Taylor and Laurent's Series

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Taylor :

For circular Region,

$$f(z) = f(a) + (z-a) f'(a) + \frac{(z-a)^2}{2!} f''(a) + \ldots\ldots$$

When all the powers of z is positive,
Then it is Taylor's series.

Laurent:

For ring type region,

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots\ldots\ldots\ldots$$

$$+ a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + \ldots\ldots$$

When the powers of z are +ve and -ve or -ve
Then it is Laurent's series.

Q. Expand $f(z) = \dfrac{1}{(z-1)(z-2)}$ in the region

(i) $0 < |z| < 1$

(ii) $1 < |z| < 2$

(iii) $|z| > 2$

Soln:

$$f(z) = \frac{1}{(z-1)(z-2)}$$

(By partial fraction)

$$= \frac{-1}{z-1} + \frac{1}{z-2} = \frac{1}{z-2} - \frac{1}{z-1}$$

(i) $0 < |z| < 1$

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1}$$

$$= \frac{1}{-2\left(1-\frac{z}{2}\right)} - \frac{1}{-1(1-z)}$$

$$= \frac{-1}{2}\left(1-\frac{z}{2}\right)^{-1} + (1-z)^{-1}$$

$$(1-D)^{-1} = 1 + D + D^2 + D^3 + \dots$$

$$= \frac{-1}{2}\left[1 + \frac{z}{2} + \frac{z^2}{2^2} + \frac{z^3}{2^3} + \dots\right] + \left[1+z+z^2+z^3+\dots\right]$$

Which is in the form of Taylor's series

(ii) $1 < |z| < 2$

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1}$$

$$= \frac{1}{-2\left(1-\frac{z}{2}\right)} - \frac{1}{z\left(1-\frac{1}{z}\right)}$$

$$= \frac{-1}{2}\left(1-\frac{z}{2}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

Expand →

$$= \frac{-1}{2}\left[1+\frac{z}{2}+\frac{z^2}{2^2}+\frac{z^3}{2^3}+\dots\right] \frac{-1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\dots\right)$$

Which is in the form of Laurent's series

(iii) $|z| > 2$

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1}$$

$$= \frac{1}{z\left(1-\frac{2}{z}\right)} - \frac{1}{z\left(1-\frac{1}{z}\right)}$$

$$= \frac{1}{z}\left(1-\frac{2}{z}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$= \frac{1}{z}\left(1+\frac{2}{z}+\frac{2^2}{z^2}+\frac{2^3}{z^3}+\dots\right) \frac{-1}{z}\left\{1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\dots\right\}$$

Which is in the form of Laurent's series

Q. Find the Laurent's series of $f(z) = \frac{7z-2}{(z+1)z(z-2)}$

in the region $1 < z+1 < 3$.

Sol^n:

$$f(z) = \frac{7z-2}{(z+1)\,z\,(z-2)}$$

Let $z+1 = u$

$\Rightarrow \quad z = u-1$

$$f(u) = \frac{7(u-1)-2}{u(u-1)(u-1-2)} = \frac{7u-9}{u(u-1)(u-3)}$$

in $1 < u < 3$

$$= \frac{-3}{u} + \frac{1}{u-1} + \frac{2}{u-3}$$

$$= \frac{-3}{u} + \frac{1}{u\left(1-\frac{1}{u}\right)} + \frac{2}{-3\left(1-\frac{u}{3}\right)}$$

$$= \frac{-3}{u} + \frac{1}{u}\left(1-\frac{1}{u}\right)^{-1} - \frac{2}{3}\left(1-\frac{u}{3}\right)^{-1}$$

$$(1-D)^{-1} = 1 + D + D^2 + D^3 + \dots\dots$$

$$= \frac{-3}{u} + \frac{1}{u}\left[1 + \frac{1}{u} + \frac{1}{u^2} + \frac{1}{u^3} + \dots\right]$$

$$- \frac{2}{3}\left[1 + \frac{u}{3} + \frac{u^2}{3^2} + \frac{u^3}{3^3} + \frac{u^4}{3^4} + \dots\right]$$

$$= \frac{-3}{u} + \frac{1}{u} + \frac{1}{u^2} + \frac{1}{u^3} + \frac{1}{u^4} + \dots$$

$$- \frac{2}{3}\left[1 + \frac{u}{3} + \frac{u^2}{3^2} + \dots\right]$$

$$= \frac{-2}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots \frac{-2}{3}\left[1 + \frac{4}{3} + \frac{4^2}{3^2} + \cdots\right]$$

$$f(z) = \frac{-2}{z+1} + \frac{1}{(z+1)^2} + \frac{1}{(z+1)^3} + \cdots \frac{-2}{3}\left[1 + \frac{(z+1)}{3} + \frac{(z+1)^2}{3^2} + \cdots\right]$$

Which is valid in the region $1 < z+1 < 3$