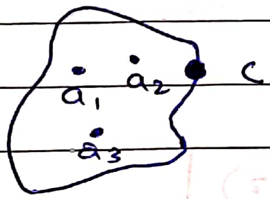
# Residue Theorem

## Residue Theorem:

If $f(z)$ is analytic in a closed curve $C$ except at finite number of singularity points within $C$, then

$$\int_C f(z)\,dz = 2\pi i \times (\text{sum of the residues at the singular points within } C)$$




$$= 2\pi i \times [\text{Res } f(a_1) + \text{Res } f(a_2) + \dots + \text{Res } f(a_n)]$$

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \dots + \int_{C_n} f(z)\,dz$$

## Formulae:

1. When poles are simple. i.e. $(z-a)$

$$\text{Res } f(a) = \lim_{z \to a} [(z-a) \cdot f(z)]$$

2. When poles are of $n^{th}$ order. i.e. $(z-a)^n$

$$\text{Res } f(a) = \frac{1}{(n-1)!} \left[ \frac{d^{n-1}}{dz^{n-1}} (z-a)^n \cdot f(z) \right]_{z=a}$$

Q. Find the residue of $f(z) = \dfrac{z^3}{(z-1)^4 (z-2)(z-3)}$

Soln:

The poles are $(z-1)^4 (z-2)(z-3) = 0$

$\therefore z = 1$ is a pole of order 4;

$z = 2$ and $z = 3$ are simple poles

The residue of $n^{th}$ order is

$$\text{Res } f(a) = \frac{1}{(n-1)!} \left[ \frac{d^{n-1}}{dz^{n-1}} (z-a)^n . f(z) \right]_{z=a}$$

$$\therefore \text{Res } f(1) = \frac{1}{3!} \left[ \frac{d^3}{dz^3} \cancel{(z-1)^4} \frac{z^3}{\cancel{(z-1)^4} (z-2)(z-3)} \right]_{z=1}$$

$$= \frac{1}{3!} \left[ \frac{d^3}{dz^3} \frac{z^3}{(z-2)(z-3)} \right]_{z=1}$$

$$= \frac{1}{3!} \left[ \frac{d^3}{dz^3} \frac{z^3}{z^2 - 5z + 6} \right]_{z=1}$$

$z^2 - 5z + 6 \,\big)\, \cancel{z^3} \,\big(\, z + 5$

$\cancel{z^3} - 5z^2 + 6z$

$(-) \quad (+) \quad (-)$

$5\cancel{z^2} - 6z$

$5\cancel{z^2} - 25z + 30$

$(-) \quad (+) \quad (-)$

$19z - 30$

$$\text{Res } f(1) = \frac{1}{3!}\left[\frac{d^3}{dz^3}\left\{(z+5) + \frac{19z - 30}{(z-2)(z-3)}\right\}\right]_{z=1}$$

(By Partial Fraction)

$$= \frac{1}{3!}\left[\frac{d^3}{dz^3}\left\{(z+5) \quad \frac{-8}{z-2} + \frac{27}{z-3}\right\}\right]_{z=1}$$

Formula:

$$\frac{d^n}{dz^n}\left(\frac{1}{n-1}\right) = \frac{(-1)^n \cdot n!}{(z-a)^{n+1}}$$

$$\text{Res } f(1) = \frac{1}{6}\left[\frac{-8(-1)^3 \cdot 3!}{(z-2)^4} + \frac{27(-1)^3 \cdot 3!}{(z-3)^4}\right]_{z=1}$$

$$\text{Res } f(1) = \frac{1}{6}\left[\frac{48}{(z-2)^4} - \frac{162}{(z-3)^4}\right]_{z=1}$$

$$= \frac{1}{6}\left[\frac{48}{(-1)^4} - \frac{162}{(-2)^4}\right]$$

$$= \frac{101}{16}$$

Residue of f(a) when $z = a$ is a simple pole

$$\text{Res } f(a) = \lim_{z \to a} \left[ (z-a) \cdot f(z) \right]$$

$$\text{Res } f(2) = \lim_{z \to 2} \left[ \cancel{(z-2)} \cdot \frac{z^3}{(z-1)^4 \cancel{(z-2)} (z-3)} \right]$$

$$= \frac{(2)^3}{(2-1)^4 (2-3)}$$

$$= -8$$

$$\text{Res } f(3) = \lim_{z \to 3} \left[ \cancel{(z-3)} \cdot \frac{z^3}{(z-1)^4 (z-2) \cancel{(z-3)}} \right]$$

$$= \frac{(3)^3}{(3-1)^4 (3-2)}$$

$$= \frac{27}{2^4}$$

$$= 1.6875$$

Q. Evaluate $\int_c \frac{z-3}{z^2+2z+5} dz$ where c is the circle.

(i) $|z| = 1$

(ii) $|z+1-i| = 2$

(iii) $|z+1+i| = 2$

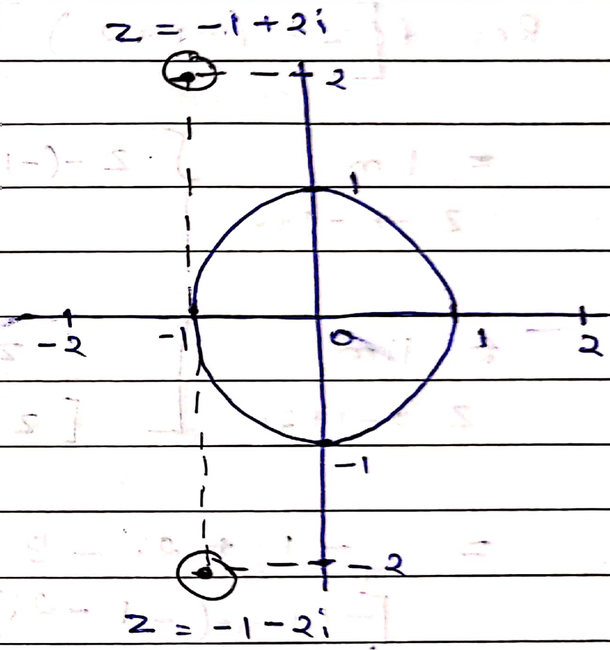
Sol$^n$:

The poles are $z^2+2z+5 = 0$

$z = -1+2i$ and $z = -1-2i$

(i) $|z| = 1$ Centre $= (0,0)$

Radius $= 1$




$\because$ Both the points are outside of the circle.

So by Cauchy's theorem,

$$\int_c f(z)\, dz = 0$$

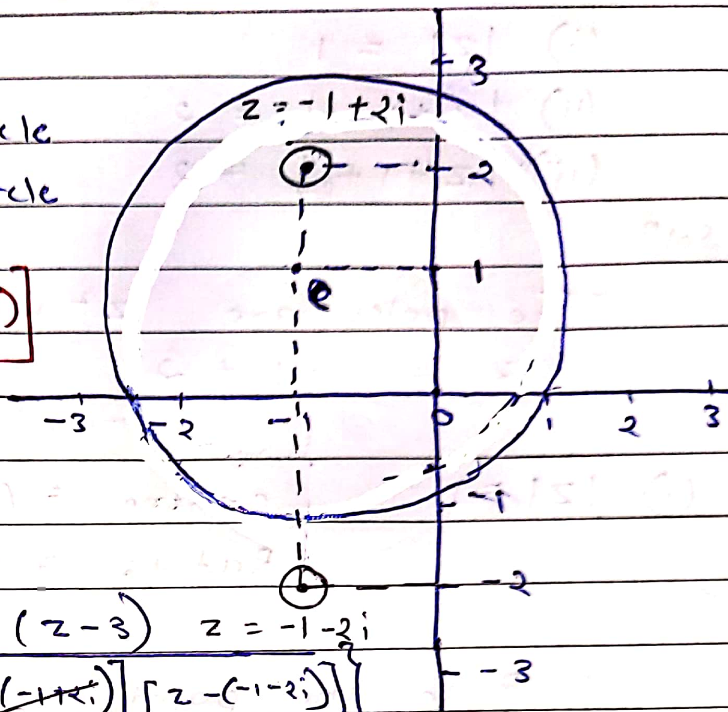
$$\Rightarrow \int_c \frac{z-3}{z^2+2z+5} dz = 0$$

(ii) $|z+1-i| = |z-(-1+i)| = 2$

Here,

$z = -1+2i$ is within the circle

$z = -1-2i$ is outside the circle




$\therefore$ Res $f(z-a) = \lim\limits_{z\to a} \left[(z-a).f(z)\right]$

Res $f\left[z-(-1+2i)\right]$

$$= \lim_{z\to -1+2i} \left\{ \frac{\cancel{z-(-1+2i)} \cdot (z-3)}{\left[\cancel{z-(-1+2i)}\right]\left[z-(-1-2i)\right]} \right\} \quad z = -1-2i$$

$$= \lim_{z\to -1+2i} \left[ \frac{z-3}{[z-(-1-2i)]} \right]$$

$$= \frac{-1+2i-3}{[-1+2i-(-1-2i)]} = \frac{-1+2i-3}{\cancel{-1}+2i+\cancel{1}+2i}$$

$$= \frac{-4+2i}{4i} = \frac{-2+i}{2i}$$

From residue theorem,

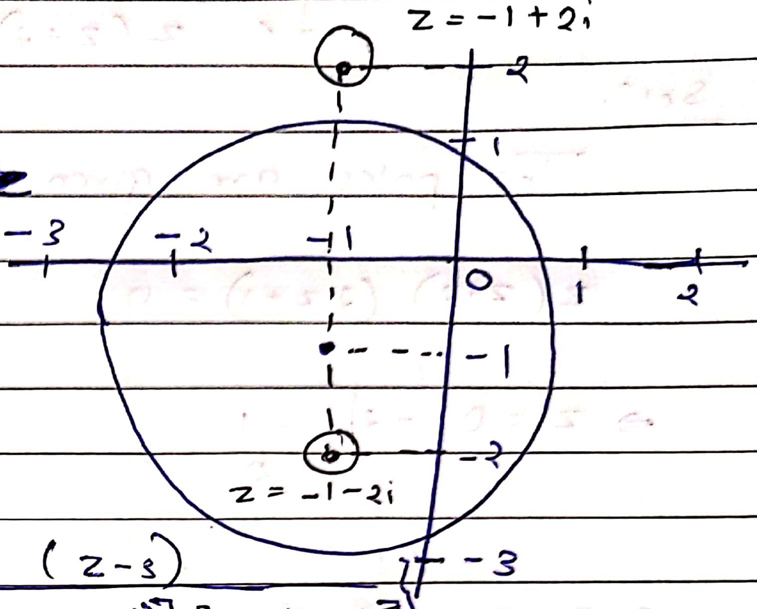
$$\int_c f(z)\,dz = 2\pi i \left[\text{Res } f(a_1) + \text{Res } f(a_2) + \dots\right]$$

$$= \int_c \frac{z-3}{z^2-2z+5}\,dz = 2\pi i \times \left\{\text{Res } f\left[z-(-1+2i)\right]\right\}$$

$$= \cancel{2}\pi \cancel{i} \times \left(\frac{-2+i}{\cancel{2i}}\right)$$

$$= \pi\,(i-2)$$

(iii) $|z+1+i| = |z-(-1-i)| = 2$




Here,

$z = -1+2i$ is outside of circle

$z = -1-2i$ is within the circle

$$\text{Res } f\left[z-(-1-2i)\right]$$

$$= \lim_{z \to (-1-2i)} \left\{ \left[\cancel{z-(-1-2i)}\right] \cdot \frac{(z-3)}{[z-(-1+2i)]\,[\cancel{z-(-1-2i)}]} \right\}$$

$$= \lim_{z \to (-1-2i)} \left[ \frac{z-3}{z-(-1+2i)} \right]$$

$$= \frac{-1-2i-3}{\cancel{-1}-2i\cancel{+1}-2i} = \frac{-4-2i}{-4i}$$

$$= \frac{2+i}{2i}$$

By residue theorem,

$$\int_C \frac{z-3}{z^2+2z+5}\,dz = 2\pi i \times \left\{ \text{Res}\left[f(z-(-1-2i))\right] \right\}$$

$$= \cancel{2}\pi \cancel{i} \times \frac{2+i}{\cancel{2i}}$$

$$= \pi(2+i)$$

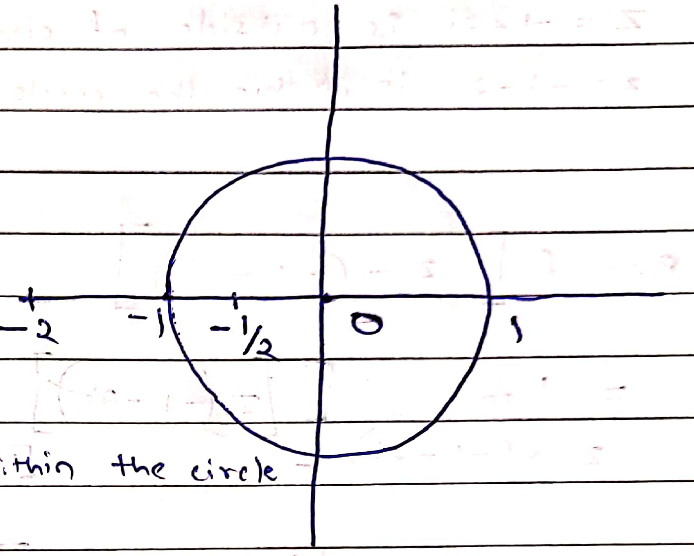
Q. Evaluate $\int_c \frac{(2z-1)}{z(z+2)(2z+1)}\, dz$ ; where $c$ is $|z| = 1$

Soln:

The poles are given by

$z(z+2)(2z+1) = 0$

$\Rightarrow z = 0, -2, -\frac{1}{2}$




$\because z = 0$ and $z = -\frac{1}{2}$ are within the circle

$z = -2$ is outside the circle

$\therefore \text{Res } f(0) = \lim_{z \to 0} \left[ \cancel{(z-0)} \frac{(2z-1)}{\cancel{z}(z+2)(2z+1)} \right]$

$= \frac{-1}{2 \times 1} = \frac{-1}{2}$

$\therefore \text{Res } f\left(\frac{-1}{2}\right) = \lim_{z \to -1/2} \left[ [z - (-1/2)] \frac{(2z-1)}{z(z+2)(2z+1)} \right]$

$= \lim_{z \to -1/2} \left[ \frac{\cancel{2z+1}}{2} \cdot \frac{(2z-1)}{z(z+2)\cancel{(2z+1)}} \right]$

$= \frac{\left(\frac{-1}{\cancel{2}} \times \cancel{2}\right) - 1}{\cancel{2} \times \left(\frac{-1}{\cancel{2}}\right)\left(\frac{-1}{2} + 2\right)} = \frac{-2}{-\left(\frac{3}{2}\right)}$

$= \frac{4}{3}$

By residue theorem,

$$\int_C f(z)\,dz = 2\pi i \left[ \operatorname{Res} f(0) + \operatorname{Res} f\left(\frac{-1}{2}\right) \right]$$

$$\int_C \frac{2z-1}{z(z+2)(2z+1)}\,dz = 2\pi i \times \left[ \frac{-1}{2} + \frac{4}{3} \right]$$

$$= 2\pi i \left[ \frac{-3+8}{6} \right]$$

$$= 2\pi i \left( \frac{5}{6} \right)$$

$$= \frac{5\pi i}{3}$$