A study and spectral analysis of the discrete Legendre polynomials

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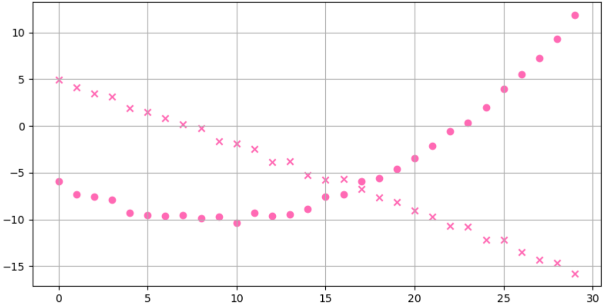
Part One: Definition of the $\mathcal{L}^{n,k}$

Fixed $n \geq 2$, a signal of dimension n will be represented with a vector of $\mathbb{R}^n$.



We are interested in geometric attributes of the signal, particularly, in knowing when it behaves like a straight line or like a parabola.

$$G_x := \{(m, x_m) : \; 0 \leq m \leq n-1\}.$$

Fixed a dimension $n \geq 2$, we are looking for a basis

$$\mathcal{L}^n = \{\mathcal{L}^{n,k} : \ \ 0 \leq k \leq n-1\}$$

of $\mathbb{R}^n$ such that

- (Size) It is an orthonormal basis of $\mathbb{R}^n$;

$$\forall x \in \mathbb{R}^n : \ \ x = \sum_{k=0}^{n-1} \langle x, \mathcal{L}^{n,k} \rangle$$

and

$$\forall x \in \mathbb{R}^n : \ \ ||x||^2 = \sum_{k=0}^{n-1} |\langle x, \mathcal{L}^{n,k} \rangle|^2$$

- (Shape) It's possible to establish a simple criteria for the shape of the graph of any signal x in terms of the coefficients $\langle x, \mathcal{L}^{n,k} \rangle$ of it with respect to $\mathcal{L}^n$.

Let $x = (x_m)_{m=0}^{n-1} \in \mathbb{R}^n$. Define al operador de discr. puntual. y a los espacios Wk

$$x \textit{ is affine} \Leftrightarrow \qquad\qquad x = (a)$$

Likewise,

$$x \text{ is constant iff } x \in W_{n,0}$$

and

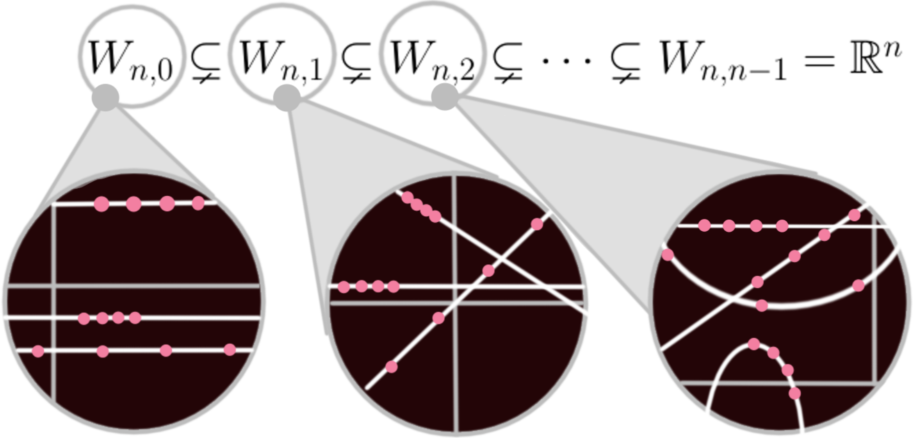
$$x \text{ is cuadratic iff } x \in W_{n,2}.$$

Proposición

Let $n \geq 2$, $0 \leq k \leq n-1$. *Let* $W_{n,k}$ *be as.*

- $dim(W_{n,k}) = k+1$,
- $\forall 0 \leq i \leq n-2$: $W_{n,i} \subseteq W_{n,i+1}$,
- $\mathbb{R}^n = W_{n,n-1}$
- *Let* $\mathcal{P} = \{t_j := t_0 + hj : \;\; 0 \leq j \leq n-1\}$ *be a uniform mesh of* n *points. For all* $x \in \mathbb{R}^n$ *and all* $0 \leq i \leq n-1$, $x \in W_{n,i}$ *if and only if there exists a polynomial* $g(x)$ *of degree at most* i *such that* $x = \Omega_{n,\mathcal{P}}(g)$.

The belonging to a subspace $W_{n,k}$ determines the shape of the graph of a signal x.



The definition of the degree of a finite signal

Proposición

Let $n \geq 2$ and $\mathcal{P}$ be a uniform mesh of n points. The function

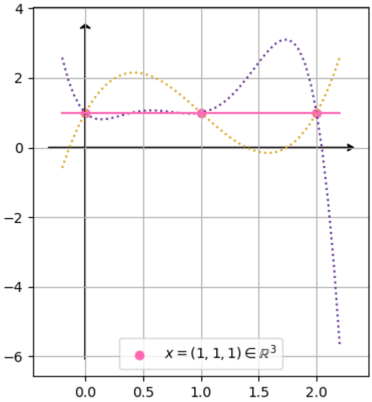
$$\Omega_{n,\mathcal{P}} : \mathbb{R}_{n-1}[t] \longrightarrow \mathbb{R}^n$$

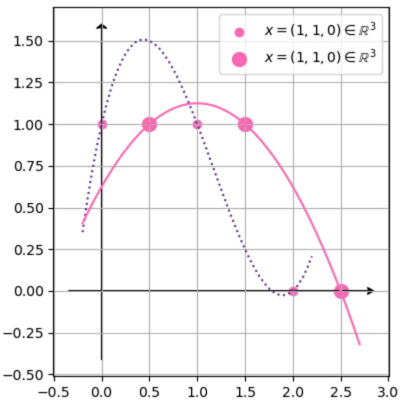
is an isomorphism of $\mathbb{R}-$vector spaces.

Notice that the mesh $\mathcal{P}$ is fixed at the beginning. Can we find two meshes $\mathcal{P}$ and $\tilde{\mathcal{P}}$ and two polynomials $f, \tilde{f}$ of degrees $0 \leq \partial(f) < partial(\tilde{f}) \leq n-1$ such that

$$\Omega_{n,\mathcal{P}}(f) = x = \Omega_{n,\tilde{\mathcal{P}}}(\tilde{f})?$$

What is sure is that we can not assure uniqueness if we remove the restriction in the degrees of the polynomials.





Proposición

Let $n \geq 2$, $x \in \mathbb{R}^n$. *If* $\mathcal{P}$ *and* $\tilde{\mathcal{P}}$ *are uniform meshes of* n *points and* $f, \tilde{f} \in \mathbb{R}_{n-1}[t]$ *are such that*

$$\Omega_{n,\mathcal{P}}(f) = x = \Omega_{n,\tilde{\mathcal{P}}}(\tilde{f}).$$

then $\partial(f) = \partial(\tilde{f})$.

Definición

Let $n \geq 2$ *and* $x \in \mathbb{R}^n$. *If* $f \in \mathbb{R}_{n-1}[t]$ *and* $\mathcal{P}$ *is a uniform mesh of* n *points such that* $x = \Omega_{n,\mathcal{P}}(f)$, *then*

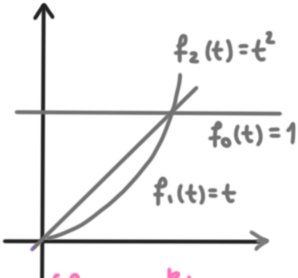
$$\partial(x) := \partial(f)$$

Relationship between the degree of a signal and the spaces $W_{n,k}$

Let $n \geq 2$ and $x \in \mathbb{R}^n$.

- x has degree 0 iff $x \in W_{n,0}$
- for all $1 \leq i \leq n-1$, x has degree i iff $x \in W_{n,i}$

Thus, $W_{n,k}$ is the space of signals of degree k.
Let's find orthonormal basis for those spaces.



$\{f_k(x)=t^k \mid 0\leq k\leq n-1\}$

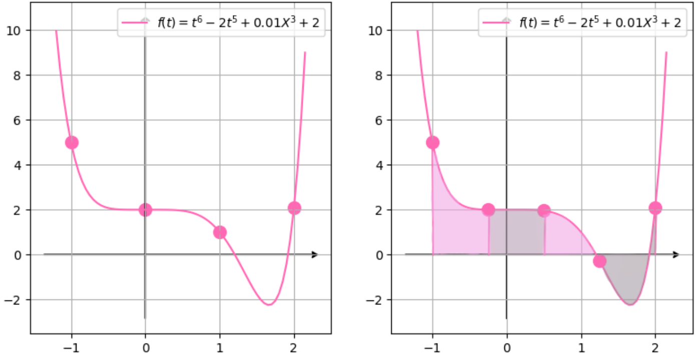
STEP I: Discretization process

$\{w_{n,k} \mid 0\leq k\leq n-1\}\subseteq \mathbb{R}^n$

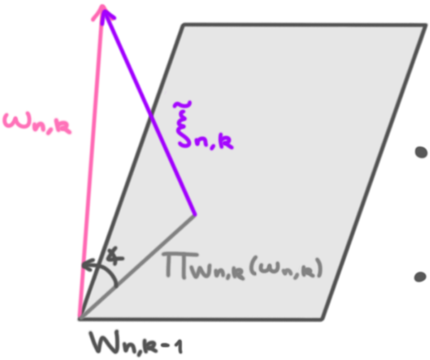
STEP II: orthonormalization process

$\{\xi_{n,k} \mid 0\leq k\leq n-1\}\subseteq \mathbb{R}^n$

DISCRETIZATION



ORTHONORMALIZATION



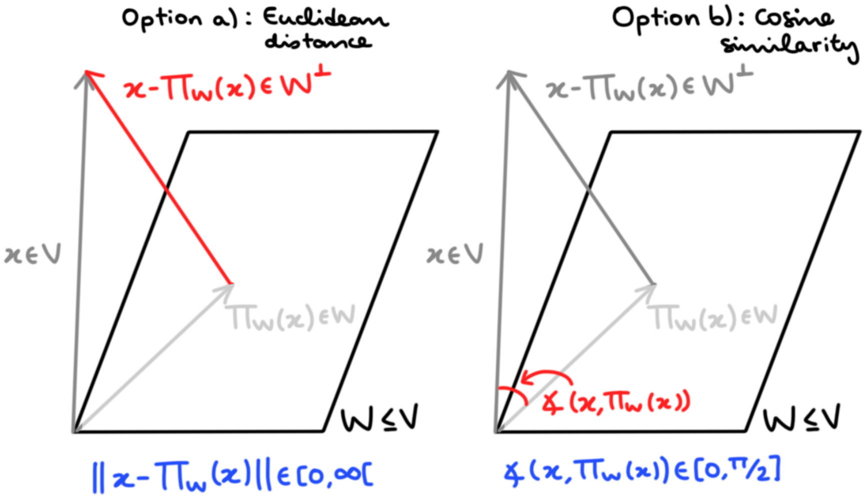
$\mathcal{L}^{n,0} = \dfrac{w_{n,0}}{\|w_{n,0}\|}$ and, for all $1 \le k \le n-1$,

- $\tilde{\xi}_{n,k} := w_{n,k} - \Pi_{W_{n,k-1}}(w_{n,k})$
  [Gram-Schmidt]
- $\mathcal{L}^{n,k} = \dfrac{\tilde{\xi}_{n,k}}{\|\tilde{\xi}_{n,k}\|}$ [Normalization]

Thus, if $x \in \mathbb{R}^n$,

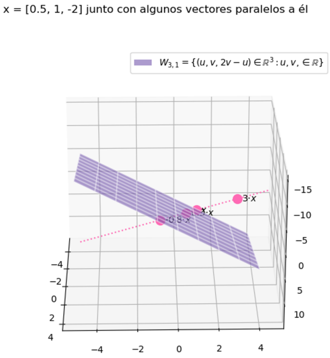
- x has degree k if and only if $x \in W_{n,k}$ (binary answer)
- We can actually measure how far apart x is from $W_{n,k}$, thus **we can measure how much x is far away from the property "having degree k"**

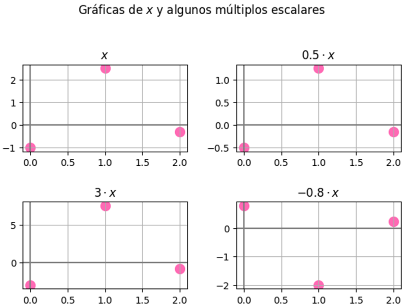
How to measure the distance to a subspace?



Why cosine similarity is better for us?

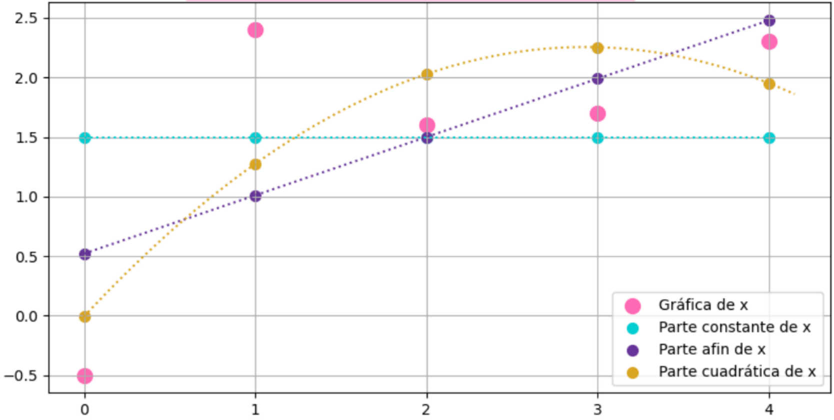
Answer: it is, as well as the shape of the graph of a signal, invariant under scalar multiplication.





Using coefficientes with respect to $\mathcal{L}^n$ to make a morphological analysis

$$x = \boxed{\boxed{\boxed{\boxed{a_0\mathcal{L}^{n,0}} + a_1\mathcal{L}^{n,1}} + a_2\mathcal{L}^{n,2}} + \dots + a_{n-1}\mathcal{L}^{n,n-1}} \in \mathbb{R}^n$$



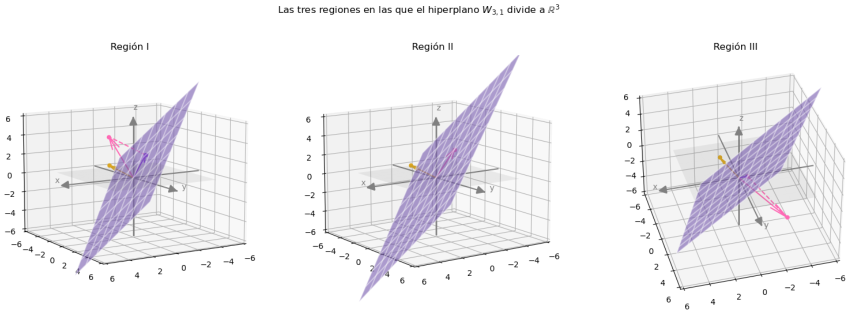
Let $x = \sum_{i=0}^{n-1} a_i \mathcal{L}^{n,i} \in \mathbb{R}^n$.

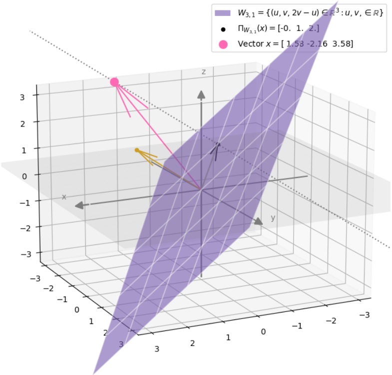
- x is approximately constant iff $a_0^2 \sim ||x||^2$
- x is approximately affine iff $a_0^2 + a_1^2 \sim ||x||^2$
- x is approximately cuadratic iff $a_0^2 + a_1^2 + a_2^2 \sim ||x||^2$

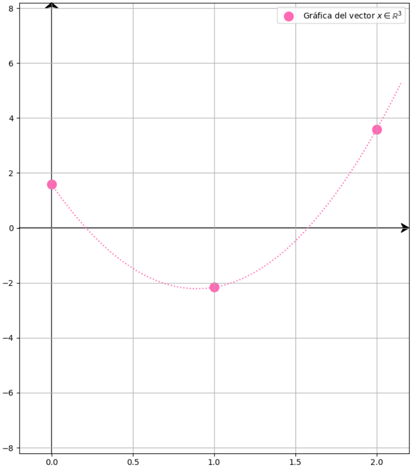
In general, x has nearly degree k iff the energy $||x||^2$ is concentrated in the first k coefficients a_k.

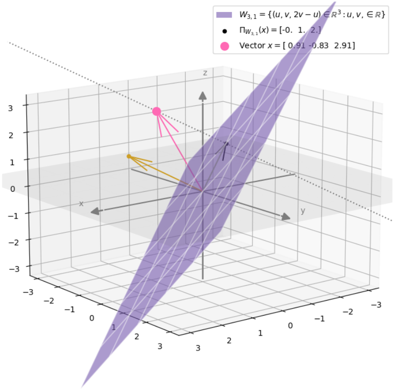
poner fórmulas

Example in $\mathbb{R}^3$




$W_{3,1} = \{(u, v, 2v - u) \in \mathbb{R}^3 : u, v, \in \mathbb{R}\}$
$\Pi_{W_{3,1}}(x) = [-0.\ \ 1.\ \ 2.]$
Vector $x = [\ 1.58\ -2.16\ \ 3.58]$
x
y
z


Gráfica del vector $x \in \mathbb{R}^3$



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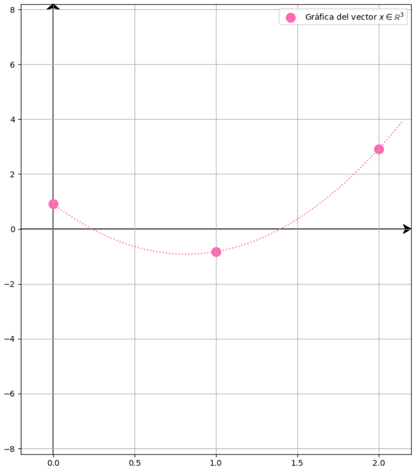
$\Pi_{W_{3,1}}(x) = $ [-0. 1. 2.]
Vector $x = $ [0.91 -0.83 2.91]



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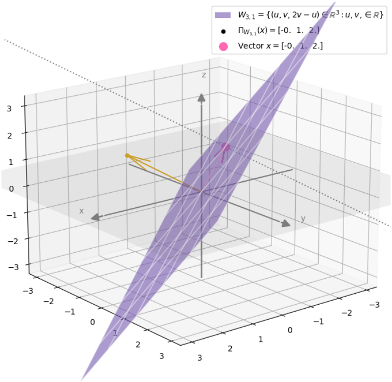
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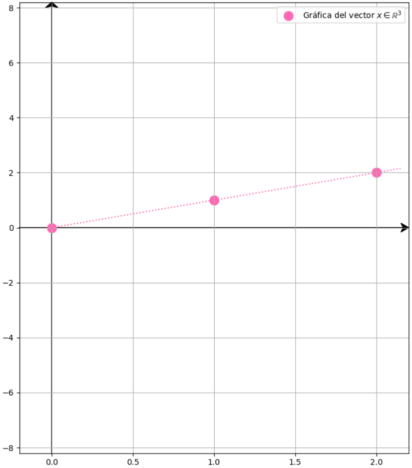
$\Pi_{W_{3,1}}(x) = [-0.\ \ 1.\ \ 2.]$
Vector $x = [-0.\ \ 1.\ \ 2.]$
z
x
y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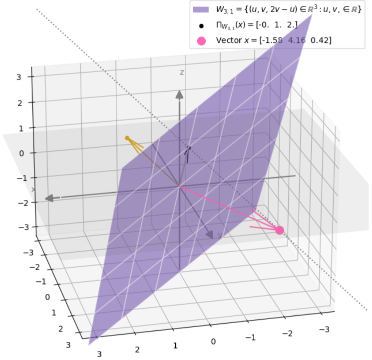
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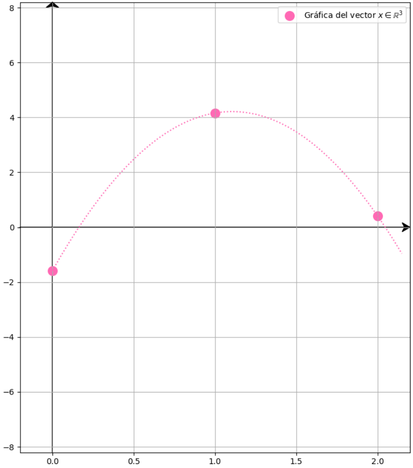
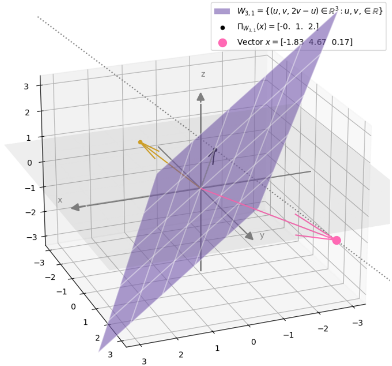
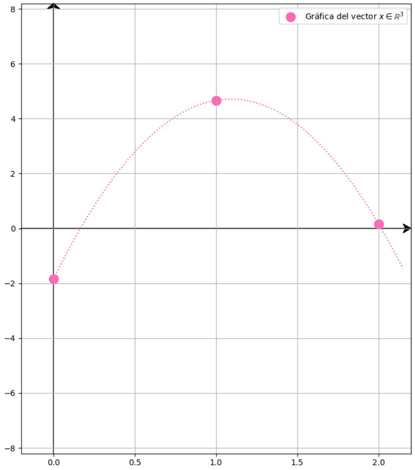
$\Pi_{W_{3,1}}(x) = [-0.\ \ 1.\ \ 2.]$
Vector $x = [-1.58\ \ 4.16\ \ 0.42]$


Gráfica del vector $x \in \mathbb{R}^3$


$W_{3,1} = \{(u, v, 2v-u) \in \mathbb{R}^3 : u, v, \in \mathbb{R}\}$
$\Pi_{W_{3,1}}(x) =$ [-0. 1. 2.]
Vector $x =$ [-1.83 4.67 0.17]
x
y
z


Gráfica del vector $x \in \mathbb{R}^3$

Relationship with the minimum square approximation method

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Part Two: Spectral analysis