graph map $M(\omega) \to M(s)$ is abasis element of the infinite dimensional space $\mathbf{Hom}_\Lambda(M(\omega), M(s)$. Similarly, each graph map $M(s) \to M(\omega)$ is a basis element of the infinite dimensional space $\mathbf{Hom}_\Lambda(M(s), M(\omega))$. From this we gather that if ω' is also an infinite non-periodic string, and s is a factor substring of ω and an image substring of ω', then graph maps of the form*

$$M(\omega) \to M(s) \to M(\omega')$$

are basis elements of the infinite dimensional space $\mathbf{Hom}_\Lambda(M(\omega), M(\omega)')$. Similarly, if ω and ω' are infinite non-periodic strings, and s is a periodic string, graph maps of the form

$$M(\omega) \to M(s) \to M(s, V), \quad M(s, V) \to M(s) \to M(\omega)$$

are basis elements of $\mathbf{Hom}_\Lambda(M(\omega)$, and $M(s, V))$, $\mathbf{Hom}_\Lambda(M(s, V), M(\omega))$, respectively.

[$\star$ *Amelie: I need to show that the above maps form a complete list of well defined maps to and from infinite dimensional string modules. The case for infinite substrings should now be well defined, thus I should throw those out as possibilities.* $\star$]

Hammock Posets

[$\star$ Amelie: In this section I will extend Schröer's "Hammocks" (see [34]). This poset has as elements strings and bands, both finite and infinite, and the order indicates when there are maps between the corresponding modules. $\star$]

[34] ; and

The Auslander-Reiten Quiver

[$\star$ Amelie: This section follows Paquette [35] and Liu, Ng, Paquette [36] to describe the Auslander-Reiten quiver for complete gentle algebras. $\star$]

[35] ; and

[36]

We would now like to extend the notions of *hooks* and *co-hooks* so that we may describe the components of the Auslander-Reiten quiver. It is important to note that for string modules $\tau M(\omega)$, the module $\tau^2 M(\omega)$ is not defined. Similarly if $M(\omega)$ is a finite dimensional string module $\tau^{-1} M(\omega)$ is infinite dimensional and $\tau^{-2} M(\omega)$ is also not defined. For finite dimensional band modules $M(\omega, V)$ given by a primitive cyclic word we have that $M(\omega, V)$ lies at the mouth of a homogeneous 1-tube as in the case of finite dimensional string algebras.

Note, there are many homomorphisms in $\mathbf{Mod}(A)$ which were described in the previous section which are not detected by the Auslander-Reiten quiver, but which still have nice combinatorial properties.

In particular we have the following,

Theorem 0.0.17. *([37]): Let Q be a quiver with countably many vertices and I an ideal of kQ such that the quotient category kQ/I is a spectroid (i.e. for $x, y \in Q_0$ we have $kQ(x,y)/I(x,y)$ is finite dimensional and $kQ(x,x)/I(x,x)$ is local). Suppose M is indecomposable in $\mathbf{rep}(Q, I)$* 37

1. *There is an almost split sequence ending in M in $\mathbf{rep}(Q, I)$ if and only if M is finitely presented and non-projective.*

2. *There is an almost split sequence starting at M in $\mathbf{rep}(Q, I)$ if and only if M is finitely co-presented and non-injective.*

For complete gentle algebras $\Lambda = kQ/I$ with finitely many simple modules (i.e. Q is a finite quiver) we can of course apply the above theorem, and for the universal cover $\overline{Q}$ from Proposition 0.0.12, and other infinite bound complete gentle quivers, we indeed have that $\overline{\Lambda} = k\overline{Q}/\overline{I}$ is a spectroid, and Theorem 0.0.17 also applies.

We would now like to describe the Auslander-Reiten sequences in $\mathbf{Mod}(\Lambda)$, for any complete gentle algebra $\Lambda = kQ/I$. To do so, it would be prudent to attempt to generalize the classic paper by Butler and Ringler [38], describing the finite dimensional case. So, let $e_i, e_j \in \Lambda$ be primitive idempotents, and let J be the arrow ideal of Λ. Given any nonzero arrow $a \in e_j J e_i$, it is clear the Λ-module $\Lambda e_i/\Lambda a$ is indecomposable and non-projective. 38

Now, for each $a \in Q_1$, let $H_-(a) = (\cdots l_3^{-1} l_2^{-1} l_1^{-1})$ be the (unique) infinite inverse string such that $H_-(a) \cdot a$ is also a string. Let $H^-(a) = (\cdots r_3^{-1} r_2^{-1} r_1^{-1})$ be the (unique) infinite inverse string such that $s \cdot H^-(a)$ is a string. Dually, define $H_+(a) = (c_1 c_2 c_3 \cdots)$ be the unique infinite direct word giving a string $H_+(a) \cdot a$, and let $H^+(a) = (d_1 d_2 d_3 \cdots)$ be the unique infinite direct word giving a string $aH^+(a)$. Now, let $U(a) = H_-(a) \cdot a \cdot H^-(a)$. We know that $H_-(a)$ is a substring of $U(a)$ and $H^-(a)$ is a factor string as defined in Definition 0.0.7, thus there are graph maps,

$$f : M(H_-(a)) \to M(U(a)) \quad \text{and} \quad g : M(U(a)) \to M(H^-(a)).$$

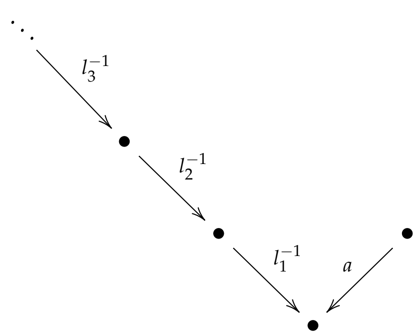
Proposition 0.0.18. *Let $\Lambda = kQ/I$ be a complete gentle algebra. For every $a \in Q_1$ we have an Auslander-Reiten sequence*

$$0 \to M(H_-(a)) \to M(U(a)) \to M(H^-(a)) \to 0.$$

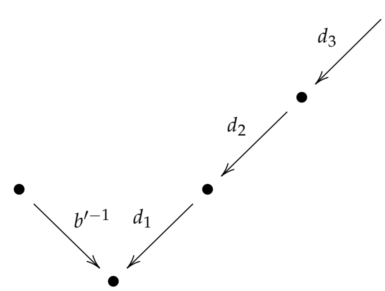
Theorem 0.0.19. *By abuse of notation, let us identify a module $M(\omega)$ with the string ω (Assuming $M(\omega)$ is not a band module) for the statement of this Theorem. Suppose $\Lambda = kQ/I$ is a complete gentle algebra and that s is a finite string. Let a be an arrow such that $a \cdot s$ is a string, and let a' be an arrow such that $s \cdot a'$ is a string. Similarly, let b, b' be arrows such that $b^{-1} \cdot s$ and $s \cdot b'^{-1}$ are strings. Such arrows $\{a, a', b, b'\}$ always exist since for any finite string s by definitions of complete gentle algebras. In particular, there is always a unique arrow, and a unique inverse arrow so that s can be extended to the right, and similarly there is a unique arrow and inverse arrow so that s can be continued to the left. Then the general component of the Auslander-Reiten quiver containing the string module $M(s)$ is of the form*

$$\begin{tikzcd}
 & & H^-(a)\cdot s\cdot H_-(a') \arrow[dr] & & \\
 & H^-(a)\cdot s \arrow[ur] \arrow[dr] & & s\cdot H_-(a') \arrow[ll, dotted, "\tau"] \arrow[dr] & \\
H^-(a)\cdot s\cdot H^+(b') \arrow[ur] \arrow[dr] & & s \arrow[ll, dotted, "\tau"] \arrow[ur] \arrow[dr] & & H_+(b)\cdot s\cdot H_-(a') \arrow[ll, dotted, "\tau"] \\
 & s\cdot H^+(b') \arrow[ur] \arrow[dr] & & H_+(b)\cdot s \arrow[ll, dotted, "\tau"] \arrow[ur] & \\
 & & H_+(b)\cdot s\cdot H^+(b') \arrow[ur] & &
\end{tikzcd}$$

where s is a finite string, $H^-(a)$ and $H^+(b')$ denotes the hooks, i.e. of the form



or



and $H_+(b)$ and $H_-(a')$ denote the co-hooks (the dual notion).

Proof. The fact that the above sequences are almost-split are again given by an direct application of the results of Butler and Ringel. From the description of the indecomposable string modules, we see that all finitely presented and finitely co-presented modules are of the form given in the above component, and those which are not finitely presented or finitely co-presented correspond to a nonperiodic infinite word which has infinite dimensional top or infinite dimensional socle, and thus is represented by a nonperiodic "zig-zag" For any such module M, τM and $\tau^{-1}M$ are not defined by [39]. □ 39

Theorem 0.0.20. *Finite dimensional band modules will again lie in homogeneous* 1*-tubes,*

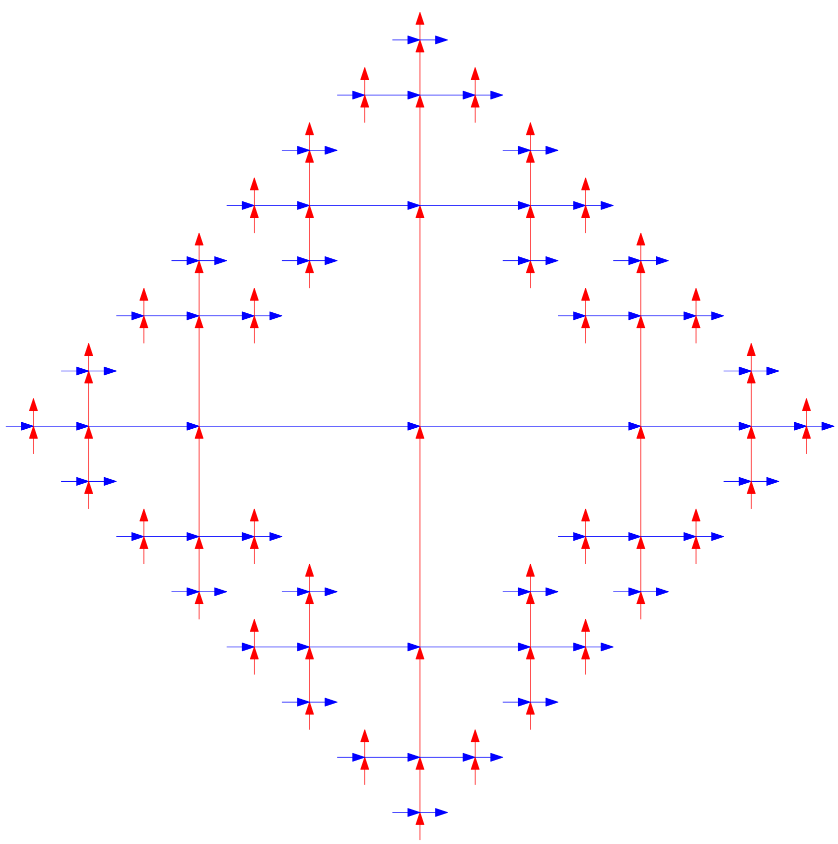
$$M(C,V) \to M(C,V') \to M(C,V).$$

Further, infinite dimensional band modules do not appear in an almost-split sequence.

Proof. The second statement is clear from Theorem 0.0.17. To prove that finite dimensional band modules lie in homogeneous 1-tubes follows from [40]. □ 40

Cantor Set Structure

Universal Cover



Part III

Derived Categories of Compact Gentle Surface Algebras

Compact Gentle Surface Orders

Orders and Burban-Drozd Gluing

Definitions and Properties

Definition 0.0.21. Let R be a complete noetherian local domain with field of fractions $\mathbb{K}$, and residue field k. An R-**Order** Λ in a k-algebra A is a unital subring of A such that

1. $\mathbb{K}\,\Lambda = A$, and

2. Λ is finitely generated as an R-module.

Let $C = [\sigma, \alpha, \phi]$ be a constellation, and let $\Gamma = \hookrightarrow \Sigma$ be the associated graph cellularly embedded in the closed Riemann surface Σ. Further, let $n(i) = n_i$ denote the length of the cycle given by σ_i[41]

1. For each cycle $\sigma_i \in \Gamma_0$ associated to the vertex i, we associate a local order $\Omega_i = \Omega(\sigma_i)$, and a regular principal ideal $\omega_i := \omega_0(\sigma_i)\Omega_i = \Omega_i\omega_0(\sigma_i)$.

2. The **hereditary**[42] order associated to σ_i is then given by

$$\mathbb{H}_i = \begin{pmatrix} \Omega_i & \omega_i & \omega_i & \cdots & \omega_i & \omega_i \\ \Omega_i & \Omega_i & \omega_i & \cdots & \omega_i & \omega_i \\ \Omega_i & \Omega_i & \Omega_i & \cdots & \omega_i & \omega_i \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \Omega_i & \Omega_i & \Omega_i & \cdots & \Omega_i & \omega_i \\ \Omega_i & \Omega_i & \Omega_i & \cdots & \Omega_i & \Omega_i \end{pmatrix}_{n(i)}$$

3. Let $\Omega_i^{(k,k)}$ denote the (k,k) entry of Ω_i in $\mathbb{H}_i$.

[41] Recall: for a constellation $C = [\sigma, \alpha, \phi]$ and the associated graph Γ on a (compact) closed Riemann surface Σ, the length of the (nonzero) cycle in the gentle medial quiver $Q(C)$, associated to σ_i is just the order of the cycle σ_i. Here we are using the notation $\sigma = \sigma_1\sigma_2\cdots\sigma_r$, where $\sigma_i \in S_{[2m]}$ is a cycle permutation.

[42] An algebra, or an order, is hereditary if no module has a minimal projective resolution greater than length 1. This means the *projective dimension* of any module is no greater than 1, and therefore the global dimension is at most 1.

4. For each $1 \leq k \leq n_i$ let

$$P_{i,k} := \begin{pmatrix} \omega_i \\ \vdots \\ \omega_i \\ \Omega_i \\ \Omega_i \\ \vdots \\ \Omega_i \\ \Omega_i \end{pmatrix}$$

where the first entry equal to Ω_i is the k^{th} row.

The modules $\{P_{i,k} : 1 \leq k \leq n_i\}$ give a complete set of non-isomorphic idecomposable projective $\mathbb{H}_i$-modules, with the natural inclusions

$$P_{i,1} \hookleftarrow P_{i,2} \hookleftarrow \cdots \hookleftarrow P_{i,n_i-1} \hookleftarrow P_{i,n_i} \hookleftarrow P_{i,1}.$$

where the final map is given by left-multiplication by $\omega_0(\sigma_i)$. If we identify $P_{i,k}$ with the edge $e^i_k = e_k(\sigma_i)$, where $\sigma_i = (e^i_1, e^i_2, ..., e^i_{n_i})$ is a cyclic permutation, then the chain of inclusions can be interpreted in terms of the cycle σ_i. From the embedding $\Gamma \hookrightarrow \Sigma$ given by the constellation $C = [\sigma, \alpha, \phi]$, this can be interpreted as walking clockwise around the vertex of σ_i. We will take $P_{i,k} = P_{i,k+n_i}$, but each e^i_k must be multiplied by the automorphism σ_i after one trip around the cycle, i.e. there is some multiplication by a power of $\omega_0(\sigma_i)$ involved. In particular, conjugation by

$$\underline{\omega_i} := \begin{pmatrix} 0 & 0 & 0\cdots & 0 & \omega_0(\sigma_i) & \\ 1 & 0 & 0\cdots & 0 & 0 & \\ 0 & 1 & 0\cdots & 0 & 0 & \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0\cdots & 0 & 0 & \\ 0 & 0 & 0\cdots & 1 & 0 & \end{pmatrix}_{n_i}$$

cyclically permutes the indecomposable projective $\mathbb{H}_i$-modules $P_{i,k}$, and it induces an automorphism of $\mathbb{H}_i$ which we also call σ_i. Now, for each cycle $\sigma_p, \sigma_q \in S_{[2m]}$ of σ, we fix an isomorphism

$$\Omega_p/\omega_p \cong \Omega_q/\omega_q.$$

Identifying all such rings, let $\overline{\Omega} = \Omega_i/\omega_i$ for all $\sigma_i \in \Gamma_0$. Let $\pi_i : \Omega_i \to \overline{\Omega}$ be a fixed epimorphism with kernel ω_i. Now, we have a pull-back diagram

$$
\begin{tikzcd}
\Omega_{p,q} \arrow[r, "\tilde{\pi}_p"] \arrow[d, "\tilde{\pi}_q"'] & \Omega_p \arrow[d, "\pi_p"] \\
\Omega_q \arrow[r, "\pi_q"'] & \overline{\Omega}.
\end{tikzcd}
$$

which is in general different and non-isomorphic for different choices of π_p and π_q.

Definition 0.0.22. Let $\mathbb{H} = \prod_{\sigma_i \in \Gamma_0} \mathbb{H}_i$. Let e_k^i be an edge around σ_i, and let $\alpha_{i,j}^{k,l} = (e_k^i, e_l^j)$ be a 2-cycle of the fixed-point free involution α of $C = [\sigma, \alpha, \phi]$ giving the end vertices σ_i and σ_j of the edge $e_k^i \equiv e_l^j$ under the gluing identifying the half-edges e_k^i and e_l^j. It is possible that $\sigma_i = \sigma_j$ if $\alpha_{i,j}^{k,l}$ defines a loop at the vertex σ_i in Γ. We replace the product $\Omega_i^k \times \Omega_j^l$ in $\mathbb{H}_i \times \mathbb{H}_j$ with $\Omega_{i,j}$. This identifies the (k,k) entry of $\mathbb{H}_i$ with the (l,l) entry of $\mathbb{H}_j$, modulo ω. Doing this for all edges of Γ, we get the **Constellation Order** or the (compact) **Gentle Surface Order** $\Lambda := \Lambda(C) = \Lambda(\Gamma)$ associated to the constellation C, or equivalently to the embedded graph $\Gamma \hookrightarrow \Sigma$.

Proposition 0.0.23. *1. The indecomposable projective Λ-modules are in bijection with the 2-cycles $(e_k^i, e_l^j) = \alpha_{k,l}^{i,j}$, of $\alpha \in S_{2m}$ for the constellation $C = [\sigma, \alpha, \phi]$. Equivalently, the indecomposable projectives are in bijection with the edges Γ_1. We label them as P_e for $\alpha_{i,j}^{k,l} = e = (e_k^i, e_l^j) \in \Gamma_1$ attached to the vertices σ_i and σ_j.*

2. Each indecomposable projective Λ-module is a pullback,

$$
\begin{tikzcd}
P_e \arrow[r] \arrow[d] & P_{i,k} \arrow[d] \\
P_{j,l} \arrow[r] & \overline{\Omega}
\end{tikzcd}
$$

where the $P_{i,k}$ is an indecomposable projective module in $\mathbb{H}_i$, and $P_{j,l}$ is an indecomposable projective of $\mathbb{H}_j$.

3. From this we obtain the following commutative diagram with exact rows *and columns,*

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & P_{i,k-1} \arrow[r] \arrow[u] & P_{i,k} \arrow[r] \arrow[u] & \overline{\Omega} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & P_{i,k-1} \arrow[r] \arrow[u] & P_e \arrow[r] \arrow[u] & P_{j,l} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[u] & P_{j,l-1} \arrow[r] \arrow[u] & P_{j,l-1} \arrow[r] \arrow[u] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

4. $\ker(P_e \to P_{i,k})$ *and* $\ker(P_e \to P_{j,l})$ *are projective* $\mathbb{H}_j$ *and* $\mathbb{H}_i$*-modules respectively, local* Λ*-modules, and are uniserial.*

A Basic Example

Example 0.0.24. Let R have maximal ideal $\mathfrak{m} = \langle m \rangle$, with residue field $k = R/\mathfrak{m}$, and field of fractions $\mathbb{K} = R_{\mathfrak{m}}$. Let Γ be the genus zero graph,

$$\sigma_1 \overset{\alpha_1}{\text{———}} \sigma_2 \overset{\alpha_2}{\text{———}} \sigma_3 \overset{\alpha_3}{\text{———}} \sigma_4$$

given by the constellation $C = [\sigma, \alpha, \phi]$ such that $\sigma = (1)(2,3)(4,5)(6) = \sigma_1\sigma_2\sigma_3\sigma_4$, and $\alpha = (1,2)(3,4)(5,6) = (e_1)(e_2)(e_3)$. We may take

$$\mathbb{H} = \left\{ \begin{pmatrix} a_{11} \end{pmatrix}, \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}, \begin{pmatrix} d_{11} \end{pmatrix} \middle| a_{ij}, b_{ij}, c_{ij}, d_{ij} \in R, b_{12}, c_{12} \in \mathfrak{m} \right\} = \prod_{i=1}^{4} \mathbb{H}_i\,.$$

We then have the congruences modulo $\mathfrak{m}$,

$$a_{11} \sim b_{11}, b_{22} \sim c_{11}, c_{22} \sim d_{11}.$$

Or in more compact notation which will be used later,

$$(1,1) \sim (2,1), (2,2) \sim (3,1), (3,2) \sim (4,1).$$

We then have

$$\Lambda = \begin{pmatrix} R & 0 & 0 & 0 & 0 & 0 \\ 0 & R & \mathfrak{m} & 0 & 0 & 0 \\ 0 & R & R & 0 & 0 & 0 \\ 0 & 0 & 0 & R & \mathfrak{m} & 0 \\ 0 & 0 & 0 & R & R & 0 \\ 0 & 0 & 0 & 0 & 0 & R \end{pmatrix}$$

such that

$$\lambda_{11} = \lambda_{22}, \lambda_{33} = \lambda_{44}, \lambda_{55} = \lambda_{66}$$

with all equalities taken modulo $\mathfrak{m}$, i.e. the residues λ_{ii} are equal in $k = R/\mathfrak{m}$. [43]

Suppose in particular that

$$\mathbb{H}_1 \cong \mathbb{H}_4 \cong k[[x]], \quad \mathbb{H}_2 \cong \mathbb{H}_3 \cong \begin{pmatrix} k[[x]] & (x)k[[x]] \\ k[[x]] & k[[x]] \end{pmatrix}$$

Letting $\mathfrak{m}_i = (x)$ for $i = 1, ..., 4$, we get a pullback diagram

$$\begin{tikzcd} \Lambda \arrow[r, "\pi_1"] \arrow[d, "\iota_1"'] & \Lambda / \mathbf{rad}(\Lambda) = k^3 \arrow[d] \\ \mathbb{H} \arrow[r, "\pi_2"'] & \mathbb{H} / \mathbf{rad}(\mathbb{H}) = k^6 \end{tikzcd}$$

where $\Omega_i = k[[x]]$ for $i = 1, ..., 4$ and $\omega_i = (x) = \mathbf{rad}(k[[x]])$, so that $\mathbf{rad}(\mathbb{H}_1) = k = \mathbf{rad}(\mathbb{H}_4)$ and $\mathbf{rad}(\mathbb{H}_2) = k \times k = \mathbf{rad}(\mathbb{H}_3)$; and $\mathbf{rad}(\mathbb{H}) = \mathbf{rad}(\Lambda)$. Then we get

$$\Lambda = \begin{pmatrix} k[[x]] & 0 & 0 & 0 & 0 & 0 \\ 0 & k[[x]] & (x)k[[x]] & 0 & 0 & 0 \\ 0 & k[[x]] & k[[x]] & 0 & 0 & 0 \\ 0 & 0 & 0 & k[[x]] & (x)k[[x]] & 0 \\ 0 & 0 & 0 & k[[x]] & k[[x]] & 0 \\ 0 & 0 & 0 & 0 & 0 & k[[x]] \end{pmatrix}$$

such that

$$\lambda_{11} = \lambda_{22}, \lambda_{33} = \lambda_{44}, \lambda_{55} = \lambda_{66}$$

with all equalities taken modulo $\mathfrak{m} = (x) = \omega_i$, i.e. the residues λ_{ii} are equal in k. The automorphisms given by σ on $\mathbb{H}_2 \cong \mathbb{H}_3$ is

$$\begin{pmatrix} 0 & x \\ 1 & 0 \end{pmatrix}$$

Further, this is exactly the algebra associated to the graph

$$\sigma_1 \overset{\alpha_1}{\text{———}} \sigma_2 \overset{\alpha_2}{\text{———}} \sigma_3 \overset{\alpha_3}{\text{———}} \sigma_4$$

Description of the Projective Objects

Description of the Injective Objects

Description of the Maps Between Projective Objects

Projective Resolutions of Simple Objects

In this section we denote by $P^\bullet$ a complex of projective modules

$$\cdots \to P^{-1} \to P^0 \to P^1 \to \cdots$$

[43] Notice, the equalities in $k = R/\mathfrak{m}$ (i.e. equalities of residues modulo $\mathfrak{m}$) are given by α in the constellation $C = [\sigma, \alpha, \phi]$.

over some algebra Λ. Let $\Lambda = \Lambda(C)$ be a gentle constellation order given by the constellation $C = [\sigma, \alpha, \phi]$. [44]

In this section we will show the following,

Proposition 0.0.25. *Classifying all 3-constellations with a fixed passport $[\lambda_1, \lambda_2, \lambda_3]$ is equivalent to classifying all* "gentle constellation orders" *with the same normalization, or equivalently all* "closed surface algebras" *with the same normalization.*

Theorem 0.0.26. *1. The indecomposable projective modules $P_e = P_{\alpha_{i,j}}$ have radical*

$$\mathbf{rad}(P_e) = U(\sigma_i^k) \oplus U(\sigma_j^l)$$

where $U(\sigma_p^q) \cong P_{p,q} \in \mathbf{Mod}(\mathbb{H}_p)$ is an indecomposable uniserial Λ-module and and indecomposable projective $\mathbb{H}_p$-module.

2. The minimal projective resolution of the simple module $S(\alpha_{i,j}^{k,l}) = S(e_k^i, e_l^j)$ of Λ, corresponding to the vertex $\alpha_{i,j}^{k,l} = (e_k^i, e_l^j)$ of Q[45], is infinite periodic. In particular the period p of the minimal resolution $P^\bullet(\alpha_{i,j}^{k,l}) = P^\bullet \to S(\alpha_{i,j}^{k,l})$ is exactly the least common multiple,

$$p(P^\bullet(i,j)) = \mathbf{lcm}\{|\mathcal{O}_\phi(e_k^i)|, |\mathcal{O}_\phi(e_l^j)|\}.$$

where $\mathcal{O}_\phi(e_k^i)$ and $\mathcal{O}_\phi(e_l^j)$ are the orbits under the action of ϕ^{-1} of e_k^i and e_l^j on the two anti-cycles (or relations in I) passing through the vertex $\alpha_{i,j}^{k,l}$.

3. The differentials in the minimal projective resolution of the simple module $S(\alpha_{i,j}^{k,l})$, $d^m : P^m \to P^{m+1}$ in the resolution

$$P^\bullet(\alpha_{i,j}^{k,l}) : \cdots \to P^m \to P^{m+1} \to \cdots$$

are given by multiplication by the matrix

$$d^m := \begin{pmatrix} a(\phi^{-m} \cdot e_k^i) & 0 \\ 0 & a(\phi^{-m} \cdot e_l^j) \end{pmatrix}.$$

where $a(\phi^{-m} e_k^i) \in Q_1$ is the arrow with $ta = \phi^{-m} e_k^i$ and $ta(\phi^{-m} \cdot e_l^j) = \phi^{-m} \cdot e_l^j$.

4. The syzygies $\Omega^m(\alpha_{k,l}^{i,j}) = \ker(d^m)$ are of the form

$$\Omega^m(\alpha_{k,l}^{i,j}) = U(\sigma(\phi^{-m}\alpha_{i,j}^{k,l}) = U(\sigma(\phi^{-m}\sigma e_k^i)) \oplus U(\sigma(\phi^{-m}\sigma e_l^j))$$

the uniserial modules at the vertex $\sigma\phi^{-m}\sigma e_k^i$ and $\sigma\phi^{-m}\sigma e_l^j$ which are annihilated by left multiplication by the arrows associated to $\phi_i^{-m}\sigma \cdot e_k^i$ and $\phi_i^{-m}\sigma \cdot e_l^j$ (so $a : ta = \sigma\phi^{-m}\sigma \cdot e_k^i, b : tb = \sigma\phi^{-m}\sigma \cdot e_l^j \in Q_1$). Note, there is exactly one such uniserial module for each orbit of the arrows $a : ta = \phi^{-m-1} e_k^i, b : tb = \phi^{-m-1} e_l^j \in Q_1$ by definition of the relations I.

[44] Question: Find all constellations with fixed monodromy group. (Problem 1.1.10) Having fixed a group $G \leq S_n$ and a passport $[K_1, ..., K_k]$, all the K_i being conjugacy classes in G, find all the constellations with the cartographic group G and with the refined passport $[K_1, ..., K_k]$ with respect to G.

[45] or equivalently the edge of the same labeling in $\Gamma(C)$ connecting vertex σ_i and σ_j

The point of the first four statements is to describe the structure of resolutions and the Cohen-Macaulay modules. This will be useful later when explaining how to recover a graph embedded in a Riemann surface entirely in terms of the projective resolutions of the simple modules. The last four statements will provide some invariants which will have some applications to constellations and graphs embedded in Riemann surface. For example, a constellation is self dual if and only if the opposite algebra of the Koszul dual is isomorphic to the algebra $\Lambda(C)$. We will prove this and other things a little later.

Proof. 1. First, $\alpha_{i,j}^{k,l} = (e_k^i, e_l^j) = e$, and with fixed labeling of the edges of $\Gamma(C)$, we have $e_k^i = \sigma_i^{k-1} \cdot e_1^i$, and $e_l^j = \sigma_j^{l-1} \cdot e_l^j$, given by the automorphism

$$\sigma_i^k := \begin{pmatrix} 0 & 0 & \cdots & 0 & \sigma_i \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}^k$$

So, σ_i acts on the algebra $\Lambda = kQ/I$ by left multiplication of e_{j-1}^i (and therefore a_{j-1}^i) by the arrow $\sigma_i a_{j-1}^i = a_j^i$ in the quiver $Q(\Lambda)$. Notice, this multiplication is always nonzero since $\sigma_i = (e_1^i, e_2^i, ..., e_{n_i}^i)$ is a cyclic permutation around the vertex it corresponds to in $\Gamma(C)$, and there is by definition a unique arrow by which σ acts on a given idempotent e_{j-1}^i (and on a_{j-1}^i) lying on this cycle corresponding to the hereditary order $\mathbb{H}_i$ in the pullback diagram defining $\Lambda(C)$.

2. Let $P(\alpha_{i,j}^{k,l})$ be the projective cover of $S(\alpha_{i,j}^{k,l})$. From the desription of the radical of $P(\alpha_{i,j}^{k,l})$ as the two uniserial module in $\mathbb{H}$ corresponding to the idempotents $\sigma \cdot e_k^i$ and $\sigma \cdot e_l^j$ in $\mathbb{H}_i$ and $\mathbb{H}_j$ respectively, the next term in the resolution is the direct sum of the two indecomposable projective covers $P(\sigma \cdot e_k^i)$ and $P(\sigma \cdot e_l^j)$ in $\mathbf{Mod}(\Lambda)$. Clearly the kernel of the covering $P(\sigma \cdot e_l^j) \to U(\sigma \cdot e_l^j)$ is exactly the uniserial $U(\sigma\alpha \cdot \sigma \cdot e_l^j)$. Now, $\sigma\alpha = \phi^{-1}$ by defintion of a constellation, so $U(\sigma\alpha \cdot \sigma \cdot e_l^j) = U(\phi^{-1} \cdot \sigma \cdot e_l^j)$, and its projective cover is $P(\phi^{-1} \cdot \sigma \cdot e_l^j)$. The kernel of this covering is $U(\sigma\alpha \cdot (\phi^{-1} \cdot \sigma \cdot e_l^j)) = U(\phi^{-1}\sigma e_k^i)$. This pattern continues also for $\sigma \ell_l^j$, and the terms P^m in the resolution are

$$P(\phi^{-m}\sigma e_k^i) \oplus P(\phi^{-m}\sigma e_l^j).$$

So the terms have indecomposable direct summands which cycle through the orbit of e_k^i and e_l^j under the action of ϕ. The orbits are anit-cycle in I, and the place at which the two cycle meet up at $\alpha_{i,j}^{k,l} = (e_k^i, e_l^j)$ is exactly $p = \mathbf{lcm}\{|\mathcal{O}_\phi(e_k^i)|, |\mathcal{O}_\phi(e_l^j)|$.

3. Since the kernel of the cover of a uniserial $P(\alpha_{i,j}^{k,l}) \to U(e_k^i)$ is exactly $U(\sigma e_k^i)$ and it is embedded in $P(\alpha_{i,j}^{k,l})$ as a submodule via multiplication by the arrow $a : ha = \sigma e_k^i$, we get that the differential is indeed,

$$d^m := \begin{pmatrix} a(\phi^{-m} \cdot e_k^i) & 0 \\ 0 & a(\phi^{-m} \cdot e_l^j) \end{pmatrix}.$$

4.

5.

6.

7.

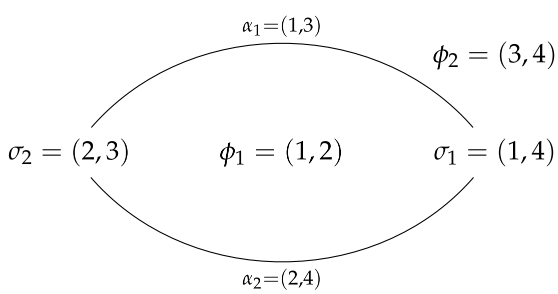
8.

$\square$

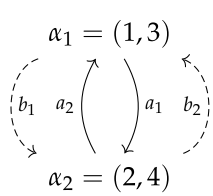
Example 0.0.27. Let $C = [\sigma, \alpha, \phi]$ be given by

$$\sigma = (1,4)(2,3), \quad \alpha = (1,3)(2,4), \quad \phi = (1,2)(3,4),$$

then $\chi(C) = 2$ and $g(C) = 0$. The embedded graph can be represented by the equator of the sphere with two vertices on it. Or, if we embed it in the plane:



The quiver which comes from this graph is



The associated matrix data is

$$\Lambda = \left\{ \begin{pmatrix} \lambda_{11} & x\cdot\lambda_{12} & 0 & 0 \\ \lambda_{21} & \lambda_{22} & 0 & 0 \\ 0 & 0 & \lambda_{33} & x\cdot\lambda_{34} \\ 0 & 0 & \lambda_{43} & \lambda_{44} \end{pmatrix} \middle| \lambda_{ij} \in k[[x]], \lambda_{22} = \lambda_{33} (\text{mod } x) \right\}$$

With normalization

$$\mathbb{H} = \left\{ \begin{pmatrix} \lambda_{11} & x\cdot\lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} \times \begin{pmatrix} \mu_{11} & x\cdot\mu_{12} \\ \mu_{21} & \mu_{22} \end{pmatrix} \middle| \lambda_{ij}, \mu_{kl} \in k[[x]] \right\}$$

and the pullback diagram is given by the relation $(1,2) \sim (2,1)$, $\overline{m} = (2,2)$. The projective resolution of the simple $S(\alpha_1)$ has the following form

$$\cdots \longrightarrow P(\alpha_2)\oplus P(\alpha_2) \xrightarrow{\begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix}} P(\alpha_1)\oplus P(\alpha_1) \xrightarrow{\begin{pmatrix} a_2 & 0 \\ 0 & b_2 \end{pmatrix}} P(\alpha_2)\oplus P(\alpha_2) \xrightarrow{\begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix}} P(\alpha_1) \longrightarrow S(\alpha_1)$$

Recovering the Surface

Give the list of projective resolutions of the simple objects, we may completely recover the constellation C, and therefore the Riemann surface and cellularly embedded graph. In fact, we don't need any other information. Neither the surface algebra, nor the medial quiver are needed to recover $C = [\sigma, \alpha, \phi]$, only the projective resolutions of the simples.

The Graded Hilbert Function

The Generalized Cartan Matrix

The Generalized Euler Form

Examples

Although it is not phrased in these terms, in [46] Kraskiewicz and Weyman study a class of finite dimensional gentle algebras $A(n)$ given by the following pullback diagram with $\mathbb{A}_n$ the equioriented type $\mathbb{A}$ quiver with $|Q_0| = n$.

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$$\begin{tikzcd} A(n) \arrow[r, "\pi_1"] \arrow[d, "\iota_1"'] & \mathbf{top}(A(n)) \arrow[d, "\iota_2"] \\ k\mathbb{A}_n \times k\mathbb{A}_n \arrow[r, "\pi_2"'] & \mathbf{top}(k\mathbb{A}_n \times k\mathbb{A}_n) \end{tikzcd}$$

The maps π_1, π_2 are the projections onto $\mathbf{top}(A(n)) = A(n)/\,\mathbf{rad}(A(n)) = \bigoplus_{i=1}^{n} P(i)/\,\mathbf{rad}(P(i)) \cong \prod_{i=1}^{n} k$, $\mathbf{top}(k\mathbb{A}_n \times k\mathbb{A}_n) = k\mathbb{A}_n/\,\mathbf{rad}(k\mathbb{A}_n) \times k\mathbb{A}_n/\,\mathbf{rad}(k\mathbb{A}_n) \cong \prod_{i=1}^{n} k \times \prod_{i=1}^{n} k$. The maps ι_1, ι_2 are idempotent embeddings given by $e_i \mapsto e'_i + e''_i$, where e_i is the i^{th} primitive idempotent of the algebra $A(n)$, e'_i is the primitive idempotent in the first copy of $k\mathbb{A}_n$ corresponding to the vertex $i \in \mathbb{A}_{n,0}$, and e''_i is the i^{th} idempotent of the second copy. We would like to study a class of algebras closely related to these algebras. Let $\tilde{\mathbb{A}}_n$ be the cyclicly oriented quiver with n arrows and vertices. For $n = 0$ (mod 2), let $\Lambda(n)$ be given by the matrix algebra

$$\Lambda(n) := \begin{pmatrix} \lambda_{11} & (x)\lambda_{12} & \cdots & (x)\lambda_{1,n-1} & (x)\lambda_{1,n} \\ \lambda_{11} & \lambda_{12} & \cdots & (x)\lambda_{1,n-1} & (x)\lambda_{1,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda_{n-1,1} & \lambda_{n-1,2} & \cdots & \lambda_{1,n-1} & (x)\lambda_{1,n} \\ \lambda_{n1} & \lambda_{n2} & \cdots & \lambda_{n,n-1} & \lambda_{n,n} \end{pmatrix} \times \begin{pmatrix} \mu_{11} & (x)\mu_{12} & \cdots & (x)\mu_{1,n-1} & (x)\mu_{1,n} \\ \mu_{11} & \mu_{12} & \cdots & (x)\mu_{1,n-1} & (x)\mu_{1,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mu_{n-1,1} & \mu_{n-1,2} & \cdots & \mu_{1,n-1} & (x)\mu_{1,n} \\ \mu_{n1} & \mu_{n2} & \cdots & \mu_{n,n-1} & \mu_{n,n} \end{pmatrix},$$

where $\lambda_{ij}, \mu_{ij} \in k[[x]], \lambda_{ii} = \mu_{ii}$ (mod x), which is given by the pullback diagram

$$\begin{tikzcd} \Lambda(n) \arrow[r, "\pi_1"] \arrow[d, "\iota_1"'] & \Lambda(n)/\,\mathbf{rad}(\Lambda(n)) \arrow[d, "\iota_2"] \\ \mathbb{H}(n) \arrow[r, "\pi_2"'] & \mathbb{H}(n)/\,\mathbf{rad}(\mathbb{H}(n)) \end{tikzcd}$$

where

$$\mathbb{H}(n) = \begin{pmatrix} \lambda_{11} & (x)\lambda_{12} & \cdots & (x)\lambda_{1,n-1} & (x)\lambda_{1,n} \\ \lambda_{11} & \lambda_{12} & \cdots & (x)\lambda_{1,n-1} & (x)\lambda_{1,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda_{n-1,1} & \lambda_{n-1,2} & \cdots & \lambda_{1,n-1} & (x)\lambda_{1,n} \\ \lambda_{n1} & \lambda_{n2} & \cdots & \lambda_{n,n-1} & \lambda_{n,n} \end{pmatrix} \times \begin{pmatrix} \mu_{11} & (x)\mu_{12} & \cdots & (x)\mu_{1,n-1} & (x)\mu_{1,n} \\ \mu_{11} & \mu_{12} & \cdots & (x)\mu_{1,n-1} & (x)\mu_{1,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mu_{n-1,1} & \mu_{n-1,2} & \cdots & \mu_{1,n-1} & (x)\mu_{1,n} \\ \mu_{n1} & \mu_{n2} & \cdots & \mu_{n,n-1} & \mu_{n,n} \end{pmatrix},$$

with $\lambda_{ij}, \mu_{ij} \in k[[x]]$. Then, $\mathbf{rad}(\mathbb{H}(n)) = \mathbf{rad}(\Lambda(n))$, and ι_1 and ι_2 are given by radical embeddings with mapping on the idempotents:

$$\Lambda(n)/\,\mathbf{rad}(\Lambda(n)) \ni e_{ii} \mapsto e'_{ii} + e''_{ii} \in \mathbb{H}(n)/\,\mathbf{rad}(\mathbb{H}(n)).$$

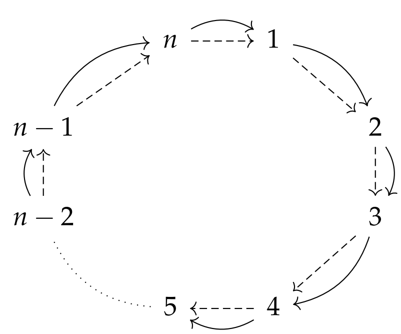
Now, if $n = 1$ (mod 2), then we take

$$\Lambda(n) := \begin{pmatrix} \lambda_{11} & (x)\lambda_{12} & \cdots & (x)\lambda_{1,n-1} & (x)\lambda_{1,n} \\ \lambda_{11} & \lambda_{12} & \cdots & (x)\lambda_{1,n-1} & (x)\lambda_{1,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda_{n-1,1} & \lambda_{n-1,2} & \cdots & \lambda_{1,n-1} & (x)\lambda_{1,n} \\ \lambda_{n1} & \lambda_{n2} & \cdots & \lambda_{n,n-1} & \lambda_{n,n} \end{pmatrix}$$

with $\lambda_{ij} \in k[[x]]$, and $\lambda_{ii} = \lambda_{2i,2i}$ (mod x), for $i = 1, 2, ..., n$. We have

$$\mathbb{H}(n) = \begin{pmatrix} \lambda_{11} & (x)\lambda_{12} & \cdots & (x)\lambda_{1,n-1} & (x)\lambda_{1,n} \\ \lambda_{11} & \lambda_{12} & \cdots & (x)\lambda_{1,n-1} & (x)\lambda_{1,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda_{n-1,1} & \lambda_{n-1,2} & \cdots & \lambda_{1,n-1} & (x)\lambda_{1,n} \\ \lambda_{n1} & \lambda_{n2} & \cdots & \lambda_{n,n-1} & \lambda_{n,n} \end{pmatrix}$$

with $\lambda_{ij} \in k[[x]]$, and the pullback diagram is the same with the new $\Lambda(n)$ and $\mathbb{H}(n)$. We call $\mathbb{H}(n)$ the **normalization** of the order $\Lambda(n)$. We will call $\Lambda(n)$ a **double simple cycle order**. In general, we may replace $k[[x]]$ by any R such that R is a complete noetherian local domain, and we then replace (x) with $\mathfrak{m}$, the maximal ideal of R. For $R = k[[x]]$ and $\mathfrak{m} = (x)$, the order $\Lambda(n)$ is isomorphic to the completion of the path algebra:



where the relations $I = \langle a_{i+1}a_i, b_{i+1}b_i \rangle_{i=1}^{n}$ is given by composition of two arrows of the same color. We then have a pullback diagram

$$\begin{tikzcd} \tilde{A}(n) \arrow[r, "\pi_1"] \arrow[d, "\iota_1"'] & k^n \arrow[d, "\iota_2"] \\ k\tilde{\mathbb{A}}_n \times k\tilde{\mathbb{A}}_n \arrow[r, "\pi_2"'] & k^n \times k^n \end{tikzcd}$$

for n odd, and

$$\begin{tikzcd} \tilde{A}(n) \arrow[r, "\pi_1"] \arrow[d, "\iota_1"'] & k^n \arrow[d, "\iota_2"] \\ k\tilde{\mathbb{A}}_{2n} \times \arrow[r, "\pi_2"'] & k^{2n} \end{tikzcd}$$

for n even.

The Homotopy Category

The Derived Category

Part IV

Representation Varieties and Rings of Relative Invariants

Background on Representation Theory of General Linear Groups

Young Tableaux and Schur Functors

A **partition** $\lambda \vdash m$, of some nonnegative integer m is a sequence of nonincreasing numbers $\lambda = (\lambda_1, ..., \lambda_s)$ such that $\sum_i \lambda_i = m$. We define a **Young diagram** corresponding to a partition $\lambda = (\lambda_1, ..., \lambda_s) \vdash m$, as the diagram with λ_i boxes in the i^{th} row. We identify partitions with their Young diagrams and speak of the two interchangeably. For a free module E over a commutative ring K with ordered basis $\{e_1, ..., e_n\}$, we associate a filling of the diagram λ by integers $\{1, ..., n\}$ to an element in the module

$$\bigwedge^{\lambda_1} E \otimes \cdots \otimes \bigwedge^{\lambda_s} E$$

as follows: If in row i and box j of λ we have the integers $t(i,j)$ for $j = 1, ..., \lambda_i$, we associate the element

$$e_{t(1,1)} \wedge \cdots \wedge e_{t(1,\lambda_1)} \otimes \cdots \otimes e_{t(s,1)} \wedge \cdots \wedge e_{t(s,\lambda_s)}.$$

We define such a filling to be a **tableaux**. A tableaux is **standard** if its rows are strictly increasing, and its columns are nondecreasing.

Fix a free module E of rank n over a commutative ring K. Let $\lambda = (\lambda_1, ..., \lambda_s) \vdash m$ be a partition. We define the module

$$L_\lambda E = \bigwedge^{\lambda_1} \otimes \cdots \otimes \bigwedge^{\lambda_s} E / R(\lambda, E)$$

where the submodule $R(\lambda, E)$ is the sum of all submodules of the form

$$\bigwedge^{\lambda_1} E \otimes \cdots \otimes \bigwedge^{\lambda_{a-1}} E \otimes R_{a,a+1} E \otimes \bigwedge^{\lambda_{a+2}} E \otimes \cdots \otimes \bigwedge^{\lambda_s} E$$

for $1 \leq a \leq s-1$. Here $R_{a,a+1}E$ is the submodule spanned by the images of the maps $\theta(\lambda, a, u, v, E)$

$$\begin{array}{c} \bigwedge^{u} E \otimes \bigwedge^{\lambda_a - u + \lambda_{a+1} - v} E \otimes \bigwedge^{v} E \quad . \\ \Big\downarrow {\scriptstyle 1 \otimes \Delta \otimes 1} \\ \bigwedge^{u} E \otimes \bigwedge^{\lambda_a - u} E \otimes \bigwedge^{\lambda_{a+1} - v} E \bigwedge^{v} E \\ \Big\downarrow {\scriptstyle m_{12} \otimes m_{34}} \\ \bigwedge^{\lambda_a} E \otimes \bigwedge^{\lambda_{a+1}} E \end{array}$$

Here, Δ is the diagonal embedding (or comultiplication), and m : $\bigwedge^{r} E \otimes \bigwedge^{s} E \to \bigwedge^{r+s} E$ is a component of the multiplication map on the exterior algebra $\bigwedge^{\bullet} E$. Combinatorially, we may use the theory of Young tableaux to describe the modules $L_{\lambda}E$. It is well known the standard tableaux form a basis of $L_{\lambda}E$. Further, the relations $R(\lambda, E)$ may be visualized as follows. Let $\lambda \vdash m$ be a partition with at most n parts. To the map $\theta(\lambda, a, u, v, E)$ we associate a **Young scheme**. The Young scheme is defined as being a diagram of shape λ with empty boxes everywhere except the a^{th} and $a+1^{st}$ rows. In row a there are u empty boxes followed by $\lambda_a - u$ filled boxes. In row $a+1$ there are $\lambda_{a+1} - v$ filled boxes followed by v empty boxes. We restrict to $u + v < \lambda_{a+1}$ as in the definition of $L_{\lambda}E$. Let $U_j = e_{t(j,1)} \wedge \cdots \wedge e_{t(j,\lambda_j)}$, $V_1 = e_{x_1} \wedge \cdots \wedge e_{x_u}$, $V_2 = e_{y_1} \wedge \cdots \wedge e_{y_{\lambda_a - u + \lambda_v - v}}$, and $V_3 = e_{z_1} \wedge \cdots \wedge e_{z_v}$. Then the image of a typical element

$$U_1 \otimes \cdots \otimes U_{a-1} \otimes V_1 \otimes V_2 \otimes V_3 \otimes U_{a+2} \otimes \cdots \otimes U_s$$

is a sum of tableaux where we put $x_1, ..., x_u$ in the empty u boxes in row a, and $z_1, ..., z_v$ in the empty v boxes of row $a+1$, and we shuffle the elements $y_1, ..., y_{\lambda_a - u + \lambda_{a+1} - v}$ between the filled boxes in row a and $a+1$. The coefficients of the tableaux in the summation are ± 1 depending on the sign of the permutations coming from the exterior diagonals.[47]

[47] From now on, we will let $L_{\lambda}F$ denote the Schur functor corresponding to the diagram λ, and we will let $S_{(\lambda-k)}F = L_{\lambda}F \otimes (\bigwedge^{\dim F} F^{*})^{\otimes k}$, where $\lambda - k$ is obtained by subtracting the integer k from every entry of the vector λ.

Standard Monomials,The Straightening Law, and Determinantal Varieties

Standard Monomials and Straightening Laws

Let us now introduce the notion of a standard monomial theory.

Definition 0.0.28. Let R be a ring and let A be a commutative R-algebra, and let $S := \{s_1, ..., s_N\} \subset R$ have a partial order $\leq$.

1. An ordered product $s_{i_1} \cdots s_{i_k}$ of elements from S is said to be a **standard monomial** if the elements appear in nondecreasing order, with respect to the partial order $\leq$.

2. We say A has a **standard monomial theory for** S if the standard monomials for a basis of A over R.

3. If $s, s' \in S$ are incomparable in the partial order $\leq$, we have a unique expression called a **straightening law**:

$$s \cdot s' = \sum_i c_i M_i, \quad c_i \in R, M_i \text{ standard.}$$

If a monomial $s_{i_1} \cdots s_{i_k}$ is nonstandard we may replace it with a standard one as follows. If there is a product $s \cdot s'$ in $s_{i_1} \cdots s_{i_k}$ with $s > s'$ we replace if with $s' \cdot s$. If s and s' are incomparable, we replace $s \cdot s'$ with $s \cdot (\sum_i c_i M_i)$.

4. We say the R-algebra A has a **straightening law** if the previous replacement algorithm always terminates after finitely many steps.

We define a **standard bi-tableau**[48], denoted $(s|t)$ as a pair of standard tableaux s and t of the same shape λ, where by convention we write s in reverse order. To a bitableau we associate a product of minors of X, where the entries of row a of s give the columns and the entries of row a of t give the rows of the minor of X. Then each pair of rows in s and t defines a minor of size λ_a, and the bi-tableau is associated to the product of these minors. It is well known that the standard bi-tableaux with at most $\min\{\dim E, \dim F\}$ columns forms a K-free basis of $K[E^* \otimes F]$, and that the standard bi-tableaux with at most k columns form a K-free basis of R/I_{k+1}. We call the associated minors to a standard bi-tableau a *standard monomial* in R or R/I_{k+1}. It can be shown that any nonstandard monomial can be written as a sum of standard monomials each of which are earlier in the partial order on monomials. In particular if M is nonstandard then we can write $M = \sum_i n_i M_i + \sum_j n_j M_j$, where M_i are standard and of the same shape as the tableau corresponding to M, and the M_j are standard but of a smaller shape in the order on tableau.

Let $\mathrm{Gr}(n, m+n)$ denote the Grassmannian variety of n-dimensional subspaces of an $(m+n)$-dimensional vector space V. Let $\{w_1, ..., w_n\}$ be a basis for an n-dimensional subspace $W \subset V$. Then the vector $w_1 \wedge \cdots \wedge w_n$ determines a point $[W]$ in the projective space $\mathbb{P}(\bigwedge^n V)$, and the map $W \mapsto [W]$ is a one-to-one correspondence between n-dimensional subspaces of V and points in $\mathbb{P}(\bigwedge^n V)$ (which are lines in $\bigwedge^n V$). For the standard basis $\{e_i\}_{i=1}^{m+n}$ of V, we have the standard basis $\{e_{i_1} \wedge \cdots \wedge e_{i_n} : i_1 < \cdots < i_n\}$ of $\bigwedge^n V$. If to any $v_1 \wedge \cdots \wedge v_n \in \bigwedge^n V$ we associate the matrix

$$X = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} v_{1,1} & v_{1,2} & \cdots & v_{1,m+n} \\ v_{2,1} & v_{2,2} & \cdots & v_{2,m+n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{n,1} & v_{n,2}\& \cdots & v_{n,m+n} & \end{pmatrix},$$

[48] An archetypal example of bi-tableaux and an algebra with straightening law and standard monomial theory is the algebra $\mathbf{Sym}(V \otimes W^*)$ for two vector spaces V and W, or more generally, let $X = (x_{ij})$ be a matrix of indeterminates so that $X \in \mathbf{Hom}_K(E, F) = E^* \otimes F$, where E and F are two free $\mathbb{Z}$-modules. Define the ring $R = \mathbb{Z}[X] = \mathbb{Z}[x_{ij}]$, to be the polynomial ring in the indeterminates (x_{ij}), and let $\{e_1, ..., e_m\}$ and $\{f_1, ..., f_n\}$ be ordered bases of E and F respectively. There is a natural action of the group $G = \mathbf{GL}(E) \times \mathbf{GL}(F)$ on R induced by the action on X define by $(A, B) \cdot X = A^{-1} X B$. If K is a field, the orbits in $\mathbf{Hom}_K(E, F)$ are classified by ranks, and the orbit closures of maps of rank k say, are all maps of rank $\leq k$. The action of G on R is more complicated. If E and F are free K-modules for a field K, and V_k is the orbit closure of rank k maps, then the defining ideal of V_k is $I_{k+1} = R(\bigwedge^{k+1} E^* \otimes \bigwedge^{k+1} F)$ generated by minors of size $k+1$, and the coordinate ring R/I_{k+1} is normal and Cohen Macaulay. A classical characteristic free approach using the Schur functors allows one to define a basis of $\mathbb{K}[E^* \otimes F]$ in terms of certain *standard monomials* given by standard bi-tableaux.

we have that in the basis $\{e_{i_1} \wedge \cdots \wedge e_{i_n} : i_1 < \cdots < i_n\}$ the coordinates of $v_1 \wedge \cdots \wedge v_n$ are given by the maximal minors of X. Let $[i_1, i_2, ..., i_n]$ denote the minor given by columns $i_1, i_2, ..., i_n$ of X. So,

$$v_1 \wedge \cdots \wedge v_n = \sum_{1 \le i_1 \le \cdots \le i_n \le m+n} [i_1, ..., i_n] e_{i_1} \wedge \cdots \wedge e_{i_n}.$$

The open set $S_{n,m+1} = \{(v_1, ..., v_n) \in V^n : v_1 \wedge \cdots \wedge v_n \neq 0\} \subset \mathbf{Hom}(K^{m+n}, K^n)$ of maximal rank $n \times (m+n)$-matrices is called the **Stiefel manifold**. Now, $\mathrm{Gr}(n, m+n)$ can be identified with the orbits in $S_{n,m+n} \subset \mathbf{Hom}(K^{m+n}, K^n)$ under the action of $\mathbf{GL}(n, K)$ by left multiplication. Further, given a homomorphism $\pi : K^r \to K^{r+s}$ of two affine spaces, of the form

$$\pi(x_1, ..., x_r) = (x_1, ..., x_r, p_1, ..., p_s)$$

where p_i are polynomials in the x_j, the image of π is a closed subvariety of K^{r+s}, and π is an isomorphism of K^r onto its image (it is the *graph* of a polynomial map). The defining ideal is $I = (x_{r+i} - p_i)$, and the inverse map is

$$(x_1, ..., x_r, ..., x_{r+s}) \mapsto (x_1, ..., x_r).$$

Now, consider the open set U of $\mathrm{Gr}(n, m+n)$ where the Plücker coordinate given by the minor coming from the last n columns is nonzero. U can be identified with the space of $n \times m$ matrices. The association is given by associating any X with the row span of the matrix

$$(X|\tilde{\mathbf{id}}_n) = \begin{pmatrix} x_{11} & \cdots & x_{1m} & 0 & \cdots & 0 & 1 \\ x_{21} & \cdots & x_{2m} & 0 & \cdots & 1 & 0 \\ \vdots & \ddots & \vdots & \vdots & & \vdots & \vdots \\ x_{n1} & \cdots & x_{nm} & 1 & \cdots & 0 & 0 \end{pmatrix}.$$

One may now identify Plücker coordinates with **bi-tableaux**.

Example 0.0.29. Let $n = 3$, $m = 5$, and write

$$X = \begin{pmatrix} x_{31} & x_{32} & x_{33} & x_{34} & x_{35} & 1 & 0 & 0 \\ x_{21} & x_{22} & x_{23} & x_{24} & x_{25} & 0 & 1 & 0 \\ x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & 0 & 0 & 1 \end{pmatrix}.$$

Now let the Plücker coordinate $[1,2,3][1,2,5][1,2,7][2,3,6][5,6,7][5,6,8]$ be represented by the tableau

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 1 & 2 & 5 \\ \hline 1 & 2 & 7 \\ \hline 2 & 3 & 8 \\ \hline 5 & 6 & 7 \\ \hline 5 & 6 & 8 \\ \hline \end{array}. \qquad (*)$$

To this we may associate the bi-tableau

$$\left(\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 1 & 2 & 3 \\ \hline 1 & 3 & \\ \cline{1-2} 1 & 2 & \\ \cline{1-2} 1 & & \\ \cline{1-1} 2 & & \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 1 & 2 & 5 \\ \hline 1 & 2 & \\ \cline{1-2} 2 & 3 & \\ \cline{1-2} 5 & & \\ \cline{1-1} 5 & & \\ \cline{1-1} \end{array}\right) \qquad (**)$$

Neither the tableau $(*)$, nor the double tableau $(**)$ are standard, and thus we may use the quadratic relations on Plücker coordinates to "straighten" the tableau $(*)$, which induces a straightening of the double tableau $(**)$ to a standard bi-tableau. These quadratic relations can be seen as the shuffling relations coming from the definition of the Schur functors from the previous sections.

Standard Plücker Coordinates

We consider the Plücker coordinate standard if and only if the associated tableau is, i.e. the tableau is strickly increasing along rows, and nondecreasing along columns. In this example, the coordinate is not standard. We have a **straightening law** on the coordinates which is given by the definition of the Schur-functors and the relations among tableaux induced by the multilinear map defining them, i.e. the **shuffling relations**. One may also view the straightening law on standard bi-tableaux as a consequence of the quadratic relations on Plücker coordinates in the Grassmannian. If we take the perspective of double tableaux, we let the left tableau be the "row tableau", with indices $j \in [1, n]$ and the right tableau as the "column tableau" with indices $i \in [1, m+n]$. Each pair of rows gives a minor of the matrix X. We think of the space of one line tableau of size k as a vector space M_k, with basis $(j_k, ..., j_i | i_1, ..., i_k)$, so that if two indices on the right or left are equal, then the symbol is zero, and the symbols are alternating separately on the left and right. For any partition $\lambda := m_1 \geq m_2 \geq \cdots \geq m_r$, the tableaux of shape λ can be thought of as a tensor product $M_{m_1} \otimes M_{m_2} \otimes \cdots \otimes M_{m_r}$. Evaluating a formal tabeleau as a product of minors gives a nontrivial kernel, which is the space spanned by the shuffling relations, or equivalently the straightening law. The action of $\mathbf{GL}(E) \times \mathbf{GL}(F)$ on $K[X]$ induces an action by the two groups of diagonal matrices, and the content of a bi-tableau is a weight vector for the product of the two algebraic tori. In particular If $(A, B) = (a_i) \times (b_j) \in T(E) \times T(F)$ is an element of the product of the tori $T(E) \subset \mathbf{GL}(E)$ and $T(F) \subset \mathbf{GL}(F)$, then the weight is $(\prod_{i+1}^{n} a_i^{-k(i)}; \prod_{j=1}^{m+n} b_j^{h(j)})$.

Partial Order on Multitableux

Varieties of Circular Complexes

Varieties of Complexes

Let us now move on to a classic example, that of the *varieties of complexes*. These are examples of orbit closures of quivers which have been studied by many authors at this point (see for example [49], [50], [51]....). Let $S = k[X^{(k)}_{i(k),j(k)}]$, where $X^{(k)}$ is the k^{th} matrix representing d_k in the complex

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$$V_n \xrightarrow{d_n} V_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} V_1 \xrightarrow{d_1} V_0$$

Let I be the ideal generated by $\mathbf{rank}(X^{(k)}) \times \mathbf{rank}(X^{(k)})$ minors of each $X^{(k)}$, and by the equations given by $X^{(k+1)}X^{(k)} = 0$. Then $R = S/I$ is the coordinate ring of the parametrizing variety of complexes with the dimension vector $(\mathbf{d}(1), ..., \mathbf{d}(n))$ and rank sequence $r = (r(1), ..., r(n-1))$.

Theorem 0.0.30. *(cite DEP, DS, PW, and T) The coordinate ring of a sequence of k-vector spaces*

$$V_n \xrightarrow{d_n} V_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} V_1 \xrightarrow{d_1} V_0$$

has a filtration by Schur functors

$$\begin{aligned} &\mathbf{Sym}(V_1 \otimes V_2^* \oplus \cdots \oplus V_{n-1} \otimes V_n^*) \\ &= \bigoplus_{(\lambda(1),...,\lambda(n-1))} L_{\lambda(1)} V_1 \otimes L_{\lambda(1)} V_2^* \otimes \cdots \otimes L_{\lambda(n-1)} V_{n-1} \otimes L_{\lambda(n-1)} V_n^* \end{aligned}$$

where each $\lambda(i)$ has at most $\min\{\dim V_i, \dim V_{i+1}\}$ columns. In the case that such a sequence is a complex of vector spaces DeConcini and Strickland show that the filtration is

$$\begin{aligned} &\mathbf{Sym}(V_1 \otimes V_2^* \oplus \cdots \oplus V(_{n-1} \otimes V_n^*)/I \\ = &\bigoplus_{(\lambda(1),...,\lambda(n-1))} L_{(\lambda(1))} V_1 \otimes L_{(\lambda(2),-\lambda(1))} V_2 \otimes \cdots \otimes L_{(\lambda(n-1),-\lambda(n-2))} V_{n-1} \otimes L_{(-\lambda(n-1))} V_n. \end{aligned}$$

Here each $\lambda(k)$ has at most $\mathbf{rank}(d_k)$ columns, where d_k are the differentials in the complex represented by the matrices $X^{(k)}$.

Theorem 0.0.31. *(cite DEP, DS, PW, and T) The algebra R has a basis given by standard multitableaux. Further, the ring R is normal and Cohen-Macaulay.*

Products of Varieties of Complexes

We may of course take products of varieties of complexes. Such varieties arise naturally when there is a partition of a quiver into subquivers with relations of length two. Let us look at an example.

Example 0.0.32. Suppose Q is the following quiver

$$\begin{tikzcd} \bullet_1 \arrow[r, dotted, shift left=2] \arrow[r] \arrow[r, squiggly, shift right=2] & \bullet_2 \arrow[r, dotted, shift left=2] \arrow[r] \arrow[r, squiggly, shift right=2] & \bullet_3 \arrow[r, dotted, shift left=2] \arrow[r] \arrow[r, squiggly, shift right=2] & \bullet_4 \end{tikzcd}$$

Let us denote the colors by the set $\{1,2,3\}$ (ordered from top to bottom). Let $\{a_1, a_2, a_3\}$ be the arrows of color "1", labeled from left to right. Similarly, label arrows of color "2" by $\{b_j\}_{j=1,2,3}$, and arrows of color "3" by $\{c_k\}_{k=1,2,3}$. Take the dimension vector $\mathbf{d} = (2,4,5,3)$, and rank maps $r_1 = (2,2,3) = r_2, r_3 = (1,3,2)$.

The associated graded object for the coordinate ring is

$$\begin{aligned} k[A_i, B_j, C_k]_{i,j,k=1,2,3}/I = {} & S_{(\lambda(a_1))}V_1 \otimes S_{(\lambda(a_2),-\lambda(a_1))}V_2 \otimes S_{(\lambda(a_3),-\lambda(a_2))}V_3 \otimes S_{(-\lambda(a_3))}V_4 \\ & \otimes S_{(\mu(b_1))}V_1 \otimes S_{(\mu(b_2),-\mu(b_1))}V_2 \otimes S_{(\mu(b_3),-\mu(b_2))}V_3 \otimes S_{(-\mu(b_3))}V_4 \\ & \otimes S_{(\nu(c_1))}V_1 \otimes S_{(\nu(c_2),-\nu(c_1))}V_2 \otimes S_{(\nu(c_3),-\nu(c_2))}V_3 \otimes S_{(-\nu(c_3))}V_4. \end{aligned}$$

Using the dimension vector $(2,4,5,3)$ and the rank sequences $r_1 = (2,2,3) = r_2$, and $r_3 = (1,3,2)$, then the maps $\lambda, \mu, \nu : Q_1 \to \mathcal{P}$, which assign a partition (or Young diagram) to each arrow, are restricted to partitions such that $\lambda(a_i)$ and $\mu(b_j)$ have no more than $r_1(i) = r_2(j)$ (for $i = j$) parts. Similarly, $\nu(c_k)$ is restricted to partitions which have no more than $r_3(k)$ parts.

Let $m_1 = \langle 3|2\rangle_1^1 \langle 2|2\rangle_1^1 \langle 3|4\rangle_2^1 \langle 2,3|2,4\rangle_3^1 \in k[\mathbf{rep}_{Q,1}(\mathbf{d}_1, r_1)]$, $m_2 = \langle 2|2\rangle_2^2 \langle 1,2|1,2\rangle_3^2 \in k[\mathbf{rep}_{Q,2}(\mathbf{d}_2, r_2)]$, and $m_3 = \langle 2,3|2,3\rangle_2^3 \langle 1,2|2,3\rangle_3^3 \in k[\mathbf{rep}_{Q,3}(\mathbf{d}_3, r_3)]$. Then the monomial $m_1 \otimes m_2 \otimes m_3 \in k[\mathbf{rep}_{Q,c}(\mathbf{d}, r)]$ corresponds to the product of multitableaux,

$$\left(\ytableaushort{2,3}, \ \ytableaushort{134,134,3}, \ \ytableaushort{1235,23}, \ \ytableaushort{13} \right) \times \left(\varnothing, \ \ytableaushort{2}, \ \ytableaushort{1345,12}, \ \ytableaushort{3} \right) \times \left(\varnothing, \ \ytableaushort{23}, \ \ytableaushort{145,12}, \ \ytableaushort{1} \right).$$

This is *not standard* since m_2 and m_3 are not standard, thus this is an element of $\Sigma(\mathbf{d}) = \Sigma(\mathbf{d}_1) + \Sigma(\mathbf{d}_2) + \Sigma(\mathbf{d}_3)$. In the partial order induce by the block order[52], we have defined, it is larger than the monomial

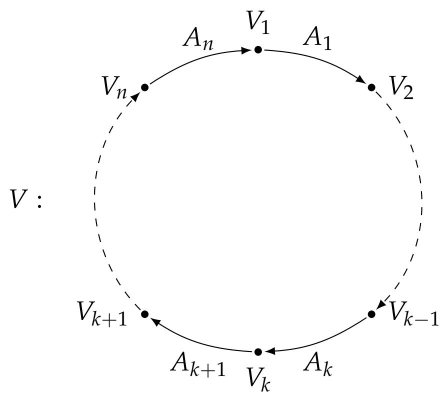
$$\langle 3|2\rangle_1^1 \langle 2|2\rangle_1^1 \langle 3|4\rangle_2^1 \langle 2,3|2,4\rangle_3^1 \otimes \langle 2|3\rangle_2^2 \langle 1,2|1,2\rangle_3^2 \otimes \langle 2,3|2,3\rangle_2^3 \langle 1,2|2,3\rangle_3^3,$$

which differs in only one place from the m_2 factor.

[52] Let $k[x_1, ..., x_k]$ have a monomial order $\leq_1$ and $k[x_{k+1}, ..., x_n]$ have a monomial order $\leq_2$. Then we define the **block order** on $k[x_1, ..., x_k] \otimes k[x_{k+1}, ..., x_n]$ by saying for any pair of monomials $m, m' \in k[x_1, ..., x_k] \otimes k[x_{k+1}, ..., x_n]$ that $m \leq m'$ if $x_1^{a_1} \cdots x_k^{a_k} \leq_1 x_1^{b_1} \cdots x^{b_k}$, or $x_1^{a_1} \cdots x_k^{a_k} = x_1^{b_1} \cdots x_k^{b_k}$, and $x_{k+1}^{a_{k+1}} \cdots x_n^{a_n} \leq_2 x_{k+1}^{b_{k+1}} \cdots x_n^{b_n}$.

Varieties of Circular Complexes

Let $\Lambda_n = k\tilde{\mathbb{A}}_n / \mathbf{rad}(k\tilde{\mathbb{A}}_n)$ be the algebra given by taking the quotient of the hereditary algebra given by the equioriented type $\tilde{\mathbb{A}}$ quiver with $|Q_0| = |Q_1| = n$, by all paths of length two. Then any representation of Λ_n will be a circular complex of vector spaces $V \in \mathcal{C}_n(\mathbf{d}, r)$,



where $\mathbf{d} = (\dim_k V_1, \dim_k V_2, ..., \dim_k V_n)$, so $\mathbf{d}(i) = \dim_k V_i$, and $r = (\mathbf{rank}\, A_1, \mathbf{rank}\, A_2, ..., \mathbf{rank}\, A_n)$ and $r(i) = \mathbf{rank}\, A_i$. It is easy to see that $r(i) + r(i+1) \leq \mathbf{d}(i+1)$ for $i \in \mathbb{Z}/n\mathbb{Z}$.

Theorem 0.0.33. *Let $\mathcal{C}_n(\mathbf{d}) = \mathbf{rep}(\Lambda_n, \mathbf{d})$ be the set of all representations of Λ_n with dimension vector $\mathbf{d}$, and let $\mathcal{C}_n(\mathbf{d}, r) = \mathbf{rep}(\Lambda_n, \mathbf{d}, r)$ be the subset of those representations with rank sequence r.*

$$\mathcal{C}_n(\mathbf{d}) = \left\{ V \in \bigoplus_{i \in \mathbb{Z}/n\mathbb{Z}} \mathbf{Hom}(V(i), V(i+1)) \middle| V(a_{i+1})V(a_i) = A_{i+1}A_i = 0 \right\}$$

can be shown to have irreducible components given by rank sequence which are maximal with respect to $\mathbf{d}$ with respect to the conditions $r(i) + r(i+1) \leq \mathbf{d}(i)$. Further, it is normal, Cohen-Macaulay, and has rational singularities. The equivariant filtration of the coordinate ring

$$\mathbb{K}[\mathcal{C}_n(\mathbf{d}, r)] = \bigoplus_{(\lambda_1, ..., \lambda_n)} \bigotimes_{i \in \mathbb{Z}/n\mathbb{Z}} S_{\lambda(i), -\lambda(i+1)} V(i)$$

holds.

Proof. [⋆ Amelie: Steps of the proof:

- Step 1: Show this for a complex $F_2 \to F_1 \to F_0$, and for $\lambda = (k)$ and $\mu = (l)$ (so $\bigwedge^k F_1^* \otimes \bigwedge^l F_1 \to S_{(l,-k)} F_1$).
- Step 2: Explain that for computing the relations defining the Schur functors, it suffices to compute on pairwise adjacent rows, and thus the simple case $\lambda = (k), \mu = (l)$, implies the more complicated case when λ and μ are not simply a row of k and l boxes respectively.

- Step 3: Explain that this computation for a three term complex is carried out on each F_i.

$\star$]

Lemma 0.0.34. *For a three term sub-complex of a circular complex,*

$$F_2 \to F_1 \to F_0$$

with dimensions $\mathbf{d} = (\mathbf{d}(2), \mathbf{d}(1), \mathbf{d}(0))$ *and rank sequence* $r = (r(2), r(1))$, *for the map*

$$\phi : \bigwedge^k F_2 \otimes \bigwedge^k F_1^* \otimes \bigwedge^l F_1 \otimes \bigwedge^l F_0^* \to \bigwedge^k F_2 \otimes S_{(l,-k)} F_1 \otimes \bigwedge^l F_0^*,$$

we have that for $X = \ker \phi$, $I = R \cdot X = \mathbf{Sym}(F_2 \otimes F_1^* \oplus F_1 \otimes F_0^*) \cdot X$ *is exactly*

$$R \cdot \left(\bigwedge^{r(2)+1} X^{(2)} \oplus \bigwedge^{r(1)+1} X^{(1)} \oplus F_2 \otimes F_0^* \right).$$

Proof. Let $F_\bullet$ be the complex

$$F_2 \to F_1 \to F_0$$

the more general case for a complex of length n follows from this argument applied locally to each F_i, which we will ellaborate on following the proof of the lemma. Let $\dim(F_i) = \mathbf{d}(i)$. Consider the case where $\lambda = (u)$, $\mu = (v)$ and take an element of

$$\bigwedge^u F_2 \otimes \bigwedge^u F_1^* \otimes \bigwedge^v F_1 \otimes \bigwedge^v F_0^*$$

of the form

$$e_{a_1} \wedge \cdots \wedge e_{a_u} \otimes f_{i_1}^* \wedge \cdots f_{i_u}^* \otimes f_{j_1} \wedge \cdots \wedge f_{j_v} \otimes g_{b_1}^* \wedge \cdots \wedge g_{b_v}^*.$$

Dualizing (i.e. tensoring with the determinant representation $\det(F_1)$) we may identify this with the element

$$e_{a_1} \wedge \cdots \wedge e_{a_u} \otimes f_{k_1} \wedge \cdots \wedge f_{k_{\mathbf{d}(1)-u}} \otimes f_{j_1} \wedge \cdots \wedge f_{j_v} \otimes g_{b_1}^* \wedge \cdots \wedge g_{b_v}^*$$

where $\{k_1, ..., k_{\mathbf{d}(1)-u}\}$ is the complement of $\{i_1, ..., i_u\}$ in $\{1, ..., \mathbf{d}(1)\}$. This element is in the space

$$\bigwedge^u F_2 \otimes \bigwedge^{\mathbf{d}(1)-u} F_1 \otimes \bigwedge^v F_1 \otimes \bigwedge^v F_0^*.$$

We may identify

$$f_{k_1} \wedge \cdots \wedge f_{k_{\mathbf{d}(1)-u}} \otimes f_{j_1} \wedge \cdots \wedge f_{j_v}$$

with the corresponding tableau, and identifying f_i with the positive integer $i \in \{1, ..., \mathbf{d}(1)\}$. We may assume such a tableau is standard, i.e the rows are strictly increasing and the columns are weakly increasing. The image of the tableau t can be computed using the diagram

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|}
\hline
 & & \cdots & & \bullet & \cdots & \bullet & \bullet & \bullet & \bullet & \cdots & \bullet & \bullet \\
\hline
\bullet & \bullet & \cdots & \bullet & \bullet & \cdots & \bullet & & \cdots & & \multicolumn{3}{c}{} \\
\cline{1-10}
\end{array}$$

with r empty boxes in the first row which is of length $\mathbf{d}(1) - u$, and s empty boxes in the second row which is of length v. This is a mnemonic device for computing the image of the following maps

$$\begin{array}{c}
\bigwedge^r F_1 \otimes \bigwedge^{\mathbf{d}(1)-u-r+v-s} F_1 \otimes \bigwedge^v F_1 \quad . \\
\Big\downarrow {\scriptstyle 1\otimes\Delta\otimes 1} \\
\bigwedge^r F_1 \otimes \bigwedge^{\mathbf{d}(1)-u-r} F_1 \otimes \bigwedge^{v-s} F_1 \otimes \bigwedge^v F_1 \\
\Big\downarrow {\scriptstyle m_{12}\otimes m_{34}} \\
\bigwedge^{\mathbf{d}(1)-u} F_1 \otimes \bigwedge^v F_1
\end{array}$$

The image of t is a sum of tableaux. Now let R be the first r entries of the first row (in the empty boxes), let Y be the remaining entries of the first row, Y' the first $v - s$ entries of the second row, and S the last s entries of the second row. Suppose that R, Y, Y', S are all disjoint sets. Then t maps to a sum of tableaux, say $t \mapsto \sum t'$, which we may identify with the corresponding tensor, and such that after dualizing $\bigwedge^{\mathbf{d}(1)-u} F_1 \otimes \bigwedge^v F_1 \to \bigwedge^u F_1^* \otimes \bigwedge^v F_1$, $\sum t' \mapsto (\sum t')^* \in \bigwedge^u F_1^* \otimes \bigwedge^v F_1$, the element

$$e_{a_1} \wedge \cdots \wedge e_{a_u} \otimes \left(\sum t'\right)^* \otimes g_{b_1}^* \wedge \cdots \wedge g_{b_v}^*$$

is an entry of $\square$

[$\star$ Amelie: Once the proof of the lemma is typed, put in Step 2 and Step 3 here, and conclude the proof. $\star$]

$\square$

Corollary 0.0.35. *The acoordinate ring of a variety of circular complexes has a basis given by standard multitableaux. Further, the coordinate ring is normal and Cohen-Macaulay.*

Examples Calculations

Let us look at a few examples of the computations mentioned in the previous section.

Example 0.0.36. Let us consider the sequence

$$F_2 \xrightarrow{A} F_1 \xrightarrow{B} F_0$$

where the sequence of dimensions is $(3,5,3)$, $\mathbf{rank}(A) \leq 2$, and $\mathbf{rank}(B) \leq 3$. Then

$$R = \mathbf{Sym}(F_2 \otimes F_1^* \otimes F_1 \otimes F_0^*)/I$$

where I is generated by $\bigwedge^3 F_2 \otimes \bigwedge^3 F_1^*$, and $F_2 \otimes F_0^*$. Suppose we take partitions $\lambda = (2)$ and $\mu = (1)$. Then we have the corresponding sequence of maps

$$\begin{array}{c} \bigwedge^4 F_1 \\ \downarrow \\ \bigwedge^2 F_2 \otimes \bigwedge^2 F_1^* \otimes F_1 \otimes F_0^* \\ \downarrow \\ \bigwedge^2 F_2 \otimes S_{(1,0,0,-1,-1)} F_1 \otimes F_3^* \end{array}$$

where $S_{(1,0,0,-1,-1)}F_1 \cong L_{(3,1)}F_1 \otimes \bigwedge^5 F_1^*$, and $\bigwedge^2 F_1 \otimes F_1^* \cong (\bigwedge^3 F_1 \otimes \bigwedge^5 F_1^*) \otimes F_1 \cong \bigwedge^3 F_1 \otimes F_1$ (in an $\mathbf{SL}(F_1)$-equivariant way). The map $\Delta : \bigwedge^4 F_1 \to \bigwedge^3 F_1 \otimes F_1$ is given by

$$f_{j_1} \wedge \cdots \wedge f_{j_4} \mapsto \sum_{\sigma \in S_4^{(3,1)}} \mathbf{sign}(\sigma) f_{\sigma(j_1)} \wedge f_{\sigma(j_2)} \wedge f_{\sigma(j_3)} \otimes f_{\sigma(j_4)}$$

The isomorphism $f_{\sigma(j_1)} \wedge f_{\sigma(j_2)} \wedge f_{\sigma(j_3)} \mapsto f^*_{j'_1(\sigma)} \wedge f^*_{j'_2(\sigma)}$ of $\bigwedge^3 F_1 \cong \bigwedge^2 F_1^*$, where $\{j'_1(\sigma), j'_2(\sigma)\}$ is the complement of $\{\sigma(j_1), \sigma(j_2), \sigma(j_3)\} \subset [1,5]$, then induces the map

$$\bigwedge^4 F_1 \to \bigwedge^2 F_1^* \otimes F_1$$

giving

$$f_{j_1} \wedge \cdots \wedge f_{j_4} \mapsto \sum_{\sigma \in S_4^{(3,1)}} \mathbf{sign}(\sigma) f^*_{j'_1(\sigma)} \wedge f^*_{j'_2(\sigma)} \otimes f_{\sigma(j_4)}$$

where we sum over all $\sigma \in S_4^{(3,1)}$ or equivalently over all the complements $\{j'_1(\sigma), j'_2(\sigma)\} = [1,5] - \{\sigma(j_1), \sigma(j_2), \sigma(j_3)\}$, one pair for each $\sigma \in S_4^{(3,1)}$. Fixing a, b, c, this then corresponds to an element

$$(e_a \wedge e_b) \otimes \sum_{\sigma \in S_4^{(3,1)}} \mathbf{sign}(\sigma) f^*_{j'_1(\sigma)} \wedge f^*_{j'_2(\sigma)} \otimes f_{\sigma(j_4)} \otimes g_c^*$$

which corresponds to matrix elements

$$\sum_{\sigma \in S_4^{(3,1)}} \mathbf{sign}(\sigma)[(A_{j'_1(\sigma),b}A_{j'_2(\sigma),a} - A_{j'_1(\sigma),a}A_{j'_2(\sigma),b}) \cdot B_{b,\sigma(j_4)}]$$
$$= \sum_{\sigma \in S_4^{(3,1)}} \mathbf{sign}(\sigma)[(BA)_{j'_1(\sigma),\sigma(j_4)}A_{j'_2(\sigma),a} - (BA)_{j'_2(\sigma),\sigma(j_4)}A_{j'_1(\sigma),a}].$$

which is in the ideal I generated by $F_2 \otimes F_0^*$ (and $\bigwedge^3 F_2 \otimes \bigwedge^3 F_1^*$).

Example 0.0.37. Let us now choose $\lambda = (2)$ and $\mu = (2)$. We have the following map,

$$\begin{array}{c} \bigwedge^2 F_2 \otimes \bigwedge^2 F_1^* \otimes \bigwedge^2 F_1 \otimes \bigwedge^2 F_0^* \ . \\ \big\downarrow \\ \bigwedge^2 F_2 \otimes S_{(1,1,0,-1,-1)} F_1 \otimes \bigwedge^2 F_0^* \end{array}$$

We compute the kernel via the following maps

1. $\bigwedge^4 F_1 \otimes F_1 \to \bigwedge^3 F_1 \otimes \bigwedge^2 F_1$,
2. $F_1 \otimes \bigwedge^4 F_1 \to \bigwedge^3 F_1 \otimes \bigwedge^2 F_1$,
3. $\bigwedge^5 F_1 \to \bigwedge^3 \otimes \bigwedge^2 F_1$.

The first map is given by

$$f_{j_1} \wedge \cdots \wedge f_{j_4} \otimes f_k \mapsto \sum_{\sigma \in S_4^{(3,1)}} \mathbf{sign}(\sigma) f_{\sigma(j_1)} \wedge f_{\sigma(j_2)} \wedge f_{\sigma(j_3)} \otimes f_{\sigma(j_4)} \wedge f_k,$$

which corresponds to the element

$$\sum_{\sigma \in S_4^{(3,1)}} \mathbf{sign}(\sigma) f^*_{j'_1(\sigma)} \wedge f^*_{j'_2(\sigma)} \otimes f_{\sigma(j_4)} \wedge f_k.$$

Fixing a, b, c, d we get a corresponding element

$$(e_a \wedge e_b) \otimes \left(\sum_{\sigma \in S_4^{(3,1)}} \mathbf{sign}(\sigma) f^*_{j'_1(\sigma)} \wedge f^*_{j'_2(\sigma)} \otimes f_{\sigma(j_4)} \wedge f_k \right) \otimes (g_c^* \wedge g_d^*).$$

where we associate the complement $\{j'_1(\sigma,k), j'_2(\sigma,k)\}$ of $\{\sigma(j_1), \sigma(j_2), \sigma(j_3)\}$ in $[1,5]$ to each $\sigma \in S_4^{(3,1)}$, and sum over all such pairs (or equivalently over all σ). This gives us the corresponding element

$$\sum_{\sigma \in S_4^{(3,1)}} \mathbf{sign}(\sigma)(A_{j'_1(\sigma),a}A_{j'_2(\sigma),b} - A_{j'_2(\sigma),a}A_{j'_1(\sigma),b}) \cdot (B_{c,\sigma(j_4)}B_{d,k} - B_{c,k}B_{d,\sigma(j_4)})$$

which is an element of $(\bigwedge^2 B) \cdot (\bigwedge^2 A) = \bigwedge^2(BA) \in \bigwedge^2(F_2 \otimes F_0^*)$, and thus in the ideal I. Similarly, the map $F_1 \otimes \bigwedge^4 F_1 \to \bigwedge^3 F_1 \otimes \bigwedge^2 F_1$ defined by

$$f_k \otimes f_{j_1} \wedge \cdots \wedge f_{j_4} \mapsto \sum_{\sigma \in S_4^{(2,2)}} \mathbf{sign}(\sigma) f_k \wedge f_{\sigma(j_1)} \wedge f_{\sigma(j_2)} \otimes f_{\sigma(j_3)} \wedge f_{\sigma(j_4)}$$

will give an element in $\bigwedge^2(F_2 \otimes F_0^*)$, as will the map $\bigwedge^5 F_1 \to \bigwedge^3 F_1 \otimes \bigwedge^2 F_1$ given by

$$f_1 \wedge \cdots \wedge f_5 \mapsto \sum_{\sigma \in S_5^{(3,2)}} \mathbf{sign}(\sigma) f_{\sigma(1)} \wedge f_{\sigma(3)} \wedge f_{\sigma(3)} \otimes f_{\sigma(4)} \wedge f_{\sigma(5)}.$$

[⋆ Amelie: I should probably explain how to identify the coordinate ring of a sequence of vector spaces with a product of open subsets of Grassmannians, and the coordinate ring of a variety of complexes with a closed subset of this space. The setup would follow Hochster's construction most likely since his exposition is very readable. ⋆]

Representation Varieties of Compact Gentle Surface Algebras

Representation Varieties

Suppose we have a constellation $C = [\sigma, \alpha, \phi]$ and that Q is the associated medial quiver, and $\Lambda = \mathbb{K}\, Q/I$ is the corresponding surface algebra. We already know we may partition the arrows Q_1, in two useful ways, either with respect to ϕ, or with respect to σ. Let us use the partition of Q_1 in terms of ϕ. Denote by ϕ_i the i^{th} cycle of ϕ and let $Q(\phi_i)$ be the corresponding subquiver of Q. Such a subquiver is a (not necessarily simple) cycle in Q. Further, the arrow set of each $Q(\phi_i)$ lies in the ideal of λ. In particular, composition of two consecutive arrows of $Q(\phi_i)$ is in the ideal. Let $\Lambda_i = kQ(\phi_i)/I$, where I is the ideal of Λ restricted to $Q(\phi_i)$.

Theorem 0.0.38. *Let $\mathbf{d}$ be a dimension vector for Λ. Let $\mathbf{rep}(\Lambda_i, \mathbf{d}_i)$ be the representation variety of Λ_i corresponding to the dimension vector $\mathbf{d}_i$, which is the restriction of $\mathbf{d}$ to $Q(\phi_i)$. Then,*

$$\mathbf{rep}(\Lambda_i, \mathbf{d}_i) \cong \mathcal{C}_i(\mathbf{d}_i)$$

$$= \left\{ V \in \bigoplus_{x \in \mathbb{Z}/n_i\mathbb{Z}} \mathbf{Hom}(V(x), V(x+1)) \middle| V(a_{i+1})V(a_i) = A_{i+1}A_i = 0 \right\}$$

is isomorphic to a variety of circular complexes, where $n_i = |\phi_i|$. Therefore,

$$\mathbf{rep}(\Lambda, \mathbf{d}) = \prod_{\phi_i} \mathbf{rep}(\Lambda_i, \mathbf{d}_i)$$

must be isomorphic to a product of such varieties.

Corollary 0.0.39. *We may thus describe the irreducible components in terms of the varieties of circular complexes. Let $r : Q_1 \to \mathbb{N}$ be a **rank map**, i.e. a function such that for every ϕ_i, $r_i : Q(\phi_i)_1 \to \mathbb{N}$ is a rank map for the dimension vector $\mathbf{d}_i$. If r_i is maximal with respect to the dimension vector $\mathbf{d}_i$ for each i, then r gives an irreducible component of $\mathbf{rep}(\Lambda, \mathbf{d})$, which will be indexed by the rank map r, and denoted*

$$\mathbf{rep}(\Lambda, \mathbf{d}, r)$$

All irreducible components are obtained in this way.

.

Filtrations and Associated Graded Algebras

In this section we will describe the coordinate rings of the spaces $\mathbf{rep}(\Lambda, \mathbf{d})$, as well as the initial ideals of the defining ideals. We will also describe a standard monomial theory on the coordinate rings, and we will construct the associated graded algebra of the coordinate ring with respect to an equivariant filtration by Schur functors.

Theorem 0.0.40. *The varieties* $\mathbf{Spec}\,(\mathfrak{R}(\Lambda, \mathbf{d}, r))$ *for compact gentle surface algebras* Λ, *are Cohen-Macaulay, normal, and in fact have rational singularities.*

Proof. Suppose Q/I is the quiver with relations corresponding to the compact gentle surface algebra Λ. Let $\mathfrak{R} = \mathbb{K}[\mathbf{rep}(\Lambda, \mathbf{d}, r)]$ be the coordinate ring of some parametrizing variety for some dimension vector $\mathbf{d} \in \mathbb{N}^{Q_0}$, and let $\mathfrak{R}_i$ be the coordinate ring of the representation variety $\mathbf{rep}(\Lambda_i, \mathbf{d}_i, r_i)$. Suppose r is maximal with respect to $\mathbf{d}$. Then each r_i is maximal with respect to $\mathbf{d}_i$ on $Q(\phi_i)$. Indeed, the restrictions on the ranks for varieties of circular complexes come from the length 2 relations on a circular complex, and thus the maximality conditions on r_i do not effect the maximality conditions on any other r_j for $i \neq j$. We may write

$$\mathfrak{R} = \bigotimes_{\phi_i} \mathfrak{R}_i$$

It follows that as a product of varieties of circular complexes, the variety $\mathbf{Spec}(\mathfrak{R})$ must be Cohen-Macaulay and normal. $\square$

Proposition 0.0.41. *For an irreducible component of* $\mathbf{rep}(\Lambda, \mathbf{d}, r)$, *we have the following associated graded algebra for the coordinate ring* $\mathfrak{R} = \mathfrak{R}(\Lambda, \mathbf{d}, r) = \mathbb{K}[\mathbf{rep}(\Lambda, \mathbf{d}, r)]$:

$$\bigoplus_{\lambda: Q_1 \to \mathcal{P}} \left(\bigotimes_{\phi_i} \left(\bigotimes_{\substack{x = h(\phi_i^{\ell} a) \in Q(\phi_i)_0 \\ \ell \in \mathbb{Z}/n_i\mathbb{Z}}} S_{\lambda(\phi_i^{\ell} a), -\lambda(\phi_i^{\ell+1} \cdot a)} V(x) \right) \right)$$

where $ha = x = t(\phi_i \cdot a)$, *and* $\phi_i \cdot a$ *denotes the action of* ϕ_i *on* $Q(\phi_i)_1$. [53]

[53] Recall: ϕ_i corresponds to an *anti-cycle* in Q, i.e. a cycle of zero relations. We define an action of ϕ_i on the arrows of this anti-cycle $Q(\phi_i)_1$, simply by defining $\phi_i \cdot a$ to be the "next" arrow in the anti-cycle, for any arrow a. This just corresponds to the cyclic order of the boundary edges of the faces of $C = [\sigma, \alpha, \phi]$.

Proof. This follows from the definitions and combinatorics we have setup for the constellations $C = [\sigma, \alpha, \phi]$, their corresponding medial quivers, and compact gentle surface algebras.[54] For clarity, we list the necessary information:

1. First, we setup a bijection between compact gentle surface algebras Λ and constellations C.

2. We then defined a partition of the corresponding gentle quiver Q, via the anti-cycles ϕ_i corresponding to the faces of the cellularly embedded graph $\Gamma \hookrightarrow \Sigma$.

3. Next, we described the representation varieties and coordinate rings of circular complexes and showed that the representation varieties of the subquivers $Q(\phi_i)$ under the partition by anti-cycles we isomorphic to varieties of circular complexes.

4. Finally, we observed that the representation variety $\mathbf{rep}(\Lambda, \mathbf{d})$ is a product of such varieties of circular complexes and inherits their geometric properties.

$\square$

[54] In particular we showed that every gentle algebra that came from a finite 2-regular quiver. We called a quiver Q "2-regular" if every vertex had in-degree and out-degree exactly 2. We showed every such quiver had a unique constellation $C(Q) = [\sigma, \alpha, \phi]$, and therefore a unique graph Γ cellularly embedded in a closed Riemann surface Σ. We also showed that all constellations $C = [\sigma, \alpha, \phi]$ have a unique medial quiver, up to a choice of orientation of the surface, which is 2-regular, and that assigning length two relations to cycles in the quiver around a face of $\Gamma \hookrightarrow \Sigma$, and that this correspondence produces a bijection between constellations and (finite) gentle 2-regular quivers, and therefore also between constellations and compact gentle surface algebras.

Standard Monomial Theory for the Coordinate Rings

Now, let us define the block order[55] on the monomials of $\mathfrak{R}$, i.e. with respect to the orders $\leq_i$ on each $\mathfrak{R}_i$ defined in §IV and IV.[56] We would like to describe a Gröbner basis for each $\mathfrak{R}_i$ and for $\mathfrak{R}$, via the standard monomials. In so doing we will show the following

Proposition 0.0.42. *The coordinate ring $\mathfrak{R}$ of the parametrizing varieties of representations of a compact gentle surface algebra Λ has a standard monomial theory.*

[55] Let $k[x_1, ..., x_k]$ have a monomial order $\leq_1$ and $k[x_{k+1}, ..., x_n]$ have a monomial order $\leq_2$. Then we define the **block order** on $k[x_1, ..., x_k] \otimes k[x_{k+1}, ..., x_n]$ by saying for any pair of monomials $m, m' \in k[x_1, ..., x_k] \otimes k[x_{k+1}, ..., x_n]$ that $m \leq m'$ if $x_1^{a_1} \cdots x_k^{a_k} \leq_1 x_1^{b_1} \cdots x^{b_k}$, or $x_1^{a_1} \cdots x_k^{a_k} = x_1^{b_1} \cdots x_k^{b_k}$, and $x_{k+1}^{a_{k+1}} \cdots x_n^{a_n} \leq_2 x_{k+1}^{b_{k+1}} \cdots x_n^{b_n}$.

[56] Remember, it is arbitrary as to how one chooses to order the the components $\mathfrak{R}_i$ in relation to each other.

Examples of Filtrations via Schur Functors and Associated Graded Algebras

Associated Graded Algebras of the Double Simple Cycle Algebras

Associated Graded Algebras for Compact Gentle Surface Algebras with One Simple Modules

The Dihedral Ringel Algebra $A_{1,1}$

Let $A_{1,1}$ be the algebra given by the following quiver with relations: $R = \langle a^2, b^2 \rangle$.

$$\begin{tikzcd} \bullet_x \arrow[loop left, "a"] \arrow[loop right, dotted, "b"] \end{tikzcd}$$

A representation for this algebra is a choice of two square matrices (A, B) with $A^2 = 0 = B^2$. The finite dimensional indecomposable representations correspond to words in the generators $\{a, b, a^{-1}, b^{-1}\}$ which do not pass through any zero relations. These are exactly the *string* and *band* modules. The description of the representation theory of infinite dimensional string algebras such as this one can be found in [57]. The coordinate ring of the parametrizing variety for the 57
representations is $k[\mathbf{rep}(Q/R, \mathbf{d}(x))] = k[A, B]/I$, where I is the ideal generated by the equations $A^2 = 0 = B^2$. The irreducible component is unique and corresponds to the maximal rank pairs $r = (r(A), r(B))$ such that $2 \cdot r(A) \leq \mathbf{d}(x)$ and $2 \cdot r(B) \leq \mathbf{d}(x)$, where $\mathbf{d}(x) = \dim V(x)$ is the dimension of the vector space assigned to the vertex x, and r is maximal in the sense that increasing the rank of either A or B would not allow the relations $A^2 = 0 = B^2$ to be satisfied. The associated graded object for the coordinate ring is,

$$\mathbf{gr}(k[\mathbf{rep}(Q/R, \mathbf{d}(x), r)]) = \bigoplus_{\lambda} \left(S_{(\lambda(a), -\lambda(a))} V(x) \otimes S_{(\lambda(b), -\lambda(b))} V(x) \right),$$

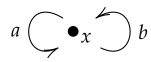
where the sum is over all functions $\lambda : Q_1 \to \mathcal{P}$ assigning a partition $\lambda(a_i)$ to each arrow in $a_i \in Q_1$ such that each row is no longer than $r(a_i)$.

For this algebra we have the associated graded object for the ring of semi-invariants for a dimension $\mathbf{d}(x)$ and rank vector $r = (r(a), r(b))$,

$$\mathbf{gr}(\mathcal{SI}(Q/R, \mathbf{d}(x), r)) = \bigoplus_{\lambda} \left(S_{(\lambda(a), -\lambda(a))} V(x) \otimes S_{(\lambda(b), -\lambda(b))} V(x) \right)^{\mathbf{SL}(V(x))}.$$

Gel'fand and Ponomarev's Algebra $A_{1,2}$

Let $A_{1,2}$ be given by the following quiver with relations: $R = \langle ab, ba \rangle$.



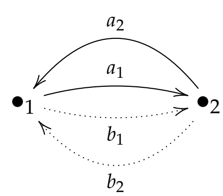
The associated graded object for the ring of semi-invariants for a given dimension $\mathbf{d}(x)$ and rank vector $r = (r(a), r(b))$ is:

$$\mathbf{gr}(\mathcal{SI}(Q/R, \mathbf{d}(x), r)) = \bigoplus_{\lambda} \left(S_{(\lambda(a), -\lambda(b))} V(x) \otimes S_{(\lambda(b), -\lambda(a))} V(x) \right)^{\mathbf{SL}(V(x))}.$$

Associated Graded Algebras for Compact Gentle Surface Algebras with Two Simple Modules

Algebra $A_{2,1}$

Let Q be the following quiver, and let the relations be $R = \langle a_2a_1, a_1a_2, b_2b_1, b_1b_2 \rangle$.

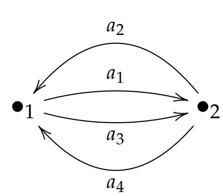


and denote this algebra $kQ/R = A_{2,1}$. We have the associated graded object for the ring of semi-invariants $\mathbf{gr}\left(\mathcal{SI}(Q/R, \mathbf{d})\right)$ is:

$$\bigoplus_{\lambda} \left(S_{(\lambda(a_1), -\lambda(a_2))} V(1) \otimes S_{(\lambda(b_1), -\lambda(b_2))} V(1) \right)^{\mathbf{SL}(V(1))} \otimes \left(S_{(\lambda(a_2), -\lambda(a_1))} V(2) \otimes S_{(\lambda(b_2), -\lambda(b_1))} V(2) \right)^{\mathbf{SL}(V(2))}.$$

Algebra $A_{2,2}$

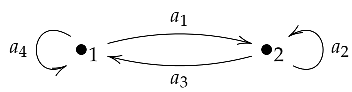
$R = \langle a_2a_1, a_3a_2, a_4a_3, a_1a_4 \rangle$.



$$\bigoplus_{\lambda} \left(S_{(\lambda(a_1),-\lambda(a_4))} V(1) \otimes S_{(\lambda(a_3),-\lambda(a_2))} V(1) \right)^{\mathbf{SL}(V(1))} \otimes \left(S_{(\lambda(a_2),-\lambda(a_1))} V(2) \otimes S_{(\lambda(a_4),-\lambda(a_3))} V(2) \right)^{\mathbf{SL}(V(2))}.$$

Algebra $A_{2,3}$

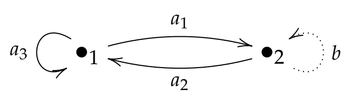
$R = \langle a_2a_1, a_3a_2, a_4a_3, a_1a_4 \rangle$.



$$\bigoplus_{\lambda} \left(S_{(\lambda(a_1),-\lambda(a_4))} V(1) \otimes S_{(\lambda(a_4),-\lambda(a_3))} V(1) \right)^{\mathbf{SL}(V(1))} \otimes \left(S_{(\lambda(a_2),-\lambda(a_1))} V(2) \otimes S_{(\lambda(a_3),-\lambda(a_2))} V(2) \right)^{\mathbf{SL}(V(2))}.$$

Algebra $A_{2,4}$

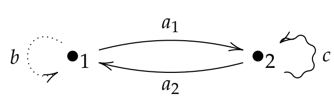
$R = \langle a_2a_1, a_3a_2, a_1a_3, b^2 \rangle$.



$$\bigoplus_{\lambda} \left(S_{(\lambda(a_1),-\lambda(a_3))} V(1) \otimes S_{(\lambda(a_3),-\lambda(a_2))} V(1) \right)^{\mathbf{SL}(V(1))} \otimes \left(S_{(\lambda(b),-\lambda(b))} V(2) \otimes S_{(\lambda(a_2),-\lambda(a_1))} V(2) \right)^{\mathbf{SL}(V(2))}.$$

Algebra $A_{2,5}$

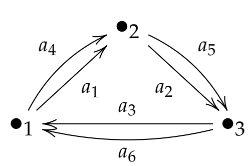
$R = \langle a_2a_1, a_1a_2, b^2, c^2 \rangle$.



$$\bigoplus_{\lambda} \left(S_{(\lambda(a_1),-\lambda(a_2))} V(1) \otimes S_{(\lambda(b),-\lambda(b))} V(1) \right)^{\mathbf{SL}(V(1))} \otimes \left(S_{(\lambda(a_2),-\lambda(a_1))} V(2) \otimes S_{(\lambda(c),-\lambda(c))} V(2) \right)^{\mathbf{SL}(V(2))}.$$

Associated Graded Algebras for Compact Gentle Surface Algebras with Three Simple Modules

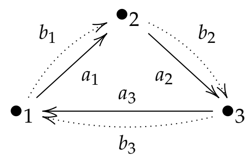
Algebra $A_{3,1}$



$$\mathcal{SI}(Q/R, \mathbf{d}) = \bigoplus_{\lambda} \bigotimes_{i \in \mathbb{Z}/6\mathbb{Z}} S_{(\lambda(a_{i+1}), -\lambda(a_i)}V(x(i))^{\mathbf{SL}(V(x(i)))}$$

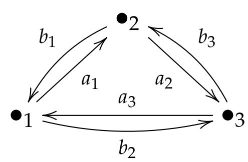
where $x(i) = ta_{i+1} = ha_i$, for each $i \in \mathbb{Z}/6\mathbb{Z}$.

Algebra $A_{3,2}$

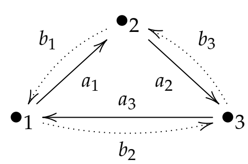


$$\mathcal{SI}(Q/R, \mathbf{d}) = \bigoplus_{\lambda} \left(\bigotimes_{i \in \mathbb{Z}/3\mathbb{Z}} S_{(\lambda(a_{i+1}), -\lambda(a_i))}V(i)^{\mathbf{SL}(V(i))} \right) \otimes \left(\bigotimes_{j \in \mathbb{Z}/3\mathbb{Z}} S_{(\lambda(b_{j+1}), -\lambda(b_j))}V(j)^{\mathbf{SL}(V(j))} \right).$$

Algebra $A_{3,3}$



Algebra $A_{3,4}$



Interestingly, the ring of semi-invariants for this algebra is isomorphic to the ring of semi-invariants of algebra $A_{3,2}$.

$$\mathcal{SI}(Q/R, \mathbf{d}) = \bigoplus_{\lambda} \left(\bigotimes_{i \in \mathbb{Z}/3\mathbb{Z}} S_{(\lambda(a_{i+1}), -\lambda(a_i))}V(i)^{\mathbf{SL}(V(i))} \right) \otimes \left(\bigotimes_{j \in \mathbb{Z}/3\mathbb{Z}} S_{(\lambda(b_{j+1}), -\lambda(b_j))}V(j)^{\mathbf{SL}(V(j))} \right).$$

Spherical Varieties and Toric Degenerations

Shmelkin's Deformation Technique and Spherical Actions

In [58], Shmelkin develops a method of calculating invariant rings $k[\mathbf{rep}(\Lambda, \mathbf{d})]^{\mathbf{SL}(\mathbf{d})}$ in the coordinate ring of the parametrizing variety of representations for a class of (not necessarily finite dimensional) gentle algebras. The calculations of such rings for finite dimensional triangular gentle algebras was studied by Kraskiewics and Weyman[59], and later by Carroll and Weyman[60], and later the associated moduli spaces were studied by Carroll and Chindris[61]. Shmelkin's method provides an alternate method which is coordinate free, making some calculations simpler.

Let k be an algebraically closed field of characteristic zero, $G = (G, \mathcal{O}_G)$ a reductive group, and $X = (X, \mathcal{O}_X)$ an affine G-variety. The action of G is called **spherical** if for some point $x \in X$ and some Borel subgroup $B \leq G$, the orbit Bx is open and dense in X. A subgroup $H \leq G$ is spherical if the homogeneous variety $G/H = (G/H, \mathcal{O}_{G/H})$ is spherical. Here $\mathcal{O}_{G/H} = \pi_* \mathcal{O}_G^H$ is the direct image of the sheaf of H-invariant regular functions with respect to the projection $\pi : G \to G/H$ and the H action of right translations on G. By [62], X is spherical if and only if $k[X]$ is a multiplicity free G-module. Let $U \leq G$ me a maximal unipontent subgroup, and T a maximal torus normalizing U. Let χ_T be the **weights** of T, i.e. the group of all characters $\chi : T \to k$, and let χ_+ be the sub-semigroup of dominant weights with respect to B. For $\lambda \in \chi_+$, let V_λ be the corresponding simple module. Define $\Gamma(X)$ to be the set of highest weights of the G-irreducible factors of $k[X]$, with respect to T, for a G-variety X. $\Gamma(X)$ is a semigroup generated by the weights of the T-homogeneous generators of the algebra of covariants $k[X]^U$. In particular $\Gamma(G/H) = \{\lambda \in \chi_+ | \ V_\lambda^H \neq 0\}$. The G-variety X is then spherical if and only if $k[X]^U \cong k[\Gamma(X)]$. Now, X is normal and Cohen-Macaulay with rational singularities if and only if $\mathbf{Spec}(k[X]^U)$ is.

A sub-semigroup $\Gamma \leq \chi_+$ is **saturated** if the subgroup $\langle\Gamma\rangle \leq \chi_T$ has intersection with χ_+, $\langle\Gamma\rangle \cap \chi_+ = \Gamma$. For $\lambda \in \chi_+$ let $\langle v_\lambda \rangle \subset V_\lambda$ be the unique U-invriant line of highest weight vectors. For a sub-

[58] D. A. Shmelkin. *Semi-invariants of Gentle Algebras by Deformation Method and Sphericity*. Preprint.

[59] W. Kraskiewicz, J. Weyman. *Generic decompositions and semi-invariants for string algebras.* Preprint https://arxiv.org/abs/1103.5415

[60] A. Carroll, J. Weyman. *Semi-Invariants for Gentle String Algebras*. https://arxiv.org/abs/1106.0774

[61] A. Carroll, C. Chindris.*On the invariant theory for acyclic gentle algebras*. https://arxiv.org/abs/1210.3579

[62]

semigroup $\Gamma \leq \chi_+$ let

$$I(\Gamma) = \{g \in G \mid gv_\lambda = v_\lambda \ \forall \ \lambda \in \Gamma\}.$$

So, $U \subseteq I(\Gamma)$, and such U are called **horospherical**, and $I(\Gamma(G/H))$ is called the **horospherical contraction** of H. Next, we provide the setup that is important for our interests.

Highest Weight Modules

Let $G = \mathbf{GL}(V) \times \mathbf{GL}(V)$ and $H = \Delta\, \mathbf{GL}(V) \hookrightarrow G$ be the diagonal subgroup. Now, let $U \leq \mathbf{GL}(V) \cong H$ be the maximal unipotent subgroup of upper triangular matrices with 1's on the diagonal, and let $T \leq \mathbf{GL}(V)$ be the maximal torus of diagonal matrices. So, $B = UT$ is the Borel subgroup of upper triangular matrices. The characters χ_T are all of the form

$$\alpha_{ij} \begin{pmatrix} t_1 & 0 & \cdots & 0 \\ 0 & t_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t_n \end{pmatrix} = t_i t_j^{-1}$$

and the weights of T in V are the characters $\chi \in \chi_T$ such that the weight spaces $V_\chi \subseteq V$,

$$V_\chi = \{v \in V \mid tv = \chi(t)v \ \forall \ t \in T\}$$

is nonzero. The vectors $v \in V_\chi$ are the weight vectors, and $V = \bigoplus_\chi V_\chi$. The root datum $\Psi = (\chi_T, R, \chi_T^*, R^*)$ where $\chi_T = \mathbb{Z}^n = \chi_T^*$, with pairing $\langle \chi, \chi^* \rangle = \chi(\chi^*(t)) = t^{\langle \chi, \chi^* \rangle}$ for $t \in k^*$. Further, $\alpha_{ij}^* : k^* \to \mathbf{GL}(V)$ with

$$\alpha_{ij}^*(t) = \begin{pmatrix} t_1 & 0 & \cdots & 0 \\ 0 & t_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t_n \end{pmatrix}$$

where $t_i = t, t_j = t^{-1}$ and $t_k = 1$ for all $k \neq i, j$. We also have $R = R^* = \{e_i - e_j \mid i \neq j\}$ and the positive roots $R^+ = \{e_i - e_j \mid 1 \leq i < j \leq n-1\}$, with corresponding basis of simple roots $D = \{e_i - e_{i+1} \mid 1 \leq i < j \leq n-1\}$ for the Weyl group $W = S_n \cong N_G(T)/Z_G(T)$, where the centralizer of T is $Z_G(T) = T$ and the normalizer of T, $N_G(T)$ is the group of matrices with exactly one nonzero entry in each row and column. Clearly, modulo T, this is the group of permutation matrices given by the map $\rho : S_n \hookrightarrow \mathbf{GL}(V)$ where the generators $\sigma_i = (i, i+1) \mapsto \rho(\sigma_i)$ where $\rho(\sigma_i)e_i = e_{i+1}$, $\rho(\sigma_i)e_{i+1} = e_i$, and $\rho(\sigma_i)e_j = e_j$ for all $j \neq i, i+1$. Now, for brevity let $we_i = \rho(w)e_i$

for $w \in W$. Then W acts on the root system R. If $\alpha = \alpha_{ij} = e_i - e_j$ is a root, then $w = (i, j) = \sigma_\alpha$ is the reflection of R about the plane orthogonal to α_{ij}. Since $\sigma_\alpha = \sigma_i\sigma_{i+1}\cdots\sigma_{j-2}\sigma_{j-1}\sigma_{j-2}\cdots\sigma_{i+1}\sigma_i$ we have $W = \langle \sigma_i | \ 1 \le i \le n-1, \sigma_i^2 = 1 = (\sigma_i\sigma_{i+1})^3, \sigma_i\sigma_j = \sigma_j\sigma_i \ (i \ne j) \rangle$. Define the weight $\lambda^* := -w_0\lambda$, where w_0 is the longest element of the Weyl group.

Let $Y = G/B$ be the flag variety and $\pi : G \to Y$ the projection. Let $\mathcal{O}_Y$ be the sheaf of local rings on Y. Let $\mu \in \chi$ be a character (equiv. weight), $\mathcal{U}$ a Zariski open set in Y, and define a coherent sheaf $\mathscr{L}(\mu)$ of $\mathcal{O}_Y$-modules by,

$$\mathscr{L}(\mu)(\mathcal{U}) = \{f \in \mathcal{O}_G(\pi^{-1}\mathcal{U}) | \ f(xb) = f(xtu) = (w_0\mu)(t^{-1})(f(x)) = \chi_\mu(b)f(x)\}$$

here $(w_0\mu) = \chi_\mu \in \chi$, $x \in \pi^{-1}(\mathcal{U}), t \in T, u \in U$ (where $B = UT$ and U is the unipotent radical of the Borel subgroup B). The weight μ determines a $\mathbf{GL}(V)$-equivariant line bundle $\mathcal{L}(\mu)$ on Y, and there is a $\mathbf{GL}(V)$-action on the space of global sections $\Gamma(Y, \mathcal{L}_\mu) = H^0(Y, \mathcal{L}(\mu)) = L(\mu)$. If μ is a dominant weight then Bott's Theorem says this is an irreducible highest weight representation of $\mathbf{GL}(V)$. Further, all $H^i(Y, \mathcal{L}(\mu)) = 0$ for $i > 0$. In particular,

$$L(\mu) = \{f \in k[G] | \ f(xb) = \chi_\mu(b)f(x)\}$$

for $\chi_\mu \in \chi^+(B)$. All irreducible $\mathbf{GL}(V)$ modules can be obtained in this way.

Now, let B^{op} be the opposite Borel subgroup of lower triangular matrices. Let U^{op} be the unipotent radical of B^{op}. For any simple $\mathbf{GL}(V)$ module, the highest weight vector with respect to U^{op} is the lowest weight vector with respect to U with weight $-\mu^*$. Then let $\Gamma(G/H) = \{(\lambda, \lambda^*) \mid \lambda \in \chi_+(H)\}$, i.e. pairs of $\mathbf{GL}(V)$ dominant weights such that $V_\lambda \otimes V_{\lambda^*} = V_\lambda \otimes V_\lambda^*$ contains a $\mathbf{GL}(V)$ invariant point (V_λ an irreducible G-module of highest weight λ). Then $\Gamma(G/H)$ is saturated. So we then have,

$$I(\Gamma(G/H)) = (U \times U^{op})\Delta T$$

where $\Delta T = \{(t, t) \in T \times T\} \le \mathbf{GL}(V) \times \mathbf{GL}(V)$. Now, let Z be some affine G-variety, $H = \Delta\,\mathbf{GL}(V)$ (a spherical subgroup).

Now, let $\preceq$ be the partial order on weights in χ_T ($\lambda \preceq \mu \iff \lambda - \mu$ is a nonnegative integer linear combination of simple roots). The highest weight λ is maximal in this order. This yields a partial order on irreducible factors of $k[Z]$ and $V_\lambda V_\mu$ is a sum of irreducible components smaller than or equal to $\lambda + \mu$ in the partial order. We then have the following

Proposition 0.0.43. [63] *An affine G-variety Z is spherical if and only if the G-module $k[Z]$ is multiplicity free. Further, if $H \le G$ is a spherical*

[63] E.B. Vinberg, B.M. Kimefeld. *Homogeneous domains on flag manifolds and spherical subgroups of semisimple Lie groups.* Funct. Analysis and Appl. 12, 168-174 (1978)

subgroup, and V an irreducible G-module with $\dim_k V < \infty$ *then* $\dim_k V^H \leq 1$.

Theorem 0.0.44. [64] *Let $H \leq G$ be a spherical subgroup of a connected reductive group G acting on the affine variety Z. Assume $\Gamma(G/H)$ is saturated. The associated graded algebra* $\mathbf{gr}\, k[Z]^H$ *given by the filtration* $k[Z] = \bigcup_{\lambda \in \chi_+} k[Z]_{\preceq \lambda}$ *is isomorphic to* $k[Z]^{I(\Gamma(G/H))}$.

[64] D. Panyushev. *On Deformation Methods in Invariant Theory.* Ann. Inst. Fourier, Grenoble 47, 1 (1997), 985-1012.

Now, let $Z = X \times Y$ be a product of affine $\mathbf{GL}(V)$-varieties and let $\mathbf{GL}(V)$ act diagonally (via restriction of the action of $G = \mathbf{GL}(V) \times \mathbf{GL}(V)$ to $H = \Delta\, \mathbf{GL}(V)$). By Theorem 2.4 of [65] we have

[65] D. Panyushev. *On Deformation Methods in Invariant Theory.* Ann. Inst. Fourier, Grenoble 47, 1 (1997), 985-1012.

$$\mathbf{gr}\, k[Z]^{\Delta\, \mathbf{GL}(V)} \cong \left(k[X]^U \otimes k[Y]^{U^{op}} \right)^{\Delta T},$$

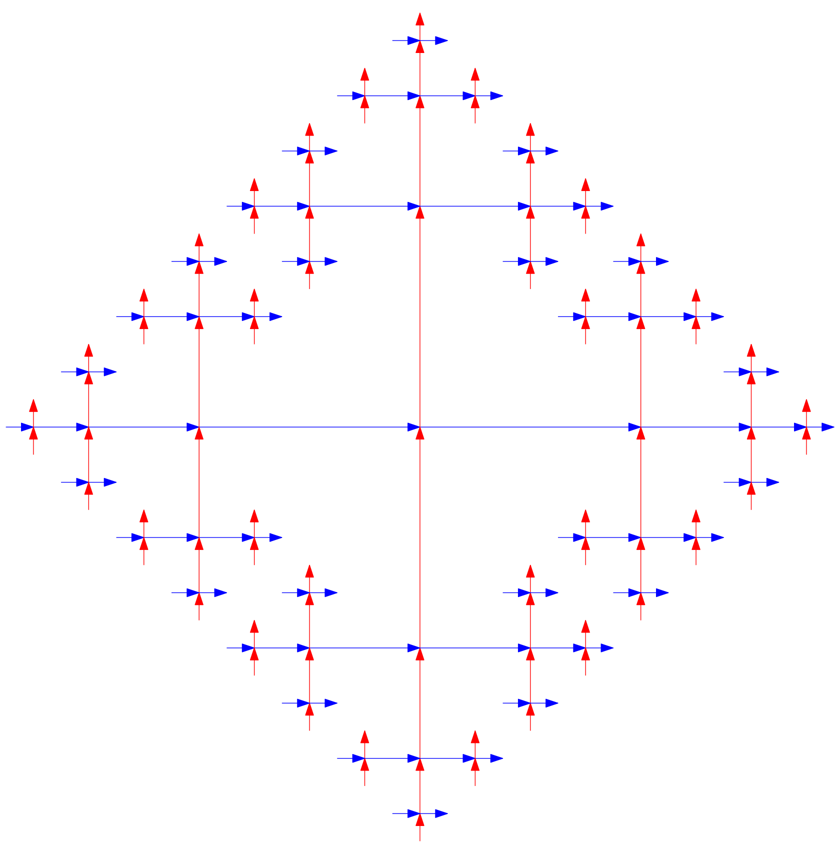
and

1. If $k[Z]^{I(\Gamma(G/H))}$ is a complete intersection, then so is $k[Z]^H$.
2. The number of generators in a minimal system of homogeneous generators of $k[Z]^{I(\Gamma(G/H))}$ is greater than or equal to that for $k[Z]^H$. In particular, if $k[Z]^{I(\Gamma(G/H))}$ is a polynomial ring, then so is $k[Z]^H$.

Now, let us assume Z is a spherical G-variety so that $k[Z] = \bigoplus_{\lambda \in \Gamma} V_\lambda$ for some sub-semigroup $\Gamma \leq \chi_+$. Further, restricting the action we get $k[Z]^U = \bigoplus_{\lambda \in \Gamma} V_\lambda^U$, a direct sum of one-dimensional subspaces corresponding to elements of Γ. Also, $k[Z]^U \cong k[\Gamma]$, where $k[\Gamma]$ is the semigroup ring of Γ.

Relative Invariants of Compact Gentle Surface Algebras

Definitions and Notation



Deformations

Examples

Algebra $A_{3,2}$

$$\begin{tikzcd}
& \bullet_2 \arrow[dr, "a_2"'] \arrow[dr, dotted, bend left, "b_2"] & \\
\bullet_1 \arrow[ur, "a_1"] \arrow[ur, dotted, bend left, "b_1"] & & \bullet_3 \arrow[ll, "a_3"'] \arrow[ll, dotted, bend left, "b_3"]
\end{tikzcd}$$

$$\mathcal{SI}(Q/R, \mathbf{d}) = \bigoplus_{\lambda} \left(\bigotimes_{i \in \mathbb{Z}/3\mathbb{Z}} S_{(\lambda(a_{i+1}), -\lambda(a_i))} V(i)^{\mathbf{SL}(V(i))} \right) \otimes \left(\bigotimes_{j \in \mathbb{Z}/3\mathbb{Z}} S_{(\lambda(b_{j+1}), -\lambda(b_j))} V(j)^{\mathbf{SL}(V(j))} \right).$$

Combinatorics of Invariants and Constellations

Relations Between Generators

Once we have a way of computing generators for the rings or relative invariants, an obvious next step would be to understand the relations between them.