Residues

Type I: Calculation of Residues at Poles

1. Determine the pole of the function $f(z) = \frac{z^2}{(z-1)^2(z+2)}$ and also find the residue at each pole

[N13/Chem/6M]

Solution:

We have, $f(z) = \frac{z^2}{(z-1)^2(z+2)}$

For singularity,

$(z-1)^2(z+2) = 0$

$\therefore z = 1,1,-2$

$\therefore z = -2$ is a simple pole and $z = 1$ is a pole of order 2

Residue of $f(z)$ at $(z = -2) = \lim_{z\to -2}(z+2)f(z)$

$$= \lim_{z\to -2}(z+2)\frac{z^2}{(z-1)^2(z+2)}$$

$$= \lim_{z\to -2}\frac{z^2}{(z-1)^2}$$

$$= \frac{(-2)^2}{(-2-1)^2}$$

$$= \frac{4}{9}$$

Residue of $f(z)$ at $(z = 1) = \frac{1}{1!}\lim_{z\to 1}\frac{d}{dz}[(z-1)^2 f(z)]$

$$= \lim_{z\to 1}\frac{d}{dz}\left[(z-1)^2\frac{z^2}{(z-1)^2(z+2)}\right]$$

$$= \lim_{z\to 1}\frac{d}{dz}\left[\frac{z^2}{z+2}\right]$$

$$= \lim_{z\to 1}\left[\frac{(z+2)(2z)-z^2(1)}{(z+2)^2}\right]$$

$$= \frac{(1+2)(2)-1^1}{(1+2)^2}$$

$$= \frac{5}{9}$$

2. Find the sum of residues at singular points of $f(z) = \frac{z}{(z-1)^2(z^2-1)}$

[N14/ChemBiot/7M][N16/ElexExtcElectBiomInst/6M]

Solution:

We have, $f(z) = \frac{z}{(z-1)^2(z^2-1)}$

For singularity,

$(z-1)^2(z^2-1) = 0$

$(z-1)^2(z-1)(z+1) = 0$

$\therefore z = 1,1,1,-1$

$\therefore z = -1$ is a simple pole and $z = 1$ is a pole of order 3

Residue of $f(z)$ at $(z = -1) = \lim_{z\to -1}(z+1)f(z)$

$$= \lim_{z\to -1}(z+1)\frac{z}{(z-1)^2(z^2-1)}$$

$$= \lim_{z\to -1}\frac{z}{(z-1)^3}$$

$$= \frac{-1}{(-1-1)^3}$$

$$= \frac{1}{8}$$

Residue of $f(z)$ at $(z = 1) = \frac{1}{2!}\lim_{z\to 1}\frac{d^2}{dz^2}[(z-1)^3 f(z)]$

$$= \frac{1}{2}\lim_{z\to 1}\frac{d^2}{dz^2}\left[(z-1)^3\frac{z}{(z-1)^3(z+1)}\right]$$

$$= \frac{1}{2}\lim_{z\to 1}\frac{d^2}{dz^2}\left[\frac{z}{z+1}\right]$$

$$= \frac{1}{2}\lim_{z\to 1}\frac{d}{dz}\left[\frac{(z+1)(1)-z(1)}{(z+1)^2}\right]$$

$$= \frac{1}{2}\lim_{z\to 1}\frac{d}{dz}\left[\frac{1}{(z+1)^2}\right]$$

$$= \frac{1}{2}\lim_{z\to 1} -\frac{2}{(z+1)^3}$$

$$= \frac{1}{2}. -\frac{2}{(1+1)^3}$$

$$= -\frac{1}{8}$$

Sum of Residues $= \frac{1}{8} - \frac{1}{8} = 0$

3. Find the residues of the function $f(z) = \frac{sin\pi\, z^2 + cos\pi\, z^2}{(z-1)(z-2)^2}$ at their poles.

[N15/AutoMechCivil/6M]

Solution:

We have, $f(z) = \frac{sin\pi\, z^2 + cos\pi\, z^2}{(z-1)(z-2)^2}$

For singularity,

$(z-1)(z-2)^2 = 0$

$\therefore z = 1, z = 2,2$

$\therefore z = 1$ is a simple pole and $z = 2$ is a pole of order 2

Residue of $f(z)$ at $(z = 1) = \lim_{z\to 1}(z-1)f(z)$

$$= \lim_{z\to 1}(z-1)\frac{sin\pi\, z^2 + cos\pi\, z^2}{(z-1)(z-2)^2}$$

$$= \lim_{z\to 1}\frac{sin\pi\, z^2 + cos\pi\, z^2}{(z-2)^2}$$

$$= \frac{sin\pi + cos\pi}{(1-2)^2}$$

$$= \frac{0-1}{(-1)^2} = -1$$

Residue of $f(z)$ at $(z = 2) = \frac{1}{1!}\lim_{z\to 2}\frac{d}{dz}[(z-2)^2 f(z)]$

$$= \lim_{z\to 2}\frac{d}{dz}\left[(z-2)^2\frac{sin\pi\, z^2 + cos\pi\, z^2}{(z-1)(z-2)^2}\right]$$

$$= \lim_{z\to 2}\frac{d}{dz}\left[\frac{sin\pi\, z^2 + cos\pi\, z^2}{(z-1)}\right]$$

$$= \lim_{z\to 2}\left[\frac{(z-1)(cos\pi\, z^2 \times 2\pi z - sin\pi\, z^2 \times 2\pi z) - (sin\pi\, z^2 + cos\pi\, z^2)(1)}{(z-1)^2}\right]$$

$$= \left[\frac{(1)(4\pi\ cos\,4\pi - 4\pi\ sin\,4\pi) - (sin\,4\pi + cos\,4\pi)}{(2-1)^2}\right]$$

$$= 4\pi - 1$$

4. Find the residues of the function $f(z) = \frac{z}{(z-1)(z+2)^2}$ at their poles.

[N15/ChemBiot/6M]

Solution:

We have, $f(z) = \frac{z}{(z-1)(z+2)^2}$

For singularity,

$(z-1)(z+2)^2 = 0$

$\therefore z = 1, z = -2, -2$

$\therefore z = 1$ is a simple pole and $z = -2$ is a pole of order 2

Residue of $f(z)$ at $(z = 1) = \lim_{z\to 1}(z-1)f(z)$

$$= \lim_{z\to 1}(z-1)\frac{z}{(z-1)(z+2)^2}$$

$$= \lim_{z\to 1}\frac{z}{(z+2)^2}$$

$$= \frac{1}{(1+2)^2}$$

$$= \frac{1}{9}$$

Residue of $f(z)$ at $(z = -2) = \frac{1}{1!}\lim_{z\to -2}\frac{d}{dz}[(z+2)^2 f(z)]$

$$= \lim_{z\to -2}\frac{d}{dz}\left[(z+2)^2\frac{z}{(z-1)(z+2)^2}\right]$$

$$= \lim_{z\to -2}\frac{d}{dz}\left[\frac{z}{(z-1)}\right]$$

$$= \lim_{z\to -2}\left[\frac{(z-1)(1)-(z)(1)}{(z-1)^2}\right]$$

$$= \frac{(-2-1)-(-2)}{(-2-1)^2}$$

$$= -\frac{1}{9}$$

5. Find the sum of residues at singular points of $f(z)=\frac{z-4}{z(z-1)(z-2)}$

[M17/ElexExtcElectBiomInst/6M]

Solution:

We have, $f(z)=\frac{z-4}{z(z-1)(z-2)}$

For singularity,

$z(z-1)(z-2)=0$

$\therefore z=0, z=1, z=2$

$\therefore$ all are simple poles

Residue of $f(z)$ at $(z=0)=\lim_{z\to 0}(z-0)f(z)$

$=\lim_{z\to 0}(z-0)\frac{z-4}{z(z-1)(z-2)}$

$=\lim_{z\to 0}\frac{z-4}{(z-1)(z-2)}$

$=\frac{-4}{(-1)(-2)}$

$=-2$

Residue of $f(z)$ at $(z=1)=\lim_{z\to 1}(z-1)f(z)$

$=\lim_{z\to 1}(z-1)\frac{z-4}{z(z-1)(z-2)}$

$=\lim_{z\to 1}\frac{z-4}{z(z-2)}$

$=\frac{-3}{(1)(-1)}$

$=3$

Residue of $f(z)$ at $(z=2)=\lim_{z\to 2}(z-2)f(z)$

$=\lim_{z\to 2}(z-2)\frac{z-4}{z(z-1)(z-2)}$

$=\lim_{z\to 2}\frac{z-4}{z(z-1)}$

$=\frac{-2}{(2)(1)}$

$=-1$

Sum of residues $=-2+3-1=0$

Type II: Cauchy's Residue Theorem

1. Evaluate $\int_c \frac{\sin\pi\, z^2+\cos\pi\, z^2}{(z-1)(z-2)} dz$ where C is the circle $|z| = 3$

[M15/ElexExtcElectBiomInst/6M]

Solution:

We have, $f(z) = \frac{\sin\pi\, z^2+\cos\pi\, z^2}{(z-1)(z-2)}$

For singularity,

$(z-1)(z-2) = 0$

$\therefore z = 1, z = 2$

We see that $z = 1$ and $z = 2$ both lies inside $C: |z| = 3$ and hence are simple poles.

Residue of $f(z)$ at $(z = 1) = \lim_{z\to 1}(z-1)f(z)$

$$= \lim_{z\to 1}(z-1)\frac{\sin\pi\, z^2+\cos\pi\, z^2}{(z-1)(z-2)}$$

$$= \lim_{z\to 1}\frac{\sin\pi\, z^2+\cos\pi\, z^2}{(z-2)}$$

$$= \frac{\sin\pi + \cos\pi}{(1-2)}$$

$$= \frac{0-1}{-1} = 1$$

Residue of $f(z)$ at $(z = 2) = \lim_{z\to 2}(z-2)f(z)$

$$= \lim_{z\to 2}(z-2)\frac{\sin\pi\, z^2+\cos\pi\, z^2}{(z-1)(z-2)}$$

$$= \lim_{z\to 2}\frac{\sin\pi\, z^2+\cos\pi\, z^2}{(z-1)}$$

$$= \frac{\sin 4\pi + \cos 4\pi}{(2-1)}$$

$$= \frac{0+1}{1} = 1$$

By Cauchys Residue Theorem,

$$\int_c f(z)dz = 2\pi i\,[sum\ of\ residues]$$

$$\int_c \frac{\sin\pi\, z^2+\cos\pi\, z^2}{(z-1)(z-2)} dz = 2\pi i\,[1+1] = 4\pi i$$

2. Evaluate using Cauchys Residue theorem $\oint_c \frac{1-2z}{z(z-1)(z-2)} dz$ where c is $|z| = 1.5$

[N15/CompIT/6M]

Solution:

We have, $f(z) = \frac{1-2z}{z(z-1)(z-2)}$

For singularity,

$z(z-1)(z-2) = 0$

$\therefore z = 0, z = 1, z = 2$

We see that $z = 0$ and $z = 1$ both lies inside $C: |z| = 1.5$ and hence are simple poles.

Residue of $f(z)$ at $(z = 0) = \lim_{z\to 0}(z-0)f(z)$

$$= \lim_{z\to 0}(z-0)\frac{1-2z}{z(z-1)(z-2)}$$

$$= \lim_{z\to 0}\frac{1-2z}{(z-1)(z-2)}$$

$$= \frac{1-0}{(0-1)(0-2)}$$

$$= \frac{1}{2}$$

Residue of $f(z)$ at $(z = 1) = \lim_{z\to 1}(z-1)f(z)$

$$= \lim_{z\to 1}(z-1)\frac{1-2z}{z(z-1)(z-2)}$$

$$= \lim_{z\to 1}\frac{1-2z}{z(z-2)}$$

$$= \frac{1-2(1)}{1(1-2)}$$

$$= \frac{-1}{-1} = 1$$

By Cauchys Residue Theorem,

$$\int_c f(z)dz = 2\pi i\,[sum\ of\ residues]$$

$$\int_c \frac{1-2z}{z(z-1)(z-2)}dz = 2\pi i\left[\frac{1}{2}+1\right] = 2\pi i\left[\frac{3}{2}\right] = 3\pi i$$

3. Evaluate using Cauchy residue theorem $\int_c \frac{12z-7}{z(2z+1)(z+2)}dz$ where C is $|z| = 1$

[M16/CompIT/6M]

Solution:

We have, $f(z) = \frac{12z-7}{z(2z+1)(z+2)}$

For singularity,

$z(2z+1)(z+2) = 0$

$\therefore z = 0, z = -\frac{1}{2}, z = -2$

We see that $z = 0$ and $z = -\frac{1}{2}$ both lies inside $C: |z| = 1$ and hence are simple poles.

Residue of $f(z)$ at $(z = 0) = \lim_{z\to 0}(z-0)f(z)$

$$= \lim_{z\to 0}(z-0)\frac{12z-7}{z(2z+1)(z+2)}$$

$$= \lim_{z\to 0}\frac{12z-7}{(2z+1)(z+2)}$$

$$= \frac{0-7}{(0+1)(0+2)}$$

$$= -\frac{7}{2}$$

Residue of $f(z)$ at $\left(z = -\frac{1}{2}\right) = \lim_{z\to -\frac{1}{2}}\left(z+\frac{1}{2}\right)f(z)$

$$= \lim_{z\to -\frac{1}{2}}\left(z+\frac{1}{2}\right)\frac{12z-7}{z(2z+1)(z+2)}$$

$$= \lim_{z\to -\frac{1}{2}}\frac{2z+1}{2}.\frac{12z-7}{z(2z+1)(z+2)}$$

$$= \frac{1}{2}\lim_{z\to -\frac{1}{2}}\frac{12z-7}{z(z+2)}$$

$$= \frac{1}{2}.\frac{12\left(-\frac{1}{2}\right)-7}{-\frac{1}{2}\left(-\frac{1}{2}+2\right)}$$

$$= \frac{1}{2}.\frac{-13}{-\frac{3}{4}}$$

$$= \frac{26}{3}$$

By Cauchys Residue Theorem,

$$\int_c f(z)dz = 2\pi i\,[sum\ of\ residues]$$

$$\int_c \frac{12z-7}{z(2z+1)(z+2)}dz = 2\pi i\left[-\frac{7}{2}+\frac{26}{3}\right] = 2\pi i\left[\frac{31}{6}\right] = \frac{31\pi i}{3}$$

4. Evaluate $\oint_c \frac{z^2}{(z-1)^2(z+1)}dz$ where c is $|z| = 2$ using residue theorem
[N16/CompIT/6M]
Solution:
We have, $f(z) = \frac{z^2}{(z-1)^2(z+1)}$
For singularity,
$(z-1)^2(z+1) = 0$
$(z-1)^2(z+1) = 0$
$\therefore z = 1,1,-1$
$\therefore z = -1$ is a simple pole and $z = 1$ is a pole of order 2
Residue of $f(z)$ at $(z = -1) = \lim_{z\to -1}(z+1)f(z)$

$$= \lim_{z\to -1}(z+1)\frac{z^2}{(z-1)^2(z+1)}$$
$$= \lim_{z\to -1}\frac{z^2}{(z-1)^2}$$
$$= \frac{(-1)^2}{(-1-1)^2}$$
$$= \frac{1}{4}$$

Residue of $f(z)$ at $(z = 1) = \frac{1}{1!}\lim_{z\to 1}\frac{d}{dz}[(z-1)^2 f(z)]$

$$= \lim_{z\to 1}\frac{d}{dz}\left[(z-1)^2\frac{z^2}{(z-1)^2(z+1)}\right]$$
$$= \lim_{z\to 1}\frac{d}{dz}\left[\frac{z^2}{z+1}\right]$$
$$= \lim_{z\to 1}\left[\frac{(z+1)(2z)-z^2(1)}{(z+1)^2}\right]$$
$$=. \frac{4-1}{(1+1)^2}$$
$$= \frac{3}{4}$$

Sum of Residues $= \frac{1}{4} + \frac{3}{4} = 1$
By Cauchys Residue Theorem,
$\int_c f(z)dz = 2\pi i\,[sum\ of\ residues]$
$\oint_c \frac{z^2}{(z-1)^2(z+1)}dz = 2\pi i\,(1) = 2\pi i$

5. Using Cauchy's residue theorem evaluate $\oint_c \frac{z^2+3}{z^2-1} dz$ where C is the circle (i) $|z-1| = 1$ (ii) $|z+1| = 1$

[N16/ElexExtcElectBiomInst/8M]

Solution:

We have, $f(z) = \frac{z^2+3}{z^2-1}$

For singularity,

$z^2 - 1 = 0$

$(z-1)(z+1) = 0$

$\therefore z = 1, -1$

(i) C is $|z-1| = 1$

We see that, $z = 1$ is a simple pole and $z = -1$ lies outside C

Residue of $f(z)$ at $(z=1) = \lim_{z\to 1}(z-1)f(z)$

$$= \lim_{z\to 1}(z-1)\frac{z^2+3}{(z+1)(z-1)}$$

$$= \lim_{z\to 1}\frac{z^2+3}{z+1}$$

$$= \frac{4}{2}$$

$$= 2$$

By Cauchys Residue Theorem,

$\int_c f(z)dz = 2\pi i\ [sum\ of\ residues]$

$\oint_c \frac{z^2+3}{z^2-1} dz = 2\pi i\ (2) = 4\pi i$

(ii) C is $|z+1| = 1$

We see that, $z = -1$ is a simple pole and $z = 1$ lies outside C

Residue of $f(z)$ at $(z=-1) = \lim_{z\to -1}(z+1)f(z)$

$$= \lim_{z\to -1}(z+1)\frac{z^2+3}{(z+1)(z-1)}$$

$$= \lim_{z\to -1}\frac{z^2+3}{z-1}$$

$$= \frac{4}{-2}$$

$$= -2$$

By Cauchys Residue Theorem,

$\int_c f(z)dz = 2\pi i\ [sum\ of\ residues]$

$\oint_c \frac{z^2+3}{z^2-1} dz = 2\pi i\ (-2) = -4\pi i$

6. Using Residue theorem, evaluate $\int_c \frac{e^z}{(z^2+\pi^2)^2} dz$ where c is $|z| = 4$

[N16/AutoMechCivil/6M]

Solution:

We have, $f(z) = \frac{e^z}{(z^2+\pi^2)^2}$

For singularity,

$(z^2 + \pi^2)^2 = 0$

$(z + \pi i)^2(z - \pi i)^2 = 0$

$\therefore z = \pm\pi i, \pm\pi i$

$\therefore z = \pi i$ and $z = -\pi i$ are poles of order 2

Residue of $f(z)$ at $(z = \pi i) = \frac{1}{1!}\lim_{z\to\pi i} \frac{d}{dz}[(z - \pi i)^2 f(z)]$

$$= \lim_{z\to\pi i} \frac{d}{dz}\left[(z - \pi i)^2 \frac{e^z}{(z+\pi i)^2(z-\pi i)^2}\right]$$

$$= \lim_{z\to\pi i} \frac{d}{dz}\left[\frac{e^z}{(z+\pi i)^2}\right]$$

$$= \lim_{z\to\pi i} \left[\frac{(z+\pi i)^2(e^z)-e^z.2(z+\pi i)}{(z+\pi i)^4}\right]$$

$$= \frac{e^{\pi i}(2\pi i-2)}{(2\pi i)^3}$$

$$= \frac{(cos\pi + isin\pi)(2\pi i-2)}{8\pi^3 i^3}$$

$$= \frac{(-1)(2\pi i-2)}{-8\pi^3 i}$$

$$= \frac{\pi i-1}{4\pi^3 i}$$

$\therefore$ Residue of $f(z)$ at $(z = -\pi i) = \frac{-\pi i-1}{-4\pi^3 i} = \frac{\pi i+1}{4\pi^3 i}$

By Cauchys Residue Theorem,

$$\int_c f(z)dz = 2\pi i\ [sum\ of\ residues]$$

$$\oint_c \frac{e^z}{(z^2+\pi^2)^2} dz = 2\pi i\left(\frac{\pi i+1}{4\pi^3 i} + \frac{\pi i-1}{4\pi^3 i}\right) = 2\pi i\left[\frac{2\pi i}{4\pi^3 i}\right] = \frac{i}{\pi}$$

7. Using Residue theorem, evaluate $\int_c \frac{e^z}{z^2+\pi^2}dz$ where c is $|z|=4$
[M17/ElexExtcElectBiomInst/6M] Ans. 0

Solution:

We have, $f(z)=\frac{e^z}{z^2+\pi^2}$

For singularity,

$z^2+\pi^2=0$

$(z+\pi i)(z-\pi i)=0$

$\therefore z=\pm\pi i$

$\therefore z=\pi i$ and $z=-\pi i$ are simple poles

Residue of $f(z)$ at $(z=\pi i)=\lim_{z\to\pi i}(z-\pi i)f(z)$

$$=\lim_{z\to\pi i}(z-\pi i)\frac{e^z}{(z+\pi i)(z-\pi i)}$$

$$=\lim_{z\to\pi i}\frac{e^z}{z+\pi i}$$

$$=\frac{e^{\pi i}}{2\pi i}$$

$\therefore$ Residue of $f(z)$ at $(z=-\pi i)=\frac{e^{-\pi i}}{-2\pi i}$

By Cauchys Residue Theorem,

$\int_c f(z)dz=2\pi i\,[sum\ of\ residues]$

$\oint_c \frac{e^z}{z^2+\pi^2}dz=2\pi i\left(\frac{e^{\pi i}-e^{-\pi i}}{2\pi i}\right)=2i\ sin\pi=0$

8. Evaluate $\int_c \frac{cotz}{z} dz$ where C is the ellipse $9x^2 + 4y^2 = 1$

[N17/CompIT/6M]

Solution:

We have, $f(z) = \frac{cotz}{z}$

For singularity,

$z = 0$

$z = 0$ is a simple pole

Residue of $f(z)$ at $(z = 0) = \lim_{z\to 0}(z - 0)f(z)$

$= \lim_{z\to 0}(z - 0)\frac{cotz}{z}$

$= \lim_{z\to 0} cotz$

$= cot0$

$= \infty$

By Cauchys Residue Theorem,

$\int_c f(z)dz = 2\pi i\ [sum\ of\ residues]$

$\oint_c \frac{cotz}{z} dz = 2\pi i\ (\infty) = does\ not\ exist$

Type III: Application of Residues

1. Evaluate $\int_0^{2\pi} \frac{d\theta}{2+cos\theta}$ using Residue theorem

   [N13/Biot/6M]

   Solution:

   We have,

   $I = \int_0^{2\pi} \frac{1}{2+cos\theta} d\theta$

   put $z = e^{i\theta}$

   $d\theta = \frac{dz}{iz}$

   $cos\theta = \frac{z^2+1}{2z}$

   $I = \int \frac{1}{2+\left(\frac{z^2+1}{2z}\right)} \frac{dz}{iz}$

   $I = \int_c \frac{1}{\left(\frac{4z+z^2+1}{2z}\right)} \frac{dz}{iz}$ where $C\ is\ |z| = 1$

   $I = \int_c \frac{2}{i(z^2+4z+1)} dz$

   $I = \int_c \frac{-2i}{(z^2+4z+1)} dz$

   Put $(z^2 + 4z + 1) = 0$

   $z = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-4\pm\sqrt{4^2-4}}{2} = \frac{-4\pm\sqrt{12}}{2} = -2 \pm \sqrt{3}$

   $z = -2 + \sqrt{3} = \alpha \& z = -2 - \sqrt{3} = \beta$

   $\therefore z = \alpha$ lies inside C and is a simple pole

   Residue of $f(z)$ at $(z = \alpha) = \lim_{z\to\alpha}(z - \alpha)\frac{-2i}{(z-\alpha)(z-\beta)}$

   $= \frac{-2i}{(\alpha-\beta)} = \frac{-2i}{2\sqrt{3}} = -\frac{i}{\sqrt{3}}$

   Now, $\int_0^{2\pi} \frac{1}{2+cos\theta} d\theta = 2\pi i\ [sum\ of\ residues] = 2\pi i\left[-\frac{i}{\sqrt{3}}\right] = \frac{2\pi}{\sqrt{3}}$

2. Evaluate $\int_0^{2\pi} \frac{d\theta}{25-16\cos^2\theta}$

[N13/AutoMechCivil/8M]

Solution:

We have, $I = \int_0^{2\pi} \frac{d\theta}{25-16\cos^2\theta}$

put $z = e^{i\theta}, d\theta = \frac{dz}{iz}, sin\theta = \frac{z^2-1}{2iz}, cos\theta = \frac{z^2+1}{2z}$

$$I = \int \frac{1}{25-16\left(\frac{z^2+1}{2z}\right)^2}\frac{dz}{iz}$$

$$I = \int_c \frac{1}{25-4\left(\frac{(z^2+1)^2}{z^2}\right)}\frac{dz}{iz} \text{ where } C \text{ is } |z| = 1$$

$$I = \int_c \frac{z^2}{25z^2-4(z^2+1)^2}\frac{dz}{iz}$$

$$I = \int_c \frac{\frac{z}{-4i}}{(z^2+1)^2-\frac{25}{4}z^2}dz$$

Put $(z^2+1)^2 - \frac{25}{4}z^2 = 0$

$$\left(z^2+1-\frac{5}{2}z\right)\left(z^2+1+\frac{5}{2}z\right) = 0$$

$$z^2 - \frac{5}{2}z + 1 = 0, z^2 + \frac{5}{2}z + 1 = 0$$

$$z = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{\frac{5}{2}\pm\sqrt{\frac{25}{4}-4}}{2} = \frac{\frac{5}{2}\pm\frac{3}{2}}{2} = 2 \; or \; \frac{1}{2}$$

$$z = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-\frac{5}{2}\pm\sqrt{\frac{25}{4}-4}}{2} = \frac{-\frac{5}{2}\pm\frac{3}{2}}{2} = -2 \; or -\frac{1}{2}$$

$\therefore z = \frac{1}{2}$ and $z = -\frac{1}{2}$ are simple poles

Residue of $f(z)$ at $\left(z = \frac{1}{2}\right) = \lim_{z\to\frac{1}{2}}(z-\frac{1}{2})\frac{\frac{z}{-4i}}{(z-2)\left(z-\frac{1}{2}\right)(z+2)\left(z+\frac{1}{2}\right)}$

$$= \frac{\frac{\frac{1}{2}}{-4i}}{\left(\frac{1}{2}-2\right)\left(\frac{1}{2}+2\right)\left(\frac{1}{2}+\frac{1}{2}\right)} = \frac{-\frac{1}{8i}}{\left(-\frac{3}{2}\right)\left(\frac{5}{2}\right)(1)} = \frac{-\frac{1}{8i}}{-\frac{15}{4}} = \frac{1}{30i}$$

Residue of $f(z)$ at $\left(z = \frac{-1}{2}\right) = \lim_{z\to-\frac{1}{2}}(z+\frac{1}{2})\frac{\frac{z}{-4i}}{(z-2)\left(z-\frac{1}{2}\right)(z+2)\left(z+\frac{1}{2}\right)}$

$$= \frac{\frac{-\frac{1}{2}}{-4i}}{\left(-\frac{1}{2}-2\right)\left(-\frac{1}{2}+2\right)\left(-\frac{1}{2}-\frac{1}{2}\right)} = \frac{\frac{1}{8i}}{\left(\frac{3}{2}\right)\left(-\frac{5}{2}\right)(-1)} = \frac{\frac{1}{8i}}{\frac{15}{4}} = \frac{1}{30i}$$

Now, $\int_0^{2\pi} \frac{d\theta}{25-16\cos^2\theta} = 2\pi i \; [sum \; of \; residues] = 2\pi i\left[\frac{1}{30i}+\frac{1}{30i}\right] = \frac{2\pi}{15}$

3. Evaluate $\int_0^{\pi} \frac{1}{3+2cos\theta} d\theta$

[N13/Chem/6M][M16/AutoMechCivil/6M][N17/ElexExtcElectBiomInst/6M]

Solution:

We have, $I = \int_0^{2\pi} \frac{1}{3+2cos\theta} d\theta$

put $z = e^{i\theta}, d\theta = \frac{dz}{iz}, sin\theta = \frac{z^2-1}{2iz}, cos\theta = \frac{z^2+1}{2z}$

$I = \int \frac{1}{3+2\left(\frac{z^2+1}{2z}\right)} \frac{dz}{iz}$

$I = \int_c \frac{1}{\left(\frac{3z+z^2+1}{z}\right)} \frac{dz}{iz}$ where $C\ is\ |z| = 1$

$I = \int_c \frac{1}{i(z^2+3z+1)} dz$

$I = \int_c \frac{-i}{(z^2+3z+1)} dz$

$I = \int_c \frac{-i}{(z^2+3z+1)} dz$

Put $(z^2 + 3z + 1) = 0$

$z = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-3 \pm \sqrt{3^2-4}}{2} = \frac{-3 \pm \sqrt{5}}{2}$

$z = \frac{-3+\sqrt{5}}{2} = \alpha \,\&\, z = \frac{-3-\sqrt{5}}{2} = \beta$

$\therefore$ $z = \alpha$ lies inside C and is a simple pole

Residue of $f(z)$ at $(z = \alpha) = \lim_{z\to\alpha} (z-\alpha) \frac{-i}{(z-\alpha)(z-\beta)}$

$$= \frac{-i}{(\alpha-\beta)} = \frac{-i}{\frac{2\sqrt{5}}{2}} = -\frac{i}{\sqrt{5}}$$

Now, $\int_0^{2\pi} \frac{1}{3+2cos\theta} d\theta = 2\pi i\ [sum\ of\ residues] = 2\pi i \left[-\frac{i}{\sqrt{5}}\right] = \frac{2\pi}{\sqrt{5}}$

$\therefore \int_0^{\pi} \frac{d\theta}{3+2cos\theta} = \frac{1}{2}\int_0^{2\pi} \frac{1}{3+2cos\theta} d\theta = \frac{1}{2} . \frac{2\pi}{\sqrt{5}} = \frac{\pi}{\sqrt{5}}$

4. Evaluate $\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta} d\theta$

[M14/CompIT/6M][M14/ChemBiot/8M][N16/ChemBiot/8M]

Solution:

We have, $I = \int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta} d\theta$

$e^{i2\theta} = \cos 2\theta + i\sin 2\theta$

$\cos 2\theta$ is a real part of $e^{i2\theta}$

$I = \int_0^{2\pi} \frac{R.P.of\ e^{i2\theta}}{5+4\cos\theta} d\theta$

put $z = e^{i\theta}, d\theta = \frac{dz}{iz}, \sin\theta = \frac{z^2-1}{2iz}, \cos\theta = \frac{z^2+1}{2z}$

$I = \int \frac{R.P.of\ z^2}{5+4\left(\frac{z^2+1}{2z}\right)} \frac{dz}{iz}$

$I = R.P.of \int_c \frac{z^3}{5z+2z^2+2} . \frac{dz}{iz}$ where $C\ is\ |z| = 1$

$I = R.P.of \int_c \frac{z^3}{2\left(z^2+\frac{5}{2}z+1\right)} \frac{dz}{iz}$

$I = R.P.of \int_c \frac{-\frac{iz^2}{2}}{\left(z^2+\frac{5}{2}z+1\right)} dz$

Put $z^2 + \frac{5}{2}z + 1 = 0$

$\therefore z = -2, z = -\frac{1}{2}$

We see that, $z = -2$ lies outside C and $z = -\frac{1}{2}$ lies inside C

Residues of $f(z)$ at $\left(z = -\frac{1}{2}\right) = \lim_{z \to -\frac{1}{2}} \left(z + \frac{1}{2}\right) . \frac{-\frac{iz^2}{2}}{(z+2)\left(z+\frac{1}{2}\right)}$

$$= \frac{-\frac{i}{2}.\frac{1}{4}}{-\frac{1}{2}+2} = \frac{-\frac{i}{8}}{\frac{3}{2}} = -\frac{i}{12}$$

Now, $\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta} d\theta = R.P.of \int_0^{2\pi} \frac{e^{i2\theta}}{5+4\cos\theta} d\theta$

$$= R.P.of\ 2\pi i\ [sum\ of\ residues]$$

$$= R.P.of\ 2\pi i \left[-\frac{i}{12}\right] = R.P.of \frac{\pi}{6}$$

$$= \frac{\pi}{6}$$

5. Using calculus of residues prove that $\int_0^{2\pi} e^{cos\theta}\ \cos(sin\theta - n\theta)\, d\theta = \frac{2\pi}{n!}$

[M14/ElexExtcElectBiomInst/6M]

We have, $I = \int_0^{2\pi} e^{cos\theta}\ \cos(sin\theta - n\theta)\ d\theta$

$I = \int_0^{2\pi} e^{cos\theta}\ \ Real\ part\ of\ e^{isin\theta\ - in\theta}\, d\theta$

$I = RP\ of\ \int_0^{2\pi} e^{cos\theta\ + isin\theta}\,.\,e^{-in\theta}\, d\theta$

$I = RP\ of\ \int_0^{2\pi} \frac{e^{e^{i\theta}}}{\left(e^{i\theta}\right)^n}\, d\theta$

Put $e^{i\theta} = z, \therefore ie^{i\theta}\, d\theta = dz, \therefore d\theta = \frac{dz}{iz}$

$I = RP\ of\ \int_c \frac{e^z}{z^n}\frac{dz}{iz}$ where $C\ is\ |z| = 1$

$I = RP\ of\ \int_c \frac{e^z}{iz^{n+1}}\, dz$

Put $z^{n+1} = 0, \therefore z = 0$ is a pole of order $n + 1$

Residue of $f(z)$ at $(z = 0) = \frac{1}{n!}\lim_{z\to 0}\frac{d^n}{dz^n}\left[(z-0)^{n+1}\frac{e^z}{iz^{n+1}}\right]$

$= \frac{1}{in!}\lim_{z\to 0} e^z$

$= \frac{1}{in!}$

Now, $\int_0^{2\pi} e^{cos\theta}\ \cos(sin\theta - n\theta)\ d\theta = R.P.of \int_0^{2\pi} e^{cos\theta\ + isin\theta}\,.\,e^{-in\theta}\, d\theta$

$= R.P.of\ 2\pi i\ [sum\ of\ residues]$

$= R.P.of\ 2\pi i\left[\frac{1}{in!}\right] = R.P.of\ \frac{2\pi}{n!}$

$= \frac{2\pi}{n!}$

6. Evaluate $\int_0^{2\pi} \frac{d\theta}{(2+cos\theta\,)^2}$

[N14/ChemBiot/6M][N14/ElexExtcElectBiomInst/6M] [N15/AutoMechCivil/8M]

Solution:

We have, $I = \int_0^{2\pi} \frac{1}{(2+cos\theta\,)^2} d\theta$

put $z = e^{i\theta}, d\theta = \frac{dz}{iz}, sin\theta = \frac{z^2-1}{2iz}, cos\theta = \frac{z^2+1}{2z}$

$I = \int \frac{1}{\left(2+\frac{z^2+1}{2z}\right)^2} \frac{dz}{iz}$

$I = \int_c \frac{1}{\left(\frac{4z+z^2+1}{2z}\right)^2} \frac{dz}{iz}$ where $C\ is\ |z| = 1$

$I = \int_c \frac{4z^2}{iz(z^2+4z+1)^2} dz$

$I = \int_c \frac{-4iz}{(z^2+4z+1)^2} dz$

Put $(z^2 + 4z + 1)^2 = 0$

$z = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-4\pm\sqrt{4^2-4}}{2} = \frac{-4\pm\sqrt{12}}{2} = \frac{-4\pm2\sqrt{3}}{2} = -2 \pm \sqrt{3}$

$z = -2 + \sqrt{3} = \alpha \& z = -2 - \sqrt{3} = \beta$

$\therefore z = \alpha$ lies inside C and is a pole of order 2

Residue of $f(z)$ at $(z = \alpha) = \frac{1}{(2-1)!} \lim_{z\to\alpha} \frac{d}{dz}\left[(z-\alpha)^2 . \frac{-4iz}{(z-\alpha)^2(z-\beta)^2}\right]$

$$= \frac{1}{1!} \lim_{z\to\alpha} \frac{d}{dz}\left[\frac{-4iz}{(z-\beta)^2}\right]$$

$$= -4i \ \lim_{z\to\alpha}\left[\frac{(z-\beta)^2(1)-z\big(2(z-\beta)\big)}{(z-\beta)^4}\right]$$

$$= -4i\left[\frac{(\alpha-\beta)^2-2\alpha(\alpha-\beta)}{(\alpha-\beta)^4}\right]$$

$$= -4i\left[\frac{\left(2\sqrt{3}\right)^2-2\left(-2+\sqrt{3}\right)\left(2\sqrt{3}\right)}{\left(2\sqrt{3}\right)^2}\right]$$

$$= -4i\left[\frac{12+8\sqrt{3}-12}{144}\right]$$

$$= -\frac{2\sqrt{3}i}{9} = -\frac{2}{3\sqrt{3}}i$$

Now, $\int_0^{2\pi} \frac{1}{(2+cos\theta\,)^2} d\theta = 2\pi i\ [sum\ of\ residues] = 2\pi i\left[-\frac{2}{3\sqrt{3}}i\right] = \frac{4\pi}{3\sqrt{3}}$

7. Evaluate $\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta} d\theta$

[N14/AutoMechCivil/8M]

Solution:

We have, $I = \int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta} d\theta$

$e^{i3\theta} = \cos 3\theta + i\sin 3\theta$

$\cos 3\theta$ is a real part of $e^{i3\theta}$

$I = \int_0^{2\pi} \frac{R.P.of\ e^{i3\theta}}{5+4\cos\theta} d\theta$

put $z = e^{i\theta}, d\theta = \frac{dz}{iz}, \sin\theta = \frac{z^2-1}{2iz}, \cos\theta = \frac{z^2+1}{2z}$

$I = \int \frac{R.P.of\ z^3}{5-4\left(\frac{z^2+1}{2z}\right)} \frac{dz}{iz}$

$I = R.P.of \int_c \frac{z^4}{5z-2z^2-2} . \frac{dz}{iz}$ where $C\ is\ |z| = 1$

$I = R.P.of \int_c \frac{z^4}{-2\left(z^2-\frac{5}{2}z+1\right)} \frac{dz}{iz}$

$I = R.P.of \int_c \frac{\frac{iz^3}{2}}{\left(z^2-\frac{5}{2}z+1\right)} dz$

Put $z^2 - \frac{5}{2}z + 1 = 0$

$\therefore z = 2, z = \frac{1}{2}$

We see that, $z = 2$ lies outside C and $z = \frac{1}{2}$ lies inside C

Residues of $f(z)$ at $\left(z = \frac{1}{2}\right) = \lim_{z\to\frac{1}{2}} \left(z - \frac{1}{2}\right) . \frac{\frac{iz^3}{2}}{(z-2)\left(z-\frac{1}{2}\right)}$

$$= \frac{\frac{i}{2}.\left(\frac{1}{8}\right)}{\frac{1}{2}-2} = \frac{\frac{i}{16}}{-\frac{3}{2}} = -\frac{i}{24}$$

Now, $\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta} d\theta = R.P.of \int_0^{2\pi} \frac{e^{i3\theta}}{5-4\cos\theta} d\theta$

$$= R.P.of\ 2\pi i\ [sum\ of\ residues]$$

$$= R.P.of\ 2\pi i \left[-\frac{i}{24}\right] = R.P.of\ \frac{\pi}{12}$$

$$= \frac{\pi}{12}$$

8. Evaluate $\int_0^{2\pi} \frac{1}{5+3sin\theta} d\theta$

[M15/AutoMechCivil/6M][M17/CompIT/6M]

Solution:

We have, $I = \int_0^{2\pi} \frac{1}{5+3sin\theta} d\theta$

put $z = e^{i\theta}, d\theta = \frac{dz}{iz}, sin\theta = \frac{z^2-1}{2iz}, cos\theta = \frac{z^2+1}{2z}$

$I = \int \frac{1}{5+3\left(\frac{z^2-1}{2iz}\right)} \frac{dz}{iz}$

$I = \int_c \frac{1}{\left(\frac{10iz+3z^2-3}{2iz}\right)} \frac{dz}{iz}$ where $C\ is\ |z| = 1$

$I = \int_c \frac{2}{(3z^2+10iz-3)} dz$

$I = \int_c \frac{2}{3\left(z^2+\frac{10i}{3}z-1\right)} dz$

$I = \int_c \frac{\frac{2}{3}}{z^2+\frac{10i}{3}z-1} dz$

Put $\left(z^2 + \frac{10i}{3} z - 1\right) = 0$

$z = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-\frac{10i}{3}\pm\sqrt{\frac{100i^2}{9}+4}}{2} = \frac{-\frac{10i}{3}\pm\sqrt{-\frac{64}{9}}}{2} = \frac{-\frac{10i}{3}\pm\frac{8i}{3}}{2} = \frac{-10i\pm 8i}{6}$

$z = \frac{-i}{3}, -3i$

$\therefore z = \frac{-i}{3}$ lies inside C and is a simple pole

Residue of $f(z)$ at $\left(z = \frac{-i}{3}\right) = \lim_{z\to\frac{-i}{3}} (z + \frac{i}{3}) \frac{\frac{2}{3}}{\left(z+\frac{i}{3}\right)(z+3i)}$

$$= \frac{\frac{2}{3}}{\left(\frac{-i}{3}+3i\right)} = \frac{\frac{2}{3}}{\frac{8i}{3}} = \frac{1}{4i}$$

Now, $\int_0^{2\pi} \frac{1}{5+3sin\theta} d\theta = 2\pi i\ [sum\ of\ residues] = 2\pi i \left[\frac{1}{4i}\right] = \frac{\pi}{2}$

9. Evaluate $\int_0^{2\pi} \frac{d\theta}{13+5\sin\theta}$

[M15/ElexExtcElectBiomInst/6M]

Solution:

We have, $I = \int_0^{2\pi} \frac{1}{13+5sin\theta} d\theta$

put $z = e^{i\theta}, d\theta = \frac{dz}{iz}, sin\theta = \frac{z^2-1}{2iz}, cos\theta = \frac{z^2+1}{2z}$

$$I = \int \frac{1}{13+5\left(\frac{z^2-1}{2iz}\right)} \frac{dz}{iz}$$

$I = \int_c \frac{1}{\left(\frac{26iz+5z^2-5}{2iz}\right)} \frac{dz}{iz}$ where $C\ is\ |z| = 1$

$$I = \int_c \frac{2}{(5z^2+26iz-5)} dz$$

$$I = \int_c \frac{2}{5\left(z^2+\frac{26i}{5}z-1\right)} dz$$

$$I = \int_c \frac{\frac{2}{5}}{\left(z^2+\frac{26i}{5}z-1\right)} dz$$

Put $\left(z^2 + \frac{26i}{5}z - 1\right) = 0$

$$z = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-\frac{26i}{5}\pm\sqrt{\left(-\frac{26i}{5}\right)^2+4}}{2} = \frac{-\frac{26i}{5}\pm\sqrt{-\frac{576}{25}}}{2} = \frac{-\frac{26i}{5}\pm\frac{24i}{5}}{2} = \frac{-26i\pm24i}{10}$$

$\therefore z = -\frac{i}{5}, -5i$

$\therefore z = -\frac{i}{5}$ lies inside C and is a simple pole

Residue of $f(z)$ at $\left(z = -\frac{i}{5}\right) = \lim_{z\to-\frac{i}{5}} (z + \frac{i}{5}) \frac{\frac{2}{5}}{\left(z+\frac{i}{5}\right)(z+5i)}$

$$= \frac{\frac{2}{5}}{\left(-\frac{i}{5}+5i\right)} = \frac{\frac{2}{5}}{\frac{24i}{5}} = -\frac{i}{12}$$

Now, $\int_0^{2\pi} \frac{1}{13+5sin\theta} d\theta = 2\pi i\ [sum\ of\ residues] = 2\pi i \left[-\frac{i}{12}\right] = \frac{\pi}{6}$

10. Evaluate $\int_0^{2\pi} \frac{\sin^2\theta}{a+bcos\theta} d\theta$ where $0 < b < a$

[N15/ElexExtcElectBiomInst/6M]

Solution:

We have, $I = \int_0^{2\pi} \frac{\sin^2\theta}{a+bcos\theta} d\theta$

put $z = e^{i\theta}, d\theta = \frac{dz}{iz}, sin\theta = \frac{z^2-1}{2iz}, cos\theta = \frac{z^2+1}{2z}$

$$I = \int \frac{\left(\frac{z^2-1}{2iz}\right)^2}{a+b\left(\frac{z^2+1}{2z}\right)} \frac{dz}{iz}$$

$$I = \int_c \frac{(z^2-1)^2}{4i^2z^2\left(\frac{2az+bz^2+b}{2z}\right)} \frac{dz}{iz}$$ where $C\ is\ |z| = 1$

$$I = \int_c \frac{(z^2-1)^2}{2i^3z^2(bz^2+2az+b)} dz$$

$$I = \int_c \frac{\frac{(z^2-1)^2}{-2ib}}{z^2\left(z^2+\frac{2a}{b}z+1\right)} dz$$

Put $z^2\left(z^2 + \frac{2a}{b}z + 1\right) = 0$

$\therefore z^2 = 0, z^2 + \frac{2a}{b}z + 1 = 0$

$z = 0,0$ and $z = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-\frac{2a}{b}\pm\sqrt{\frac{4a^2}{b^2}-4}}{2} = \frac{-a\pm\sqrt{a^2-b^2}}{b}$

$\therefore z = 0$ is a pole of order 2

$z = \frac{-a+\sqrt{a^2-b^2}}{b} = \alpha$ lies inside C

$z = \frac{-a-\sqrt{a^2-b^2}}{b} = \beta$ lies outside C

Also, we see that $\alpha.\beta = \frac{-a+\sqrt{a^2-b^2}}{b}.\frac{-a-\sqrt{a^2-b^2}}{b} = \frac{a^2-a^2+b^2}{b^2} = \frac{b^2}{b^2} = 1$

$\therefore \alpha = \frac{1}{\beta}$ or $\beta = \frac{1}{\alpha}$

And, $\alpha - \beta = \frac{2\sqrt{a^2-b^2}}{b}$

Residue of $f(z)$ at $(z = \alpha) = \lim_{z\to\alpha}(z-\alpha)\frac{\frac{(z^2-1)^2}{-2ib}}{z^2(z-\alpha)(z-\beta)}$

$$= -\frac{1}{2ib}.\frac{(\alpha^2-1)^2}{\alpha^2(\alpha-\beta)}$$

$$= -\frac{1}{2ib}.\frac{(\alpha^2-1)^2}{\alpha^2}.\frac{1}{\alpha-\beta}$$

$$= -\frac{1}{2ib}.\left(\alpha-\frac{1}{\alpha}\right)^2.\frac{1}{\alpha-\beta}$$

$$= -\frac{1}{2ib}.(\alpha-\beta)^2.\frac{1}{\alpha-\beta}$$
$$= -\frac{1}{2ib}.(\alpha-\beta)$$
$$= \frac{-\sqrt{a^2-b^2}}{ib^2}$$

Residue of $f(z)$ at $(z=0) = \frac{1}{1!}\lim_{z\to 0}\frac{d}{dz}\left[z^2.\frac{\frac{(z^2-1)^2}{-2ib}}{z^2\left(z^2+\frac{2a}{b}z+1\right)}\right]$

$$= -\frac{1}{2ib}\lim_{z\to 0}\frac{d}{dz}\left[\frac{z^4-2z^2+1}{z^2+\frac{2a}{b}z+1}\right]$$

$$= -\frac{1}{2ib}\lim_{z\to 0}\left[\frac{\left(z^2+\frac{2a}{b}z+1\right)(4z^3-4z)-(z^4-2z^2+1)\left(2z+\frac{2a}{b}\right)}{\left(z^2+\frac{2a}{b}z+1\right)^2}\right]$$
$$= -\frac{1}{2ib}\left[\frac{0-\frac{2a}{b}}{1^2}\right]$$
$$= \frac{a}{ib^2}$$

Now, $\int_0^{2\pi}\frac{\sin^2\theta}{a+bcos\theta}d\theta = 2\pi i\,[sum\ of\ residues]$

$$= 2\pi i\left[\frac{a}{ib^2}-\frac{\sqrt{a^2-b^2}}{ib^2}\right]$$
$$= \frac{2\pi}{b^2}\left(a-\sqrt{a^2-b^2}\right)$$

11. Evaluate $\int_0^{2\pi} \frac{\cos^2\theta}{5+4cos\theta} d\theta$
[M16/ElexExtcElectBiomInst/6M]
Solution:
We have, $I = \int_0^{2\pi} \frac{\cos^2\theta}{5+4cos\theta} d\theta$
put $z = e^{i\theta}, d\theta = \frac{dz}{iz}, sin\theta = \frac{z^2-1}{2iz}, cos\theta = \frac{z^2+1}{2z}$

$$I = \int \frac{\left(\frac{z^2+1}{2z}\right)^2}{5+4\left(\frac{z^2+1}{2z}\right)} \frac{dz}{iz}$$

$$I = \int_c \frac{(z^2+1)^2}{4z^2\left(\frac{5z+2z^2+2}{z}\right)} \frac{dz}{iz} \text{ where } C \text{ is } |z| = 1$$

$$I = \int_c \frac{(z^2+1)^2}{4iz^2(2z^2+5z+2)} dz$$

$$I = \int_c \frac{\frac{(z^2+1)^2}{8i}}{z^2\left(z^2+\frac{5}{2}z+1\right)} dz$$

Put $z^2\left(z^2+\frac{5}{2}z+1\right) = 0$

$$z = 0,0 \text{ and } z = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-\frac{5}{2}\pm\sqrt{\frac{25}{4}-4}}{2} = \frac{-\frac{5}{2}\pm\sqrt{\frac{9}{4}}}{2} = \frac{-\frac{5}{2}\pm\frac{3}{2}}{2} = \frac{-5\pm3}{4}$$

$\therefore z = 0$ is a pole of order 2, $z = -\frac{1}{2}$ lies inside C, $z = -2$ lies outside C

Residue of $f(z)$ at $\left(z = -\frac{1}{2}\right) = \lim_{z\to-\frac{1}{2}}(z+\frac{1}{2}) \frac{\frac{(z^2+1)^2}{8i}}{z^2\left(z+\frac{1}{2}\right)(z+2)}$

$$= \frac{\left(\frac{1}{4}+1\right)^2}{8i.\left(-\frac{1}{2}\right)^2\left(-\frac{1}{2}+2\right)} = \frac{\frac{25}{16}}{8i.\frac{1}{4}.\frac{3}{2}} = \frac{25}{48i}$$

Residue of $f(z)$ at $(z = 0) = \frac{1}{1!}\lim_{z\to0} \frac{d}{dz}\left[z^2.\frac{\frac{(z^2+1)^2}{8i}}{z^2\left(z^2+\frac{5}{2}z+1\right)}\right]$

$$= \frac{1}{8i}\lim_{z\to0} \frac{d}{dz}\left[\frac{z^4+2z^2+1}{z^2+\frac{5}{2}z+1}\right]$$

$$= \frac{1}{8i}\lim_{z\to0}\left[\frac{\left(z^2+\frac{5}{2}z+1\right)(4z^3+4z)-(z^4+2z^2+1)\left(2z+\frac{5}{2}\right)}{\left(z^2+\frac{5}{2}z+1\right)^2}\right] = \frac{1}{8i}\left[\frac{0-\frac{5}{2}}{1^2}\right] = \frac{-5}{16i}$$

Now, $\int_0^{2\pi} \frac{\cos^2\theta}{5+4cos\theta} d\theta = 2\pi i\,[sum\ of\ residues]$

$$= 2\pi i\left[\frac{25}{48i} - \frac{5}{16i}\right] = 2\pi i\left[\frac{10}{48i}\right] = \frac{5\pi}{12}$$

12. Using Residue theorem evaluate $\int_0^{2\pi} \frac{d\theta}{5-3cos\theta}$

[M16/ChemBiot/6M]

Solution:

We have, $I = \int_0^{2\pi} \frac{1}{5-3cos\theta} d\theta$

put $z = e^{i\theta}, d\theta = \frac{dz}{iz}, sin\theta = \frac{z^2-1}{2iz}, cos\theta = \frac{z^2+1}{2z}$

$I = \int \frac{1}{5-3\left(\frac{z^2+1}{2z}\right)} \frac{dz}{iz}$

$I = \int_c \frac{1}{\left(\frac{10z-3z^2-3}{2z}\right)} \frac{dz}{iz}$ where $C\ is\ |z| = 1$

$I = \int_c \frac{2}{i(-3z^2+10z-3)} dz$

$I = \int_c \frac{2}{-3i\left(z^2-\frac{10}{3}z+1\right)} dz$

$I = \int_c \frac{\frac{2i}{3}}{z^2-\frac{10}{3}z+1} dz$

Put $\left(z^2 - \frac{10}{3}z + 1\right) = 0$

$z = \frac{1}{3}, 3$

$\therefore z = \frac{1}{3}$ lies inside C and is a simple pole

Residue of $f(z)$ at $\left(z = \frac{1}{3}\right) = \lim_{z\to\frac{1}{3}} (z - \frac{1}{3}) \frac{\frac{2i}{3}}{\left(z-\frac{1}{3}\right)(z-3)}$

$$= \frac{\frac{2i}{3}}{\left(\frac{1}{3}-3\right)} = \frac{\frac{2i}{3}}{-\frac{8}{3}} = -\frac{i}{4}$$

Now, $\int_0^{2\pi} \frac{1}{5-3cos\theta} d\theta = 2\pi i\ [sum\ of\ residues] = 2\pi i \left[-\frac{i}{4}\right] = \frac{\pi}{2}$

13. Evaluate $\int_0^{2\pi} \frac{\cos 3\theta}{5+4\cos\theta} d\theta$
[N16/ElexExtcElectBiomInst/6M]
Solution:
We have, $I = \int_0^{2\pi} \frac{\cos 3\theta}{5+4\cos\theta} d\theta$
$e^{i3\theta} = \cos 3\theta + i\sin 3\theta$
$\cos 3\theta$ is a real part of $e^{i3\theta}$
$I = \int_0^{2\pi} \frac{R.P. of\ e^{i3\theta}}{5+4\cos\theta} d\theta$
put $z = e^{i\theta}, d\theta = \frac{dz}{iz}, \sin\theta = \frac{z^2-1}{2iz}, \cos\theta = \frac{z^2+1}{2z}$
$I = \int \frac{R.P. of\ z^3}{5+4\left(\frac{z^2+1}{2z}\right)} \frac{dz}{iz}$
$I = R.P. of \int_c \frac{z^4}{5z+2z^2+2} . \frac{dz}{iz}$ where $C\ is\ |z| = 1$
$I = R.P. of \int_c \frac{z^4}{2\left(z^2+\frac{5}{2}z+1\right)} \frac{dz}{iz}$
$I = R.P. of \int_c \frac{-\frac{iz^3}{2}}{\left(z^2+\frac{5}{2}z+1\right)} dz$
Put $z^2 + \frac{5}{2}z + 1 = 0$
$\therefore z = -2, z = -\frac{1}{2}$
We see that, $z = -2$ lies outside C and $z = -\frac{1}{2}$ lies inside C
Residues of $f(z)$ at $\left(z = -\frac{1}{2}\right) = \lim_{z \to -\frac{1}{2}} \left(z + \frac{1}{2}\right) . \frac{\frac{-iz^3}{2}}{(z+2)\left(z+\frac{1}{2}\right)}$
$$= \frac{\frac{-i}{2}.\left(-\frac{1}{8}\right)}{-\frac{1}{2}+2} = \frac{i}{24}$$
Now, $\int_0^{2\pi} \frac{\cos 3\theta}{5+4\cos\theta} d\theta = R.P. of \int_0^{2\pi} \frac{e^{i3\theta}}{5+4\cos\theta} d\theta$
$$= R.P. of\ 2\pi i\ [sum\ of\ residues]$$
$$= R.P. of\ 2\pi i \left[\frac{i}{24}\right] = R.P. of -\frac{\pi}{12}$$
$$= -\frac{\pi}{12}$$

14. Using Residue theorem evaluate $\int_0^{2\pi} \frac{d\theta}{5+4cos\theta}$

[N17/AutoMechCivil/4M]

Solution:

We have, $I = \int_0^{2\pi} \frac{1}{5+4cos\theta} d\theta$

put $z = e^{i\theta}, d\theta = \frac{dz}{iz}, sin\theta = \frac{z^2-1}{2iz}, cos\theta = \frac{z^2+1}{2z}$

$I = \int \frac{1}{5+4\left(\frac{z^2+1}{2z}\right)} \frac{dz}{iz}$

$I = \int_c \frac{1}{\left(\frac{5z+2z^2+2}{z}\right)} \frac{dz}{iz}$ where $C\ is\ |z| = 1$

$I = \int_c \frac{1}{i(2z^2+5z+2)} dz$

$I = \int_c \frac{2}{2i\left(z^2+\frac{5}{2}z+1\right)} dz$

$I = \int_c \frac{-\frac{i}{2}}{z^2+\frac{5}{2}z+1} dz$

Put $\left(z^2 + \frac{5}{2}z + 1\right) = 0$

$z = -\frac{1}{2}, -2$

$\therefore z = -\frac{1}{2}$ lies inside C and is a simple pole

Residue of $f(z)$ at $\left(z = -\frac{1}{2}\right) = \lim_{z\to -\frac{1}{2}} (z + \frac{1}{2}) \frac{-\frac{i}{2}}{\left(z+\frac{1}{2}\right)(z+2)}$

$$= \frac{-\frac{i}{2}}{(-\frac{1}{2}+2)} = \frac{-\frac{i}{2}}{\frac{3}{2}} = -\frac{i}{3}$$

Now, $\int_0^{2\pi} \frac{1}{5+4cos\theta} d\theta = 2\pi i\ [sum\ of\ residues] = 2\pi i \left[-\frac{i}{3}\right] = \frac{2\pi}{3}$

Type IV: Contour Integration

1. Evaluate $\int_0^\infty \frac{x^3 sinx}{(x^2+a^2)^2} dx$ using contour integration

[M14/ElexExtcElectBiomInst/6M]

Consider, $I = \int_{-\infty}^\infty \frac{x^3 sinx}{(x^2+a^2)^2} dx$

We know that, $e^{ix} = cosx + isinx$

i.e. $sinx$ is an I.P. of e^{ix}

$I = I.P. of \int_{-\infty}^\infty \frac{x^3 e^{ix}}{(x^2+a^2)^2} dx$

Consider the contour to be a very large semi circle of radius ∞ lying in the upper half of the plane.

Put $(x^2 + a^2)^2 = 0$

$x = ai, -ai, ai, -ai$

We see that, $x = ai$ is a pole of order 2.

Residue of $f(x)$ at $(x = ai) = \frac{1}{1!} \lim_{x \to ai} \frac{d}{dx}\left[(x-ai)^2 \frac{x^3 e^{ix}}{(x+ai)^2(x-ai)^2}\right]$

$= \lim_{x \to ai} \frac{d}{dx}\left[\frac{x^3 e^{ix}}{(x+ai)^2}\right]$

$= \lim_{x \to ai} \left[\frac{(x+ai)^2\left(x^3 ie^{ix} + e^{ix}.3x^2\right) - x^3 e^{ix}.2(x+ai)}{(x+ai)^4}\right]$

$= \frac{(2ai)^2\left(a^3 i^3 i.e^{i^2 a} + e^{i^2 a}.3a^2 i^2\right) - (ai)^3 e^{i^2 a} 2(2ai)}{(2ai)^4}$

$= \frac{(4a^2 i^2)(a^3 i^4 e^{-a} + 3a^2 i^2 e^{-a}) - 4a^4 i^4 e^{-a}}{16a^4 i^4}$

$= \frac{-4a^2 e^{-a}(a^3 - 3a^2) - 4a^4 e^{-a}}{16a^4 i^4} = \frac{-4a^4 e^{-a}(a-3+1)}{16a^4}$

$= -\frac{e^{-a}(a-2)}{4}$

Now, $\int_{-\infty}^\infty \frac{x^3 sinx}{(x^2+a^2)^2} dx = I.P. of \int_{-\infty}^\infty \frac{x^3 e^{ix}}{(x^2+a^2)^2} dx$

$= I.P. of\ 2\pi i[sum\ of\ residues]$

$= I.P. of\ 2\pi i\left[-\frac{e^{-a}(a-2)}{4}\right] = -e^{-a}(a-2)\frac{\pi}{2}$

2. Evaluate $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)} dx$ using contour integration.
[M14/AutoMechCivil/8M][M17/ElexExtcElectBiomInst/6M]
Solution:
Consider the contour to be a very large semi circle of radius ∞ lying in the upper half of the plane.

$I = \int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)} dx$

Put $(x^2 + a^2)(x^2 + b^2) = 0$

$x = ai, -ai, bi, -bi$

We see that, $x = ai$ and $x = bi$ lies inside C and are simple poles.

Residue of $f(x)$ at $(x = ai) = \lim_{x \to ai} (x - ai) \frac{x^2}{(x+ai)(x-ai)(x^2+b^2)}$

$= \frac{a^2 i^2}{2ai(a^2 i^2 + b^2)}$

$= \frac{-a^2}{(2ai)(-a^2+b^2)} = \frac{a}{(2i)(a^2-b^2)}$

Residue of $f(x)$ at $(x = bi) = \lim_{x \to bi} (x - bi) \frac{x^2}{(x+bi)(x-bi)(x^2+a^2)}$

$= \frac{b^2 i^2}{2bi(b^2 i^2 + a^2)}$

$= \frac{-b^2}{(2bi)(-b^2+a^2)} = \frac{-b}{(2i)(a^2-b^2)}$

Now, $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)} dx = 2\pi i[sum\ of\ residues]$

$= 2\pi i \left[\frac{a}{(2i)(a^2-b^2)} + \frac{-b}{(2i)(a^2-b^2)}\right]$

$= \frac{2\pi i}{2i} \left[\frac{a-b}{a^2-b^2}\right]$

$= \frac{\pi}{a+b}$

3. Evaluate $\int_{-\infty}^{\infty} \frac{\cos 3x}{(x^2+1)(x^2+4)} dx$

[N14/ElexExtcElectBiomInst/6M]

Solution:

Consider $I = \int_{-\infty}^{\infty} \frac{\cos 3x}{(x^2+1)(x^2+4)} dx$

$I = \int_{-\infty}^{\infty} \frac{R.P.of\ e^{i3x}}{(x^2+1)(x^2+4)} dx$

Consider the contour to be a very large semi circle of radius ∞ lying in the upper half of the plane.

$I = R.P.of \int_{-\infty}^{\infty} \frac{e^{i3x}}{(x^2+1)(x^2+4)} dx$

Put $(x^2+1)(x^2+4) = 0$

$x = i, -i, 2i, -2i$

We see that, $x = i$ and $x = 2i$ lies inside C and are simple poles.

Residue of $f(x)$ at $(x = i) = \lim_{x \to i} (x - i) \frac{e^{i3x}}{(x+i)(x-i)(x^2+4)}$

$= \frac{e^{i^2 3}}{2i(i^2+4)}$

$= \frac{e^{-3}}{6i}$

Residue of $f(x)$ at $(x = 2i) = \lim_{x \to 2i} (x - 2i) \frac{e^{i3x}}{(x+2i)(x-2i)(x^2+1)}$

$= \frac{e^{i^2 6}}{4i(4i^2+1)}$

$= \frac{e^{-6}}{-12i}$

Now, $\int_{-\infty}^{\infty} \frac{\cos 3x}{(x^2+1)(x^2+4)} dx = R.P.of \int_{-\infty}^{\infty} \frac{e^{i3x}}{(x^2+1)(x^2+4)} dx$

$= R.P.of\ 2\pi i[sum\ of\ residues]$

$= R.P.of\ 2\pi i \left[\frac{e^{-3}}{6i} + \frac{e^{-6}}{-12i}\right]$

$= R.P\ of\ \frac{2\pi i}{2i} \left[\frac{e^{-3}}{3} - \frac{e^{-6}}{6}\right]$

$= R.P.of\ \pi \left[\frac{e^{-3}}{3} - \frac{e^{-6}}{6}\right]$

$= \pi \left[\frac{e^{-3}}{3} - \frac{e^{-6}}{6}\right]$

4. Show that $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)} dx = \frac{\pi}{3}$

[N14/CompIT/6M][M16/CompIT/6M]

Solution:

Consider

$$I = \int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)} dx$$

Consider the contour to be a very large semi circle of radius ∞ lying in the upper half of the plane.

Put $(x^2+1)(x^2+4) = 0$

$x = i, -i, 2i, -2i$

We see that, $x = i$ and $x = 2i$ lies inside C and are simple poles.

Residue of $f(x)$ at $(x = i) = \lim_{x \to i} (x - i) \frac{x^2}{(x+i)(x-i)(x^2+4)}$

$$= \frac{i^2}{2i(i^2+4)} = \frac{-1}{6i}$$

Residue of $f(x)$ at $(x = 2i) = \lim_{x \to 2i} (x - 2i) \frac{x^2}{(x+2i)(x-2i)(x^2+1)}$

$$= \frac{4i^2}{4i(4i^2+1)} = \frac{-4}{-12i} = \frac{2}{6i}$$

Now, $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)} dx = 2\pi i[sum\ of\ residues]$

$$= 2\pi i \left[\frac{-1}{6i} + \frac{2}{6i}\right]$$

$$= 2\pi i \left[\frac{1}{6i}\right]$$

$$= \frac{\pi}{3}$$

5. Evaluate $\int_0^\infty \frac{1}{(x^2+a^2)^2} dx$

[M15/CompIT/6M]

Solution:

Consider

$$I = \int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)^2} dx$$

Consider the contour to be a very large semi circle of radius ∞ lying in the upper half of the plane.

Put $(x^2 + a^2)^2 = 0$

$x = ai, -ai, ai, -ai$

We see that, $x = ai$ is a pole of order 2.

Residue of $f(x)$ at $(x = ai) = \frac{1}{1!} \lim_{x \to ai} \frac{d}{dx} \left[(x - ai)^2 \frac{1}{(x+ai)^2 (x-ai)^2} \right]$

$$= \lim_{x \to ai} \frac{d}{dx} \left[\frac{1}{(x+ai)^2} \right]$$

$$= \lim_{x \to ai} \left[-\frac{2}{(x+ai)^3} \right]$$

$$= -\frac{2}{(2ai)^3}$$

$$= -\frac{2}{8a^3 i^3} = \frac{1}{4a^3 i}$$

Now, $\int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)^2} dx = 2\pi i [sum\ of\ residues]$

$$= 2\pi i \left[\frac{1}{4a^3 i} \right] = \frac{\pi}{2a^3}$$

Thus, $\int_0^\infty \frac{1}{(x^2+a^2)^2} dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)^2} dx = \frac{1}{2} \left[\frac{\pi}{2a^3} \right] = \frac{\pi}{4a^3}$

6. Evaluate $\int_{-\infty}^{\infty} \frac{x^2}{x^6+1} dx$

[N15/ElexExtcElectBiomInst/6M]

Solution:

Consider, $I = \int_{-\infty}^{\infty} \frac{x^2}{x^6+1} dx$

Put $x^3 = t, \therefore 3x^2 dx = dt, \therefore x^2 dx = \frac{dt}{3}$

$I = \int_{-\infty}^{\infty} \frac{1}{t^2+1} \frac{dt}{3}$

$I = \frac{1}{3} \int_{-\infty}^{\infty} \frac{dt}{t^2+1}$

i.e. $I = \frac{1}{3} \int_{-\infty}^{\infty} \frac{dx}{x^2+1}$

Consider the contour to be a very large semi circle of radius ∞ lying in the upper half of the plane.

Put $(x^2 + 1) = 0$

$x = i, -i$

We see that, $x = i$ lies inside C and is a simple pole.

Residue of $f(x)$ at $(x = i) = \lim_{x \to i} (x - i) \frac{1}{(x+i)(x-i)}$

$= \frac{1}{2i}$

Now, $\frac{1}{3} \int_{-\infty}^{\infty} \frac{1}{(x^2+1)} dx = \frac{2\pi i}{3} [sum\ of\ residues]$

$= \frac{2\pi i}{3} \left[\frac{1}{2i}\right]$

$\therefore \int_{-\infty}^{\infty} \frac{x^2}{x^6+1} dx = \frac{1}{3} \int_{-\infty}^{\infty} \frac{1}{(x^2+1)} dx = \frac{\pi}{3}$

7. Evaluate $\int_{-\infty}^{\infty}\frac{x^2-x+2}{x^4+10x^2+9}dx$ using contour integration.

[M16/ElexExtcElectBiomInst/6M]

Solution:

Consider the contour to be a very large semi-circle of radius ∞ lying in the upper half of the plane.

$I=\int_{-\infty}^{\infty}\frac{x^2-x+2}{x^4+10x^2+9}dx$

Put $x^4+10x^2+9=0$

$(x^2+9)(x^2+1)=0$

$x=3i,-3i,i,-i$

We see that, $x=3i$ and $x=i$ lies inside C and are simple poles.

Residue of $f(x)$ at $(x=3i)=\lim_{x\to 3i}(x-3i)\frac{x^2-x+2}{(x+3i)(x-3i)(x^2+1)}$

$=\frac{9i^2-3i+2}{6i(9i^2+1)}$

$=\frac{-7-3i}{-48i}=\frac{7+3i}{48i}$

Residue of $f(x)$ at $(x=i)=\lim_{x\to i}(x-i)\frac{x^2-x+2}{(x+i)(x-i)(x^2+9)}$

$=\frac{i^2-i+2}{2i(i^2+9)}$

$=\frac{1-i}{16i}=\frac{3-3i}{48i}$

Now, $\int_{-\infty}^{\infty}\frac{x^2-x+2}{x^4+10x^2+9}dx=2\pi i[sum\ of\ residues]$

$=2\pi i\left[\frac{7+3i}{48i}+\frac{3-3i}{48i}\right]$

$=2\pi i\left[\frac{10}{48i}\right]$

$=\frac{5\pi}{12}$

8. Evaluate $\int_{-\infty}^{\infty} \frac{x^2+x+2}{x^4+10x^2+9} dx$ using contour integration.
[N16/CompIT/6M]
Solution:
Consider the contour to be a very large semi-circle of radius ∞ lying in the upper half of the plane.
$I = \int_{-\infty}^{\infty} \frac{x^2+x+2}{x^4+10x^2+9} dx$
Put $x^4 + 10x^2 + 9 = 0$
$(x^2 + 9)(x^2 + 1) = 0$
$x = 3i, -3i, i, -i$
We see that, $x = 3i$ and $x = i$ lies inside C and are simple poles.
Residue of $f(x)$ at $(x = 3i) = \lim_{x \to 3i} (x - 3i) \frac{x^2+x+2}{(x+3i)(x-3i)(x^2+1)}$
$= \frac{9i^2+3i+2}{6i(9i^2+1)}$
$= \frac{-7+3i}{-48i} = \frac{7-3i}{48i}$
Residue of $f(x)$ at $(x = i) = \lim_{x \to i} (x - i) \frac{x^2+x+2}{(x+i)(x-i)(x^2+9)}$
$= \frac{i^2+i+2}{2i(i^2+9)}$
$= \frac{1+i}{16i} = \frac{3+3i}{48i}$
Now, $\int_{-\infty}^{\infty} \frac{x^2+x+2}{x^4+10x^2+9} dx = 2\pi i[sum\ of\ residues]$
$= 2\pi i \left[\frac{7-3i}{48i} + \frac{3+3i}{48i}\right]$
$= 2\pi i \left[\frac{10}{48i}\right]$
$= \frac{5\pi}{12}$

9. Evaluate $\int_0^\infty \frac{dx}{x^2+1}$

[M17/AutoMechCivil/6M]

Solution:

Consider, $I = \int_{-\infty}^{\infty} \frac{dx}{x^2+1}$

Consider the contour to be a very large semi circle of radius ∞ lying in the upper half of the plane.

Put $(x^2 + 1) = 0$

$x = i, -i$

We see that, $x = i$ lies inside C and is a simple pole.

Residue of $f(x)$ at $(x = i) = \lim_{x\to i}(x - i)\frac{1}{(x+i)(x-i)}$

$= \frac{1}{2i}$

Now, $\int_{-\infty}^{\infty} \frac{1}{(x^2+1)} dx = 2\pi i[sum\ of\ residues]$

$= 2\pi i\left[\frac{1}{2i}\right] = \pi$

$\therefore \int_0^\infty \frac{dx}{x^2+1} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{(x^2+1)} dx = \frac{\pi}{2}$

10. Evaluate $\int_0^\infty \frac{1}{(x^2+1)(x^2+9)} dx$

[N17/ElexExtcElectBiomInst/6M]

Solution:

Consider the contour to be a very large semi circle of radius ∞ lying in the upper half of the plane.

$I = \int_{-\infty}^{\infty} \frac{1}{(x^2+9)(x^2+1)} dx$

Put $(x^2+9)(x^2+1) = 0$

$x = 3i, -3i, i, -i$

We see that, $x = 3i$ and $x = 2i$ lies inside C and are simple poles.

Residue of $f(x)$ at $(x = 3i) = \lim_{x\to 3i}(x-3i)\frac{1}{(x+3i)(x-3i)(x^2+1)}$

$= \frac{1}{6i(9i^2+1)}$

$= \frac{i}{48}$

Residue of $f(x)$ at $(x = i) = \lim_{x\to i}(x-i)\frac{1}{(x+i)(x-i)(x^2+9)}$

$= \frac{1}{2i(i^2+9)}$

$= -\frac{i}{16}$

Now, $\int_{-\infty}^{\infty} \frac{1}{(x^2+9)(x^2+1)} dx = 2\pi i[sum\ of\ residues]$

$= 2\pi i\left[\frac{i}{48} - \frac{i}{16}\right]$

$= \frac{\pi}{12}$

Thus, $\int_0^\infty \frac{1}{(x^2+1)(x^2+9)} dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{(x^2+9)(x^2+1)} dx = \frac{1}{2}.\frac{\pi}{12} = \frac{\pi}{24}$

11. Evaluate $\int_0^\infty \frac{1}{(x^2+a^2)^3} dx$

[N17/CompIT/6M]

Solution:

Consider

$I = \int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)^3} dx$

Consider the contour to be a very large semi circle of radius ∞ lying in the upper half of the plane.

Put $(x^2 + a^2)^3 = 0$

$x = ai, -ai, ai, -ai, ai, -ai$

We see that, $x = ai$ is a pole of order 2.

Residue of $f(x)$ at $(x = ai) = \frac{1}{2!} \lim_{x \to ai} \frac{d^2}{dx^2} \left[(x - ai)^3 \frac{1}{(x+ai)^3 (x-ai)^3} \right]$

$$= \frac{1}{2} \lim_{x \to ai} \frac{d^2}{dx^2} \left[\frac{1}{(x+ai)^3} \right]$$
$$= \frac{1}{2} \lim_{x \to ai} \frac{d}{dx} \left[-\frac{3}{(x+ai)^4} \right]$$
$$= \frac{1}{2} \lim_{x \to ai} \left[\frac{12}{(x+ai)^5} \right]$$
$$= \frac{1}{2} . \frac{12}{(2ai)^5}$$
$$= \frac{6}{32a^5 i^5} = -\frac{3i}{16a^5}$$

Now, $\int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)^3} dx = 2\pi i[sum\ of\ residues]$

$$= 2\pi i \left[-\frac{3i}{16a^5} \right] = \frac{3\pi}{8a^5}$$

Thus, $\int_0^\infty \frac{1}{(x^2+a^2)^3} dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)^3} dx = \frac{1}{2} \left[\frac{3\pi}{8a^5} \right] = \frac{3\pi}{16a^5}$

12. Using Residue Theorem, evaluate $\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2} dx$

[N17/AutoMechCivil/4M]

Solution:

Consider

$$I = \int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2} dx$$

Consider the contour to be a very large semi circle of radius ∞ lying in the upper half of the plane.

Put $(x^2+1)^2 = 0$

$x = i, -i, i, -i$

We see that, $x = i$ is a pole of order 2.

Residue of $f(x)$ at $(x = i) = \frac{1}{1!} \lim_{x \to i} \frac{d}{dx} \left[(x-i)^2 \frac{1}{(x+i)^2 (x-i)^2} \right]$

$$= \lim_{x \to i} \frac{d}{dx} \left[\frac{1}{(x+i)^2} \right]$$

$$= \lim_{x \to i} \left[-\frac{2}{(x+i)^3} \right]$$

$$= -\frac{2}{(2i)^3}$$

$$= -\frac{2}{8i^3} = \frac{1}{4i}$$

Now, $\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2} dx = 2\pi i [sum\ of\ residues]$

$$= 2\pi i \left[\frac{1}{4i} \right] = \frac{\pi}{2}$$