Vehicle Routing Problem

Logistics is a major industry, with some estimates valuing it at USD 8183 billion globally in 2015. Most service providers operate a number of vehicles (e.g., trucks and container ships), a number of depots, where the vehicles are based overnight, and serve a number of client locations with each vehicle during each day. There are many optimization and control problems that consider these parameters.

Computationally, the key challenge is how to design routes from depots to a number of client locations and back to the depot, so as to minimize vehicle-miles traveled, time spent, or similar objective functions.

In this project, I have formalized an idealized version of the problem and showcase its solution using the quantum approximate optimization approach of Farhi, Goldstone, and Gutmann (2014).

VRP formulation

The VRP can be formulated as:

$$(VRP) \quad f = \min_{\{x_{ij}\}_{i\sim j}\in\{0,1\},\{u_i\}_{i=1,\dots,n}\in\mathbb{R}} \quad \sum_{i\sim j} w_{ij}x_{ij}$$

subject to the node-visiting constraint:

$$\sum_{j\in\delta(i)^+} x_{ij} = 1, \ \sum_{j\in\delta(i)^-} x_{ji} = 1, \forall i \in \{1,\dots,n\},$$

the depot-visiting constraints:

$$\sum_{i\in\delta(0)^+} x_{0i} = K, \ \sum_{j\in\delta(0)^+} x_{j0} = K,$$

and the sub-tour elimination constraints:

$$u_i - u_j + Qx_{ij} \le Q - q_j, \forall i\sim j, \, i,j \ne 0, \quad q_i \le u_i \le Q, \forall i, i \ne 0.$$

In particular,

- The cost function is linear in the cost functions and weighs the different arches based on a positive weight $w_{ij} > 0$ (typically the distance between node i and node j);
- The first set of constraints enforce that from and to every client, only one link is allowed;
- The second set of constraints enforce that from and to the depot, exactly K links are allowed;
- The third set of constraints enforce the sub-tour elimination constraints and are bounds on u_i, with $Q > q_j > 0$, and $Q, q_i \in \mathbb{R}$.

Quantum solution using Quantum Approximate Optimization Algorithm

I have followed the quantum approximate optimization approach of Farhi, Goldstone, and Gutmann (2014).
The algorithm can be summarized as follows:

State Preparation:
1.First, we transform the combinatorial problem into a binary polynomial optimization problem with equality constraints only.
2.Map the resulting problem into a Ising Hamiltonian (H) for basis Z.
3.We choose a set of control parameters ‘theta’ and make a trial state |psi(theta)>.
built using a quantum circuit made of C-Phase gates and single-qubit Y rotations.

Algorithm:
1. We evaluate the expectation value of Hamiltonian in the Z-basis.
2. We use a classical optimizer to control the parameters ‘theta’ and calculate the ground state energy of the Hamiltonian.
3. Lastly, by using the optimized value of parameters, we find the ground state which describes the solution to the problem.

To solve the problem using Qiskit, we first convert the problem into a Quadratic Program before passing it to the variational circuit.

$$(IH) \quad H = \sum_{i\sim j} w_{ij}x_{ij} + A \sum_{i\in\{1,\dots,n\}} \Big(\sum_{j\in\delta(i)^+} x_{ij} - 1\Big)^2 + A \sum_{i\in\{1,\dots,n\}} \Big(\sum_{j\in\delta(i)^-} x_{ji} - 1\Big)^2 + A\Big(\sum_{i\in\delta(0)^+} x_{0i} - K\Big)^2 + A\Big(\sum_{j\in\delta(0)^+} x_{j0} - K\Big)^2$$

where A is a big enough parameter.

In the vector $\mathbf{z}$, and for a complete graph ($\delta(i)^+ = \delta(i)^- = \{0, 1, \dots, i-1, i+1, \dots, n\}$), H can be written as follows.

$$\min_{\mathbf{z}\in\{0,1\}^{n(n+1)}} \mathbf{w}^T\mathbf{z} + A \sum_{i\in\{1,\dots,n\}} \left(\mathbf{e}_i \otimes \mathbf{1}_n^T\mathbf{z} - 1\right)^2 + A \sum_{i\in\{1,\dots,n\}} \left(\mathbf{v}_i^T\mathbf{z} - 1\right)^2 + A\left((\mathbf{e}_0 \otimes \mathbf{1}_n)^T\mathbf{z} - K\right)^2 + A\left(\mathbf{v}_0^T\mathbf{z} - K\right)^2.$$

That is:

$$\min_{\mathbf{z}\in\{\mathbf{0},\mathbf{1}\}^{\mathbf{n}(\mathbf{n}+\mathbf{1})}} \mathbf{z}^{\mathbf{T}}\mathbf{Q}\mathbf{z} + \mathbf{g}^{\mathbf{T}}\mathbf{z} + \mathbf{c},$$

Where: first term:

$$\mathbf{Q} = A \sum_{i\in\{0,1,\dots,n\}} \left[(\mathbf{e}_i \otimes \mathbf{1}_n)(\mathbf{e}_i \otimes \mathbf{1}_n)^T + \mathbf{v}_i\mathbf{v}_i^T\right]$$

Second term:

$$\mathbf{g} = \mathbf{w} - 2A \sum_{i\in\{1,\dots,n\}} \left[(\mathbf{e}_i \otimes \mathbf{1}_n) + \mathbf{v}_i\right] - 2AK\left[(\mathbf{e}_0 \otimes \mathbf{1}_n) + \mathbf{v}_0\right]$$

Third term:

$$c = 2An + 2AK^2.$$

References

1.E. Farhi, J. Goldstone, S. Gutmann e-print arXiv 1411.4028, 2014
2.https://qiskit.org/documentation/optimization/tutorials/07_examples_vehicle_routing.html