Taylor's & Laurent's Series

1. Obtain two distinct Laurent's series for $\frac{2z-3}{z^2-4z+3}$ in powers of (z – 4) indicating the regions of convergence.
[N13/AutoMechCivil/8M][M15/AutoMechCivil/8M]
[M15/ElexExtcElectBiomInst/8M][M17/ElexExtcElectBiomInst/8M]
Solution:
We have, $f(z) = \frac{2z-3}{z^2-4z+3} = \frac{2z-3}{(z-3)(z-1)}$
Let $\frac{2z-3}{(z-3)(z-1)} = \frac{A}{z-3} + \frac{B}{z-1}$
$2z - 3 = A(z-1) + B(z-3)$
On comparing the coefficients, we get
$A + B = 2$
$-A - 3B = -3$
On solving, we get
$A = \frac{3}{2}, B = \frac{1}{2}$
$f(z) = \frac{\frac{3}{2}}{z-3} + \frac{\frac{1}{2}}{z-1}$
Now, put $z - 4 = u$ i.e $z = u + 4$
$\therefore f(z) = \frac{\frac{3}{2}}{u+4-3} + \frac{\frac{1}{2}}{u+4-1} = \frac{\frac{3}{2}}{u+1} + \frac{\frac{1}{2}}{u+3}$
The ROCs are as follows
(i) $|u| < 1$ (ii) $1 < |u| < 3$ (iii) $|u| > 3$

Now, Laurents series are as follows
(a) For $1 < |u| < 3$ i.e. $1 < |z-4| < 3$
$f(z) = \frac{\frac{3}{2}}{u+1} + \frac{\frac{1}{2}}{3+u}$
$f(z) = \frac{\frac{3}{2}}{u\left[1+\frac{1}{u}\right]} + \frac{\frac{1}{2}}{3\left[1+\frac{u}{3}\right]}$
$f(z) = \frac{3}{2u}\left[1+\frac{1}{u}\right]^{-1} + \frac{1}{6}\left[1+\frac{u}{3}\right]^{-1}$
$f(z) = \frac{3}{2u}\left[1 - \frac{1}{u} + \frac{1}{u^2} - \frac{1}{u^3} + \cdots \ldots\right] + \frac{1}{6}\left[1 - \frac{u}{3} + \frac{u^2}{9} - \frac{u^3}{27} + \cdots \ldots\right]$
$f(z) = \frac{3}{2}\left[\frac{1}{u} - \frac{1}{u^2} + \frac{1}{u^3} - \cdots \ldots..\right] + \frac{1}{6}\left[1 - \frac{u}{3} + \frac{u^2}{9} - \cdots \ldots\right]$
$f(z) = \frac{3}{2}\left[\frac{1}{z-4} - \frac{1}{(z-4)^2} + \frac{1}{(z-4)^3} - \cdots ..\right] + \frac{1}{6}\left[1 - \frac{z-4}{3} + \frac{(z-4)^2}{9} - \cdots ..\right]$

(b) For $|u| > 3$ i.e. $|z-4| > 3$

$$f(z) = \frac{\frac{3}{2}}{u+1} + \frac{\frac{1}{2}}{u+3}$$

$$f(z) = \frac{\frac{3}{2}}{u\left[1+\frac{1}{u}\right]} + \frac{\frac{1}{2}}{u\left[1+\frac{3}{u}\right]}$$

$$f(z) = \frac{3}{2u}\left[1+\frac{1}{u}\right]^{-1} + \frac{1}{2u}\left[1+\frac{3}{u}\right]^{-1}$$

$$f(z) = \frac{3}{2u}\left[1-\frac{1}{u}+\frac{1}{u^2}-\frac{1}{u^3}+\cdots\ldots\right] + \frac{1}{2u}\left[1-\frac{3}{u}+\frac{9}{u^2}-\frac{27}{u^3}+\cdots\ldots\right]$$

$$f(z) = \frac{3}{2}\left[\frac{1}{u}-\frac{1}{u^2}+\frac{1}{u^3}-\cdots\ldots..\right] + \frac{1}{2}\left[\frac{1}{u}-\frac{3}{u^2}+\frac{9}{u^3}-\cdots\ldots\right]$$

$$f(z) = \frac{3}{2}\left[\frac{1}{z-4}-\frac{1}{(z-4)^2}+\frac{1}{(z-4)^3}-\cdots..\right] + \frac{1}{2}\left[\frac{1}{z-4}-\frac{3}{(z-4)^2}+\frac{9}{(z-4)^3}-\cdots..\right]$$

2. Obtain all Taylors and Laurents expansions of function $\frac{(z-2)(z+2)}{(z+1)(z+4)}$ about $z=0$

[M14/CompIT/8M]

Solution:

We have, $f(z)=\frac{(z-2)(z+2)}{(z+1)(z+4)}=\frac{z^2-4}{z^2+5z+4}$

$f(z)=\frac{z^2+5z+4-5z-4-4}{z^2+5z+4}$

$f(z)=1+\frac{-5z-8}{z^2+5z+4}$

Let $\frac{-5z-8}{(z+1)(z+4)}=\frac{A}{z+1}+\frac{B}{z+4}$

$-5z-8=A(z+4)+B(z+1)$

Comparing the coefficients, we get

$A+B=-5$

$4A+B=-8$

On solving, we get

$A=-1, B=-4$

$f(z)=1-\frac{1}{z+1}-\frac{4}{z+4}$

The ROCs are as follows

(i) $|z|<1$ (ii) $1<|z|<4$ (iii) $|z|>4$

The Taylors series is given by

(i) For $|z|<1$

$f(z)=1-\frac{1}{1+z}-\frac{4}{4+z}$

$f(z)=1-\frac{1}{1+z}-\frac{4}{4\left(1+\frac{z}{4}\right)}$

$f(z)=1-[1+z]^{-1}-\left[1+\frac{z}{4}\right]^{-1}$

$f(z)=1-[1-z+z^2-z^3+\cdots..]-\left[1-\frac{z}{4}+\frac{z^2}{16}-\frac{z^3}{64}+\cdots.\right]$

The Laurents series is given by

(ii) For $1<|z|<4$

$f(z)=1-\frac{1}{z+1}-\frac{4}{4+z}$

$f(z)=1-\frac{1}{z\left(1+\frac{1}{z}\right)}-\frac{4}{4\left(1+\frac{z}{4}\right)}$

$f(z)=1-\frac{1}{z}\left[1+\frac{1}{z}\right]^{-1}-\left[1+\frac{z}{4}\right]^{-1}$

$f(z)=1-\frac{1}{z}\left[1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\cdots..\right]-\left[1-\frac{z}{4}+\frac{z^2}{16}-\frac{z^3}{64}+\cdots.\right]$

The second Laurents series is given by

(iii) For $|z| > 4$

$$f(z) = 1 - \frac{1}{z+1} - \frac{4}{z+4}$$

$$f(z) = 1 - \frac{1}{z\left(1+\frac{1}{z}\right)} - \frac{4}{z\left(1+\frac{4}{z}\right)}$$

$$f(z) = 1 - \frac{1}{z}\left[1+\frac{1}{z}\right]^{-1} - \frac{1}{z}\left[1+\frac{4}{z}\right]^{-1}$$

$$f(z) = 1 - \frac{1}{z}\left[1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots..\right] - \frac{1}{z}\left[1 - \frac{4}{z} + \frac{16}{z^2} - \frac{64}{z^3} + \cdots.\right]$$

3. Expand $f(z) = \frac{z^2-1}{z^2+5z+6}$ about $z = 0$
[M14/ElexExtcElectBiomInst/8M][M17/CompIT/8M]
Solution:
We have, $f(z) = \frac{z^2-1}{z^2+5z+6}$

$f(z) = \frac{z^2+5z+6-5z-6-1}{z^2+5z+6}$

$f(z) = 1 + \frac{-5z-7}{z^2+5z+6}$

Let $\frac{-5z-7}{(z+2)(z+3)} = \frac{A}{z+2} + \frac{B}{z+3}$

$-5z - 7 = A(z+3) + B(z+2)$

Comparing the coefficients, we get

$A + B = -5$

$3A + 2B = -7$

On solving, we get

$A = 3, B = -8$

$f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}$

The ROCs are as follows

(i) $|z| < 2$ (ii) $2 < |z| < 3$ (iii) $|z| > 3$

The Taylors series is given by

(i) For $|z| < 2$

$f(z) = 1 + \frac{3}{2+z} - \frac{8}{3+z}$

$f(z) = 1 + \frac{3}{2\left(1+\frac{z}{2}\right)} - \frac{8}{3\left(1+\frac{z}{3}\right)}$

$f(z) = 1 + \frac{3}{2}\left[1 + \frac{z}{2}\right]^{-1} - \frac{8}{3}\left[1 + \frac{z}{3}\right]^{-1}$

$f(z) = 1 + \frac{3}{2}\left[1 - \frac{z}{2} + \frac{z^2}{4} - \frac{z^3}{8} + \cdots..\right] - \frac{8}{3}\left[1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \cdots.\right]$

The Laurents series is given by

(ii) For $2 < |z| < 3$

$f(z) = 1 + \frac{3}{z+2} - \frac{8}{3+z}$

$f(z) = 1 + \frac{3}{z\left(1+\frac{2}{z}\right)} - \frac{8}{3\left(1+\frac{z}{3}\right)}$

$f(z) = 1 + \frac{3}{z}\left[1 + \frac{2}{z}\right]^{-1} - \frac{8}{3}\left[1 + \frac{z}{3}\right]^{-1}$

$f(z) = 1 + \frac{3}{z}\left[1 - \frac{2}{z} + \frac{4}{z^2} - \frac{8}{z^3} + \cdots..\right] - \frac{8}{3}\left[1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \cdots.\right]$

The Laurents series is given by

(ii) For $|z| > 3$

$$f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

$$f(z) = 1 + \frac{3}{z\left(1+\frac{2}{z}\right)} - \frac{8}{z\left(1+\frac{3}{z}\right)}$$

$$f(z) = 1 + \frac{3}{z}\left[1+\frac{2}{z}\right]^{-1} - \frac{8}{z}\left[1+\frac{3}{z}\right]^{-1}$$

$$f(z) = 1 + \frac{3}{z}\left[1 - \frac{2}{z} + \frac{4}{z^2} - \frac{8}{z^3} + \cdots..\right] - \frac{8}{z}\left[1 - \frac{3}{z} + \frac{9}{z^2} - \frac{27}{z^3} + \cdots.\right]$$

4. Find all possible Laurent's expansions of the function $\frac{7z-2}{z(z-2)(z+1)}$ about z = -1 indicating the region of convergence.

[M14/AutoMechCivil/8M][M15/CompIT/8M]

Solution:

We have, $f(z) = \frac{7z-2}{z(z-2)(z+1)}$

Let $\frac{7z-2}{z(z-2)(z+1)} = \frac{A}{z} + \frac{B}{z-2} + \frac{C}{z+1}$

$7z - 2 = A(z-2)(z+1) + Bz(z+1) + Cz(z-2)$

$7z - 2 = A(z^2 - z - 2) + B(z^2 + z) + C(z^2 - 2z)$

On comparing the coefficients, we get

$A + B + C = 0$

$-A + B - 2C = 7$

$-2A = -2$

On solving, we get

$A = 1, B = 2, C = -3$

$f(z) = \frac{1}{z} + \frac{2}{z-2} - \frac{3}{z+1}$

Now, put $z + 1 = u$ i.e $z = u - 1$

$\therefore f(z) = \frac{1}{u-1} + \frac{2}{u-1-2} - \frac{3}{u-1+1}$

$\therefore f(z) = \frac{1}{u-1} + \frac{2}{u-3} - \frac{3}{u}$

The ROCs are as follows

(i) $|u| < 1$ (ii) $1 < |u| < 3$ (iii) $|u| > 3$

The Taylors Series is given by

(i) For $|u| < 1$ i.e. $|z+1| < 1$

$f(z) = \frac{1}{-1+u} + \frac{2}{-3+u} - \frac{3}{u}$

$f(z) = \frac{1}{-1(1-u)} + \frac{2}{-3\left(1-\frac{u}{3}\right)} - \frac{3}{u}$

$f(z) = -1[1-u]^{-1} - \frac{2}{3}\left[1 - \frac{u}{3}\right]^{-1} - \frac{3}{u}$

$f(z) = -[1 + u + u^2 + u^3 + \cdots.] - \frac{2}{3}\left[1 + \frac{u}{3} + \frac{u^2}{9} + \frac{u^3}{27} + \cdots.\right] - \frac{3}{u}$

$f(z) = -[1 + (z+1) + (z+1)^2 + \cdots.] - \frac{2}{3}\left[1 + \frac{z+1}{3} + \frac{(z+1)^2}{9} + \cdots\right] - \frac{3}{z+1}$

The first Laurents Series is given by

(ii) For $1 < |u| < 3$ i.e. $1 < |z+1| < 3$

$f(z) = \frac{1}{u-1} + \frac{2}{-3+u} - \frac{3}{u}$

$$f(z)=\frac{1}{u\left(1-\frac{1}{u}\right)}+\frac{2}{-3\left(1-\frac{u}{3}\right)}-\frac{3}{u}$$

$$f(z)=\frac{1}{u}\left[1-\frac{1}{u}\right]^{-1}-\frac{2}{3}\left[1-\frac{u}{3}\right]^{-1}-\frac{3}{u}$$

$$f(z)=\frac{1}{u}\left[1+\frac{1}{u}+\frac{1}{u^2}+\frac{1}{u^3}+\cdots.\right]-\frac{2}{3}\left[1+\frac{u}{3}+\frac{u^2}{9}+\frac{u^3}{27}+\cdots.\right]-\frac{3}{u}$$

$$f(z)=\frac{1}{z+1}\left[1+\frac{1}{(z+1)}+\frac{1}{(z+1)^2}+\cdots.\right]-\frac{2}{3}\left[1+\frac{z+1}{3}+\frac{(z+1)^2}{9}+\cdots\right]-\frac{3}{z+1}$$

The second Laurents Series is given by

(iii) For $|u|>3$ i.e. $|z+1|>3$

$$f(z)=\frac{1}{u-1}+\frac{2}{u-3}-\frac{3}{u}$$

$$f(z)=\frac{1}{u\left(1-\frac{1}{u}\right)}+\frac{2}{u\left(1-\frac{3}{u}\right)}-\frac{3}{u}$$

$$f(z)=\frac{1}{u}\left[1-\frac{1}{u}\right]^{-1}+\frac{2}{u}\left[1-\frac{3}{u}\right]^{-1}-\frac{3}{u}$$

$$f(z)=\frac{1}{u}\left[1+\frac{1}{u}+\frac{1}{u^2}+\frac{1}{u^3}+\cdots.\right]+\frac{2}{u}\left[1+\frac{3}{u}+\frac{9}{u^2}+\frac{27}{u^3}+\cdots.\right]-\frac{3}{u}$$

$$f(z)=\frac{1}{z+1}\left[1+\frac{1}{(z+1)}+\frac{1}{(z+1)^2}+\cdots.\right]+\frac{2}{(z+1)}\left[1+\frac{3}{z+1}+\frac{9}{(z+1)^2}+\cdots\right]-\frac{3}{z+1}$$

5. Find Laurent's Series which represents the function $f(z)=\frac{2}{(z-1)(z-2)}$ where (i) $|z|<1$ (ii) $1<|z|<2$ (iii) $|z|>2$
[N14/CompIT/8M][N15/CompIT/8M][N16/CompIT/8M]
Solution:
We have, $f(z)=\frac{2}{(z-1)(z-2)}$
Let $\frac{2}{(z-1)(z-2)}=\frac{A}{z-1}+\frac{B}{z-2}$
$2=A(z-2)+B(z-1)$
Comparing the coefficients, we get
$A+B=0$
$-2A-B=2$
On solving, we get
$A=-2, B=2$
$f(z)=-\frac{2}{z-1}+\frac{2}{z-2}$
(i) $|z|<1$
$f(z)=-\frac{2}{-1+z}+\frac{2}{-2+z}$
$f(z)=-\frac{2}{-[1-z]}+\frac{2}{-2\left[1-\frac{z}{2}\right]}$
$f(z)=2[1-z]^{-1}-1\left[1-\frac{z}{2}\right]^{-1}$
$f(z)=2[1+z+z^2+z^3+\cdots.]-\left[1+\frac{z}{2}+\frac{z^2}{2^2}+\frac{z^3}{2^3}+\cdots\right]$
(ii) $1<|z|<2$
$f(z)=-\frac{2}{z-1}+\frac{2}{-2+z}$
$f(z)=-\frac{2}{z\left[1-\frac{1}{z}\right]}+\frac{2}{-2\left[1-\frac{z}{2}\right]}$
$f(z)=-\frac{2}{z}\left[1-\frac{1}{z}\right]^{-1}-1\left[1-\frac{z}{2}\right]^{-1}$
$f(z)=-\frac{2}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots.\right]-\left[1+\frac{z}{2}+\frac{z^2}{2^2}+\frac{z^3}{2^3}+\cdots\right]$
(iii) $|z|>2$
$f(z)=-\frac{2}{z-1}+\frac{2}{z-2}$
$f(z)=-\frac{2}{z\left[1-\frac{1}{z}\right]}+\frac{2}{z\left[1-\frac{2}{z}\right]}$
$f(z)=-\frac{2}{z}\left[1-\frac{1}{z}\right]^{-1}+\frac{2}{z}\left[1-\frac{2}{z}\right]^{-1}$
$f(z)=-\frac{2}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots.\right]+\frac{2}{z}\left[1+\frac{2}{z}+\frac{2^2}{z^2}+\frac{2^3}{z^3}+\cdots\right]$

6. Obtain Taylor's and Laurent's series for $f(z)=\frac{1}{(1+z^2)(z+2)}$ for

(i) $1<|z|<2$ (ii) $|z|>2$

[N14/AutoMechCivil/8M]

Solution:

We have, $f(z)=\frac{1}{(1+z^2)(z+2)}$

Let $\frac{1}{(1+z^2)(z+2)}=\frac{Az+B}{1+z^2}+\frac{C}{z+2}$

$1=(Az+B)(z+2)+C(1+z^2)$

$1=A(z^2+2z)+B(z+2)+C(z^2+1)$

Comparing the coefficients, we get

$A+C=0$

$2A+B=0$

$2B+C=1$

On solving, we get

$A=-\frac{1}{5},B=\frac{2}{5},C=\frac{1}{5}$

$f(z)=\frac{-\frac{z}{5}+\frac{2}{5}}{1+z^2}+\frac{\frac{1}{5}}{z+2}=\frac{1}{5}\left[\frac{2-z}{1+z^2}+\frac{1}{z+2}\right]$

(i) $1<|z|<2$

$f(z)=\frac{1}{5}\left[\frac{2-z}{z^2+1}+\frac{1}{2+z}\right]$

$f(z)=\frac{1}{5}\left[\frac{2-z}{z^2\left[1+\frac{1}{z^2}\right]}+\frac{1}{2\left[1+\frac{z}{2}\right]}\right]$

$f(z)=\frac{2-z}{5z^2}\left[1+\frac{1}{z^2}\right]^{-1}+\frac{1}{10}\left[1+\frac{z}{2}\right]^{-1}$

$f(z)=\frac{2-z}{5z^2}\left[1-\frac{1}{z^2}+\frac{1}{z^4}-\frac{1}{z^6}+\cdots.\right]+\frac{1}{10}\left[1-\frac{z}{2}+\frac{z^2}{2^2}-\frac{z^3}{2^3}+\cdots\right]$

(ii) $|z|>2$

$f(z)=\frac{1}{5}\left[\frac{2-z}{z^2+1}+\frac{1}{z+2}\right]$

$f(z)=\frac{1}{5}\left[\frac{2-z}{z^2\left[1+\frac{1}{z^2}\right]}+\frac{1}{z\left[1+\frac{2}{z}\right]}\right]$

$f(z)=\frac{2-z}{5z^2}\left[1+\frac{1}{z^2}\right]^{-1}+\frac{1}{5z}\left[1+\frac{2}{z}\right]^{-1}$

$f(z)=\frac{2-z}{5z^2}\left[1-\frac{1}{z^2}+\frac{1}{z^4}-\frac{1}{z^6}+\cdots.\right]+\frac{1}{5z}\left[1-\frac{2}{z}+\frac{2^2}{z^2}-\frac{2^3}{z^3}+\cdots\right]$

7. Obtain all Taylor's and Laurent's expansions of $f(z)=\frac{z-1}{z^2-2z-3}$ indicating regions of convergence.
[N14/ElexExtcElectBiomInst/8M][M16/ElexExtcElectBiomInst/8M]
Solution:
We have, $f(z)=\frac{z-1}{z^2-2z-3}=\frac{z-1}{(z-3)(z+1)}$
Let $\frac{z-1}{(z-3)(z+1)}=\frac{A}{z-3}+\frac{B}{z+1}$
$z-1=A(z+1)+B(z-3)$
Comparing the coefficients, we get
$A+B=1$
$A-3B=-1$
On solving, we get
$A=\frac{1}{2}, B=\frac{1}{2}$
$f(z)=\frac{\frac{1}{2}}{z-3}+\frac{\frac{1}{2}}{z+1}$
The Region of Convergence are
(i) $|z|<1$ (ii) $1<|z|<3$ (iii) $|z|>3$

The Taylors series is given by
(i) $|z|<1$
$$f(z)=\frac{\frac{1}{2}}{-3+z}+\frac{\frac{1}{2}}{1+z}$$
$$f(z)=\frac{\frac{1}{2}}{-3\left[1-\frac{z}{3}\right]}+\frac{\frac{1}{2}}{1+z}$$
$$f(z)=-\frac{1}{6}\left[1-\frac{z}{3}\right]^{-1}+\frac{1}{2}[1+z]^{-1}$$
$$f(z)=-\frac{1}{6}\left[1+\frac{z}{3}+\frac{z^2}{3^2}+\frac{z^3}{3^3}+\cdots.\right]+\frac{1}{2}[1-z+z^2-z^3+\cdots..]$$

The first Laurents Series is given by
(ii) $1<|z|<3$
$$f(z)=\frac{\frac{1}{2}}{-3+z}+\frac{\frac{1}{2}}{z+1}$$
$$f(z)=\frac{\frac{1}{2}}{-3\left[1-\frac{z}{3}\right]}+\frac{\frac{1}{2}}{z\left[1+\frac{1}{z}\right]}$$
$$f(z)=-\frac{1}{6}\left[1-\frac{z}{3}\right]^{-1}+\frac{1}{2z}\left[1+\frac{1}{z}\right]^{-1}$$
$$f(z)=-\frac{1}{6}\left[1+\frac{z}{3}+\frac{z^2}{3^2}+\frac{z^3}{3^3}+\cdots.\right]+\frac{1}{2z}\left[1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\cdots..\right]$$

The second Laurents Series is given by

(iii) $|z| > 3$

$$f(z) = \frac{\frac{1}{2}}{z-3} + \frac{\frac{1}{2}}{z+1}$$

$$f(z) = \frac{\frac{1}{2}}{z\left[1-\frac{3}{z}\right]} + \frac{\frac{1}{2}}{z\left[1+\frac{1}{z}\right]}$$

$$f(z) = \frac{1}{2z}\left[1-\frac{3}{z}\right]^{-1} + \frac{1}{2z}\left[1+\frac{1}{z}\right]^{-1}$$

$$f(z) = \frac{1}{2z}\left[1+\frac{3}{z}+\frac{3^2}{z^2}+\frac{3^3}{z^3}+\cdots.\right] + \frac{1}{2z}\left[1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\cdots..\right]$$

8. Find expansion of $f(z)=\frac{1}{(1+z^2)(z+2)}$ indicating region of convergence

[N15/ElexExtcElectBiomInst/8M]

Solution:

We have, $f(z)=\frac{1}{(1+z^2)(z+2)}$

Let $\frac{1}{(1+z^2)(z+2)}=\frac{Az+B}{1+z^2}+\frac{C}{z+2}$

$1=(Az+B)(z+2)+C(1+z^2)$

$1=A(z^2+2z)+B(z+2)+C(z^2+1)$

Comparing the coefficients, we get

$A+C=0$

$2A+B=0$

$2B+C=1$

On solving, we get

$A=-\frac{1}{5},B=\frac{2}{5},C=\frac{1}{5}$

$f(z)=\frac{-\frac{z}{5}+\frac{2}{5}}{1+z^2}+\frac{\frac{1}{5}}{z+2}=\frac{1}{5}\left[\frac{2-z}{1+z^2}+\frac{1}{z+2}\right]$

The region of convergence are

(i) $|z|<1$ (ii) $1<|z|<2$ (iii) $|z|>2$

(i) $|z|<1$

$f(z)=\frac{1}{5}\left[\frac{2-z}{1+z^2}+\frac{1}{2+z}\right]$

$f(z)=\frac{1}{5}\left[\frac{2-z}{1+z^2}+\frac{1}{2\left[1+\frac{z}{2}\right]}\right]$

$f(z)=\frac{2-z}{5}[1+z^2]^{-1}+\frac{1}{10}\left[1+\frac{z}{2}\right]^{-1}$

$f(z)=\frac{2-z}{5}[1-z^2+z^4-z^6+\cdots.]+\frac{1}{10}\left[1-\frac{z}{2}+\frac{z^2}{2^2}-\frac{z^3}{2^3}+\cdots\right]$

(ii) $1<|z|<2$

$f(z)=\frac{1}{5}\left[\frac{2-z}{z^2+1}+\frac{1}{2+z}\right]$

$f(z)=\frac{1}{5}\left[\frac{2-z}{z^2\left[1+\frac{1}{z^2}\right]}+\frac{1}{2\left[1+\frac{z}{2}\right]}\right]$

$f(z)=\frac{2-z}{5z^2}\left[1+\frac{1}{z^2}\right]^{-1}+\frac{1}{10}\left[1+\frac{z}{2}\right]^{-1}$

$f(z)=\frac{2-z}{5z^2}\left[1-\frac{1}{z^2}+\frac{1}{z^4}-\frac{1}{z^6}+\cdots.\right]+\frac{1}{10}\left[1-\frac{z}{2}+\frac{z^2}{2^2}-\frac{z^3}{2^3}+\cdots\right]$

(iii) $|z| > 2$

$$f(z) = \frac{1}{5}\left[\frac{2-z}{z^2+1} + \frac{1}{z+2}\right]$$

$$f(z) = \frac{1}{5}\left[\frac{2-z}{z^2\left[1+\frac{1}{z^2}\right]} + \frac{1}{z\left[1+\frac{2}{z}\right]}\right]$$

$$f(z) = \frac{2-z}{5z^2}\left[1 + \frac{1}{z^2}\right]^{-1} + \frac{1}{5z}\left[1 + \frac{2}{z}\right]^{-1}$$

$$f(z) = \frac{2-z}{5z^2}\left[1 - \frac{1}{z^2} + \frac{1}{z^4} - \frac{1}{z^6} + \cdots.\right] + \frac{1}{5z}\left[1 - \frac{2}{z} + \frac{2^2}{z^2} - \frac{2^3}{z^3} + \cdots\right]$$

9. Expand $f(z)=\frac{1}{z^2(z-1)(z+2)}$ about z = 0 for (i) $|z|<1$ (ii) $1<|z|<2$

[N15/AutoMechCivil/8M]

Solution:

We have, $f(z)=\frac{1}{z^2(z-1)(z+2)}$

Let $\frac{1}{z^2(z-1)(z+2)}=\frac{A}{z}+\frac{B}{z^2}+\frac{C}{z-1}+\frac{D}{z+2}$

$1=Az(z-1)(z+2)+B(z-1)(z+2)+Cz^2(z+2)+Dz^2(z-1)$

$1=A(z^3+z^2-2z)+B(z^2+z-2)+C(z^3+2z^2)+D(z^3-z^2)$

Comparing the coefficients, we get

$A+C+D=0$

$A+B+2C-D=0$

$-2A+B=0$

$-2B=1$

On solving, we get

$A=-\frac{1}{4},B=-\frac{1}{2},C=\frac{1}{3},D=-\frac{1}{12}$

$f(z)=\frac{-\frac{1}{4}}{z}-\frac{\frac{1}{2}}{z^2}+\frac{\frac{1}{3}}{z-1}-\frac{\frac{1}{12}}{z+2}$

(i) $|z|<1$

$f(z)=-\frac{1}{4z}-\frac{1}{2z^2}+\frac{\frac{1}{3}}{-1+z}-\frac{\frac{1}{12}}{2+z}$

$f(z)=-\frac{1}{4z}-\frac{1}{2z^2}+\frac{\frac{1}{3}}{-(1-z)}-\frac{\frac{1}{12}}{2\left(1+\frac{z}{2}\right)}$

$f(z)=-\frac{1}{4z}-\frac{1}{2z^2}-\frac{1}{3}[1-z]^{-1}-\frac{1}{24}\left[1+\frac{z}{2}\right]^{-1}$

$f(z)=-\frac{1}{4z}-\frac{1}{2z^2}-\frac{1}{3}[1+z+z^2+z^3+\cdots.]-\frac{1}{24}\left[1-\frac{z}{2}+\frac{z^2}{2^2}-\frac{z^3}{2^3}+\cdots\right]$

(ii) $1<|z|<2$

$f(z)=-\frac{1}{4z}-\frac{1}{2z^2}+\frac{\frac{1}{3}}{z-1}-\frac{\frac{1}{12}}{2+z}$

$f(z)=-\frac{1}{4z}-\frac{1}{2z^2}+\frac{\frac{1}{3}}{z\left(1-\frac{1}{z}\right)}-\frac{\frac{1}{12}}{2\left(1+\frac{z}{2}\right)}$

$f(z)=-\frac{1}{4z}-\frac{1}{2z^2}+\frac{1}{3z}\left[1-\frac{1}{z}\right]^{-1}-\frac{1}{24}\left[1+\frac{z}{2}\right]^{-1}$

$f(z)=-\frac{1}{4z}-\frac{1}{2z^2}+\frac{1}{3z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots.\right]-\frac{1}{24}\left[1-\frac{z}{2}+\frac{z^2}{2^2}-\frac{z^3}{2^3}+\cdots\right]$

10. Expand $f(z) = \frac{1}{z(z+1)(z-2)}$ in Laurent's series when

(i) |z| < 1 (ii) $1 <$ |z| < 2 (iii) |z| > 2

[M16/CompIT/8M]

Solution:

We have, $f(z) = \frac{1}{\mathrm{z(z-2)(z+1)}}$

Let $\frac{1}{\mathrm{z(z-2)(z+1)}} = \frac{A}{z} + \frac{B}{z-2} + \frac{C}{z+1}$

$1 = A(z-2)(z+1) + Bz(z+1) + Cz(z-2)$

$1 = A(z^2 - z - 2) + B(z^2 + z) + C(z^2 - 2z)$

On comparing the coefficients, we get

$A + B + C = 0$

$-A + B - 2C = 0$

$-2A = 7$

On solving, we get

$A = -\frac{7}{2}, B = \frac{7}{6}, C = \frac{7}{3}$

$f(z) = \frac{-\frac{7}{2}}{z} + \frac{\frac{7}{6}}{z-2} + \frac{\frac{7}{3}}{z+1}$

(i) |z| < 1

$f(z) = -\frac{7}{2z} + \frac{\frac{7}{6}}{-2+z} + \frac{\frac{7}{3}}{1+z}$

$f(z) = -\frac{7}{2z} + \frac{\frac{7}{6}}{-2\left[1-\frac{z}{2}\right]} + \frac{\frac{7}{3}}{[1+z]}$

$f(z) = -\frac{7}{2z} - \frac{7}{12}\left[1 - \frac{z}{2}\right]^{-1} + \frac{7}{3}[1+z]^{-1}$

$f(z) = -\frac{7}{2z} - \frac{7}{12}\left[1 + \frac{z}{2} + \frac{z^2}{2^2} + \frac{z^3}{2^3} + \cdots.\right] + \frac{7}{3}[1 - z + z^2 - z^3 + \cdots.]$

(ii) $1 <$ |z| < 2

$f(z) = -\frac{7}{2z} + \frac{\frac{7}{6}}{-2+z} + \frac{\frac{7}{3}}{z+1}$

$f(z) = -\frac{7}{2z} + \frac{\frac{7}{6}}{-2\left[1-\frac{z}{2}\right]} + \frac{\frac{7}{3}}{z\left[1+\frac{1}{z}\right]}$

$f(z) = -\frac{7}{2z} - \frac{7}{12}\left[1 - \frac{z}{2}\right]^{-1} + \frac{7}{3z}\left[1 + \frac{1}{z}\right]^{-1}$

$f(z) = -\frac{7}{2z} - \frac{7}{12}\left[1 + \frac{z}{2} + \frac{z^2}{2^2} + \frac{z^3}{2^3} + \cdots.\right] + \frac{7}{3z}\left[1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots.\right]$

(iii) $|z| > 2$

$$f(z) = -\frac{7}{2z} + \frac{\frac{7}{6}}{z-2} + \frac{\frac{7}{3}}{z+1}$$

$$f(z) = -\frac{7}{2z} + \frac{\frac{7}{6}}{z\left[1-\frac{2}{z}\right]} + \frac{\frac{7}{3}}{z\left[1+\frac{1}{z}\right]}$$

$$f(z) = -\frac{7}{2z} + \frac{7}{6z}\left[1 - \frac{2}{z}\right]^{-1} + \frac{7}{3z}\left[1 + \frac{1}{z}\right]^{-1}$$

$$f(z) = -\frac{7}{2z} + \frac{7}{6z}\left[1 + \frac{2}{z} + \frac{2^2}{z^2} + \frac{2^3}{z^3} + \cdots.\right] + \frac{7}{3z}\left[1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots.\right]$$

11. Obtain Laurents series for the function $f(z) = \frac{-7z-2}{z(z-2)(z+1)}$ about $z = -1$

[M16/AutoMechCivil/8M]

Solution:

We have, $f(z) = \frac{-7z-2}{z(z-2)(z+1)}$

Let $\frac{-7z-2}{z(z-2)(z+1)} = \frac{A}{z} + \frac{B}{z-2} + \frac{C}{z+1}$

$-7z - 2 = A(z-2)(z+1) + Bz(z+1) + Cz(z-2)$

$-7z - 2 = A(z^2 - z - 2) + B(z^2 + z) + C(z^2 - 2z)$

On comparing the coefficients, we get

$A + B + C = 0$

$-A + B - 2C = -7$

$-2A = -2$

On solving, we get

$A = 1, B = -\frac{8}{3}, C = \frac{5}{3}$

$f(z) = \frac{1}{z} + \frac{-\frac{8}{3}}{z-2} + \frac{\frac{5}{3}}{z+1}$

Now, put $z + 1 = u$ i.e $z = u - 1$

$\therefore f(z) = \frac{1}{u-1} - \frac{\frac{8}{3}}{u-1-2} + \frac{\frac{5}{3}}{u-1+1}$

$\therefore f(z) = \frac{1}{u-1} - \frac{\frac{8}{3}}{u-3} + \frac{\frac{5}{3}}{u}$

The ROCs are as follows

(i) $|u| < 1$ (ii) $1 < |u| < 3$ (iii) $|u| > 3$

The Taylors Series is given by

(i) For $|u| < 1$ i.e. $|z + 1| < 1$

$f(z) = \frac{1}{-1+u} - \frac{\frac{8}{3}}{-3+u} + \frac{\frac{5}{3}}{u}$

$f(z) = \frac{1}{-1(1-u)} - \frac{\frac{8}{3}}{-3\left(1-\frac{u}{3}\right)} + \frac{5}{3u}$

$f(z) = -1[1-u]^{-1} + \frac{8}{9}\left[1 - \frac{u}{3}\right]^{-1} + \frac{5}{3u}$

$f(z) = -[1 + u + u^2 + u^3 + \cdots.] + \frac{8}{9}\left[1 + \frac{u}{3} + \frac{u^2}{9} + \frac{u^3}{27} + \cdots.\right] + \frac{5}{3u}$

$f(z) = -[1 + (z+1) + (z+1)^2 + \cdots] + \frac{8}{9}\left[1 + \frac{z+1}{3} + \frac{(z+1)^2}{9} + \cdots\right] + \frac{5}{3(z+1)}$

The first Laurents Series is given by

(ii) For $1 < |u| < 3$ i.e. $1 < |z+1| < 3$

$$f(z) = \frac{1}{u-1} - \frac{\frac{8}{3}}{-3+u} + \frac{\frac{5}{3}}{u}$$

$$f(z) = \frac{1}{u\left(1-\frac{1}{u}\right)} - \frac{\frac{8}{3}}{-3\left(1-\frac{u}{3}\right)} + \frac{5}{3u}$$

$$f(z) = \frac{1}{u}\left[1-\frac{1}{u}\right]^{-1} + \frac{8}{9}\left[1-\frac{u}{3}\right]^{-1} + \frac{5}{3u}$$

$$f(z) = \frac{1}{u}\left[1+\frac{1}{u}+\frac{1}{u^2}+\frac{1}{u^3}+\cdots.\right] + \frac{8}{9}\left[1+\frac{u}{3}+\frac{u^2}{9}+\frac{u^3}{27}+\cdots.\right] + \frac{5}{3u}$$

$$f(z) = \frac{1}{z+1}\left[1+\frac{1}{(z+1)}+\frac{1}{(z+1)^2}+\cdots.\right] + \frac{8}{9}\left[1+\frac{z+1}{3}+\frac{(z+1)^2}{9}+\cdots\right] + \frac{5}{3(z+1)}$$

The second Laurents Series is given by

(iii) For $|u| > 3$ i.e. $|z+1| > 3$

$$f(z) = \frac{1}{u-1} - \frac{\frac{8}{3}}{u-3} + \frac{\frac{5}{3}}{u}$$

$$f(z) = \frac{1}{u\left(1-\frac{1}{u}\right)} - \frac{\frac{8}{3}}{u\left(1-\frac{3}{u}\right)} + \frac{5}{3u}$$

$$f(z) = \frac{1}{u}\left[1-\frac{1}{u}\right]^{-1} - \frac{8}{3u}\left[1-\frac{3}{u}\right]^{-1} + \frac{5}{3u}$$

$$f(z) = \frac{1}{u}\left[1+\frac{1}{u}+\frac{1}{u^2}+\frac{1}{u^3}+\cdots.\right] - \frac{8}{3u}\left[1+\frac{3}{u}+\frac{9}{u^2}+\frac{27}{u^3}+\cdots.\right] + \frac{5}{3u}$$

$$f(z) = \frac{1}{z+1}\left[1+\frac{1}{(z+1)}+\frac{1}{(z+1)^2}+\cdots\right] - \frac{8}{3(z+1)}\left[1+\frac{3}{z+1}+\frac{9}{(z+1)^2}+\cdots\right] + \frac{5}{3(z+1)}$$

12. Find all possible Laurents series expansion of $\frac{z}{(z-1)(z-2)}$ about $z=-2$ indicating region of convergence

[N16/ElexExtcElectBiomInst/8M]

Solution:

We have, $f(z)=\frac{z}{(z-1)(z-2)}$

Let $\frac{z}{(z-1)(z-2)}=\frac{A}{z-1}+\frac{B}{z-2}$

$z=A(z-2)+B(z-1)$

On comparing the coefficients, we get

$A+B=1$

$-2A-B=0$

On solving, we get

$A=-1, B=2$

$f(z)=\frac{2}{z-2}-\frac{1}{z-1}$

Now, put $z+2=u$ i.e $z=u-2$

$\therefore f(z)=\frac{2}{u-2-2}-\frac{1}{u-2-1}$

$\therefore f(z)=\frac{2}{u-4}-\frac{1}{u-3}$

The ROCs are as follows

(i) $|u|<3$ (ii) $3<|u|<4$ (iii) $|u|>4$

The Taylors Series is given by

(i) For $|u|<3$ i.e. $|z+2|<3$

$f(z)=\frac{2}{-4+u}-\frac{1}{-3+u}$

$f(z)=\frac{2}{-4\left(1-\frac{u}{4}\right)}-\frac{1}{-3\left(1-\frac{u}{3}\right)}$

$f(z)=-\frac{1}{2}\left[1-\frac{u}{4}\right]^{-1}+\frac{1}{3}\left[1-\frac{u}{3}\right]^{-1}$

$f(z)=-\frac{1}{2}\left[1+\frac{u}{4}+\frac{u^2}{16}+\frac{u^3}{64}+\cdots.\right]+\frac{1}{3}\left[1+\frac{u}{3}+\frac{u^2}{9}+\frac{u^3}{27}+\cdots.\right]$

$f(z)=-\frac{1}{2}\left[1+\frac{(z+2)}{4}+\frac{(z+2)^2}{4}+\cdots\right]+\frac{1}{3}\left[1+\frac{z+2}{3}+\frac{(z+2)^2}{9}+\cdots\right]$

The first Laurents Series is given by

(ii) For $3<|u|<4$ i.e. $3<|z+2|<4$

$f(z)=\frac{2}{-4+u}-\frac{1}{u-3}$

$f(z)=\frac{2}{-4\left(1-\frac{u}{4}\right)}-\frac{1}{u\left(1-\frac{3}{u}\right)}$

$$f(z) = -\frac{1}{2}\left[1 - \frac{u}{4}\right]^{-1} - \frac{1}{u}\left[1 - \frac{3}{u}\right]^{-1}$$
$$f(z) = -\frac{1}{2}\left[1 + \frac{u}{4} + \frac{u^2}{16} + \frac{u^3}{64} + \cdots.\right] - \frac{1}{u}\left[1 + \frac{3}{u} + \frac{9}{u^2} + \frac{27}{u^3} + \cdots.\right]$$
$$f(z) = -\frac{1}{2}\left[1 + \frac{(z+2)}{4} + \frac{(z+2)^2}{4} + \cdots\right] - \frac{1}{z+2}\left[1 + \frac{3}{z+2} + \frac{9}{(z+2)^2} + \cdots\right]$$

The second Laurents Series is given by

(iii) For $|u| > 4$ i.e. $|z + 2| > 4$

$$f(z) = \frac{2}{u-4} - \frac{1}{u-3}$$
$$f(z) = \frac{2}{u\left(1-\frac{4}{u}\right)} - \frac{1}{u\left(1-\frac{3}{u}\right)}$$
$$f(z) = \frac{2}{u}\left[1 - \frac{4}{u}\right]^{-1} - \frac{1}{u}\left[1 - \frac{3}{u}\right]^{-1}$$
$$f(z) = \frac{2}{u}\left[1 + \frac{4}{u} + \frac{16}{u^2} + \frac{64}{u^3} + \cdots.\right] - \frac{1}{u}\left[1 + \frac{3}{u} + \frac{9}{u^2} + \frac{27}{u^3} + \cdots.\right]$$
$$f(z) = \frac{2}{z+2}\left[1 + \frac{4}{z+2} + \frac{16}{(z+2)^2} + \cdots\right] - \frac{1}{z+2}\left[1 + \frac{3}{z+2} + \frac{9}{(z+2)^2} + \cdots\right]$$

13. Expand $\mathrm{f(z)} = \frac{z^2-1}{z^2+5z+6}$ about $z = 1$

[N16/AutoMechCivil/8M]

Solution:

We have, $f(z) = \frac{z^2-1}{z^2+5z+6}$

$f(z) = \frac{z^2+5z+6-5z-6-1}{z^2+5z+6}$

$f(z) = 1 + \frac{-5z-7}{z^2+5z+6}$

Let $\frac{-5z-7}{(z+2)(z+3)} = \frac{A}{z+2} + \frac{B}{z+3}$

$-5z - 7 = A(z+3) + B(z+2)$

Comparing the coefficients, we get

$A + B = -5$

$3A + 2B = -7$

On solving, we get

$A = 3, B = -8$

$f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}$

Now, put $z - 1 = u$ i.e $z = u + 1$

$\therefore f(z) = 1 + \frac{3}{u+1+2} - \frac{8}{u+1+3}$

$\therefore f(z) = 1 + \frac{3}{u+3} - \frac{8}{u+4}$

The ROCs are as follows

(i) $|u| < 3$ (ii) $3 < |u| < 4$ (iii) $|u| > 4$

The Taylors series is given by

(i) For $|u| < 3$ i.e. $|z-1| < 3$

$f(z) = 1 + \frac{3}{3+u} - \frac{8}{4+u}$

$f(z) = 1 + \frac{3}{3\left(1+\frac{u}{3}\right)} - \frac{8}{4\left(1+\frac{u}{4}\right)}$

$f(z) = 1 + \left[1 + \frac{u}{3}\right]^{-1} - 2\left[1 + \frac{u}{4}\right]^{-1}$

$f(z) = 1 + \left[1 - \frac{u}{3} + \frac{u^2}{9} - \frac{u^3}{27} + \cdots..\right] - 2\left[1 - \frac{u}{4} + \frac{u^2}{16} - \frac{u^3}{64} + \cdots.\right]$

$f(z) = 1 + \left[1 - \frac{z-1}{3} + \frac{(z-1)^2}{9} - \cdots.\right] - 2\left[1 - \frac{z-1}{4} + \frac{(z-1)^2}{16} - \cdots..\right]$

The Laurents series is given by

(ii) For $3 < |u| < 4$ i.e. $3 < |z-1| < 4$

$f(z) = 1 + \frac{3}{u+3} - \frac{8}{4+u}$

$f(z) = 1 + \frac{3}{u\left(1+\frac{3}{u}\right)} - \frac{8}{4\left(1+\frac{u}{4}\right)}$

$f(z) = 1 + \frac{3}{u}\left[1 + \frac{3}{u}\right]^{-1} - 2\left[1 + \frac{u}{4}\right]^{-1}$

$f(z) = 1 + \frac{3}{u}\left[1 - \frac{3}{u} + \frac{9}{u^2} - \frac{27}{u^3} + \cdots..\right] - 2\left[1 - \frac{u}{4} + \frac{u^2}{16} - \frac{u^3}{64} + \cdots.\right]$

$f(z) = 1 + \frac{3}{z-1}\left[1 - \frac{3}{z-1} + \frac{9}{(z-1)^2} - \cdots.\right] - 2\left[1 - \frac{z-1}{4} + \frac{(z-1)^2}{16} - \cdots..\right]$

The Laurents series is given by

(ii) For $|u| > 4$ i.e. $|z-1| > 4$

$f(z) = 1 + \frac{3}{u+3} - \frac{8}{u+4}$

$f(z) = 1 + \frac{3}{u\left(1+\frac{3}{u}\right)} - \frac{8}{u\left(1+\frac{4}{u}\right)}$

$f(z) = 1 + \frac{3}{u}\left[1 + \frac{3}{u}\right]^{-1} - \frac{8}{u}\left[1 + \frac{4}{u}\right]^{-1}$

$f(z) = 1 + \frac{3}{u}\left[1 - \frac{3}{u} + \frac{9}{u^2} - \frac{27}{u^3} + \cdots..\right] - \frac{8}{u}\left[1 - \frac{4}{u} + \frac{16}{u^2} - \frac{64}{u^3} + \cdots.\right]$

$f(z) = 1 + \frac{3}{z-1}\left[1 - \frac{3}{z-1} + \frac{9}{(z-1)^2} - \cdots.\right] - \frac{8}{z-1}\left[1 - \frac{4}{z-1} + \frac{16}{(z-1)^2} - \cdots..\right]$

14. Obtain the Taylor's and Laurent series which represent the function $\frac{z^2-1}{(z+3)(z+4)}$ in the regions (i) $|z| < 3$ (ii) $3 < |z| < 4$ (iii) $|z| > 4$

[M17/AutoMechCivil/8M]

Solution:

We have, $f(z) = \frac{z^2-1}{(z+3)(z+4)}$

$f(z) = \frac{z^2-1}{z^2+7z+12}$

$f(z) = \frac{z^2+7z+12-7z-12-1}{z^2+7z+12}$

$f(z) = 1 + \frac{-7z-13}{z^2+7z+12}$

$f(z) = 1 + \frac{-7z-13}{(z+3)(z+4)}$

Let $\frac{-7z-13}{(z+3)(z+4)} = \frac{A}{z+3} + \frac{B}{z+4}$

$-7z - 13 = A(z+4) + B(z+3)$

On comparing the coefficients, we get

$A + B = -7$

$4A + 3B = -13$

On solving, we get

$A = 8, B = -15$

$f(z) = 1 + \frac{8}{z+3} - \frac{15}{z+4}$

(i) $|z| < 3$

$f(z) = 1 + \frac{8}{3+z} - \frac{15}{4+z}$

$f(z) = 1 + \frac{8}{3\left(1+\frac{z}{3}\right)} - \frac{15}{4\left(1+\frac{z}{4}\right)}$

$f(z) = 1 + \frac{8}{3}\left[1+\frac{z}{3}\right]^{-1} - \frac{15}{4}\left[1+\frac{z}{4}\right]^{-1}$

$f(z) = 1 + \frac{8}{3}\left[1 - \frac{z}{3} + \frac{z^2}{3^2} - \frac{z^3}{3^3} + \cdots..\right] - \frac{15}{4}\left[1 - \frac{z}{4} + \frac{z^2}{4^2} - \frac{z^3}{4^3} + \cdots\right]$

(ii) $3 < |z| < 4$

$f(z) = 1 + \frac{8}{z+3} - \frac{15}{4+z}$

$f(z) = 1 + \frac{8}{z\left(1+\frac{3}{z}\right)} - \frac{15}{4\left(1+\frac{z}{4}\right)}$

$f(z) = 1 + \frac{8}{z}\left[1+\frac{3}{z}\right]^{-1} - \frac{15}{4}\left[1+\frac{z}{4}\right]^{-1}$

$f(z) = 1 + \frac{8}{z}\left[1 - \frac{3}{z} + \frac{3^2}{z^2} - \frac{3^3}{z^3} + \cdots..\right] - \frac{15}{4}\left[1 - \frac{z}{4} + \frac{z^2}{4^2} - \frac{z^3}{4^3} + \cdots\right]$

(iii) $|z| > 4$

$$f(z) = 1 + \frac{8}{z+3} - \frac{15}{z+4}$$

$$f(z) = 1 + \frac{8}{z\left(1+\frac{3}{z}\right)} - \frac{15}{z\left(1+\frac{4}{z}\right)}$$

$$f(z) = 1 + \frac{8}{z}\left[1 + \frac{3}{z}\right]^{-1} - \frac{15}{z}\left[1 + \frac{4}{z}\right]^{-1}$$

$$f(z) = 1 + \frac{8}{z}\left[1 - \frac{3}{z} + \frac{3^2}{z^2} - \frac{3^3}{z^3} + \cdots..\right] - \frac{15}{z}\left[1 - \frac{4}{z} + \frac{4^2}{z^2} - \frac{4^3}{z^3} + \cdots\right]$$

15. Expand $f(z)=\frac{1}{z(1+z)(z-2)}$ (i) within the unit circle about the origin, (ii) within the annulus region between the concentric circles about the origin having radii 1 and 2 respectively, (iii) in the exterior of the circle with the centre at the origin and the radius 2.

[N17/CompIT/8M]

Solution:

We have, $f(z)=\frac{1}{z(z-2)(z+1)}$

Let $\frac{1}{z(z-2)(z+1)}=\frac{A}{z}+\frac{B}{z-2}+\frac{C}{z+1}$

$1=A(z-2)(z+1)+Bz(z+1)+Cz(z-2)$

$1=A(z^2-z-2)+B(z^2+z)+C(z^2-2z)$

On comparing the coefficients, we get

$A+B+C=0$

$-A+B-2C=0$

$-2A=7$

On solving, we get

$A=-\frac{7}{2},B=\frac{7}{6},C=\frac{7}{3}$

$f(z)=\frac{-\frac{7}{2}}{z}+\frac{\frac{7}{6}}{z-2}+\frac{\frac{7}{3}}{z+1}$

(i) $|z|<1$

$f(z)=-\frac{7}{2z}+\frac{\frac{7}{6}}{-2+z}+\frac{\frac{7}{3}}{1+z}$

$f(z)=-\frac{7}{2z}+\frac{\frac{7}{6}}{-2\left[1-\frac{z}{2}\right]}+\frac{\frac{7}{3}}{[1+z]}$

$f(z)=-\frac{7}{2z}-\frac{7}{12}\left[1-\frac{z}{2}\right]^{-1}+\frac{7}{3}[1+z]^{-1}$

$f(z)=-\frac{7}{2z}-\frac{7}{12}\left[1+\frac{z}{2}+\frac{z^2}{2^2}+\frac{z^3}{2^3}+\cdots.\right]+\frac{7}{3}[1-z+z^2-z^3+\cdots.]$

(ii) $1<|z|<2$

$f(z)=-\frac{7}{2z}+\frac{\frac{7}{6}}{-2+z}+\frac{\frac{7}{3}}{z+1}$

$f(z)=-\frac{7}{2z}+\frac{\frac{7}{6}}{-2\left[1-\frac{z}{2}\right]}+\frac{\frac{7}{3}}{z\left[1+\frac{1}{z}\right]}$

$f(z)=-\frac{7}{2z}-\frac{7}{12}\left[1-\frac{z}{2}\right]^{-1}+\frac{7}{3z}\left[1+\frac{1}{z}\right]^{-1}$

$f(z)=-\frac{7}{2z}-\frac{7}{12}\left[1+\frac{z}{2}+\frac{z^2}{2^2}+\frac{z^3}{2^3}+\cdots.\right]+\frac{7}{3z}\left[1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\cdots.\right]$

(iii) $|z| > 2$

$$f(z) = -\frac{7}{2z} + \frac{\frac{7}{6}}{z-2} + \frac{\frac{7}{3}}{z+1}$$

$$f(z) = -\frac{7}{2z} + \frac{\frac{7}{6}}{z\left[1-\frac{2}{z}\right]} + \frac{\frac{7}{3}}{z\left[1+\frac{1}{z}\right]}$$

$$f(z) = -\frac{7}{2z} + \frac{7}{6z}\left[1-\frac{2}{z}\right]^{-1} + \frac{7}{3z}\left[1+\frac{1}{z}\right]^{-1}$$

$$f(z) = -\frac{7}{2z} + \frac{7}{6z}\left[1+\frac{2}{z}+\frac{2^2}{z^2}+\frac{2^3}{z^3}+\cdots.\right] + \frac{7}{3z}\left[1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\cdots.\right]$$

16. Obtain the Taylor's and Laurent's series which represent the function $f(z) = \frac{z}{(z-1)(z-2)}$ in the regions (i) $|z| < 1$ (ii) $1 < |z| < 2$

[N17/AutoMechCivil/6M]

Solution:

We have, $f(z) = \frac{z}{(z-1)(z-2)}$

Let $\frac{z}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$

$z = A(z-2) + B(z-1)$

Comparing the coefficients, we get

$A + B = 1$

$-2A - B = 2$

On solving, we get

$A = -3, B = 4$

$f(z) = -\frac{3}{z-1} + \frac{4}{z-2}$

(i) $|z| < 1$

$f(z) = -\frac{3}{-1+z} + \frac{4}{-2+z}$

$f(z) = -\frac{3}{-[1-z]} + \frac{4}{-2\left[1-\frac{z}{2}\right]}$

$f(z) = 3[1-z]^{-1} - 2\left[1-\frac{z}{2}\right]^{-1}$

$f(z) = 3[1 + z + z^2 + z^3 + \cdots.] - 2\left[1 + \frac{z}{2} + \frac{z^2}{2^2} + \frac{z^3}{2^3} + \cdots\right]$

(ii) $1 < |z| < 2$

$f(z) = -\frac{3}{z-1} + \frac{4}{-2+z}$

$f(z) = -\frac{3}{z\left[1-\frac{1}{z}\right]} + \frac{4}{-2\left[1-\frac{z}{2}\right]}$

$f(z) = -\frac{3}{z}\left[1-\frac{1}{z}\right]^{-1} - 2\left[1-\frac{z}{2}\right]^{-1}$

$f(z) = -\frac{3}{z}\left[1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots.\right] - 2\left[1 + \frac{z}{2} + \frac{z^2}{2^2} + \frac{z^3}{2^3} + \cdots\right]$