More Important Distributions

Type I: Binomial Distribution

1. 7 dice are thrown 729 times, how many times do you expect at least 4 dice to show 3 or 5?
[N13/Chem/6M][M15/AutoMechCivil/6M][N16/AutoMechCivil/6M][M17/AutoMechCivil/6M]

Solution:

$p =$ probability to show 3 or 5 $= \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$

$q = 1 - p = \frac{2}{3}$

$n = 7$

$N = 729$

By Binomial Distribution,

$$P(X = r) = {}^{n}C_r . p^r . q^{n-r} = {}^{7}C_r . \left(\frac{1}{3}\right)^r . \left(\frac{2}{3}\right)^{7-r}$$

$$\begin{aligned} P(atleast\ 4) &= P(X \geq 4) \\ &= P(X = 4) + P(X = 5) + P(X = 6) + P(X = 7) \\ &= \frac{280}{2187} + \frac{28}{729} + \frac{14}{2187} + \frac{1}{2187} \\ &= \frac{379}{2187} \\ &= 0.1733 \end{aligned}$$

$$\begin{aligned} \text{Expected number of times} &= N \times P \\ &= 729 \times 0.1733 \\ &= 126.33 \approx 126 \end{aligned}$$

2. The ratio of the probability of 3 successes in 5 independent trials to the probability of 2 successes in 5 independent trials is 1/4. What is the probability of 4 successes in 6 independent trials?

[M14/AutoMechCivil/6M][N14/ChemBiot/7M]

Solution:

By Binomial Distribution,

$P(X = r) = {}^{n}C_r . p^r . q^{n-r}$

It is given that,

$\frac{P(3\ success\ \ in\ 5\ trials\ )}{P(2\ success\ \ in\ 5\ trials\ )} = \frac{1}{4}$

$\frac{P(X=3,n=5)}{P(X=2,n=5)} = \frac{1}{4}$

$\frac{{}^{5}C_3\ p^3\ q^2}{{}^{5}C_2\ p^2\ q^3} = \frac{1}{4}$

$\frac{p}{q} = \frac{1}{4}$

$q = 4p$

We also know that,

$p + q = 1$

$\therefore p + 4p = 1$

$5p = 1$

$\therefore p = \frac{1}{5}$

$\therefore q = \frac{4}{5}$

Now,

$$\begin{aligned} P(4\ success\ in\ 6\ trials) &= P(X = 4, n = 6) \\ &= {}^{6}C_4\ p^4\ q^2 \\ &= {}^{6}C_4\ \left(\frac{1}{5}\right)^4\ \left(\frac{4}{5}\right)^2 \\ &= \frac{48}{3125} \\ &= 0.01536 \end{aligned}$$

3. The probability that a man, aged 60 will live upto 70 is 0.65. what is the probability that out of 10 such men now at 60 atleast 7 will live upto 70?
[M14/ChemBiot/5M][N16/ChemBiot/5M]
Solution:
$p = 0.65$
$q = 0.35$
$n = 10$
By Binomial Distribution,
$P(X = r) = {}^{n}C_r . p^r . q^{n-r} = {}^{10}C_r . (0.65)^r . (0.35)^{10-r}$
$P(atleast\ 7) = P(X \geq 7)$
$= P(X = 7) + P(X = 8) + P(X = 9) + P(X = 10)$
$= 0.2522 + 0.1757 + 0.0725 + 0.0135$
$= 0.5139$

4. If 10% bolts manufactured by a machine are defective. Find the probability that out of 5 bolts chosen at most one will be defective.
[N14/AutoMechCivil/6M]
Solution:
$p = 10\% = 0.1$
$q = 90\% = 0.9$
$n = 5$
By Binomial Distribution,
$P(X = r) = {}^{n}C_r . p^r . q^{n-r} = {}^{5}C_r . (0.1)^r . (0.9)^{5-r}$
$P(atmost\ 1) = P(X \leq 1)$
$= P(X = 0) + P(X = 1)$
$= 0.5905 + 0.3281$
$= 0.9186$

5. For a special security in a certain protected area it was decided to put three lighting bulbs on each pole. If each bulb has a probability p of burning out in the first 100 hours of service, calculate the probability that atleast one of them is still good after 100 hours, given $p = 0.3$.
How many bulbs would be needed on each pole to ensure 99% safety that atleast one is good after 100 hours?
[N14/CompIT/6M]
Solution:
$p = 0.3$ (probability of burning out)
$q = 0.7$ (probability of burning on)
$n = 3$
By Binomial Distribution,
$P(X = r) = {}^{n}C_r . p^r . q^{n-r} = {}^{3}C_r . (0.3)^r . (0.7)^{3-r}$

$$
\begin{aligned}
P(atleast\ 1\ still\ burning) &= P(atmost\ 2\ burned\ out) \\
&= P(X \leq 2) \\
&= 1 - P(X > 2) \\
&= 1 - P(X = 3) \\
&= 1 - 0.027 \\
&= 0.973
\end{aligned}
$$

Now,
let probability of the bulb being good after 100 hours be $p = 0.7$
thus, the probability of the bulb burned out after 100 hours will be $q = 0.3$
it is given that,
$P(atleast\ 1\ good) = 99\%$
$P(X \geq 1) = 0.99$
$1 - P(X = 0) = 0.99$
$1 - 0.99 = P(X = 0)$
$P(X = 0) = 0.01$
${}^{n}C_0 (0.7)^0 (0.3)^n = 0.01$
$(0.3)^n = 0.01$
$nlog(0.3) = \log(0.01)$
$n = \frac{\log(0.01)}{\log(0.3)}$
$n = 3.825 \approx 4$

6. The incidence of an occupational disease in an industry is such that the workers have 20% chance of suffering from it. What is the probability that out of 6 workers 4 or more will catch the disease?

[N15/ChemBiot/5M][M16/ChemBiot/5M]

Solution:

$p = 20\% = 0.2$

$q = 80\% = 0.8$

$n = 6$

By Binomial Distribution,

$P(X = r) = {}^{n}C_r . p^r . q^{n-r} = {}^{6}C_r . (0.2)^r . (0.8)^{6-r}$

$$
\begin{aligned}
P(4\ or\ more) = P(X \geq 4) \\
&= P(X = 4) + P(X = 5) + P(X = 6) \\
&= \frac{48}{3125} + \frac{24}{15625} + \frac{1}{15625} \\
&= \frac{53}{3125} \\
&= 0.01696
\end{aligned}
$$

7. The probability that at any moment one telephone line out of 10 will be busy is 0.2. (i) what is the probability that 5 lines are busy? (ii) find the expected number of busy lines and also find the probability of this number (iii) what is the probability that all lines are busy?

[N15/AutoMechCivil/6M]

Solution:

$p = 0.2$

$q = 0.8$

$n = 10$

By Binomial Distribution,

$P(X = r) = {}^{n}C_{r}.p^{r}.q^{n-r} = {}^{10}C_{r}.(0.2)^{r}.(0.8)^{10-r}$

(i) $P(5\ lines\ are\ busy) = P(X = 5)$

$= 0.0264$

(ii) $E(X) = np = 10 \times 0.2 = 2$

$P(X = 2) = 0.302$

(iii) $P(all\ lines\ are\ busy) = P(X = 10)$

$= (0.2)^{10} = \frac{1}{9765625} = 0.000000102$

8. The probability of a man hitting the target is 1/4 (i) if he fires 7 times what is the probability of his hitting the target at least twice? (ii) How many times must he fire so that the probability of his hitting the target at least once is greater than 2/3?

[M16/AutoMechCivil/5M]

Solution:

$p = \frac{1}{4}$

$q = \frac{3}{4}$

By Binomial Distribution,

$P(X = r) = {}^nC_r . p^r . q^{n-r} = {}^nC_r . \left(\frac{1}{4}\right)^r . \left(\frac{3}{4}\right)^{n-r}$

(i) if $n = 7$,

$$\begin{aligned} P(atleast\ twice) &= P(X \geq 2) \\ &= 1 - P(X < 2) \\ &= 1 - \{P(X = 0) + P(X = 1)\} \\ &= 1 - \left\{\frac{2187}{16384} + \frac{5103}{16384}\right\} \\ &= 1 - \frac{3645}{8192} \\ &= 0.5550 \end{aligned}$$

(ii) $P(atleast\ once) > \frac{2}{3}$

$P(X \geq 1) > \frac{2}{3}$

$1 - P(X = 0) > \frac{2}{3}$

$1 - \frac{2}{3} > P(X = 0)$

$\frac{1}{3} > {}^nC_0 . \left(\frac{1}{4}\right)^0 . \left(\frac{3}{4}\right)^{n-0}$

$\left(\frac{3}{4}\right)^n < \frac{1}{4}$

$n \log\left(\frac{3}{4}\right) < \log\left(\frac{1}{4}\right)$

$n < \frac{\log\left(\frac{1}{4}\right)}{\log\left(\frac{3}{4}\right)}$

$n < 4.818$

i.e. $n = 4$

9. Fit a binomial distribution to the following data:

X	0	1	2	3	4	5	6
F	5	18	28	12	7	6	4

[M16/CompIT/6M]

Solution:

$n = 6$

$N = \sum F = 5 + 18 + 28 + 12 + 7 + 6 + 4 = 80$

Mean $= \frac{\sum F.x}{\sum F} = \frac{5(0)+18(1)+28(2)+12(3)+7(4)+6(5)+4(6)}{5+18+28+12+7+6+4} = 2.4$

$np = 2.4$

$6p = 2.4$

$p = 0.4$

$q = 0.6$

By Binomial Distribution,

$P(X = r) = {}^{n}C_r . p^r . q^{n-r} = {}^{6}C_r . (0.4)^r . (0.6)^{6-r}$

X	$P(X)$	$F(X) = NP = 80P$
0	$\frac{729}{15625}$	$3.73 \approx 4$
1	$\frac{2916}{15625}$	$14.9299 \approx 15$
2	$\frac{972}{3125}$	$24.883 \approx 25$
3	$\frac{864}{3125}$	$22.118 \approx 22$
4	$\frac{432}{3125}$	$11.059 \approx 11$
5	$\frac{576}{15625}$	$2.949 \approx 3$
6	$\frac{64}{15625}$	$0.3276 \approx 0$

10. If X is binomially distributed with E(X)=2 and Var(X)=4/3. Find the probability distribution of X.
[N16/CompIT/6M][N17/CompIT/6M]
Solution:
$E(X) = np = 2$
$V(X) = npq = \frac{4}{3}$
$\frac{np}{npq} = \frac{2}{\frac{4}{3}}$
$\frac{1}{q} = \frac{3}{2}$
$\therefore q = \frac{2}{3}$
$p = 1 - q$
$\therefore p = \frac{1}{3}$
$np = 2$
$n\left(\frac{1}{3}\right) = 2$
$\therefore n = 6$
By Binomial Distribution,
$P(X = r) = {}^{n}C_r . p^r . q^{n-r} = {}^{6}C_r . \left(\frac{1}{3}\right)^r . \left(\frac{2}{3}\right)^{6-r}$

X	0	1	2	3	4	5	6
$P(X)$	$\frac{64}{729}$	$\frac{64}{243}$	$\frac{80}{243}$	$\frac{160}{729}$	$\frac{20}{243}$	$\frac{4}{243}$	$\frac{1}{729}$

Type II: Poisson distribution

1. The proofs of a 500 page book contain 500 misprints. Find the probability that there are at least 4 misprints in a randomly chosen page.

   [N13/Biot/5M]

   Solution:

   $m = \frac{500}{500} = 1$

   By Poisson distribution,

   $P(X = r) = \frac{e^{-m}.m^{r}}{r!} = \frac{(e^{-1}.1^{r})}{r!}$

   $$\begin{aligned} P(atleast\ 4) &= P(X \geq 4) \\ &= 1 - P(X < 4) \\ &= 1 - \{P(0) + P(1) + P(2) + P(3)\} \\ &= 1 - \left\{\frac{e^{-1}}{0!} + \frac{e^{-1}}{1!} + \frac{e^{-1}}{2!} + \frac{e^{-1}}{3!}\right\} \\ &= 1 - \frac{8}{3}e^{-1} \\ &= 0.019 \end{aligned}$$

2. Find the probability that at most 4 defective bulbs will be found in a box of 200 bulbs, if it is known that 2% of the bulbs are defective.
[M14/ChemBiot/6M]
Solution:

$n = 200$

$p = 2\% = \frac{2}{100}$

$m = np = 200 \times \frac{2}{100} = 4$

By Poisson distribution,

$P(X = r) = \frac{e^{-m}.m^r}{r!} = \frac{(e^{-4}.4^r)}{r!}$

$$\begin{aligned} P(atmost\ 4) &= P(X \leq 4) \\ &= P(0) + P(1) + P(2) + P(3) + P(4) \\ &= \frac{e^{-4}.4^0}{0!} + \frac{e^{-4}.4^1}{1!} + \frac{e^{-4}.4^2}{2!} + \frac{e^{-4}.4^3}{3!} + \frac{e^{-4}.4^4}{4!} \\ &= e^{-4}\left[1 + 4 + 8 + \frac{32}{2} + \frac{32}{3}\right] \\ &= \frac{103}{3} e^{-4} \\ &= 0.6288 \end{aligned}$$

3. A car hire firm has two cars which it hires out day by day. The number of demands for a car on each day is distributed as poisson variate with mean 1.5. calculate the proportion of days on which (i) neither car is used (ii) some demand is refused.

[M14/AutoMechCivil/6M]

Solution:

$m = 1..5$

By Poisson distribution,

$$P(X = r) = \frac{e^{-m}.m^r}{r!} = \frac{e^{-1.5}.1.5^r}{r!}$$

(i) $P(neither\ car\ is\ used) = P(X = 0)$

$$= \frac{e^{-1.5}.1.5^0}{0!}$$
$$= 0.2231$$

(ii) $P(some\ demand\ is\ refused) = P(X > 2)$

$$= 1 - P(X \leq 2)$$
$$= 1 - \{P(0) + P(1) + P(2)\}$$
$$= 1 - \left\{\frac{e^{-1.5}.1.5^0}{0!} + \frac{e^{-1.5}.1.5^1}{1!} + \frac{e^{-1.5}.1.5^2}{2!}\right\}$$
$$= 1 - \{0.2231 + 0.3347 + 0.2510\}$$
$$= 0.1912$$

4. A random variable X follows poisson distribution with variance 3 calculate $P(X = 2)$ and $P(X \geq 4)$

[N14/AutoMechCivil/6M]

Solution:

$variance = m = 3$

By Poisson distribution,

$$P(X = r) = \frac{e^{-m}.m^r}{r!} = \frac{e^{-3}.3^r}{r!}$$

$$P(X = 2) = \frac{e^{-3}.3^2}{2!} = 0.2240$$

$$\begin{aligned} P(X \geq 4) &= 1 - P(X < 4) \\ &= 1 - \{P(0) + P(1) + P(2) + P(3)\} \\ &= 1 - \left\{\frac{e^{-3}.3^0}{0!} + \frac{e^{-3}.3^1}{1!} + \frac{e^{-3}.3^2}{2!} + \frac{e^{-3}.3^3}{3!}\right\} \\ &= 1 - \{0.0498 + 0.1494 + 0.2240 + 0.2240\} \\ &= 0.3528 \end{aligned}$$

5. A hospital switch board receives an average of 4 emergency calls in a 10 minutes interval. What is the probability that (i) there are atleast 2 emergency calls (ii) there are exactly 3 emergency call in an interval of 10 minutes?

[M15/ChemBiot/6M]

Solution:

$m = 4$

By Poisson distribution,

$P(X = r) = \frac{e^{-m}.m^r}{r!} = \frac{e^{-4}.4^r}{r!}$

(i) $P(atleast\ 2) = P(X \geq 2)$

$= 1 - P(X < 2)$

$= 1 - \{P(0) + P(1)\}$

$= 1 - \{0.0183 + 0.0733\}$

$= 0.9084$

(ii) $P(exactly\ 3) = P(X = 3) = 0.1954$

6. In sampling a large number of parts manufactured by a machine, the mean number of defectives in a sample of 20 is 2. Out of 100 such samples, how many would you expect to have 3 defectives, using (i) Binomial distribution (ii) Poisson distribution

[N15/CompIT/6M]

Solution:

$m = 2$

$n = 20$

$m = np$

$p = \frac{m}{n} = \frac{2}{20} = \frac{1}{10}$

$q = 1 - p = \frac{9}{10}$

$N = 100$

(i) By Binomial Distribution

$P(X = r) = {}^{n}C_{r}.p^{r}.q^{n-r}$

$P(X = 3) = {}^{20}C_{3}.\left(\frac{1}{10}\right)^{3}.\left(\frac{9}{10}\right)^{17} = 0.1901$

Expected number of defectives $= N \times P$

$= 100 \times 0.1901$

$= 19.01 \approx 19$

(ii) By Poisson distribution,

$P(X = r) = \frac{e^{-m}.m^{r}}{r!}$

$P(X = 3) = \frac{e^{-2}.2^{3}}{3!} = 0.1804$

Expected number of defectives $= N \times P$

$= 100 \times 0.1804$

$= 18.04 \approx 18$

7. Fit a poisson distribution to the following data:

X	0	1	2	3	4	5	6	7	8
F	56	156	132	92	37	22	4	0	1

[N15/CompIT/6M]

Solution:

$N = \sum F = 56 + 156 + 132 + 92 + 37 + 22 + 4 + 0 + 1 = 500$

Mean $= \frac{\sum F.x}{\sum F} = \frac{56(0)+156(1)+132(2)+92(3)+37(4)+22(5)+4(6)+0(7)+1(8)}{500} = 1.972$

$m = 1.972$

By Poisson distribution,

$P(X = r) = \frac{e^{-m}.m^r}{r!} = \frac{e^{-1.972}.1.972^r}{r!}$

X	$P(X)$	$F(X) = NP = 500P$
0	0.1391	$69.589 \approx 70$
1	0.2745	$137.23 \approx 137$
2	0.2706	$135.31 \approx 135$
3	0.1779	$88.94 \approx 89$
4	0.0877	$43.85 \approx 44$
5	0.0346	$17.29 \approx 17$
6	0.0114	$5.68 \approx 6$
7	0.0032	$1.60 \approx 2$
8	0.0008	0

8. If a random variable X follows Poisson distribution such that P(X=1)=2P(X=2), find the mean, variance of the distribution. Also find P(X=3).
[M16/CompIT/6M][N16/CompIT/6M]
Solution:
By Poisson distribution,
$$P(X = r) = \frac{e^{-m}.m^r}{r!}$$
It is given that,
$$P(X = 1) = 2P(X = 2)$$
$$\frac{e^{-m}.m^1}{1!} = 2.\frac{e^{-m}.m^2}{2!}$$
$$m = m^2$$
$$m = 1$$
$$P(X = 3) = \frac{e^{-1}.1^3}{3!} = 0.0613$$

9. It is known that the probability of an item produced by a certain machine will be defective is 0.05. if the produced items are sent to the market in the packets of 20, find the number of packets containing (i) at least (ii) exactly (iii) at most 2 defective items in a consignment of 1000 packets using Poisson distribution approximation to the Binomial distribution.
[N16/AutoMechCivil/6M] [M17/AutoMechCivil/6M]
Solution:
$p = 0.05$
$n = 20$
$m = np = 1$
$N = 1000$
By Poisson distribution,
$P(X = r) = \frac{e^{-m}.m^r}{r!} = \frac{e^{-1}.1^r}{r!}$
(i) $P(at\ least\ 2) = P(X \geq 2)$
$= 1 - P(X < 2)$
$= 1 - \{P(0) + P(1)\}$
$= 1 - \{0.3679 + 0.3679\}$
$= 0.2642$
Expected number of packets $= N \times P = 1000 \times 0.2642 = 264.2 \approx 264$
(ii) $P(exactly\ 2) = P(X = 2)$
$= \frac{e^{-1}.1^2}{2!}$
$= 0.1839$
Expected number of packets $= N \times P = 1000 \times 0.1839 = 183.9 \approx 184$
(iii) $P(atmost\ 2) = P(X \leq 2)$
$= P(0) + P(1) + P(2)$
$= 0.3679 + 0.3679 + 0.1839$
$= 0.9197$
Expected number of packets $= N \times P = 1000 \times 0.9197 = 919.7 \approx 920$

10. If x is a Poisson variable such that $P(x = 1) = P(x = 2)$ find $E(x^2)$
[N17/AutoMechCivil/6M]
Solution:
We have,
$$P(X = x) = \frac{e^{-m} m^x}{x!}$$
Given that,
$$P(x = 1) = P(x = 2)$$
$$\frac{e^{-m} m^1}{1!} = \frac{e^{-m} m^2}{2!}$$
$$m = \frac{m^2}{2}$$
$$m = 2$$
$$\therefore mean = E(x) = 2$$
$$\therefore var = V(x) = 2$$
Now,
$$V(x) = E(x^2) - [E(x)]^2$$
$$2 = E(x^2) - [2]^2$$
$$E(x^2) = 6$$

Type III: Normal Distribution

1. The life of Army shoes normally distributed with mean 8 months and sd 2 months. If 5000 pairs are issued how many pairs would be expected to need replacement after 12 months?

   [N13/ChemBiot/6M]

   Solution:

   $\mu = 8$

   $\sigma = 2$

   $z = \frac{x-\mu}{\sigma} = \frac{x-8}{2}$

   $N = 2000$

   $$\begin{aligned} P(atleast\ 12\ months) &= P(X \geq 12) \\ &= P\left(z \geq \frac{12-8}{2}\right) \\ &= P(z \geq 2) \\ &= Area\ from\ z = 2\ to\ z = \infty \\ &= A(\infty) - A(2) \\ &= 0.5 - 0.4772 \\ &= 0.0228 \end{aligned}$$

   Expected number of pairs $= N \times P = 5000 \times 0.0228 = 114$

2. In a competitive examination, the top 15% of the students appeared will get grade 'A', while the bottom 20% will be declared fail. If the grades are normally distributed with mean % of marks 75 and SD 10, determine the lowest % of marks to receive grade A and lowest % of marks that passes.

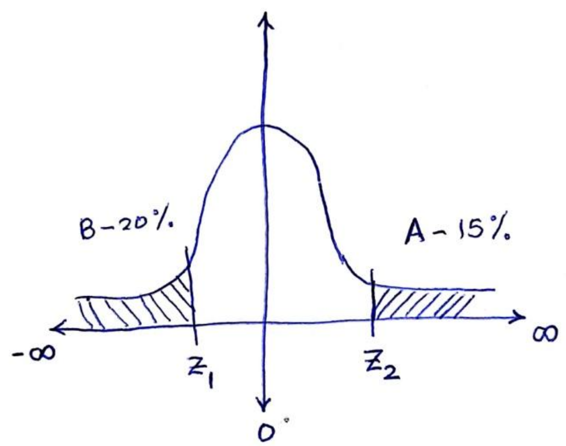
[M14/CompIT/8M]

Solution:

$\mu = 75$

$\sigma = 10$

$z = \frac{x-\mu}{\sigma} = \frac{x-75}{10}$



$A(0\ to\ z_1) = 30\% = 0.3$

From table, $z_1 = -0.84$

Thus, $z_1 = \frac{x_1-\mu}{\sigma}$

$-0.84 = \frac{x_1-75}{10}$

$-8.4 = x_1 - 75$

$x_1 = 66.6$

Thus, the highest of grade B is 66.6 i.e. the lowest percentage of marks that passes is $66.6 \approx 67$

$A(0\ to\ z_2) = 35\% = 0.35$

From table, $z_2 = 1.04$

Thus, $z_2 = \frac{x_2-\mu}{\sigma}$

$1.04 = \frac{x_2-75}{10}$

$10.4 = x_2 - 75$

$x_2 = 85.4$

Thus, the lowest percentage of marks of grade A is $85.4 \approx 85$

3. The marks obtained by students in a college are normally distributed with mean 65 and variance 25. If 3 students are selected at random from this college what is the probability that atleast one of them would have scored more than 75 marks?
[M14/ChemBiot/6M][N16/ChemBiot/6M]
Solution:
$\mu = 65$
$\sigma^2 = 25$
$z = \frac{x-\mu}{\sigma} = \frac{x-65}{5}$
$P(scoring\ more\ than\ 75) = P(X > 75)$
$= P\left(z > \frac{75-65}{5}\right)$
$= P(z > 2)$
$= Area\ from\ z = 2\ to\ z = \infty$
$= A(\infty) - A(2)$
$= 0.5 - 0.4772$
$= 0.0228$
Thus, P(scoring more than 75) $= p = 0.0228$
P(scoring not more than 75) $= q = 1 - p = 0.9772$
$n = 3$
By Binomial Distribution
$P(X = r) = {}^{n}C_r.p^r.q^{n-r} = {}^{3}C_r.(0.0228)^r.(0.9772)^{3-r}$
$P(atleast\ 1) = P(X \geq 1)$
$= 1 - P(X = 0)$
$= 1 - 0.9331$
$= 0.0669$

4. The marks of 1000 students of a university are normally distributed with mean 70 and standard deviation 5. Estimate the number of students whose marks will be (i) between 60 and 75 (ii) more than 75

[M14/AutoMechCivil/6M][M16/CompIT/8M]

Solution:

$\mu = 70$

$\sigma = 5$

$z = \frac{x-\mu}{\sigma} = \frac{x-70}{5}$

$N = 1000$

(i) $P(between\ 60\ and\ 75) = P(60 < X < 75)$

$$= P\left(\frac{60-70}{5} < z < \frac{75-70}{5}\right)$$
$$= P(-1 < z < 1)$$
$$= Area\ from\ z = -1\ to\ z = 1$$
$$= A(-1) + A(1)$$
$$= 2\ A(1)$$
$$= 2 \times 0.3413$$
$$= 0.6826$$

Expected number of students $= N \times P = 1000 \times 0.6826 = 682.6 \approx 683$

(ii) $P(more\ than\ 75) = P(X > 75)$

$$= P\left(z > \frac{75-70}{5}\right)$$
$$= P(z > 1)$$
$$= Area\ from\ z = 1\ to\ z = \infty$$
$$= A(\infty) - A(1)$$
$$= 0.5 - 0.3413$$
$$= 0.1587$$

Expected number of students $= N \times P = 1000 \times 0.1587 = 158.7 \approx 159$

5. The mean inside diameter of a sample of 200 washers produced by a machine is 0.502 inches and the standard deviation is 0.005 inches. The purpose for which these washers are intended allows a maximum tolerance in the diameter of 0.496 to 0.508 inches, otherwise the washers are considered defective. Determine the percentage of defective washers produced by the machinery assuming the diameters are normally distributed.
[N14/CompIT/8M]

Solution:

$N = 200$

$\mu = 0.502$

$\sigma = 0.005$

$z = \frac{x-\mu}{\sigma} = \frac{x-0.502}{0.005}$

$$
\begin{aligned}
P(non - defective\ washers) &= P(0.496 < X < 0.508) \\
&= P\left(\frac{0.496-0.502}{0.005} < z < \frac{0.508-0.502}{0.005}\right) \\
&= P(-1.2 < z < 1.2) \\
&= Area\ from\ z = -1.2\ to\ z = 1.2 \\
&= 2\ A(1.2) \\
&= 2 \times 0.3849 \\
&= 0.7698 \\
&= 76.98\%
\end{aligned}
$$

Thus,

$P(defective\ washers) = 1 - 0.7698 = 0.2302 = 23.02\%$

6. The weights of 4000 students are found to be normally distributed with mean 50 kgs and standard deviation 5 kgs. Find the probability that the students selected at random will have weight (i) less than 45 (ii) between 45 and 60.

[N14/AutoMechCivil/6M]

Solution:

$N = 4000$

$\mu = 50$

$\sigma = 5$

$z = \frac{x-\mu}{\sigma} = \frac{x-50}{5}$

(i) $P(less\ than\ 45) = P(X < 45)$

$= P\left(z < \frac{45-50}{5}\right)$

$= P(z < -1)$

$= Area\ from\ z = -\infty\ to\ z = -1$

$= A(\infty) - A(1)$

$= 0.5 - 0.3413$

$= 0.1587$

(ii) $P(between\ 45\ and\ 60) = P(45 < X < 60)$

$= P\left(\frac{45-50}{5} < z < \frac{60-50}{5}\right)$

$= P(-1 < z < 2)$

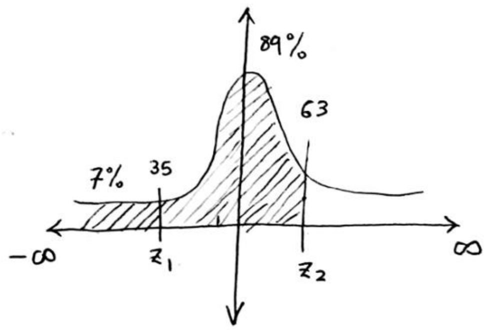
$= Area\ from\ z = -1\ to\ z = 2$

$= A(1) + A(2)$

$= 0.3413 + 0.4772$

$= 0.8185$

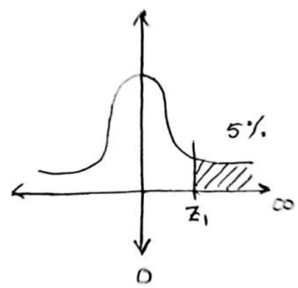
7. In a normal distribution 7 % items are under 35 & 89% are under 63. Find probability of items lies between 45 and 56.
[M15/CompIT/8M]
Solution:



$A(0\ to\ z_1) = 43\% = 0.43$
From table, $z_1 = -1.48$
$z_1 = \frac{x_1-\mu}{\sigma}$
$-1.48 = \frac{35-\mu}{\sigma}$
$-1.48\sigma + \mu = 35$ ……………….(1)
$A(0\ to\ z_2) = 39\% = 0.39$
From table, $z_2 = 1.23$
$z_2 = \frac{x_2-\mu}{\sigma}$
$1.23 = \frac{63-\mu}{\sigma}$
$1.23\sigma + \mu = 63$ ………………….(2)
Solving (1) & (2), we get
$\sigma = 10.33, \mu = 50.29$
$\therefore z = \frac{x-\mu}{\sigma} = \frac{x-50.29}{10.33}$

$$\begin{aligned} P(between\ 45\ and\ 56) &= P(45 < x < 56) \\ &= P\left(\frac{45-50.29}{10.33} < z < \frac{56-50.29}{10.33}\right) \\ &= P(-0.51 < z < 0.55) \\ &= A(0.51) +\ A(0.55) \\ &= 0.1950 + 0.2088 \\ &= 0.4038 \end{aligned}$$

8. Monthly salary in a big organization is normally distributed with mean Rs. 3000 & s.d Rs. 250. What should be the minimum salary of workers in this organization so that the probability that he belongs to top 5% workers?
[M15/AutoMechCivil/6M][N17/AutoMechCivil/6M]
Solution:
$\mu = 3000$
$\sigma = 250$
$z = \frac{x-\mu}{\sigma} = \frac{x-3000}{250}$

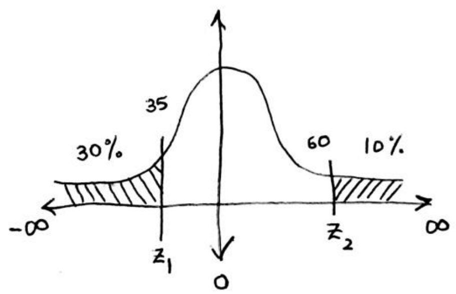


$A(0\ to\ z_1) = 45\% = 0.45$
From table, $z_1 = \frac{1.64+1.65}{2} = 1.645$
$z_1 = \frac{x_1-3000}{250}$
$1.645 = \frac{x_1-3000}{250}$
$\therefore x_1 = 3411.25 \approx 3411$
Thus, the minimum salary of workers should be Rs. 3411/- so that they belong to the top 5% workers.

9. For a normal variate with mean 2.5 and sd 3.5, find the probability that
(i) $2 \leq x \leq 4.5$ (ii) $-1.5 \leq x \leq 5.5$
[N15/AutoMechCivil/6M]
Solution:
$\mu = 2.5$
$\sigma = 3.5$
$z = \frac{x-\mu}{\sigma} = \frac{x-2.5}{3.5}$
(i) $P(2 \leq x \leq 4.5) = P\left(\frac{2-2.5}{3.5} \leq z \leq \frac{4.5-2.5}{3.5}\right)$
$= P(-0.14 \leq z \leq 0.57)$
$= A(0.14) + A(0.57)$
$= 0.0557 + 0.2157$
$= 0.2714$
(ii) $P(-1.5 \leq x \leq 5.5) = P\left(\frac{-1.5-2.5}{3.5} \leq z \leq \frac{5.5-2.5}{3.5}\right)$
$= P(-1.14 \leq z \leq 0.86)$
$= A(1.14) + A(0.86)$
$= 0.3729 + 0.3051$
$= 0.6780$

10. If the heights of 500 students is normally distributed with mean 68 inches and standard deviation 4 inches. Find the expected number of students having heights: (i) less than 62 inches (ii) between 65 and 71 inches.
[N15/CompIT/8M]
Solution:
$N = 500$
$\mu = 68$
$\sigma = 4$
$z = \frac{x-\mu}{\sigma} = \frac{x-68}{4}$
(i) $P(less\ than\ 62) = P(X < 62)$
$= P\left(z < \frac{62-68}{4}\right)$
$= P(z < -1.5)$
$= Area\ from\ z = -\infty\ to\ z = -1.5$
$= A(\infty) - A(1.5)$
$= 0.5 - 0.4332$
$= 0.0668$
Expected number of students $= N \times P = 500 \times 0.0668 = 33.4 \approx 33$
(ii) $P(between\ 65\ and\ 71) = P(65 < X < 71)$
$= P\left(\frac{65-68}{4} < z < \frac{71-68}{4}\right)$
$= P(-0.75 < z < 0.75)$
$= Area\ from\ z = -0.75\ to\ z = 0.75$
$= A(0.75) + A(0.75)$
$= 2 \times 0.2734$
$= 0.5468$
Expected number of students $= N \times P = 500 \times 0.5468 = 273.4 \approx 273$

11. Find the mean & s.d of a normal distribution of marks in an exam where 30% of the candidates obtained marks below 35, 10% got above 60.
[M16/AutoMechCivil/6M]
Solution:



$A(0\ to\ z_1) = 20\% = 0.20$
From table, $z_1 = -0.52$
$z_1 = \frac{x_1 - \mu}{\sigma}$
$-0.52 = \frac{35 - \mu}{\sigma}$
$-0.52\sigma + \mu = 35$ ……………….(1)
$A(0\ to\ z_2) = 40\% = 0.40$
From table, $z_2 = 1.28$
$z_2 = \frac{x_2 - \mu}{\sigma}$
$1.28 = \frac{60 - \mu}{\sigma}$
$1.28\sigma + \mu = 60$ ………………….(2)
Solving (1) & (2), we get
$\sigma = 13.88, \mu = 42.22$

12. If X is a normal variable with mean 10 & standard deviation 4. Find
(i) $P(5 \leq X \leq 18)$ (ii) $P(X \leq 12)$ (iii) $P(|X-14|<1)$
[N16/CompIT/8M][N17/CompIT/8M]
Solution:
$\mu = 10$
$\sigma = 4$
$z = \frac{x-\mu}{\sigma} = \frac{x-10}{4}$
(i) $P(5 \leq x \leq 18) = P\left(\frac{5-10}{4} \leq z \leq \frac{18-10}{4}\right)$
$= P(-1.25 \leq z \leq 2)$
$= A(1.25) + A(2)$
$= 0.3944 + 0.4772$
$= 0.8716$
(ii) $P(x \leq 12) = P\left(z \leq \frac{12-10}{4}\right)$
$= P(z \leq 0.5)$
$= A(\infty) + A(0.5)$
$= 0.5 + 0.1915$
$= 0.6915$
(iii) $P(|x-14|<1) = P(-1<(x-14)<1)$
$= P(-1+14<x<1+14)$
$= P(13<x<15)$
$= P\left(\frac{13-10}{4}<z<\frac{15-10}{4}\right)$
$= P(0.75<z<1.25)$
$= A(1.25) - A(0.75)$
$= 0.3944 - 0.2734$
$= 0.1210$

13. In an intelligence test administered to 1000 students the average score was 42 and standard deviation 24. Find the number of students (i) exceeding the score 50 (ii) between 30 and 54

[N16/AutoMechCivil/6M] [M17/AutoMechCivil/6M]

Solution:

$N = 1000$

$\mu = 42$

$\sigma = 24$

$z = \frac{x-\mu}{\sigma} = \frac{x-42}{24}$

(i) $P(exceeding\ 50) = P(X > 50)$

$$= P\left(z > \frac{50-42}{24}\right)$$
$$= P(z > 0.33)$$
$$= A(\infty) - A(0.33)$$
$$= 0.5 - 0.1293$$
$$= 0.3707$$

Expected number of students $= N \times P = 1000 \times 0.3707 = 370.7 \approx 371$

(ii) $P(between\ 30\ and\ 54) = P(30 < X < 54)$

$$= P\left(\frac{30-42}{24} < z < \frac{54-42}{24}\right)$$
$$= P(-0.5 < z < 0.5)$$
$$= Area\ from\ z = -0.5\ to\ z = 0.5$$
$$= A(0.5) + A(0.5)$$
$$= 2 \times 0.1915$$
$$= 0.383$$

Expected number of students $= N \times P = 1000 \times 0.383 = 383$

14. A manufacturer knows from his past experience that the resistance of resistors he produces is normal with mean 100 ohms and standard deviation 2 ohms. What percentage of resistors will have resistance between 98 ohms and 102 ohms?

[M17/CompIT/8M]

Solution:

$\mu = 100$

$\sigma = 2$

$z = \frac{x-\mu}{\sigma} = \frac{x-100}{2}$

$$
\begin{aligned}
P(between\ 98\ and\ 102) &= P(98 < X < 102) \\
&= P\left(\frac{98-100}{2} < z < \frac{102-100}{2}\right) \\
&= P(-1 < z < 1) \\
&= Area\ from\ z = -1\ to\ z = 1 \\
&= A(1) + A(1) \\
&= 2 \times 0.3413 \\
&= 0.6826 \\
&= 68.26\%
\end{aligned}
$$

Theory Questions:

1. Find the moment generating function of a Poisson distribution. Hence, find its mean and variance.
[M14/CompIT/6M]
2. Find first two moments about the origin of Poisson distribution and hence find mean and variance
[M17/CompIT/6M]

Solution:

We have, m.g.f. given by $M_0(t) = E(e^{tx}) = \sum P(x)e^{tx}$

$$M_0(t) = \sum \frac{e^{-m} m^x}{x!} e^{tx} = e^{-m} \sum \frac{(me^t)^x}{x!}$$

$$M_0(t) = e^{-m}\left[1 + \frac{me^t}{1!} + \frac{(me^t)^2}{2!} + \frac{(me^t)^3}{3!} + ..\right] = e^{-m} e^{me^t} = e^{m(e^t-1)}$$

$$\therefore \mu_1' = \left[\frac{d}{dt} M_0(t)\right]_{t=0} = \left[e^{m(e^t-1)}.me^t\right]_{t=0} = m$$

$\therefore$ mean = m

$$\mu_2' = \left[\frac{d^2}{dt^2} M_0(t)\right]_{t=0} = \left[e^{m(e^t-1)}.me^t + me^t.e^{m(e^t-1)}.me^t\right]_{t=0}$$

$$\mu_2' = m + m^2$$

$\therefore$ variance $= \mu_2' - \mu_1'^2 = (m + m^2) - (m)^2 = m$

3. Find mean and variance of binomial distribution
[M15/CompIT/6M]
Solution:
We have the first moment about origin as
$\mu_1' = E(X) = \sum P.x$
$\therefore \mu_1' = \sum_{x=0} {}^nC_x p^x q^{n-x}.x$ [For a binomial distribution, $P = {}^nC_x p^x q^{n-x}$]
$\therefore \mu_1' = {}^nC_0 p^0 q^n.0 + {}^nC_1 p^1 q^{n-1}.1 + {}^nC_2 p^2 q^{n-2}.2 + \cdots .. + {}^nC_n p^n q^0.n$
$\therefore \mu_1' = npq^{n-1} + \frac{n(n-1)}{2!}.p^2 q^{n-2}.2 + \cdots \dots \dots + p^n.n$
$$\therefore \mu_1' = np\left[q^{n-1} + (n-1)q^{n-2}.p + \frac{(n-1)(n-2)}{2!}q^{n-3}p^3 + \cdots \dots + p^{n-1}\right]$$
$\therefore \mu_1' = np[q+p]^{n-1} = np$ $[\because p + q = 1]$
$\therefore$ mean $= np$
We have the second moment about origin as
$\mu_2' = E(X^2) = \sum P.x^2 = \sum_{x=0} {}^nC_x p^x q^{n-x}.x^2$
But x^2 can be written as $x^2 = x + x(x-1)$
$\therefore \mu_2' = \sum[x + x(x-1)]\, {}^nC_x p^x q^{n-x}$
$\therefore \mu_2' = \sum_{x=0} {}^nC_x p^x q^{n-x}.x + \sum_{x=0} {}^nC_x p^x q^{n-x}.x\,(x-1)$
$\therefore \mu_2' = np + [\,{}^nC_2 p^2 q^{n-2}.2.1 + {}^nC_3 p^3 q^{n-3}.3.2 + {}^nC_4 p^4 q^{n-4}.4.3 + \cdots]$
$\therefore \mu_2' = np + \frac{n(n-1)}{2!}p^2 q^{n-2}.2 + \frac{n(n-1)(n-2)}{3!}p^3 q^{n-3}.6 + \frac{n(n-1)(n-2)(n-3)}{4!}p^4 q^{n-4}.12 + \cdots$
$\therefore \mu_2' = np + n(n-1)p^2 q^{n-2} + n(n-1)(n-2)p^3 q^{n-3} + \frac{n(n-1)(n-2)(n-3)}{2}p^4 q^{n-4} + \cdots$
$\therefore \mu_2' = np + n(n-1)p^2\left[q^{n-2} + (n-2)pq^{n-3} + \frac{(n-2)(n-3)}{2!}p^2 q^{n-4} + \cdots\right]$
$\therefore \mu_2' = np + n(n-1)p^2(q+p)^{n-2} = np + n(n-1)p^2$
$\therefore \mu_2' = np[1 + (n-1)p] = np[1 - p + np]$
$\therefore \mu_2' = np[q + np] = npq + n^2p^2$
$\therefore$ Variance $= \mu_2' - {\mu_1'}^2 = npq + n^2p^2 - (np)^2 = npq$

4. Find mean and standard deviation of binomial distribution.
[M15/CompIT/6M]
Solution:
We have the first moment about origin as
$\mu_1^{'} = E(X) = \sum P.x$
$\therefore \mu_1^{'} = \sum_{x=0} {}^nC_x p^x q^{n-x}.x$ [For a binomial distribution, $P = {}^nC_x p^x q^{n-x}$]
$\therefore \mu_1^{'} = {}^nC_0 p^0 q^n.0 + {}^nC_1 p^1 q^{n-1}.1 + {}^nC_2 p^2 q^{n-2}.2 + \cdots .. + {}^nC_n p^n q^0.n$
$\therefore \mu_1^{'} = npq^{n-1} + \frac{n(n-1)}{2!}.p^2 q^{n-2}.2 + \cdots \ldots \ldots + p^n.n$
$\therefore \mu_1^{'} = np\left[q^{n-1} + (n-1)q^{n-2}.p + \frac{(n-1)(n-2)}{2!} q^{n-3}p^3 + \cdots \ldots + p^{n-1}\right]$
$\therefore \mu_1^{'} = np[q+p]^{n-1} = np$ [$\because p + q = 1$]
$\therefore$ mean $= np$
We have the second moment about origin as
$\mu_2^{'} = E(X^2) = \sum P.x^2 = \sum_{x=0} {}^nC_x p^x q^{n-x}.x^2$
But x^2 can be written as $x^2 = x + x(x-1)$
$\therefore \mu_2^{'} = \sum[x + x(x-1)]\, {}^nC_x p^x q^{n-x}$
$\therefore \mu_2^{'} = \sum_{x=0} {}^nC_x p^x q^{n-x}.x + \sum_{x=0} {}^nC_x p^x q^{n-x}.x\,(x-1)$
$\therefore \mu_2^{'} = np + [\,{}^nC_2 p^2 q^{n-2}.2.1 + {}^nC_3 p^3 q^{n-3}.3.2 + {}^nC_4 p^4 q^{n-4}.4.3 + \cdots]$
$\therefore \mu_2^{'} = np + \frac{n(n-1)}{2!} p^2 q^{n-2}.2 + \frac{n(n-1)(n-2)}{3!} p^3 q^{n-3}.6 + \frac{n(n-1)(n-2)(n-3)}{4!} p^4 q^{n-4}.12 + \cdots$
$\therefore \mu_2^{'} = np + n(n-1)p^2 q^{n-2} + n(n-1)(n-2)p^3 q^{n-3} + \frac{n(n-1)(n-2)(n-3)}{2} p^4 q^{n-4} + \cdots$
$\therefore \mu_2^{'} = np + n(n-1)p^2\left[q^{n-2} + (n-2)pq^{n-3} + \frac{(n-2)(n-3)}{2!} p^2 q^{n-4} + \cdots\right]$
$\therefore \mu_2^{'} = np + n(n-1)p^2 (q+p)^{n-2} = np + n(n-1)p^2$
$\therefore \mu_2^{'} = np[1 + (n-1)p] = np[1 - p + np]$
$\therefore \mu_2^{'} = np[q + np] = npq + n^2p^2$
$\therefore$ Variance $= \mu_2^{'} - {\mu_1^{'}}^2 = npq + n^2p^2 - (np)^2 = npq$
$\therefore$ standard deviation $= \sqrt{variance} = \sqrt{npq}$