Vectors in R^n

Type I: Vector Arithmetic

No question from this part has been asked yet

Type II: Inner Product Spaces

No question from this part has been asked yet

Type III: Orthogonality & Orthonormality

1. Find a vector orthogonal to both $u = (-6,4,2), v = (3,1,5)$
[M14/ElexExtcElectBiomInst/5M][M15/ElexExtcElectBiomInst/5M]
Solution:
We have,
$u = (-6,4,2), v = (3,1,5)$
Let $w = (x, y, z)$ be orthogonal to both u and v
If u is orthogonal to w then,
$u \cdot w = 0$
$-6x + 4y + 2z = 0$ …………(1)
Similarly, v is orthogonal to w then,
$v \cdot w = 0$
$3x + y + 5z = 0$ ………………(2)
Solving eqn (1) and (2) by Crammer's rule, we get
$$\frac{x}{\begin{vmatrix} 4 & 2 \\ 1 & 5 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -6 & 2 \\ 3 & 5 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -6 & 4 \\ 3 & 1 \end{vmatrix}}$$
$$\frac{x}{18} = \frac{-y}{-36} = \frac{z}{-18}$$
$$\frac{x}{1} = \frac{y}{2} = \frac{z}{-1}$$
$\therefore w = (1,2,-1)$

2. State Cauchy-Schwartz inequality and hence show that $(x^2 + y^2 + z^2)^{\frac{1}{2}} \geq \frac{1}{13}(3x + 4y + 12z), x, y, z$ are positive

[M14/ElexExtcElectBiomInst/6M]

Solution:

Cauchy-Schwartz inequality:

If u and v are any vectors in R^n, then

$|u \cdot v| \leq \|u\|\|v\|$

Let $u = (x, y, z), v = (3,4,12)$

$|u \cdot v| = |3x + 4y + 12z|$

$\|u\| = \sqrt{x^2 + y^2 + z^2}$

$\|v\| = \sqrt{3^2 + 4^2 + 12^2} = 13$

By Cauchy-Schwartz inequality,

$|u \cdot v| \leq \|u\|\|v\|$

$|3x + 4y + 12z| \leq 13\sqrt{x^2 + y^2 + z^2}$

$(x^2 + y^2 + z^2)^{\frac{1}{2}} \geq \frac{1}{13}(3x + 4y + 12z)$

3. State and prove Cauchy-Schwartz inequality for the vectors and verify it
$u = (-4,2,1), v = (8,-4,-2)$
[N14/ElexExtcElectBiomInst/5M][N15/ElexExtcElectBiomInst/6M]
Solution:
Cauchy-Schwartz inequality:
If u and v are any vectors in R^n, then
$|u \cdot v| \leq \|u\|\|v\|$
We have,
$u \cdot v = \|u\|\|v\|cos\theta$
$|u \cdot v| = \|u\|\|v\||cos\theta|$
But $|cos\theta| \leq 1$
$\therefore |u \cdot v| \leq \|u\|\|v\|$
Now, $u = (-4,2,1), v = (8,-4,-2)$
$u \cdot v = (-4)(8) + (2)(-4) + (1)(-2) = -42$
$\therefore |u \cdot v| = 42$
$\|u\| = \sqrt{(-4)^2 + 2^2 + 1^2} = \sqrt{21}$
$\|v\| = \sqrt{8^2 + (-4)^2 + (-2)^2} = 2\sqrt{21}$
$\therefore \|u\|\|v\| = \sqrt{21} \times 2\sqrt{21} = 42$
Thus, $|u \cdot v| = \|u\|\|v\|$
Hence, Cauchy-Schwartz inequality is verified

4. Find a unit vector in orthogonal to both $(1,1,0), v = (0,1,1)$
[M15/ ElexExtcElectBiomInst/5M]
Solution:
We have,
$u = (1,1,0), v = (0,1,1)$
Let $w = (x, y, z)$ be orthogonal to both u and v
If u is orthogonal to w then,
$u \cdot w = 0$
$x + y + 0z = 0$ …………(1)
Similarly, v is orthogonal to w then,
$v \cdot w = 0$
$0x + y + z = 0$ ………………(2)
Solving eqn (1) and (2) by Crammer's rule, we get

$$\frac{x}{\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}}$$

$$\frac{x}{1} = \frac{-y}{1} = \frac{z}{1}$$

$$\frac{x}{1} = \frac{y}{-1} = \frac{z}{1}$$

$\therefore w = (1, -1, 1)$

Unit vector $= \pm \frac{w}{\|w\|} = \pm \frac{(1,-1,1)}{\sqrt{1^2 + (-1)^2 + 1^2}} = \pm \frac{(1,-1,1)}{\sqrt{3}}$

5. Verify Cauchy Schwartz inequality for $u = (1,2,1)$ and $v = (3,0,4)$. Also find the angle between u & v

[M16/ElexExtcElectBiomInst/5M]

Solution:

Now, $u = (1,2,1), v = (3,0,4)$

$u \cdot v = (1)(3) + (2)(0) + (1)(4) = 7$

$\therefore |u \cdot v| = 7$ ……..(1)

$\|u\| = \sqrt{1^2 + 2^2 + 1^2} = \sqrt{6}$

$\|v\| = \sqrt{3^2 + 0^2 + 4^2} = 5$

$\therefore \|u\|\|v\| = \sqrt{6} \times 5 = 5\sqrt{6} = 12.25$ ……..(2)

Thus,

From (1) & (2), $|u \cdot v| < \|u\|\|v\|$

Hence, Cauchy-Schwartz inequality is verified

Now,

$cos\theta = \frac{u \cdot v}{\|u\|\|v\|}$

$cos\theta = \frac{7}{5\sqrt{6}}$

$\theta = \cos^{-1}\left(\frac{7}{5\sqrt{6}}\right)$

6. Find a unit vector in R^3 orthogonal to both $u = (1,0,1), v = (0,1,1)$
[N16/ElexExtcElectBiomInst/4M]
Solution:
We have,
$u = (1,0,1), v = (0,1,1)$
Let $w = (x, y, z)$ be orthogonal to both u and v
If u is orthogonal to w then,
$u \cdot w = 0$
$x + 0y + z = 0$ …………(1)
Similarly, v is orthogonal to w then,
$v \cdot w = 0$
$0x + y + z = 0$ ………………(2)
Solving eqn (1) and (2) by Crammer's rule, we get
$$\frac{x}{\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}}$$
$$\frac{x}{-1} = \frac{-y}{1} = \frac{z}{1}$$
$$\frac{x}{-1} = \frac{y}{-1} = \frac{z}{1}$$
$\therefore w = (1,1,-1)$
Unit vector $= \pm \frac{w}{\|w\|} = \pm \frac{(1,1,-1)}{\sqrt{1^2+1^2+(-1)^2}} = \pm \frac{(1,1,-1)}{\sqrt{3}}$

7. State and prove Cauchy-Schwartz inequality. Hence show that for real values of a, b and θ, $(acos\theta + bsin\theta)^2 \leq a^2 + b^2$
[N17/ElexExtcElectBiomInst/6M]
Solution:
Solution:
Cauchy-Schwartz inequality:
If u and v are any vectors in R^n, then
$|u \cdot v| \leq \|u\|\|v\|$
We have,
$u \cdot v = \|u\|\|v\|cos\theta$
$|u \cdot v| = \|u\|\|v\||cos\theta|$
But $|cos\theta| \leq 1$
$\therefore |u \cdot v| \leq \|u\|\|v\|$
Let $u = (a, b), v = (cos\theta, sin\theta)$
$|u \cdot v| = |acos\theta + bsin\theta|$
$\|u\| = \sqrt{a^2 + b^2}$
$\|v\| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$
By Cauchy-Schwartz inequality,
$|u \cdot v| \leq \|u\|\|v\|$
$|acos\theta + bsin\theta| \leq 1\sqrt{a^2 + b^2}$
$(acos\theta + bsin\theta)^2 \leq a^2 + b^2$

8. Find a unit vector orthogonal to both $u = (-3,2,1)$ and $v = (3,1,5)$
[N17 ElexExtcElectBiomInst/5M]
Solution:
We have,
$u = (-3,2,1), v = (3,1,5)$
Let $w = (x, y, z)$ be orthogonal to both u and v
If u is orthogonal to w then,
$u \cdot w = 0$
$-3x + 2y + z = 0$ …………(1)
Similarly, v is orthogonal to w then,
$v \cdot w = 0$
$3x + y + 5z = 0$ ………………(2)
Solving eqn (1) and (2) by Crammer's rule, we get

$$\frac{x}{\begin{vmatrix} 2 & 1 \\ 1 & 5 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -3 & 1 \\ 3 & 5 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -3 & 2 \\ 3 & 1 \end{vmatrix}}$$

$$\frac{x}{9} = \frac{-y}{-18} = \frac{z}{-9}$$

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{-1}$$

$\therefore w = (1,2,-1)$

Unit vector $= \pm \frac{w}{\|w\|} = \pm \frac{(1,2,-1)}{\sqrt{1^2+(2)^2+(-1)^2}} = \pm \frac{(1,2,-1)}{\sqrt{6}}$

Type IV: Gram-Schmidt process

1. Find an orthonormal basis of the following subspace of R^3,
   $S = \{[1,2,0],[0,3,1]\}$
   [M14/ElexExtcElectBiomInst/6M][N15/ElexExtcElectBiomInst/5M]
   Solution:
   Let $u_1 = (1,2,0)$ and $u_2 = (0,3,1)$
   By Gram Schmidt orthogonalization process, we get

$$v_1 = u_1 = (1,2,0)$$

$$v_2 = u_2 - \frac{\langle u_2, v_1\rangle}{\|v_1\|^2}\, v_1$$

$$= (0,3,1) - \frac{(0,3,1)\cdot(1,2,0)}{1^2+2^2+0}\,(1,2,0)$$

$$= (0,3,1) - \frac{6}{5}(1,2,0)$$

$$= (0,3,1) - \left(\frac{6}{5}, \frac{12}{5}, 0\right)$$

$$v_2 = \left(-\frac{6}{5}, \frac{3}{5}, 1\right)$$

Norms of these vectors are,

$$\|v_1\| = \sqrt{5}$$

$$\|v_2\| = \sqrt{\frac{36}{25} + \frac{9}{25} + 1} = \sqrt{\frac{14}{5}}$$

Hence, the orthonomal basis are

$$q_1 = \frac{v_1}{\|v_1\|} = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0\right)$$

$$q_2 = \frac{v_2}{\|v_2\|} = \sqrt{\frac{5}{14}}\left(-\frac{6}{5}, \frac{3}{5}, 1\right) = \left(-\frac{6}{\sqrt{70}}, \frac{3}{\sqrt{70}}, \frac{5}{\sqrt{70}}\right)$$

2. Construct an orthonormal basis of R^2 by applying Gram Schmidt orthogonalization to $S = \{[3,1],[2,2]\}$
[N14/ElexExtcElectBiomInst/6M]
Solution:
Let $u_1 = (3,1)$ and $u_2 = (2,2)$
By Gram Schmidt orthogonalization process, we get

$$v_1 = u_1 = (3,1)$$
$$v_2 = u_2 - \frac{\langle u_2, v_1\rangle}{\|v_1\|^2}\, v_1$$
$$= (2,2) - \frac{(2,2)\cdot(3,1)}{3^2+1^2}\,(3,1)$$
$$= (2,2) - \frac{8}{10}(3,1)$$
$$= (2,2) - \left(\frac{12}{5}, \frac{4}{5}\right)$$
$$v_2 = \left(-\frac{2}{5}, \frac{6}{5}\right)$$

Norms of these vectors are,

$$\|v_1\| = \sqrt{10}$$
$$\|v_2\| = \sqrt{\frac{4}{25} + \frac{36}{25}} = \sqrt{\frac{8}{5}} = 2\sqrt{\frac{2}{5}} = \frac{2\sqrt{10}}{5}$$

Hence, the orthonomal basis are

$$q_1 = \frac{v_1}{\|v_1\|} = \left(\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right)$$
$$q_2 = \frac{v_2}{\|v_2\|} = \frac{5}{2\sqrt{10}}\left(-\frac{2}{5}, \frac{6}{5}\right) = \left(-\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right)$$

3. Let R^3 have the Euclidean inner product. Use Gram-Schmidt process to transform the basis $\{u_1, u_2, u_3\}$ into orthonormal bases where $u_1 = (1,1,1)$, $u_2 = (0,1,1)$, $u_3 = (0,0,1)$
[M16/ElexExtcElectBiomInst/6M]
Solution:
Let $u_1 = (1,1,1)$ and $u_2 = (0,1,1)$ and $u_3 = (0,0,1)$
By Gram Schmidt orthogonalization process, we get

$$v_1 = u_1 = (1,1,1)$$

$$v_2 = u_2 - \frac{\langle u_2, v_1\rangle}{\|v_1\|^2} v_1$$

$$= (0,1,1) - \frac{(0,1,1)\cdot(1,1,1)}{1^2+1^2+1^2}(1,1,1)$$

$$= (0,1,1) - \frac{2}{3}(1,1,1)$$

$$= (0,1,1) - \left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right)$$

$$v_2 = \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

$$v_3 = u_3 - \frac{\langle u_3, v_1\rangle}{\|v_1\|^2} v_1 - \frac{\langle u_3, v_2\rangle}{\|v_2\|^2} v_2$$

$$= (0,0,1) - \frac{(0,0,1)\cdot(1,1,1)}{1^2+1^2+1^2}(1,1,1) - \frac{(0,0,1)\cdot\left(-\frac{2}{3},\frac{1}{3},\frac{1}{3}\right)}{\frac{4}{9}+\frac{1}{9}+\frac{1}{9}}\left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

$$= (0,0,1) - \frac{1}{3}(1,1,1) - \frac{\frac{1}{3}}{\frac{2}{3}}\left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

$$= (0,0,1) - \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) - \left(-\frac{1}{3}, \frac{1}{6}, \frac{1}{6}\right)$$

$$= \left(0, -\frac{1}{2}, \frac{1}{2}\right)$$

Norms of these vectors are,

$$\|v_1\| = \sqrt{3}$$

$$\|v_2\| = \sqrt{\frac{4}{9} + \frac{1}{9} + \frac{1}{9}} = \sqrt{\frac{2}{3}} = \frac{\sqrt{6}}{3}$$

$$\|v_3\| = \sqrt{0 + \frac{1}{4} + \frac{1}{4}} = \sqrt{\frac{1}{2}}$$

Hence, the orthonomal basis are

$$q_1 = \frac{v_1}{\|v_1\|} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$q_2 = \frac{v_2}{\|v_2\|} = \frac{3}{\sqrt{6}}\left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right) = \left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$$

$$q_3 = \frac{v_3}{\|v_3\|} = \sqrt{2}\left(0, -\frac{1}{2}, \frac{1}{2}\right) = \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

4. Construct an orthonormal basis of R^3 using Gram Schmidt process to $S = \{(3,0,4), (-1,0,7), (2,9,11)\}$
[N16/ElexExtcElectBiomInst/6M][M17/ElexExtcElectBiomInst/6M]
Solution:
Let $u_1 = (3,0,4)$ and $u_2 = (-1,0,7)$ and $u_3 = (2,9,11)$
By Gram Schmidt orthogonalization process, we get

$$v_1 = u_1 = (3,0,4)$$
$$v_2 = u_2 - \frac{\langle u_2, v_1\rangle}{\|v_1\|^2} v_1$$
$$= (-1,0,7) - \frac{(-1,0,7)\cdot(3,0,4)}{9+0+16} (3,0,4)$$
$$= (-1,0,7) - \frac{25}{25}(3,0,4)$$
$$= (-1,0,7) - (3,0,4)$$
$$v_2 = (-4,0,3)$$
$$v_3 = u_3 - \frac{\langle u_3, v_1\rangle}{\|v_1\|^2} v_1 - \frac{\langle u_3, v_2\rangle}{\|v_2\|^2} v_2$$
$$= (2,9,11) - \frac{(2,9,11)\cdot(3,0,4)}{9+0+16}(3,0,4) - \frac{(2,9,11)\cdot(-4,0,3)}{16+0+9}(-4,0,3)$$
$$= (2,9,11) - \frac{50}{25}(3,0,4) - \frac{25}{25}(-4,0,3)$$
$$= (2,9,11) - 2(3,0,4) - (-4,0,3)$$
$$= (0,9,0)$$

Norms of these vectors are,

$$\|v_1\| = \sqrt{25} = 5$$
$$\|v_2\| = \sqrt{16+0+9} = \sqrt{25} = 5$$
$$\|v_3\| = \sqrt{0+9+0} = \sqrt{9} = 3$$

Hence, the orthonomal basis are

$$q_1 = \frac{v_1}{\|v_1\|} = \left(\frac{3}{5}, 0, \frac{4}{5}\right)$$
$$q_2 = \frac{v_2}{\|v_2\|} = \left(-\frac{4}{5}, 0, \frac{3}{5}\right)$$
$$q_3 = \frac{v_3}{\|v_3\|} = (0,3,0)$$

5. Let R^3 have the Euclidean inner product. Use Gram-Schmidt process to transform the basis $\{u_1, u_2, u_3\}$ into orthonormal bases where $u_1 = (1,1,1)$, $u_2 = (-1,1,0)$, $u_3 = (1,2,1)$

[N17/ElexExtcElectBiomInst/6M]

Solution:

Let $u_1 = (1,1,1)$ and $u_2 = (-1,1,0)$ and $u_3 = (1,2,1)$

By Gram Schmidt orthogonalization process, we get

$$v_1 = u_1 = (1,1,1)$$

$$v_2 = u_2 - \frac{\langle u_2, v_1\rangle}{\|v_1\|^2} v_1$$

$$= (-1,1,0) - \frac{(-1,1,0)\cdot(1,1,1)}{1^2+1^2+1^2}(1,1,1)$$

$$= (-1,1,0) - (0)(1,1,1)$$

$$v_2 = (-1,1,0)$$

$$v_3 = u_3 - \frac{\langle u_3, v_1\rangle}{\|v_1\|^2} v_1 - \frac{\langle u_3, v_2\rangle}{\|v_2\|^2} v_2$$

$$= (1,2,1) - \frac{(1,2,1)\cdot(1,1,1)}{1^2+1^2+1^2}(1,1,1) - \frac{(1,2,1)\cdot(-1,1,0)}{(-1)^2+1^2+0^2}(-1,1,0)$$

$$= (1,2,1) - \frac{4}{3}(1,1,1) - \frac{1}{2}(-1,1,0)$$

$$= (1,2,1) - \left(\frac{4}{3},\frac{4}{3},\frac{4}{3}\right) - \left(-\frac{1}{2},\frac{1}{2},0\right)$$

$$v_3 = \left(\frac{1}{6},\frac{1}{6},-\frac{1}{3}\right)$$

Norms of these vectors are,

$$\|v_1\| = \sqrt{3}$$

$$\|v_2\| = \sqrt{1+1+0} = \sqrt{2}$$

$$\|v_3\| = \sqrt{\frac{1}{36}+\frac{1}{36}+\frac{1}{9}} = \sqrt{\frac{1}{6}}$$

Hence, the orthonomal basis are

$$q_1 = \frac{v_1}{\|v_1\|} = \left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$$

$$q_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{2}}(-1,1,0) = \left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right)$$

$$q_3 = \frac{v_3}{\|v_3\|} = \sqrt{6}\left(\frac{1}{6},\frac{1}{6},-\frac{1}{3}\right) = \left(\frac{1}{\sqrt{6}},\frac{1}{\sqrt{6}},-\frac{2}{\sqrt{6}}\right)$$