Type III: Application of Residues

Put $z = e^{i\theta}$

$\therefore dz = i\, e^{i\theta}\, d\theta,$

$i.e. d\theta = \frac{dz}{iz}$

$\therefore sin\theta = \frac{z^2-1}{2iz}$

$\therefore\ cos\theta = \frac{z^2+1}{2z}$

1. Using Residue theorem evaluate $\int_0^{2\pi} \frac{d\theta}{5-3cos\theta}$

[M16/ChemBiot/6M]

Solution:

We have, $I = \int_0^{2\pi} \frac{1}{5-3cos\theta} d\theta$

put $z = e^{i\theta}, d\theta = \frac{dz}{iz}, sin\theta = \frac{z^2-1}{2iz}, cos\theta = \frac{z^2+1}{2z}$

$I = \int \frac{1}{5-3\left(\frac{z^2+1}{2z}\right)} \frac{dz}{iz}$

$I = \int_c \frac{1}{\left(\frac{10z-3z^2-3}{2z}\right)} \frac{dz}{iz}$ where $C\ is\ |z| = 1$

$I = \int_c \frac{2}{i(-3z^2+10z-3)} dz$

$I = \int_c \frac{2}{-3i\left(z^2-\frac{10}{3}z+1\right)} dz$

$I = \int_c \frac{\frac{2i}{3}}{z^2-\frac{10}{3}z+1} dz$

Put $\left(z^2 - \frac{10}{3}z + 1\right) = 0$

$z = \frac{1}{3}, 3$

$\therefore z = \frac{1}{3}$ lies inside C and is a simple pole

Residue of $f(z)$ at $\left(z = \frac{1}{3}\right) = \lim_{z\to\frac{1}{3}} (z - \frac{1}{3}) \frac{\frac{2i}{3}}{\left(z-\frac{1}{3}\right)(z-3)}$

$$= \frac{\frac{2i}{3}}{\left(\frac{1}{3}-3\right)} = \frac{\frac{2i}{3}}{-\frac{8}{3}} = -\frac{i}{4}$$

Now, $\int_0^{2\pi} \frac{1}{5-3cos\theta} d\theta = 2\pi i\ [sum\ of\ residues] = 2\pi i \left[-\frac{i}{4}\right] = \frac{\pi}{2}$

2. Using Residue theorem evaluate $\int_0^{2\pi} \frac{d\theta}{5+3\sin\theta}$
[M15/AutoMechCivil/6M][M17/CompIT/6M]
Solution:

We have, $I = \int_0^{2\pi} \frac{1}{5+3\sin\theta} d\theta$

put $z = e^{i\theta}, d\theta = \frac{dz}{iz}, \sin\theta = \frac{z^2-1}{2iz}, \cos\theta = \frac{z^2+1}{2z}$

$$I = \int \frac{1}{5+3\left(\frac{z^2-1}{2iz}\right)} \frac{dz}{iz}$$

$I = \int_c \frac{1}{\left(\frac{10iz+3z^2-3}{2iz}\right)} \frac{dz}{iz}$ where C is $|z| = 1$

$$I = \int_c \frac{2}{(3z^2+10iz-3)} dz$$

$$I = \int_c \frac{2}{3\left(z^2+\frac{10i}{3}z-1\right)} dz$$

$$I = \int_c \frac{\frac{2}{3}}{z^2+\frac{10i}{3}z-1} dz$$

Put $\left(z^2 + \frac{10i}{3}z - 1\right) = 0$

$$z = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-\frac{10i}{3}\pm\sqrt{\frac{100i^2}{9}+4}}{2} = \frac{-\frac{10i}{3}\pm\sqrt{-\frac{64}{9}}}{2} = \frac{-\frac{10i}{3}\pm\frac{8i}{3}}{2} = \frac{-10i\pm8i}{6}$$

$$z = \frac{-i}{3}, -3i$$

$\therefore z = \frac{-i}{3}$ lies inside C and is a simple pole

Residue of $f(z)$ at $\left(z = \frac{-i}{3}\right) = \lim_{z\to\frac{-i}{3}} (z + \frac{i}{3}) \frac{\frac{2}{3}}{\left(z+\frac{i}{3}\right)(z+3i)}$

$$= \frac{\frac{2}{3}}{\left(\frac{-i}{3}+3i\right)} = \frac{\frac{2}{3}}{\frac{8i}{3}} = \frac{1}{4i}$$

Now, $\int_0^{2\pi} \frac{1}{5+3\sin\theta} d\theta = 2\pi i\ [sum\ of\ residues] = 2\pi i \left[\frac{1}{4i}\right] = \frac{\pi}{2}$

3. Using Residue theorem evaluate $\int_0^{\pi} \frac{d\theta}{3+2cos\theta}$
[N13/Chem/6M][M16/AutoMechCivil/6M][N17/ElexExtcElectBiomInst/6M]
Solution:

We have, $I = \int_0^{2\pi} \frac{1}{3+2cos\theta} d\theta$

put $z = e^{i\theta}, d\theta = \frac{dz}{iz}, sin\theta = \frac{z^2-1}{2iz}, cos\theta = \frac{z^2+1}{2z}$

$I = \int \frac{1}{3+2\left(\frac{z^2+1}{2z}\right)} \frac{dz}{iz}$

$I = \int_c \frac{1}{\left(\frac{3z+z^2+1}{z}\right)} \frac{dz}{iz}$ where $C\ is\ |z| = 1$

$I = \int_c \frac{1}{i(z^2+3z+1)} dz$

$I = \int_c \frac{-i}{(z^2+3z+1)} dz$

$I = \int_c \frac{-i}{(z^2+3z+1)} dz$

Put $(z^2 + 3z + 1) = 0$

$z = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-3\pm\sqrt{3^2-4}}{2} = \frac{-3\pm\sqrt{5}}{2}$

$z = \frac{-3+\sqrt{5}}{2} = \alpha \& z = \frac{-3-\sqrt{5}}{2} = \beta$

$\therefore z = \alpha$ lies inside C and is a simple pole

Residue of $f(z)$ at $(z = \alpha) = \lim_{z\to\alpha}(z-\alpha)\frac{-i}{(z-\alpha)(z-\beta)}$

$$= \frac{-i}{(\alpha-\beta)} = \frac{-i}{\frac{2\sqrt{5}}{2}} = -\frac{i}{\sqrt{5}}$$

Now, $\int_0^{2\pi} \frac{1}{3+2cos\theta} d\theta = 2\pi i\ [sum\ of\ residues] = 2\pi i \left[-\frac{i}{\sqrt{5}}\right] = \frac{2\pi}{\sqrt{5}}$

$\therefore \int_0^{\pi} \frac{d\theta}{3+2cos\theta} = \frac{1}{2}\int_0^{2\pi} \frac{1}{3+2cos\theta} d\theta = \frac{1}{2}.\frac{2\pi}{\sqrt{5}} = \frac{\pi}{\sqrt{5}}$

4. Evaluate $\int_0^{2\pi} \frac{d\theta}{(2+\cos\theta)^2}$

[N14/ChemBiot/6M][N14/ElexExtcElectBiomInst/6M]
[N15/AutoMechCivil/8M]

Solution:

We have, $I = \int_0^{2\pi} \frac{1}{(2+\cos\theta\ )^2} d\theta$

put $z = e^{i\theta}, d\theta = \frac{dz}{iz}, \sin\theta = \frac{z^2-1}{2iz}, \cos\theta = \frac{z^2+1}{2z}$

$I = \int \frac{1}{\left(2+\frac{z^2+1}{2z}\right)^2} \frac{dz}{iz}$

$I = \int_c \frac{1}{\left(\frac{4z+z^2+1}{2z}\right)^2} \frac{dz}{iz}$ where $C\ is\ |z| = 1$

$I = \int_c \frac{4z^2}{iz(z^2+4z+1)^2} dz$

$I = \int_c \frac{-4iz}{(z^2+4z+1)^2} dz$

Put $(z^2 + 4z + 1)^2 = 0$

$z = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-4\pm\sqrt{4^2-4}}{2} = \frac{-4\pm\sqrt{12}}{2} = \frac{-4\pm2\sqrt{3}}{2} = -2 \pm \sqrt{3}$

$z = -2 + \sqrt{3} = \alpha \& z = -2 - \sqrt{3} = \beta$

$\therefore z = \alpha$ lies inside C and is a pole of order 2

Residue of $f(z)$ at $(z = \alpha) = \frac{1}{(2-1)!} \lim_{z\to\alpha} \frac{d}{dz}\left[(z-\alpha)^2 . \frac{-4iz}{(z-\alpha)^2(z-\beta)^2}\right]$

$= \frac{1}{1!} \lim_{z\to\alpha} \frac{d}{dz}\left[\frac{-4iz}{(z-\beta)^2}\right]$

$= -4i \ \lim_{z\to\alpha} \left[\frac{(z-\beta)^2(1)-z\left(2(z-\beta)\right)}{(z-\beta)^4}\right]$

$= -4i\left[\frac{(\alpha-\beta)^2-2\alpha(\alpha-\beta)}{(\alpha-\beta)^4}\right]$

$= -4i\ \left[\frac{\left(2\sqrt{3}\right)^2-2\left(-2+\sqrt{3}\right)\left(2\sqrt{3}\right)}{\left(2\sqrt{3}\right)^2}\right]$

$= -4i\ \left[\frac{12+8\sqrt{3}-12}{144}\right]$

$= -\frac{2\sqrt{3}i}{9} = -\frac{2}{3\sqrt{3}} i$

Now, $\int_0^{2\pi} \frac{1}{(2+\cos\theta\ )^2} d\theta = 2\pi i\ [sum\ of\ residues] = 2\pi i\left[-\frac{2}{3\sqrt{3}} i\right] = \frac{4\pi}{3\sqrt{3}}$

5. Using Residue theorem evaluate $\int_0^{2\pi} \frac{d\theta}{5-4\sin\theta}$

Solution:

We have, $I = \int_0^{2\pi} \frac{1}{5-4\sin\theta} d\theta$

put $z = e^{i\theta}, d\theta = \frac{dz}{iz}, \sin\theta = \frac{z^2-1}{2iz}, \cos\theta = \frac{z^2+1}{2z}$

$I = \int \frac{1}{5-4\left(\frac{z^2-1}{2iz}\right)} \frac{dz}{iz}$

$I = \int_c \frac{1}{\left(\frac{10iz-4z^2+4}{2iz}\right)} \frac{dz}{iz}$ where C is $|z| = 1$

$I = \int_c \frac{2}{(-4z^2+10iz+4)} dz$

$I = \int_c \frac{2}{-4\left(z^2-\frac{5i}{2}z-1\right)} dz$

$I = \int_c \frac{-\frac{1}{2}}{z^2-\frac{5i}{2}z-1} dz$

Put $\left(z^2 - \frac{5i}{2}z - 1\right) = 0$

$z = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{\frac{5i}{2}\pm\sqrt{\frac{25i^2}{4}+4}}{2} = \frac{\frac{5i}{2}\pm\sqrt{-\frac{9}{4}}}{2} = \frac{\frac{5i}{2}\pm\frac{3i}{2}}{2} = \frac{5i\pm3i}{4}$

$z = 2i, \frac{i}{2}$

$\therefore z = \frac{i}{2}$ lies inside C and is a simple pole

Residue of $f(z)$ at $\left(z = \frac{i}{2}\right) = \lim_{z\to\frac{i}{2}} (z - \frac{i}{2}) \frac{-\frac{1}{2}}{\left(z-\frac{i}{2}\right)(z-2i)}$

$$= \frac{-\frac{1}{2}}{\left(\frac{i}{2}-2i\right)} = \frac{-\frac{1}{2}}{-\frac{3i}{2}} = \frac{1}{3i}$$

Now, $\int_0^{2\pi} \frac{1}{5-4\sin\theta} d\theta = 2\pi i\ [sum\ of\ residues] = 2\pi i \left[\frac{1}{3i}\right] = \frac{2\pi}{3}$

6. Using Residue theorem evaluate $\int_0^{2\pi}\frac{d\theta}{5+4cos\theta}$
[N17/AutoMechCivil/4M]
Solution:

We have, $I=\int_0^{2\pi}\frac{1}{5+4cos\theta}d\theta$

put $z=e^{i\theta}, d\theta=\frac{dz}{iz}, sin\theta=\frac{z^2-1}{2iz}, cos\theta=\frac{z^2+1}{2z}$

$I=\int\frac{1}{5+4\left(\frac{z^2+1}{2z}\right)}\frac{dz}{iz}$

$I=\int_c \frac{1}{\left(\frac{5z+2z^2+2}{z}\right)}\frac{dz}{iz}$ where $C\ is\ |z|=1$

$I=\int_c \frac{1}{i(2z^2+5z+2)}dz$

$I=\int_c \frac{2}{2i\left(z^2+\frac{5}{2}z+1\right)}dz$

$I=\int_c \frac{-\frac{i}{2}}{z^2+\frac{5}{2}z+1}dz$

Put $\left(z^2+\frac{5}{2}z+1\right)=0$

$z=-\frac{1}{2},-2$

$\therefore z=-\frac{1}{2}$ lies inside C and is a simple pole

Residue of $f(z)$ at $\left(z=-\frac{1}{2}\right)=\lim_{z\to-\frac{1}{2}}(z+\frac{1}{2})\frac{-\frac{i}{2}}{\left(z+\frac{1}{2}\right)(z+2)}$

$$=\frac{-\frac{i}{2}}{(-\frac{1}{2}+2)}=\frac{-\frac{i}{2}}{\frac{3}{2}}=-\frac{i}{3}$$

Now, $\int_0^{2\pi}\frac{1}{5+4cos\theta}d\theta=2\pi i\ [sum\ of\ residues]=2\pi i\left[-\frac{i}{3}\right]=\frac{2\pi}{3}$

7. Evaluate $\int_0^{2\pi} \frac{d\theta}{13-5cos\theta}$

Solution:

We have, $I = \int_0^{2\pi} \frac{1}{13-5cos\theta} d\theta$

put $z = e^{i\theta}, d\theta = \frac{dz}{iz}, sin\theta = \frac{z^2-1}{2iz}, cos\theta = \frac{z^2+1}{2z}$

$I = \int \frac{1}{13-5\left(\frac{z^2+1}{2z}\right)} \frac{dz}{iz}$

$I = \int_c \frac{1}{\left(\frac{26z-5z^2-5}{2z}\right)} \frac{dz}{iz}$ where $C\ is\ |z| = 1$

$I = \int_c \frac{2}{i(-5z^2+26z-5)} dz$

$I = \int_c \frac{-2i}{-5\left(z^2-\frac{26}{5}z+1\right)} dz$

$I = \int_c \frac{\frac{2i}{5}}{\left(z^2-\frac{26}{5}z+1\right)} dz$

Put $\left(z^2 - \frac{26}{5}z + 1\right) = 0$

$z = 5, \frac{1}{5}$

$\therefore z = \frac{1}{5}$ lies inside C and is a simple pole

Residue of $f(z)$ at $\left(z = \frac{1}{5}\right) = \lim_{z \to \frac{1}{5}} (z - \frac{1}{5}) \frac{\frac{2i}{5}}{\left(z-\frac{1}{5}\right)(z-5)}$

$$= \frac{\frac{2i}{5}}{\left(\frac{1}{5}-5\right)} = \frac{\frac{2i}{5}}{-\frac{24}{5}} = -\frac{i}{12}$$

Now, $\int_0^{2\pi} \frac{1}{13-5cos\theta} d\theta = 2\pi i\ [sum\ of\ residues] = 2\pi i \left[-\frac{i}{12}\right] = \frac{\pi}{6}$

8. **Evaluate** $\int_0^{2\pi} \frac{d\theta}{13+5cos\theta}$

Solution:

We have, $I = \int_0^{2\pi} \frac{1}{13+5cos\theta} d\theta$

put $z = e^{i\theta}, d\theta = \frac{dz}{iz}, sin\theta = \frac{z^2-1}{2iz}, cos\theta = \frac{z^2+1}{2z}$

$I = \int \frac{1}{13+5\left(\frac{z^2+1}{2z}\right)} \frac{dz}{iz}$

$I = \int_c \frac{1}{\left(\frac{26z+5z^2+5}{2z}\right)} \frac{dz}{iz}$ where $C\ is\ |z| = 1$

$I = \int_c \frac{2}{i(5z^2+26z+5)} dz$

$I = \int_c \frac{-2i}{5\left(z^2+\frac{26}{5}z+1\right)} dz$

$I = \int_c \frac{-\frac{2i}{5}}{\left(z^2+\frac{26}{5}z+1\right)} dz$

Put $\left(z^2 + \frac{26}{5}z + 1\right) = 0$

$z = -5, -\frac{1}{5}$

$\therefore z = -\frac{1}{5}$ lies inside C and is a simple pole

Residue of $f(z)$ at $\left(z = -\frac{1}{5}\right) = \lim_{z\to-\frac{1}{5}} (z + \frac{1}{5}) \frac{-\frac{2i}{5}}{\left(z+\frac{1}{5}\right)(z+5)}$

$$= \frac{-\frac{2i}{5}}{\left(-\frac{1}{5}+5\right)} = \frac{-\frac{2i}{5}}{\frac{24}{5}} = -\frac{i}{12}$$

Now, $\int_0^{2\pi} \frac{1}{13+5cos\theta} d\theta = 2\pi i\ [sum\ of\ residues] = 2\pi i \left[-\frac{i}{12}\right] = \frac{\pi}{6}$

9. Evaluate $\int_0^{2\pi} \frac{d\theta}{13+5sin\theta}$
[M15/ElexExtcElectBiomInst/6M]

Solution:

We have, $I = \int_0^{2\pi} \frac{1}{13+5sin\theta} d\theta$

put $z = e^{i\theta}, d\theta = \frac{dz}{iz}, sin\theta = \frac{z^2-1}{2iz}, cos\theta = \frac{z^2+1}{2z}$

$$I = \int \frac{1}{13+5\left(\frac{z^2-1}{2iz}\right)} \frac{dz}{iz}$$

$I = \int_c \frac{1}{\left(\frac{26iz+5z^2-5}{2iz}\right)} \frac{dz}{iz}$ where $C\ is\ |z| = 1$

$$I = \int_c \frac{2}{(5z^2+26iz-5)} dz$$

$$I = \int_c \frac{2}{5\left(z^2+\frac{26i}{5}z-1\right)} dz$$

$$I = \int_c \frac{\frac{2}{5}}{\left(z^2+\frac{26i}{5}z-1\right)} dz$$

Put $\left(z^2 + \frac{26i}{5} z - 1\right) = 0$

$$z = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-\frac{26i}{5}\pm\sqrt{\left(-\frac{26i}{5}\right)^2+4}}{2} = \frac{-\frac{26i}{5}\pm\sqrt{-\frac{576}{25}}}{2} = \frac{-\frac{26i}{5}\pm\frac{24i}{5}}{2} = \frac{-26i\pm24i}{10}$$

$\therefore z = -\frac{i}{5}, -5i$

$\therefore z = -\frac{i}{5}$ lies inside C and is a simple pole

Residue of $f(z)$ at $\left(z = -\frac{i}{5}\right) = \lim_{z\to-\frac{i}{5}} (z + \frac{i}{5}) \frac{\frac{2}{5}}{\left(z+\frac{i}{5}\right)(z+5i)}$

$$= \frac{\frac{2}{5}}{\left(-\frac{i}{5}+5i\right)} = \frac{\frac{2}{5}}{\frac{24i}{5}} = -\frac{i}{12}$$

Now, $\int_0^{2\pi} \frac{1}{13+5sin\theta} d\theta = 2\pi i\ [sum\ of\ residues] = 2\pi i \left[-\frac{i}{12}\right] = \frac{\pi}{6}$

10. Evaluate $\int_0^{2\pi} \frac{d\theta}{2+cos\theta}$ using Residue theorem
[N13/Biot/6M]
Solution:

We have, $I = \int_0^{2\pi} \frac{1}{2+cos\theta} d\theta$

put $z = e^{i\theta}, d\theta = \frac{dz}{iz}, sin\theta = \frac{z^2-1}{2iz}, cos\theta = \frac{z^2+1}{2z}$

$I = \int \frac{1}{2+\left(\frac{z^2+1}{2z}\right)} \frac{dz}{iz}$

$I = \int_c \frac{1}{\left(\frac{4z+z^2+1}{2z}\right)} \frac{dz}{iz}$ where $C\ is\ |z| = 1$

$I = \int_c \frac{2}{i(z^2+4z+1)} dz$

$I = \int_c \frac{-2i}{(z^2+4z+1)} dz$

Put $(z^2 + 4z + 1) = 0$

$z = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-4\pm\sqrt{4^2-4}}{2} = \frac{-4\pm\sqrt{12}}{2} = -2 \pm \sqrt{3}$

$z = -2 + \sqrt{3} = \alpha \& z = -2 - \sqrt{3} = \beta$

$\therefore z = \alpha$ lies inside C and is a simple pole

Residue of $f(z)$ at $(z = \alpha) = \lim_{z\to\alpha}(z - \alpha) \frac{-2i}{(z-\alpha)(z-\beta)}$

$= \frac{-2i}{(\alpha-\beta)} = \frac{-2i}{2\sqrt{3}} = -\frac{i}{\sqrt{3}}$

Now, $\int_0^{2\pi} \frac{1}{2+cos\theta} d\theta = 2\pi i\ [sum\ of\ residues] = 2\pi i \left[-\frac{i}{\sqrt{3}}\right] = \frac{2\pi}{\sqrt{3}}$

11. Evaluate $\int_0^{\pi} \frac{d\theta}{(2+cos\theta)^2}$

Solution:

We have, $I = \int_0^{2\pi} \frac{1}{(2+cos\theta\,)^2} d\theta$

put $z = e^{i\theta}, d\theta = \frac{dz}{iz}, sin\theta = \frac{z^2-1}{2iz}, cos\theta = \frac{z^2+1}{2z}$

$I = \int \frac{1}{\left(2+\frac{z^2+1}{2z}\right)^2} \frac{dz}{iz}$

$I = \int_c \frac{1}{\left(\frac{4z+z^2+1}{2z}\right)^2} \frac{dz}{iz}$ where $C\ is\ |z| = 1$

$I = \int_c \frac{4z^2}{iz(z^2+4z+1)^2} dz$

$I = \int_c \frac{-4iz}{(z^2+4z+1)^2} dz$

Put $(z^2 + 4z + 1)^2 = 0$

$z = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-4\pm\sqrt{4^2-4}}{2} = \frac{-4\pm\sqrt{12}}{2} = \frac{-4\pm2\sqrt{3}}{2} = -2 \pm \sqrt{3}$

$z = -2 + \sqrt{3} = \alpha \& z = -2 - \sqrt{3} = \beta$

$\therefore z = \alpha$ lies inside C and is a pole of order 2

Residue of $f(z)$ at $(z = \alpha) = \frac{1}{(2-1)!} \lim_{z\to\alpha} \frac{d}{dz}\left[(z-\alpha)^2 . \frac{-4iz}{(z-\alpha)^2(z-\beta)^2}\right]$

$= \frac{1}{1!} \lim_{z\to\alpha} \frac{d}{dz}\left[\frac{-4iz}{(z-\beta)^2}\right]$

$= -4i\ \ \lim_{z\to\alpha}\left[\frac{(z-\beta)^2(1)-z\big(2(z-\beta)\big)}{(z-\beta)^4}\right]$

$= -4i\left[\frac{(\alpha-\beta)^2-2\alpha(\alpha-\beta)}{(\alpha-\beta)^4}\right]$

$= -4i\ \left[\frac{\left(2\sqrt{3}\right)^2-2\left(-2+\sqrt{3}\right)\left(2\sqrt{3}\right)}{\left(2\sqrt{3}\right)^2}\right]$

$= -4i\ \left[\frac{12+8\sqrt{3}-12}{144}\right]$

$= -\frac{2\sqrt{3}i}{9} = -\frac{2}{3\sqrt{3}}i$

Now, $\int_0^{2\pi} \frac{1}{(2+cos\theta\,)^2} d\theta = 2\pi i\ [sum\ of\ residues] = 2\pi i\left[-\frac{2}{3\sqrt{3}}i\right] = \frac{4\pi}{3\sqrt{3}}$

Thus, $\int_0^{\pi} \frac{d\theta}{(2+cos\theta\,)^2} = \frac{1}{2}\int_0^{2\pi} \frac{1}{(2+cos\theta\,)^2} d\theta = \frac{1}{2}.\frac{4\pi}{3\sqrt{3}} = \frac{2\pi}{3\sqrt{3}}$

12. Evaluate $\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta} d\theta$
[M14/CompIT/6M][M14/ChemBiot/8M][N16/ChemBiot/8M]
Solution:

We have, $I = \int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta} d\theta$

$e^{i2\theta} = \cos 2\theta + i\sin 2\theta$

$\cos 2\theta$ is a real part of $e^{i2\theta}$

$I = \int_0^{2\pi} \frac{R.P.of\ e^{i2\theta}}{5+4\cos\theta} d\theta$

put $z = e^{i\theta}, d\theta = \frac{dz}{iz}, \sin\theta = \frac{z^2-1}{2iz}, \cos\theta = \frac{z^2+1}{2z}$

$I = \int \frac{R.P.of\ z^2}{5+4\left(\frac{z^2+1}{2z}\right)} \frac{dz}{iz}$

$I = R.P.of \int_c \frac{z^3}{5z+2z^2+2} . \frac{dz}{iz}$ where $C\ is\ |z| = 1$

$I = R.P.of \int_c \frac{z^3}{2\left(z^2+\frac{5}{2}z+1\right)} \frac{dz}{iz}$

$I = R.P.of \int_c \frac{-\frac{iz^2}{2}}{\left(z^2+\frac{5}{2}z+1\right)} dz$

Put $z^2 + \frac{5}{2}z + 1 = 0$

$\therefore z = -2, z = -\frac{1}{2}$

We see that, $z = -2$ lies outside C and $z = -\frac{1}{2}$ lies inside C

Residues of $f(z)$ at $\left(z = -\frac{1}{2}\right) = \lim_{z\to -\frac{1}{2}} \left(z + \frac{1}{2}\right) . \frac{-\frac{iz^2}{2}}{(z+2)\left(z+\frac{1}{2}\right)}$

$$= \frac{-\frac{i}{2}.\frac{1}{4}}{-\frac{1}{2}+2} = \frac{-\frac{i}{8}}{\frac{3}{2}} = -\frac{i}{12}$$

Now, $\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta} d\theta = R.P.of \int_0^{2\pi} \frac{e^{i2\theta}}{5+4\cos\theta} d\theta$

$= R.P.of\ 2\pi i\ [sum\ of\ residues]$

$= R.P.of\ 2\pi i \left[-\frac{i}{12}\right] = R.P.of \frac{\pi}{6}$

$= \frac{\pi}{6}$

13. Evaluate $\int_0^{2\pi}\frac{cos3\theta}{5-4cos\theta}d\theta$
[N14/AutoMechCivil/8M]
Solution:

We have, $I=\int_0^{2\pi}\frac{cos\,3\theta}{5-4cos\theta}d\theta$

$e^{i3\theta}=cos3\theta+isin3\theta$

$cos3\theta$ is a real part of $e^{i3\theta}$

$I=\int_0^{2\pi}\frac{R.P.of\ e^{i3\theta}}{5+4cos\theta}d\theta$

put $z=e^{i\theta},d\theta=\frac{dz}{iz},sin\theta=\frac{z^2-1}{2iz},cos\theta=\frac{z^2+1}{2z}$

$I=\int\frac{R.P.of\ z^3}{5-4\left(\frac{z^2+1}{2z}\right)}\frac{dz}{iz}$

$I=R.P.of\int_c\ \frac{z^4}{5z-2z^2-2}.\frac{dz}{iz}$ where $C\ is\ |z|=1$

$I=R.P.of\int_c\ \frac{z^4}{-2\left(z^2-\frac{5}{2}z+1\right)}\frac{dz}{iz}$

$I=R.P.of\int_c\ \frac{\frac{iz^3}{2}}{\left(z^2-\frac{5}{2}z+1\right)}dz$

Put $z^2-\frac{5}{2}z+1=0$

$\therefore z=2,z=\frac{1}{2}$

We see that, $z=2$ lies outside C and $z=\frac{1}{2}$ lies inside C

Residues of $f(z)$ at $\left(z=\frac{1}{2}\right)=\lim_{z\to\frac{1}{2}}\left(z-\frac{1}{2}\right).\frac{\frac{iz^3}{2}}{(z-2)\left(z-\frac{1}{2}\right)}$

$$=\frac{\frac{i}{2}.\left(\frac{1}{8}\right)}{\frac{1}{2}-2}=\frac{\frac{i}{16}}{-\frac{3}{2}}=-\frac{i}{24}$$

Now, $\int_0^{2\pi}\frac{cos\,3\theta}{5-4cos\theta}d\theta=R.P.of\int_0^{2\pi}\frac{e^{i3\theta}}{5-4cos\theta}d\theta$

$$=R.P.of\ 2\pi i\ [sum\ of\ residues]$$

$$=R.P.of\ 2\pi i\left[-\frac{i}{24}\right]=R.P.of\ \frac{\pi}{12}$$

$$=\frac{\pi}{12}$$

14. Evaluate $\int_0^{2\pi} \frac{cos3\theta}{5+4cos\theta} d\theta$
[N16/ElexExtcElectBiomInst/6M]
Solution:

We have, $I = \int_0^{2\pi} \frac{cos\,3\theta}{5+4cos\theta} d\theta$

$e^{i3\theta} = cos3\theta + isin3\theta$

$cos3\theta$ is a real part of $e^{i3\theta}$

$I = \int_0^{2\pi} \frac{R.P.of\ e^{i3\theta}}{5+4cos\theta} d\theta$

put $z = e^{i\theta}, d\theta = \frac{dz}{iz}, sin\theta = \frac{z^2-1}{2iz}, cos\theta = \frac{z^2+1}{2z}$

$I = \int \frac{R.P.of\ z^3}{5+4\left(\frac{z^2+1}{2z}\right)} \frac{dz}{iz}$

$I = R.P.of \int_c \frac{z^4}{5z+2z^2+2} . \frac{dz}{iz}$ where $C\ is\ |z| = 1$

$I = R.P.of \int_c \frac{z^4}{2\left(z^2+\frac{5}{2}z+1\right)} \frac{dz}{iz}$

$I = R.P.of \int_c \frac{-\frac{iz^3}{2}}{\left(z^2+\frac{5}{2}z+1\right)} dz$

Put $z^2 + \frac{5}{2}z + 1 = 0$

$\therefore z = -2, z = -\frac{1}{2}$

We see that, $z = -2$ lies outside C and $z = -\frac{1}{2}$ lies inside C

Residues of $f(z)$ at $\left(z = -\frac{1}{2}\right) = \lim_{z \to -\frac{1}{2}} \left(z + \frac{1}{2}\right) . \frac{-\frac{iz^3}{2}}{(z+2)\left(z+\frac{1}{2}\right)}$

$$= \frac{-\frac{i}{2}.-\frac{1}{8}}{-\frac{1}{2}+2} = \frac{\frac{i}{16}}{\frac{3}{2}} = \frac{i}{24}$$

Now, $\int_0^{2\pi} \frac{cos\,3\theta}{5+4cos\theta} d\theta = R.P.of \int_0^{2\pi} \frac{e^{i3\theta}}{5+4cos\theta} d\theta$

$$= R.P.of\ 2\pi i\ [sum\ of\ residues]$$

$$= R.P.of\ 2\pi i \left[\frac{i}{24}\right] = R.P.of - \frac{\pi}{12}$$

$$= -\frac{\pi}{12}$$

15. Evaluate $\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta} d\theta$

Solution:

We have, $I = \int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta} d\theta$

$e^{i2\theta} = \cos 2\theta + i\sin 2\theta$

$\cos 2\theta$ is a real part of $e^{i2\theta}$

$I = \int_0^{2\pi} \frac{R.P.of\ e^{2i\theta}}{5+4\cos\theta} d\theta$

put $z = e^{i\theta}, d\theta = \frac{dz}{iz}, \sin\theta = \frac{z^2-1}{2iz}, \cos\theta = \frac{z^2+1}{2z}$

$I = \int \frac{R.P.of\ z^2}{5+4\left(\frac{z^2+1}{2z}\right)} \frac{dz}{iz}$

$I = R.P.of \int_c \frac{z^3}{5z+2z^2+2} . \frac{dz}{iz}$ where $C\ is\ |z| = 1$

$I = R.P.of \int_c \frac{z^3}{2\left(z^2+\frac{5}{2}z+1\right)} \frac{dz}{iz}$

$I = R.P.of \int_c \frac{-\frac{iz^2}{2}}{\left(z^2+\frac{5}{2}z+1\right)} dz$

Put $z^2 + \frac{5}{2}z + 1 = 0$

$\therefore z = -2, z = -\frac{1}{2}$

We see that, $z = -2$ lies outside C and $z = -\frac{1}{2}$ lies inside C

Residues of $f(z)$ at $\left(z = -\frac{1}{2}\right) = \lim_{z\to -\frac{1}{2}} \left(z + \frac{1}{2}\right) . \frac{-\frac{iz^2}{2}}{(z+2)\left(z+\frac{1}{2}\right)}$

$$= \frac{-\frac{i}{2}.\left(-\frac{1}{2}\right)^2}{-\frac{1}{2}+2} = \frac{-\frac{i}{8}}{\frac{3}{2}} = \frac{-i}{12}$$

Now, $\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta} d\theta = R.P.of \int_0^{2\pi} \frac{e^{i2\theta}}{5+4\cos\theta} d\theta$

$= R.P.of\ 2\pi i\ [sum\ of\ residues]$

$= R.P.of\ 2\pi i \left[\frac{-i}{12}\right] = R.P.of\ \frac{\pi}{6}$

$= \frac{\pi}{6}$

16. Evaluate $\int_0^{2\pi} \frac{\cos^2\theta}{5+4cos\theta} d\theta$
[M16/ElexExtcElectBiomInst/6M]
Solution:

We have, $I = \int_0^{2\pi} \frac{\cos^2\theta}{5+4cos\theta} d\theta$

put $z = e^{i\theta}, d\theta = \frac{dz}{iz}, sin\theta = \frac{z^2-1}{2iz}, cos\theta = \frac{z^2+1}{2z}$

$$I = \int \frac{\left(\frac{z^2+1}{2z}\right)^2}{5+4\left(\frac{z^2+1}{2z}\right)} \frac{dz}{iz}$$

$I = \int_c \frac{(z^2+1)^2}{4z^2\left(\frac{5z+2z^2+2}{z}\right)} \frac{dz}{iz}$ where $C\ is\ |z| = 1$

$$I = \int_c \frac{(z^2+1)^2}{4iz^2(2z^2+5z+2)} dz$$

$$I = \int_c \frac{\frac{(z^2+1)^2}{8i}}{z^2\left(z^2+\frac{5}{2}z+1\right)} dz$$

Put $z^2\left(z^2+\frac{5}{2}z+1\right) = 0$

$z = 0,0$ and $z = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-\frac{5}{2}\pm\sqrt{\frac{25}{4}-4}}{2} = \frac{-\frac{5}{2}\pm\sqrt{\frac{9}{4}}}{2} = \frac{-\frac{5}{2}\pm\frac{3}{2}}{2} = \frac{-5\pm3}{4}$

$\therefore z = 0$ is a pole of order 2, $z = -\frac{1}{2}$ lies inside C, $z = -2$ lies outside C

Residue of $f(z)$ at $\left(z = -\frac{1}{2}\right) = \lim_{z\to-\frac{1}{2}}(z+\frac{1}{2}) \frac{\frac{(z^2+1)^2}{8i}}{z^2\left(z+\frac{1}{2}\right)(z+2)}$

$$= \frac{\left(\frac{1}{4}+1\right)^2}{8i.\left(-\frac{1}{2}\right)^2\left(-\frac{1}{2}+2\right)} = \frac{\frac{25}{16}}{8i.\frac{1}{4}.\frac{3}{2}} = \frac{25}{48i}$$

Residue of $f(z)$ at $(z = 0) = \frac{1}{1!}\lim_{z\to0} \frac{d}{dz}\left[z^2.\frac{\frac{(z^2+1)^2}{8i}}{z^2\left(z^2+\frac{5}{2}z+1\right)}\right]$

$$= \frac{1}{8i}\lim_{z\to0} \frac{d}{dz}\left[\frac{z^4+2z^2+1}{z^2+\frac{5}{2}z+1}\right]$$

$$= \frac{1}{8i}\lim_{z\to0}\left[\frac{\left(z^2+\frac{5}{2}z+1\right)(4z^3+4z)-(z^4+2z^2+1)\left(2z+\frac{5}{2}\right)}{\left(z^2+\frac{5}{2}z+1\right)^2}\right] = \frac{1}{8i}\left[\frac{0-\frac{5}{2}}{1^2}\right] = \frac{-5}{16i}$$

Now, $\int_0^{2\pi} \frac{\cos^2\theta}{5+4cos\theta} d\theta = 2\pi i\ [sum\ of\ residues]$

$$= 2\pi i\left[\frac{25}{48i} - \frac{5}{16i}\right] = 2\pi i\left[\frac{10}{48i}\right] = \frac{5\pi}{12}$$

17. Evaluate $\int_0^{2\pi} \frac{\sin^2\theta}{a+bcos\theta} d\theta$ where $0 < b < a$
[N15/ElexExtcElectBiomInst/6M]
Solution:

We have, $I = \int_0^{2\pi} \frac{\sin^2\theta}{a+bcos\theta} d\theta$

put $z = e^{i\theta}, d\theta = \frac{dz}{iz}, sin\theta = \frac{z^2-1}{2iz}, cos\theta = \frac{z^2+1}{2z}$

$$I = \int \frac{\left(\frac{z^2-1}{2iz}\right)^2}{a+b\left(\frac{z^2+1}{2z}\right)} \frac{dz}{iz}$$

$$I = \int_c \frac{(z^2-1)^2}{4i^2z^2\left(\frac{2az+bz^2+b}{2z}\right)} \frac{dz}{iz} \text{ where } C \text{ } is \text{ } |z| = 1$$

$$I = \int_c \frac{(z^2-1)^2}{2i^3z^2(bz^2+2az+b)} dz$$

$$I = \int_c \frac{\frac{(z^2-1)^2}{-2ib}}{z^2\left(z^2+\frac{2a}{b}z+1\right)} dz$$

Put $z^2\left(z^2 + \frac{2a}{b}z + 1\right) = 0$

$\therefore z^2 = 0, z^2 + \frac{2a}{b}z + 1 = 0$

$z = 0,0$ and $z = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-\frac{2a}{b}\pm\sqrt{\frac{4a^2}{b^2}-4}}{2} = \frac{-a\pm\sqrt{a^2-b^2}}{b}$

$\therefore z = 0$ is a pole of order 2

$z = \frac{-a+\sqrt{a^2-b^2}}{b} = \alpha$ lies inside C

$z = \frac{-a-\sqrt{a^2-b^2}}{b} = \beta$ lies outside C

Also, we see that $\alpha.\beta = \frac{-a+\sqrt{a^2-b^2}}{b} . \frac{-a-\sqrt{a^2-b^2}}{b} = \frac{a^2-a^2+b^2}{b^2} = \frac{b^2}{b^2} = 1$

$\therefore \alpha = \frac{1}{\beta}$ or $\beta = \frac{1}{\alpha}$

And, $\alpha - \beta = \frac{2\sqrt{a^2-b^2}}{b}$

Residue of $f(z)$ at $(z = \alpha) = \lim_{z\to\alpha}(z-\alpha) \frac{\frac{(z^2-1)^2}{-2ib}}{z^2(z-\alpha)(z-\beta)}$

$$= -\frac{1}{2ib} . \frac{(\alpha^2-1)^2}{\alpha^2(\alpha-\beta)}$$

$$= -\frac{1}{2ib} . \frac{(\alpha^2-1)^2}{\alpha^2} . \frac{1}{\alpha-\beta}$$

$$= -\frac{1}{2ib} . \left(\alpha - \frac{1}{\alpha}\right)^2 . \frac{1}{\alpha-\beta}$$

$$= -\frac{1}{2ib}.(\alpha-\beta)^2.\frac{1}{\alpha-\beta}$$
$$= -\frac{1}{2ib}.(\alpha-\beta)$$
$$= \frac{-\sqrt{a^2-b^2}}{ib^2}$$

Residue of $f(z)$ at $(z=0) = \frac{1}{1!}\lim_{z\to 0}\frac{d}{dz}\left[z^2.\frac{\frac{(z^2-1)^2}{-2ib}}{z^2\left(z^2+\frac{2a}{b}z+1\right)}\right]$

$$= -\frac{1}{2ib}\lim_{z\to 0}\frac{d}{dz}\left[\frac{z^4-2z^2+1}{z^2+\frac{2a}{b}z+1}\right]$$

$$= -\frac{1}{2ib}\lim_{z\to 0}\left[\frac{\left(z^2+\frac{2a}{b}z+1\right)(4z^3-4z)-(z^4-2z^2+1)\left(2z+\frac{2a}{b}\right)}{\left(z^2+\frac{2a}{b}z+1\right)^2}\right]$$
$$= -\frac{1}{2ib}\left[\frac{0-\frac{2a}{b}}{1^2}\right]$$
$$= \frac{a}{ib^2}$$

Now, $\int_0^{2\pi}\frac{\sin^2\theta}{a+bcos\theta}d\theta = 2\pi i\,[sum\ of\ residues]$

$$= 2\pi i\left[\frac{a}{ib^2}-\frac{\sqrt{a^2-b^2}}{ib^2}\right]$$
$$= \frac{2\pi}{b^2}\left(a-\sqrt{a^2-b^2}\right)$$

18. Evaluate $\int_0^{2\pi} \frac{d\theta}{25-16\cos^2\theta}$
[N13/AutoMechCivil/8M]
Solution:
We have, $I = \int_0^{2\pi} \frac{d\theta}{25-16\cos^2\theta}$
put $z = e^{i\theta}, d\theta = \frac{dz}{iz}, sin\theta = \frac{z^2-1}{2iz}, cos\theta = \frac{z^2+1}{2z}$

$$I = \int \frac{1}{25-16\left(\frac{z^2+1}{2z}\right)^2}\frac{dz}{iz}$$

$$I = \int_c \frac{1}{25-4\left(\frac{(z^2+1)^2}{z^2}\right)}\frac{dz}{iz} \text{ where } C \ is \ |z| = 1$$

$$I = \int_c \frac{z^2}{25z^2-4(z^2+1)^2}\frac{dz}{iz}$$

$$I = \int_c \frac{\frac{z}{-4i}}{(z^2+1)^2-\frac{25}{4}z^2}dz$$

Put $(z^2+1)^2 - \frac{25}{4}z^2 = 0$

$$\left(z^2+1-\frac{5}{2}z\right)\left(z^2+1+\frac{5}{2}z\right) = 0$$

$$z^2 - \frac{5}{2}z + 1 = 0, z^2 + \frac{5}{2}z + 1 = 0$$

$$z = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{\frac{5}{2}\pm\sqrt{\frac{25}{4}-4}}{2} = \frac{\frac{5}{2}\pm\frac{3}{2}}{2} = 2 \ or \ \frac{1}{2}$$

$$z = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-\frac{5}{2}\pm\sqrt{\frac{25}{4}-4}}{2} = \frac{-\frac{5}{2}\pm\frac{3}{2}}{2} = -2 \ or -\frac{1}{2}$$

$\therefore z = \frac{1}{2}$ and $z = -\frac{1}{2}$ are simple poles

Residue of $f(z)$ at $\left(z = \frac{1}{2}\right) = \lim_{z\to\frac{1}{2}}(z-\frac{1}{2})\frac{\frac{z}{-4i}}{(z-2)\left(z-\frac{1}{2}\right)(z+2)\left(z+\frac{1}{2}\right)}$

$$= \frac{\frac{\frac{1}{2}}{-4i}}{\left(\frac{1}{2}-2\right)\left(\frac{1}{2}+2\right)\left(\frac{1}{2}+\frac{1}{2}\right)} = \frac{-\frac{1}{8i}}{\left(-\frac{3}{2}\right)\left(\frac{5}{2}\right)(1)} = \frac{-\frac{1}{8i}}{-\frac{15}{4}} = \frac{1}{30i}$$

Residue of $f(z)$ at $\left(z = \frac{-1}{2}\right) = \lim_{z\to-\frac{1}{2}}(z+\frac{1}{2})\frac{\frac{z}{-4i}}{(z-2)\left(z-\frac{1}{2}\right)(z+2)\left(z+\frac{1}{2}\right)}$

$$= \frac{\frac{-\frac{1}{2}}{-4i}}{\left(-\frac{1}{2}-2\right)\left(-\frac{1}{2}+2\right)\left(-\frac{1}{2}-\frac{1}{2}\right)} = \frac{\frac{1}{8i}}{\left(\frac{3}{2}\right)\left(-\frac{5}{2}\right)(-1)} = \frac{\frac{1}{8i}}{\frac{15}{4}} = \frac{1}{30i}$$

Now, $\int_0^{2\pi} \frac{d\theta}{25-16\cos^2\theta} = 2\pi i \ [sum \ of \ residues] = 2\pi i\left[\frac{1}{30i}+\frac{1}{30i}\right] = \frac{2\pi}{15}$

19. Evaluate $\int_0^{2\pi} \frac{d\theta}{1+a\cos\theta}$, $a^2 < 1$

Solution:

We have, $I = \int_0^{2\pi} \frac{d\theta}{1+a\cos\theta}$

put $z = e^{i\theta}, d\theta = \frac{dz}{iz}, \sin\theta = \frac{z^2-1}{2iz}, \cos\theta = \frac{z^2+1}{2z}$

$I = \int \frac{1}{1+a\left(\frac{z^2+1}{2z}\right)} \frac{dz}{iz}$

$I = \int_c \frac{2z}{2z+az^2+a} \frac{dz}{iz}$ where $C\ is\ |z| = 1$

$I = \int_c \frac{2z}{az^2+2z+a} \frac{dz}{iz}$

$I = \int_c \frac{\frac{2}{ai}}{z^2+\frac{2}{a}z+1} dz$

Put $z^2 + \frac{2}{a}z + 1 = 0$

$z = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-\frac{2}{a}\pm\sqrt{\frac{4}{a^2}-4}}{2} = \frac{-1\pm\sqrt{1-a^2}}{a}$

We see that, $z = \frac{-1+\sqrt{1-a^2}}{a} = \alpha$ lies inside C

And $z = \frac{-1-\sqrt{1-a^2}}{a} = \beta$ lies outside C

Residues of $f(z)$ at $(z = \alpha) = \lim_{z\to\alpha}(z-\alpha).\frac{\frac{2}{ai}}{(z-\alpha)(z-\beta)}$

$$= \frac{\frac{2}{ai}}{\alpha-\beta} = \frac{2}{(ai)\left(\frac{2\sqrt{1-a^2}}{a}\right)} = \frac{1}{i\sqrt{1-a^2}}$$

Now, $\int_0^{2\pi} \frac{d\theta}{1+a\cos\theta} = 2\pi i\ [sum\ of\ residues]$

$$= 2\pi i\left[\frac{1}{i\sqrt{1-a^2}}\right] = \frac{2\pi}{\sqrt{1-a^2}}$$

20. Show by method of residue that $\int_0^{\pi} \frac{a}{a^2+\sin^2\theta} d\theta = \frac{\pi}{\sqrt{1+a^2}}$, $0 < a < 1$

Solution:

We have, $I = \int_0^{2\pi} \frac{a}{a^2+\sin^2\theta} d\theta$

put $z = e^{i\theta}, d\theta = \frac{dz}{iz}, sin\theta = \frac{z^2-1}{2iz}, cos\theta = \frac{z^2+1}{2z}$

$I = \int \frac{a}{a^2+\left(\frac{z^2-1}{2iz}\right)^2} \frac{dz}{iz}$

$I = \int_c \frac{4i^2z^2a}{4i^2z^2a^2+(z^2-1)^2} \frac{dz}{iz}$ where $C\ is\ |z| = 1$

$I = \int_c \frac{4iaz}{(z^2-1)^2-4a^2z^2} dz$

Put $(z^2-1)^2 - 4a^2z^2 = 0$

$\therefore (z^2-1-2az)(z^2-1+2az) = 0$

$\therefore z^2 - 2az - 1 = 0 \Rightarrow z = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{2a\pm\sqrt{4a^2+4}}{2} = a \pm \sqrt{a^2+1}$

$\therefore z^2 + 2az - 1 = 0 \Rightarrow z = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-2a\pm\sqrt{4a^2+4}}{2} = -a \pm \sqrt{a^2+1}$

We see that, $z = a + \sqrt{a^2+1} = \alpha$ lies outside C

And $z = a - \sqrt{a^2+1} = \beta$ lies inside C

And $z = -a + \sqrt{a^2+1} = \gamma$ lies inside C

And $z = -a - \sqrt{a^2+1} = \delta$ lies outside C

Residues of $f(z)$ at $(z = \beta) = \lim_{z\to\beta}(z-\beta).\frac{4iaz}{(z-\alpha)(z-\beta)(z-\gamma)(z-\delta)}$

$= \frac{4ia\beta}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} = \frac{4ia\left(a-\sqrt{a^2+1}\right)}{\left(-2\sqrt{a^2+1}\right)\left(2a-2\sqrt{a^2+1}\right)(2a)}$

$= -\frac{i}{2\sqrt{a^2+1}}$

Residues of $f(z)$ at $(z = \gamma) = \lim_{z\to\gamma}(z-\gamma).\frac{4iaz}{(z-\alpha)(z-\beta)(z-\gamma)(z-\delta)}$

$= \frac{4ia\gamma}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} = \frac{4ia\left(-a+\sqrt{a^2+1}\right)}{(-2a)\left(-2a+2\sqrt{a^2+1}\right)\left(2\sqrt{a^2+1}\right)}$

$= -\frac{i}{2\sqrt{a^2+1}}$

Now, $\int_0^{2\pi} \frac{a}{a^2+\sin^2\theta} d\theta = 2\pi i\ [sum\ of\ residues]$

$= 2\pi i\left[-\frac{i}{2\sqrt{a^2+1}} - \frac{i}{2\sqrt{a^2+1}}\right] = \frac{\pi}{\sqrt{a^2+1}}$

21. Evaluate using residues: $\int_0^{2\pi} \frac{\cos 2\theta}{1-2a\cos\theta+a^2} d\theta$ **,** $-1 < a < 1$

Solution:

We have, $I = \int_0^{2\pi} \frac{\cos 2\theta}{1-2a\cos\theta + a^2} d\theta$

$e^{i2\theta} = \cos 2\theta + i\sin 2\theta$

$\cos 2\theta$ is a real part of $e^{i2\theta}$

$I = \int_0^{2\pi} \frac{R.P.of\ e^{i2\theta}}{1-2a\cos\theta + a^2} d\theta$

put $z = e^{i\theta}, d\theta = \frac{dz}{iz}, \sin\theta = \frac{z^2-1}{2iz}, \cos\theta = \frac{z^2+1}{2z}$

$I = \int \frac{R.P.of\ z^2}{1-2a\left(\frac{z^2+1}{2z}\right)+a^2} \frac{dz}{iz}$

$I = R.P.of \int_c \frac{z^3}{z-az^2-a+a^2z} \frac{dz}{iz}$ where $C\ is\ |z| = 1$

$I = R.P.of \int_c \frac{z^3}{-a\left(z^2-az-\frac{1}{a}z+1\right)} \frac{dz}{iz}$

$I = R.P.of \int_c \frac{\frac{z^2}{-ia}}{\left(z^2-az-\frac{1}{a}z+1\right)} dz$

Put $z^2 - az - \frac{1}{a}z + 1 = 0$

$\therefore z(z-a) - \frac{1}{a}(z-a) = 0$

$\therefore (z-a)\left(z-\frac{1}{a}\right) = 0$

$\therefore z = a, z = \frac{1}{a}$

We see that, $z = a$ lies inside C and $z = \frac{1}{a}$ lies outside C

Residues of $f(z)$ at $(z = a) = \lim_{z\to a}(z-a).\frac{\frac{z^2}{-ia}}{(z-a)\left(z-\frac{1}{a}\right)}$

$$= \frac{\left(\frac{a^2}{-ia}\right)}{a-\frac{1}{a}} = \frac{\frac{ia}{a^2-1}}{a} = \frac{ia^2}{a^2-1}$$

Now, $\int_0^{2\pi} \frac{\cos 2\theta}{1-2a\cos\theta + a^2} d\theta = R.P.of \int_0^{2\pi} \frac{e^{i2\theta}}{1-2a\cos\theta + a^2} d\theta$

$$= R.P.of\ 2\pi i\ [sum\ of\ residues]$$

$$= R.P.of\ 2\pi i \left[\frac{ia^2}{a^2-1}\right] = R.P.of\ \frac{2\pi a^2}{1-a^2}$$

$$= \frac{2\pi a^2}{1-a^2}$$

22. Using calculus of residues prove that $\int_0^{2\pi} e^{cos\theta}\cos(sin\theta - n\theta)\,d\theta = \frac{2\pi}{n!}$
[M14/ElexExtcElectBiomInst/6M]

Solution:

We have, $I = \int_0^{2\pi} e^{cos\theta}\ \cos(sin\theta - n\theta)\ d\theta$

$I = \int_0^{2\pi} e^{cos\theta}\ \ Real\ part\ of\ e^{isin\theta\ -in\theta}\, d\theta$

$I = RP\ of\ \int_0^{2\pi} e^{cos\theta\ +isin\theta}\,.e^{-in\theta}\, d\theta$

$I = RP\ of\ \int_0^{2\pi} \frac{e^{e^{i\theta}}}{\left(e^{i\theta}\right)^n}\, d\theta$

Put $e^{i\theta} = z, \therefore ie^{i\theta}\, d\theta = dz, \therefore d\theta = \frac{dz}{iz}$

$I = RP\ of\ \int_c \frac{e^z}{z^n}\frac{dz}{iz}$ where $C\ is\ |z| = 1$

$I = RP\ of\ \int_c \frac{e^z}{iz^{n+1}}\, dz$

Put $z^{n+1} = 0, \therefore z = 0$ is a pole of order $n+1$

Residue of $f(z)$ at $(z = 0) = \frac{1}{n!}\lim_{z\to 0}\frac{d^n}{dz^n}\left[(z-0)^{n+1}\frac{e^z}{iz^{n+1}}\right]$

$= \frac{1}{in!}\lim_{z\to 0} e^z$

$= \frac{1}{in!}$

Now, $\int_0^{2\pi} e^{cos\theta}\ \cos(sin\theta - n\theta)\ d\theta = R.P.of \int_0^{2\pi} e^{cos\theta\ +isin\theta}\,.e^{-in\theta}\, d\theta$

$= R.P.of\ 2\pi i\ [sum\ of\ residues]$

$= R.P.of\ 2\pi i\left[\frac{1}{in!}\right] = R.P.of\ \frac{2\pi}{n!}$

$= \frac{2\pi}{n!}$

23. Evaluate $\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}$ where $a > b > 0$, using residues.

Solution:

We have, $I = \int_0^{2\pi} \frac{1}{a+b\cos\theta} d\theta$

put $z = e^{i\theta}, d\theta = \frac{dz}{iz}, \sin\theta = \frac{z^2-1}{2iz}, \cos\theta = \frac{z^2+1}{2z}$

$$I = \int \frac{1}{a+b\left(\frac{z^2+1}{2z}\right)} \frac{dz}{iz}$$

$$I = \int_c \frac{1}{\left(\frac{2az+bz^2+b}{2z}\right)} \frac{dz}{iz} \text{ where } C \; is \; |z| = 1$$

$$I = \int_c \frac{\frac{2}{i}}{(bz^2+2az+b)} dz$$

$$I = \int_c \frac{\frac{2}{bi}}{\left(z^2+\frac{2a}{b}z+1\right)} dz$$

Put $\left(z^2 + \frac{2a}{b} z + 1\right) = 0$

$\therefore \; z^2 + \frac{2a}{b} z + 1 = 0$

$$z = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-\frac{2a}{b} \pm \sqrt{\frac{4a^2}{b^2}-4}}{2} = \frac{-a \pm \sqrt{a^2-b^2}}{b}$$

$z = \frac{-a+\sqrt{a^2-b^2}}{b} = \alpha$ lies inside C

$z = \frac{-a-\sqrt{a^2-b^2}}{b} = \beta$ lies outside C. And, $\alpha - \beta = \frac{2\sqrt{a^2-b^2}}{b}$

Residue of $f(z)$ at $(z = \alpha) = \lim_{z\to\alpha} (z-\alpha) \frac{\frac{2}{bi}}{(z-\alpha)(z-\beta)}$

$$= \frac{2}{bi(\alpha-\beta)} = \frac{2}{bi\left[\frac{2\sqrt{a^2-b^2}}{b}\right]} = \frac{1}{i\sqrt{a^2-b^2}}$$

Now, $\int_0^{2\pi} \frac{1}{a+b\cos\theta} d\theta = 2\pi i \; [sum \; of \; residues]$

$$= 2\pi i \left[\frac{1}{i\sqrt{a^2-b^2}}\right]$$

$$= \frac{2\pi}{\sqrt{a^2-b^2}}$$

24. Evaluate $\int_0^{2\pi} \frac{sin^2\theta}{5-4cos\theta} d\theta$

Solution:

We have, $I = \int_0^{2\pi} \frac{\sin^2\theta}{5-4cos\theta} d\theta$

put $z = e^{i\theta}, d\theta = \frac{dz}{iz}, sin\theta = \frac{z^2-1}{2iz}, cos\theta = \frac{z^2+1}{2z}$

$$I = \int \frac{\left(\frac{z^2-1}{2iz}\right)^2}{5-4\left(\frac{z^2+1}{2z}\right)} \frac{dz}{iz}$$

$$I = \int_c \frac{(z^2-1)^2}{4i^2z^2\left(\frac{10z-4z^2-4}{2z}\right)} \frac{dz}{iz}$$ where $C\ is\ |z| = 1$

$$I = \int_c \frac{(z^2-1)^2}{2i^3z^2(-4z^2+10z-4)} dz$$

$$I = \int_c \frac{\frac{(z^2-1)^2}{8i}}{z^2\left(z^2-\frac{5}{2}z+1\right)} dz$$

Put $z^2\left(z^2 - \frac{5}{2}z + 1\right) = 0$

$\therefore z^2 = 0, z^2 - \frac{5}{2}z + 1 = 0$

$z = 0,0$ and $z = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{\frac{5}{2}\pm\sqrt{\frac{25}{4}-4}}{2} = \frac{\frac{5}{2}\pm\frac{3}{2}}{2} = \frac{5\pm3}{4} = 2, \frac{1}{2}$

$\therefore z = 0$ is a pole of order 2, $z = \frac{1}{2}$ lies inside C, $z = 2$ lies outside C

Residue of $f(z)$ at $\left(z = \frac{1}{2}\right) = \lim_{z\to\frac{1}{2}}(z - \frac{1}{2}) \frac{\frac{(z^2-1)^2}{8i}}{z^2\left(z-\frac{1}{2}\right)(z-2)}$

$$= \frac{1}{8i}.\frac{\left(\left(\frac{1}{2}\right)^2-1\right)^2}{\left(\frac{1}{2}\right)^2\left(\frac{1}{2}-2\right)} = \frac{1}{8i}.\frac{\left(\frac{9}{16}\right)}{\left(\frac{1}{4}\right)\left(-\frac{3}{2}\right)} = -\frac{3}{16i}$$

Residue of $f(z)$ at $(z = 0) = \frac{1}{1!}\lim_{z\to0} \frac{d}{dz}\left[z^2.\frac{\frac{(z^2-1)^2}{8i}}{z^2\left(z^2-\frac{5}{2}z+1\right)}\right]$

$$= \frac{1}{8i}\lim_{z\to0} \frac{d}{dz}\left[\frac{z^4-2z^2+1}{z^2-\frac{5}{2}z+1}\right]$$

$$= \frac{1}{8i}\lim_{z\to0}\left[\frac{\left(z^2-\frac{5}{2}z+1\right)(4z^3-4z)-(z^4-2z^2+1)\left(2z-\frac{5}{2}\right)}{\left(z^2-\frac{5}{2}z+1\right)^2}\right] = \frac{1}{8i}\left[\frac{0+\frac{5}{2}}{1^2}\right] = \frac{5}{16i}$$

Now, $\int_0^{2\pi} \frac{\sin^2\theta}{5-4cos\theta} d\theta = 2\pi i\ [sum\ of\ residues] = 2\pi i\left[-\frac{3}{16i} + \frac{5}{16i}\right] = \frac{\pi}{4}$

25. State the residue theorem. Hence evaluate $\int_0^{2\pi}\frac{d\theta}{3-2cos\theta+sin\theta}$

Solution:

We have, $I = \int_0^{2\pi}\frac{1}{3-2cos\theta + sin\theta}d\theta$

put $z = e^{i\theta}, d\theta = \frac{dz}{iz}, sin\theta = \frac{z^2-1}{2iz}, cos\theta = \frac{z^2+1}{2z}$

$I = \int \frac{1}{3-2\left(\frac{z^2+1}{2z}\right)+\left(\frac{z^2-1}{2iz}\right)}\frac{dz}{iz}$

$I = \int_c \frac{1}{\frac{6iz-2iz^2-2i+z^2-1}{2iz}}\frac{dz}{iz}$ where $C\ is\ |z| = 1$

$I = \int_c \frac{2}{(1-2i)z^2+6iz-(1+2i)}dz$

Put $(1-2i)z^2+6iz-(1+2i) = 0$

$z = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-6i\pm\sqrt{36i^2+4(1-2i)(1+2i)}}{2(1-2i)} = \frac{-6i\pm\sqrt{36i^2+4-16i^2}}{2(1-2i)} = \frac{-6i\pm\sqrt{-16}}{2(1-2i)}$

$z = \frac{-6i\pm 4i}{2(1-2i)} = \frac{-3i\pm 2i}{1-2i}$

$z = -\frac{i}{1-2i}, z = -\frac{5i}{1-2i}$

$z = -\frac{i(1+2i)}{1-4i^2}, z = -\frac{5i(1+2i)}{1-4i^2}$

$z = \frac{-i-2i^2}{5}, z = \frac{-5i-10i^2}{5}$

$z = \frac{2}{5}-\frac{i}{5}, z = 2-i$

$z = \frac{2}{5}-\frac{i}{5} = \alpha$ lies inside C

$z = 2-i = \beta$ lies outside C

And, $\alpha-\beta = -\frac{8}{5}+\frac{4i}{5}$

Residue of $f(z)$ at $(z=\alpha) = \lim_{z\to\alpha}(z-\alpha)\frac{2}{(1-2i)(z-\alpha)(z-\beta)}$

$$= \frac{2}{(1-2i)(\alpha-\beta)}$$

$$= \frac{2}{(1-2i)\left(-\frac{8}{5}+\frac{4i}{5}\right)} = \frac{10}{-8+4i+16i-8i^2}$$

$$= \frac{10}{20i} = \frac{1}{2i}$$

Now, $\int_0^{2\pi}\frac{1}{3-2cos\theta + sin\theta}d\theta = 2\pi i\ [sum\ of\ residues]$

$$= 2\pi i\left[\frac{1}{2i}\right]$$

$$= \pi$$

26. Evaluate $\int_0^{2\pi} \frac{d\theta}{a+bsin\theta}$ where $a > b$, using residues.

Solution:

We have, $I = \int_0^{2\pi} \frac{1}{a+bsin\theta} d\theta$

put $z = e^{i\theta}, d\theta = \frac{dz}{iz}, sin\theta = \frac{z^2-1}{2iz}, cos\theta = \frac{z^2+1}{2z}$

$$I = \int \frac{1}{a+b\left(\frac{z^2-1}{2iz}\right)} \frac{dz}{iz}$$

$$I = \int_c \frac{1}{\left(\frac{2iaz+bz^2-b}{2iz}\right)} \frac{dz}{iz} \text{ where } C \text{ is } |z| = 1$$

$$I = \int_c \frac{2}{(bz^2+2aiz-b)} dz$$

$$I = \int_c \frac{\frac{2}{b}}{\left(z^2+\frac{2ai}{b}z-1\right)} dz$$

Put $\left(z^2 + \frac{2ai}{b} z - 1\right) = 0$

$$z = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-\frac{2ia}{b} \pm \sqrt{\frac{4i^2a^2}{b^2}+4}}{2} = \frac{-ia \pm \sqrt{i^2a^2+b^2}}{b} = \frac{-ia \pm i\sqrt{a^2-b^2}}{b}$$

$z = \frac{-ia+i\sqrt{a^2-b^2}}{b} = \alpha$ lies inside C

$z = \frac{-ia-i\sqrt{a^2-b^2}}{b} = \beta$ lies outside C

And, $\alpha - \beta = \frac{2i\sqrt{a^2-b^2}}{b}$

Residue of $f(z)$ at $(z = \alpha) = \lim_{z\to\alpha}(z-\alpha) \frac{\frac{2}{b}}{(z-\alpha)(z-\beta)}$

$$= \frac{2}{b(\alpha-\beta)} = \frac{2}{b\left[\frac{2i\sqrt{a^2-b^2}}{b}\right]} = \frac{1}{i\sqrt{a^2-b^2}}$$

Now, $\int_0^{2\pi} \frac{1}{a+bcos\theta} d\theta = 2\pi i \ [sum\ of\ residues]$

$$= 2\pi i \left[\frac{1}{i\sqrt{a^2-b^2}}\right]$$

$$= \frac{2\pi}{\sqrt{a^2-b^2}}$$

Type IV: Contour Integration

1. Evaluate $\int_{-\infty}^{\infty} \frac{x^2+x+2}{x^4+10x^2+9} dx$ using contour integration.
[N16/CompIT/6M]
Solution:

Consider the contour to be a very large semicircle of radius ∞ lying in the upper half of the plane.

$I = \int_{-\infty}^{\infty} \frac{x^2+x+2}{x^4+10x^2+9} dx$

Put $x^4 + 10x^2 + 9 = 0$

$(x^2 + 9)(x^2 + 1) = 0$

$x = 3i, -3i, i, -i$

We see that, $x = 3i$ and $x = i$ lies inside C and are simple poles.

Residue of $f(x)$ at $(x = 3i) = \lim_{x \to 3i} (x - 3i) \frac{x^2+x+2}{(x+3i)(x-3i)(x^2+1)}$

$= \frac{9i^2+3i+2}{6i(9i^2+1)}$

$= \frac{-7+3i}{-48i} = \frac{7-3i}{48i}$

Residue of $f(x)$ at $(x = i) = \lim_{x \to i} (x - i) \frac{x^2+x+2}{(x+i)(x-i)(x^2+9)}$

$= \frac{i^2+i+2}{2i(i^2+9)}$

$= \frac{1+i}{16i} = \frac{3+3i}{48i}$

Now, $\int_{-\infty}^{\infty} \frac{x^2+x+2}{x^4+10x^2+9} dx = 2\pi i[sum\ of\ residues]$

$= 2\pi i \left[\frac{7-3i}{48i} + \frac{3+3i}{48i}\right]$

$= 2\pi i \left[\frac{10}{48i}\right]$

$= \frac{5\pi}{12}$

2. Evaluate $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)} dx$ using contour integration.
[M14/AutoMechCivil/8M] [M17/ElexExtcElectBiomInst/6M]
Solution:

Consider the contour to be a very large semi circle of radius ∞ lying in the upper half of the plane.

$I = \int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)} dx$

Put $(x^2 + a^2)(x^2 + b^2) = 0$

$x = ai, -ai, bi, -bi$

We see that, $x = ai$ and $x = bi$ lies inside C and are simple poles.

Residue of $f(x)$ at $(x = ai) = \lim_{x\to ai}(x - ai) \frac{x^2}{(x+ai)(x-ai)(x^2+b^2)}$

$= \frac{a^2 i^2}{2ai(a^2 i^2+b^2)}$

$= \frac{-a^2}{(2ai)(-a^2+b^2)} = \frac{a}{(2i)(a^2-b^2)}$

Residue of $f(x)$ at $(x = bi) = \lim_{x\to bi}(x - bi) \frac{x^2}{(x+bi)(x-bi)(x^2+a^2)}$

$= \frac{b^2 i^2}{2bi(b^2 i^2+a^2)}$

$= \frac{-b^2}{(2bi)(-b^2+a^2)} = \frac{-b}{(2i)(a^2-b^2)}$

Now, $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)} dx = 2\pi i[sum\ of\ residues]$

$= 2\pi i \left[\frac{a}{(2i)(a^2-b^2)} + \frac{-b}{(2i)(a^2-b^2)}\right]$

$= \frac{2\pi i}{2i}\left[\frac{a-b}{a^2-b^2}\right]$

$= \frac{\pi}{a+b}$

3. Evaluate $\int_{-\infty}^{\infty}\frac{\boldsymbol{cosx}}{(\boldsymbol{x^2+a^2})(\boldsymbol{x^2+b^2})}\boldsymbol{dx}$

Solution:

Consider $I=\int_{-\infty}^{\infty}\frac{cosx}{(x^2+a^2)(x^2+b^2)}dx$

$I=\int_{-\infty}^{\infty}\frac{R.P.of\ e^{ix}}{(x^2+a^2)(x^2+b^2)}dx$

Consider the contour to be a very large semi circle of radius ∞ lying in the upper half of the plane.

$I=R.P.of\int_{-\infty}^{\infty}\frac{e^{ix}}{(x^2+a^2)(x^2+b^2)}dx$

Put $(x^2+a^2)(x^2+b^2)=0$

$x=ai,-ai,bi,-bi$

We see that, $x=ai$ and $x=bi$ lies inside C and are simple poles.

Residue of $f(x)$ at $(x=ai)=\lim_{x\to ai}(x-ai)\frac{e^{ix}}{(x+ai)(x-ai)(x^2+b^2)}$

$$=\frac{e^{i^2a}}{2ai(a^2i^2+b^2)}$$

$$=\frac{e^{-a}}{(2ai)(-a^2+b^2)}=\frac{-e^{-a}}{(2ai)(a^2-b^2)}$$

Residue of $f(x)$ at $(x=bi)=\lim_{x\to bi}(x-bi)\frac{e^{ix}}{(x+bi)(x-bi)(x^2+a^2)}$

$$=\frac{e^{i^2b}}{2bi(b^2i^2+a^2)}$$

$$=\frac{e^{-b}}{(2bi)(-b^2+a^2)}=\frac{e^{-b}}{(2bi)(a^2-b^2)}$$

Now, $\int_{-\infty}^{\infty}\frac{cosx}{(x^2+a^2)(x^2+b^2)}dx=R.P.of\int_{-\infty}^{\infty}\frac{e^{ix}}{(x^2+a^2)(x^2+b^2)}dx$

$$=R.P.of\ 2\pi i[sum\ of\ residues]$$

$$=R.P.of\ 2\pi i\left[\frac{-e^{-a}}{(2ai)(a^2-b^2)}+\frac{e^{-b}}{(2bi)(a^2-b^2)}\right]$$

$$=R.P\ of\ \frac{2\pi i}{2i(a^2-b^2)}\left[\frac{e^{-b}}{b}-\frac{e^{-a}}{a}\right]$$

$$=R.P.of\ \frac{\pi}{a^2-b^2}\left[\frac{e^{-b}}{b}-\frac{e^{-a}}{a}\right]$$

$$=\frac{\pi}{a^2-b^2}\left[\frac{e^{-b}}{b}-\frac{e^{-a}}{a}\right]$$

4. Evaluate $\int_{-\infty}^{\infty}\frac{\cos 3x}{(x^2+1)(x^2+4)}dx$

[N14/ElexExtcElectBiomInst/6M]

Solution:

Consider $I=\int_{-\infty}^{\infty}\frac{\cos 3x}{(x^2+1)(x^2+4)}dx$

$I=\int_{-\infty}^{\infty}\frac{R.P.of\ e^{i3x}}{(x^2+1)(x^2+4)}dx$

Consider the contour to be a very large semi circle of radius ∞ lying in the upper half of the plane.

$I=R.P.of\int_{-\infty}^{\infty}\frac{e^{i3x}}{(x^2+1)(x^2+4)}dx$

Put $(x^2+1)(x^2+4)=0$

$x=i,-i,2i,-2i$

We see that, $x=i$ and $x=2i$ lies inside C and are simple poles.

Residue of $f(x)$ at $(x=i)=\lim_{x\to i}(x-i)\frac{e^{i3x}}{(x+i)(x-i)(x^2+4)}$

$$=\frac{e^{i^2 3}}{2i(i^2+4)}$$

$$=\frac{e^{-3}}{6i}$$

Residue of $f(x)$ at $(x=2i)=\lim_{x\to 2i}(x-2i)\frac{e^{i3x}}{(x+2i)(x-2i)(x^2+1)}$

$$=\frac{e^{i^2 6}}{4i(4i^2+1)}$$

$$=\frac{e^{-6}}{-12i}$$

Now, $\int_{-\infty}^{\infty}\frac{\cos 3x}{(x^2+1)(x^2+4)}dx=R.P.of\int_{-\infty}^{\infty}\frac{e^{i3x}}{(x^2+1)(x^2+4)}dx$

$$=R.P.of\ 2\pi i[sum\ of\ residues]$$

$$=R.P.of\ 2\pi i\left[\frac{e^{-3}}{6i}+\frac{e^{-6}}{-12i}\right]$$

$$=R.P\ of\ \frac{2\pi i}{2i}\left[\frac{e^{-3}}{3}-\frac{e^{-6}}{6}\right]$$

$$=R.P.of\ \pi\left[\frac{e^{-3}}{3}-\frac{e^{-6}}{6}\right]$$

$$=\pi\left[\frac{e^{-3}}{3}-\frac{e^{-6}}{6}\right]$$

5. Evaluate $\int_{-\infty}^{\infty}\frac{x^2-x+2}{x^4+10x^2+9}dx$ using contour integration. [M16/ElexExtcElectBiomInst/6M]

Solution:

Consider the contour to be a very large semi-circle of radius ∞ lying in the upper half of the plane.

$I=\int_{-\infty}^{\infty}\frac{x^2-x+2}{x^4+10x^2+9}dx$

Put $x^4+10x^2+9=0$

$(x^2+9)(x^2+1)=0$

$x=3i,-3i,i,-i$

We see that, $x=3i$ and $x=i$ lies inside C and are simple poles.

Residue of $f(x)$ at $(x=3i)=\lim_{x\to 3i}(x-3i)\frac{x^2-x+2}{(x+3i)(x-3i)(x^2+1)}$

$=\frac{9i^2-3i+2}{6i(9i^2+1)}$

$=\frac{-7-3i}{-48i}=\frac{7+3i}{48i}$

Residue of $f(x)$ at $(x=i)=\lim_{x\to i}(x-i)\frac{x^2-x+2}{(x+i)(x-i)(x^2+9)}$

$=\frac{i^2-i+2}{2i(i^2+9)}$

$=\frac{1-i}{16i}=\frac{3-3i}{48i}$

Now, $\int_{-\infty}^{\infty}\frac{x^2-x+2}{x^4+10x^2+9}dx=2\pi i[sum\ of\ residues]$

$=2\pi i\left[\frac{7+3i}{48i}+\frac{3-3i}{48i}\right]$

$=2\pi i\left[\frac{10}{48i}\right]$

$=\frac{5\pi}{12}$

6. Evaluate $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+9)(x^2+4)} dx$ using contour integration

Solution:

Consider the contour to be a very large semi circle of radius ∞ lying in the upper half of the plane.

$I = \int_{-\infty}^{\infty} \frac{x^2}{(x^2+9)(x^2+4)} dx$

Put $(x^2+9)(x^2+4) = 0$

$x = 3i, -3i, 2i, -2i$

We see that, $x = 3i$ and $x = 2i$ lies inside C and are simple poles.

Residue of $f(x)$ at $(x = 3i) = \lim_{x \to 3i} (x - 3i) \frac{x^2}{(x+3i)(x-3i)(x^2+4)}$

$= \frac{9i^2}{6i(9i^2+4)}$

$= \frac{-9}{-30i} = \frac{3}{10i}$

Residue of $f(x)$ at $(x = 2i) = \lim_{x \to 2i} (x - 2i) \frac{x^2}{(x+2i)(x-2i)(x^2+9)}$

$= \frac{4i^2}{4i(4i^2+9)}$

$= \frac{-4}{20i} = -\frac{2}{10i}$

Now, $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+9)(x^2+4)} dx = 2\pi i[sum\ of\ residues]$

$= 2\pi i \left[\frac{3}{10i} - \frac{2}{10i}\right]$

$= \frac{2\pi i}{10i}$

$= \frac{\pi}{5}$

7. Evaluate $\int_{-\infty}^{\infty} \frac{x^2+x+3}{x^4+5x^2+4} dx$ using contour integration

Solution:

Consider $I = \int_{-\infty}^{\infty} \frac{x^2+x+3}{x^4+5x^2+4} dx$

$I = \int_{-\infty}^{\infty} \frac{x^2+x+3}{(x^2+1)(x^2+4)} dx$

Consider the contour to be a very large semi circle of radius ∞ lying in the upper half of the plane.

Put $(x^2+1)(x^2+4) = 0$

$x = i, -i, 2i, -2i$

We see that, $x = i$ and $x = 2i$ lies inside C and are simple poles.

Residue of $f(x)$ at $(x = i) = \lim_{x\to i}(x-i)\frac{x^2+x+3}{(x+i)(x-i)(x^2+4)}$

$= \frac{i^2+i+3}{2i(i^2+4)}$

$= \frac{2+i}{6i}$

Residue of $f(x)$ at $(x = 2i) = \lim_{x\to 2i}(x-2i)\frac{x^2+x+3}{(x+2i)(x-2i)(x^2+1)}$

$= \frac{4i^2+2i+3}{4i(4i^2+1)}$

$= \frac{-1+2i}{-12i} = \frac{1-2i}{12i}$

Now, $\int_{-\infty}^{\infty} \frac{x^2+x+3}{(x^2+1)(x^2+4)} dx = 2\pi i[sum\ of\ residues]$

$= 2\pi i\left[\frac{2+i}{6i} + \frac{1-2i}{12i}\right]$

$= 2\pi i\left[\frac{4+2i+1-2i}{12i}\right]$

$= \frac{10\pi}{12}$

$= \frac{5\pi}{6}$

8. **Show that** $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)} dx = \frac{\pi}{3}$

[N14/CompIT/6M][M16/CompIT/6M]

Solution:

Consider

$I = \int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)} dx$

Consider the contour to be a very large semi circle of radius ∞ lying in the upper half of the plane.

Put $(x^2+1)(x^2+4) = 0$

$x = i, -i, 2i, -2i$

We see that, $x = i$ and $x = 2i$ lies inside C and are simple poles.

Residue of $f(x)$ at $(x = i) = \lim_{x \to i} (x-i) \frac{x^2}{(x+i)(x-i)(x^2+4)}$

$$= \frac{i^2}{2i(i^2+4)} = \frac{-1}{6i}$$

Residue of $f(x)$ at $(x = 2i) = \lim_{x \to 2i} (x-2i) \frac{x^2}{(x+2i)(x-2i)(x^2+1)}$

$$= \frac{4i^2}{4i(4i^2+1)} = \frac{-4}{-12i} = \frac{2}{6i}$$

Now, $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)} dx = 2\pi i[sum\ of\ residues]$

$$= 2\pi i \left[\frac{-1}{6i} + \frac{2}{6i}\right]$$

$$= 2\pi i \left[\frac{1}{6i}\right]$$

$$= \frac{\pi}{3}$$

9. Evaluate $\int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)(x^2+b^2)} dx$ using contour integration.

Solution:

Consider the contour to be a very large semi circle of radius ∞ lying in the upper half of the plane.

$I = \int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)(x^2+b^2)} dx$

Put $(x^2 + a^2)(x^2 + b^2) = 0$

$x = ai, -ai, bi, -bi$

We see that, $x = ai$ and $x = bi$ lies inside C and are simple poles.

Residue of $f(x)$ at $(x = ai) = \lim_{x \to ai} (x - ai) \frac{1}{(x+ai)(x-ai)(x^2+b^2)}$

$$= \frac{1}{2ai(a^2i^2+b^2)}$$

$$= \frac{1}{(2ai)(-a^2+b^2)} = \frac{-1}{(2ai)(a^2-b^2)}$$

Residue of $f(x)$ at $(x = bi) = \lim_{x \to bi} (x - bi) \frac{1}{(x+bi)(x-bi)(x^2+a^2)}$

$$= \frac{1}{2bi(b^2i^2+a^2)}$$

$$= \frac{1}{(2bi)(-b^2+a^2)} = \frac{1}{(2bi)(a^2-b^2)}$$

Now, $\int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)(x^2+b^2)} dx = 2\pi i[sum\ of\ residues]$

$$= 2\pi i \left[\frac{-1}{(2ai)(a^2-b^2)} + \frac{1}{(2bi)(a^2-b^2)}\right]$$

$$= \frac{2\pi i}{2i(a^2-b^2)}\left[-\frac{1}{a} + \frac{1}{b}\right]$$

$$= \frac{\pi}{(a-b)(a+b)}\left[\frac{a-b}{ab}\right]$$

$$= \frac{\pi}{ab(a+b)}$$

10. Evaluate $\int_0^\infty \frac{1}{(x^2+1)(x^2+9)} dx$

[N17/ElexExtcElectBiomInst/6M]

Solution:

Consider the contour to be a very large semi circle of radius ∞ lying in the upper half of the plane.

$I = \int_{-\infty}^{\infty} \frac{1}{(x^2+9)(x^2+1)} dx$

Put $(x^2+9)(x^2+1) = 0$

$x = 3i, -3i, i, -i$

We see that, $x = 3i$ and $x = 2i$ lies inside C and are simple poles.

Residue of $f(x)$ at $(x = 3i) = \lim_{x\to 3i}(x-3i)\frac{1}{(x+3i)(x-3i)(x^2+1)}$

$= \frac{1}{6i(9i^2+1)}$

$= \frac{i}{48}$

Residue of $f(x)$ at $(x = i) = \lim_{x\to i}(x-i)\frac{1}{(x+i)(x-i)(x^2+9)}$

$= \frac{1}{2i(i^2+9)}$

$= -\frac{i}{16}$

Now, $\int_{-\infty}^{\infty} \frac{1}{(x^2+9)(x^2+1)} dx = 2\pi i[sum\ of\ residues]$

$= 2\pi i\left[\frac{i}{48} - \frac{i}{16}\right]$

$= \frac{\pi}{12}$

Thus, $\int_0^\infty \frac{1}{(x^2+1)(x^2+9)} dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{(x^2+9)(x^2+1)} dx = \frac{1}{2}.\frac{\pi}{12} = \frac{\pi}{24}$

11. Evaluate $\int_0^\infty \frac{dx}{x^2+1}$
[M17/AutoMechCivil/6M]
Solution:
Consider
$I = \int_{-\infty}^{\infty} \frac{1}{(x^2+1)} dx$
Consider the contour to be a very large semi circle of radius ∞ lying in the upper half of the plane.
Put $(x^2 + 1) = 0$
$x = i, -i$
We see that, $x = i$ lies inside C and is a simple pole.
Residue of $f(x)$ at $(x = i) = \lim_{x \to i} (x - i) \frac{1}{(x+i)(x-i)}$
$= \frac{1}{2i}$
Now, $\int_{-\infty}^{\infty} \frac{1}{(x^2+1)} dx = 2\pi i[sum\ of\ residues]$
$= 2\pi i \left[\frac{1}{2i}\right]$
$\therefore \int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \pi$
Thus, $\int_0^\infty \frac{dx}{x^2+1} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \frac{\pi}{2}$

12. Evaluate $\int_{-\infty}^{\infty}\frac{dx}{x^4+1}$

Solution:

Consider

$I=\int_{-\infty}^{\infty}\frac{1}{(x^4+1)}dx$

Consider the contour to be a very large semi circle of radius ∞ lying in the upper half of the plane.

Put $(x^4+1)=0$

$x^4=-1$

$x=[-1]^{\frac{1}{4}}$

$x=[cos\pi+isin\pi]^{\frac{1}{4}}$

$x_k=[\cos(2k\pi+\pi)+isin(2k\pi+\pi)]^{\frac{1}{4}}$

$x_k=\cos\frac{(2k+1)\pi}{4}+i\sin\frac{(2k+1)\pi}{4}$

Putting $k=0,1,2,3$

$x_0=\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}=e^{\frac{i\pi}{4}}$

$x_1=\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}=e^{\frac{i3\pi}{4}}$

$x_2=\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}=e^{\frac{i5\pi}{4}}$

$x_3=\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}=e^{\frac{i7\pi}{4}}$

We see that, $x=e^{\frac{i\pi}{4}},x=e^{\frac{i3\pi}{4}}$ lies inside C and are simple poles.

In general, let $x=\alpha$ be a simple pole.

Residue of $f(x)$ at $(x=\alpha)=\lim_{x\to\alpha}(x-\alpha)\frac{1}{x^4+1}\left[\frac{0}{0}\right]$

$=\lim_{x\to\alpha}\frac{1}{4x^3}$

$=\frac{1}{4\alpha^3}$

Residue of $f(x)$ at $\left(x=e^{\frac{i\pi}{4}}\right)=\frac{1}{4e^{\frac{i3\pi}{4}}}=\frac{1}{4}e^{-\frac{3i\pi}{4}}$

$=\frac{1}{4}\left[\cos\frac{3\pi}{4}-i\sin\frac{3\pi}{4}\right]=\frac{1}{4}\left[-\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right]$

Residue of $f(x)$ at $\left(x=e^{\frac{i3\pi}{4}}\right)=\frac{1}{4e^{\frac{i9\pi}{4}}}=\frac{1}{4}e^{-\frac{9i\pi}{4}}$

$=\frac{1}{4}\left[\cos\frac{9\pi}{4}-i\sin\frac{9\pi}{4}\right]=\frac{1}{4}\left[\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right]$

Now, $\int_{-\infty}^{\infty}\frac{1}{(x^4+1)}dx=2\pi i[sum\ of\ residues]$

$$= 2\pi i\left[\frac{1}{4}\left(-\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)+\frac{1}{4}\left(\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)\right]$$

$$= \frac{2\pi i}{4}\left[-\frac{2i}{\sqrt{2}}\right]$$

$$= \frac{\pi}{\sqrt{2}}$$

13. Evaluate $\int_0^\infty \frac{dx}{x^4+16}$

Solution:

Consider

$I = \int_{-\infty}^{\infty} \frac{1}{(x^4+16)} dx$

Consider the contour to be a very large semi circle of radius ∞ lying in the upper half of the plane.

Put $(x^4+16) = 0$

$x^4 = -16$

$x = [-16]^{\frac{1}{4}} = (16)^{\frac{1}{4}}(-1)^{\frac{1}{4}}$

$x = 2[cos\pi + isin\pi]^{\frac{1}{4}}$

$x_k = 2[\cos(2k\pi+\pi) + isin(2k\pi+\pi)]^{\frac{1}{4}}$

$x_k = 2\left[\cos\frac{(2k+1)\pi}{4} + i\sin\frac{(2k+1)\pi}{4}\right]$

Putting $k = 0,1,2,3$

$x_0 = 2\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right] = 2e^{\frac{i\pi}{4}}$

$x_1 = 2\left[\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right] = 2e^{\frac{i3\pi}{4}}$

$x_2 = 2\left[\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right] = 2e^{\frac{i5\pi}{4}}$

$x_3 = 2\left[\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right] = 2e^{\frac{i7\pi}{4}}$

We see that, $x = 2e^{\frac{i\pi}{4}}, x = 2e^{\frac{i3\pi}{4}}$ lies inside C and are simple poles.

In general, let $x = \alpha$ be a simple pole.

Residue of $f(x)$ at $(x = \alpha) = \lim_{x\to\alpha}(x-\alpha)\frac{1}{x^4+1}\left[\frac{0}{0}\right]$

$= \lim_{x\to\alpha}\frac{1}{4x^3}$

$= \frac{1}{4\alpha^3}$

Residue of $f(x)$ at $\left(x = 2e^{\frac{i\pi}{4}}\right) = \frac{1}{32e^{\frac{i3\pi}{4}}} = \frac{1}{32}e^{-\frac{3i\pi}{4}}$

$= \frac{1}{32}\left[\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\right] = \frac{1}{32}\left[-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right]$

Residue of $f(x)$ at $\left(x = e^{\frac{i3\pi}{4}}\right) = \frac{1}{32e^{\frac{i9\pi}{4}}} = \frac{1}{32}e^{-\frac{9i\pi}{4}}$

$= \frac{1}{32}\left[\cos\frac{9\pi}{4} - i\sin\frac{9\pi}{4}\right] = \frac{1}{32}\left[\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right]$

Now, $\int_{-\infty}^{\infty}\frac{1}{(x^4+16)}dx = 2\pi i[sum\ of\ residues]$

$$= 2\pi i\left[\frac{1}{32}\left(-\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)+\frac{1}{32}\left(\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)\right]$$

$$= \frac{2\pi i}{32}\left[-\frac{2i}{\sqrt{2}}\right]$$

$$= \frac{\pi}{8\sqrt{2}}$$

$$\therefore \int_{-\infty}^{\infty}\frac{dx}{x^4+16} = \frac{\pi}{8\sqrt{2}}$$

Thus, $\int_0^{\infty}\frac{dx}{x^4+16} = \frac{1}{2}\int_{-\infty}^{\infty}\frac{dx}{x^4+16} = \frac{\pi}{16\sqrt{2}}$

14. Evaluate $\int_{-\infty}^{\infty} \frac{x^2}{x^6+1} dx$
[N15/ElexExtcElectBiomInst/6M]
Solution:

Consider, $I = \int_{-\infty}^{\infty} \frac{x^2}{x^6+1} dx$

Put $x^3 = t, \therefore 3x^2 dx = dt, \therefore x^2 dx = \frac{dt}{3}$

$I = \int_{-\infty}^{\infty} \frac{1}{t^2+1} \frac{dt}{3}$

$I = \frac{1}{3} \int_{-\infty}^{\infty} \frac{dt}{t^2+1}$

i.e. $I = \frac{1}{3} \int_{-\infty}^{\infty} \frac{dx}{x^2+1}$

Consider the contour to be a very large semi circle of radius ∞ lying in the upper half of the plane.

Put $(x^2 + 1) = 0$

$x = i, -i$

We see that, $x = i$ lies inside C and is a simple pole.

Residue of $f(x)$ at $(x = i) = \lim_{x \to i} (x - i) \frac{1}{(x+i)(x-i)}$

$= \frac{1}{2i}$

Now, $\frac{1}{3} \int_{-\infty}^{\infty} \frac{1}{(x^2+1)} dx = \frac{2\pi i}{3} [sum\ of\ residues]$

$= \frac{2\pi i}{3} \left[\frac{1}{2i}\right]$

$\therefore \int_{-\infty}^{\infty} \frac{x^2}{x^6+1} dx = \frac{1}{3} \int_{-\infty}^{\infty} \frac{1}{(x^2+1)} dx = \frac{\pi}{3}$

15. Evaluate using contour integration $\int_0^\infty \frac{dx}{x^4+1}$

Solution:

Consider

$$I = \int_{-\infty}^{\infty} \frac{1}{(x^4+1)} dx$$

Consider the contour to be a very large semi circle of radius ∞ lying in the upper half of the plane.

Put $(x^4+1)=0$

$$x^4 = -1$$

$$x = [-1]^{\frac{1}{4}}$$

$$x = [cos\pi + isin\pi]^{\frac{1}{4}}$$

$$x_k = [\cos(2k\pi+\pi) + isin(2k\pi+\pi)]^{\frac{1}{4}}$$

$$x_k = \cos\frac{(2k+1)\pi}{4} + i\sin\frac{(2k+1)\pi}{4}$$

Putting $k = 0,1,2,3$

$$x_0 = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = e^{\frac{i\pi}{4}}$$

$$x_1 = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} = e^{\frac{i3\pi}{4}}$$

$$x_2 = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4} = e^{\frac{i5\pi}{4}}$$

$$x_3 = \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4} = e^{\frac{i7\pi}{4}}$$

We see that, $x = e^{\frac{i\pi}{4}}, x = e^{\frac{i3\pi}{4}}$ lies inside C and are simple poles.

In general, let $x = \alpha$ be a simple pole.

Residue of $f(x)$ at $(x = \alpha) = \lim_{x\to\alpha}(x-\alpha)\frac{1}{x^4+1}\left[\frac{0}{0}\right]$

$$= \lim_{x\to\alpha}\frac{1}{4x^3}$$

$$= \frac{1}{4\alpha^3}$$

Residue of $f(x)$ at $\left(x = e^{\frac{i\pi}{4}}\right) = \frac{1}{4e^{\frac{i3\pi}{4}}} = \frac{1}{4}e^{-\frac{3i\pi}{4}}$

$$= \frac{1}{4}\left[\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\right] = \frac{1}{4}\left[-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right]$$

Residue of $f(x)$ at $\left(x = e^{\frac{i3\pi}{4}}\right) = \frac{1}{4e^{\frac{i9\pi}{4}}} = \frac{1}{4}e^{-\frac{9i\pi}{4}}$

$$= \frac{1}{4}\left[\cos\frac{9\pi}{4} - i\sin\frac{9\pi}{4}\right] = \frac{1}{4}\left[\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right]$$

Now, $\int_{-\infty}^{\infty} \frac{1}{(x^4+1)} dx = 2\pi i[sum\ of\ residues]$

$$= 2\pi i\left[\frac{1}{4}\left(-\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)+\frac{1}{4}\left(\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)\right]$$

$$= \frac{2\pi i}{4}\left[-\frac{2i}{\sqrt{2}}\right]$$

$$= \frac{\pi}{\sqrt{2}}$$

$\therefore \int_{-\infty}^{\infty}\frac{dx}{x^4+1} = \frac{\pi}{\sqrt{2}}$

Thus, $\int_{0}^{\infty}\frac{dx}{x^4+1} = \frac{1}{2}\int_{-\infty}^{\infty}\frac{dx}{x^4+1} = \frac{\pi}{2\sqrt{2}}$

16. Evaluate $\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+4)^2}$

Solution:

Consider

$I = \int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+4)^2} dx$

Consider the contour to be a very large semi circle of radius ∞ lying in the upper half of the plane.

Put $(x^2+1)(x^2+4^2) = 0$

$x = i, -i, 2i, -2i, 2i, -2i$

We see that, $x = i$ is a simple pole and $x = 2i$ is a pole of order 2.

Residue of $f(x)$ at $(x = i) = \lim_{x\to i}(x-i)\frac{1}{(x+i)(x-i)(x^2+4)^2}$

$= \frac{1}{2i(i^2+4)^2} = \frac{1}{18i} = -\frac{i}{18}$

Residue of $f(x)$ at $(x = 2i) = \frac{1}{1!}\lim_{x\to 2i}\frac{d}{dx}\left[(x-2i)^2\frac{1}{(x+2i)^2(x-2i)^2(x^2+1)}\right]$

$= \lim_{x\to 2i}\frac{d}{dx}\left[\frac{1}{(x+2i)^2(x^2+1)}\right]$

$= \lim_{x\to 2i}\frac{-1}{(x+2i)^4(x^2+1)^2}\times\{(x+2i)^2(2x)+(x^2+1).2(x+2i)\}$

$= \frac{-\{(4i)^2(4i)+(4i^2+1).2(4i)\}}{(4i)^4(4i^2+1)^2}$

$= \frac{-64i^3+24i}{256\times 9} = \frac{88i}{2304} = \frac{11i}{288}$

Now, $\int_{-\infty}^{\infty}\frac{1}{(x^2+1)(x^2+4)^2}dx = 2\pi i[sum\ of\ residues]$

$= 2\pi i\left[\frac{-i}{18}+\frac{11i}{288}\right]$

$= 2\pi i\left[-\frac{5i}{288}\right]$

$= \frac{5\pi}{144}$

17. Evaluate $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)^2} dx$

Solution:

Consider

$$I = \int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)^2} dx$$

Consider the contour to be a very large semi circle of radius ∞ lying in the upper half of the plane.

Put $(x^2 + a^2)^2 = 0$

$x = ai, -ai, ai, -ai$

We see that, $x = ai$ is a pole of order 2.

Residue of $f(x)$ at $(x = ai) = \frac{1}{1!} \lim_{x \to ai} \frac{d}{dx}\left[(x - ai)^2 \frac{x^2}{(x+ai)^2(x-ai)^2}\right]$

$$= \lim_{x \to ai} \frac{d}{dx}\left[\frac{x^2}{(x+ai)^2}\right]$$

$$= \lim_{x \to ai} \left[\frac{(x+ai)^2(2x) - x^2 2(x+ai)}{(x+ai)^4}\right]$$

$$= \frac{(2ai)^2(2ai) - (ai)^2 2(2ai)}{(2ai)^4}$$

$$= \frac{8a^3i^3 - 4a^3i^3}{16a^4i^4} = \frac{4a^3i^3}{16a^4i^4} = \frac{1}{4ai}$$

Now, $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)^2} dx = 2\pi i[sum\ of\ residues]$

$$= 2\pi i\left[\frac{1}{4ai}\right] = \frac{\pi}{2a}$$

18. Evaluate $\int_0^\infty \frac{x^3 sinx}{(x^2+a^2)^2} dx$ using contour integration
[M14/ElexExtcElectBiomInst/6M]
Solution:

Consider, $I = \int_{-\infty}^{\infty} \frac{x^3 sinx}{(x^2+a^2)^2} dx$

We know that, $e^{ix} = cosx + isinx$

i.e. $sinx$ is an I.P. of e^{ix}

$I = I.P. of\ \int_{-\infty}^{\infty} \frac{x^3 e^{ix}}{(x^2+a^2)^2} dx$

Consider the contour to be a very large semi circle of radius ∞ lying in the upper half of the plane.

Put $(x^2 + a^2)^2 = 0$

$x = ai, -ai, ai, -ai$

We see that, $x = ai$ is a pole of order 2.

Residue of $f(x)$ at $(x = ai) = \frac{1}{1!} \lim_{x \to ai} \frac{d}{dx}\left[(x - ai)^2 \frac{x^3 e^{ix}}{(x+ai)^2 (x-ai)^2}\right]$

$$= \lim_{x \to ai} \frac{d}{dx}\left[\frac{x^3 e^{ix}}{(x+ai)^2}\right]$$

$$= \lim_{x \to ai}\left[\frac{(x+ai)^2 (x^3 ie^{ix} + e^{ix}.3x^2) - x^3 e^{ix}.2(x+ai)}{(x+ai)^4}\right]$$

$$= \frac{(2ai)^2 \left(a^3 i^3 i.e^{i^2 a} + e^{i^2 a}.3a^2 i^2\right) - (ai)^3 e^{i^2 a} 2(2ai)}{(2ai)^4}$$

$$= \frac{(4a^2 i^2)(a^3 i^4 e^{-a} + 3a^2 i^2 e^{-a}) - 4a^4 i^4 e^{-a}}{16a^4 i^4}$$

$$= \frac{-4a^2 e^{-a}(a^3 - 3a^2) - 4a^4 e^{-a}}{16a^4 i^4} = \frac{-4a^4 e^{-a}(a-3+1)}{16a^4}$$

$$= -\frac{e^{-a}(a-2)}{4}$$

Now, $\int_{-\infty}^{\infty} \frac{x^3 sinx}{(x^2+a^2)^2} dx = I.P. of \int_{-\infty}^{\infty} \frac{x^3 e^{ix}}{(x^2+a^2)^2} dx$

$$= I.P. of\ 2\pi i[sum\ of\ residues]$$

$$= I.P. of\ \ 2\pi i\left[-\frac{e^{-a}(a-2)}{4}\right] = -e^{-a}(a-2)\frac{\pi}{2}$$

19. Evaluate $\int_0^\infty \frac{x^2}{(x^2+1)^3} dx$

Solution:

Consider

$I = \int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^3} dx$

Consider the contour to be a very large semi circle of radius ∞ lying in the upper half of the plane.

Put $(x^2+1)^3 = 0$

$x = i, -i, i, -i, i, -i$

We see that, $x = i$ is a pole of order 3.

Residue of $f(x)$ at $(x = i) = \frac{1}{2!} \lim_{x \to i} \frac{d^2}{dx^2} \left[(x-i)^3 \frac{x^2}{(x+i)^3 (x-i)^3} \right]$

$$= \frac{1}{2} \lim_{x \to i} \frac{d^2}{dx^2} \left[\frac{x^2}{(x+i)^3} \right]$$

$$= \frac{1}{2} \lim_{x \to i} \frac{d}{dx} \left[\frac{(x+i)^3 (2x) - x^2 3(x+i)^2}{(x+i)^6} \right]$$

$$= \frac{1}{2} \lim_{x \to i} \frac{d}{dx} \left[\frac{(x+i)(2x) - 3x^2}{(x+i)^4} \right]$$

$$= \frac{1}{2} \lim_{x \to i} \frac{d}{dx} \left[\frac{2ix - x^2}{(x+i)^4} \right]$$

$$= \frac{1}{2} \lim_{x \to i} \frac{(x+i)^4 (2i - 2x) - (2ix - x^2).4(x+i)^3}{(x+i)^8}$$

$$= \frac{1}{2} . \frac{(2i)^4 (2i - 2i) - (2i^2 - i^2).4(2i)^3}{(2i)^8}$$

$$= \frac{1}{2} . \frac{0 - i^2 .32 i^3}{256 i^8} = -\frac{1}{16 i^3} = \frac{1}{16 i}$$

Now, $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^3} dx = 2\pi i \left[\frac{1}{16i} \right] = \frac{\pi}{8}$

20. Evaluate $\int_0^\infty \frac{1}{(x^2+a^2)^3} dx$

[N17/CompIT/6M]

Solution:

Consider

$I = \int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)^3} dx$

Consider the contour to be a very large semi circle of radius ∞ lying in the upper half of the plane.

Put $(x^2 + a^2)^3 = 0$

$x = ai, -ai, ai, -ai, ai, -ai$

We see that, $x = ai$ is a pole of order 2.

Residue of $f(x)$ at $(x = ai) = \frac{1}{2!} \lim_{x\to ai} \frac{d^2}{dx^2}\left[(x - ai)^3 \frac{1}{(x+ai)^3(x-ai)^3}\right]$

$$= \frac{1}{2} \lim_{x\to ai} \frac{d^2}{dx^2}\left[\frac{1}{(x+ai)^3}\right]$$

$$= \frac{1}{2} \lim_{x\to ai} \frac{d}{dx}\left[-\frac{3}{(x+ai)^4}\right]$$

$$= \frac{1}{2} \lim_{x\to ai} \left[\frac{12}{(x+ai)^5}\right]$$

$$= \frac{1}{2} . \frac{12}{(2ai)^5}$$

$$= \frac{6}{32a^5 i^5} = -\frac{3i}{16a^5}$$

Now, $\int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)^3} dx = 2\pi i[sum\ of\ residues]$

$$= 2\pi i\left[-\frac{3i}{16a^5}\right] = \frac{3\pi}{8a^5}$$

Thus, $\int_0^\infty \frac{1}{(x^2+a^2)^3} dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)^3} dx = \frac{1}{2}\left[\frac{3\pi}{8a^5}\right] = \frac{3\pi}{16a^5}$

21. Evaluate $\int_0^\infty \frac{1}{(x^2+a^2)^2}dx$

[M15/CompIT/6M]

Solution:

Consider

$I = \int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)^2}dx$

Consider the contour to be a very large semi circle of radius ∞ lying in the upper half of the plane.

Put $(x^2 + a^2)^2 = 0$

$x = ai, -ai, ai, -ai$

We see that, $x = ai$ is a pole of order 2.

Residue of $f(x)$ at $(x = ai) = \frac{1}{1!}\lim_{x\to ai} \frac{d}{dx}\left[(x - ai)^2 \frac{1}{(x+ai)^2(x-ai)^2}\right]$

$= \lim_{x\to ai} \frac{d}{dx}\left[\frac{1}{(x+ai)^2}\right]$

$= \lim_{x\to ai} \left[-\frac{2}{(x+ai)^3}\right]$

$= -\frac{2}{(2ai)^3}$

$= -\frac{2}{8a^3i^3} = \frac{1}{4a^3i}$

Now, $\int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)^2}dx = 2\pi i[sum\ of\ residues]$

$= 2\pi i\left[\frac{1}{4a^3i}\right] = \frac{\pi}{2a^3}$

Thus, $\int_0^\infty \frac{1}{(x^2+a^2)^2}dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)^2}dx = \frac{1}{2}\left[\frac{\pi}{2a^3}\right] = \frac{\pi}{4a^3}$

22. Evaluate $\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^3} dx$

Solution:

Consider

$$I = \int_{-\infty}^{\infty} \frac{1}{(x^2+1)^3} dx$$

Consider the contour to be a very large semi circle of radius ∞ lying in the upper half of the plane.

Put $(x^2+1)^3 = 0$

$x = i, -i, i, -i, i, -i$

We see that, $x = i$ is a pole of order 3.

Residue of $f(x)$ at $(x = i) = \frac{1}{2!} \lim_{x \to i} \frac{d^2}{dx^2} \left[(x-i)^3 \frac{1}{(x+i)^3 (x-i)^3} \right]$

$$= \frac{1}{2} \lim_{x \to i} \frac{d^2}{dx^2} \left[\frac{1}{(x+i)^3} \right]$$

$$= \frac{1}{2} \lim_{x \to i} \frac{d}{dx} \left[-\frac{3}{(x+i)^4} \right]$$

$$= \frac{1}{2} \lim_{x \to i} \frac{12}{(x+i)^5}$$

$$= \frac{1}{2} . \frac{12}{(2i)^5} = \frac{1}{2} . \frac{12}{32 i^5} = \frac{3}{16i}$$

Now, $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^3} dx = 2\pi i \left[\frac{3}{16i} \right] = \frac{3\pi}{8}$

23. Using Residue Theorem, evaluate $\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2} dx$

[N17/AutoMechCivil/4M]

Solution:

Consider

$I = \int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2} dx$

Consider the contour to be a very large semi circle of radius ∞ lying in the upper half of the plane.

Put $(x^2+1)^2 = 0$

$x = i, -i, i, -i$

We see that, $x = i$ is a pole of order 2.

Residue of $f(x)$ at $(x = i) = \frac{1}{1!} \lim_{x \to i} \frac{d}{dx}\left[(x-i)^2 \frac{1}{(x+i)^2(x-i)^2}\right]$

$$= \lim_{x \to i} \frac{d}{dx}\left[\frac{1}{(x+i)^2}\right]$$

$$= \lim_{x \to i}\left[-\frac{2}{(x+i)^3}\right]$$

$$= -\frac{2}{(2i)^3}$$

$$= -\frac{2}{8i^3} = \frac{1}{4i}$$

Now, $\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2} dx = 2\pi i[sum\ of\ residues]$

$$= 2\pi i\left[\frac{1}{4i}\right] = \frac{\pi}{2}$$