Matrices

Type I: Eigen Values & Eigen Vectors

1. Find the Eigen values and Eigen vectors of the matrix $A = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$

[N13/Chem/5M][N16/AutoMechCivil/8M][M17/AutoMechCivil/8M]

Solution:

We have,

$A = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}, |A| = 6$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 8-\lambda & -8 & -2 \\ 4 & -3-\lambda & -2 \\ 3 & -4 & 1-\lambda \end{vmatrix} = 0$

$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$

$\lambda^3 - [8-3+1]\lambda^2 + \left[\begin{vmatrix} -3 & -2 \\ -4 & 1 \end{vmatrix} + \begin{vmatrix} 8 & -2 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 8 & -8 \\ 4 & -3 \end{vmatrix}\right]\lambda - 6 = 0$

$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

$\lambda = 1,2,3$

(i) For $\lambda = 1, [A - \lambda I]X = 0$ gives

$\begin{bmatrix} 7 & -8 & -2 \\ 4 & -4 & -2 \\ 3 & -4 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$7x_1 - 8x_2 - 2x_3 = 0$

$4x_1 - 4x_2 - 2x_3 = 0$

Solving the above equations by Crammers rule, we get

$\frac{x_1}{\begin{vmatrix} -8 & -2 \\ -4 & -2 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} 7 & -2 \\ 4 & -2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 7 & -8 \\ 4 & -4 \end{vmatrix}}$

$\frac{x_1}{8} = -\frac{x_2}{-6} = \frac{x_3}{4}$

$\frac{x_1}{4} = \frac{x_2}{3} = \frac{x_3}{2}$

Hence, corresponding to $\lambda = 1$ the eigen vector is $X_1 = [4,3,2]'$

(ii) For $\lambda = 2, [A - \lambda I]X = 0$ gives

$$\begin{bmatrix} 6 & -8 & -2 \\ 4 & -5 & -2 \\ 3 & -4 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$6x_1 - 8x_2 - 2x_3 = 0$

$4x_1 - 5x_2 - 2x_3 = 0$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} -8 & -2 \\ -5 & -2 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} 6 & -2 \\ 4 & -2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 6 & -8 \\ 4 & -5 \end{vmatrix}}$$

$$\frac{x_1}{6} = -\frac{x_2}{-4} = \frac{x_3}{2}$$

$$\frac{x_1}{3} = \frac{x_2}{2} = \frac{x_3}{1}$$

Hence, corresponding to $\lambda = 2$ the eigen vector is $X_2 = [3,2,1]'$

(iii) For $\lambda = 3, [A - \lambda I]X = 0$ gives

$$\begin{bmatrix} 5 & -8 & -2 \\ 4 & -6 & -2 \\ 3 & -4 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$5x_1 - 8x_2 - 2x_3 = 0$

$4x_1 - 6x_2 - 2x_3 = 0$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} -8 & -2 \\ -6 & -2 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} 5 & -2 \\ 4 & -2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 5 & -8 \\ 4 & -6 \end{vmatrix}}$$

$$\frac{x_1}{4} = -\frac{x_2}{-2} = \frac{x_3}{2}$$

$$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{1}$$

Hence, corresponding to $\lambda = 3$ the eigen vector is $X_3 = [2,1,1]'$

2. If $A=\begin{bmatrix}3 & 1 & -1\\ 2 & 2 & -1\\ 2 & 2 & 0\end{bmatrix}$, find characteristics roots and vectors of A^2

[N13/Biot/6M]

Solution:

We have,

$$A=\begin{bmatrix}3 & 1 & -1\\ 2 & 2 & -1\\ 2 & 2 & 0\end{bmatrix}, |A|=4$$

The characteristic equation,

$$|A-\lambda I|=0$$

$$\begin{vmatrix}3-\lambda & 1 & -1\\ 2 & 2-\lambda & -1\\ 2 & 2 & 0-\lambda\end{vmatrix}=0$$

$$\lambda^3-[sum\ of\ diagonals]\lambda^2+[sum\ of\ minors\ of\ diagonals]\lambda-|A|=0$$

$$\lambda^3-[3+2+0]\lambda^2+\left[\begin{vmatrix}2 & -1\\ 2 & 0\end{vmatrix}+\begin{vmatrix}3 & -1\\ 2 & 0\end{vmatrix}+\begin{vmatrix}3 & 1\\ 2 & 2\end{vmatrix}\right]\lambda-4=0$$

$$\lambda^3-5\lambda^2+8\lambda-4=0$$

$$(\lambda-1)(\lambda-2)(\lambda-2)=0$$

$$\lambda=1,2,2$$

(i) For $\lambda=1, [A-\lambda I]X=0$ gives

$$\begin{bmatrix}2 & 1 & -1\\ 2 & 1 & -1\\ 2 & 2 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$2x_1+x_2-x_3=0$$

$$2x_1+2x_2-x_3=0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix}1 & -1\\ 2 & -1\end{vmatrix}}=-\frac{x_2}{\begin{vmatrix}2 & -1\\ 2 & -1\end{vmatrix}}=\frac{x_3}{\begin{vmatrix}2 & 1\\ 2 & 2\end{vmatrix}}$$

$$\frac{x_1}{1}=-\frac{x_2}{0}=\frac{x_3}{2}$$

Hence, corresponding to $\lambda=1$ the eigen vector is $X_1=[1,0,2]'$

(ii) For $\lambda=2, [A-\lambda I]X=0$ gives

$$\begin{bmatrix}1 & 1 & -1\\ 2 & 0 & -1\\ 2 & 2 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$x_1+x_2-x_3=0$$

$$2x_1+0x_2-x_3=0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix}1 & -1\\ 0 & -1\end{vmatrix}}=-\frac{x_2}{\begin{vmatrix}1 & -1\\ 2 & -1\end{vmatrix}}=\frac{x_3}{\begin{vmatrix}1 & 1\\ 2 & 0\end{vmatrix}}$$

$\frac{x_1}{-1} = -\frac{x_2}{1} = \frac{x_3}{-2}$
Hence, corresponding to $\lambda = 2$ the eigen vector is $X_2 = [1,1,2]'$
Thus, the characteristic roots of A is 1,2,2
And, the characteristic roots of A^2 is $1^2, 2^2, 2^2$ i.e. 1,4,4
Characteristic vectors of A^2 is same as that of A

3. Find the eigen values and eigen vectors of the matrix $\begin{bmatrix} -2 & 5 & 4 \\ 5 & 7 & 5 \\ 4 & 5 & -2 \end{bmatrix}$

[M14/ElexExtcElectBiomInst/6M]

Solution:

We have,

$$A = \begin{bmatrix} -2 & 5 & 4 \\ 5 & 7 & 5 \\ 4 & 5 & -2 \end{bmatrix}, |A| = 216$$

The characteristic equation,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} -2-\lambda & 5 & 4 \\ 5 & 7-\lambda & 5 \\ 4 & 5 & -2-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$$

$$\lambda^3 - [-2+7-2]\lambda^2 + \left[\begin{vmatrix} 7 & 5 \\ 5 & -2 \end{vmatrix} + \begin{vmatrix} -2 & 4 \\ 4 & -2 \end{vmatrix} + \begin{vmatrix} -2 & 5 \\ 5 & 7 \end{vmatrix}\right]\lambda - 216 = 0$$

$$\lambda^3 - 3\lambda^2 - 90\lambda - 216 = 0$$

$$(\lambda - 12)(\lambda + 3)(\lambda + 6) = 0$$

$$\lambda = 12, -3, -6$$

(i) For $\lambda = 12$, $[A - \lambda I]X = 0$ gives

$$\begin{bmatrix} -14 & 5 & 4 \\ 5 & -5 & 5 \\ 4 & 5 & -14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-14x_1 + 5x_2 + 4x_3 = 0$$

$$5x_1 - 5x_2 + 5x_3 = 0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} 5 & 4 \\ -5 & 5 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} -14 & 4 \\ 5 & 5 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -14 & 5 \\ 5 & -5 \end{vmatrix}}$$

$$\frac{x_1}{45} = -\frac{x_2}{-90} = \frac{x_3}{45}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{1}$$

Hence, corresponding to $\lambda = 12$ the eigen vector is $X_1 = [1,2,1]'$

(ii) For $\lambda = -3$, $[A - \lambda I]X = 0$ gives

$$\begin{bmatrix} 1 & 5 & 4 \\ 5 & 10 & 5 \\ 4 & 5 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + 5x_2 + 4x_3 = 0$$

$$5x_1 + 10x_2 + 5x_3 = 0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix}5 & 4\\ 10 & 5\end{vmatrix}} = -\frac{x_2}{\begin{vmatrix}1 & 4\\ 5 & 5\end{vmatrix}} = \frac{x_3}{\begin{vmatrix}1 & 5\\ 5 & 10\end{vmatrix}}$$

$$\frac{x_1}{-15} = -\frac{x_2}{-15} = \frac{x_3}{-15}$$

$$\frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{1}$$

Hence, corresponding to $\lambda = -3$ the eigen vector is $X_2 = [1, -1, 1]'$

(iii) For $\lambda = -6$, $[A - \lambda I]X = 0$ gives

$$\begin{bmatrix}4 & 5 & 4\\ 5 & 13 & 5\\ 4 & 5 & 4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$4x_1 + 5x_2 + 4x_3 = 0$$

$$5x_1 + 13x_2 + 5x_3 = 0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix}5 & 4\\ 13 & 5\end{vmatrix}} = -\frac{x_2}{\begin{vmatrix}4 & 4\\ 5 & 5\end{vmatrix}} = \frac{x_3}{\begin{vmatrix}4 & 5\\ 5 & 13\end{vmatrix}}$$

$$\frac{x_1}{-27} = -\frac{x_2}{0} = \frac{x_3}{27}$$

$$\frac{x_1}{-1} = \frac{x_2}{0} = \frac{x_3}{1}$$

Hence, corresponding to $\lambda = -6$ the eigen vector is $X_3 = [-1, 0, 1]'$

4. If $A = \begin{bmatrix} x & 4x \\ 2 & y \end{bmatrix}$ has eigen values 5 and – 1 then find values of x and y.

[M14/CompIT/5M]

Solution:

We have,

$$A = \begin{bmatrix} x & 4x \\ 2 & y \end{bmatrix}$$

The characteristic equation,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} x-\lambda & 4x \\ 2 & y-\lambda \end{vmatrix} = 0$$

$$(x-\lambda)(y-\lambda) - 8x = 0$$

$$xy - x\lambda - y\lambda + \lambda^2 - 8x = 0$$

$$\lambda^2 - (x+y)\lambda + (xy - 8x) = 0$$

The eigen values are the roots of the above equation which is given as $5 \,\&\, -1$.

Sum of roots $= (x+y)$

$$5 + (-1) = x + y$$

$$x + y = 4$$

$$y = 4 - x$$

Product of roots $= (xy - 8x)$

$$(5) \times (-1) = xy - 8x$$

$$xy - 8x = -5$$

$$x(y-8) + 5 = 0$$

But $y = 4 - x$

$$x(4 - x - 8) + 5 = 0$$

$$x(-x-4) + 5 = 0$$

$$x^2 + 4x - 5 = 0$$

$$\therefore x = -5, 1$$

$$\therefore y = 9, 3$$

5. Find eigen values and eigen vectors of the matrix A where $A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$

[M14/ChemBiot/8M][N14/ChemBiot/5M][N14/ElexExtcElectBiomInst/6M]

Solution:

We have,

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}, |A| = 7$$

The characteristic equation,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 2 & 3-\lambda & 2 \\ 3 & 3 & 4-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$$

$$\lambda^3 - [2+3+4]\lambda^2 + \left[\begin{vmatrix} 3 & 2 \\ 3 & 4 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix}\right]\lambda - 7 = 0$$

$$\lambda^3 - 9\lambda^2 + 15\lambda - 7 = 0$$

$$(\lambda - 1)(\lambda - 1)(\lambda - 7) = 0$$

$$\lambda = 1,1,7$$

(i) For $\lambda = 1$, $[A - \lambda I]X = 0$ gives

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_2 - 2R_1, R_3 - 3R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore x_1 + x_2 + x_3 = 0$$

The rank (r) of the matrix is 1 and number of unknowns (n) is 3

Thus, $n - r = 3 - 1 = 2$ linearly independent solution

Let $x_3 = t$ & $x_2 = s$

$$\therefore x_1 = -s - t$$

$$\therefore X = \begin{bmatrix} -s-t \\ s \\ t \end{bmatrix} = \begin{bmatrix} -s-t \\ s+0t \\ 0s+t \end{bmatrix} = \begin{bmatrix} -s \\ s \\ 0s \end{bmatrix} + \begin{bmatrix} -t \\ 0t \\ t \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Hence, corresponding to $\lambda = 1$ the eigen vectors are

$X_1 = [-1,1,0]'$ & $X_2 = [-1,0,1]'$

(ii) For $\lambda = 7, [A - \lambda I]X = 0$ gives

$$\begin{bmatrix} -5 & 1 & 1 \\ 2 & -4 & 2 \\ 3 & 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-5x_1 + x_2 + x_3 = 0$$

$$2x_1 - 4x_2 + 2x_3 = 0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} 1 & 1 \\ -4 & 2 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} -5 & 1 \\ 2 & 2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -5 & 1 \\ 2 & -4 \end{vmatrix}}$$

$$\frac{x_1}{6} = -\frac{x_2}{-12} = \frac{x_3}{18}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{3}$$

Hence, corresponding to $\lambda = 7$ the eigen vector is $X_3 = [1,2,3]'$

6. Find the value of μ which satisfy the equation $A^{100}X = \mu X$ where
$$A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & -2 & -2 \\ 1 & 1 & 0 \end{bmatrix}$$
[N14/ElexExtcElectBiomInst/5M]
Solution:
$$A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & -2 & -2 \\ 1 & 1 & 0 \end{bmatrix}, |A| = 0$$
The characteristic equation,
$$|A - \lambda I| = 0$$
$$\begin{vmatrix} 2-\lambda & 1 & -1 \\ 0 & -2-\lambda & -2 \\ 1 & 1 & 0-\lambda \end{vmatrix} = 0$$
$$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$$
$$\lambda^3 - [2-2+0]\lambda^2 + \left[\begin{vmatrix} -2 & -2 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & -2 \end{vmatrix}\right]\lambda - 0 = 0$$
$$\lambda^3 + 0\lambda^2 - 1\lambda = 0$$
$$\lambda(\lambda^2 - 1) = 0$$
$$\lambda(\lambda - 1)(\lambda + 1) = 0$$
$$\lambda = 0, 1, -1$$
We know that, if λ is an eigen value of A and X is an eigen vector then
$$[A - \lambda I]X = 0$$
$$AX - \lambda X = 0$$
$$AX = \lambda X$$
Multiplying by A on both sides,
$$A(AX) = A(\lambda X)$$
$$A^2X = \lambda(AX)$$
$$A^2X = \lambda(\lambda X)$$
$A^2X = \lambda^2 X$ i.e. λ^2 is an eigen value of A^2
Similarly,
$$A^3X = \lambda^3 X$$
$A^4X = \lambda^4 X$ ……… $A^{100}X = \lambda^{100}X$
It is given that, $A^{100}X = \mu X$
Thus,
$$\mu = \lambda^{100}$$
Since, $\lambda = 0, 1, -1$
$$\mu = 0^{100}, 1^{100}, (-1)^{100}$$
$$\mu = 0, 1, 1$$

7. If $A = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 3 & 5 \\ 0 & 0 & -2 \end{bmatrix}$ find the eigen values of $A^3 + 5A + 8I$

[N14/CompIT/5M]

Solution:

We have,

$$A = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 3 & 5 \\ 0 & 0 & -2 \end{bmatrix}, |A| = 6$$

The characteristic equation,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} -1-\lambda & 2 & 3 \\ 0 & 3-\lambda & 5 \\ 0 & 0 & -2-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$$

$$\lambda^3 - [-1+3-2]\lambda^2 + \left[\begin{vmatrix} 3 & 5 \\ 0 & -2 \end{vmatrix} + \begin{vmatrix} -1 & 3 \\ 0 & -2 \end{vmatrix} + \begin{vmatrix} -1 & 2 \\ 0 & 3 \end{vmatrix}\right]\lambda - 6 = 0$$

$$\lambda^3 - 0\lambda^2 - 7\lambda - 6 = 0$$

$$\lambda = -1, -2, 3$$

The eigen values of A is $-1, -2, 3$

The eigen values of A^3 is $(-1)^3, (-2)^3, 3^3$ i.e $-1, -8, 27$

The eigen values of $5A$ is $5(-1), 5(-2), 5(3)$ i.e. $-5, -10, 15$

The eigen values of I is 1,1,1

The eigen values of $8I$ is 8,8,8

Thus, the eigen values of $A^3 + 5A + 8I$ is

$$-1 + (-5) + 8; \quad -8 + (-10) + 8; \quad 27 + 15 + 8$$

i.e. $2, -10, 50$

8. Determine the eigen values and eigen vectors of $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

[M15/ChemBiot/6M]

Solution:

We have,

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}, |A| = 0$$

The characteristic equation,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 8-\lambda & -6 & 4 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$$

$$\lambda^3 - [8+7+3]\lambda^2 + \left[\begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}\right]\lambda - 0 = 0$$

$$\lambda^3 - 18\lambda^2 + 45\lambda = 0$$

$$\lambda(\lambda - 15)(\lambda - 3) = 0$$

$$\lambda = 0,3,15$$

(i) For $\lambda = 0$, $[A - \lambda I]X = 0$ gives

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$8x_1 - 6x_2 + 2x_3 = 0$$

$$-6x_1 + 7x_2 - 4x_3 = 0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} -6 & 2 \\ 7 & -4 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} 8 & 2 \\ -6 & -4 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}}$$

$$\frac{x_1}{10} = -\frac{x_2}{-20} = \frac{x_3}{20}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{2}$$

Hence, corresponding to $\lambda = 0$ the eigen vector is $X_1 = [1,2,2]'$

(ii) For $\lambda = 3$, $[A - \lambda I]X = 0$ gives

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$5x_1 - 6x_2 + 2x_3 = 0$$

$$2x_1 - 4x_2 + 0x_3 = 0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix}-6 & 2\\ -4 & 0\end{vmatrix}} = -\frac{x_2}{\begin{vmatrix}5 & 2\\ 2 & 0\end{vmatrix}} = \frac{x_3}{\begin{vmatrix}5 & -6\\ 2 & -4\end{vmatrix}}$$

$$\frac{x_1}{8} = -\frac{x_2}{-4} = \frac{x_3}{-8}$$

$$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2}$$

Hence, corresponding to $\lambda = 3$ the eigen vector is $X_2 = [2,1,-2]'$

(iii) For $\lambda = 15$, $[A - \lambda I]X = 0$ gives

$$\begin{bmatrix}-7 & -6 & 2\\ -6 & -8 & -4\\ 2 & -4 & -12\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$-7x_1 - 6x_2 + 2x_3 = 0$$

$$-6x_1 - 8x_2 - 4x_3 = 0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix}-6 & 2\\ -8 & -4\end{vmatrix}} = -\frac{x_2}{\begin{vmatrix}-7 & 2\\ -6 & -4\end{vmatrix}} = \frac{x_3}{\begin{vmatrix}-7 & -6\\ -6 & -8\end{vmatrix}}$$

$$\frac{x_1}{40} = -\frac{x_2}{40} = \frac{x_3}{20}$$

$$\frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1}$$

Hence, corresponding to $\lambda = 15$ the eigen vector is $X_3 = [2,-2,1]'$

9. Find the eigen values and eigen vectors of $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

[M15/ElexExtcElectBiomInst/6M]

Solution:

We have,

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}, |A| = 5$$

The characteristic equation,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$$

$$\lambda^3 - [2+3+2]\lambda^2 + \left[\begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix}\right]\lambda - 5 = 0$$

$$\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$$

$$(\lambda - 1)(\lambda - 1)(\lambda - 5) = 0$$

$$\lambda = 1,1,5$$

(i) For $\lambda = 1, [A - \lambda I]X = 0$ gives

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_2 - R_1, R_3 - R_1$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore x_1 + 2x_2 + x_3 = 0$$

The rank (r) of the matrix is 1 and number of unknowns (n) is 3

Thus, $n - r = 3 - 1 = 2$ linearly independent solution

Let $x_3 = t$ & $x_2 = s$

$$\therefore x_1 = -2s - t$$

$$\therefore X = \begin{bmatrix} -2s - t \\ s \\ t \end{bmatrix} = \begin{bmatrix} -2s - t \\ s + 0t \\ 0s + t \end{bmatrix} = \begin{bmatrix} -2s \\ s \\ 0s \end{bmatrix} + \begin{bmatrix} -t \\ 0t \\ t \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Hence, corresponding to $\lambda = 1$ the eigen vectors are

$X_1 = [-2,1,0]'$ & $X_2 = [-1,0,1]'$

(ii) For $\lambda = 5, [A - \lambda I]X = 0$ gives

$$\begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-3x_1 + 2x_2 + x_3 = 0$$

$$x_1 - 2x_2 + x_3 = 0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} 2 & 1 \\ -2 & 1 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} -3 & 1 \\ 1 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -3 & 2 \\ 1 & -2 \end{vmatrix}}$$

$$\frac{x_1}{4} = -\frac{x_2}{-4} = \frac{x_3}{4}$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

Hence, corresponding to $\lambda = 5$ the eigen vector is $X_3 = [1,1,1]'$

10. If $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$, find characteristic roots and characteristic vectors of $A^3 + I$.

[M15/AutoMechCivil/5M]

Solution:

We have,

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}, |A| = 5$$

The characteristic equation,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$$

$$\lambda^3 - [2+3+2]\lambda^2 + \left[\begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix}\right]\lambda - 5 = 0$$

$$\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$$

$$(\lambda - 1)(\lambda - 1)(\lambda - 5) = 0$$

$$\lambda = 1,1,5$$

(i) For $\lambda = 1$, $[A - \lambda I]X = 0$ gives

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_2 - R_1, R_3 - R_1$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore x_1 + 2x_2 + x_3 = 0$$

The rank (r) of the matrix is 1 and number of unknowns (n) is 3

Thus, $n - r = 3 - 1 = 2$ linearly independent solution

Let $x_3 = t$ & $x_2 = s$

$$\therefore x_1 = -2s - t$$

$$\therefore X = \begin{bmatrix} -2s - t \\ s \\ t \end{bmatrix} = \begin{bmatrix} -2s - t \\ s + 0t \\ 0s + t \end{bmatrix} = \begin{bmatrix} -2s \\ s \\ 0s \end{bmatrix} + \begin{bmatrix} -t \\ 0t \\ t \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Hence, corresponding to $\lambda = 1$ the eigen vectors are

$X_1 = [-2,1,0]'$ & $X_2 = [-1,0,1]'$

(ii) For $\lambda = 5$, $[A - \lambda I]X = 0$ gives

$$\begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-3x_1 + 2x_2 + x_3 = 0$$

$$x_1 - 2x_2 + x_3 = 0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} 2 & 1 \\ -2 & 1 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} -3 & 1 \\ 1 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -3 & 2 \\ 1 & -2 \end{vmatrix}}$$

$$\frac{x_1}{4} = -\frac{x_2}{-4} = \frac{x_3}{4}$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

Hence, corresponding to $\lambda = 5$ the eigen vector is $X_3 = [1,1,1]'$

The characteristic roots of A is $1,1,5$

The characteristic roots of A^3 is $1^3, 1^3, 5^3$ i.e. $1,1,125$

The characteristic roots of I is $1,1,1$

Thus, The characteristic roots of $A^3 + I$ is

$1 + 1$; $1 + 1$; $125 + 1$

i.e. $2, 2, 126$

The characteristic vectors of $A^3 + I$ is same as that of A

11. If $A=\begin{bmatrix}1 & 0\\ 2 & 4\end{bmatrix}$ then find the eigen values of $4A^{-1}+3A+2I$

[M15/CompIT/5M]

Solution:

We have,

$A=\begin{bmatrix}1 & 0\\ 2 & 4\end{bmatrix}, |A|=4$

The characteristic equation,

$|A-\lambda I|=0$

$\begin{vmatrix}1-\lambda & 0\\ 2 & 4-\lambda\end{vmatrix}=0$

$(1-\lambda)(4-\lambda)-0=0$

$\lambda=1,4$

The eigen values of A is $1,4$

The eigen values of A^{-1} is $1^{-1},4^{-1}$ i.e. $1,\frac{1}{4}$

The eigen values of $4A^{-1}$ is $4(1),4\left(\frac{1}{4}\right)$ i.e. $4,1$

The eigen values of $3A$ is $3(1),3(4)$ i.e. $3,12$

The eigen values of I is $1,1$

The eigen values of $2I$ is $2,2$

Thus, the eigen values of $4A^{-1}+3A+2I$ is

$4+3+2\,;1+12+2$

i.e. $9;15$

12. Find the eigen values of adjoint of $\begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$

[N15/AutoMechCivil/5M][N17/CompIT/5M]

Solution:

We have,

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}, |A| = 6$$

The characteristic equation,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 0 & -1 \\ 0 & 2-\lambda & 0 \\ -1 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$$

$$\lambda^3 - [2+2+2]\lambda^2 + \left[\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix}\right]\lambda - 6 = 0$$

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

$$(\lambda - 1)(\lambda - 1)(\lambda - 5) = 0$$

$$\lambda = 1,3,2$$

The eigen values of A is 1,3,2

The eigen values of $adjA$ is $3 \times 2, 1 \times 2, 1 \times 3$ i.e 6,2,3

13. Find the eigen values and eigen vectors of $A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$.

[N15/ChemBiot/6M]

Solution:

We have,

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}, |A| = 6$$

The characteristic equation,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & -1 & 1 \\ 1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$$

$$\lambda^3 - [2+2+2]\lambda^2 + \left[\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix}\right]\lambda - 6 = 0$$

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

$$(\lambda - 1)(\lambda - 2)(\lambda - 3) = 0$$

$$\lambda = 1,2,3$$

(i) For $\lambda = 1, [A - \lambda I]X = 0$ gives

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 - x_2 + x_3 = 0$$

$$x_1 + x_2 - x_3 = 0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}}$$

$$\frac{x_1}{0} = -\frac{x_2}{-2} = \frac{x_3}{2}$$

$$\frac{x_1}{0} = \frac{x_2}{1} = \frac{x_3}{1}$$

Hence, corresponding to $\lambda = 1$ the eigen vector is $X_1 = [0,1,1]'$

(ii) For $\lambda = 2, [A - \lambda I]X = 0$ gives

$$\begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$0x_1 - x_2 + x_3 = 0$$

$$x_1 + 0x_2 - x_3 = 0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix}-1 & 1\\ 0 & -1\end{vmatrix}} = -\frac{x_2}{\begin{vmatrix}0 & 1\\ 1 & -1\end{vmatrix}} = \frac{x_3}{\begin{vmatrix}0 & -1\\ 1 & 0\end{vmatrix}}$$

$$\frac{x_1}{1} = -\frac{x_2}{-1} = \frac{x_3}{1}$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

Hence, corresponding to $\lambda = 2$ the eigen vector is $X_2 = [1,1,1]'$

(iii) For $\lambda = 3, [A - \lambda I]X = 0$ gives

$$\begin{bmatrix}-1 & -1 & 1\\ 1 & -1 & -1\\ 1 & -1 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$-x_1 - x_2 + x_3 = 0$$

$$x_1 - x_2 - x_3 = 0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix}-1 & 1\\ -1 & -1\end{vmatrix}} = -\frac{x_2}{\begin{vmatrix}-1 & 1\\ 1 & -1\end{vmatrix}} = \frac{x_3}{\begin{vmatrix}-1 & -1\\ 1 & -1\end{vmatrix}}$$

$$\frac{x_1}{2} = -\frac{x_2}{0} = \frac{x_3}{2}$$

$$\frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{1}$$

Hence, corresponding to $\lambda = 3$ the eigen vector is $X_3 = [1,0,1]'$

14. Find the eigen values and eigen vectors of A^3 where $A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{bmatrix}$

[N15/ElexExtcElectBiomInst/6M]

Solution:

We have,

$$A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{bmatrix}, |A| = 4$$

The characteristic equation,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 4-\lambda & 6 & 6 \\ 1 & 3-\lambda & 2 \\ -1 & -5 & -2-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$$

$$\lambda^3 - [4+3-2]\lambda^2 + \left[\begin{vmatrix} 3 & 2 \\ -5 & -2 \end{vmatrix} + \begin{vmatrix} 4 & 6 \\ -1 & -2 \end{vmatrix} + \begin{vmatrix} 4 & 6 \\ 1 & 3 \end{vmatrix}\right]\lambda - 4 = 0$$

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

$$(\lambda - 1)(\lambda - 2)(\lambda - 2) = 0$$

$$\lambda = 1,2,2$$

(i) For $\lambda = 1, [A - \lambda I]X = 0$ gives

$$\begin{bmatrix} 3 & 6 & 6 \\ 1 & 2 & 2 \\ -1 & -5 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + 2x_2 + 2x_3 = 0$$

$$-x_1 - 5x_2 - 3x_3 = 0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} 2 & 2 \\ -5 & -3 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} 1 & 2 \\ -1 & -3 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & 2 \\ -1 & -5 \end{vmatrix}}$$

$$\frac{x_1}{4} = -\frac{x_2}{-1} = \frac{x_3}{-3}$$

Hence, corresponding to $\lambda = 1$ the eigen vector is $X_1 = [4,1,-3]'$

(ii) For $\lambda = 2, [A - \lambda I]X = 0$ gives

$$\begin{bmatrix} 2 & 6 & 6 \\ 1 & 1 & 2 \\ -1 & -5 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$2x_1 + 6x_2 + 6x_3 = 0$$

$$-x_1 - 5x_2 - 4x_3 = 0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} 6 & 6 \\ -5 & -4 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} 2 & 6 \\ -1 & -4 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 2 & 6 \\ -1 & -5 \end{vmatrix}}$$

$\frac{x_1}{6} = -\frac{x_2}{-2} = \frac{x_3}{-4}$

Hence, corresponding to $\lambda = 2$ the eigen vector is $X_2 = [3,1,-2]'$

Thus, the eigen values of A is 1,2,2

And, the eigen values of A^3 is $1^3, 2^3, 2^3$ i.e. 1,8,8

Eigen vectors of A^3 is same as that of A

15. Find the eigen values and eigen vectors of $A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{bmatrix}$

[N15/CompIT/6M][N16/ChemBiot/6M]

Solution:

We have,

$$A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{bmatrix}, |A| = 4$$

The characteristic equation,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 4-\lambda & 6 & 6 \\ 1 & 3-\lambda & 2 \\ -1 & -5 & -2-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$$

$$\lambda^3 - [4+3-2]\lambda^2 + \left[\begin{vmatrix} 3 & 2 \\ -5 & -2 \end{vmatrix} + \begin{vmatrix} 4 & 6 \\ -1 & -2 \end{vmatrix} + \begin{vmatrix} 4 & 6 \\ 1 & 3 \end{vmatrix}\right]\lambda - 4 = 0$$

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

$$(\lambda - 1)(\lambda - 2)(\lambda - 2) = 0$$

$$\lambda = 1,2,2$$

(i) For $\lambda = 1, [A - \lambda I]X = 0$ gives

$$\begin{bmatrix} 3 & 6 & 6 \\ 1 & 2 & 2 \\ -1 & -5 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + 2x_2 + 2x_3 = 0$$

$$-x_1 - 5x_2 - 3x_3 = 0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} 2 & 2 \\ -5 & -3 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} 1 & 2 \\ -1 & -3 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & 2 \\ -1 & -5 \end{vmatrix}}$$

$$\frac{x_1}{4} = -\frac{x_2}{-1} = \frac{x_3}{-3}$$

Hence, corresponding to $\lambda = 1$ the eigen vector is $X_1 = [4,1,-3]'$

(ii) For $\lambda = 2, [A - \lambda I]X = 0$ gives

$$\begin{bmatrix} 2 & 6 & 6 \\ 1 & 1 & 2 \\ -1 & -5 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$2x_1 + 6x_2 + 6x_3 = 0$$

$$-x_1 - 5x_2 - 4x_3 = 0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} 6 & 6 \\ -5 & -4 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} 2 & 6 \\ -1 & -4 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 2 & 6 \\ -1 & -5 \end{vmatrix}}$$

$\frac{x_1}{6} = -\frac{x_2}{-2} = \frac{x_3}{-4}$

Hence, corresponding to $\lambda = 2$ the eigen vector is $X_2 = [3,1,-2]'$

16. Find the eigen values and eigen vectors of $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$.

[M16/ChemBiot/6M]

Solution:

We have,

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}, |A| = 36$$

The characteristic equation,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$$

$$\lambda^3 - [3+5+3]\lambda^2 + \left[\begin{vmatrix} 5 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & -1 \\ -1 & 5 \end{vmatrix}\right]\lambda - 36 = 0$$

$$\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$$

$$\lambda = 2,3,6$$

(i) For $\lambda = 2$, $[A - \lambda I]X = 0$ gives

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 - x_2 + x_3 = 0$$

$$-x_1 + 3x_2 - x_3 = 0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} -1 & 1 \\ 3 & -1 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & -1 \\ -1 & 3 \end{vmatrix}}$$

$$\frac{x_1}{-2} = -\frac{x_2}{0} = \frac{x_3}{2}$$

$$\frac{x_1}{-1} = \frac{x_2}{0} = \frac{x_3}{1}$$

Hence, corresponding to $\lambda = 2$ the eigen vector is $X_1 = [-1,0,1]'$

(ii) For $\lambda = 3$, $[A - \lambda I]X = 0$ gives

$$\begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$0x_1 - x_2 + x_3 = 0$$

$$-x_1 + 2x_2 - x_3 = 0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} 0 & 1 \\ -1 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 0 & -1 \\ -1 & 2 \end{vmatrix}}$$

$$\frac{x_1}{-1} = -\frac{x_2}{1} = \frac{x_3}{-1}$$
$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

Hence, corresponding to $\lambda = 3$ the eigen vector is $X_2 = [1,1,1]'$

(iii) For $\lambda = 6, [A - \lambda I]X = 0$ gives

$$\begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-3x_1 - x_2 + x_3 = 0$$
$$-x_1 - x_2 - x_3 = 0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} -1 & 1 \\ -1 & -1 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} -3 & 1 \\ -1 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -3 & -1 \\ -1 & -1 \end{vmatrix}}$$
$$\frac{x_1}{2} = -\frac{x_2}{4} = \frac{x_3}{2}$$
$$\frac{x_1}{1} = \frac{x_2}{-2} = \frac{x_3}{1}$$

Hence, corresponding to $\lambda = 6$ the eigen vector is $X_3 = [1,-2,1]'$

17. Find eigen values and eigen vectors of A^3 where $A=\begin{bmatrix}2&1&1\\2&3&2\\3&3&4\end{bmatrix}$

[M16/ElexExtcElectBiomInst/6M]

Solution:

We have,

$$A=\begin{bmatrix}2&1&1\\2&3&2\\3&3&4\end{bmatrix}, |A|=7$$

The characteristic equation,

$$|A-\lambda I|=0$$

$$\begin{vmatrix}2-\lambda&1&1\\2&3-\lambda&2\\3&3&4-\lambda\end{vmatrix}=0$$

$$\lambda^3-[sum\ of\ diagonals]\lambda^2+[sum\ of\ minors\ of\ diagonals]\lambda-|A|=0$$

$$\lambda^3-[2+3+4]\lambda^2+\left[\begin{vmatrix}3&2\\3&4\end{vmatrix}+\begin{vmatrix}2&1\\3&4\end{vmatrix}+\begin{vmatrix}2&1\\2&3\end{vmatrix}\right]\lambda-7=0$$

$$\lambda^3-9\lambda^2+15\lambda-7=0$$

$$(\lambda-1)(\lambda-1)(\lambda-7)=0$$

$$\lambda=1,1,7$$

(i) For $\lambda=1$, $[A-\lambda I]X=0$ gives

$$\begin{bmatrix}1&1&1\\2&2&2\\3&3&3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

By R_2-2R_1, R_3-3R_1

$$\begin{bmatrix}1&1&1\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\therefore x_1+x_2+x_3=0$$

The rank (r) of the matrix is 1 and number of unknowns (n) is 3

Thus, $n-r=3-1=2$ linearly independent solution

Let $x_3=t$ & $x_2=s$

$$\therefore x_1=-s-t$$

$$\therefore X=\begin{bmatrix}-s-t\\s\\t\end{bmatrix}=\begin{bmatrix}-s-t\\s+0t\\0s+t\end{bmatrix}=\begin{bmatrix}-s\\s\\0s\end{bmatrix}+\begin{bmatrix}-t\\0t\\t\end{bmatrix}=s\begin{bmatrix}-1\\1\\0\end{bmatrix}+t\begin{bmatrix}-1\\0\\1\end{bmatrix}$$

Hence, corresponding to $\lambda=1$ the eigen vectors are

$X_1=[-1,1,0]'$ & $X_2=[-1,0,1]'$

(ii) For $\lambda = 7, [A - \lambda I]X = 0$ gives

$$\begin{bmatrix} -5 & 1 & 1 \\ 2 & -4 & 2 \\ 3 & 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-5x_1 + x_2 + x_3 = 0$

$2x_1 - 4x_2 + 2x_3 = 0$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} 1 & 1 \\ -4 & 2 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} -5 & 1 \\ 2 & 2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -5 & 1 \\ 2 & -4 \end{vmatrix}}$$

$$\frac{x_1}{6} = -\frac{x_2}{-12} = \frac{x_3}{18}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{3}$$

Hence, corresponding to $\lambda = 7$ the eigen vector is $X_3 = [1,2,3]'$

Thus, the eigen values of A is 1,1,7

And, the eigen values of A^3 is $1^3, 1^3, 7^3$ i.e. $1,1,343$

Eigen vectors of A^3 is same as that of A

18. Find the characteristic roots of A and $A^2 + I$ where $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

[M16/AutoMechCivil/5M]

Solution:

We have,

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}, |A| = 45$$

The characteristic equation,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$$

$$\lambda^3 - [-2+1+0]\lambda^2 + \left[\begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix} + \begin{vmatrix} -2 & -3 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} -2 & 2 \\ 2 & 1 \end{vmatrix}\right]\lambda - 45 = 0$$

$$\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$$

$$(\lambda - 5)(\lambda + 3)(\lambda + 3) = 0$$

$$\lambda = 5, -3, -3$$

The characteristic roots of A is $5, -3, -3$

The characteristic roots of A^2 is $5^2, (-3)^2, (-3)^2$ i.e. $25, 9, 9$

The characteristic roots of I is $1, 1, 1$

Thus, the characteristic roots of $A^2 + I$ is

$25 + 1 \,;\, 9 + 1 \,;\, 9 + 1$

i.e. $26 \,;\, 10 \,;\, 10$

19. Find the eigen values of $A^2 + 2I$ where $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -2 & 0 \\ 3 & 5 & 3 \end{bmatrix}$ and I is the Identity matrix of order 3

[M16/CompIT/5M]

Solution:

We have,

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -2 & 0 \\ 3 & 5 & 3 \end{bmatrix}, |A| = -6$$

The characteristic equation,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 2 & -2-\lambda & 0 \\ 3 & 5 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$$

$$\lambda^3 - [1 - 2 + 3]\lambda^2 + \left[\begin{vmatrix} -2 & 0 \\ 5 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 3 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 2 & -2 \end{vmatrix}\right]\lambda - (-6) = 0$$

$$\lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0$$

$$(\lambda - 1)(\lambda + 2)(\lambda - 3) = 0$$

$$\lambda = 1, -2, 3$$

The eigen values of A is $1, -2, 3$

The eigen values of A^2 is $1^2, 2^2, (-3)^2$ i.e. $1,4,9$

The eigen values of I is $1,1,1$

The eigen values of $2I$ is $2,2,2$

Thus, the characteristic roots of $A^2 + 2I$ is

$1 + 2 ; 4 + 2 ; 9 + 2$

i.e. $3 ; 6 ; 11$

20. Find the eigen values and eigen vectors of $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

[N16/CompIT/5M]

Solution:

We have,

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}, |A| = 8$$

The characteristic equation,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$$

$$\lambda^3 - [2+2+2]\lambda^2 + \left[\begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix}\right]\lambda - 8 = 0$$

$$\lambda^3 - 6\lambda^2 + 12\lambda - 8 = 0$$

$$\lambda = 2,2,2$$

For $\lambda = 2, [A - \lambda I]X = 0$ gives

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$0x_1 + x_2 + 0x_3 = 0$$

$$0x_1 + 0x_2 + x_3 = 0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}}$$

$$\frac{x_1}{1} = -\frac{x_2}{0} = \frac{x_3}{0}$$

Hence, corresponding to $\lambda = 2$ the eigen vector is $X_1 = [1,0,0]'$

21. If $A = \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix}$ then find the eigen values of $6A^{-1} + A^3 + 2I$

[M17/CompIT/5M]

Solution:

We have,

$A = \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix}$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 2-\lambda & 4 \\ 0 & 3-\lambda \end{vmatrix} = 0$

$(2-\lambda)(3-\lambda) - 0 = 0$

$\lambda = 2,3$

The eigen values of A is 2,3

The eigen values of A^{-1} is $2^{-1}, 3^{-1}$ i.e. $\frac{1}{2}, \frac{1}{3}$

The eigen values of $6A^{-1}$ is $6\left(\frac{1}{2}\right), 6\left(\frac{1}{3}\right)$ i.e. 3,2

The eigen values of A^3 is $2^3, 3^3$ i.e. 8,27

The eigen values of I is 1,1

The eigen values of $2I$ is 2,2

Thus, the eigen values of $6A^{-1} + A^3 + 2I$ is

$3 + 8 + 2\ ;\ 2 + 27 + 2$

i.e. $13; 31$

22. Find eigen values and eigen vectors of A^2+2I where $A=\begin{bmatrix}8 & -8 & -2\\ 4 & -3 & -2\\ 3 & -4 & 1\end{bmatrix}$

[N17/ElexExctElectBiomInst/6M] Ans. 3, 6, 11 & [4,3,2]’,[3,2,1]’,[2,1,1]’

Solution:

We have,

$$A=\begin{bmatrix}8 & -8 & -2\\ 4 & -3 & -2\\ 3 & -4 & 1\end{bmatrix}, |A|=6$$

The characteristic equation,

$$|A-\lambda I|=0$$

$$\begin{vmatrix}8-\lambda & -8 & -2\\ 4 & -3-\lambda & -2\\ 3 & -4 & 1-\lambda\end{vmatrix}=0$$

$$\lambda^3-[sum\ of\ diagonals]\lambda^2+[sum\ of\ minors\ of\ diagonals]\lambda-|A|=0$$

$$\lambda^3-[8-3+1]\lambda^2+\left[\begin{vmatrix}-3 & -2\\ -4 & 1\end{vmatrix}+\begin{vmatrix}8 & -2\\ 3 & 1\end{vmatrix}+\begin{vmatrix}8 & -8\\ 4 & -3\end{vmatrix}\right]\lambda-6=0$$

$$\lambda^3-6\lambda^2+11\lambda-6=0$$

$$\lambda=1,2,3$$

(i) For $\lambda=1, [A-\lambda I]X=0$ gives

$$\begin{bmatrix}7 & -8 & -2\\ 4 & -4 & -2\\ 3 & -4 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$7x_1-8x_2-2x_3=0$$

$$4x_1-4x_2-2x_3=0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix}-8 & -2\\ -4 & -2\end{vmatrix}}=-\frac{x_2}{\begin{vmatrix}7 & -2\\ 4 & -2\end{vmatrix}}=\frac{x_3}{\begin{vmatrix}7 & -8\\ 4 & -4\end{vmatrix}}$$

$$\frac{x_1}{8}=-\frac{x_2}{-6}=\frac{x_3}{4}$$

$$\frac{x_1}{4}=\frac{x_2}{3}=\frac{x_3}{2}$$

Hence, corresponding to $\lambda=1$ the eigen vector is $X_1=[4,3,2]'$

(ii) For $\lambda=2, [A-\lambda I]X=0$ gives

$$\begin{bmatrix}6 & -8 & -2\\ 4 & -5 & -2\\ 3 & -4 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$6x_1-8x_2-2x_3=0$$

$$4x_1-5x_2-2x_3=0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix}-8 & -2\\ -5 & -2\end{vmatrix}}=-\frac{x_2}{\begin{vmatrix}6 & -2\\ 4 & -2\end{vmatrix}}=\frac{x_3}{\begin{vmatrix}6 & -8\\ 4 & -5\end{vmatrix}}$$

$\frac{x_1}{6} = -\frac{x_2}{-4} = \frac{x_3}{2}$

$\frac{x_1}{3} = \frac{x_2}{2} = \frac{x_3}{1}$

Hence, corresponding to $\lambda = 2$ the eigen vector is $X_2 = [3,2,1]'$

(iii) For $\lambda = 3, [A - \lambda I]X = 0$ gives

$$\begin{bmatrix} 5 & -8 & -2 \\ 4 & -6 & -2 \\ 3 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$5x_1 - 8x_2 - 2x_3 = 0$

$4x_1 - 6x_2 - 2x_3 = 0$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} -8 & -2 \\ -6 & -2 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} 5 & -2 \\ 4 & -2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 5 & -8 \\ 4 & -6 \end{vmatrix}}$$

$\frac{x_1}{4} = -\frac{x_2}{-2} = \frac{x_3}{2}$

$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{1}$

Hence, corresponding to $\lambda = 3$ the eigen vector is $X_3 = [2,1,1]'$

The eigen values of A is $1,2,3$

The eigen values of A^2 is $1^2, 2^2, 3^2$ i.e. $1,4,9$

The eigen values of I is $1,1,1$

The eigen values of $2I$ is $2,2,2$

Thus, the eigen values of $A^2 + 2I$ is

$1 + 2\ ; 4 + 2; 9 + 2$

i.e. $3; 6; 11$

the eigen vectors of $A^2 + 2I$ is same as that of A

Type II: Transformation and Diagonalising of Matrix

1. Show that the matrix $A = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$ is diagonalizable. Hence find diagonal and transforming matrix

[N13/Chem/7M][N14/ChemBiot/7M][N15/ElexExtcElectBiomInst/8M][M16/ChemBiot/8M][M16/CompIT/8M][M17/CompIT/8M]

Solution:

We have,

$$A = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}, |A| = 6$$

The characteristic equation,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 8-\lambda & -8 & -2 \\ 4 & -3-\lambda & -2 \\ 3 & -4 & 1-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$$

$$\lambda^3 - [8-3+1]\lambda^2 + \left[\begin{vmatrix} -3 & -2 \\ -4 & 1 \end{vmatrix} + \begin{vmatrix} 8 & -2 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 8 & -8 \\ 4 & -3 \end{vmatrix}\right]\lambda - 6 = 0$$

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

$$\lambda = 1,2,3$$

(i) For $\lambda = 1, [A - \lambda I]X = 0$ gives

$$\begin{bmatrix} 7 & -8 & -2 \\ 4 & -4 & -2 \\ 3 & -4 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$7x_1 - 8x_2 - 2x_3 = 0$$

$$4x_1 - 4x_2 - 2x_3 = 0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} -8 & -2 \\ -4 & -2 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} 7 & -2 \\ 4 & -2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 7 & -8 \\ 4 & -4 \end{vmatrix}}$$

$$\frac{x_1}{8} = -\frac{x_2}{-6} = \frac{x_3}{4}$$

$$\frac{x_1}{4} = \frac{x_2}{3} = \frac{x_3}{2}$$

Hence, corresponding to $\lambda = 1$ the eigen vector is $X_1 = [4,3,2]'$

(ii) For $\lambda = 2, [A - \lambda I]X = 0$ gives

$$\begin{bmatrix} 6 & -8 & -2 \\ 4 & -5 & -2 \\ 3 & -4 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$6x_1 - 8x_2 - 2x_3 = 0$$

$4x_1 - 5x_2 - 2x_3 = 0$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} -8 & -2 \\ -5 & -2 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} 6 & -2 \\ 4 & -2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 6 & -8 \\ 4 & -5 \end{vmatrix}}$$

$$\frac{x_1}{6} = -\frac{x_2}{-4} = \frac{x_3}{2}$$

$$\frac{x_1}{3} = \frac{x_2}{2} = \frac{x_3}{1}$$

Hence, corresponding to $\lambda = 2$ the eigen vector is $X_2 = [3,2,1]'$

(iii) For $\lambda = 3, [A - \lambda I]X = 0$ gives

$$\begin{bmatrix} 5 & -8 & -2 \\ 4 & -6 & -2 \\ 3 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$5x_1 - 8x_2 - 2x_3 = 0$

$4x_1 - 6x_2 - 2x_3 = 0$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} -8 & -2 \\ -6 & -2 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} 5 & -2 \\ 4 & -2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 5 & -8 \\ 4 & -6 \end{vmatrix}}$$

$$\frac{x_1}{4} = -\frac{x_2}{-2} = \frac{x_3}{2}$$

$$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{1}$$

Hence, corresponding to $\lambda = 3$ the eigen vector is $X_3 = [2,1,1]'$

Thus, the matrix $A = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$ is diagonalised to $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ by the transformation $M^{-1}AM = D$ where $M = \begin{bmatrix} 4 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$

2. Is the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ diagonalisable? If so, find diagonal form and transforming matrix.

[M14/ElexExtcElectBiomInst/8M][N17/CompIT/8M]

Solution:

We have,

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}, |A| = 3$$

The characteristic equation,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$$

$$\lambda^3 - [2+2+1]\lambda^2 + \left[\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}\right]\lambda - 3 = 0$$

$$\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$$

$$(\lambda - 1)(\lambda - 1)(\lambda - 3) = 0$$

$$\lambda = 1,1,3$$

The Algebraic Multiplicity of $\lambda = 1$ is 2 and that of $\lambda = 3$ is 1

(i) For $\lambda = 1$, $[A - \lambda I]X = 0$ gives

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_2 - R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore x_1 + x_2 + x_3 = 0$

The rank (r) of the matrix is 1 and number of unknowns (n) is 3

Thus, $n - r = 3 - 1 = 2$ linearly independent solution

The Geometric Multiplicity of $\lambda = 1$ is 2

Since, Algebraic Multiplicity = Geometric Multiplicity,

matrix A is diagonalizable.

Let $x_3 = t$ & $x_2 = s$

$\therefore x_1 = -s - t$

$$\therefore X = \begin{bmatrix} -s-t \\ s \\ t \end{bmatrix} = \begin{bmatrix} -s-t \\ s+0t \\ 0s+t \end{bmatrix} = \begin{bmatrix} -s \\ s \\ 0s \end{bmatrix} + \begin{bmatrix} -t \\ 0t \\ t \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Hence, corresponding to $\lambda = 1$ the eigen vectors are

$X_1 = [-1,1,0]'$ & $X_2 = [-1,0,1]'$

(ii) For $\lambda = 3$, $[A - \lambda I]X = 0$ gives

$$\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x_1 + x_2 + x_3 = 0$$

$$x_1 - x_2 + x_3 = 0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix}}$$

$$\frac{x_1}{2} = -\frac{x_2}{-2} = \frac{x_3}{0}$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{0}$$

Hence, corresponding to $\lambda = 3$ the eigen vector is $X_3 = [1,1,0]'$

Thus, the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ is diagonalised to $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ by the transformation $M^{-1}AM = D$ where $M = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

3. Show that the matrix $A=\begin{bmatrix}8 & -6 & 2\\ -6 & 7 & -4\\ 2 & -4 & 3\end{bmatrix}$ is diagonalizable. Hence find diagonal and transforming matrix
[M14/AutoMechCivil/8M][N14/AutoMechCivil/8M]
[M15/AutoMechCivil/6M][M15/CompIT/8M]

Solution:

We have,

$$A=\begin{bmatrix}8 & -6 & 2\\ -6 & 7 & -4\\ 2 & -4 & 3\end{bmatrix}, |A|=0$$

The characteristic equation,

$$|A-\lambda I|=0$$

$$\begin{vmatrix}8-\lambda & -6 & 4\\ -6 & 7-\lambda & -4\\ 2 & -4 & 3-\lambda\end{vmatrix}=0$$

$$\lambda^3-[sum\ of\ diagonals]\lambda^2+[sum\ of\ minors\ of\ diagonals]\lambda-|A|=0$$

$$\lambda^3-[8+7+3]\lambda^2+\left[\begin{vmatrix}7 & -4\\ -4 & 3\end{vmatrix}+\begin{vmatrix}8 & 2\\ 2 & 3\end{vmatrix}+\begin{vmatrix}8 & -6\\ -6 & 7\end{vmatrix}\right]\lambda-0=0$$

$$\lambda^3-18\lambda^2+45\lambda=0$$

$$\lambda(\lambda-15)(\lambda-3)=0$$

$$\lambda=0,3,15$$

(i) For $\lambda=0$, $[A-\lambda I]X=0$ gives

$$\begin{bmatrix}8 & -6 & 2\\ -6 & 7 & -4\\ 2 & -4 & 3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$8x_1-6x_2+2x_3=0$$

$$-6x_1+7x_2-4x_3=0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix}-6 & 2\\ 7 & -4\end{vmatrix}}=-\frac{x_2}{\begin{vmatrix}8 & 2\\ -6 & -4\end{vmatrix}}=\frac{x_3}{\begin{vmatrix}8 & -6\\ -6 & 7\end{vmatrix}}$$

$$\frac{x_1}{10}=-\frac{x_2}{-20}=\frac{x_3}{20}$$

$$\frac{x_1}{1}=\frac{x_2}{2}=\frac{x_3}{2}$$

Hence, corresponding to $\lambda=0$ the eigen vector is $X_1=[1,2,2]'$

(ii) For $\lambda=3$, $[A-\lambda I]X=0$ gives

$$\begin{bmatrix}5 & -6 & 2\\ -6 & 4 & -4\\ 2 & -4 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$5x_1-6x_2+2x_3=0$$

$2x_1 - 4x_2 + 0x_3 = 0$

Solving the above equations by Crammers rule, we get

$\frac{x_1}{\begin{vmatrix}-6 & 2\\-4 & 0\end{vmatrix}} = -\frac{x_2}{\begin{vmatrix}5 & 2\\2 & 0\end{vmatrix}} = \frac{x_3}{\begin{vmatrix}5 & -6\\2 & -4\end{vmatrix}}$

$\frac{x_1}{8} = -\frac{x_2}{-4} = \frac{x_3}{-8}$

$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2}$

Hence, corresponding to $\lambda = 3$ the eigen vector is $X_2 = [2,1,-2]'$

(iii) For $\lambda = 15$, $[A - \lambda I]X = 0$ gives

$$\begin{bmatrix}-7 & -6 & 2\\-6 & -8 & -4\\2 & -4 & -12\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$-7x_1 - 6x_2 + 2x_3 = 0$

$-6x_1 - 8x_2 - 4x_3 = 0$

Solving the above equations by Crammers rule, we get

$\frac{x_1}{\begin{vmatrix}-6 & 2\\-8 & -4\end{vmatrix}} = -\frac{x_2}{\begin{vmatrix}-7 & 2\\-6 & -4\end{vmatrix}} = \frac{x_3}{\begin{vmatrix}-7 & -6\\-6 & -8\end{vmatrix}}$

$\frac{x_1}{40} = -\frac{x_2}{40} = \frac{x_3}{20}$

$\frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1}$

Hence, corresponding to $\lambda = 15$ the eigen vector is $X_3 = [2,-2,1]'$

Thus, the matrix $A = \begin{bmatrix}8 & -6 & 2\\-6 & 7 & -4\\2 & -4 & 3\end{bmatrix}$ is diagonalised to $D = \begin{bmatrix}0 & 0 & 0\\0 & 3 & 0\\0 & 0 & 15\end{bmatrix}$ by the transformation $M^{-1}AM = D$ where $M = \begin{bmatrix}1 & 2 & 2\\2 & 1 & -2\\2 & -2 & 1\end{bmatrix}$

4. Show that the matrix $A=\begin{bmatrix}-9 & 4 & 4\\ -8 & 3 & 4\\ -16 & 8 & 7\end{bmatrix}$ is diagonalizable. Hence find diagonal and transforming matrix

[M14/CompIT/8M][N14/CompIT/8M][N14/ElexExtcElectBiomInst/8M][N15/CompIT/8M][N17/ElexExtcElectBiomInst/8M]

Solution:

We have,

$$A=\begin{bmatrix}-9 & 4 & 4\\ -8 & 3 & 4\\ -16 & 8 & 7\end{bmatrix}, |A|=3$$

The characteristic equation,

$$|A-\lambda I|=0$$

$$\begin{vmatrix}-9-\lambda & 4 & 4\\ -8 & 3-\lambda & 4\\ -16 & 8 & 7-\lambda\end{vmatrix}=0$$

$$\lambda^3-[sum\ of\ diagonals]\lambda^2+[sum\ of\ minors\ of\ diagonals]\lambda-|A|=0$$

$$\lambda^3-[-9+3+7]\lambda^2+\left[\begin{vmatrix}3 & 4\\ 8 & 7\end{vmatrix}+\begin{vmatrix}-9 & 4\\ -16 & 7\end{vmatrix}+\begin{vmatrix}-9 & 4\\ -8 & 3\end{vmatrix}\right]\lambda-3=0$$

$$\lambda^3-\lambda^2-5\lambda-3=0$$

$$(\lambda+1)(\lambda+1)(\lambda-3)=0$$

$$\lambda=-1,-1,3$$

The Algebraic Multiplicity of $\lambda=-1$ is 2 and that of $\lambda=3$ is 1

(i) For $\lambda=-1, [A-\lambda I]X=0$ gives

$$\begin{bmatrix}-8 & 4 & 4\\ -8 & 4 & 4\\ -16 & 8 & 8\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

By R_2-R_1, R_3-2R_1

$$\begin{bmatrix}-8 & 4 & 4\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$\therefore -8x_1+4x_2+4x_3=0$$

The rank (r) of the matrix is 1 and number of unknowns (n) is 3

Thus, $n-r=3-1=2$ linearly independent solution

The Geometric Multiplicity of $\lambda=1$ is 2

Since, Algebraic Multiplicity = Geometric Multiplicity,

matrix A is diagonalizable.

Let $x_3=2t$ & $x_2=2s$

$$\therefore x_1=s+t$$

$$\therefore X = \begin{bmatrix} s+t \\ 2s \\ 2t \end{bmatrix} = \begin{bmatrix} s+t \\ 2s+0t \\ 0s+2t \end{bmatrix} = \begin{bmatrix} s \\ 2s \\ 0s \end{bmatrix} + \begin{bmatrix} t \\ 0t \\ 2t \end{bmatrix} = s\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

Hence, corresponding to $\lambda = -1$ the eigen vectors are

$X_1 = [1,2,0]'$ & $X_2 = [1,0,2]'$

(ii) For $\lambda = 3$, $[A - \lambda I]X = 0$ gives

$$\begin{bmatrix} -12 & 4 & 4 \\ -8 & 0 & 4 \\ -16 & 8 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-12x_1 + 4x_2 + 4x_3 = 0$$

$$-8x_1 + 0x_2 + 4x_3 = 0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} 4 & 4 \\ 0 & 4 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} -12 & 4 \\ -8 & 4 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -12 & 4 \\ -8 & 0 \end{vmatrix}}$$

$$\frac{x_1}{16} = -\frac{x_2}{-16} = \frac{x_3}{32}$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{2}$$

Hence, corresponding to $\lambda = 3$ the eigen vector is $X_3 = [1,1,2]'$

Thus, the matrix $A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$ is diagonalised to $D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ by the transformation $M^{-1}AM = D$ where $M = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 2 & 0 & 2 \end{bmatrix}$

5. Show that the matrix $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ is diagonalizable. Hence find diagonal and transforming matrix

[M15/ElexExtcElectBiomInst/8M]

Solution:

We have,

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}, |A| = 32$$

The characteristic equation,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$$

$$\lambda^3 - [6+3+3]\lambda^2 + \left[\begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 6 & -2 \\ -2 & 3 \end{vmatrix}\right]\lambda - 32 = 0$$

$$\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$$

$$(\lambda - 2)(\lambda - 2)(\lambda - 8) = 0$$

$$\lambda = 2,2,8$$

The Algebraic Multiplicity of $\lambda = 2$ is 2 and that of $\lambda = 8$ is 1

(i) For $\lambda = 2$, $[A - \lambda I]X = 0$ gives

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_2 + \frac{1}{2}R_1, R_3 - \frac{1}{2}R_1$

$$\begin{bmatrix} 4 & -2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore 4x_1 - 2x_2 + 2x_3 = 0$$

The rank (r) of the matrix is 1 and number of unknowns (n) is 3

Thus, $n - r = 3 - 1 = 2$ linearly independent solution

The Geometric Multiplicity of $\lambda = 2$ is 2

Since, Algebraic Multiplicity = Geometric Multiplicity,

matrix A is diagonalizable.

Let $x_3 = 2t$ & $x_2 = 2s$, $\therefore x_1 = s - t$

$$\therefore X = \begin{bmatrix} s-t \\ 2s \\ 2t \end{bmatrix} = \begin{bmatrix} s-t \\ 2s+0t \\ 0s+2t \end{bmatrix} = \begin{bmatrix} s \\ 2s \\ 0s \end{bmatrix} + \begin{bmatrix} -t \\ 0t \\ 2t \end{bmatrix} = s\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

Hence, corresponding to $\lambda = 2$ the eigen vectors are
$X_1 = [1,2,0]'$ & $X_2 = [-1,0,2]'$

(ii) For $\lambda = 8$, $[A - \lambda I]X = 0$ gives

$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x_1 - 2x_2 + 2x_3 = 0$$
$$-2x_1 - 5x_2 - 1x_3 = 0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} -2 & 2 \\ -5 & -1 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} -2 & 2 \\ -2 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -2 & -2 \\ -2 & -5 \end{vmatrix}}$$

$$\frac{x_1}{12} = -\frac{x_2}{6} = \frac{x_3}{6}$$

$$\frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{1}$$

Hence, corresponding to $\lambda = 8$ the eigen vector is $X_3 = [2,-1,1]'$

Thus, the matrix $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ is diagonalised to $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$ by the transformation $M^{-1}AM = D$ where $M = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 0 & -1 \\ 0 & 2 & 1 \end{bmatrix}$

6. Show that the matrix $A = \begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}$ is similar to a diagonal matrix. Also find the transforming matrix and diagonal matrix.

[M16/ElexExtcElectBiomInst/8M]

Solution:

We have,

$$A = \begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}, |A| = 0$$

The characteristic equation,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & -6 & -4 \\ 0 & 4-\lambda & 2 \\ 0 & -6 & -3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$$

$$\lambda^3 - [1+4-3]\lambda^2 + \left[\begin{vmatrix} 4 & 2 \\ -6 & -3 \end{vmatrix} + \begin{vmatrix} 1 & -4 \\ 0 & -3 \end{vmatrix} + \begin{vmatrix} 1 & -6 \\ 0 & 4 \end{vmatrix}\right]\lambda - 32 = 0$$

$$\lambda^3 - 2\lambda^2 + \lambda = 0$$

$$\lambda(\lambda - 1)(\lambda - 1) = 0$$

$$\lambda = 0,1,1$$

The Algebraic Multiplicity of $\lambda = 1$ is 2 and that of $\lambda = 0$ is 1

(i) For $\lambda = 1, [A - \lambda I]X = 0$ gives

$$\begin{bmatrix} 0 & -6 & -4 \\ 0 & 3 & 2 \\ 0 & -6 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_2 + \frac{1}{2}R_1, R_3 - R_1$

$$\begin{bmatrix} 0 & -6 & -4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore 0x_1 - 6x_2 - 4x_3 = 0$

The rank (r) of the matrix is 1 and number of unknowns (n) is 3

Thus, $n - r = 3 - 1 = 2$ linearly independent solution

The Geometric Multiplicity of $\lambda = 1$ is 2

Since, Algebraic Multiplicity = Geometric Multiplicity,

matrix A is diagonalizable.

Let $x_1 = 1, x_2 = 2 \therefore x_3 = -3$

Let $x_1 = 1, x_2 = -2 \therefore x_3 = 3$

Hence, corresponding to $\lambda = 1$ the eigen vectors are

$X_1 = [1,2,-3]'$ & $X_2 = [1,-2,3]'$

(ii) For $\lambda = 0, [A - \lambda I]X = 0$ gives

$$\begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 - 6x_2 - 4x_3 = 0$

$0x_1 + 4x_2 + 2x_3 = 0$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} -6 & -4 \\ 4 & 2 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} 1 & -4 \\ 0 & 2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & -6 \\ 0 & 4 \end{vmatrix}}$$

$$\frac{x_1}{4} = -\frac{x_2}{2} = \frac{x_3}{4}$$

$$\frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{2}$$

Hence, corresponding to $\lambda = 0$ the eigen vector is $X_3 = [2,-1,2]'$

Thus, the matrix $A = \begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}$ is diagonalised to $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ by the transformation $M^{-1}AM = D$ where $M = \begin{bmatrix} 1 & 1 & 2 \\ 2 & -2 & -1 \\ -3 & 3 & 2 \end{bmatrix}$

7. Show that $A=\begin{bmatrix}2 & 2 & 1\\1 & 3 & 1\\1 & 2 & 2\end{bmatrix}$ is diagonalizable. Find the transforming and diagonal form.

[M16/AutoMechCivil/8M]

Solution:

We have,

$$A=\begin{bmatrix}2 & 2 & 1\\1 & 3 & 1\\1 & 2 & 2\end{bmatrix}, |A|=5$$

The characteristic equation,

$$|A-\lambda I|=0$$

$$\begin{vmatrix}2-\lambda & 2 & 1\\1 & 3-\lambda & 1\\1 & 2 & 2-\lambda\end{vmatrix}=0$$

$$\lambda^3-[sum\ of\ diagonals]\lambda^2+[sum\ of\ minors\ of\ diagonals]\lambda-|A|=0$$

$$\lambda^3-[2+3+2]\lambda^2+\left[\begin{vmatrix}3 & 1\\2 & 2\end{vmatrix}+\begin{vmatrix}2 & 1\\1 & 2\end{vmatrix}+\begin{vmatrix}2 & 2\\1 & 3\end{vmatrix}\right]\lambda-5=0$$

$$\lambda^3-7\lambda^2+11\lambda-5=0$$

$$(\lambda-1)(\lambda-1)(\lambda-5)=0$$

$$\lambda=1,1,5$$

The Algebraic Multiplicity of $\lambda=1$ is 2 and that of $\lambda=5$ is 1

(i) For $\lambda=1$, $[A-\lambda I]X=0$ gives

$$\begin{bmatrix}1 & 2 & 1\\1 & 2 & 1\\1 & 2 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

By R_2-R_1, R_3-R_1

$$\begin{bmatrix}1 & 2 & 1\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\therefore x_1+2x_2+x_3=0$$

The rank (r) of the matrix is 1 and number of unknowns (n) is 3

Thus, $n-r=3-1=2$ linearly independent solution

The Geometric Multiplicity of $\lambda=1$ is 2

Since, Algebraic Multiplicity = Geometric Multiplicity,

matrix A is diagonalizable.

Let $x_3=t$ & $x_2=s$, $\therefore x_1=-2s-t$

$$\therefore X=\begin{bmatrix}-2s-t\\s\\t\end{bmatrix}=\begin{bmatrix}-2s-t\\s+0t\\0s+t\end{bmatrix}=\begin{bmatrix}-2s\\s\\0s\end{bmatrix}+\begin{bmatrix}-t\\0t\\t\end{bmatrix}=s\begin{bmatrix}-2\\1\\0\end{bmatrix}+t\begin{bmatrix}-1\\0\\1\end{bmatrix}$$

Hence, corresponding to $\lambda = 1$ the eigen vectors are

$X_1 = [-2,1,0]'$ & $X_2 = [-1,0,1]'$

(ii) For $\lambda = 5$, $[A - \lambda I]X = 0$ gives

$$\begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-3x_1 + 2x_2 + x_3 = 0$$

$$x_1 - 2x_2 + x_3 = 0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} 2 & 1 \\ -2 & 1 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} -3 & 1 \\ 1 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -3 & 2 \\ 1 & -2 \end{vmatrix}}$$

$$\frac{x_1}{4} = -\frac{x_2}{-4} = \frac{x_3}{4}$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

Hence, corresponding to $\lambda = 5$ the eigen vector is $X_3 = [1,1,1]'$

Thus, the matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ is diagonalised to $D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ by the transformation $M^{-1}AM = D$ where $M = \begin{bmatrix} -2 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

8. Find a matrix P that diagonalises the matrix $A=\begin{bmatrix}-1 & 4 & -2\\ -3 & 4 & 0\\ -3 & 1 & 3\end{bmatrix}$ and determine $P^{-1}AP$

[N16/ChemBiot/8M]

Solution:

We have,

$$A=\begin{bmatrix}-1 & 4 & -2\\ -3 & 4 & 0\\ -3 & 1 & 3\end{bmatrix}, |A|=6$$

The characteristic equation,

$$|A-\lambda I|=0$$

$$\begin{vmatrix}-1-\lambda & 4 & -2\\ -3 & 4-\lambda & 0\\ -3 & 1 & 3-\lambda\end{vmatrix}=0$$

$$\lambda^3-[sum\ of\ diagonals]\lambda^2+[sum\ of\ minors\ of\ diagonals]\lambda-|A|=0$$

$$\lambda^3-[-1+4+3]\lambda^2+\left[\begin{vmatrix}4 & 0\\ 1 & 3\end{vmatrix}+\begin{vmatrix}-1 & -2\\ -3 & 3\end{vmatrix}+\begin{vmatrix}-1 & 4\\ -3 & 4\end{vmatrix}\right]\lambda-6=0$$

$$\lambda^3-6\lambda^2+11\lambda-6=0$$

$$\lambda=1,3,2,$$

(i) For $\lambda=1, [A-\lambda I]X=0$ gives

$$\begin{bmatrix}-2 & 4 & -2\\ -3 & 3 & 0\\ -3 & 1 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$-2x_1+4x_2-2x_3=0$$

$$-3x_1+3x_2+0x_3=0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix}4 & -2\\ 3 & 0\end{vmatrix}}=-\frac{x_2}{\begin{vmatrix}-2 & -2\\ -3 & 0\end{vmatrix}}=\frac{x_3}{\begin{vmatrix}-2 & 4\\ -3 & 3\end{vmatrix}}$$

$$\frac{x_1}{6}=-\frac{x_2}{-6}=\frac{x_3}{6}$$

$$\frac{x_1}{1}=\frac{x_2}{1}=\frac{x_3}{1}$$

Hence, corresponding to $\lambda=1$ the eigen vector is $X_1=[1,1,1]'$

(ii) For $\lambda=3, [A-\lambda I]X=0$ gives

$$\begin{bmatrix}-4 & 4 & -2\\ -3 & 1 & 0\\ -3 & 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$-4x_1+4x_2-2x_3=0$$

$$-3x_1+x_2+0x_3=0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} 4 & -2 \\ 1 & 0 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} -4 & -2 \\ -3 & 0 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -4 & 4 \\ -3 & 1 \end{vmatrix}}$$

$$\frac{x_1}{2} = -\frac{x_2}{-6} = \frac{x_3}{8}$$

$$\frac{x_1}{1} = \frac{x_2}{3} = \frac{x_3}{4}$$

Hence, corresponding to $\lambda = 3$ the eigen vector is $X_2 = [1,3,4]'$

(iii) For $\lambda = 2, [A - \lambda I]X = 0$ gives

$$\begin{bmatrix} -3 & 4 & -2 \\ -3 & 2 & 0 \\ -3 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-3x_1 + 4x_2 - 2x_3 = 0$$

$$-3x_1 + 2x_2 + 0x_3 = 0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} 4 & -2 \\ 2 & 0 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} -3 & -2 \\ -3 & 0 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -3 & 4 \\ -3 & 2 \end{vmatrix}}$$

$$\frac{x_1}{4} = -\frac{x_2}{-6} = \frac{x_3}{6}$$

$$\frac{x_1}{2} = \frac{x_2}{3} = \frac{x_3}{3}$$

Hence, corresponding to $\lambda = 2$ the eigen vector is $X_3 = [2,3,3]'$

Thus, the matrix $A = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$ is diagonalised to $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ by the transformation $P^{-1}AP = D$ where $P = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 3 \\ 1 & 4 & 3 \end{bmatrix}$

9. Show that the matrix $A=\begin{bmatrix}4&2&-2\\-5&3&2\\-2&4&1\end{bmatrix}$ is diagonalizable. Hence find diagonal and transforming matrix
[N17/AutoMechCivil/6M]
Solution:
We have,
$$A=\begin{bmatrix}4&2&-2\\-5&3&2\\-2&4&1\end{bmatrix}, |A|=10$$
The characteristic equation,
$$|A-\lambda I|=0$$
$$\begin{vmatrix}4-\lambda&2&-2\\-5&3-\lambda&2\\-2&4&1-\lambda\end{vmatrix}=0$$
$$\lambda^3-[sum\ of\ diagonals]\lambda^2+[sum\ of\ minors\ of\ diagonals]\lambda-|A|=0$$
$$\lambda^3-[4+3+1]\lambda^2+\left[\begin{vmatrix}3&2\\4&1\end{vmatrix}+\begin{vmatrix}4&-2\\-2&1\end{vmatrix}+\begin{vmatrix}4&2\\-5&3\end{vmatrix}\right]\lambda-10=0$$
$$\lambda^3-8\lambda^2+17\lambda-10=0$$
$$\lambda=1,2,5$$
(i) For $\lambda=1$, $[A-\lambda I]X=0$ gives
$$\begin{bmatrix}3&2&-2\\-5&2&2\\-2&4&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$
$$3x_1+2x_2-2x_3=0$$
$$-5x_1+2x_2+2x_3=0$$
Solving the above equations by Crammers rule, we get
$$\frac{x_1}{\begin{vmatrix}2&-2\\2&2\end{vmatrix}}=-\frac{x_2}{\begin{vmatrix}3&-2\\-5&2\end{vmatrix}}=\frac{x_3}{\begin{vmatrix}3&2\\-5&2\end{vmatrix}}$$
$$\frac{x_1}{8}=-\frac{x_2}{-4}=\frac{x_3}{16}$$
$$\frac{x_1}{2}=\frac{x_2}{1}=\frac{x_3}{4}$$
Hence, corresponding to $\lambda=1$ the eigen vector is $X_1=[2,1,4]'$
(ii) For $\lambda=2$, $[A-\lambda I]X=0$ gives
$$\begin{bmatrix}2&2&-2\\-5&1&2\\-2&4&-1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$
$$2x_1+2x_2-2x_3=0$$
$$-5x_1+x_2+2x_3=0$$
Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} 2 & -2 \\ 1 & 2 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} 2 & -2 \\ -5 & 2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 2 & 2 \\ -5 & 1 \end{vmatrix}}$$

$$\frac{x_1}{6} = -\frac{x_2}{-6} = \frac{x_3}{12}$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{2}$$

Hence, corresponding to $\lambda = 2$ the eigen vector is $X_2 = [1,1,2]'$

(iii) For $\lambda = 5, [A - \lambda I]X = 0$ gives

$$\begin{bmatrix} -1 & 2 & -2 \\ -5 & -2 & 2 \\ -2 & 4 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x_1 + 2x_2 - 2x_3 = 0$$

$$-5x_1 - 2x_2 + 2x_3 = 0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} 2 & -2 \\ -2 & 2 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} -1 & -2 \\ -5 & 2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -1 & 2 \\ -5 & -2 \end{vmatrix}}$$

$$\frac{x_1}{0} = -\frac{x_2}{-12} = \frac{x_3}{12}$$

$$\frac{x_1}{0} = \frac{x_2}{1} = \frac{x_3}{1}$$

Hence, corresponding to $\lambda = 5$ the eigen vector is $X_3 = [0,1,1]'$

Thus, the matrix $A = \begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix}$ is diagonalised to $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ by the transformation $M^{-1}AM = D$ where $M = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix}$

Type III: Cayley-Hamilton Theorem

1. State Cayley-Hamilton theorem. Use it to express $2A^5 - 3A^4 + A^2 - 4I$ as a linear polynomial in A, when $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

[N13/Biot/5M]

Solution:

Cayley-Hamilton Theorem: Every square matrix satisfies its characteristic equation.

We have,

$$A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

The characteristic equation,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 3-\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)(2-\lambda) + 1 = 0$$

$$\lambda^2 - 5\lambda + 7 = 0$$

By C-H theorem,

$$A^2 - 5A + 7I = 0$$

Dividing $2\lambda^5 - 3\lambda^4 + \lambda^2 - 4$ by $\lambda^2 - 5\lambda + 7$

$$\begin{array}{r|l}
 & 2\lambda^3 + 7\lambda^2 + 21\lambda + 57 \\
\lambda^2 - 5\lambda + 7 & \overline{2\lambda^5 - 3\lambda^4 + \lambda^2 - 4} \\
 & 2\lambda^5 - 10\lambda^4 + 14\lambda^3 \\
 & - \quad + \qquad - \\
 & \overline{\quad 7\lambda^4 - 14\lambda^3 + \lambda^2 - 4} \\
 & \quad 7\lambda^4 - 35\lambda^3 + 49\lambda^2 \\
 & \quad - \quad + \qquad - \\
 & \overline{\qquad 21\lambda^3 - 48\lambda^2 - 4} \\
 & \qquad 21\lambda^3 - 105\lambda^2 + 147\lambda \\
 & \qquad - \quad + \qquad - \\
 & \overline{\qquad\quad 57\lambda^2 - 147\lambda - 4} \\
 & \qquad\quad 57\lambda^2 - 285\lambda + 399 \\
 & \qquad\quad - \quad + \qquad - \\
 & \overline{\qquad\qquad 138\lambda - 403}
\end{array}$$

Thus,

$$2A^5 - 3A^4 + A^2 - 4I = (2A^3 + 7A^2 + 21A + 57I)(A^2 - 5A + 7I) + 138A - 403I$$

$$2A^5 - 3A^4 + A^2 - 4I = (2A^3 + 7A^2 + 21A + 57I)(0) + 138A - 403I$$

$$2A^5 - 3A^4 + A^2 - 4I = 138A - 403I$$

2. Find the characteristic equation of the matrix A given below and hence find the matrix represented by

$$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I \text{ where } A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

[N13/ChemBiot/7M]

Solution:

We have,

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}, |A| = 3$$

The characteristic equation,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$$

$$\lambda^3 - [2+1+2]\lambda^2 + \left[\begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}\right]\lambda - 3 = 0$$

$$\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$$

By Cayley Hamilton theorem,

$$A^3 - 5A^2 + 7A - 3I = 0$$

Dividing $\lambda^8 - 5\lambda^7 + 7\lambda^6 - 3\lambda^5 + \lambda^4 - 5\lambda^3 + 8\lambda^2 - 2\lambda + 1$ by $\lambda^3 - 5\lambda^2 + 7\lambda - 3$

$$\lambda^5 + \lambda$$

$$\lambda^3 - 5\lambda^2 + 7\lambda - 3 \,\big|\, \overline{\lambda^8 - 5\lambda^7 + 7\lambda^6 - 3\lambda^5 + \lambda^4 - 5\lambda^3 + 8\lambda^2 - 2\lambda + 1}$$

$$\lambda^8 - 5\lambda^7 + 7\lambda^6 - 3\lambda^5$$

$$- \quad + \quad - \quad +$$

$$\lambda^4 - 5\lambda^3 + 8\lambda^2 - 2\lambda + 1$$

$$\lambda^4 - 5\lambda^4 + 7\lambda^2 - 3\lambda$$

$$- \quad + \quad - \quad +$$

$$\lambda^2 + \lambda + 1$$

Thus,

$$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I = (A^5 + A)(A^3 - 5A^2 + 7A - 3I) + A^2 + A + I$$

$$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I = (A^5 + A)(0) + A^2 + A + I$$

$$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I = A^2 + A + I$$

$$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I = \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

3. Verify Cayley Hamilton theorem for the matrix A and hence find A^{-1} where

$$A=\begin{bmatrix}1 & 2 & -2\\ -1 & 3 & 0\\ 0 & -2 & 1\end{bmatrix}$$

[M14/ChemBiot/5M][N16/ChemBiot/5M][N17/ElexExtcElectBiomInst/6M]

Solution:

We have,

$$A=\begin{bmatrix}1 & 2 & -2\\ -1 & 3 & 0\\ 0 & -2 & 1\end{bmatrix}, |A|=1$$

The characteristic equation,

$$|A-\lambda I|=0$$

$$\begin{vmatrix}1-\lambda & 2 & -2\\ -1 & 3-\lambda & 0\\ 0 & -2 & 1-\lambda\end{vmatrix}=0$$

$$\lambda^3-[sum\ of\ diagonals]\lambda^2+[sum\ of\ minors\ of\ diagonals]\lambda-|A|=0$$

$$\lambda^3-[1+3+1]\lambda^2+\left[\begin{vmatrix}3 & 0\\ -2 & 1\end{vmatrix}+\begin{vmatrix}1 & -2\\ 0 & 1\end{vmatrix}+\begin{vmatrix}1 & 2\\ -1 & 3\end{vmatrix}\right]\lambda-1=0$$

$$\lambda^3-5\lambda^2+9\lambda-1=0$$

By Cayley Hamilton theorem,

$$A^3-5A^2+9A-I=0$$

Consider,

$$A^2=A.A=\begin{bmatrix}1 & 2 & -2\\ -1 & 3 & 0\\ 0 & -2 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & -2\\ -1 & 3 & 0\\ 0 & -2 & 1\end{bmatrix}=\begin{bmatrix}-1 & 12 & -4\\ -4 & 7 & 2\\ 2 & -8 & 1\end{bmatrix}$$

$$A^3=A^2.A=\begin{bmatrix}-1 & 12 & -4\\ -4 & 7 & 2\\ 2 & -8 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & -2\\ -1 & 3 & 0\\ 0 & -2 & 1\end{bmatrix}=\begin{bmatrix}-13 & 42 & -2\\ -11 & 9 & 10\\ 10 & -22 & -3\end{bmatrix}$$

L.H.S. $=A^3-5A^2+9A-I$

$$=\begin{bmatrix}-13 & 42 & -2\\ -11 & 9 & 10\\ 10 & -22 & -3\end{bmatrix}-5\begin{bmatrix}-1 & 12 & -4\\ -4 & 7 & 2\\ 2 & -8 & 1\end{bmatrix}+9\begin{bmatrix}1 & 2 & -2\\ -1 & 3 & 0\\ 0 & -2 & 1\end{bmatrix}-\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}= \text{R.H.S.}$$

Thus, Cayley Hamilton theorem is verified

Now,

$$A^3-5A^2+9A-I=0$$

Pre-multiplying by A^{-1}, we get

$$A^2-5A+9I-A^{-1}=0$$

$$A^{-1}=A^2-5A+9I$$

$$A^{-1} = \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} + 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

4. Apply C-H Theorem to $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ and deduce that $A^8 = 625$ I.

[M14/ElexExtcElectBiomInst/6M]

Solution:

We have,

$$A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

The characteristic equation,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(-1-\lambda) - 4 = 0$$

$$\lambda^2 - 5 = 0$$

By C-H theorem,

$$A^2 - 5I = 0$$

$$A^2 = 5I$$

Squaring both the sides,

$$A^4 = 25I$$

Squaring both the sides,

$$A^8 = 625I$$

5. Verify Cayley Hamilton theorem and find $A^{-1}$for $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$. Hence find $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$ in terms of A.

[M14/AutoMechCivil/5M][N14/AutoMechCivil/5M]

Solution:

We have,

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

The characteristic equation,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(3-\lambda) - 8 = 0$$

$$\lambda^2 - 4\lambda - 5 = 0$$

By C-H theorem,

$$A^2 - 4A - 5I = 0$$

Consider,

$$A^2 = A.A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix}$$

L.H.S. $= A^2 - 4A - 5I$

$$= \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} - 4\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \text{R.H.S}$$

Thus, Cayley Hamilton theorem is verified

Now,

$$A^2 - 4A - 5I = 0$$

Pre-multiplying by A^{-1}, we get

$$A - 4I - 5A^{-1} = 0$$

$$5A^{-1} = A - 4I$$

$$5A^{-1} = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$5A^{-1} = \begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{5}\begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix}$$

Dividing $\lambda^5 - 4\lambda^4 - 7\lambda^3 + 11\lambda^2 - \lambda - 10$ by $\lambda^2 - 4\lambda - 5$

$$
\begin{array}{l|l}
 & \lambda^3 - 2\lambda + 3 \\
\hline
\lambda^2 - 4\lambda - 5 & \lambda^5 - 4\lambda^4 - 7\lambda^3 + 11\lambda^2 - \lambda - 10 \\
 & \lambda^5 - 4\lambda^4 - 5\lambda^3 \\
 & - \quad + \qquad + \\
\hline
 & \qquad -2\lambda^3 + 11\lambda^2 - \lambda - 10 \\
 & \qquad -2\lambda^3 + 8\lambda^2 + 10\lambda \\
 & \qquad + \qquad - \qquad - \\
\hline
 & \qquad\qquad 3\lambda^2 - 11\lambda - 10 \\
 & \qquad\qquad 3\lambda^2 - 12\lambda - 15 \\
 & \qquad\qquad - \quad + \qquad + \\
\hline
 & \qquad\qquad\qquad \lambda + 5
\end{array}
$$

Thus,

$A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I = (A^3 - 2A + 3I)(A^2 - 4A - 5I) + A + 5I$

$A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I = (A^3 - 2A + 3I)(0) + A + 5I$

$A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I = A + 5I$

6. Verify that the matrix $A=\begin{bmatrix}1 & 2 & 0\\ 2 & -1 & 0\\ 0 & 0 & -1\end{bmatrix}$ satisfies the characteristic equation, Hence find A^{-2}

[N14/ElexExtcElectBiomInst/6M]

Solution:

We have,

$$A=\begin{bmatrix}1 & 2 & 0\\ 2 & -1 & 0\\ 0 & 0 & -1\end{bmatrix}, |A|=5$$

The characteristic equation,

$$|A-\lambda I|=0$$

$$\begin{vmatrix}1-\lambda & 2 & 0\\ 2 & -1-\lambda & 0\\ 0 & 0 & -1-\lambda\end{vmatrix}=0$$

$$\lambda^3-[sum\ of\ diagonals]\lambda^2+[sum\ of\ minors\ of\ diagonals]\lambda-|A|=0$$

$$\lambda^3-[1-1-1]\lambda^2+\left[\begin{vmatrix}-1 & 0\\ 0 & -1\end{vmatrix}+\begin{vmatrix}1 & 0\\ 0 & -1\end{vmatrix}+\begin{vmatrix}1 & 2\\ 2 & -1\end{vmatrix}\right]\lambda-5=0$$

$$\lambda^3+\lambda^2-5\lambda-5=0$$

By Cayley Hamilton theorem,

$$A^3+A^2-5A-5I=0$$

Consider,

$$A^2=A.A=\begin{bmatrix}1 & 2 & 0\\ 2 & -1 & 0\\ 0 & 0 & -1\end{bmatrix}\begin{bmatrix}1 & 2 & 0\\ 2 & -1 & 0\\ 0 & 0 & -1\end{bmatrix}=\begin{bmatrix}5 & 0 & 0\\ 0 & 5 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$A^3=A^2.A=\begin{bmatrix}5 & 0 & 0\\ 0 & 5 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & -2\\ -1 & 3 & 0\\ 0 & -2 & 1\end{bmatrix}=\begin{bmatrix}5 & 10 & 0\\ 10 & -5 & 0\\ 0 & 0 & -1\end{bmatrix}$$

L.H.S. $=A^3+A^2-5A-5I$

$$=\begin{bmatrix}5 & 10 & 0\\ 10 & -5 & 0\\ 0 & 0 & -1\end{bmatrix}+\begin{bmatrix}5 & 0 & 0\\ 0 & 5 & 0\\ 0 & 0 & 1\end{bmatrix}-5\begin{bmatrix}1 & 2 & 0\\ 2 & -1 & 0\\ 0 & 0 & -1\end{bmatrix}-5\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}= \text{R.H.S.}$$

Thus, Cayley Hamilton theorem is verified

Now,

$$A^3+A^2-5A-5I=0$$

Pre-multiplying by A^{-1}, we get

$$A^2+A-5I-5A^{-1}=0$$

$$5A^{-1}=A^2+A-5I$$

$$5A^{-1} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

Also,

$$A^3 + A^2 - 5A - 5I = 0$$

Pre-multiplying by A^{-2}, we get

$$A + I - 5A^{-1} - 5A^{-2} = 0$$

$$5A^{-2} = A + I - 5A^{-1}$$

$$5A^{-2} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

$$A^{-2} = \frac{1}{5}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

7. Verify Cayley Hamilton Theorem for $A = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$ and hence evaluate

$2A^4 - 5A^3 - 7A + 6I$

[M15/ChemBiot/6M]

Solution:

We have,

$A = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 2-\lambda \end{vmatrix} = 0$

$(1-\lambda)(2-\lambda) - 4 = 0$

$\lambda^2 - 3\lambda - 2 = 0$

By C-H theorem, $A^2 - 3A - 2I = 0$

Consider, $A^2 = A.A = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 6 & 8 \end{bmatrix}$

L.H.S. $= A^2 - 3A - 2I$

$= \begin{bmatrix} 5 & 6 \\ 6 & 8 \end{bmatrix} - 3\begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ = R.H.S

Thus, Cayley Hamilton theorem is verified

Dividing $2\lambda^4 - 5\lambda^3 - 7\lambda + 6$ by $\lambda^2 - 3\lambda - 2$

$$
\begin{array}{r|l}
 & 2\lambda^2 + \lambda + 7 \\
\hline
\lambda^2 - 3\lambda - 2 & 2\lambda^4 - 5\lambda^3 - 7\lambda + 6 \\
 & 2\lambda^4 - 6\lambda^3 - 4\lambda^2 \\
 & - \quad + \quad + \\
\hline
 & \lambda^3 + 4\lambda^2 - 7\lambda + 6 \\
 & \lambda^3 - 3\lambda^2 - 2\lambda \\
 & - \quad + \quad + \\
\hline
 & 7\lambda^2 - 5\lambda + 6 \\
 & 7\lambda^2 - 21\lambda - 14 \\
 & + \quad + \\
\hline
 & 16\lambda + 20
\end{array}
$$

Thus,

$2A^4 - 5A^3 - 7A + 6I = (2A^2 + A + 7I)(A^2 - 3A - 2I) + 16A + 20I$

$2A^4 - 5A^3 - 7A + 6I = (2A^2 + A + 7I)(0) + 16A + 20I$

$2A^4 - 5A^3 - 7A + 6I = 16A + 20I$

$2A^4 - 5A^3 - 7A + 6I = 16\begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} + 20\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 36 & 32 \\ 32 & 52 \end{bmatrix}$

8. Verify Cayley Hamilton theorem for the matrix A and hence find A^{-1} and A^4 where $A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

[M15/ElexExtcElectBiomInst/8M]

Solution:

We have,

$$A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}, |A| = 1$$

The characteristic equation,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 2 & -2 \\ -1 & 3-\lambda & 0 \\ 0 & -2 & 1-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$$

$$\lambda^3 - [1+3+1]\lambda^2 + \left[\begin{vmatrix} 3 & 0 \\ -2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix}\right]\lambda - 1 = 0$$

$$\lambda^3 - 5\lambda^2 + 9\lambda - 1 = 0$$

By Cayley Hamilton theorem,

$$A^3 - 5A^2 + 9A - I = 0$$

Consider,

$$A^2 = A.A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix}$$

$$A^3 = A^2.A = \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{bmatrix}$$

L.H.S. $= A^3 - 5A^2 + 9A - I$

$$= \begin{bmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{bmatrix} - 5\begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix} + 9\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{R.H.S.}$$

Thus, Cayley Hamilton theorem is verified

Now,

$$A^3 - 5A^2 + 9A - I = 0$$

Pre-multiplying by A^{-1}, we get

$$A^2 - 5A + 9I - A^{-1} = 0$$

$$A^{-1} = A^2 - 5A + 9I$$

$$A^{-1} = \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} + 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

Also,

$$A^3 - 5A^2 + 9A - I = 0$$

Pre-multiplying by A, we get

$$A^4 - 5A^3 + 9A^2 - A = 0$$

$$A^4 = 5A^3 - 9A^2 + A$$

$$A^4 = 5\begin{bmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{bmatrix} - 9\begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} -55 & 104 & 24 \\ -20 & -15 & 32 \\ 32 & -40 & -23 \end{bmatrix}$$

9. State Cayley-Hamilton theorem & verify the same for $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$

[N15/CompIT/5M]

Solution:

Cayley-Hamilton Theorem: Every square matrix satisfies its characteristic equation.

We have,

$A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 1-\lambda & 3 \\ 2 & 2-\lambda \end{vmatrix} = 0$

$(1-\lambda)(2-\lambda) - 6 = 0$

$\lambda^2 - 3\lambda - 4 = 0$

By C-H theorem,

$A^2 - 3A - 4I = 0$

Consider,

$A^2 = A.A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 9 \\ 6 & 10 \end{bmatrix}$

L.H.S. $= A^2 - 3A - 4I$

$= \begin{bmatrix} 7 & 9 \\ 6 & 10 \end{bmatrix} - 3\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ = R.H.S

Thus, Cayley Hamilton theorem is verified

10. Verify Cayley Hamilton Theorem and find A^{54} where $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$

[N15/ElexExtcElectBiomInst/6M]

Solution:

We have,

$A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 1-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = 0$

$(1-\lambda)(-1-\lambda) - 4 = 0$

$\lambda^2 - 5 = 0$

By C-H theorem,

$A^2 - 5I = 0$

Consider,

$A^2 = A.A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$

L.H.S. $= A^2 - 5I$

$= \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} =$ R.H.S

Thus, Cayley Hamilton theorem is verified

Now,

$A^2 - 5I = 0$

$A^2 = 5I$

Raise to 27 on both sides, we get

$(A^2)^{27} = (5I)^{27}$

$A^{54} = 5^{27} I$

11. Find the characteristic equation of the matrix A given below and hence find the matrix represented by $A^6-6A^5+9A^4+4A^3-12A^2+2A-I$ where $A=\begin{bmatrix}3&10&5\\-2&-3&-4\\3&5&7\end{bmatrix}$

[N15/AutoMechCivil/6M][N16/ElexExtcElectBiomInst/6M]

Solution:

We have,

$$A=\begin{bmatrix}3&10&5\\-2&-3&-4\\3&5&7\end{bmatrix},|A|=12$$

The characteristic equation,

$$|A-\lambda I|=0$$

$$\begin{vmatrix}3-\lambda&10&5\\-2&-3-\lambda&-4\\3&5&7-\lambda\end{vmatrix}=0$$

$$\lambda^3-[sum\ of\ diagonals]\lambda^2+[sum\ of\ minors\ of\ diagonals]\lambda-|A|=0$$

$$\lambda^3-[3-3+7]\lambda^2+\left[\begin{vmatrix}-3&-4\\5&7\end{vmatrix}+\begin{vmatrix}3&5\\3&7\end{vmatrix}+\begin{vmatrix}3&10\\-2&-3\end{vmatrix}\right]\lambda-12=0$$

$$\lambda^3-7\lambda^2+16\lambda-12=0$$

By Cayley Hamilton theorem,

$$A^3-7A^2+16A-12I=0$$

Dividing $\lambda^6-6\lambda^5+9\lambda^4+4\lambda^3-12\lambda^2+2\lambda-1$ by $\lambda^3-7\lambda^2+16\lambda-12$

$$\begin{array}{r|l}
 & \lambda^3+\lambda^2 \\
\lambda^3-7\lambda^2+16\lambda-12 & \lambda^6-6\lambda^5+9\lambda^4+4\lambda^3-12\lambda^2+2\lambda-1 \\
 & \lambda^6-7\lambda^5+16\lambda^4-12\lambda^3 \\
 & -\quad+\quad-\quad+ \\
 & \lambda^5-7\lambda^4+16\lambda^3-12\lambda^2+2\lambda-1 \\
 & \lambda^5-7\lambda^4+16\lambda^3-12\lambda^2 \\
 & -\quad+\quad-\quad+ \\
 & 2\lambda-1
\end{array}$$

Thus,

$$A^6-6A^5+9A^4+4A^3-12A^2+2A-I=(A^3+A^2)(A^3-7A^2+16A-12I)+2A-I$$

$$A^6-6A^5+9A^4+4A^3-12A^2+2A-I=(A^3+A^2)(0)+2A-I$$

$$A^6-6A^5+9A^4+4A^3-12A^2+2A-I=2A-I$$

$$A^6-6A^5+9A^4+4A^3-12A^2+2A-I=2\begin{bmatrix}3&10&5\\-2&-3&-4\\3&5&7\end{bmatrix}-\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$A^6 - 6A^5 + 9A^4 + 4A^3 - 12A^2 + 2A - I = \begin{bmatrix} 5 & 20 & 10 \\ -4 & -7 & -8 \\ 6 & 10 & 13 \end{bmatrix}$$

12. Show that the matrix $A=\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}$ satisfies Cayley Hamilton theorem and hence find A^{-1} if exists.

[M16/ElexExtcElectBiomInst/6M]

Solution:

We have,

$$A=\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}, |A|=4$$

The characteristic equation,

$$|A-\lambda I|=0$$

$$\begin{vmatrix}2-\lambda & -1 & 1\\ -1 & 2-\lambda & -1\\ 1 & -1 & 2-\lambda\end{vmatrix}=0$$

$$\lambda^3-[sum\ of\ diagonals]\lambda^2+[sum\ of\ minors\ of\ diagonals]\lambda-|A|=0$$

$$\lambda^3-[2+2+2]\lambda^2+\left[\begin{vmatrix}2 & -1\\ -1 & 2\end{vmatrix}+\begin{vmatrix}2 & 1\\ 1 & 2\end{vmatrix}+\begin{vmatrix}2 & -1\\ -1 & 2\end{vmatrix}\right]\lambda-4=0$$

$$\lambda^3-6\lambda^2+9\lambda-4=0$$

By Cayley Hamilton theorem,

$$A^3-6A^2+9A-4I=0$$

Consider,

$$A^2=A.A=\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}=\begin{bmatrix}6 & -5 & 5\\ -5 & 6 & -5\\ 5 & -5 & 6\end{bmatrix}$$

$$A^3=A^2.A=\begin{bmatrix}6 & -5 & 5\\ -5 & 6 & -5\\ 5 & -5 & 6\end{bmatrix}\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}=\begin{bmatrix}22 & -21 & 21\\ -21 & 22 & -21\\ 21 & -21 & 22\end{bmatrix}$$

L.H.S. $=A^3-6A^2+9A-4I$

$$=\begin{bmatrix}22 & -21 & 21\\ -21 & 22 & -21\\ 21 & -21 & 22\end{bmatrix}-6\begin{bmatrix}6 & -5 & 5\\ -5 & 6 & -5\\ 5 & -5 & 6\end{bmatrix}+9\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}-4\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}=\text{R.H.S.}$$

Thus, Cayley Hamilton theorem is verified

Now,

$$A^3-6A^2+9A-4I=0$$

Pre-multiplying by A^{-1}, we get

$$A^2-6A+9I-4A^{-1}=0$$

$$4A^{-1}=A^2-6A+9I$$

$$4A^{-1} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

13. Find the characteristic equation of A where $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$. Show that the matrix A satisfies the characteristic equation and hence find A^{-1} and A^4

[M16/ChemBiot/6M]

Solution:

We have,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}, |A| = 40$$

The characteristic equation,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & -1-\lambda & 4 \\ 3 & 1 & -1-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$$

$$\lambda^3 - [1-1-1]\lambda^2 + \left[\begin{vmatrix} -1 & 4 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 3 & -1 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix}\right]\lambda - 40 = 0$$

$$\lambda^3 + \lambda^2 - 18\lambda - 40 = 0$$

By Cayley Hamilton theorem,

$$A^3 + A^2 - 18A - 40I = 0$$

Consider,

$$A^2 = A.A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 14 & 3 & -2 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix}$$

$$A^3 = A^2.A = \begin{bmatrix} 14 & 3 & -2 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 44 & 33 & 46 \\ 24 & 13 & 74 \\ 52 & 14 & 8 \end{bmatrix}$$

L.H.S. $= A^3 + A^2 - 18A - 40I$

$$= \begin{bmatrix} 44 & 33 & 46 \\ 24 & 13 & 74 \\ 52 & 14 & 8 \end{bmatrix} + \begin{bmatrix} 14 & 3 & -2 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix} - 18\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix} - 40\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{R.H.S.}$$

Thus, Cayley Hamilton theorem is verified

Now,

$$A^3 + A^2 - 18A - 40I = 0$$

Pre-multiplying by A^{-1}, we get

$$A^2 + A - 18I - 40A^{-1} = 0$$

$$40A^{-1} = A^2 + A - 18I$$

$$40A^{-1} = \begin{bmatrix} 14 & 3 & -2 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix} - 18\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{40}\begin{bmatrix} -3 & 5 & 11 \\ 14 & -10 & 2 \\ 5 & 5 & -5 \end{bmatrix}$$

Also,

$$A^3 + A^2 - 18A - 40I = 0$$

Pre-multiplying by$A$, we get

$$A^4 + A^3 - 18A^2 - 40A = 0$$

$$A^4 = -A^3 + 18A^2 + 40A$$

$$A^4 = -\begin{bmatrix} 44 & 33 & 46 \\ 24 & 13 & 74 \\ 52 & 14 & 8 \end{bmatrix} + 18\begin{bmatrix} 14 & 3 & -2 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix} + 40\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 248 & 101 & 218 \\ 272 & 109 & 50 \\ 104 & 98 & 204 \end{bmatrix}$$

14. Use Cayley Hamilton Theorem to find $2A^4 - 5A^3 - 7A + 6I$ where
$A = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$
[N16/AutoMechCivil/5M][M17/AutoMechCivil/5M]
Solution:
We have,
$A = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$
The characteristic equation,
$|A - \lambda I| = 0$
$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 2-\lambda \end{vmatrix} = 0$
$(1-\lambda)(2-\lambda) - 4 = 0$
$\lambda^2 - 3\lambda - 2 = 0$
By C-H theorem, $A^2 - 3A - 2I = 0$
Dividing $2\lambda^4 - 5\lambda^3 - 7\lambda + 6$ by $\lambda^2 - 3\lambda - 2$

$$
\begin{array}{r|l}
 & 2\lambda^2 + \lambda + 7 \\
\lambda^2 - 3\lambda - 2 & 2\lambda^4 - 5\lambda^3 - 7\lambda + 6 \\
 & 2\lambda^4 - 6\lambda^3 - 4\lambda^2 \\
 & - \quad + \quad + \\
 & \lambda^3 + 4\lambda^2 - 7\lambda + 6 \\
 & \lambda^3 - 3\lambda^2 - 2\lambda \\
 & - \quad + \quad + \\
 & 7\lambda^2 - 5\lambda + 6 \\
 & 7\lambda^2 - 21\lambda - 14 \\
 & + \quad + \\
 & 16\lambda + 20
\end{array}
$$

Thus,
$2A^4 - 5A^3 - 7A + 6I = (2A^2 + A + 7I)(A^2 - 3A - 2I) + 16A + 20I$
$2A^4 - 5A^3 - 7A + 6I = (2A^2 + A + 7I)(0) + 16A + 20I$
$2A^4 - 5A^3 - 7A + 6I = 16A + 20I$
$2A^4 - 5A^3 - 7A + 6I = 16\begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} + 20\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 36 & 32 \\ 32 & 52 \end{bmatrix}$

15. Verify Cayley Hamilton theorem for $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ and hence find A^{-1} & A^4

[M17/ElexExtcElectBiomInst/6M]

Solution:

We have,

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}, |A| = 3$$

The characteristic equation,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$$

$$\lambda^3 - [2+1+2]\lambda^2 + \left[\begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}\right]\lambda - 3 = 0$$

$$\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$$

By Cayley Hamilton theorem,

$$A^3 - 5A^2 + 7A - 3I = 0$$

Consider,

$$A^2 = A.A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$$

$$A^3 = A^2.A = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{bmatrix}$$

L.H.S. $= A^3 - 5A^2 + 7A - 3I$

$$= \begin{bmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{bmatrix} - 5\begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + 7\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{R.H.S.}$$

Thus, Cayley Hamilton theorem is verified

Now,

$$A^3 - 5A^2 + 7A - 3I = 0$$

Pre-multiplying by A^{-1}, we get

$$A^2 - 5A + 7I - 3A^{-1} = 0$$

$$3A^{-1} = A^2 - 5A + 7I$$

$$3A^{-1} = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} - 5\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$3A^{-1} = \begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & 0 \\ -1 & -1 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & 0 \\ -1 & -1 & 2 \end{bmatrix}$$

Also,

$$A^3 - 5A^2 + 7A - 3I = 0$$

Pre-multiplying by A, we get

$$A^4 - 5A^3 + 7A^2 - 3A = 0$$

$$A^4 = 5A^3 - 7A^2 + 3A$$

$$A^4 = 5\begin{bmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{bmatrix} - 7\begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + 3\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 41 & 40 & 40 \\ 0 & 1 & 0 \\ 40 & 40 & 41 \end{bmatrix}$$

16. If $A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$, then prove that $A^{-1} = A^2 - 5A + 9I$

[N17/AutoMechCivil/5M]

Solution:

We have,

$$A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}, |A| = 1$$

The characteristic equation,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 2 & -2 \\ -1 & 3-\lambda & 0 \\ 0 & -2 & 1-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$$

$$\lambda^3 - [1+3+1]\lambda^2 + \left[\begin{vmatrix} 3 & 0 \\ -2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix}\right]\lambda - 1 = 0$$

$$\lambda^3 - 5\lambda^2 + 9\lambda - 1 = 0$$

By Cayley Hamilton theorem,

$$A^3 - 5A^2 + 9A - I = 0$$

Now,

$$A^3 - 5A^2 + 9A - I = 0$$

Pre-multiplying by A^{-1}, we get

$$A^2 - 5A + 9I - A^{-1} = 0$$

$$A^{-1} = A^2 - 5A + 9I$$

Type IV: Functions of a Square matrix

1. If $A = \begin{bmatrix} -1 & 4 \\ 2 & 1 \end{bmatrix}$ then prove that: 3 tan A = A tan 3

[M14/AutoMechCivil/6M][M17/ElexExtcElectBiomInst/6M]

Solution:

We have,

$A = \begin{bmatrix} -1 & 4 \\ 2 & 1 \end{bmatrix}$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} -1-\lambda & 4 \\ 2 & 1-\lambda \end{vmatrix} = 0$

$(-1-\lambda)(1-\lambda) - 8 = 0$

$\lambda^2 - 9 = 0$

$\lambda = 3, -3$

Let $3tanA = \alpha_1 A + \alpha_0 I$

By C-H theorem,

$3\ tan\lambda = \alpha_1 \lambda + \alpha_0$

Putting $\lambda = 3$ and $\lambda = -3$, we get

$3\ tan3 = 3\alpha_1 + \alpha_0$ ………………(1)

$-3\ tan3 = -3\alpha_1 + \alpha_0$ …………(2)

Adding (1) & (2), we get

$2\alpha_0 = 0$

$\therefore \alpha_0 = 0$

Thus, $\alpha_1 = tan3$

Therefore,

$\boxed{3\ tanA = A\ tan3}$

2. If $A = \begin{bmatrix} \frac{\pi}{2} & 3\frac{\pi}{2} \\ \pi & \pi \end{bmatrix}$, find sinA

[M14/ElexExtcElectBiomInst/4M]

Solution:

We have,

$$A = \begin{bmatrix} \frac{\pi}{2} & 3\frac{\pi}{2} \\ \pi & \pi \end{bmatrix}$$

The characteristic equation,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} \frac{\pi}{2} - \lambda & \frac{3\pi}{2} \\ \pi & \pi - \lambda \end{vmatrix} = 0$$

$$\left(\frac{\pi}{2} - \lambda\right)(\pi - \lambda) - \frac{3\pi^2}{2} = 0$$

$$\lambda^2 - \frac{3\pi}{2}\lambda - \pi^2 = 0$$

$$\lambda = 2\pi, -\frac{\pi}{2}$$

Let $sinA = \alpha_1 A + \alpha_0 I$

By C-H theorem,

$$sin\lambda = \alpha_1\lambda + \alpha_0$$

Putting $\lambda = 2\pi$ and $\lambda = -\frac{\pi}{2}$, we get

$0 = 2\pi\alpha_1 + \alpha_0$ ………………(1)

$-1 = -\frac{\pi}{2}\alpha_1 + \alpha_0$ …………(2)

Subtracting (1) & (2), we get

$$\frac{5\pi}{2}\alpha_1 = 1$$

$$\therefore \alpha_1 = \frac{2}{5\pi}$$

Thus, $\alpha_0 = -\frac{4}{5}$

Therefore,

$$sinA = \frac{2}{5\pi}A - \frac{4}{5}I$$

$$sinA = \frac{2}{5\pi}\begin{bmatrix} \frac{\pi}{2} & 3\frac{\pi}{2} \\ \pi & \pi \end{bmatrix} - \frac{4}{5}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$sinA = \begin{bmatrix} \frac{1}{5} & \frac{3}{5} \\ \frac{2}{5} & \frac{2}{5} \end{bmatrix} - \begin{bmatrix} \frac{4}{5} & 0 \\ 0 & \frac{4}{5} \end{bmatrix}$$

$$sinA = \begin{bmatrix} -\frac{3}{5} & \frac{3}{5} \\ \frac{2}{5} & -\frac{2}{5} \end{bmatrix}$$

3. If $A = \begin{bmatrix} 2 & 3 \\ -3 & -4 \end{bmatrix}$, find A ^{50}

[N14/AutoMechCivil/6M][M16/ElexExtcElectBiomInst/6M]

Solution:

We have,

$A = \begin{bmatrix} 2 & 3 \\ -3 & -4 \end{bmatrix}$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 2-\lambda & 3 \\ -3 & 4-\lambda \end{vmatrix} = 0$

$(2-\lambda)(4-\lambda) + 9 = 0$

$\lambda^2 + 2\lambda + 1 = 0$

$\lambda = -1, -1$

Let $A^{50} = \alpha_1 A + \alpha_0 I$

By C-H theorem,

$\lambda^{50} = \alpha_1 \lambda + \alpha_0$ ……………(A)

Putting $\lambda = -1$, we get

$1 = -\alpha_1 + \alpha_0$ ………………(1)

Differentiating eqn (A) w.r.t λ, we get

$50\lambda^{49} = \alpha_1$

Putting $\lambda = -1$, we get

$\alpha_1 = -50$

Thus, $\alpha_0 = -49$

Therefore,

$A^{50} = -50A - 49I$

$A^{50} = -50 \begin{bmatrix} 2 & 3 \\ -3 & -4 \end{bmatrix} - 49 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$A^{50} = \begin{bmatrix} -149 & -150 \\ 150 & 151 \end{bmatrix}$

4. If $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, then show that $A^{50} = \frac{1}{2}\begin{bmatrix} 1+3^{50} & -1+3^{50} \\ -1+3^{50} & 1+3^{50} \end{bmatrix}$

[M15/ElexExtcElectBiomInst/6M][M15/AutoMechCivil/6M]
[N16/CompIT/8M][N17/AutoMechCivil/6M]

Solution:

We have,

$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$

$(2-\lambda)(2-\lambda) - 1 = 0$

$\lambda^2 - 4\lambda + 3 = 0$

$\lambda = 1, 3$

Let $A^{50} = \alpha_1 A + \alpha_0 I$

By C-H theorem,

$\lambda^{50} = \alpha_1 \lambda + \alpha_0$

Putting $\lambda = 3$ and $\lambda = 1$, we get

$3^{50} = 3\alpha_1 + \alpha_0$ ………………(1)

$1 = \alpha_1 + \alpha_0$ …………(2)

Subtracting (1) & (2), we get

$2\alpha_1 = 3^{50} - 1$

$\therefore \alpha_1 = \frac{3^{50}-1}{2}$

And,

$1 = \frac{3^{50}-1}{2} + \alpha_0$

$1 - \frac{3^{50}-1}{2} = \alpha_0$

$\alpha_0 = \frac{3-3^{50}}{2}$

Thus,

$A^{50} = \frac{3^{50}-1}{2} A + \frac{3-3^{50}}{2} I$

$A^{50} = \frac{3^{50}-1}{2}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + \frac{3-3^{50}}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$A^{50} = \frac{1}{2}\begin{bmatrix} 2.3^{50} - 2 & 3^{50} - 1 \\ 3^{50} - 1 & 2.3^{50} - 2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 3 - 3^{50} & 0 \\ 0 & 3 - 3^{50} \end{bmatrix}$

$A^{50} = \frac{1}{2}\begin{bmatrix} 1+3^{50} & -1+3^{50} \\ -1+3^{50} & 1+3^{50} \end{bmatrix}$

5. If $A = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$ find $A^7 - 9A^2 + I$

[N15/ChemBiot/5M][M16/ChemBiot/5M]

Solution:

We have,

$A = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 1-\lambda & 4 \\ 1 & 1-\lambda \end{vmatrix} = 0$

$(1-\lambda)(1-\lambda) - 4 = 0$

$\lambda^2 - 2\lambda - 3 = 0$

$\lambda = -1, 3$

Let $A^7 - 9A^2 + I = \alpha_1 A + \alpha_0 I$

By C-H theorem,

$\lambda^7 - 9\lambda^2 + 1 = \alpha_1 \lambda + \alpha_0$

Putting $\lambda = 3$ and $\lambda = -1$, we get

$2107 = 3\alpha_1 + \alpha_0$ ………………(1)

$-9 = -\alpha_1 + \alpha_0$ …………(2)

Subtracting (1) & (2), we get

$4\alpha_1 = 2116$

$\therefore \alpha_1 = 529$

And,

$-9 = -529 + \alpha_0$

$\alpha_0 = 520$

Thus,

$A^7 - 9A^2 + I = 529A + 520I$

$A^7 - 9A^2 + I = 529 \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} + 520 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$A^7 - 9A^2 + I = \begin{bmatrix} 1049 & 2116 \\ 529 & 1049 \end{bmatrix}$

6. Find e^A & 4^A if $A = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}$ with the help of modal matrix

[N15/ElexExtcElectBiomInst/8M]

Solution:

We have,

$$A = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}$$

The characteristic equation,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} \frac{3}{2} - \lambda & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} - \lambda \end{vmatrix} = 0$$

$$\left(\frac{3}{2} - \lambda\right)\left(\frac{3}{2} - \lambda\right) - \frac{1}{4} = 0$$

$$\lambda^2 - 3\lambda + 2 = 0$$

$$\lambda = 1, 2$$

(i) For $\lambda = 1$, $[A - \lambda I]X = 0$ gives

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\frac{x_1}{2} + \frac{x_2}{2} = 0$$

$$x_1 = -x_2$$

$$\frac{x_1}{-1} = \frac{x_2}{1}$$

Thus, eigen vectors for $\lambda = 1$ is $[-1,1]'$

(ii) For $\lambda = 2$, $[A - \lambda I]X = 0$ gives

$$\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-\frac{x_1}{2} + \frac{x_2}{2} = 0$$

$$x_1 = x_2$$

$$\frac{x_1}{1} = \frac{x_2}{1}$$

Thus, eigen vectors for $\lambda = 1$ is $[1,1]'$

Therefore, the matrix $A = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}$ is diagonalised to $D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ by the transformation $M^{-1}AM = D$ where $M = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$

And, $M^{-1} = \frac{1}{|M|} adjM = \frac{1}{-2}\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$

Now,

$f(A) = Mf(D)M^{-1}$

$e^A = Me^D M^{-1}$

$e^A = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^1 & 0 \\ 0 & e^2 \end{bmatrix}\frac{1}{2}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$

$e^A = \frac{1}{2}\begin{bmatrix} -e & e^2 \\ e & e^2 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$

$e^A = \frac{1}{2}\begin{bmatrix} e + e^2 & -e + e^2 \\ -e + e^2 & e + e^2 \end{bmatrix}$

Now,

$4^A = \frac{1}{2}\begin{bmatrix} 4 + 4^2 & -4 + 4^2 \\ -4 + 4^2 & 4 + 4^2 \end{bmatrix}$

$4^A = \frac{1}{2}\begin{bmatrix} 20 & 12 \\ 12 & 20 \end{bmatrix}$

$4^A = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix}$

7. Find e^A & 4^A if $A = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}$

[N15/AutoMechCivil/8M]

Solution:

We have,

$$A = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}$$

The characteristic equation,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} \frac{3}{2} - \lambda & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} - \lambda \end{vmatrix} = 0$$

$$\left(\frac{3}{2} - \lambda\right)\left(\frac{3}{2} - \lambda\right) - \frac{1}{4} = 0$$

$$\lambda^2 - 3\lambda + 2 = 0$$

$$\lambda = 1, 2$$

Let $e^A = \alpha_1 A + \alpha_0 I$

By C-H theorem, $e^\lambda = \alpha_1 \lambda + \alpha_0$

Putting $\lambda = 2$ and $\lambda = 1$, we get

$e^2 = 2\alpha_1 + \alpha_0$ ………………(1)

$e = \alpha_1 + \alpha_0$ …………(2)

Subtracting (1) & (2), we get

$$\alpha_1 = e^2 - e$$

And, $e = e^2 - e + \alpha_0$

$$\alpha_0 = 2e - e^2$$

Thus,

$$e^A = (e^2 - e)A + (2e - e^2)I$$

$$e^A = (e^2 - e)\begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix} + (2e - e^2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$e^A = \frac{1}{2}\begin{bmatrix} 3e^2 - e & e^2 - e \\ e^2 - e & 3e^2 - e \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 4e - 2e^2 & 0 \\ 0 & 4e - 2e^2 \end{bmatrix}$$

$$e^A = \begin{bmatrix} e + e^2 & -e + e^2 \\ -e + e^2 & e + e^2 \end{bmatrix}$$

Putting $e = 4$, we get $4^A = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix}$

8. If $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$, prove that $A^{50} - 5A^{49} = \begin{bmatrix} 4 & -4 \\ -2 & 2 \end{bmatrix}$

[M16/AutoMechCivil/6M][N16/ElexExtcElectBiomInst/6M]

Solution:

We have,

$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = 0$

$(1-\lambda)(3-\lambda) - 8 = 0$

$\lambda^2 - 4\lambda - 5 = 0$

$\lambda = -1, 5$

Let $A^{50} - 5A^{49} = \alpha_1 A + \alpha_0 I$

By C-H theorem,

$\lambda^{50} - 5\lambda^{49} = \alpha_1 \lambda + \alpha_0$

Putting $\lambda = 5$ and $\lambda = -1$, we get

$0 = 5\alpha_1 + \alpha_0$ ………………(1)

$6 = -\alpha_1 + \alpha_0$ …………(2)

Subtracting (1) & (2), we get

$6\alpha_1 = -6$

$\therefore \alpha_1 = -1$

And,

$6 = 1 + \alpha_0$

$\alpha_0 = 5$

Thus,

$A^{50} - 5A^{49} = -A + 5I$

$A^{50} - 5A^{49} = -\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$A^{50} - 5A^{49} = \begin{bmatrix} 4 & -4 \\ -2 & 2 \end{bmatrix}$

9. Find 5^A if $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

[N17/ElexExtcElectBiomInst/6M]

Solution:

We have,

$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = 0$

$(3-\lambda)(3-\lambda) - 1 = 0$

$\lambda^2 - 6\lambda + 8 = 0$

$\lambda = 2, 4$

Let $5^A = \alpha_1 A + \alpha_0 I$

By C-H theorem, $5^\lambda = \alpha_1 \lambda + \alpha_0$

Putting $\lambda = 2$ and $\lambda = 4$, we get

$5^2 = 2\alpha_1 + \alpha_0$ ………………(1)

$5^4 = 4\alpha_1 + \alpha_0$ …………(2)

Solving (1) & (2), we get

$\alpha_1 = 300, \alpha_0 = -575$

Thus,

$5^A = 300A - 575I$

$5^A = 300 \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} - 575 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$5^A = \begin{bmatrix} 325 & 300 \\ 300 & 325 \end{bmatrix}$

Type V: Derogatory or Non-Derogatory Matrix

1. Is $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$ derogatory? Explain.

[N13/Biot/6M][M17/ElexExtcElectBiomInst/5M][N17/CompIT/6M]

Solution:

We have,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}, |A| = 0$$

The characteristic equation,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & 3-\lambda & 4 \\ 3 & 4 & 5-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$$

$$\lambda^3 - [1+3+5]\lambda^2 + \left[\begin{vmatrix} 3 & 4 \\ 4 & 5 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 3 & 5 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix}\right]\lambda - 0 = 0$$

$$\lambda^3 - 9\lambda^2 - 6\lambda = 0$$

We see that the eigen values are distinct.

Thus, minimal polynomial is $f(x) = x^3 - 9x^2 - 6x$

Since, the degree of minimal polynomial is same as the order of the matrix, matrix A is not derogatory

2. Show that $A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$ is derogatory.

[M14/ElexExtcElectBiomInst/4M][N14/CompIT/6M][N15/ChemBiot/6M][M16/CompIT/6M][M17/CompIT/6M]

Solution:

We have,

$$A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}, |A| = 4$$

The characteristic equation,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 5-\lambda & -6 & -6 \\ -1 & 4-\lambda & 2 \\ 3 & -6 & -4-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$$

$$\lambda^3 - [5+4-4]\lambda^2 + \left[\begin{vmatrix} 4 & 2 \\ -6 & -4 \end{vmatrix} + \begin{vmatrix} 5 & -6 \\ 3 & -4 \end{vmatrix} + \begin{vmatrix} 5 & -6 \\ -1 & 4 \end{vmatrix}\right]\lambda - 4 = 0$$

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

$$(\lambda - 1)(\lambda - 2)(\lambda - 2) = 0$$

Thus, minimal polynomial is $f(x) = (x-1)(x-2) = x^2 - 3x + 2$

Consider,

$$f(A) = A^2 - 3A + 2I$$

$$f(A) = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}\begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix} - 3\begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$f(A) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, $f(x) = x^2 - 3x + 2$ annihilates the matrix A

Since, the degree of minimal polynomial is less than the order of the matrix, matrix A is derogatory

3. Determine whether A is derogatory, $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

[M14/CompIT/6M]

Solution:

We have,

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}, |A| = 8$$

The characteristic equation,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$$

$$\lambda^3 - [2+2+2]\lambda^2 + \left[\begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix}\right]\lambda - 8 = 0$$

$$\lambda^3 - 6\lambda^2 + 12\lambda - 8 = 0$$

$$(\lambda - 2)(\lambda - 2)(\lambda - 2) = 0$$

Now, consider the minimal polynomial is $f(x) = (x - 2)$

$$f(A) = A - 2I$$

$$f(A) = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$f(A) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, $f(x) = x - 2$ does not annihilate the matrix A

Thus, matrix A is not derogatory

Also, consider the minimal polynomial is

$$f(x) = (x - 2)(x - 2) = x^2 - 4x + 4$$

$$f(A) = A^2 - 4A + 4I$$

$$f(A) = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} - 4\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} + 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$f(A) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, $f(x) = x^2 - 4x + 4$ does not annihilate the matrix A

Thus, matrix A is not derogatory

4. Define minimal polynomial and derogatory matrix. Test whether the matrix $A = \begin{bmatrix} 7 & 4 & -1 \\ 4 & 7 & -1 \\ -4 & -4 & 4 \end{bmatrix}$ is derogatory.

[N14/ElexExtcElectBiomInst/6M][M15/ElexExtcElectBiomInst/6M] [M15/CompIT/6M]

Solution:

The monic polynomial of lowest degree that annihilates the matrix is called a minimal polynomial.

If the degree of the minimal polynomial is less than the order of the matrix and if the minimal polynomial annihilates the matrix then the matrix is said to be derogatory.

We have,

$$A = \begin{bmatrix} 7 & 4 & -1 \\ 4 & 7 & -1 \\ -4 & -4 & 4 \end{bmatrix}, |A| = 108$$

The characteristic equation,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 7-\lambda & 4 & -1 \\ 4 & 7-\lambda & -1 \\ -4 & -4 & 4-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$$

$$\lambda^3 - [7+7+4]\lambda^2 + \left[\begin{vmatrix} 7 & -1 \\ -4 & 4 \end{vmatrix} + \begin{vmatrix} 7 & -1 \\ -4 & 4 \end{vmatrix} + \begin{vmatrix} 7 & 4 \\ 4 & 7 \end{vmatrix}\right]\lambda - 108 = 0$$

$$\lambda^3 - 18\lambda^2 + 81\lambda - 108 = 0$$

$$(\lambda - 12)(\lambda - 3)(\lambda - 3) = 0$$

Thus, minimal polynomial is $f(x) = (x-12)(x-3) = x^2 - 15x + 36$

Consider,

$$f(A) = A^2 - 15A + 36I$$

$$f(A) = \begin{bmatrix} 7 & 4 & -1 \\ 4 & 7 & -1 \\ -4 & -4 & 4 \end{bmatrix}\begin{bmatrix} 7 & 4 & -1 \\ 4 & 7 & -1 \\ -4 & -4 & 4 \end{bmatrix} - 15\begin{bmatrix} 7 & 4 & -1 \\ 4 & 7 & -1 \\ -4 & -4 & 4 \end{bmatrix} + 36\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$f(A) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, $f(x) = x^2 - 15x + 36$ annihilates the matrix A

Since, the degree of minimal polynomial is less than the order of the matrix, matrix A is derogatory

5. Show that $A = \begin{bmatrix} 2 & -3 & 3 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$ is derogatory and find its minimal polynomial.

[N16/AutoMechCivil/6M][M17/AutoMechCivil/6M]

Solution:

We have,

$$A = \begin{bmatrix} 2 & -3 & 3 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}, |A| = 16$$

The characteristic equation,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & -3 & 3 \\ 0 & 3-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$$

$$\lambda^3 - [2+3+3]\lambda^2 + \left[\begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 2 & 3 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 2 & -3 \\ 0 & 3 \end{vmatrix}\right]\lambda - 16 = 0$$

$$\lambda^3 - 8\lambda^2 + 20\lambda - 16 = 0$$

$$(\lambda - 4)(\lambda - 2)(\lambda - 2) = 0$$

Thus, minimal polynomial is $f(x) = (x-4)(x-2) = x^2 - 6x + 8$

Consider,

$$f(A) = A^2 - 6A + 8I$$

$$f(A) = \begin{bmatrix} 2 & -3 & 3 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}\begin{bmatrix} 2 & -3 & 3 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix} - 6\begin{bmatrix} 2 & -3 & 3 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix} + 8\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$f(A) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, $f(x) = x^2 - 6x + 8$ annihilates the matrix A

Since, the degree of minimal polynomial is less than the order of the matrix, matrix A is derogatory

6. Show that $A = \begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$ is non – derogatory.

[N16/CompIT/6M]

Solution:

We have,

$$A = \begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}, |A| = -6$$

The characteristic equation,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & -2 & 3 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$$

$$\lambda^3 - [2+1-1]\lambda^2 + \left[\begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} + \begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix}\right]\lambda - (-6) = 0$$

$$\lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0$$

$$(\lambda + 2)(\lambda - 3)(\lambda - 1) = 0$$

We see that the eigen values are distinct.

Thus, minimal polynomial is $f(x) = x^3 - 2x^2 - 5x + 6$

Since, the degree of minimal polynomial is same as the order of the matrix, matrix A is not derogatory

Type VI: Congruent Transformations (Canonical Form)

1. Reduce the following quadratic form to canonical form and find its rank and signature $x_1^2 + 2x_2^2 + 3x_3^2 + 2x_1x_2 - 2x_1x_3 + 2x_2x_3$
   [N13/Biot/8M]
   Solution:
   We have,
   $X'AX = x_1^2 + 2x_2^2 + 3x_3^2 + 2x_1x_2 - 2x_1x_3 + 2x_2x_3$
   $$\therefore A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$
   Consider,
   $A = I_3 A I_3$
   $$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
   By $R_2 - R_1, R_3 + R_1, C_2 - C_1, C_3 + C_1$
   $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
   By $R_3 - 2R_2, C_3 - 2C_2$
   $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$
   By $\frac{1}{\sqrt{2}} R_3, \frac{1}{\sqrt{2}} C_3$
   $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ \frac{3}{\sqrt{2}} & -\sqrt{2} & \frac{1}{\sqrt{2}} \end{bmatrix} A \begin{bmatrix} 1 & -1 & \frac{3}{\sqrt{2}} \\ 0 & 1 & -\sqrt{2} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$
   Thus, $B = P'AP$
   Rank $(r) = 3$
   Index $(s) = 2$
   Signature $= 2s - r = 2(2) - 3 = 1$

2. Reduce the matrix of the quadratic form $2x_1^2 + x_2^2 - 3x_3^2 - 8x_2x_3 - 4x_3x_1 + 12x_1x_2$ to canonical form and find its rank, index, signature and value class. Also write linear transformation which brings about the normal reduction.

[M14/ChemBiot/8M]

Solution:

We have,

$$X^{'}AX = 2x_1^2 + x_2^2 - 3x_3^2 - 8x_2x_3 - 4x_3x_1 + 12x_1x_2$$

$$\therefore A = \begin{bmatrix} 2 & 6 & -2 \\ 6 & 1 & -4 \\ -2 & -4 & -3 \end{bmatrix}$$

Consider,

$$A = I_3AI_3$$

$$\begin{bmatrix} 2 & 6 & -2 \\ 6 & 1 & -4 \\ -2 & -4 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $R_2 - 3R_1, R_3 + R_1, C_2 - 3C_1, C_3 + C_1$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -17 & 2 \\ 0 & 2 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -3 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $R_3 + \frac{2}{17}R_2, C_3 + \frac{2}{17}C_2$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -17 & 0 \\ 0 & 0 & -\frac{81}{17} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ \frac{11}{17} & \frac{2}{17} & 1 \end{bmatrix} A \begin{bmatrix} 1 & -3 & \frac{11}{17} \\ 0 & 1 & \frac{2}{17} \\ 0 & 0 & 1 \end{bmatrix}$$

By $\frac{1}{\sqrt{2}}R_1, \frac{1}{\sqrt{17}}R_2, \frac{\sqrt{17}}{9}R_3, \frac{1}{\sqrt{2}}C_1, \frac{1}{\sqrt{17}}C_2, \frac{\sqrt{17}}{9}C_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ -\frac{3}{\sqrt{17}} & \frac{1}{\sqrt{17}} & 0 \\ \frac{11}{9\sqrt{17}} & \frac{2}{9\sqrt{17}} & \frac{\sqrt{17}}{9} \end{bmatrix} A \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{3}{\sqrt{17}} & \frac{11}{9\sqrt{17}} \\ 0 & \frac{1}{\sqrt{17}} & \frac{2}{9\sqrt{17}} \\ 0 & 0 & \frac{\sqrt{17}}{9} \end{bmatrix}$$

Thus, $B = P^{'}AP$

Rank $(r) = 3$

Index $(s) = 1$

Signature $= 2s - r = 2(1) - 3 = -1$

Value Class = indefinite

Thus,

$X^{'}AX = 2x_1^2 + x_2^2 - 3x_3^2 - 8x_2x_3 - 4x_3x_1 + 12x_1x_2$ is reduced to

$Y'BY = y_1^2 - y_2^2 - y_3^2$

By the transformation $X = PY$

Where, $x_1 = \frac{1}{\sqrt{2}} y_1 - \frac{3}{\sqrt{17}} y_2 + \frac{11}{9\sqrt{17}} y_3$

$x_2 = \frac{1}{\sqrt{17}} y_2 + \frac{2}{9\sqrt{17}} y_3$

$x_3 = \frac{\sqrt{17}}{9} y_3$

3. Reduce the following quadratic form $6x_1^2 + 3x_2^2 + 14x_3^2 + 4x_1x_2 + 18x_1x_3 + 4x_2x_3$ to diagonal form through congruent transformations. **[M14/ChemBiot/6M]**

Solution:

We have,

$X'AX = 6x_1^2 + 3x_2^2 + 14x_3^2 + 4x_1x_2 + 18x_1x_3 + 4x_2x_3$

$$\therefore A = \begin{bmatrix} 6 & 2 & 9 \\ 2 & 3 & 2 \\ 9 & 2 & 14 \end{bmatrix}$$

Consider,

$A = I_3AI_3$

$$\begin{bmatrix} 6 & 2 & 9 \\ 2 & 3 & 2 \\ 9 & 2 & 14 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $R_2 - \frac{2}{6}R_1, R_3 - \frac{9}{6}R_1, C_2 - \frac{2}{6}C_1, C_3 - \frac{9}{6}C_1$

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & \frac{7}{3} & -1 \\ 0 & -1 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{3} & 1 & 0 \\ -\frac{3}{2} & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -\frac{1}{3} & -\frac{3}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $R_3 + \frac{3}{7}R_2, C_3 + \frac{3}{7}C_2$

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & \frac{7}{3} & 0 \\ 0 & 0 & \frac{1}{14} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{3} & 1 & 0 \\ -\frac{23}{14} & \frac{3}{7} & 1 \end{bmatrix} A \begin{bmatrix} 1 & -\frac{1}{3} & -\frac{23}{14} \\ 0 & 1 & \frac{3}{7} \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore B = P'AP$

Thus,

$X'AX = 6x_1^2 + 3x_2^2 + 14x_3^2 + 4x_1x_2 + 18x_1x_3 + 4x_2x_3$ is reduced to

$Y'BY = 6y_1^2 + \frac{7}{3}y_2^2 + \frac{1}{14}y_3^2$

By the transformation $X = PY$

Where, $x_1 = y_1 - \frac{1}{3}y_2 - \frac{23}{14}y_3$

$x_2 = y_2 + \frac{3}{7}y_3$

$x_3 = y_3$

4. Reduce the quadratic form $Q = x^2 + y^2 - 2z^2 - 4xy - 2yz + 10xz$ to canonical form through congruent transformation
[M14/ElexExtcElectBiomInst/6M]
Solution:
We have,
$X'AX = x^2 + y^2 - 2z^2 - 4xy - 2yz + 10xz$
$\therefore A = \begin{bmatrix} 1 & -2 & 5 \\ -2 & 1 & -1 \\ 5 & -1 & -2 \end{bmatrix}$
Consider,
$A = I_3 A I_3$
$$\begin{bmatrix} 1 & -2 & 5 \\ -2 & 1 & -1 \\ 5 & -1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
By $R_2 + 2R_1, R_3 - 5R_1, C_2 + 2C_1, C_3 - 5C_1$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 9 \\ 0 & 9 & -27 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 2 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
By $R_3 + 3R_2, C_3 + 3C_2$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$
By $\frac{1}{\sqrt{3}} R_2, \frac{1}{\sqrt{3}} C_2$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\ 1 & 3 & 1 \end{bmatrix} A \begin{bmatrix} 1 & \frac{2}{\sqrt{3}} & 1 \\ 0 & \frac{1}{\sqrt{3}} & 3 \\ 0 & 0 & 1 \end{bmatrix}$$
Thus, $B = P'AP$
Thus,
$X'AX = x^2 + y^2 - 2z^2 - 4xy - 2yz + 10xz$ is reduced to
$Y'BY = u^2 - v^2$
By the transformation $X = PY$
Where, $x = u + \frac{2v}{\sqrt{3}} + w$
$y = \frac{v}{\sqrt{3}} + 3w$
$z = w$

5. Reduce the following matrix of the quadratic form $3x_1^2+5x_2^2+3x_3^2-2x_1x_2+2x_1x_3-2x_2x_3$ to canonical form and find its rank, index, signature and value class

[N14/ElexExtcElectBiomInst/6M][M16/ElexExtcElectBiomInst/8M]

Solution:

We have,

$$X'AX=3x_1^2+5x_2^2+3x_3^2-2x_1x_2+2x_1x_3-2x_2x_3$$

$$\therefore A=\begin{bmatrix}3&-1&1\\-1&5&-1\\1&-1&3\end{bmatrix}$$

Consider,

$$A=I_3AI_3$$

$$\begin{bmatrix}3&-1&1\\-1&5&-1\\1&-1&3\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}A\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

By $R_2+\frac{1}{3}R_1, R_3-\frac{1}{3}R_1, C_2+\frac{1}{3}C_1, C_3-\frac{1}{3}C_1$

$$\begin{bmatrix}3&0&0\\0&\frac{14}{3}&-\frac{2}{3}\\0&-\frac{2}{3}&\frac{8}{3}\end{bmatrix}=\begin{bmatrix}1&0&0\\\frac{1}{3}&1&0\\-\frac{1}{3}&0&1\end{bmatrix}A\begin{bmatrix}1&\frac{1}{3}&-\frac{1}{3}\\0&1&0\\0&0&1\end{bmatrix}$$

By $R_3+\frac{2}{14}R_2, C_3+\frac{2}{14}C_2$

$$\begin{bmatrix}3&0&0\\0&\frac{14}{3}&0\\0&0&\frac{18}{7}\end{bmatrix}=\begin{bmatrix}1&0&0\\\frac{1}{3}&1&0\\-\frac{2}{7}&\frac{1}{7}&1\end{bmatrix}A\begin{bmatrix}1&\frac{1}{3}&-\frac{2}{7}\\0&1&\frac{1}{7}\\0&0&1\end{bmatrix}$$

By $\frac{1}{\sqrt{3}}R_1, \frac{\sqrt{3}}{\sqrt{14}}R_2, \frac{\sqrt{7}}{\sqrt{18}}R_3, \frac{1}{\sqrt{3}}C_1, \frac{\sqrt{3}}{\sqrt{14}}C_2, \frac{\sqrt{7}}{\sqrt{18}}C_3$

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}\frac{1}{\sqrt{3}}&0&0\\\frac{1}{\sqrt{42}}&\frac{\sqrt{3}}{\sqrt{14}}&0\\-\frac{2}{\sqrt{126}}&\frac{1}{\sqrt{126}}&\frac{\sqrt{7}}{\sqrt{18}}\end{bmatrix}A\begin{bmatrix}\frac{1}{\sqrt{3}}&\frac{1}{\sqrt{42}}&-\frac{2}{\sqrt{126}}\\0&\frac{\sqrt{3}}{\sqrt{14}}&\frac{1}{\sqrt{126}}\\0&0&\frac{\sqrt{7}}{\sqrt{18}}\end{bmatrix}$$

Thus, $B=P'AP$

Rank $(r)=3$

Index $(s)=3$

Signature $=2s-r=2(3)-3=3$

Value Class = positive definite

6. Reduce to diagonal form the following symmetric matrix A by congruent transformation and find the rank, index signature where $A = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 2 & 3 \\ -1 & 3 & 1 \end{bmatrix}$

[M15/ChemBiot/8M]

Solution:

We have,

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 2 & 3 \\ -1 & 3 & 1 \end{bmatrix}$$

Consider,

$$A = I_3 A I_3$$

$$\begin{bmatrix} 3 & 2 & -1 \\ 2 & 2 & 3 \\ -1 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $R_2 - \frac{2}{3}R_1, R_3 + \frac{1}{3}R_1, C_2 - \frac{2}{3}C_1, C_3 + \frac{1}{3}C_1$

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & \frac{2}{3} & \frac{11}{3} \\ 0 & \frac{11}{3} & \frac{2}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ \frac{1}{3} & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -\frac{2}{3} & \frac{1}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $R_3 - \frac{11}{2}R_2, C_3 - \frac{11}{2}C_2$

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & \frac{2}{3} & 0 \\ 0 & 0 & -\frac{39}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ 4 & -\frac{11}{2} & 1 \end{bmatrix} A \begin{bmatrix} 1 & -\frac{2}{3} & 4 \\ 0 & 1 & -\frac{11}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

Thus, $B = P^{'}AP$

Rank $(r) = 3$

Index $(s) = 2$

Signature $= 2s - r = 2(2) - 3 = 1$

7. Reduce the following quadratic form into canonical form and hence find its rank, index, signature and value class by using congruent transformation
$6x^2 + 3y^2 + 3z^2 - 4xy + 4xz - 2yz$
[M15/ElexExtcElectBiomInst/8M]
Solution:
We have,
$X'AX = 6x^2 + 3y^2 + 3z^2 - 4xy + 4xz - 2yz$
$$\therefore A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$
Consider,
$A = I_3 A I_3$
$$\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
By $R_2 + \frac{2}{6}R_1, R_3 - \frac{2}{6}R_1, C_2 + \frac{2}{6}C_1, C_3 - \frac{2}{6}C_1$
$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & \frac{7}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{3} & \frac{7}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ -\frac{1}{3} & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
By $R_3 + \frac{1}{7}R_2, C_3 + \frac{1}{7}C_2$
$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & \frac{7}{3} & 0 \\ 0 & 0 & \frac{16}{7} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ -\frac{2}{7} & \frac{1}{7} & 1 \end{bmatrix} A \begin{bmatrix} 1 & \frac{1}{3} & -\frac{2}{7} \\ 0 & 1 & \frac{1}{7} \\ 0 & 0 & 1 \end{bmatrix}$$
By $\frac{1}{\sqrt{6}}R_1, \frac{1}{\sqrt{6}}C_1, \frac{\sqrt{7}}{\sqrt{3}}R_2, \frac{\sqrt{7}}{\sqrt{3}}C_2, \frac{\sqrt{7}}{4}R_3, \frac{\sqrt{7}}{4}C_3$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{6}} & 0 & 0 \\ \frac{\sqrt{7}}{3\sqrt{3}} & \frac{\sqrt{7}}{\sqrt{3}} & 0 \\ -\frac{1}{2\sqrt{7}} & \frac{1}{4\sqrt{7}} & \frac{\sqrt{7}}{4} \end{bmatrix} A \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{\sqrt{7}}{3\sqrt{3}} & -\frac{1}{2\sqrt{7}} \\ 0 & \frac{\sqrt{7}}{\sqrt{3}} & \frac{1}{4\sqrt{7}} \\ 0 & 0 & \frac{\sqrt{7}}{4} \end{bmatrix}$$
Thus, $B = P'AP$
Rank $= 3$
Index $(s) = 3$
Signature $= 2s - r = 2(3) - 3 = 3$
Value class $=$ positive definite

8. Reduce the matrix of the quadratic form $x^2 + 2y^2 + 2z^2 - 2xy - 2yz + zx$ to canonical form and find its rank and signature. Also write linear transformation which brings about the normal reduction.
[N15/ChemBiot/8M]
Solution:
We have,
$X'AX = x^2 + 2y^2 + 2z^2 - 2xy - 2yz + zx$

$$\therefore A = \begin{bmatrix} 1 & -1 & \frac{1}{2} \\ -1 & 2 & -1 \\ \frac{1}{2} & -1 & 2 \end{bmatrix}$$

Consider,
$A = I_3 A I_3$

$$\begin{bmatrix} 1 & -1 & \frac{1}{2} \\ -1 & 2 & -1 \\ \frac{1}{2} & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $R_2 + R_1, R_3 - \frac{1}{2}R_1, C_2 + C_1, C_3 - \frac{1}{2}C_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{7}{4} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -\frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $R_3 + \frac{1}{2}R_2, C_3 + \frac{1}{2}C_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{3}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

By $\frac{\sqrt{2}}{\sqrt{3}}R_3, \frac{\sqrt{2}}{\sqrt{3}}C_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & \frac{1}{\sqrt{6}} & \frac{\sqrt{2}}{\sqrt{3}} \end{bmatrix} A \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{\sqrt{2}}{\sqrt{3}} \end{bmatrix}$$

Thus, $B = P'AP$
Rank $= 3$
Index $(s) = 3$
Signature $= 2s - r = 2(3) - 3 = 3$
Value class = positive definite

Thus,
$X^{'}AX = x^2 + 2y^2 + 2z^2 - 2xy - 2yz + zx$ is reduced to
$Y^{'}BY = u^2 + v^2 + w^2$
By the transformation $X = PY$
Where, $x = u + w$
$y = v + \frac{w}{\sqrt{6}}$
$z = \frac{\sqrt{2}w}{\sqrt{3}}$

9. Reduce the Quadratic form $xy + yz + zx$ to diagonal form through congruent transformation
[N15/ElexExtcElectBiomInst/6M]
Solution:
We have,
$X'AX = xy + yz + zx$
$$\therefore A = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$
Consider,
$A = I_3 A I_3$
$$\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
By $R_1 + R_2, C_1 + C_2$
$$\begin{bmatrix} 1 & \frac{1}{2} & 1 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
By $R_2 - \frac{1}{2}R_1, R_3 - R_1, C_2 - \frac{1}{2}C_1, C_3 - C_1$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{4} & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ -1 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -\frac{1}{2} & -1 \\ 1 & \frac{1}{2} & -1 \\ 0 & 0 & 1 \end{bmatrix}$$
Thus, $B = P'AP$

10. Reduce the matrix of the quadratic form $2x_1^2 + x_2^2 - 3x_3^2 - 8x_2x_3 - 4x_3x_1 + 12x_1x_2$ to canonical form and find its rank, index, signature and value class.

[M16/ChemBiot/8M][N16/ChemBiot/8M]

Solution:

We have,

$$X'AX = 2x_1^2 + x_2^2 - 3x_3^2 - 8x_2x_3 - 4x_3x_1 + 12x_1x_2$$

$$\therefore A = \begin{bmatrix} 2 & 6 & -2 \\ 6 & 1 & -4 \\ -2 & -4 & -3 \end{bmatrix}$$

Consider,

$$A = I_3AI_3$$

$$\begin{bmatrix} 2 & 6 & -2 \\ 6 & 1 & -4 \\ -2 & -4 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $R_2 - 3R_1, R_3 + R_1, C_2 - 3C_1, C_3 + C_1$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -17 & 2 \\ 0 & 2 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -3 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $R_3 + \frac{2}{17}R_2, C_3 + \frac{2}{17}C_2$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -17 & 0 \\ 0 & 0 & -\frac{81}{17} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ \frac{11}{17} & \frac{2}{17} & 1 \end{bmatrix} A \begin{bmatrix} 1 & -3 & \frac{11}{17} \\ 0 & 1 & \frac{2}{17} \\ 0 & 0 & 1 \end{bmatrix}$$

By $\frac{1}{\sqrt{2}}R_1, \frac{1}{\sqrt{17}}R_2, \frac{\sqrt{17}}{9}R_3, \frac{1}{\sqrt{2}}C_1, \frac{1}{\sqrt{17}}C_2, \frac{\sqrt{17}}{9}C_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ -\frac{3}{\sqrt{17}} & \frac{1}{\sqrt{17}} & 0 \\ \frac{11}{9\sqrt{17}} & \frac{2}{9\sqrt{17}} & \frac{\sqrt{17}}{9} \end{bmatrix} A \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{3}{\sqrt{17}} & \frac{11}{9\sqrt{17}} \\ 0 & \frac{1}{\sqrt{17}} & \frac{2}{9\sqrt{17}} \\ 0 & 0 & \frac{\sqrt{17}}{9} \end{bmatrix}$$

Thus, $B = P'AP$

Rank $(r) = 3$

Index $(s) = 1$

Signature $= 2s - r = 2(1) - 3 = -1$

Value Class = indefinite

11. Reduce the quadratic form $2x^2 - 2y^2 + 2z^2 - 2xy - 8yz + 6xz$ to canonical and hence find its rank, index, signature and value class
[N16/ElexExtcElectBiomInst/6M]
Solution:
We have,
$X'AX = 2x^2 - 2y^2 + 2z^2 - 2xy - 8yz + 6xz$

$$\therefore A = \begin{bmatrix} 2 & -1 & 3 \\ -1 & -2 & -4 \\ 3 & -4 & 2 \end{bmatrix}$$

Consider,
$A = I_3 A I_3$

$$\begin{bmatrix} 2 & -1 & 3 \\ -1 & -2 & -4 \\ 3 & -4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $R_2 + \frac{1}{2}R_1, R_3 - \frac{3}{2}R_1, C_2 + \frac{1}{2}C_1, C_3 - \frac{3}{2}C_1$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -\frac{5}{2} & -\frac{5}{2} \\ 0 & -\frac{5}{2} & -\frac{5}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ -\frac{3}{2} & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & \frac{1}{2} & -\frac{3}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $R_3 - R_2, C_3 - C_2$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -\frac{5}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ -2 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & \frac{1}{2} & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

By $\frac{1}{\sqrt{2}}R_1, \frac{1}{\sqrt{2}}C_1, \frac{\sqrt{2}}{\sqrt{5}}R_2, \frac{\sqrt{2}}{\sqrt{5}}C_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{10}} & \frac{\sqrt{2}}{\sqrt{5}} & 0 \\ -2 & -1 & 1 \end{bmatrix} A \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{10}} & -2 \\ 0 & \frac{\sqrt{2}}{\sqrt{5}} & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus, $B = P'AP$
Rank $= 2$
Index $(s) = 1$
Signature $= 2s - r = 2(1) - 2 = 0$
Value class $=$ indefinite

12. Find the linear transformation $Y = AX$ which carries $X_1 = (1,1,-1)'$, $X_2 = (1,-1,1)'$, $X_3 = (-1,1,1)'$ onto $Y_1 = (2,1,3)'$, $Y_2 = (2,3,1)'$, $Y_3 = (4,1,3)'$
[M17/ElexExtcElectBiomInst/6M]
Solution:
Let the transformation matrix be $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

$Y = AX$ gives $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Now, $Y_1 = AX_1$ gives

$$\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$2 = a_1 + b_1 - c_1$ ………….(1)
$1 = a_2 + b_2 - c_2$ ………….(2)
$3 = a_3 + b_3 - c_3$ ………….(3)

Now, $Y_2 = AX_2$ gives

$$\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$2 = a_1 - b_1 + c_1$ ………….(4)
$3 = a_2 - b_2 + c_2$ ………….(5)
$1 = a_3 - b_3 + c_3$ ………….(6)

Now, $Y_3 = AX_3$ gives

$$\begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$4 = -a_1 + b_1 + c_1$ ………….(7)
$1 = -a_2 + b_2 + c_2$ ………….(8)
$3 = -a_3 + b_3 + c_3$ ………….(9)

Solving (1), (4), (7), we get $a_1 = 2, b_1 = 3, c_1 = 3$
Solving (2), (5), (8), we get $a_2 = 2, b_2 = 1, c_2 = 2$
Solving (3), (6), (9), we get $a_3 = 2, b_3 = 3, c_3 = 2$

$$\therefore A = \begin{bmatrix} 2 & 3 & 3 \\ 2 & 1 & 2 \\ 2 & 3 & 2 \end{bmatrix}$$

13. Reduce the quadratic form $x^2 - 2y^2 + 10z^2 - 10xy + 4xz - 2zy$ to canonical form and hence find its rank, index, signature and value class.
[M17/ElexExtcElectBiomInst/8M]
Solution:
We have,
$X'AX = x^2 - 2y^2 + 10z^2 - 10xy + 4xz - 2zy$
$\therefore A = \begin{bmatrix} 1 & -5 & 2 \\ -5 & -2 & -1 \\ 2 & -1 & 10 \end{bmatrix}$
Consider,
$A = I_3 A I_3$
$$\begin{bmatrix} 1 & -5 & 2 \\ -5 & -2 & -1 \\ 2 & -1 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
By $R_2 + 5R_1, R_3 - 2R_1, C_2 + 5C_1, C_3 - 2C_1$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -27 & 9 \\ 0 & 9 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 5 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
By $R_3 + \frac{1}{3}R_2, C_3 + \frac{1}{3}C_2$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -27 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ -\frac{1}{3} & \frac{1}{3} & 1 \end{bmatrix} A \begin{bmatrix} 1 & 5 & -\frac{1}{3} \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix}$$
By $\frac{1}{\sqrt{27}}R_2, \frac{1}{\sqrt{27}}C_2, \frac{1}{3}R_3, \frac{1}{3}C_3$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{5}{3\sqrt{3}} & \frac{1}{3\sqrt{3}} & 0 \\ -\frac{1}{9} & \frac{1}{9} & \frac{1}{3} \end{bmatrix} A \begin{bmatrix} 1 & \frac{5}{3\sqrt{3}} & -\frac{1}{9} \\ 0 & \frac{1}{3\sqrt{3}} & \frac{1}{9} \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$
Thus, $B = P'AP$
Rank $= 3$
Index $(s) = 2$
Signature $= 2s - r = 2(2) - 3 = 1$
Value class $=$ indefinite

14. Reduce the matrix of the quadratic form $5x_1^2 + 26x_2^2 + 10x_3^2 + 4x_2x_3 + 14x_3x_1 + 6x_1x_2$ to canonical form and find its rank, index, signature and value class.

[N17/ElexExtcElectBiomInst/8M]

Solution:

We have,

$$X'AX = 5x_1^2 + 26x_2^2 + 10x_3^2 + 4x_2x_3 + 14x_3x_1 + 6x_1x_2$$

$$\therefore A = \begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix}$$

Consider,

$$A = I_3 A I_3$$

$$\begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $R_2 - \frac{3}{5}R_1, R_3 - \frac{7}{5}R_1, C_2 - \frac{3}{5}C_1, C_3 - \frac{7}{5}C_1$

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & \frac{121}{5} & -\frac{11}{5} \\ 0 & -\frac{11}{5} & \frac{1}{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{5} & 1 & 0 \\ -\frac{7}{5} & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -\frac{3}{5} & -\frac{7}{5} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $R_3 + \frac{1}{11}R_2, C_3 + \frac{1}{11}C_2$

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & \frac{121}{5} & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{5} & 1 & 0 \\ -\frac{16}{11} & \frac{1}{11} & 1 \end{bmatrix} A \begin{bmatrix} 1 & -\frac{3}{5} & -\frac{16}{11} \\ 0 & 1 & \frac{1}{11} \\ 0 & 0 & 1 \end{bmatrix}$$

By $\frac{1}{\sqrt{5}}R_1, \frac{\sqrt{5}}{11}R_2, \frac{1}{\sqrt{5}}C_1, \frac{\sqrt{5}}{11}C_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} & 0 & 0 \\ -\frac{3}{11\sqrt{5}} & \frac{\sqrt{5}}{11} & 0 \\ -\frac{16}{11} & \frac{1}{11} & 1 \end{bmatrix} A \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{3}{11\sqrt{5}} & -\frac{16}{11} \\ 0 & \frac{\sqrt{5}}{11} & \frac{1}{11} \\ 0 & 0 & 1 \end{bmatrix}$$

Thus, $B = P'AP$

Rank $(r) = 2$

Index $(s) = 2$

Signature $= 2s - r = 2(2) - 2 = 2$

Value Class = positive semi definite

Type VII: Orthogonal Transformations

1. Find the orthogonal matrix that will diagonalise the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$

[N13/Chem/7M]

Solution:

We have,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}, |A| = 0$$

The characteristic equation,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & 4-\lambda & 6 \\ 3 & 6 & 9-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$$

$$\lambda^3 - [1+4+9]\lambda^2 + \left[\begin{vmatrix} 4 & 6 \\ 6 & 9 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 3 & 9 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix}\right]\lambda - (0) = 0$$

$$\lambda^3 - 14\lambda^2 + 0\lambda - 0 = 0$$

$$\lambda = 0,0,14$$

(i) For $\lambda = 14$, $[A - \lambda I]X = 0$ gives

$$\begin{bmatrix} -13 & 2 & 3 \\ 2 & -10 & 6 \\ 3 & 6 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-13x_1 + 2x_2 + 3x_3 = 0$$

$$2x_1 - 10x_2 + 6x_3 = 0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} 2 & 3 \\ -10 & 6 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} -13 & 3 \\ 2 & 6 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -13 & 2 \\ 2 & -10 \end{vmatrix}}$$

$$\frac{x_1}{42} = -\frac{x_2}{-84} = \frac{x_3}{126}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{3}$$

Hence, corresponding to $\lambda = 14$ the eigen vector is $X_1 = [1,2,3]'$

(ii) For $\lambda = 0$, $[A - \lambda I]X = 0$ gives

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_2 - 2R_1, R_3 - 3R_1$

$$\begin{bmatrix}1 & 2 & 3\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$x_1+2x_2+3x_3=0$

Let $x_3=1, x_2=0$

$\therefore x_1=-3$

Hence, corresponding to $\lambda=0$ the eigen vector is $X_2=[-3,0,1]'$

To find the third eigen vector let $X_3=[x,y,z]'$

Since, X_3 satisfies the equation $x_1+2x_2+3x_3=0$ and since the vector X_3 is orthogonal to the vector $X_2=[-3,0,1]'$

Thus, solving $x+2y+3z=0;\ -3x+0y+z=0$ by Crammer's rule,

$$\frac{x}{\begin{vmatrix}2 & 3\\0 & 1\end{vmatrix}}=-\frac{y}{\begin{vmatrix}1 & 3\\-3 & 1\end{vmatrix}}=\frac{z}{\begin{vmatrix}1 & 2\\-3 & 0\end{vmatrix}}$$

$$\frac{x}{2}=-\frac{y}{10}=\frac{z}{6}$$

$\therefore X_3=[1,-5,3]'$

Now, we normalize the vectors X_1, X_2, X_3, we find unit vectors S_1, S_2, S_3 which are scalar multiples of X_1, X_2, X_3 respectively

$$S_1=\frac{X_1}{\|X_1\|}=\frac{X_1}{\sqrt{1+4+9}}=\frac{X_1}{\sqrt{14}}$$

$$S_2=\frac{X_2}{\|X_2\|}=\frac{X_2}{\sqrt{9+0+1}}=\frac{X_2}{\sqrt{10}}$$

$$S_3=\frac{X_3}{\|X_3\|}=\frac{X_3}{\sqrt{1+25+9}}=\frac{X_3}{\sqrt{35}}$$

$$P=[S_1,S_2,S_3]=\begin{bmatrix}\frac{1}{\sqrt{14}} & \frac{-3}{\sqrt{10}} & \frac{1}{\sqrt{35}}\\ \frac{2}{\sqrt{14}} & 0 & \frac{-5}{\sqrt{35}}\\ \frac{3}{\sqrt{14}} & \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{35}}\end{bmatrix}$$

Then P is an orthogonal matrix and $P^{-1}AP=\begin{bmatrix}14 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$ where $P^{-1}=P'$

2. Find the orthogonal matrix that will diagonalise the matrix

$$A = \begin{bmatrix} 7 & 0 & -2 \\ 0 & 5 & -2 \\ -2 & -2 & 6 \end{bmatrix}$$

[N14/ChemBiot/7M][N16/ChemBiot/8M]

Solution:

We have,

$$A = \begin{bmatrix} 7 & 0 & -2 \\ 0 & 5 & -2 \\ -2 & -2 & 6 \end{bmatrix}, |A| = 162$$

The characteristic equation,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 7-\lambda & 0 & -2 \\ 0 & 5-\lambda & -2 \\ -2 & -2 & 6-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$$

$$\lambda^3 - [7+5+6]\lambda^2 + \left[\begin{vmatrix} 5 & -2 \\ -2 & 6 \end{vmatrix} + \begin{vmatrix} 7 & -2 \\ -2 & 6 \end{vmatrix} + \begin{vmatrix} 7 & 0 \\ 0 & 54 \end{vmatrix}\right]\lambda - 162 = 0$$

$$\lambda^3 - 18\lambda^2 + 99\lambda - 162 = 0$$

$$(\lambda - 3)(\lambda - 9)(\lambda - 6) = 0$$

$$\lambda = 3,9,6$$

(i) For $\lambda = 3$, $[A - \lambda I]X = 0$ gives

$$\begin{bmatrix} 4 & 0 & -2 \\ 0 & 2 & -2 \\ -2 & -2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$4x_1 + 0x_2 - 2x_3 = 0$$

$$0x_1 + 2x_2 - 2x_3 = 0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} 0 & -2 \\ -2 & 3 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} 4 & -2 \\ 0 & -2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 4 & 0 \\ 0 & 2 \end{vmatrix}}$$

$$\frac{x_1}{-4} = -\frac{x_2}{-8} = \frac{x_3}{8}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{2}$$

Hence, corresponding to $\lambda = 3$ the eigen vector is $X_1 = [1,2,2]'$

(ii) For $\lambda = 9$, $[A - \lambda I]X = 0$ gives

$$\begin{bmatrix} -2 & 0 & -2 \\ 0 & -4 & -2 \\ -2 & -2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x_1 + 0x_2 - 2x_3 = 0$$

$$0x_1 - 4x_2 - 2x_3 = 0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} 0 & -2 \\ -4 & -2 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} -2 & -2 \\ 0 & -2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -2 & 0 \\ 0 & -4 \end{vmatrix}}$$

$$\frac{x_1}{-8} = -\frac{x_2}{4} = \frac{x_3}{8}$$

$$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2}$$

Hence, corresponding to $\lambda = 9$ the eigen vector is $X_2 = [2,1,-2]'$

(iii) For $\lambda = 6, [A - \lambda I]X = 0$ gives

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & -1 & -2 \\ -2 & -2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + 0x_2 - 2x_3 = 0$$

$$0x_1 - x_2 - 2x_3 = 0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} 0 & -2 \\ -1 & -2 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} 1 & -2 \\ 0 & -2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}}$$

$$\frac{x_1}{-2} = -\frac{x_2}{-2} = \frac{x_3}{-1}$$

$$\frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1}$$

Hence, corresponding to $\lambda = 6$ the eigen vector is $X_3 = [2,-2,1]'$

Now, we normalize the vectors X_1, X_2, X_3, we find unit vectors S_1, S_2, S_3 which are scalar multiples of X_1, X_2, X_3 respectively

$$S_1 = \frac{X_1}{\|X_1\|} = \frac{X_1}{\sqrt{1+4+4}} = \frac{X_1}{3}$$

$$S_2 = \frac{X_2}{\|X_2\|} = \frac{X_2}{\sqrt{4+1+4}} = \frac{X_2}{3}$$

$$S_3 = \frac{X_3}{\|X_3\|} = \frac{X_3}{\sqrt{4+4+1}} = \frac{X_3}{3}$$

$$P = [S_1, S_2, S_3] = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

Then P is an orthogonal matrix and $P^{-1}AP = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 6 \end{bmatrix}$ where $P^{-1} = P'$

3. Find the orthogonal matrix P that diagonalises $A=\begin{bmatrix}4&2&2\\2&4&2\\2&2&4\end{bmatrix}$

[N16/ElexExtcElectBiomInst/6M]

Solution:

We have,

$A=\begin{bmatrix}4&2&2\\2&4&2\\2&2&4\end{bmatrix}, |A|=32$

The characteristic equation,

$|A-\lambda I|=0$

$\begin{vmatrix}4-\lambda&2&2\\2&4-\lambda&2\\2&2&4-\lambda\end{vmatrix}=0$

$\lambda^3-[sum\ of\ diagonals]\lambda^2+[sum\ of\ minors\ of\ diagonals]\lambda-|A|=0$

$\lambda^3-[4+4+4]\lambda^2+\left[\begin{vmatrix}4&2\\2&4\end{vmatrix}+\begin{vmatrix}4&2\\2&4\end{vmatrix}+\begin{vmatrix}4&2\\2&4\end{vmatrix}\right]\lambda-32=0$

$\lambda^3-12\lambda^2+36\lambda-32=0$

$(\lambda-8)(\lambda-2)(\lambda-2)=0$

$\lambda=8,2,2$

(i) For $\lambda=8$, $[A-\lambda I]X=0$ gives

$\begin{bmatrix}-4&2&2\\2&-4&2\\2&2&-4\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

$-4x_1+2x_2+2x_3=0$

$2x_1-4x_2+2x_3=0$

Solving the above equations by Crammers rule, we get

$\frac{x_1}{\begin{vmatrix}2&2\\-4&2\end{vmatrix}}=-\frac{x_2}{\begin{vmatrix}-4&2\\2&2\end{vmatrix}}=\frac{x_3}{\begin{vmatrix}-4&2\\2&-4\end{vmatrix}}$

$\frac{x_1}{12}=-\frac{x_2}{-12}=\frac{x_3}{12}$

$\frac{x_1}{1}=\frac{x_2}{1}=\frac{x_3}{1}$

Hence, corresponding to $\lambda=8$ the eigen vector is $X_1=[1,1,1]'$

(ii) For $\lambda=2$, $[A-\lambda I]X=0$ gives

$\begin{bmatrix}2&2&2\\2&2&2\\2&2&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

By R_2-R_1, R_3-R_1

$$\begin{bmatrix}2&2&2\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$2x_1+2x_2+2x_3=0$

$x_1+x_2+x_3=0$

Let $x_3=1, x_2=0$

$\therefore x_1=-1$

Hence, corresponding to $\lambda=2$ the eigen vector is $X_2=[-1,0,1]'$

To find the third eigen vector let $X_3=[x,y,z]'$

Since, X_3 satisfies the equation $x_1+x_2+x_3=0$ and since the vector X_3 is orthogonal to the vector $X_2=[-1,0,1]'$

Thus, solving $x+y+z=0$; $-x+0y+z=0$ by Crammer's rule,

$$\frac{x}{\begin{vmatrix}1&1\\0&1\end{vmatrix}}=-\frac{y}{\begin{vmatrix}1&1\\-1&1\end{vmatrix}}=\frac{z}{\begin{vmatrix}1&1\\-1&0\end{vmatrix}}$$

$$\frac{x}{1}=-\frac{y}{2}=\frac{z}{1}$$

$\therefore X_3=[1,-2,1]'$

Now, we normalize the vectors X_1, X_2, X_3, we find unit vectors S_1, S_2, S_3 which are scalar multiples of X_1, X_2, X_3 respectively

$$S_1=\frac{X_1}{\|X_1\|}=\frac{X_1}{\sqrt{1+1+1}}=\frac{X_1}{\sqrt{3}}$$

$$S_2=\frac{X_2}{\|X_2\|}=\frac{X_2}{\sqrt{1+0+1}}=\frac{X_2}{\sqrt{2}}$$

$$S_3=\frac{X_3}{\|X_3\|}=\frac{X_3}{\sqrt{1+4+1}}=\frac{X_3}{\sqrt{6}}$$

$$P=[S_1,S_2,S_3]=\begin{bmatrix}\frac{1}{\sqrt{3}}&\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{6}}\\\frac{1}{\sqrt{3}}&0&-\frac{2}{\sqrt{6}}\\\frac{1}{\sqrt{3}}&-\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{6}}\end{bmatrix}$$

Then P is an orthogonal matrix and $P^{-1}AP=\begin{bmatrix}8&0&0\\0&2&0\\0&0&2\end{bmatrix}$ where $P^{-1}=P'$

4. Find the orthogonal matrix P that diagonalises $A = \begin{bmatrix} 7 & 4 & -4 \\ 4 & -8 & -1 \\ -4 & -1 & -8 \end{bmatrix}$

[N16/ChemBiot/8M]

Solution:

We have,

$$A = \begin{bmatrix} 7 & 4 & -4 \\ 4 & -8 & -1 \\ -4 & -1 & -8 \end{bmatrix}, |A| = 729$$

The characteristic equation,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 7-\lambda & 4 & -4 \\ 4 & -8-\lambda & -1 \\ -4 & -1 & -8-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$$

$$\lambda^3 - [7-8-8]\lambda^2 + \left[\begin{vmatrix} -8 & -1 \\ -1 & -8 \end{vmatrix} + \begin{vmatrix} 7 & -4 \\ -4 & -8 \end{vmatrix} + \begin{vmatrix} 7 & 4 \\ 4 & -8 \end{vmatrix}\right]\lambda - 729 = 0$$

$$\lambda^3 + 9\lambda^2 - 81\lambda - 729 = 0$$

$$\lambda = 9, -9, -9$$

(i) For $\lambda = 9$, $[A - \lambda I]X = 0$ gives

$$\begin{bmatrix} -2 & 4 & -4 \\ 4 & -17 & -1 \\ -4 & -1 & -17 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x_1 + 4x_2 - 4x_3 = 0$$

$$4x_1 - 17x_2 - x_3 = 0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} 4 & -4 \\ -17 & -1 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} -2 & -4 \\ 4 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -2 & 4 \\ 4 & -17 \end{vmatrix}}$$

$$\frac{x_1}{-72} = -\frac{x_2}{18} = \frac{x_3}{18}$$

$$\frac{x_1}{4} = \frac{x_2}{1} = \frac{x_3}{-1}$$

Hence, corresponding to $\lambda = 9$ the eigen vector is $X_1 = [4, 1, -1]'$

(ii) For $\lambda = -9$, $[A - \lambda I]X = 0$ gives

$$\begin{bmatrix} 16 & 4 & -4 \\ 4 & 1 & -1 \\ -4 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_2 - \frac{1}{4}R_1, R_3 + \frac{1}{4}R_1$

$$\begin{bmatrix} 16 & 4 & -4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$16x_1 + 4x_2 - 4x_3 = 0$

$4x_1 + x_2 - x_3 = 0$

Let $x_3 = 4, x_2 = 0$

$\therefore x_1 = 1$

Hence, corresponding to $\lambda = -9$ the eigen vector is $X_2 = [1,0,4]'$

To find the third eigen vector let $X_3 = [x, y, z]'$

Since, X_3 satisfies the equation $4x_1 + x_2 - x_3 = 0$ and since the vector X_3 is orthogonal to the vector $X_2 = [1,0,4]'$

Thus, solving $4x + y - z = 0$; $x + 0y + 4z = 0$ by Crammer's rule,

$$\frac{x}{\begin{vmatrix} 1 & -1 \\ 0 & 4 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} 4 & -1 \\ 1 & 4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 4 & 1 \\ 1 & 0 \end{vmatrix}}$$

$$\frac{x}{4} = -\frac{y}{17} = \frac{z}{-1}$$

$\therefore X_3 = [4, -17, -1]'$

Now, we normalize the vectors X_1, X_2, X_3, we find unit vectors S_1, S_2, S_3 which are scalar multiples of X_1, X_2, X_3 respectively

$$S_1 = \frac{X_1}{\|X_1\|} = \frac{X_1}{\sqrt{16+1+1}} = \frac{X_1}{\sqrt{18}}$$

$$S_2 = \frac{X_2}{\|X_2\|} = \frac{X_2}{\sqrt{1+0+16}} = \frac{X_2}{\sqrt{17}}$$

$$S_3 = \frac{X_3}{\|X_3\|} = \frac{X_3}{\sqrt{16+289+1}} = \frac{X_3}{3\sqrt{34}}$$

$$P = [S_1, S_2, S_3] = \begin{bmatrix} \frac{4}{\sqrt{18}} & \frac{1}{\sqrt{17}} & \frac{4}{3\sqrt{34}} \\ \frac{1}{\sqrt{18}} & 0 & -\frac{17}{3\sqrt{34}} \\ \frac{-1}{\sqrt{18}} & \frac{4}{\sqrt{17}} & \frac{-1}{3\sqrt{34}} \end{bmatrix}$$

P is an orthogonal matrix and $P^{-1}AP = \begin{bmatrix} 9 & 0 & 0 \\ 0 & -9 & 0 \\ 0 & 0 & -9 \end{bmatrix}$ where $P^{-1} = P'$

Type VIII: Singular Value Decomposition

1. Find the singular value decomposition of the matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}$

[M14/ElexExtcElectBiomInst/8M]

Solution:

We have,

$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}$ which is 3×2

$\therefore U$ is 3×2, D is 2×2 and V is 2×2

$A' = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

$\therefore B = A'A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

The characteristic equation,

$|B - \lambda I| = 0$

$\begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = 0$

$(3-\lambda)(3-\lambda) - 1 = 0$

$\lambda^2 - 6\lambda + 8 = 0$

$(\lambda - 4)(\lambda - 2) = 0$

$\lambda = 4,2$

Singular values of A is $\sigma_1 = \sqrt{4} = 2, \sigma_2 = \sqrt{2}$

(i) If $\lambda = 4$, $[B - \lambda I]X = 0$ gives

$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

By $R_2 + R_1$

$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore -x_1 + x_2 = 0$

$x_1 = x_2$

Let $x_1 = x_2 = t$

$\therefore X = \begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Eigen vector corresponding to $\lambda = 4$ is $X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\therefore v_1 = \frac{X_1}{\|X_1\|} = \frac{X_1}{\sqrt{1+1}} = \frac{X_1}{\sqrt{2}} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

(ii) If $\lambda = 2$, $[B - \lambda I]X = 0$ gives

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

By $R_2 - R_1$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore x_1 + x_2 = 0$

$x_1 = -x_2$

Let $x_1 = -x_2 = t$

$$\therefore X = \begin{bmatrix} t \\ -t \end{bmatrix} = t \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Eigen vector corresponding to $\lambda = 2$ is $X_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\therefore v_2 = \frac{X_2}{\|X_2\|} = \frac{X_2}{\sqrt{1+1}} = \frac{X_2}{\sqrt{2}} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$v = [v_1 \quad v_2] = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Further,

Let $u = [u_1 \quad u_2 \quad u_3]$

Where, $u_1 = \frac{1}{\sigma_1} A\, v_1 = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \frac{2}{\sqrt{2}} \\ \frac{2}{\sqrt{2}} \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

Where, $u_2 = \frac{1}{\sigma_2} A\, v_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 0 \\ \frac{2}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Let $u_3 = [a \quad b \quad c]$

$u_1' u_3 = 0$

$$[1 \quad 1 \quad 0] \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

$a + b + 0c = 0$ ………………(1)

$u_2' u_3 = 0$

$$[0 \quad 0 \quad 1] \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

$0a + 0b + c = 0$ ………………(2)

Solving eqn (1) & (2) by Crammers rule,

$$\frac{a}{\begin{vmatrix}1 & 0\\0 & 1\end{vmatrix}} = -\frac{b}{\begin{vmatrix}1 & 0\\0 & 1\end{vmatrix}} = \frac{c}{\begin{vmatrix}1 & 1\\0 & 0\end{vmatrix}}$$

$$\frac{a}{1} = -\frac{b}{1} = \frac{c}{0}$$

$$X_3 = \begin{bmatrix}1\\-1\\0\end{bmatrix}$$

$$u_3 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\\0\end{bmatrix}$$

$$\therefore u = [u_1 \quad u_2 \quad u_3] = \begin{bmatrix}\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}}\\ 0 & 1 & 0\end{bmatrix}$$

Thus, SVD of A is

$$A = UDV' = \begin{bmatrix}\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}}\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}2 & 0\\0 & \sqrt{2}\\0 & 0\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix}$$

2. Find the singular value decomposition of the matrix $\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$

[N15/ElexExtcElectBiomInst/8M]

Solution:

We have,

$A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ which is 2×2

$\therefore U$ is 2×2, D is 2×2 and V is 2×2

$A^{'} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

$\therefore B = A^{'}A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix}$

The characteristic equation,

$|B - \lambda I| = 0$

$\begin{vmatrix} 2-\lambda & 4 \\ 4 & 8-\lambda \end{vmatrix} = 0$

$(2-\lambda)(8-\lambda) - 16 = 0$

$\lambda^2 - 10\lambda = 0$

$(\lambda)(\lambda - 10) = 0$

$\lambda = 0,10$

Singular values of A is $\sigma_1 = 0, \sigma_2 = \sqrt{10}$

(i) If $\lambda = 0$, $[B - \lambda I]X = 0$ gives

$\begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

By $R_2 - 2R_1$

$\begin{bmatrix} 2 & 4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore 2x_1 + 4x_2 = 0$

$x_1 = -2x_2$

Let $\frac{x_1}{-2} = \frac{x_2}{1} = t$

$\therefore X = \begin{bmatrix} -2t \\ t \end{bmatrix} = t\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

Eigen vector corresponding to $\lambda = 0$ is $X_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$\therefore v_1 = \frac{X_1}{\|X_1\|} = \frac{X_1}{\sqrt{4+1}} = \frac{X_1}{\sqrt{5}} = \begin{bmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$

(ii) If $\lambda = 10$, $[B - \lambda I]X = 0$ gives

$\begin{bmatrix} -8 & 4 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

By $R_2 + \frac{1}{2}R_1$

$\begin{bmatrix} -8 & 4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore -8x_1 + 4x_2 = 0$

$2x_1 = x_2$

Let $\frac{x_1}{1} = \frac{x_2}{2} = t$

$\therefore X = \begin{bmatrix} t \\ 2t \end{bmatrix} = t\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Eigen vector corresponding to $\lambda = 10$ is $X_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\therefore v_2 = \frac{X_2}{\|X_2\|} = \frac{X_2}{\sqrt{1+4}} = \frac{X_2}{\sqrt{5}} = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$

$v = [v_1 \quad v_2] = \begin{bmatrix} -\frac{2}{\sqrt{2}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}$

Further,

Let $u = [u_1 \quad u_2]$

Where, $u_2 = \frac{1}{\sigma_2} A\, v_2 = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix} = \frac{1}{\sqrt{10}}\begin{bmatrix} \frac{5}{\sqrt{5}} \\ \frac{5}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

Since, $\sigma_1 = 0$ we cannot use the formula $u_1 = \frac{1}{\sigma_1} A\, v_1$

We shall use the concept of orthogonality,

Let $u_1 = [a \quad b]$

$u_2' u_1 = 0$

$[1 \quad 1]\begin{bmatrix} a \\ b \end{bmatrix} = 0$

$a + b = 0$

$a = -b$

$\frac{a}{-1} = \frac{b}{1}$

$\therefore X = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\therefore u_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

Thus, $u = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

Thus, SVD of A is

$$A = UDV^{'} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{10} \end{bmatrix} \begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$$

3. Find the singular value decomposition of the matrix $\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$

[N16/ElexExtcElectBiomInst/8M][M17/ElexExtcElectBiomInst/8M]

Solution:

We have,

$A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$ which is 2×2

$\therefore U$ is 2×2, D is 2×2 and V is 2×2

$A' = \begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix}$

$\therefore B = A'A = \begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 6 & 13 \end{bmatrix}$

The characteristic equation,

$|B - \lambda I| = 0$

$\begin{vmatrix} 4-\lambda & 6 \\ 6 & 13-\lambda \end{vmatrix} = 0$

$(4-\lambda)(13-\lambda) - 36 = 0$

$\lambda^2 - 17\lambda + 16 = 0$

$(\lambda - 1)(\lambda - 16) = 0$

$\lambda = 1,16$

Singular values of A is $\sigma_1 = 1, \sigma_2 = \sqrt{16} = 4$

(i) If $\lambda = 1$, $[B - \lambda I]X = 0$ gives

$\begin{bmatrix} 3 & 6 \\ 6 & 12 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

By $R_2 - 2R_1$

$\begin{bmatrix} 3 & 6 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore 3x_1 + 6x_2 = 0$

$x_1 = -2x_2$

Let $\frac{x_1}{-2} = \frac{x_2}{1} = t$

$\therefore X = \begin{bmatrix} -2t \\ t \end{bmatrix} = t\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

Eigen vector corresponding to $\lambda = 1$ is $X_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$\therefore v_1 = \frac{X_1}{\|X_1\|} = \frac{X_1}{\sqrt{4+1}} = \frac{X_1}{\sqrt{5}} = \begin{bmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$

(ii) If $\lambda = 16$, $[B - \lambda I]X = 0$ gives

$\begin{bmatrix} -12 & 6 \\ 6 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

By $R_2 + \frac{1}{2}R_1$

$$\begin{bmatrix} -12 & 6 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore -12x_1 + 6x_2 = 0$

$2x_1 = x_2$

Let $\frac{x_1}{1} = \frac{x_2}{2} = t$

$$\therefore X = \begin{bmatrix} t \\ 2t \end{bmatrix} = t\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Eigen vector corresponding to $\lambda = 16$ is $X_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$\therefore v_2 = \frac{X_2}{\|X_2\|} = \frac{X_2}{\sqrt{1+4}} = \frac{X_2}{\sqrt{5}} = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$$

$$v = [v_1 \quad v_2] = \begin{bmatrix} -\frac{2}{\sqrt{2}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}$$

Further,

Let $u = [u_1 \quad u_2]$

Where, $u_1 = \frac{1}{\sigma_1} A\, v_1 = \frac{1}{1}\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$

Where, $u_2 = \frac{1}{\sigma_2} A\, v_2 = \frac{1}{4}\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} \frac{8}{\sqrt{5}} \\ \frac{4}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$

Thus, $u = \begin{bmatrix} -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$

Thus, SVD of A is

$$A = UDV' = \begin{bmatrix} -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$$

Theory Questions:

1. Prove that the eigen values of a hermitian matrix are real.
[M14/ElexExtcElectBiomInst/5M]
Solution:
Let A be a Hermitian matrix with λ as an eigen value and X as a corresponding eigen vector. Then,
$AX = \lambda X$
Premultiplying by X^{θ} we get
$X^{\theta} AX = X^{\theta} \lambda X = \lambda X^{\theta} X$
$\left(X^{\theta} AX\right)^{\theta} = \left(\lambda X^{\theta} X\right)^{\theta}$
$X^{\theta} A^{\theta} \left(X^{\theta}\right)^{\theta} = \bar{\lambda} X^{\theta} \left(X^{\theta}\right)^{\theta}$
$X^{\theta} AX = \bar{\lambda} X^{\theta} X$ since$A^{\theta} = A$
$\lambda X^{\theta} X = \bar{\lambda} X^{\theta} X$ since$X^{\theta} AX = \lambda X^{\theta} X$
$(\lambda - \bar{\lambda}) X^{\theta} X = 0$
But X is not a zero vector, $X^{\theta} X \neq 0$
$\therefore\ \lambda - \bar{\lambda} = 0$
$\therefore \lambda = \bar{\lambda}$.
Hence λ is real

2. If λ is an eigen value of A then λ^n is an eigen value of A^n corresponding to the same eigen vectors X, where n is a natural number.
[M15/ElexExtcElectBiomInst/5M][N17/ElexExtcElectBiomInst/5M]
Solution:
Consider,

$$[A - \lambda I]X = 0$$
$$AX - \lambda X = 0$$
$$AX = \lambda X$$

Where λ is an eigen value of A & X is the corresponding eigen vector
Now, Multiplying the above equation by A,

$$A(AX) = A(\lambda X)$$
$$A^2X = \lambda(AX)$$
$$A^2X = \lambda(\lambda X)$$
$$A^2X = \lambda^2 X$$

$\therefore$ $\lambda$is an eigen value of A^2

Similarly, $A^3X = \lambda^3 X \dots\dots\dots\dots\dots\dots A^nX = \lambda^n X$

$\therefore$ λ^n is an eigen value of A^n corresponding to the same eigen vectors X

3. Prove that the eigen values of a unitary matrix are of unit modulus.
[N15/ElexExtcElectBiomInst/5M]
Solution:
Let A be aunitary matrix and λ be an eigen value.
$\therefore AA^{\theta} = A^{\theta}A = I$
Let X is the eigen vector corresponding to λ then
$AX = \lambda X$
$(AX)^{\theta} = (\lambda X)^{\theta}$
$X^{\theta}A^{\theta} = \lambda X^{\theta}$
Now,
$(X^{\theta}A^{\theta})(AX) = (\lambda X^{\theta})(\lambda X)$
$X^{\theta}(A^{\theta}A)X = \lambda^2 X^{\theta}X$
$X^{\theta}IX = \lambda^2 X^{\theta}X$
$X^{\theta}X = \lambda^2 X^{\theta}X$
$X^{\theta}X(\lambda^2 - 1) = 0$
But $X^{\theta}X \neq 0, \therefore \lambda^2 - 1 = 0, i.e. \lambda = \pm 1$

4. If λ & X are eigen values and eigen vectors of A then prove that $\frac{1}{\lambda}$ and X are eigen values and eigen vectors of A^{-1}, provided A is non singular matrix **[M16/ElexExtcElectBiomInst/5M]**

Solution:

Consider,

$$[A - \lambda I]X = 0$$
$$AX - \lambda X = 0$$
$$AX = \lambda X$$

Where λ is an eigen value of A & X is the corresponding eigen vector

Now, Multiplying the above equation by A^{-1},

$$A^{-1}(AX) = A^{-1}(\lambda X)$$
$$X = \lambda(A^{-1}X)$$
$$\frac{1}{\lambda}X = A^{-1}X$$

$\therefore \frac{1}{\lambda}$ is an eigen value of A^{-1}

5. If $f(x)$ is an algebraic polynomial in x and λ is an eigen value and X is the corresponding eigen vector of a square matrix A then $f(\lambda)$ is an eigen value and X is the corresponding eigen vector of $f(A)$.

[N16/ElexExtcElectBiomInst/5M]

Solution:

Consider,

$$[A - \lambda I]X = 0$$
$$AX - \lambda X = 0$$
$$AX = \lambda X$$

Where λ is an eigen value of A & X is the corresponding eigen vector

Now, Multiplying the above equation by A,

$$A(AX) = A(\lambda X)$$
$$A^2X = \lambda(AX)$$
$$A^2X = \lambda(\lambda X)$$
$$A^2X = \lambda^2 X$$

$\therefore \lambda$ is an eigen value of A^2

Similarly, $A^3X = \lambda^3 X \ldots\ldots\ldots\ldots\ldots\ldots A^nX = \lambda^n X$

$\therefore \lambda^n$ is an eigen value of A^n corresponding to the same eigen vectors X

Let $f(x) = a_0 + a_1x + a_2x^2 + \cdots \ldots . + a_nx^n$

Then $f(A) = a_0I + a_1A + a_2A^2 + \cdots \ldots . + a_nA^n$

$$\therefore f(A)X = (a_0I + a_1A + \cdots \ldots .. + a_nA^n)X$$
$$= a_0X + a_1AX + \cdots \ldots . + a_nA^nX$$
$$= a_0X + a_1\lambda X + \cdots \ldots .. + a_n\lambda^nX$$
$$= (a_0 + a_1\lambda + \cdots \ldots \ldots + a_n\lambda^n)X$$

$\therefore f(A)X = f(\lambda)X$

Thus, $f(\lambda)$ is an eigen value and X is the corresponding eigen vector of $f(A)$