Calculus of Variations

Type I: Euler's Differential Equation

1. Find the extremal of $\int_{x_1}^{x_2}(y^2 - y'^2 - 2ycoshx)dx$

[N14/ElexExtcElectBiomInst/5M][N15/ElexExtcElectBiomInst/5M]

Solution:

We have,

$F = y^2 - y'^2 - 2y\, coshx$

By Euler's Differential Equation,

$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$

$(2y - 2coshx) - \frac{d}{dx}(-2y') = 0$

$2y - 2coshx + 2y'' = 0$

$y'' + y = coshx$

$D^2y + y = coshx$

$(D^2 + 1)y = coshx$

C.F.:

Put $D^2 + 1 = 0$

$D = \pm i$

Thus, C.F. is given by

$y = c_1 cosx + c_2 sinx$

P.I.:

$y = \frac{1}{f(D)}X$

$y = \frac{1}{D^2+1}\, coshx$

$y = \frac{cos\,hx}{2}$

G.S.:

$y = C.F. + P.I.$

$y = c_1 cosx + c_2 sinx + \frac{1}{2}coshx$

2. Determine the function that gives the shortest distance between two given points

[N14/ElexExtcElectBiomInst/6M][M16/ElexExtcElectBiomInst/5M]

Solution:

We know that the distance between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by,

$$ds = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Thus,

$$I = \int_A^B ds$$

$$I = \int_A^B \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$$I = \int_A^B \sqrt{(dx)^2 + (dy)^2}$$

$$I = \int_A^B \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$I = \int_A^B \sqrt{1 + y'^2}\, dx$$

Now,

$$F = \sqrt{1 + y'^2}$$

By Euler's Differential Equation,

$$\frac{\partial F}{\partial y'} = c$$

$$\frac{1}{2\sqrt{1+y'^2}} \times 2y' = c$$

$$\frac{y'}{\sqrt{1+y'^2}} = c$$

Squaring, we get

$$\frac{y'^2}{1+y'^2} = c^2$$

$$\frac{1+y'^2}{y'^2} = \frac{1}{c^2}$$

$$\frac{1}{y'^2} + 1 = \frac{1}{c^2}$$

$$\frac{1}{y'^2} = \frac{1}{c^2} - 1$$

$$y'^2 = \frac{1}{\frac{1}{c^2} - 1}$$

$$y' = \sqrt{\frac{1}{\frac{1}{c^2} - 1}}$$

$$y' = a$$

Integrating, we get

$$y = ax + b$$

3. Find the extremals of $\int_{x_1}^{x_2} \frac{\sqrt{1+y'^2}}{x} dx$

[M15/ElexExtcElectBiomInst/5M]

Solution:

We have,

$F = \frac{\sqrt{1+y'^2}}{x}$

By Euler's Differential Equation,

$\frac{\partial F}{\partial y'} = c$

$\frac{1}{x} . \frac{1}{2\sqrt{1+y'^2}} \times 2y' = c$

$\frac{y'}{\sqrt{1+y'^2}} = cx$

Squaring, we get

$\frac{y'^2}{1+y'^2} = c^2 x^2$

$\frac{1}{c^2 x^2} = \frac{1+y'^2}{y'^2}$

$\frac{1}{c^2 x^2} = \frac{1}{y'^2} + 1$

$\frac{c_1^2}{x^2} - 1 = \frac{1}{y'^2}$

$\frac{c_1^2 - x^2}{x^2} = \frac{1}{y'^2}$

$y'^2 = \frac{x^2}{c_1^2 - x^2}$

$y' = \frac{x}{\sqrt{c_1^2 - x^2}}$

$\frac{dy}{dx} = \frac{x}{\sqrt{c_1^2 - x^2}}$

$dy = \frac{x}{\sqrt{c_1^2 - x^2}} \, dx$

$\int dy = -\frac{1}{2} \int \frac{-2x}{\sqrt{c_1^2 - x^2}} dx$

$y = -\frac{1}{2} . 2\sqrt{c_1^2 - x^2} + c_2$

$y - c_2 = -\sqrt{c_1^2 - x^2}$

Squaring, we get

$(y - c_2)^2 = c_1^2 - x^2$

$x^2 + (y - c_2)^2 = c_1^2$

4. Find the curve on which the functional $\int_0^1 (y'^2 + 12xy)dx$ with $y(0) = 0$ and $y(1) = 1$ is extremal.

[M15/ElexExtcElectBiomInst/6M][M16/ElexExtcElectBiomInst/5M]

Solution:

We have,

$F = y'^2 + 12xy$

By Euler's Differential Equation,

$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$

$12x - \frac{d}{dx}(2y') = 0$

$12x - 2y'' = 0$

$y'' = 6x$

Integrating, we get

$y' = 6\frac{x^2}{2} + c_1$

Integrating, we get

$y = 3\frac{x^3}{3} + c_1x + c_2$ ……………………….(1)

Putting $x = 0, y = 0$ in eqn (1), we get

$0 = 0 + 0 + c_2$

$\therefore c_2 = 0$

Putting $x = 1, y = 1$ in eqn (1), we get

$1 = 1 + c_1 + c_2$

$0 = c_1 + c_2$

$c_2 = -c_1 = 0$

Thus,

$y = x^3$

5. Find the extremals of $\int_{x_1}^{x_2}(x+y')y'\,dx$

[N16/ElexExtcElectBiomInst/5M]

Solution:

We have,

$F=(x+y')y'=xy'+y'^2$

By Euler's Differential Equation,

$\frac{\partial F}{\partial y'}=c$

$x+2y'=c$

$2y'=c-x$

$y'=\frac{c}{2}-\frac{x}{2}$

Integrating, we get

$y=\frac{c}{2}x-\frac{x^2}{4}+c_1$

$y=c_1+c_2x-\frac{x^2}{4}$

6. Find the extremal of the function $\int_0^{\frac{\pi}{2}}(2xy + y^2 - y'^2)dx$ with $y(0) = 0$, $y\left(\frac{\pi}{2}\right) = 0$.

[N16/ElexExtcElectBiomInst/6M]

Solution:

We have,

$F = 2xy + y^2 - y'^2$

By Euler's Differential Equation,

$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$

$(2x + 2y) - \frac{d}{dx}(-2y') = 0$

$2x + 2y + 2y'' = 0$

$y'' + y = -x$

$D^2y + y = -x$

$(D^2 + 1)y = -x$

C.F.:

Put $D^2 + 1 = 0$

$D = \pm i$

Thus, C.F. is given by

$y = c_1 cosx + c_2 sinx$

P.I.:

$y = \frac{1}{f(D)}X$

$y = \frac{1}{D^2+1} .-x$

$y = [1 + D^2]^{-1}(-x)$

$y = -x$

G.S.:

$y = C.F. + P.I.$

$y = c_1 cosx + c_2 sinx - x$

When $x = 0, y = 0$

$0 = c_1$

When $x = \frac{\pi}{2}, y = 0$

$0 = c_2 - \frac{\pi}{2}$

$c_2 = \frac{\pi}{2}$

Thus, the solution becomes

$y = \frac{\pi}{2} sinx - x$

7. Find the extremals of $\int_{x_1}^{x_2} \frac{1+y^2}{y'^2} dx$

[M17/ElexExtcElectBiomInst/5M]

Solution:

We have,

$F = \frac{1+y^2}{y'^2}$

By Euler's Differential Equation,

$F - y' \frac{\partial F}{\partial y'} = c$

$\frac{1+y^2}{y'^2} - y' \left[(1+y^2)\left(-\frac{2}{y'^3}\right)\right] = c$

$\frac{1+y^2}{y'^2} + \frac{2(1+y^2)}{y'^2} = c$

$\frac{3(1+y^2)}{y'^2} = c$

$\frac{3(1+y^2)}{c} = y'^2$

$y' = \sqrt{\frac{3}{c}(1+y^2)}$

$\frac{dy}{dx} = \sqrt{\frac{3}{c}} . \sqrt{1+y^2}$

$\frac{dy}{\sqrt{1+y^2}} = c_1 dx \qquad c_1 = \sqrt{\frac{3}{c}}$

$\int \frac{dy}{\sqrt{1+y^2}} = c_1 \int dx$

$\sinh^{-1} y = c_1 x + c_2$

$y = \sinh(c_1 x + c_2)$

8. Show that the functional $\int_0^{\frac{\pi}{2}}\left\{2xy+\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right\}dt$ such that $x(0)=0, x\left(\frac{\pi}{2}\right)=-1, y(0)=0, y\left(\frac{\pi}{2}\right)=1$ is stationary if $x=-sint, y=sint$

[M17/ElexExtcElectBiomInst/6M]

Solution:

We have,

$f=2xy+x'^2+y'^2$

The condition for extremum are,

$\frac{\partial f}{\partial x}-\frac{d}{dt}\left(\frac{\partial f}{\partial x'}\right)=0$, $\frac{\partial f}{\partial y}-\frac{d}{dt}\left(\frac{\partial f}{\partial y'}\right)=0$

$2y-\frac{d}{dt}(2x')=0$, $2x-\frac{d}{dt}(2y')=0$

$2y-2\frac{d^2x}{dt^2}=0$, $2x-2\frac{d^2y}{dt^2}=0$

$\frac{d^2x}{dt^2}=y$, $\frac{d^2y}{dt^2}=x$ ……………………….(1)

$\frac{d^3x}{dt^3}=\frac{dy}{dt}$

$\frac{d^4x}{dt^4}=\frac{d^2y}{dt^2}$

$\frac{d^4x}{dt^4}=x$ From (1)

$\frac{d^4x}{dt^4}-x=0$

$D^4x-x=0$

$(D^4-1)x=0$

$(D^4-1)=0$

$(D^2-1)(D^2+1)=0$

$D=1,-1,i,-i$

Its solution,

$x=c_1e^t+c_2e^{-t}+c_3cost+c_4sint$ ……………..(2)

$\frac{dx}{dt}=c_1e^t-c_2e^{-t}-c_3sint+c_4cost$

$\frac{d^2x}{dt^2}=c_1e^t+c_2e^{-t}-c_3cost-c_4sint$

But $\frac{d^2x}{dt^2}=y$

$\therefore y=c_1e^t+c_2e^{-t}-c_3cost-c_4sint$ ………………(3)

Now, when $t=0, x=0$ in eqn (1) gives,

$0=c_1+c_2+c_3$ ……………….(A)

When $t=0, y=0$ in eqn (3) gives,

$0=c_1+c_2-c_3$ ……………….(B)

When $t = \frac{\pi}{2}, x = -1$ in eqn (2) gives,

$-1 = c_1 e^{\frac{\pi}{2}} + c_2 e^{-\frac{\pi}{2}} + c_4$ ………………(C)

When $t = \frac{\pi}{2}, y = 1$ in eqn (3) gives,

$1 = c_1 e^{\frac{\pi}{2}} + c_2 e^{-\frac{\pi}{2}} - c_4$ …………….(D)

Adding (A) & (B), we get $c_1 + c_2 = 0$ …………(E)

Subtracting (A) & (B), we get $c_3 = 0$

Adding (C) & (D), we get $c_1 e^{\frac{\pi}{2}} + c_2 e^{-\frac{\pi}{2}} = 0$ …….(F)

Subtracting (C) & (D), we get $c_4 = -1$

Solving (E) & (F), we get $c_1 = 0, c_2 = 0$

Thus, the solutions are

$x = -sint, y = sint$

9. Find the extremals of $\int_{x_1}^{x_2}(1 + x^2y')y'dx$

[N17/ElexExtcElectBiomInst/5M]

Solution:

We have,

$F = (1 + x^2y')y' = y' + x^2y'^2$

By Euler's Differential Equation,

$\frac{\partial F}{\partial y'} = c$

$1 + x^2 2y' = c$

$2x^2y' = c - 1$

$y' = \frac{c-1}{2x^2}$

$y' = \frac{c_1}{x^2}$

Integrating, we get

$y = -\frac{c_1}{x} + c_2$

Type II: Functionals of second order derivatives

1. Find the extremal of $\int_{x_0}^{x_1}(16y^2 - y''^2 + x^2)dx$

[M14/ElexExtcElectBiomInst/5M]

Solution:

We have,

$F = 16y^2 - y''^2 + x^2$

By Euler's Differential Equation,

$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) = 0$

$32y - \frac{d}{dx}(0) + \frac{d^2}{dx^2}(-2y'') = 0$

$32y - 2y^{iv} = 0$

$y^{iv} - 16y = 0$

$D^4 y - 16y = 0$

$(D^4 - 16)y = 0$

$D^4 - 16 = 0$

$(D^2 - 4)(D^2 + 4) = 0$

$D = 2, -2, 2i, -2i$

Thus, the solution is given by

$y = c_1 e^{2x} + c_2 e^{-2x} + c_3 cos2x + c_4 sin2x$

2. Find the extremal of $\int_{x_0}^{x_1}(2xy - y''^2)dx$

[M15/ElexExtcElectBiomInst/6M][N15/ElexExtcElectBiomInst/5M]
[M17/ElexExtcElectBiomInst/6M]

Solution:

We have,

$F = 2xy - y''^2$

By Euler's Differential Equation,

$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) = 0$

$2x - \frac{d}{dx}(0) + \frac{d^2}{dx^2}(-2y'') = 0$

$2x - 2y^{iv} = 0$

$y^{iv} = x$

Integrating, we get

$y''' = \frac{x^2}{2} + c_1$

Integrating, we get

$y'' = \frac{x^3}{6} + c_1 x + c_2$

Integrating, we get

$y' = \frac{x^4}{24} + c_1\frac{x^2}{2} + c_2 x + c_3$

Integrating, we get

$y = \frac{x^5}{120} + c_1\frac{x^3}{6} + c_2\frac{x^2}{2} + c_3\, x + c_4$

$y = \frac{x^5}{5!} + c_1\frac{x^3}{3!} + c_2\frac{x^2}{2!} + c_3 x + c_4$

3. Find the extremal of $\int_{x_0}^{x_1}(16y^2 - y''^2)dx$

[M16/ElexExtcElectBiomInst/6M]

Solution:

We have,

$F = 16y^2 - y''^2$

By Euler's Differential Equation,

$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) = 0$

$32y - \frac{d}{dx}(0) + \frac{d^2}{dx^2}(-2y'') = 0$

$32y - 2y^{iv} = 0$

$y^{iv} - 16y = 0$

$D^4y - 16y = 0$

$(D^4 - 16)y = 0$

$D^4 - 16 = 0$

$(D^2 - 4)(D^2 + 4) = 0$

$D = 2, -2, 2i, -2i$

Thus, the solution is given by

$y = c_1e^{2x} + c_2e^{-2x} + c_3cos2x + c_4sin2x$

4. Find the extremal of $\int_{x_0}^{x_1}(y''^2 - y^2)dx$

[N17/ElexExctElectBiomInst/6M]

Solution:

We have,

$F = y''^2 - y^2$

By Euler's Differential Equation,

$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) = 0$

$-2y - \frac{d}{dx}(0) + \frac{d^2}{dx^2}(2y'') = 0$

$-2y + 2y^{iv} = 0$

$y^{iv} - y = 0$

$D^4y - y = 0$

$(D^4 - 1)y = 0$

$D^4 - 1 = 0$

$(D^2 - 1)(D^2 + 1) = 0$

$D = 1, -1, i, -i$

Thus, the solution is given by

$y = c_1e^x + c_2e^{-x} + c_3cosx + c_4sinx$

Type III: Isoperimetric Problems

1. Find the curve $y = f(x)$ for which $\int_{x_1}^{x_2} y\sqrt{1+y'^2}dx$ is minimum subject to the constraint $\int_{x_1}^{x_2} \sqrt{1+y'^2}dx = l$

[M14/ElexExtcElectBiomInst/6M]

Solution:

We have,

$F = y\sqrt{1+y'^2}$ & $G = \sqrt{1+y'^2}$

$H = F + \lambda G = y\sqrt{1+y'^2} + \lambda\sqrt{1+y'^2} = (y+\lambda)\sqrt{1+y'^2}$

By Euler's Differential Equation,

$$H - y'\frac{\partial H}{\partial y'} = c$$

$$(y+\lambda)\sqrt{1+y'^2} - y'\left(\frac{y+\lambda}{2\sqrt{1+y'^2}} \times 2y'\right) = c$$

$$(y+\lambda)\left[\sqrt{1+y'^2} - \frac{y'^2}{\sqrt{1+y'^2}}\right] = c$$

$$(y+\lambda)\left[\frac{1+y'^2-y'^2}{\sqrt{1+y'^2}}\right] = c$$

$$\frac{y+\lambda}{\sqrt{1+y'^2}} = c$$

Squaring, we get

$$(y+\lambda)^2 = c^2(1+y'^2)$$

$$\frac{(y+\lambda)^2}{c^2} = 1 + y'^2$$

$$y'^2 = \frac{(y+\lambda)^2}{c^2} - 1$$

$$y'^2 = \frac{(y+\lambda)^2 - c^2}{c^2}$$

$$y' = \frac{\sqrt{(y+\lambda)^2 - c^2}}{c}$$

$$\frac{dy}{dx} = \frac{\sqrt{(y+\lambda)^2 - c^2}}{c}$$

$$\frac{1}{\sqrt{(y+\lambda)^2 - c^2}}dy = \frac{1}{c}dx$$

$$\int \frac{1}{\sqrt{(y+\lambda)^2 - c^2}}dy = \frac{1}{c}\int dx$$

$$\cosh^{-1}\left(\frac{y+\lambda}{c}\right) = \frac{1}{c}.x + c_1$$

$$\frac{y+\lambda}{c} = \cosh\left(\frac{x+cc_1}{c}\right)$$

$$y = c\cosh\left(\frac{x+c'}{c}\right) - \lambda$$

2. Find the plane curve with fixed perimeter and maximum area
[M14/ElexExtcElectBiomInst/8M][N14/ElexExtcElectBiomInst/6M]
[N15/ElexExtcElectBiomInst/5M]
3. Show that a closed curve C of given fixed length (perimeter) which encloses maximum area is a circle.
[M16/ElexExtcElectBiomInst/8M]
Solution:
By Greens theorem, the area is given by

$$A = \frac{1}{2}\int xdy - ydx$$

$$A = \frac{1}{2}\int \left(x\frac{dy}{dx} - y\right)dx$$

$$A = \frac{1}{2}\int (xy' - y)dx$$

Let the perimeter or length be,

$$\int \sqrt{1+y'^2}dx = l$$

Now,

$$F = \frac{1}{2}(xy' - y) \ \& \ G = \sqrt{1+y'^2}$$

$$H = F + \lambda G$$

$$H = \frac{1}{2}(xy' - y) + \lambda\sqrt{1+y'^2}$$

By Euler's Differential Equation,

$$\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) = 0$$

$$-\frac{1}{2} - \frac{d}{dx}\left(\frac{x}{2} + \frac{\lambda}{2\sqrt{1+y'^2}}.2y'\right) = 0$$

$$-\frac{1}{2} - \frac{1}{2} + \frac{d}{dx}\left(\frac{\lambda y'}{\sqrt{1+y'^2}}\right) = 0$$

$$\frac{d}{dx}\left(\frac{\lambda y'}{\sqrt{1+y'^2}}\right) = 1$$

Integrating, we get

$$\frac{\lambda y'}{\sqrt{1+y'^2}} = x + c_1$$

Squaring, we get

$$\frac{y'^2}{1+y'^2} = \frac{(x+c_1)^2}{\lambda^2}$$

$$\frac{1+y'^2}{y'^2} = \frac{\lambda^2}{(x+c_1)^2}$$

$$\frac{1}{y'^2} + 1 = \frac{\lambda^2}{(x+c_1)^2}$$

$$\frac{1}{y'^2} = \frac{\lambda^2}{(x+c_1)^2} - 1$$

$\frac{1}{y'^2} = \frac{\lambda^2-(x+c_1)^2}{(x+c_1)^2}$

$y'^2 = \frac{(x+c_1)^2}{\lambda^2-(x+c_1)^2}$

$y' = \frac{x+c_1}{\sqrt{\lambda^2-(x+c_1)^2}}$

$\frac{dy}{dx} = \frac{x+c_1}{\sqrt{\lambda^2-(x+c_1)^2}}$

$dy = \frac{x+c_1}{\sqrt{\lambda^2-(x+c_1)^2}}\ dx$

$\int dy = -\frac{1}{2}\int \frac{-2(x+c_1)}{\sqrt{\lambda^2-(x+c_1)^2}} dx$

$y = -\frac{1}{2}.2\sqrt{\lambda^2-(x+c_1)^2} + c_2$

$(y-c_2) = -\sqrt{\lambda^2-(x+c_1)^2}$

Squaring, we get

$(y-c_2)^2 = \lambda^2-(x+c_1)^2$

$(x+c_1)^2 + (y-c_2)^2 = \lambda^2$, an equation of a circle

4. Find the curve $y = f(x)$ for which $\int_0^{\pi} (y'^2 - y^2)dx$ is extremum if $\int_0^{\pi} ydx = 1$

[N16/ElexExtcElectBiomInst/6M]

Solution:

We have,

$F = y'^2 - y^2$ & $G = y$

$H = F + \lambda G = y'^2 - y^2 + \lambda y$

By Euler's Differential Equation,

$\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) = 0$

$-2y + \lambda - \frac{d}{dx}(2y') = 0$

$-2y + \lambda - 2y'' = 0$

$y'' + y = \frac{\lambda}{2}$

$(D^2 + 1)y = \frac{\lambda}{2}$

C.F.:

Put $D^2 + 1 = 0$

$D = \pm i$

Thus, C.F. is given by

$y = c_1 cosx + c_2 sinx$

P.I.:

$y = \frac{1}{f(D)} X$

$y = \frac{1}{D^2+1} . \frac{\lambda}{2}$

$y = \frac{1}{D^2+1} . \frac{\lambda}{2} . e^{0x}$

$y = \frac{\lambda}{2}$

G.S.:

$y = C.F. + P.I.$

$y = c_1 cosx + c_2 sinx + \frac{\lambda}{2}$

It is given that,

$\int_0^{\pi} ydx = 1$

$\int_0^{\pi} c_1 cosx + c_2 sinx + \frac{\lambda}{2} . dx = 1$

$\left[c_1 sinx - c_2 cosx + \frac{\lambda}{2} x\right]_0^{\pi} = 1$

$0 - c_2 cos\pi + \frac{\lambda}{2}\pi - 0 + c_2 cos0 - 0 = 1$

$2c_2 = 1 - \frac{\lambda\pi}{2}$

$2c_2 = 1 - \frac{\lambda\pi}{2}$

$c_2 = \frac{1}{2} - \frac{\lambda\pi}{4}$

Thus, the solution becomes,

$y = c_1 cosx + \left(\frac{1}{2} - \frac{\lambda\pi}{4}\right) sinx + \frac{\lambda}{2}$

$y = c_2\ cosc_1\ sinx + c_2 sinc_1\ cosx + \frac{\lambda}{2}$

5. Show that the extremal of the isoperimetric problem $I[y(x)] = \int_{x_1}^{x_2} y'^2 dx$ subject to the condition $\int_{x_1}^{x_2} y dx = k$ is a parabola.

[N17/ElexExtcElectBiomInst/8M]

Solution:

We have,

$F = y'^2$ & $G = y$

$H = F + \lambda G = y'^2 + \lambda y$

By Euler's Differential Equation,

$\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) = 0$

$\lambda - \frac{d}{dx}(2y') = 0$

$\lambda - 2y'' = 0$

$y'' = \frac{\lambda}{2}$

Integrating, we get

$y' = \frac{\lambda}{2}.x + c_1$

Integrating, we get

$y = \frac{\lambda}{2}.\frac{x^2}{2} + c_1.x + c_2$

Type IV: Rayleigh-Ritz method

1. Using Rayleigh-Ritz method, find an appropriate solution for the extremal of the functional $I[y(x)] = \int_0^1 \left[xy + \frac{1}{2}y'^2\right] dx$ subject to $y(0) = y(1) = 0$

[M14/ElexExtcElectBiomInst/6M][M16/ElexExtcElectBiomInst/6M]

Solution:

Let the required solution be,

$y = a + bx + cx^2$

When $x = 0, y = 0$

$0 = a + 0 + 0$

$\therefore a = 0$

When $x = 1, y = 0$

$0 = a + b + c$

$b + c = 0$

$b = -c$

Thus, $y = -cx + cx^2$

$y = -c(x - x^2)$

$y' = -c(1 - 2x)$

Now,

$I = \int_0^1 \left[xy + \frac{1}{2}y'^2\right] dx$

$I = \int_0^1 \left[-cx(x - x^2) + \frac{1}{2}c^2(1 - 2x)^2\right] dx$

$I = \int_0^1 \left[-cx^2 + cx^3 + \frac{c^2}{2} - 2c^2x + 2c^2x^2\right] dx$

$I = \int_0^1 \left[cx^3 + 2c^2x^2 - cx^2 - 2c^2x + \frac{c^2}{2}\right] dx$

$I = \left[c\frac{x^4}{4} + 2c^2\frac{x^3}{3} - c\frac{x^3}{3} - 2c^2\frac{x^2}{2} + \frac{c^2}{2}x\right]_0^1$

$I = \frac{c}{4} + \frac{2c^2}{3} - \frac{c}{3} - c^2 + \frac{c^2}{2}$

$I = \frac{c^2}{6} - \frac{c}{12}$

For extremum,

$\frac{dI}{dc} = 0$

$\frac{2c}{6} - \frac{1}{12} = 0$

$c = \frac{1}{4}$

Thus, the solution is $y = -\frac{1}{4}(x - x^2)$

2. Solve the boundary value problem $I = \int_0^1 [2xy - y^2 - y'^2]dx$, given $y(0) = y(1) = 0$ by Rayleigh Ritz method
[N14/ElexExtcElectBiomInst/8M]
Solution:
Let the required solution be,
$y = a + bx + cx^2$
When $x = 0, y = 0$
$0 = a + 0 + 0$
$\therefore a = 0$
When $x = 1, y = 0$
$0 = a + b + c$
$b + c = 0$
$b = -c$
Thus, $y = -cx + cx^2$
$y = -c(x - x^2)$
$y' = -c(1 - 2x)$
Now,
$I = \int_0^1 [2xy - y^2 - y'^2]dx$
$I = \int_0^1 [-2cx(x - x^2) - c^2(x - x^2)^2 - c^2(1 - 2x)^2]dx$
$I = \int_0^1 [-2cx^2 + 2cx^3 - c^2(x^2 - 2x^3 + x^4) - c^2(1 - 4x + 4x^2)]\, dx$
$I = \int_0^1 [-2cx^2 + 2cx^3 - c^2x^2 + 2c^2x^3 - c^2x^4 - c^2 + 4c^2x - 4c^2x^2]dx$
$I = \int_0^1 [-c^2x^4 + (2c^2 + 2c)x^3 + (-2c - 5c^2)x^2 + 4c^2x - c^2]dx$
$I = \left[-c^2\frac{x^5}{5} + (2c^2 + 2c)\frac{x^4}{4} + (-2c - 5c^2)\frac{x^3}{3} + 4c^2\frac{x^2}{2} - c^2x\right]_0^1$
$I = -\frac{c^2}{5} + \frac{c^2}{2} + \frac{c}{2} - \frac{2c}{3} - \frac{5c^2}{3} + 2c^2 - c^2$
$I = -\frac{11}{30}c^2 - \frac{1}{6}c$
For extremum,
$\frac{dI}{dc} = 0$
$-\frac{22}{30}c - \frac{1}{6} = 0$
$c = -\frac{5}{22}$
Thus, the solution is $y = \frac{5}{22}(x - x^2)$

3. Using Rayleigh Ritz method, find an approximate solution for the extremal of the functional $I(y) = \int_0^1 (y'^2 - 2y - 2xy)\, dx$ subject to $y(0) = 2, y(1) = 1$
[M15/ElexExtcElectBiomInst/8M][N17/ElexExtcElectBiomInst/6M]
Solution:
Let the required solution be,
$y = a + bx + cx^2$
When $x = 0, y = 2$
$2 = a + 0 + 0$
$\therefore a = 2$
When $x = 1, y = 1$
$1 = a + b + c$
$b + c = 1 - 2$
$b = -c - 1$
Thus, $y = 2 - (c + 1)x + cx^2$
$y' = -(c + 1) + 2cx$
Now,
$$I = \int_0^1 [y'^2 - 2y - 2xy]dx$$
$$I = \int_0^1 [\{-(c + 1) + 2cx\}^2 - 2\{2 - (c + 1)x + cx^2\} - 2x\{2 - (c + 1)x + cx^2\}]dx$$
$$I = \int_0^1 [(c + 1)^2 - 4c(c + 1)x + 4c^2x^2 - 4 + 2(c + 1)x - 2cx^2 - 4x + 2(c + 1)x^2 - 2cx^3]\, dx$$
$$I = \left[(c + 1)^2x - 4c(c + 1)\frac{x^2}{2} + 4c^2\frac{x^3}{3} - 4x + 2(c + 1)\frac{x^2}{2} - 2c\frac{x^3}{3} - 4\frac{x^2}{2} + 2(c + 1)\frac{x^3}{3} - 2c\frac{x^4}{4}\right]_0^1$$
$$I = (c + 1)^2 - 2c(c + 1) + \frac{4c^2}{3} + (c + 1) - \frac{2c}{3} - 2 + \frac{2(c+1)}{3} - \frac{c}{2}$$
$$I = c^2 + 2c + 1 - 2c^2 - 2c + \frac{4c^2}{3} + c + 1 - \frac{2c}{3} - 2 + \frac{2c}{3} + \frac{2}{3} - \frac{c}{2}$$
$$I = \frac{c^2}{3} + \frac{c}{2} + \frac{2}{3}$$
For extremum,
$$\frac{dI}{dc} = 0$$
$$\frac{2c}{3} + \frac{1}{2} = 0, \quad \therefore c = -\frac{3}{4}$$
Thus, the solution is $y = 2 - \left(-\frac{3}{4} + 1\right)x - \frac{3}{4}x^2$
$$y = 2 - \frac{x}{4} - \frac{3x^2}{4}$$

4. Solve the boundary value problem $I = \int_0^1 [2xy + y^2 - y'^2]dx$, given $y(0) = y(1) = 0$ by Rayleigh Ritz method
[N15/ElexExtcElectBiomInst/6M]
Solution:
Let the required solution be,
$y = a + bx + cx^2$
When $x = 0, y = 0$
$0 = a + 0 + 0$
$\therefore a = 0$
When $x = 1, y = 0$
$0 = a + b + c$
$b + c = 0$
$b = -c$
Thus, $y = -cx + cx^2$
$y = -c(x - x^2)$
$y' = -c(1 - 2x)$
Now,
$I = \int_0^1 [2xy + y^2 - y'^2]dx$
$I = \int_0^1 [-2cx(x - x^2) + c^2(x - x^2)^2 - c^2(1 - 2x)^2]dx$
$I = \int_0^1 [-2cx^2 + 2cx^3 + c^2(x^2 - 2x^3 + x^4) - c^2(1 - 4x + 4x^2)]\, dx$
$I = \int_0^1 [-2cx^2 + 2cx^3 + c^2x^2 - 2c^2x^3 + c^2x^4 - c^2 + 4c^2x - 4c^2x^2]dx$
$I = \int_0^1 [c^2x^4 + (-2c^2 + 2c)x^3 + (-2c - 3c^2)x^2 + 4c^2x - c^2]dx$
$I = \left[c^2\frac{x^5}{5} + (-2c^2 + 2c)\frac{x^4}{4} + (-2c - 3c^2)\frac{x^3}{3} + 4c^2\frac{x^2}{2} - c^2x\right]_0^1$
$I = \frac{c^2}{5} - \frac{c^2}{2} + \frac{c}{2} - \frac{2c}{3} - \frac{3c^2}{3} + 2c^2 - c^2$
$I = -\frac{3}{10}c^2 - \frac{1c}{6}$
For extremum,
$\frac{dI}{dc} = 0$
$-\frac{6}{10}c - \frac{1}{6} = 0$
$c = -\frac{5}{18}$
Thus, the solution is $y = \frac{5}{18}(x - x^2)$

5. Solve the boundary value problem $I = \int_0^1 [y'^2 - y^2 - 2xy]dx$, given $y(0) = y(1) = 0$ by Rayleigh Ritz method

[N16/ElexExtcElectBiomInst/8M]

Solution:

Let the required solution be,

$y = a + bx + cx^2$

When $x = 0, y = 0$

$0 = a + 0 + 0$

$\therefore a = 0$

When $x = 1, y = 0$

$0 = a + b + c$

$b + c = 0$

$b = -c$

Thus, $y = -cx + cx^2$

$y = -c(x - x^2)$

$y' = -c(1 - 2x)$

Now,

$I = \int_0^1 [y'^2 - y^2 - 2xy]dx$

$I = -\int_0^1 [2xy + y^2 - y'^2]dx$

$I = -\int_0^1 [-2cx(x - x^2) + c^2(x - x^2)^2 - c^2(1 - 2x)^2]dx$

$I = -\int_0^1 [-2cx^2 + 2cx^3 + c^2(x^2 - 2x^3 + x^4) - c^2(1 - 4x + 4x^2)]\,dx$

$I = -\int_0^1 [-2cx^2 + 2cx^3 + c^2x^2 - 2c^2x^3 + c^2x^4 - c^2 + 4c^2x - 4c^2x^2]dx$

$I = -\int_0^1 [c^2x^4 + (-2c^2 + 2c)x^3 + (-2c - 3c^2)x^2 + 4c^2x - c^2]dx$

$I = -\left[c^2\frac{x^5}{5} + (-2c^2 + 2c)\frac{x^4}{4} + (-2c - 3c^2)\frac{x^3}{3} + 4c^2\frac{x^2}{2} - c^2x\right]_0^1$

$I = -\frac{c^2}{5} + \frac{c^2}{2} - \frac{c}{2} + \frac{2c}{3} + \frac{3c^2}{3} - 2c^2 + c^2$

$I = \frac{3}{10}c^2 + \frac{1c}{6}$

For extremum,

$\frac{dI}{dc} = 0$

$\frac{6}{10}c + \frac{1}{6} = 0$

$c = -\frac{5}{18}$

Thus, the solution is $y = \frac{5}{18}(x - x^2)$