MLE Problem Set 3

Pengjia Cui

1. Binomial Likelihood

(a) Log-Likelihood Function

Given that $X \sim \mathrm{Bin}(n, p)$, the probability mass function (PMF) is:

$$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$$

Taking the natural logarithm:

$$\log \mathcal{L}(p) = \log \binom{n}{x} + x \log p + (n - x) \log(1 - p)$$

Since the binomial coefficient $\binom{n}{x}$ does not depend on p, it is omitted in differentiation:

$$\log \mathcal{L}(p) = x \log p + (n - x) \log(1 - p)$$

(b) Score Function

The **score function** is:

$$S(p) = \frac{\partial}{\partial p} \log \mathcal{L}(p) = \frac{x}{p} - \frac{n - x}{1 - p}$$

(c) Maximum Likelihood Estimator (MLE)

Setting the score function to zero:

$$\frac{x}{p} - \frac{n-x}{1-p} = 0$$

Solving for p:

$$\hat{p} = \frac{x}{n}$$

Thus, the **MLE of** p is:

$$\hat{p} = \frac{x}{n}$$

(d) Observed Fisher Information

$$I(p) = -\frac{\partial S(p)}{\partial p} = \frac{x}{p^2} + \frac{n-x}{(1-p)^2}$$

Evaluating at $\hat{p}$:

$$I(\hat{p}) = \frac{n}{\hat{p}(1-\hat{p})}$$

(e) Relation to Bernoulli Variance

For a **Bernoulli** random variable $Y \sim \mathrm{Bern}(p)$, the variance is:

$$\mathrm{Var}(Y) = p(1-p)$$

Thus, the Fisher information:

$$I(p) = \frac{n}{p(1-p)}$$

is the **inverse of the Bernoulli variance scaled by** n.

---

2. Clinton Impeachment Vote

(a) Constructing the Binary Variable

```r
impeach_data <- read.csv("impeach.csv", header = TRUE)

# Remove rows with missing values
impeach_data <- na.omit(impeach_data)

# Define binary impeachment variable
impeach_data$impch <- ifelse(impeach_data$votesum > 0, 1, 0)
```

(b) Data Summary

```r
summary(impeach_data)
```

```
    votesum         clint96         partyid          aflcio97     
 Min.   :0.000   Min.   :26.00   Min.   :0.0000   Min.   :  0.00  
 1st Qu.:0.000   1st Qu.:42.00   1st Qu.:0.0000   1st Qu.:  0.00  
 Median :2.000   Median :48.00   Median :1.0000   Median : 50.00  
 Mean   :1.844   Mean   :50.25   Mean   :0.5245   Mean   : 51.41  
 3rd Qu.:4.000   3rd Qu.:57.00   3rd Qu.:1.0000   3rd Qu.:100.00  
 Max.   :4.000   Max.   :94.00   Max.   :1.0000   Max.   :100.00  
    ccoal98           impch       
 Min.   :  0.00   Min.   :0.0000  
 1st Qu.:  0.00   1st Qu.:0.0000  
 Median : 58.00   Median :1.0000  
 Mean   : 53.16   Mean   :0.5268  
 3rd Qu.:100.00   3rd Qu.:1.0000  
 Max.   :100.00   Max.   :1.0000  
```

(c) Logistic Regression Model

```r
model1 <- glm(impch ~ partyid + clint96, data = impeach_data, family = binomial)
summary(model1)
```

```

Call:
glm(formula = impch ~ partyid + clint96, family = binomial, data = impeach_data)

Coefficients:
            Estimate Std. Error z value Pr(>|z|)    
(Intercept) 11.22019    3.25291   3.449 0.000562 ***
partyid      7.95961    1.03158   7.716 1.20e-14 ***
clint96     -0.31520    0.07542  -4.179 2.92e-05 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

(Dispersion parameter for binomial family taken to be 1)

    Null deviance: 593.487  on 428  degrees of freedom
Residual deviance:  57.753  on 426  degrees of freedom
AIC: 63.753

Number of Fisher Scoring iterations: 9
```

(d) Second Model Including Conservatism Measure

```r
model2 <- glm(impch ~ partyid + clint96 + ccoal98, data = impeach_data, family = binomial)
summary(model2)
```

```

Call:
glm(formula = impch ~ partyid + clint96 + ccoal98, family = binomial,
    data = impeach_data)

Coefficients:
```

```
            Estimate Std. Error z value Pr(>|z|)
(Intercept)  5.01373    4.15379   1.207  0.22742
partyid      5.77896    1.11994   5.160 2.47e-07 ***
clint96     -0.22278    0.08957  -2.487  0.01287 *
ccoal98      0.05572    0.01776   3.137  0.00171 **
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

(Dispersion parameter for binomial family taken to be 1)

    Null deviance: 593.487  on 428  degrees of freedom
Residual deviance:  45.216  on 425  degrees of freedom
AIC: 53.216

Number of Fisher Scoring iterations: 9
```

```r
# Compare models using Likelihood Ratio Test
lr_test <- 1 - pchisq(2 * (logLik(model2) - logLik(model1)), df = 1)
lr_test
```

```
'log Lik.' 0.0003990683 (df=4)
```

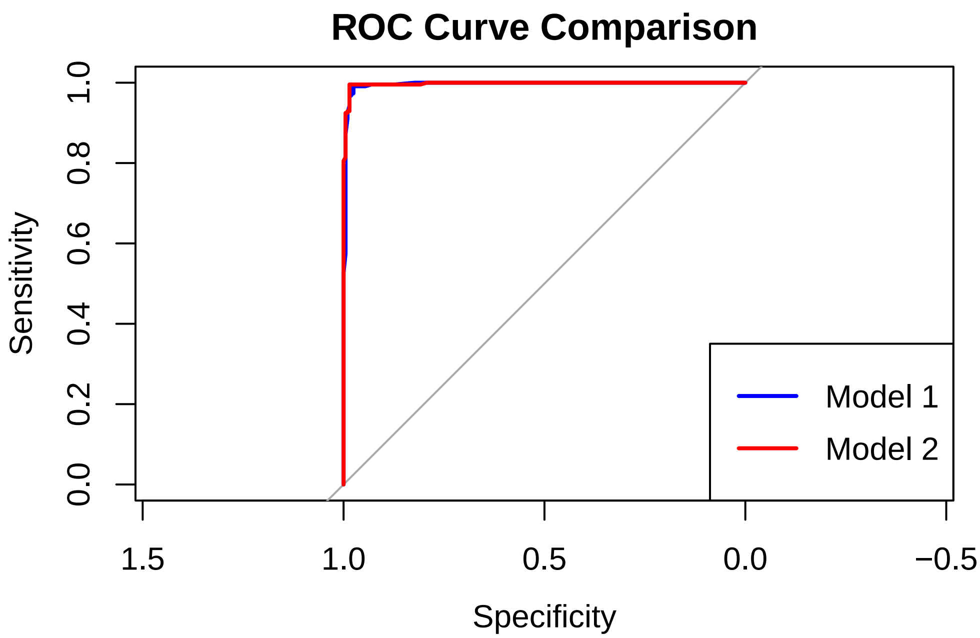
(e) ROC Curve Comparison

```r
library(pROC)

# Ensure response and predictions have the same length
predicted_prob1 <- predict(model1, type = "response")
predicted_prob2 <- predict(model2, type = "response")

roc1 <- roc(impeach_data$impch, predicted_prob1)
roc2 <- roc(impeach_data$impch, predicted_prob2)

plot(roc1, col = "blue", main = "ROC Curve Comparison", lwd = 2)
lines(roc2, col = "red", lwd = 2)
legend("bottomright", legend = c("Model 1", "Model 2"), col = c("blue", "red"), lty = 1, lwd
```




```r
auc1 <- auc(roc1)
auc2 <- auc(roc2)
print(c(AUC_Model1 = auc1, AUC_Model2 = auc2))
```

```
AUC_Model1 AUC_Model2
 0.9959022  0.9975260
```

```r
roc_test <- roc.test(roc1, roc2)
roc_test
```

```

    DeLong's test for two correlated ROC curves

data:  roc1 and roc2
Z = -0.80567, p-value = 0.4204
alternative hypothesis: true difference in AUC is not equal to 0
95 percent confidence interval:
 -0.005574273  0.002326529
sample estimates:
AUC of roc1 AUC of roc2
  0.9959022   0.9975260
```

Conclusion

- The logistic regression results confirm **party affiliation** as the strongest predictor.
- Clinton's vote share in a district also negatively correlates with impeachment support.
- The likelihood ratio test assesses whether including conservatism improves model fit.
- The ROC curve comparison evaluates the classification performance of both models.

Appendix

I certify that we did not use any LLM or generative AI tool in this assignment.