LORAS AND GSA-NETS FOR OCR

AMELIE SCHREIBER

Abstract. The following is a merging of two theories, that of LoRAs or "Low Rank Adaptations" introduced in [Hu et. al.], and that of group equivariant self-attention in transformers or other neural networks that include attention. We prove three things. First, LoRAs do not disrupt translation equivariance when included in translation equivairant attention with relative positional encodings. Second, LoRAs do not disrupt the lifting self-attention layers described in [RC]. And third, LoRAs do not disrupt equivariance of group self-attention layers as defined in [RC]. We also include a correction to the proof of equivariance of group self-attention by fixing the definition of the relative positional encodings defined in Proposition 5.2 and its proof in [RC].

Contents

1. Introduction

Consider the task of transcribing and translating a complex and lengthy text – a human might spend months or even years to accomplish this task, depending on the language and the intricacy of the text. Now, imagine a deep learning model that can accomplish the same task in a fraction of the time. That's the potential efficiency of the new method we introduced in this paper.

Optical Character Recognition (OCR) has been a crucial technology in digitizing printed text, enabling us to convert scanned images of printed text into machine-encoded text. Traditional OCR techniques have relied on various methods, including feature extraction, segmentation, and pattern recognition, to detect and identify characters in an image. However, these methods have struggled with complex texts and poor-quality images, necessitating the development of more sophisticated models.

The current state-of-the-art in OCR technology is the Transformer OCR (TrOCR) model, introduce in [Li et. al.]. This model leverages the strengths of both Vision Transformers (ViT) and large language models. The ViT component is responsible for extracting features from images of text,

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which are then fed into a large language model for decoding into machine-encoded text. This combination enables TrOCR to handle a wide range of text images and has resulted in unprecedented performance in the OCR field. In this paper, we propose a novel approach to further improve the performance and efficiency of OCR models using Low-Rank Adaptation (LoRA). LoRA is a method that significantly reduces the cost of fine-tuning large models. Instead of retraining all model parameters, which is computationally expensive, LoRA freezes the pre-trained model weights and injects trainable rank decomposition matrices into each layer of the architecture. This approach can reduce the number of trainable parameters by several orders of magnitude, making the fine-tuning process much more efficient.

LoRA can be applied to both the language model and vision parts of the TrOCR model, making it a versatile tool for model fine-tuning. Importantly, LoRA prevents catastrophic forgetting, a phenomenon where a deep learning model quickly forgets previously learned information when new data is introduced. This feature of LoRA is particularly valuable for large models that are trained on a diverse set of tasks. Furthermore, LoRA modularizes large models, making it easier to adapt them to multiple downstream tasks, such as transcribing and translating text in different languages and from different time periods.

Our approach also incorporates the concept of equivariant neural networks. Equivariance in a neural network refers to the property where the output of the network changes in the same way as the input. For instance, if the input image is rotated, the output feature map also rotates in the same way. This property is beneficial for handling variations in data, such as different orientations and positions of text in OCR tasks.

We propose the use of group self-attention networks (GSA-Nets), a type of equivariant transformer neural network, in place of the vision transformer part of the TrOCR model. GSA-Nets are designed to be equivariant to arbitrary symmetry groups, improving the model's ability to capture semantic and syntactic information in images of text. The equivariance property of GSA-Nets enhances generalizability, allowing the model to better handle unseen variations in the data.

Furthermore, GSA-Nets improve parameter efficiency, meaning that they can achieve comparable or better performance than non-equivariant models with fewer parameters. This efficiency can lead to faster training times and less computational resources needed. Moreover, since GSA-Nets are naturally robust to transformations such as rotations and translations, they require significantly less data augmentation, a commonly used technique in training deep learning models that involves artificially expanding the training dataset by creating modified versions of existing data.

In this paper, we propose a novel approach that integrates LoRA and GSA-Nets to enhance the performance and efficiency of the TrOCR model. We hypothesize that by using GSA-Nets in place of the vision part of the TrOCR model and LoRAs for fine-tuning we can achieve better, state-of-the-art results.

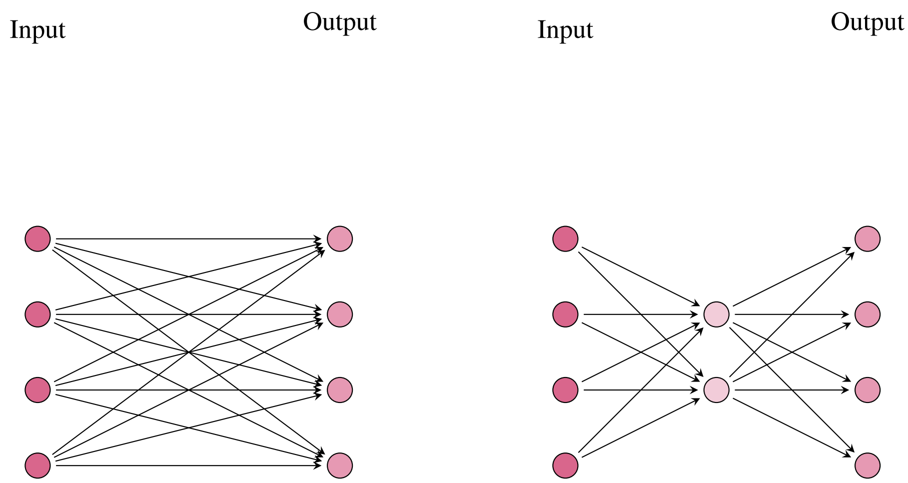
2. What is...a LoRA?

In the realm of deep learning, the concept of Low Rank Adaptations (LoRAs) was first introduced by [Hu et. al.]. These LoRAs provide an efficient alternative to the traditional fine-tuning of neural networks. The process begins by freezing the pre-existing weights of a layer in the neural network. For instance, in the context of a transformer's attention mechanism, this could involve freezing the weights of the query, key, or value matrices—often represented as W_Q, W_K, and W_V.

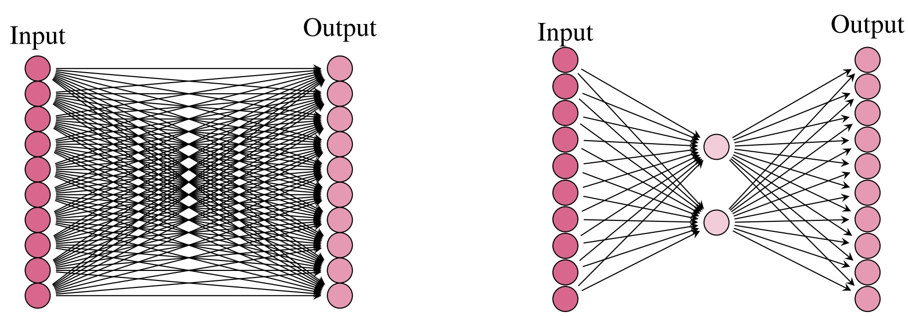
Following this, a LoRA layer is introduced to one or more of these pre-trained weight matrices. If we consider W to be a frozen weight matrix, the LoRA layer would take the form of $W+\Delta W$, wherein $\Delta W = BA$ constitutes the LoRA. Typically, these are low-rank decompositions, with $A \in \mathbb{R}^{r \times d_{in}}$ and $B \in \mathbb{R}^{d_{out} \times r}$, where the original weight matrix is $W \in \mathbb{R}^{d_{out} \times d_{in}}$. It is common for r to be significantly less than $\min\{d_{in}, d_{out}\}$.

The application of LoRAs only provides significant benefits when r is much smaller than the input and output dimension. Nevertheless, we can still opt for a smaller r and implement a LoRA in lieu of conventional fine-tuning. Empirical evidence suggests that in many cases, selecting $r = 4$ or $r = 8$ is sufficient—even for large weight matrices such as the query, key, and value matrices of a transformer's attention mechanism.

Let's now explore a scenario where the application of a LoRA does not yield any substantial benefits in terms of reducing the number of parameters:



Here, we see that the number of parameters for the LoRA layer $\Delta W = BA$ is the same as the original layer W, where we have $4 \times 2 \times 2 = 16$ parameters for the LoRA (on the right), and $4 \times 4 = 16$ parameters for the original frozen weight matrix (on the left). Next, let's look at an example that gives us 40% the parameters of the frozen weight matrix:



Here we see the original (frozen) weight matrix has 10^2 parameters, and the LoRA has only $10 \times 2 \times 2 = 40$ parameters. In most cases, we have that the rank (ideally this is the number of neurons in the middle layer of the LoRA) of the frozen matrix is much smaller than the input and output dimensions, and there is in fact a drastic reduction in parameter count. As an example, we might have an input and output dimension of say 100, in which case the weight matrix has $100^2 = 10,000$ parameters. However, the rank of this matrix is very often much lower than 100. In practice, it was shown that choosing $r = 4$ for the query, key, and value matrices is often more than sufficient for a LoRA as the middle dimension. In this case, we would get $100 \times 4 \times 2 = 800$ parameters in the LoRA, which is less than one tenth the original parameter count. Once we have such a LoRA in place, we can train it on some downstream task, and then add the LoRA weight matrix BA to the original (frozen) weight matrix W to obtain a model that performs well on this new task. Now, let us look at how LoRAs can be added into a translation equivariant attention mechanism without any loss of translation equivariance.

3. Incldung LoRAs Does Not Disrupt Translation Equivariance

The this section we explore the intersection of two major developments in transformer architectures - Low-Rank Adaptations (LoRAs) and Group Self-Attention Networks (GSA-Nets). Both of these innovations aim to enhance the efficiency and effectiveness of transformer models, but they each approach the problem from different angles. While LoRAs focus on reducing the computational and storage demands of fine-tuning large-scale models, GSA-Nets aim to improve model performance by incorporating symmetry groups into self-attention mechanisms.

The introduction of LoRAs, which freeze pre-trained model weights and inject trainable rank decomposition matrices into each layer of the Transformer architecture, greatly reduces the number of trainable parameters for downstream tasks. However, the question arises: does the inclusion of LoRAs affect the translation equivariance of GSA-Nets? Translation equivariance is a key property of neural networks, which describes the fact that if a pattern is translated, its numerical descriptors are also translated but not modified. This property is central to the functioning of GSA-Nets.

This section thus delves into the compatibility of these two advancements in transformer architecture. It seeks to examine whether the inclusion of LoRAs disrupts the translation equivariance property that is crucial to the performance of GSA-Nets. By exploring this question, we can gain a deeper understanding of how different innovations in transformer architecture can be effectively combined, allowing us to develop more efficient and powerful deep learning models.

In the following proof, we follow notation very similar to [RC]. Suppose we are given an input matrix $X \in \mathbb{R}^{d_{in} \times d}$, with columns representing the embedding vectors of d tokens. We can formulate the self-attention matrix in a transformer with relative positional encoding as

$$A_{i,j} = (W_Q X)(W_K(X + P_{x(j)-x(i)}))^T \tag{1}$$

We leave out the scaling factor $1/\sqrt{d_k}$ for simplicity, but it can easily be included without disrupting any of our arguments or proofs. It will be more beneficial for our purposes to formulate self-attention in a function theoretic way. In particular, we can view the input matrix X as a vector valued function $f : S \to \mathbb{R}^{d_{in}}$, that is $f \in L_{\mathbb{R}^{d_{in}}}(S)$, for the index set $S = \{1, 2, ..., d\}$. We then view the query and key matrices as maps $\varphi_{qry} : L_{\mathbb{R}^{d_{in}}}(S) \to L_{\mathbb{R}^{d_k}}(S)$ and $\varphi_{key} : L_{\mathbb{R}^{d_{in}}}(S) \to L_{\mathbb{R}^{d_k}}(S)$. There is also a value function $\varphi_{val} : L_{\mathbb{R}^{d_{in}}}(S) \to L_{\mathbb{R}^{d_v}}(S)$. With these in hand, we can express the attention map with positional encoding as

$$A_{i,j} = \alpha[f](i, j) = \langle \varphi_{qry}(f(i)), \varphi_{key}(f(j) + \rho(i, j)) \rangle \tag{2}$$

Here, we have written $\rho(i, j)$ for the positional encoding. The map $\alpha[f] : S \times S \to \mathbb{R}$ maps pairs of elements $i, j \in S$ to the attention score of j relative to i. We can then write the attention mechanism as

$$\zeta[f](i) = \sum_{j \in S} \sigma_j \left(\alpha[f](i, j)\right) \varphi_{val}(f(j)) \tag{3}$$

Next, we would like to include a LoRA for the query, key, and value maps. We will formulate this as

$$\Delta\varphi_{qry}(f(i)) = (\varphi^A_{qry} \circ \varphi^B_{qry})(f(i)) = \varphi^B_{qry}(\varphi^A_{qry}(f(i))) \tag{4}$$

$$\Delta\varphi_{key}(f(i)) = (\varphi^A_{key} \circ \varphi^B_{key})(f(i)) = \varphi^B_{key}(\varphi^A_{key}(f(i))) \tag{5}$$

$$\Delta\varphi_{val}(f(i)) = (\varphi^A_{val} \circ \varphi^B_{val})(f(i)) = \varphi^B_{val}(\varphi^A_{val}(f(i))) \tag{6}$$

Here, we have

$$\varphi^A_{qry} : L_{\mathbb{R}^{d_k}}(S) \to L_{\mathbb{R}^{r(A)}}(S) \qquad \varphi^B_{qry} : L_{\mathbb{R}^{r(A)}}(S) \to L_{\mathbb{R}^{d_k}}(S) \tag{7}$$

$$\varphi^A_{key} : L_{\mathbb{R}^{d_k}}(S) \to L_{\mathbb{R}^{r(A)}}(S) \qquad \varphi^B_{key} : L_{\mathbb{R}^{r(A)}}(S) \to L_{\mathbb{R}^{d_k}}(S) \tag{8}$$

$$\varphi^A_{val} : L_{\mathbb{R}^{d_v}}(S) \to L_{\mathbb{R}^{r(A)}}(S) \qquad \varphi^B_{val} : L_{\mathbb{R}^{r(A)}}(S) \to L_{\mathbb{R}^{d_v}}(S) \tag{9}$$

Next, including this in the attention mechanism, we get

$$\alpha^{LoRA}[f](i, j) = \langle \varphi_{qry}(f(i)) + \Delta\varphi_{qry}(f(i)), \varphi_{key}(f(j) + \rho(i, j)) + \Delta\varphi_{key}(f(j) + \rho(i, j))\rangle \tag{10}$$

and then

$$\zeta^{LoRA}[f](i) = \sum_{j\in S} \sigma_j \left(\alpha^{LoRA}[f](i, j)\right) (\varphi_{val}(f(j)) + \Delta\varphi_{val}(f(j))) \tag{11}$$

Now, in order to have translation equivariance of a LoRA multihead self-attention with relative positional encoding, we need the following equation to hold,

$$m^r_{LoRA}[L_y[f], \rho](i) = L_y[m^r_{LoRA}[f, \rho]](i) \tag{12}$$

where $L_y[f](i) = f(x^{-1}(x(i) - y))$. The LoRA multihead self-attention with relative positional encodings on $L_y[f]$ is given by

$$
\begin{aligned}
&m^{r}_{LoRA}[L_y[f],\rho](i)\\
&=\varphi_{out}\Bigg(\bigcup_{h\in[H]}\sum_{j\in N(i)}\sigma_j\Bigg(\Big\langle\varphi^{(h)}_{qry}(L_y[f](i))+\Delta\varphi^{(h)}_{qry}(f(i)),\\
&\quad\varphi^{(h)}_{key}(L_y[f](j)+\rho(i,j))+\Delta\varphi^{(h)}_{key}(f(j)+\rho(i,j))\Big\rangle\Bigg)\Bigg(\varphi^{(h)}_{val}(L_y[f](j))+\Delta\varphi^{(h)}_{val}(f(j))\Bigg)\Bigg)\\
&=\varphi_{out}\Bigg(\bigcup_{h\in[H]}\sum_{j\in N(i)}\sigma_j\Bigg(\Big\langle\varphi^{(h)}_{qry}(f(x^{-1}(x(i)-y)))+\Delta\varphi^{(h)}_{qry}(f(x^{-1}(x(i)-y))),\\
&\quad\varphi^{(h)}_{key}(f(x^{-1}(x(j)-y))+\rho(i,j))+\Delta\varphi^{(h)}_{key}(f(x^{-1}(x(j)-y))+\rho(i,j))\Big\rangle\Bigg)\\
&\quad\times\Bigg(\varphi^{(h)}_{val}(f(x^{-1}(x(j)-y)))+\Delta\varphi^{(h)}_{val}(f(x^{-1}(x(j)-y)))\Bigg)\Bigg)\\
&=\varphi_{out}\Bigg(\bigcup_{h\in[H]}\sum_{x^{-1}(x(\bar{j})+y)\in N(x^{-1}(x(\bar{i})+y))}\sigma_{x^{-1}(x(\bar{j})+y)}\Bigg(\Big\langle\varphi^{(h)}_{qry}(f(\bar{i}))+\Delta\varphi^{(h)}_{qry}(f(\bar{i})),\\
&\quad\varphi^{(h)}_{key}(f(\bar{j})+\rho(x^{-1}(x(\bar{i})+y),x^{-1}(x(\bar{j})+y)))+\Delta\varphi^{(h)}_{key}(f(\bar{j})+\rho(x^{-1}(x(\bar{i})+y),x^{-1}(x(\bar{j})+y)))\Big\rangle\Bigg)\\
&\quad\times\Bigg(\varphi^{(h)}_{val}(f(\bar{j}))+\Delta\varphi^{(h)}_{val}(f(\bar{j}))\Bigg)\Bigg)
\end{aligned}
$$

Here we have used the substitution $\bar{i}=x^{-1}(x(i)-y)\implies i=x^{-1}(x(\bar{i})+y)$ and $\bar{j}=x^{-1}(x(j)-y)\implies j=x^{-1}(x(\bar{j})+y)$. We can further reduce the equations using the defintion of $\rho(i,j)=\rho^{P}(x(j)-x(i))$:

$$
\begin{aligned}
&=\varphi_{out}\Bigg(\bigcup_{h\in[H]}\sum_{x^{-1}(x(\bar{j})+y)\in N(x^{-1}(x(\bar{i})+y))}\sigma_{x^{-1}(x(\bar{j})+y)}\Bigg(\Big\langle\varphi^{(h)}_{qry}(f(\bar{i}))+\Delta\varphi^{(h)}_{qry}(f(\bar{i})),\\
&\varphi^{(h)}_{key}(f(\bar{j})+\rho^{P}(x(\bar{j})+y-(x(\bar{i})+y)))+\Delta\varphi^{(h)}_{key}(f(\bar{j})+\rho^{P}(x(\bar{j})+y-(x(\bar{i})+y)))\Big\rangle\Bigg)\Bigg(\varphi^{(h)}_{val}(f(\bar{j}))+\Delta\varphi^{(h)}_{val}(f(\bar{j}))\Bigg)\Bigg)\\
&=\varphi_{out}\Bigg(\bigcup_{h\in[H]}\sum_{x^{-1}(x(\bar{j})+y)\in N(x^{-1}(x(\bar{i})+y))}\sigma_{x^{-1}(x(\bar{j})+y)}\Bigg(\Big\langle\varphi^{(h)}_{qry}(f(\bar{i}))+\Delta\varphi^{(h)}_{qry}(f(\bar{i})),\\
&\varphi^{(h)}_{key}(f(\bar{j})+\rho^{P}(x(\bar{j})-x(\bar{i})))+\Delta\varphi^{(h)}_{key}(f(\bar{j})+\rho^{P}(x(\bar{j})-x(\bar{i})))\Big\rangle\Bigg)\Bigg(\varphi^{(h)}_{val}(f(\bar{j}))+\Delta\varphi^{(h)}_{val}(f(\bar{j}))\Bigg)\Bigg)\\
&=\varphi_{out}\Bigg(\bigcup_{h\in[H]}\sum_{x^{-1}(x(\bar{j})+y)\in N(x^{-1}(x(\bar{i})+y))}\sigma_{x^{-1}(x(\bar{j})+y)}\Bigg(\Big\langle\varphi^{(h)}_{qry}(f(\bar{i}))+\Delta\varphi^{(h)}_{qry}(f(\bar{i})),\\
&\varphi^{(h)}_{key}(f(\bar{j})+\rho(\bar{i},\bar{j}))+\Delta\varphi^{(h)}_{key}(f(\bar{j})+\rho(\bar{i},\bar{j}))\Big\rangle\Bigg)\Bigg(\varphi^{(h)}_{val}(f(\bar{j}))+\Delta\varphi^{(h)}_{val}(f(\bar{j}))\Bigg)\Bigg)
\end{aligned}
$$

For any translation $y\in\mathbb{R}^{d_{in}}$, where d_{in} is the dimension of $f(i)$ and $f(j)$, the summation remains the same, so we have:

$$
\sum_{x^{-1}(x(\bar{j})+y)\in N(x^{-1}(x(\bar{i})+y))}[\bullet]=\sum_{x^{-1}(x(\bar{j}))\in N(x^{-1}(x(\bar{i})))}[\bullet]=\sum_{\bar{j}\in N(\bar{i})}[\bullet]\tag{13}
$$

As we can see, $m^r_{LoRA}[L_y[f],\rho](i) = L_y[m^r_{LoRA}[f,\rho]](i)$. We can thus conclude that relative positional encodings, coupled with LoRAs for the query, key, and value weight matrices gives a translation equivariant multihead self-attention mechanism. In particular, addition of LoRAs in a translation equivariant model with relative positional encodings does not disrupt the translation equivariance, which means we can further reduce the expression as

$$\begin{aligned}
m^r_{LoRA}[L_y[f],\rho](i) = \varphi_{out}\Bigg(\bigcup_{h\in[H]} \sum_{x^{-1}(x(\bar{j})+y)\in N(x^{-1}(x(\bar{i})+y))} \sigma_{x^{-1}(x(\bar{j})+y)} \Bigg(\Big\langle \varphi^{(h)}_{qry}(f(\bar{i})) + \Delta\varphi^{(h)}_{qry}(f(\bar{i})),& \\
\varphi^{(h)}_{key}(f(\bar{j}) + \rho(\bar{i},\bar{j})) + \Delta\varphi^{(h)}_{key}(f(\bar{j}) + \rho(\bar{i},\bar{j})) \Big\rangle \Bigg) \Big(\varphi^{(h)}_{val}(f(\bar{j})) + \Delta\varphi^{(h)}_{val}(f(\bar{j})) \Big) \Bigg)& \\
= m^r_{LoRA}[f,\rho](\bar{i})& \\
= m^r_{LoRA}[f,\rho](x^{-1}(x(i)-y))& \\
= L_y[m^r_{LoRA}[f,\rho]](i)&
\end{aligned}$$

To reiterate in slightly different notation, for translation equivariance of a LoRA multihead self-attention with relative positional encoding, the following equation needs to hold,

$$m^r_{LoRA}[L_y[f],\rho](i) = L_y[m^r_{LoRA}[f,\rho]](i) \tag{14}$$

where $L_y[f](i) = f(x^{-1}(x(i) - y))$. The LoRA multihead self-attention with relative positional encodings on $L_y[f]$ is:

$$\begin{aligned}
&m^r_{LoRA}[L_y[f],\rho](i) \\
&= \varphi_{out}\Bigg(\bigcup_{h\in[H]} \sum_{j\in N(i)} \frac{\exp\Big\langle \varphi^{(h)}_{qry}(L_y[f](i)) + \Delta\varphi^{(h)}_{qry}(f(i)), \varphi^{(h)}_{key}(L_y[f](j) + \rho(i,j)) + \Delta\varphi^{(h)}_{key}(f(j) + \rho(i,j)) \Big\rangle}{\sum_{k\in N(i)} \exp\Big\langle \varphi^{(h)}_{qry}(L_y[f](i)) + \Delta\varphi^{(h)}_{qry}(f(i)), \varphi^{(h)}_{key}(L_y[f](k) + \rho(i,k)) + \Delta\varphi^{(h)}_{key}(f(k) + \rho(i,k)) \Big\rangle} \\
&\times \Big(\varphi^{(h)}_{val}(L_y[f](j)) + \Delta\varphi^{(h)}_{val}(f(j)) \Big) \Bigg)
\end{aligned}$$

We use the substitution $\bar{i} = x^{-1}(x(i) - y) \implies i = x^{-1}(x(\bar{i}) + y)$ and $\bar{j} = x^{-1}(x(j) - y) \implies j = x^{-1}(x(\bar{j}) + y)$, and the definition of $\rho(i,j) = \rho^P(x(j) - x(i))$:

$$\begin{aligned}
&m^r_{LoRA}[L_y[f],\rho](i) \\
&= \varphi_{out}\Bigg(\bigcup_{h\in[H]} \sum_{\bar{j}\in N(\bar{i})} \frac{\exp\Big\langle \varphi^{(h)}_{qry}(f(\bar{i})) + \Delta\varphi^{(h)}_{qry}(f(\bar{i})), \varphi^{(h)}_{key}(f(\bar{j}) + \rho(\bar{i},\bar{j})) + \Delta\varphi^{(h)}_{key}(f(\bar{j}) + \rho(\bar{i},\bar{j})) \Big\rangle}{\sum_{\bar{k}\in N(\bar{i})} \exp\Big\langle \varphi^{(h)}_{qry}(f(\bar{i})) + \Delta\varphi^{(h)}_{qry}(f(\bar{i})), \varphi^{(h)}_{key}(f(\bar{k}) + \rho(\bar{i},\bar{k})) + \Delta\varphi^{(h)}_{key}(f(\bar{k}) + \rho(\bar{i},\bar{k})) \Big\rangle} \\
&\times \Big(\varphi^{(h)}_{val}(f(\bar{j})) + \Delta\varphi^{(h)}_{val}(f(\bar{j})) \Big) \Bigg) \\
&= m^r_{LoRA}[f,\rho](\bar{i}) \\
&= m^r_{LoRA}[f,\rho](x^{-1}(x(i)-y)) \\
&= L_y[m^r_{LoRA}[f,\rho]](i)
\end{aligned}$$

The last equations conclude that relative positional encodings, coupled with LoRAs for the query, key, and value weight matrices, result in a translation equivariant multihead self-attention mechanism. More specifically, addition of LoRAs in a translation equivariant model with relative positional encodings does not disrupt the translation equivariance.

4. LoRAs and Lifting Self-Attention

Next, we study the lifting self-attention layers defined in [RC]. For the lifting self-attention layer to be G-equivariant, we must have that $m^r_{G\uparrow}[L_g[f],\rho](i,h) = L_g[m^r_{G\uparrow}[f,\rho]](i,h)$. Consider a g-transformed input signal $L_g[f](i) = L_yL_{h_3}[f](i) = f(x^{-1}(h_3^{-1}(x(i)-y)))$. Here $g = (y, h_3) \in \mathbb{R}^{d_{in}} \rtimes \mathcal{H}$. The lifting self-attention layer with LoRAs, applied to $L_g[f]$ is:

$$
\begin{aligned}
&m^{r,LoRA}_{G\uparrow}[L_yL_{h_3}[f],\rho](i,h)\\
&= \varphi_{out}\Bigg(\bigcup_{head\in[H]}\sum_{j\in N(i)}\sigma_j\Bigg(\langle\varphi^{head}_{qry}(L_yL_{h_3}[f](i)) + \Delta\varphi^{head}_{qry}(L_yL_{h_3}[f](i)), \varphi^{head}_{key}(L_yL_{h_3}[f](j)\\
&\quad + L_h[\rho](i,j)) + \Delta\varphi^{head}_{key}(L_yL_{h_3}[f](j) + L_h[\rho](i,j))\rangle\Bigg)(\varphi^{head}_{val}(L_yL_{h_3}[f](j)) + \Delta\varphi^{head}_{val}(L_yL_{h_3}[f](j)))\Bigg)\\
&= \varphi_{out}\Bigg(\bigcup_{head\in[H]}\sum_{j\in N(i)}\sigma_j\Bigg(\langle\varphi^{head}_{qry}(f(x^{-1}(h_3^{-1}(x(i)-y))) + \Delta\varphi^{head}_{qry}(f(x^{-1}(h_3^{-1}(x(i)-y))), \varphi^{head}_{key}(f(x^{-1}(h_3^{-1}(x(j)-y)))\\
&\quad + L_h[\rho](i,j) + \Delta\varphi^{head}_{key}(f(x^{-1}(h_3^{-1}(x(j)-y))) + L_h[\rho](i,j))\rangle\Bigg)\\
&\quad \times(\varphi^{head}_{val}(f(x^{-1}(h_3^{-1}(x(j)-y)))) + \Delta\varphi^{head}_{val}(f(x^{-1}(h_3^{-1}(x(j)-y)))))\Bigg)\\
&= \varphi_{out}\Bigg(\bigcup_{head\in[H]}\sum_{x^{-1}(h_3^{-1}x(\bar{j})+y)\in N(x^{-1}(h_3^{-1}x(\bar{i})+y))}\sigma_j\Bigg(\langle\varphi^{head}_{qry}(f(\bar{i})) + \Delta\varphi^{head}_{qry}(f(\bar{i})), \varphi^{head}_{key}(f(\bar{j}))\\
&\quad + L_h[\rho](x^{-1}(h_3x(\bar{i})+y), x^{-1}(h_3x(\bar{j})+y)) + \Delta\varphi^{head}_{key}(f(\bar{j}) + L_h[\rho](x^{-1}(h_3x(\bar{i})+y), x^{-1}(h_3x(\bar{j})+y)))\rangle\Bigg)\\
&\quad \times(\varphi^{head}_{val}(f(\bar{j})) + \Delta\varphi^{head}_{val}(f(\bar{j})))\Bigg)
\end{aligned}
$$

Here we have used $\bar{i} = x^{-1}(h_3^{-1}(x(i)-y)) \implies i = x^{-1}(h_3x(\bar{i})+y)$ and $\bar{j} = x^{-1}(h_3^{-1}(x(j)-y)) \implies j = x^{-1}(h_3x(\bar{j})+y)$. By using the defintion of $\rho(i,j)$ we can further reduce the expression:

$$= \varphi_{out}\Bigg(\bigcup_{h\in[H]} \sum_{x^{-1}(h_3^{-1}x(\bar{j})+y)\in N(x^{-1}(h_3^{-1}x(\bar{i})+y))} \sigma_{x^{-1}(h_3^{-1}x(\bar{j})+y)}\Big(\langle \varphi_{qry}^{(h)} f(\bar{i}) + \Delta\varphi_{qry}^{(h)} f(\bar{i}), \varphi_{key}^{(h)}(f(\bar{j})$$

$$+ \rho^P(h^{-1}(h_3^{-1}x(\bar{j}) + y) - h^{-1}(h_3 x(\bar{i}) + y))) + \Delta\varphi_{key}^{(h)}(f(\bar{j}) + \rho^P(h^{-1}(h_3^{-1}x(\bar{j}) + y) - h^{-1}(h_3 x(\bar{i}) + y)))\rangle\Big)$$

$$\times (\varphi_{val}^{(h)}(f(\bar{j})) + \Delta\varphi_{val}^{(h)}(f(\bar{j})))\Bigg)$$

$$= \varphi_{out}\Bigg(\bigcup_{h\in[H]} \sum_{x^{-1}(h_3^{-1}x(\bar{j})+y)\in N(x^{-1}(h_3^{-1}x(\bar{i})+y))} \sigma_{x^{-1}(h_3^{-1}x(\bar{j})+y)}\Big(\langle \varphi_{qry}^{(h)} f(\bar{i}) + \Delta\varphi_{qry}^{(h)} f(\bar{i}), \varphi_{key}^{(h)}(f(\bar{j})$$

$$+ \rho^P(h^{-1}h_3(x(\bar{j}) - x(\bar{i})))) + \Delta\varphi_{key}^{(h)}(f(\bar{j}) + \rho^P(h^{-1}h_3(x(\bar{j}) - x(\bar{i}))))\rangle\Big)$$

$$\times (\varphi_{val}^{(h)}(f(\bar{j})) + \Delta\varphi_{val}^{(h)}(f(\bar{j})))\Bigg)$$

$$= \varphi_{out}\Bigg(\bigcup_{h\in[H]} \sum_{x^{-1}(h_3^{-1}x(\bar{j})+y)\in N(x^{-1}(h_3^{-1}x(\bar{i})+y))} \sigma_{x^{-1}(h_3^{-1}x(\bar{j})+y)}\Big(\langle \varphi_{qry}^{(h)} f(\bar{i}), \varphi_{key}^{(h)}(f(\bar{j})$$

$$+ L_{h_3^{-1}h}(\bar{i}, \bar{j}) + \Delta\varphi_{key}^{(h)}(f(\bar{j}) + L_{h_3^{-1}h}(\bar{i}, \bar{j}))\rangle\Big)$$

$$\times (\varphi_{val}^{(h)}(f(\bar{j})) + \Delta\varphi_{val}^{(h)}(f(\bar{j})))\Bigg)$$

Since for unimodular (or compact) groups the area of summation remains equal for any $g \in G$, we have:

$$\sum_{x^{-1}(h_3 x(\bar{j})+y)\in N(x^{-1}(h_3 x(\bar{i})+y))} [\bullet] = \sum_{x^{-1}(h_3 x(\bar{j}))\in N(x^{-1}(h_3 x(\bar{i})))} [\bullet] = \sum_{x^{-1}(x(\bar{j}))\in N(x(\bar{i})))} [\bullet] = \sum_{\bar{j}\in N(\bar{i})} [\bullet] \tag{15}$$

As a result, we can further reduce the expression above as:

$$m_{G\uparrow}^{r,LoRA}[L_y L_{h_3}[f], \rho](i, h)$$

$$= \varphi_{out}\Bigg(\bigcup_{h\in[H]} \sum_{\bar{j}\in N(\bar{i})} \sigma_{\bar{j}}\Big(\langle \varphi_{qry}^{(h)}(f(\bar{i})) + \Delta\varphi_{qry}^{(h)}(f(\bar{i})), \varphi_{key}^{(h)}(f(\bar{j}) + L_{h_3^{-1}h}[\rho](\bar{i}, \bar{j})) + \Delta\varphi_{key}^{(h)}(f(\bar{j}) + L_{h_3^{-1}h}[\rho](\bar{i}, \bar{j}))\rangle\Big)$$

$$\times (\varphi_{val}^{(h)}(f(\bar{j})) + \Delta\varphi_{val}^{(h)}(f(\bar{j})))\Bigg)$$

$$= m_{G\uparrow}^{r,LoRA}[r, \rho](\bar{i}, h_3^{-1}h)$$

$$= m_{G\uparrow}^{r,LoRA}(x^{-1}(h_3^{-1}(x(i) - y)), h_3^{-1}h)$$

$$= L_y L_{h_3}[m_{G\uparrow}^{r,LoRA}[f, \rho]](i, h)$$

So, we see that adding LoRAs does not disrupt equivariance of the lifting self-attention layer.

5. LoRAs and Group Self-Attention

Next, we can also show that the inclusion of LoRAs does not disrupt equivariance of group self-attention.

$$
\begin{aligned}
&m^r_{G,LoRA}[L_y L_{h_3}[f],\rho](i,h_1)\\
&= \varphi_{out}\Bigg(\bigcup_{head\in[H]}\sum_{h\in\mathcal{H}}\sum_{(j,h_2)\in N(i,h_1)}\sigma_{(j,h_2)}\Bigg(\langle\varphi^{head}_{qry}L_yL_{h_3}[f](i,h_1)+\Delta\varphi^{head}_{qry}L_yL_{h_3}[f](i,h_1),\varphi^{head}_{key}(L_yL_{h_3}[f](j,h_2)\\
&\quad + L_h[\rho]((i,h_1),(j,h_2)))+\Delta\varphi^{head}_{key}(L_yL_{h_3}[f](j,h_2)+L_h[\rho]((i,h_1),(j,h_2)))\rangle\Bigg)\\
&\quad \times(\varphi^{head}_{val}(L_yL_{h_3}[f](j,h_2))+\Delta\varphi^{head}_{val}(L_yL_{h_3}[f](j,h_2)))\Bigg)\\
&= \varphi_{out}\Bigg(\bigcup_{head\in[H]}\sum_{h\in\mathcal{H}}\sum_{(j,h_2)\in N(i,h_1)}\sigma_{(j,h_2)}\Bigg(\langle\varphi^{head}_{qry}f(x^{-1}(h_3^{-1}(x(i)-y)),h_3^{-1}h_1)+\Delta\varphi^{head}_{qry}f(x^{-1}(h_3^{-1}(x(i)-y)),h_3^{-1}h_1),\\
&\quad \varphi^{head}_{key}(f(x^{-1}(h_3^{-1}(x(i)-y)),h_3^{-1}h_2)\\
&\quad + L_h[\rho]((i,h_1),(j,h_2)))+\Delta\varphi^{head}_{key}(f(x^{-1}(h_3^{-1}(x(i)-y)),h_3^{-1}h_2)+L_h[\rho]((i,h_1),(j,h_2)))\rangle\Bigg)\\
&\quad (\varphi^{head}_{val}(f(x^{-1}(h_3^{-1}(x(i)-y)),h_3^{-1}h_2))+\Delta\varphi^{head}_{val}(f(x^{-1}(h_3^{-1}(x(i)-y)),h_3^{-1}h_2)))\Bigg)\\
&= \varphi_{out}\Bigg(\bigcup_{head\in[H]}\sum_{h_3h\in\mathcal{H}}\sum_{(x^{-1}(h_3x(\bar{j})+y),h_3h_2')\in N((x^{-1}(h_3x(\bar{i})+y),h_3h_1'))}\sigma_{(x^{-1}(h_3x(\bar{j})+y),h_3h_2')}\Bigg(\langle\varphi^{head}_{qry}f(\bar{i},h_1')+\Delta\varphi^{head}_{qry}f(\bar{i},h_1'),\varphi^{head}_{key}(f(\bar{j},h_2')\\
&\quad + L_h[\rho]((x^{-1}(h_3^{-1}x(\bar{i})+y),h_3h_1'),(x^{-1}(h_3^{-1}x(\bar{j})+y),h_3h_2')))\\
&\quad + \Delta\varphi^{head}_{key}(f(\bar{j},h_2')+L_h[\rho]((x^{-1}(h_3^{-1}x(\bar{i})+y),h_3h_1'),(x^{-1}(h_3^{-1}x(\bar{j})+y),h_3h_2')))\rangle\Bigg)\\
&\quad \times(\varphi^{head}_{val}(f(\bar{j},h_2'))+\Delta\varphi^{head}_{val}(f(\bar{j},h_2')))\Bigg)
\end{aligned}
$$

Next, we use the definition of $L_h[\rho]((i,h_1),(j,h_2)) = \rho^P(h^{-1}(x(j)-x(i)),h_1^{-1}h_2)$ to reduce the equations further:

$$
\begin{aligned}
&= \varphi_{out}\Bigg(\bigcup_{head\in[H]} \sum_{h_3 h\in\mathcal{H}} \sum_{(x^{-1}(h_3 x(\bar{j})+y), h_3 h_2')\in N((x^{-1}(h_3 x(\bar{i})+y), h_3 h_1'))} \sigma_{(x^{-1}(h_3 x(\bar{j})+y), h_3 h_2')}\Bigg(\langle \varphi_{qry}^{head} f(\bar{i}, h_1') + \Delta\varphi_{qry}^{head} f(\bar{i}, h_1'), \varphi_{key}^{head}(f(\bar{j}, h_2') \\
&\quad + \rho^P(h^{-1}(h_3 x(\bar{j}) + y - (h_3 x(\bar{i}) + y)), (h_3 h_1')^{-1}(h_3 h_2')) \\
&\quad + \Delta\varphi_{key}^{head}(f(\bar{j}, h_2') + \rho^P(h^{-1}(h_3 x(\bar{j}) + y - (h_3 x(\bar{i}) + y)), (h_3 h_1')^{-1}(h_3 h_2')))\rangle\Bigg) \\
&\quad \times (\varphi_{val}^{head}(f(\bar{j}, h_2')) + \Delta\varphi_{val}^{head}(f(\bar{j}, h_2')))\Bigg) \\
&= \varphi_{out}\Bigg(\bigcup_{head\in[H]} \sum_{h_3 h\in\mathcal{H}} \sum_{(x^{-1}(h_3 x(\bar{j})+y), h_3 h_2')\in N((x^{-1}(h_3 x(\bar{i})+y), h_3 h_1'))} \sigma_{(x^{-1}(h_3 x(\bar{j})+y), h_3 h_2')}\Bigg(\langle \varphi_{qry}^{head} f(\bar{i}, h_1') + \Delta\varphi_{qry}^{head} f(\bar{i}, h_1'), \varphi_{key}^{head}(f(\bar{j}, h_2') \\
&\quad + \rho^P(h^{-1}(h_3 x(\bar{j}) - h_3 x(\bar{i})), {h'}_1^{-1} h_2') + \Delta\varphi_{key}^{head}(f(\bar{j}, h_2') + \rho^P(h^{-1}(h_3 x(\bar{j}) - h_3 x(\bar{i})), {h'}_1^{-1} h_2'))\rangle\Bigg) \\
&\quad \times (\varphi_{val}^{head}(f(\bar{j}, h_2')) + \Delta\varphi_{val}^{head}(f(\bar{j}, h_2')))\Bigg) \\
&= \varphi_{out}\Bigg(\bigcup_{head\in[H]} \sum_{h_3 h\in\mathcal{H}} \sum_{(x^{-1}(h_3 x(\bar{j})+y), h_3 h_2')\in N((x^{-1}(h_3 x(\bar{i})+y), h_3 h_1'))} \sigma_{(x^{-1}(h_3 x(\bar{j})+y), h_3 h_2')}\Bigg(\langle \varphi_{qry}^{head} f(\bar{i}, h_1') + \Delta\varphi_{qry}^{head} f(\bar{i}, h_1'), \varphi_{key}^{head}(f(\bar{j}, h_2') \\
&\quad + L_{h_3^{-1}h}[\rho](\bar{i}, h_1'), (\bar{j}, h_2')) + \Delta\varphi_{key}^{head}(f(\bar{j}, h_2') + L_{h_3^{-1}h}[\rho](\bar{i}, h_1'), (\bar{j}, h_2')))\rangle\Bigg)(\varphi_{val}^{head}(f(\bar{j}, h_2')) + \Delta\varphi_{val}^{head}(f(\bar{j}, h_2')))\Bigg)
\end{aligned}
$$

Now, since G is assumed unimodular (or compact) we have that the following summations are equal,

(16)

$$
\sum_{(x^{-1}(h_3 x(\bar{j})+y), h_3 h_2')\in N((x^{-1}(h_3 x(\bar{i})+y), h_3 h_1'))} [\bullet] = \sum_{(x^{-1}(h_3 x(\bar{j})), h_3 h_2')\in N((x^{-1}(h_3 x(\bar{i})), h_3 h_1'))} [\bullet] = \sum_{(x^{-1}(x(\bar{j})), h_2')\in N(x^{-1}(x(\bar{i})), h_1')} [\bullet] = \sum_{(\bar{j}, h_2')\in N(\bar{i}, h_1')} [\bullet]
$$

We also have that $\sum_{h_3 h\in\mathcal{H}}[\bullet] = \sum_{h\in\mathcal{H}}[\bullet]$. Finally, we make the following observations,

$$
\begin{aligned}
&m_{G,LoRA}^r[L_y L_{h_3}[f], \rho](i, h_1) \\
&= \varphi_{out}\Bigg(\bigcup_{head\in[H]} \sum_{h\in\mathcal{H}} \sum_{(\bar{j}, h_2')\in N(\bar{i}, h_1')} \sigma_{(\bar{j}, h_2')}\Bigg(\langle \varphi_{qry}^{head}(f(\bar{i}, h_1')) + \Delta\varphi_{qry}^{head}(f(\bar{i}, h_1')), \varphi_{key}^{head}(f(\bar{j}, h_2') + \\
&\quad L_{h_3^{-1}h}[\rho]((\bar{i}, h_1'), (\bar{j}, h_2')) + \Delta\varphi_{key}^{head}(f(\bar{j}, h_2') + L_{h_3^{-1}h}[\rho]((\bar{i}, h_1'), (\bar{j}, h_2'))))\rangle\Bigg)(\varphi_{val}^{head}(f(\bar{j}, h_2')) + \Delta\varphi_{val}^{head}(f(\bar{j}, h_2')))\Bigg) \\
&= m_{G,LoRA}^r[f, \rho](\bar{i}, h_3^{-1} h_1) \\
&= m_{G,LoRA}^r[f, \rho](x^{-1}(h_3^{-1}(x(i) - y)), h_3^{-1} h_1) \\
&= L_y L_{h_3}[m_{G,LoRA}^r[f, \rho]](i, h_1)
\end{aligned}
$$

So, we see that in fact $m_{G,LoRA}^r[L_y L_{h_3}[f], \rho](i, h) = L_y L_{h_3}[m_{G,LoRA}^r[f, \rho]](i, h)$, and LoRAs do not disrupt equivariance of group self-attention.

We can see for the following setup, that the same arguments work if we include a LoRA $\Delta\varphi_{out}$ for φ_{out}. In particular, including such a LoRA does not disrupt the equivariance of the group self-attention layers.

$$
\begin{aligned}
m^r_{G,LoRA}[L_yL_{h_3}[f],\rho](i,h_1) = \varphi_{out}\Bigg(&\bigcup_{head\in[H]}\sum_{h\in\mathcal{H}}\sum_{(j,h_2)\in N(i,h_1)}\sigma_{(j,h_2)}\Bigg(\langle\varphi^{head}_{qry}L_yL_{h_3}[f](i,h_1)+\Delta\varphi^{head}_{qry}L_yL_{h_3}[f](i,h_1),\\
&\varphi^{head}_{key}(L_yL_{h_3}[f](j,h_2)\\
&+L_h[\rho]((i,h_1),(j,h_2)))+\Delta\varphi^{head}_{key}(L_yL_{h_3}[f](j,h_2)+L_h[\rho]((i,h_1),(j,h_2)))\rangle\Bigg)\\
&(\varphi^{head}_{val}(L_yL_{h_3}[f](j,h_2))+\Delta\varphi^{head}_{val}(L_yL_{h_3}[f](j,h_2)))\Bigg)\\
&+\Delta\varphi_{out}\Bigg(\bigcup_{head\in[H]}\sum_{h\in\mathcal{H}}\sum_{(j,h_2)\in N(i,h_1)}\sigma_{(j,h_2)}\Bigg(\langle\varphi^{head}_{qry}L_yL_{h_3}[f](i,h_1)+\Delta\varphi^{head}_{qry}L_yL_{h_3}[f](i,h_1),\\
&\varphi^{head}_{key}(L_yL_{h_3}[f](j,h_2)\\
&+L_h[\rho]((i,h_1),(j,h_2)))+\Delta\varphi^{head}_{key}(L_yL_{h_3}[f](j,h_2)+L_h[\rho]((i,h_1),(j,h_2)))\rangle\Bigg)\\
&(\varphi^{head}_{val}(L_yL_{h_3}[f](j,h_2))+\Delta\varphi^{head}_{val}(L_yL_{h_3}[f](j,h_2)))\Bigg)
\end{aligned}
$$

Note, a similar argument holds for translation equivariance, and for lifting self-attention layers as well. Thus, we may include LoRAs $\Delta\varphi_{out}$ in translation equivariant models, and in the lifting self-attention layers as well without disrupting equivariance.

6. Appendix

In [RC] there are a few calculation errors in the proof of the equivariance of group self-attention layers. Below is an explanation of why the proof of the equivariance property does not work and how to fix it. Let $L_g[f](i,h_1) = L_yL_{h_3}[f](i,h_1) = f(x^{-1}(h_3^{-1}(x(i)-y)), h_3^{-1}h_1)$ be a g-transformed input signal, where $g=(y,h_3)\in G=\mathbb{R}^{d_{in}}\rtimes\mathcal{H}$. The group self-attention operation on $L_g[f]$ is given by:

$$
\begin{aligned}
m_G^r[L_yL_{h_3}[f],\rho](i,h_1) &= \varphi_{out}\Bigg(\bigcup_{head\in[H]}\sum_{h\in\mathcal{H}}\sum_{(j,h_2)\in N(i,h_1)}\sigma_{(j,h_2)}\Big(\langle\varphi_{qry}^{head}(L_yL_{h_3}[f](i,h_1)),\varphi_{key}^{head}(L_yL_{h_3}[f](j,h_2)\\
&\quad + L_h[\rho]((i,h_1),(j,h_2)))\rangle\Big)\varphi_{val}^{head}(L_yL_{h_3}[f](j,h_2))\Bigg)\\
&= \varphi_{out}\Bigg(\bigcup_{head\in[H]}\sum_{h\in\mathcal{H}}\sum_{(j,h_2)\in N(i,h_1)}\sigma_{(j,h_2)}\Big(\langle\varphi_{qry}^{head}(f(x^{-1}(h_3^{-1}(x(i)-y))),h_3^{-1}h_1),\\
&\quad \varphi_{key}^{head}(f(x^{-1}(h_3^{-1}(x(i)-y)),h_3^{-1}h_2)\\
&\quad + L_h[\rho]((i,h_1),(j,h_2)))\rangle\Big)\varphi_{val}^{head}(f(x^{-1}(h_3^{-1}(x(i)-y)),h_3^{-1}h_2))\Bigg)\\
&= \varphi_{out}\Bigg(\bigcup_{head\in[H]}\sum_{h_3h\in\mathcal{H}}\sum_{(x^{-1}(h_3x(\bar{j})+y),h_3h_2')\in N((x^{-1}(h_3x(\bar{i})+y),h_3h_1'))}\sigma_{(x^{-1}(h_3x(\bar{j})+y),h_3h_2')}\Big(\langle\varphi_{qry}^{head}(f(\bar{i},h_1')),\varphi_{key}^{head}(f(\bar{j},h_2')\\
&\quad + L_h[\rho]((x^{-1}(h_3^{-1}x(\bar{i})+y),h_3h_1'),(x^{-1}(h_3^{-1}x(\bar{j})+y),h_3h_2')))\rangle\Big)\varphi_{val}^{head}(f(\bar{j},h_2'))\Bigg)
\end{aligned}
$$

Here, we have used $\bar{i} = x^{-1}(h_3^{-1}(x(i)-y)) \implies i = x^{-1}(h_3^{-1}x(\bar{i})+y)$, and $\bar{j} = x^{-1}(h_3^{-1}(x(j)-y)) \implies j = x^{-1}(h_3^{-1}x(\bar{j})+y)$, and $h_1' = h_3^{-1}h_1$ and $h_2' = h_3^{-1}h_2$. By using the definition of $L_h\rho((i,h_1),(j,h_2)) = \rho^P(h^{-1}(x(j)-x(i)),h^{-1}h_1^{-1}h_2)$ we can further reduce the equations:

$$
\begin{aligned}
&= \varphi_{out}\Bigg(\bigcup_{head\in[H]}\sum_{h_3h\in\mathcal{H}}\sum_{(x^{-1}(h_3x(\bar{j})+y),h_3h_2')\in N((x^{-1}(h_3x(\bar{i})+y),h_3h_1'))}\sigma_{(x^{-1}(h_3x(\bar{j})+y),h_3h_2')}\Big(\langle\varphi_{qry}^{head}(f(\bar{i},h_1')),\varphi_{key}^{head}(f(\bar{j},h_2')\\
&\quad + \rho^P(h^{-1}(h_3x(\bar{j})+y-(h_3x(\bar{i})+y)),h^{-1}(h_3h_1')^{-1}(h_3h_2')))\rangle\Big)\varphi_{val}^{head}(f(\bar{j},h_2'))\Bigg)\\
&= \varphi_{out}\Bigg(\bigcup_{head\in[H]}\sum_{h_3h\in\mathcal{H}}\sum_{(x^{-1}(h_3x(\bar{j})+y),h_3h_2')\in N((x^{-1}(h_3x(\bar{i})+y),h_3h_1'))}\sigma_{(x^{-1}(h_3x(\bar{j})+y),h_3h_2')}\Big(\langle\varphi_{qry}^{head}(f(\bar{i},h_1')),\varphi_{key}^{head}(f(\bar{j},h_2')\\
&\quad + \rho^P(h^{-1}(h_3x(\bar{j})-h_3x(\bar{i})),h^{-1}{h'}_1^{-1}h_2'))\rangle\Big)\varphi_{val}^{head}(f(\bar{j},h_2'))\Bigg) \quad \text{notice the lack of a factor of } h_3^{-1} \text{ in front of } {h'}_1^{-1}h_2' \text{ here}\\
&\neq \varphi_{out}\Bigg(\bigcup_{head\in[H]}\sum_{h_3h\in\mathcal{H}}\sum_{(x^{-1}(h_3x(\bar{j})+y),h_3h_2')\in N((x^{-1}(h_3x(\bar{i})+y),h_3h_1'))}\sigma_{(x^{-1}(h_3x(\bar{j})+y),h_3h_2')}\Big(\langle\varphi_{qry}^{head}(f(\bar{i},h_1')),\varphi_{key}^{head}(f(\bar{j},h_2')\\
&\quad + L_{h_3^{-1}h}[\rho](\bar{i},h_1'),(\bar{j},h_2')))\rangle\Big)\varphi_{val}^{head}(f(\bar{j},h_2'))\Bigg)
\end{aligned}
$$

So, we see that $m_G^r[L_yL_{h_3}[f],\rho](i,h) \neq L_yL_{h_3}[m_G^r[f,\rho]](i,h)$. It is also said in [RC] that the equivariance of group self-attention is due to the relative positional encoding being **invariant** to the group action. That is, it is claimed to follow from $L_g[\rho](i,j) = \rho(i,j)$. This however is false. In particular, the positional encoding $\rho(i,j)$ is not used in the proof, it is the positional encoding $\rho((i,h_1),(j,h_2))$ that is used. Furthermore, we see that it is not G-invariant unless $x(j)-x(i)$ is an H-invariant vector.

$$
\begin{aligned}
L_g[\rho]((i,h_1),(j,h_2)) &= L_y L_h[\rho]((i,h_1),(j,h_2)) \\
&= L_h[L_y[\rho]]((i,h_1),(j,h_2)) \\
&= L_h[\rho]((x^{-1}(x(i)-y),h_1),(x^{-1}(x(j)-y),h_2)) \\
&= \rho((x^{-1}(h^{-1}(x(i)-y)),h^{-1}h_1),(x^{-1}(h^{-1}(x(j)-y)),h^{-1}h_2)) \\
&= \rho^P(h^{-1}(x(j)-y)-h^{-1}(x(i)-y),(h^{-1}h_1)^{-1}(h^{-1}h_2)) \\
&= \rho^P(h^{-1}(x(j)-x(i)),(h^{-1}h_1)^{-1}(h^{-1}h_2)) \\
&= \rho^P(h^{-1}(x(j)-x(i)),h_1^{-1}hh^{-1}h_2) \\
&= \rho^P(h^{-1}(x(j)-x(i)),h_1^{-1}h_2) \\
&\neq \rho((i,h_1),(j,h_2)) \quad \text{unless } h^{-1}(x(j)-x(i)) = x(j)-x(i)
\end{aligned}
$$

This means that group self-attention is not group equivariant, as is. The other proofs in the paper do in fact work, however. Moreover, if we redefine $L_h[\rho]((i,h_1),(j,h_2))$ as $\rho^P(h^{-1}(x(j)-x(i)),h_1^{-1}h_2)$ we see that equivariance does in fact hold, and that the positional encoding is $\mathcal{H}$-invariant in the group component.

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Email address: amelie.schreiber.math@gmail.com