## Lattice-Theoretic Framework for Data-Flow Analysis

**Last time**

- Generalizing data-flow analysis
- Reaching definitions vs. reaching constants

**Today**

- Introduce lattice-theoretic framework for data-flow analysis



## Context

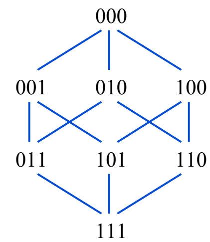
**Goals**

- Provide a single formal model that describes all data-flow analyses
- Formalize the notions of **safe**, **conservative**, and **optimistic**
- Place bounds on time complexity of data-flow analysis

**Approach**

- Define **domain** of program properties (flow values) computed by data-flow analysis, and organize the domain of elements as a **lattice**
- Define flow functions and a merge function over this domain using lattice operations
- Exploit lattice theory to achieving goals

## Lattices




**Define lattice** L = (V, ^ )

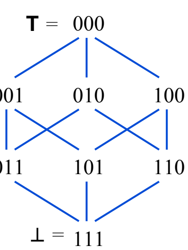
- V is a set of elements of the lattice
- ^ is a binary relation over the elements of V (**meet** or **greatest lower bound**)

**Properties of ^**

- $x,y \in V \Rightarrow x \wedge y \in V$ (closure)
- $x,y \in V \Rightarrow x \wedge y = y \wedge x$ (commutativity)
- $x,y,z \in V \Rightarrow (x \wedge y) \wedge z = x \wedge (y \wedge z)$ (associativity)



## Lattices (cont)




**Under (≤)**

- Imposes a partial order on V
- $x \le y \Leftrightarrow x \wedge y = x$

**Top (⊤)**

- A unique "greatest" element of V (if it exists)
- $\forall x \in V - \{\top\}, x < \top$

**Bottom (⊥)**

- A unique "least" element of V (if it exists)
- $\forall x \in V - \{\perp\}, \perp < x$

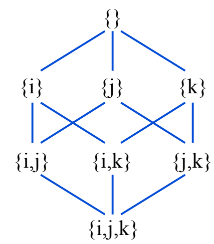
**Height of lattice L**

- The longest path through the partial order from greatest to least element (top to bottom)

## Data-Flow Analysis via Lattices

**Relationship**

- Elements of the lattice (V) represent flow values (in[] and out[] sets)
  - *e.g.*, Sets of live variables for liveness
- ⊤ represents "best-case" information (initial flow value)
  - *e.g.*, Empty set
- ⊥ represents "worst-case" information
  - *e.g.*, Universal set
- ^ (meet) merges flow values
  - *e.g.*, Set union
- If x ≤ y, then x is a conservative approximation of y
  - *e.g.*, Superset

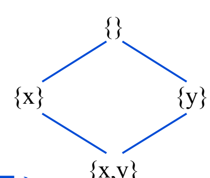




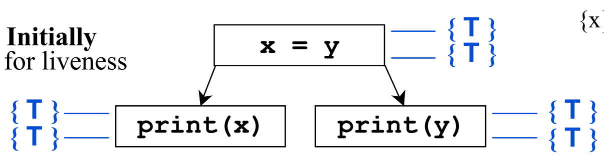

## Data-Flow Analysis via Lattices (cont)

**Remember what these flow values represent**

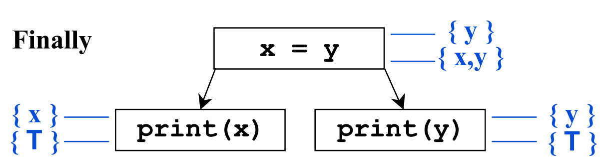
- At each program point a lattice element represents an in[] set or an out[] set




**Initially**
for liveness




**Finally**



## Data-Flow Analysis Frameworks

**Data-flow analysis framework**

- A set of **flow values** (V)
- A binary **meet operator** (^)
- A set of **flow functions** (F) (also known as **transfer functions**)

**Flow Functions**

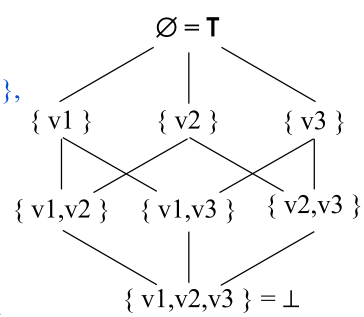
- F = {f: V→V}
  f describes how each node in CFG affects the flow values
- Flow functions map program behavior onto lattices



## Visualizing DFA Frameworks as Lattices

**Example**: Liveness analysis with 3 variables
S = {v1, v2, v3}

- V: $2^S =$ {{v1,v2,v3},
  {v1,v2},{v1,v3},{v2,v3},
  {v1},{v2},{v3}, ∅}
- Meet (^): ∪
  - ≤: ⊇
  - Top(T): ∅
  - Bottom (⊥): υ
- F: $\{f_n(X) = Gen_n \cup (X - Kill_n), \forall n\}$




Inferior solutions are lower on the lattice
More conservative solutions are lower on the lattice

## More Examples

**Reaching definitions**

- V: $2^S$ (S = set of all defs)
- ^: $\cup$
  - ≤: $\supseteq$
  - Top(T): $\varnothing$
  - Bottom (⊥): S
- F: . . .

**Reaching Constants**

- V: $2^{v \times c}$, variables v and constants c
- ^: $\cap$
  - ≤: $\subseteq$
  - Top(T): v×c
  - Bottom (⊥): $\varnothing$
- F: . . .



## Tuples of Lattices

**Problem**

- Simple analyses may require very complex lattices (*e.g.,* Reaching constants)

**Solution**

- Use a tuple of lattices, one per variable

$\mathbf{L = (V, \wedge) \;\equiv\; (L_T = (V_T, \wedge_T))^N}$

- $V = (V_T)^N$
- Meet (^): point-wise application of $\wedge_T$
- $(\ldots, v_i, \ldots) \leq (\ldots, u_i, \ldots) \equiv v_i \leq_T u_i, \forall\, i$
- Top (T): tuple of tops ($T_T$)
- Bottom (⊥): tuple of bottoms ($\perp_T$)
- Height (L) = N * height($L_T$)

## Tuples of Lattices Example

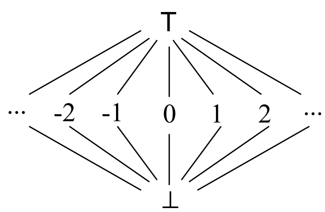
**Reaching constants (previously)**

– P = v×c, for variables v & constants c

– V: $2^P$

**Alternatively**

– V = c ∪ {⊤, ⊥}




**The whole problem is a tuple of lattices, one for each variable**



## Examples of Lattice Domains

**Two-point lattice** (⊤ and ⊥)

– Examples?

– Implementation?

**Set of incomparable values** (and ⊤ and ⊥)

– Examples?

**Powerset lattice** ($2^S$)

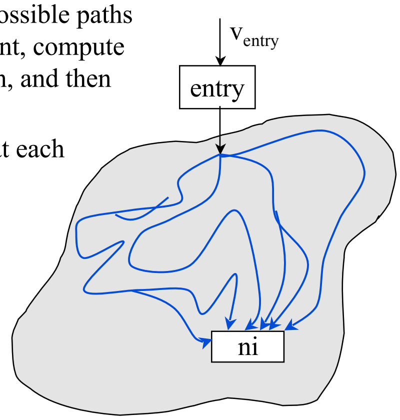
– ⊤ = ∅ and ⊥ = S, or vice versa

– Isomorphic to tuple of two-point lattices

## Solving Data-Flow Analyses

**Goal**

- For a forward problem, consider all possible paths from the entry to a given program point, compute the flow values at the end of each path, and then meet these values together
- Meet-over-all-paths (MOP) solution at each program point
- $\wedge_{\text{all paths } n1, n2, ..., ni} (f_{ni}(...f_{n2}(f_{n1}(v_{entry}))))$






## Solving Data-Flow Analyses (cont)

**Problems**

- Loops result in an infinite number of paths
- Statements following merge must be analyzed for all preceding paths
  - Exponential blow-up

**Solution**

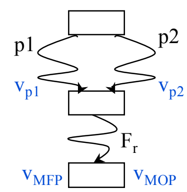
- Compute meets early (at merge points) rather than at the end
- Maximum fixed-point (MFP)

**Questions**

- Is this legal?
- Is this efficient?
- Is this accurate?

## Legality

**"Is $v_{MFP}$ legal?" $\equiv$ "Is $v_{MFP} \leq v_{MOP}$?"**




**Look at Merges**

$v_{MOP} = F_r(v_{p1}) \wedge F_r(v_{p2})$
$v_{MFP} = F_r(v_{p1} \wedge v_{p2})$
$v_{MFP} \leq v_{MOP} \equiv F_r(v_{p1} \wedge v_{p2}) \leq F_r(v_{p1}) \wedge F_r(v_{p2})$

**Observation**

**$\forall x,y \in V$**

**$f(x \wedge y) \leq f(x) \wedge f(y) \Leftrightarrow x \leq y \Rightarrow f(x) \leq f(y)$**

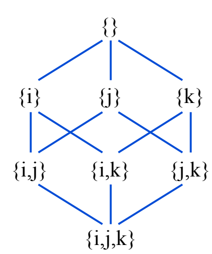
$\therefore$ $v_{MFP}$ legal when $F_r$ (really, the flow functions) are monotonic



## Monotonicity

**Monotonicity: $(\forall x,y \in V)[x \leq y \Rightarrow f(x) \leq f(y)]$**

- If the flow function f is applied to two members of V, the result of applying f to the "lesser" of the two members will be under the result of applying f to the "greater" of the two
- Giving a flow function more conservative inputs leads to more conservative outputs (never more optimistic outputs)




**Why else is monotonicity important?**

**For monotonic F over domain V**

- The maximum number of times F can be applied to self w/o reaching a fixed point is height(V) – 1
- IDFA is guaranteed to terminate if the flow functions are monotonic and the lattice has finite height

## Efficiency

**Parameters**

- n: Number of nodes in the CFG
- k: Height of lattice
- t: Time to execute one flow function

**Complexity**

- O(nkt)

**Example**

- Reaching definitions?



## Accuracy

**Distributivity**

- $f(u \wedge v) = f(u) \wedge f(v)$
- $v_{MFP} \leq v_{MOP} \equiv F_r(v_{p1} \wedge v_{p2}) \leq F_r(v_{p1}) \wedge F_r(v_{p2})$
- If the flow functions are distributive, MFP = MOP

**Examples**

- Reaching definitions?
- Reaching constants?

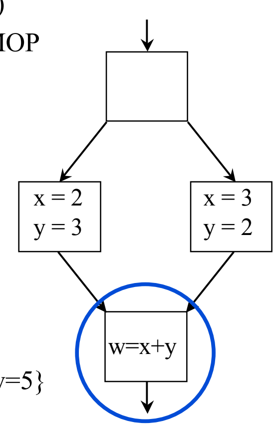
$f(u \wedge v) = f(\{x=2,y=3\} \wedge \{x=3,y=2\})$

$\quad = f(\varnothing) = \varnothing$

$f(u) \wedge f(v) = f(\{x=2,y=3\}) \wedge f(\{x=3,y=2\})$

$\quad = [\{x=2,y=3,w=5\} \wedge \{x=3,y=2,w=5\}] = \{w=5\}$

$\Rightarrow$ MFP $\neq$ MOP



## Another Example

**Integer range analysis**

- Calculate an approximation to the set of integer values that integer variables can take on
- Uses
  - Array-bounds-check elimination
  - Array access dependence testing
  - Overflow check elimination

**What is the domain?**

- What is its height?

**What are the flow functions?**

- Are they monotonic?



## Concepts

**Lattices**

- Conservative approximation
- Optimistic (initial guess)
- Data-flow analysis frameworks
- Tuples of lattices

**Data-flow analysis**

- Fixed point
- Meet-over-all-paths (MOP)
- Maximum fixed point (MFP)
- Legal/safe (monotonic)
- Efficient
- Accurate (distributive)

## Next Time

**Lecture**

– Program representations (static single assignment)